The Onsager Equation with Curvature and Source Terms
A Finite Element Solution

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Approval Sheet

The thesis

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Nomenclature

Coordinates

r – radial position
η – nondimensionalized radial position
x – stretched radial coordinate
z – axial position
y – nondimensionalized axial position

Flow variables

ρ – density
p – pressure
v_r – radial velocity
v_θ – circumferential velocity
v_z – axial velocity
T – temperature
$\widetilde{H}$ – auxiliary variable; see (3.9)
$\widetilde{\Phi}$ – auxiliary variable; see (3.12)
H – auxiliary variable; see (3.21) and (3.45)
Φ – auxiliary variable; see (3.22) and (3.46)

Centrifuge parameters and physical constants

a – centrifuge radius
L – centrifuge length
ρ_w – density at the wall
p_w – pressure at the wall
Ω – angular speed of rotation
c_p – specific heat capacity
R – gas constant
μ – dynamic viscosity
k – thermal conductivity

Other terms from the conservation equations

$\vec{v}$ – velocity vector
$\vec{q}$ – heat flux vector
$\vec{f}$ – body force vector
$\bar{\bar{\tau}}$ – shear stress tensor
δ_{ij} – kronecker delta
$\bar{\bar{\delta}}$ – kronecker delta tensor
e – specific energy
u – specific internal energy
h – specific enthalpy
ϕ_v – viscous dissipation function
$\dot{m}_s$ – localized rate of addition of mass from a mass source
$\mathcal{S}_M$ – collective localized effect of all sources of mass
$\mathcal{S}_R$ – collective localized effect of all sources of radial momentum
$\mathcal{S}_\theta$ – collective localized effect of all sources of circumferential momentum
$\mathcal{S}_Z$ – collective localized effect of all sources of axial momentum
$\mathcal{S}_E$ – collective localized effect of all sources of energy
$\bar{\mathcal{S}}$ – non-homogeneous terms in the Onsager equation

Potential functions

ψ – stream function
χ – Onsager's master potential function
$\mathcal{X}$ – modified master potential function

Important dimensionless quantities

Pr – Prandtl number; see the sentence following (3.9)
A – stratification parameter; see (3.19)
Z – aspect ratio; see (3.36)
Re – Reynolds number; see (3.37)
$\widehat{K}$ – Brinkman number; see (3.38)

Terms related to the boundary conditions

$\theta(y)$ – prescribed wall temperature gradient
$f(y)$ – prescribed mass flow high in the atmosphere
$g(x)$ – part of the Carrier-Maslen condition
$h(y)$ – non-homogeneous boundary condition high in the atmosphere
$\chi^{(ref)}$ – reference function that satisfies all radial boundary conditions
$\mathcal{H}(x, y)$ – non-homogeneous term that arises after introducing the modified master potential
$G(x)$ – part of the modified Carrier-Maslen condition

Terms related to the finite element method

M – subdivisions in the x direction
N – subdivisions in the y direction
$\tilde{\mathcal{X}}$ – test function ("weighting" function) for Galerkin method
$\mathcal{B}$ – part of the simplified Galerkin form; see (5.15) and (5.16)
$\mathcal{F}$ – part of the simplified Galerkin form; see (5.15) and (5.17)
Φ_k – two-dimensional basis functions; see (5.18)
s_i – cubic spline basis functions
σ_i – modified cubic spline basis functions
λ_j – linear spline basis functions

Accents and modifiers

overbar – nondimensionalized variable
subscript '0' – value at the reference state
subscript '1' – perturbation from the reference state
subscript 'r' – radial direction
subscript 'θ' – circumferential direction
subscript 'z' – axial direction
subscript 's' – source term
subscript 'T' – value at the top of the atmosphere
superscript '*' – flow in the Ekman layers
superscript '-' – prescribed value at the bottom boundary
superscript '+' – prescribed value at the top boundary

Chapter One

Background

Introduction

The protagonist in that which follows is the gas centrifuge; the plot is the enrichment of uranium. The narrative itself is more documentary than drama, the object being to chart the course, faithfully and meticulously, of molecules whirling inside a centrifuge cylinder. Another author could – and many have[1] – with the same characters tell an equally true but more cinema-friendly account, chronicling instances of corporate theft, international espionage, the internment of scientists, and worse. Surely, the gas centrifuge is a testament of mankind's remarkable ability to unearth the secrets of nature; yet, eventually, even the noblest enthusiasm for certain elusive atoms must be reconciled with the "awful arithmetic of the atomic bomb." [2] The hope is that this work will support those committed to the responsible use of this extraordinary machine.

The basics

Gas centrifuges are used to separate the molecules of a gas by isotope; the most common application is uranium enrichment. Natural uranium consists of two primary isotopes, ^{238}U (99.3%), and ^{235}U (0.7%). Generally, the goal is to increase the concentration of the fissile isotope, ^{235}U. Varying levels of enrichment are required for different purposes – some examples include nuclear power (~5%), research reactors (~20%), medical devices (~60%), naval propulsion (~20-90%) and military applications (~90%). To achieve these enrichment concentrations, multiple centrifuges are arranged in cascades.

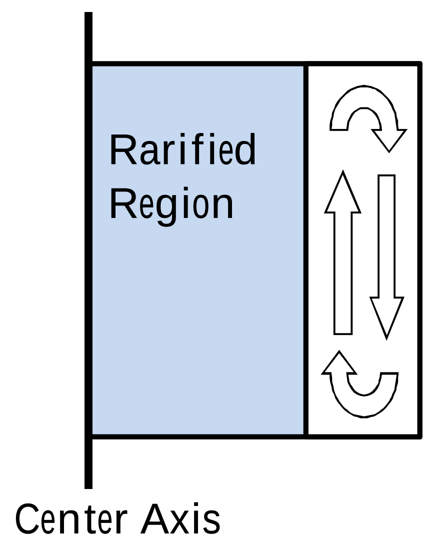

Figure 1-1. A cross-section of a centrifuge rotor. Centrifuges spin so rapidly that most of the gas is compressed against the outer wall. A counter-current circulation is induced to improve separating performance.

In essence, a centrifuge is a fast-rotating cylinder which contains the gas to be separated. As the centrifuge spins, molecules with greater mass drift outwards (away from the axis of rotation), pushing the lighter molecules inwards. To increase separation efficiency, engineers employ one of several techniques to induce countercurrent circulation along the axis (i.e.

[1] See, for example, [23] and [24].
[2] The quoted language is from President Eisenhower's "Atoms for Peace" speech delivered to the UN General Assembly in 1953.

the flow circulates from top to bottom). This practice produces a large concentration difference between the top and bottom of the cylinder.

The precise pattern of gas flow inside a centrifuge is difficult to determine. The velocity at the outer wall reaches several hundred meters per second, resulting in extremely large gradients in pressure and density. In addition, the great majority of gas molecules compress into the region immediately adjacent to the outer wall, creating a rarified inner region for which the Navier-Stokes equations are poorly suited. A variety of analysis techniques have been applied to this problem; the purpose of this thesis is to investigate the Onsager-Maslen equation, a sixth-order partial differential equation which approximates the fluid dynamics inside a centrifuge rotor.

Brief literature review

Early attempts to investigate the hydrodynamics inside a gas centrifuge included a variety of analytical models, each based on its own set of assumptions. A survey of some of these models is given by Soubbaramayer in [1]. Undoubtedly, one of the most important examples of this kind is the Onsager pancake model, a theory first posited by the Nobel laureate Lars Onsager. Beginning in 1961, he led a group of scientists under the auspices of the United States Atomic Energy Commission charged with studying the fundamental aspects of the fluid flow in a centrifuge. Wood and Morton outline the basic theory developed by this team in [2], which is primarily based on an unpublished report by Onsager and on work by George Carrier and Stephen Maslen presented in [3] and [4]. The results in [2] are reported in a form amenable to numerical calculations.

A good deal of the research performed by the centrifuge theory group took place at the University of Virginia. Several documents published by this group underpin the work presented in this thesis. First are two illuminating reports by Maslen, [5] and [6], which account for the effect of curvature terms originally neglected in [2]. In recognition of this contribution, the modified version of the Onsager equation which includes curvature terms is typically referred to as the Onsager-Maslen equation. Other important documents focus on the “source” terms introduced in [2]. Of note are [7], in which Barbarsky and Wood clarify the origin and meaning of these terms, and [8], in which Wood and Painter describe their effects in more detail.

Numerous publications introduce methods for solving the Onsager equation in its various forms. For the basic equation with a linear wall temperature gradient, an eigenfunction expansion solution is presented in [2] and a finite element solution is given by Gunzburger and Wood in [9]. Ribando uses a finite difference scheme in [10] to solve the same model without condensing it into a single equation. Wood, Jordan, and Gunzburger solve the Onsager-Maslen equation (which includes curvature terms) in [11] via a finite element method. Wood and

Sanders investigate the effect of sources of mass, momentum, and energy with an eigenfunction expansion solution in [12], and Gunzburger, Wood, and Jordan solve a very similar problem with a finite element technique in [13].

In the intervening years, a scattering of publications related to the original pancake model has appeared. In [14], Wood investigates the impact of feed effects on a single-stage centrifuge cascade. The problem of multi-isotope separation is considered by Wood, Mason, and Soubbaramayer in [15]. Bourn, Peterson, and Wood solve the pancake model equations with a temperature potential rather than Onsager's master potential in [16]. Babarsky, Herbst, and Wood investigate more fully in [17] the relationship between this temperature potential and the master potential. In [18], Doneddu, Roblin, and Wood present the results of an optimization study for gas centrifuge design. Wood, Ying, Zeng, Nie, and Shang estimate the overall separation factor for various multicomponent mixtures in [19]. Finally, Pradhan and Kumaran introduce a generalized Onsager model and compare its predictions with Direct Simulation Monte Carlo calculations in [20].

Content and motivation

The essential component of this thesis is a solution of the Onsager-Maslen equation which allows for sources and sinks of mass, momentum, and energy. Other authors have considered the effects of curvature terms and source terms separately; the aim here is to account for both phenomena at once. The solution is obtained with a finite element algorithm.

One goal is to provide a model which can accurately predict the flow characteristics of small, relatively low speed gas centrifuges, such as those whose designs became available through the A.Q. Khan network. United States centrifuges (those for which the pancake model were developed) are tall and operate at very high speeds. Under these conditions, the errors introduced by the pancake approximation are least severe. The analysis method demonstrated in the following chapters – because it preserves effects of curvature terms *and* permits the use of source terms to model the introduction and withdrawal of feed – should lead to more accurate performance estimates for small, low speed gas centrifuge machines.

Chapter Two

Conservation Laws for a Compressible Viscous Fluid

Introduction

This chapter examines the principles of conservation of mass, linear momentum, and energy for a compressible, viscous fluid. The discussion is based on special conservation equations derived by Barbarsky and Wood in [7] that account for possible sources and sinks of mass, momentum, and energy within the flow field. The basic equations are reduced to more practicable forms with a combination of the approaches given in [7], [21], and [22].[3]

Conservation of mass

The principle of conservation of mass in an infinitesimally small control volume is expressed by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = \dot{m}_s \ , \tag{2.1}$$

where $\dot{m}_s$ is the rate of addition of mass (per unit volume) emanating from a point source within the control volume. After expanding the $\nabla \cdot (\rho \vec{v})$ term as $\vec{v} \cdot \nabla \rho + \rho(\nabla \cdot \vec{v})$, the equation may be rewritten as

$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \vec{v}) + \dot{m}_s \ , \tag{2.2}$$

in which D/Dt is the material derivative. This new form is the *equation of change for density* and has the following physical interpretation: the density of the fluid within the infinitesimal control volume changes with a rate determined by the two terms on the right-hand side. The first influencing factor is the divergence of the velocity vector. If this value is positive, spatial changes in the velocity field have a net effect of "whisking away" mass faster than it is being replaced, driving the density of fluid flowing through the point to decrease. In the opposite case (a negative divergence), spatial variation in the flow speed tends to accumulate mass in the tiny control volume, increasing the density of fluid flowing through its confines. The second influencing factor is the increase (or decrease) in density caused by the mass source (or sink).

[3] Reference [21] and references [7] and [22] use opposite sign conventions for the stress tensor. The convention adopted here is that adopted in [7], in [22], and in most textbooks on fluid dynamics.

Conservation of momentum

The principle of conservation of linear momentum at a point is expressed by

$$\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla\cdot(\rho\vec{v}\vec{v}) = \rho\vec{f} + \nabla\cdot\bar{\bar{\tau}} - \nabla p + \dot{m}_s\vec{v}_s + \vec{f}_s \qquad (2.3)$$

where $\rho\vec{v}\vec{v}$ is of tensorial nature, $\vec{v}_s$ is the velocity of the flow emanating from the mass source and $\vec{f}_s$ is a source term equivalent to a body force. The first step in reducing this equation is to expand the $\nabla\cdot(\rho\vec{v}\vec{v})$ term on the left-hand side as $\vec{v}\cdot\nabla(\rho\vec{v}) + \rho\vec{v}(\nabla\cdot\vec{v})$. Then, rearranging the equation and introducing the material derivative yields

$$\frac{D}{Dt}(\rho\vec{v}) = -\rho\vec{v}(\nabla\cdot\vec{v}) + \rho\vec{f} + \nabla\cdot\bar{\bar{\tau}} - \nabla p + \dot{m}_s\vec{v}_s + \vec{f}_s\,. \qquad (2.4)$$

This form, the *equation of change for momentum*, shows that the fluid momentum varies with a rate determined by the six influencing factors listed on the right-hand side. The left-hand side of (2.4) is further expanded using

$$\frac{D}{Dt}(\rho\vec{v}) = \rho\frac{D\vec{v}}{Dt} + \vec{v}\frac{D\rho}{Dt}\,, \qquad (2.5)$$

which gives the relationship between the rates of change of the fluid's momentum and of its constituent properties: density and velocity. This result suggests subtracting from equation (2.4) the product of $\vec{v}$ and (2.2), which, after some simplification, yields the *equation of change for velocity*,

$$\rho\frac{D\vec{v}}{Dt} = \rho\vec{f} + \nabla\cdot\bar{\bar{\tau}} - \nabla p + \dot{m}_s(\vec{v}_s - \vec{v}) + \vec{f}_s\,. \qquad (2.6)$$

The right hand side lists the five influencing factors that determine the rate of change of the fluid's velocity.

Conservation of energy

The principle of conservation of total energy (internal and kinetic) in an infinitesimally small control volume is expressed as

$$\frac{\partial}{\partial t}(\rho e) + \nabla\cdot(\rho e\vec{v}) = \rho\vec{v}\cdot\vec{f} + \nabla\cdot(\bar{\bar{\tau}}\cdot\vec{v}) - \nabla\cdot(p\vec{v}) - \nabla\cdot\vec{q} + \dot{m}_s\left(e_s + \frac{p_s}{\rho_s}\right) + \dot{q}_s + \dot{w}_s \qquad (2.7)$$

where e_s is the specific total energy of the source flow, the term p_s/ρ_s is used to represent the flow work associated with introducing the source fluid, $\dot{q}_s$ is the heat source per unit time and volume, and $\dot{w}_s$ is the work rate per unit volume (excluding flow work) due to the source. The first step in reducing this equation is, again, to expand the second term on the left-hand side, this time as $\nabla \cdot (\rho e\vec{v}) = \vec{v} \cdot \nabla(\rho e) + \rho e(\nabla \cdot \vec{v})$. Rearranging and introducing the material derivative leads to the *equation of change for total energy*,

$$\frac{D}{Dt}(\rho e) = -\rho e(\nabla \cdot \vec{v}) + \rho\vec{v} \cdot \vec{f} + \nabla \cdot (\bar{\bar{\tau}} \cdot \vec{v}) - \nabla \cdot (p\vec{v}) - \nabla \cdot \vec{q} + \dot{m}_s\left(e_s + \frac{p_s}{\rho_s}\right) + \dot{q}_s + \dot{w}_s\ , \tag{2.8}$$

from which one may calculate the rate of change of energy in the fluid.

Next, noting that total energy can be represented by $e = u + \vec{v} \cdot \vec{v}/2$, with u being the specific internal energy, and also recognizing that $(D/Dt)(\vec{v} \cdot \vec{v}/2) = \vec{v} \cdot (D\vec{v}/Dt)$, the left-hand side of (2.8) may be expanded as

$$\frac{D}{Dt}(\rho e) = \rho\frac{Du}{Dt} + \rho\vec{v} \cdot \frac{D\vec{v}}{Dt} + e\frac{D\rho}{Dt}\,. \tag{2.9}$$

The energy of the fluid contained in the infinitesimally small control volume changes with a rate that is inextricably linked to the rates of change of the fluid's internal energy, velocity, and density. This result suggests subtracting both the dot product of $\vec{v}$ and (2.6) and the product of e and (2.2) from (2.8). The result, after simplification, is the *equation of change for internal energy*,

$$\rho\frac{Du}{Dt} = -p(\nabla \cdot \vec{v}) - \nabla \cdot \vec{q} + \bar{\bar{\tau}}{:}\nabla\vec{v} + \dot{m}_s\left(u_s - u + \frac{p_s}{\rho_s} + \frac{1}{2}|\vec{v} - \vec{v}_s|^2\right) - \vec{v} \cdot \vec{f}_s + \dot{q}_s + \dot{w}_s \tag{2.10}$$

in which the right-hand side lists each factor that contributes to changes in the fluid's internal energy. Note that $\bar{\bar{\tau}}{:}\nabla\vec{v}$ is the irreversible rate of internal energy increase by viscous dissipation and is equal to $\nabla \cdot (\bar{\bar{\tau}} \cdot \vec{v}) - \vec{v} \cdot (\nabla \cdot \bar{\bar{\tau}})$.

The reduction of the energy equation includes one additional step, which is to express the internal energy in terms of the fluid temperature and its heat capacity. After using (2.2) to replace the $(\nabla \cdot \vec{v})$ term in (2.10), and rearranging, one obtains,

$$\rho\frac{Du}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \vec{q} + \bar{\bar{\tau}}{:}\nabla\vec{v} + \dot{m}_s\left(u_s + \frac{p_s}{\rho_s} - u - \frac{p}{\rho} + \frac{1}{2}|\vec{v} - \vec{v}_s|^2\right) - \vec{v} \cdot \vec{f}_s + \dot{q}_s + \dot{w}_s\ . \tag{2.11}$$

Then, by introducing enthalpy, $h = u + p/\rho$, and recognizing that $\rho(Dh/Dt) = \rho(Du/Dt) - (p/\rho)(D\rho/Dt) + (Dp/Dt)$, equation (2.11) is rewritten

$$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} - \nabla\cdot\vec{q} + \bar{\bar{\tau}}:\nabla\vec{v} \qquad (2.12)$$
$$+\dot{m}_s\left(h_s - h + \frac{1}{2}|\vec{v} - \vec{v}_s|^2\right) - \vec{v}\cdot\vec{f}_s + \dot{q}_s + \dot{w}_s$$

For an ideal gas, assuming constant specific heat capacity, the term Dh/Dt can be expressed as $c_p(DT/Dt)$, finally leading to an *equation of change for temperature*,

$$\rho c_p\frac{DT}{Dt} = \frac{Dp}{Dt} - \nabla\cdot\vec{q} + \bar{\bar{\tau}}:\nabla\vec{v} \qquad (2.13)$$
$$+\dot{m}_s\left(c_p(T_s - T) + \frac{1}{2}|\vec{v} - \vec{v}_s|^2\right) - \vec{v}\cdot\vec{F}_s + \dot{q}_s + \dot{w}_s\ .$$

Final steps

To complete the reduction process, the stress tensor is evaluated assuming a Newtonian fluid,

$$\bar{\bar{\tau}} = \mu\left[\nabla\vec{v} + (\nabla\vec{v})^\dagger - \frac{2}{3}(\nabla\cdot\vec{v})\bar{\bar{\delta}}\right], \qquad (2.14)$$

and Fourier's law is employed to model temperature-driven heat flux,

$$\vec{q} = -k\nabla T\ . \qquad (2.15)$$

Under these assumptions, and for constant viscosity and constant thermal conductivity, the reduced momentum equation – equation (2.6) – becomes

$$\rho\frac{D\vec{v}}{Dt} = \rho\vec{f} + \mu\left[\nabla^2\vec{v} + \frac{1}{3}\nabla(\nabla\cdot\vec{v})\right] - \nabla p + \dot{m}_s(\vec{v}_s - \vec{v}) + \vec{f}_s\ , \qquad (2.16)$$

and the reduced energy equation – equation (2.13) – becomes,

$$\rho c_p\frac{DT}{Dt} = \frac{Dp}{Dt} + k\nabla^2 T\ + \mu\phi_v \qquad (2.17)$$
$$+\dot{m}_s\left(c_p(T_s - T) + \frac{1}{2}|\vec{v} - \vec{v}_s|^2\right) - \vec{v}\cdot\vec{f}_s + \dot{q}_s + \dot{w}_s\ ,$$

where the dissipation function Φ_v is

$$\phi_v = \frac{1}{2}\sum_i\sum_j\left[\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3}(\nabla\cdot\vec{v})\delta_{ij}\right]^2. \qquad (2.18)$$

Chapter Three

The Onsager-Maslen Equation with Sources

Introduction

The Onsager-Maslen equation is the modification of Onsager's basic equation that arises when curvature terms are retained in the derivation. The purpose of this chapter is to develop a version of that equation which accounts for the effects of sources (and sinks) of mass, momentum, and energy in the flow field. The derivation is informed primarily by [2], [6], and the equations from the previous chapter, but also with aid from [11], [12], [5], [7], [8], and [20].

The domain for the problem is a right circular cylinder traversed by a cylindrical coordinate system with components (r, θ, z). Its origin is the intersection of the cylinder axis and bottom surface. The problem is reduced to two spatial dimensions by assuming axisymmetric flow.

For a compressible, viscous fluid, the ideal gas law and the principles of conservation of mass, momentum, and energy make up a complete system in the flow variables p, ρ, $\vec{v}$, and T. The equations expressing these principles (derived in the previous chapter) which account for possible sources and sinks of mass, momentum, and energy within the flow field are

$$p = \rho R T \ , \tag{3.1}$$

$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \vec{v}) + \mathcal{S}_M \ , \tag{3.2}$$

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \mu \left[\nabla^2 \vec{v} + \frac{1}{3}\nabla(\nabla \cdot \vec{v})\right] + \vec{\mathcal{S}}_{R\theta Z} \ , \tag{3.3}$$

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + k\nabla^2 T + \mu \phi_v + \mathcal{S}_E \ . \tag{3.4}$$

In these equations, the effects of gravity are neglected under the assumption that it has a negligible impact on high speed rotating flows. The appropriate viscous dissipation term is

$$\begin{aligned} \phi_v = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \frac{v_r^2}{r^2} + \left(\frac{\partial v_z}{\partial z}\right)^2 + \frac{\partial v_r}{\partial z}\frac{\partial v_z}{\partial r}\right] \\ + \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2 + \left(\frac{\partial v_z}{\partial r}\right)^2 + \left(\frac{\partial v_r}{\partial z}\right)^2 + \left(\frac{\partial v_\theta}{\partial z}\right)^2 - \frac{2}{3}[\nabla \cdot \vec{v}]^2 \ . \end{aligned} \tag{3.5}$$

Finally, the scalar components of the momentum equation are, in cylindrical coordinates and for steady, axisymmetric flow,

$$\rho\left(\frac{Dv_r}{Dt} - \frac{{v_\theta}^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{v_r}{r^2} + \frac{1}{3}\frac{\partial}{\partial r}(\nabla \cdot \vec{v})\right] + \mathcal{S}_R \ , \tag{3.6}$$

$$\rho\left(\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r}\right) = \mu\left[\nabla^2 v_\theta - \frac{v_\theta}{r^2}\right] + \mathcal{S}_\theta \ , \tag{3.7}$$

$$\rho \frac{Dv_z}{Dt} = -\frac{\partial p}{\partial z} + \mu\left[\nabla^2 v_z + \frac{1}{3}\frac{\partial}{\partial z}(\nabla \cdot \vec{v})\right] + \mathcal{S}_Z \ . \tag{3.8}$$

Recasting the equations

It is useful to recast the governing set (3.1)-(3.4) into an alternate form. Two new equations are developed which replace the azimuthal momentum and energy equations. The first of these equations is based on the auxiliary variable, $\widetilde{H}$, defined as

$$\widetilde{H} = \frac{T}{T_0} + \frac{Pr}{2c_p T_0}\left(v_r^2 + v_\theta(v_\theta - r\Omega) + v_z^2\right)\,, \tag{3.9}$$

where Pr is the Prandtl number, $Pr = \mu c_p / k$, which relates the ease of diffusion of momentum to the ease of diffusion of heat. The initial step is to add (3.4) to the product of $\vec{v}$ and (3.3), giving[4]

$$\begin{aligned}\rho\frac{D}{Dt}\left(c_pT + \tfrac{1}{2}\vec{v}\cdot\vec{v}\right) &= \nabla^2\left(kT + \mu\tfrac{1}{2}\vec{v}\cdot\vec{v}\right) \\ &+\mu\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2 + 2\frac{\partial v_r}{\partial z}\frac{\partial v_z}{\partial r} - 2\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{3}\vec{v}\cdot\nabla(\nabla\cdot\vec{v}) - \frac{2}{3}[\nabla\cdot\vec{v}]^2\right] \\ &+\mathcal{S}_E + \vec{v}\cdot\vec{\mathcal{S}}_{R\theta Z}\,;\end{aligned} \tag{3.10}$$

then, subtracting from this result the quantity $r\Omega/2$ multiplied by (3.7) and dividing every term by a constant temperature T_0 yields[5]

$$\begin{aligned}&\rho c_p\frac{D}{Dt}\left(\frac{T}{T_0} + \frac{1}{2c_pT_0}\left(v_r^2 + v_\theta(v_\theta - r\Omega) + v_z^2\right)\right) = k\nabla^2\widetilde{H} \\ &+\frac{\mu}{T_0}\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2 + 2\frac{\partial v_r}{\partial z}\frac{\partial v_z}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}\left(v_\theta(v_\theta - r\Omega)\right) + \frac{1}{3}\vec{v}\cdot\nabla(\nabla\cdot\vec{v}) - \frac{2}{3}[\nabla\cdot\vec{v}]^2\right] \\ &+\frac{1}{T_0}\left[\mathcal{S}_E + \vec{v}\cdot\vec{\mathcal{S}}_{R\theta Z} - \frac{r\Omega}{2}\mathcal{S}_\theta\right].\end{aligned} \tag{3.11}$$

Equation (3.11) is the first of the new equations. Notice that if $Pr = 1$, then from (3.9) most of the terms to the left of the equals sign may be replaced by $\widetilde{H}$.

The second equation is based on another auxiliary variable, $\widetilde{\Phi}$, defined as

$$\widetilde{\Phi} = \frac{T}{T_0} + \frac{Pr}{2c_pT_0}\left(v_r^2 + v_z^2\right) - \frac{v_\theta^2}{r^2\Omega^2}\,. \tag{3.12}$$

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[4] Recall that $\vec{v}\cdot\frac{D\vec{v}}{Dt} = \frac{D}{Dt}\left(\frac{1}{2}\vec{v}\cdot\vec{v}\right)$. Additionally, in a cylindrical coordinate system with axisymmetric flow, $\vec{v}\cdot\nabla^2\vec{v} = \nabla^2\left(\frac{1}{2}\vec{v}\cdot\vec{v}\right) - \left(\frac{v_r}{r}\right)^2 - \left(\frac{v_\theta}{r}\right)^2 - (\nabla v_r)\cdot(\nabla v_r) - (\nabla v_r)\cdot(\nabla v_r) - (\nabla v_r)\cdot(\nabla v_r)$.

[5] Note that $\frac{r\Omega}{2}\frac{Dv_\theta}{Dt} = \frac{1}{2}\frac{D}{Dt}(r\Omega v_\theta) - \frac{\Omega}{2}v_r v_\theta$ and $\frac{r\Omega}{2}\nabla^2 v_\theta = \nabla^2\left(\frac{1}{2}r\Omega v_\theta\right) - \frac{\Omega}{2r}v_\theta - \Omega\frac{\partial v_\theta}{\partial r}$.

The development of this equation requires first multiplying Pr/c_pT_0 by the difference of (3.10) and the product of v_θ and (3.7), which is[6]

$$\rho\frac{D}{Dt}\left(Pr\left(\frac{T}{T_0}+\frac{v_r^2+v_z^2}{2c_pT_0}\right)\right)-\frac{\rho Pr}{c_pT_0}\frac{v_rv_\theta^2}{r}= \qquad (3.13)$$

$$\mu\nabla^2\left(\frac{T}{T_0}+\frac{Pr}{2c_pT_0}(v_r^2+v_z^2)\right)+\frac{\mu Pr}{c_pT_0}\left[\left(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)^2+\left(\frac{\partial v_\theta}{\partial z}\right)^2\right]$$
$$+\frac{\mu Pr}{c_pT_0}\left[\left(\frac{\partial v_r}{\partial r}\right)^2+\left(\frac{v_r}{r}\right)^2+\left(\frac{\partial v_z}{\partial z}\right)^2+2\frac{\partial v_r}{\partial z}\frac{\partial v_z}{\partial r}+\frac{1}{3}\vec{v}\cdot\nabla(\nabla\cdot\vec{v})-\frac{2}{3}[\nabla\cdot\vec{v}]^2\right]$$
$$+\frac{Pr}{c_pT_0}[\mathcal{S}_E+v_r\mathcal{S}_R+v_z\mathcal{S}_Z]\,,$$

then finding the product of (3.7) and the quantity $2v_\theta/r^2\Omega^2$, which is[7]

$$\rho\frac{D}{Dt}\left(\frac{v_\theta^2}{r^2\Omega^2}\right)+4\rho\frac{v_r}{r}\left(\frac{v_\theta}{r\Omega}\right)^2= \qquad (3.14)$$

$$\mu\left[\nabla^2\left(\frac{v_\theta^2}{r^2\Omega^2}\right)+\frac{4v_\theta}{r^2\Omega^2}\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)-\frac{2}{r^2\Omega^2}\left[\left(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)^2+\left(\frac{\partial v_\theta}{\partial z}\right)^2\right]\right]$$
$$+\frac{2v_\theta}{r^2\Omega^2}\mathcal{S}_\theta\,,$$

and finally subtracting (3.14) from (3.13) to obtain

$$\rho\frac{D}{Dt}\left(Pr\left(\frac{T}{T_0}+\frac{v_r^2+v_z^2}{2c_pT_0}\right)-\left(\frac{v_\theta}{r\Omega}\right)^2\right)-\left(1+\frac{r^2\Omega^2Pr}{4c_pT_0}\right)\frac{4\rho v_r}{r}\left(\frac{v_\theta}{r\Omega}\right)^2= \qquad (3.15)$$

$$\mu\nabla^2\widetilde{\Phi}-\frac{4\mu v_\theta}{r^2\Omega^2}\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{2\mu}{r^2\Omega^2}\left(1+\frac{r^2\Omega^2Pr}{2c_pT_0}\right)\left[\left(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)^2+\left(\frac{\partial v_\theta}{\partial z}\right)^2\right]$$
$$+\frac{\mu Pr}{c_pT_0}\left[\left(\frac{\partial v_r}{\partial r}\right)^2+\left(\frac{v_r}{r}\right)^2+\left(\frac{\partial v_z}{\partial z}\right)^2+2\frac{\partial v_r}{\partial z}\frac{\partial v_z}{\partial r}+\frac{1}{3}\vec{v}\cdot\nabla(\nabla\cdot\vec{v})-\frac{2}{3}[\nabla\cdot\vec{v}]^2\right]$$
$$+\frac{Pr}{c_pT_0}[\mathcal{S}_E+v_r\mathcal{S}_R+v_z\mathcal{S}_Z]-\frac{2v_\theta}{r^2\Omega^2}\mathcal{S}_\theta\,.$$

Here again, if $Pr=1$, the variable $\widetilde{\Phi}$ may replace most of the terms to the left of the equals sign.

Equations (3.1), (3.2), (3.6), (3.8), (3.11), and (3.15), along with the auxiliary relationships (3.9) and (3.12) make up a complete system.

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[6] Note the identities $v_\theta\frac{Dv_\theta}{Dt}=\frac{D}{Dt}\left(\frac{1}{2}v_\theta^2\right)$ and $v_\theta\nabla^2v_\theta=\nabla^2\left(\frac{1}{2}v_\theta^2\right)-(\nabla v_\theta)\cdot(\nabla v_\theta)$.

[7] Note that $\frac{2v_\theta}{r^2\Omega^2}\frac{Dv_\theta}{Dt}=\frac{D}{Dt}\left(\frac{v_\theta^2}{r^2\Omega^2}\right)+2\frac{v_r}{r}\left(\frac{v_\theta}{r\Omega}\right)^2$ and $\frac{2v_\theta}{r^2\Omega^2}\nabla^2v_\theta=\nabla^2\left(\frac{v_\theta^2}{r^2\Omega^2}\right)-\frac{2}{r^2\Omega^2}(\nabla v_\theta)\cdot(\nabla v_\theta)+\frac{8v_\theta}{r^3\Omega^2}\frac{\partial v_\theta}{\partial r}-\frac{4v_\theta^2}{r^4\Omega^2}$.

The reference solution

The flow in a gas centrifuge is assumed to vary only slightly from the flow pattern exhibited by a fluid in solid body rotation. This reference state – nicknamed isothermal "wheel flow" – has velocities

$$\begin{aligned} v_{r0} &= 0 \,, \\ v_{\theta 0} &= r\Omega \,, \\ v_{z0} &= 0 \,, \end{aligned} \tag{3.16}$$

and a constant temperature, T_0. The corresponding pressure and density are determined from the radial component of the momentum equation,

$$\frac{\partial p_0}{\partial r} = \rho_0 \frac{{v_{\theta 0}}^2}{r} = \rho_0 r \Omega^2 \,, \tag{3.17}$$

and the ideal gas law, leading to

$$\frac{p_0}{p_w} = \frac{\rho_0}{\rho_w} = e^{-A^2\left(1-\left(\frac{r}{a}\right)^2\right)} \,. \tag{3.18}$$

In this expression, p_w and ρ_w are the pressure and density at the wall, a is the radius of the cylinder, and the variable A is the stratification parameter, defined as the ratio of the wall speed to the most probable molecular speed of the gas,

$$A = \frac{a\Omega}{\sqrt{2RT_0}} \,. \tag{3.19}$$

It is important to emphasize that this reference solution satisfies the system of equations (3.1)-(3.4) and therefore also the alternate system made up of (3.1), (3.2), (3.6), (3.8), (3.11), and (3.15).

The linearized equations

Assuming the flow circulating in a gas centrifuge is a linear perturbation of the reference state described above, the flow variables may be written

$$\begin{gathered} v_r = v_{r1} \,, \quad v_\theta = r\Omega + v_{\theta 1} \,, \quad v_z = v_{z1} \,, \\ p = p_0 + p_1 \,, \quad \rho = \rho_0 + \rho_1 \,, \quad T = T_0 + T_1 \,, \\ \widetilde{H} = \widetilde{H}_0 + H, \quad \widetilde{\Phi} = \widetilde{\Phi}_0 + \Phi \,. \end{gathered} \tag{3.20}$$

Note that $\widetilde{H}_0 = 1$ and

$$H = \frac{T_1}{T_0} + \frac{r^2\Omega^2 Pr}{2c_p T_0}\frac{v_{\theta 1}}{r\Omega}\,, \tag{3.21}$$

while $\widetilde{\Phi}_0 = 0$, and

$$\Phi = \frac{T_1}{T_0} - \frac{2v_{\theta 1}}{r\Omega}\,. \tag{3.22}$$

These expressions are inserted into (3.1), (3.2), (3.6), (3.8), (3.11), and (3.15). Linearized equations for the perturbation variables (e.g., p_1) are developed by subtracting from each equation the same equation evaluated at the reference state, and then retaining only the linear terms. The resulting system is given by (3.23)-(3.28),

$$\frac{p_1}{p_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0}\,, \tag{3.23}$$

$$v_{r1}\frac{\partial \rho_0}{\partial r} = -\rho_0(\nabla\cdot\vec{v}_1) + \mathcal{S}_M\,, \tag{3.24}$$

$$-2\rho_0\Omega v_{\theta 1} - \rho_1 r\Omega^2 = -\frac{\partial p_1}{\partial r} + \mu\left[\nabla^2 v_{r1} - \frac{v_{r1}}{r^2} + \frac{1}{3}\frac{\partial}{\partial r}(\nabla\cdot\vec{v}_1)\right] + \mathcal{S}_R\,, \tag{3.25}$$

$$0 = -\frac{\partial p_1}{\partial z} + \mu\left[\nabla^2 v_{z1} + \frac{1}{3}\frac{\partial}{\partial z}(\nabla\cdot\vec{v}_1)\right] + \mathcal{S}_Z\,, \tag{3.26}$$

$$0 = \nabla^2 H - \frac{\Omega Pr}{c_p T_0}\frac{1}{r}\frac{\partial}{\partial r}(r v_{\theta 1}) + \frac{1}{kT_0}\left[\mathcal{S}_E + \frac{r\Omega}{2}\mathcal{S}_\theta\right], \tag{3.27}$$

$$-4\left(1 + \frac{r^2\Omega^2 Pr}{4c_p T_0}\right)\frac{\rho_0 v_{r1}}{r} = \mu\nabla^2\Phi - \frac{4\mu}{r}\frac{\partial}{\partial r}\left(\frac{v_{\theta 1}}{r\Omega}\right) + \frac{Pr}{c_p T_0}\mathcal{S}_E - \frac{2}{r\Omega}\mathcal{S}_\theta\,. \tag{3.28}$$

Nondimensionalization

It is useful at this stage to introduce dimensionless versions of each variable. These include dimensionless coordinates,

$$\eta = \frac{r}{a}\,, \quad y = \frac{z}{L}\,, \tag{3.29}$$

dimensionless pressures and densities,

$$\bar{p}_0 = \frac{p_0}{p_w}\,, \quad \bar{p} = \frac{p_1}{p_w}\,, \quad \bar{\rho}_0 = \frac{\rho_0}{\rho_w}\,, \quad \bar{\rho} = \frac{\rho_1}{\rho_w} \tag{3.30}$$

dimensionless velocities,

$$\bar{v}_r = \frac{v_{r1}}{a\Omega}\,, \quad \bar{v}_z = \frac{v_{z1}}{a\Omega}\,, \quad \bar{v}_\theta = \frac{v_{\theta 1}}{r\Omega}\,, \tag{3.31}$$

a dimensionless temperature,

$$\bar{T} = \frac{T_1}{T_0}\,, \tag{3.32}$$

and dimensionless source terms,

$$\bar{S}_M = \frac{S_M}{\rho_w \Omega}\,,\quad \bar{S}_R = \frac{S_R}{\rho_w a \Omega^2}\,,\quad \bar{S}_\theta = \frac{S_\theta}{\rho_w a \Omega^2}\,,\quad \bar{S}_Z = \frac{S_Z}{\rho_w a \Omega^2}\,,\quad \bar{S}_E = \frac{S_E}{\rho_w a^2 \Omega^3}\,. \tag{3.33}$$

In addition, a dimensionless Laplacian operator is defined as

$$\bar{\nabla}^2 = a^2\nabla^2 = \frac{1}{\eta}\frac{\partial}{\partial \eta}\eta\frac{\partial}{\partial \eta} + \frac{1}{Z^2}\frac{\partial^2}{\partial y^2}\,, \tag{3.34}$$

and a dimensionless divergence is

$$\bar{\nabla}\cdot\bar{v} = \frac{\nabla\cdot\vec{v}_1}{\Omega} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\eta\bar{v}_r + \frac{1}{Z}\frac{\partial \bar{v}_z}{\partial y}\,. \tag{3.35}$$

Finally, there are five dimensionless numbers of interest: the Prandtl number and stratification parameter (A), each defined previously, the cylinder aspect ratio,

$$Z = \frac{L}{a}\,, \tag{3.36}$$

the Reynolds number,

$$Re = \frac{\rho_w a^2 \Omega}{\mu}\,, \tag{3.37}$$

and the parameter $\hat{K}$ – sometimes called the Brinkman number – where

$$\hat{K} = \frac{a^2\Omega^2 Pr}{4c_p T_0}\,. \tag{3.38}$$

Employing (3.29)-(3.38), the system of (3.23)-(3.28) may be written in dimensionless form,

$$\bar{p} = \bar{\rho} + \bar{\rho}_0\bar{T}\,, \tag{3.39}$$

$$\bar{v}_r\frac{\partial\bar{\rho}_0}{\partial \eta} = -\bar{\rho}_0(\bar{\nabla}\cdot\bar{v}) + \bar{S}_M\,, \tag{3.40}$$

$$-2\eta\bar{\rho}_0\bar{v}_\theta - \eta\bar{\rho} = -\frac{1}{2A^2}\frac{\partial\bar{p}}{\partial\eta} + \frac{1}{Re}\left[\bar{\nabla}^2\bar{v}_r - \frac{\bar{v}_r}{\eta^2} + \frac{1}{3}\frac{\partial}{\partial\eta}(\bar{\nabla}\cdot\bar{v})\right] + \bar{S}_R\,, \tag{3.41}$$

$$0 = -\frac{1}{2A^2}\frac{\partial\bar{p}}{\partial y} + \frac{Z}{Re}\left[\bar{\nabla}^2\bar{v}_z + \frac{1}{3Z}\frac{\partial}{\partial y}(\bar{\nabla}\cdot\bar{v})\right] + Z\bar{S}_Z\,, \tag{3.42}$$

$$0 = \frac{1}{Re}\bar{\nabla}^2 H - \frac{4\hat{K}}{Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta^2\bar{v}_\theta) + 4\hat{K}\left[\bar{S}_E + \frac{1}{2}\eta\bar{S}_\theta\right], \tag{3.43}$$

$$-4(1 + \hat{K}\eta^2)\frac{\bar{\rho}_0\bar{v}_r}{\eta} = \frac{1}{Re}\bar{\nabla}^2\Phi - \frac{4}{Re}\frac{1}{\eta}\frac{\partial\bar{v}_\theta}{\partial\eta} + 4\left[\hat{K}\bar{S}_E - \frac{1}{2\eta}\bar{S}_\theta\right]. \tag{3.44}$$

Note from (3.21) and (3.22) that H and Φ are already dimensionless; moreover,

$$H = \bar{T} + 2\hat{K}\eta^2\bar{v}_\theta \;, \tag{3.45}$$

$$\Phi = \bar{T} - 2\bar{v}_\theta \;. \tag{3.46}$$

The combined momentum equation

In this section, the radial and axial momentum equations will be combined into a single equation in such a way that all dependence on $\bar{\rho}$ and $\bar{p}$ is eliminated. Before proceeding, it is advantageous to note that the continuity equation may be used to eliminate the divergence terms in each momentum equation. Employing (3.40) and a helpful identity developed from (3.17),

$$\frac{1}{\bar{\rho}_0}\frac{\partial\bar{\rho}_0}{\partial\eta} = 2A^2\eta \;, \tag{3.47}$$

the momentum equations (3.41) and (3.42) become

$$-2\eta\bar{\rho}_0\bar{v}_\theta - \eta\bar{\rho} = -\frac{1}{2A^2}\frac{\partial\bar{p}}{\partial\eta} + \frac{1}{Re}\left[\bar{\nabla}^2\bar{v}_r - \frac{\bar{v}_r}{\eta^2} - \frac{2A^2}{3}\frac{\partial}{\partial\eta}(\eta\bar{v}_r)\right] + \bar{\mathcal{S}}_R + \frac{1}{3Re}\frac{\partial}{\partial\eta}\left[\frac{1}{\bar{\rho}_0}\bar{\mathcal{S}}_M\right], \tag{3.48}$$

$$0 = -\frac{1}{2A^2}\frac{\partial\bar{p}}{\partial y} + \frac{Z}{Re}\left[\bar{\nabla}^2\bar{v}_z - \frac{2A^2}{3Z}\frac{\partial}{\partial y}(\eta\bar{v}_r)\right] + Z\bar{\mathcal{S}}_Z + \frac{1}{3Re}\frac{\partial}{\partial y}\left[\frac{1}{\bar{\rho}_0}\bar{\mathcal{S}}_M\right]. \tag{3.49}$$

The process of combining the two begins by using (3.39) to eliminate $\bar{\rho}$ in (3.48) which allows for the introduction of Φ, leading to

$$\eta\bar{\rho}_0\Phi - \eta\bar{p} = -\frac{1}{2A^2}\frac{\partial\bar{p}}{\partial\eta} + \frac{1}{Re}\left[\bar{\nabla}^2\bar{v}_r - \frac{\bar{v}_r}{\eta^2} - \frac{2A^2}{3}\frac{\partial}{\partial\eta}(\eta\bar{v}_r)\right] + \bar{\mathcal{S}}_R + \frac{1}{3Re}\frac{\partial}{\partial\eta}\left[\frac{1}{\bar{\rho}_0}\bar{\mathcal{S}}_M\right]; \tag{3.50}$$

then, after dividing by $\bar{\rho}_0$, (3.47) may be used to rewrite the previous result as

$$\eta\Phi + \frac{1}{2A^2}\frac{\partial}{\partial\eta}\left(\frac{\bar{p}}{\bar{\rho}_0}\right) = \frac{1}{\bar{\rho}_0 Re}\left[\bar{\nabla}^2\bar{v}_r - \frac{\bar{v}_r}{\eta^2} - \frac{2A^2}{3}\frac{\partial}{\partial\eta}(\eta\bar{v}_r)\right] + \frac{1}{\bar{\rho}_0}\bar{\mathcal{S}}_R + \frac{1}{3\bar{\rho}_0 Re}\frac{\partial}{\partial\eta}\left[\frac{1}{\bar{\rho}_0}\bar{\mathcal{S}}_M\right]. \tag{3.51}$$

Similarly, dividing (3.49) by $\bar{\rho}_0$ yields

$$0 = -\frac{1}{2A^2}\frac{\partial}{\partial y}\left(\frac{\bar{p}}{\bar{\rho}_0}\right) + \frac{Z}{\bar{\rho}_0 Re}\left[\bar{\nabla}^2\bar{v}_z - \frac{2A^2}{3Z}\frac{\partial}{\partial y}(\eta\bar{v}_r)\right] + \frac{Z}{\bar{\rho}_0}\bar{\mathcal{S}}_Z + \frac{1}{3\bar{\rho}_0 Re}\frac{\partial}{\partial y}\left[\frac{1}{\bar{\rho}_0}\bar{\mathcal{S}}_M\right]. \tag{3.52}$$

Finally, differentiating (3.51) with respect to z and differentiating (3.52) with respect to η makes it possible to eliminate the pressure terms between them, resulting in

$$\eta\frac{\partial\Phi}{\partial y} = \frac{1}{Re}\frac{\partial}{\partial y}\left[\frac{1}{\bar{\rho}_0}\left[\overline{\nabla}^2\bar{v}_r - \frac{\bar{v}_r}{\eta^2} - \frac{2A^2}{3}\frac{\partial}{\partial\eta}(\eta\bar{v}_r)\right]\right] - \frac{Z}{Re}\frac{\partial}{\partial\eta}\left[\frac{1}{\bar{\rho}_0}\left[\overline{\nabla}^2\bar{v}_z - \frac{2A^2}{3Z}\frac{\partial}{\partial y}(\eta\bar{v}_r)\right]\right] \qquad (3.53)$$

$$+\frac{\partial}{\partial y}\left[\frac{1}{\bar{\rho}_0}\bar{\mathcal{S}}_R\right] - Z\frac{\partial}{\partial\eta}\left[\frac{1}{\bar{\rho}_0}\bar{\mathcal{S}}_Z\right] + \frac{1}{3Re}\frac{\partial}{\partial y}\left[\frac{1}{\bar{\rho}_0}\frac{\partial}{\partial\eta}\left[\frac{1}{\bar{\rho}_0}\bar{\mathcal{S}}_M\right]\right] - \frac{1}{3Re}\frac{\partial}{\partial\eta}\left[\frac{1}{\bar{\rho}_0}\frac{\partial}{\partial y}\left[\frac{1}{\bar{\rho}_0}\bar{\mathcal{S}}_M\right]\right].$$

After rearranging and rewriting (3.53) using operator notation,

$$\eta\frac{\partial\Phi}{\partial y} = \frac{1}{\bar{\rho}_0 Re}\left(\overline{\nabla}^2 - \frac{1}{\eta^2} - \frac{4A^4}{3}\eta^2\right)\frac{\partial}{\partial y}\bar{v}_r - \frac{Z}{Re}\frac{\partial}{\partial\eta}\frac{1}{\bar{\rho}_0}\overline{\nabla}^2\bar{v}_z \qquad (3.54)$$

$$+\frac{1}{\bar{\rho}_0}\frac{\partial}{\partial y}\bar{\mathcal{S}}_R - Z\frac{\partial}{\partial\eta}\frac{1}{\bar{\rho}_0}\bar{\mathcal{S}}_Z + \frac{2A^2}{3Re}\frac{\eta}{\bar{\rho}_0^2}\frac{\partial}{\partial y}\bar{\mathcal{S}}_M ,$$

it becomes clear that the right hand side (excluding the source terms) is a function solely of the radial and axial velocities.

Defining the stream function

Equation (3.40) rewritten in an alternate form,

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta\bar{\rho}_0\bar{v}_r) + \frac{1}{Z}\frac{\partial}{\partial y}(\bar{\rho}_0\bar{v}_z) = \bar{\mathcal{S}}_M , \qquad (3.55)$$

suggests a stream function, ψ, defined such that

$$\eta\bar{\rho}_0\bar{v}_r = -\frac{\partial\psi}{\partial y} - \int_\eta^1 \eta'\bar{\mathcal{S}}_M d\eta' , \qquad (3.56)$$

$$\bar{\rho}_0\bar{v}_z = \frac{Z}{\eta}\frac{\partial\psi}{\partial\eta} . \qquad (3.57)$$

Integrating (3.56) from 0 to y, evaluating the result at $\eta = 1$, and then requiring that no flow passes through the rotor wall, gives

$$\psi|_{\eta=1} = 0 , \qquad (3.58)$$

in which $\psi|_{\eta=1,y=0}$ has been set to zero which is permissible because any constant may be added to the stream function without affecting the primitive velocities. Then, integrating (3.57) from η to 1 and inserting the result from (3.58) yields

$$\psi = -\frac{1}{Z}\int_\eta^1 \eta'\bar{\rho}_0\bar{v}_z d\eta' . \qquad (3.59)$$

This last expression relates the stream function to the net axial mass flux between η and the wall.

A system of three equations

Noting that

$$\bar{v}_\theta = \frac{H-\Phi}{2(1+\hat{K}\eta^2)} \tag{3.60}$$

and utilizing the stream function introduced in (3.56)-(3.57), equations (3.43) and (3.44) may be reformed into

$$\frac{1}{Re}\overline{\nabla}^2 H = \frac{2\hat{K}}{Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta^2\frac{H-\Phi}{1+\hat{K}\eta^2}\right) - 4\hat{K}\left[\bar{S}_E + \frac{1}{2}\eta\bar{S}_\theta\right], \tag{3.61}$$

$$\frac{1}{Re}\overline{\nabla}^2\Phi - 4\frac{1+\hat{K}\eta^2}{\eta^2}\frac{\partial\psi}{\partial y} = \frac{2}{Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{H-\Phi}{1+\hat{K}\eta^2}\right) \tag{3.62}$$
$$+4\frac{1+\hat{K}\eta^2}{\eta^2}\int_\eta^1 \eta'\bar{S}_M d\eta' - 4\left[\hat{K}\bar{S}_E - \frac{1}{2\eta}\bar{S}_\theta\right].$$

Then, inserting the stream function into equation (3.54) yields

$$\frac{\partial\Phi}{\partial y} = -\frac{1}{Re\bar{\rho}_0\eta}\left(\overline{\nabla}^2 - \frac{1}{\eta^2} - \frac{4A^4}{3}\eta^2\right)\frac{1}{\bar{\rho}_0\eta}\frac{\partial^2\psi}{\partial y^2} - \frac{Z^2}{Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}\frac{1}{\bar{\rho}_0}\overline{\nabla}^2\frac{1}{\bar{\rho}_0\eta}\frac{\partial\psi}{\partial\eta} \tag{3.63}$$
$$+\frac{1}{\bar{\rho}_0\eta}\frac{\partial}{\partial y}\bar{S}_R - \frac{Z}{\eta}\frac{\partial}{\partial\eta}\frac{1}{\bar{\rho}_0}\bar{S}_Z + \frac{2A^2}{3Re}\frac{1}{\bar{\rho}_0^2}\frac{\partial}{\partial y}\bar{S}_M$$
$$-\frac{1}{Re\bar{\rho}_0\eta}\left(\overline{\nabla}^2 - \frac{1}{\eta^2} - \frac{4A^4}{3}\eta^2\right)\frac{1}{\bar{\rho}_0\eta}\int_\eta^1 \eta'\frac{\partial\bar{S}_M}{\partial y}d\eta'.$$

Equations (3.61)-(3.63) make up a complete system in the variables H, Φ, and ψ. Thus far, no approximations have been made aside from the linearization itself.

Simplifying assumptions

To advance further towards the Onsager-Maslen equation, it is necessary to make additional simplifications. If the Reynolds number is high, the viscous terms in (3.61)-(3.63) are expected to have minimal impact on the flow except within boundary layers. Furthermore, away from the end walls, radial diffusion is expected to be of greater significance than axial diffusion. For these reasons, it is assumed that viscous terms with derivatives in the y coordinate may be safely neglected; the result after this simplification is given below in (3.64)-(3.66).[8] The validity of this assumption is reinforced for cylinders with a high aspect ratio – see (3.34) and consider the interaction between Z^2 and Re in the third term in (3.63).

[8] This same result would have arisen if two separate assumptions had been made from the beginning: (1) to neglect all viscous terms in the radial momentum equation, and (2) to neglect viscous terms with derivatives in the axial direction in the remaining conservation equations.

$$\frac{1}{Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial H}{\partial\eta}\right) = \frac{2\hat{K}}{Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta^2\frac{H-\Phi}{1+\hat{K}\eta^2}\right) - 4\hat{K}\left[\bar{\mathcal{S}}_E + \frac{1}{2}\eta\bar{\mathcal{S}}_\theta\right], \tag{3.64}$$

$$\frac{1}{Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\Phi}{\partial\eta}\right) - 4\frac{1+\hat{K}\eta^2}{\eta^2}\frac{\partial\psi}{\partial y} = \frac{2}{Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{H-\Phi}{1+\hat{K}\eta^2}\right) + 4\frac{1+\hat{K}\eta^2}{\eta^2}\int_\eta^1 \eta'\bar{\mathcal{S}}_M d\eta' - 4\left[\hat{K}\bar{\mathcal{S}}_E - \frac{1}{2\eta}\bar{\mathcal{S}}_\theta\right], \tag{3.65}$$

$$\frac{\partial\Phi}{\partial y} = -\frac{Z^2}{Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{\rho}_0\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{\rho}_0\eta}\frac{\partial\psi}{\partial\eta}\right)\right)\right) + \frac{1}{\bar{\rho}_0\eta}\frac{\partial\bar{\mathcal{S}}_R}{\partial y} - \frac{Z}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{\rho}_0}\bar{\mathcal{S}}_Z\right) + \frac{2A^2}{3Re}\frac{1}{\bar{\rho}_0^2}\frac{\partial\bar{\mathcal{S}}_M}{\partial y} - \frac{1}{Re\bar{\rho}_0\eta}\bar{\nabla}^2\left(\frac{1}{\bar{\rho}_0\eta}\int_\eta^1 \eta'\frac{\partial\bar{\mathcal{S}}_M}{\partial y}d\eta'\right) + \frac{1}{Re}\left(1+\frac{4A^4\eta^4}{3}\right)\frac{1}{\bar{\rho}_0^2\eta^4}\int_\eta^1 \eta'\frac{\partial\bar{\mathcal{S}}_M}{\partial y}d\eta'\ . \tag{3.66}$$

The source terms involving $\bar{\mathcal{S}}_M$ in (3.66) have been retained for completeness; however, the equivalent terms are neglected in [2] and [7] because these (viscous) terms are scaled by $\bar{\rho}_0$ in such a way that they are expected to be non-negligible only high in the atmosphere where viscous effects are fairly unimportant.

Reducing the system to two equations

In this section, equations (3.64) and (3.65) are combined into a single equation that is a function of Φ. Multiplying equation (3.64) by η and then integrating from η_T to η yields

$$\frac{1}{Re}\eta\frac{\partial H}{\partial\eta} = \frac{2\hat{K}}{Re}\eta^2\frac{H-\Phi}{1+\hat{K}\eta^2} + \frac{\eta_0}{Re}\frac{\partial\bar{T}}{\partial\eta}\bigg|_{\eta=\eta_T} + \frac{2\hat{K}\eta_0^3}{Re}\frac{\partial\bar{v}_\theta}{\partial\eta}\bigg|_{\eta=\eta_T} - 4\hat{K}\int_{\eta_T}^{\eta}\eta'\left[\bar{\mathcal{S}}_E + \frac{1}{2}\eta'\bar{\mathcal{S}}_\theta\right]d\eta'\ ; \tag{3.67}$$

however, if η_T is chosen sufficiently far from the rotor wall such that the temperature and circumferential velocity gradients have decayed to zero (or if η_T is set to zero), then[9]

$$\frac{1}{Re}\eta\frac{\partial H}{\partial\eta} = \frac{2\hat{K}}{Re}\eta^2\frac{H-\Phi}{1+\hat{K}\eta^2} - 4\hat{K}\int_{\eta_T}^{\eta}\eta'\left[\bar{\mathcal{S}}_E + \frac{1}{2}\eta'\bar{\mathcal{S}}_\theta\right]d\eta'\ . \tag{3.68}$$

After dividing by $\eta^2(1+\hat{K}\eta^2)$ and rearranging, equation (3.68) may be rewritten

$$\frac{1}{Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{H}{1+\hat{K}\eta^2}\right) = -\frac{2}{Re}\frac{\hat{K}}{(1+\hat{K}\eta^2)^2}\Phi - \frac{4\hat{K}}{\eta^2(1+\hat{K}\eta^2)}\int_{\eta_T}^{\eta}\eta'\left[\bar{\mathcal{S}}_E + \frac{1}{2}\eta'\bar{\mathcal{S}}_\theta\right]d\eta'\ . \tag{3.69}$$

Next, using the identity

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\Phi}{\partial\eta}\right) = \frac{1+\lambda\eta^2}{\eta^3}\frac{\partial}{\partial\eta}\left(\frac{\eta^3}{1+\lambda\eta^2}\frac{\partial\Phi}{\partial\eta}\right) - \frac{2}{\eta(1+\lambda\eta^2)}\frac{\partial\Phi}{\partial\eta}, \tag{3.70}$$

[9] See Chapter Four for more a more detailed explanation of the boundary requirements at $\eta = \eta_T$.

equation (3.65) may be reformed into

$$\frac{1}{Re}\frac{1+\hat{K}\eta^2}{\eta^3}\frac{\partial}{\partial\eta}\left(\frac{\eta^3}{1+\hat{K}\eta^2}\frac{\partial\Phi}{\partial\eta}\right) - 4\frac{1+\hat{K}\eta^2}{\eta^2}\frac{\partial\psi}{\partial y} = \frac{2}{Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{H-\Phi}{1+\hat{K}\eta^2}\right) + \frac{2}{Re}\frac{1}{\eta(1+\hat{K}\eta^2)}\frac{\partial\Phi}{\partial\eta} + 4\frac{1+\hat{K}\eta^2}{\eta^2}\int_\eta^1 \eta'\bar{S}_M d\eta' - 4\left[\hat{K}\bar{S}_E - \frac{1}{2\eta}\bar{S}_\theta\right]; \tag{3.71}$$

then, using equation (3.69) to eliminate the H term in (3.71) leads to, after multiplying by η^2 and dividing by $4(1+\hat{K}\eta^2)$,

$$\frac{1}{4Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{\eta^3}{1+\hat{K}\eta^2}\frac{\partial\Phi}{\partial\eta}\right) - \frac{\partial\psi}{\partial y} = + \int_\eta^1 \eta'\bar{S}_M d\eta' - \frac{\eta^2}{1+\hat{K}\eta^2}\left[\hat{K}\bar{S}_E - \frac{1}{2\eta}\bar{S}_\theta\right] - \frac{2\hat{K}}{(1+\hat{K}\eta^2)^2}\int_{\eta_T}^{\eta}\eta'\left[\bar{S}_E + \frac{1}{2}\eta'\bar{S}_\theta\right]d\eta' \,. \tag{3.72}$$

This result and (3.66) make up a system in the variables ψ and Φ.

Onsager's master potential

Onsager's master potential function, χ, which fully encapsulates the flow variables ψ and Φ (and therefore $\bar{p}$, $\bar{\rho}$, $\bar{v}_r$, $\bar{v}_z$, and the combination $\bar{T} - 2\bar{v}_\theta$) is defined by three equations. The first two are, using (3.72),

$$\psi = \frac{1}{\eta}\frac{\partial\chi}{\partial\eta}, \tag{3.73}$$

$$\frac{\partial\Phi}{\partial\eta} = 4Re\frac{1+\hat{K}\eta^2}{\eta^3}\frac{\partial\chi}{\partial y} + 4Re\frac{1+\hat{K}\eta^2}{\eta^3}\int_{\eta_T}^{\eta}\eta'\int_{\eta'}^{1}\eta''\bar{S}_M d\eta''\,d\eta' - 4Re\frac{1+\hat{K}\eta^2}{\eta^3}\int_{\eta_T}^{\eta}\frac{\eta'^3}{1+\hat{K}\eta'^2}\left[\hat{K}\bar{S}_E - \frac{1}{2\eta'}\bar{S}_\theta\right]d\eta' - 8Re\hat{K}\frac{1+\hat{K}\eta^2}{\eta^3}\int_{\eta_T}^{\eta}\frac{\eta'}{\left(1+\hat{K}\eta'^2\right)^2}\int_{\eta_T}^{\eta'}\eta''\left[\bar{S}_E + \frac{1}{2}\eta''\bar{S}_\theta\right]d\eta''\,d\eta' \,, \tag{3.74}$$

and the third is, using (3.66) and (3.73),

$$\frac{\partial\Phi}{\partial y} = -\frac{Z^2}{Re}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{\rho}_0\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{\rho}_0\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial\chi}{\partial\eta}\right)\right)\right)\right) + \frac{1}{\bar{\rho}_0\eta}\frac{\partial\bar{S}_R}{\partial y} - \frac{Z}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{\rho}_0}\bar{S}_Z\right) + \frac{2A^2}{3Re}\frac{1}{\bar{\rho}_0^2}\frac{\partial\bar{S}_M}{\partial y} - \frac{1}{Re\bar{\rho}_0\eta}\bar{\nabla}^2\left(\frac{1}{\bar{\rho}_0\eta}\int_\eta^1\eta'\frac{\partial\bar{S}_M}{\partial y}d\eta'\right) + \frac{1}{Re}\left(1+\frac{4A^4\eta^4}{3}\right)\frac{1}{\bar{\rho}_0^2\eta^4}\int_\eta^1\eta'\frac{\partial\bar{S}_M}{\partial y}d\eta' \,. \tag{3.75}$$

Note that each of the primitive variables, except for $\partial\Phi/\partial\eta$, is obtained by taking radial derivatives of the master potential.

The Onsager-Maslen equation

Differentiating equation (3.74) with respect to y and differentiating (3.75) with respect to η, then eliminating the Φ term between them yields

$$\begin{aligned}
&\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{\rho}_0\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{\rho}_0\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial\chi}{\partial\eta}\right)\right)\right)\right)\right)+\frac{4Re^2}{Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\frac{\partial^2\chi}{\partial y^2}= \\
&\quad-\frac{4Re^2}{Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\int_{\eta_T}^{\eta}\eta'\int_{\eta'}^{1}\eta''\frac{\partial\bar{S}_M}{\partial y}d\eta''\,d\eta' \\
&\quad+\frac{Re}{Z^2}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{\rho}_0\eta}\frac{\partial\bar{S}_R}{\partial y}\right) \\
&\quad-\frac{Re}{Z}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{\rho}_0}\bar{S}_Z\right)\right) \\
&\quad+\frac{4Re^2}{Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\int_{\eta_T}^{\eta}\frac{\eta'^3}{1+\hat{K}\eta'^2}\frac{\partial}{\partial y}\left[\hat{K}\bar{S}_E-\frac{1}{2\eta'}\bar{S}_\theta\right]d\eta' \\
&\quad+\frac{8Re^2\hat{K}}{Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\int_{\eta_T}^{\eta}\frac{\eta'}{\left(1+\hat{K}\eta'^2\right)^2}\int_{\eta_T}^{\eta'}\eta''\frac{\partial}{\partial y}\left[\bar{S}_E+\frac{1}{2}\eta''\bar{S}_\theta\right]d\eta''\,d\eta' \\
&\quad+\frac{2A^2}{3Z^2}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{\rho}_0^2}\frac{\partial\bar{S}_M}{\partial y}\right) \\
&\quad-\frac{1}{Z^2}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\bar{\rho}_0\eta}\bar{\nabla}^2\left(\frac{1}{\bar{\rho}_0\eta}\int_{\eta}^{1}\eta'\frac{\partial\bar{S}_M}{\partial y}d\eta'\right)\right) \\
&\quad+\frac{1}{Z^2}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\left(1+\frac{4A^4\eta^4}{3}\right)\frac{1}{\bar{\rho}_0^2\eta^4}\int_{\eta}^{1}\eta'\frac{\partial\bar{S}_M}{\partial y}d\eta'\right),
\end{aligned} \tag{3.76}$$

which is a partial differential equation of sixth order in η and of second order in y. The final step is to introduce the stretched radial coordinate, x, defined as

$$x = A^2(1-\eta^2)\,, \tag{3.77}$$

From (3.77), it is clear that[10]

$$\frac{1}{\eta}\frac{\partial}{\partial\eta} = -2A^2\frac{\partial}{\partial x}\,, \tag{3.78}$$

and that (3.18) may be written

$$\frac{1}{\bar{\rho}_0} = e^x\,. \tag{3.79}$$

[10] An alternate form of this identity is useful for converting the integrals: $\eta d\eta = -\frac{1}{2A^2}dx$.

Then, applying (3.77)-(3.79) to equation (3.76) produces, at last, the Onsager-Maslen equation,

$$\frac{\partial^2}{\partial x^2}\left(e^x\frac{\partial}{\partial x}\left(\eta^2\frac{\partial}{\partial x}\left(e^x\frac{\partial^2\chi}{\partial x^2}\right)\right)\right)+\frac{Re^2}{16A^{12}Z^2}\frac{1+\widehat{K}\eta^2}{\eta^4}\frac{\partial^2\chi}{\partial y^2}= \tag{3.80}$$

$$\begin{aligned}
&+\frac{Re^2}{32A^{14}Z^2}\frac{1+\widehat{K}\eta^2}{\eta^4}\int_x^{x_T}\frac{\eta'^2}{1+\widehat{K}\eta'^2}\frac{\partial}{\partial y}\left[\widehat{K}\bar{S}_E-\frac{1}{2\eta'}\bar{S}_\theta\right]dx' \\
&-\frac{Re^2}{64A^{16}Z^2}\frac{1+\widehat{K}\eta^2}{\eta^4}\int_x^{x_T}\int_0^{x'}\frac{\partial\bar{S}_M}{\partial y}dx''\,dx' \\
&-\frac{Re}{32A^{10}Z^2}\left[\frac{\partial}{\partial x}\left(\frac{e^x}{\eta}\frac{\partial\bar{S}_R}{\partial y}\right)+2A^2Z\frac{\partial^2}{\partial x^2}(e^x\bar{S}_Z)\right] \\
&+\frac{Re^2\widehat{K}}{32A^{16}Z^2}\frac{1+\widehat{K}\eta^2}{\eta^4}\int_x^{x_T}\frac{1}{\left(1+\widehat{K}\eta'^2\right)^2}\int_{x'}^{x_T}\frac{\partial}{\partial y}\left[\bar{S}_E+\frac{1}{2}\eta''\bar{S}_\theta\right]dx''\,dx' \\
&-\frac{1}{48A^8Z^2}\frac{\partial}{\partial x}\left(e^{2x}\frac{\partial\bar{S}_M}{\partial y}\right) \\
&+\frac{1}{16A^8Z^2}\frac{\partial}{\partial x}\left(\frac{e^x}{\eta}\frac{\partial}{\partial x}\left(\eta^2\frac{\partial}{\partial x}\left(\frac{e^x}{\eta}\int_0^x\frac{\partial\bar{S}_M}{\partial y}dx'\right)\right)\right) \\
&+\frac{1}{64A^{12}Z^4}\frac{\partial}{\partial x}\left(\frac{e^{2x}}{\eta^2}\int_0^x\frac{\partial^3\bar{S}_M}{\partial y^3}dx'\right) \\
&-\frac{1}{64A^{12}Z^2}\frac{\partial}{\partial x}\left(\left(1+\frac{4A^4\eta^4}{3}\right)\frac{e^{2x}}{\eta^4}\int_0^x\frac{\partial\bar{S}_M}{\partial y}dx'\right).
\end{aligned}$$

If the curvature term is set to one (the pancake approximation), equation (3.80) reduces to Onsager's equation as derived in [2] and elsewhere.[11] The first three source terms are equivalent to the source terms derived in [2] (as clarified by [7]), the fourth appears because curvature terms were included in the derivation, and the last four are considered in [7] but deemed negligible.[12]

Note that if integration by parts is used to simplify the fourth source term, it may be combined with the first, resulting in a slightly simplified overall source function,[13]

$$\begin{aligned}
\bar{S}=\;&-\frac{Re^2}{64A^{16}Z^2}\frac{1+\widehat{K}\eta^2}{\eta^4}\int_x^{x_T}\int_0^{x'}\frac{\partial\bar{S}_M}{\partial y}dx''\,dx'-\frac{Re}{32A^{10}Z^2}\frac{\partial}{\partial x}\left(\frac{e^x}{\eta}\frac{\partial\bar{S}_R}{\partial y}\right) \\
&-\frac{Re^2}{64A^{14}Z^2}\frac{1}{\eta^4}\int_x^{x_T}\eta'\frac{\partial\bar{S}_\theta}{\partial y}dx'-\frac{Re}{16A^8Z}\frac{\partial^2}{\partial x^2}(e^x\bar{S}_Z)+\frac{Re^2\widehat{K}}{32A^{14}Z^2}\frac{1}{\eta^2}\int_x^{x_T}\frac{\partial\bar{S}_E}{\partial y}dx' \\
&-\frac{1}{48A^8Z^2}\frac{\partial}{\partial x}\left(e^{2x}\frac{\partial\bar{S}_M}{\partial y}\right)+\frac{1}{16A^8Z^2}\frac{\partial}{\partial x}\left(\frac{e^x}{\eta}\frac{\partial}{\partial x}\left(\eta^2\frac{\partial}{\partial x}\left(\frac{e^x}{\eta}\int_0^x\frac{\partial\bar{S}_M}{\partial y}dx'\right)\right)\right) \\
&+\frac{1}{64A^{12}Z^4}\frac{\partial}{\partial x}\left(\frac{e^{2x}}{\eta^2}\int_0^x\frac{\partial^3\bar{S}_M}{\partial y^3}dx'\right)-\frac{1}{64A^{12}Z^2}\frac{\partial}{\partial x}\left(\left(1+\frac{4A^4\eta^4}{3}\right)\frac{e^{2x}}{\eta^4}\int_0^x\frac{\partial\bar{S}_M}{\partial y}dx'\right).
\end{aligned} \tag{3.81}$$

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[11] Note that in [2], Wood and Morton scale the axial coordinate with a rather than with L. One advantage of using L is that the cylinder aspect ratio, Z, appears explicitly in the resulting equation. To see the result of scaling by a instead, set Z to one everywhere it appears in (3.81)
[12] See the sentences following (3.66) above.
[13] This form is better suited for comparison with the equivalent expression given in [20]. A few discrepancies are apparent, which may be traced to minor differences (as well as minor errors) in the derivation presented in [20].

Chapter Four

Boundary Conditions

Introduction

Before solving the Onsager-Maslen equation, it is necessary to specify eight total boundary conditions. The Onsager-Maslen equation has six derivatives in the radial coordinate. Three of the radial boundary conditions are specified at the cylinder's outer wall (at $x = 0$) and the remaining three specify the flow high in the atmosphere (at $x = x_T$). The two axial boundary conditions (one for each end of the cylinder; i.e., at $y = 0$ and $y = 1$) are more complex and are referred to as Carrier-Maslen conditions. The purpose of this chapter is to describe the origins and nature of each of the eight boundary conditions.

Revisiting the stream function and master potential

In this section, a variety of expressions for the stream function and master potential are developed which will aid in transforming the physical requirements of the problem to the appropriate radial boundary conditions. To that end, equations (3.56) and (3.57), written in terms of x and ψ, are

$$\eta\bar{\rho}_0\bar{v}_r = -\frac{\partial\psi}{\partial y} - \frac{1}{2A^2}\int_0^x \bar{S}_M dx' \,, \tag{4.1}$$

$$\bar{\rho}_0\bar{v}_z = -2A^2 Z\frac{\partial\psi}{\partial x} \,. \tag{4.2}$$

and, written alternatively in terms of x and χ, are

$$\frac{1}{2A^2}\eta\bar{\rho}_0\bar{v}_r = \frac{\partial}{\partial y}\left(\frac{\partial\chi}{\partial x}\right) - \frac{1}{4A^4}\int_0^x \bar{S}_M dx' \,, \tag{4.3}$$

$$\bar{\rho}_0\bar{v}_z = 4A^4 Z\frac{\partial^2\chi}{\partial x^2} \,. \tag{4.4}$$

Integrating (4.1) with respect to y and integrating (4.2) with respect to x results in

$$\psi = \psi|_{y=0} - \eta\int_0^y \bar{\rho}_0\bar{v}_r dy' - \frac{1}{2A^2}\int_0^x\int_0^y \bar{S}_M dy'\, dx' \,, \tag{4.5}$$

$$\psi = \psi|_{x=0} - \frac{1}{2A^2 Z}\int_0^x \bar{\rho}_0\bar{v}_z dx' \,. \tag{4.6}$$

Setting $\psi|_{x=0,y=0}$ to zero, which is permissible because the associated flow velocities depend only on derivatives of ψ, the equations (4.5) and (4.6) may be combined in two separate ways to yield two alternate but equivalent expressions for the stream function,

$$\psi = -\frac{1}{2A^2 Z}\int_0^x \bar{\rho}_0\bar{v}_z|_{y=0}dx' - \eta\int_0^y \bar{\rho}_0\bar{v}_r dy' - \frac{1}{2A^2}\int_0^x\int_0^y \bar{S}_M dy'\,dx' \,, \tag{4.7}$$

$$\psi = -\int_0^y \bar{v}_r|_{x=0}dy' - \frac{1}{2A^2 Z}\int_0^x \bar{\rho}_0\bar{v}_z dx' \,. \tag{4.8}$$

Equations (4.7) and (4.8) written in terms of the master potential are,

$$\frac{\partial\chi}{\partial x} = \frac{1}{4A^4 Z}\int_0^x \bar{\rho}_0\bar{v}_z|_{y=0}dx' + \frac{\eta}{2A^2}\int_0^y \bar{\rho}_0\bar{v}_r dy' + \frac{1}{4A^4}\int_0^x\int_0^y \bar{S}_M dy'\,dx' \,, \tag{4.9}$$

$$\frac{\partial\chi}{\partial x} = \frac{1}{2A^2}\int_0^y \bar{v}_r|_{x=0}dy' + \frac{1}{4A^4 Z}\int_0^x \bar{\rho}_0\bar{v}_z dx' \,; \tag{4.10}$$

then, integrating from x to x_T yields

$$\begin{aligned}\chi = \chi|_{x=x_T} &- \frac{1}{4A^4 Z}\int_x^{x_T}\int_0^{x'} \bar{\rho}_0\bar{v}_z|_{y=0}dx''\,dx' - \frac{1}{2A^2}\int_x^{x_T}\eta'\int_0^y \bar{\rho}_0\bar{v}_r dy'\,dx' \\ &- \frac{1}{4A^4}\int_x^{x_T}\int_0^{x'}\int_0^y \bar{S}_M dy'\,dx''\,dx' \,,\end{aligned} \tag{4.11}$$

$$\chi = \chi|_{x=x_T} - \frac{x_T - x}{2A^2}\int_0^y \bar{v}_r|_{x=0}dy' - \frac{1}{4A^4 Z}\int_x^{x_T}\int_0^{x'} \bar{\rho}_0\bar{v}_z dx''\,dx' \,. \tag{4.12}$$

The subsequent analysis will also require (3.75) written in terms of x,

$$\begin{aligned}\frac{\partial\Phi}{\partial y} = &\frac{32A^{10}Z^2}{Re}\frac{\partial}{\partial x}\left(e^x\frac{\partial}{\partial x}\left(\eta^2\frac{\partial}{\partial x}\left(e^x\frac{\partial^2\chi}{\partial x^2}\right)\right)\right) \\ &+ \frac{e^x}{\eta}\frac{\partial\bar{S}_R}{\partial y} + 2A^2 Z\frac{\partial}{\partial x}\left(e^x\bar{S}_Z\right) \\ &+ \frac{2A^2}{3Re}e^{2x}\frac{\partial\bar{S}_M}{\partial y} - \frac{2A^2}{Re}\frac{e^x}{\eta}\frac{\partial}{\partial x}\left(\eta^2\frac{\partial}{\partial x}\left(\frac{e^x}{\eta}\int_0^x\frac{\partial\bar{S}_M}{\partial y}dx'\right)\right) \\ &- \frac{1}{2A^2 Re Z^2}\frac{e^{2x}}{\eta^2}\int_0^x\frac{\partial^3\bar{S}_M}{\partial y^3}dx' + \frac{1}{2A^2 Re}\left(1 + \frac{4A^4\eta^4}{3}\right)\frac{e^{2x}}{\eta^4}\int_0^x\frac{\partial\bar{S}_M}{\partial y}dx' \,,\end{aligned} \tag{4.13}$$

and the integral of (3.74) written in terms of x,[14]

$$\begin{aligned}\int_0^y\frac{\partial\Phi}{\partial x}dy' = &-\frac{2Re}{A^2}\frac{1+\hat{K}\eta^2}{\eta^4}\left[\chi - \chi|_{y=0}\right] \\ &- \frac{Re}{2A^6}\frac{1+\hat{K}\eta^2}{\eta^4}\int_x^{x_T}\int_0^{x'}\int_0^y \bar{S}_M dy'\,dx''\,dx' \\ &- \frac{Re}{2A^4}\frac{1}{\eta^4}\int_x^{x_T}\eta'\int_0^y \bar{S}_\theta dy'\,dx' \\ &+ \frac{Re\hat{K}}{A^4}\frac{1}{\eta^2}\int_x^{x_T}\int_0^y \bar{S}_E dy'\,dx' \,.\end{aligned} \tag{4.14}$$

[14] The source terms have been condensed; see the final paragraph of Chapter Four.

Boundary conditions at the rotor wall

At the rotor wall, the boundary requirements on the primitive variables are: no mass through the wall,

$$\bar{v}_r|_{x=0} = 0\,, \tag{4.15}$$

the no slip condition,

$$\bar{v}_\theta|_{x=0} = 0\,, \quad \bar{v}_z|_{x=0} = 0\,, \tag{4.16}$$

and a prescribed temperature gradient,

$$\left.\frac{\partial \bar{T}}{\partial y}\right|_{x=0} = \theta(y)\,. \tag{4.17}$$

These specifications lead to three requirements on the master potential. The first is, from (4.10) and (4.15),

$$\left.\frac{\partial \chi}{\partial x}\right|_{x=0} = 0\,. \tag{4.18}$$

Next, from (4.4) and (4.16), is

$$\left.\frac{\partial^2 \chi}{\partial x^2}\right|_{x=0} = 0\,. \tag{4.19}$$

The third condition is slightly more complicated. It is obtained by evaluating (4.13) at the wall and then using (4.16), (4.17), and (3.22) to write

$$\begin{aligned}\theta(y) = \frac{32\mathrm{A}^{10}Z^2}{Re}\frac{\partial}{\partial x}\left(e^x\frac{\partial}{\partial x}\left(\eta^2\frac{\partial}{\partial x}\left(e^x\frac{\partial^2\chi}{\partial x^2}\right)\right)\right)\Bigg|_{x=0} \\ + \left.\frac{\partial \bar{S}_R}{\partial y}\right|_{x=0} + 2A^2 Z\left(\bar{S}_Z + \frac{\partial \bar{S}_Z}{\partial x}\right)\Bigg|_{x=0} \\ - \frac{10A^2}{3Re}\left.\frac{\partial \bar{S}_M}{\partial y}\right|_{x=0} - \frac{2A^2}{Re}\frac{\partial}{\partial x}\left(\frac{\partial \bar{S}_M}{\partial y}\right)\Bigg|_{x=0}\,.\end{aligned} \tag{4.20}$$

Typically, the source terms are assumed zero at the wall, resulting in

$$\frac{\partial}{\partial x}\left(e^x\frac{\partial}{\partial x}\left(\eta^2\frac{\partial}{\partial x}\left(e^x\frac{\partial^2\chi}{\partial x^2}\right)\right)\right)\Bigg|_{x=0} = \frac{Re}{32A^{10}Z^2}\theta(y)\,. \tag{4.21}$$

Boundary conditions high in the atmosphere

High in the atmosphere, the boundary requirements are: prescribed radial mass flow (which is frequently zero),

$$\bar{\rho}_0 \bar{v}_r \big|_{x=x_T} = f(y) \,, \tag{4.22}$$

no shear stress,

$$\left.\frac{\partial \bar{v}_\theta}{\partial x}\right|_{x=x_T} = 0 \,, \quad \left.\frac{\partial \bar{v}_z}{\partial x}\right|_{x=x_T} = 0 \,, \tag{4.23}$$

and no heat transfer,

$$\left.\frac{\partial \bar{T}}{\partial x}\right|_{x=x_T} = 0 \,. \tag{4.24}$$

These specifications again lead to three requirements on the master potential. The first comes from (4.9) and (4.22),

$$\left.\frac{\partial \chi}{\partial x}\right|_{x=x_T} = \frac{1}{4A^4 Z} \int_0^{x_T} \bar{\rho}_0 \bar{v}_z \big|_{y=0} dx' + \frac{1}{2A^2} \eta \big|_{x=xT} \int_0^{y} f(y') dy' + \frac{1}{4A^4} \int_0^{x_T} \int_0^{y} \bar{S}_M dy' \, dx' \,. \tag{4.25}$$

Note that the first term on the right-hand side of (4.25) represents the net flow through the bottom boundary, which is zero in many cases.

The second condition is obtained by rearranging and differentiating (4.4) then applying (4.23), leading to,

$$\left.\frac{\partial}{\partial x}\left(e^x \frac{\partial^2 \chi}{\partial x^2}\right)\right|_{x=x_T} = 0 \,. \tag{4.26}$$

Lastly, evaluating (4.14) at the top of the atmosphere and then applying (4.23), (4.24), and (3.22) yields

$$\chi \big|_{x=x_T} = \chi \big|_{x=x_T, y=0} \,. \tag{4.27}$$

It is permissible to set $\chi|_{x=x_T,y=0}$ to zero because the primitive variables all depend on derivatives of the master potential. Doing so produces the third condition,

$$\chi \big|_{x=x_T} = 0 \,. \tag{4.28}$$

Carrier-Maslen boundary conditions

The boundary conditions at the top and bottom of the cylinder are constructed so that the flow in the vicinity of the end caps is constrained to match the Ekman layer solution first presented by Carrier and Maslen in [3]. The derivation of these boundary conditions given here is informed by [5], [6], and especially [11].

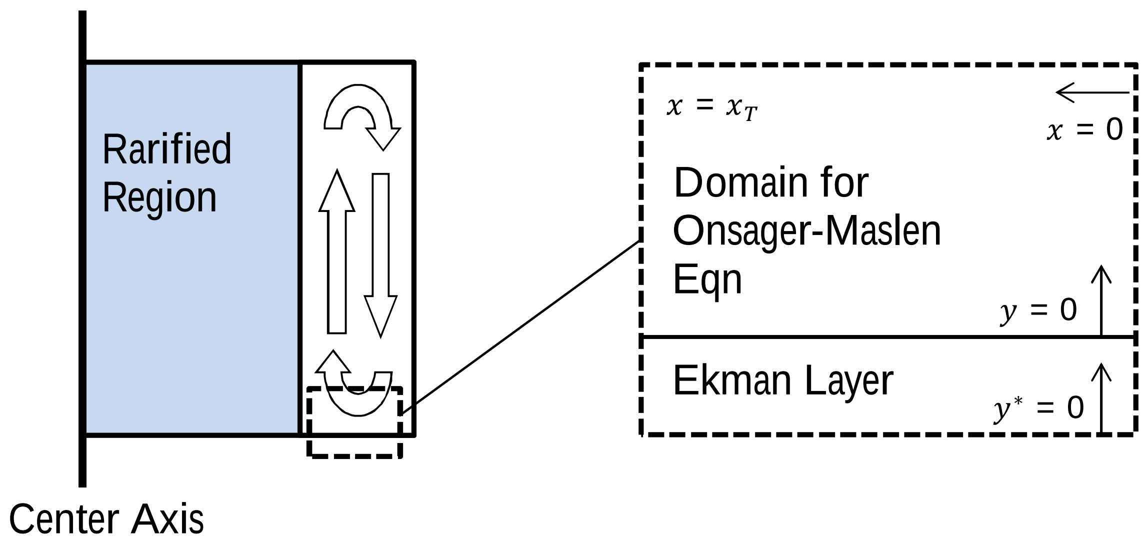


Figure 4-1. The Carrier-Maslen condition at the bottom surface matches the Ekman layer solution to the value of the master potential at $y = 0$.

To begin, it is useful to reproduce some of the equations developed in the previous chapter which are valid within the axial boundary layers. To that end, equation (3.54) may be used to write, in operator notation,

$$\frac{\partial \Phi^*}{\partial y^*} = \frac{1}{Re\bar{\rho}_0\eta}\left(\bar{\nabla}^2 - \frac{1}{\eta^2} - \frac{4A^4}{3}\eta^2\right)\frac{\partial \bar{v}_r^*}{\partial y^*} - \frac{Z}{Re}\frac{1}{\eta}\frac{\partial}{\partial \eta}\frac{1}{\bar{\rho}_0}\bar{\nabla}^2\bar{v}_z^* \,. \tag{4.29}$$

and equations (3.56), (3.60), and (3.62) may be combined to yield

$$\frac{1}{Re}\bar{\nabla}^2\Phi^* + 4\frac{1+\hat{K}\eta^2}{\eta}\bar{\rho}_0\bar{v}_r^* = \frac{4}{Re}\frac{1}{\eta}\frac{\partial \bar{v}_\theta^*}{\partial \eta} \,. \tag{4.30}$$

Note that the flow variables as well as the y^* coordinate have been marked with a star to identify them as part of the boundary layer solution in the regions below or above $y = 0$ and $y = 1$, respectively. In addition, the source terms are not considered outside of the interior region.

Retaining only the terms most highly differentiated in the axial direction, equations (4.29) and (4.30) become

$$\frac{\partial \Phi^*}{\partial y^*} = \frac{1}{ReZ^2\bar{\rho}_0\eta}\frac{\partial^3 \bar{v}_r^*}{\partial y^{*3}} , \tag{4.31}$$

$$\frac{\partial^2 \Phi^*}{\partial y^{*2}} = -4ReZ^2\,\frac{1+\hat{K}\eta^2}{\eta}\,\bar{\rho}_0\bar{v}_r^* . \tag{4.32}$$

Then, differentiating (4.31) to combine it with (4.32) yields

$$\frac{\partial^4 \bar{v}_r^*}{\partial y^{*4}} = -4Re^2Z^4\left(1+\hat{K}\eta^2\right)\bar{\rho}_0^2\bar{v}_r^* . \tag{4.33}$$

Considering first the flow near the bottom boundary (between $y^* = 0$, which represents the bottom surface, and $y = 0$), solutions to (4.33) may be written in the form $v_r^* = e^{my^*}$. The solutions which decay for large y^* have

$$m_1, m_2 = -(1 \pm i)\sqrt{ReZ^2\bar{\rho}_0\sqrt{1+\hat{K}\eta^2}} \tag{4.34}$$

and so the general solution may be written

$$v_r^* = C_1(\eta)e^{m_1 y^*} + C_2(\eta)e^{m_2 y^*} . \tag{4.35}$$

The next step is to find expressions for Φ^* and a stream function ψ^*. First, integrating (4.31) once yields

$$\Phi^* = \frac{1}{ReZ^2\bar{\rho}_0\eta}\frac{\partial^2 \bar{v}_r^*}{\partial y^{*2}} + C_3(\eta) ; \tag{4.36}$$

then, after defining a stream function in the same manner as in the previous chapter, i.e., [15]

$$\eta\bar{\rho}_0\bar{v}_r^* = -\frac{\partial \psi^*}{\partial y^*} , \tag{4.37}$$

$$\bar{\rho}_0\bar{v}_z^* = \frac{Z}{\eta}\frac{\partial \psi^*}{\partial \eta} , \tag{4.38}$$

inserting it into the right-hand side of (4.33), and integrating once, the result is

$$\psi^* = \frac{1}{4Re^2Z^4\bar{\rho}_0}\frac{\eta}{1+\hat{K}\eta^2}\frac{\partial^3 \bar{v}_r^*}{\partial y^{*3}} + C_4(\eta) . \tag{4.39}$$

Far from the bottom boundary ($Re^{1/2}Zy^* \to \infty$), the requirement is that

$$\Phi^* \to \Phi|_{y=0} , \quad \psi^* \to \psi|_{y=0} , \tag{4.40}$$

[15] See (3.55)-(3.59) and recall that mass sources/sinks are not considered in the axial boundary layers.

which leads to

$$C_3(\eta) = \Phi|_{y=0} \,, \quad C_4(\eta) = \psi|_{y=0} \,. \tag{4.41}$$

Evaluating (4.35), (4.36), and (4.39) at the bottom boundary ($y^* = 0$) yields, after some rearranging,[16]

$$v_r^- = C_1(\eta) + C_2(\eta) \,, \tag{4.42}$$

$$\frac{\eta}{2\sqrt{1+\hat{K}\eta^2}}\left(\Phi^- - \Phi|_{y=0}\right) = iC_1(\eta) - iC_2(\eta) \,, \tag{4.43}$$

$$\frac{2Z}{\eta}\sqrt{\frac{Re\sqrt{1+\hat{K}\eta^2}}{\bar{\rho}_0}}\left(\psi^- - \psi|_{y=0}\right) = (1-i)C_1(\eta) + (1+i)C_2(\eta) \,, \tag{4.44}$$

where the variables marked with a minus sign are the prescribed values at the bottom surface. Then, subtracting (4.43) and (4.44) from (4.42) gives the Carrier-Maslen condition for the bottom boundary,

$$v_r^- - \frac{\eta}{2\sqrt{1+\hat{K}\eta^2}}\left(\Phi^- - \Phi|_{y=0}\right) - \frac{2Z}{\eta}\sqrt{\frac{Re\sqrt{1+\hat{K}\eta^2}}{\bar{\rho}_0}}\left(\psi^- - \psi|_{y=0}\right) = 0 \,. \tag{4.45}$$

The remaining task is to write this equation in terms of the master potential. Equation (3.74) written in terms of x and evaluated at $y = 0$ is[17]

$$\begin{aligned} \left.\frac{\partial\Phi}{\partial x}\right|_{y=0} = -\frac{2Re}{A^2}\frac{1+\hat{K}\eta^2}{\eta^4}\left.\frac{\partial\chi}{\partial y}\right|_{y=0} & \\ -\frac{Re}{2A^6}\frac{1+\hat{K}\eta^2}{\eta^4}\int_x^{x_T}\int_0^{x'}\bar{S}_M|_{y=0}dx''\,dx' & \\ -\frac{Re}{2A^4}\frac{1}{\eta^4}\int_x^{x_T}\eta'\bar{S}_\theta|_{y=0}dx' & \\ +\frac{Re\hat{K}}{A^4}\frac{1}{\eta^2}\int_x^{x_T}\bar{S}_E|_{y=0}dx' \,, & \end{aligned} \tag{4.46}$$

Then, rearranging (4.45) as

$$\frac{2\sqrt{1+\hat{K}\eta^2}}{\eta}v_r^- - \left(\Phi^- - \Phi|_{y=0}\right) - 4Re^{1/2}Z\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2}e^{x/2}\left(\psi^- - \psi|_{y=0}\right) = 0 \,, \tag{4.47}$$

differentiating with respect to x, and applying (3.73) and (4.46) yields the required form of the Carrier-Maslen relationship,

$$\frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\left.\frac{\partial\chi}{\partial y}\right|_{y=0} = -\frac{Re^{3/2}}{4A^8Z}\frac{\partial}{\partial x}\left(\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2}e^{x/2}\left.\frac{\partial\chi}{\partial x}\right|_{y=0}\right) + g^-(x) \,, \tag{4.48}$$

[16] The variables v_r^-, Φ^-, and ψ^- are equal to $v_r^*|_{y^*=0}$, $\Phi^*|_{y^*=0}$, and $\psi^*|_{y^*=0}$, respectively.
[17] The source terms have been condensed; see the final paragraph of Chapter Four.

where

$$g^{-}(x) = \frac{Re}{16A^{10}Z^2}\frac{\partial}{\partial x}\left(\frac{\sqrt{1+\hat{K}\eta^2}}{\eta}v_r^-\right) - \frac{Re}{32A^{10}Z^2}\frac{\partial\Phi^-}{\partial x} - \frac{Re^{3/2}}{8A^{10}Z}\frac{\partial}{\partial x}\left(\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2}e^{x/2}\psi^-\right) \tag{4.49}$$

$$- \frac{Re^2}{64A^{16}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\int_x^{x_T}\int_0^{x'}\bar{S}_M\Big|_{y=0}dx''\,dx' - \frac{Re^2}{64A^{14}Z^2}\frac{1}{\eta^4}\int_x^{x_T}\eta'\bar{S}_\theta\Big|_{y=0}dx'$$

$$+ \frac{Re^2\hat{K}}{32A^{14}Z^2}\frac{1}{\eta^2}\int_x^{x_T}\bar{S}_E\Big|_{y=0}dx' \,.$$

A similar analysis for the top boundary condition yields the second Carrier-Maslen relationship:

$$\frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\frac{\partial\chi}{\partial y}\Big|_{y=1} = \frac{Re^{3/2}}{4A^8Z}\frac{\partial}{\partial x}\left(\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2}e^{x/2}\frac{\partial\chi}{\partial x}\Big|_{y=1}\right) + g^+(x)\,, \tag{4.50}$$

where

$$g^{+}(x) = \frac{Re}{16A^{10}Z^2}\frac{\partial}{\partial x}\left(\frac{\sqrt{1+\hat{K}\eta^2}}{\eta}v_r^+\right) - \frac{Re}{32A^{10}Z^2}\frac{\partial\Phi^+}{\partial x} + \frac{Re^{3/2}}{8A^{10}Z}\frac{\partial}{\partial x}\left(\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2}e^{x/2}\psi^+\right) \tag{4.51}$$

$$- \frac{Re^2}{64A^{16}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\int_x^{x_T}\int_0^{x'}\bar{S}_M\Big|_{y=1}dx''\,dx' - \frac{Re^2}{64A^{14}Z^2}\frac{1}{\eta^4}\int_x^{x_T}\eta'\bar{S}_\theta\Big|_{y=1}dx'$$

$$+ \frac{Re^2\hat{K}}{32A^{14}Z^2}\frac{1}{\eta^2}\int_x^{x_T}\bar{S}_E\Big|_{y=1}dx' \,.$$

Reduction to homogeneous boundary conditions

The application of the finite element method to solve the Onsager-Maslen equation (Chapter Five) is greatly simplified if the essential boundary conditions of the problem are homogeneous. Of the four essential boundary conditions derived above, three – (4.18), (4.19), and (4.28) – are already homogeneous, while the fourth – (4.25) – is not. The purpose of this section is to reduce the current problem to one with four homogeneous essential boundary conditions.

The first step is to introduce a reference function, given by

$$\chi^{(ref)} = \frac{h(y)}{3x_T^2}\left(x^3 - x_T^3\right) \tag{4.52}$$

where

$$h(y) = \frac{1}{4A^4Z}\int_0^{x_T}\bar{\rho}_0\bar{v}_z\Big|_{y=0}dx' + \frac{1}{2A^2}\eta\Big|_{x=xT}\int_0^y f(y')dy' + \frac{1}{4A^4}\int_0^{x_T}\int_0^y\bar{S}_M dy'\,dx'\,, \tag{4.53}$$

which is used to define a modified master potential, $\mathcal{X}$, such that

$$\mathcal{X} = \chi - \chi^{(ref)} \, . \tag{4.54}$$

Replacing χ with the quantity $\mathcal{X} + \chi^{(ref)}$ in the Onsager-Maslen equation and in its associated boundary conditions generates an alternate problem with the desired qualities.

The modified problem

The reformulated Onsager-Maslen equation is

$$\frac{\partial^2}{\partial x^2}\left(e^x \frac{\partial}{\partial x}\left(\eta^2 \frac{\partial}{\partial x}\left(e^x \frac{\partial^2 \mathcal{X}}{\partial x^2}\right)\right)\right) + \frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\frac{\partial^2 \mathcal{X}}{\partial y^2} = \bar{\mathcal{S}} + \mathcal{H} \tag{4.55}$$

where $\bar{\mathcal{S}}$ is given by (3.81) and $\mathcal{H}$ is given by

$$\mathcal{H} = -\frac{2h(y)}{x_T^2}\frac{\partial^2}{\partial x^2}\left(e^x \frac{\partial}{\partial x}\left(\eta^2 \frac{\partial}{\partial x}(xe^x)\right)\right) - \frac{Re^2}{48A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\frac{x^3 - x_T^3}{x_T^2}h''(y) \, . \tag{4.56}$$

At the wall, the essential radial boundary conditions are

$$\left.\frac{\partial \mathcal{X}}{\partial x}\right|_{x=0} = 0 \, , \tag{4.57}$$

$$\left.\frac{\partial^2 \mathcal{X}}{\partial x^2}\right|_{x=0} = 0 \, , \tag{4.58}$$

and the third boundary condition is

$$\begin{aligned} \left.\frac{\partial}{\partial x}\left(e^x \frac{\partial}{\partial x}\left(\eta^2 \frac{\partial}{\partial x}\left(e^x \frac{\partial^2 \mathcal{X}}{\partial x^2}\right)\right)\right)\right|_{x=0} &= \frac{Re}{32A^{10}Z^2}\theta(y) - \frac{10(1-1/A^2)}{x_T^2}h(y) \\ &\quad - \frac{Re}{32A^{10}Z^2}\left.\frac{\partial \bar{\mathcal{S}}_R}{\partial y}\right|_{x=0} - \frac{Re}{16A^8 Z}\left.\left(\bar{\mathcal{S}}_Z + \frac{\partial \bar{\mathcal{S}}_Z}{\partial x}\right)\right|_{x=0} \\ &\quad + \frac{5}{48A^8Z^2}\left.\frac{\partial \bar{\mathcal{S}}_M}{\partial y}\right|_{x=0} + \frac{1}{16A^8Z^2}\left.\frac{\partial}{\partial x}\left(\frac{\partial \bar{\mathcal{S}}_M}{\partial y}\right)\right|_{x=0} . \end{aligned} \tag{4.59}$$

At the top of the atmosphere, the essential radial boundary conditions are

$$\chi|_{x=x_T} = 0\,, \tag{4.60}$$

$$\left.\frac{\partial \chi}{\partial x}\right|_{x=x_T} = 0\,, \tag{4.61}$$

and the third boundary condition is

$$\left.\frac{\partial}{\partial x}\left(e^x \frac{\partial^2 \chi}{\partial x^2}\right)\right|_{x=x_T} = -\frac{2(x_T+1)}{x_T^2} e^{x_T} h(y)\,. \tag{4.62}$$

Finally, the Carrier-Maslen conditions are, at the bottom surface,

$$\frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\left.\frac{\partial \chi}{\partial y}\right|_{y=0} = -\frac{Re^{3/2}}{4A^8 Z}\frac{\partial}{\partial x}\left(\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2} e^{x/2}\left.\frac{\partial \chi}{\partial x}\right|_{y=0}\right) + G^-(x)\,, \tag{4.63}$$

where

$$\begin{aligned} G^-(x) = & -\frac{Re^2}{48A^{12}x_T^2 Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}(x^3 - x_T^3)h'(0) - \frac{Re^{3/2}}{4A^8 x_T^2 Z}\frac{\partial}{\partial x}\left(\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2} x^2 e^{x/2}\right)h(0) \\ & + \frac{Re}{16A^{10}Z^2}\frac{\partial}{\partial x}\left(\frac{\sqrt{1+\hat{K}\eta^2}}{\eta} v_r^-\right) - \frac{Re}{32A^{10}Z^2}\frac{\partial \Phi^-}{\partial x} - \frac{Re^{3/2}}{8A^{10}Z}\frac{\partial}{\partial x}\left(\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2} e^{x/2}\psi^-\right) \\ & - \frac{Re^2}{64A^{16}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\int_x^{x_T}\int_0^{x'} \bar{\mathcal{S}}_M|_{y=0}\, dx''\, dx' - \frac{Re^2}{64A^{14}Z^2}\frac{1}{\eta^4}\int_x^{x_T}\eta' \bar{\mathcal{S}}_\theta|_{y=0}\, dx' \\ & + \frac{Re^2 \hat{K}}{32A^{14}Z^2}\frac{1}{\eta^2}\int_x^{x_T}\bar{\mathcal{S}}_E|_{y=0}\, dx'\,, \end{aligned} \tag{4.64}$$

and, at the top surface,

$$\frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\left.\frac{\partial \chi}{\partial y}\right|_{y=1} = \frac{Re^{3/2}}{4A^8 Z}\frac{\partial}{\partial x}\left(\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2} e^{x/2}\left.\frac{\partial \chi}{\partial x}\right|_{y=1}\right) + G^+(x)\,, \tag{4.65}$$

where

$$\begin{aligned} G^+(x) = & -\frac{Re^2}{48A^{12}x_T^2 Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}(x^3 - x_T^3)h'(1) + \frac{Re^{3/2}}{4A^8 x_T^2 Z}\frac{\partial}{\partial x}\left(\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2} x^2 e^{x/2}\right)h(1) \\ & \frac{Re}{16A^{10}Z^2}\frac{\partial}{\partial x}\left(\frac{\sqrt{1+\hat{K}\eta^2}}{\eta} v_r^+\right) - \frac{Re}{32A^{10}Z^2}\frac{\partial \Phi^+}{\partial x} + \frac{Re^{3/2}}{8A^{10}Z}\frac{\partial}{\partial x}\left(\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2} e^{x/2}\psi^+\right) \\ & - \frac{Re^2}{64A^{16}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\int_x^{x_T}\int_0^{x'} \bar{\mathcal{S}}_M|_{y=1}\, dx''\, dx' - \frac{Re^2}{64A^{14}Z^2}\frac{1}{\eta^4}\int_x^{x_T}\eta' \bar{\mathcal{S}}_\theta|_{y=1}\, dx' \\ & + \frac{Re^2 \hat{K}}{32A^{14}Z^2}\frac{1}{\eta^2}\int_x^{x_T}\bar{\mathcal{S}}_E|_{y=1}\, dx'\,. \end{aligned} \tag{4.66}$$

Chapter Five

Finite Element Solution

Introduction

This chapter describes the application of the Galerkin finite element method to solve the sixth-order Onsager-Maslen differential equation first derived in Chapter Three and subject to the boundary conditions and modifications described in Chapter Four. The approach taken here is informed primarily by [11], [9], and [13].

Basic Galerkin formulation

Consider an approximate solution to the Onsager-Maslen equation for which each point has an associated residual, such that

$$(e^x(\eta^2(e^x\mathcal{X}_{xx})_x)_x)_{xx} + \frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\mathcal{X}_{yy} - \bar{\mathcal{S}}(x,y) - \mathcal{H}(x,y) = R(x,y)\ . \tag{5.1}$$

The weak form, a relaxed version of the original differential equation, is developed by requiring that the residual function vanish when integrated over the domain; i.e.,

$$\begin{aligned}\int_0^{xT}\int_0^1 R dy\, dx &= 0 \\ &= \int_0^{xT}\int_0^1 \left[(e^x(\eta^2(e^x\mathcal{X}_{xx})_x)_x)_{xx} + \frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\mathcal{X}_{yy} - \bar{\mathcal{S}} - \mathcal{H}\right] dy\, dx\ .\end{aligned} \tag{5.2}$$

The Galerkin method uses a specialized weak form in which the residual function is "weighted" by a function, $\tilde{\mathcal{X}}$, such that

$$\int_0^{xT}\int_0^1 \tilde{\mathcal{X}}\left[(e^x(\eta^2(e^x\mathcal{X}_{xx})_x)_x)_{xx} + \frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\mathcal{X}_{yy} - \bar{\mathcal{S}} - \mathcal{H}\right] dy\, dx = 0\ . \tag{5.3}$$

The weighting function is frequently referred to as the "test function."

Applying the boundary conditions

The first term in (5.3), after integrating by parts three times, is

$$\begin{aligned}&\int_0^{xT}\int_0^1 \tilde{\mathcal{X}}(e^x(\eta^2(e^x\mathcal{X}_{xx})_x)_x)_{xx} dy\, dx = \\ &\quad -\int_0^{xT}\int_0^1 \eta^2(e^x\mathcal{X}_{xx})_x\left(e^x\tilde{\mathcal{X}}_{xx}\right)_x dy\, dx + \int_0^1 \tilde{\mathcal{X}}(e^x(\eta^2(e^x\mathcal{X}_{xx})_x)_x)_x\Big|_{x=0}^{x_T} dy \\ &\quad -\int_0^1 e^x\tilde{\mathcal{X}}_x(\eta^2(e^x\mathcal{X}_{xx})_x)_x\Big|_{x=0}^{x_T} dy + \int_0^1 \eta^2 e^x\tilde{\mathcal{X}}_{xx}(e^x\mathcal{X}_{xx})_x\Big|_{x=0}^{x_T} dy\ .\end{aligned} \tag{5.4}$$

Requiring that the weighting function satisfy the problem's essential boundary conditions in the x coordinate; i.e.,

$$\tilde{\mathcal{X}}_x\big|_{x=0} = \tilde{\mathcal{X}}_{xx}\big|_{x=0} = \tilde{\mathcal{X}}\big|_{x=x_T} = \tilde{\mathcal{X}}_x\big|_{x=x_T} = 0\,, \tag{5.5}$$

the term in (5.4) reduces to

$$\begin{aligned}\int_0^{xT}\int_0^1 \tilde{\mathcal{X}}(e^x(\eta^2(e^x\mathcal{X}_{xx})_x)_x)_{xx}dy\,dx = \\ -\int_0^{xT}\int_0^1 \eta^2(e^x\mathcal{X}_{xx})_x\left(e^x\tilde{\mathcal{X}}_{xx}\right)_x dy\,dx \\ -\int_0^1 \tilde{\mathcal{X}}(e^x(\eta^2(e^x\mathcal{X}_{xx})_x)_x)_x\big|_{x=0}\,dy + \int_0^1 \eta^2 e^x\tilde{\mathcal{X}}_{xx}(e^x\mathcal{X}_{xx})_x\big|_{x=x_T}dy\,.\end{aligned} \tag{5.6}$$

Then, after applying the problem's natural boundary conditions in the x coordinate,[18]

$$(e^x(\eta^2(e^x\mathcal{X}_{xx})_x)_x)_x|_{x=0} = \frac{Re}{32A^{10}Z^2}\theta(y) - \frac{10\left(1-1/A^2\right)}{x_T^2}h(y)\,, \tag{5.7}$$

$$(e^x\mathcal{X}_{xx})_x|_{x=x_T} = -\frac{2(x_T+1)}{x_T^2}e^{x_T}h(y)\,, \tag{5.8}$$

equation (5.6) simplifies further to

$$\begin{aligned}\int_0^{xT}\int_0^1 \tilde{\mathcal{X}}(e^x(\eta^2(e^x\mathcal{X}_{xx})_x)_x)_{xx}dy\,dx = \\ -\int_0^{xT}\int_0^1 \eta^2(e^x\mathcal{X}_{xx})_x\left(e^x\tilde{\mathcal{X}}_{xx}\right)_x dy\,dx \\ -\frac{Re}{32A^{10}Z^2}\int_0^1 \tilde{\mathcal{X}}\big|_{x=0}\theta(y)\,dy + \frac{10\left(1-1/A^2\right)}{x_T^2}\int_0^1 \tilde{\mathcal{X}}\big|_{x=0}h(y)\,dy \\ -\frac{2(x_T+1)}{x_T^2}e^{2x_T}\int_0^1 \eta^2\tilde{\mathcal{X}}_{xx}\big|_{x=x_T}h(y)dy\,.\end{aligned} \tag{5.9}$$

The second term in (5.3) becomes, after integrating by parts once,

$$\begin{aligned}\int_0^{x_T}\int_0^1 \frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\tilde{\mathcal{X}}\mathcal{X}_{yy}dy\,dx = \\ -\int_0^{x_T}\int_0^1 \frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\mathcal{X}_y\tilde{\mathcal{X}}_y dy\,dx + \frac{Re^2}{16A^{12}Z^2}\int_0^{x_T}\frac{1+\hat{K}\eta^2}{\eta^4}\tilde{\mathcal{X}}\mathcal{X}_y\Big|_{y=0}^1 dx\,.\end{aligned} \tag{5.10}$$

The Carrier-Maslen boundary conditions with curvature terms included are

$$\frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\mathcal{X}_y\Big|_{y=0} = -\frac{Re^{3/2}}{4A^8Z}\left(\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2}e^{x/2}\mathcal{X}_x|_{y=0}\right)_x + G^-(x)\,, \tag{5.11}$$

$$\frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\mathcal{X}_y\Big|_{y=1} = \frac{Re^{3/2}}{4A^8Z}\left(\frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2}e^{x/2}\mathcal{X}_x|_{y=1}\right)_x + G^+(x)\,. \tag{5.12}$$

[18] For simplicity, it is assumed here that the source terms are zero at the boundary.

Applying these to (5.10) yields

$$\int_0^{x_T}\int_0^1 \frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\tilde{\mathcal{X}}\mathcal{X}_{yy}dy\,dx = -\int_0^{x_T}\int_0^1 \frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\mathcal{X}_y\tilde{\mathcal{X}}_y dy\,dx \tag{5.13}$$
$$+\frac{Re^{3/2}}{4A^8Z}\int_0^{x_T}\left[\tilde{\mathcal{X}}\Big|_{y=0}\left(\frac{(1+\hat{K}\eta^2)^{3/4}}{\eta^2}e^{x/2}\mathcal{X}_x|_{y=0}\right)_x + \tilde{\mathcal{X}}\Big|_{y=1}\left(\frac{(1+\hat{K}\eta^2)^{3/4}}{\eta^2}e^{x/2}\mathcal{X}_x|_{y=1}\right)_x\right]dx$$
$$+\int_0^{x_T}\left[\tilde{\mathcal{X}}\Big|_{y=1}G^+(x)-\tilde{\mathcal{X}}\Big|_{y=0}G^-(x)\right]dx\;.$$

Then, after integrating the second and third terms by parts and applying (5.5), equation (5.13) reduces to

$$\int_0^{x_T}\int_0^1 \frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\tilde{\mathcal{X}}\mathcal{X}_{yy}dy\,dx = -\int_0^{x_T}\int_0^1 \frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\mathcal{X}_y\tilde{\mathcal{X}}_y dy\,dx \tag{5.14}$$
$$-\frac{Re^{3/2}}{4A^8Z}\int_0^{x_T}\frac{(1+\hat{K}\eta^2)^{3/4}}{\eta^2}e^{x/2}\left(\mathcal{X}_x\tilde{\mathcal{X}}_x\Big|_{y=0}+\mathcal{X}_x\tilde{\mathcal{X}}_x\Big|_{y=1}\right)dx$$
$$+\int_0^{x_T}\left[\tilde{\mathcal{X}}\Big|_{y=1}G^+(x)-\tilde{\mathcal{X}}\Big|_{y=0}G^-(x)\right]dx\;.$$

Simplified Galerkin form

Collecting the terms in (5.9) and (5.14), equation (5.3) may be written

$$\mathcal{B}(\mathcal{X},\tilde{\mathcal{X}}) = \mathcal{F}(\tilde{\mathcal{X}})\,, \tag{5.15}$$

where

$$\mathcal{B}(\mathcal{X},\tilde{\mathcal{X}}) = \tag{5.16}$$
$$\int_0^{xT}\int_0^1 \eta^2(e^x\mathcal{X}_{xx})_x(e^x\tilde{\mathcal{X}}_{xx})_x dy\,dx + \int_0^{x_T}\int_0^1 \frac{Re^2}{16A^{12}Z^2}\frac{1+\hat{K}\eta^2}{\eta^4}\mathcal{X}_y\tilde{\mathcal{X}}_y dy\,dx$$
$$+\frac{Re^{3/2}}{4A^8Z}\int_0^{x_T}\frac{(1+\hat{K}\eta^2)^{3/4}}{\eta^2}e^{x/2}\left(\mathcal{X}_x\tilde{\mathcal{X}}_x\Big|_{y=0}+\mathcal{X}_x\tilde{\mathcal{X}}_x\Big|_{y=1}\right)dx\,,$$

$$\mathcal{F}(\tilde{\mathcal{X}}) = \tag{5.17}$$
$$-\int_0^{xT}\int_0^1\tilde{\mathcal{X}}\bar{\mathcal{S}}dy\,dx - \int_0^{xT}\int_0^1\tilde{\mathcal{X}}\mathcal{H}dy\,dx$$
$$-\frac{Re}{32A^{10}Z^2}\int_0^1\tilde{\mathcal{X}}\Big|_{x=0}\theta(y)\,dy + \int_0^{x_T}\left[\tilde{\mathcal{X}}\Big|_{y=1}G^+(x)-\tilde{\mathcal{X}}\Big|_{y=0}G^-(x)\right]dx$$
$$+\frac{10(1-1/A^2)}{x_T^2}\int_0^1\tilde{\mathcal{X}}\Big|_{x=0}h(y)\,dy - \frac{2(x_T+1)}{x_T^2}e^{2x_T}\int_0^1\eta^2\tilde{\mathcal{X}}_{xx}\Big|_{x=x_T}h(y)dy\;.$$

The remainder of this chapter is devoted to constructing a finite element solution to the equation described by (5.15)-(5.17).

Approximating the solution

The general form of a finite-element method FEM) solution is a linear combination of basis functions, $\Phi_k(x,y)$, such that

$$\mathcal{X}(x,y) \approx \sum_{k=1}^{K} a_k \Phi_k(x,y) \; , \tag{5.18}$$

where K is the total degrees of freedom, which is related to the total number of grid points. Each constant (a_k) and basis function (Φ_k) pair is associated with a grid point. The two-dimensional basis functions are the product of separate one-dimensional basis functions such that

$$\Phi_k(x,y) = \sigma_i(x)\lambda_j(y) \; , \tag{5.19}$$

where each i corresponds to a gridline in the x-direction and each j corresponds to a gridline in the y-direction. Hence, the general form of the solution may be written

$$\mathcal{X}(x,y) \approx \sum_{k=1}^{K} a_k \sigma_i(x)\lambda_j(y) \approx \sum_i \sum_j a_k \, \sigma_i(x)\lambda_j(y) \; , \tag{5.20}$$

where k is a function of i, and j. The exact relationship between i, j, and k is discussed in greater detail later in the chapter.

The x-direction

In the x-direction, the domain is partitioned into M subintervals with divisions given by

$$0 = x_0 < x_1 < \cdots < x_{M-1} < x_M = x_T \; . \tag{5.21}$$

Cubic polynomials are appropriate splining functions because the simplified weak form has three derivatives in the x coordinate. In addition, choosing cubic splines introduces $M + 3$ degrees of freedom in this direction.[19]

[19] A total of M cubic functions of the form $Ax^3 + Bx^2 + Cx + D$ are required. Without constraints, the spline function would have a total of $4M$ degrees of freedom; however, requiring that the function and its first two derivatives be continuous eliminates $3(M-1)$ degrees of freedom.

Cubic spline basis functions

Any cubic spline function may be expressed as a linear combination of B-spline basis functions. In this case, the spline may be written

$$s(x) = \sum_{i=-1}^{M+1} c_i^{(s)} s_i(x) \; . \tag{5.22}$$

Each $s_i(x)$ is restricted to be nonzero only over the interval $[x_{i-2}, x_{i+2}]$ which surrounds the node point x_i and is divided into four regions such that $x_{i-2} < x_{i-1} < x_i < x_{i+1} < x_{i+2}$. The $s_i(x)$ are defined piecewise as follows. Let

$$\begin{aligned} \varsigma_{i1} &= \alpha_i \left(\frac{x-x_{i-2}}{x_{i+2}-x_{i-2}}\right)^3 , \quad \varsigma_{i2} = \beta_i \left(\frac{x-x_{i-1}}{x_{i+2}-x_{i-1}}\right)^3 , \\ \varsigma_{i3} &= \gamma_i \left(\frac{x-x_i}{x_{i+2}-x_i}\right)^3 , \quad \varsigma_{i4} = \delta_i \left(\frac{x-x_{i+1}}{x_{i+2}-x_{i+1}}\right)^3 ; \end{aligned} \tag{5.23}$$

then,

$$s_i(x) = \begin{cases} \varsigma_{i1} & x_{i-2} \le x \le x_{i-1} \\ \varsigma_{i1} + \varsigma_{i2} & x_{i-1} \le x \le x_i \\ \varsigma_{i1} + \varsigma_{i2} + \varsigma_{i3} & x_i \le x \le x_{i+1} \\ \varsigma_{i1} + \varsigma_{i2} + \varsigma_{i3} + \varsigma_{i4} & x_{i+1} \le x \le x_{i+2} \end{cases} . \tag{5.24}$$

Notice that the function and its first two derivatives are automatically continuous over the interval for which it is defined.

Evaluating $s_i(x)$ at the five nodes helps identify the first two of the four constraints on $s_i(x)$ which are required to solve for the coefficients α_i, β_i, γ_i, and δ_i. The nodal values are

$$s_i(x_{i-2}) = 0 \; , \tag{5.25}$$

$$s_i(x_{i-1}) = \alpha_i \left(\frac{x_{i-1}-x_{i-2}}{x_{i+2}-x_{i-2}}\right)^3 , \tag{5.26}$$

$$s_i(x_i) = \alpha_i \left(\frac{x_i-x_{i-2}}{x_{i+2}-x_{i-2}}\right)^3 + \beta_i \left(\frac{x_i-x_{i-1}}{x_{i+2}-x_{i-1}}\right)^3 , \tag{5.27}$$

$$s_i(x_1) = \alpha_i \left(\frac{x_{i+1}-x_{i-2}}{x_{i+2}-x_{i-2}}\right)^3 + \beta_i \left(\frac{x_{i+1}-x_{i-1}}{x_{i+2}-x_{i-1}}\right)^3 + \gamma_i \left(\frac{x_{i+1}-x_i}{x_{i+2}-x_i}\right)^3 , \tag{5.28}$$

$$s_i(x_2) = \alpha_i + \beta_i + \gamma_i + \delta_i \; . \tag{5.29}$$

The first constraint is to set the value of $s_i(x)$ to one at the middle node ($x = x_i$),[20]

$$\alpha_i \left(\frac{x_i - x_{i-2}}{x_{i+2} - x_{i-2}}\right)^3 + \beta_i \left(\frac{x_i - x_{i-1}}{x_{i+2} - x_{i-1}}\right)^3 = 1 , \tag{5.30}$$

and the second pins the value of $s_i(x)$ to zero at the rightmost end of the interval ($x = x_{i+2}$),

$$\alpha_i + \beta_i + \gamma_i + \delta_i = 0 . \tag{5.31}$$

The remaining constraints are motivated by the observation that the first and second derivatives at the leftmost end ($x = x_{i-2}$) are zero, whereas the first and second derivatives at the rightmost end ($x = x_{i+2}$) are

$$s_i{}'(x_{i+2}) = \alpha_i \frac{3}{x_{i+2} - x_{i-2}} + \beta_i \frac{3}{x_{i+2} - x_{i-1}} + \gamma_i \frac{3}{x_{i+2} - x_i} + \delta_i \frac{3}{x_{i+2} - x_{i+1}} , \tag{5.32}$$

$$s_i{}''(x_{i+2}) = \alpha_i \frac{6}{(x_{i+2} - x_{i-2})^2} + \beta_i \frac{6}{(x_{i+2} - x_{i-1})^2} + \gamma_i \frac{6}{(x_{i+2} - x_i)^2} + \delta_i \frac{6}{(x_{i+2} - x_{i+1})^2} . \tag{5.33}$$

Requiring that these be zero gives the two additional requirements,

$$\frac{\alpha_i}{x_{i+2} - x_{i-2}} + \frac{\beta_i}{x_{i+2} - x_{i-1}} + \frac{\gamma_i}{x_{i+2} - x_i} + \frac{\delta_i}{x_{i+2} - x_{i+1}} = 0 , \tag{5.34}$$

$$\frac{\alpha_i}{(x_{i+2} - x_{i-2})^2} + \frac{\beta_i}{(x_{i+2} - x_{i-1})^2} + \frac{\gamma_i}{(x_{i+2} - x_i)^2} + \frac{\delta_i}{(x_{i+2} - x_{i+1})^2} = 0 . \tag{5.35}$$

The four constraints may be written together in a matrix equation as

$$\begin{bmatrix} \left(\frac{x_i - x_{i-2}}{x_{i+2} - x_{i-2}}\right)^3 & \left(\frac{x_i - x_{i-1}}{x_{i+2} - x_{i-1}}\right)^3 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ (x_{i+2} - x_{i-2})^{-1} & (x_{i+2} - x_{i-1})^{-1} & (x_{i+2} - x_i)^{-1} & (x_{i+2} - x_{i+1})^{-1} \\ (x_{i+2} - x_{i-2})^{-2} & (x_{i+2} - x_{i-1})^{-2} & (x_{i+2} - x_i)^{-2} & (x_{i+2} - x_{i+1})^{-2} \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \\ \delta_i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{5.36}$$

The solution to this problem defines a cubic spline basis function corresponding to the i^{th} gridline in the x-direction. Each of these basis functions has support over four subintervals, meaning that the spline function may be written

$$s(x) = c_{i-2}^{(s)} s_{i-2}(x) + c_{i-1}^{(s)} s_{i-1}(x) + c_i^{(s)} s_i(x) + c_{i+1}^{(s)} s_{i+1}(x) , \tag{5.37}$$
$$\text{on } [x_{i-1}, x_i], \text{ for } i = 1,2, \ldots , M ,$$

[20] Note that, depending on the node spacing, the peak of the curve can occur away from $x = x_i$. In these cases, the function value at $x = x_i$ will be one and the peak value of the function will be greater than one.

which is an alternate form of (5.22). It is helpful to remember that the α_i, β_i, γ_i, and δ_i are determined entirely by the grid spacing and are not degrees of freedom in the same sense as the constants in (5.22) and (5.37).

The first, second, and third derivatives of $s(x)$ are

$$s'(x) = \sum_{i=-1}^{M+1} c_i^{(s)} s_i'(x) \, , \tag{5.38}$$

$$s''(x) = \sum_{i=-1}^{M+1} c_i^{(s)} s_i''(x) \, , \tag{5.39}$$

$$s'''(x) = \sum_{i=-1}^{M+1} c_i^{(s)} s_i'''(x) \, , \tag{5.40}$$

where

$$s_i'(x) = \begin{cases} \varsigma_{i1}' & x_{i-2} \le x \le x_{i-1} \\ \varsigma_{i1}' + \varsigma_{i2}' & x_{i-1} \le x \le x_i \\ \varsigma_{i1}' + \varsigma_{i2}' + \varsigma_{i3}' & x_i \le x \le x_{i+1} \\ \varsigma_{i1}' + \varsigma_{i2}' + \varsigma_{i3}' + \varsigma_{i4}' & x_{i+1} \le x \le x_{i+2} \end{cases} \, , \tag{5.41}$$

with ς_{i1}', ς_{i2}', ς_{i3}', and ς_{i4}' defined as

$$\varsigma_{i1}' = \frac{3\alpha_i}{(x_{i+2}-x_{i-2})^3}(x - x_{i-2})^2 \, , \quad \varsigma_{i2}' = \frac{3\beta_i}{(x_{i+2}-x_{i-1})^3}(x - x_{i-1})^2 \, , \tag{5.42}$$
$$\varsigma_{i3}' = \frac{3\gamma_i}{(x_{i+2}-x_i)^3}(x - x_i)^2 \, , \quad \varsigma_{i4}' = \frac{3\delta_i}{(x_{i+2}-x_{i+1})^3}(x - x_{i+1})^2 \, ;$$

where

$$s_i''(x) = \begin{cases} \varsigma_{i1}'' & x_{i-2} \le x \le x_{i-1} \\ \varsigma_{i1}'' + \varsigma_{i2}'' & x_{i-1} \le x \le x_i \\ \varsigma_{i1}'' + \varsigma_{i2}'' + \varsigma_{i3}'' & x_i \le x \le x_{i+1} \\ \varsigma_{i1}'' + \varsigma_{i2}'' + \varsigma_{i3}'' + \varsigma_{i4}'' & x_{i+1} \le x \le x_{i+2} \end{cases} \, , \tag{5.43}$$

with ς_{i1}'', ς_{i2}'', ς_{i3}'', and ς_{i4}'' defined as

$$\varsigma_{i1}'' = \frac{6\alpha_i}{(x_{i+2}-x_{i-2})^3}(x - x_{i-2}) \, , \quad \varsigma_{i2}'' = \frac{6\beta_i}{(x_{i+2}-x_{i-1})^3}(x - x_{i-1}) \, , \tag{5.44}$$
$$\varsigma_{i3}'' = \frac{6\gamma_i}{(x_{i+2}-x_i)^3}(x - x_i) \, , \quad \varsigma_{i4}'' = \frac{6\delta_i}{(x_{i+2}-x_{i+1})^3}(x - x_{i+1}) \, ;$$

and where

$$s_i'''(x) = \begin{cases} \varsigma_{i1}''' & x_{i-2} \le x \le x_{i-1} \\ \varsigma_{i1}''' + \varsigma_{i2}''' & x_{i-1} \le x \le x_i \\ \varsigma_{i1}''' + \varsigma_{i2}''' + \varsigma_{i3}''' & x_i \le x \le x_{i+1} \\ \varsigma_{i1}''' + \varsigma_{i2}''' + \varsigma_{i3}''' + \varsigma_{i4}''' & x_{i+1} \le x \le x_{i+2} \end{cases} , \tag{5.45}$$

with ς_{i1}''', ς_{i2}''', ς_{i3}''', and ς_{i4}''' defined as

$$\varsigma_{i1}''' = \frac{6\alpha_i}{(x_{i+2}-x_{i-2})^3}, \quad \varsigma_{i2}''' = \frac{6\beta_i}{(x_{i+2}-x_{i-1})^3}, \\ \varsigma_{i3}''' = \frac{6\gamma_i}{(x_{i+2}-x_i)^3}, \quad \varsigma_{i4}''' = \frac{6\delta_i}{(x_{i+2}-x_{i+1})^3}. \tag{5.46}$$

Modified cubic spline basis functions

A set of constrained basis functions are required which automatically satisfy the essential boundary conditions of the problem at x-direction boundaries. The spline composed of these basis functions, $\sigma(x)$, must satisfy $\sigma'(0) = \sigma''(0) = \sigma(x_T) = \sigma'(x_T) = 0$. The four constraints eliminate four degrees of freedom so that the modified spline may be written

$$\sigma(x) = \sum_{i=1}^{M-1} c_i^{(\sigma)} \sigma_i(x) . \tag{5.47}$$

Its basis functions are, in terms of the B-spline functions,

$$\sigma_i(x) = \begin{cases} s_1(x) - \mu_{-1} s_0(x) + \mu_0 s_{-1}(x) & i = 1 \\ s_i(x) & i = 2,3, \dots, M-2 \\ s_{M-1}(x) - \nu_{-1} s_M(x) + \nu_0 s_{M+1}(x) & i = M-1 \end{cases} , \tag{5.48}$$

where, for $k = 0, -1$, the constants μ_k and ν_k are

$$\mu_k = \frac{s_k''(0)s_1'(0) - s_k'(0)s_1''(0)}{s_0'(0)s_{-1}''(0) - s_{-1}'(0)s_0''(0)}, \tag{5.49}$$

$$\nu_k = \frac{s_{M-k}'(x_T)s_{M-1}(x_T) - s_{M-k}(x_T)s_{M-1}'(x_T)}{s_M(x_T)s_{M+1}'(x_T) - s_{M+1}(x_T)s_M'(x_T)}. \tag{5.50}$$

Equation (5.47) may be written alternatively as

$$\sigma(x) = \begin{cases} c_1^{(\sigma)}[s_1(x) - \mu_{-1}s_0(x) + \mu_0 s_{-1}(x)] + c_2^{(\sigma)}s_2(x) & [x_0, x_1] \\ c_1^{(\sigma)}[s_1(x) - \mu_{-1}s_0(x)] + c_2^{(\sigma)}s_2(x) + c_3^{(\sigma)}s_3(x) & [x_1, x_2] \\ c_{i-2}^{(\sigma)}s_{i-2}(x) + c_{i-1}^{(\sigma)}s_{i-1}(x) + c_i^{(\sigma)}s_i(x) + c_{i+1}^{(\sigma)}s_{i+1}(x) & [x_{i-1}, x_i] \\ c_{M-3}^{(\sigma)}s_{M-3}(x) + c_{M-2}^{(\sigma)}s_{M-2}(x) + c_{M-1}^{(\sigma)}[s_{M-1}(x) - \nu_{-1}s_M(x)] & [x_{M-2}, x_{M-1}] \\ c_{M-2}^{(\sigma)}s_{M-2}(x) + c_{M-1}^{(\sigma)}[s_{M-1}(x) - \nu_{-1}s_M(x) + \nu_0 s_{M+1}(x)] & [x_{M-1}, x_M] \end{cases} , \quad (5.51)$$

in which $i = 3,4 \dots, M-2$.

The y-direction

In the y-direction, the domain is partitioned into N intervals such that

$$0 = y_0 < y_1 < \cdots < y_j < \cdots < y_{N-1} < y_N = y_T \quad (5.52)$$

The weak form has only one derivative in this direction, so linear splines are acceptable basis functions. Choosing a linear spline introduces $N+1$ degrees of freedom.[21]

Linear basis functions

A linear spline may be expressed as a linear combination of "hat" functions, in which each "hat" function is the basis function corresponding to a gridline in the y coordinate. Therefore, the spline may be written

$$\lambda(y) = \sum_{j=0}^{N} c_j^{(\lambda)}\lambda_j(y) . \quad (5.53)$$

[21] A total of N functions of the form $Ax + B$ are required. Without constraints, the spline function would have a total of $2N$ degrees of freedom; however, requiring that the function be continuous eliminates $N-1$ degrees of freedom.

where the "hat" functions, $\lambda_j(y)$, are given by

$$\lambda_j(y) = \begin{cases} \frac{y_1 - y}{y_1}, & j = 0, & 0 \le y \le y_1 \\ \frac{y - y_{j-1}}{y_j - y_{j-1}}, & j = 1,2,\dots,N-1, & y_{j-1} \le y \le y_j \\ \frac{y_{j+1} - y}{y_{j+1} - y_j}, & j = 1,2,\dots,N-1, & y_j \le y \le y_{j+1} \\ \frac{y - y_{N-1}}{y_N - y_{N-1}}, & j = N, & y_{N-1} \le y \le y_N \end{cases}, \tag{5.54}$$

The alternate form of (5.53) is

$$\lambda(y) = c_{j-1}^{(\lambda)}\lambda_{j-1}(y) + c_j^{(\lambda)}\lambda_j(y) = \frac{c_j^{(\lambda)} - c_{j-1}^{(\lambda)}}{y_j - y_{j-1}}\, y + \frac{c_{j-1}^{(\lambda)} y_j - c_j^{(\lambda)} y_{j-1}}{y_j - y_{j-1}} \tag{5.55}$$

on $[y_{j-1}, y_j]$, for $j = 1,2,\dots,N$.

The derivative of $\lambda(y)$ is

$$\lambda'(y) = \sum_{j=0}^{N} c_j^{(\lambda)} \lambda_j'(y) , \tag{5.56}$$

where

$$\lambda_j'(y) = \begin{cases} \frac{-1}{y_1}, & j = 0, & 0 \le y < y_1 \\ \frac{1}{y_j - y_{j-1}}, & j = 1,2,\dots,N-1, & y_{j-1} < y < y_j \\ \frac{-1}{y_{j+1} - y_j}, & j = 1,2,\dots,N-1, & y_j < y < y_{j+1} \\ \frac{1}{y_N - y_{N-1}}, & j = N, & y_{N-1} < y < y_N \end{cases}. \tag{5.57}$$

Two-dimensional basis functions

At this point it is clear that (5.18) and (5.20) may be refined into

$$\mathcal{X}(x, y) \approx \sum_{k=1}^{K} a_k \Phi_k(x, y) \approx \sum_{i=1}^{M-1} \sum_{j=0}^{N} a_k \sigma_i(x) \lambda_j(y) , \tag{5.58}$$

where

$$k = (N + 1)(i - 1) + (j + 1) \tag{5.59}$$

and $K = (M - 1)(N + 1)$, which is the total amount of degrees of freedom. The summation variable, k, is defined in such a way that it increments from bottom to top along each x gridline.

Returning to the Galerkin form

This section evaluates the Galerkin statement of the problem – equations (5.15)-(5.17) – using the approximating spline functions developed above. Assuming that $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ may be represented by

$$\mathcal{X}(x,y) \approx \sum_{k=1}^{K} a_k \Phi_k(x,y) \; , \tag{5.60}$$

$$\widetilde{\mathcal{X}}(x,y) \approx \sum_{k^*=1}^{K} b_{k^*} \Phi_{k^*}(x,y) \; , \tag{5.61}$$

equation (5.15) may be written

$$\mathcal{B}(\sum_{k=1}^{K} a_k \Phi_k(x,y) \, , \sum_{k^*=1}^{K} b_{k^*} \Phi_{k^*}(x,y)) = \mathcal{F}(\sum_{k^*=1}^{K} b_{k^*} \Phi_{k^*}(x,y)) \; . \tag{5.62}$$

After some rearranging, (5.62) becomes

$$\sum_{k^*=1}^{K} b_{k^*} \sum_{k=1}^{K} a_k \, B(\Phi_k, \Phi_{k^*}) = \sum_{k^*=1}^{K} b_{k^*} \, F(\Phi_{k^*}) \; , \tag{5.63}$$

from which one may cancel the leading terms, resulting in

$$\sum_{k=1}^{K} a_k \, B(\Phi_k, \Phi_{k^*}) = F(\Phi_{k^*}) \; . \tag{5.64}$$

This final equation is valid for each $k^* = 1,2, \ldots, K$. In addition, it may be written as a matrix problem to solve for the coefficients a_k. The matrix problem is

$$[\mathrm{A}]\vec{\mathrm{C}} = \vec{\mathrm{D}} \; , \tag{5.65}$$

where, for $k = 1,2, \ldots, K$ and $k^* = 1,2, \ldots, K$,

$$\mathrm{A}_{k^*k} = B(\Phi_k, \Phi_{k^*}) \; , \tag{5.66}$$

$$\mathrm{C}_{k^*} = a_{k^*} \; , \tag{5.67}$$

$$\mathrm{D}_{k^*} = F(\Phi_{k^*}) \; . \tag{5.68}$$

Building the numerical solution

To solve (5.65), it is necessary to evaluate $B(\Phi_k, \Phi_{k^*})$ and $F(\Phi_{k^*})$. Using (5.16), equation (5.66) may be written

$$\mathrm{A}_{k^*k} = I_1 + I_2 + I_3 \; , \tag{5.69}$$

and, using (5.17), equation (5.68) may be written,

$$\mathsf{D}_{k^*} = I_4 + I_5 + I_6 + I_7 + I_8 \,, \tag{5.70}$$

where

$$I_1 = \int_0^{xT} \int_0^1 \eta^2 \left(e^x \Phi_{k_{xx}}\right)_x \left(e^x \Phi_{k^*{}_{xx}}\right)_x dy\, dx \,, \tag{5.71}$$

$$I_2 = \int_0^{x_T} \int_0^1 \frac{Re^2}{16A^{12}Z^2} \frac{1+\hat{K}\eta^2}{\eta^4} \Phi_{k_y} \Phi_{k^*{}_y} dy\, dx \,, \tag{5.72}$$

$$I_3 = \frac{Re^{3/2}}{4A^8 Z} \int_0^{x_T} \frac{\left(1+\hat{K}\eta^2\right)^{3/4}}{\eta^2} e^{x/2} \left(\Phi_{k_x} \Phi_{k^*{}_x}\Big|_{y=0} + \Phi_{k_x} \Phi_{k^*{}_x}\Big|_{y=1} \right) dx \,, \tag{5.73}$$

$$I_4 = -\int_0^{xT} \int_0^1 \Phi_{k^*} \bar{\mathcal{S}}\, dy\, dx \,, \tag{5.74}$$

$$I_5 = -\int_0^{xT} \int_0^1 \Phi_{k^*} \mathcal{H} dy\, dx \,, \tag{5.75}$$

$$I_6 = -\frac{Re}{32A^{10}Z^2} \int_0^1 \Phi_{k^*}\big|_{x=0} \theta(y)\, dy \,, \tag{5.76}$$

$$I_7 = \int_0^{x_T} \left[\Phi_{k^*}\big|_{y=1} G^+(x) - \Phi_{k^*}\big|_{y=0} G^-(x) \right] dx \,, \tag{5.77}$$

$$I_8 = \frac{10\left(1-1/A^2\right)}{x_T^2} \int_0^1 \Phi_{k^*}\big|_{x=0} h(y)\, dy - \frac{2(x_T+1)}{x_T^2} e^{2x_T} \int_0^1 \eta^2 \Phi_{k^*{}_{xx}}\Big|_{x=x_T} h(y) dy \,. \tag{5.78}$$

These integrals are evaluated with products of Gaussian quadrature rules in each direction.

Chapter Six

Verification

Introduction

As a means of verification, the algorithm described in the previous chapter has been implemented for several test cases to permit comparison with results published by other authors. The calculations were made with the geometry and operating conditions given in Table 6-1.

Diameter (cm)	18.29
Length (cm)	335.3
Temperature (C)	300
Wall Pressure (kPa)	13.3

Table 6-1. Centrifuge parameters for verification calculations.

Calculations were made at one of three speeds: 700 m/s, 500 m/s, or 400 m/s. The values of the Reynolds number (Re), stratification parameter (A), and Brinkman number ($\widehat{K}$) at each of these speeds are given in Table 6-2.

	Re	A	$\widehat{K}$
700 m/s	6.7e6	5.9	1.1
500 m/s	4.8e6	4.2	0.55
400 m/s	3.8e6	3.4	0.35

Table 6-2. Values of non-dimensional quantities at three speeds.

The basic pancake model

The first comparison is made with results presented in [2]. These calculations employ the pancake approximation. The driving mechanism used to induce countercurrent flow is a linear wall temperature gradient. The calculations were made using a temperature difference of one degree Celsius along the outer wall. Figure 6-1 shows the axial mass flux at the height $y = 0.5$ for the three speeds listed in Table 6-2. A 24 element by 24 element grid was employed with gridlines concentrated near each of the three walls in the manner suggested by [9]. The top of the atmosphere (x_T) was set at 8. Figure 6-1 matches Figure 7 in [2], which was created from an eigenfunction expansion solution.

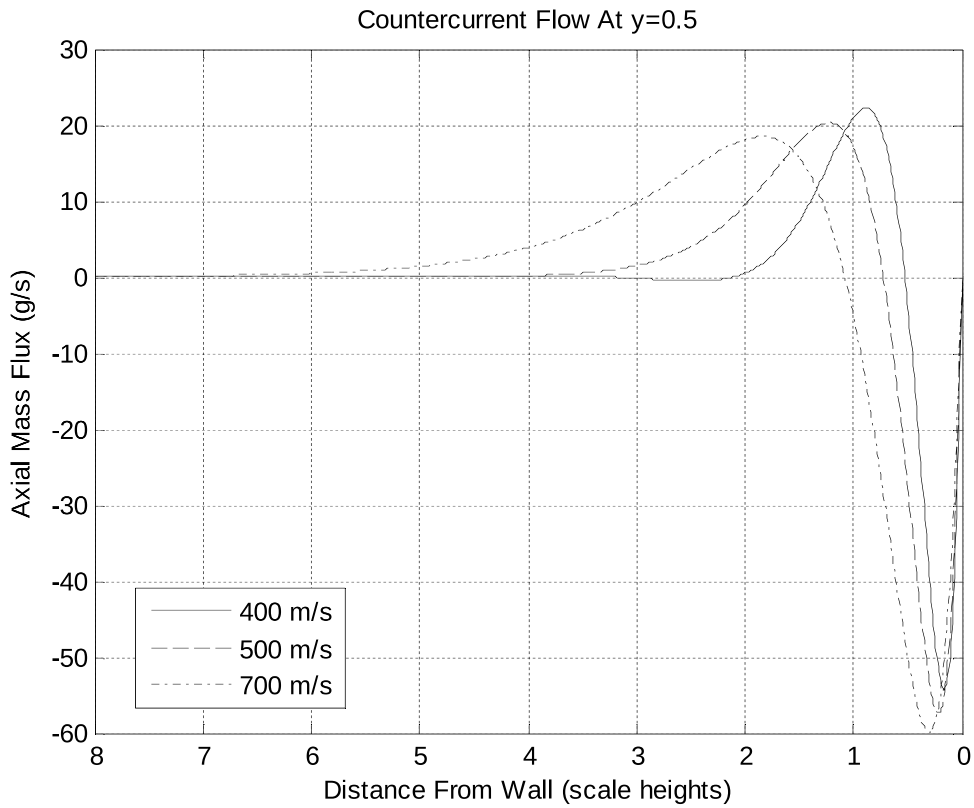
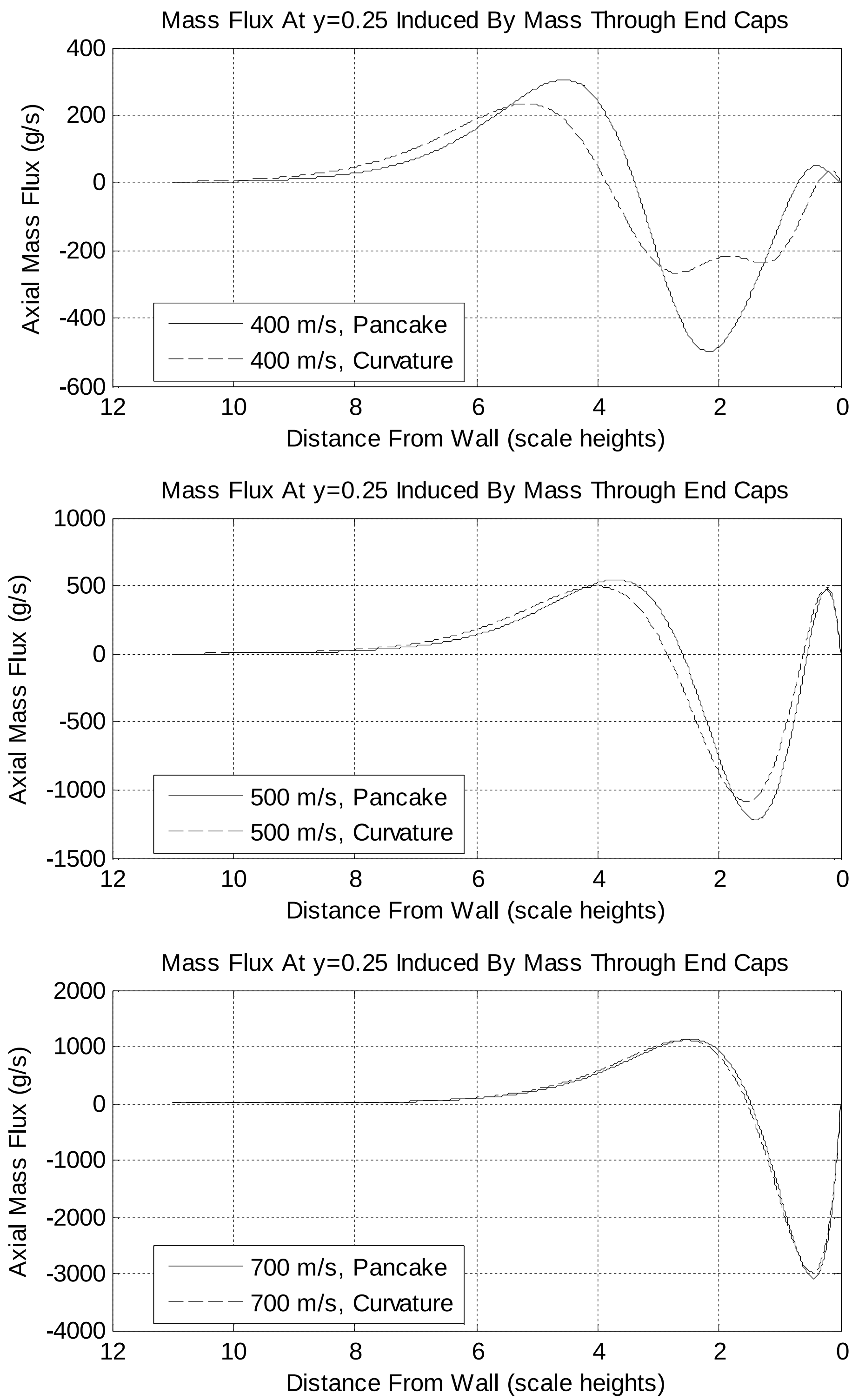


Figure 6-1. Countercurrent flow at the midplane ($y = 0.5$) induced by a linear wall temperature gradient for three different speeds.

Curvature term effects

The second comparison is with results given in [11] which examine the effect of the curvature terms. The countercurrent flow in this case is induced by the introduction and removal of mass through the top and bottom end caps; see [11] for details. Figures 6-2a, 6-2b, and 6-2c – which are equivalent to Figures 3b, 4b, and 5b in [11] – show the flow profiles calculated both with and without curvature terms for three rotor speeds. At the slowest slow speed (400 m/s), there is a dramatic difference between the two curves. However, for higher speeds the deviation between the two solutions vanishes. The calculations described in this section were made with a 50 element by 50 element grid concentrated near the three walls in the same manner as above. The top of the atmosphere (x_T) was set to 11.



Figures 6-2a, 6-2b, and 6-2c. Countercurrent flow at $y = 0.25$ induced by the addition and removal of mass through the end caps for rotor speeds of 400 m/s, 500 m/s, and 700 m/s.

Source term effects

This section examines the effects of each of the various types of source terms. All results are reported for a rotor speed of 700 m/s. Figure 6-3 shows the countercurrent flow at $y = 0.25$ generated by a point mass source of 1 gram per second located high in the atmosphere and halfway along the centrifuge ($x = 8, y = 0.5$). The mass is removed through holes in the top and bottom boundaries at $x = 8$. Figures 6-4a and 6-4b show the flow profile at $y = 0.25$ induced by point sources of radial momentum and axial momentum, respectively, while Figures 6-5a and 6-5b show the flow induced by a point circumferential momentum source and a point energy source, respectively. The strength of each momentum source is one dyne (10^{-5} N) and the strength of the energy source is one Watt. Equivalent results obtained using the pancake approximation are reported in [12] – see Figures 2a-2e in that reference – and the results given here agree nicely. At such a high speed (700 m/s), the solutions with curvature terms do not deviate dramatically from those without them. The calculations in this section were made with various sized grids and the top of the atmosphere (x_T) was set to 11.

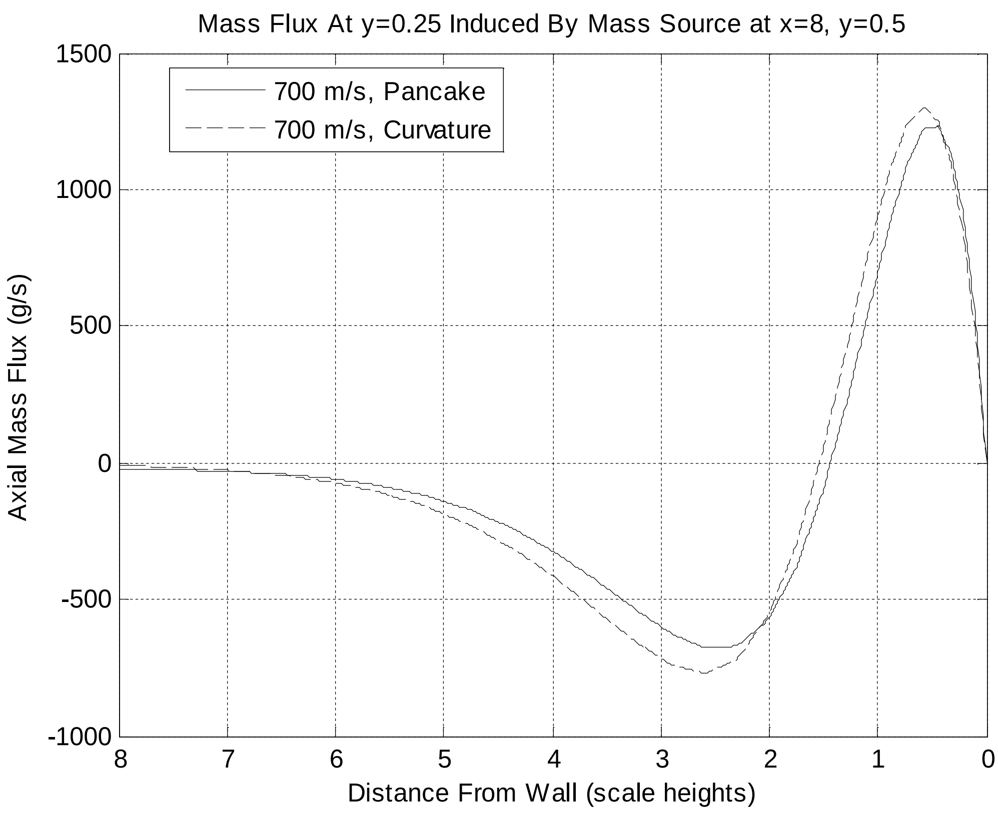
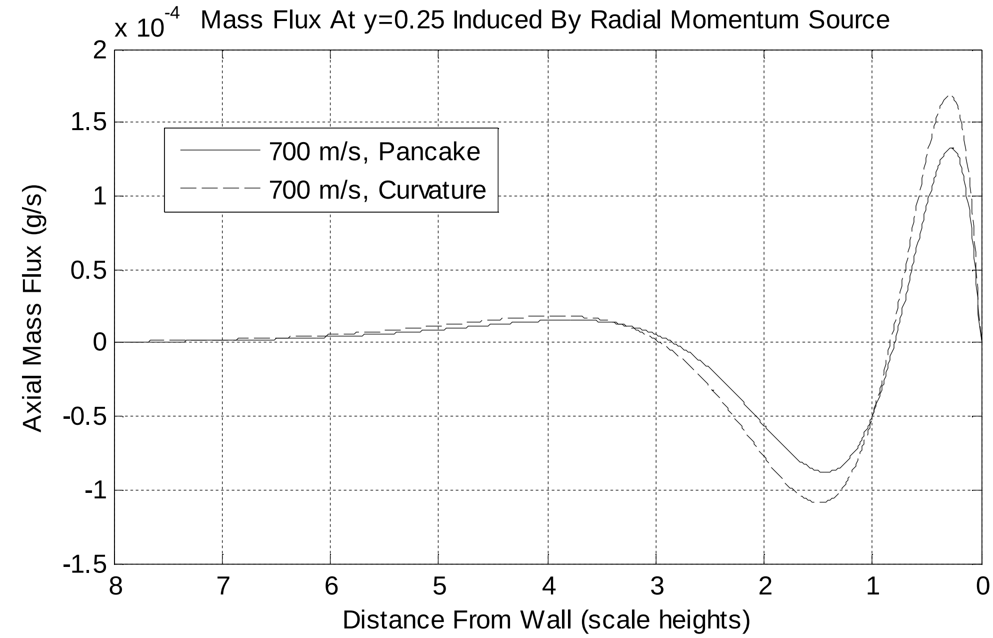
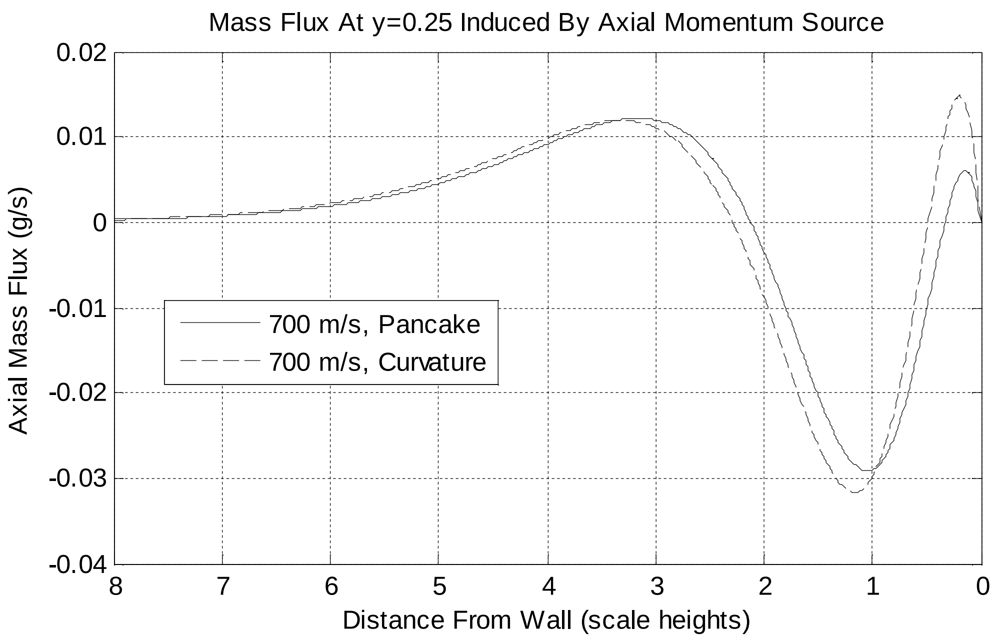
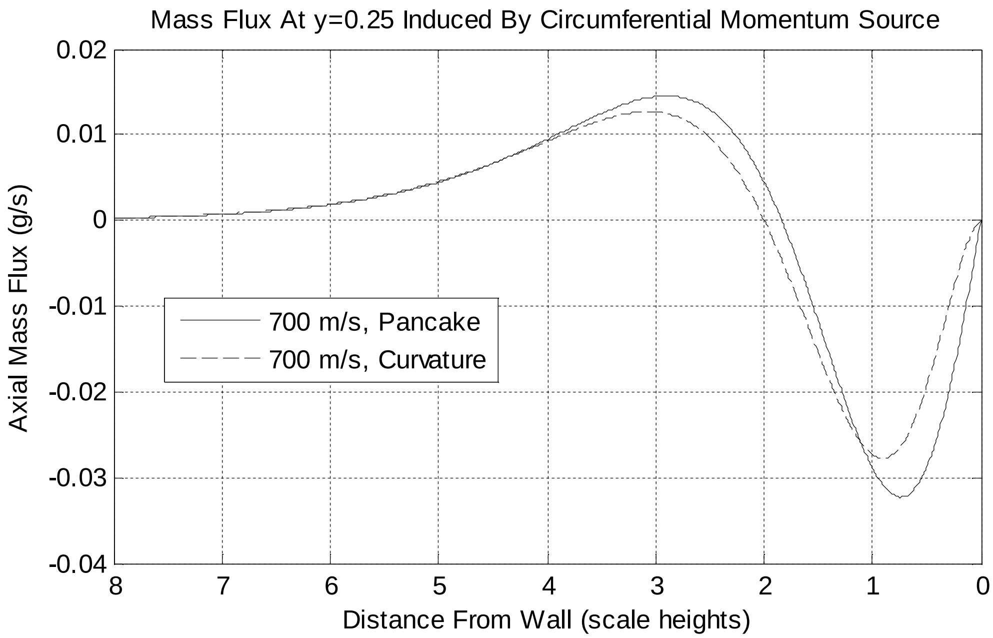


Figure 6-3. Countercurrent flow at $y = 0.25$ induced by a point mass source at $x = 8$ and $y = 0.5$. The rotor speed is 700 m/s. Half of the mass is removed through a hole in the top boundary and half through a hole in the bottom boundary, each at $x = 8$.





Figures 6-4a and 6-4b. Countercurrent flow at $y = 0.25$ for a point source of radial momentum and a point source of axial momentum, each at $x = 8$, $y = 0.5$. The strength of each source is one dyne. The calculations are for a high speed (700 m/s) and the curvature effects are relatively mild.



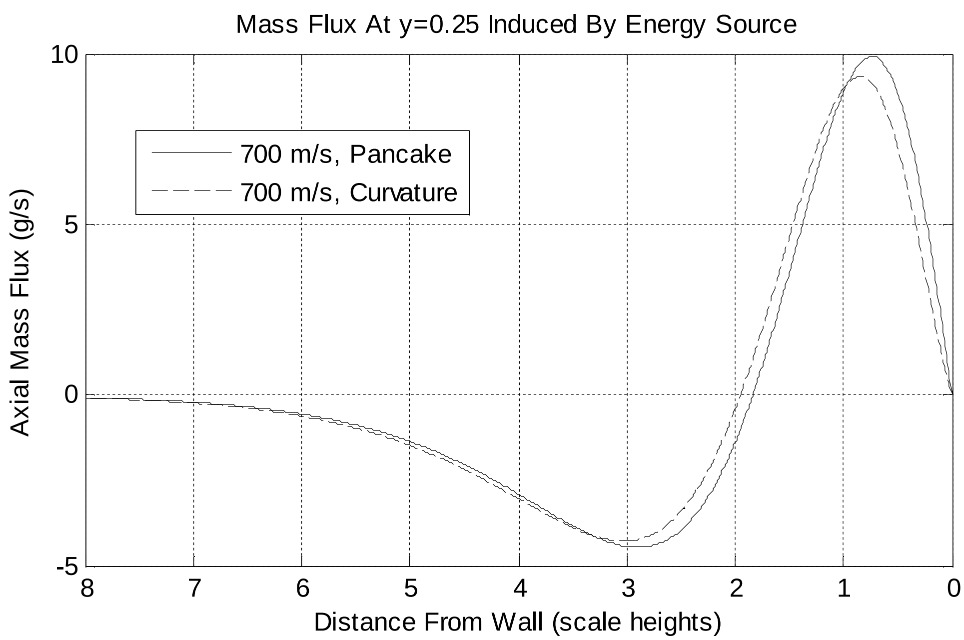


Figure 6-5a and 6-5b. Countercurrent flow at $y = 0.25$ for a point source of circumferential momentum and a point source of energy, each at $x = 8$, $y = 0.5$. The strength of the momentum source is one dyne and the strength of the energy source is one Watt. The calculations are for a high speed (700 m/s) and the curvature effects are relatively mild.

The effects of curvature and source terms at a lower speed

In the previous section, only a single (high) rotor speed was considered (700 m/s). This section explores the countercurrent flow induced by each of the source terms at a lower speed (500 m/s). Figure 6-6 shows the countercurrent flow generated by the point mass source described above. Figures 6-7a, 6-7b, 6-8a, and 6-8b report the expected flow induced by the three momentum sources and the energy source described above. For all source types, the flow profiles generated with the inclusion of curvature terms differ to a greater extent at the lower speed than at the higher speed from those calculated using the pancake approximation.

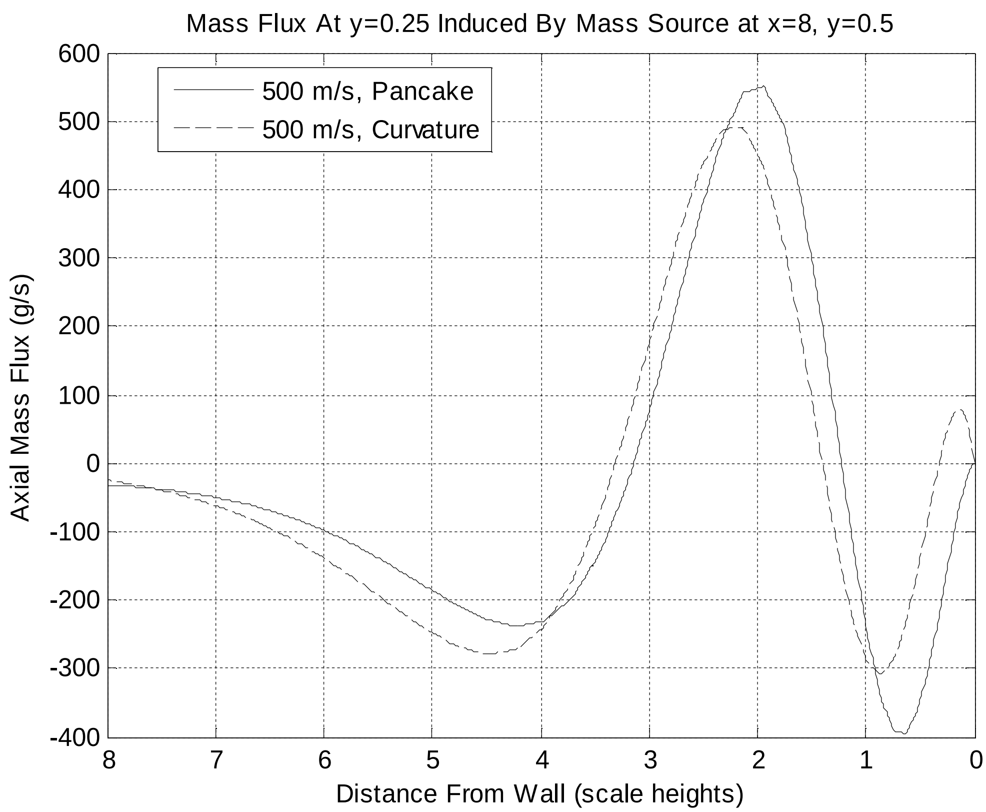
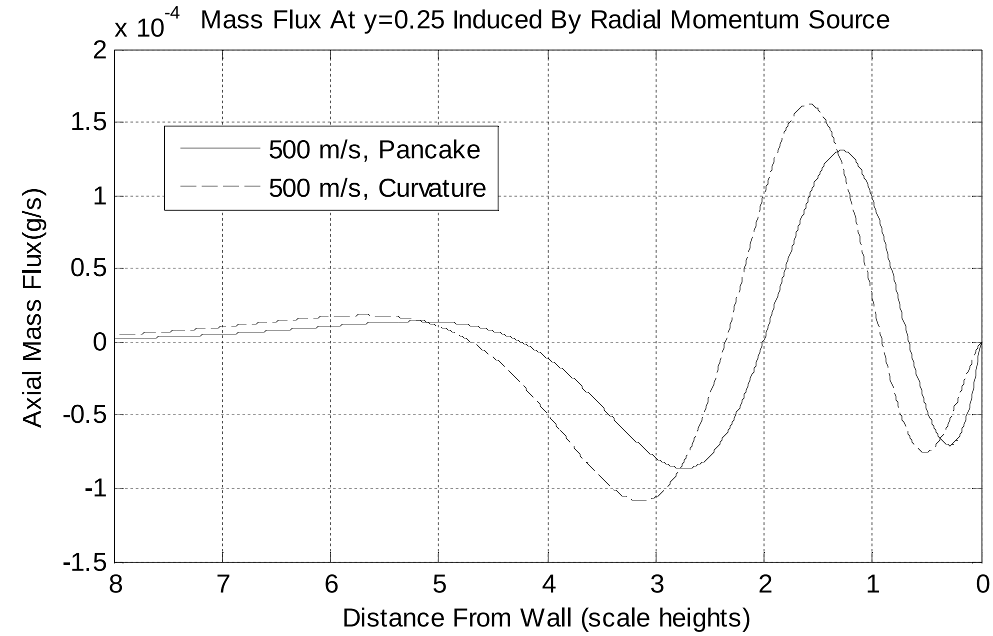
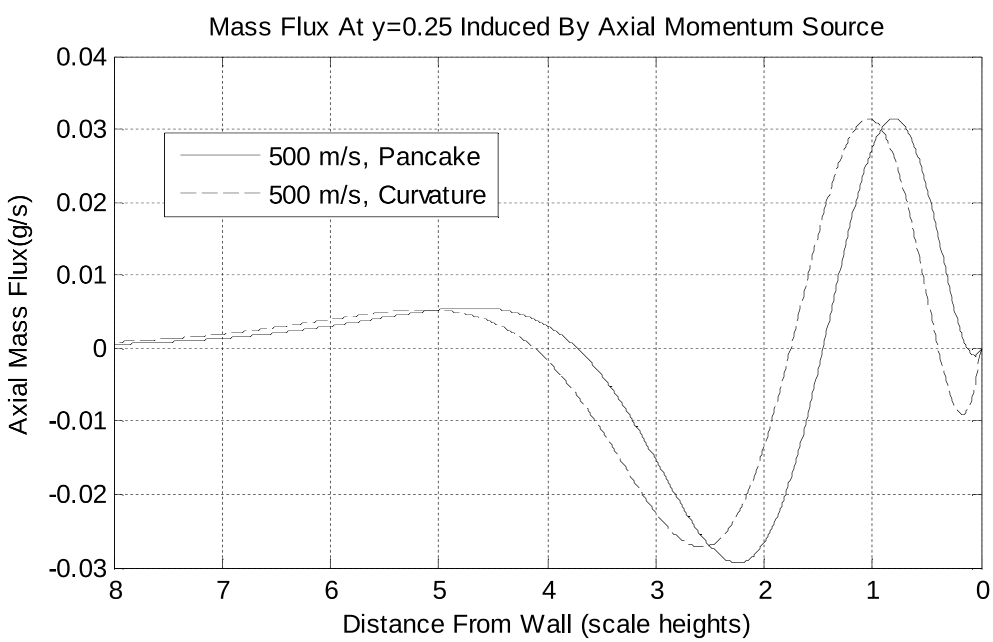
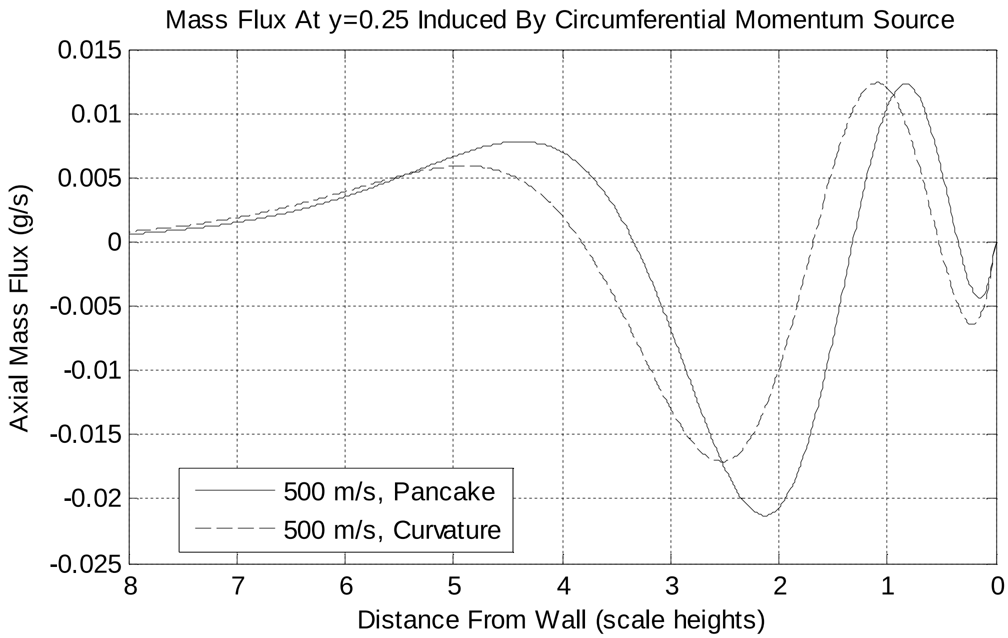


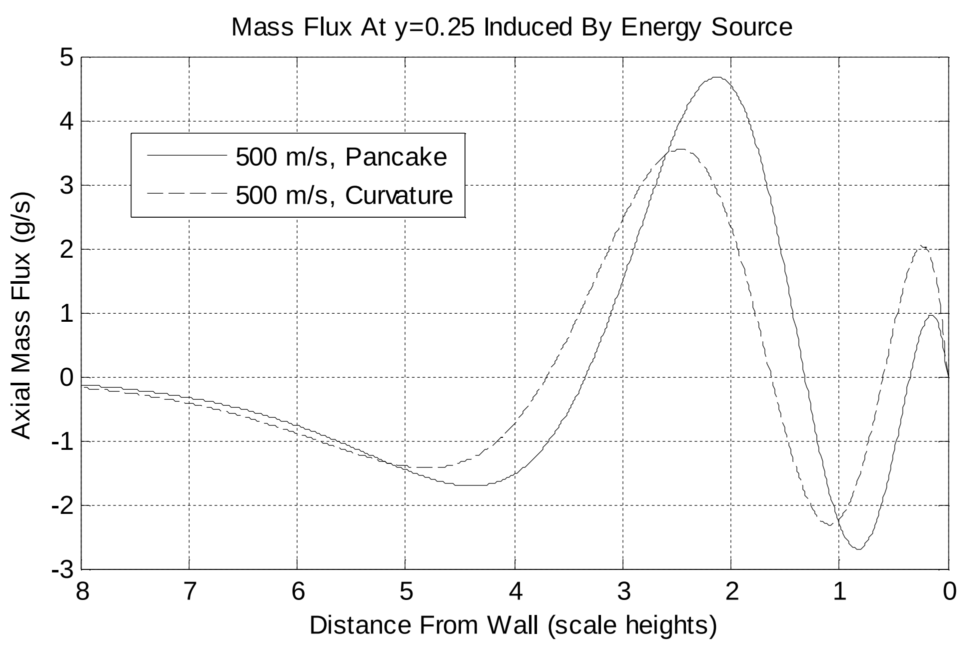
Figure 6-6. Countercurrent flow at $y = 0.25$ induced by a mass source at $x = 8$ and $y = 0.5$. The rotor speed is 500 m/s. The mass is removed through holes in the top and bottom boundaries at $x = 8$.





Figures 6-7a and 6-7b. Countercurrent flow at $y = 0.25$ for a point source of radial momentum and a point source of axial momentum, each at $x = 8$, $y = 0.5$. The strengths of the sources are one dyne. The calculations are for a medium speed (500 m/s) and the curvature effects are more pronounced (compare with Figure 6-3).





Figures 6-8a and 6-8b. Countercurrent flow at $y = 0.25$ for a point source of circumferential momentum and a point source energy, each at $x = 8$, $y = 0.5$. The strength of the momentum source is one dyne and the strength of the energy source is one Watt. The calculations are for a medium speed (500 m/s) and the curvature effects are more pronounced (compare with Figure 6-4).

Conclusions

This chapter has demonstrated that the finite element solution of the Onsager equation developed in the previous chapters is able to accurately reproduce results published by other authors. In particular, it captures the effects of both curvature terms and source terms in the Onsager equation. In general, prior authors have considered these phenomena separately; the present work accounts for both at once.

The results presented here support the assertion that the impact of curvature terms is more pronounced in the analysis of relatively lower-speed centrifuges. Further work is warranted – a logical next step is to analyze a low-speed machine using a more realistic feed model. In addition, it will be important to calculate – using the fluid dynamics solution – what effect the inclusion of curvature terms has on the expected separating performance of a low-speed centrifuge.

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