

Essentially Normal Composition Operators

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Abstract

In this dissertation we prove a simple criterion for essential normality of composition operators C_φ induced by maps φ in a large class of functions. Additionally, we construct essentially normal composition operators which have arbitrary even order of contact with the unit circle at one point. To do so, we rely on results from three distinct areas. We use results and techniques of Agler-Lykova-Young to construct rational analytic self-maps of the unit disk with specified Taylor coefficients at a given boundary point. This allows us to decompose a composition operator modulo the ideal of compact operators into a sum of rationally induced composition operators based on results of Kriete-Moorhouse. The adjoint of each rationally induced composition operator is then studied using results and ideas of Bourdon-Shapiro. Essential normality is then characterized, beginning with a single summand in the decomposition and continuing to the more general case. Finally, we construct essentially normal composition operators which have arbitrary even order of contact with the unit circle at one point.

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Chapter 1

Introduction

For φ an analytic self-map of the unit disk \mathbb{D} , the composition operator $C_\varphi: f \rightarrow f \circ \varphi$ induced by φ is a bounded operator on the Hardy space H^2 . Composition operators have been widely studied and many results can be found in [13] and [25].

Recently, several authors have investigated C^* -algebras generated by one or more composition operators. One aim in studying such C^* -algebras is to understand the properties of the operators they contain. In particular, there is much interest in determining the spectrum of algebraic combinations of composition operators and their adjoints.

For C_φ an essentially normal composition operator, the quotient algebra of the C^* -algebra generated by C_φ by the ideal of the compact operators \mathcal{K} is commutative, and so spectral theory in this quotient algebra can in principle be understood via Gelfand Theory. In [6], Bourdon-Levi-Narayan-Shapiro characterize the class of non-trivially essentially normal composition operators induced by linear fractional maps and provide additional examples of essentially normal composition operators induced by maps which have order of contact 2 with $\partial\mathbb{D}$ at one point. To the best of the

author's knowledge, no non-trivially essentially normal composition operators having order of contact $n > 2$ with the unit circle were known prior to the present work.

In this dissertation we prove a simple criterion for essential normality of composition operators induced by maps φ in a large class of functions. Additionally, we construct essentially normal composition operators which have arbitrary even order of contact with the unit circle at one point. To do so, we rely on results from three distinct areas, presented in Chapters 3-5 after a review of preliminary material in Chapter 2.

In Chapter 3 we discuss a boundary version of the Carathéodory-Fejér problem studied by Agler-Lykova-Young in [1, 2], of determining whether a given finite sequence of complex numbers comprises the initial Taylor coefficients of an analytic self-map of the upper half-plane, \mathbb{H} , at a boundary point. We review several of the results and techniques used in [1, 2], then derive similar results for a slightly different case of interest. We apply this to obtain a parametrization of all analytic self-maps of \mathbb{H} having specified order of contact with the real line at 0, and to construct rational analytic self-maps of \mathbb{H} that are analytic at 0 and have specified Taylor coefficients there.

In Chapter 4 we review some definitions and results of Kriete-Moorhouse [19] regarding compactness of linear combinations of composition operators. We then apply our results from Chapter 3 to obtain a decomposition of a composition operator modulo \mathcal{K} into a sum of composition operators induced by rational functions. We

define basic functions as rational self-maps of the unit disk satisfying some additional properties, and prove a version of the decomposition theorem with basic functions. Additionally, we review a result from [19] regarding weighted composition operators in the Calkin algebra and use this result in the proof of a similar result concerning weighted adjoints of composition operators.

In Chapter 5 we use the pointwise formula for the adjoint of a rationally induced composition operator C_φ^* (introduced in [12, 15]) to obtain an operator formula for $C_\psi C_\varphi^*$ where ψ satisfies some conditions. We use a variant of the formula which appears in Bourdon-Shapiro [7] and ideas from [7] to reduce our formula modulo \mathcal{K} for the special case where φ is a basic function.

Finally, in Chapter 6 we investigate essential normality. We begin by proving a necessary and sufficient condition for essential normality for the case where the inducing map φ is a basic function. We then decompose C_φ into a sum of basic composition operators modulo \mathcal{K} and show that essential normality can only occur when this sum consists of a single term. Lastly, using the parametrization of functions with given order of contact proved in Chapter 3, we construct essentially normal composition operators which have arbitrary even order of contact with the unit circle at one point.

Chapter 2

Preliminary Material

2.1 Bounded Operators on a Hilbert Space

For any Hilbert space \mathcal{H} , we denote the space of all bounded linear operators from \mathcal{H} to \mathcal{H} by $\mathcal{B}(\mathcal{H})$. The operator norm on $\mathcal{B}(\mathcal{H})$ is defined by

$$\|T\| = \sup\{\|Th\| : h \in \mathcal{H}, \|h\|_{\mathcal{H}} \leq 1\},$$

for all $T \in \mathcal{B}(\mathcal{H})$, and $\mathcal{B}(\mathcal{H})$ is complete in this norm.

We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is finite rank if its range is finite dimensional.

An operator T is compact if the following equivalent conditions hold.

Proposition 2.1.1. *[23, Theorem 3.3.3] The following conditions are equivalent.*

1. T is in the closure of the set of finite rank operators in $\mathcal{B}(\mathcal{H})$
2. The closure of the image of the unit ball of \mathcal{H} under T is compact in \mathcal{H}
3. Each net in the unit ball of \mathcal{H} has a subnet whose image under T converges in \mathcal{H}

Note that all finite rank operators are compact. Additionally, from the third condition in Proposition 2.1.1 it follows that:

Proposition 2.1.2. *$T \in \mathcal{B}(\mathcal{H})$ is compact and $S \in \mathcal{B}(\mathcal{H})$ satisfies $\|Sh\| \leq \|Th\|$ for all $h \in \mathcal{H}$, then S is compact as well.*

We denote the collection of all compact operators on \mathcal{H} by \mathcal{K} and note that this collection forms a closed, two-sided ideal in $\mathcal{B}(\mathcal{H})$. The quotient algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$ is called the Calkin algebra [4, 23]. If $T - S \in \mathcal{K}$, i.e., T and S are in the same coset in the Calkin algebra, we write

$$T \equiv S \pmod{\mathcal{K}}.$$

For each operator $T \in \mathcal{B}(\mathcal{H})$ there exists a unique operator $T^* \in \mathcal{B}(\mathcal{H})$, called the adjoint of T , that satisfies

$$\langle Tg, h \rangle = \langle g, T^*h \rangle$$

for all $g, h \in \mathcal{H}$. An operator T is normal if it commutes with its adjoint, that is, $TT^* = T^*T$, and essentially normal if the self-commutator $[T, T^*] = TT^* - T^*T$ is compact, that is,

$$T^*T \equiv TT^* \pmod{\mathcal{K}}.$$

2.2 The Hardy Space H^2 , Composition Operators and Toeplitz Operators

The Hardy space of the disk, $H^2 := H^2(\mathbb{D})$, is the space of all analytic functions f on \mathbb{D} whose power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfy

$$\|f\|_{H^2}^2 := \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

The Hardy space is a Hilbert space under the inner product

$$\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle_{H^2} := \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

More information about the Hardy space can be found in [13].

We can identify $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in H^2 with the boundary function $f(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}$ in $H^2(\partial\mathbb{D})$. Note that

$$f(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$$

for almost every θ [21].

For φ in $L^\infty(\partial\mathbb{D})$, we define the multiplication operator M_φ acting on $L^2(\partial\mathbb{D})$ by

$$(M_\varphi f)(e^{i\theta}) = \varphi(e^{i\theta})f(e^{i\theta}),$$

and the Toeplitz operator T_φ acting on $H^2(\partial\mathbb{D})$ by $T_\varphi = PM_\varphi|_{H^2}$, where P is the projection of $L^2(\partial\mathbb{D})$ onto $H^2(\partial\mathbb{D})$.

For φ an analytic self map of \mathbb{D} , we define the composition operator C_φ on H^2 by

$$C_\varphi f = f \circ \varphi.$$

By Littlewood's subordination principle, C_φ is a bounded linear operator on H^2 [21].

The space H^∞ consisting of all the functions that are analytic and bounded on \mathbb{D} arises in the study of operators on H^2 . H^∞ is a Banach algebra with pointwise operations and the sup norm, and is contained in H^2 .

2.3 More About Composition Operators

2.3.1 Compactness for Composition Operators

Compactness of the composition operator C_φ is closely related to the boundary behavior of φ . The following two results suggest that C_φ is compact when $\varphi(z)$ does not get too close to the boundary too often.

Proposition 2.3.1. *[25, p 23] If $\|\varphi\|_\infty < 1$ then C_φ is compact.*

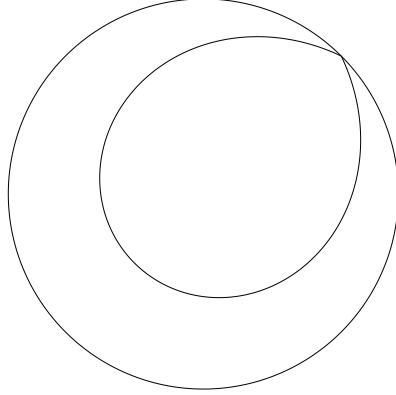
Proposition 2.3.2. *[25, p 32] If the set $\{\zeta \in \partial\mathbb{D}: |\varphi(\zeta)| = 1\}$ has positive Lebesgue measure then C_φ is not compact*

In order to state additional results relating the boundary behavior of φ and compactness of C_φ , we introduce the notion of non-tangential approach regions, angular derivatives and the Denjoy-Wolff point.

For $\zeta \in \partial\mathbb{D}$ and $\alpha > 1$ we define the non-tangential approach region at ζ by

$$\Gamma_\alpha(\zeta) = \{z \in \mathbb{D}: |z - \zeta| < \alpha(1 - |z|)\}.$$

Figure 2.1: A typical non-tangential approach region ($\alpha = 3, \zeta = e^{\frac{\pi}{4}i}$)



We say a function f on \mathbb{D} has non-tangential limit $L \in \mathbb{C}$ at ζ and write

$$\angle \lim_{z \rightarrow \zeta} f(z) = L$$

if $f(z) \rightarrow L$ as $z \rightarrow \zeta$ along any path contained in a non-tangential approach region at ζ .

Let φ be an analytic self-map of \mathbb{D} . Then φ has a (finite) angular derivative at ζ in $\partial\mathbb{D}$ provided $\varphi(\zeta)$, the non-tangential limit of φ at ζ , exists and has modulus one, and

$$\varphi'(\zeta) := \angle \lim_{z \rightarrow \zeta} \frac{\varphi(z) - \varphi(\zeta)}{z - \zeta}$$

exists as a finite complex number. If the angular derivative $\varphi'(\zeta)$ fails to exist, we write $|\varphi'(\zeta)| = \infty$.

Theorem 2.3.3 (Julia Carathéodory [13]). *For φ an analytic self-map of \mathbb{D} and $\zeta \in \partial\mathbb{D}$, the following are equivalent*

1. $d(\zeta) = \liminf_{z \rightarrow \zeta} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty$

2. φ has finite angular derivative $\varphi'(\zeta)$ at ζ

3. Both φ and φ' have (finite) non-tangential limits at ζ , with $|\eta| = 1$ for $\eta = \lim_{r \rightarrow 1} \varphi(r\zeta)$.

Moreover, when these conditions hold, we have $\lim_{r \rightarrow 1} \varphi'(r\zeta) = \varphi'(\zeta) = d(\zeta)\bar{\zeta}\eta$ and

$$d(\zeta) = \angle \lim_{z \rightarrow \zeta} \frac{1-|\varphi(z)|}{1-|z|}.$$

The angular derivative criterion for compactness is given by the following result.

Proposition 2.3.4. [25, p 57] *Suppose φ is an analytic self-map of \mathbb{D}*

1. *If C_φ is compact then φ does not have an angular derivative at any point in $\partial\mathbb{D}$.*

2. *If φ is univalent and does not have an angular derivative at any point of $\partial\mathbb{D}$, then C_φ is compact.*

Theorem 2.3.5 (The Denjoy-Wolff Theorem). [25, p 75] *If φ is an analytic self-map of \mathbb{D} with no fixed point in \mathbb{D} then there is a point $w \in \partial\mathbb{D}$ (“the Denjoy-Wolff point of φ ”) such that the n th iterate $\varphi_n = \varphi \circ \varphi \circ \dots \circ \varphi$ (n times) converges to w uniformly on compact subsets of \mathbb{D} .*

The following is contained in “The Grand Iteration Theorem”.

Theorem 2.3.6. [25, p 78] *Suppose φ is an analytic self-map of \mathbb{D} that is not an elliptic automorphism.*

1. *If φ has no fixed point in \mathbb{D} , then the Denjoy-Wolff point $w \in \partial\mathbb{D}$ of φ is a fixed point for φ and the angular derivative $\varphi'(w)$ exists and satisfies $0 < \varphi'(w) \leq 1$.*

2. Conversely, if φ has a boundary fixed point w at which $\varphi'(w) \leq 1$, then φ has no fixed point in \mathbb{D} , and w is the Denjoy-Wolff point of φ .

We can conclude the following corollary.

Corollary 2.3.7. [25, p 84] *If C_φ is compact then φ has a fixed point in \mathbb{D} .*

2.3.2 Adjoint Formula For The Linear Fractional Case

Let $\varphi = \frac{az + b}{cz + d}$ be a linear-fractional self-map of \mathbb{D} . Then the Krein adjoint of φ is defined by

$$\sigma(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}.$$

Cowen's adjoint formula (see [11]), relates the adjoint of C_φ with C_σ by

$$C_\varphi^* = T_g C_\sigma T_h^*,$$

where $g(z) = (-\bar{b}z + \bar{d})^{-1}$, $h(z) = cz + d$ are in H^∞ . Building on this, in [20] Kriete-MacCluer-Moorhouse provide a simple expression for C_φ^* in the Calkin algebra for the following linear fractional case.

Proposition 2.3.8. [20, Theorem 3, Proposition 3] *Suppose that $\varphi = \frac{az + b}{cz + d}$ is a linear-fractional self-map of \mathbb{D} , not an automorphism, which satisfies $\varphi(\zeta) = \lambda$ for some $\zeta, \lambda \in \partial\mathbb{D}$. Then*

$$C_\varphi^* \equiv \frac{1}{|\varphi'(\zeta)|} C_\sigma \pmod{\mathcal{K}}$$

where σ is the Krein adjoint of φ .

In Proposition 5.2.3, we generalize this formula to the bigger class of functions \mathcal{A} defined in Section 4.1.

2.3.3 Essentially Normal Composition Operators

Recall that an operator T is normal if it commutes with its adjoint, and essentially normal if $[T, T^*] = TT^* - T^*T$ is compact. A composition operator C_φ is normal if and only if φ is of the form $\varphi(z) = \alpha z$ with $|\alpha| \leq 1$ [21, p 171] (result due to Howard Schwartz [21, p 194]). We say that C_φ is non-trivially essentially normal if it is essentially normal, not compact and not normal.

In [6], Bourdon-Levi-Narayan-Shapiro determined that a composition operator induced by a linear fractional map $\varphi(z) = \frac{az+b}{cz+d}$ is non-trivially essential normal if and only if φ is a parabolic non-automorphism. In particular φ has contact with the unit circle $\partial\mathbb{D}$ at one point ζ and satisfies $\varphi'(\zeta) = 1$. Additional examples of essentially normal composition operators induced by maps which have order of contact 2 with $\partial\mathbb{D}$ at one point are provided in [6, Section 7].

To the best of the author's knowledge, no non-trivially essentially normal composition operators having order of contact $n > 2$ with $\partial\mathbb{D}$ were known prior to the present work.

2.4 Order of Contact

We define the notion of order of contact with the boundary both in the context of the unit disk \mathbb{D} and in the context of the upper half-plane \mathbb{H} .

Definition 2.1. We say that φ has contact with $\partial\mathbb{D}$ at ζ of order $c > 0$ if there is some neighborhood V of ζ such that φ is analytic on $V \cap \mathbb{D}$ and maps $V \cap \mathbb{D}$ into \mathbb{D} , and

1. $\varphi(\zeta) := \lim_{z \rightarrow \zeta, z \in \mathbb{D}} \varphi(z)$ exists and $\varphi(\zeta) \in \partial\mathbb{D}$
2. $\frac{1 - |\varphi(e^{i\theta})|^2}{|\varphi(\zeta) - \varphi(e^{i\theta})|^c}$ is essentially bounded above and away from zero as $e^{i\theta} \rightarrow \zeta$.

Definition 2.2. We say that f has contact with \mathbb{R} at 0 of order $c > 0$ if there is some neighborhood W of 0 such that f is analytic on $W \cap \mathbb{H}$ and maps $W \cap \mathbb{H}$ into \mathbb{H} , and

1. $f(0) := \lim_{z \rightarrow 0, z \in \mathbb{H}} f(z)$ exists and $f(0) \in \mathbb{R}$
2. $\frac{\text{Im } f(x)}{|f(0) - f(x)|^c}$ is essentially bounded above and away from zero as $x \rightarrow 0$ in \mathbb{R} .

The following proposition provides some insight and intuition to the definition of order of contact in the upper half-plane by exploring the case where the function is analytic at the point of contact.

Proposition 2.4.1. *Suppose that f is analytic in a neighborhood W of 0 with Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ there, and that f maps 0 into \mathbb{R} and $W \cap \mathbb{H}$ into \mathbb{H} .*

Then f maps an interval in \mathbb{R} containing 0 into \mathbb{R} if and only if all the coefficients a_k are real. Otherwise, f has order of contact $n = \min\{k: \operatorname{Im} a_k \neq 0\}$ with \mathbb{R} at 0, n is even and $\operatorname{Im} a_n > 0$.

Proof. For all $x \in \mathbb{R}$ we have

$$\operatorname{Im} f(x) = \sum_{k=0}^{\infty} \operatorname{Im}(a_k)x^k.$$

If all the coefficients a_k are real then f maps an interval in \mathbb{R} containing 0 into \mathbb{R} .

Otherwise, let n be the minimal index for which a_n is non-real. Then

$$\operatorname{Im} f(x) \sim \operatorname{Im}(a_n)x^n$$

and so f does not map an interval in \mathbb{R} containing 0 into \mathbb{R} . Additionally, we have

$$|f(0) - f(x)| \sim |a_1| \cdot |x|,$$

and since f maps $W \cap \mathbb{H}$ into \mathbb{H} we have that $a_1 > 0$ and $\operatorname{Im} f(x) > 0$ for $x \in \mathbb{R}$ near 0. Thus

$$0 < \frac{\operatorname{Im} f(x)}{|f(0) - f(x)|^n} \sim \frac{\operatorname{Im}(a_n)x^n}{|a_1|^n \cdot |x|^n}$$

and so n is even and $\operatorname{Im}(a_n) > 0$. We get that $\frac{\operatorname{Im} f(x)}{|f(0) - f(x)|^n} \sim \frac{\operatorname{Im}(a_n)}{|a_1|^n}$ and so f has order of contact n with \mathbb{R} at 0. \square

We show that the definition in \mathbb{D} corresponds to that in \mathbb{H} . To transfer \mathbb{D} to \mathbb{H} we use the family of conformal maps $\tau_\alpha: \mathbb{D} \rightarrow \mathbb{H}$ for $\alpha \in \partial\mathbb{D}$, defined by

$$\tau_\alpha(z) = i \frac{\alpha - z}{\alpha + z}.$$

Proposition 2.4.2. *Suppose φ is analytic at ζ . Then φ has order of contact c with $\partial\mathbb{D}$ at ζ if and only if $f = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_{\zeta}^{-1}$ has order of contact c with \mathbb{R} at 0.*

Proof. Note that f fixes 0 and that $x = \tau_{\zeta}(e^{i\theta})$ tends to 0 in \mathbb{R} if and only if $e^{i\theta}$ tends to ζ in $\partial\mathbb{D}$. We calculate:

$$\operatorname{Im} f(x) = \operatorname{Im} (\tau_{\varphi(\zeta)} (\varphi(e^{i\theta}))) = \operatorname{Im} \left(i \frac{\varphi(\zeta) - \varphi(e^{i\theta})}{\varphi(\zeta) + \varphi(e^{i\theta})} \right) = \frac{1 - |\varphi(e^{i\theta})|^2}{|\varphi(\zeta) + \varphi(e^{i\theta})|^2},$$

and

$$|f(x)| = |\tau_{\varphi(\zeta)} (\varphi(e^{i\theta}))| = \frac{|\varphi(\zeta) - \varphi(e^{i\theta})|}{|\varphi(\zeta) + \varphi(e^{i\theta})|}$$

Thus for any $c > 0$,

$$\frac{\operatorname{Im} f(x)}{|f(x)|^c} = \frac{1 - |\varphi(e^{i\theta})|^2}{|\varphi(\zeta) - \varphi(e^{i\theta})|^c} \cdot \frac{|\varphi(\zeta) + \varphi(e^{i\theta})|^c}{|\varphi(\zeta) + \varphi(e^{i\theta})|^2}$$

Note that

$$\lim_{e^{i\theta} \rightarrow \zeta} \frac{|\varphi(\zeta) + \varphi(e^{i\theta})|^c}{|\varphi(\zeta) - \varphi(e^{i\theta})|^2} = \frac{|2\varphi(\zeta)|^c}{|2\varphi(\zeta)|^2} = 2^{c-2}$$

and so φ has order of contact c with $\partial\mathbb{D}$ at ζ if and only if f has order of contact c with \mathbb{R} at 0. □

Order of contact with \mathbb{D} and \mathbb{H} is illustrated in Fig. 2.2, Fig. 2.3 and Fig. 2.4. The maps f_1, f_2 and f_3 are self-maps of \mathbb{H} constructed using Proposition 3.3.3, with contact with \mathbb{R} at 0 of order 2, 4 and 6 respectively and given by

$$\begin{aligned} f_1(z) &= \frac{-z}{iz-1} = z + iz^2 + O(z^3) \\ f_2(z) &= \frac{iz^2 - z}{z^2 + iz - 1} = z + z^3 + iz^4 + O(z^6) \\ f_3(z) &= \frac{z^3 + iz^2 - z}{-iz^3 + 2z^2 + iz - 1} = z + z^3 + 2z^5 + iz^6 + O(z^7). \end{aligned}$$

The maps $\varphi_1, \varphi_2, \varphi_3$ are the corresponding self-maps of \mathbb{D} with contact at 1, and are given by

$$\begin{aligned}\varphi_1(z) &= \frac{z+1}{-z+3} \\ \varphi_2(z) &= \frac{z^2+2z+1}{z^2-2z+5} \\ \varphi_3(z) &= \frac{1}{2} \cdot \frac{z^3+z^2+z+1}{z^2-z+2}.\end{aligned}$$

2.5 Rational Functions

Rational functions will have a significant role in this thesis. We review some basic definitions and properties and discuss the notion of branches of the inverse function.

Let r be a rational function, and write

$$r(z) = \frac{p(z)}{q(z)}$$

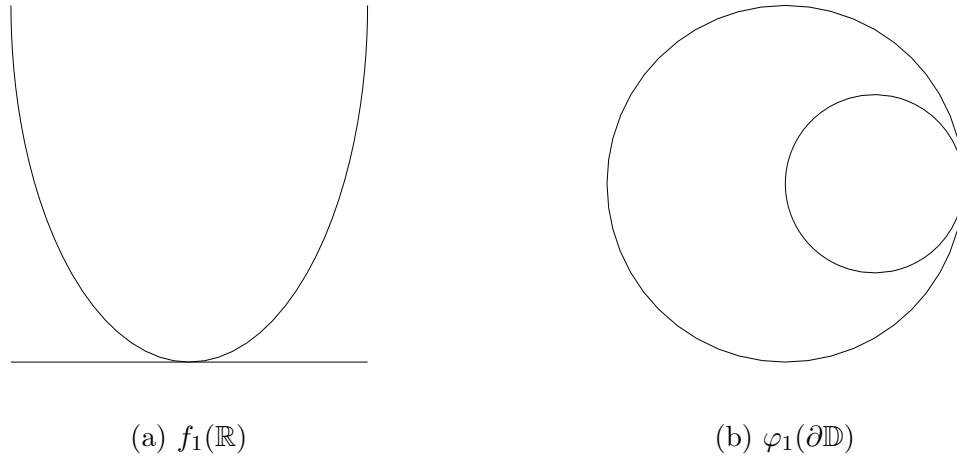
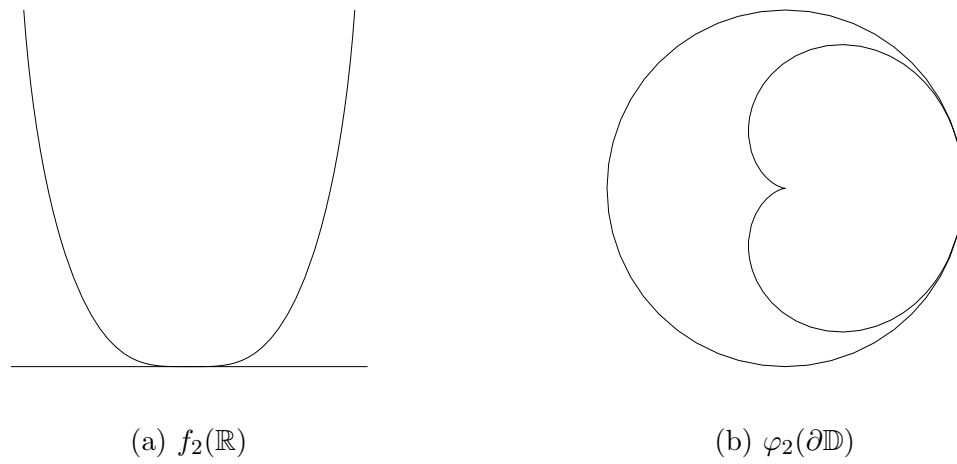
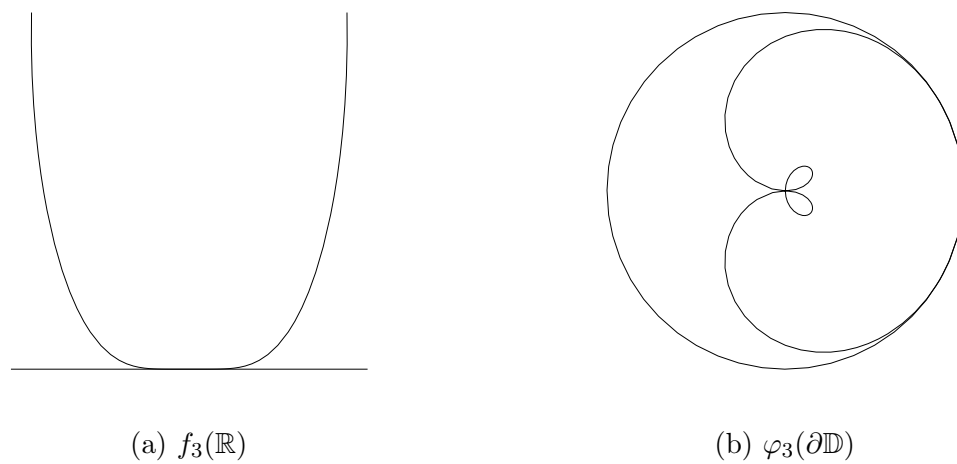
where p and q are relatively prime polynomials. Then the degree d of r is defined to be the maximal degree of p and q . We think of r as mapping the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ into itself. For each point $w \in \widehat{\mathbb{C}}$ the inverse image $r^{-1}(\{w\})$ has d points counting multiplicities [14, Problems 28-32, pages 181-182].

We say w is a regular value of r if $r^{-1}(\{w\})$ consists of d distinct points, and that w is a critical value otherwise. By elementary function theory, w is a critical value for r if and only if $r^{-1}(\{w\})$ contains at least one critical point - a point that has no neighborhood on which r is univalent. For a point $z \in \mathbb{C}$, z is a critical point if and only if $r'(z) = 0$. Thus, all but finitely many points of $\widehat{\mathbb{C}}$ are regular values for r .

Let w be a regular value of r , so that $r^{-1}(\{w\})$ consists of d distinct points $\{z_1, z_2, \dots, z_d\}$. Then each point z_j is the center of an open disc Δ_j on which r has a univalent restriction. So the restriction of r to Δ_j has an analytic inverse s_j on $r(\Delta_j)$. It is called a branch of r^{-1} defined on $r(\Delta)$. We take the intersection of the r -images of the discs $\Delta_1, \dots, \Delta_d$ to obtain a neighborhood W of w on which all d distinct branches s_1, \dots, s_d are defined. By the following proposition, we can extend these d branches to any simply connected domain V containing W and consisting of regular values of r .

Proposition 2.5.1. *[7, Corollary 3] Suppose r is a rational function and s is a branch of r^{-1} defined on some disc Δ consisting of regular values of r . Suppose W is a simply connected domain consisting of regular values of r and containing Δ . Then s has an analytic continuation to a branch of r^{-1} defined on W .*

Thus, for any rational function r of degree d and any simply connected domain V consisting of regular values of r , there are d distinct branches s_1, \dots, s_d of r^{-1} defined on V . Furthermore, distinct branches defined on the same connected open set have disjoint images [7, Proposition 1], and so $s_1(V), \dots, s_d(V)$ are pairwise disjoint.

Figure 2.2: Order of contact 2 in \mathbb{D} and \mathbb{H} Figure 2.3: Order of contact 4 in \mathbb{D} and \mathbb{H} Figure 2.4: Order of contact 6 in \mathbb{D} and \mathbb{H} 

Chapter 3

The Boundary Carathéodory-Fejér Problem

The Carathéodory-Fejér problem [8, 5] is to determine whether a given finite sequence of complex numbers comprises the initial Taylor coefficients of an analytic map f mapping the unit disk \mathbb{D} to the upper half-plane \mathbb{H} . In [1], Agler-Lykova-Young provide a solvability criterion for the boundary version, denoted ∂CFP , and give a parametrization of all solutions in the real case, denoted $\partial CFP(\mathbb{R})$.

We review several of the results and techniques used in [1] and the subsequent paper [2], then apply these to obtain a parametrization of all solutions to a slightly different special case of interest - $\partial CFP(\text{contact-}n)$. We apply this to obtain a parametrization of all analytic self-maps of \mathbb{H} having specified order of contact with \mathbb{R} at 0, and with varying initial Taylor coefficients. Finally, we construct rational analytic self-maps of the upper half-plane \mathbb{H} that are analytic at 0 and have specified Taylor coefficients there.

3.1 The Boundary Carathéodory-Fejér Problem

In [1], Agler-Lykova-Young study a boundary version of the Carathéodory-Fejér problem, denoted ∂CFP , for functions in the Pick class \mathcal{P} where the interpolation node x lies on the boundary \mathbb{R} . Here \mathcal{P} is defined to be the set of analytic functions f mapping the upper half-plane \mathbb{H} into $\mathbb{H} \cup \mathbb{R}$.

Problem 3.1.1 (∂CFP). *Given a point $x \in \mathbb{R}$, a non-negative integer n and numbers $a_{-1}, a_0, \dots, a_n \in \mathbb{C}$, determine whether there exists a function $f \in \mathcal{P}$ such that f is analytic in a deleted neighborhood of x and*

$$L_k(f, x) = a_k, \quad k = -1, 0, 1, \dots, n,$$

where $L_k(f, x)$ is the k th Laurent coefficient of f at x .

A solvability criterion for this general boundary problem is given in [1, Theorem 1.2], which we partially present in Theorem 3.1.3. Parametrization of all solutions for the real case, $\partial CFP(\mathbb{R})$, where $a_{-1}, a_0, \dots, a_n \in \mathbb{R}$, is given in [1, Theorem 8.1, Theorem 8.2, Corollary 8.6].

In the subsequent paper, [2], weaker solutions to ∂CFP , having non-tangential pseudo-Taylor expansions, are considered. For our purposes, unrestricted pseudo-Taylor expansions will suffice, and so we will say a function $f \in \mathcal{P}$ is a weak solution of Problem ∂CFP if f has a pseudo-Taylor expansion

$$f(z) = a_0 + a_1(z - x) + \dots + a_n(z - x)^n + o(|z - x|^n)$$

as $z \rightarrow 0$ unrestrictedly in \mathbb{H} .

Theorem 3.1.2. *[2, Theorem 5.2] Problem ∂CFP with $a_{-1} = 0$ has a weak solution if and only if it has a solution that is analytic at x .*

We are interested in a boundary version of the Carathéodory-Fejér problem where the interpolation node is taken to be $x = 0$. We restrict attention to functions f in the Pick class \mathcal{P} which extend analytically to a neighborhood of 0 and denote this subset of the Pick class by \mathcal{P}_0 . Recall from Proposition 2.4.1 that for $f \in \mathcal{P}_0$, f has order of contact n with the real line at 0 if and only if n is even and the initial Taylor coefficient of f at 0 satisfy $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$. We denote this special case of ∂CFP where $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$ by $\partial CFP(\text{contact-}n)$. The solvability criterion for $\partial CFP(\text{contact-}n)$ case is as follows.

Theorem 3.1.3. *[1, Theorem 1.2(2)] Let $n = 2m$ be an even positive integer, $a_0, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$, and let $H_m(a_1, \dots, a_{n-1})$ be the Hankel matrix defined by*

$$H_m(a_1, \dots, a_{n-1}) = \begin{bmatrix} a_1 & a_2 & \dots & a_m \\ a_2 & a_3 & \dots & a_{m+1} \\ \cdot & \cdot & \dots & \cdot \\ a_m & a_{m+1} & \dots & a_{n-1} \end{bmatrix}$$

Then there exists a function $f \in \mathcal{P}_0$ with initial Taylor coefficients a_0, \dots, a_n at 0 if and only if $H_m(a_1, \dots, a_{n-1}) > 0$, i.e., this matrix is positive definite.

Our goal is to derive a parametrization of all the solutions to $\partial CFP(\text{contact-}n)$.

We will use the methods used in [1] for the real case and arrive at a similar parametrization.

3.2 Reduction and Augmentation

The main tool we will use is a technique for passing from a function in the Pick class to a simpler one and back again due to G. Julia [18]. We define the reduction and the augmentation (at 0) of a function as follows.

Definition 3.1. For any non-constant function $f \in \mathcal{P}_0$ such that $f(0) \in \mathbb{R}$, we define the **reduction** of f (at 0) to be the function g on \mathbb{H} given by the equation

$$g(z) = -\frac{1}{f(z) - f(0)} + \frac{1}{f'(0)z}$$

Definition 3.2. For any function $g \in \mathcal{P}_0$ and any $a_0 \in \mathbb{R}$, $a_1 > 0$, we define the **augmentation** of g (at 0) by a_0, a_1 to be the function f on \mathbb{H} given by

$$f(z) = a_0 + \frac{1}{\frac{1}{a_1 z} - g(z)}$$

The important property of the operations of reduction and augmentation is that they preserve the Pick class.

Proposition 3.2.1. [3, Theorem 3.4]

1. If $f \in \mathcal{P}_0$ is non-constant and $f(0) \in \mathbb{R}$, then the reduction of f is in \mathcal{P}_0 as well.
2. If $g \in \mathcal{P}_0$ and $a_0 \in \mathbb{R}$, $a_1 > 0$ then the augmentation f of g by a_0, a_1 is in \mathcal{P}_0 as well and satisfies $f(0) = a_0$, $f'(0) = a_1$.

Note that reduction and augmentation are inverse operations on \mathcal{P}_0 , that is,

1. if $f \in \mathcal{P}_0$ and if g is the reduction of f then f is the augmentation of g by $f(0), f'(0)$;
2. if $g \in \mathcal{P}_0$ and if f is the augmentation of g by a_0, a_1 then g is the reduction of f .

We will need the following results relating the properties of the Taylor coefficients of a function and those of its reduction.

Proposition 3.2.2. *[1, Proposition 2.5] Let $f \in \mathcal{P}_0$ satisfy $f'(0) > 0$, and let g be the reduction of f . Let the Taylor expansions of f and g about 0 be*

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad g(z) = \sum_{j=0}^{\infty} b_j z^j.$$

Then the Taylor coefficients a_j and b_j are related by

$$\begin{bmatrix} a_1 & 0 & 0 & \dots \\ a_2 & a_1 & 0 & \dots \\ a_3 & a_2 & a_1 & \dots \\ \cdot & \cdot & \cdot & \dots \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \cdot \end{bmatrix} = \frac{1}{a_1} \begin{bmatrix} a_2 \\ a_3 \\ a_4 \\ \cdot \end{bmatrix}.$$

Corollary 3.2.3. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ and $g(z) = \sum_{j=0}^{\infty} b_j z^j$ be as in Proposition 3.2.2. Then for any integer $n \geq 2$,*

$$a_1, \dots, a_n \quad \text{determine} \quad b_0, \dots, b_{n-2},$$

and in the other direction

$$a_1, b_0, \dots, b_{n-2} \quad \text{determine} \quad a_2, \dots, a_n.$$

Furthermore, if $a_0, a_1 \in \mathbb{R}$, then

$$N = \min\{k: \operatorname{Im} a_k \neq 0\} \quad \text{if and only if} \quad N - 2 = \min\{k: \operatorname{Im} b_k \neq 0\}.$$

Proof. For $k = 1, 2, \dots$, the k th row of the matrix equation in Proposition 3.2.2 is

$$a_k b_0 + a_{k-1} b_1 + \dots + a_1 b_{k-1} = \frac{a_{k+1}}{a_1}.$$

Adjusting the index k and solving for the appropriate terms, we get

$$b_k = \frac{1}{a_1} \left(\frac{a_{k+2}}{a_1} - a_{k+1} b_0 - a_k b_1 - \dots - a_2 b_{k-1} \right) \quad \text{for } k = 0, 1, 2, \dots \quad (3.2.1)$$

$$a_k = a_1 (a_{k-1} b_0 + a_{k-2} b_1 + \dots + a_1 b_{k-2}) \quad \text{for } k = 2, 3, 4, \dots \quad (3.2.2)$$

Let $n \geq 2$. Using Eq. (3.2.1) and by induction on k , we see that a_1, \dots, a_n determine b_0, \dots, b_{n-2} . Using Eq. (3.2.2) and again by induction, we see that a_1 together with b_0, \dots, b_{n-2} determine a_2, \dots, a_n .

Now suppose that $a_0, a_1, \dots, a_{N-1} \in \mathbb{R}$ and $a_N \in \mathbb{H}$. Then using Eq. (3.2.1) and by induction on $k = 0, \dots, N-3$ we see that $b_0, \dots, b_{N-3} \in \mathbb{R}$. Now, taking imaginary parts of Eq. (3.2.1) for $k = N-2$, we get that $\operatorname{Im}(b_{N-2}) = \frac{1}{a_1^2} \operatorname{Im}(a_N)$, so that $b_{N-2} \in \mathbb{H}$. To other direction of the implication follows similarly from Eq. (3.2.2). \square

Proposition 3.2.4. [1, Corollary 3.3] Let $f(z) = \sum_{j=0}^{j=\infty} a_j z^j$, $g(z) = \sum_{j=0}^{j=\infty} b_j z^j$ be as in Proposition 3.2.2. Then $H_m(a_1, \dots, a_{2m-1}) > 0$ if and only if $H_{m-1}(b_1, \dots, b_{2m-3}) > 0$.

3.3 Parametrization of Solutions to the Contact- n Case

We can obtain an alternative expression for the augmentation of a function by introducing the following notation. For a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, denote the corresponding linear fractional transformation by $L[A]$:

$$L[A]h = \frac{a_{11}h + a_{12}}{a_{21}h + a_{22}}.$$

Note that composition of linear fractional transformations corresponds to matrix multiplication, i.e., $L[A]L[B]h = L[AB]h$.

By the definition, the augmentation f of g by a_0, a_1 is

$$f(z) = a_0 + \frac{1}{\frac{1}{a_1 z} - g(z)} = a_0 - \frac{a_1 z}{a_1 z g(z) - 1} = \frac{a_0 a_1 z g(z) - a_0 - a_1 z}{a_1 z g(z) - 1},$$

and so $f(z) = L[A(a_0, a_1)(z)]g(z)$, where $A(a_0, a_1)(z)$ is defined by

$$A(a_0, a_1)(z) = \begin{bmatrix} a_0 a_1 z & -a_0 - a_1 z \\ a_1 z & -1 \end{bmatrix}.$$

Theorem 3.3.1. *Let $n = 2m$ be an even positive integer and let $a_0, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$ be such that $H_m(a_1, \dots, a_{n-1}) > 0$. Then all solutions to $\partial CFP(\text{contact-}n)$ are given by*

$$f(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]g(z)$$

for any $g \in \mathcal{P}_0$ satisfying $g(0) = a_0^{(m)}$, where

$$a_0^{(0)} = a_0, a_0^{(1)}, \dots, a_0^{(m-1)} \in \mathbb{R}, \quad a_0^{(m)} \in \mathbb{H}, \quad a_1^{(0)} = a_1, a_1^{(1)}, \dots, a_1^{(m-1)} > 0$$

are determined by a_0, \dots, a_n .

Proof. Recall that by Theorem 3.1.3, there exists a solution to $\partial CF\mathcal{P}$ (contact- n) with initial Taylor coefficients a_0, \dots, a_n , and fix a solution f_0 . We inductively define $f_{k+1} \in \mathcal{P}_0$ to be the reduction of f_k for $k = 1, \dots, m$, and denote the Taylor coefficients of f_k at 0 by $a_0^{(k)}, a_1^{(k)}, \dots$. Notice that f_k is the augmentation of f_{k+1} by $a_0^{(k)}, a_1^{(k)}$, so that

$$f_k(z) = L[A(a_0^{(k)}, a_1^{(k)})]f_{k+1}(z),$$

and f_0 can be written as

$$\begin{aligned} f_0(z) &= L[A(a_0, a_1)]f_1(z) \\ &= L[A(a_0^{(0)}, a_1^{(0)})(z)A(a_0^{(1)}, a_1^{(1)})(z)]f_2(z) = \dots \\ &= L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]f_m(z). \end{aligned}$$

By Corollary 3.2.3, all the Taylor coefficients listed below are determined by a_0, \dots, a_n and do not depend on our choice of f_0 .

$$\begin{array}{ccccccc} f_0 & \xrightarrow{\text{red}} & f_1 & \xrightarrow{\text{red}} & \dots & \xrightarrow{\text{red}} & f_{m-1} & \xrightarrow{\text{red}} & f_m \\ a_0 & & a_0^{(1)} & & \dots & & a_0^{(m-1)} & & \mathbf{a_0^{(m)}} \\ a_1 & & a_1^{(1)} & & \dots & & a_1^{(m-1)} & & \\ a_2 & & a_2^{(1)} & & \dots & & \mathbf{a_2^{(m-1)}} & & \\ \vdots & & \vdots & & \ddots & & & & \\ a_{n-2} & & \mathbf{a_{n-2}^{(1)}} & & & & & & \\ a_{n-1} & & & & & & & & \\ \mathbf{a_n} & & & & & & & & \end{array}$$

In general, for $k = 1, \dots, m$ we have that $a_0^{(k)}, \dots, a_{n-2k}^{(k)}$ are determined. Corollary 3.2.3 further shows that $a_0^{(k)}, \dots, a_{n-2k-1}^{(k)} \in \mathbb{R}$ and $a_{n-2k}^{(k)} \in \mathbb{H}$, i.e., all the non-bold coefficients

above are in \mathbb{R} and all the bold coefficient are in \mathbb{H} . In particular, we get that

$$a_0^{(0)} = a_0, a_0^{(1)}, \dots, a_0^{(m-1)} \in \mathbb{R}, \quad a_0^{(m)} \in \mathbb{H}.$$

Additionally, by Proposition 3.2.4, for $k = 1, \dots, m-1$ the Hankel matrix $H_{m-k}(a_1^{(k)}, \dots, a_{n-2k-1}^{(k)})$ is positive and so in particular we get that

$$a_1^{(0)} = a_1, a_1^{(1)}, \dots, a_1^{(m-1)} > 0.$$

Note that since f_0, \dots, f_{m-1} are all real valued at 0 and non-constant, taking the above reductions makes sense.

Now let F_0 be any solution of $\partial CFP(\text{contact-}n)$ with a_0, \dots, a_n , and let F_k denote the k th reduction of F_0 . As discussed above, the first $n - 2k + 1$ Taylor coefficients of F_k are $a_0^{(k)}, \dots, a_{n-2k}^{(k)}$, and so F_0 can be written as

$$F_0(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]F_m(z),$$

where F_m satisfies $F_m(0) = a_0^{(m)}$.

Conversely, for any $F_m \in \mathcal{P}_0$ satisfying $F_m(0) = a_0^{(m)}$ we inductively define F_k for $k = m - 1, \dots, 0$ to be the augmentation of F_{k+1} by $a_0^{(k)}, a_1^{(k)}$, i.e.,

$$F_k(z) = L[A(a_0^{(k)}, a_1^{(k)})]F_{k+1}(z).$$

By the second assertion in Corollary 3.2.3, the first $n - 2k + 1$ Taylor coefficients of F_k are again $a_0^{(k)}, \dots, a_{n-2k}^{(k)}$, and so the function

$$F_0(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]F_m(z),$$

is indeed a solution to $\partial CFP(\text{contact-}n)$ with a_0, \dots, a_n . □

Theorem 3.3.2. *Let $n = 2m$ be an even positive integer. Let $a_0, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$ be such that $H_m(a_1, \dots, a_{n-1}) > 0$. Then all solutions for $\partial CFP(\text{contact-}n)$ are parametrized by*

$$f(z) = \frac{p(z)h(z) + q(z)}{r(z)h(z) + s(z)}$$

where p, q, r, s are polynomials with real coefficients of degree at most m determined by a_0, \dots, a_n and satisfying for some $K > 0$,

$$(ps - qr)(z) = Kz^n$$

and $h \in \mathcal{P}_0$ satisfies $h(0) = h_0$, where h_0 is determined by a_0, \dots, a_n .

Proof. Let $a_0^{(0)}, \dots, a_0^{(m)}$ and $a_1^{(0)}, \dots, a_1^{(m-1)}$ be as in Theorem 3.3.1, so that all solutions are parametrized by

$$f(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]h(z),$$

where h is any function in \mathcal{P}_0 with $h(0) = a_0^{(m)}$.

We define the polynomials p, q, r, s by

$$\begin{bmatrix} p(z) & q(z) \\ r(z) & s(z) \end{bmatrix} = A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z),$$

so that

$$f(z) = L \begin{bmatrix} p(z) & q(z) \\ r(z) & s(z) \end{bmatrix} h(z) = \frac{p(z)h(z) + q(z)}{r(z)h(z) + s(z)}.$$

Recall that

$$A(\alpha_0, \alpha_1)(z) = \begin{bmatrix} \alpha_0 \alpha_1 z & -\alpha_0 - \alpha_1 z \\ \alpha_1 z & -1 \end{bmatrix},$$

and so p, q, r, s are real polynomials of degree at most m . Notice that

$$\det A(\alpha_0, \alpha_1)(z) = -\alpha_0\alpha_1z + \alpha_0\alpha_1z + \alpha_1^2z^2 = \alpha_1^2z^2,$$

so by taking determinants we get that

$$(ps - qr)(z) = \left(a_1^{(0)}\right)^2 z^2 \cdots \left(a_1^{(m-1)}\right)^2 z^2 = Kz^n.$$

□

As a corollary to Theorem 3.3.1, we can now provide a parametrization of all contact n functions.

Proposition 3.3.3. *All functions $f \in \mathcal{P}_0$ with order of contact $n = 2m$ with \mathbb{R} at 0 are parametrized by*

$$f(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]g(z),$$

where $g \in \mathcal{P}_0$ satisfies $g(0) \in \mathbb{H}$ and

$$a_0^{(0)}, a_0^{(1)}, \dots, a_0^{(m-1)} \in \mathbb{R}, \quad a_1^{(0)}, a_1^{(1)}, \dots, a_1^{(m-1)} > 0.$$

Furthermore, for any $f \in \mathcal{P}_0$, these parameters are uniquely determined.

Proof. Let $f \in \mathcal{P}_0$ have order of contact $n = 2m$ with \mathbb{R} at 0 with initial Taylor coefficients a_0, \dots, a_n . Then by Theorem 3.1.3, $H_m(a_1, \dots, a_{n-1}) > 0$ and so by Theorem 3.3.1 f is of the desired form.

For the other direction note that any f of this form is obtained by applying m augmentations to $g \in \mathcal{P}_0$, and so $f \in \mathcal{P}_0$. Recall that by Corollary 3.2.3, augmentation

increases the order of contact with \mathbb{R} by 2, and so $g(0) \in \mathbb{H}$ implies that f has order of contact $2m = n$.

To see uniqueness, notice that if

$$f(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]g(z)$$

then $a_0^{(k)} = f_k(0), a_1^{(k)} = f'_k(0)$ and $g = f_m$ where f_k denotes the k th reduction of f . □

3.4 Rational Functions with Specified Taylor Coefficients

We turn our attention to construction of simple solutions to ∂CFP . Initially we construct rational maps $f \in \mathcal{P}_0$ which map 0 into \mathbb{R} and the rest of $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ into \mathbb{H} and have specified initial Taylor coefficients at 0. We then add the condition that 0 is a regular value for f , i.e., $f^{-1}(\{0\})$ consists of d distinct points where d is the degree of f . We begin by showing that augmentation preserves the properties we are interested in.

Proposition 3.4.1. *Let $g \in \mathcal{P}_0$ be a rational function of degree d . Then any augmentation f of g is a rational function of degree $d + 1$.*

Proof. Let $g = \frac{p(z)}{q(z)} \in \mathcal{P}_0$ be a rational function of degree d , with $p(z)$ and $q(z)$ are relatively prime, and let $f \in \mathcal{P}_0$ be the augmentation of g by a_0, a_1 . Then by the

definition of augmentation,

$$f(z) = a_0 + \frac{1}{\frac{1}{a_1 z} - \frac{p(z)}{q(z)}},$$

and so it suffices to show that the rational function

$$\frac{1}{a_1 z} - \frac{p(z)}{q(z)} = \frac{q(z) - a_1 z p(z)}{a_1 z q(z)} = \frac{r(z)}{s(z)}$$

has degree $d + 1$. We see that $\max\{\deg r, \deg s\} = d + 1$, and it remains to show that $r(z)$ and $s(z)$ are relatively prime.

First note that $q(z)$ is not divisible by z since g is analytic at 0, and so z is not a common divisor of $r(z)$ and $s(z)$. Now assume that $z - \alpha$ is a common divisor of $r(z)$ and $s(z)$ for some $\alpha \neq 0$. Then $z - \alpha$ divides $a_1 z q(z)$ implies $z - \alpha$ divides $q(z)$, and $z - \alpha$ divides both $q(z)$ and $q(z) - a_1 z p(z)$ implies $z - \alpha$ divides $p(z)$. This contradicts the assumption that $p(z)$ and $q(z)$ are relatively prime, and so $r(z)$ and $s(z)$ are indeed relatively prime. \square

Proposition 3.4.2. *Let $g \in \mathcal{P}_0$ mapping $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} . Let $f = L[A(a_0, a_1)]g$ with $\text{Im } a_0 \geq 0$ and $a_1 > 0$ (note that for a_0 real, f is an augmentation of g). Then f also maps $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} .*

Proof. By the definition of $A(a_0, a_1)$,

$$f(z) = \frac{a_0 a_1 z g(z) - a_0 - a_1 z}{a_1 z g(z) - 1} = a_0 + \frac{1}{\frac{1}{a_1 z} - g(z)}.$$

Let $x \in \widehat{\mathbb{R}} \setminus \{0\}$, so that $g(x) \in \mathbb{H}$. Then

$$\text{Im} \left(\frac{1}{a_1 x} - g(x) \right) = -\text{Im } g(x) < 0,$$

and so

$$\operatorname{Im} f(x) = \operatorname{Im} a_0 + \operatorname{Im} \left(\frac{1}{\frac{1}{a_1 x} - g(x)} \right) > 0.$$

□

Proposition 3.4.3. *Let f be in \mathcal{P} and suppose that for some $n = 2m$, f has pseudo Taylor coefficients*

$$\lim_{z \rightarrow 0} \frac{f^{(k)}(z)}{k!} = a_k \quad \text{for } k = 0, 1, \dots, n,$$

where the limits are taken unrestrictedly in \mathbb{H} , such that $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$.

Then there exists a degree m rational function $F \in \mathcal{P}_0$ that maps $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} and has initial Taylor coefficients a_0, \dots, a_n at $z = 0$.

Proof. By Theorem 3.1.2 and Theorem 3.1.3, $H_m(a_1, \dots, a_{n-1}) > 0$ and so by Theorem 3.3.1 with $g(z) \equiv a_0^{(m)}$, we have that

$$F(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]a_0^{(m)},$$

is in \mathcal{P}_0 with the desired Taylor coefficients. Here $a_0^{(0)} = a_0, a_0^{(1)}, \dots, a_0^{(m-1)} \in \mathbb{R}$, $a_0^{(m)} \in \mathbb{H}$ and $a_1^{(0)} = a_1, a_1^{(1)}, \dots, a_1^{(m-1)} > 0$ are determined by a_0, \dots, a_n .

Note that F is obtained by applying m augmentations to $g(z) \equiv a_0^{(m)}$. Thus by Proposition 3.4.1, F is a degree m rational function, and by Proposition 3.4.2 applied m times, F maps $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} . □

Proposition 3.4.4. *Let f be in \mathcal{P} and suppose that for some $n = 2m$, f has pseudo Taylor coefficients*

$$\lim_{z \rightarrow 0} \frac{f^{(k)}(z)}{k!} = a_k \quad \text{for } k = 0, 1, \dots, n,$$

where the limits are taken unrestrictedly in \mathbb{H} , such that $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$.

Then there exists a degree $m + 1$ rational function $f_0 \in \mathcal{P}_0$ that has 0 as a regular value, maps $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} and has initial Taylor coefficients a_0, \dots, a_n at $z = 0$.

Proof. By Theorem 3.1.2 and Theorem 3.1.3, $H_m(a_1, \dots, a_{n-1}) > 0$ and so by Theorem 3.3.1 we have that for any $g \in \mathcal{P}_0$ with $g(0) = a_0^{(m)}$,

$$F(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]g(z),$$

is in \mathcal{P}_0 with the desired Taylor coefficients. Here $a_0^{(0)} = a_0, a_0^{(1)}, \dots, a_0^{(m-1)} \in \mathbb{R}$, $a_0^{(m)} \in \mathbb{H}$ and $a_1^{(0)} = a_1, a_1^{(1)}, \dots, a_1^{(m-1)} > 0$ are determined by a_0, \dots, a_n .

For any $w \in \mathbb{H}$, we define $g_w(z) = L[A(a_0^{(m)}, 1)(z)]w = a_0^{(m)} + \frac{z}{1 - wz}$. Then g is a degree 1 rational function in \mathcal{P}_0 and by Proposition 3.4.2, g_w maps $\widehat{\mathbb{R}}$ into \mathbb{H} .

Let F_w denote the function F above resulting from the choice $g = g_w$ and note that F_w is obtained by applying m augmentations to g_w . Thus by Proposition 3.4.1, F_w is a degree $m + 1$ rational function, and by Proposition 3.4.2 applied m times, F_w maps $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} .

It remains to find a $w \in \mathbb{H}$ such that 0 is a regular value for F_w , i.e., $F_w(z) = 0$

has $m + 1$ distinct solutions. Let p, q, r, s be defined by

$$\begin{bmatrix} p(z) & q(z) \\ r(z) & s(z) \end{bmatrix} = A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z) A(a_0^{(m)}, 1)(z),$$

so that p, q, r, s are polynomials of degree at most $m + 1$, and

$$F_w(z) = \frac{p(z)w + q(z)}{r(z)w + s(z)}.$$

Since the degree of F_w is $m + 1$, $p(z)w + q(z)$ and $r(z)w + s(z)$ have no common roots and so it suffices to find w such that $p(z)w + q(z) = 0$ has $m + 1$ distinct roots. Note that p and q have no non-zero roots in common, since by taking determinants, we have that

$$(ps - qr)(z) = (a_1^{(0)})^2 z^2 \cdots (a_1^{(m)})^2 z^2 = K z^{n+2}, \quad K \neq 0.$$

This equation also shows that at least one of p and q has degree $m + 1$. Suppose for the moment that $F_w(0) = a_0 \neq 0$, i.e $z = 0$ is not a common root for p and q . Then the function $-\frac{q(z)}{p(z)}$ is a rational function of degree $m + 1$ and so all but finitely many $w \in \mathbb{C}$ are regular values. For any choice $w \in \mathbb{H}$ that is a regular value of $-\frac{q(z)}{p(z)}$, F_w is the desired function.

If $F_w(0) = a_0 = 0$ then $z = 0$ is a common root for p and q , however $F'_w(0) = a_1 \neq 0$ and so $z = 0$ is a solution of multiplicity one to $p(z)w + q(z) = 0$. Thus the function $-\frac{q(z)}{p(z)}$ is a rational function of degree m , and so we can find $w \in \mathbb{H}$ with pre-image consisting of m points. These m points and the point $z = 0$ are then the $m + 1$ distinct solutions to $F_w(z) = 0$. □

Chapter 4

Relations In The Calkin Algebra

In [19], Kriete-Moorhouse investigate compactness of linear combinations of composition operators where the inducing maps have certain properties of analyticity. We review some definitions and results from [19], then apply our results from Chapter 3 to obtain a decomposition of a composition operator in the Calkin algebra into a sum of composition operators induced by rational or basic functions (see Definition 4.1).

Additionally, we review a result from [19] regarding weighted composition operators in the Calkin algebra and use this result in the proof of a similar result concerning weighted adjoints of composition operators.

4.1 The Class of Functions \mathcal{A}

We will work in the class of functions \mathcal{A} (denoted \mathcal{S} in [19]) consisting of analytic self-maps φ of \mathbb{D} with certain properties of analyticity. The model for a function in \mathcal{A} is an analytic self-map of \mathbb{D} which extends analytically to a neighborhood of \mathbb{D} and is not a Blaschke product. In particular, we restrict the number of points of contact

with the unit circle, and require relatively nice behavior at the points of contact. To make a precise definition, we first discuss Clark measures.

4.1.1 Clark Measures

Let φ be an analytic self-map of \mathbb{D} . If $|\alpha| = 1$, there exists a finite positive Borel measure μ_α on $\partial\mathbb{D}$ such that

$$\frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \operatorname{Re} \left(\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} \right) = \int_{\partial\mathbb{D}} P_z(e^{it}) d\mu_\alpha(t)$$

for z in \mathbb{D} , where

$$P_z(e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2}$$

is the Poisson kernel at z . The existence of μ_α follows since the left side the equation above is a positive harmonic function. The measures μ_α are called the Clark measures of φ . More information about Clark measures can be found in [9] and Sarason [24].

The singular part of the measure, μ_α^s , is carried by $\varphi^{-1}(\{\alpha\})$, the set of those $\zeta \in \partial\mathbb{D}$ where $\varphi(\zeta)$, the non-tangential limit of φ at ζ , exists and equals α . The measure μ_α^s is the sum of the pure point measure

$$\mu_\alpha^{pp} = \sum_{\varphi(\zeta)=\alpha} \frac{1}{|\varphi'(\zeta)|} \delta_\zeta$$

where δ_ζ is the unit point mass at ζ , and a continuous singular measure μ_α^{cs} .

We write

$$E(\varphi) = \bigcup_{|\alpha|=1} \overline{\operatorname{spt}(\mu_\alpha^s)},$$

where $\text{spt}(\mu)$ denotes the closed support of a measure μ . Note that for any φ ,

$$F(\varphi) = \{\zeta : \varphi \text{ has finite angular derivative at } \zeta\}$$

is a subset of $E(\varphi)$. If $E(\varphi)$ is a finite set then the continuous singular measure μ_α^{cs} all vanish, and we get that $E(\varphi) = F(\varphi)$.

4.1.2 Definition of \mathcal{A}

We define the class \mathcal{A} to be the set of analytic self-maps φ of \mathbb{D} with $E(\varphi) = F(\varphi)$ and such that for each point $\zeta \in F(\varphi)$, there exist an n such that φ has order of contact n at ζ and complex numbers a_0, a_1, \dots, a_n such that

$$\varphi(z) = a_0 + a_1(z - \zeta) + \dots + a_n(z - \zeta)^n + o(|z - \zeta|^n)$$

as $z \rightarrow \zeta$ unrestrictedly in \mathbb{D} .

For $\varphi \in \mathcal{A}$ with order of contact n at ζ , we define the derivatives of φ at ζ by

$$\varphi^{(j)}(\zeta) := \angle \lim_{z \rightarrow \zeta} \varphi^{(j)}(z) = j!a_j$$

for $j = 1, \dots, n$ and note that these limits do exist (see, the argument in [24, p 47]).

Proposition 4.1.1. *\mathcal{A} contains all self-maps of \mathbb{D} that extend analytically to a neighborhood of \mathbb{D} that are not finite Blaschke products.*

Proof. Let φ be a self-map of \mathbb{D} which extends analytically to $V \supset \mathbb{D}$ and is not a finite Blaschke product. Let $S = \{\zeta \in \partial\mathbb{D} : |p(\zeta)| = 1\}$ be the set of points where φ

has contact with $\partial\mathbb{D}$. Then S consists of zeros of the analytic function

$$f(z) := \varphi(z) - \varphi_e(z),$$

where $\varphi_e = \rho \circ \varphi \circ \rho$, and ρ is the inversion in the unit circle defined by $\rho(z) := 1/\bar{z}$.

If S is infinite then the set of zeros of f has an accumulation point in V and so f is identically 0 on V . In this case $\varphi(e^{i\theta}) = 1/\overline{\varphi(e^{i\theta})}$ for all $e^{i\theta}$, i.e φ is inner. But since

φ is an inner function analytic in a neighborhood of \mathbb{D} , φ must be a finite Blaschke product, contradicting our assumption. Thus S is finite and so $E(\varphi)$ is finite. Since

φ is analytic at each point $\zeta \in F(\varphi) = E(\varphi)$ and maps a small arc containing ζ onto a curve with contact with $\partial\mathbb{D}$ at exactly one point, φ has finite order of contact at ζ .

Lastly, for n equal to the order of contact of φ with $\partial\mathbb{D}$ at ζ , we can write

$$\varphi(z) = a_0 + a_1(z - \zeta) + \dots + a_n(z - \zeta)^n + o(|z - \zeta|^n)$$

using the Taylor coefficients of φ at ζ . Thus, φ is indeed in \mathcal{A} . □

4.2 Linear Relations for Composition Operators in the Calkin Algebra

In [19], Kriete-Moorhouse show that information relating to compactness of a linear combination of compositions operators

$$c_1 C_{\varphi_1} + \dots + c_r C_{\varphi_r}$$

where $\varphi_1, \dots, \varphi_r \in \mathcal{A}$, is carried by the behavior of the functions φ_j at their points of contact with the unit circle. More precisely, the relevant information for φ at a point of contact ζ is

$$D_n(\varphi, \zeta) = (\varphi(\zeta), \varphi'(\zeta), \dots, \varphi^{(n)}(\zeta))$$

where n is the order of contact of φ with the unit circle at ζ .

The following result determines compactness of a linear combination of composition operators for operators induced by functions in \mathcal{A} .

Theorem 4.2.1. [19, Theorem 5.13] *Let $\varphi_1, \dots, \varphi_r$ in \mathcal{A} and write F for the union $F(\varphi_1) \cup \dots \cup F(\varphi_r)$, a finite set. For ζ in F and $k = 2, 4, 6, \dots$, let*

$$\mathbb{N}_k(\zeta) = \{j : F(\varphi_j) \text{ contains } \zeta \text{ and } k \text{ is the order of contact of } \varphi_j \text{ at } \zeta\}$$

and let

$$\mathcal{E}_k(\zeta) = \{D_k(\varphi_j, \zeta) : j \text{ is in } \mathbb{N}_k(\zeta)\}.$$

Given complex numbers c_1, \dots, c_r , the following are equivalent:

1. $c_1 C_{\varphi_1} + \dots + c_r C_{\varphi_r}$ is compact
2. for every ζ in F , every even $k \geq 2$ and every \mathbf{d} in $\mathcal{E}_k(\zeta)$,

$$\sum_{\substack{j \in \mathbb{N}_k(\zeta) \\ D_k(\varphi_j, \zeta) = \mathbf{d}}} c_j = 0.$$

Our goal is to decompose C_φ , modulo the ideal \mathcal{K} of compact operators, into a sum of composition operators induced by rational functions, each having contact with the unit circle at exactly one point. We begin by proving the following lemma.

Lemma 4.2.2. *Let $\varphi \in \mathcal{A}$ and $\zeta \in F(\varphi)$. Let $n = 2m$ denote the order of contact of φ with $\partial\mathbb{D}$ at ζ . Then there exists a degree m rational function φ_0 with order of contact n with $\partial\mathbb{D}$ at ζ and mapping the rest of $\overline{\mathbb{D}}$ into \mathbb{D} which satisfies $D_n(\varphi_0, \zeta) = D_n(\varphi, \zeta)$.*

Proof. Let $\lambda = \varphi(\zeta)$ and define

$$f = \tau_\lambda \circ \varphi \circ \tau_\zeta^{-1},$$

where $\tau_\alpha(z) = i\frac{\alpha-z}{\alpha+z}$. Then $f \in \mathcal{P}$ fixes 0 and since $\varphi \in \mathcal{A}$, f has pseudo Taylor coefficients

$$\lim_{z \rightarrow 0} \frac{f^{(k)}(z)}{k!} = a_k \quad \text{for } k = 0, 1, \dots, n$$

such that $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$.

Thus, by Proposition 3.4.3, there exists a degree m rational function $F \in \mathcal{P}_0$ that maps $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} and has initial Taylor coefficients a_0, \dots, a_n at $z = 0$. We let φ_0 be the self-map of \mathbb{D} defined by

$$\varphi_0 = \tau_\lambda^{-1} \circ F \circ \tau_\zeta.$$

Note that τ_α preserves the degree of a rational function, and so φ_0 is a degree m rational function. As a consequence of the boundary behavior of F , φ_0 maps ζ to λ and the rest of $\overline{\mathbb{D}}$ into \mathbb{D} . Since $D_n(\varphi_0, \zeta)$ is determined by $D_n(\tau_\zeta, \zeta)$, $D_n(F, 0)$ and $D_n(\tau_\lambda^{-1}, 0)$ (see Theorem 5.3.1), φ_0 satisfies $D_n(\varphi_0, \zeta) = D_n(\varphi, \zeta)$. Lastly, since F has order of contact n with \mathbb{R} at 0 and by Proposition 2.4.2, φ_0 has order of contact n with $\partial\mathbb{D}$ at ζ . □

We combine this result with Theorem 4.2.1 to obtain our first decomposition theorem.

Theorem 4.2.3. *Let $\varphi \in \mathcal{A}$ with $F(\varphi) = \{\zeta_1, \dots, \zeta_r\}$ and let $n_j = 2m_j$ denote the order of contact of φ with the unit circle at ζ_j . Then there exists a decomposition,*

$$C_\varphi \equiv C_{\varphi_1} + \dots + C_{\varphi_r} \pmod{\mathcal{K}}.$$

where for each $j = 1, \dots, r$, φ_j is a rational self-map of \mathbb{D} of degree m_j which maps ζ_j into the unit circle $\partial\mathbb{D}$ and the rest of $\partial\mathbb{D}$ into \mathbb{D} and satisfies $D_{n_j}(\varphi_j, \zeta_j) = D_{n_j}(\varphi, \zeta_j)$.

Proof. Existence of $\varphi_1, \dots, \varphi_r$ follows from Lemma 4.2.2. The result follows by applying Theorem 4.2.1 to $\varphi, \varphi_1, \dots, \varphi_n$ with constants $1, -1, \dots, -1$. \square

In the following chapters, we will require that the inducing rational maps have an additional property of regularity. We define basic functions as follows, and proceed to prove our second decomposition theorem where each term is a composition operator induced by a basic function.

Definition 4.1. A function φ analytic on \mathbb{D} is a basic function with contact at ζ if the following hold.

1. φ is a rational function mapping the unit disk \mathbb{D} into itself.
2. $\varphi(\zeta)$ is on the unit circle, and φ maps the rest of the unit circle into \mathbb{D} .
3. $\varphi(\zeta)$ is a regular value for φ .

Lemma 4.2.4. *Let $\varphi \in \mathcal{A}$ and $\zeta \in F(\varphi)$. Let $n = 2m$ denote the order of contact of φ with $\partial\mathbb{D}$ at ζ . Then there exists a degree $m + 1$ basic function φ_0 with order of contact n with $\partial\mathbb{D}$ at ζ which satisfies $D_n(\varphi_0, \zeta) = D_n(\varphi, \zeta)$.*

Proof. We begin as in the proof of Lemma 4.2.2, by defining $\lambda = \varphi(\zeta)$ and $f = \tau_\lambda \circ \varphi \circ \tau_\zeta^{-1}$. Here, by Proposition 3.4.4, there exists a degree $m + 1$ rational function $G \in \mathcal{P}_0$ that maps $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} , has initial Taylor coefficients a_0, \dots, a_n at $z = 0$, and has $z = 0$ as a regular value.

Again, we define $\varphi_0 = \tau_\lambda^{-1} \circ G \circ \tau_\zeta$ and get that φ_0 is a degree $m + 1$ rational function with the desired boundary behavior at ζ . Since $D_n(\varphi_0, \zeta)$ is determined by $D_n(\tau_\zeta, \zeta)$, $D_n(G, 0)$ and $D_n(\tau_\lambda^{-1}, 0)$ (see Theorem 5.3.1), φ_0 satisfies $D_n(\varphi_0, \zeta) = D_n(\varphi, \zeta)$. Additionally, since G has order of contact n with \mathbb{R} at 0 and by Proposition 2.4.2, φ_0 also has order of contact n with $\partial\mathbb{D}$ at ζ .

Now, since 0 is a regular value for G , we have that

$$G(w) = (\tau_\lambda \circ \varphi_0 \circ \tau_\zeta^{-1})(w) = 0$$

has $m + 1$ distinct solutions. So

$$\varphi_0(\tau_\zeta^{-1}(w)) = (\tau_\lambda^{-1})(0) = \lambda$$

has $m + 1$ distinct solutions, i.e., λ is a regular value for φ_0 . □

Theorem 4.2.5. *Let $\varphi \in \mathcal{A}$ with $F(\varphi) = \{\zeta_1, \dots, \zeta_r\}$ and let $n_j = 2m_j$ denote the order of contact of φ with the unit circle at ζ_j . Then there exists a decomposition,*

$$C_\varphi \equiv C_{\varphi_1} + \dots + C_{\varphi_r} \pmod{\mathcal{K}}.$$

where for each $j = 1, \dots, r$, φ_j is a basic function of degree $m_j + 1$ which has contact at ζ_j and satisfies $D_{n_j}(\varphi_j, \zeta_j) = D_{n_j}(\varphi, \zeta_j)$.

Proof. Existence of $\varphi_1, \dots, \varphi_r$ follows from Lemma 4.2.4. The result follows by applying Theorem 4.2.1 to $\varphi, \varphi_1, \dots, \varphi_n$ with constants $1, -1, \dots, -1$. \square

4.3 Weighted Composition Operators and Adjoints in the Calkin Algebra

In [19], it is shown that the coset of the weighted composition operator $M_w C_\varphi$ in the Calkin algebra is determined by the values of w on $E(\varphi)$ in the following sense.

Proposition 4.3.1. [19, Theorem 3.1] *Let φ be an analytic self-map of \mathbb{D} with $|\varphi(e^{i\theta})| < 1$ a.e. Suppose w is a bounded measurable function on $\partial\mathbb{D}$ such that $|w|$ is continuous at every point of $E(\varphi)$. Then*

$$M_w C_\varphi \text{ is compact if and only if } w \equiv 0 \text{ on } E(\varphi).$$

Here we think of the operator $M_w C_\varphi$ as mapping H^2 to L^2 , however it can also be considered as mapping L^2 to L^2 and if $w \in H^\infty$ as mapping H^2 to H^2 .

For the special case that $\varphi \in \mathcal{A}$ has contact with $\partial\mathbb{D}$ at exactly one point, we have the following corollary.

Corollary 4.3.2. *Let $\varphi \in \mathcal{A}$ be such that $F(\varphi) = \{\zeta\}$. Suppose w is a bounded*

measurable function on $\partial\mathbb{D}$ such that w is continuous at ζ . Then

$$M_w C_\varphi \equiv w(\zeta) C_\varphi \pmod{\mathcal{K}}$$

Proof. Let $v(z) := w(z) - w(\zeta)$. Then v is bounded on $\partial\mathbb{D}$, continuous at ζ and satisfies $v(\zeta) = 0$ so by Proposition 4.3.1, $M_v C_\varphi = M_w C_\varphi - w(\zeta) C_\varphi$ is compact. \square

We will prove a similar result for weighted adjoints of composition operators. The proof relies on existence of an H^∞ function which satisfies several boundary conditions. Before we turn to constructing this H^∞ function, we state and prove the following fact.

Proposition 4.3.3. *If $f \in L^1(\partial\mathbb{D})$ is real valued, then the map h defined by*

$$h(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} f(e^{i\theta}) \frac{d\theta}{2\pi},$$

is analytic on \mathbb{D} , and has non-tangential boundary values a.e. on $\partial\mathbb{D}$ which satisfy

$$\operatorname{Re} h(e^{it}) = f(e^{it}) \text{ a.e. on } \partial\mathbb{D}.$$

Furthermore, if f is continuously differentiable on an open arc Γ contained in $\partial\mathbb{D}$, then h extends continuously to that arc.

Proof. We write

$$h(z) = \int_0^{2\pi} P_z(e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi} + i \int_0^{2\pi} Q_z(e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi},$$

where $P_z(e^{i\theta}) = \operatorname{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z}$ and $Q_z(e^{i\theta}) = \operatorname{Im} \frac{e^{i\theta} + z}{e^{i\theta} - z}$ are the Poisson and the conjugate

Poisson kernels. It is well known that the function

$$u(z) = \int_0^{2\pi} P_z(e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi}$$

has non-tangential limits a.e. which satisfy

$$\operatorname{Re} h(e^{it}) = u(e^{it}) = f(e^{it}) \text{ a.e. on } \partial\mathbb{D}.$$

Furthermore, if f is continuous on Γ then, by standard techniques, the maps $u_r(e^{it}) = u(re^{it})$ converge uniformly to their boundary function as $r \rightarrow 1$ on any closed arc contained in Γ . Since u is the uniform limit of continuous functions on Γ , u is also continuous on Γ . By [16, pp 78-79], similar results hold for the conjugate function v , defined by

$$v(z) = \int_0^{2\pi} Q_z(e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Namely, v has non-tangential limits a.e., and if f is continuously differentiable on Γ , then the maps $v_r(e^{it}) = v(re^{it})$ converge uniformly to their boundary functions as $r \rightarrow 1$ on any closed arc contained in Γ . Since v is the uniform limit of continuous functions on Γ , v is also continuous on Γ . \square

Lemma 4.3.4. *Let $\lambda \in \partial\mathbb{D}$ and v be a non-negative bounded function on $\partial\mathbb{D}$ which is continuous at λ , continuously differentiable in a deleted neighborhood of λ and satisfies $v(\lambda) = 0$. Then there exists a function $b \in H^\infty$ which extends continuously to $\partial\mathbb{D}$ and satisfies $b(\lambda) = 0$ and $|b(e^{i\theta})| \geq v(e^{i\theta})$ on $\partial\mathbb{D}$.*

Proof. We begin by constructing a continuously differentiable real valued function u on $\partial\mathbb{D}$. Let Γ be a closed arc centered at λ such that v is continuously differentiable on $\Gamma \setminus \{\lambda\}$. We define u on Γ by

$$u(e^{i\theta}) = v(e^{i\theta}) + |e^{i\theta} - \lambda|,$$

and extend u to $\partial\mathbb{D} \setminus \Gamma$ so that the resulting function u is continuously differentiable on $\partial\mathbb{D} \setminus \{\lambda\}$ and satisfies $u(e^{i\theta}) \geq v(e^{i\theta}) + |e^{i\theta} - \lambda|$ on $\partial\mathbb{D}$.

We let $f = \log u$, and note that f is continuously differentiable on $\partial\mathbb{D} \setminus \{\lambda\}$ and satisfies $f(e^{i\theta}) \geq \log |e^{i\theta} - \lambda|$, so that $f \in L^1(\partial\mathbb{D})$. Therefore, by Proposition 4.3.3, h defined by

$$h(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log u(e^{i\theta}) \frac{d\theta}{2\pi},$$

is analytic on \mathbb{D} , and extends continuously to $\partial\mathbb{D} \setminus \{\lambda\}$. Furthermore, on $\partial\mathbb{D} \setminus \{\lambda\}$,

$$\operatorname{Re} h(e^{it}) = \log u(e^{it}).$$

We define b on $\overline{\mathbb{D}}$ by $b(z) = e^{h(z)}$ for $z \neq \lambda$ and $b(\lambda) = 0$. Then on $\partial\mathbb{D} \setminus \{\lambda\}$,

$$|b(e^{it})| = e^{\operatorname{Re} h(e^{it})} = u(e^{it}),$$

so that $b \in H^\infty$ extends continuously to $\overline{\mathbb{D}} \setminus \{\lambda\}$, with $|b(e^{it})| \geq v(e^{it})$ on $\partial\mathbb{D}$. It remains to show that $b(z) \rightarrow 0$ as $z \rightarrow \lambda$ unrestrictedly in $\overline{\mathbb{D}}$.

Note that

$$|b(z)| = \exp \left(\int_0^{2\pi} P_z(e^{i\theta}) \log(u(e^{i\theta})) \frac{d\theta}{2\pi} \right),$$

where $P_z(e^{i\theta}) = \operatorname{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z}$ is the Poisson kernel. Thus, it suffices to show that

$$\int_0^{2\pi} P_z(e^{i\theta}) \log(u(e^{i\theta})) \frac{d\theta}{2\pi} \rightarrow -\infty$$

as $z \rightarrow \lambda$ unrestrictedly in \mathbb{D} . Let $M > 0$. We find ϵ such that for $|z - \lambda| < \epsilon$ the integral above is less than $-M$. Note that $\log u(e^{i\theta}) \rightarrow -\infty$ as $e^{i\theta} \rightarrow \lambda$, and let $\delta > 0$

be such that on the arc $\Gamma_\delta = \partial\mathbb{D} \cap \{|z - \lambda| < \delta\}$ we have $\log u(e^{i\theta}) < -4M$. Then

$$\int_{\Gamma_\delta} P_z(e^{i\theta}) \log(u(e^{i\theta})) \frac{d\theta}{2\pi} < -4M \int_{\Gamma_\delta} P_z(e^{i\theta}) \frac{d\theta}{2\pi}$$

Note that u is bounded from above so that $\log u < C$ for some constant C . Now choose $\epsilon < \delta$ such that for $|z - \lambda| < \epsilon$ we have both

$$\int_{\Gamma_\delta} P_z(e^{i\theta}) \frac{d\theta}{2\pi} > \frac{1}{2}$$

and $P_z(e^{i\theta}) < \frac{M}{C}$ for all $e^{i\theta} \in \partial\mathbb{D} \setminus \Gamma_\delta$.

Then for all $z \in \mathbb{D}$ such that $|z - \lambda| < \epsilon$, we have

$$\begin{aligned} \int_0^{2\pi} P_z(e^{i\theta}) \log u(e^{i\theta}) \frac{d\theta}{2\pi} &= \int_{\Gamma_\delta} P_z(e^{i\theta}) \log u(e^{i\theta}) \frac{d\theta}{2\pi} + \int_{\partial\mathbb{D} \setminus \Gamma_\delta} P_z(e^{i\theta}) \log u(e^{i\theta}) \frac{d\theta}{2\pi} \\ &< -4M \int_{\Gamma_\delta} P_z(e^{i\theta}) \frac{d\theta}{2\pi} + \int_{\partial\mathbb{D} \setminus \Gamma_\delta} \frac{M}{C} \cdot C \frac{d\theta}{2\pi} \\ &< -4M \cdot \frac{1}{2} + M = -M. \end{aligned}$$

This completes the proof. □

Proposition 4.3.5. *Let $\varphi \in \mathcal{A}$ be such that $F(\varphi) = \{\zeta\}$. Suppose w is a bounded measurable function on $\partial\mathbb{D}$ which is continuously differentiable in a deleted neighborhood of $\lambda = \varphi(\zeta)$. Then*

$$M_w C_\varphi^* \equiv w(\lambda) C_\varphi^* \pmod{\mathcal{K}}$$

where M_w is viewed as an operator from H^2 to L^2 , and in particular,

$$T_w C_\varphi^* \equiv w(\lambda) C_\varphi^* \pmod{\mathcal{K}}.$$

Proof. We can write

$$M_w C_\varphi^* = (M_{w-w(\lambda)} + M_{w(\lambda)}) C_\varphi^* = M_{w-w(\lambda)} C_\varphi^* + w(\lambda) C_\varphi^*.$$

Thus, it suffices to prove that $M_v C_\varphi^*$ is compact where $v = w - w(\lambda)$.

Note that $|v|$ is continuously differentiable in some deleted neighborhood of λ , and so by Lemma 4.3.4 there exists a $b \in H^\infty$ which extends continuously to $\partial\mathbb{D}$ and satisfies $b(\lambda) = 0$ and $|b(e^{i\theta})| \geq v(e^{i\theta})$ on $\partial\mathbb{D}$. We get that for all $f \in H^2$,

$$\begin{aligned} \|M_v C_\varphi^* f\|^2 &= \int_0^{2\pi} |v(e^{i\theta})|^2 |(C_\varphi^* f)(e^{i\theta})|^2 \frac{d\theta}{2\pi} \\ &\leq \int_0^{2\pi} |\overline{b(e^{i\theta})}|^2 |(C_\varphi^* f)(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \|M_{\bar{b}} C_\varphi^* f\|^2, \end{aligned}$$

so by Proposition 2.1.2, it suffices to show that $M_{\bar{b}} C_\varphi^*$ is a compact operator from H^2 to L^2 . We write

$$M_{\bar{b}} C_\varphi^* = P M_{\bar{b}} C_\varphi^* + (I - P) M_{\bar{b}} C_\varphi^*$$

and show that both terms on the right hand side are compact.

First note that since \bar{b} is a continuous function on $\partial\mathbb{D}$, the L^2 operator $(I - P) M_{\bar{b}} P$ is compact (see the version of Hartman's theorem in [22, p 214, Theorem 2.2.5]). Thus, the term $(I - P) M_{\bar{b}} C_\varphi^* = (I - P) M_{\bar{b}} P C_\varphi^*$ is compact from H^2 to L^2 .

We now show $P M_{\bar{b}} C_\varphi^* = T_{\bar{b}} C_\varphi^*$ is compact on H^2 by looking at its adjoint $(T_{\bar{b}} C_\varphi^*)^* = C_\varphi T_b$. Since $b \in H^\infty$, we have

$$C_\varphi T_b = C_\varphi M_b = M_{b \circ \varphi} C_\varphi.$$

Since $\varphi \in \mathcal{A}$ with $F(\varphi) = \{\zeta\}$ and $b(\varphi(\zeta)) = b(\lambda) = 0$, by Corollary 4.3.2 we have

$$M_{b \circ \varphi} C_\varphi \equiv b(\varphi(\zeta)) C_\varphi \equiv 0 \cdot C_\varphi \equiv 0 \pmod{\mathcal{K}},$$

provided that $|b \circ \varphi|$ considered as a function on $\partial\mathbb{D}$ is continuous at ζ . Recall that b is continuous on $\overline{\mathbb{D}}$, and that $\varphi \in \mathcal{A}$ with $F(\varphi) = \{\zeta\}$ so that $\varphi(e^{i\theta})$ is in \mathbb{D} and tends to λ as $e^{i\theta} \rightarrow \zeta$. Thus $|b \circ \varphi|$ is indeed continuous at ζ . \square

Chapter 5

Adjoint Formula for Rationally Induced Composition Operators

Recent work of Cowen-Gallardo [12], Hammond-Moorhouse-Robbins [15] and Bourdon-Shapiro [7] has produced pointwise formulas for C_φ^* , where the inducing map φ is rational. The constituent parts of these pointwise formulas contain multiple-valued analytic functions which do not necessarily represent well-defined operators individually. In Section 5.1 we show how to work with these pointwise formulas to produce legitimate operator equations involving C_φ^* for the rational case. Then, in Section 5.2, we consider the case where φ is basic and reduce our equations to the Calkin algebra.

5.1 From Pointwise Formula To Operator Equation

Let φ be a rational self-map of \mathbb{D} of degree d . We associate with φ its exterior map $\varphi_e := \rho \circ \varphi \circ \rho$, where ρ is the inversion in the unit circle defined by $\rho(z) := 1/\bar{z}$. Then φ_e maps $\mathbb{D}_e := \{z \in \mathbb{C} : |z| > 1\}$ into itself, and so $\varphi_e^{-1}(\mathbb{D}) \subset \mathbb{D}$.

Recall from Section 2.5 that for any simply connected domain V consisting of regular values of φ_e , there exist d distinct branches $\sigma_1, \dots, \sigma_d$ of φ_e^{-1} defined on V . Note that one possible choice of V is the unit disk with radial slits from each critical value of φ_e to the unit circle removed. A choice that may be much smaller but sufficient for our needs is a sufficiently small neighborhood of a regular value of φ_e .

The work of Cowen-Gallardo [12] and Hammond-Moorhouse-Robbins [15] has produced the following pointwise formula for the adjoint of a rationally-induced composition operator.

Proposition 5.1.1. *[15, Corollary 8] Suppose that V is a set on which d distinct branches $\sigma_1, \dots, \sigma_d$ of φ_e^{-1} are defined. Then for all $f \in H^2$ and all $z \in V \cap \mathbb{D}$,*

$$C_{\varphi}^* f(z) = \frac{f(0)}{1 - \overline{\varphi(\infty)}z} + \sum_{j=1}^d \frac{z\sigma'_j(z)}{\sigma_j(z)} f(\sigma_j(z))$$

In [7], Bourdon-Shapiro give an elementary proof for this original formula and introduce the following variant which we will use.

Proposition 5.1.2. *[7, Corollary 8] Suppose that V is a set on which d distinct branches $\sigma_1, \dots, \sigma_d$ of φ_e^{-1} are defined. Then for all $f \in H^2$ and all $z \in V \cap \mathbb{D}$,*

$$C_{\varphi}^* f(z) = \frac{f(0)}{1 - \overline{\varphi(0)}z} + \sum_{j=1}^d z\sigma'_j(z) S^* f(\sigma_j(z)), \quad (5.1.1)$$

where S^* is the adjoint of the shift operator S defined by $(Sf)(z) = zf(z)$.

5.1.1 Wishful Thinking - The Outer Regular Case

Note that Eq. (5.1.1) can be rewritten, at least formally, as

$$C_\varphi^* = \Lambda + \sum_{j=1}^d M_{h_j} C_{\sigma_j} S^*, \quad (5.1.2)$$

where $h_j(z) = z\sigma'_j(z)$, and Λ is the rank one operator defined by $\Lambda(f) := \frac{f(0)}{1-\varphi(0)z}$.

Notice that the terms $M_{h_j} C_{\sigma_j} S^*$ are reminiscent of Cowen's original formula for the linear-fractional case.

Bourdon-Shapiro define φ as outer regular when its critical values all lie in \mathbb{D} . Note that $w \in \mathbb{C}$ is a regular value for φ_e if and only if $\rho(w)$ is a regular value for φ , so that for outer regular functions φ we can choose $V = r\mathbb{D}$ for some $r > 1$. Then, restricting domains to \mathbb{D} , we have that $\sigma_1, \dots, \sigma_d$ are analytic self-maps of \mathbb{D} and h_1, \dots, h_d are H^∞ functions. Thus, the wishful thinking equation Eq. (5.1.2) is a legitimate operator equation in the outer regular case [7, Theorem 13(a)].

Example 5.1. [7, Example 9] Let φ be the analytic self-map of \mathbb{D} fixing 1 defined by

$$\varphi(z) = \frac{1}{3 - z - z^2}.$$

Then φ has a critical point at $z = -\frac{1}{2}$ so $\varphi(-\frac{1}{2}) = \frac{4}{13}$ is a critical value. Note that $\varphi(\infty) = 0$ is also a critical since $\varphi^{-1}(\{0\})$ contains only one point. Since both critical values of φ lie in \mathbb{D} , φ is outer regular and so Eq. (5.1.2) is a legitimate operator equation.

We calculate $\varphi_e = \rho \circ \varphi \circ \rho$ and solve the equation $\varphi_e(w) = z$ to obtain σ_1, σ_2 .

$$\varphi_e(w) = 3 - \frac{1}{w} - \frac{1}{w^2}$$

$$\sigma_1(z) = \frac{1 + \sqrt{13 - 4z}}{2(3 - z)} \quad \text{and} \quad \sigma_2(z) = \frac{1 - \sqrt{13 - 4z}}{2(3 - z)},$$

where $\sqrt{}$ denotes the principal branch of the square root. Indeed, σ_1 and σ_2 are analytic on $\mathbb{C} \setminus [\frac{13}{4}, \infty)$ except for an isolated singularity at $z = 3$.

Thus Eq. (5.1.2) provides the following legitimate operator formula for C_φ^* :

$$C_\varphi^* = \Lambda + M_{h_1} C_{\sigma_1} S^* + M_{h_2} C_{\sigma_2} S^*.$$

Note that $\tilde{\varphi} = \tau_1 \circ \varphi \circ \tau_1^{-1}$ is

$$\tilde{\varphi}(z) = \frac{iz^2 - 3z}{2z^2 + 5iz - 1}$$

and has Taylor expansion

$$\tilde{\varphi}(z) = 3z + 14iz^2 + O(z^3)$$

and so φ has order of contact 2 with the unit circle at $\zeta = 1$.

The fact that the outer regular function in the example has order of contact 2 with the unit circle is no coincidence. We prove the following proposition using results from Section 5.3.

Proposition 5.1.3. *If φ is rational such that φ has order of contact $n > 2$ with the unit circle at ζ , then φ is not outer regular.*

Proof. Suppose that φ is outer regular and let σ be a branch of φ_e^{-1} mapping $\varphi(\zeta)$ to ζ and defined on \mathbb{D} . We transfer the problem to the upper half-plane and work with

$$\tilde{\varphi} = \tau_{\varphi(\zeta)}^{-1} \circ \varphi \circ \tau_\zeta, \quad \tilde{\varphi}_e = \tau_{\varphi(\zeta)}^{-1} \circ \varphi_e \circ \tau_\zeta, \quad \tilde{\sigma} = \tau_\zeta^{-1} \circ \sigma \circ \tau_{\varphi(\zeta)}$$

which are all analytic self-maps of the upper half-plane according to our assumption.

Let a_0, a_1, a_2, a_3 denote the initial Taylor coefficients of $\tilde{\varphi}$ at $z = 0$, and b_0, b_1, b_2, b_3 denote those of $\tilde{\sigma}$. In Proposition 5.3.6, we show that σ has order of contact n at λ . Thus by Theorem 3.1.3 the Hankel matrices $H_m(a_1, \dots, a_{n-1})$ and $H_m(b_1, \dots, b_{n-1})$ are both positive, and in particular, their 2nd leading principal minors are positive, i.e.,

$$a_1 a_3 - a_2^2 > 0, \quad b_1 b_3 - b_2^2 > 0.$$

In Proposition 5.3.4 we show that $\tilde{\varphi}$ and $\tilde{\varphi}_e$ have equal Taylor coefficient a_0, \dots, a_{n-1} . Noting that $\tilde{\varphi}_e \circ \tilde{\sigma} = id$, we can therefore express b_1, b_2, b_3 in terms of a_1, a_2, a_3 using the product rule (or Faà di Bruno's formula, Theorem 5.3.1). We get the equations

$$\begin{aligned} 1 &= (\tilde{\varphi}_e \circ \tilde{\sigma})'(z) = \tilde{\varphi}_e'(\tilde{\sigma}(z))\tilde{\sigma}'(z) \\ 0 &= (\tilde{\varphi}_e \circ \tilde{\sigma})''(z) = \tilde{\varphi}_e''(\tilde{\sigma}(z))\tilde{\sigma}'(z)^2 + \tilde{\varphi}_e'(\tilde{\sigma}(z))\tilde{\sigma}''(z) \\ 0 &= (\tilde{\varphi}_e \circ \tilde{\sigma})'''(z) = \tilde{\varphi}_e'''(\tilde{\sigma}(z))\tilde{\sigma}'(z)^3 + 3\tilde{\varphi}_e''(\tilde{\sigma}(z))\tilde{\sigma}'(z)\tilde{\sigma}''(z) + \tilde{\varphi}_e'(\tilde{\sigma}(z))\tilde{\sigma}'''(z), \end{aligned}$$

and evaluating at $z = 0$, we have

$$1 = a_1 b_1, \quad 0 = 2a_2 b_1^2 + a_1 \cdot 2b_2, \quad 0 = 6a_3 b_1^3 + 3 \cdot 2a_2 b_1 \cdot 2b_2 + a_1 \cdot 6b_3.$$

Solving for b_1, b_2, b_3 we see that

$$b_1 = \frac{1}{a_1}, \quad b_2 = -\frac{a_2}{a_1^3}, \quad b_3 = \frac{2a_2^2 - a_1 a_3}{a_1^5}.$$

Thus

$$b_1 b_3 - b_2^2 = \frac{1}{a_1} \cdot \frac{2a_2^2 - a_1 a_3}{a_1^5} - \frac{a_2^2}{a_1^6} = \frac{-(a_1 a_3 - a_2^2)}{a_1^6} < 0$$

which is a contradiction. □

5.1.2 Operator Equation Involving C_φ^*

In general, the set V on which $\sigma_1, \dots, \sigma_d$ are analytic can not be chosen to contain all of \mathbb{D} , and so the wishful thinking operators $C_{\sigma_1}, \dots, C_{\sigma_d}$ in Eq. (5.1.2) are not legitimate operators. We overcome this difficulty by pre-composing with a map ψ with image contained in $V \cap \mathbb{D}$ to obtain analytic self-maps of \mathbb{D} , $\sigma_1 \circ \psi, \dots, \sigma_d \circ \psi$. This will enable us to write a legitimate operator formula for $C_\psi C_\varphi^*$.

Proposition 5.1.4. *Let ψ be an analytic self-map of \mathbb{D} satisfying $\overline{\psi(\mathbb{D})} \subset V$. Then*

$$C_\psi C_\varphi^* = C_\psi \Lambda + \sum_{j=1}^d M_{h_j \circ \psi} C_{\sigma_j \circ \psi} S^*$$

where $h_j(z) = z\sigma'_j(z)$, S^* is the adjoint of the shift operator and Λ is the rank one operator defined by $\Lambda(f) := \frac{f(0)}{1-\varphi(0)z}$.

In particular, $h_j \circ \psi$ are H^∞ functions and $\sigma_j \circ \psi$ are analytic self-maps of \mathbb{D} .

Proof. Since ψ maps \mathbb{D} into V and $\sigma_1, \dots, \sigma_d$ are analytic on V , we get that $\sigma_1 \circ \psi, \dots, \sigma_d \circ \psi$ are analytic self-maps of \mathbb{D} (recall that $\varphi_e^{-1}(\mathbb{D}) \subset \mathbb{D}$). The functions $\sigma'_1, \dots, \sigma'_d$ are analytic on V and so bounded on $\psi(\mathbb{D})$, so $h_1 \circ \psi, \dots, h_d \circ \psi$ are H^∞ functions. By the pointwise formula given in Proposition 5.1.2, for all $f \in H^2$ and all

$z \in \mathbb{D}$ we have

$$\begin{aligned}
C_\psi C_\varphi^* f(z) &= C_\varphi^* f(\psi(z)) = \frac{f(0)}{1 - \overline{\varphi(0)}\psi(z)} + \sum_{j=1}^d \psi(z) \sigma_j'(\psi(z)) S^* f(\sigma_j(\psi(z))) \\
&= (\Lambda f)(\psi(z)) + \sum_{j=1}^d h_j(\psi(z)) S^* f((\sigma_j \circ \psi)(z)) \\
&= (C_\psi \Lambda f)(z) + \sum_{j=1}^d (M_{h_j \circ \psi} C_{\sigma_j \circ \psi} S^* f)(z).
\end{aligned}$$

Therefore, the operator formula in the statement holds. \square

5.2 The Basic Case

We turn to the case where φ is a basic function of degree d with order of contact n at ζ . Denote $\lambda = \varphi(\zeta)$ and recall that $\lambda \in \partial\mathbb{D}$ is a regular value for φ and so $\rho(\lambda) = \lambda$ is a regular value for φ_e . Thus we can choose V to be a neighborhood of λ consisting of regular values of φ_e .

Recall from Section 2.5 that $\sigma_1(V), \dots, \sigma_d(V)$ are pairwise disjoint, and let σ denote the unique branch of φ_e^{-1} that maps λ to ζ .

Proposition 5.2.1. *Let ψ be an analytic self map of \mathbb{D} satisfying $\overline{\psi(\mathbb{D})} \subset V$. Then*

$$C_\psi C_\varphi^* \equiv M_{h \circ \psi} C_{\sigma \circ \psi} S^* \pmod{\mathcal{K}}$$

where $h(z) := z\sigma'(z)$.

In particular, $h \circ \psi$ is an $H^\infty(\mathbb{D})$ function and $\sigma \circ \psi$ is an analytic self-map of \mathbb{D} .

Proof. Notice that since ρ is the identity on $\partial\mathbb{D}$, φ_e maps exactly one point in $\partial\mathbb{D}$ (the point ζ) into $\partial\mathbb{D}$ and so $\varphi_e^{-1}(\partial\mathbb{D}) = \{\zeta\}$. We also have that the branches $\sigma_1, \dots, \sigma_d$ map $\overline{\psi(\mathbb{D})}$ to pairwise disjoint subsets of $\overline{\mathbb{D}}$ (see Section 2.5), and these sets are closed as continuous images of compact sets. Since $\lambda \in \overline{\psi(\mathbb{D})}$ and by the definition of σ , we see that $\zeta \in \sigma(\overline{\psi(\mathbb{D})})$, and so for $\sigma_j \neq \sigma$, the closed set $\sigma(\overline{\psi(\mathbb{D})})$ does not intersect $\partial\mathbb{D}$. Thus, we have $\|\sigma_j \circ \psi\|_\infty < 1$ and so the composition operator $\mathbb{C}_{\sigma_j \circ \psi}$ is compact.

Since Λ in Proposition 5.1.4 is rank one, reducing Proposition 5.1.4 modulo the compacts gives

$$C_\psi C_\varphi^* \equiv M_{h \circ \psi} C_{\sigma \circ \psi} S^* \pmod{\mathcal{K}}$$

where h is defined by $h(z) = z\sigma'(z)$. Proposition 5.1.4 also shows that $h \circ \psi$ is an $H^\infty(\mathbb{D})$ function and $\sigma \circ \psi$ is an analytic self-map of \mathbb{D} . \square

By continuity of σ and since $\sigma(\lambda) = \zeta$, there exists a neighborhood $V_0 \subset V$ of λ on which σ is bounded away from zero.

Proposition 5.2.2. *Let ψ be an analytic self map of \mathbb{D} mapping $\overline{\mathbb{D}}$ into V_0 . Then*

$$C_\psi C_\varphi^* \equiv M_{g \circ \psi} \mathbb{C}_{\sigma \circ \psi} \pmod{\mathcal{K}}$$

where $g(z) := \frac{z\sigma'(z)}{\sigma(z)}$.

In particular, $g \circ \psi$ is an $H^\infty(\mathbb{D})$ function and $\sigma \circ \psi$ is an analytic self-map of \mathbb{D} .

Proof. By Proposition 5.2.1,

$$C_\psi C_\varphi^* \equiv M_{h \circ \psi} C_{\sigma \circ \psi} S^* \pmod{\mathcal{K}}$$

where h is defined by $h(z) = z\sigma'(z)$, and $h \circ \psi \in H^\infty(\mathbb{D})$ and $\sigma \circ \psi$ is an analytic self-map of \mathbb{D} . Note that for any $f \in H^2$ and any nonzero $z \in \mathbb{D}$, we have that

$$(S^*f)(z) = \frac{f(z) - f(0)}{z}.$$

Thus, there exists a $K_1 \in \mathcal{K}$ such that for any $f \in H^2$ and $z \in \mathbb{D}$, we have

$$\begin{aligned} (C_\psi C_\varphi^* f)(z) &= (M_{h \circ \psi} C_{\sigma \circ \psi} S^* f)(z) + (K_1 f)(z) \\ &= \psi(z)\sigma'(\psi(z)) \frac{f(\sigma(\psi(z))) - f(0)}{\sigma(\psi(z))} + (K_1 f)(z) \\ &= \frac{\psi(z)\sigma'(\psi(z))}{\sigma(\psi(z))} f(\sigma(\psi(z))) - \frac{\psi(z)\sigma'(\psi(z))}{\sigma(\psi(z))} f(0) + (K_1 f)(z) \\ &= g(\psi(z))f(\sigma(\psi(z))) - g(\psi(z))f(0) + (K_1 f)(z) \end{aligned}$$

Since ψ maps $\overline{\mathbb{D}}$ into V_0 , $\sigma \circ \psi$ is bounded away from zero on $\overline{\mathbb{D}}$, and so $g \circ \psi$ is an $H^\infty(\mathbb{D})$ function. We can define the rank one operator $(K_2 f)(z) := g(\psi(z))f(0)$, and rewrite the above equation as

$$(C_\psi C_\varphi^* f)(z) = (M_{g \circ \psi} C_{\sigma \circ \psi} f)(z) - (K_2 f)(z) + (K_1 f)(z).$$

Thus $C_\psi C_\varphi^* f \equiv M_{g \circ \psi} C_{\sigma \circ \psi} f \pmod{\mathcal{K}}$, as desired. \square

5.2.1 Generalized Adjoint Formula in the Calkin Algebra

If ψ satisfies stronger conditions, we get the following formula which generalizes the adjoint formula modulo \mathcal{K} , developed by Kriete-MacCluer-Moorhouse [20], of a composition operator induced by a linear fractional map (see Proposition 2.3.8).

Recall that $f(\xi)$ denotes the non-tangential limit of f at ξ for a function f of \mathbb{D} and $\xi \in \partial\mathbb{D}$.

Proposition 5.2.3. *There exists a neighborhood W of λ such that for any self-map ψ of \mathbb{D} that satisfies:*

1. $\psi(D) \subset W$
2. ψ is analytic at λ and fixes it
3. $\psi^{-1}(\{\lambda\}) := \{\beta \in \partial\mathbb{D} : \psi(\beta) \text{ exists and is equal to } \lambda\} = \{\lambda\}$

we have that the map $\sigma \circ \psi$ is in the class \mathcal{A} with $F(\sigma \circ \psi) = \{\lambda\}$ and that

$$C_\psi C_\varphi^* \equiv \frac{1}{|\varphi'(\zeta)|} C_{\sigma \circ \psi} \pmod{\mathcal{K}}.$$

Proof. Let W be a neighborhood of λ such that the closure of W is contained in V_0 . Then for any ψ as above, $\overline{\psi(\mathbb{D})}$ is contained in V_0 , and so by Proposition 5.2.2 we have

$$C_\psi C_\varphi^* \equiv M_{g \circ \psi} C_{\sigma \circ \psi} \pmod{\mathcal{K}}.$$

To show that $\sigma \circ \psi \in \mathcal{A}$, first recall that

$$E(\sigma \circ \psi) = \overline{\bigcup_{|\alpha|=1} \text{spt}(\mu_\alpha^s)}$$

and that for each $\alpha \in \partial\mathbb{D}$, μ_α^s is carried by

$$(\sigma \circ \psi)^{-1}(\{\alpha\}) = \{\beta \in \partial\mathbb{D} : (\sigma \circ \psi)(\beta) \text{ exists and is equal to } \alpha\}.$$

Suppose $(\sigma \circ \psi)(\beta) = \alpha$ for some $\alpha, \beta \in \partial\mathbb{D}$. Then applying φ_e we get that $\psi(\beta)$ exists and $\sigma(\psi(\beta)) = \alpha$. Recall that $\varphi_e^{-1}(\partial\mathbb{D}) = \{\zeta\}$, $\sigma(\lambda) = \zeta$ and σ is univalent on V . Thus $\sigma(\psi(\beta)) = \alpha$ implies that $\alpha = \zeta$ and $\psi(\beta) = \lambda$, and so by the third condition on ψ we have $\beta = \lambda$. Therefore $spt(\mu_\alpha^s)$ is empty for $\alpha \neq \zeta$, and $spt(\mu_\zeta^s) \subset \{\lambda\}$. We conclude that $E(\sigma \circ \psi) \subset \{\lambda\}$.

We have that $\sigma \circ \psi$ is analytic at λ and maps λ to ζ , so that $F(\sigma \circ \psi) = \{\lambda\}$. Since $\sigma \circ \psi$ is analytic at λ and does not map an arc of $\partial\mathbb{D}$ containing λ into $\partial\mathbb{D}$, $\sigma \circ \psi$ has finite order of contact at λ and a Taylor expansion to that order about λ . This shows that $\sigma \circ \psi$ is indeed in the class \mathcal{A} .

Recall that due to the definition of V_0 , for $g(z) = \frac{z\sigma'(z)}{\sigma(z)}$ the map $g \circ \psi$ is an $H^\infty(\mathbb{D})$ function. Since $g \circ \psi$ is also continuous at λ , we can apply Corollary 4.3.2 to get:

$$C_\psi C_\varphi^* \equiv (g \circ \psi)(\lambda) C_{\sigma \circ \psi} \pmod{\mathcal{K}}.$$

We calculate

$$(g \circ \psi)(\lambda) = g(\lambda) = \frac{\lambda\sigma'(\lambda)}{\sigma(\lambda)} = \frac{\lambda}{\zeta\varphi'(\zeta)} = \frac{1}{\zeta\bar{\lambda}\varphi'(\zeta)},$$

where $\sigma'(\lambda) = 1/\varphi'(\zeta)$ follows by applying the chain rule for $\sigma \circ \varphi_e$ at $z = \zeta$, and substituting $\varphi_e'(\zeta) = \varphi'(\zeta)$ which holds by Proposition 5.3.4 or direct calculation.

To complete the proof, note that by Theorem 2.3.3, $\varphi'(\zeta) = d(\zeta)\bar{\zeta}\lambda$, where $d(\zeta) > 0$, and so $|\varphi'(\zeta)| = d(\zeta) = \zeta\bar{\lambda}\varphi'(\zeta)$. □

Note that for a linear fractional map $\varphi(z) = \frac{az+b}{cz+d}$, we have that $\varphi_e(z)$ is invertible, and $\varphi_e^{-1}(z) = \sigma(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}$ is the Krein adjoint of φ . Recall that by

Proposition 2.3.8,

$$C_\varphi^* \equiv \frac{1}{|\varphi'(\zeta)|} C_\sigma \pmod{\mathcal{K}},$$

and so Proposition 5.2.3 can be viewed as a generalization of Proposition 2.3.8.

Note also that if $\tau \in \mathcal{A}$ with $F(\tau) = \{\zeta\}$, Proposition 2.3.8 can be extended to produce a formula for C_τ^* provided τ has order of contact 2 with $\partial\mathbb{D}$ at ζ , whereas Proposition 5.2.3 can be extended to produce a formula for $C_\psi C_\tau^*$ provided that ψ satisfies the conditions of Proposition 5.2.3, and regardless of the order of contact of τ with $\partial\mathbb{D}$ at ζ .

5.3 Relationship Between φ and σ

Let φ be a rational function with order of contact n with the unit circle at ζ , mapping ζ to λ . Let σ be a branch of φ_e^{-1} mapping λ to ζ . Then $\sigma \circ \varphi_e = id$ near ζ and $\varphi_e \circ \sigma = id$ near λ . Thus

$$D_n(\sigma \circ \varphi_e, \zeta) = (\zeta, 1, 0, \dots, 0), \quad \text{and} \quad D_n(\varphi_e \circ \sigma, \lambda) = (\lambda, 1, 0, \dots, 0).$$

Although φ and σ are not inverse functions, we will show that they are “almost inverse” in the sense that

$$D_{n-1}(\sigma \circ \varphi, \zeta) = (\zeta, 1, 0, \dots, 0), \quad \text{and} \quad D_{n-1}(\varphi \circ \sigma, \lambda) = (\lambda, 1, 0, \dots, 0).$$

Throughout this section, we will use Faà di Bruno’s formula, an identity generalizing the chain rule that has been known since 1800. The following is the statement of the formula in combinatorial form.

Theorem 5.3.1 (Faà di Bruno's formula). [17] *If g is analytic at z and f is analytic at $g(z)$, then*

$$(f \circ g)^{(k)}(z) = \sum_{\pi \in \Pi} f^{(|\pi|)}(g(z)) \cdot \prod_{B \in \pi} g^{(|B|)}(z),$$

where Π is the set of partitions of $\{1, \dots, k\}$.

Note that as a consequence of Theorem 5.3.1, if g is analytic at z and f is analytic at $g(z)$ then $D_k(g, z)$ and $D_k(f, g(z))$ determine $D_k(f \circ g, z)$. We rewrite Faà di Bruno's formula in the following way.

Lemma 5.3.2. *If g is analytic at z and f is analytic at $g(z)$, then*

$$(f \circ g)^{(k)}(z) = f^{(k)}(g(z))g'(z)^k + F(D_{k-1}(f, g(z)), D_{k-1}(g, z)) + f'(g(z))g^{(k)}(z),$$

where

$$F((a_0, a_1, \dots, a_{k-1}), (b_0, b_1, \dots, b_{k-1})) = \sum_{\substack{\pi \in \Pi \\ 1 < |\pi| < k}} a_{|\pi|} \cdot \prod_{B \in \pi} b_{|B|}.$$

Proof. This follows from Theorem 5.3.1, with the first term originating from the partition $\pi = \{\{1\}, \{2\}, \dots, \{k\}\}$, and the last term originating from the partition $\pi = \{\{1, 2, \dots, k\}\}$. □

We transfer φ, φ_e and σ to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$, and analyze their relationships there. The corresponding self-maps of \mathbb{H} are

$$\widetilde{\varphi} = \tau_\lambda \circ \varphi \circ \tau_\zeta^{-1}, \quad \widetilde{\varphi}_e = \tau_\lambda \circ \varphi_e \circ \tau_\zeta^{-1}, \quad \widetilde{\sigma} = \tau_\zeta \circ \sigma \circ \tau_\lambda^{-1}$$

where $\tau_\alpha(z) = i\frac{\alpha-z}{\alpha+z}$ is a conformal map which maps \mathbb{D} onto \mathbb{H} and α to 0. Note that

$\widetilde{\varphi}, \widetilde{\varphi}_e$ and $\widetilde{\sigma}$ all fix 0 and are analytic in a neighborhood of 0.

Lemma 5.3.3. $\widetilde{\varphi}_e$ is the upper half-plane exterior map associated with $\widetilde{\varphi}$, i.e.,

$$\widetilde{\varphi}_e(z) = \overline{\widetilde{\varphi}(\bar{z})}.$$

Proof. Recall that $\varphi_e = \rho \circ \varphi \circ \rho$ where $\rho: z \rightarrow 1/\bar{z}$, and so

$$\widetilde{\varphi}_e(z) = (\tau_\lambda \circ \rho \circ \varphi \circ \rho \circ \tau_\zeta^{-1})(z).$$

We denote $r(z) = \bar{z}$, so that

$$\overline{\widetilde{\varphi}(\bar{z})} = (r \circ \widetilde{\varphi} \circ r)(z) = (r \circ \tau_\lambda \circ \varphi \circ \tau_\zeta^{-1} \circ r)(z).$$

It therefore suffices to show that $\tau_\alpha \circ \rho = r \circ \tau_\alpha$ for any $\alpha \in \partial\mathbb{D}$. Indeed

$$\tau_\alpha \left(\frac{1}{\bar{z}} \right) = i \frac{\alpha - \frac{1}{\bar{z}}}{\alpha + \frac{1}{\bar{z}}} = i \frac{\bar{z} - \bar{\alpha}}{\bar{z} + \bar{\alpha}} = -i \frac{\bar{\alpha} - \bar{z}}{\bar{\alpha} + \bar{z}} = \overline{\tau_\alpha(z)}$$

□

Proposition 5.3.4. $D_{n-1}(\varphi, \zeta) = D_{n-1}(\varphi_e, \zeta)$ and $\varphi^{(n)}(\zeta) \neq \varphi_e^{(n)}(\zeta)$.

Proof. By Theorem 5.3.1, it suffices show that

$$D_{n-1}(\widetilde{\varphi}, 0) = D_{n-1}(\widetilde{\varphi}_e, 0) \quad \text{and} \quad \widetilde{\varphi}^{(n)}(0) \neq \widetilde{\varphi}_e^{(n)}(0).$$

Let $\widetilde{\varphi}(z) = \sum_0^\infty a_k z^k$ be the Taylor expansion of $\widetilde{\varphi}$ about 0. Then by Lemma 5.3.3, we can write $\widetilde{\varphi}_e(z) = \overline{\widetilde{\varphi}(\bar{z})} = \sum_0^\infty \bar{a}_k z^k$, and so we have

$$\widetilde{\varphi}^{(k)}(0) = k! a_k \quad \text{and} \quad \widetilde{\varphi}_e^{(k)}(0) = k! \bar{a}_k$$

for all k . To complete the proof, recall that φ has order of contact n with the unit circle, and so $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ and $\text{Im } a_n > 0$. □

Lemma 5.3.5. *The n 'th order data for $\sigma \circ \varphi$ and $\varphi \circ \sigma$ is given by*

$$D_n(\sigma \circ \varphi, \zeta) = (\zeta, 1, 0, \dots, 0, c \cdot \sigma'(\lambda))$$

$$D_n(\varphi \circ \sigma, \lambda) = (\lambda, 1, 0, \dots, 0, c \cdot \sigma'(\lambda)^n),$$

where c is a non zero constant given by $c = \varphi^{(n)}(\zeta) - \varphi_e^{(n)}(\zeta)$.

Proof. By Lemma 5.3.2, $(\sigma \circ \varphi)^{(n)}(\zeta)$ and $(\sigma \circ \varphi_e)^{(n)}(\zeta)$ can be written as follows.

$$(\sigma \circ \varphi)^{(n)}(\zeta) = \sigma^{(n)}(\lambda) \varphi'(\zeta)^n + F(D_{n-1}(\sigma, \lambda), D_{n-1}(\varphi, \zeta)) + \sigma'(\lambda) \varphi^{(n)}(\zeta)$$

$$(\sigma \circ \varphi_e)^{(n)}(\zeta) = \sigma^{(n)}(\lambda) \varphi_e'(\zeta)^n + F(D_{n-1}(\sigma, \lambda), D_{n-1}(\varphi_e, \zeta)) + \sigma'(\lambda) \varphi_e^{(n)}(\zeta)$$

Recall that $\sigma \circ \varphi_e = id$ and so $(\sigma \circ \varphi_e)^{(n)}(\zeta) = 0$, and that by Proposition 5.3.4

$D_{n-1}(\varphi, \zeta) = D_{n-1}(\varphi_e, \zeta)$. Thus subtracting the above equations yields:

$$(\sigma \circ \varphi)^{(n)}(\zeta) = \sigma'(\lambda) \cdot (\varphi^{(n)}(\zeta) - \varphi_e^{(n)}(\zeta)).$$

Similarly, applying Theorem 5.3.1 to $(\varphi \circ \sigma)^{(n)}(\lambda)$ and $(\varphi_e \circ \sigma)^{(n)}(\lambda)$, we get

$$(\varphi \circ \sigma)^{(n)}(\zeta) = \varphi^{(n)}(\zeta) \sigma'(\lambda)^n + F(D_{n-1}(\varphi, \zeta), D_{n-1}(\sigma, \lambda)) + \varphi'(\zeta) \sigma^{(n)}(\lambda)$$

$$(\varphi_e \circ \sigma)^{(n)}(\zeta) = \varphi_e^{(n)}(\zeta) \sigma'(\lambda)^n + F(D_{n-1}(\varphi_e, \zeta), D_{n-1}(\sigma, \lambda)) + \varphi_e'(\zeta) \sigma^{(n)}(\lambda).$$

Again, since $D_{n-1}(\varphi, \zeta) = D_{n-1}(\varphi_e, \zeta)$ and $(\sigma \circ \varphi_e)^{(n)}(\zeta) = 0$, subtracting the above equations yields:

$$(\varphi \circ \sigma)^{(n)}(\zeta) = \sigma'(\lambda)^n \cdot (\varphi^{(n)}(\zeta) - \varphi_e^{(n)}(\zeta)).$$

Recall that by Proposition 5.3.4, $\varphi_e^{(n)}(\zeta) \neq \varphi^{(n)}(\zeta)$, and so $c \neq 0$. □

As additional applications of Faà di Bruno's Formula, we calculate the order of contact of σ with $\partial\mathbb{D}$ at λ and determine the order of contact of composite maps under some assumptions.

Proposition 5.3.6. *The map σ has order of contact n with $\partial\mathbb{D}$ at λ .*

Proof. Let $\tilde{\varphi}(z) = \sum_0^\infty a_k z^k$ and $\tilde{\sigma}(z) = \sum_0^\infty b_k z^k$ be the Taylor expansions about 0 of $\tilde{\varphi}$ and $\tilde{\sigma}$ respectively. We have that $a_0, \dots, a_{n-1} \in \mathbb{R}$ and $\text{Im } a_n > 0$ since φ has order of contact n with \mathbb{D} , and it suffices to show that $b_0 = 0, b_1, \dots, b_{n-1} \in \mathbb{R}$ and $\text{Im } b_n > 0$ (see Proposition 2.4.2, Proposition 2.4.1).

Recall that $\tilde{\varphi}_e(z) = \overline{\tilde{\varphi}(\bar{z})} = \sum_0^\infty \bar{a}_k z^k$ by Lemma 5.3.3. Notice that $\tilde{\varphi}$ fixes 0 and maps $W \cap \mathbb{H}$ into \mathbb{H} for some neighborhood W of 0, and so $a_1 > 0$ and $b_1 = \tilde{\sigma}'(0) = 1/\tilde{\varphi}_e'(0) = 1/a_1 > 0$. Using Lemma 5.3.2 for $\tilde{\sigma} \circ \tilde{\varphi}_e = id$ and $k > 1$, we get

$$0 = \tilde{\sigma}^{(k)}(0)\tilde{\varphi}_e'(0)^k + F(D_{k-1}(\tilde{\sigma}, 0), D_{k-1}(\tilde{\varphi}_e, 0)) + \tilde{\sigma}'(0)\tilde{\varphi}_e^{(k)}(0).$$

Thus, by induction on $k = 2, \dots, n-1$, we see that b_0, \dots, b_k are real valued. For $k = n$, taking imaginary parts we get

$$0 = \text{Im}(\tilde{\sigma}^{(n)}(0))\tilde{\varphi}_e'(0)^n + \tilde{\sigma}'(0) \text{Im}(\tilde{\varphi}_e^{(n)}(0)).$$

or

$$\text{Im}(b_n) = -\frac{b_1}{a_1^n} \text{Im}(\bar{a}_n) = \frac{b_1}{a_1^n} \text{Im}(a_n) > 0.$$

□

Proposition 5.3.7. *Let ψ_1 be analytic at α_1 such that for some neighborhood W_1 of α_1 , ψ_1 maps $W_1 \cap \mathbb{D}$ into \mathbb{D} , and let ψ_2 be analytic at $\alpha_2 = \psi_1(\alpha_1)$ such that for some neighborhood W_2 of α_2 , ψ_2 maps $W_2 \cap \mathbb{D}$ into \mathbb{D} .*

We define N_1 and N_2 by

$$N_i = \begin{cases} c & \psi_i \text{ has order of contact } c \text{ with } \partial\mathbb{D} \text{ at } \alpha_i \\ \infty & \psi_i \text{ maps an arc of } \partial\mathbb{D} \text{ containing } \alpha_i \text{ into } \partial\mathbb{D} \end{cases}$$

Let K be the minimum of M and N . Then if K is finite, $\psi_2 \circ \psi_1$ has order of contact K at α_1 , and if K is infinite then $\psi_2 \circ \psi_1$ maps an arc of $\partial\mathbb{D}$ containing α_1 into $\partial\mathbb{D}$.

Proof. The corresponding maps on the upper half-plane are

$$f = \widetilde{\psi_2} = \tau_{\psi_2(\alpha_2)} \circ \psi_2 \circ \tau_{\alpha_2}^{-1}, \quad g = \widetilde{\psi_1} = \tau_{\alpha_2} \circ \psi_1 \circ \tau_{\alpha_1}^{-1}$$

and

$$(\widetilde{\psi_2 \circ \psi_1}) = \tau_{(\psi_2 \circ \psi_1)(\alpha_1)} \circ \psi_2 \circ \psi_1 \circ \tau_{\alpha_1}^{-1} = f \circ g.$$

Note that these maps are all analytic in a neighborhood of 0, and let $f(z) = \sum_{j=0}^{j=\infty} c_j z^j$ and $g(z) = \sum_{j=0}^{j=\infty} d_j z^j$ be the Taylor series of f and g there. Note that if N_2 is finite then by Proposition 2.4.2 and Proposition 2.4.1, $c_1, \dots, c_{N-1} \in \mathbb{R}$ and $c_N \in \mathbb{H}$ and if N_2 is infinite then $c_1, c_2, \dots \in \mathbb{R}$. The corresponding statement for g holds similarly. Note that since f and g both map some neighborhood of zero intersected with \mathbb{H} into \mathbb{H} , we have that $c_1, d_1 > 0$.

If $c_1, \dots, c_k, d_1, \dots, d_k \in \mathbb{R}$, then by Theorem 5.3.1 and induction on $j = 1, \dots, k$, we see that $(f \circ g)'(0), \dots, (f \circ g)^{(j)}(0)$ are all real valued. Thus, $(f \circ g)'(0), \dots, (f \circ g)^{(k)}(0)$

are real valued in this case.

In the case where K is infinite, we get that all Taylor coefficients of $f \circ g$ at 0 are real valued and so $f \circ g$ maps an interval in \mathbb{R} containing 0 into \mathbb{R} . Thus $\psi_2 \circ \psi_1$ maps an arc of $\partial\mathbb{D}$ containing α_1 into $\partial\mathbb{D}$.

In the case where K is finite, we get by the argument above that

$$c_1, \dots, c_{K-1}, d_1, \dots, d_{K-1} \in \mathbb{R}.$$

By Lemma 5.3.2 at $z = 0$ we have

$$(f \circ g)^{(K)}(0) = f^{(K)}(0)g'(0)^K + F(D_{K-1}(f, 0), D_{K-1}(g, 0)) + f'(0)g^{(K)}(0).$$

and taking imaginary parts we get

$$\operatorname{Im}(f \circ g)^{(K)}(0) = g'(0)^K \operatorname{Im} f^{(K)}(0) + f'(0) \operatorname{Im} g^{(K)}(0).$$

Since $c_1, d_1 > 0$, and $\operatorname{Im} c_K, \operatorname{Im} d_K \geq 0$ with at least one of them positive by the definition of K , it follows that $\operatorname{Im}(f \circ g)^{(K)}(0) > 0$. Thus $f \circ g$ has order of contact K with \mathbb{R} at 0, and so by Proposition 2.4.2, $\psi_2 \circ \psi_1$ has order of contact K at α_1 . \square

Chapter 6

Essential normality

We begin by applying our results from Section 5.2 to characterize essential normality for the basic case. We then decompose C_φ into a sum of basic composition operators and show that essential normality can only occur when this sum consists of a single term. Lastly, using the parametrization of functions with given order of contact proved in Chapter 3, we construct essentially normal composition operators which have arbitrary even order of contact with the unit circle at one point.

Definition 6.1. For $\epsilon > 0$ we define $\psi_{\lambda,\epsilon}$ to be the Riemann mapping from \mathbb{D} onto $\{|z - \lambda| < \epsilon\} \cap \mathbb{D}$, which fixes λ . Note that $\psi = \psi_{\lambda,\epsilon}$ extends continuously to $\partial\mathbb{D}$ and analytically across $\partial\mathbb{D}$ in a neighborhood of λ .

6.1 Essential Normality For Basic Composition Operators

Let φ be a basic function with contact at ζ which fixes ζ . Let n be the order of contact of φ with $\partial\mathbb{D}$ at ζ , and let σ be the unique branch of φ_e^{-1} defined on a neighborhood

of ζ which fixes ζ .

Proposition 6.1.1. *There exists $\epsilon > 0$ such that for $\psi = \psi_{\zeta, \epsilon}$, the following conditions are equivalent.*

1. $C_\psi[C_\varphi, C_\varphi^*]$ is compact
2. $D_n(\sigma \circ \varphi \circ \psi, \zeta) = D_n(\varphi \circ \sigma \circ \psi, \zeta)$
3. $D_n(\sigma \circ \varphi, \zeta) = D_n(\varphi \circ \sigma, \zeta)$
4. $\varphi'(\zeta) = 1$

Proof. (1) \iff (2): Let W be the neighborhood of ζ guaranteed by Proposition 5.2.3. Notice that $\varphi^{-1}(W) \cap W$ is open and contains ζ , and choose $\epsilon > 0$ such that $\{|z - \zeta| \leq \epsilon\} \cap \mathbb{D}$ is contained in $\varphi^{-1}(W) \cap W$.

Denote $\psi = \psi_{\zeta, \epsilon}$ and note that both ψ and $\varphi \circ \psi$ map \mathbb{D} into W , are analytic at ζ and fix ζ . Additionally, both ψ and φ extend continuously to $\overline{\mathbb{D}}$ and satisfy

$$\varphi^{-1}(\{\zeta\}) = \{\zeta\}, \quad (\varphi \circ \psi)^{-1}(\{\zeta\}) = \{\zeta\}.$$

Thus, by Proposition 5.2.3, both of the following equations in the Calkin algebra hold:

$$C_\psi C_\varphi^* \equiv \frac{1}{|\varphi'(\zeta)|} C_{\sigma \circ \psi} \pmod{\mathcal{K}}, \quad C_{\varphi \circ \psi} C_\varphi^* \equiv \frac{1}{|\varphi'(\zeta)|} C_{\sigma \circ \varphi \circ \psi} \pmod{\mathcal{K}},$$

Therefore, we can express $C_\psi[C_\varphi, C_\varphi^*]$ in the Calkin algebra by

$$\begin{aligned} C_\psi[C_\varphi, C_\varphi^*] &\equiv C_\psi C_\varphi C_\varphi^* - C_\psi C_\varphi^* C_\varphi \equiv C_{\varphi \circ \psi} C_\varphi^* - C_\psi C_\varphi^* C_\varphi \\ &\equiv \frac{1}{|\varphi'(\zeta)|} C_{\sigma \circ \varphi \circ \psi} - \frac{1}{|\varphi'(\zeta)|} C_{\sigma \circ \psi} C_\varphi \equiv \frac{1}{|\varphi'(\zeta)|} (C_{\sigma \circ \varphi \circ \psi} - C_{\varphi \circ \sigma \circ \psi}) \pmod{\mathcal{K}}. \end{aligned}$$

Note that ψ maps an arc of $\partial\mathbb{D}$ containing ζ into $\partial\mathbb{D}$, and recall that by Proposition 5.3.6 σ has order of contact n at ζ . Thus, by Proposition 5.3.7 both $\sigma \circ \varphi \circ \psi$ and $\varphi \circ \sigma \circ \psi$ have order of contact n at ζ . Note also that $\sigma \circ \varphi \circ \psi$ and $\varphi \circ \sigma \circ \psi$ are in the class \mathcal{A} with $F = \{\zeta\}$. Therefore, by Theorem 4.2.1,

$$C_{\sigma \circ \varphi \circ \psi} - C_{\varphi \circ \sigma \circ \psi} \in \mathcal{K} \iff D_n(\sigma \circ \varphi \circ \psi, \zeta) = D_n(\varphi \circ \sigma \circ \psi, \zeta).$$

(2) \iff (3): Recall that as a consequence of Theorem 5.3.1, if g is analytic at z and f is analytic at $g(z)$ then $D_k(g, z)$ and $D_k(f, g(z))$ determine $D_k(f \circ g, z)$. Since ψ is invertible in a neighborhood of ζ , we get that

$$D_n(\sigma \circ \varphi \circ \psi, \zeta) = D_n(\varphi \circ \sigma \circ \psi, \zeta) \iff D_n(\sigma \circ \varphi, \zeta) = D_n(\varphi \circ \sigma, \zeta)$$

(3) \iff (4): By Lemma 5.3.5, the n 'th order data for $\sigma \circ \varphi$ and $\varphi \circ \sigma$ is given by

$$D_n(\sigma \circ \varphi, \zeta) = (\zeta, 1, 0, \dots, 0, c \cdot \sigma'(\zeta))$$

$$D_n(\varphi \circ \sigma, \zeta) = (\zeta, 1, 0, \dots, 0, c \cdot \sigma'(\zeta)^n),$$

where c is a non zero constant. Hence the n 'th order data is equal if and only if $\sigma'(\zeta) = \sigma'(\zeta)^n$. Recall that $\sigma'(\zeta) = \frac{1}{\varphi'_e(\zeta)} = \frac{1}{\varphi'(\zeta)}$, and that $\varphi'(\zeta) > 0$ since φ fixes ζ . Thus $\sigma'(\zeta) = \sigma'(\zeta)^n$ if and only if $\varphi'(\zeta) = 1$. \square

If C_ψ were bounded below, compactness of $C_\psi A$ would imply compactness of A for any operator A . We prove a somewhat weaker result using a technique from [10, section 4]. We denote the characteristic function on $\partial\mathbb{D}$ for the arc $\Gamma_\delta = \{|z - \zeta| < \delta\} \cap \partial\mathbb{D}$ by χ_δ .

Lemma 6.1.2. *Let $\epsilon > 0$ and $\psi = \psi_{\zeta, \epsilon}$. Then for any $\chi = \chi_\delta$ with $0 < \delta < \epsilon$, and any operator A we have*

$$C_\psi A \text{ is compact} \implies T_\chi A \text{ is compact.}$$

Proof. Suppose that $C_\psi A$ is compact. It suffices to show that the H^2 to L^2 operator $M_\chi A$ is compact. Let t_1, t_2 be such that e^{it} parametrizes the curve Γ_δ for $t_1 \leq t \leq t_2$, and let α_1, α_2 be such that $\psi(e^{it})$ parametrizes the curve Γ_δ for $\alpha_1 \leq t \leq \alpha_2$. Then

$$\begin{aligned} \|M_\chi f\|_{L^2} &= \int_{t_1}^{t_2} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \int_{\Gamma_\delta} \frac{|f(e^{i\theta})|^2}{2\pi iz} dz = \int_{\alpha_1}^{\alpha_2} |f(\psi(e^{it}))|^2 |\psi'(e^{it})| \frac{dt}{2\pi} \\ &\leq M \int_{\alpha_1}^{\alpha_2} |f(\psi(e^{it}))|^2 \frac{dt}{2\pi} \leq M \|C_\psi f\|_{L^2}^2, \end{aligned}$$

where $M = \max\{|\psi'(e^{it})| : \alpha_1 \leq t \leq \alpha_2\}$ is finite since ψ can be analytically extended to a neighborhood of Γ .

Note that $C_\psi A$ is compact from H^2 to H^2 and so it is compact from H^2 to L^2 , and that for all $f \in H^2$ we have that $\|M_\chi A f\|_{L^2} \leq \sqrt{M} \|C_\psi A f\|_{L^2}$. Thus $M_\chi A$ is compact from H^2 to L^2 . \square

Theorem 6.1.3. *Suppose φ is a basic function with contact at ζ which fixes ζ . Then C_φ is essentially normal if and only if $\varphi'(\zeta) = 1$.*

Proof. By Proposition 6.1.1, there exists $\epsilon > 0$ such that for $\psi = \psi_{\zeta, \epsilon}$, we have $C_\psi[C_\varphi, C_\varphi^*]$ is compact if and only if $\varphi'(\zeta) = 1$. Thus, it suffices to show that compactness of $C_\psi[C_\varphi, C_\varphi^*]$ implies compactness of $[C_\varphi, C_\varphi^*]$.

Suppose that $C_\psi[C_\varphi, C_\varphi^*]$ is compact. Then by Lemma 6.1.2 with $\delta = \epsilon/2$ and $\chi = \chi_\delta$, we get that $T_\chi[C_\varphi, C_\varphi^*]$ is compact. Now since $\varphi \in \mathcal{A}$ with $F = \{\zeta\}$ and χ is continuously differentiable at ζ , by Corollary 4.3.2 and Proposition 4.3.5,

$$T_\chi C_\varphi \equiv \chi(\zeta)C_\varphi \equiv C_\varphi \pmod{\mathcal{K}} \quad \text{and} \quad T_\chi C_\varphi^* \equiv \chi(\varphi(\zeta))C_\varphi^* \equiv C_\varphi^* \pmod{\mathcal{K}}.$$

Therefore,

$$T_\chi[C_\varphi, C_\varphi^*] \equiv T_\chi C_\varphi C_\varphi^* - T_\chi C_\varphi^* C_\varphi \equiv [C_\varphi, C_\varphi^*] \pmod{\mathcal{K}},$$

and so $[C_\varphi, C_\varphi^*]$ is compact as well. \square

6.2 General Essential Normality

In this section we will identify the essentially normal composition operators induced by a general function in the class \mathcal{A} . We let φ be in \mathcal{A} and denote $F(\varphi) = \{\zeta_1, \dots, \zeta_r\}$ and the order of contact of φ with $\partial\mathbb{D}$ at ζ_j by n_j . Then by Theorem 4.2.5, there exists a decomposition,

$$C_\varphi \equiv C_{\varphi_1} + \dots + C_{\varphi_r} \pmod{\mathcal{K}}.$$

where φ_j is a basic function which has contact at ζ_j and satisfies

$$D_{n_j}(\varphi_j, \zeta_j) = D_{n_j}(\varphi, \zeta_j).$$

For each j , we denote $\lambda_j := \varphi(\zeta_j)$, and let σ_j be the unique branch of $(\varphi_j)_e^{-1}$ defined in some neighborhood of λ_j which maps λ_j to ζ_j .

Lemma 6.2.1. *Let φ_1 and φ_2 be basic functions with contact at ζ_1 and ζ_2 respectively, and denote $\lambda_1 = \varphi_1(\zeta_1)$ and $\lambda_2 = \varphi_2(\zeta_2)$. Then the following holds:*

1. *If $\zeta_1 \neq \zeta_2$ then $C_{\varphi_1}^* C_{\varphi_2} \equiv 0 \pmod{\mathcal{K}}$*

2. *If $\lambda_1 \neq \lambda_2$ then $C_{\varphi_2} C_{\varphi_1}^* \equiv 0 \pmod{\mathcal{K}}$*

Proof. For the first part, suppose that $\zeta_1 \neq \zeta_2$ and let b be defined by $b(z) = \frac{z-\zeta_1}{\zeta_2-\zeta_1}$, so that b is continuous at ζ_1 and ζ_2 and satisfies $b(\zeta_1) = 0$ and $b(\zeta_2) = 1$. Then by Corollary 4.3.2,

$$T_b C_{\varphi_2} \equiv b(\zeta_2) C_{\varphi_2} \equiv C_{\varphi_2} \pmod{\mathcal{K}}, \quad \text{and} \quad T_{\bar{b}} C_{\varphi_1} \equiv \overline{b(\zeta_1)} C_{\varphi_1} \equiv 0 \pmod{\mathcal{K}},$$

and combining these two equations, we get

$$C_{\varphi_1}^* C_{\varphi_2} \equiv C_{\varphi_1}^* (T_b C_{\varphi_2}) \equiv C_{\varphi_1}^* T_b^* C_{\varphi_2} \equiv (T_{\bar{b}} C_{\varphi_1})^* C_{\varphi_2} \equiv 0 \pmod{\mathcal{K}}.$$

For the second part, suppose $\lambda_1 \neq \lambda_2$ and let c be defined by $c(z) = \frac{z-\lambda_2}{\lambda_1-\lambda_2}$, so that c is continuously differentiable in neighborhoods of λ_1 and λ_2 and satisfies $c(\lambda_1) = 1$ and $c(\lambda_2) = 0$. Then by Proposition 4.3.5,

$$T_c C_{\varphi_1}^* \equiv c(\lambda_1) C_{\varphi_1}^* \equiv C_{\varphi_1}^* \pmod{\mathcal{K}}, \quad \text{and} \quad T_{\bar{c}} C_{\varphi_2}^* \equiv \overline{c(\lambda_2)} C_{\varphi_2}^* \equiv 0 \pmod{\mathcal{K}},$$

and combining these two equations, we get

$$C_{\varphi_2} C_{\varphi_1}^* \equiv C_{\varphi_2} (T_c C_{\varphi_1}^*) \equiv (C_{\varphi_2}^*)^* (T_{\bar{c}})^* C_{\varphi_1}^* \equiv (T_{\bar{c}} C_{\varphi_2}^*)^* C_{\varphi_1}^* \equiv 0 \pmod{\mathcal{K}}.$$

□

Proposition 6.2.2. *Let φ in \mathcal{A} be such that C_φ is essentially normal. Let $\zeta_1, \dots, \zeta_r, n_1, \dots, n_r, \lambda_1, \dots, \lambda_r, \varphi_1, \dots, \varphi_r$ and $\sigma_1, \dots, \sigma_r$ be as described above. Then for any i in $\{1, \dots, r\}$ there exists a unique j such that $\varphi(\zeta_j) = \zeta_i$. Furthermore, for this j , $n_j = n_i = n$ and*

$$D_n(\sigma_i \circ \varphi_i, \zeta_i) = D_n(\varphi_j \circ \sigma_j, \lambda_j).$$

Proof. By relabeling the functions, it suffices to show this for $i = 1$. By Lemma 6.2.1, we see that

$$\begin{aligned} [C_\varphi, C_\varphi^*] &\equiv [C_{\varphi_1} + \dots + C_{\varphi_r}, C_{\varphi_1}^* + \dots + C_{\varphi_r}^*] \\ &\equiv \sum_{j,k=1}^r C_{\varphi_k} C_{\varphi_j}^* - \sum_{j,k=1}^r C_{\varphi_j}^* C_{\varphi_k} \equiv \sum_{\substack{j,k=1,\dots,r \\ \lambda_k=\lambda_j}} C_{\varphi_k} C_{\varphi_j}^* - \sum_{j=1}^r C_{\varphi_j}^* C_{\varphi_j} \pmod{\mathcal{K}}. \end{aligned}$$

Now let b be a polynomial such that $b(\zeta_1) = 1$ and b is 0 at all the points in $\{\zeta_2, \dots, \zeta_r, \lambda_1, \dots, \lambda_r\} \setminus \{\zeta_1\}$. Then b is continuously differentiable everywhere and so by Corollary 4.3.2 and Proposition 4.3.5,

$$\begin{aligned} M_b C_{\varphi_1} &\equiv C_{\varphi_1} \pmod{\mathcal{K}}, & M_b C_{\varphi_i} &\equiv 0 \pmod{\mathcal{K}} \quad \text{for } i \neq 1, \\ M_b C_{\varphi_i}^* &\equiv C_{\varphi_i}^* \pmod{\mathcal{K}} \quad \text{if } \lambda_i = \zeta_1, & M_b C_{\varphi_i}^* &\equiv 0 \pmod{\mathcal{K}} \quad \text{if } \lambda_i \neq \zeta_1. \end{aligned}$$

Therefore, we have

$$\begin{aligned} M_b [C_\varphi, C_\varphi^*] &\equiv \sum_{\substack{j,k=1\dots r \\ \lambda_j=\lambda_k}} M_b C_{\varphi_k} C_{\varphi_j}^* - \sum_{j=1}^r M_b C_{\varphi_j}^* C_{\varphi_j} \\ &\equiv \sum_{\substack{j=1\dots r \\ \lambda_j=\lambda_1}} C_{\varphi_1} C_{\varphi_j}^* - \sum_{\substack{j=1\dots r \\ \lambda_j=\zeta_1}} C_{\varphi_j}^* C_{\varphi_j} \pmod{\mathcal{K}} \end{aligned}$$

By Proposition 5.2.3, for each φ_j there exists a neighborhood W_j of λ_j such that

$$C_\psi C_{\varphi_j}^* \equiv \frac{1}{|\varphi_j'(\zeta_j)|} C_{\sigma_j \circ \psi} \pmod{\mathcal{K}},$$

for any self-map ψ of \mathbb{D} that is analytic at λ_j , satisfies $\psi^{-1}(\{\lambda_j\}) = \{\lambda_j\}$ and maps \mathbb{D} into W_j .

Notice that if $\lambda_j = \zeta_1$ then W_j is a neighborhood of ζ_1 and if $\lambda_j = \lambda_1$ then $\lambda_1 \in W_j$ and so that $\varphi_1^{-1}(W_j)$ is a neighborhood of ζ_1 . We can therefore define a neighborhood W of ζ_1 by

$$W = \left(\bigcap_{\lambda_j = \lambda_1} \varphi_1^{-1}(W_j) \right) \cap \left(\bigcap_{\lambda_j = \zeta_1} W_j \right).$$

Now let $\epsilon > 0$ be such that $W_0 := \{|z - \zeta_1| \leq \epsilon\} \cap \mathbb{D}$ is contained in W , and choose $\psi = \psi_{\zeta_1, \epsilon}$.

Now for j such that $\lambda_j = \lambda_1$, we see that $\varphi_1 \circ \psi$ maps $\overline{\mathbb{D}}$ into $\varphi_1(W) \subset W_j$ and so by Proposition 5.2.3,

$$C_\psi C_{\varphi_1} C_{\varphi_j}^* \equiv C_{\varphi_1 \circ \psi} C_{\varphi_j}^* \equiv \frac{1}{|\varphi_j'(\zeta_j)|} C_{\sigma_j \circ \varphi_1 \circ \psi} \pmod{\mathcal{K}}.$$

Similarly, for j such that $\lambda_j = \zeta_1$, we see that ψ maps $\overline{\mathbb{D}}$ into $V \subset W_j$ so by Proposition 5.2.3,

$$C_\psi C_{\varphi_j}^* C_{\varphi_j} \equiv \frac{1}{|\varphi_j'(\zeta_j)|} C_{\sigma_j \circ \psi} C_{\varphi_j} \equiv \frac{1}{|\varphi_j'(\zeta_j)|} C_{\varphi_j \circ \sigma_j \circ \psi} \pmod{\mathcal{K}}.$$

By our assumption that C_φ is essentially normal, the operator $C_\psi M_b[C_\varphi, C_\varphi^*]$ is compact. We can now express this operator as a linear combination of composition

operators by

$$0 \equiv C_\psi M_b[C_\varphi, C_\varphi^*] \equiv \sum_{\substack{j=1\dots r \\ \lambda_j=\lambda_1}} C_\psi C_{\varphi_1} C_{\varphi_j}^* - \sum_{\substack{j=1\dots r \\ \lambda_j=\zeta_1}} C_\psi C_{\varphi_j}^* C_{\varphi_j} \quad (6.2.1)$$

$$\equiv \sum_{\lambda_k=\lambda_1} \frac{1}{|\varphi'(\zeta_k)|} C_{\sigma_k \circ \varphi_1 \circ \psi} - \sum_{\lambda_j=\zeta_1} \frac{1}{|\varphi'(\zeta_j)|} C_{\varphi_j \circ \sigma_j \circ \psi} \pmod{\mathcal{K}} \quad (6.2.2)$$

In order to draw some conclusions from Theorem 4.2.1, we discuss some properties of the inducing maps above. First note that ψ maps an arc of $\partial\mathbb{D}$ containing ζ_1 to an arc of $\partial\mathbb{D}$, and recall that by Proposition 5.3.6 σ_j has order of contact n_j at $\lambda_j = \lambda_1$. Thus, by Proposition 5.3.7, $\sigma_j \circ \varphi_1 \circ \psi$ has order of contact $\min(n_1, n_j)$ at ζ_1 . We also note that $\sigma_j \circ \varphi_1 \circ \psi$ is in the class \mathcal{A} with $F(\sigma_j \circ \varphi_1 \circ \psi) = \{\zeta_1\}$ and maps ζ_1 to ζ_j (indeed $\zeta_1 \xrightarrow{\psi} \zeta_1 \xrightarrow{\varphi_1} \lambda_1 = \lambda_j \xrightarrow{\sigma_j} \zeta_j$). Similarly, by Proposition 5.3.7, $\varphi_j \circ \sigma_j \circ \psi$ has order of contact n_j at ζ_1 , and $\varphi_j \circ \sigma_j \circ \psi$ is in the class \mathcal{A} with $F(\varphi_j \circ \sigma_j \circ \psi) = \{\zeta_1\}$ and fixes ζ_1 (indeed $\zeta_1 \xrightarrow{\psi} \zeta_1 = \lambda_j \xrightarrow{\sigma_j} \zeta_j \xrightarrow{\varphi_j} \lambda_j = \zeta_1$).

Thus, all the inducing maps above are in \mathcal{A} with $F = \{\zeta_1\}$, and each inducing map in the first sum maps ζ_1 to ζ_j , while all inducing maps in the second sum fix ζ_1 . Since the ζ_j 's are distinct, any term in the first sum originating from $j \neq 1$ would prevent compactness by Theorem 4.2.1. Thus for all $j \neq 1$ we must have that $\lambda_j \neq \lambda_1$ and so Eq. (6.2.1) becomes

$$0 \equiv \frac{1}{|\varphi'(\zeta_1)|} C_{\sigma_1 \circ \varphi_1 \circ \psi} - \sum_{\lambda_j=\zeta_1} \frac{1}{|\varphi'(\zeta_j)|} C_{\varphi_j \circ \sigma_j \circ \psi} \pmod{\mathcal{K}}.$$

Note that by symmetry $\lambda_1, \dots, \lambda_r$ are distinct, and so there is at most one j such that $\lambda_j = \zeta_1$. Thus, the second sum contains at most one term. Again by Theorem 4.2.1,

the second sum cannot be empty, so there is a unique j such that $\lambda_j = \zeta_1$. Furthermore, for this j , $|\varphi'(\zeta_1)| = |\varphi'(\zeta_j)|$ and $\sigma_1 \circ \varphi_1 \circ \psi$ and $\varphi_j \circ \sigma_j \circ \psi$ have equal order of contact, $n = n_1 = n_j$, at ζ_1 and

$$D_n(\sigma_1 \circ \varphi_1 \circ \psi, \zeta_1) = D_n(\varphi_j \circ \sigma_j \circ \psi, \zeta_1).$$

Recall that as a consequence of Theorem 5.3.1, if g is analytic at z and f is analytic at $g(z)$ then $D_k(g, z)$ and $D_k(f, g(z))$ determine $D_k(f \circ g, z)$. Since ψ is invertible in a neighborhood of ζ , we get that

$$D_n(\sigma_1 \circ \varphi_1, \zeta_1) = D_n(\varphi_j \circ \sigma_j, \zeta_1).$$

□

Lemma 6.2.3. *Let φ be in \mathcal{A} be such that C_φ is essentially normal and suppose that φ fixes $\zeta \in F(\varphi)$. Then ζ is the unique Denjoy-Wolff point of φ .*

Proof. We label the points of $F(\varphi)$ so that $\zeta_1 = \zeta$ and let φ_1 and σ_1 be as described above. Then by Proposition 6.2.2,

$$D_n(\sigma_1 \circ \varphi_1, \zeta_1) = D_n(\varphi_1 \circ \sigma_1, \zeta_1).$$

In particular, the n th derivatives are equal and so by Lemma 5.3.5,

$$c \cdot \sigma_1'(\zeta_1) = c \cdot \varphi_1'(\zeta_1)^n,$$

where $c \neq 0$. Using the fact that $\sigma_1'(\zeta_1) = 1/\varphi_1'(\zeta_1) > 0$, we get that

$$\varphi'(\zeta_1) = \varphi_1'(\zeta_1) = 1$$

and so, by Theorem 2.3.6, ζ_1 is the unique Denjoy-Wolff point of φ . □

Theorem 6.2.4. *Let φ be in \mathcal{A} with $F(\varphi) = \{\zeta_1, \dots, \zeta_r\}$. Then C_φ is essentially normal if and only if $r = 1$, φ fixes ζ_1 and $\varphi'(\zeta_1) = 1$.*

Proof. If $r = 1$, φ fixes ζ_1 and $\varphi'(\zeta_1) = 1$ then $[C_\varphi, C_\varphi^*] \equiv [C_{\varphi_1}, C_{\varphi_1}^*]$ is compact by Theorem 6.1.3.

For the other direction, we suppose that C_φ is essentially normal. In order to obtain a contradiction, suppose that $r > 1$. By Lemma 6.2.3, $F(\varphi)$ contains at most one fixed point, so there exists some index i such that $\zeta_i \in F(\varphi)$ is not fixed.

Recall that by Proposition 6.2.2, for each $i \in \{1, \dots, r\}$ there is a unique j such that $\varphi(\zeta_j) = \zeta_i$. Let $i_1 = i$ and let i_2 be the unique index for i_1 as above. Continue to define i_{l+1} as the unique index for i_l as above until a cycle is closed, $i_1 \rightarrow \dots \rightarrow i_k \rightarrow i_1$.

Let $\varphi_{(k)} = \varphi \circ \dots \circ \varphi$ and note that the operator $C_{\varphi_{(k)}} = C_\varphi^k$ is essentially normal since C_φ and C_φ^* commute modulo \mathcal{K} . We get that $\varphi_{(k)} \in \mathcal{A}$ induces an essentially normal composition operator and fixes $k > 1$ distinct points, $\zeta_{i_1}, \dots, \zeta_{i_k}$, in $F(\varphi_{(k)})$. This is a contradiction by Lemma 6.2.3, and so we must have that $r = 1$. By Theorem 6.1.3 we get that φ fixes ζ_1 and $\varphi'(\zeta_1) = 1$. \square

Corollary 6.2.5. *Let φ be a self-map of \mathbb{D} which extends analytically to a neighborhood of \mathbb{D} and that is not a finite Blaschke product. Then C_φ is non-trivially essentially normal if and only if there exists $\zeta \in \partial\mathbb{D}$ such that φ fixes ζ and $\varphi'(\zeta) = 1$, and φ maps $\partial\mathbb{D} \setminus \{\zeta\}$ into \mathbb{D} .*

Proof. By Proposition 4.1.1, $\varphi \in \mathcal{A}$. If $F(\varphi)$ is empty then $\|\varphi\|_\infty < 1$ and so $C_\varphi \in \mathcal{K}$ is trivially essentially normal. Otherwise, the claim follows from Theorem 6.2.4. \square

6.3 Construction of Essentially Normal Composition Operators

We combine the criterion for essential normality given in Theorem 6.2.4 with several results from Chapter 3 to construct essentially normal composition operators which have arbitrary even order of contact with $\partial\mathbb{D}$ at a point ζ .

Proposition 6.3.1. *Let $\zeta \in \partial\mathbb{D}$, $n = 2m$ and*

$$s_1, \dots, s_{m-1} \in \mathbb{R}, \quad t_1, \dots, t_{m-1} > 0, \quad w \in \mathbb{H}.$$

Let φ be defined by $\varphi = \tau_\zeta^{-1} \circ f \circ \tau_\zeta$ where

$$f(z) = L[A(0, 1)(z)A(s_1, t_1)(z) \cdots A(s_{m-1}, t_{m-1})(z)]w,$$

and

$$A(s, t)(z) = \begin{bmatrix} stz & -s - tz \\ tz & -1 \end{bmatrix}.$$

Then φ is a degree m rational function with order of contact n at ζ which induces an essentially normal composition operator.

Proof. By Proposition 3.3.3, f has order of contact n with \mathbb{R} at $z = 0$, and so φ has order of contact n with $\partial\mathbb{D}$ at ζ . Note that f is obtained by taking m augmentations of the constant function $g(z) \equiv w$. By Proposition 3.4.1 and Proposition 3.4.2, f is a rational function of degree m mapping $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} . Since the last augmentation performed to obtain f has parameters $a_0 = 0, a_1 = 1$, f satisfies $f(0) = 0, f'(0) = 1$.

Note that τ_ζ preserves the degree of a rational function, and so $\varphi = \tau_\zeta^{-1} \circ f \circ \tau_\zeta$ is a degree m rational function. Additionally, the properties of f ensure that φ is in \mathcal{A} with $F(\varphi) = \{\zeta\}$ and φ fixes ζ and satisfies $\varphi'(\zeta) = 1$. Thus, by Theorem 6.2.4, the operator C_φ is essentially normal. \square

For order of contact $n = 4$ we define $f(s, t, w)$ and $\varphi(\zeta, s, t, w)$ for any $s \in \mathbb{R}$, $t > 0$ and $w \in \mathbb{H}$ and $\zeta \in \partial\mathbb{D}$ by

$$\begin{aligned} f(s, t, w)(z) &= L[A(0, 1)(z)A(s, t)(z)]w \\ &= L \left[\begin{bmatrix} 0 & -z \\ z & -1 \end{bmatrix} \begin{bmatrix} sz & -s - tz \\ tz & -1 \end{bmatrix} \right] w = \frac{-tz^2w + z}{(stz^2 - tz)w - sz - tz^2 + 1}, \end{aligned}$$

and

$$\varphi(\zeta, s, t, w) = \tau_\zeta^{-1} \circ f(s, t, w) \circ \tau_\zeta.$$

Then $\varphi(\zeta, s, t, w)$ has order of contact 4 with $\partial\mathbb{D}$ at ζ and induces an essentially normal composition operator.

We examine the order of contact 2 case for motivation. It is known that for a linear fractional φ the operator C_φ is non-trivially essentially normal if and only if φ is a parabolic non-automorphisms [6]. In this case, φ corresponds to a translation in the right half-plane.

We transfer the contact 2 version of the formula above to the right half-plane using the conformal mapping $T(z) = \frac{i}{z}$ which maps \mathbb{H} to the right half-plane and 0

to ∞ . For any $w \in \mathbb{H}$ and $\zeta \in \partial\mathbb{D}$ the self-map of \mathbb{H}

$$f_w(z) = L[A(0,1)(z)]w = L \begin{bmatrix} 0 & -z \\ z & -1 \end{bmatrix} w = \frac{-z}{wz - 1} = \frac{1}{\frac{1}{z} - w}$$

corresponds to the self map of the right half-plane

$$(T \circ f_w \circ T)(z) = \frac{i}{f_w(i/z)} = i\left(\frac{z}{i} - w\right) = z - iw.$$

Applying this same procedure to our order of contact 4 functions

$$\begin{aligned} f(s, t, w)(z) &= \frac{-tz^2w + z}{(stz^2 - tz)w - sz - tz^2 + 1} \\ &= \frac{(1 - tzw)z}{(wtz - 1)sz + (1 - twz) - tz^2} \\ &= \frac{1}{-s + \frac{1}{z} - \frac{tz}{1 - twz}} = \frac{1}{-s + \frac{1}{z} + \frac{1}{w - \frac{1}{tz}}}, \end{aligned}$$

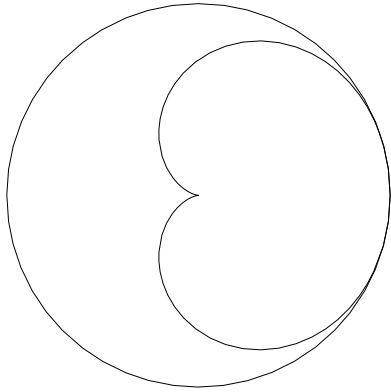
we calculate

$$\begin{aligned} (T \circ f(s, t, w) \circ T)(z) &= \frac{i}{f(s, t, w)(i/z)} = i\left(-s + \frac{z}{i} + \frac{1}{w - \frac{z}{it}}\right) \\ &= z - is + \frac{1}{\frac{1}{i}z - iw}. \end{aligned}$$

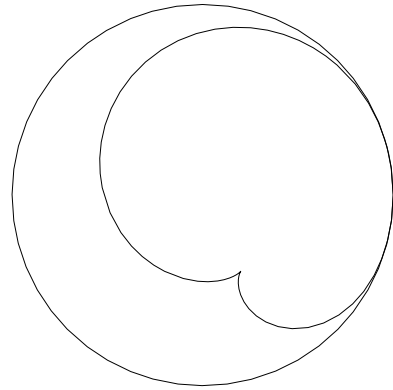
In Fig. 6.1, we sketch the image of $\partial\mathbb{D}$ under $\varphi = \varphi(1, s, t, w)$ for various parameters.

Figure 6.1: Examples of inducing maps of essentially normal composition operators

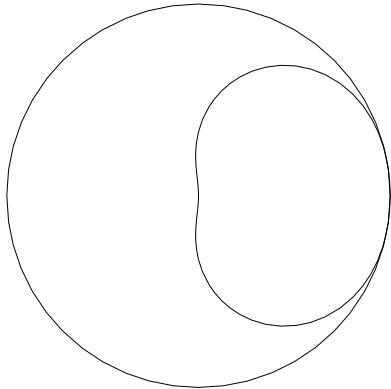
$$(a) \varphi(1, 0, 1, i) = \frac{z^2 + 2z + 1}{z^2 - 2z + 5}$$



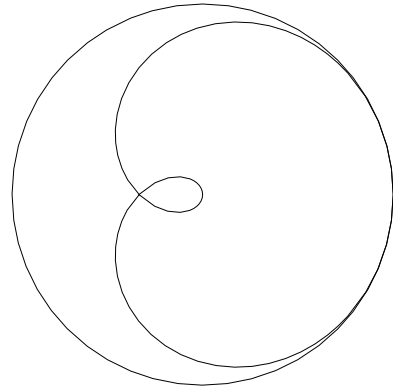
$$(b) \varphi(1, 1, 1, i) = \frac{z^2 + 2(1+i)z + 1 - 2i}{z^2 - 2(1-i)z + 5 - 2i}$$



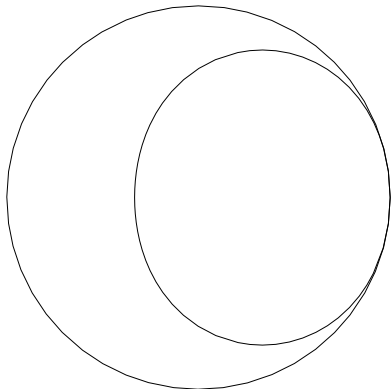
$$(c) \varphi(1, 0, 1/2, i) = \frac{z + 1}{z^2 - 3z + 4}$$



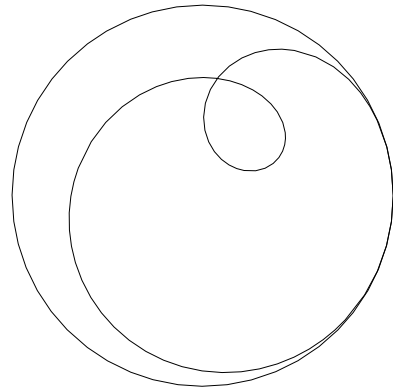
$$(d) \varphi(1, 0, 2, i) = \frac{z^2 + 7}{3z^2 + 4z + 1}$$



$$(e) \varphi(1, 0, 1, 2i) = \frac{-z^2 + 4z + 1}{z^2 - 4z + 7}$$



$$(f) \varphi(1, 0, 1, i/2 + 1) = \frac{(-2+2i)z^2 + (2+i)z + i}{iz^2 - (4+2i)z + 2+4i}$$



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