Trace and center of the twisted Heisenberg category

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Abstract

Khovanov's categorification of the Heisenberg algebra has many interesting representation theoretic and algebro-combinatorial properties. This Heisenberg category was constructed so that its Grothendieck group contains (and is conjecturally equal to) the Heisenberg algebra, but applying alternative decategorification functors reveals additional information. One such functor, the trace, yields $W_{1+\infty}$ at level one, a large and rich algebra which contains the Heisenberg algebra. Another such functor, the center, gives an algebra of shifted symmetric functions with connections to the asymptotic representation theory of symmetric groups.

In this dissertation, we investigate the trace and center of a twisted version of the Heisenberg category, which was defined by Cautis and Sussan to categorify the twisted Heisenberg algebra. We show that its trace is isomorphic to a distinguished subalgebra of $W_{1+\infty}$ at level one introduced by Kac, Wang, and Yan. The center of the category is then shown to be a subalgebra of the symmetric functions generated by odd power sums. There is a natural action of the trace of a category on its center. We describe this action for the twisted Heisenberg category, which is a twisted version of a representation of $W_{1+\infty}$ introduced by Lascoux and Thibon.

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Contents

Chapter 1

Introduction

1.1 Categorification

Categorification is the process of enriching an algebraic object by increasing its categorical dimension by one, e.g. passing from a set to a category or from a category to a 2-category. The original object can be recovered through the inverse process of decategorification. For example, the natural numbers $\mathbb{N}$ and the integers $\mathbb{Z}$ are both categorified by $\mathrm{Vect}_{\mathbb{C}}$, the category of finite-dimensional complex vector spaces, and the decategorification maps are taking the dimension dim and the Grothendieck group K_0, respectively. Each categorification preserves certain algebraic properties of $\mathbb{N}$ and $\mathbb{Z}$—for instance, addition is categorified to direct sum and multiplication to tensor product—while adding additional detail which does not appear on the set level, in the form of maps between vector spaces.

Many important objects in representation theory have interesting categorifications, and the idea of studying an object and its representation theory by passing to a categorification has proved to be fundamental. Most famously, [KL09] and [Rou08] independently constructed a 2-category which categorified the quantum group associated to arbitrary root data; later, [Bru16] showed that these constructions were equivalent. This categorification provides new perspectives on several phenomena which are surprising when viewed on the quantum group level, including the positivity of the structure constants of Lusztig's canonical basis.

Another important example of a categorification is the Heisenberg category $\mathcal{H}$ constructed in [Kho14], the Grothendieck group of which contains (and is conjecturally isomorphic to) the Heisenberg algebra $\mathfrak{h}$. The category $\mathcal{H}$ has objects monoidally generated by objects P and Q, and its morphisms are given by a graphical calculus of planar string diagrams. An interesting feature of this categorification is the connection between its graphical calculus and the representation theory of symmetric groups. There is a categorical action of $\mathcal{H}$ on $\mathbb{C}S_n$-modules for $n > 0$ by induction and restriction functors, and the categorification of the Heisenberg algebra relation $PQ = QP + 1$ corresponds in this action to the Mackey theorem for symmetric groups.

The Grothendieck group K_0 is the most commonly used decategorification in representation theory, and both the KLR 2-category and Khovanov's Heisenberg category are constructed to be decategorified with K_0. It is natural to ask whether there are alternative decategorification functors which provide interesting information, and whether these alternative functors can be applied to existing categorifications in order to yield new perspectives.

One such alternative decategorification is given by the trace. For a $\mathbb{C}$-linear category $\mathcal{C}$, define the trace of $\mathcal{C}$ to be the vector space

$$\operatorname{Tr}(\mathcal{C}) := \left(\bigoplus_{x \in \operatorname{Ob}(\mathcal{C})} \operatorname{End}_{\mathcal{C}}(X) \right) \Big/ \operatorname{span}_{\mathbb{C}}\{fg - gf\},$$

where f and g run through all pairs of morphisms $f : x \to y$ and $g : y \to x$ with $x, y \in \operatorname{Ob}(\mathcal{C})$. If $\mathcal{C}$ additionally carries a monoidal structure, then $\operatorname{Tr}(\mathcal{C})$ is naturally a $\mathbb{C}$-algebra.

The trace has several advantages over the Grothendieck group. First, there is a natural map $K_0(\mathcal{C}) \to \operatorname{Tr}(\mathcal{C})$, taking the isomorphism class of an object to the class of its identity morphism, which is injective but rarely surjective. Hence, in some sense, the trace gives more and deeper information about the structure of the category than the Grothendieck group. In the basic example of $\mathcal{C} = \operatorname{Vect}_{\mathbb{C}}$, the space of commutators $\{fg - gf\}$ is precisely the subspace of matrices with trace 0. Hence the class of a morphisms ϕ is $[\phi] = \operatorname{tr}(\phi)[1_{\mathcal{C}}]$, where $\operatorname{tr}(\phi)$ is the usual trace of ϕ as a matrix. Hence $\operatorname{Tr}(\mathcal{C}) \cong \mathbb{C}$, whereas $K_0(\mathcal{C}) \cong \mathbb{Z}$.

This pattern holds for more sophisticated examples of categorifications as well. The Grothendieck group of the KLR 2-categorification of a quantum group is isomorphic to the idempotented form of the quantum group, while the trace of this categorification is shown in [BHLW17] to be isomorphic to the current algebra, a much larger and more sophisticated structure.

The Lie algebra $W_{1+\infty}$ of differential operators on the circle is important in conformal field theory and representation theory. It is an infinite dimensional Lie algebra closely related to $\mathfrak{gl}_\infty$ (cf. [FKRW00]). In [CLLS16], the trace of the Heisenberg category $\operatorname{Tr}(\mathcal{H})$ is shown to be isomorphic to $W_{1+\infty}$ at level one. In particular, it properly contains the Heisenberg algebra $\mathfrak{h}$. Hence $\operatorname{Tr}(\mathcal{H})$ likely contains more information than $K_0(\mathcal{H})$.

The trace also has some technical advantages: it is invariant under passage to the Karoubi envelope, or idempotent completion, of $\mathcal{C}$ ([BGHL14, Proposition 3.2]), and it is defined for linear categories, whereas K_0 requires the category to be additive.

1.2 Hecke algebras and spin versions

The Hecke algebra is a q-deformation of a group algebra $\mathbb{C}W$ which has appeared in many contexts in representation theory. There is a graded (or degenerate) affine version $\mathfrak{H}_W$, which was developed in type A in [Dri86] and in all types in [Lus89].

The structure of Khovanov's Heisenberg category $\mathcal{H}$ is closely related to the structure of the degenerate affine Hecke algebra of type A, $\mathfrak{H}_A$. In particular, the endomorphism algebras of objects of the form P^m and Q^n are isomorphic to $\mathfrak{H}_{A_m}$ and $\mathfrak{H}_{A_n}^{op}$, respectively. Furthermore, a key ingredient in the computation of the trace of $\mathcal{H}$ is a triangular decomposition ([CLLS16, Lemma 35])

$$\mathrm{Tr}(\mathcal{H}) \cong \bigoplus_{m,n\in\mathbb{Z}_{\geq 0}} \mathrm{Tr}((\mathfrak{H}_{A_m})^{op} \otimes \mathfrak{H}_{A_n} \otimes \mathbb{C}[d_0, d_2, d_4 \ldots .])$$

where d_i are certain endomorphisms of the identity and, for an algebra $\mathcal{A}$, we define the trace (or zeroth Hochschild homology) to be

$$\mathrm{Tr}(\mathcal{A}) := \mathcal{A}/[\mathcal{A}, \mathcal{A}].$$

Hence, understanding the trace of $\mathfrak{H}_{A_{n-1}}$ is essential to the study of $\mathrm{Tr}(\mathcal{H})$. The complete Hochschild homology of $\mathfrak{H}_{A_{n-1}}$, was determined in [Sol10]; later, the trace was described directly in [CH16]. The latter paper determined a linear basis for $\mathrm{Tr}(\mathfrak{H}_{A_{n-1}})$, the Weyl group part of which is labeled by partitions of n.

Many fundamental structures in representation theory have natural twisted, or spin, analogues, which have some additional non-commutativity. Schur described a spin symmetric group algebra $\mathbb{C}S_n^-$ in [Sch11] as the quotient of a double cover $\tilde{S_n}$ of S_n. The linear representation theory of $\mathbb{C}S_n^-$ is equivalent to the spin representation theory of S_n. The representation theory of $\mathbb{C}S_n^-$ was systematically described in [Sch11]; see [Józ88a] for a modern exposition of this theory. Spin Weyl group algebras $\mathbb{C}W^-$ in all finite types were introducted by Morris [Mor76].

An important tool in the spin representation theory of W is the degenerate affine Hecke-Clifford algebra associated to W, $\mathfrak{H}^{\mathfrak{c}}_W$, (sometimes called the affine Sergeev algebra in type A). These algebras were introduced by [Naz97] in type A. A spin analogue of these type A algebras (which has the same super representation theory as $\mathfrak{H}^{\mathfrak{c}}_W$ but excludes the Clifford algebra) was then introduced in [Wan09]. The Hecke-Clifford and spin Hecke algebras in all classical types were then described by [KW08].

The Clifford algebra naturally arises when studying spin (or projective) representation theory of Weyl groups: indeed, we have (cf. [KW08]) an isomorphism $\mathcal{C}_W \rtimes \mathbb{C}W \xrightarrow{\sim} \mathcal{C}_W \otimes \mathbb{C}W^-$, where $\mathcal{C}_W$ is the Clifford algebra associated to the reflection representation of W. The algebra $\mathfrak{H}^{\mathfrak{c}}_W$ reflects this important connection in its definition: as a vector space, we have $\mathfrak{H}^{\mathfrak{c}}_W \cong S(V) \otimes \mathcal{C}_W \otimes \mathbb{C}W$, where V is the natural (i.e., reflection) representation of W.

Two questions arise naturally from this definition:

1. is there a way to lift the calculation of $\mathrm{Tr}(\mathfrak{H}_A)$ in [CH16] to an appropriate spin setting?

2. is there a corresponding twisted or spin Heisenberg category whose trace is controlled by this twisted algebra?

In the first part of this dissertation, we compute a linear basis for the trace of the degenerate affine Hecke-Clifford algebra in types ABD. This work appeared in [Ree17]. Our work showed that the trace of these algebras is connected to a certain set of conjugacy classes of the Weyl group, called the even split conjugacy classes. Leveraging the combinatorics of these classes, we found a description of $\mathrm{Tr}(\mathfrak{H}^{\mathfrak{c}}_W)$ which is analogous to the enumeration of the even split conjugacy classes. In particular, we prove the following (refer to Theorem 2.4.4 for precise detail).

Theorem A. [Theorem 2.4.4] *The even traces of the degenerate affine Hecke–Clifford algebras in types ABD have linear bases $\{w_C f_{C;i}\}$ where w_C are conjugacy class representatives of even split conjugacy classes in the Weyl groups of types ABD, and $f_{C;i}$ is the basis of a subspace of $S(V)$, the symmetric algebra of the natural representation, fixed by parabolic subgroups of the Weyl group.*

Note that $\mathfrak{H}^{\mathfrak{c}}_W$ is a superalgebra, i.e. a $\mathbb{Z}_2$-graded algebra, in which the generators of the Clifford algebra have degree 1 and all other generators have degree 0. When studying the trace of a superalgebra, the correct space to consider is actually the even trace, $\mathrm{Tr}(\mathfrak{H}^{\mathfrak{c}}_W)_{\bar{0}}$. Because odd elements act with zero trace on any $\mathbb{Z}_2$-graded $\mathfrak{H}^{\mathfrak{c}}_W$-module, the even trace contains all of the interesting representation-theoretic information about $\mathfrak{H}^{\mathfrak{c}}_W$, and is thus the meaningful portion of the trace in this context. See Section 2.2.6 for details on this restriction.

The proof that $\{w_C f_{J_C;i}\}$ is a spanning set for $\mathrm{Tr}(\mathfrak{H}^{\mathfrak{c}})_X$ relies on several reduction results. We show that an arbitrary element of $\mathfrak{H}^{\mathfrak{c}}_X$ can be reduced modulo $[\mathfrak{H}^{\mathfrak{c}}_X, \mathfrak{H}^{\mathfrak{c}}_X]$ to an element containing no instances of generators of the Clifford algebra. We then show that Weyl group elements belonging to certain conjugacy classes, which vary between types, vanish in the cocenter. Then we take advantage of a filtration of $\mathfrak{H}^{\mathfrak{c}}_X$ to pass to the associated graded object, and use methods developed in [CH16] to prove that $\{w_C f_{J_C;i}\}$ spans in that setting. Finally, we can lift the the spanning set to the ungraded object.

To prove linear independence, we establish a trace formula for parabolically induced $\mathfrak{H}^{\mathfrak{c}}_X$-modules . This trace formula allows us to separate the Weyl group elements by their action on subspaces of $S(V)$. By applying this trace formula to the action of the Weyl group on a set of irreducible modules of the parabolic subalgebras, we obtain the linear independence result and hence Theorem A.

1.3 The trace of the twisted Heisenberg category

A variation of the Heisenberg category in which morphisms are controlled by the degenerate affine Hecke-Clifford algebra was introduced in [CS15]. The twisted Heisenberg category, $\mathcal{H}_{tw}$, has objects monoidally generated by P and Q, and additional morphisms corresponding to generators of the Clifford algebra. The endomorphism algebra of P^m is isomorphic to $\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}$, in analogy to the untwisted setting. The category $\mathcal{H}_{tw}$ contains a twisted version of the Heisenberg algebra $\mathfrak{h}_{tw}$ in its Grothendieck group, and conjecturally $K_0(\mathcal{H}_{tw}) \cong \mathfrak{h}_{tw}$.

The algebra $\mathrm{Tr}(\mathcal{H}_{tw})$ has a triangular decomposition similar to that of $\mathrm{Tr}(\mathcal{H})$: namely, we showed in [OR17] that

$$\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}} \cong \bigoplus_{m,n\in\mathbb{Z}_{\geq 0}} \mathrm{Tr}((\mathfrak{H}^C_m)^{op} \otimes \mathfrak{H}^C_n \otimes \mathbb{C}[d_0, d_2, d_4 ....])_{\bar{0}}.$$

We should therefore expect that the description of $\mathrm{Tr}(\mathfrak{H}^C_n)_{\bar{0}}$ given in [Ree17] will enable us to calculate $\mathrm{Tr}(\mathcal{H}_{tw})$.

The second part of this dissertation focuses on the details on this computation (work which appeared in [OR17]). We establish a connection between the even trace of $\mathcal{H}_{tw}$ and a subalgebra of $W_{1+\infty}$. This algebra, $W^- \subset W_{1+\infty}$, is the fixed-point subalgebra of one of two degree-preserving anti-involutions of $W_{1+\infty}$, and was introduced in [KWY98]. Whereas $W_{1+\infty}$ is closely related to $\mathfrak{gl}_\infty$, the algebra W^- is related to classical type (i.e., type BCD) subalgebras of $\mathfrak{gl}_\infty$. We prove the following result:

Theorem B. [Theorem 3.5.10] *The even trace of $\mathcal{H}_{tw}$ is isomorphic to W^- at level one.*

To prove Theorem B, we first compute sets of algebra generators and relations for both W^- and $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, adapting arguments used in [CLLS16] to accommodate the new supercommutative elements arising from the twisting. We then study actions of each algebra on its canonical level one Fock space representation. These Fock space representations are isomorphic, and so induce a linear map $\Phi : \mathrm{Tr}(\mathcal{H}_{tw}) \to W^-$. We prove that Φ is an algebra homomorphism by studying the actions of both W^- and $\mathrm{Tr}(\mathcal{H}_{tw})$ on their Heisenberg subalgebras. Finally, we check that the actions of the generators are identified under Φ, and deduce that Φ is an algebra isomorphism.

1.4 The center of the twisted Heisenberg category

Another decategorification functor is the center $Z(\mathcal{C})$. This is by definition the space of endomorphisms of the monoidal identity object $1_{\mathcal{C}}$. In the graphical calculi of $\mathcal{H}$ and $\mathcal{H}_{tw}$, the center is spanned by closed diagrams, called bubbles. As a consequence of the triangular decomposition of each category, the centers $Z(\mathcal{H})$ and $Z(\mathcal{H}_{tw})$ are isomorphic to polynomial algebras in infinitely many variables. Because of the close connection between these algebras and representations of the symmetric group and spin symmetric group, respectively, it is interesting to ask whether these polynomial algebras can in fact be realized as an algebra of symmetric polynomials.

This question was answered in the affirmative for $\mathcal{H}$ in [KLM16]. It was shown that $Z(\mathcal{H})$ is isomorphic to an algebra Λ^* of shifted symmetric functions, which are symmetric in the shifted variables $(x_i + i)$, $i > 0$. Furthermore, this isomorphism is canonical in the following sense: as a result of the categorical action of $\mathcal{H}$ on symmetric group modules, there are surjective algebra homomorphisms

$$f_n^{\mathcal{H}} : \mathrm{End}_{\mathcal{H}}(1) \longrightarrow Z(\mathbb{C}S_n)$$

to the center of each symmetric group algebra. There are also natural surjective algebra homomorphisms

$$f_n^{\Lambda^*} : \Lambda^* \longrightarrow Z(\mathbb{C}S_n)$$

and the isomorphism of [KLM16] intertwines these maps.

We will provide the corresponding answer for the twisted case in the third part of this dissertation. The center of the twisted Heisenberg category is isomorphic to the subalgebra of the symmetric functions generated by power sum symmetric functions associated to odd partitions. This algebra is sometimes known as the supersymmetric (or doubly symmetric) functions, and is also generated by Schur Q- and factorial Schur Q-functions. In particular, $Z(\mathcal{H}_{tw})$ is generated by the closures of diagrams corresponding to n-cycles, and we prove that such a diagram corresponds to a shifted (or factorial) Schur Q-function described by [Iva01].

Theorem C. [Theorem 4.5.2] *There is an isomorphism $Z(\mathcal{H}_{tw}) \cong \Gamma$, where Γ is the algebra generated by odd power sum symmetric functions, which sends the closure of an n-cycle in $Z(\mathcal{H}_{tw})$ to the shifted Schur-Q function $Q_{(n)}$.*

One interesting feature of the center of the non-twisted Heisenberg category $\mathcal{H}$ is that, as shifted symmetric functions, the bubble generators are best understood in terms of moments of Kerov's transition and co-transition measures on Young diagrams; fundamental tools used to answer probabilistic questions related to the asymptotic representation theory of symmetric groups [Ker93b]. In this paper we show that this connection to asymptotic representation theory extends to the twisted Heisenberg category. Specifically, we identify the clockwise bubble generators $\{d_{2k}\}_{k\geq 0}$ and counterclockwise bubble generators $\{\bar{d}_{2k}\}_{k\geq 1}$ with two sets of algebraically independent generators for Γ discovered by Petrov [Pet09], $\{\mathbf{g}_k^{\downarrow}\}_{k\geq 0}$ and $\{\mathbf{g}_k^{\uparrow}\}_{k\geq 0}$ respectively. The functions $\{\mathbf{g}_k^{\downarrow}\}_{k\geq 0}$ (respectively $\{\mathbf{g}_k^{\uparrow}\}_{k\geq 0}$) encode the down (resp. up) transition kernels for a Markov process on the graph of all strict partitions (also known as the Schur graph). Hence, the difference between up and down transition functions manifests itself graphically in $Z(\mathcal{H}_{tw})$ as a difference in orientation of diagrams.

The trace of a category with a graphical calculus of morphisms can be defined diagramatically as the algebra of diagrams on the annulus, while the center consists of closed diagrams on the plane. There is a natural action of the trace of a category on its center, which can be diagrammatically defined as gluing annular diagrams (elements of the trace) around planar ones (elements of the center). In the case of Khovanov's Heisenberg category, the results of [CLLS16] and [KLM16] give rise to an action of $W_{1+\infty}$ on the algebra of shifted symmetric functions. This representation of $W_{1+\infty}$ was described in terms of symmetric group representation theory by Lascoux and Thibon in [LT01]. Theorem B along with Theorem C gives a representation of W^- on Γ. We describe this representation, which is a twisted version of the representation described in [LT01].

1.5 Organization

The dissertation is organized as follows. In Chapter 2, we describe the degenerate affine Hecke-Clifford and spin Hecke algebras in types ABD. We then compute the traces of these algebras as vector spaces.

In Chapter 3, we define the twisted Heisenberg category and describe the graphical calculus of its morphisms. We discuss the vertex algebra W^- and its Fock space representation. Using the Fock space representations on either side, we define a linear map $W^- \to \mathcal{H}_{tw}$ and prove that it is an isomorphism.

Finally, in Chapter 4, we describe the center of $\mathcal{H}_{tw}$ as an algebra of symmetric functions generated by the odd power sums. We also describe the natural action of the trace on the center in terms of a known vertex algebra action on the symmetric functions.

Chapter 2

Traces of Hecke-Clifford and spin Hecke algebras

The degenerate affine Hecke-Clifford algebra was introduced in type A_{n-1} in [Naz97], and in all classical types in [KW08]. These algebras are variations on the degenerate (or graded) affine Hecke algebras, which were introduced independently in [Dri86] (to study Yangians) and in [Lus89] (to study representations of reductive p-adic groups). The degenerate affine spin Hecke algebras were introduced in type A_{n-1} in [Wan09], and in all classical types in [KW08]. These are degenerate affine Hecke algebras associated to the spin Weyl groups. Hecke-Clifford algebras and spin Hecke algebras are closely related to the study of the spin representation theory of classical Weyl groups [Józ88a].

In this chapter, we determine a basis of the trace (i.e., the cocenter or zeroth Hochschild homology) of the degenerate affine Hecke-Clifford and spin Hecke algebras in classical types. The chapter is organized as follows. In Section 1, we establish notations and describe the degenerate affine Hecke-Clifford algebras in types A, B, and D. In Section 2, we prove a series of lemmas to reduce an arbitrary element in the cocenter of $\mathfrak{H}^{\mathfrak{c}}_X$ to a corresponding element with no Clifford algebra generators, and then prove that Weyl group elements not belonging to certain distinguished conjugacy classes vanish in the cocenter. In Section 3, we establish a spanning set of the associated graded object $\mathrm{Tr}(\mathfrak{H}^0)_X$ and lift it to $\mathrm{Tr}(\mathfrak{H}^{\mathfrak{c}})_X$ in each type. We then proceed in Section 4 to prove that these spanning sets are linearly independent by establishing a trace formula for parabolically induced module. In Section 5, we construct the degenerate spin affine Hecke algebra in each type. Section 6 contains reduction formulas similar to those in section 3, with proofs adapted to the new setting. Finally, in Section 7, we establish a spanning set for $\overline{\mathfrak{H}}^{\mathsf{sp}}_X$ and then take advantage of the Morita superequivalence to prove that it is linearly independent.

The work in this chapter initially appeared in [Ree17].

2.1 Preliminaries on Hecke-Clifford algebras

We establish basic notations and definitions, and then recall the definition of the degenerate affine Hecke-Clifford algebra in types A, B, and D. We then recall some basic facts about these algebras, including a PBW property and a filtration, and finally define the trace.

2.1.1 Root systems and the Weyl group

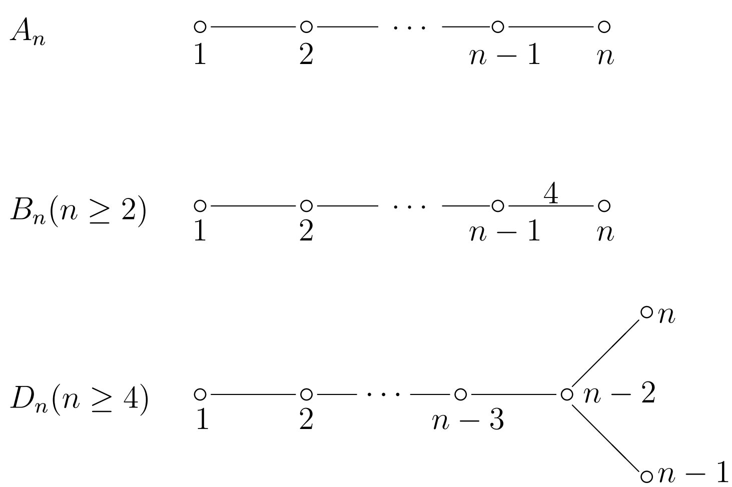
Let $\Phi = (V_0, R, V_0^\vee, R^\vee)$ be a semisimple real root system: V_0 and $V_0^\vee$ are finite dimensional real vector spaces, R and $R^\vee$ span V_0 and $V_0^\vee$ respectively and, there is a bijection $R \leftrightarrow R^\vee$ such that $(\alpha, \alpha^\vee) = 2$, and R and $R^\vee$ are preserved by the reflections $s_\alpha : v \mapsto (v - (v, \alpha^\vee)\alpha)$. Set

$$V = \mathbb{C} \otimes_{\mathbb{R}} V_0 \text{ and } V^\vee = \mathbb{C} \otimes_{\mathbb{R}} V_0^\vee.$$

Let W be the finite Weyl group of Φ, the subgroup of $GL(V)$ generated by s_α, $\alpha \in R$. Fix a choice of positive roots R^+ and positive coroots $(R^+)^\vee$, and let $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset R^+$ be a basis, the set of simple roots. Then W is a finite Coxeter group with presentation

$$\langle s_1, \dots, s_n | (s_i s_j)^{m_{ij}} = 1,\ m_{ii} = 1,\ m_{ij} = m_{ji} \in \mathbb{Z}_{\geq 2}, \text{for } i \neq j \rangle \tag{2.1}$$

where $m_{ij} \in \{1, 2, 3, 4, 6\}$ is specified by the Coxeter-Dynkin diagrams, wherein the vertices correspond to generators of W. Two generators s_i and s_j, $i \neq j$, have $m_{ij} = 2$ if there is no edge between i and j, $m_{ij} = 3$ if i and j are connected by an unmarked edge, and $m_{ij} = \ell$ if the edge connecting i and j is labeled with an $\ell \geq 4$.



(2.2)

For every subset $J \subset \Pi$, denote by W_J the parabolic subgroup of W, generated by $\{s_i = s_{\alpha_i} | \alpha \in J\}$. Denote by $V_J, R_J, V_J^\vee, R_J^\vee$ the corresponding vector spaces.

2.1.2 The Clifford algebra

The reflection representation V carries a W-invariant nondegenerate bilinear form $(-, -)$, which gives rise to an identification $V^* \cong V$. We identify V^* with a suitable subspace of $\mathbb{C}^n$ and choose a standard orthonormal basis $\{e_i\}$ of $\mathbb{C}^n$.

Denote by $\mathcal{C}_n$ the Clifford algebra assosciated to $(\mathbb{C}^n, (-, -))$. It is an associative $\mathbb{C}$-algebra with identity which contains $\mathbb{C}^n$ as a subspace and is generated by elements of $\mathbb{C}^n$ subject to the relation

$$uv + vu = (u, v) \qquad u, v \in \mathbb{C}^n. \tag{2.3}$$

Set $c_i = \sqrt{2}e_i$ for each i. Then $\mathcal{C}_V$ is generated by elements $c_1, \ldots, c_n$ subject to relations

$$c_i^2 = 1, \quad c_i c_j = -c_j c_i \quad i \neq j. \tag{2.4}$$

Let $\mathcal{C}_V$ be the Clifford algebra associated to $(V, (-,-))$, which is a subalgebra of $\mathcal{C}_n$. The algebra $\mathcal{C}_V$ has generators β_i corresponding to the simple roots α_i of the Lie algebra corresponding to W; note that, in this chapter, we always choose to work with the Lie algebra $\mathfrak{gl}_n$ in type A_{n-1}, rather than $\mathfrak{sl}_n$. Note that $\mathcal{C}_V$ is naturally a superalgebra with each β_i odd. The explicit generators are given in the following table for types A_{n-1}, B_n, and D_n:

Type of W	N	Generators for $\mathcal{C}_W$
A_{n-1}	n	$\beta_i = \frac{1}{\sqrt{2}}(c_i - c_{i+1}), 1 \leq i \leq n-1$
B_n	n	$\beta_i = \frac{1}{\sqrt{2}}(c_i - c_{i+1}), 1 \leq i \leq n-1$, $\beta_n = c_n$
D_n	n	$\beta_i = \frac{1}{\sqrt{2}}(c_i - c_{i+1}), 1 \leq i \leq n-1$, $\beta_n = \frac{1}{\sqrt{2}}(c_{n-1} + c_n)$

The action of W on V preserves the bilinear form $(\, , \,)$, so W acts on $\mathcal{C}_V$ by automorphisms. This allows us to form the semidirect product $\mathcal{C}_V \rtimes \mathbb{C}W$, which is also naturally a superalgebra with $\mathbb{C}W$ even.

2.1.3 The degenerate affine Hecke-Clifford algebras

We recall the degenerate affine Hecke-Clifford algebras of types A_{n-1}, B_n, and D_n, following the descriptions of [Naz97] in type A_{n-1} and [KW08] in types B_n and D_n. Let $S(V)$ be the symmetric algebra of V. Then $S(V) \cong \mathbb{C}[x_1, \ldots, x_n]$, where $\{x_1, \ldots, x_n\}$ is a basis of V^*. Let $u \in \mathbb{C}$ and set $W = S_n$, the Weyl group of type A_{n-1}.

The degenerate affine Hecke Clifford algebra of type A_{n-1}, $\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}$, is the $\mathbb{C}$-algebra generated by $x_1, \ldots, x_n$, $c_1, \ldots, c_n$, and S_n, subject to relations making $\mathbb{C}[x_1, \ldots, x_n]$, $\mathcal{C}_V$, and $\mathbb{C}S_n$ subalgebras, along with the additional relations:

$$x_i c_i = -c_i x_i, \quad x_i c_j = c_j x_i \quad (i \neq j), \tag{2.5}$$
$$\sigma c_i = c_{\sigma(i)} \sigma \quad (1 \leq i \leq n, \sigma \in S_n), \tag{2.6}$$
$$x_{i+1} s_i - s_i x_i = u(1 - c_{i+1} c_i), \tag{2.7}$$
$$x_j s_i = s_i x_j \quad (j \neq i, i+1). \tag{2.8}$$

Denote the action of S_n on $S(V)$ by $f \mapsto f^\sigma$, $f \in S(V)$, $\sigma \in S_n$.

Next, let $W = W_{D_n}$, the Weyl group of type D_n. It is generated by elements $s_1, \ldots, s_n$ where $s_1, \ldots, s_{n-1}$ are subject to the defining relations of S_n, and there are the additional relations:

$$s_i s_n = s_n s_i \quad (i \neq n-2), \tag{2.9}$$

$$s_{n-2}s_ns_{n-2} = s_ns_{n-2}s_n, \quad s_n^2 = 1. \tag{2.10}$$

The degenerate affine Hecke-Clifford algebra of type D_n, $\mathfrak{H}^{\mathfrak{c}}_{D_n}$, is generated by x_i, c_i, s_i, $1 \le i \le n$, subject to relations making $\mathbb{C}[x_1, \dots, x_n]$, $\mathcal{C}_V$, and $\mathbb{C}W$ subalgebras, along with the relations (2.5) – (2.8) and the additional relations:

$$\begin{aligned} s_nc_n &= -c_{n-1}s_n, \\ s_nc_i &= c_is_n \quad (i \neq n-1, n), \\ s_nx_n + x_{n-1}s_n &= -u(1 + c_{n-1}c_n), \\ s_nx_i &= x_is_n \quad (i \neq n-1, n). \end{aligned} \tag{2.11}$$

Finally, let $W = W_{B_n}$, the Weyl group of type B_n. It is generated by elements $s_1, \dots, s_n$, where $s_1, \dots, s_{n-1}$ are subject to the defining relations on S_n, and there are the additional relations:

$$s_is_n = s_ns_i \quad (1 \le i \le n-2) \tag{2.12}$$

$$(s_{n-1}s_n)^4 = 1, \quad s_n^2 = 1. \tag{2.13}$$

The simple reflections of W lie in two different conjugacy classes: s_n is not conjugate to $s_1, \dots, s_{n-1}$.

Let $u, v \in \mathbb{C}$. The degenerate affine Hecke-Clifford algebra of type B_n, $\mathfrak{H}^{\mathfrak{c}}_{B_n}$, is generated by x_i, c_i, s_i, $1 \le i \le n$, subject to relations subject to relations making $\mathbb{C}[x_1, \dots, x_n]$, $\mathcal{C}_V$, and $\mathbb{C}W$ subalgebras, along with the relations 2.5 – 2.8, and the additional relations:

$$\begin{aligned} s_nc_n &= -c_ns_n, \\ s_nc_i &= c_is_n \quad (i \neq n), \\ s_nx_n + x_ns_n &= -\sqrt{2}\, v, \\ s_nx_i &= x_is_n \quad (i \neq n). \end{aligned}$$

The PBW theorems for the degenerate affine Hecke-Clifford algebras in type A_{n-1} were proved in [Naz97] and in [KW08] using different methods, and in types B_n and D_n in [KW08]. The even center of these algebras- the subalgebra of even central elements - was also established in [KW08].

Proposition 2.1.1. *Let $X = A_{n-1}$, D_n or B_n.*

1. *The multiplication of subalgebras $\mathbb{C}[x_1, \dots, x_n], \mathcal{C}_V$, and $\mathbb{C}W$ induces a vector space isomorphism*
$$\mathbb{C}[x_1, \dots, x_n] \otimes \mathcal{C}_V \otimes \mathbb{C}W \longrightarrow \mathfrak{H}^{\mathfrak{c}}_X.$$
Equivalently, the elements $\{x^\alpha c^\epsilon w | \alpha \in \mathbb{Z}^n_+, \epsilon \in \mathbb{Z}^n_2, w \in W\}$ form a linear basis for $\mathfrak{H}^{\mathfrak{c}}_X$.

2. *Let $X = A_{n-1}, B_n$ or D_n. Then*
$$Z(\mathfrak{H}^{\mathfrak{c}}_X)_{\bar{0}} \cong \mathbb{C}[x_1^2, \dots, x_n^2]^{W_X}.$$

Each of these algebras is naturally a superalgebra with even generators from $S(V)$ and $\mathbb{C}W$ and odd generators from $\mathcal{C}_V$.

Denote by $S(V^2)$ the subspace of $S(V)$ spanned by the squares of the basis elements in $S(V)$. Thus $Z(\mathfrak{H}^{\mathfrak{c}}_X)_{\bar{0}} \cong S(V^2)^{W_X}$.

2.1.4 A filtration of $\mathfrak{H}^{\mathfrak{c}}_X$

In any of the algebras defined in Section 2.3, we can define a notion of degree as follows. From the various PBW basis theorems, we see that every $h \in \mathfrak{H}^{\mathfrak{c}}_X$, for $X = A_{n-1}, B_n$, or D_n, can be written

$$h = \sum_{w \in W} a_w c_w w$$

where $a_w \in S(V)$ and $c_w \in \mathcal{C}_V$. Set

$$|h| = \max_{w \in W}\{|a_w|\}$$

where $|a_w|$ denotes degree in $S(V)$. Set $\mathcal{F}^j\mathfrak{H}^{\mathfrak{c}}_X = \{h \in \mathfrak{H}^{\mathfrak{c}}_X | \ |h| \leq j\}$; then we have a filtration

$$\mathcal{C}_V \rtimes \mathbb{C}W = \mathcal{F}^0\mathfrak{H}^{\mathfrak{c}}_X \subset \mathcal{F}^1\mathfrak{H}^{\mathfrak{c}}_X \subset \dots$$

Let $\mathrm{gr}(\mathfrak{H}^{\mathfrak{c}}_X)$ be the associated graded algebra. It is clear from the defining relations for $\mathfrak{H}^{\mathfrak{c}}_X$ that $\mathrm{gr}(\mathfrak{H}^{\mathfrak{c}}_X) \cong \mathfrak{H}^0_X$, the degenerate affine Hecke-Clifford algebra with parameter $u = 0$.

2.1.5 Parabolic Subalgebras

For any $J \subset \Pi$, define the parabolic subalgebra $\mathfrak{H}^{\mathfrak{c}}_{X,J}$ to be the subalgebra of $\mathfrak{H}^{\mathfrak{c}}_X$ generated by W_J, $\mathcal{C}_V$, and $\mathbb{C}[x_1, \dots, x_n]$. For every $\mathfrak{H}^{\mathfrak{c}}_{X,J}$-module M, define the parabolically induced module

$$\mathrm{Ind}^{\mathfrak{H}^{\mathfrak{c}}_X}_{\mathfrak{H}^{\mathfrak{c}}_{X,J}} M := \mathfrak{H}^{\mathfrak{c}}_X \otimes_{\mathfrak{H}^{\mathfrak{c}}_{X,J}} M.$$

2.1.6 The trace

For any $h, h' \in \mathfrak{H}^{\mathfrak{c}}_X$, define the commutator $[h, h'] = hh' - h'h$. Let $[\mathfrak{H}^{\mathfrak{c}}_X, \mathfrak{H}^{\mathfrak{c}}_X]$ be the submodule of $\mathfrak{H}^{\mathfrak{c}}_X$ generated by all commutators. The *trace* of $\mathfrak{H}^{\mathfrak{c}}_X$ is the space

$$\mathrm{Tr}(\mathfrak{H}^{\mathfrak{c}}_X) := \left(\frac{\mathfrak{H}^{\mathfrak{c}}_X}{[\mathfrak{H}^{\mathfrak{c}}_X, \mathfrak{H}^{\mathfrak{c}}_X]}\right)_{\bar{0}}.$$

The main goal of this chapter is to find a linear basis for the trace.

Note that we restrict our definition to only the *even* trace. Referring to the example of the trace of the finite Hecke-Clifford algebra, as studied in [WW12a, Section 4.1], gives intuition as to why this is the correct notion of trace.

Wan and Wang study the space of trace functions on the finite Hecke-Clifford algebra $\mathcal{H}_n$: linear functions $\phi : \mathcal{H}_n \to \mathbb{C}$ such that $\phi([h, h']) = 0$ for all $h, h' \in \mathcal{H}_n$, and $\phi(h) = 0$ for all $h \in (\mathcal{H}_n)_{\bar{1}}$. This latter requirement encodes the information that odd elements act with zero trace on any $\mathbb{Z}_2$-graded $\mathcal{H}_n$-module (because multiplication by an odd element results in a shift in degree). The space of such trace functions is clearly canonically isomorphic to the dual of the even trace, rather than of the full trace. Moreover, since the even trace of $\mathcal{H}_n$ has dimension equal to the number of irreducible $\mathbb{Z}_2$-graded representations of $\mathcal{H}_n$, this restriction sets up the desired linear isomorphism between the space of trace functions and the linear span of the irreducible representations (the matrix of this isomorphism is the character table of the algebra).

In the affine case, we see that the trace of the action of an odd element on any $\mathfrak{H}^{\mathfrak{c}}_X$-module is still zero, due to the same degree shift. Hence we deduce that the interesting information about traces of $\mathfrak{H}^{\mathfrak{c}}_X$ (and, thus, much of the interesting representation-theoretic information about $\mathfrak{H}^{\mathfrak{c}}_X$) is contained in the even trace.

2.2 Reduction

The goal of this section is to show that an element $h = \sum_{w\in W} a_w c_w w \in \mathfrak{H}^{\mathfrak{c}}_X$ is congruent in the trace to a (possibly differently indexed) linear combination $h = \sum_i a_i w_i$ without any Clifford algebra elements, and to show that certain conjugacy classes of Weyl group elements vanish in the trace.

2.2.1 Clifford reduction in type A_{n-1}

We adapt the procedure in [WW12a], where similar formulas are developed in the finite and non-degenerate case, with appropriate modifications. Let $w_{(n)} = s_1 s_2 \dots s_{n-1} = (1\ 2\ \dots\ n)$. The following follows directly from the defining relations in $\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}$.

Lemma 2.2.1. *In $\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}$, we have*

$$\begin{aligned} w_{(n)} c_i &= c_{i+1} w_{(n)} \qquad \textit{for } 1 \le i \le n-1, \\ w_{(n)} c_n &= c_1 w_{(n)}. \end{aligned}$$

For $n \in \mathbb{Z}^{>0}$, let $[n] = \{1, 2, \dots, n\}$. For any subset $I \subseteq [n]$, let $c_I = \Pi_{i\in I} c_i$. Note that it suffices to consider only elements wc_I where $|I|$ is even, since $|c_i| = 1$ for all i and we are studying the even trace.

Lemma 2.2.2. *For $I \subseteq [n]$ with $|I|$ even, we have*

$$w_{(n)} c_I \equiv \pm w_{(n)} \qquad \mathrm{mod}\ [\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}, \mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}].$$

Proof. Write $I = \{i_1, \dots, i_k\}$. Then

$$\begin{aligned} w_{(n)} c_I &= (1\ 2\ \dots\ n) c_{i_1} \dots c_{i_k} \\ &= c_{i_1+1} (1\ 2\ \dots\ n) c_{i_2} \dots c_{i_k} \\ &\equiv (1\ 2\ \dots\ n) c_{i_2} \dots c_{i_k} c_{i_1+1} \qquad \mathrm{mod}\ [\mathfrak{H}^{\mathfrak{c}}_A, \mathfrak{H}^{\mathfrak{c}}_A] \\ &= \begin{cases} (-1)^{k-1} w_{(n)} c_{i_1+1} c_{i_2} \dots c_{i_k} & i_1 + 1 < i_2 \\ (-1)^{k-2} w_{(n)} c_{i_3} \dots c_{i_k} & i_1 + 1 = i_2. \end{cases} \end{aligned} \tag{2.14}$$

Now we have either reduced the size of I by two or increased i_1 by one. Since $|I|$ is even, we can continue in this way until no c_i remain. □

If $\gamma = (\gamma_1, \ldots, \gamma_k)$ is a sequence of (not necessarily decreasing) positive integers such that $\sum_{i=1}^k \gamma_i = n$, call γ a composition of n. For such a composition γ of n, set $w_\gamma = w_{\gamma_1} \ldots w_{\gamma_k}$.

Lemma 2.2.3. *Let $\gamma = (\gamma_1, \gamma_2)$ be a composition of n with $\gamma_1, \gamma_2 > 0$. Let $I_1 = \{i_1, \ldots, i_a\} \subseteq \{1, \ldots, \gamma_1\}$ and $I_2 = \{j_1, \ldots, j_b\} \subseteq \{\gamma_1 + 1, \ldots, \gamma_2\}$, and assume that $a + b$ is even. Then we have*

$$w_\gamma c_{I_1} c_{I_2} \equiv \begin{cases} 0 & a, b \text{ odd} \\ \pm w_\gamma & a, b \text{ even} \end{cases} \qquad \mod [\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}, \mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}].$$

Proof. Note that a and b must have the same parity if their sum is even. We have $w_\gamma = w_{\gamma_1} w_{\gamma_2} = w_{\gamma_2} w_{\gamma_1}$. Suppose that a and b are both odd. Then

$$\begin{aligned} w_\gamma c_{I_1} c_{I_2} &= w_{\gamma_1} c_{I_1} w_{\gamma_2} c_{I_2} \\ &\equiv w_{\gamma_2} c_{I_2} w_{\gamma_1} c_{I_1} \mod [\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}, \mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}] \\ &= w_\gamma c_{I_2} c_{I_1} \\ &= -w_\gamma c_{I_1} c_{I_2}. \end{aligned}$$

since commuting c_{I_1} past c_{I_2} yields a sign of $(-1)^{ab}$. Hence $w_\gamma c_{I_1} c_{I_2} \equiv 0 \mod [\mathfrak{H}^{\mathfrak{c}}_A, \mathfrak{H}^{\mathfrak{c}}_A]$.

Next, suppose a and b are even. Note that $\gamma_1 + 1 \leq j_1 \leq n - 1$, so c_{j_1} anticommutes with all c_{i_s}. We have

$$\begin{aligned} w_\gamma c_{I_1} c_{I_2} &= w_{\gamma_1} w_{\gamma_2} c_{i_1} \ldots c_{i_a} c_{j_1} \ldots c_{j_b} \\ &= -w_{\gamma_1} c_{i_1} w_{\gamma_2} c_{j_1} c_{i_2} \ldots c_{i_a} c_{j_2} \ldots c_{j_b} \\ &= -c_{i_1+1} c_{j_1+1} w_\gamma c_{i_2} \ldots c_{i_a} c_{j_2} \ldots c_{j_b} \\ &\equiv -w_\gamma c_{i_2} \ldots c_{i_a} c_{j_2} \ldots c_{j_b} c_{i_1+1} c_{j_1+1} \mod [\mathfrak{H}^{\mathfrak{c}}_A, \mathfrak{H}^{\mathfrak{c}}_A]. \end{aligned}$$

Now, commuting c_{i_1+1} and c_{j_1+1} has four possible results, depending on which of the two (if either) cancels with the second Clifford element in their subset. In particular, we have

$$w_\gamma c_{I_1} c_{I_2} \equiv \begin{cases} w_\gamma c_{i_1+1} c_{i_2} \ldots c_{i_a} c_{j_1+1} c_{j_2} \ldots c_{j_b} & i_1 + 1 < i_2, \ j_1 + 1 < j_2 \\ -w_\gamma c_{i_1+1} c_{i_2} \ldots c_{i_a} c_{j_3} \ldots c_{j_b} & i_1 + 1 < i_2, \ j_1 + 1 = j_2 \\ -w_\gamma c_{i_3} \ldots c_{i_a} c_{j_1+1} c_{j_2} \ldots c_{j_b} & i_1 + 1 = i_2, \ j_1 + 1 < j_2 \\ w_\gamma c_{i_3} \ldots c_{i_a} c_{j_3} \ldots c_{j_b} & i_1 + 1 = i_2, \ j_1 + 1 = j_2. \end{cases}$$

In any case, we have either reduced the length of c_{I_1} or c_{I_2} or increased the index of the first element. Continuing in this manner gives the result. □

By induction, we have the following:

Proposition 2.2.4. *If $\gamma = (\gamma_1, \ldots, \gamma_k)$ is a composition of n, $I \subset [n]$ is an even subset, and $I_k = I \cap \{\sum_{i=1}^{k-1} \gamma_i + 1, \ldots, \sum_{i=1}^{k} \gamma_i\}$, then we have*

$$w_\gamma c_I \equiv \begin{cases} \pm w_\gamma & \text{if every } |I_k| \text{ is even} \\ 0 & \text{else} \end{cases} \quad \mod [\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}, \mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}]. \tag{2.15}$$

The sign is determined by the structure of each subset. Finally, specializing γ to a partition of n, we obtain the desired result.

2.2.2 Clifford reduction in types B_n and D_n

We can extend Proposition 2.2.4 to types B and D. The commutation relations between elements of W and elements of $\mathcal{C}_n$ in types B_n and D_n differs from that in type A_{n-1} only in that we have the extra relations $s_n c_n = -c_n s_n$ and $s_n c_n = -c_{n-1} s_n$, respectively. Let $w_{(n)} = s_1 \dots s_n$. We have the following versions of Lemmas 2.2.1 and 2.2.2:

Lemma 2.2.5. *In $\mathfrak{H}^{\mathfrak{c}}_{B_n}$ and $\mathfrak{H}^{\mathfrak{c}}_{D_n}$, we have*

$$\begin{aligned} w_{(n)} c_i &= c_{i+1} w_{(n)} \qquad \text{for } 1 \le i \le n-1, \\ w_{(n)} c_n &= -c_1 w_{(n)}. \end{aligned}$$

Lemma 2.2.6. *Let $X = B_n$ or D_n. For $I \subseteq [n]$ with $|I|$ even, we have*

$$w_{(n)} c_I \equiv \pm w_{(n)} \qquad \mod [\mathfrak{H}^{\mathfrak{c}}_X, \mathfrak{H}^{\mathfrak{c}}_X].$$

The proofs are identical, with an additional (-1) added in equation (2.14) if $n \in I$. We also have

Lemma 2.2.7. *Let $\gamma = (\gamma_1, \gamma_2)$ be a composition of n with $\gamma_1, \gamma_2 > 0$. Let $I_1 = \{i_1, \dots, i_a\} \subseteq \{1, \dots, \gamma_1\}$ and $I_2 = \{j_1, \dots, j_b\} \subseteq \{\gamma_1 + 1, \dots, \gamma_2\}$, and assume that $a + b$ is even. Then for $X = B_n$ or D_n, we have*

$$w_\gamma c_{I_1} c_{I_2} \equiv \begin{cases} 0 & a, b \text{ odd} \\ \pm w_\gamma & a, b \text{ even} \end{cases} \qquad \mod [\mathfrak{H}^{\mathfrak{c}}_X, \mathfrak{H}^{\mathfrak{c}}_X].$$

There are only two modifications to the proof. In Lemma 2.2.3, the only problem occurs if I_2 contains both $n-1$ and n, so that c_{I_2} ends with $\dots c_{n-1} c_n$. Then commuting c_{I_2} past w_{γ_2} gives in type D_n that

$$w_{\gamma_2} \dots c_{n-1} c_n = \dots s_{n-1} (c_n c_{n-1}) s_n = \dots (c_{n-1} c_n) w_\gamma = c_{I_2} w_{\gamma_2}$$

Hence there is no impact on the proof. In type B_n, there is a sign change which cancels out: we have

$$\begin{aligned} w_{\gamma_2} \dots c_{n-1} c_n &= \dots s_{n-1} (-c_{n-1} c_n) s_n \\ &= \dots (-c_n c_{n-1}) w_{\gamma_2} \\ &= \dots (c_{n-1} c_n) w_{\gamma_2}. \end{aligned}$$

Thus, we have the following proposition.

Proposition 2.2.8. *Let $X = B_n$ or D_n. If $\gamma = (\gamma_1, \dots, \gamma_k)$ is a composition of n, $I \subset [n]$ is an even subset, and $I_k = I \cap \{\sum_{i=1}^{k-1} \gamma_i + 1, \dots, \sum_{i=1}^{k} \gamma_i\}$, then we have*

$$w_\gamma c_I \equiv \begin{cases} \pm w_\gamma & \text{if every } |I_k| \text{ is even} \\ 0 & \text{else} \end{cases} \quad \mod \ [\mathfrak{H}^{\mathfrak{c}}_X, \mathfrak{H}^{\mathfrak{c}}_X]. \tag{2.16}$$

2.2.3 Conjugacy classes in type A_{n-1}

Though we can apply the reduction formulas from the previous section to remove Clifford algebra generators from our basis elements, they still restrict the Weyl group elements that can appear.

Let $\mathcal{OP}_n$ be the set of partitions of n with all odd parts. It is proved in [BW13] that $\mathcal{OP}_n$ parametrizes the even split conjugacy classes of $\mathbb{C}W$ in type A_{n-1}. These are the even conjugacy classes in $\mathbb{C}W$ which split into two separate conjugacy classes in the double cover $\mathbb{C}\widetilde{W}$. It is proved in [Józ88b] that the number of even split conjugacy classes is the number of simple $\mathbb{C}W^-$-modules, so we should expect the combinatorics of these classes to play a role in our bases.

Proposition 2.2.9. *If λ is a partition of n with $\lambda \notin \mathcal{OP}_n$ and $w \in S_n$ has cycle type λ, then $w \equiv 0 \mod [\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}, \mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}]$.*

Proof. Since elements which are conjugate in $\mathfrak{H}^{\mathfrak{c}}_A$ are congruent in the trace, we may take

$$w_\lambda = (1 \dots \lambda_1)(\lambda_1 + 1 \dots \lambda_1 + \lambda_2) \dots (\lambda_1 + \dots + \lambda_{n-1} + 1 \dots n).$$

Suppose that λ has an even part, and take it without loss of generality to be λ_1. Then

$$\begin{aligned} w_\lambda &\equiv c_1 \dots c_{\lambda_1} w_\lambda c_{\lambda_1} \dots c_1 \mod [\mathfrak{H}^{\mathfrak{c}}_A, \mathfrak{H}^{\mathfrak{c}}_A] \\ &= c_1 \dots c_{\lambda_1} c_1 c_{\lambda_1} \dots c_2 w_\lambda \\ &= (-1)^{\lambda_1 - 1} w_\lambda. \end{aligned}$$

In the last step, we have commuted one of the c_1's past each other Clifford element (a total of $\lambda_1 - 2$ inversions), after which each c_i cancels. Hence, we have $w_\lambda \equiv -w_\lambda \mod [\mathfrak{H}^{\mathfrak{c}}_A, \mathfrak{H}^{\mathfrak{c}}_A]_{\bar{0}}$, whence $w_\lambda \equiv 0 \mod [\mathfrak{H}^{\mathfrak{c}}_A, \mathfrak{H}^{\mathfrak{c}}_A]_{\bar{0}}$. □

2.2.4 Conjugacy classes in types B_n and D_n

Conjugacy classes in the Weyl group in type B_n correspond to bipartitions (λ, μ), $|\lambda| + |\mu| = n$ (cf. [Mac15]). For a partition λ, denote by $\ell(\lambda)$ the number of parts of λ. Let $\mathcal{OP}$ denote the set of partitions (of any n) with all odd parts, and $\mathcal{EP}$ denote the set of partitions with all even parts. The even split conjugacy classes of the spin Weyl group of type B_n are parametrized by bipartitions of n $(\lambda, \mu) \in (\mathcal{OP}, \mathcal{EP})$, cf. [BW13].

Proposition 2.2.10. *Let (λ, μ) be a bipartition of n and $w \in W_{B_n}$ an element in the conjugacy class corresponding to (λ, μ). If $(\lambda, \mu) \notin (\mathcal{OP}, \mathcal{EP})$, then $w \equiv 0 \mod [\mathfrak{H}^{\mathfrak{c}}_{B_n}, \mathfrak{H}^{\mathfrak{c}}_{B_n}]$.*

Proof. For a bipartition $(\lambda = (\lambda_1, \dots, \lambda_r), \mu = (\mu_1, \dots, \mu_s))$, let

$$w_{\lambda,\mu} = (1, \dots, \lambda_1) \dots (\sum_{j=1}^{r-1} \lambda_j + 1, \dots, |\lambda|)(|\lambda| + 1, \dots, |\lambda| + \mu_1) \dots (|\lambda| + \sum_{j=1}^{s-1} \mu_j + 1, \dots, n),$$

where the λ-cycles are understood to be positive, and the μ-cycles negative. We claim that unless $(\lambda,\mu)\in(\mathcal{OP},\mathcal{EP})$ with $\ell(\mu)$ even, $w_{\lambda,\mu}\equiv 0 \mod [\mathfrak{H}^{\mathfrak{c}}_{B_n},\mathfrak{H}^{\mathfrak{c}}_{B_n}]$. Indeed, if λ has even part λ_i, let $c=c_{\lambda_i+1}c_{\lambda_i+2}\dots c_{\lambda_{i+1}}$. Then, as in type A,

$$cw_{\lambda,\mu}c^{-1}=-w_{\lambda,\mu}.$$

If μ has an odd part μ_i, we may assume without loss of generality that it corresponds to a cycle containing n, adjusting $w_{\lambda,\mu}$ if necessary. Let $c=c_{\mu_i+1}\dots c_n$ (the length of c is μ_i-1). Then $cw_{\lambda,\mu}c^{-1}=(-1)^{\mu_i-1}cc^{-1}w_{\lambda,\mu}=-w_{\lambda,\mu}$.

For example, if $(\lambda,\mu)=(\{2\},\{3\})$, $w_{\lambda,\mu}=(12)(345)$. We have

$$\begin{aligned}c_1c_2(12)(345)c_2c_1&=c_1c_2c_1c_2(12)(345)\\&=-c_1^2c_2^2(12)(345)\\&=-(12)(345).\end{aligned}$$

Also,

$$\begin{aligned}c_3c_4c_5(12)(345)c_5c_4c_3&=(-1)c_3c_4c_5c_3c_5c_4(12)(345)\\&=(-1)^3c_3^2c_4^2c_5^2(12)(345)\\&=-(12)(345).\end{aligned}$$

Finally, if $\ell(\mu)$ is odd, conjugating by c_n yields $w_{\lambda,\mu}\equiv -w_{\lambda,\mu}$. □

Conjugacy classes in the Weyl group in type D_n also correspond to bipartitions. Let $\mathcal{SOP}$ denote the set of partitions (of any n) with distinct odd parts; the set $\mathcal{SOP}$ parametrizes the even split conjugacy classes of the spin Weyl group of type D_n, cf. [BW13].

Proposition 2.2.11. *Let (λ,μ) be a bipartition of n and $w\in W_{D_n}$ an element in the conjugacy class corresponding to (λ,μ). If n is odd and $(\lambda,\mu)\notin(\mathcal{OP},\mathcal{EP})$ with $\ell(\mu)$ even, then $w\equiv 0 \mod [\mathfrak{H}^{\mathfrak{c}}_{D_n},\mathfrak{H}^{\mathfrak{c}}_{D_n}]$. If n is even and $(\lambda,\mu)\notin(\mathcal{EP},\mathcal{OP})$ with $\ell(\mu)$ even and $(\lambda,\mu)\notin(\emptyset,\mathcal{SOP})$, then $w\equiv 0 \mod [\mathfrak{H}^{\mathfrak{c}}_{D_n},\mathfrak{H}^{\mathfrak{c}}_{D_n}]$.*

Proof. For a bipartition (λ,μ), let $w_{\lambda,\mu}$ be as above. If n is odd, we have that $w_{\lambda,\mu}\equiv 0$ mod $[\mathfrak{H}^{\mathfrak{c}}_{D_n},\mathfrak{H}^{\mathfrak{c}}_{D_n}]_{\bar{0}}$ unless $(\lambda,\mu)\in(\mathcal{OP},\mathcal{EP})$ with $\ell(\mu)$ even by the same arguments as in type B. If n is even and $(\lambda,\mu)\in(\emptyset,\mathcal{SOP})$, conjugation by c as above does not fix $w_{\lambda,\mu}$ up to sign, so this case does not vanish. □

2.3 Spanning sets of the trace in $\mathfrak{H}^{\mathfrak{c}}_X$

We pass to the associated graded object of the Hecke-Clifford algebra, which is isomorphic to the Hecke-Clifford algebra with parameter identically 0. We establish a spanning set of the trace in this case, and then lift it to $\mathrm{Tr}(\mathfrak{H}^{\mathfrak{c}}_X)$ using an algebraic argument.

2.3.1 Spanning set of $\mathrm{Tr}(\mathfrak{H}_X^0)$

Let C be a conjugacy class of W_J, $J \subseteq I$. We say that C is elliptic in W_J if $W_{J'} \cap C = \emptyset$ for all proper subsets $J' \subset J$. Any element of W_J which is a member of an elliptic conjugacy class is called an elliptic element in W_J.

Example. In $W_{A_{n-1}} = S_n$, there is a unique elliptic conjugacy class, which corresponds to the partition (n). The elliptic elements are the n-cycles. For any connected subset J of the root system, the elliptic elements in W_J are the $(|J|+1)$-cycles.

In W_{B_n} and W_{D_n}, the unique elliptic conjugacy class corresponds to the bipartition $(\emptyset, (n))$, and the elliptic elements are the negative n-cycles.

We say that two subsets $J_1, J_2 \subset I$ are W-equivalent, $J_1 \sim_W J_2$, if there exists a $w \in W$ such that $w(J_1) = J_2$. Set $\mathcal{I} = 2^I / \sim_W$, the set of equivalence classes of subsets of I for the equivalence relation $\sim_W$. For any conjugacy class C of W, set J_C to be the minimal element (with respect to cardinality) of $\mathcal{I}$ such that $C \cap J_C \neq \emptyset$– since there is exactly one element of $\mathcal{I}$ of each cardinality, such a J_C must exist. Note that if C is an elliptic conjugacy class, $J_C = I$.

Any $w \in C \cap J_C$ is by definition elliptic in W_{J_C}. Fix one such elliptic element, $w_C \in W_{J_C}$, and let ${}^J W^J$ be a set of minimal length representatives for $W_J/W\backslash W_J$. We have the following result due to [CH16] linking centralizers of elliptic elements in parabolic subgroups to the normalizers of the parabolic subgroups:

Proposition 2.3.1. *[CH16, Proposition 2.4.3] Let $J \subset I$ and let $w \in W_J$ be an elliptic element. Let $Z = \{z \in^J W^J | z(J) = J\}$. Then we have*

$$W_J C_W(w) = N_W(W_J) = W_J Z W_J.$$

Now we establish spanning sets of the trace in each type.

Recall that $S(V^2)$ is the subspace of $S(V)$ spanned by the squares of basis elements, that $\mathfrak{H}_X^0$ is the affine Hecke-Clifford algebra of type X with parameter identically 0, that $\mathrm{gr}(\mathfrak{H}_X^{\mathfrak{c}}) \cong \mathfrak{H}_X^0$. Thus we have $\mathfrak{H}_X^0 \cong \mathbb{C}W \ltimes (\mathcal{C}_n \otimes S(V^2))$ as $\mathbb{C}$-algebras. Hence we certainly have $\mathrm{Tr}(\mathfrak{H}_X^0) \subset \mathrm{span}\{wc^\alpha S(V^2)\}$, where $w \in W$ and $\alpha \in \mathbb{Z}_2^n$.

Proposition 2.3.2. *We have* $\mathrm{Tr}(\mathfrak{H}_X^0) = \mathrm{span}\{wS(V^2)\}$.

Proof. Apply Proposition 2.2.3 and the corresponding results for types B_n and D_n to each element in the spanning set. Every element will thus either be congruent to 0 or to an element in $w_C S(V^2)$ for some C in the trace. □

Next, let $x, y \in W$ and $f \in S(V^2)$. Then

$$xyx^{-1}f \equiv yx^{-1}fx = yf^{x^{-1}} \mod [\mathfrak{H}_X^0, \mathfrak{H}_X^0]$$

where f^σ denotes the action of σ on f by conjugation. Hence $\mathrm{span}\{wS(V^2)\} = \{w_C S(V^2)\}$, where C is the conjugacy class of w and w_C is a representative.

Now we restrict the conjugacy classes of Weyl group elements which may appear.

Proposition 2.3.3. *1. We have* $\mathrm{Tr}(\mathfrak{H}^0_{A_{n-1}}) = \mathrm{span}\{w_\lambda S(V^2)\}_{\lambda\in\mathcal{OP}_n}$.

2. *We have* $\mathrm{Tr}(\mathfrak{H}^0_{B_n}) = \mathrm{span}\{w_{\lambda,\mu} S(V^2)\}_{(\lambda,\mu)\in(\mathcal{OP},\mathcal{EP}),\ \ell(\mu)\ \mathit{even}}$.

3. *If n is odd,* $\mathrm{Tr}(\mathfrak{H}^0_{D_n}) = \mathrm{span}\{w_{\lambda,\mu} S(V^2)\}_{(\lambda,\mu)\in(\mathcal{OP},\mathcal{EP}),\ \ell(\mu)\ \mathit{even}}$. *If n is odd,* $\mathrm{Tr}(\mathfrak{H}^0_{D_n}) =$ $\mathrm{span}\{w_{\lambda,\mu} S(V^2)\}$ *with* $(\lambda,\mu) \in (\mathcal{OP},\mathcal{EP})$ *or* $(\lambda,\mu) \in (\emptyset, \mathcal{SOP})$, $\ell(\mu)$ *even.*

Proof. By Propositions 2.2.9, 2.2.10, and 2.2.11, every element not in these conjugacy classes is congruent to 0 in the trace, so removing them from a set does not change the span. □

Finally, we restrict to a subspace of the symmetric algebra.

Proposition 2.3.4. *Fix a conjugacy class C of W, and let $J = J_C$. Then we have*

$$w_C S(V^2) \equiv w_C S((V^2)^{W_J})^{N_W(W_J)} \mod [\mathfrak{H}^0_X, \mathfrak{H}^0_X].$$

Proof. We follow [CH16, Section 6]. We have $V^2 = (V^2)^{W_J} \oplus U$ as a W_J-module, where U is spanned by $\{x_i^2 | i \in J\}$. Since w_C is elliptic in W_J, it acts faithfully on U, so $1 - w_C$ is invertible on U. Let $f \in S(V^2)$ and $u \in U$. Since $1 - w_C$ has full rank, there exists a $v \in U$ such that $v - w_C(v) = u$. Thus, we see that

$$uw_C f = vw_C f - w_C(v) w_C f = vw_C f - w_C f v = [v, w_C f].$$

Hence $U w_C S(V^2) \in [\mathfrak{H}^0_X, \mathfrak{H}^0_X]$. Therefore we have

$$w_C S(V^2) = w_C S(U) S((V^2)^{W_J}) = S(U) w_C S((V^2)^{W_J}) \subset w_C S((V^2)^{W_J}) + [\mathfrak{H}^0_X, \mathfrak{H}^0_X].$$

Let $f \in S((V^2)^{W_J})$ and $x \in C_W(w_C)$. Then we have

$$\begin{aligned} w_C f \equiv x w_C f x^{-1} &= w_C x f x^{-1} \\ &= w_C f^x \mod [\mathfrak{H}^0_X, \mathfrak{H}^0_X]. \end{aligned}$$

Hence we can average over the centralizer of w_C to obtain that

$$w_C f = \frac{1}{|C_W(w_C)|} \sum_{x \in C_W(w_C)} w_C f^x \in w_C S((V^2)^{W_J})^{C_W(w_J)}.$$

Finally, apply Proposition 2.3.1 to get

$$w_C f \in w_C S((V^2)^{W_J})^{N_W(W_J)}$$

using the fact that $S((V^2)^{W_J})^{C_W(w)} = S((V^2)^{W_J})^{W_J C_W(w)}$. □

For each C, let $\{f_{J_C;i}\}$ be a basis of the vector space $S((V^2)^{W_{J_C}})^{N_W(W_{J_C})}$. Propositions 2.3.3 and 2.3.4 give us the following.

Proposition 2.3.5. *1. The set* $\{w_\lambda f_{J_\lambda;i}\}_{\lambda\in\mathcal{OP}_n}$ *spans* $\mathrm{Tr}(\mathfrak{H}^0_{A_{n-1}})$.

2. *The set* $\{w_{\lambda,\mu} f_{J_{\lambda,\mu};i}\}_{(\lambda,\mu)\in(\mathcal{OP},\mathcal{EP}),\ \ell(\mu)\ \mathit{even}}$ *spans* $\mathrm{Tr}(\mathfrak{H}^0_{B_n})$.

3. *If n is odd, the set* $\{w_{\lambda,\mu} f_{J_{\lambda,\mu};i}\}_{\substack{(\lambda,\mu)\in(\mathcal{OP},\mathcal{EP}),\\ \ell(\mu)\ \mathit{even}}}$ *spans* $\mathrm{Tr}(\mathfrak{H}^0_{D_n})$.

   If n is even, the set $\{w_{\lambda,\mu} f_{J_{\lambda,\mu};i}\}$ *with* $(\lambda,\mu) \in (\mathcal{OP},\mathcal{EP})$ *or* $(\lambda,\mu) \in (\emptyset, \mathcal{SOP}_n)$ *with* $\ell(\mu)$ *even spans* $\mathrm{Tr}(\mathfrak{H}^0_{D_n})$.

2.3.2 Spanning set of $\mathrm{Tr}(\mathfrak{H}_X^{\mathfrak{c}})$

The goal of this section is to lift the spanning set constructed above for $\mathrm{Tr}(\mathfrak{H}_X^0)$ to $\mathrm{Tr}(\mathfrak{H}_X^{\mathfrak{c}})$. The proof is motivated by [CH16, Section 6.2], with appropriate modifications.

Lemma 2.3.6. *If S spans $\mathrm{Tr}(\mathfrak{H}_X^0)$, then its image in $\mathrm{Tr}(\mathfrak{H}_X^{\mathfrak{c}})$ spans $\mathrm{Tr}(\mathfrak{H}_X^{\mathfrak{c}})$.*

Proof. We proceed by induction on degree (the base case being precisely $\mathfrak{H}_X^0$). Commutators preserve degree, and, in particular, if f_1 and f_2 are homogenous elements of $S(V)$ of degree k and j, respectively and $\epsilon_1, \epsilon_2 \in \mathbb{Z}_2^n$, then the top degree term of $[w_1c^{\epsilon_1}x_1, w_2c^{\epsilon_2}x_2]$ is given by

$$y := \pm w_1 w_2 {w_2}^{-1} x_1 x_2 w_2^{-1}(c^{\epsilon_1})c^{\epsilon_2} - w_2 w_1 w_1^{-1}(x_2) x_1 w_1^{-1}(c^{\epsilon_2})c^{\epsilon_1} \tag{2.17}$$

where the signs are determined by the number of nontrivial c_i crossing over x_i terms in f_1 or f_2. This has degree $j+k$. Hence we have

$$[w_1c^{\epsilon_1}f_1, w_2c^{\epsilon_2}f_2] \in y + \mathcal{F}^{j+k-1}.$$

It suffices to show that we can write homogenous elements of $\mathfrak{H}_X^{\mathfrak{c}}$ as a linear combination of elements in S, commutators, and elements of lower degree. Let $h \in \mathfrak{H}_X^{\mathfrak{c}}$ be homogenous of degree k, and write $h = \sum_w a_w w$, $a_w \in S(V)$. Let $h_0 = \sum_w a_w w$ be the corresponding element in $\mathfrak{H}_X^0$. We have a spanning set for $\mathfrak{H}_X^0$, so we may write

$$h_0 = \sum_{x \in S} c_x x_0$$

where x is the element of $\mathfrak{H}_X^{\mathfrak{c}}$ represented by x_0. Without loss of generality, we may choose these representatives to have maximal degree in $\mathfrak{H}_X^{\mathfrak{c}}$. Hence we write

$$h_0 = \sum_{x\in S} c_x x_0 + \sum_i [a_{0,i}, b_{0,i}] \quad a_{0,i}b_{0,i} \in \mathfrak{H}_X^0, [a_{0,i}, b_{0,i}] \in \mathcal{F}^k. \tag{2.18}$$

Here $a_{0,i}$ and $b_{0,i}$ are representatives of some $a_i, b_i \in \mathfrak{H}_X^{\mathfrak{c}}$, with $[a_i, b_i] \in \mathcal{F}^k$ for all i. By (2.17), we have $[a_i, b_i] - [a_{0,i}, b_{0,i}] \in \mathcal{F}^{k-1}$. Then $h - \sum_{x\in S} c_x x - [a_i, b_i] \in \mathcal{F}^{k-1}$, i.e. the difference between h and its corresponding element in $\mathfrak{H}_X^0$ has degree less than k. Thus we can write h as a linear combination of elements in S up to an element of $\mathcal{F}^{k-1}$; by induction, we are done. □

The following is an immediate consequence of Lemma 2.3.6 and Proposition 2.3.5.

Proposition 2.3.7. *1. The set $\{w_\lambda f_{J_\lambda;i}\}_{\lambda\in\mathcal{OP}_n}$ spans $\mathrm{Tr}(\mathfrak{H}_{A_{n-1}}^{\mathfrak{c}})$.*

2. The set $\{w_{\lambda,\mu} f_{J_{\lambda,\mu;i}}\}_{(\lambda,\mu)\in(\mathcal{OP},\mathcal{EP})}$ spans $\mathrm{Tr}(\mathfrak{H}_{B_n}^{\mathfrak{c}})$.

3. If n is odd, the set $\{w_{\lambda,\mu} f_{J_{\lambda,\mu;i}}\}_{\substack{(\lambda,\mu)\in(\mathcal{OP},\mathcal{EP}),\\ \ell(\mu)\ \text{even}}}$ spans $\mathrm{Tr}(\mathfrak{H}_{D_n}^{\mathfrak{c}})$.

If n is even, the set $\{w_{\lambda,\mu} f_{J_{\lambda,\mu;i}}\}$ with $(\lambda,\mu) \in (\mathcal{OP},\mathcal{EP})$ or $(\lambda,\mu) \in (\emptyset, \mathcal{SOP}_n)$ with $\ell(\mu)$ even spans $\mathrm{Tr}(\mathfrak{H}_{D_n}^{\mathfrak{c}})$.

2.4 Linear independence in $\mathfrak{H}_X^{\mathfrak{c}}$

We now show that our sets from Proposition 2.3.7 are linearly independent, and hence form bases. We first require some results about induction and restriction functors in $\mathfrak{H}_X^{\mathfrak{c}} - \mathrm{mod}$ in order to prove a trace formula for parabolically induced $\mathfrak{H}_X^{\mathfrak{c}}$-modules. The trace formula will allow us to separate the elements of the spanning sets.

Define the trace pairing $\mathrm{Tr} : \mathfrak{H}_X^{\mathfrak{c}} \times R(\mathfrak{H}_X^{\mathfrak{c}}) \to \mathbb{C}$ by

$$\mathrm{Tr}(h, \pi) = \mathrm{tr}\, \pi(h).$$

For each $J \subset I$, let $i_J : (\mathfrak{H}_X^{\mathfrak{c}})_J \to \mathfrak{H}_X^{\mathfrak{c}}$ be the inclusion. We define r_J as in [CH16]: for $h \in \mathfrak{H}_X^{\mathfrak{c}}$, let ψ_h be the right $(\mathfrak{H}_X^{\mathfrak{c}})_J$-module morphism given by left multiplication by h. Set $r_J(h) = \mathrm{tr}(\psi_h)$ - we can define $\mathrm{tr}\, \psi_h$ because $\mathfrak{H}_X^{\mathfrak{c}}$ is a free right $(\mathfrak{H}_X^{\mathfrak{c}})_J$-module with finite basis W^J. We record two results from [CH16] needed to prove a trace formula. The following is [CH16, Lemma 5.5.1], and the proof extends easily to the Hecke-Clifford case.

Lemma 2.4.1. *Let $J \subset I$.*

1. *For each $h \in (\mathfrak{H}_X^{\mathfrak{c}})_J$ and $\pi \in R(\mathfrak{H}_X^{\mathfrak{c}})$, we have $\mathrm{Tr}(i_J(h), \pi) = \mathrm{Tr}(h, r_J(\pi))$.*
2. *For each $h \in \mathfrak{H}_X^{\mathfrak{c}}$ and $\pi \in R((\mathfrak{H}_X^{\mathfrak{c}})_J)$, we have $\mathrm{Tr}(h, i_J(\pi)) = \mathrm{Tr}(r_j(h), \pi)$.*

Note that i_J and r_J restrict to well-defined maps $\overline{i_J}$ and $\overline{r_J}$, respectively, on the trace; the lemma holds for these maps as well. We also have the following formula for $r_J(wf)$.

Lemma 2.4.2. *[CH16, Proposition 6.3.1] Let $J, J' \subset I$. Let $w \in W_J$ be elliptic and let C be the conjugacy class of w in W. Then, for any $f \in S((V^2)^{W_J})^{N_W(W_J)}$, we have*

$$\overline{r_{J'}}(wf) = \sum_{x \in {}^J W^{J'}, x^{-1}(J) \subset J'} x^{-1} \circ (wf).$$

Remark Note that if $x^{-1}wx \in W_{J'}$ for $x \in W$, we must have that $x \in^J W^{J'}$ and that $x^{-1}(J) \subset J'$ since w is an elliptic element. Conversely, if $x \in^J W^{J'}$ satisfies $x^{-1}(J) \subset J'$, it must also be true that $xwx^{-1} \in W_{J'}$, again because w is elliptic. Hence, if $C \cap W_{J'} = \emptyset$, the above sum is empty, so $r_J(wf) = 0$ for any f.

Proposition 2.4.3. *Let $J, J' \subset I$, let $w \in W_J$ be elliptic and let C be the conjugacy class of w in W. Let M be an $(\mathfrak{H}_X^{\mathfrak{c}})_{J'}$-module and $f \in S((V^2)^{W_J})^{N_W(W_J)}$. Then we have*

$$\mathrm{Tr}(wf, \mathrm{Ind}_{(\mathfrak{H}_X^{\mathfrak{c}})_{J'}}^{\mathfrak{H}_X^{\mathfrak{c}}} M) = \begin{cases} 0 & \textit{if } C \cap W_{J'} = \emptyset \\ |N_W(W_J)/W_J| \, \mathrm{Tr}(wf, M) & \textit{else.} \end{cases}$$

Proof. If $C \cap W_{J'} = \emptyset$, the statement follows by Lemma 2.4.2 and Remark 2.4. Assume $C \cap W_{J'} \neq \emptyset$; since w is elliptic in W_J, J must be J_C. By Remark 2.4, there exists an $x \in^J W^{J'}$ such that $x^{-1}(J) \subset J'$, so we must have $|J| \leq |J'|$. Thus $J \sim J'$ by the minimality of J_C with respect to cardinality. By Proposition 2.3.1,

$$|\{z \in^J W^{J'} | z^{-1}(J) = J'\}| = |\{z \in^J W^J | z^{-1}(J) = J\}| = |N_W(W_J)/W_J|$$

where the first equality follows because the second set is sent to the first by x^{-1}. The proposition now follows from Lemma 2.4.2. □

Recall that a conjugacy class C of W, J_C is a minimal subset of I such that $C \cap W_{J_C} \neq \emptyset$, w_C is an elliptic element in W_{J_C}, and $\{f_{J_C,i}\}$ is a basis of the vector space $S((V^2)^{W_{J_C}})^{N_W(W_{J_C})}$. The following is the first main result of the chapter.

Theorem 2.4.4. *For $X = A_{n-1}$, B_n, or D_n, the spanning set of* $\mathrm{Tr}(\mathfrak{H}_X^{\mathfrak{c}})$ *given in Proposition 2.3.7 is linearly independent, and hence forms a basis of* $\mathrm{Tr}(\mathfrak{H}_X^{\mathfrak{c}})$.

Proof. We proceed by induction on cardinality of subsets $J \subset I$. Our goal is to apply Proposition 2.4.3 to separate the elements of the spanning set into linearly independent subsets. Suppose that

$$\sum_{C,i} a_{C,i} w_C f_{J_C;i} = 0$$

where $a_{C,i} \in \mathbb{C}$. First, set $J = \emptyset$. We have $(\mathfrak{H}_X^{\mathfrak{c}})_\emptyset = S(V)$, so every character of $(\mathfrak{H}_X^{\mathfrak{c}})_\emptyset$ is parametrized by an element $v \in V^\vee$. Fix such a v and its corresponding character χ_v, and consider $\mathrm{Ind}_{(\mathfrak{H}_X^{\mathfrak{c}})_\emptyset}^{\mathfrak{H}_X^{\mathfrak{c}}}(\chi_v)$. Since $W_\emptyset = \{1\}$, by Proposition 2.4.3,

$$\mathrm{Tr}\left(w_C f_{J_C;i}, \mathrm{Ind}_{(\mathfrak{H}_X^{\mathfrak{c}})_\emptyset}^{\mathfrak{H}_X^{\mathfrak{c}}}(\chi_v)\right) = 0$$

for all $C \neq \{1\}$ and all i. Thus, we have

$$\begin{aligned}\mathrm{Tr}\left(\sum_{C,i} a_{C,i} w_C f_{J_C;i}, \mathrm{Ind}_{(\mathfrak{H}_X^{\mathfrak{c}})_\emptyset}^{\mathfrak{H}_X^{\mathfrak{c}}}(\chi_v)\right) &= \mathrm{Tr}\left(\sum_{i} a_{1,i} f_{\emptyset;i}, \mathrm{Ind}_{(\mathfrak{H}_X^{\mathfrak{c}})_\emptyset}^{\mathfrak{H}_X^{\mathfrak{c}}}(\chi_v)\right)\\ &= |W| \sum_i a_{1,i}(f_{\emptyset;i}, v)\\ &= 0.\end{aligned}$$

Hence $\sum_i a_{1,i} f_{\emptyset;i}$ vanishes on each W orbit of $V^\vee$, so the polynomial function $\sum a_{1,i} f_{\emptyset;i}$ vanishes on its entire domain. Thus

$$\sum_i a_{1,i} f_{\emptyset,i} = 0.$$

But $\{f_{\emptyset;i}\}$ was taken to be a basis. Hence we must have $a_{1,i} = 0$ for all i.

Now let $J \subset I$. We assume by induction that $a_{C',i'} = 0$ for all i' and for every $J_{C'} \subset J$ (up to equivalence with respect to $\sim$). For any module M of $(\mathfrak{H}_X^{\mathfrak{c}})_J$, the nonzero summands of $\mathrm{Tr}\left(\sum_{C,i} a_{C,i} w_C f_{J_C;i}, M\right)$ are parametrized by those C such that $J_C \cap J \neq \emptyset$; by the induction hypothesis, we can assume that $J_C = J$ for all such C. Let M be an irreducible $(\mathfrak{H}_X^{\mathfrak{c}})_J$-module, with irreducible character χ_v parametrized by $v \in N_W(W_J)/(V^\vee)^{W_J}$. Then applying $\mathrm{Tr}(-, \mathrm{Ind}_{(\mathfrak{H}_X^{\mathfrak{c}})_J}^{\mathfrak{H}_X^{\mathfrak{c}}} M)$ to the linear combination gives that

$$\begin{aligned}\mathrm{Tr}\left(\sum_{J_C=J,i} a_{C,i} w_C f_{J_C;i}, \mathrm{Ind}_{(\mathfrak{H}_X^{\mathfrak{c}})_J}^{\mathfrak{H}_X^{\mathfrak{c}}} M\right) &= |N_W(W_J)/W_J| \,\mathrm{Tr}\left(\sum_{J_C=J,i} a_{C,i} w_C f_{J_C;i}, M\right)\\ &= |N_W(W_J)/W_J| \sum_{J_C=J,i} a_{C,i}(f_{J_C;i}, v),\end{aligned}$$

by Proposition 2.4.3. By hypothesis, we therefore have

$$|N_W(W_J)/W_J| \sum_{J_C=J,i} a_{C,i}(f_{J_C,i}, v) = 0.$$

As above, this implies that the polynomial $\sum a_{C,i} f_{J_C;i}$ vanishes on its domain, contradicting the linear independence of $\{f_{J_C;i}\}$. Hence, $a_{C,i} = 0$ for all $J_C = J$ (up to equivalence with respect to $\sim$) and all i. By induction $a_{C,i} = 0$ for all C and all i, as desired.

□

2.5 Degenerate spin affine Hecke algebras

We now aim to develop a result on bases of traces analogous to Theorem 2.4.4 for a closely related class of algebras, the degenerate spin affine Hecke algebras.

2.5.1 The skew polynomial algebra

Let $\mathcal{C}\langle b_1, \ldots, b_n\rangle$ be the algebra generated by $b_1, \ldots, b_n$ subject to the relations

$$b_i b_j + b_j b_i = 0 \ (i \neq j). \tag{2.19}$$

This is the *skew polynomial algebra*. It has a subalgebra $\mathcal{C}\langle b_1^2, \ldots, b_n^2\rangle$; these algebras will take the place of $S(V)$ and $S(V^2)$, respectively, in our discussion of spin affine Hecke algebras.

The skew polynomial algebra has a natural superalgebra structure by letting each b_i be odd.

2.5.2 Spin Weyl groups

Let W be a finite Weyl group. There is a distinguished double cover $\widetilde{W}$ for W:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{W} \longrightarrow W \longrightarrow 1. \tag{2.20}$$

We denote by $\mathbb{Z}_2 = \{1, z\}$, and by $\tilde{t}_i$ a fixed preimage of the generators s_i of W for each i. The group $\widetilde{W}$ is generated by $z, \tilde{t}_1, \ldots, \tilde{t}_n$ with relations

$$z^2 = 1, \qquad (\tilde{t}_i\tilde{t}_j)^{m_{ij}} = \begin{cases} 1, & \text{if } m_{ij} = 1, 3 \\ z, & \text{if } m_{ij} = 2, 4, 6. \end{cases}$$

The quotient algebra $\mathbb{C}W^- := \mathbb{C}\widetilde{W}/\langle z + 1\rangle$ of $\mathbb{C}\widetilde{W}$ by the ideal generated by $z + 1$ is called the spin Weyl group algebra associated to W. Denote by $t_i \in \mathbb{C}W^-$ the image of $\tilde{t}_i$. The spin Weyl group algebra $\mathbb{C}W^-$ has the following presentation: $\mathbb{C}W^-$ is the algebra generated by $t_i, 1 \leq i \leq n$, subject to the relations

$$(t_i t_j)^{m_{ij}} = (-1)^{m_{ij}+1} \equiv \begin{cases} 1, & \text{if } m_{ij} = 1, 3 \\ -1, & \text{if } m_{ij} = 2, 4, 6. \end{cases} \tag{2.21}$$

The algebra $\mathbb{C}W^-$ is naturally a superalgebra by letting each t_i be odd.

In particular, let W be the Weyl group of type A_{n-1}, B_n, or D_n. Then the spin Weyl group algebra $\mathbb{C}W^-$ is generated by $t_1, \dots, t_n$ with the labeling as in the Coxeter-Dynkin diagrams (cf. (2.2)) and the explicit relations:

Type of W	Defining Relations for $\mathbb{C}W^-$
A_{n-1}	$t_i^2 = 1$, $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$, $(t_i t_j)^2 = -1$ if $\lvert i-j\rvert > 1$
B_n	$t_1, \dots, t_{n-1}$ satisfy the relations for $\mathbb{C}W^-_{A_{n-1}}$, $t_n^2 = 1, (t_i t_n)^2 = -1$ if $i \neq n-1, n$, $(t_{n-1} t_n)^4 = -1$
D_n	$t_1, \dots, t_{n-1}$ satisfy the relations for $\mathbb{C}W^-_{A_{n-1}}$, $t_n^2 = 1, (t_i t_n)^2 = -1$ if $i \neq n-2, n$, $t_{n-2} t_n t_{n-2} = t_n t_{n-2} t_n$

2.5.3 The degenerate spin affine Hecke algebra

Let $W = W_{A_{n-1}}$. The degenerate spin affine Hecke algebra of type A_{n-1}, $\mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}$, was constructed in [Wan09]. It is the $\mathbb{C}$-algebra generated by $\mathcal{C}\langle b_1, \dots, b_n\rangle$ and $\mathbb{C}W^-$ subject to the additional relations:

$$b_{i+1} t_i + t_i b_i = 1 \tag{2.22}$$

$$t_j b_i + b_i t_j = 0, \quad (i \neq j, j+1). \tag{2.23}$$

Note that $\mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}$ contains the skew polynomial algebra and $\mathbb{C}W^-$ as subalgebras. The algebra $\mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}$ has a superalgebra structure with all generators being odd.

The degenerate spin affine Hecke algebraas in types B_n and D_n were constructed in [KW08]. Let $W = W_{D_n}$. The degenerate spin affine Hecke algebra of type D_n, $\mathfrak{H}^{\mathsf{sp}}_{D_n}$, is is the algebra generated by $\mathcal{C}\langle b_1, \dots, b_n\rangle$ and $\mathbb{C}W^-$ subject to the additional relations:

$$\begin{aligned} t_i b_i + b_{i+1} t_i &= 1 \quad (1 \le i \le n-1) \\ t_i b_j &= -b_j t_i \quad (j \neq i, i+1,\ 1 \le i \le n-1) \\ t_n b_n + b_{n-1} t_n &= 1 \\ t_n b_i &= -b_i t_n \quad (i \neq n-1, n). \end{aligned}$$

In particular, the subalgebra generated by $t_1, \dots, t_{n-1}$ and $b_1, \dots, b_n$ is isomorphic to $\mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}$.

Finally, let $u \in \mathbb{C}$ and let $W = W_{B_n}$. Then $\mathfrak{H}^{\mathsf{sp}}_{B_n}$ is the algebra generated by $\mathcal{C}\langle b_1, \dots, b_n\rangle$ and $\mathbb{C}W^-$ subject to the following relations:

$$\begin{aligned} t_i b_i + b_{i+1} t_i &= 1 \quad (1 \le i \le n-1) \\ t_i b_j &= -b_j t_i \quad (j \neq i, i+1,\ 1 \le i \le n-1) \\ t_n b_n + b_n t_n &= u \end{aligned}$$

$$t_n b_i = -b_i t_n \quad (i \neq n).$$

These algebras have a PBW property, and we have a description of their even centers.

Proposition 2.5.1. *[Wan09],[KW08]*

1. *Let $X = A_{n-1},\ D_n$, or B_n. The multiplication of the subalgebras $\mathbb{C}W^-$ and $\mathbb{C}[b_1, \dots, b_n]$ induces a vector space isomorphism*

$$\mathcal{C}\langle b_1, \dots, b_n\rangle \otimes \mathbb{C}W_X^- \xrightarrow{\sim} \mathfrak{H}_X^{\mathsf{sp}}.$$

2. *Let $X = A_{n-1}, B_n$ or D_n. The even center of $\mathfrak{H}_X^{\mathsf{sp}}$ is isomorphic to $\mathcal{C}\langle b_1^2, \dots, b_n^2\rangle^{W_X}$.*

2.5.4 A Morita superequivalence

The degenerate spin affine Hecke algebras are closely related to the degenerate affine Hecke-Clifford algebras, via the following isomorphism.

Proposition 2.5.2. *[Wan09],[KW08] Let $X = A_{n-1},\ D_n$, or B_n. Then there exists an isomorphism of superalgebras*

$$\Phi : \mathfrak{H}_X^{\mathfrak{c}} \xrightarrow{\sim} \mathcal{C}_V \otimes \mathfrak{H}_X^{\mathsf{sp}}.$$

Since $\mathcal{C}_V$ is a simple superalgebra, this isomorphism defines a Morita superequivalence in the sense of [Wan09]. In particular, when n is even, $\mathcal{C}_V \cong M(2^{n-1}|2^{n-1})$, and we have a usual Morita equivalence. When n is odd, $\mathcal{C}_V \cong Q(2^{n-1})$, and the categories $\mathfrak{H}_X^{\mathfrak{c}}$-smod and $\mathfrak{H}_X^{\mathsf{sp}}$-smod are equivalent up to a parity shift.

In the non-$\mathbb{Z}_2$ graded setting, a Morita equivalence $A \to B$ induces an isomorphism of traces $\overline{A} \xrightarrow{\sim} \overline{B}$, because $\overline{X} \cong \mathrm{HH}_0(X)$ for any algebra X (where HH_* is Hochschild homology) and Hochschild homology is Morita-invariant(cf. [Kas04]). This result does not extend directly to the superalgebra case - if n is odd, the Morita superequivalence does not necessarily preserve homology - but the superequivalence of $\mathfrak{H}_X^{\mathfrak{c}}$ and $\mathfrak{H}_X^{\mathsf{sp}}$ nonetheless suggests a connection between their traces.

We aim to compute a basis of

$$\overline{\mathfrak{H}_X^{\mathsf{sp}}} = \left(\frac{\mathfrak{H}_X^{\mathsf{sp}}}{[\mathfrak{H}_X^{\mathsf{sp}}, \mathfrak{H}_X^{\mathsf{sp}}]}\right)_{\bar{0}}.$$

using methods similar to the Hecke-Clifford case. As a consequence, we will show that the traces are in fact isomorphic in this case.

2.6 Reduction for the spin affine Hecke algebra

This section is analogous to section 3. We prove a variety of reduction results to restrict the types of Weyl group elements that appear in the trace.

2.6.1 Reduction in type A_{n-1}

We adapt the procedure in [WW12a], where a basis for the space of trace functions for the spin Hecke algebra in type A_{n-1} is computed. For $w \in S_n$, fix a reduced expression $w = s_{i_1} \dots s_{i_n}$. Denote $t_w = t_{i_1} \dots t_{i_n} \in \mathbb{C}W^-$. As in section 3, set $t_{(n)} = t_1 t_2 \dots t_{n-1}$, and for $\gamma = (\gamma_1, \dots, \gamma_\ell)$ a composition of n, let $t_\gamma = t_{\gamma_1} t_{\gamma_2} \dots t_{\gamma_\ell}$.

Proposition 2.6.1. *Let $\gamma = (\gamma_1, \dots, \gamma_\ell)$ be a composition of n with $\ell(t_\gamma)$ even and let μ be the partition of n corresponding to γ. Then we have*

$$t_\gamma \equiv \begin{cases} 0, & \text{if } \mu \notin \mathcal{OP}_n \\ \pm t_\mu, & \text{if } \mu \in \mathcal{OP}_n \end{cases} \mod [\mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}, \mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}].$$

Proof. Suppose $\mu \notin \mathcal{OP}_n$. Let a be the least integer such that γ_a is even, and let b be the least integer such that $b > a$ and γ_b is even; such a b must exist because $\ell(t_\gamma)$ is even. Set $t_{y,k} = t_{\gamma_1 + \dots + \gamma_{k-1} + 1} \dots t_{\gamma_1 + \dots + \gamma_k}$ (the cycle corresponding to γ_k in t_γ). Thus $t_\gamma = t_{\gamma,1} \dots t_{\gamma,\ell}$. Commuting $t_{\gamma,k}$ over $t_{\gamma,j}$ results in a sign of $(-1)^{(j-1)(k-1)}$, which is negative only if j and k are both even. Thus we have

$$\begin{aligned} t_\gamma &\equiv t_{\gamma,a} t_{\gamma,a+1} \dots t_{\gamma,\ell} t_{\gamma,1} \dots t_{\gamma,a-1} \\ &= t_{\gamma,a} t_{\gamma_b} t_{\gamma,a+1} \dots t_{\gamma,a-1} \\ &= -t_{\gamma,b} t_{\gamma,a} \dots t_{\gamma,a-1} \\ &\equiv -t_{\gamma,a} \dots t_{\gamma,a-1} t_{\gamma,b} \\ &= -t_\gamma \end{aligned}$$

where the equivalences are mod $[\mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}, \mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}]$. Hence $t_\gamma \equiv 0 \mod [\mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}, \mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}]$.

If $\mu \in \mathcal{OP}_n$, the images of t_γ and t_μ in $\overline{\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}}$ are equal; since the isomorphism Φ restricts to an injective map on $\overline{\mathbb{C}W^-}$, they must be equal in $\overline{\mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}}$ as well. □

Proposition 2.6.2. *Let w_C be a minimal length representative of a conjugacy class C corresponding to the cycle type $\mu = (\mu_1, \dots, \mu_\ell) \vdash n$. Then we have*

$$t_{w_C} \equiv \begin{cases} \pm t_\mu, & \text{if } \mu \in \mathcal{OP}_n \\ 0, & \text{otherwise} \end{cases} \mod [\mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}, \mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}].$$

Proof. The minimal length representative must have the form

$$w_C = (s_{i^1_1} s_{i^1_2} \dots s_{i^1_{\gamma_1 - 1}})(s_{i^2_1} s_{i^2_2} \dots s_{i^2_{\gamma_2 - 1}}) \dots (s_{i^\ell_1} \dots s_{i^\ell_{\gamma_\ell - 1}})$$

where $\gamma = (\gamma_1, \dots, \gamma_\ell)$ is a composition of n given by rearragning the parts of μ, and $i^k_j = \gamma_1 + \dots + \gamma_{k-1} + j$. We claim that $t_{w_C} \equiv \pm t_\gamma \mod [\mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}, \mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}]$; the lemma will then follow from Proposition 2.6.1.

It suffices to consider the case $\gamma = (n)$ (by dealing with each cycle separately). Thus $w_C = s_{i_1} \dots s_{i_{n-1}}$. If $i_j = j$ for all $1 \le j \le n-1$, then $w_C = w_\gamma$. Otherwise, there is at least one a such that $i_a \neq a$; choose the smallest such a (note that we must have $i_a > a$). We proceed by induction on a. Observe that

$$t_{w_C} = (-1)^{a-1} t_{i_a} t_1 t_2 \dots t_{a-1} t_{i_{a+1}} \dots t_{i_{n-1}}$$

$$\equiv (-1)^{a-1} t_1 t_2 \dots t_{a-1} t_{i_{a+1}} \dots t_{i_{n-1}} t_{i_a} \mod [\mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}, \mathfrak{H}^{\mathsf{sp}}_{A_{n-1}}] \tag{2.24}$$

Now $t_{i_{a+1}}$ is in the ath position; repeat this process until $t_{i_{a+k}} = t_a$ is in the ath position.

Hence

$$t_{w_C} = \pm t_1 t_2 \dots t_a t_{i'_{a+1}} \dots t_{i'_{n-1}}.$$

By the inductive hypothesis, we are done. □

2.6.2 Reduction in types B_n and D_n

Set $t^-_{(n)} = t_1 t_2 \dots t_{n-1} t_n = t_{(n)} t_n$. For compositions $\gamma = (\gamma_1, \dots, \gamma_\ell)$ and $\nu = (\nu_1, \dots, \nu_k)$ of n, let $t_{(\gamma,\nu)} = t_{\gamma_1} \dots t_{\gamma_\ell} t^-_{\nu_1} \dots t^-_{\nu_k}$.

Proposition 2.6.3. *1. Let $\gamma = (\gamma_1, \dots, \gamma_\ell)$ and $\nu = (\nu_1, \dots, \nu_k)$ be compositions of n with $\ell(t_\gamma) + \ell(t_\nu)$ even and let (λ, μ) be the bipartition of n corresponding to (γ, ν). Then we have*

$$t_{(\gamma,\nu)} \equiv \begin{cases} 0, & \text{if } (\lambda,\mu) \notin (\mathcal{OP}, \mathcal{EP}) \\ \pm t_{(\lambda,\mu)}, & \text{if } (\lambda,\mu) \in (\mathcal{OP}, \mathcal{EP}) \end{cases} \mod \ [\mathfrak{H}^{\mathsf{sp}}_{B_n}, \mathfrak{H}^{\mathsf{sp}}_{B_n}].$$

2. Let γ, ν, λ, and μ be as in the previous part. If n is odd, we have

$$t_{(\gamma,\nu)} \equiv \begin{cases} 0, & \text{if } (\lambda,\mu) \notin (\mathcal{OP}, \mathcal{EP}) \\ \pm t_{(\lambda,\mu)}, & \text{if } (\lambda,\mu) \in (\mathcal{OP}_n, \mathcal{EP}_n) \end{cases} \mod \ [\mathfrak{H}^{\mathsf{sp}}_{D_n}, \mathfrak{H}^{\mathsf{sp}}_{D_n}].$$

If n is even, we have

$$t_{(\gamma,\nu)} \equiv \begin{cases} \pm t_{(\lambda,\mu)}, & \text{if } (\lambda,\mu) \in (\mathcal{OP}, \mathcal{EP}) \text{ or} (\lambda,\mu) \in (\emptyset, \mathcal{SOP}_n) \\ 0, & \text{otherwise} \end{cases} \mod \ [\mathfrak{H}^{\mathsf{sp}}_{D_n}, \mathfrak{H}^{\mathsf{sp}}_{D_n}].$$

Proof. 1. If $\lambda \notin \mathcal{OP}$, we can repeat the proof of Proposition 2.6.1 to show that $t_\gamma \equiv 0$, and hence $t_{(\gamma,\nu)} \equiv 0$. Suppose $\mu \notin \mathcal{EP}$. Let a be the smallest integer so that ν_a is odd. If $\ell(\nu)$ is even, let b be the smallest integer such that $b > a$ and ν_b is odd. Then we have, as in Proposition 2.6.1,

$$\begin{aligned} t^-_\nu &\equiv t^-_{\nu,a} t^-_{\nu,a+1} \dots t^-_{nu,\ell} t^-_{\nu,1} \dots t^-_{\nu,a-1} \\ &= -t^-_{\nu,b} t^-_{\nu,a} \dots t^-_{\nu,a-1} \\ &\equiv (-1)^{2(k-b-1)+1} t^-_\nu. \end{aligned}$$

Here the extra signs come from commuting $t^-_{\nu,b}$ past the t_n in each term. Hence $t^-_\nu \equiv 0$ mod $[\mathfrak{H}^{\mathsf{sp}}_X, \mathfrak{H}^{\mathsf{sp}}_X]$.

If $(\gamma, \nu) \in (\mathcal{OP}, \mathcal{EP})$ the equality follows as in the type A_{n-1} case.

2. If n is odd or n is even and $(\lambda, \mu) \notin (\emptyset, \mathcal{SOP}_n)$, the proof follows as in type B_n. For $(\lambda, \mu) \in \mathcal{SOP}_n$, following the proof as in type B_n gives

$$t^-_\nu \equiv t^-_{\nu,a} t^-_{\nu,a+1} \dots t^-_{nu,\ell} t^-_{\nu,1} \dots t^-_{\nu,a-1}$$

$$
\begin{aligned}
&= -t^-_{\nu,b} t^-_{\nu,a} \dots t^-_{\nu,a-1} \\
&\equiv (-1)^{1+2(2)} t_{\nu,a} \dots t_{\nu,b} t_n t_{n-2} t_n \dots t_{\nu,a-1} \\
&= (-1)^5 t_{\nu,a} \dots t_{\nu,b} tn - 2 t_n t_{n-2} \dots t_{\nu,a-1} \\
&= (-1)^{k-b-1+5} t^-_\nu .
\end{aligned}
$$

But since $\ell(\mu)$ is even, $k-b-1$ is odd, so $k-b-1+5$ is even. Hence we have no sign change. Indeed, since the image of t^-_ν is nonzero and equal to t_μ in $\mathfrak{H}^{\mathfrak{c}}_{D_n}$, we must have the same equality in $\mathfrak{H}^{\mathsf{sp}}_{D_n}$.

□

Proposition 2.6.4. *1. Let w_C be a minimal length representative of a conjugacy class C corresponding to the bipartition (λ, μ) in W_{B_n}. Then we have*

$$
t_{w_C} \equiv \begin{cases} \pm t_{(\lambda,\mu)}, & \text{if } (\lambda,\mu) \in (\mathcal{OP}, \mathcal{EP}) \\ 0, & \text{otherwise,} \end{cases} \quad \mod \; [\mathfrak{H}^{\mathsf{sp}}_{B_n}, \mathfrak{H}^{\mathsf{sp}}_{B_n}].
$$

2. Let w_C be a minimal length representative of a conjugacy class C corresponding to the bipartition (λ, μ) in W_{D_n}. Then if n is odd we have

$$
t_{w_C} \equiv \begin{cases} \pm t_{(\lambda,\mu)}, & \text{if } (\lambda,\mu) \in (\mathcal{OP}, \mathcal{EP}) \\ 0, & \text{otherwise.} \end{cases} \quad \mod \; [\mathfrak{H}^{\mathsf{sp}}_{D_n}, \mathfrak{H}^{\mathsf{sp}}_{D_n}].
$$

If n is even, we have

$$
t_{w_C} \equiv \begin{cases} \pm t_{(\lambda,\mu)}, & \text{if } (\lambda,\mu) \in (\mathcal{OP}, \mathcal{EP}) \text{ or} (\lambda,\mu) \in (\emptyset, \mathcal{SOP}_n) \\ 0, & \text{otherwise.} \end{cases} \quad \mod \; [\mathfrak{H}^{\mathsf{sp}}_{D_n}, \mathfrak{H}^{\mathsf{sp}}_{D_n}].
$$

Proof. Both cases follow immediately from the proof of Proposition 2.6.2, using Proposition 2.6.3 in place of 2.6.1 and modifying Equation 2.24 to use the appropriate relations. □

2.7 Bases for the trace of the spin affine Hecke algebra

We establish spanning sets for $\overline{\mathfrak{H}}^{\mathsf{sp}}_X$, and then use the Morita superequivalence to prove that these sets are linearly independent.

2.7.1 Spanning sets

Let C_X be the set of conjugacy classes labelled by $\mathcal{OP}_n$ if $X = A_{n-1}$, by $(\mathcal{OP}, \mathcal{EP})$ if $X = B_n$, and by $(\mathcal{OP}, \mathcal{EP})$ if $X = D_n$ with n odd, or $(\mathcal{OP}, \mathcal{EP}) \cup (\emptyset, \mathcal{SOP}_n)$ if $X = D_n$ with n even. The following is similar to [WW12a, Theorem 6.6].

Lemma 2.7.1. *Let $w \in W_X$ with $\ell(w)$ even. Then there exist $f^-_{w,\nu} \in \mathbb{C}$ such that*

$$
t_w \equiv \sum_{\nu \in \mathsf{C}_X} f^-_{w,\nu} t_\nu .
$$

Proof. The type A_{n-1} case is proved in [WW12a]. The type B_n and D_n cases follow from a similar argument using Proposition 2.6.4. □

As before, filter $\mathfrak{H}_X^{\mathsf{sp}}$ by degree in $\mathcal{C}\langle b_1, \ldots, b_n\rangle$ and let $\mathfrak{H}_X^{-,0}$ be the associated graded object, which is isomorphic to the degenerate spin affine Hecke algebra with all parameters identically 0. Now we follow the procedure in Section 2.4.

Lemma 2.7.2. *For $X = A_{n-1}$, B_n, or D_n, we have*

$$\overline{\mathfrak{H}_X^{\mathsf{sp},0}} \subset \operatorname{span}\{t_{w_C}\mathbb{C}[b_1^2, \ldots, b_n^2]\}_{C \in \mathsf{C}_X}.$$

Proof. Apply Propositions 7.1.2 and 7.2.2 to each element in the (trivial) spanning set $W_X^-\mathcal{C}\langle b_1^2, \ldots, b_n^2\rangle$. Thus every element is either congruent to 0 or to t_{w_C} mod $[\mathfrak{H}_X^{\mathsf{sp}}, \mathfrak{H}_X^{\mathsf{sp}}]$. □

For a conjugacy class C of W^-, define J_C and w_C as before, using the natural action of W^- on $\{1, 2, \ldots, n\}$. Now, for convenience, denote $\mathcal{C}\langle \mathbf{b}^2\rangle = \mathcal{C}\langle b_1^2, \ldots, b_n^2\rangle$.

Lemma 2.7.3. *Fix a conjugacy class C of W, and let $J = J_C$. Then we have*

$$t_{w_C}\mathcal{C}\langle \mathbf{b}^2\rangle \equiv t_{w_C}\mathcal{C}\langle (\mathbf{b}^2)^{W_J^-}\rangle^{N_{W^-}(W_J^-)}.$$

Proof. The proof of Proposition 2.3.4 extends to this case without modification except possible the addition of signs: it depends only on the action of W_X on $S(V)$, which is the same as the action of W_X^- on $\mathcal{C}\langle \mathbf{b}^2\rangle$ with parameter 0 up to a possible change in sign. □

For each $C \in \mathsf{C}_X$ and W of types A_{n-1}, B_n, or D_n, let $\{f_{J_C;i}^-\}$ be a basis of the vector space $\mathcal{C}\langle (\mathbf{b}^2)^{W_J^-}\rangle^{N_{W^-}(W_J^-)}$. Combining Lemmae 8.0.2 and 8.0.3 gives:

Proposition 2.7.4. *For $X = A_{n-1}$, B_n, or D_n, we have*

$$\overline{\mathfrak{H}_X^{\mathsf{sp},0}} = \operatorname{span}\{t_{w_C} f_{J_C;i}^-\}_{C \in \mathsf{C}_X}.$$

Finally, we lift the spanning set to the ungraded object, as before.

Proposition 2.7.5. *For $X = A_{n-1}$, B_n, or D_n, we have that*

$$\overline{\mathfrak{H}_X^{\mathsf{sp}}} = \operatorname{span}\{t_{w_C} f_{J_C;i}^-\}_{C \in \mathsf{C}_X}.$$

Proof. Equations (2.17) and (2.18) lift to the spin case up to a change in sign; but these equations were already agnostic to sign, so this does not affect the proof. Hence any spanning set of $\overline{\mathfrak{H}_X^{\mathsf{sp},0}}$ is also a spanning set of $\overline{\mathfrak{H}_X^{\mathsf{sp}}}$. □

2.7.2 Linear independence

The following is the second main result of the chapter.

Theorem 2.7.6. *For $X = A_{n-1}$, B_n, or D_n, the set $\{t_{w_C} f_{J_C;i}^-\}_{C \in C_X}$ forms a linear basis of $\overline{\mathfrak{H}_X^{\mathsf{sp}}}$.*

Proof. It suffices to prove that these sets are linearly independent. We take advantage of Theorem 2.4.4 and the isomorphism in Proposition 2.5.2. The inverse Φ^{-1} of the isomorphism in Proposition 2.5.2 induces an injective map $\mathfrak{H}_X^{\mathsf{sp}} \to \mathfrak{H}_X^{\mathsf{c}}$. We claim that it restricts to an inclusion

$$\overline{\Phi^{-1}} : \overline{\mathfrak{H}_X^{\mathsf{sp}}} \to \quad \mathrm{Tr}(\mathfrak{H}_X^{\mathsf{c}}).$$

Indeed, by Proposition 2.7.5, the set $\{t_{w_C} f^-_{J_C;i}\}_{C \in \mathsf{C}_X}$ spans $\overline{\mathfrak{H}_X^{\mathsf{sp}}}$. The image of an element in this set under Φ^{-1} is

$$\Phi^{-1}(t_{w_C} f^-) = w_C f$$

where $f \in S(V^2)$ is obtained from $f^- \in \mathcal{C}\langle \mathbf{b}^2 \rangle$ by replacing all b_i's with x_i's. But the images of the elements $w_C f_{J_C;i}$ for $C \in \mathsf{C}_X$ were shown to be linearly independent in Theorem 5.0.2, so the map $\overline{\Phi^{-1}}$ is an inclusion. The set $\{t_{w_C} f^-_{J_C;i}\}_{C \in \mathsf{C}_X}$ is then the preimage under an inclusion of a linearly independent set, and must therefore be linearly independent.

□

Chapter 3

Trace of the twisted Heisenberg category

In [Kho14], Khovanov describes a linear monoidal category $\mathcal{H}$ which conjecturally categorifies the Heisenberg algebra. The morphisms of $\mathcal{H}$ are governed by a graphical calculus of planar diagrams. This category has connections to many interesting areas of representation theory and combinatorics.

Recall that the trace, or zeroth Hochschild homology, of a $\Bbbk$-linear additive category $\mathcal{C}$ is the $\Bbbk$-vector space given by

$$\mathrm{Tr}(\mathcal{C}) := \left(\bigoplus_{x \in \mathrm{Ob}(\mathcal{C})} \mathrm{End}_{\mathcal{C}}(X) \right) \Big/ \mathrm{span}_{\Bbbk}\{fg - gf\},$$

where f and g run through all pairs of morphisms $f : x \to y$ and $g : y \to x$ with $x, y \in \mathrm{Ob}(\mathcal{C})$. If a $\Bbbk$-linear category $\mathcal{C}$ carries a monoidal structure, then $\mathrm{span}\{fg - gf\}$ is an ideal, and $\mathrm{Tr}(\mathcal{C})$ becomes an algebra where multiplication in the trace is induced from tensor product of $\mathcal{C}$.

The trace of $\mathcal{H}$, which can be defined diagrammatically as the algebra of diagrams on the annulus, was shown in [CLLS16] to be isomorphic to a quotient of the W-algebra $W_{1+\infty}$.

The twisted Heisenberg algebra $\mathfrak{h}_{tw}$ is a unital associative algebra generated by $h_{m/2}$, $m \in 2\mathbb{Z} + 1$, subject to the relations

$$\left[h_{\frac{n}{2}}, h_{\frac{m}{2}}\right] = \frac{n}{2}\delta_{n,-m}.$$

In [CS15], a twisted version of the Heisenberg category, denoted $\mathcal{H}_{tw}$, is introduced. It is also a $\mathbb{C}$-linear additive monoidal category, with an additional $\mathbb{Z}/2\mathbb{Z}$-grading. It is proved that $K_0(\mathcal{H}_{tw})$ contains $\mathfrak{h}_{tw}$ (they are conjecturally isomorphic).

The goal of this chapter is to study the trace $\mathrm{Tr}(\mathcal{H}_{tw})$, and determine additional structure. We show that the even part of $\mathrm{Tr}(\mathcal{H}_{tw})$ is isomorphic as an algebra to a quotient of a subalgebra of $W_{1+\infty}$ that we will denote by W^-. We give explicit descriptions of $W_{1+\infty}$ and W^- in Section 4.6.1. This confirms the expectation in [CLLS16] that there should be a relationship between $\mathcal{H}_{tw}$ and one of two subalgebras of $W_{1+\infty}$ defined in [KWY98]. Even though the isomorphism between $K_0(\mathcal{H}_{tw})$ and the twisted Heisenberg algebra $\mathfrak{h}_{tw}$ is still conjectural, we are able to completely characterize $\mathrm{Tr}(\mathcal{H}_{tw})$.

The structure of the chapter is as follows. In Section 1, we describe the W-algebra W^- of interest, describe its gradings and a set of generators, and study its Fock space representation. In Section 2, we describe trace decategorification in more detail and present the twisted Heisenberg category studied in [CS15], as well as its gradings. We also identify a copy of the degenerate affine Hecke-Clifford algebra within the trace. In Section 3, we study a subalgebra of $\mathrm{Tr}(\mathcal{H}_{tw})$ consisting of circular diagrams called bubbles, and describe how they interact with other elements of the trace. Section 4 contains a number of calculations of diagrammatic relations in the trace that are useful for computing a generating set of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$. Finally, in Section 5, we describe a triangular decomposition of the trace, and then establish a generating set. This allows us to prove the desired isomorphism by using the action of each algebra on its Fock space.

Much of the work in this chapter originally appeared in [OR17] and is joint with Can Ozan Oğuz.

3.1 W-algebra

In this section, we will recall the W-algebra we are interested in, its structure as a $\mathbb{Z}$-graded and $\mathbb{N}$-filtered algebra, and one of its subalgebras – the twisted Heisenberg algebra – as well as their Fock space representations.

3.1.1 Twisted Heisenberg algebra $\mathfrak{h}_{tw}$

We recall the definition of the twisted Heisenberg algebra. The twisted Heisenberg algebra $\mathfrak{h}_{tw}$ is a unital associative algebra generated by h_n for $n \in \mathbb{Z} + \frac{1}{2}$ subject to the relation that $[h_n, h_m] = n\delta_{n,-m}$.

3.1.2 W-algebra W^-

Let $\mathcal{D}$ denote the Lie algebra of differential operators on the circle. The central extension $\hat{\mathcal{D}}$ of $\mathcal{D}$ is described in [KWY98]. It is generated by by C and by $w_{k,l} = t^k D^l$ for $l \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$ where t is a variable over $\mathbb{C}$, and $D = t\frac{d}{dt}$, subject to relations that C and $w_{0,0}$ are central, and:

$$[t^r f(D), t^s g(D)] = t^{r+s}(f(D+s)g(D) - f(D)g(D+r)) + \psi(t^r f(D), t^s g(D))C, \tag{3.1}$$

where

$$\psi(t^r f(D), t^s g(D)) = \begin{cases} \sum\limits_{-r \leq j \leq -1} f(j)g(j+r) & r = -s \geq 0 \\ 0 & r+s \neq 0 \end{cases} \tag{3.2}$$

for f, g polynomials.

The W-algebra $W_{1+\infty}$ is the universal enveloping algebra of $\hat{\mathcal{D}}$. It is shown in [CLLS16] that trace of Khovanov's Heisenberg category is isomorphic to $W_{1+\infty}$ at level one.

In this chapter, we are interested in the universal enveloping algebra of a central extension of a Lie subalgebra of $\mathcal{D}$ fixed by a degree preserving anti-involution. This algebra was introduced in [KWY98]. Define the map:

$$\begin{aligned} \sigma : \ \mathcal{D} &\longrightarrow \mathcal{D} \\ 1 &\mapsto \sigma(1) = -1 \\ t &\mapsto \sigma(t) = -t \\ D &\mapsto \sigma(D) = -D. \end{aligned}$$

This is a degree preserving anti-involution of $\mathcal{D}$, and the Lie subalgebra fixed by $-\sigma$ is

$$\mathcal{D}^- := \{a \in \mathcal{D} | \sigma(a) = -a\}.$$

Let $\hat{\mathcal{D}}^-$ be a central extension of $\mathcal{D}^-$ where the 2-cocycle is the restriction of the 2-cocycle ψ given above. Therefore $\hat{\mathcal{D}}^-$ is a Lie subalgebra of $\hat{\mathcal{D}}$.

More explicitly, $\hat{\mathcal{D}}^-$ is the Lie algebra over the vector space spanned by $\{C\} \cup \{t^{2k-1}g(D+(2k-1)/2);\ g \text{ even}\} \cup \{t^{2k}f(D+k);\ f \text{ odd}\}$ where $k \in \mathbb{Z}$ and even and odd refer to even and odd polynomial functions. Its Lie bracket is given by Equation (3.1).

Denote by W^- the universal enveloping algebra of $\hat{\mathcal{D}}^-$. Our main result relates the trace of twisted Heisenberg category to a quotient of W^-.

$$\begin{array}{cccccc} & \mathcal{D} & \xrightarrow{\text{central extension}} & \widehat{\mathcal{D}} & \xrightarrow{\text{enveloping algebra}} & W_{1+\infty} \\ \text{fixed by } -\sigma & \cup & & \cup & & \cup \\ & \mathcal{D}^- & \xrightarrow{\text{central extension}} & \widehat{\mathcal{D}^-} & \xrightarrow{\text{enveloping algebra}} & W^- \end{array}$$

Note that not all $w_{k,\ell}$ are contained in W^-.

Example 1.1 When $k-\ell$ is an even integer, $w_{k,\ell} \notin W^-$. Moreover, the difference $k-\ell$ being odd is not sufficient. For example, $t^2D = w_{2,1} \notin W^-$ since an element starting with t^2 should be followed by $f(D+1)$ where f is an odd polynomial function. Hence $t^2D = w_{2,1} \notin W^-$ but $t^2(D+1) = t^2D + t^2 = w_{2,1} + w_{2,0} \in W^-$ (and, indeed, $\sigma(t^2(D+1)) = t^2(-D-1) = -t^2(D+1)$).

3.1.3 Gradings on W^-

There is a natural $\mathbb{Z}^{\geq 0}$ filtration of W^- called the *differential filtration* with $w_{k,\ell}$ in degree ℓ; denote this filtration by $|\cdot|_{dot}$. It is convenient to define an additional filtration: the *difference filtration*, where $w_{k,\ell}$ is in degree $\ell - k$, denoted $|\cdot|_{diff}$. That this is a filtration follows from the fact that W^- also carries a filtration with $w_{k,\ell}$ in degree k.

These filtrations are compatible, so we have a $(\mathbb{Z}\times\mathbb{Z}^{\geq 0})$-filtration with with an element $f = t^j g(D-j/2) \in W^-$ in bidegree $\leq (|f|_{diff}, |f|_{dot}) = (\deg(g)-j, \deg(g))$, where $\deg(g)$ is the polynomial degree of $g(w)\in\mathbb{C}[w]$. Define the following subalgebras of W^-:

$$W^{-,>} = \mathbb{C}\langle t^j g(D-j/2) | \deg(g) - j \geq 1\rangle;$$

$$W^{-,<} = \mathbb{C}\langle t^j g(D-j/2) | \deg(g) - j \leq 1\rangle;$$

$$W^{-,0} = \mathbb{C}\langle g(D) | \deg(g) \text{ odd}\rangle.$$

Let $W^{-,\omega}[\leq r, \leq k]$ denote the set of elements in difference degree $\leq r$ and differential degree $\leq k$, with $\omega \in \{>, <, 0\}$.

Denote by $\operatorname{gr} W^-$ the associated graded object with respect to this filtration. Hence $\operatorname{gr} W^-$ is $(\mathbb{Z}\times\mathbb{Z}^{\geq 0})$-graded with $|w_{k,\ell}| = (\ell - k, \ell)$. For $\omega\in\{>,<,0\}$, define a generating series for the graded dimension of $\operatorname{gr}(W^-)^\omega$ by

$$P_{\operatorname{gr}(\mathcal{W}^-)^\omega}(t,q) = \sum_{r\in\mathbb{Z}}\sum_{k\in\mathbb{Z}, k\geq 0} \dim \operatorname{gr}(W^-)^\omega[r,k] t^r q^k.$$

Proposition 3.1.1. *The graded dimensions of* $\operatorname{gr}(W^-)^>$ *and* $\operatorname{gr}(W^-)^<$ *are given by:*

$$P_{\operatorname{gr}(\mathcal{D}^-)^>} = \prod_{r\geq 0}\prod_{k>0} \frac{1}{1-t^{2r+1}q^k};$$

$$P_{\operatorname{gr}(\mathcal{D}^-)^<} = \prod_{r\leq 0}\prod_{k>0} \frac{1}{1-t^{2r-1}q^k}.$$

Proof. The algebra W^- is generated by elements of the form $t^j g(D-j/2)$, where $\deg(g)-j$ is odd. Hence $\operatorname{gr}(\mathcal{D}^-)^>$ is freely generated by elements $w_{k,\ell}$ with $k-\ell$ odd; such elements have bidgegree $(k-\ell,\ell)$. The proposition follows. □

Let $W^-_{r,s}$ denote the rank r, differential filtration s part of W^-. It is easy to see that the differential filtration zero part of W^-, namely $\bigcup_{r\in\mathbb{Z}} W^-_{r,0}$, is spanned as a vector space by $\{C\}\cup\{t^{2n+1}\}_{n\in\mathbb{Z}}$. As an algebra, we have that

$$[t^{2n+1}, t^{2m+1}] = (2n+1)\delta_{n,-m} \tag{3.3}$$

Hence we have an isomorphism between the differential filtration zero part of W^- and the twisted Heisenberg algebra $\mathfrak{h}_{tw}$ given by:

$$\begin{aligned} \phi: \quad \mathfrak{h}_{tw} &\longrightarrow \bigcup_{r\in\mathbb{Z}} W^-_{r,0} \\ h_{\frac{2n+1}{2}} &\mapsto \frac{1}{\sqrt{2}} t^{2n+1} \end{aligned}$$

where $n\in\mathbb{Z}$.

3.1.4 Generators of the algebra W^-

The following lemma describes a generating set for W^- as an algebra.

Lemma 3.1.2. *The algebra $W^-/\langle w_{0,0}, C\rangle$ is generated by $w_{1,0}$, $w_{0,3}$, and $w_{\pm 2,1} \pm w_{\pm 2,0}$.*

Proof. Let $t^k g(D-k/2)$ be an arbitrary element of W^-. Without loss of generality, we may assume g is a monic monomial of the form $g(w) = w^\ell$ with $\ell - k$ odd, since lower terms in g are just monomials of this form with lower degree, and thus can be generated separately. Therefore, we have

$$t^k g(D-k/2) = \sum_{i=0}^{\ell} \binom{\ell}{i} (-1)^{\ell-i} (k/2)^{\ell-i} t^k D^i. \tag{3.4}$$

The leading term of this element with respect to differential degree is $t^k D^\ell$. We will generate the leading term first, and address lower terms afterwards. There are two cases, depending on the parities of k and ℓ.

First, suppose that $k = 2n$ is even and $\ell = 2m+1$ is odd (recall that k and ℓ must have opposite parity in W^-). Hence, we must generate $w_{\pm 2n, 2m+1}$. The following calculations are easy, using Formula 3.1:

$$\begin{aligned}
[w_{-2,1} - w_{-2,0}, w_{1,0}] &= w_{-1,0}, \\
[w_{1,0}, w_{0,3}] &= -3(w_{1,2} + w_{1,1}) - w_{1,0}, \\
[w_{1,2b}, w_{0,3}] &= -3w_{1,2b+2} + O(w_{1,2b+1}),
\end{aligned} \tag{3.5}$$

where $O(\omega)$ refers to terms with lower differential degree than ω. Hence, starting with $w_{1,2} - w_{1,1}$, we can use the Equation (3.5) above to generate $w_{1,2b}$ for any b. Now we have:

$$[w_{\pm 2a,1}, w_{1,0}] = w_{\pm 2a+1,0}, \tag{3.6}$$

$$[w_{\pm 2a+1,0}, w_{1,2} - w_{1,1}] = -(4a+2)w_{2a+2,1} - (2a+1)(2a+2)w_{2a+2,0}. \tag{3.7}$$

Thus, starting from $w_{2,1} + w_{2,0}$, we can generate $w_{2a,1}$ for any a. Finally, we have:

$$[w_{-1,0}, w_{1,2b}] = \sum_{i=0}^{2b-1} \binom{2b}{i} (-1)^{2b-i+1} w_{0,i} = w_{0,2b-1} + O(w_{0,2b-2}), \tag{3.8}$$

$$\begin{aligned}
[w_{\pm 2a,1}, w_{0,2b-1}] &= -\sum_{i=0}^{2b-2} \binom{2b-1}{i} (\pm 1)^{2b-i} (2)^{2b-2-i} t^{2a} D^{i+1} \\
&= w_{2a,2b-1} + O(w_{2a,2b-2}).
\end{aligned}$$

So, we can generate a polynomial with leading term $w_{\pm 2n,2m+1}$.

Next, suppose that $k = 2n+1$ is odd and positive and $\ell = 2m$ is even. Using Formula (3.1), we have:

$$\begin{aligned}[w_{2a+1,0}, w_{0,2b+1}] &= t^{2a+1} \sum_{i=0}^{2b} \binom{2b+1}{i} (2a+1)^{2b+1-i} D^i \\ &= w_{2a+1,2b} + O(w_{2a+1,2b-1}).\end{aligned}$$

Now Equations (3.6) and (3.8) give that we can generate $w_{2a+1,0}$ and $w_{0,2b+1}$. Hence we can generate a polynomial with leading term $w_{2a+1,2b}$.

Finally, assume that $k = -(2n+1)$ is odd and $n = 2m$ is even. Using Formula (3.1), we have:

$$[w_{-2a,1}, w_{1,0}] = w_{1-2a,0}.$$

By Equation (3.7), we can therefore generate $w_{-(2a+1),0}$ for any a. Next, note that:

$$[w_{-1,0}, w_{1,2b}] = -\sum_{i=0}^{2b-1} \binom{2b-1}{i} (-1)^{2b-1-i} D^i = w_{0,2b-1} + O(w_{0,2b-2}).$$

By Equation (3.5), we can generator $w_{0,2b+1}$ for any b. Finally, we have

$$\begin{aligned}[w_{-(2a+1),0}, w_{0,2b-1}] &= t^{-(2a+1)} \sum_{i=0}^{2n-2} \binom{2n-1}{i} (-1)^{2n-i} (2a+1)^{2n-1-i} D^i \\ &= w_{-(2a+1),2b-2} + O(w_{-(2a+1),2b-3}).\end{aligned}$$

Thus, we can generate a polynomial with leading term $w_{-(2n+1),2m}$.

It remains to adjust the lower terms of these equations so that they match those in Equation (3.4). But note that each equation used above to generate the leading term results in lower terms which lie in different filtrations of W^-. Therefore we can adjust the coefficients of lower terms by scaling individual equations above. Since there is no dependency between these equations, we can choose constant coefficients for the generators so that our generated polynomial has the correct lower terms. □

3.1.5 Fock space representation of W^-

The algebra W^- inherits a Fock space representation from $W_{1+\infty}$. Let $W^{-,\geq} = W^{-,0} \oplus W^{-,>}$. For parameters $c, d \in \mathbb{C}$, let $\mathbb{C}_{c,d}$ be a one-dimensional module for $W^{-,\geq}$ on which each $w_{k,\ell}$ with $(k,\ell) \neq (0,0)$ acts as zero, C acts as c, and $w_{0,0}$ acts as d. Let $\mathcal{M}_{c,d} := \mathrm{Ind}_{W^{-,\geq}}^{W^-} \mathbb{C}_{c,d}$. This induced module possesses the following properties:

Proposition 3.1.3. *[AFMO94, FKRW00] The W^--module $\mathcal{M}_{c,d}$ has a unique irreducible quotient $\mathcal{V}_{c,d}$, which is isomorphic as a vector space to $\mathbb{C}[w_{-1,0}, w_{-2,0}, w_{-3,0}, \ldots]$.*

Proposition 3.1.4. *[SV13] The action of $W^-/(C-1, w_{0,0})$ is faithful on $\mathcal{V}_{1,0}$.*

Proof. This follows immmediately from the argument in [SV13] for $W_{1+\infty}$ because W^- is a subalgebra. □

Proposition 3.1.3 allows us to the compute the action of the generators on $\mathcal{V}_{1,0}$, which we record for convenience below.

Proposition 3.1.5. *Let k be a positive integer. The generators of W^- act on $\mathcal{V}_{1,0}$ as follows:*

$$\begin{aligned}
[w_{1,0}, w_{-k,0}] &= \delta_{1,k}, \\
[w_{-2,1} - w_{-2,0}, w_{-k,0}] &= (k+2)w_{-(k+2),0}, \\
[w_{2,1} + w_{2,0}, w_{-k,0}] &= -(k+2)w_{2-k,0}, \\
[w_{0,3}, w_{-k,0}] &= 3kw_{-k,2} - 3k^2 w_{-k,1} + k^3 w_{-k,0}.
\end{aligned}$$

3.2 Twisted Heisenberg category

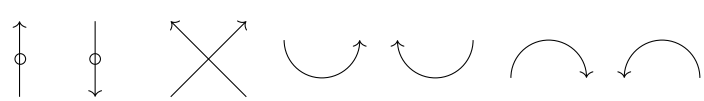
We will now describe the main object of interest in this chapter, the twisted Heisenberg category $\mathcal{H}_{tw}$. After defining the category, we recall the trace decategorification functor and some of its properties. We then describe some filtrations of $\mathrm{Tr}(\mathcal{H}_{tw})$, identify a copy of the degenerate affine Hecke-Clifford algebra $\mathfrak{H}_n^C$, and describe the trace of $\mathfrak{H}_n^C$. Finally, we identify a set of distinguished elements in $\mathrm{Tr}(\mathcal{H}_{tw})$ which generate the nonzero filtration degrees of the algebra.

3.2.1 Definition of $\mathcal{H}_{tw}$

The twisted Heisenberg category $\mathcal{H}^t$ is defined in [CS15] as the Karoubi envelope of a $\mathbb{C}$-linear $\mathbb{Z}/2\mathbb{Z}$-graded additive monoidal category, whose moprhisms are described diagrammatically. There is an injective algebra homomorpshim from $\mathfrak{h}_{tw}$ to the split Grothendieck group of the twisted Heisenberg caterogy $K_0(\mathcal{H}^t)$. As in the untwisted case, this map is conjecturally surjective.

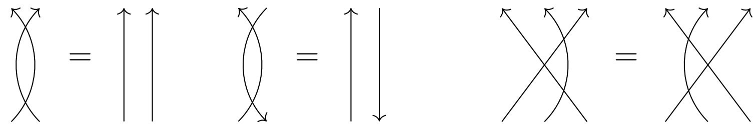
The object of our main interest is the trace decategorification or zeroth Hochschild homology of $\mathcal{H}^t$. It is shown in [BGHL14, Proposition 3.2] that trace of an additive category is isomorphic to the trace of its Karoubi envelope,. Therefore, we can work with the non-idempotent completed version of $\mathcal{H}^t$. We will denote it by $\mathcal{H}_{tw}$. Focusing our attention to $\mathcal{H}_{tw}$ allows us to work with the diagrammatics introduced in [CS15].

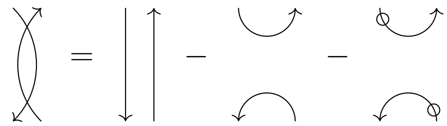
The category $\mathcal{H}_{tw}$ is the $\mathbb{C}$-linear, $\mathbb{Z}/2\mathbb{Z}$-graded monoidal additive category whose objects are generated by P and Q. A generic object is a sequence of P's and Q's. The morphisms of $\mathcal{H}_{tw}$ are generated by oriented planar diagrams up to boundary fixing isotopies, with generators

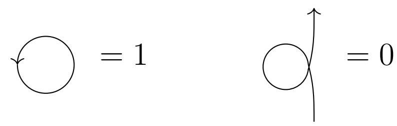


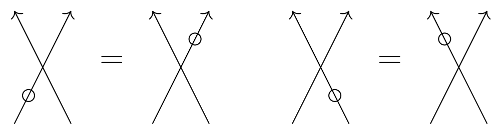
where the first diagram corresponds to a map $P \to P\{1\}$ and the second diagram corresponds to a map $Q \to Q\{1\}$, where $\{1\}$ denotes the $\mathbb{Z}/2\mathbb{Z}$-grading shift. The first two diagrams above have degree one, and the last five have degree zero. The identity morphisms of P

and Q are indicated by an undecorated upward and downward pointing arrow, respectively. These generators satisfy the following relations:


(3.9)

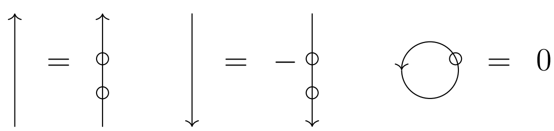

(3.10)


(3.11)


(3.12)

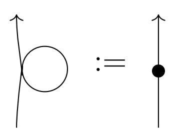

(3.13)


(3.14)

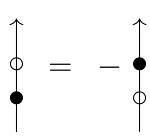

(3.15)

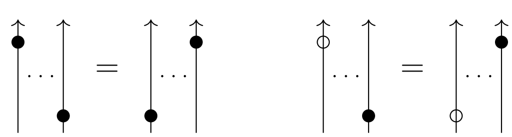

(3.16)

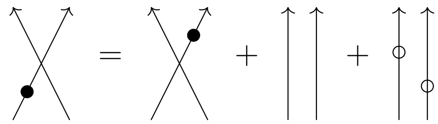
Also, if we let

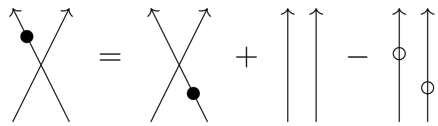

(3.17)

we get the following relations:


(3.18)


(3.19)

 (3.20)

 (3.21)

If x and y are morphisms, the diagram corresponding to $x \otimes y$ is obtained by placing the diagram of y to the right of the diagram of x. Since the relative positions of the hollow dots are important, we will work with the convention that the hollow dots in the diagram of y will be placed below the height of hollow dots in the diagram of x.

3.2.2 Trace decategorification

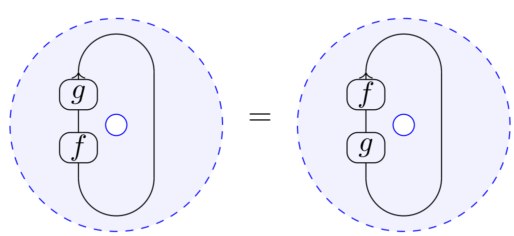
In [BGHL14], the trace or zeroth Hochschild homology of a $\Bbbk$-linear additive category $\mathcal{C}$ is proposed as an alternative decategorification functor. Here we will recall its definition, and point out one subtlety occuring in our case due to the supercommutative nature of hollow dots and solid dots.

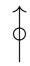
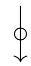
Let $\mathcal{C}$ be a $\Bbbk$-linear additive category. Then its trace decategorification, denoted $\mathrm{Tr}(\mathcal{C})$, is defined as follows:

$$\mathrm{Tr}(\mathcal{C}) \simeq \left(\bigoplus_{x \in \mathcal{O}b(\mathcal{C})} \mathrm{End}(x) \right) / \mathcal{I},$$

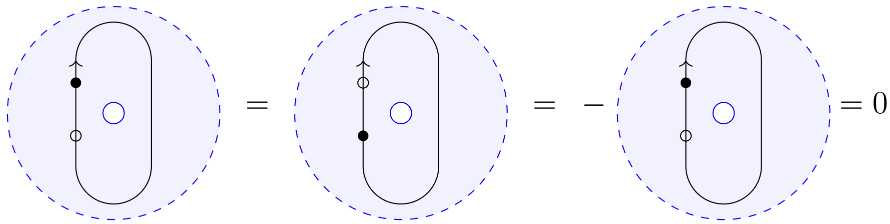
where $\mathcal{I}$ is the ideal generated by $\mathrm{span}_{\Bbbk}\{fg - gf\}$ for all $f : x \to y$ and $g : y \to x$, $x, y \in \mathcal{O}b(\mathcal{C})$. Note that here we quotient out by an ideal, so $\mathrm{Tr}(\mathcal{C})$ has an algebra structure.

Trace decategorification has a nice diagrammatic interpretation, in which we consider our string diagrams to be drawn on an annulus instead of a plane. The annulus recaptures the trace relation $fg = gf$ diagrammatically since we can slide f or g around the annulus to change their composition order.



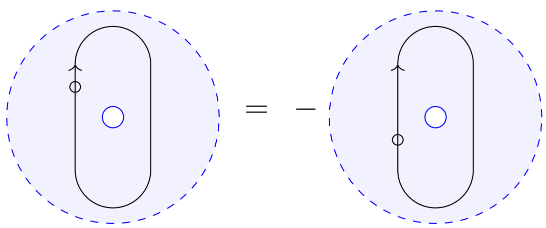
As described in Section 3.2, $\mathcal{H}_{tw}$ has a $\mathbb{Z}/2\mathbb{Z}$-grading where  and  have degree one, and other generating diagrams have degree zero. We also have supercommutativity relations (3.16) and (3.18) and supercyclicity relations (3.13) and (3.14). These relations have several interesting diagrammatic consequences.

Example 2.1 Working with relation (3.18), we have the following compuation:



Here the first equality is obtained by sending the solid dot around the annulus using trace relation, and the second equality is a consequence of relation (3.18). Therefore the above diagram is equal to zero in the trace.

Example 2.2 To demonstrate the subtlety with supercyclicity relations (3.13) and (3.14), consider the following situation:



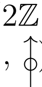
If we denote  by f, with the usual trace relation, we would get $f \circ id = id \circ f$. However, in this case, we gain an extra negative sign from the supercyclicity relations. So, we must replace the usual trace relation $fg = gf$ with the supertrace relation $fg = (-1)^{|f||g|} gf$ in the ideal $\mathcal{I}$, where $|f|, |g|$ are the degrees of f and g with respect to the $\mathbb{Z}/2\mathbb{Z}$ grading. This example can be generalized to show that composition of an an odd morphism with a cycle of odd length is zero in the supertrace, since it will be equal to its negative when a hollow dot travels around the annulus and arrives to its original position.

We wish to restrict our study to the following subalgebra of the trace.

Definition 2.1 The *even trace* of $\mathcal{H}_{tw}$ is defined by

$$\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}} \simeq \Big(\bigoplus_{x \in \mathcal{O}b(\mathcal{H}_{tw})} \mathrm{End}_{\bar{0}}(x) \Big) / \mathcal{I}_{\bar{0}}$$

where $\mathrm{End}_{\bar{0}}(x)$ consists of even degree endomorphisms and $\mathcal{I}_{\bar{0}}$ is its ideal generated by $\mathrm{span}_{\mathbb{C}}\{fg - gf\}$ for all $f : x \to y$ and $g : y \to x$, $x, y \in \mathcal{O}b(\mathcal{H}_{tw})$.

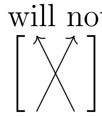
This is the restriction of the trace to only the *even* part (with respect to the $\mathbb{Z}/2\mathbb{Z}$ grading induced by $\deg(c_i) = 1$). The odd part of the trace is not zero (it contains, e.g., ), but is not interesting from a representation theoretic viewpoint as explained above. The example of trace functions on the finite Hecke-Clifford algebra in [WW12a, Section 4.1] demonstrates the importance of the even trace.

Wan and Wang study the space of trace functions on the finite Hecke-Clifford algebra $\mathcal{H}_n$: linear functions $\phi : \mathcal{H}_n \to \mathbb{C}$ such that $\phi([h, h']) = 0$ for all $h, h' \in \mathcal{H}_n$, and $\phi(h) = 0$ for all $h \in (\mathcal{H}_n)_{\bar{1}}$. This latter requirement encodes the information that odd elements act with zero trace on any $\mathbb{Z}_2$-graded $\mathcal{H}_n$-module (because multiplication by an odd element results in a shift in degree). The space of such trace functions is clearly canonically isomorphic to the dual of the even trace, rather than of the full trace. The same observation holds for the trace of the affine Hecke-Clifford algebra, as studied in [Ree17].

We will see in Section 4 that the structure of $\mathrm{Tr}(\mathcal{H}_{tw})$ is largely controlled by the even trace of the degenerate affine Hecke-Clifford algebra in type A; we therefore do not lose interesting representation-theoretic information by restricting to $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, and greatly simplify our calculations by doing so. For instance, the ambiguity in the supercyclicity relations identified in Example 3.2 does not interfere with calculations in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$.

Since $\mathcal{I}_{\bar{0}}$ is an ideal of $\bigoplus_{x \in \mathcal{O}b(\mathcal{H}_{tw})} \mathrm{End}_{\bar{0}}(x)$, the compositions fg and gf must be even morphisms, even though individually f and g may be odd morphisms. This situation is analogous to the even trace of the degenerate affine Hecke-Clifford algebra studied in [Ree17], where Clifford generators c_i do no appear individually (as they are odd generators), but still have an impact on the trace via the relation $c_i^2 = -1$.

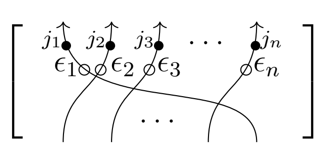
Diagrammatically, the above definition means that we will have an even number of hollow dots on our diagrams. In a diagram with $2n$ hollow dots, sliding one around the annulus from top to the bottom will multiply the diagram by $(-1)^{2n-1}(-1) = 1$ where $(-1)^{2n-1}$ is a result of changing relative height with the remaining $2n-1$ hollow dots using relation (3.16) and (-1) is the result of sliding it through a clockwise cup using relation (3.14).

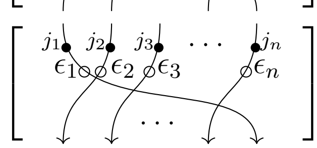
Remark For the sake of clarity, when working with diagrams in the even trace we will not draw them on an annulus, but will instead draw them inside square brackets, e.g. . This notation refers to the equivalence class of the diagram in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$.

Our main theorem will relate $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ and W^-. In particular, we will establish that the correspondence in Table 1 gives an isomorphism between $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ and W^-. Recall that $w_{k,\ell} = t^\ell D^k \in W^-$.

3.2.3 Distinguished elements h_n

Define the elements:

$h_n^{(x_1^{j_1} \cdots x_n^{j_n})(c_1^{\epsilon_1} \cdots c_n^{\epsilon_n})} :=$  ,

$h_{-n}^{(x_1^{j_1} \cdots x_n^{j_n})(c_1^{\epsilon_1} \cdots c_n^{\epsilon_n})} :=$  ,

where $\epsilon_i \in \{0, 1\}$. In both of these elements, we consider the hollow dots to be descending in height from left to right, so that the dot labeled ϵ_1 is the highest.

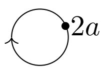
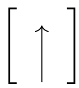
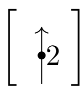
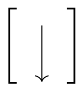
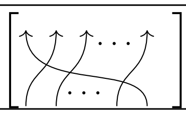
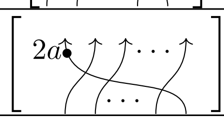
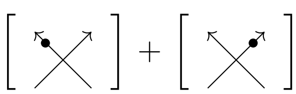
$\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$	bidegree (k-l,k)	values of (l,k)	W^-
	(2a+1,2a+1)	(0,2a+1)	$-2w_{0,2a+1}$
	(1,0)	(−1,0)	$\sqrt{2}w_{-1,0}$
	(3,2)	(−1,2)	$\sqrt{2}(w_{-1,2} - w_{-1,1})$
	(−1,0)	(1,0)	$\sqrt{2}w_{1,0}$
	(n,0)	(−n,0)	$\sqrt{2}w_{-n,0}$
	(n+a,a)	(−n,a)	$\sqrt{2}(w_{-n,a} + O(w_{-n,a-1}))$
	(3,1)	(−2,1)	$2\sqrt{2}(w_{-2,1} - w_{-2,0})$

Table 3.1: Correspondence between $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ and W^-

Remark These elements are analogues to those denoted $h_{\pm n} \otimes (x_1^{j_1} \cdots x_n^{j_n})$ in [CLLS16].

Additionally, set

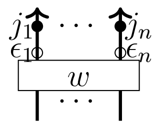
$$h_n^{\sum x_i^{j_i}} = \sum h_n^{x_i^{j_i}}.$$

Lemma 3.2.1. *For $n \geq 1$ and $1 \leq i \leq n-1$ we have*

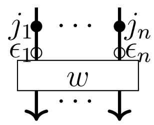
1. $h_{\pm n}^{x_i} = h_{\pm n}^{x_{i+1}} \pm h_{\pm i} h_{\pm(n-i)}$.
2. $h_{\pm n}^{x_i c_j} = -h_{\pm n}^{x_{i+1} c_{j+1}}$.

Proof. Part (1) is just [CLLS16, Lemma 14], except our solid dot sliding relation through crossing involves an extra term with hollow bubbles. But cycles with single hollow dot are zero since sending the hollow dot around the annulus gives us the same diagram with a negative sign. For the above calculations, our n-cycles split into smaller cycles with single hollow dot at least on one of them. The proof of part 2 depends on the relative position of i and j, but is a straightforward computation. □

Let $w \in S_n$, and define the elements:

$f_{w;j_1,\ldots,j_n;\epsilon_1,\ldots,\epsilon_n} =$ 

and

$f_{w;j_1,\ldots,j_n;\epsilon_1,\ldots,\epsilon_n} =$  .

Lemma 3.2.2. *Let $w \in S_n$ and $(n_1, \ldots, n_r)$ be a composition of n. Then*

$$[f_{\pm w;j_1,\ldots,j_n;\epsilon_1,\ldots,\epsilon_n}] = \sum d_{n_1,\ldots,n_r} h_{n_1}^{p_{n_1} c_{n_1}} \ldots h_{n_r}^{p_{n_r} c_{n_r}}$$

for constants $d_{n_1,\ldots,n_r} \in \mathbb{C}$, polynomials p_{n_i} in i variables, and elements c_{n_i} consisting of at most i Clifford generators (e.g. $c_{n_3} = \{c_1^{\epsilon_1} c_2^{\epsilon_2} c_3^{\epsilon_3} | \epsilon_i \in \{0,1\}\}$).

Proof. We proceed by induction on $\sum \epsilon_i$. The base case is $\sum \epsilon_i = 0$; then

$$[f_{\pm w;j_1,\ldots,j_n;\epsilon_1,\ldots,\epsilon_n}] = [f_{\pm w;j_1,\ldots,j_n}]$$

and we apply [CLLS16, Lemma 15].

Now assume the statement is true for $\sum \epsilon_i = k$ for all $k < m \leq n$. Take $(\epsilon_1, \ldots, \epsilon_n)$ so that $\sum \epsilon_i = m$. Choose $g \in S_n$ such that $gwg^{-1} = w_\lambda$, where λ is the cycle type of w (so $gwg^{-1} = (s_1 \ldots s_{n_1-1}) \ldots (s_{n_1+\ldots+n_{r-1}} \ldots s_{n_1+\ldots+n_r-1})$). Let $p = x_1^{j_1} \ldots x_n^{j_n}$ and $c = c_1^{\epsilon_1} \ldots c_n^{\epsilon_n}$. Then we have

$$f_{\pm w;j_1,\ldots,j_n;\epsilon_1,\ldots,\epsilon_n} = pcw = (-1)^\epsilon cpw$$

where

$$\epsilon = \sum_{\epsilon_i = 1} j_i.$$

Thus conjugating by g gives that

$$\begin{aligned} gpcwg^{-1} &= (-1)^\epsilon gcpwg^{-1} \\ &= (-1)^\epsilon (g.c)gpwg^{-1} \\ &= (-1)^\epsilon \left[(g.c)(g.p)gwg^{-1} + (g.c)p_L wg^{-1}\right], \end{aligned}$$

where p_L is a polynomial of degree less than $j_1 + \ldots + j_n$. Note that gwg^{-1} is a product of cycles, so the first term in the above expression has the correct form. In the second term, we have $\{i | \epsilon_{g(i)} = 1\} \leq m$ (strict inequality can occur if g has fixed points). If $\{i | \epsilon_{g(i)} = 1\} < m$, we are done by induction, so assume that we have equality.

Now repeat the process on the second term, choosing a $g' \in S_n$ such that $g'(wg^{-1})(g')^{-1}$ is a product of cycles, and conjugating $(g.c)p_L wg^{-1}$. Each application of this process results in one term in which the symmetric group element is a product of cycles (which has the desired form), and one term with the degree of the polynomial part strictly lesser and the degree of the Clifford part weakly lesser.

If the degree of the Clifford part ever strictly decreases, we are done. If not, the conjugation will eventually reduce the degree of the polynomial part to 0, so we have an element of the form $c'\sigma$, $c' \in C\ell_n$ and $\sigma \in S_n$. Choose a $g'' \in S_n$ such that $g''\sigma(g'')^{-1}$ is a product of cycles; then

$$g''c\sigma(g'')^{-1} = (g''c)g''\sigma(g'')^{-1}.$$

This now has the desired form. □

Proposition 3.2.3. *Let $w \in S_n$ and $(n_1, \ldots, n_r)$ a composition of n. Then*

$$[f_{\pm w;j_1,\ldots,j_n;\epsilon_1,\ldots,\epsilon_n}] = \sum d_{n_1,\ldots,n_r} h_{\pm n_1}^{x_1^{\ell_1} c_1^{k_1}} \ldots h_{\pm n_r}^{x_1^{\ell_r} c_1^{k_r}}$$

where $d_{n_1,\ldots,n_r} \in \mathbb{C}$ and $\ell_1, \ldots, \ell_r, k_1, \ldots, k_r \in \mathbb{N}$.

Proof. This follows immediately from the preceding lemmas. □

Proposition 3.2.3 allows us to write any element in $\mathrm{Tr}^{>}(\mathcal{H}_{tw})_{\bar{0}}$ or $\mathrm{Tr}^{<}(\mathcal{H}_{tw})_{\bar{0}}$ as a linear combination of the elements h_n. We will therefore direct our attention to these elements in future computations.

3.2.4 Gradings in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$

The next lemma follows from diagrammatic computations in the next section. We record it here for convenience of terminology.

Lemma 3.2.4. *The algebra $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ is $\mathbb{Z}$-filtered where $\deg\left(h_n^{x_1^{2a}}\right) \leq n$ for any $a \geq 0$.*

This is called the rank filtration. Denote by $\mathrm{Tr}^{>}(\mathcal{H}_{tw})_{\bar{0}}$ (resp. $\mathrm{Tr}^{<}(\mathcal{H}_{tw})_{\bar{0}}$) the subalgebra of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ generated by $h_n^{x_1^{2a}}$, $n \geq 1$ (resp. $n \leq 1$).

Lemma 3.2.5. *The algebra $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ is $\mathbb{Z}^{\geq 0}$-filtered where $\deg\left(h_n^{x_1^{2a}}\right) \leq a$ for any $a \geq 0$.*

Proof. Dots can slide through crossings modulo a correction term containing fewer dots. □

This is called the dot filtration, and corresponds to the differential filtration (given by $\deg(w_{\ell,k}) = k$) in W^-.

These filtrations are compatible, so $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ is $(\mathbb{Z} \times \mathbb{Z}^{\geq 0})$-filtered with $h_n^{x_1^{2a}}$ in bidegree (n, a). For $\omega \in \{>, <, 0\}$ denote the associated graded object by $\mathrm{gr}\,\mathrm{Tr}^{\omega}(\mathcal{H}_{tw})_{\bar{0}}$. Define a generating series for the graded dimension of $\mathrm{gr}\,\mathrm{Tr}^{\omega}(\mathcal{H}_{tw})_{\bar{0}}$ by

$$P_{\mathrm{gr}\,\mathrm{Tr}^{\omega}(\mathcal{H}_{tw})_{\bar{0}}}(t, q) = \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}, k \geq 0} \dim \mathrm{gr}\,\mathrm{Tr}^{\omega}(\mathcal{H}_{tw})_{\bar{0}}[r, k] t^r q^k.$$

The following is an easy calculation using Proposition 3.5.4 and Theorem 2.4.4. They are not used in the proof of the main result, but we record them here for convenience.

Proposition 3.2.6. *The graded dimensions of* $\operatorname{gr}\operatorname{Tr}^{>}(\mathcal{H}_{tw})_{\bar{0}}$ *and* $\operatorname{gr}\operatorname{Tr}^{<}(\mathcal{H}_{tw})_{\bar{0}}$ *are given by:*

$$P_{\operatorname{gr}(\mathcal{D}^{-})^{>}} = \prod_{r\geq 0}\prod_{k>0}\frac{1}{1-t^{2r+1}q^{k}};$$

$$P_{\operatorname{gr}(\mathcal{D}^{-})^{<}} = \prod_{r\leq 0}\prod_{k>0}\frac{1}{1-t^{2r-1}q^{k}}.$$

Note that the rank grading and dot gradings are shifted by 1 for clockwise bubbles (so d_2 is in bidegree $(1,2)$ and d_4 is in bidegree $(1,3)$). This is a consequence of the decomposition formula in Lemma 3.3.3.

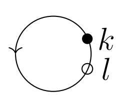
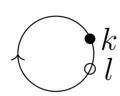
3.3 Bubbles

We investigate the endomorphisms of 1 in $\operatorname{Tr}(\mathcal{H}_{tw})$, known as bubbles. We prove that all bubbles can be written in terms of clockwise bubbles, and deduce formulas for moving bubbles past strands in the trace.

3.3.1 Definition and basic properties

Elements of $\operatorname{End}_{\mathcal{H}_{tw}}(1)$ are $\mathbb{C}$-linear combinations of possibly intersecting or nested closed diagrams, which may have dots. We can always separate the nested pieces, and resolve any crossing that occur between different closed diagrams using the defining relations and end up with non intersecting, not nested closed oriented diagrams. Each one can be deformed into an oriented circle, possibly with dots, via an isotopy. A single closed, oriented, non self intersecting diagram is called a bubble. They are the building blocks of endomorphisms of the identity object in $\mathcal{H}_{tw}$.

We define

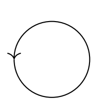
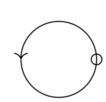
$\bar{d}_{k,l} :=$  and $d_{k,l} :=$  for $k,l \in \mathbb{Z}_{\geq 0}$.

Given any closed diagram with any configuration of dots, it is possible to collect the hollow dots and the solid dots together, possibly after multiplying the diagram by -1, by using relation (3.18). Solid dots move freely along caps and cups, and hollow dots may capture a negative sign while moving along caps or cups, depending on the orientation. After regrouping, we may assume that the dots are placed on the right middle side of the diagram as above.

Moreover, using the left two equations in relation (3.15), we can erase a pair of hollow dots, possibly by changing the sign of the diagram.

Therefore the set $\{d_{k,l}, \bar{d}_{k,l} | k \in \mathbb{Z}_{\geq 0}, l \in \{0,1\}\}$ is a spanning set for $\operatorname{End}_{\mathcal{H}_{tw}}(1)$.

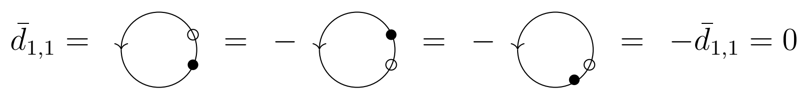
In our defining relations, we have that

$\bar{d}_{0,0} =$  $= 1$ and $\bar{d}_{0,1} =$  $= 0.$

Further, we have the following.

Lemma 3.3.1. *We have that $\bar{d}_{k,1} = 0$ and $d_{k,1} = 0$ for all non-negative integers k.*

Proof. An example computation shows that



where in the second equality, the negative sign comes from Relation (3.16), and the third equality comes from sliding the solid dot around.

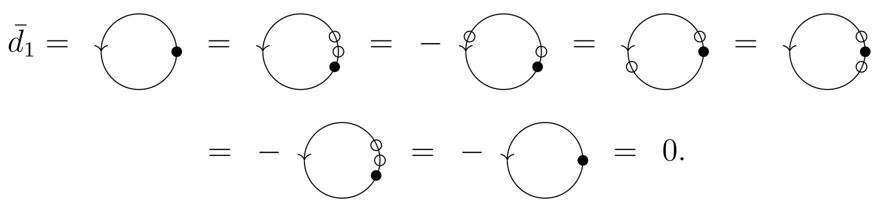
More generally, if we have k solid dots where k is an even integer, then sliding the hollow dot around the circle and passing it through k solid dots multiplies the diagram by $(-1)^{k+1}$, so the diagram is zero. If k is an odd number, sliding a solid dot around the circle and passing it through a hollow dot catches a minus sign, so these diagrams are zero as well.

These arguments do not depend on the orientation of the bubble, hence the result follows. □

From now on, we will assume that the second index in $\bar{d}_{k,l}$ and $d_{k,l}$ is always zero. We will omit it from our notation and write d_k instead of $d_{k,0}$, and $\bar{d}_k$ instead of $\bar{d}_{k,0}$.

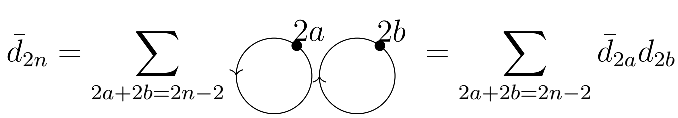
Lemma 3.3.2. *We have that $d_{2n+1} = \bar{d}_{2n+1} = 0$ for all non-negative integers n.*

Proof. Note that



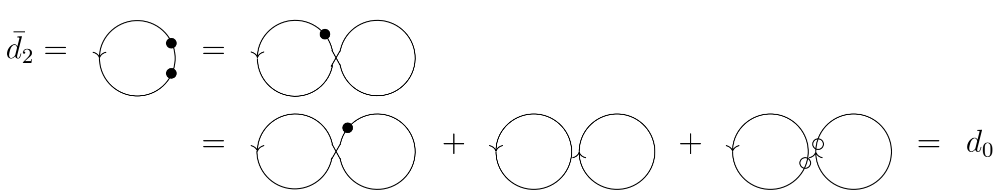
The same arguments works for any odd number of solid dots and works for clockwise oriented bubbles. □

Lemma 3.3.3. *We have that*



for any integers a, b and $n \geq 1$.

Proof. For the $n = 1$ case, we have the following computation:



where the first diagram on right hand side is zero since it contains a left curl, the second term is $\bar{d}_0 d_0 = d_0$ and the last term is zero by Lemma 3.3.1.

For general n, if you replace one of the solid dots with a right-twist curl, and slide the remaining $2n-1$ dots through the crossings using relations 3.20 and 3.21 repeatedly, we will get many resolution terms, consisting of a sum of product of counterclockwise and clockwise bubbles, some with only solid dots, some with hollow dots as well. The terms with hollow dots are zero, and so are the terms with an odd number of solid dots. Also, the figure eight shape contains a left twist curl, so it is zero as well, which proves the statement. □

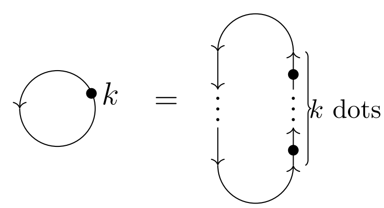
3.3.2 Algebraic independence of bubbles

A categorified Fock space action for $\mathcal{H}_{tw}$ is described in [CS15, Section 6.3]. $\mathcal{H}_{tw}$ acts on the category $\mathfrak{S}$, whose objects are induction and restriction functors between $\mathbb{Z}/2\mathbb{Z}$-graded finite dimensional $\mathbb{S}_n$-modules, for all $n \geq 1$. Morphisms of $\mathfrak{S}$ are natural transformations between the induction and restriction functors.

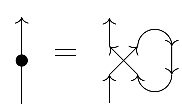
Following Khovanov's approach from [Kho14], let $\mathfrak{S}_n$ be the subcategory of $\mathfrak{S}$, whose objects start with induction or restriction functors from $\mathbb{Z}/2\mathbb{Z}$-graded finite dimensional $\mathbb{S}_n$-modules. For every $n \in \mathbb{Z}_{\geq 1}$, we have a functor $F_n : \mathcal{H}_{tw} \to \mathfrak{S}_n$ sending P to Ind_n^{n+1} and sending Q to Res_n^{n-1}.

Note that F_n sends $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ to the center of $\mathbb{S}_n$, which is same as the center of $\mathbb{C}[S_n]$.

Explicit descriptions of the actions of a crossing, a cup and a cap are provided in [CS15]. We would like to study the action of clockwise bubbles to show their algebraic independence. Note that d_{2k} is obtained as composition of a cup, k copies of $h_1^{x_1}$ and a cap.



Therefore to study the action of d_{2k}, we need to know the action of $h_1^{x_1}$ in addition to actions of cups and caps. Now $h_1^{x_1}$ is defined as a combination of caps, cups and crossings:



(3.22)

Using the explicit description of Fock space representation of $\mathcal{H}$ in [CS15], we compute the required actions. These computations give that $h_{-1}^{x_1}$ acts by sending

$$1 \mapsto J_{n+1} = \sum_{i=1}^{n} (1 - c_{n+1} c_i)(i, n+1).$$

This is the $(n+1)$-st even Jucys-Murphy element. Therefore $h_1^{x_1^2}$ acts by multiplication by J_{n+1}^2. This is analogous to the untwisted case where the same element acts as multiplication by a Jucys-Murphy element.

Finally, the action of the bubble d_{2k} is given by multiplication by

$$\sum_{i=1}^{n}(i \leftrightarrow n+1)J_{n+1}^{2k}(n+1 \leftrightarrow i) - c_n(i \leftrightarrow n+1)J_{n+1}^{2k}(n+1 \leftrightarrow i)c_1,$$

where $(i \leftrightarrow n)$ denotes the n-cycle $s_i s_{i+1} \ldots . s_{n-1}$.

Here we can apply the filtration argument on the number of disturbances of permutations as done in [Kho14, Section 4] to obtain the following.

Proposition 3.3.4. *The elements $\{d_{2k}\}_{k\geq 0}$ are algebraically independent, i.e. there is an isomorphism*

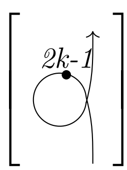
$$\mathrm{End}_{\mathcal{H}_{tw}}(1) \cong k[d_0, d_2, d_4, \ldots].$$

Therefore the bubbles are algebraically independent, and they form of a copy of a polynomial ring in infinitely many variables.

3.3.3 Counter-clockwise bubble slide moves

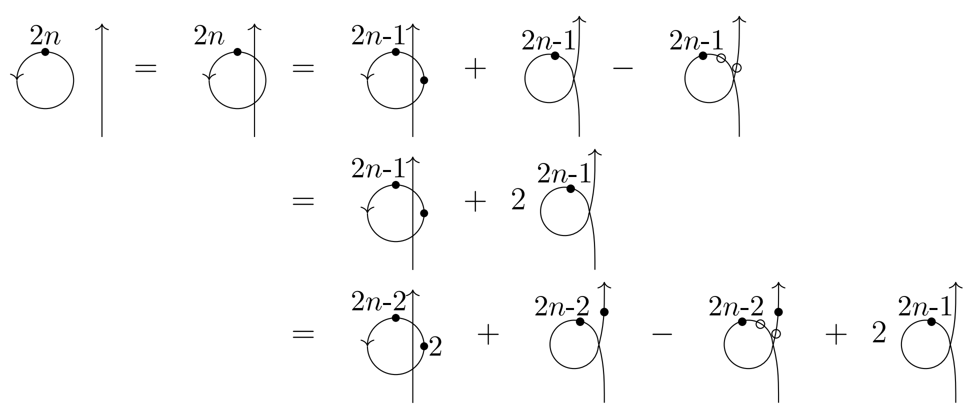
In order to describe $\mathrm{Tr}(\mathcal{H}_{tw})$ as a vector space, it would be convenient to have a standard form for our diagrams in the trace. In particular, we want to collect all the bubbles appearing in a diagram on the rightmost part of the diagram. In order to do so, we must describe how bubbles slide through upward and downward strands. Note that since we can work with local relations, the bubbles don't have to interact with solid dots or crossings, they can simply slide through under a crossing or under a solid dot.

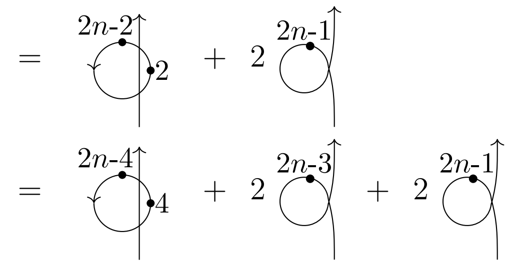
All calculations in this section take place in the trace, though we omit the brackets in some situations for readability.

Lemma 3.3.5. *We have that* $[\bar{d}_{2n}, h_1] = 2\sum_{k=1}^{n}$ 

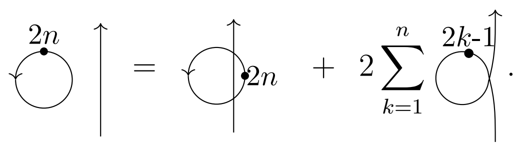
in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ *for any positive integer* n.

Proof. The proof is a direct computation, given below:



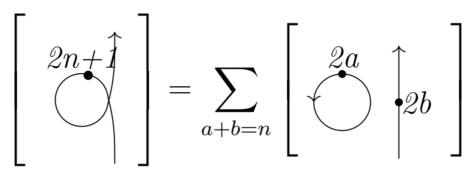


Continuing to slide dots in the first term in this way, we obtain:



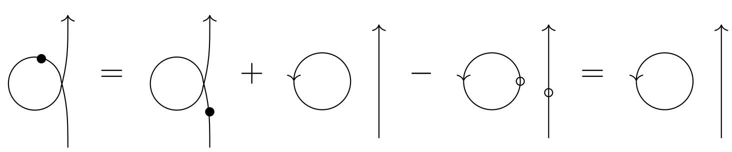
□

Lemma 3.3.6. *We have that*



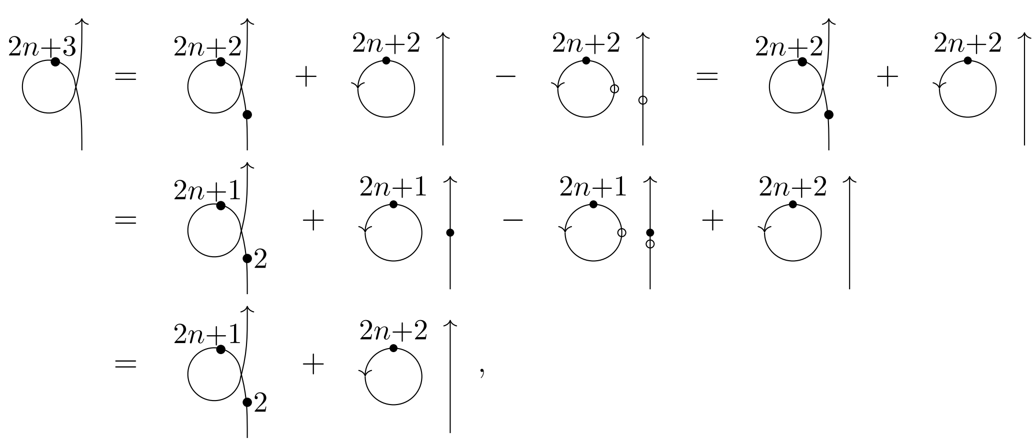
in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ *for any non-negative integer* n.

Proof. This is an easy computation using induction on n. The base case is

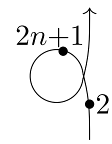


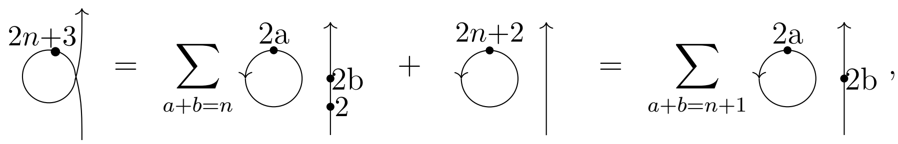
where the first term after the first equality contains a left twist curl, and the last term is zero since a bubble with a hollow dot is zero.

For the induction step, suppose the statement holds for $n \geq 1$. Then



where on the second line, we know that counter-clockwise bubbles with odd number of hollow dots are zero by Proposition 3.3.2, and the terms with hollow dots are zero by Proposition 3.3.1.

Now we can apply our induction hypothesis to the upper part of  to get that



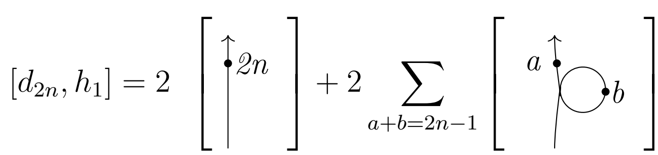
as desired. □

Obtaining an explicit formula for sliding counter-clockwise bubbles is difficult since we express their commutators in terms of left twist curls with some dots on the curl, whose resolution terms still leave us with counter-clockwise bubbles on the left side of $h_1^{x_i^a}$. However, the situation is better with clockwise oriented bubbles.

3.3.4 Clockwise bubble slide moves

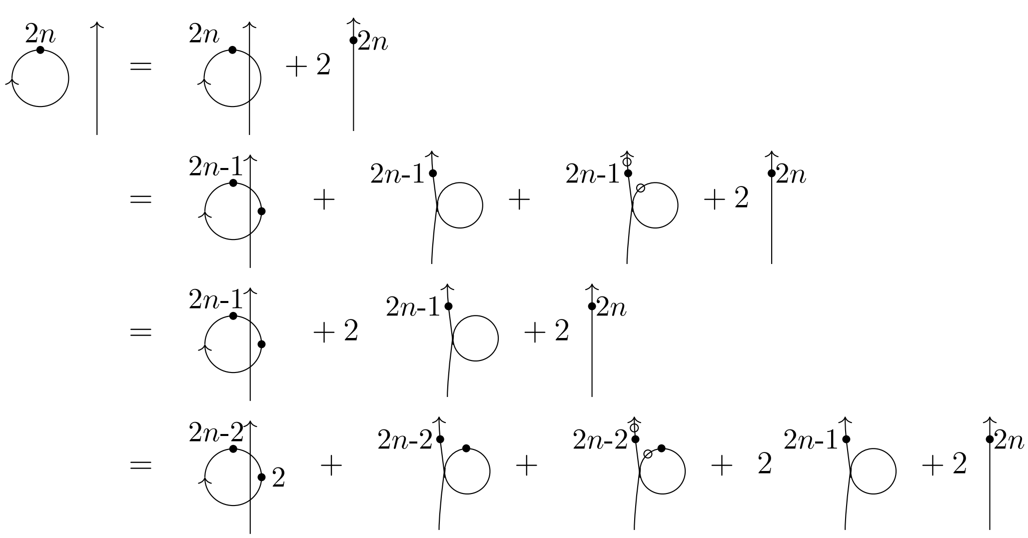
We can compute an explicit formula for clockwise bubble slides.

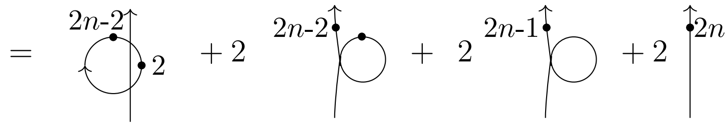
Lemma 3.3.7. *We have*



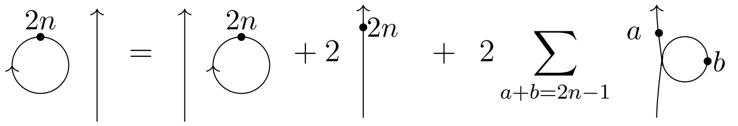
in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ *for all* $n \geq 0$.

Proof. This is a direct computation, given below:





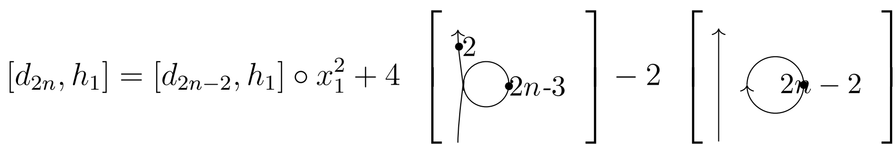
Continuing to slide dots in the first term in this way, we obtain:



∎

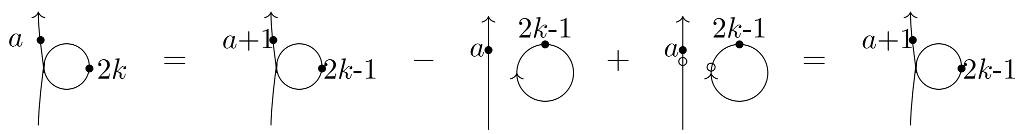
In particular, we can refine this statement to obtain the following recursive formula for computing $[d_{2n}, h_1]$.

Lemma 3.3.8. *We have*



in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ *for all* $n \geq 0$.

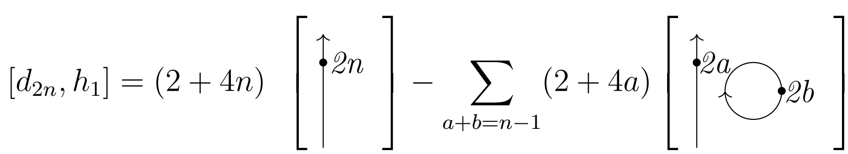
Proof. This lemma follows from the observation that



where the second term after the first equality is zero by Lemma 3.3.2, and the third term is zero by Lemma 3.3.1. Applying this result to the summands in the statement of Lemma 3.3.7 yields the result. ∎

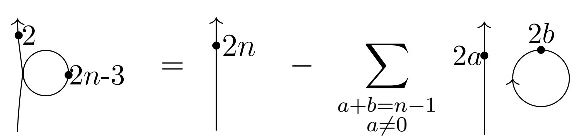
Finally, we obtain an explicit formula for computing $[d_{2n}, h_1]$.

Proposition 3.3.9. *We have*

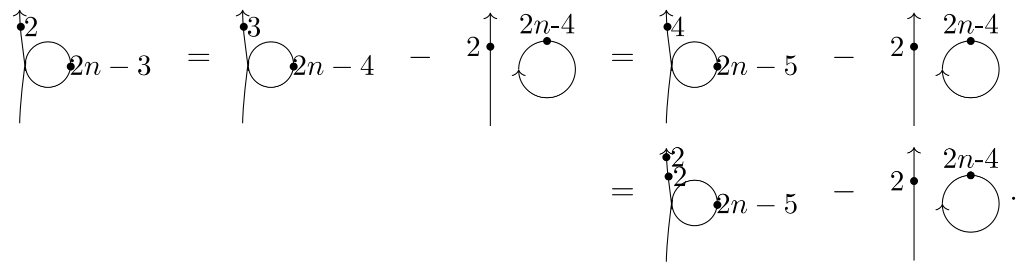


in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ *for all* $n \geq 0$.

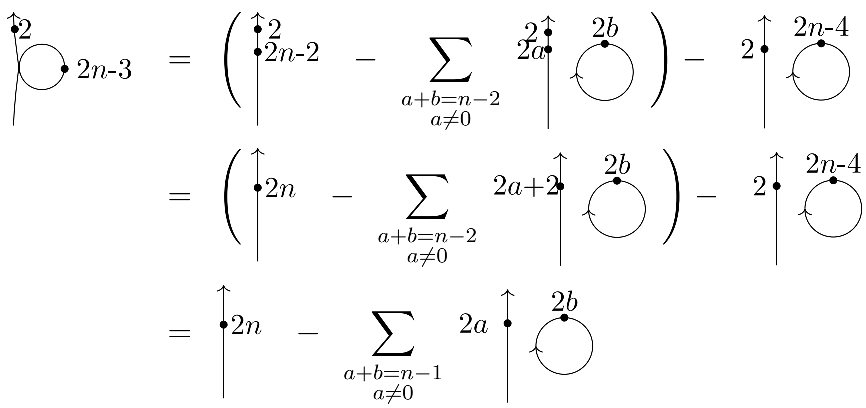
Proof. We claim that



for $n \geq 2$. We proceed via induction on n. The base case $n = 2$ is a direct computation. Now suppose the formula holds for some $n \geq 2$. Then



Now we can apply the induction hypothesis to the lower part of the first term in the last expression. This gives us:

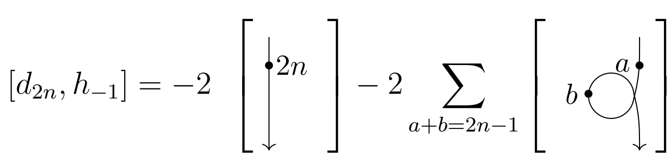


Applying this result to the recursive formula in Lemma 3.3.8 proves the statement.

□

Commutators of bubbles with downward strands are similar to those of bubbles with upward strands.

Lemma 3.3.10. *We have*

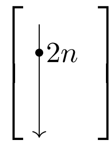
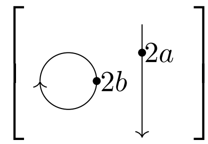


in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ *for all* $n \geq 0$.

Proof. This follows from a computation similar to those in the proofs of Lemmas 3.3.7 and 3.3.8. □

Finally we have an explicit formula for commutators of clockwise oriented bubbles and a single downward strand.

Proposition 3.3.11. *We have*

$$[d_{2n}, h_{-1}] = -(2+4n)$$  $$+ \sum_{a+b=n-1} (2+4a)$$ 

in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ *for* $n \geq 0$.

Proof. This follows from Lemma 3.3.10, using a similar argument as in the proof of Proposition 3.3.9. □

Note that in this formula, we are still left with clockwise bubbles on the left side of a downward strand, but with fewer dots on it. Hence the formula may be applied inductively in order to move all the bubbles to the rightmost part of the diagram.

3.4 Diagrammatic lemmas

This section contains some technical computations to derive relations between diagrams consisting of up and down strands. These relations allow us to find a generating set of $\mathrm{Tr}(\mathcal{H}_{tw})$ in Section 6.

3.4.1 Differential degree zero part of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$

The differential degree zero part of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ consists of elements $\{h_n\}_{n\in\mathbb{Z}}$. First, we have the following basic fact.

Proposition 3.4.1. *[Ree17, Proposition 3.9] We have*

$$h_{2n} \cong 0$$

for any $n \in \mathbb{Z}$.

Proof. By Proposition 3.5.4, the proof in the Hecke-Clifford algebra applies here, as well. □

The elements of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ satisfy the following relations.

Lemma 3.4.2. *The following commutators are zero for all non-negative integers* n, m*:*

1. $[h_n, h_m] = 0$,
2. $[h_{-n}, h_{-m}] = 0$,
3. $[h_{2n}, h_{-2n}] = 0$.

Proof. Parts (1) and (2) follow from the fact that similarly oriented strands can be split apart when they cross twice. Part (3) follows immediately from Proposition 3.4.1. □

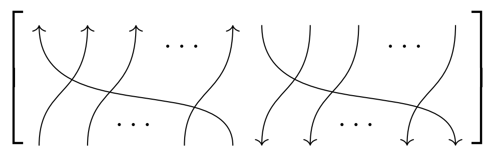
To obtain a copy of the twisted Heisenberg algebra in the $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, we need to look at commutators between elements with odd numbers of oppositely oriented strands.

Lemma 3.4.3. *We have, for any $n, m \in \mathbb{Z}^{\geq 0}$,*

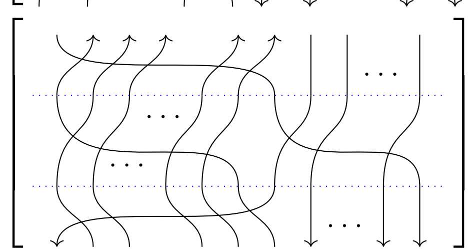
$$[h_{2n+1}, h_{-2m+1}] = (\delta_{n,-m})(-2(2n+1)).$$

Proof. First note that [CLLS16, Lemma 19] and [CLLS16, Lemma 20] holds in our twisted case with a small modification, since all the arguments in their proofs use the fact that the resolution terms contain left twist curls, hence are zero. There are extra resolution terms with hollow dots due to relation (3.10), but two hollow dots on a diagram containing a left twist curl still gives zero. The only modification comes in the case $m = n$ where we get two copies of counter-clockwise bubbles instead of one, since a two hollow dots on a counter-clockwise bubble end up canceling each other without changing the sign of the diagram. We immediately get that when $m \neq n$, our commutator is zero since we have no solid dots. Therefore we have

$h_{2n+1}h_{-2m+1} =$



$=$



$$= h_{-(2m+1)}h_{2n+1}(-2\bar{d}_0(2n+1)).$$

Hence $[h_{(2n+1)}, h_{-(2m+1)}] = \delta_{n,-m}(-2(2n+1))$. □

Therefore the subset $A = \{h_{(2n+1)}\}_{n\in\mathbb{Z}}$ of the filtration degree zero part of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ is isomorphic to the twisted Heisenberg algebra via

$$\begin{aligned} \phi: \quad \mathfrak{h}_{tw} &\xrightarrow{\sim} A \\ h_{\frac{2n+1}{2}} &\mapsto \frac{1}{2}h_{-(2n+1)}. \end{aligned}$$

In the W-algebra W^-, we have an isomophic copy of the twisted Heisenberg algebra as well, given by $B = \{\omega_{2n+1,0}\}_{n\in\mathbb{Z}}$, with the isomorphism given by

$$\begin{aligned} \psi: \quad \mathfrak{h}_{tw} &\xrightarrow{\sim} B \\ h_{\frac{2n+1}{2}} &\mapsto \frac{1}{\sqrt{2}}\,\omega_{2n+1,0}. \end{aligned}$$

Therefore we have an isomorphism between the degree zero part of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ and the degree zero part of W^{-}:

$$\begin{aligned} \psi \circ \phi^{-1}: \quad & A & \xrightarrow{\sim} & B \\ & h_{-(2n+1)} & \mapsto & \sqrt{2}\, w_{2n+1,0}. \end{aligned}$$

3.4.2 Nonzero differential degree part of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$

We have the following basic facts about diagrams in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}^{\geq}$, which we may copy from the corresponding facts in the trace of the affine Hecke-Clifford algebra because of the triangular decomposition of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ described in Proposition 3.5.4.

Proposition 3.4.4. *[Ree17, Propositions 3.9, 4.2] In $Tr(\mathfrak{H}^{C})$ for any $m, n \in \mathbb{Z}$, we have*

$$h_{2n+1}^{x_1^{2m+1}} = 0,$$

$$h_{2n}^{x_1^{2m}} = 0.$$

Hence any diagram containing an odd cycle with an odd number of dots or an even cycle with an even number of dots is zero. Therefore, the difference of the number of strands and number of solid dots must be odd. This agrees with the fact that in the W-algebra W^{-}, $l-k$ has to be an odd number for $w_{l,k}$.

The generators of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}^{\geq}$ satisfy the following relations.

Lemma 3.4.5. *For $m, n \in \mathbb{Z}$ with $mn > 0$, we have*

1. $[h_{2m}^{x_1}, h_{2n}^{x_1}] = 2(n-m)h_{2n+2m}^{x_1}$.
2. $[h_{m}^{c_1}, h_{n}^{c_1}] = -2h_{n}^{c_1}$.

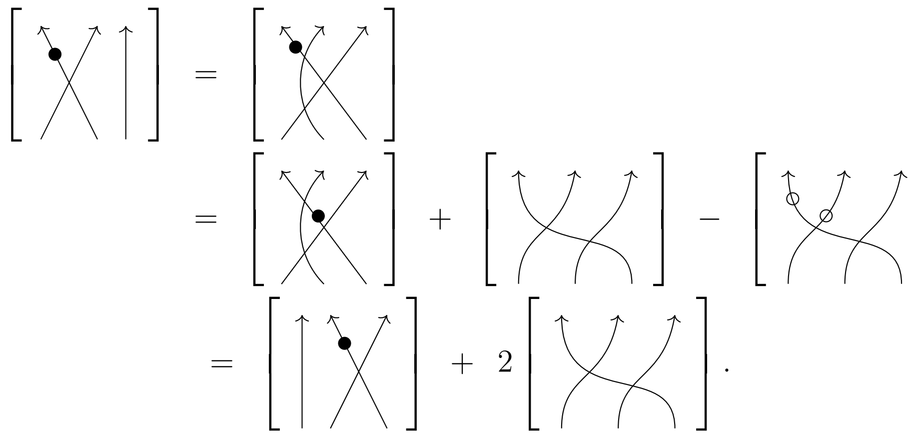
Proof. Part (1) is a slight modification of [CLLS16, Lemma 23]. By Proposition 3.4.4, if at least one of the indices inside the commutator is odd, the commutator will be zero. Hence we will work with the case where both indices are even numbers. The modification we need in [CLLS16, Lemma 23] is a result of us having two resolution terms in our relations (4.33) and (4.34). As a consequence of having even number of strands in both of our elements, canceling the two empty dots in our resolution terms give rise to the same sign as the other resolution term, hence we have a coefficient of two in front of our result.

Part (2) follows easily the proof of [CLLS16, Lemma 23] since moving an empty dot through a crossing is for free in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, and we get a negative sign from changing relative heights of hollow dots. □

Lemma 3.4.6. *For $n \geq 0$, we have*

$$[h_{\pm 2n}^{(x_1+\ldots+x_{2n})}, h_1] = \pm 4n h_{\pm(2n+1)}.$$

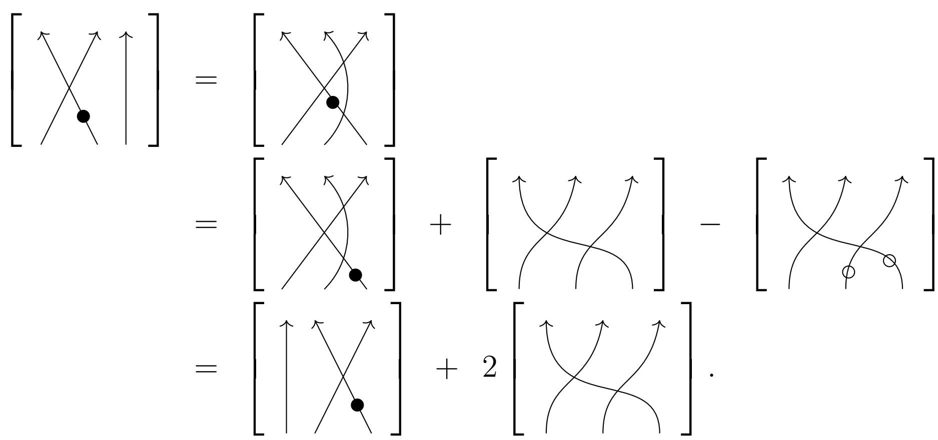
Proof. First note that we have:



(3.23)

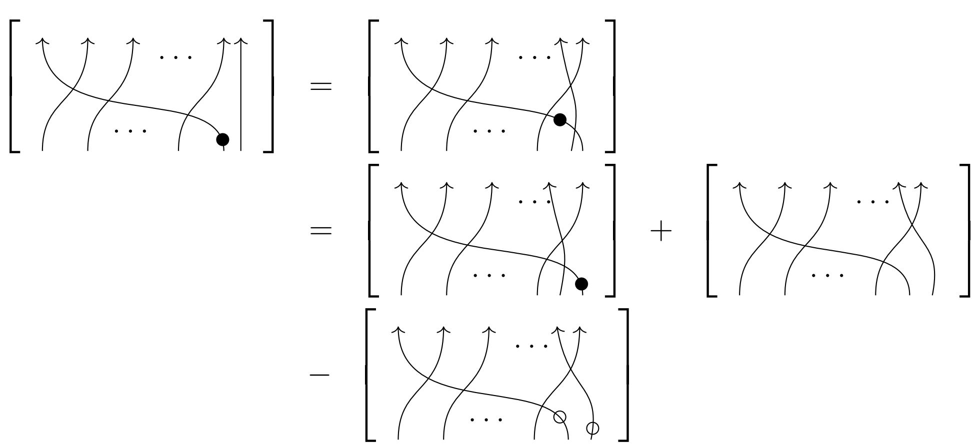
Hence $[h_2^{x_1}, h_1] = 2h_3$.

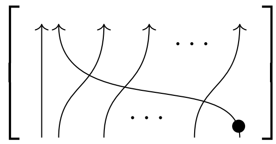
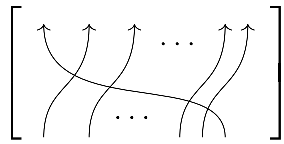
Next, moving the solid dot in $h_2^{x_2}$ around to the bottom of the crossing using the trace relation gives:



So, $[h_2^{(x_1+x_2)}, h_1] = 4h_3$.

Next, we claim that $[h_{2n}^{x_{2n}}, h_1] = 2h_{2n+1}$ for any n. Indeed, we have:



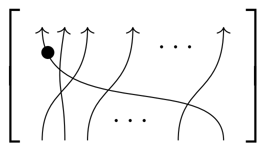
$$=$$  $$+\ 2$$  ,

where the last equality is obtained by pushing the crossings at the bottom of the diagrams without dots to the top. Indeed, diagrammatic calculations similar to the above give that

$$[h_{2n}^{x_a}, h_1] = 2h_{2n+1}$$

for any $1 < a \leq 2n$.

Finally, note that

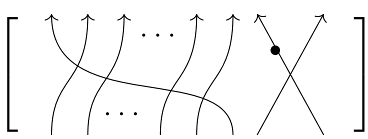
$$h_{2n}^{x_1} h_1 \quad =$$  .

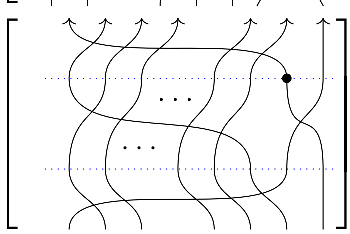
The dot will slide over the top-leftmost crossing in the same manner as in Equation (3.23), meaning the correction terms will cancel out. Hence, we have the desired result. □

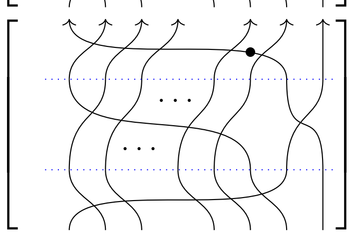
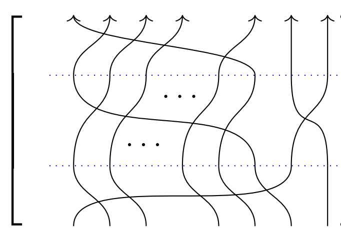
Lemma 3.4.7. *Let m be an odd integer. We have*

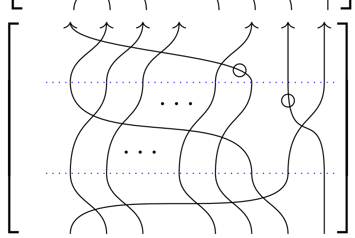
$$[h_2^{(x_1+x_2)}, h_m] = 4mh_{m+2}.$$

Proof. We compute directly:

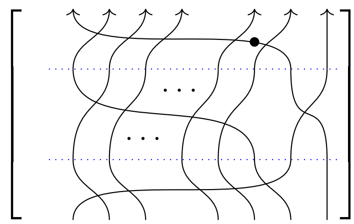
$$h_m h_2^{x_1} \quad =$$ 

$$=$$ 

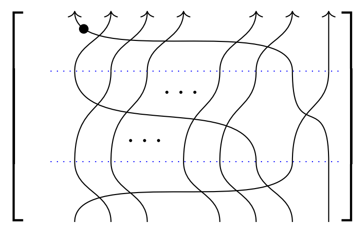
$$=$$  $$-$$ 

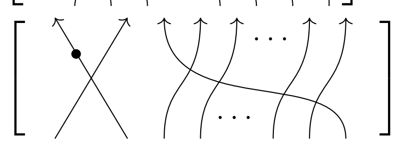
$$+$$  .

Cancelling the empty dots in the last term results in a change in sign, and both of the latter diagrams are $(m+2)$-cycles. Hence we have:

$=$  $- \ 2h_{m+2}$

Sliding the solid dot in the first diagram all the way to the left results in m total crossing resolutions, each of which yieds a term of $-2h_{m+2}$. So,

$=$  $- \ 2mh_{m+2}$

$=$  $- \ 2mh_{m+2}$

Hence we have

$$[h_2^{x_1}, h_m] = 2mh_{m+2}.$$

A similar computation gives that

$$[h_2^{x_2}, h_m] = 2mh_{m+2},$$

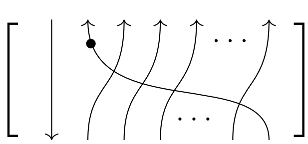
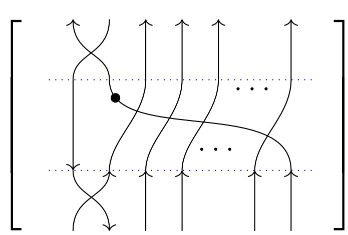
giving the desired result. □

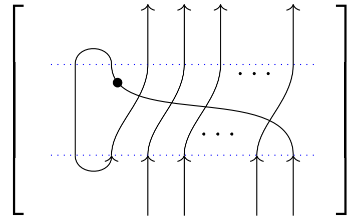
Lemma 3.4.8. *We have*

$$[h_{2n}^{(x_1+x_2+\ldots+x_{2n})}, h_{-(2m+1)}] = \begin{cases} -4(2m+1)h_{2n-2m-1} & \text{if } n > m \geq 1 \\ 0 & \text{if } n = m \geq 1 \\ -2(2m+1)h_{2n-2m-1} & \text{if } 1 \leq n < m. \end{cases}$$

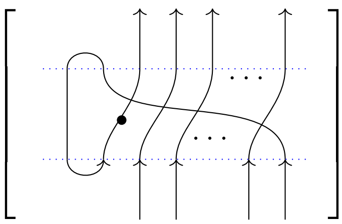
Proof. We follow the methods of [CLLS16, Lemma 26], substituting our new relations as necessary.

As in that case, let $\beta_n = h_{2n}^{x_1}$ and $\alpha_m = h_{2m+1}^{x_1}$, and proceed by induction on m. When $m = 1$, we can compute directly:

 $\overset{(3.9)}{=}$  (3.24)

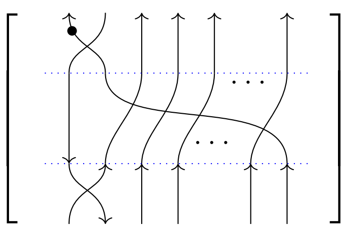
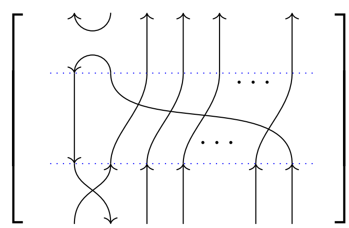
$+\ 2$  (3.25)

where the trailing terms arising from relation (3.9) have the same sign after cancelling the empty dots, and thus add together. We claim that the diagram in the second term is h_{2n-1}. Indeed, sliding the dot gives:

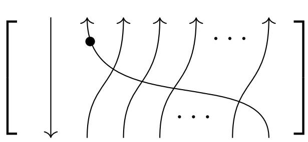
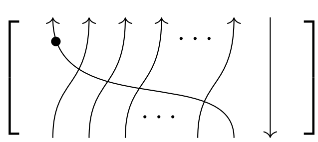
 $\overset{(4.33)}{=} \quad \overline{d}_{0,0}h_{2n-1} + \overline{d}_{0,1}h_{2n-1} = h_{2n-1}$

by relations (3.11) and (3.15).

Now, sliding the solid dot over the crossing on the right hand side of Equation (3.24) gives:

$\overset{(4.33)}{=}$  $+\ 2$ 

where the trailing terms arising from relation (4.33) have the same sign after canceling the empty dots, and thus add together. We can use the trace relation to slide the top cup in the second term to the bottom; after simplication, this term is therefore equal to h_{n-1}. The first term is equal to $\beta_n\alpha_{-1}$ as in [CLLS16, Lemma 26]. Thus,

 $=$  $+\ 4h_{2n-1}$

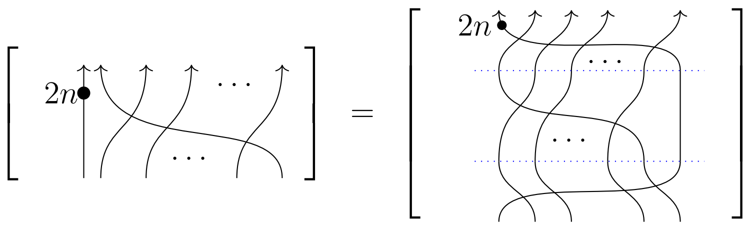
as desired. The base case of the induction is proved. The induction step follows from examination of the Jacobi identity, exactly as in [CLLS16, Lemma 26], using our Lemma 3.4.6 in place of [CLLS16, Lemma 24].

□

Lemma 3.4.9. *Let $n \in \mathbb{Z}$. We have*

$$[h_1^{x_1^2}, h_{2n-1}] = 2h_{2n}^{x_1+\ldots+x_{2n}} + 2h_{2n}^{x_2+\ldots+x_{2n-1}}.$$

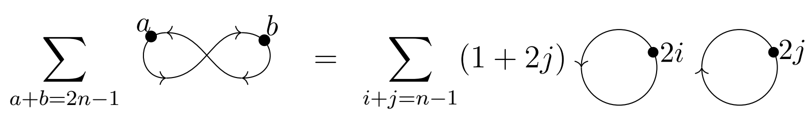
Proof. This is a straightforward diagrammatic calculation similar to Lemmas 3.4.7 and 3.4.8. We have



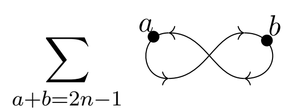
Sliding the dots all the way to the right side of the diagram results in $2(2n-1)$ resolution terms. Each of these resolution terms contains a $2n$-cycle and a single solid dot - there are 2 resolution terms containing a solid dot on the first strand and 2 containing a solid dot on the last strand, and 4 resolution terms with a dot on each other strand. All empty dots cancel in such a way that no resolution terms cancel with each other. The result follows. $\square$

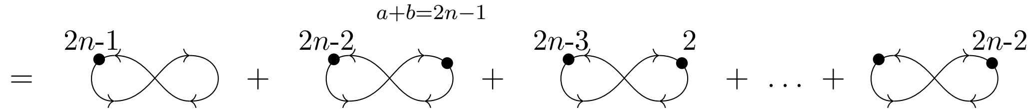
The following lemmas will allow us to generate bubbles with arbitrary numbers of dots using just $h_{\pm 1}^{x_1^2}$.

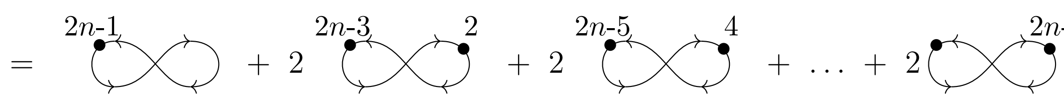
Lemma 3.4.10. *We have*



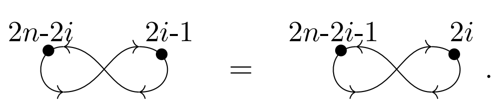
Proof. We compute:



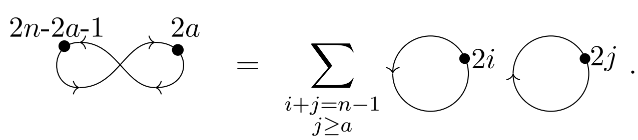




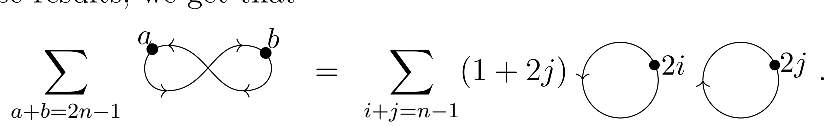
because we have



Moreover, we can decompose these figure eights into a linear combination of products of two bubbles using dot slide relations 4.33 and 4.34 as follows:



Combining these results, we get that



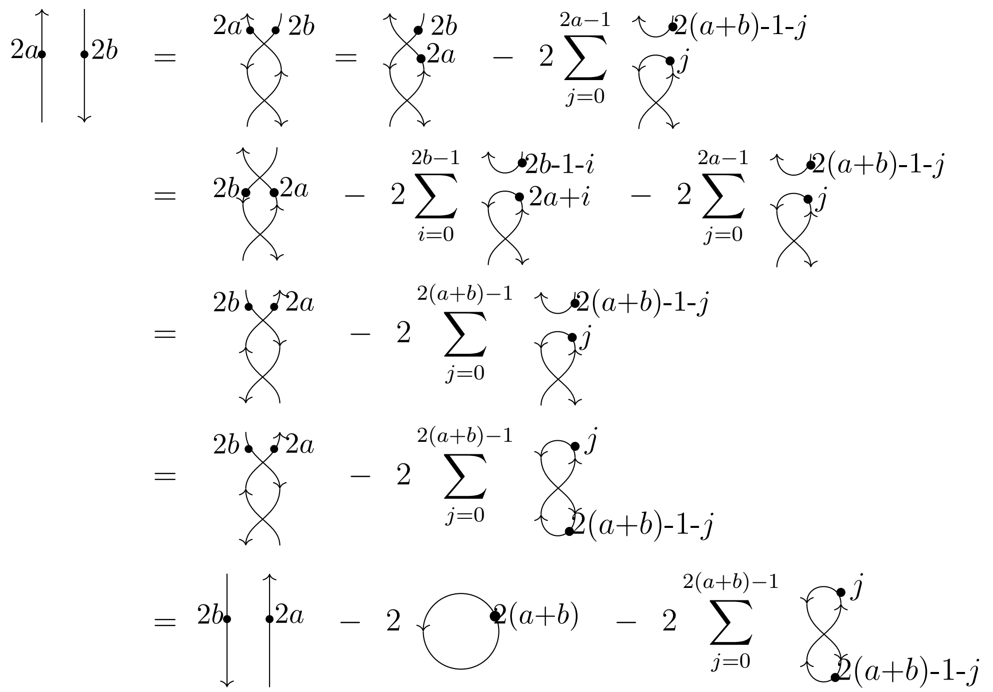
$\square$

Lemma 3.4.11. *We have*

$$[h_1^{x_1^{2a}}, h_{-1}^{x_1^{2b}}] = -2\bar{d}_{2(a+b)} - \sum_{i+j=2(a+b)-1} (2+4j)\bar{d}_{2i}d_{2j}$$

for $a, b \in \mathbb{Z}_{\geq 0}$.

Proof. We compute:



Therefore $[h_1^{x_1^{2a}}, h_{-1}^{x_1^{2b}}] = -2\bar{d}_{2(a+b)} - \sum_{i+j=2(a+b)-1} (2+4j)\bar{d}_{2i}d_{2j}$. □

3.5 Algebra isomorphism

In this section, we will study the structure of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, first as a vector space and then as an algebra. We show that $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ has a triangular decomposition into two copies of the trace of $\mathfrak{H}_n^C$ and a polynomial algebra. We then describe a generating set for $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, which allows us to define the algebra homomorphism to W^-. Finally, we prove that this homomorphism is an isomorphism.

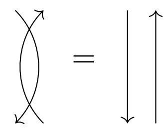
3.5.1 Trace of $\mathcal{H}_{tw}$ as a vector space

Let $m, n \geq 0$ and define $J_{m,n}$ to be the 2-sided ideal in $\mathrm{End}_{\mathcal{H}_{tw}}(P^m Q^n)$ generated by diagrams which contain at least one arc connecting a pair of upper points.

Lemma 3.5.1. *There exists a split short exact sequence*

$$0 \to J_{m,n} \to \mathrm{End}_{\mathcal{H}_{tw}}(P^m Q^n) \to (\mathfrak{H}^C)_m^{op} \otimes \mathfrak{H}_n^C \otimes \mathbb{C}[d_0, d_2, d_4....] \to 0.$$

Proof. In $\mathrm{End}_{\mathcal{H}_{tw}}(P^mQ^n)$, due to the middle diagram in relation (3.9), we can assume our diagrams have no crossing between opposite oriented strands. Taking the quotient $\mathrm{End}_{\mathcal{H}_{tw}}(P^mQ^n)/J_{m,n}$ kills diagrams with cups connecting two upper points, and those with caps connecting two lower points. Therefore we are left with diagrams, possibly with bubbles, which have no caps or cups and have crossings only among like-oriented strands. Note that in the quotient $\mathrm{End}_{\mathcal{H}_{tw}}(P^mQ^n)/J_{m,n}$, the diagram in relation (3.10) simplifies to



and therefore we can move the bubbles to the rightmost part of our diagrams for free. This gives us a short exact sequence

$$0 \to J_{m,n} \to \mathrm{End}_{\mathcal{H}_{tw}}(P^mQ^n) \to \mathrm{End}_{\mathcal{H}_{tw}}(P^m) \otimes \mathrm{End}_{\mathcal{H}_{tw}}(Q^n) \otimes \mathrm{End}_{\mathcal{H}_{tw}}(1) \to 0.$$

By [CS15, Proposition 7.1], we have that $\mathrm{End}_{\mathcal{H}_{tw}}(P^m)$ is isomorphic to $(\mathfrak{H}^C)^{op}_m$ and that $\mathrm{End}_{\mathcal{H}_{tw}}(Q^n)$ is isomoprhic to $\mathfrak{H}^C_n$. By Proposition 3.3.4, it follows that $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ is isomorphic to $\mathbb{C}[d_0, d_2, d_4....]$. Hence the result follows. □

Lemma 3.5.2. *If $f, g \in \mathfrak{H}^C_n$ such that $fg = 1$, then $f, g \in \mathcal{C}\ell_n \rtimes \mathbb{C}[S_n] \subset \mathfrak{H}^C_n$.*

Proof. There is an $\mathbb{N}$-filtration on $\mathfrak{H}^C_n$ given by $\deg(x_i) = 1$ for $i \in \{1, ..., n\}$ and other generators have degree zero. Under this filtration, the degree zero part of $\mathfrak{H}^C_n$ is the semidirect product $Cl_n \rtimes \mathbb{C}[S_n]$. Therefore, in the associated graded object, we see that if $fg = 1$, $\deg(gr(f)gr(g)) = \deg(gr(f)) + \deg(gr(g)) = \deg(1) = 0$, hence $gr(f), gr(g)$ are in degree zero part. Therefore $f, g \in \mathcal{C}\ell_n \rtimes \mathbb{C}[S_n]$. □

Lemma 3.5.3. *The indecomposable objects of $\mathcal{H}_{tw}$ are of the form P^mQ^n for $m, n \in \mathbb{Z}_{\geq 0}$.*

Proof. First, note that if QP appears in an object, that object can be decomposed into more components using the diagram in relation (3.10). Hence all indecomposable objects must be of the form P^mQ^n.

On the other hand, to see that every sequence of the form P^mQ^n is an indecomposable object, we will show that any idempotent in $\mathrm{End}(P^mQ^n)$ has to be the identity.

Let f, g be two maps as mentioned in Lemma 3.5.2. Note that gf is an idempotent since $(gf)(gf) = g(fg)f = gf$. Since we had the splitting short exact sequence $0 \to J_{m,n} \to \mathrm{End}_{\mathcal{H}_{tw}}(P^mQ^n) \to \mathrm{End}(P^m) \otimes \mathrm{End}(Q^n) \otimes \mathrm{End}(id) \to 0$ in Lemma 3.5.1, we know that the maps f and g will decompose into (f_1, f_2) and (g_1, g_2) where $f_1, g_1 : P^m \to P^m$ and $(f_2, g_2) : Q^n \to Q^n$. Now g_1f_1 is the identity map in $\mathrm{End}(P^m)$, and by the above lemma $g_1, f_1 \in \mathcal{C}\ell_n \rtimes \mathbb{C}[S_n]$. Similarly, $f_2, g_2 \in \mathcal{C}\ell_n \rtimes \mathbb{C}[S_n]$.

But in $\mathcal{C}\ell_n \rtimes \mathbb{C}[S_n]$, $g_1f_1 = 1$ implies that $f_1g_1 = 1$ as well. To see this, consider the diagrams corresponding to g_1 and f_1 which consist of a permutation and some hollow dots on top. After composing these diagrams, we can collect all the hollow dots on the top since hollow dots can pass through crossing for free, possibly gaining a sign. Furthermore, each strand has an even number of hollow dots, since this composition is the identity map. So, the hollow dots cancel with each other. This shows that the corresponding permutations of f_1 and g_1 are inverses of each other, and in particular they commute. Therefore $f_1 g_1 = 1$. Similarly, $f_2 g_2 = 1$. Thus we have that $fg = 1$. □

Proposition 3.5.4. *We have the triangular decomposition of* $\mathrm{Tr}(\mathcal{H}_{tw})$*:*

$$\mathrm{Tr}(\mathcal{H}_{tw}) \cong \bigoplus_{m,n \in \mathbb{Z}_{\geq 0}} \mathrm{Tr}((\mathfrak{H}_m^C)^{op} \otimes \mathfrak{H}_n^C \otimes \mathbb{C}[d_0, d_2, d_4 ....]).$$

Proof. As shown in [BGHL14], to find $\mathrm{Tr}(\mathcal{H}_{tw})$, it is enough to consider the direct sum over indecomposable objects of endomorphism spaces of objects of H_{tw}. Let $I = \mathrm{span}_{\mathbb{C}}\{fg - gf\}$ where $f : x \to y$ and $g : y \to x$ for x, y objects of a $\mathbb{C}$-linear category. Therefore by Lemma 3.5.3 we have

$$\mathrm{Tr}(\mathcal{H}_{tw}) \cong \Big(\bigoplus_{m,n \in \mathbb{Z}_{\geq 0}} \mathrm{End}_{\mathcal{H}_{tw}}(P^m Q^n) \Big) / I.$$

By Lemma 3.5.1, this gives us

$$\mathrm{Tr}(\mathcal{H}_{tw}) \cong \Big(\bigoplus_{m,n \in \mathbb{Z}_{\geq 0}} ((\mathfrak{H}_m^C)^{op} \otimes \mathfrak{H}_n^C \otimes \mathbb{C}[d_0, d_2, d_4 ....]) \oplus J_{m,n} \Big) / I.$$

Recall that the ideal $J_{m,n}$ is generated by diagrams containing at least one cup connecting two upper points. Therefore, the diagrams in $J_{m,n}$ must also contain caps, since they are dealing with endomorphisms. Using the trace relation and the relations in $\mathcal{H}_{tw}$, we can express the elements of $J_{m,n}$ as direct sum of endomorphisms of $P^{m'} Q^{n'}$ for $m' \leq m$ and $n' \leq n$. Hence we have

$$\begin{aligned} \mathrm{Tr}(\mathcal{H}_{tw}) &\cong \bigoplus_{m,n \in \mathbb{Z}_{\geq 0}} \mathrm{Tr}((\mathfrak{H}_m^C)^{op} \otimes \mathfrak{H}_n^C \otimes \mathbb{C}[d_0, d_2, d_4 ....]) \\ &\cong \Big(\bigoplus_{m,n \in \mathbb{Z}_{\geq 0}} \mathrm{Tr}((\mathfrak{H}_m^C)^{op} \otimes \mathfrak{H}_n^C) \Big) \otimes \mathbb{C}[d_0, d_2, d_4 ....]. \end{aligned}$$

□

3.5.2 Generators of the algebra $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$

The following gives a generating set for $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ as an algebra.

Lemma 3.5.5. *The algebra* $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ *is generated by* h_{-1}, $h_{\pm 2}^{(x_1+x_2)}$, *and* $d_0 + d_2$.

Proof. First, Proposition 3.3.11 implies that h_1 and $(d_0 + d_2)$ allow us generate a differential degree two element $h_1^{x_1^2}$; since all relations in $\mathcal{H}_{tw}$ are local, we can evaluate the commutator $[h_1^{x_1^2}, (d_0 + d_2)]$ by moving the dot to the bottom of the upward strand and sliding the bubbles over the upper portion. We can therefore apply Lemma 3.3.11 repeatedly to show that $\mathrm{ad}(d_0 + d_2)^n h_1$ has a leading term of $h_1^{x_1^{2n}}$.

By Lemma 3.4.7, the elements h_{-1} and $h_2^{x_1+x_2}$ are sufficient to generate h_{2m+1} for all integers $m > 0$. Then we can generate $h_{2n}^{x_1+\ldots+x_n}$ from $h_1^{x_1^2}$ and h_{2m+1} by using Lemma 3.4.9. Lemma 3.4.8, h_{-1} and $h_{2n}^{x_1+x_2+\ldots+x_n}$ allow us to generate h_{2r+1} for all integers r.

Proposition 3.2.3 implies that all elements with nonzero rank degree can be written as a sum of elements of the form $h_{\pm n}^{x_1^\ell c_1^k}$. By Propositions 3.4.1 and 3.4.4, all elements of this form except for the ones generated in the preceding paragraphs are 0 in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, so we have generated all of $\mathrm{Tr}^>(\mathcal{H}_{tw})_{\bar{0}}$ and $\mathrm{Tr}^<(\mathcal{H}_{tw})_{\bar{0}}$.

Finally, Lemma 3.4.11 allows us to generate d_{2n}, applying Lemma 3.3.3 to split up the $\overline{d}_{2n}$ terms. □

3.5.3 The isomorphism

There is an obvious isomorphism of vector spaces between the Fock space representations of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ and W^-:

$$\phi : V = \mathbb{C}[h_1, h_2, \ldots] \to \mathbb{C}[w_{-1,0}, w_{-2,0}, \ldots] = \mathcal{V}_{1,0}.$$

Recall that each algebra acts faithfully on its Fock space representation.

Lemma 3.5.6. *The map ϕ in Equation* (3.5.3) *commutes with the action of the twisted Heisenberg subalgebras in V and $\mathcal{V}_{1,0}$, i.e.:*

$$\phi(h_r v) = \sqrt{2} w_{-r,0} \phi(v).$$

Proof. The vector space realizations of V and $\mathcal{V}_{1,0}$ in Equation (3.5.3) imply that the action of h_r on V is simply the adjoint action of h_r on the subalgebra $\mathrm{Tr}^>(\mathcal{H}_{tw})_{\bar{0}}$, and the action of $w_{-r,0}$ on $\phi(v)$ is the adjoint action of $w_{-r,0}$ on $(W^-)^-$. The Lemma follows from our computation of these twisted Heisenberg relations in Propositions 3.1.5 and 3.4.3. □

Lemma 3.5.7. *For any $v \in V$ we have $\phi((d_0 + d_2)v) = -2w_{0,3}\phi(v)$.*

Proof. Propositions 3.1.5 and 3.3.9 give that $w_{0,3}$ maps $w_{-1,0}$ to an element with leading term $w_{-1,2}$, and $(d_0 + d_2)$ maps h_1 to an element with leading term $h_1^{x_1^2}$. Comparision of the actions of these terms on the twisted Heisenberg subalgebras on either side gives that that their images in the endomorphisms of the Fock space are identical. □

Lemma 3.5.8. *For any $v \in V$ we have $\phi(h_{\pm 2}^{(x_1+x_2)} v) = 2\sqrt{2}(w_{\mp 2,1} + w_{\mp 2,0})\phi(v)$.*

Proof. This follows from comparision of Lemma 3.4.7 and Proposition 3.1.5. □

Now extend ϕ to a map

$$\Phi : \mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}} \longrightarrow W^-/\langle w_{0,0}, C-1 \rangle$$

by mapping

$$h_1 \mapsto \sqrt{2} w_{-1,0} \qquad h_{\pm 2}^{(x_1+x_2)} \to 2\sqrt{2} w_{\mp 2,1} + w_{\mp 2,0} \qquad d_2 + d_0 \mapsto -2w_{0,3}$$

and extending algebraically, i.e.

$$\Phi(a_1 \dots a_k) = \Phi(a_1) \dots \Phi(a_k)$$

for generators $a_1, \dots, a_k$ of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$.

Lemma 3.5.9. *The map Φ above is well defined.*

Proof. Suppose $A \in \mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ has two representations in terms of generators, $A = a_{i_1} \dots a_{i_k} = a_{j_1} \dots a_{j_\ell}$. Then $a_{i_1} \dots a_{i_k}.V = a_{j_1} \dots a_{j_\ell}.V$, so applying Φ gives $\Phi(a_{i_1} \dots a_{i_k}).\mathcal{V}_{1,0} = \Phi(a_{j_1} \dots a_{j_\ell}).\mathcal{V}_{1,0}$. Hence $\Phi(a_{i_1} \dots a_{i_k}) = \Phi(a_{j_1} \dots a_{j_\ell})$ by the faithfulness of the Fock space representation for W^-. □

Theorem 3.5.10. *The map Φ is an isomorphism of algebras.*

Proof. We immediately have that Φ is surjective, because it maps generators to generators. Thus, it remains to show that Φ is injective. Let $A := a_{i_1} \dots a_{i_k} \in \mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ and assume that $\Phi(A).\mathcal{V}_{1,0} = 0$. Then $\Phi(A) = 0$ by the faithfulness of the representation. But then $\Phi(a_{i_1}) \dots \Phi(a_{i_k}).\mathcal{V}_{1,0} = 0$. Then, by Lemmas 3.5.6, 3.5.7, and 3.5.8, we have $\Phi(a_{i_1}) \dots \Phi(a_{i_k}).\mathcal{V}_{1,0} = \phi(a_{i_1} \dots a_{i_k}.V) = \phi(A.v) = 0$. But ϕ is an isomorphism, so this implies that $A.V = 0$. Hence $A = 0$ by the faithfulness of the Fock space representation of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$. □

Chapter 4

Center of the twisted Heisenberg category

Recall that the trace of Khovanov's Heisenberg category $\mathcal{H}$ was shown in [CLLS16] to be isomorphic to a quotient of $W_{1+\infty}$. The center of $\mathcal{H}$, which is the algebra $\mathrm{End}_{\mathcal{H}}(\mathbb{1})$ of endomorphisms of the monoidal identity, was shown in [KLM16] to be isomorphic to the algebra of shifted symmetric functions Λ^* of Okounkov and Olshanski [OO97].

In Chapter 3, we saw that as a commutative $\mathbb{C}$-algebra, the center of $\mathcal{H}_{tw}$ is isomorphic to a polynomial algebra:

$$\mathrm{End}_{\mathcal{H}_{tw}}(1) \cong \mathbb{C}[d_0, d_2, d_4, \dots] \cong \mathbb{C}[\bar{d}_2, \bar{d}_4, \bar{d}_6, \dots],$$

where d_{2k} and $\bar{d}_{2k}$ correspond to certain clockwise and counterclockwise bubble generators respectively. While symmetric groups play a central role for $\mathcal{H}$ in [Kho14], finite Sergeev superalgebras $\{\mathbb{S}_n\}_{n\geq 0}$ (also known as finite Hecke–Clifford algebras of type A) play the central role for $\mathcal{H}_{tw}$. In particular, Cautis and Sussan construct a family of functors $\{F_n^{\mathcal{H}_{tw}}\}_{n\geq 0}$ from $\mathcal{H}_{tw}$ to bimodule categories of Sergeev algebras in order to categorify the Fock space representation of the twisted Heisenberg algebra, as discussed in Section 3.4.2. When restricted to $\mathrm{End}_{\mathcal{H}_{tw}}(1)$, each $F_n^{\mathcal{H}_{tw}}$ gives a surjective algebra homomorphism $F_n^{\mathcal{H}_{tw}} : \mathrm{End}_{\mathcal{H}_{tw}}(1) \twoheadrightarrow Z(\mathbb{S}_n)_{\bar{0}}$ where $Z(\mathbb{S}_n)_{\bar{0}}$ is the even center of $\mathbb{S}_n$.

In this chapter, we study the combinatorial and representation theoretic properties of $\mathrm{End}_{\mathcal{H}_{tw}}(1)$. Our main result is an isomorphism $\varphi : \mathrm{End}_{\mathcal{H}_{tw}}(1) \xrightarrow{\sim} \Gamma$, where Γ is a subalgebra of the algebra of symmetric functions $\Gamma = \mathbb{C}[p_1, p_3, p_5, \dots]$ (Γ is sometimes known as the algebra of supersymmetric [Iva01] or doubly symmetric [Pet09] functions). The construction of φ relies on the fact that there are embeddings of both $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ and Γ into the algebra of functions on strict partitions, $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$.

In our proof of Theorem 4.5.2 we identify the images of certain algebraically independent generators of these algebras in $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$ – the closures of n-cycles from $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ and inhomogeneous analogues of odd power sums $\mathfrak{p}_{(n)}$ in Γ. The latter were first investigated by Ivanov in his study of the asymptotic behavior of characters of projective representations of symmetric groups [Iva01]. We go on to identify the closure of idempotents of $\mathbb{S}_n$ with scalar multiples of Ivanov's factorial Schur Q-functions. Intriguingly, the coefficients that appear on the image of idempotent closures when written in terms of factorial Schur Q-functions count the number of paths between specific vertices in the Schur graph. A similar phenomenon was observed in [KLM16]. A dictionary between Γ and $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ is found in Table 4.1.

In parallel to the surjective homomorphisms $\{F_n^{\mathcal{H}_{tw}}\}_{n\geq 0}$ from $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ to $\{Z(\mathbb{S}_n)_{\bar{0}}\}_{n\geq 0}$, for all $n \geq 0$ one can also construct surjective homomorphisms $F_n^{\Gamma} : \Gamma \twoheadrightarrow Z(\mathbb{S}_n)_{\bar{0}}$ [Iva01]. Our isomorphism φ is canonical in the sense that it intertwines the pair $F_n^{\mathcal{H}_{tw}}$ and F_n^{Γ} for each $n \geq 0$.

The chapter is structured as follows. In Sections 4.1 and 4.2 we describe necessary background material on Schur's graph and the representation theory of Sergeev algebras. In Section 4.3 we describe the subalgebra Γ of the symmetric functions and several of its bases. In Sections 4.4 we define the twisted Heisenberg category $\mathcal{H}_{tw}$ and review the functors $\{F_n^{\mathcal{H}_{tw}}\}_{n\geq 0}$. In Section 4.5 we establish the isomorphism between $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ and Γ. Finally, Section 4.6 then describes the W-algebra W^- and its induced action on Γ.

This work originally appeared in [KOR17] and is joint with H. Kvinge and C. Ozan Oğuz.

4.1 Transition functions on the Schur graph

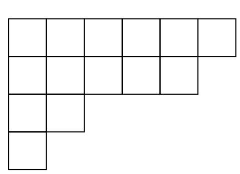
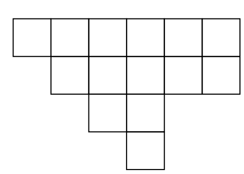
Let $\mathcal{P}_n$ be the set of all partitions of n and set

$$\mathcal{P} := \bigcup_{n\geq 0} \mathcal{P}_n.$$

We freely identify a partition ρ with its corresponding Young diagram. If $\rho \in \mathcal{P}_n$ then we write $|\rho| = n$. If $\rho = (\rho_1, \rho_2, \ldots, \rho_r)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_t) \in \mathcal{P}$ then we write $\eta \subset \rho$ when $\eta_i \leq \rho_i$ for all $i \geq 1$. We denote the number of parts (or length) of a partition ρ by $\ell(\rho)$. A partition $\mu = (\mu_1, \ldots, \mu_r) \in \mathcal{P}_n$ is called an *odd partition* if μ_i is odd for all $1 \leq i \leq r$. We denote the collection of odd partitions of n by $\mathcal{OP}_n$ and set $\mathcal{OP} := \bigcup_{n\geq 0} \mathcal{OP}_n$.

We call a partition $\lambda \in \mathcal{P}_n$ *strict* if all its nonzero parts are distinct. Let $\mathcal{SP}_n$ be the set of all strict partitions of n and set $\mathcal{SP} := \bigcup_{n\geq 0} \mathcal{SP}_n$. To a strict partition λ we can associate its *shifted Young diagram* $S(\lambda)$ which is obtained from the Young diagram (using English notation) by shifting all rows so that the ith row is shifted rightward by $(i-1)$ cells.

Example 1.1 Let $\lambda = (6, 5, 2, 1) \in \mathcal{SP}_{14}$, then

λ =  , $S(\lambda)$ =  .

Henceforth we reserve the variables λ and ν for strict partitions and the variables μ and γ for odd partitions.

For $\nu, \lambda \in \mathcal{SP}$, we write $\nu \nearrow \lambda$ (respectively $\nu \searrow \lambda$) when we can obtain λ from ν by adding (resp. removing) a single cell $\square$. Set

$$\kappa(\nu,\lambda) := \begin{cases} 2 & \text{if } \nu \nearrow \lambda,\ \ell(\lambda)=\ell(\nu), \\ 1 & \text{if } \nu \nearrow \lambda \text{ and } \ell(\lambda)=\ell(\nu)+1, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.2 The *Schur graph* $\mathbb{G}$ is the graded graph such that:

- the vertex set of $\mathbb{G}$ corresponds to $\mathcal{SP}$ and the nth graded component is $\mathcal{SP}_n$,
- the number of edges from ν to λ is given by $\kappa(\nu,\lambda,)$.

The version of $\mathbb{G}$ that we consider here is the same as that studied in [Pet09]. Another version of the Schur graph without edge multiplicity was investigated in [Bor97]. Both graphs have the same down transition functions (see (4.1) below) so in principal we could have chosen to use either.

A *standard shifted Young tableau* of shape $\lambda \in \mathcal{SP}_n$ is a bijective labeling of the cells of $S(\lambda)$ by the integers $\{1,\dots,n\}$ such that entries increase from left to right across rows and down columns. Let g_λ be the number of standard shifted Young tableaux of shape λ. g_λ can be computed explicitly as

$$g_\lambda = \frac{n!}{\lambda_1!\lambda_2!\dots\lambda_r!}\prod_{1\le i<j\le \ell(\lambda)}\frac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j}.$$

Following [Pet09] we denote the number of paths from $\emptyset$ to λ in $\mathbb{G}$ by $h(\lambda)$. Then

$$h(\lambda) = 2^{|\lambda|-\ell(\lambda)}g_\lambda.$$

In [BO09] Borodin and Olshanski used coherent families of measures on partitions to construct infinite-dimensional diffusion processes. Petrov studied analogous processes on the Schur graph [Pet09]. We review some basic definitions related to the latter of these works below.

The *down transition function* $p^\downarrow : \mathbb{G}\times\mathbb{G}\to\mathbb{Q}$ on $\mathbb{G}$ is defined so that for $\nu,\lambda\in\mathcal{SP}$,

$$p^\downarrow(\lambda,\nu) := \frac{h(\nu)}{h(\lambda)}\kappa(\nu,\lambda). \tag{4.1}$$

In particular, when the first argument of $p^\downarrow$ is restricted to $\mathcal{SP}_n$ and the second to $\mathcal{SP}_{n-1}$ the function $p^\downarrow$ gives a Markov transition kernel from $\mathbb{G}_n$ to $\mathbb{G}_{n-1}$. Hence for $\lambda\in\mathcal{SP}_n$, $p^\downarrow(\lambda,\cdot)$ defines a probability measure on $\mathcal{SP}_{n-1}$.

A *coherent system* on $\mathbb{G}$ with respect to down transition function $p^\downarrow$ is a collection of probability measures $\{M_n\}_{n\ge 0}$, with M_n a probability measure on $\mathbb{G}_n$, such that if $\nu\in\mathcal{SP}_{n-1}$, then

$$M_{n-1}(\nu) = \sum_{\lambda\searrow\nu} p^\downarrow(\lambda,\nu)M_n(\lambda).$$

One choice of coherent system with respect to the $p^{\downarrow}$ defined by (4.1) is the collection of *Plancherel measures* $\{Pl_n\}_{n\geq 0}$ where for $\lambda \in \mathcal{SP}_n$

$$Pl_n(\lambda) := \frac{2^{\ell(\lambda)-n}h(\lambda)^2}{n!}.$$

Given the coherent system $\{Pl_n\}_{n\geq 0}$ and the down transition function $p^{\downarrow}$, the corresponding *up transition function* $p^{\uparrow} : \mathbb{G} \times \mathbb{G} \to \mathbb{Q}$ is defined as

$$p^{\uparrow}(\nu, \lambda) := \frac{Pl_{n+1}(\lambda)}{Pl_n(\nu)} p^{\downarrow}(\lambda, \nu) = \frac{h(\lambda)}{h(\nu)(|\nu|+1)}$$

(in [Pet09] these are denoted by $p^{\uparrow}_{\infty}$).

In the next section we will make a connection between induction and restriction of simple Sergeev supermodules and $p^{\uparrow}$, $p^{\downarrow}$.

4.2 The Sergeev algebra and the twisted hyperoctahedral group

Let S_n be the symmetric group on n elements with $s_1, s_2, \ldots, s_{n-1}$ the Coxeter generators of S_n. We will work with the Clifford algebra $\mathcal{C}\ell_n$ which is the unital associative algebra with n generators:

$$\mathcal{C}\ell_n := \mathbb{C}\langle c_1, \ldots, c_n \mid c_i^2 = -1, c_i c_j = -c_j c_i \text{ for } i \neq j\rangle. \tag{4.2}$$

Remark There is another common presentation of $\mathcal{C}\ell_n$ in which $\mathcal{C}\ell_n$ is generated by $c'_1, \ldots, c'_n$ subject to the relations $c'^2_i = 1$ and $c'_i c'_j = -c'_j c'_i$ for $i \neq j$ (see [WW12b] and [Kle05] for example). An equivalence between this presentation and (4.2) can be obtained by setting $c_i = \sqrt{-1}c'_i$.

Definition 2.3 The *finite Sergeev algebra* (also known as the *finite Hecke-Clifford algebra of type A*) is

$$\mathbb{S}_n \simeq \mathcal{C}\ell_n \rtimes \mathbb{C}[S_n] \tag{4.3}$$

where the action of S_n on the Clifford generators is by permuting indices, i.e.

$$s_i c_i = c_{i+1} s_i, \quad s_i c_{i+1} = c_i s_i, \quad \text{and} \quad s_i c_j = c_j s_i \quad \text{for } j \neq i,\ i+1.$$

$\mathbb{S}_n$ is a superalgebra via the $\mathbb{Z}_2$-grading in which the S_n generators $\{s_i, 1 \leq i \leq n-1\}$ are even and the $\mathcal{C}\ell_n$ generators $\{c_j, 1 \leq j \leq n\}$ are odd. For homogeneous element $x \in \mathbb{S}_n$ we write $|x|$ for the degree of x. The Sergeev algebras form a tower of superalgebras via the embedding $\mathbb{S}_{n-1} \hookrightarrow \mathbb{S}_n$ which sends $s_i \mapsto s_i$ and $c_i \mapsto c_i$. We call this the *standard embedding* and use it implicitly throughout this paper. We set $\mathbb{S}_0 = \mathbb{C}$.

It will be convenient to realize $\mathbb{S}_n$ as the quotient of a group algebra. Let C_2 denote the cyclic group of order two. Define the group

$$\Pi_n := \langle z, a_1, \ldots, a_n \mid a_i^2 = z, a_i a_j = z a_j a_i, z a_i = a_i z, z^2 = 1 \rangle.$$

Π_n is a double cover of C_2^n via the short exact sequence:

$$1 \longrightarrow C_2 \longrightarrow \Pi_n \longrightarrow C_2^n \longrightarrow 1 \tag{4.4}$$

in which C_2 is mapped to the subgroup $\{1, z\} \subset \Pi_n$ and $z \in \Pi_n$ is mapped to 1.

Definition 2.4 The *twisted hyperoctahedral group* is defined as $\widehat{B_n} := \Pi_n \rtimes S_n$ where S_n acts on $\{a_i\}$ by permuting their indices, and acts trivially on z.

$\mathbb{C}[\widehat{B_n}]$ is also a superalgebra via the $\mathbb{Z}/2\mathbb{Z}$ grading which sets $\deg(a_j) = 1$ for $1 \leq j \leq n$ and $\deg(z) = \deg(s_i) = 0$ for $1 \leq i \leq n-1$. Using (4.4) one can show that $\widehat{B}_n$ is a double cover of the hyperoctahedral group $B_n = C_2^n \rtimes S_n$ (i.e. the type B Weyl group) via the short exact sequence:

$$1 \longrightarrow C_2 \longrightarrow \widehat{B}_n \stackrel{f}{\longrightarrow} B_n \longrightarrow 1$$

where f sends z to 1. On the other hand from a comparison of generators and relations it is clear that

$$\mathbb{S}_n \simeq \mathbb{C}[\widehat{B_n}]/\langle z+1 \rangle. \tag{4.5}$$

We denote the corresponding projection by $\pi_n : \mathbb{C}[\widehat{B_n}] \to \mathbb{S}_n$.

Since z is central and $z^2 = 1$, for any $\mathbb{C}[\widehat{B}_n]$-supermodule L, we have that z must act by multiplication by either 1 or -1. Hence studying $\mathbb{S}_n$-supermodules is equivalent to studying $\mathbb{C}[\widehat{B}_n]$-supermodules where z acts as multiplication by -1 (these are commonly referred to as spin representations of $\widehat{B}_n$). Furthermore, via the super Wedderburn Theorem it follows that

$$\mathbb{C}[\widehat{B}_n] \;\cong\; \mathbb{C}[\widehat{B}_n]/\langle z-1 \rangle \oplus \mathbb{C}[\widehat{B}_n]/\langle z+1 \rangle \;\cong\; \mathbb{C}[B_n] \oplus \mathbb{S}_n. \tag{4.6}$$

The group algebras $\mathbb{C}[\widehat{B}_n]$ also form a tower of algebras with the embedding $\mathbb{C}[\widehat{B}_{n-1}] \hookrightarrow \mathbb{C}[\widehat{B}_n]$ which sends $s_i \mapsto s_i$, $a_i \mapsto a_i$, and $z \mapsto z$. Note that this maps the subalgebra $\mathbb{C}[S_{n-1}]$ into the subalgebra $\mathbb{C}[S_n]$ in the usual way, and projected down to $\mathbb{S}_{n-1}$ and $\mathbb{S}_n$ this becomes the standard embedding described above. We set $\mathbb{C}[\widehat{B}_0]$ to be the subalgebra generated by z.

Lemma 4.2.1. *For $n \geq 2$,*

$$\{\, s_i \ldots s_{n-1} a_n^{\epsilon} \mid 1 \leq i \leq n, \epsilon \in \{0,1\}\} \tag{4.7}$$

is a collection of left coset representatives of $\widehat{B}_{n-1}$ in $\widehat{B}_n$.

Note that we follow the convention that the elements corresponding to $i = n$ are a_n^{ϵ} for $\epsilon \in \{0,1\}$ in Lemma 4.2.1.

Proof. The set $\{ s_i \dots s_{n-1} \mid 1 \leq i \leq n \}$ forms a collection of minimal length left coset representatives of S_{n-1} in S_n. It follows from this and the fact that $\widehat{B}_n := \Pi_n \rtimes S_n$ that any element $g \in \widehat{B}_n$ can be written as $g = s_i \dots s_{n-1}\omega a_n^{\epsilon} a_J z^{\beta}$ where $1 \leq i \leq n$, $\omega \in S_{n-1}$, $a_J = a_{j_1} \dots a_{j_t}$ for some $J = \{j_1, \dots, j_t\} \subseteq \{1, 2, \dots, n-1\}$, and $\epsilon, \beta \in \{0, 1\}$. Since a_n commutes with S_{n-1} we have $x = s_i \dots s_{n-1} a_n^{\epsilon} \omega a_J z^{\beta}$. Since $\omega a_J z^{\beta} \in \widehat{B}_{n-1}$, the set (4.7) contains a set of left coset representatives. The result then follows from the observation that the size of (4.7) is $2n$ while $|\widehat{B}_n| = 2^{n+1} n!$ and $|\widehat{B}_{n-1}| = 2^n (n-1)!$. □

Remark When $g = s_i \dots s_{n-1} a_n^{\epsilon}$ for $\epsilon \in \{0, 1\}$ then $g^{-1} = a_n^{\epsilon} z^{\epsilon} s_{n-1} \dots s_i$ and consequently while $\pi_n(g) = s_i \dots s_{n-1} c_n^{\epsilon}$,

$$\pi_n(g^{-1}) = (-1)^{\epsilon} c_n^{\epsilon} s_{n-1} \dots s_i = (-1)^{|g|} c_n^{\epsilon} s_{n-1} \dots s_i.$$

We use the inclusions $\widehat{B}_1 \subset \widehat{B}_2 \subset \dots \subset \widehat{B}_{n-1} \subset \widehat{B}_n \subset \dots$ to iterate Lemma 4.2.1 to get that for all $1 \leq k < n$,

$$\widehat{\mathcal{LC}}_k^n := \{ (s_{i_n} \dots s_{n-1} a_n^{\epsilon_n})(s_{i_{n-1}} \dots s_{n-2} a_{n-1}^{\epsilon_{n-1}}) \dots (s_{i_{k+1}} \dots s_k a_{k+1}^{\epsilon_{k+1}})$$
$$\mid 1 \leq i_j \leq j, \ \epsilon_j \in \{0, 1\} \}$$

is a collection of left coset representatives of $\widehat{B}_k$ in $\widehat{B}_n$. Note in particular that

$$|\widehat{\mathcal{LC}}_k^n| = \frac{|\widehat{B}_n|}{|\widehat{B}_k|} = n^{\downarrow k} 2^{n-k} \tag{4.8}$$

where $n^{\downarrow k}$ is the falling factorial

$$n^{\downarrow k} := \frac{n!}{(n-k)!} = n(n-1) \dots (n-k+1)$$

for $1 \leq k < n$. The projection $\pi_n : \mathbb{C}[\widehat{B}_n] \to \mathbb{S}_n$ sends the elements of $\widehat{\mathcal{LC}}_k^n$ to distinct non-zero elements of $\mathbb{S}_n$ and we set

$$\mathcal{LC}_k^n := \pi_n(\widehat{\mathcal{LC}}_k^n).$$

The set of conjugacy classes of $\widehat{B}_n$ are indexed by pairs of partitions (ρ_+, ρ_-) such that $|\rho_+| + |\rho_-| = n$, plus an additional parameter $\epsilon \in \{0, 1\}$ when either $(\rho_+, \rho_-) = (\mu, \emptyset)$ with $\mu \in \mathcal{OP}_n$ or $(\rho_+, \rho_-) = (\emptyset, \lambda)$ with $\lambda \in \mathcal{SP}_n$. We denote this indexing set by Conj. A detailed description of the conjugacy class structure of $\widehat{B}_n$ can be obtained by analyzing the conjugacy class structure of B_n (which follows from the basic theory for the conjugacy class structure of wreath products [Mac15, Appendix B]) and investigating how the inverse image of these sets under the map $\gamma : \widehat{B}_n \twoheadrightarrow B_n$ split into new conjugacy classes [Rea76]. The additional parameter $\epsilon \in \{0, 1\}$ appears precisely when a conjugacy class in B_n splits into two conjugacy classes in $\widehat{B}_n$. Since we are ultimately interested in the center of $\mathbb{S}_n$ which can be described using information about the conjugacy classes indexed by $(\mu, \emptyset, \epsilon)$ for $\mu \in \mathcal{OP}_n$ and $\epsilon \in \{0, 1\}$, we limit ourselves to considering these classes. We call this set of conjugacy classes $\mathrm{Conj}_{odd} \subset \mathrm{Conj}$. For $\beta \in \mathrm{Conj}$, we write $\mathrm{Conj}(\beta)$ for the corresponding conjugacy class.

We introduce a family of elements of $\mathbb{S}_n$ which will be useful for constructing representatives for the conjugacy classes from Conj_{odd}. For $\mu = (\mu_1, \ldots, \mu_r) \in \mathcal{OP}_k$, set $\pi_\mu = 1$ if $\mu = (1^k)$ and otherwise

$$\begin{aligned}\pi_\mu &:= (s_{k-1} \ldots s_{k-\mu_r+1}) \ldots (s_{\mu_1+\mu_2-1} \ldots s_{\mu_1+1})(s_{\mu_1-1} \ldots s_2 s_1) \\ &= (k, k-1, \ldots, k-\mu_r+1) \ldots (\mu_1+\mu_2, \ldots, \mu_1+1)(\mu_1, \ldots, 2, 1) \in S_k. \end{aligned} \tag{4.9}$$

For $n \geq k$ we define $\sigma_{\mu;n} := \tau_0 \pi_\mu \tau_0^{-1}$, where τ_0 is the longest element of S_n by Coxeter length. Notice that $\sigma_{\mu;n}$ has cycle type $(\mu, 1^{n-k}) \in \mathcal{OP}_n$ and fixes $1, 2, \ldots, n-k$ pointwise.

Example 2.1 Let $\mu = (5, 3) \in \mathcal{OP}_8$, then

$$\pi_\mu = (s_7 s_6)(s_4 s_3 s_2 s_1) = (8, 7, 6)(5, 4, 3, 2, 1)$$

and π_μ has cycle type μ. For $n = 12$,

$$\sigma_{\mu;12} = (s_5 s_6)(s_8 s_9 s_{10} s_{11}) = (5, 6, 7)(8, 9, 10, 11, 12)$$

while for $n = 15$,

$$\sigma_{\mu;15} = (s_8 s_9)(s_{11} s_{12} s_{13} s_{14}).$$

Proposition 4.2.2. *The elements* $\{ \sigma_{\mu;n}, \; z\sigma_{\mu;n} \mid \mu \in \mathcal{OP}_n \}$ *form a complete set of conjugacy class representatives for the conjugacy classes* Conj_{odd} *in* $\widehat{B}_n$ *with* $\sigma_{\mu;n}$ *corresponding to* $(\mu, \emptyset, 0) \in \mathrm{Conj}_{odd}$ *and* $z\sigma_{\mu;n}$ *corresponding to* $(\mu, \emptyset, 1) \in \mathrm{Conj}_{odd}$.

Proof. This follows from the description of the conjugacy classes of B_n and results on conjugacy class splitting in $\widehat{B}_n$ [Rea76] (see [WW12b, Section 2.5] for an overview). □

Note that under the projection map $\pi_n : \mathbb{C}[\widehat{B}_n] \to \mathbb{S}_n$, the two sets of conjugacy classes $\{\sigma_{\mu;n}\}_{\mu \in \mathcal{SP}_n}$ and $\{z\sigma_{\mu;n}\}_{\mu \in \mathcal{SP}_n}$ are identified since $\pi_n(z) = -1$.

The size of the conjugacy classes $\mathrm{Conj}(\mu, \emptyset, \epsilon)$ will be important to us later. For $\rho \in \mathcal{P}_n$, we denote by z_ρ the size of the stabilizer of an element of S_n of cycle type ρ under the conjugation action. Recall that

$$z_\rho = \prod_{i \in \mathbb{Z}_{\geq 0}} i^{m_i(\rho)} m_i(\rho)!$$

where $m_i(\rho)$ is the number of parts of size i in ρ.

Lemma 4.2.3. *[Iva01] For* $\mu \in \mathcal{OP}_n$, $\epsilon \in \{0, 1\}$

$$|\mathrm{Conj}(\mu, \emptyset, \epsilon)| = \frac{n!}{z_\mu} 2^{n-\ell(\mu)}.$$

$\mathbb{S}_n$ has analogs to the classical Jucys-Murphy elements of $\mathbb{C}[S_n]$. These elements $\{J_i\}_{i=1}^n$, which we also call Jucys-Murphy elements, are defined by

$$J_1 := 0, \qquad J_k := \sum_{j=1}^{k}(1 + c_j c_k)(j, k).$$

They generate a commutative subalgebra of $\mathbb{S}_n$ and their spectra have a combinatorial interpretation analogous to that of the classical Jucys-Murphy elements ([Naz97], [VS08], [HKS11] [Wan10]).

4.2.1 The super representation theory of $\mathbb{S}_n$ and $\widehat{B_n}$

In this section we will review basic facts about the super representation theory of $\mathbb{S}_n$. Recall that any $\mathbb{S}_n$-supermodule is by definition a spin representation of $\widehat{B}_n$, so all statements about $\mathbb{S}_n$-supermodules also hold for $\widehat{B}_n$ spin representations. We refer the reader to [Kle05] and [WW12b] for thorough accounts of these topics as well as a review of super representation theory.

Let $\delta : \mathcal{SP} \to \{0, 1\}$ be defined by

$$\delta(\lambda) := \begin{cases} 0 & \ell(\lambda) \text{ is even} \\ 1 & \ell(\lambda) \text{ is odd.} \end{cases}$$

The function δ will be useful for describing quantities related to the representation theory of $\mathbb{S}_n$.

Theorem 4.2.4. *[Ser84]*

1. *The set of simple $\mathbb{S}_n$-supermodules are indexed by $\mathcal{SP}_n$, and the simple $\mathbb{S}_n$-supermodule L^λ corresponding to $\lambda \in \mathcal{SP}_n$ is of type M if $\ell(\lambda)$ is even and of type Q if $\ell(\lambda)$ is odd.*

2. *Let $\lambda \in \mathcal{SP}_n$, then*
$$\dim(L^\lambda) = 2^{n - \frac{\ell(\lambda)-\delta(\lambda)}{2}} g_\lambda.$$

The algebras $\{\mathbb{S}_n\}_{n\geq 0}$ are semisimple. When N and M are $\mathbb{S}_n$-supermodules we write

$$[M : N] := \dim(\mathrm{Hom}_{\mathbb{S}_n}(M, N)).$$

The theorem below describes the branching for $\{\mathbb{S}_n\}_{n\geq 0}$.

Theorem 4.2.5. *[Kle05] Let $\lambda \in \mathcal{SP}_n$ and $\nu \in \mathcal{SP}_{n-1}$, then:*

1. $[L^\nu : \mathrm{Res}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}} L^\lambda] = \begin{cases} 2^{\frac{2+\ell(\nu)-\delta(\nu)-\ell(\lambda)+\delta(\lambda)}{2}} & \lambda \searrow \nu \\ 0 & \textit{otherwise.} \end{cases}$

2. $[\mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}} L^\nu : L^\lambda] = \begin{cases} 2^{\frac{2+\ell(\nu)-\delta(\nu)-\ell(\lambda)+\delta(\lambda)}{2}} & \nu \nearrow \lambda, \\ 0 & \textit{otherwise.} \end{cases}$

The next theorem describes how $\mathbb{S}_n$ as a left $\mathbb{S}_n$-supermodule decomposes into a direct sum of simple $\mathbb{S}_n$-supermodules.

Proposition 4.2.6. *$\mathbb{S}_n$ as a left $\mathbb{S}_n$-supermodule decomposes as*

$$\mathbb{S}_n \cong \bigoplus_{\lambda \in \mathcal{SP}_n} (L^\lambda)^{\oplus \dim(L^\lambda)/2^{\delta(\lambda)}}.$$

Proof. It follows from the definition of $\mathbb{S}_n$ that for $\lambda \in \mathcal{SP}_n$, the multiplicity of L^λ in $\mathbb{S}_n$ is equal to the multiplicity of the corresponding spin representation of $\widehat{B}_n$, which we denote by $\widehat{L^\lambda}$, in the left regular super representation of $\widehat{B}_n$. If $\widehat{L^\lambda}$ is of type M (i.e. $\delta(\lambda) = 0$), then considered as an ungraded $\mathbb{C}[\widehat{B}_n]$-module $\widehat{L^\lambda}$ remains simple [Kle05, Section 12.2], and hence it follows from the representation theory of finite groups that $\widehat{L^\lambda}$ has multiplicity $\dim(\widehat{L^\lambda})$.

When $\widehat{L^\lambda}$ is of type Q (i.e. $\delta(\lambda) = 1$) then considered as an ungraded $\mathbb{C}[\widehat{B}_n]$-module, $\widehat{L^\lambda}$ splits into a direct sum of two simple $\mathbb{C}[\widehat{B}_n]$-modules

$$\widehat{L^\lambda} = \widehat{L^\lambda}_{\bar{0}} \oplus \widehat{L^\lambda}_{\bar{1}}$$

the two summands corresponding to the even and odd components [Kle05, Section 12.2]. Each of these have dimension $\dim(\widehat{L^\lambda})/2$ and hence the multiplicity of each in the regular representation of $\widehat{B}_n$ will be $\dim(\widehat{L^\lambda})/2$. It follows that there will be $\dim(\widehat{L^\lambda})/2$ copies of $\widehat{L^\lambda}$ in $\widehat{B}_n$ and hence $\dim(L^\lambda)/2$ copies of L^λ in $\mathbb{S}_n$. □

We can now relate $p^\downarrow(\cdot,\cdot)$ and $p^\uparrow(\cdot,\cdot)$ to the representation theory of the algebras $\{\mathbb{S}_n\}_{n\geq 0}$.

Proposition 4.2.7. *Let $\lambda \in \mathcal{SP}_n$, $\nu \in \mathcal{SP}_{n-1}$. Then*

1. $p^\downarrow(\lambda,\nu) = \dfrac{[L^\nu : \mathrm{Res}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}} L^\lambda]\dim(L^\nu)}{\dim(L^\lambda)}$,
2. $2^{\delta(\lambda)-\delta(\nu)} p^\uparrow(\nu,\lambda) = \dfrac{[\mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}} L^\nu : L^\lambda]\dim(L^\lambda)}{\dim(\mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}} L^\nu)}$.

Proof. Both 1 and 2 follow from the branching rules for Sergeev algebras in Theorem 4.2.5 and the dimension formula in Theorem 2 □

For $\lambda \in \mathcal{SP}_n$, we denote by $\widehat{\chi}^\lambda$ the character corresponding to simple $\mathbb{C}[\widehat{B}_n]$-supermodule $\widehat{L}^\lambda$. This descends to a character χ^λ for simple $\mathbb{S}_n$-supermodule L^λ with

$$\chi^\lambda(\pi_n(g)) = \widehat{\chi}^\lambda(g).$$

The *normalized character* $\widetilde{\chi}^\lambda$ is defined such that for $x \in \mathbb{S}_n$

$$\widetilde{\chi}^\lambda(x) := \frac{\chi^\lambda(x)}{\dim(L^\lambda)} = \frac{\chi^\lambda(x)}{\chi^\lambda(1)}.$$

Proposition 4.2.8. *For $\lambda \in \mathcal{SP}_n$, the character χ^λ is uniquely determined by its value on the elements $\{\sigma_{\mu;n} \mid \mu \in \mathcal{OP}_n\}$.*

Proof. This follows from a similar statement [Iva01, Proposition 1.9] where each element $\sigma_{\mu;n}$ is replaced by an element of $\mathbb{S}_n$ that is conjugate to it. Since characters are constant across conjugacy classes, the result follows. □

Given Proposition 4.2.8, for $\mu \in \mathcal{OP}_k$ with $k \leq n$, it is convenient to write $\chi^\lambda(\mu \cup 1^{n-k}) := \chi^\lambda(\sigma_{\mu;n})$.

4.2.2 The centers of $\mathbb{S}_n$ and $\mathbb{C}[\widehat{B}_n]$

As a superalgebra the center of $\mathbb{S}_n$ breaks up into even and odd components of super-commutative elements, $Z(\mathbb{S}_n) = Z(\mathbb{S}_n)_{\bar{0}} \oplus Z(\mathbb{S}_n)_{\bar{1}}$. In this chapter we will focus on $Z(\mathbb{S}_n)_{\bar{0}}$, which corresponds to the center of $\mathbb{S}_n$ after the $(\mathbb{Z}/2\mathbb{Z})$-grading has been forgotten. It will later be important that, $Z(\mathbb{S}_n)_{\bar{0}}$ is exactly those elements that act on all simple $\mathbb{S}_n$-modules as multiplication by a scalar. Following [Iva01] we will construct a basis for $Z(\mathbb{S}_n)_{\bar{0}}$ via the surjection $\pi : \mathbb{C}[\widehat{B}_n] \twoheadrightarrow \mathbb{S}_n$.

Recall that the set Conj indexes the conjugacy classes of $\widehat{B}_n$. For $\beta \in \text{Conj}$ set

$$\widehat{C}_\beta := \sum_{g \in \text{Conj}(\beta)} g.$$

It is clear that $\{\widehat{C}_\beta\}_{\beta \in \text{Conj}}$ is a basis for the ungraded center of $\mathbb{C}[\widehat{B}_n]$. In [Iva01], Ivanov uses the subset of this basis corresponding to elements of Conj_{odd} of the form $(\mu, \emptyset, 0)$ to construct a basis for $Z(\mathbb{S}_n)_{\bar{0}}$. For $\mu \in \mathcal{OP}_n$ let

$$C_\mu := \pi_n(\widehat{C}_{(\mu,\emptyset,0)}).$$

Proposition 4.2.9. *[Iva01] The set $\{C_\mu \mid \mu \in \mathcal{OP}_n\}$ is a linear basis for $Z(\mathbb{S}_n)_{\bar{0}}$.*

We now define a scaled version of Ivanov's basis of $Z(\mathbb{S}_n)_{\bar{0}}$ which naturally appears from the center of the twisted Heisenberg category.

Definition 2.5 For $k \leq n$ and $\mu \in \mathcal{OP}_k$, define

$$\widehat{A}_{\mu;n} := \sum_{g \in \widehat{\mathcal{LC}}^n_{n-k}} g \sigma_{\mu;n} g^{-1}$$

and

$$A_{\mu;n} := \pi(\widehat{A}_{\mu;n}).$$

Proposition 4.2.10. *Let $k \leq n$ and $\mu \in \mathcal{OP}_k$ then:*

1. *$\widehat{A}_{\mu;n} \in Z(\mathbb{C}[\widehat{B}_n])$ and $A_{\mu;n} \in Z(\mathbb{S}_n)_{\bar{0}}$.*

2. $\widehat{A}_{\mu;n} = 2^{k-n+\ell(\mu)} \dfrac{z_{\mu\cup 1^{n-k}}}{(n-k)!}\widehat{C}_{(\mu\cup 1^{n-k},\emptyset,0)}$.

3. Let $h \in \widehat{B}_n$ be an element not belonging to the same conjugacy class as $\sigma_{\mu;n}$ or $z\sigma_{\mu;n}$ for some $\mu \in \mathcal{OP}_n$ (i.e. h does not belong to a conjugacy class indexed by $(\mu, \emptyset, \epsilon)$, $\epsilon \in \{0,1\}$). Then

$$\pi_n\Big(\sum_{g\in\widehat{\mathcal{LC}}^n_{n-k}} ghg^{-1}\Big) = 0.$$

Proof. 1. Recall that we defined $\sigma_{\mu;n}$ as a distinguished element from the conjugacy class of $\widehat{B}_n$ indexed by $(\mu\cup 1^{n-k}, \emptyset, 0)$. Since $\sigma_{\mu;n}$ is by definition a product of $s_{n-1}, \ldots, s_{n-k+1}$ it commutes with $\widehat{B}_{n-k}$. Since $\widehat{\mathcal{LC}}^n_{n-k}$ is a collection of left coset representatives of $\widehat{B}_{n-k}$ in $\widehat{B}_n$ any element $g \in \widehat{B}_n$ can be written uniquely as $g = \sigma h$ for $\sigma \in \widehat{\mathcal{LC}}^n_{n-k}$ and $h \in \widehat{B}_{n-k}$. Thus $g\sigma_{\mu;n}g^{-1} = \sigma h\sigma_{\mu;n}h^{-1}\sigma^{-1} = \sigma\sigma_{\mu;n}\sigma^{-1}$ and hence $g\sigma_{\mu;n}g^{-1}$ is completely determined by the left coset to which g belongs. It follows that

$$\sum_{g\in\widehat{B}_n} g\sigma_{\tilde{\mu};n}g^{-1} = |\widehat{B}_{n-k}| \sum_{g\in\widehat{\mathcal{LC}}^n_{n-k}} g\sigma_{\mu;n}g^{-1} = |\widehat{B}_{n-k}|\widehat{A}_{\mu;n} \tag{4.10}$$

and $\widehat{A}_{\mu;n} \in Z(\mathbb{C}[\widehat{B}_n])$ since $\widehat{A}_{\mu;n}$ is a scalar multiple of a central element.

Finally, note that π_n is a degree-preserving homomorphism and $\widehat{A}_{\mu;n}$ is even, so $\pi(\widehat{A}_{\mu;n}) = A_{\mu;n} \in Z(\mathbb{S}_n)_{\bar{0}}$.

2. It follows from Lemma 4.2.3 and the orbit stabilizer theorem that

$$\sum_{g\in\widehat{B}_n} g\sigma_{\mu;n}g^{-1} = 2^{\ell(\mu)+1} z_{\mu\cup 1^{n-k}}\widehat{C}_{(\mu\cup 1^{n-k},\emptyset,0)}.$$

Then (4.10) implies that

$$|\widehat{B}_{n-k}|\widehat{A}_{\mu;n} = 2^{\ell(\mu)+1} z_{\mu\cup 1^{n-k}}\widehat{C}_{(\mu\cup 1^{n-k},\emptyset,0)}.$$

The result follows.

3. We show that for any element $h \in \widehat{B}_n$ which belongs to a conjugacy class indexed by $\beta \neq (\mu, \emptyset, \epsilon)$, for $\epsilon \in \{0,1\}$, $\mu \in \mathcal{OP}_n$, $\pi_n(h) = x$ is zero in the trace of $\mathbb{S}_n$, which is the algebra defined by

$$\overline{\mathbb{S}_n} = \frac{\mathbb{S}_n}{[\mathbb{S}_n, \mathbb{S}_n]}.$$

If $x \equiv 0 \bmod [\mathbb{S}_n, \mathbb{S}_n]$, then x is either 0 or conjugate to its negative. In $\overline{\mathbb{S}_n}$, each element is equal to its conjugates, so $x \equiv 0 \bmod [\mathbb{S}_n, \mathbb{S}_n]$ implies that for every $h' \in \mathrm{Conj}(\beta)$ with $\pi_n(h') = x'$ either $x' = 0$ or there is another element $h'' \in \mathrm{Conj}(\beta)$ such that $\pi_n(h'') = -x'$. Either case implies that $\pi_n(\widehat{C}_\beta) = 0$, and hence by the same argument as 2,

$$\pi_n\Big(\sum_{g\in\widehat{\mathcal{LC}}^n_{n-k}} ghg^{-1}\Big) = 0. \tag{4.11}$$

When $h \in S_n \subset \widehat{B}_n$ and h has cycle type $\rho \notin \mathcal{OP}_n$, so that h is in the conjugacy class labeled $(\rho, \emptyset)$ then by [Ree17, Proposition 3.9] $\pi_n(h) \equiv 0 \mod [\mathbb{S}_n, \mathbb{S}_n]$. On the other hand when $h \in \widehat{B}_n$ is a member of the conjugacy class labeled (ρ, η) for $\eta \neq \emptyset$ then by [Ree17, Proposition 3.4] $\pi_n(h) \equiv 0 \mod [\mathbb{S}_n, \mathbb{S}_n]$. The result then follows from the previous paragraph.

□

It follows from Proposition 4.2.10.2 that $\{A_{\mu;n} \mid \mu \in \mathcal{OP}_n\}$ is also a linear basis of $Z(\mathbb{S}_n)_{\bar{0}}$.

For a spin representation $\widehat{L}^\lambda$ of $\widehat{B}_n$, the corresponding character $\widehat{\chi}^\lambda$ is a homomorphism when restricted to $Z(\mathbb{C}[\widehat{B}_n])_0$ and χ^λ is a homomorphism on $Z(\mathbb{S}_n)_{\bar{0}}$.

Proposition 4.2.11. *Let $\lambda \in \mathcal{SP}_n$ and $\mu \in \mathcal{OP}_k$. Then*

$$\widetilde{\chi}^\lambda(A_{\mu;n}) = 2^k n^{\downarrow k} \frac{\chi^\lambda(\sigma_{\mu;n})}{\chi^\lambda(1)}.$$

Proof. This follows from the fact that characters are invariant under conjugation and (4.8).

□

Another basis for $Z(\mathbb{S}_n)_{\bar{0}}$ is given by the set of central idempotents of $\mathbb{S}_n$ corresponding to the simple $\mathbb{S}_n$-supermodules. We denote these central idempotents by $\{e_\lambda \mid \lambda \in \mathcal{SP}_n\}$.

Lemma 4.2.12. *For $\lambda \in \mathcal{SP}_n$, the central idempotent $e_\lambda \in \mathbb{S}_n$ corresponding to simple $\mathbb{S}_n$-supermodule L^λ can be written as*

$$e_\lambda = 2^{\frac{-\ell(\lambda)-\delta(\lambda)}{2}} \frac{g_\lambda}{n!} \sum_{\mu \in \mathcal{OP}_n} \chi^\lambda(\mu) C_\mu.$$

Proof. The definition of $\mathbb{S}_n$ implies that e_λ is the image of the corresponding central idempotent $\widehat{e_\lambda}$ in $\widehat{B}_n$ under the projection map π_n. There are two cases to consider: that in which L^λ is of type M (i.e. $\delta(\lambda) = 0$) and that in which L^λ is of type Q (i.e. $\delta(\lambda) = 1$). Consider the case where L^λ is of type M. Then $\widehat{L}^\lambda$ is of type M also and $\widehat{L}^\lambda$ viewed as an ungraded $\mathbb{C}[\widehat{B}_n]$-module remains simple. Since $\widehat{B}_n$ is a finite group, the central idempotent corresponding to $\widehat{L}^\lambda$ is

$$\widehat{e_\lambda} = \frac{\dim(\widehat{L}^\lambda)}{|\widehat{B}_n|} \sum_{\beta \in \mathrm{Conj}} \widehat{\chi}^\lambda(g) \widehat{C}_\beta.$$

In the case where L^λ is of type Q, $\widehat{L}^\lambda$ viewed as an ungraded $\mathbb{C}[\widehat{B}_n]$-module breaks into the direct sum of two simple $\mathbb{C}[\widehat{B}_n]$-modules $\widehat{L}_0^\lambda$ and $\widehat{L}_1^\lambda$ of equal dimension. Then the central idempotent corresponding to $\widehat{L}^\lambda$ is given by

$$\widehat{e_\lambda} = \frac{\dim(\widehat{L}^\lambda)}{2|\widehat{B}_n|} \sum_{\beta \in \mathrm{Conj}} \widehat{\chi}^\lambda(g) \widehat{C}_\beta$$

where $\widehat{\chi}^\lambda(g) = \widehat{\chi}^{\lambda_0}(g) + \widehat{\chi}^{\lambda_1}(g)$. Thus in general

$$\widehat{e_\lambda} = \frac{\dim(\widehat{L}^\lambda)}{2^{\delta(\lambda)}|\widehat{B}_n|} \sum_{\beta \in \mathrm{Conj}} \widehat{\chi}^\lambda(g) \widehat{C}_\beta.$$

Applying π_n to $\widehat{e_\lambda}$, Proposition 4.2.10.3 implies that most terms go to zero and we are left with

$$e_\lambda = 2^{\frac{-\ell(\lambda)-\delta(\lambda)}{2}-1}\frac{g_\lambda}{n!}\sum_{\beta\in \mathrm{Conj}_{odd}} \chi^\lambda(\beta)\pi_n(\widehat{C}_\beta).$$

Recall that $\beta \in \mathrm{Conj}_{odd}$ contains pairs $\beta = (\mu, \emptyset, 0)$ and $\bar{\beta} = (\mu, \emptyset, 1)$ for $\mu \in \mathcal{OP}_n$ such that if $x \in \mathrm{Conj}(\beta)$ then $zx \in \mathrm{Conj}(\bar{\beta})$. It follows that $\pi_n(\widehat{C}_{\bar{\beta}}) = -\pi_n(\widehat{C}_\beta)$. At the same time, since z acts as multiplication by -1 on $\widehat{L}^\lambda$ then $\chi^\lambda(z\sigma_{\mu;n}) = -\chi^\lambda(\sigma_{\mu;n})$. It follows that

$$2^{\frac{-\ell(\lambda)-\delta(\lambda)}{2}-1}\frac{g_\lambda}{n!}\sum_{\beta\in \mathrm{Conj}_{odd}} \chi^\lambda(\beta)\pi_n(\widehat{C}_\beta) = 2^{\frac{-\ell(\lambda)-\delta(\lambda)}{2}}\frac{g_\lambda}{n!}\sum_{\mu\in \mathcal{OP}_n} \chi^\lambda(\mu)C_\mu.$$

□

4.2.3 Interlacing coordinates for strict partitions

In [Ker00] Kerov developed a useful way to parametrize Young diagrams via their interlacing coordinates. Petrov showed that shifted strict Young diagrams can be similarly parametrized with only slight modification [Pet09]. For $\lambda \in \mathcal{SP}$, and $\square \in S(\lambda)$ with coordinates (i, j), the *content* of $\square$ is defined to be

$$\mathrm{cont}(\square) := i - j.$$

Note that when $\square$ comes from a shifted diagram, $\mathrm{cont}(\square)$ is always nonnegative. Let

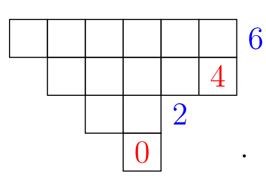
1. $X(\lambda)$ be the set of contents for cells that we can add to $S(\lambda)$ to get another shifted strict partition.

2. $Y(\lambda)$ be the set of contents for cells that we can remove from $S(\lambda)$ to get another shifted strict partition.

The set $(X(\lambda), Y(\lambda))$ uniquely characterizes $S(\lambda)$ and is called the *Kerov coordinates* $S(\lambda)$. We follow [Pet09] and denote the shifted diagram obtained by adding a cell $\square$ with content $x \in X(\lambda)$ to $S(\lambda)$ by $S(\lambda) + \square(x)$ and the shifted diagram obtained by removing a cell $\square$ from $S(\lambda)$ with content $y \in Y(\lambda)$ by $S(\lambda) - \square(y)$.

Example 2.2 In the case of $\lambda = (6, 5, 2, 1) \in \mathcal{SP}_{14}$, $Y(\lambda) = \{0, 4\}$, $X(\lambda) = \{2, 6\}$, and

$S(\lambda) =$



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Proposition 4.2.13. *[Pet09, Proposition 3.2] Let $\lambda \in \mathcal{SP}_n$.*

1. If 1 is a component of λ then for some integer $d \geq 1$ we have

$$X(\lambda) = \{x_1, \dots, x_d\}, \qquad Y(\lambda) = \{0, y_2, \dots, y_d\}.$$

and

$$0 = y_1 < x_1 < y_2 < x_2 < \cdots < y_d < x_d.$$

2. If 1 is not a component of λ then for some integer $d \geq 0$ we have

$$X(\lambda) = \{0, x_1, \dots, x_d\}, \qquad Y(\lambda) = \{y_1, \dots, y_d\}.$$

and

$$0 = x_0 < y_1 < x_1 < y_2 < x_2 < \cdots < y_d < x_d.$$

As in [Pet09], we define $X'(\lambda) := X(\lambda) \setminus \{0\}$. Then it is clear from Proposition 4.2.13 that for all $\lambda \in \mathcal{SP}$, $|X'(\lambda)| = |Y(\lambda)|$.

We set

$$s(i) := i(i+1).$$

The following proposition connects $p^{\downarrow}(\lambda, \cdot,)$ and $p^{\uparrow}(\lambda, \cdot)$ with the interlacing coordinates for λ.

Proposition 4.2.14. *[Pet09, Proposition 3.6, 3.7] Let $\lambda \in \mathcal{SP}_n$, then*

1.

$$\sum_{x \in X(\lambda)} \frac{p^{\uparrow}(\lambda, \lambda + \square(x))}{z - s(x)} = \frac{\prod_{y \in Y(\lambda)}(z - s(y))}{z \prod_{x \in X'(\lambda)}(z - s(x))},$$

2.

$$1 - \sum_{y \in Y(\lambda)} \frac{2|\lambda| p^{\downarrow}(\lambda, \lambda - \square(y))}{z - s(y)} = \frac{\prod_{x \in X'(\lambda)}(z - s(x))}{\prod_{y \in Y(\lambda)}(z - s(y))}.$$

The left sides of (1) and (2) in Proposition 4.2.14 can be rewritten as

$$\sum_{x \in X(\lambda)} \frac{p^{\uparrow}(\lambda, \lambda + \square(x))}{z - s(x)} = \sum_{k=0}^{\infty} \sum_{x \in X(\lambda)} p^{\uparrow}(\lambda, \lambda + \square(x)) s(x)^k z^{-k-1}, \tag{4.12}$$

and

$$1 - \sum_{y \in Y(\lambda)} \frac{p^{\downarrow}(\lambda, \lambda - \square(y))}{z - s(y)} = 1 - 2|\lambda| \sum_{k=0}^{\infty} \sum_{y \in Y(\lambda)} p^{\downarrow}(\lambda, \lambda - \square(y)) s(y)^k z^{-k-1}. \tag{4.13}$$

Petrov defined functions on $\mathcal{SP}$ corresponding to the coefficients on the right side of (4.12) and (4.13)

$$\mathbf{g}_k^{\uparrow}(\lambda) := \sum_{x \in X(\lambda)} p^{\uparrow}(\lambda, \lambda + \square(x)) s(x)^k$$

and

$$\mathbf{g}_{k+1}^{\downarrow}(\lambda) := 2|\lambda| \sum_{y \in Y(\lambda)} p^{\downarrow}(\lambda, \lambda - \square(y)) s(y)^k$$

and investigated their properties in [Pet09]. Note that $\mathbf{g}_0^{\uparrow} = 1$.

Remark $\mathbf{g}_k^{\uparrow}(\lambda)$ and $\mathbf{g}_k^{\downarrow}(\lambda)$ are the strict partition analog to moments of Kerov's transition and co-transition measure [Ker00]. They will play a similar role to the one they played in [KLM16].

Proposition 4.2.15. *[Pet09, Proposition 5.4] For $\lambda \in \mathcal{SP}_n$,*

$$\mathbf{g}_k^{\uparrow} = \mathbf{g}_k^{\downarrow} + \sum_{\substack{i,j>0,\\ i+j=k}} \mathbf{g}_i^{\uparrow}(\lambda)\mathbf{g}_j^{\downarrow}(\lambda).$$

We now give algebraic interpretations of $\mathbf{g}_k^{\uparrow}(\lambda)$ and $\mathbf{g}_k^{\downarrow}(\lambda)$ analogous to those found by Biane for Kerov's transition and co-transition measure on Young diagrams [Bia98]. Let $\mathrm{pr}_{n-1} : \mathbb{S}_n \to \mathbb{S}_{n-1}$ be the linear map defined such that for $x \in \mathbb{S}_n$

$$\mathrm{pr}_{n-1}(x) := \begin{cases} x & \text{if } x \in \mathbb{S}_{n-1} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.2.16. *Let $\lambda \in \mathcal{SP}_n$ for $n \geq 1$ and $k \geq 0$, then*

1. $\widetilde{\chi}^{\lambda}(\mathrm{pr}_n(J_{n+1}^{2k})) = \mathbf{g}_k^{\uparrow}(\lambda)$.
2. $\widetilde{\chi}^{\lambda}\Big(\sum_{x \in \mathcal{LC}_{n-1}^n} xJ_n^r x^{-1}\Big) = \begin{cases} \mathbf{g}_{k+1}^{\downarrow}(\lambda) & \textit{if } r = 2k \textit{ is even} \\ 0 & \textit{otherwise.} \end{cases}$

Proof. 1. Consider the character $\tau_n : \mathbb{S}_n \to \mathbb{C}$ corresponding to $\mathbb{S}_n$ acting on itself by left multiplication. For $x \in \mathbb{S}_n$,

$$\tau_n(x) := \begin{cases} 2^n n! & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

It follows from (4.14) that

$$2(n+1)\tau_n(\mathrm{pr}_n(x)) = \tau_{n+1}(x).$$

Also note that if $y \in \mathbb{S}_n$ and $x \in \mathbb{S}_{n+1}$ then

$$\mathrm{pr}_n(yx) = y\mathrm{pr}_n(x).$$

Recall that e_λ is the central idempotent of $\mathbb{S}_n$ corresponding to simple $\mathbb{S}_n$-supermodule L^λ. Then by Lemma 4.2.6 there are $2^{-\delta(\lambda)}\dim(L^\lambda)$ copies of L^λ in the $\mathbb{S}_n$-supermodule $\mathbb{S}_n$ so that

$$2(n+1)\tau_n(\mathrm{pr}_n(e_\lambda J_{n+1}^{2k})) = 2(n+1)\tau_n(e_\lambda \mathrm{pr}_n(J_{n+1}^{2k})) \tag{4.15}$$
$$= 2^{1-\delta(\lambda)}(n+1)\dim(L^\lambda)\chi^\lambda(\mathrm{pr}_n(J_{n+1}^{2k})). \tag{4.16}$$

On the other hand, the weight space decomposition for the Jucys-Murphy operators on $\mathbb{S}_n$-supermodules implies that

$$2(n+1)\tau_n(\mathrm{pr}_n(e_\lambda J_{n+1}^{2k})) = \tau_{n+1}(e_\lambda J_n^{2k})$$

$$= \sum_{x \in X(\lambda)} \left[L^\lambda : \operatorname{Res}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}} L^{\lambda+\square(x)} \right] \frac{\dim(L^\lambda)\dim(L^{\lambda+\square(x)})}{2^{\delta(\lambda+\square(x))}} s(x)^k.$$

Thus, taking the normalized character gives

$$\widetilde{\chi}^\lambda(\mathrm{pr}_{n-1}(J_n^{2k}))$$
$$= \sum_{x \in X(\lambda)} 2^{\delta(\lambda+\square(x))-\delta(\lambda)} \left[L^\lambda : \operatorname{Res}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}} L^{\lambda+\square(x)} \right] \frac{\dim(L^{\lambda+\square(x)})}{2(n+1)\dim(L^\lambda)} s(x)^k.$$

As $\dim(\operatorname{Ind}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}} L^\lambda) = 2(n+1)\dim(L^\lambda)$ and by Frobenious reciprocity

$$\left[L^\lambda : \operatorname{Res}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}} L^{\lambda+\square(x)} \right] = \left[\operatorname{Ind}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}} L^\lambda : L^{\lambda+\square(x)} \right],$$

applying Lemma 4.2.7.2 gives the desired result.

2. The elements c_i and the Jucys-Murphy elements J_i satisfy $J_i c_i = -c_i J_i$ and for $x = s_i \dots s_{n-1} c_n^\epsilon$ we have $x^{-1} = (-1)^\epsilon c_n s_{n-1} \dots s_i$. Therefore

$$\sum_{x \in \mathcal{LC}_{n-1}^n} x J_n^{2k} x^{-1}$$

$$= \sum_{i=1}^n s_i \dots s_{n-1} J_n^r s_{n-1} \dots s_i \quad - \quad s_i \dots s_{n-1} c_n J_n^r c_n s_{n-1} \dots s_i$$

$$= \sum_{i=1}^n s_i \dots s_{n-1} J_n^r s_{n-1} \dots s_i \quad - \quad (-1)^{r+1} s_i \dots s_{n-1} J_n^r s_{n-1} \dots s_i. \tag{4.17}$$

When r is odd, this is then equal to zero. When $r = 2k$, (4.17) is equal to

$$2 \sum_{i=1}^n s_i \dots s_{n-1} J_n^{2k} s_{n-1} \dots s_i.$$

Since characters are invariant under conjugation, we have

$$\widetilde{\chi}^\lambda \Big(2 \sum_{i=1}^n s_i \dots s_{n-1} J_n^{2k} s_{n-1} \dots s_i \Big) = 2n \widetilde{\chi}^\lambda(J_n^{2k}).$$

Decomposing J_n into its weight spaces then gives

$$2n\widetilde{\chi}^\lambda(J_n^{2k}) = 2n \sum_{y \in Y(\lambda)} \frac{\left[L^{\lambda-\square(y)}, \operatorname{Res}_{\mathbb{S}_{n-1}}^{\mathbb{S}_n} L^\lambda \right] \dim(L^{\lambda-\square(y)}) s(y)^k}{\dim(L^\lambda)}$$
$$= 2n \sum_{y \in Y(\lambda)} p^\downarrow(\lambda, \lambda - \square(y)) s(y)^k$$

where the last equality uses Lemma 4.2.7.1

□

4.3 The subalgebra Γ

We recall relevant facts about the algebra Γ following [Mac15]. Let p_k be the kth power sum symmetric function,

$$p_k(x_1, \ldots, x_n) = x_1^k + \ldots + x_n^k$$

and recall that for $\rho \in \mathcal{P}$,

$$p_\rho(x_1, \ldots, x_n) := \prod_{k=1}^{\ell(\rho)} p_{\rho_k}(x_1, \ldots, x_n).$$

Define Γ_n to be the subalgebra of the symmetric polynomials in $x_1, x_2, \ldots, x_n$ generated by $\{p_\lambda \mid \lambda \in \mathcal{OP}_n\}$.

For each n, there is a surjective homomorphism

$$\Gamma_{n+1} \twoheadrightarrow \Gamma_n$$

given by setting $x_{n+1} = 0$. Define Γ to be the projective limit of these algebras with respect to these homomorphisms

$$\Gamma := \varprojlim \Gamma_n.$$

Alternatively, Γ can be described as the subalgebra of the symmetric functions generated by the odd power sum symmetric functions

$$\Gamma = \mathbb{C}[p_1, p_3, p_5, \ldots].$$

Elements of Γ can be evaluated on partitions in the following way. Let $f \in \Gamma$ and $\rho \in \mathcal{P}$, and define

$$f(\rho) = f(\rho_1, \rho_2, \ldots, \rho_{\ell(\rho)}, 0, \ldots). \tag{4.18}$$

Let $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$ denote the algebra of functions from $\mathcal{SP}$ to $\mathbb{C}$ with pointwise multiplication.

Proposition 4.3.1. *[IK99, Proposition 6.2]) The algebra Γ embeds into Fun($\mathcal{SP}, \mathbb{C}$) via the evaluation map* (4.18).

Example 3.1 For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{SP}_n$ we have

$$p_1(\lambda) = \lambda_1 + \lambda_2 + \cdots + \lambda_r = n.$$

We recall an important linear basis of Γ, the Schur Q-functions [Mac15, Section III.8]. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{SP}$, then

$$Q_{\lambda|N}(x_1, \ldots, x_N) = \frac{2^{\ell(\lambda)}}{(N-r)!} \sum_{\omega \in S_N} \omega \left(x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_r^{\lambda_r} \prod_{\substack{1 \le i \le r \\ i < j \le N}} \frac{x_i + x_j}{x_i - x_j} \right),$$

where we say $Q_{\lambda|N} = 0$ if $\ell(\lambda) > N$. The sequence $(Q_{\lambda|N})_{N=1,2,\ldots}$ defines an element $Q_\lambda \in \Gamma$ known as the *Schur Q-functions*. The set $\{Q_\lambda\}_{\lambda\in\mathcal{SP}}$ forms a linear basis of Γ.

Define numbers X^λ_μ for $\lambda \in \mathcal{SP}_n$, $\mu \in \mathcal{OP}_n$, via

$$p_\mu = \sum_{\lambda\in\mathcal{SP}_n} 2^{-\ell(\lambda)} X^\lambda_\mu Q_\lambda. \tag{4.19}$$

There is a "factorial" version of the Schur Q-functions, defined in [Iva04]. For $\lambda \in \mathcal{SP}$, the *factorial Schur Q-polynomial* corresponding to λ is defined as:

$$Q^*_{\lambda|N}(x_1,\ldots,x_N) := \frac{2^{\ell(\lambda)}}{(N-l)!} \sum_{\omega\in S_N} \omega \left(x_1^{\downarrow\lambda_1} x_2^{\downarrow\lambda_2} \ldots x_l^{\downarrow\lambda_l} \prod_{\substack{1\le i\le l \\ i<j\le N}} \frac{x_i+x_j}{x_i-x_j} \right). \tag{4.20}$$

If $\ell(\lambda) > N$, then $Q^*_{\lambda|N}$ is defined to be 0. The collection $(Q^*_{\lambda|N})_{N=1,2,\ldots}$ defines an element of Γ, the *factorial Schur Q-function* Q^*_λ. Factorial Schur Q-functions have the following useful properties.

Proposition 4.3.2. *[Iva01] Let $\lambda, \nu \in \mathcal{SP}$.*

1. *There exists $g \in \Gamma$ of degree less than $|\lambda|$ such that*
$$Q^*_\lambda = Q_\lambda + g.$$
2. *The collection $\{Q^*_\lambda\}_{\lambda\in\mathcal{SP}}$ is a linear basis of Γ.*
3. *If $\nu \in \mathcal{SP}_k$, $\lambda \in \mathcal{SP}_n$ for $k \le n$ and $\nu \not\subseteq \lambda$, $Q^*_\lambda(\nu) = 0$.*

Let $\psi : \Gamma \to \Gamma$ be the linear map that sends $Q_\lambda \mapsto Q^*_\lambda$. For any $\mu \in \mathcal{OP}$, define the inhomogeneous analogue of the power sum $\mathfrak{p}_\mu := \psi(p_\mu) \in \Gamma$. Applying ψ to both sides of (4.19) gives

$$\mathfrak{p}_\mu = \sum_{\lambda\in\mathcal{SP}_k} 2^{-\ell(\lambda)} X^\lambda_\mu Q^*_\lambda.$$

It also follows from the fact that $X^\lambda_\mu = 2^{-\ell(\mu)+\frac{\ell(\lambda)-\delta(\lambda)}{2}} \chi^\lambda(\mu)$ [Iva01, Proposition 3.3] and

$$Q_\lambda = \sum_{\mu\in\mathcal{OP}_n} \frac{2^{\ell(\mu)}}{z_\mu} X^\lambda_\mu p_\mu$$

that

$$Q^*_\lambda = 2^{\frac{\ell(\lambda)-\delta(\lambda)}{2}} \sum_{\mu\in\mathcal{OP}_n} \frac{\chi^\lambda(\mu)}{z_\mu} \mathfrak{p}_\mu. \tag{4.21}$$

The elements $\{\mathfrak{p}_\mu\}_{\mu\in\mathcal{SP}}$ were first studied in [Iva01], where Ivanov proves that they satisfy the following properties.

Proposition 4.3.3. *[Iva01] Let $\mu \in \mathcal{OP}_k$ and $\lambda \in \mathcal{SP}_n$.*

1. *There exists* $g \in \Gamma$ *of degree less than* $|\mu|$ *such that*

$$\mathfrak{p}_\mu = p_\mu + g.$$

2. *The family* $(\mathfrak{p}_\mu)_{\mu \in \mathcal{OP}}$ *is a linear basis of* Γ.

3.
$$\mathfrak{p}_\mu(\lambda) = \begin{cases} n^{\downarrow k} \cdot \dfrac{X^\lambda_{\mu \cup (1^{n-k})}}{g_\lambda} & \text{if } |\lambda| \geq |\mu|, \\ 0 & \text{otherwise} \end{cases}$$

where in particular $g_\lambda = X^\lambda_{1^{|\lambda|}}$.

4. *Let* $\gamma \in \mathcal{OP}$. *Define* $\mu \cup \gamma$ *to be the partition formed by taking the disjoint union of parts of* μ *and* γ *and rearranging them in decreasing order. Then there exists* $g \in \Gamma$ *of degree less than* $|\mu \cup \gamma|$ *such that*

$$\mathfrak{p}_\mu \cdot \mathfrak{p}_\gamma = \mathfrak{p}_{\mu \cup \gamma} + g.$$

As a Corollary to part 3 of the above Proposition, we have another formula for the value of $\mathfrak{p}_\rho$.

Corollary 4.3.4. *[Iva01] Let* $\mu \in \mathcal{OP}_k$ *and* $\lambda \in \mathcal{SP}_n$. *We have*

$$\mathfrak{p}_\mu(\lambda) = 2^{k-\ell(\mu)} n^{\downarrow k} \frac{\chi^\lambda(\mu \cup 1^{n-k})}{\chi^\lambda(1^n)}.$$

Corollary 4.3.5. *The elements* $\{\mathfrak{p}_{2k+1}\}_{k \geq 0}$ *are algebraically independent and generate* Γ.

It is shown in [Pet09] that viewed as elements of $\text{Fun}(\mathcal{SP}, \mathbb{C})$, $\{\mathbf{g}_k^\uparrow\}_{k\geq 1}$ and $\{\mathbf{g}_k^\downarrow\}_{k \geq 1}$ belong to Γ.

Proposition 4.3.6. *[Pet09, Corollary 4.7] The elements* $\{\mathbf{g}_k^\uparrow\}_{k\geq 1}$ *and* $\{\mathbf{g}_k^\downarrow\}_{k\geq 1}$ *are each sets of algebraically independent generators of* Γ

$$\Gamma \cong \mathbb{C}[\mathbf{g}_1^\uparrow, \mathbf{g}_2^\uparrow, \dots] \cong \mathbb{C}[\mathbf{g}_1^\downarrow, \mathbf{g}_2^\downarrow, \dots]$$

and

$$\deg(\mathbf{g}_k^\uparrow) = \deg(\mathbf{g}_k^\downarrow) = 2k - 1.$$

4.4 The twisted Heisenberg category

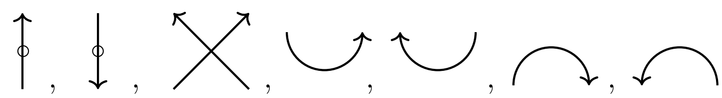
4.4.1 The definition of $\mathcal{H}_{tw}$

The twisted Heisenberg category $\mathcal{H}^t$ was discovered by Cautis-Sussan in [CS15]. It can be defined as the idempotent completion of a $\mathbb{C}$-linear $\mathbb{Z}/2\mathbb{Z}$-graded additive monoidal category

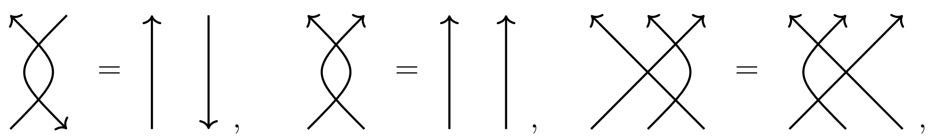
$\mathcal{H}_{tw}$, whose morphisms are described diagrammatically as oriented compact 1-manifolds immersed in $\mathbb{R}\times[0,1]$. There is an injective algebra homomorphism from a twisted Heisenberg algebra into the split Grothendieck group of $K_0(\mathcal{H}^t)$. As in the untwisted case, this map is conjecturally surjective.

Since we are interested in the center of this category which remains invariant under passage to the idempotent completion, we choose to work with $\mathcal{H}_{tw}$. All results related to the center of $\mathcal{H}_{tw}$ also hold for the center of $\mathcal{H}^t$.

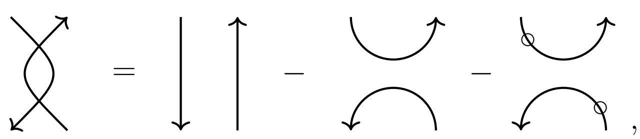
The objects of $\mathcal{H}_{tw}$ are generated by P and Q so that a generic object in $\mathcal{H}_{tw}$ is a direct sum of sequences of P's and Q's. We denote the empty sequence, which is the unit object of $\mathcal{H}_{tw}$, by $\mathbb{1}$. The morphisms of $\mathcal{H}_{tw}$ are generated by oriented planar diagrams up to boundary fixing isotopies, with generators



where the first diagram corresponds to a map $P \to P\{1\}$ and the second diagram corresponds to a map $Q \to Q\{1\}$, where $\{1\}$ denotes the $\mathbb{Z}/2\mathbb{Z}$-grading shift. The first two diagrams above have degree one, and the last five have degree zero. These generators satisfy the following relations:



(4.22)



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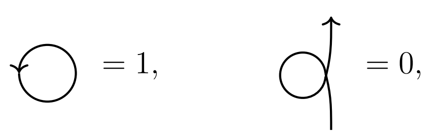

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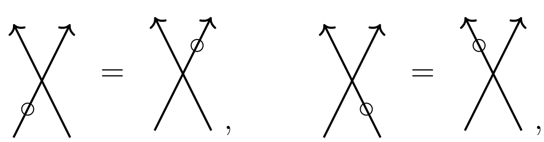

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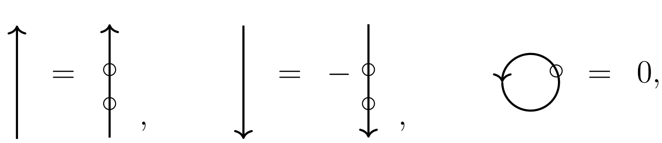
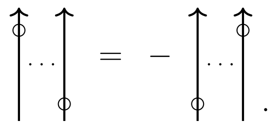
(4.26)

(4.27)

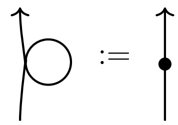


(4.28)



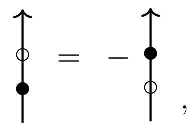
(4.29)

If we denote a right-twist circle by a dot

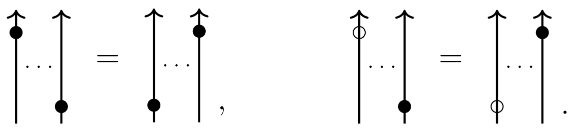


(4.30)

then we have the following relations:

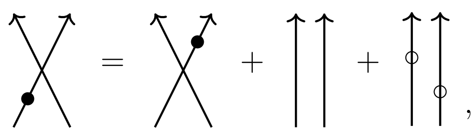


(4.31)

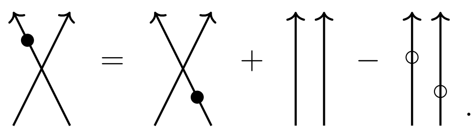


(4.32)

From [OR17] we have the following "dot sliding" relations,



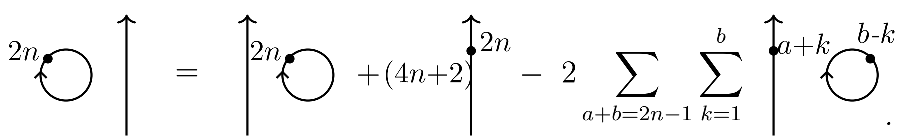
(4.33)



(4.34)

We can also move clockwise "bubbles" with dots on them through strands.

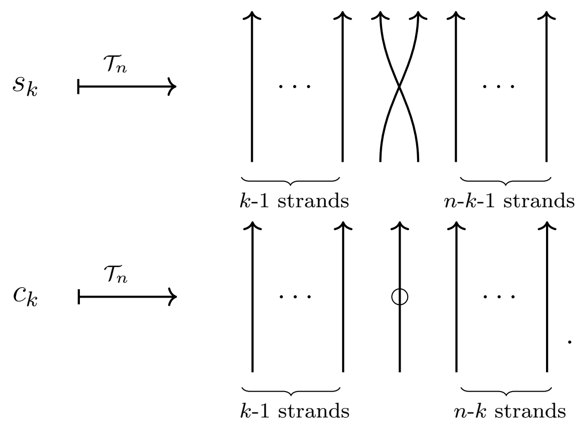
Lemma 4.4.1. *Let $n \geq 0$, then*



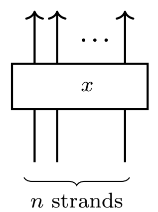
Proof. This follows from the proof of Lemma 4.7 in [OR17] along with the dot sliding relation (4.34).

□

Because of relations (4.22), (4.25), (4.28), and (4.29), there are homomorphisms $\mathcal{T}_n : \mathbb{S}_n^{opp} \to \mathrm{Hom}_{\mathcal{H}_{tw}}(P^n)$ which send



In order to simplify our diagrams we write the image of $x \in \mathbb{S}_n$ under $\mathcal{T}_n$ as

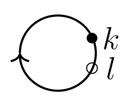
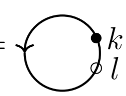
$\mathcal{T}_n(x) \quad =:$ 

(4.35)

4.4.2 The center of $\mathcal{H}_{tw}$

The center of a $\Bbbk$-linear monoidal category $\mathcal{C}$ is defined to be the endomorphism algebra of the monoidal unit $\mathbb{1}$ of $\mathcal{C}$, that is $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$. In a diagrammatic category such as $\mathcal{H}_{tw}$, $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ is then by definition the commutative algebra of closed diagrams where multiplication of two closed diagrams corresponds to placing them next to each other.

It is shown in [Kho14] that clockwise or counterclockwise bubbles with solid dots on them form an algebraically independent generating set for the center of Khovanov's Heisenberg category. We consider the bubble analogs in $\mathrm{End}_{\mathcal{H}_{tw}}(1)$:

$d_{n,l} :=$  and $\bar{d}_{n,l} :=$  . (4.36)

These elements of $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ were studied in [OR17] using techniques analogous to those from [Kho14]. Because we can cancel pairs of hollow dots (the first relation from (4.28)) it is clear that the only cases that need to be considered are $d_{n,0}$ and $d_{n,1}$ and $\bar{d}_{n,0}$ and $\bar{d}_{n,1}$. Actually it is shown in [OR17] that due to relations (4.26) and (4.27) $d_{n,1} = \bar{d}_{n,1} = 0$. Therefore we simplify our notation to

$$d_n := d_{n,0} \qquad \text{and} \qquad \bar{d}_n := \bar{d}_{n,0}.$$

Lemma 4.4.2. *[OR17] If n is odd, then $d_n = \bar{d}_n = 0$.*

The remaining nonzero bubbles serve as the analog to Khovanov's bubble generators.

Proposition 4.4.3. *[OR17] The elements $\{d_{2k}\}_{k\geq 0}$ are algebraically independent generators of* $\mathrm{End}_{\mathcal{H}_{tw}}(1)$*, i.e. there is an isomorphism*

$$\mathrm{End}_{\mathcal{H}_{tw}}(1) \cong \mathbb{C}[d_0, d_2, d_4, \dots].$$

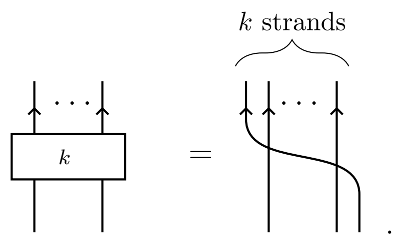
The elements $\{d_{2k}\}_{k\geq 0}$ and $\{\bar{d}_{2k}\}_{k\geq 0}$ can be related via a recursive relation.

Proposition 4.4.4. *[OR17] For $n \geq 1$,*

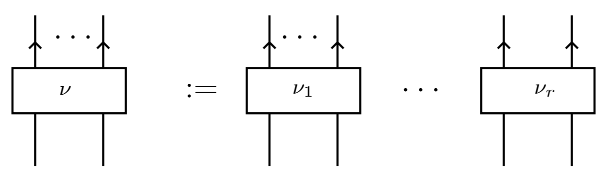
$$\bar{d}_{2n} = \sum_{2a+2b=2n-2} \bar{d}_{2a} d_{2b}.$$

Corollary 4.4.5. *The elements $\{\bar{d}_{2k}\}_{k\geq 1}$ are another algebraically independent generating set of* $\mathrm{End}_{\mathcal{H}_{tw}}(1)$.

Another natural set of diagrams in $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ come from the closure of permutations. We define

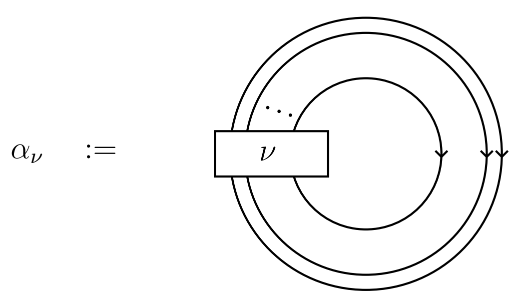


For $\nu = (\nu_1, \dots, \nu_r) \in \mathcal{P}_k$, let



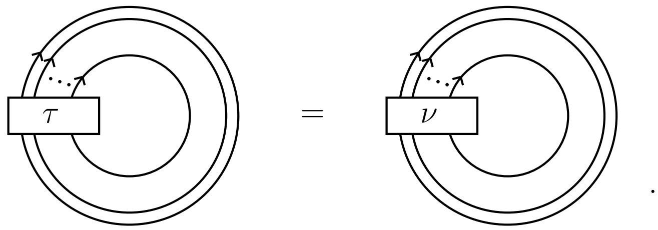
(4.37)

then we define



We set $\alpha_k := \alpha_{(k)}$.

Remark This notation is consistent with (4.35) in the sense that it is shown in [KLM16] that if $\tau \in S_k$ has cycle type ν then



One can impose a grading on $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ by setting:

$$\deg(d_0) = 0 \qquad \text{and} \qquad \deg(d_{2k}) = 2k + 1. \tag{4.38}$$

Lemma 4.4.6. *In terms of the grading defined by* (4.38)*,*

$$\alpha_{2k+1} = d_{2k} + l.o.t.$$

Proof. We can reduce the diagram α_{2k+1} to a polynomial in $d_0, d_2, d_4, \dots$ via repeated application of the dot sliding moves (4.33)-(4.34) and clockwise bubble sliding move from Lemma 4.4.1. The goal of each move is to increase the number of crossings coming from solid dots and separate nested diagrams. Each application of these rules will result in a single connected diagram D whose total number of crossings is $2k$ (including those from solid dots), plus additional terms whose total number of crossings (including those from solid dots) is strictly less than $2k$ (this can be seen by examining (4.33)-(4.34) and (4.4.1)). At the end of this process we have a single bubble with $2k$ dots plus additional terms each of which has total number of dots strictly less that $2k$. □

Corollary 4.4.7. $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ *is generated by* $\{\alpha_{2k+1}\}_{k\geq 0}$ *and these elements are algebraically independent.*

4.4.3 Diagrams as bimodule homomorphisms

An action of $\mathcal{H}_{tw}$ on the category $\mathfrak{S}$ whose objects are compositions of induction and restriction functors between $\mathbb{Z}/2\mathbb{Z}$-graded finite dimensional $\mathbb{S}_n$-supermodules, for all $n \geq 0$, is described in [CS15, Section 6.3]. Because induction and restriction functors for the algebras $\mathbb{S}_n$ can be written as

$$\mathrm{Ind}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}}(-) = \mathbb{S}_{n+1} \otimes_{\mathbb{S}_n} - \qquad \text{and} \qquad \mathrm{Res}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}}(-) = {}_{\mathbb{S}_n}\mathbb{S}_{n+1} \otimes_{\mathbb{S}_{n+1}} -$$

(where ${}_{\mathbb{S}_n}\mathbb{S}_{n+1}$ is $\mathbb{S}_{n+1}$ as a left $\mathbb{S}_n$-module) the objects of $\mathfrak{S}$ can alternatively be described as tensor products of certain $(\mathbb{S}_{k_1}, \mathbb{S}_{k_2})$-bimodules for all $k_1, k_2 \geq 0$. We will use this interpretation extensively below. Let $k_1, k_2 \leq n$, then we write

- (n) for $\mathbb{S}_n$ considered as a $(\mathbb{S}_n, \mathbb{S}_n)$-bimodule,
- $(n)_{k_2}$ for $\mathbb{S}_n$ considered as a $(\mathbb{S}_n, \mathbb{S}_{k_2})$-bimodule,
- ${}_{k_1}(n)$ for $\mathbb{S}_n$ considered as a $(\mathbb{S}_{k_1}, \mathbb{S}_n)$-bimodule,

- ${}_{k_1}(n)_{k_2}$ for $\mathbb{S}_n$ considered as a $(\mathbb{S}_{k_1}, \mathbb{S}_{k_2})$-bimodule.

The morphisms in $\mathfrak{S}$ are certain natural transformations of these compositions of induction/restriction functors (or, equivalently, certain bimodule homomorphisms). Like $\mathcal{H}_{tw}$, morphisms in $\mathfrak{S}$ can be presented diagrammatically as oriented compact 1-manifolds immersed in $\mathbb{R} \times [0,1]$. Unlike $\mathcal{H}_{tw}$, in $\mathfrak{S}$ we label the regions of the strip $\mathbb{R} \times [0,1]$ by non-negative integers, so that if there is an upwards oriented line separating two regions and the right region is labeled by n, then the left region must be labeled by $n+1$. The diagram

$$n+1 \quad \Big\uparrow \quad n$$

denotes the identity endomorphism of the induction functor $\mathrm{Ind}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}}$ or alternatively the identity endomorphism of the bimodule $(n+1)_n$.

If there is a downward oriented line separating two regions and the right is labeled by $n+1$ then the left must be labeled by n. The diagram

$$n \quad \Big\downarrow \quad n+1$$

denotes the identity endomorphism of the restriction functor $\mathrm{Res}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}}$ or alternatively the identity endomorphism of the bimodule ${}_n(n+1)$. Descriptions of other morphisms in $\mathfrak{S}$ are most easily given in terms of bimodules, so we henceforth use this language exclusively.

The hollow dots correspond to multiplication by Clifford elements with

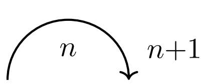
$$n+1 \quad \Big\uparrow\!\!\circ \quad n$$

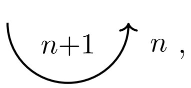
being the bimodule map $(n+1)_n \to (n+1)_n$ which sends $x \mapsto (-1)^{|x|} x c_{n+1}$ and

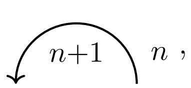
$$n \quad \Big\downarrow\!\!\circ \quad n+1$$

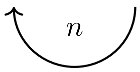
the bimodule map ${}_n(n+1) \to_n (n+1)$ which sends $x \mapsto c_{n+1} x$.

The bimodule maps associated to the four cups and caps are:

 $$, \quad (n+1)_n(n+1) \to (n+1), \quad x \otimes y \mapsto xy, \quad x, y \in \mathbb{S}_{n+1}, \tag{4.39}$$

 $$, \quad (n) \to {}_n(n+1)_n, \quad x \mapsto x, \quad x \in \mathbb{S}_n, \tag{4.40}$$

 $$, \quad {}_n(n+1)_n \to (n), \quad x \mapsto \mathrm{pr}_n(x) = \begin{cases} x & x \in \mathbb{S}_n \\ 0 & \text{otherwise}, \end{cases} \tag{4.41}$$

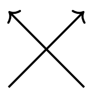


$n{+}1$, $(n+1) \to (n+1)_n(n+1),$ (4.42)

where (4.42) is determined by the condition that

$$1 \;\mapsto\; \sum_{i=1}^{n+1} s_i \cdots s_n \otimes s_n \cdots s_i \;-\; s_i \ldots s_n c_{n+1} \otimes c_{n+1} s_n \ldots s_i$$
$$= \sum_{x \in \mathcal{LC}_n^{n+1}} x \otimes x^{-1}. \tag{4.43}$$

Finally, the upward crossing is the bimodule map



n , $(n+2)_n \to (n+2)_n, \quad x \mapsto x s_{n+1}, \quad x \in \mathbb{S}_{n+2}.$ (4.44)

Any diagram that has a region labeled with a negative number is set to **0**. It is shown in [CS15] that all diagrams are compatible with isotopy.

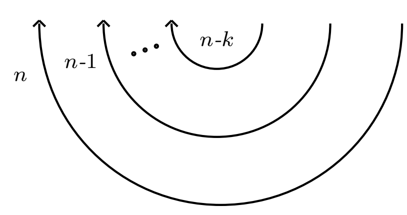
Remark The action of $\mathcal{H}_{tw}$ on $\mathfrak{S}$ can be lifted to the idempotent closures of these categories. This then becomes a categorification of the Fock space representation [CS15].

Following Khovanov's approach from [Kho14], let $\mathfrak{S}_k$ be the subcategory of $\mathfrak{S}$ whose objects are right $\mathbb{S}_k$-modules. For every $k \in \mathbb{Z}_{\geq 0}$ there is a functor $F_k^{\mathcal{H}_{tw}} : \mathcal{H}_{tw} \to \mathfrak{S}_k$ where for any product of P's and Q's, P is sent to $(n+1)_n$ and Q is sent to ${}_{n-1}(n)$. The value of n in the rightmost of these bimodules is k and all other values are determined by this. Under $F_k^{\mathcal{H}_{tw}}$ a morphism (or diagram) is mapped to a morphism in $\mathfrak{S}_k$ by labeling the rightmost region by k which determines the labelings of all other regions.

Note that the image of a closed diagram D under $F_n^{\mathcal{H}_{tw}}$ will be an $(\mathbb{S}_n, \mathbb{S}_n)$-bimodule endomorphim of $\mathbb{S}_n$ which we denote as $f : \mathbb{S}_n \to \mathbb{S}_n$. f is fully determined by the value $f(1)$ since for any $x \in \mathbb{S}_n$, $f(x) = xf(1)$. Furthermore, $f(1)$ is an element of $Z(\mathbb{S}_n)_{\bar{0}}$ because $xf(1) = f(x) = f(1)x$. In this way we can identify the image of $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ under $F_n^{\mathcal{H}_{tw}}$ with elements of $Z(\mathbb{S}_n)_{\bar{0}}$.

We next study the image of some of the elements of $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ from Section 4.4.2 under the functor $F_n^{\mathcal{H}_{tw}}$.

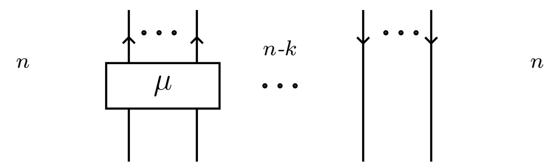
Lemma 4.4.8. *1. The diagram*



corresponds to the $(\mathbb{S}_n, \mathbb{S}_n)$*-bimodule homomorphism* $(n) \to (n)_{n-k}(n)$ *which sends*

$$1 \mapsto \sum_{x \in \mathcal{LC}^n_{n-k}} x \otimes x^{-1}.$$

2. *For* $\mu \in \mathcal{OP}_k$ *with* $k \leq n$*, the diagram*



corresponds to the $(\mathbb{S}_n, \mathbb{S}_n)$*-bimodule homomorphism* $(n)_{n-k}(n) \to (n)_{n-k}(n)$ *which for* $x, y \in \mathbb{S}_n$ *sends*

$$x \otimes y \mapsto x\sigma_{\mu;n} \otimes y.$$

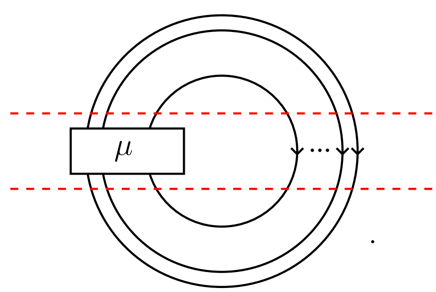
Proof. Both 1 and 2 follow from calculations using the definitions of cups (4.42) and crossings (4.44). □

Proposition 4.4.9. *For* $\mu \in \mathcal{OP}_k$,

$$F_n^{\mathcal{H}_{tw}}(\alpha_\mu) = \begin{cases} A_{\mu;n} & \text{if } k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The diagram for α_μ can be broken into three components

$\alpha_\mu \quad :=$



Reading from bottom to top, the first component corresponds to Lemma 4.4.8.1, and the second corresponds to Lemma 4.4.8.2 The composition of these two maps sends

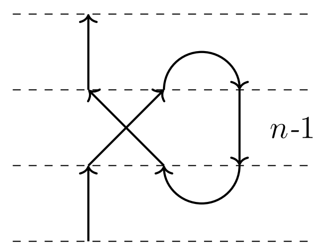
$$1 \mapsto \sum_{x \in \mathcal{LC}^n_{n-k}} x\sigma_{\mu;n} \otimes x^{-1}.$$

The top component of k nested caps is the multiplication map which sends

$$\sum_{x \in \mathcal{LC}^n_{n-k}} x\sigma_{\mu;n} \otimes x^{-1} \mapsto \sum_{x \in \mathcal{LC}^n_{n-k}} x\sigma_{\mu;n} x^{-1} = A_{\mu;n}.$$

□

Lemma 4.4.10. *[OR17] For $n-1 \geq 0$, the right twist curl*



corresponds to the $(\mathbb{S}_n, \mathbb{S}_{n-1})$-bimodule homomorphism, $(n)_{n-1} \to (n)_{n-1}$ which multiplies $x \in \mathbb{S}_n$ on the right by the Jucys-Murphy element J_n

$$x \mapsto xJ_n.$$

Proposition 4.4.11. *Let $k \geq 0$ and $n \geq 1$, then*

1. $F_n^{\mathcal{H}_{tw}}(\bar{d}_{2k}) = \mathrm{pr}_n(J_{n+1}^{2k})$,
2. $F_n^{\mathcal{H}_{tw}}(d_{2k}) = \sum_{x \in \mathcal{LC}_{n-1}^n} xJ_n^{2k}x^{-1}$.

Proof. These follow from direct calculation using the definitions of cups and caps (4.39)-(4.42) and Lemma 4.4.10.

□

4.5 An isomorphism between $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ and Γ

In this section we establish an isomorphism between $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ and Γ. The key step in the construction of this map will be identifying the elements of $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ with functions on $\mathcal{SP}$, i.e. as elements of $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$. To do this let $\lambda \in \mathcal{SP}_n$ and $x \in \mathrm{End}_{\mathcal{H}_{tw}}(1)$, then we evaluate x on λ by

$$x(\lambda) := \widetilde{\chi}^\lambda(F_n^{\mathcal{H}_{tw}}(x)).$$

Because $F_n^{\mathcal{H}_{tw}}$ is a homomorphism on $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ which maps into $Z(\mathbb{S}_n)_{\bar{0}}$ and $\widetilde{\chi}^\lambda$ is a homomorphism when restricted to $Z(\mathbb{S}_n)_{\bar{0}}$, this defines a homomorphism into $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$.

Proposition 4.5.1. *For $\mu \in \mathcal{OP}_k$ and $\lambda \in \mathcal{SP}_n$ we have*

$$\alpha_\mu(\lambda) = \begin{cases} 2^k n^{\downarrow k} \frac{\chi^\lambda(\mu \cup 1^{n-k})}{\chi^\lambda(1^n)} & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases}$$

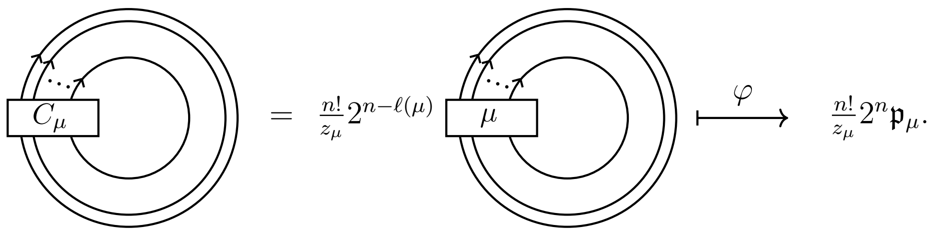
Proof. This follows from Proposition 4.2.11 and Proposition 4.4.9. □

Theorem 4.5.2. *There is an algebra isomorphism $\varphi : \mathrm{End}_{\mathcal{H}_{tw}}(1) \to \Gamma$ which for any $\mu \in \mathcal{OP}$, sends*

$$\alpha_\mu \mapsto 2^{\ell(\mu)}\mathfrak{p}_\mu.$$

Proof. It is clear from Proposition 4.5.1 and Corollary 4.3.4 that $2^{-\ell(\mu)}\alpha_\mu$ and $\mathfrak{p}_\mu$ map to the same function in $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$. Furthermore the collection of functions which are the image of $\{\mathfrak{p}_{2k+1}\}_{k\geq 0}$ are algebraically independent by Proposition 4.3.1 and Corollary 4.3.5. By Proposition 4.4.7 $\mathrm{End}_{\mathcal{H}_{tw}}(1)$ is generated by the algebraically independent elements $\{\alpha_{2k+1}\}_{k\geq 0}$. It then follows that the map that sends $\alpha_\mu \mapsto 2^{\ell(\mu)}\mathfrak{p}_\mu$ is an isomorphism. □

Let $\mu \in \mathcal{OP}_n$. It follows from Lemma 4.2.3, Remark 4.4.2, and Theorem 4.5.2 that



(4.45)

Theorem 4.5.3. *Let $\lambda \in \mathcal{SP}_n$. Under the isomorphism $\varphi : \mathrm{End}_{\mathcal{H}_{tw}}(1) \to \Gamma$, the closure of the central idempotent e_λ of $\mathbb{S}_n$ maps to $h(\lambda)Q^*_\lambda$.*

Proof. Recall from Lemma 4.2.12 that

$$e_\lambda = 2^{\frac{-\ell(\lambda)-\delta(\lambda)}{2}} \frac{g_\lambda}{n!} \sum_{\mu \in \mathcal{OP}_n} \chi^\lambda(\mu) C_\mu$$

while by (4.21)

$$Q^*_\lambda = 2^{\frac{\ell(\lambda)-\delta(\lambda)}{2}} \sum_{\mu \in \mathcal{OP}_n} \frac{\chi^\lambda(\mu)}{z_\mu} \mathfrak{p}_\mu.$$

Combining these facts with Theorem 4.5.2 and (4.45) it follows that the closure of e_λ is equal to $2^{n-\ell(\lambda)} g_\lambda Q^*_\lambda = h(\lambda)Q^*_\lambda$. □

Remark Recall that the Schur Q-functions are related to the Schur P-functions by $P_\lambda = 2^{-\ell(\lambda)}Q_\lambda$. Ivanov also studied factorial Schur P-functions $\{P^*_\lambda\}_{\lambda \in \mathcal{SP}_n}$ where $P^*_\lambda = 2^{-\ell(\lambda)}Q^*_\lambda$ [Iva01]. Then one alternative description of the closure of e_λ in Γ is as $2^n g_\lambda P^*_\lambda$.

Moving in the opposite direction, we can also identify the elements of Γ corresponding to the generators $\{d_{2k}\}_{k\geq 0}$ and $\{\bar{d}_{2k}\}_{k\geq 0}$.

Theorem 4.5.4. *For $k \geq 0$,*

1. $\psi(\bar{d}_{2k}) = \mathbf{g}^\uparrow_k(\cdot)$,
2. $\psi(d_{2k}) = \mathbf{g}^\downarrow_{k+1}(\cdot)$.

Proof. This follows from Proposition 4.2.16 and Proposition 4.4.11. □

Remark In light of Theorem 4.5.4, Proposition 4.4.4 can be seen as a diagrammatic manifestation of Proposition 4.2.15.

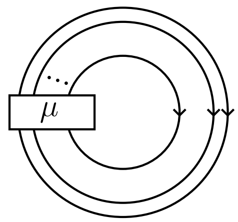
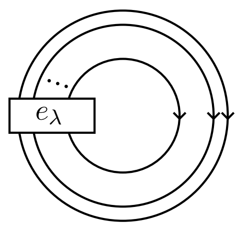
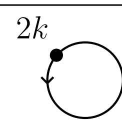
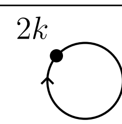
Γ	diagram in $\mathrm{End}_{\mathcal{H}_{tw}}(1)$
$\mathfrak{p}_\mu,\ \mu \in \mathcal{OP}$	$\frac{1}{2^{\ell(\mu)}}$ 
$Q^*_\lambda,\ \lambda \in \mathcal{SP}$	$\frac{1}{h(\lambda)}$ 
$\mathbf{g}^\uparrow_k$	
$\mathbf{g}^\downarrow_{k+1}$	

Table 4.1: A dictionary between Γ and diagrams in $\mathrm{End}_{\mathcal{H}_{tw}}(1)$. The notation $h(\lambda)$ denotes the number of paths in the Schur graph from $\emptyset$ to λ

4.6 An action of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ on Γ

Aside from taking the Grothendieck group or center, another method for decategorifing a category $\mathcal{C}$ is taking the categorical trace of $\mathcal{C}$, $\mathrm{Tr}(\mathcal{C})$ (also known as the zeroth Hochschild homology of $\mathcal{C}$). See [BGHL14] for a discussion of this method of decategorification. In [OR17], it is shown that the even part of the trace of $\mathcal{H}_{tw}$, $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$, is isomorphic to a quotient of the vertex algebra W^-, a subalgebra of $W_{1+\infty}$ defined by Kac, Wang, and Yan [KWY98].

In a diagrammatic setting such as this, the trace can be realized as the algebra of closed diagrams on an annulus. There is a natural action of $\mathrm{Tr}(\mathcal{C})$ on the center of the category $\mathcal{C}$, $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$, where diagrammatically a closed diagram on an annulus acts on a closed diagram in a disk by plugging the annulus with the disk, resulting in a new diagram in the disk. The results of [OR17] along with Theorem 4.5.2 imply that W^- acts on Γ. This action is similar to the action of $W_{1+\infty}$ on the centers of symmetric group algebras described in [LT01]. In this section we will first review W^- and then describe the action of the generators of W^- on basis elements of Γ.

4.6.1 The W-algebra W^-

We review the vertex algebra W^-, a quotient of which appears in the trace of $\mathcal{H}_{tw}$.

Let $\mathcal{D}$ denote the Lie algebra of differential operators on the circle. The central extension $\hat{\mathcal{D}}$ of $\mathcal{D}$ is described in [KWY98]. It is generated by C and by $w_{k,l} = t^k D^l$ for $l \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$ where t is a variable over $\mathbb{C}$, and $D = t\frac{d}{dt}$, subject to relations that C and $w_{0,0}$ are central, and:

$$[t^r f(D), t^s g(D)] = t^{r+s}(f(D+s)g(D) - f(D)g(D+r)) + \psi(t^r f(D), t^s g(D))C, \tag{4.46}$$

where

$$\psi(t^r f(D), t^s g(D)) = \begin{cases} \sum_{-r\leq j\leq -1} f(j)g(j+r) & r = -s \geq 0 \\ 0 & r+s \neq 0 \end{cases} \tag{4.47}$$

for f, g polynomials. The W-algebra $W_{1+\infty}$ is the universal enveloping algebra of $\hat{\mathcal{D}}$. It is shown in [CLLS16] that the trace of Khovanov's Heisenberg category is isomorphic to $W_{1+\infty}/\langle \omega_{0,0}, C-1\rangle$.

The twisted Heisenberg category $\mathcal{H}_{tw}$ is related to the universal enveloping algebra of a central extension of a Lie subalgebra of $\mathcal{D}$ fixed by a degree preserving anti-involution. Define the map:

$$\begin{aligned} \zeta :\ & \mathcal{D} \longrightarrow \mathcal{D} \\ & 1 \mapsto \zeta(1) = -1 \\ & t \mapsto \zeta(t) = -t \\ & D \mapsto \zeta(D) = -D. \end{aligned}$$

This is a polynomial-degree preserving anti-involution of $\mathcal{D}$, and the Lie subalgebra fixed by $-\zeta$ is

$$\mathcal{D}^- := \{a \in \mathcal{D} | \zeta(a) = -a\}.$$

Let $\hat{\mathcal{D}}^-$ be a central extension of $\mathcal{D}^-$ where the 2-cocycle is the restriction of the 2-cocycle ψ given above. Then $\hat{\mathcal{D}}^-$ is a Lie subalgebra of $\hat{\mathcal{D}}$.

More explicitly, $\hat{\mathcal{D}}^-$ is the Lie algebra over the vector space spanned by $\{C\} \cup \{t^{2k-1} g(D + (2k-1)/2);\ g \text{ even}\} \cup \{t^{2k} f(D+k);\ f \text{ odd}\}$ where $k \in \mathbb{Z}$ and even and odd refer to even and odd polynomial functions. Its Lie bracket is given by equation (4.46).

Denote by W^- the universal enveloping algebra of $\hat{\mathcal{D}}^-$. The trace of $\mathcal{H}_{tw}$ was shown in [OR17] to be isomorphic to the quotient $W^-/\langle \omega_{0,0}, C-1\rangle$.

Note that not all $w_{k,\ell}$ are contained in W^-.

Example 6.1 When $k-\ell$ is an even integer, $w_{k,\ell} \notin W^-$. Moreover, the difference $k-\ell$ being odd is not sufficient. For example, $t^2 D = w_{2,1} \notin W^-$ since an element starting with t^2 should be followed by $f(D+1)$ where f is an odd polynomial function. Hence $t^2 D = w_{2,1} \notin W^-$ but $t^2(D+1) = t^2 D + t^2 = w_{2,1} + w_{2,0} \in W^-$ (and, indeed, $\zeta(t^2(D+1)) = t^2(-D-1) = -t^2(D+1)$).

A generating set for W^- as an algebra was described in [OR17].

Proposition 4.6.1. *[OR17, Lemma 2.2] The algebra $W^-/\langle \omega_{0,0}, C-1\rangle$ is generated by $\omega_{1,0}$, $\omega_{0,3}$, and $\omega_{\pm 2,1} \pm \omega_{\pm 2,0}$.*

In order to explicitly write down an action of the algebra W^- on Γ, we will work with a more convenient generating set for W^-.

Proposition 4.6.2. *The algebra $W^-/\langle \omega_{0,0}, C-1\rangle$ is also generated by $\omega_{1,0}$, $\omega_{-1,0}$ and $\omega_{0,3}$.*

Proof. Let $A := \{\omega_{1,0}, \omega_{-1,0}, \omega_{0,3}\}$. We will show that we can obtain the generators in Proposition 4.6.1 using the elements of A. This amounts to obtaining $\omega_{\pm 2,1} \pm \omega_{\pm 2,0}$ via elements of A. It is a straightforward computation that

$$\omega_{0,1} = -\frac{1}{20}[[\omega_{0,3}, \omega_{-1,0}], \omega_{1,0}] + \frac{1}{5}\omega_{-1,0}\omega_{1,0}$$

and using $\omega_{0,1}$, we can obtain $\omega_{-1,2} - \omega_{-1,1}$ as follows

$$\omega_{-1,2} - \omega_{-1,1} = \frac{1}{6}[\omega_{0,3}, \omega_{-1,0}] + \frac{1}{3}\omega_{-1,0}\omega_{0,1}.$$

Then one of the elements we are looking for is given by

$$\omega_{-2,1} - \omega_{-2,0} = \frac{1}{2}[\omega_{-1,2} - \omega_{-1,1}, \omega_{-1,0}].$$

To obtain $\omega_{2,1} + \omega_{2,0}$, we follow a very similar computation:

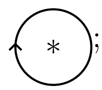
$$\omega_{1,2} + \omega_{1,1} = -\frac{1}{6}[\omega_{0,3}, \omega_{1,0}] + \frac{1}{3}\omega_{0,1}\omega_{1,0}$$

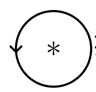
and finally

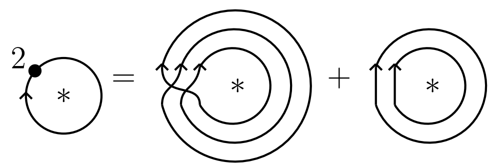
$$\omega_{2,1} + \omega_{2,0} = -\frac{1}{2}[\omega_{1,2} + \omega_{1,1}, \omega_{1,0}].$$

□

The images of these generators under the isomorphism $W^-/\langle \omega_{0,0}, C-1 \rangle \to \mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ from [OR17] are as follows:

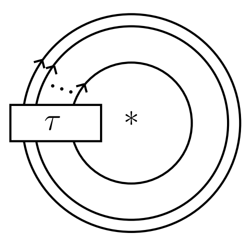
$$\sqrt{2}\omega_{-1,0} \longmapsto$$  ;

$$\sqrt{2}\omega_{1,0} \longmapsto$$  ;

$$-2\omega_{0,3} \longmapsto$$  .

Additionally we will use the elements $\omega_{-(2n+1),0}$ and their images in $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$:

$$\sqrt{2}\omega_{-(2n+1),0} \longmapsto \qquad (4.48)$$



where τ is a $2n+1$ cycle.

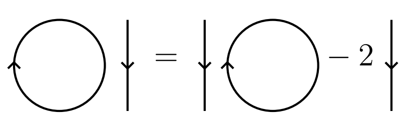
4.6.2 A description of the action

We describe the action of the W^- generating set $\{\omega_{1,0}, \omega_{-1,0}, \omega_{0,3}\}$ on the vector space basis $\{\mathfrak{p}_\mu\}_{\mu \in \mathcal{OP}}$ of Γ. We achieve this by describing the action of the corresponding generators of $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ on the basis $\{\alpha_\mu\}_{\mu \in \mathcal{OP}}$ of $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$.

Lemma 4.6.3. *We have*

$$\alpha_{(\mu,1)} = \alpha_\mu \alpha_1 - 2|\mu|\alpha_\mu.$$

Proof. This simply follows from the local bubble sliding relation

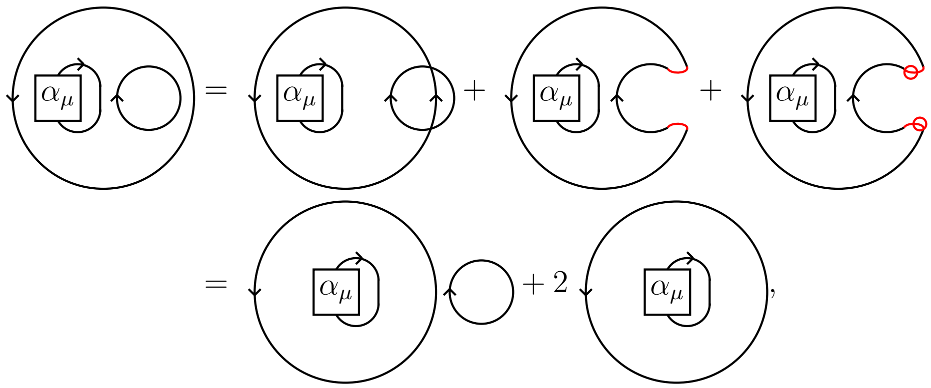


applied $|\mu|$ times to the diagram $\alpha_{(\mu,1)}$, as we pull the clockwise bubble α_1 from within α_μ. □

Lemma 4.6.4. *We have*

$$\omega_{1,0} \cdot \alpha_\mu \alpha_1 = (\alpha_1 + 2)\omega_{1,0} \cdot \alpha_\mu.$$

Proof. We compute:

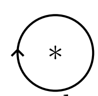


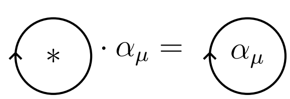
as desired. □

Theorem 4.6.5. *The generators* $\mathrm{Tr}(\mathcal{H}_{tw})_{\bar{0}}$ *act on the basis elements* $\{\mathfrak{p}_\mu\}_{\mu \in \mathcal{OP}}$ *of* Γ *as follows:*

$$\omega_{-1,0} \cdot \mathfrak{p}_\mu = \sqrt{2}\mathfrak{p}_{(\mu,1)},$$
$$\omega_{1,0} \cdot \mathfrak{p}_\mu = \frac{1}{\sqrt{2}}\mathfrak{p}_\mu + \frac{k}{\sqrt{2}}\mathfrak{p}_{\hat{\mu}},$$
$$\omega_{0,3} \cdot \mathfrak{p}_\mu = -\mathfrak{p}_3\mathfrak{p}_\mu - 2\mathfrak{p}_{(1,1)}\mathfrak{p}_\mu$$

where k *is the number of parts of size 1 of* μ *and* $\hat{\mu}$ *stands for the partition obtained by removing one part of size 1 from* μ *if this is possible. When* $\mu = (1)$ *then* $\mathfrak{p}_{\widehat{(1)}} = 1$.

Proof. For the action of $\omega_{-1,0}$, note that the action of  on α_μ is diagrammatically just enclosing the diagram of α_μ by a clockwise oriented strand:

 (4.49)

and the resulting diagram is the diagram of $\alpha_{(\mu,1)}$. Replacing α_μ by $2^{\ell(\mu)}\mathfrak{p}_\mu$ and the clockwise bubble by $\sqrt{2}\omega_{-1,0}$, we get

$$\omega_{-1,0} \cdot \mathfrak{p}_\mu = \sqrt{2}\mathfrak{p}_{(\mu,1)}.$$

We also know that $\omega_{-(2n+1),0} \cdot \mathfrak{p}_\mu = \sqrt{2}\mathfrak{p}_{(\mu,2n+1)}$ from (4.48). To calculate the action of $\omega_{1,0}$, we will use the the commutator relations

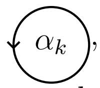
$$[\omega_{-1,0}, \omega_{1,0}] = -1$$

$$[\omega_{-(2n+1),0}, \omega_{1,0}] = 0 \quad \text{for } n \geq 0$$

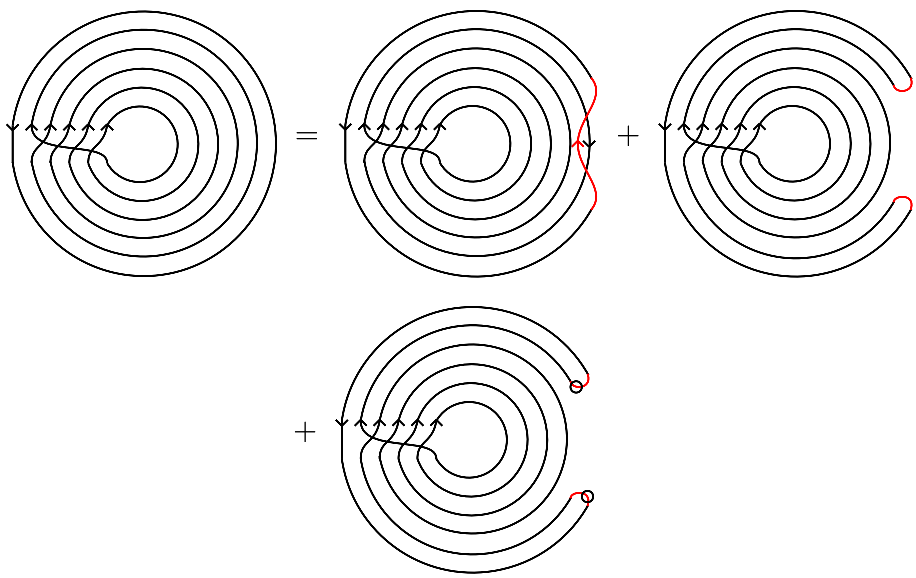
and (4.49).

To simplify the notation in the following computations, we will use $\omega_+ := \sqrt{2}\omega_{1,0}$.

We start by showing that if the partition μ doesn't contain any parts of size one, then $\omega_+ \cdot \alpha_\mu = \alpha_\mu$ by induction on $\ell(\mu)$. We provide a diagrammatic proof for the base case $\ell(\mu) = 1$ (i.e. $\alpha_\mu = \alpha_k$ for $k \neq 1$ odd).

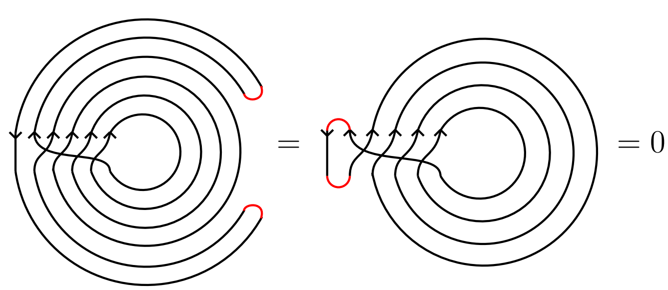
In the diagram , we claim that we can pass α_k through the outer strand for free, meaning that all the resolution terms that appear as a result of relation 4.23 are zero.

We provide the computation for the case of $\alpha_k = \alpha_5$, and explain how the arguments generalize to any α_k. We have



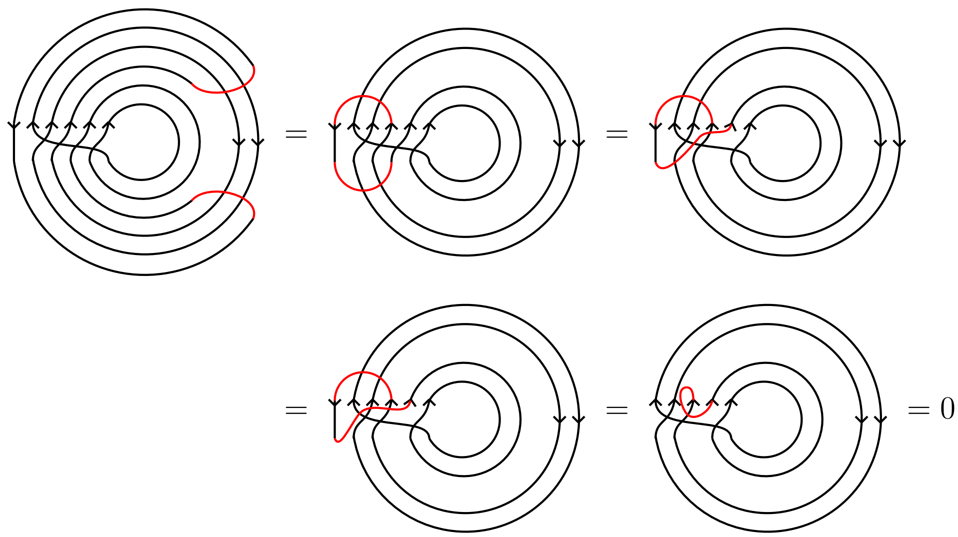
and the two hollow dots appearing in the last term cancel with each other if we slide them along the outermost strand. This observation will hold for the rest of the computation, so we will omit drawing the second resolution term and instead write the first resolution term with coefficient 2. We will show that all resolution terms coming from crossings on the: outermost strand, innermost strand, and intermediate strands are zero.

For the resolution term coming from the crossing of outermost strands, we have



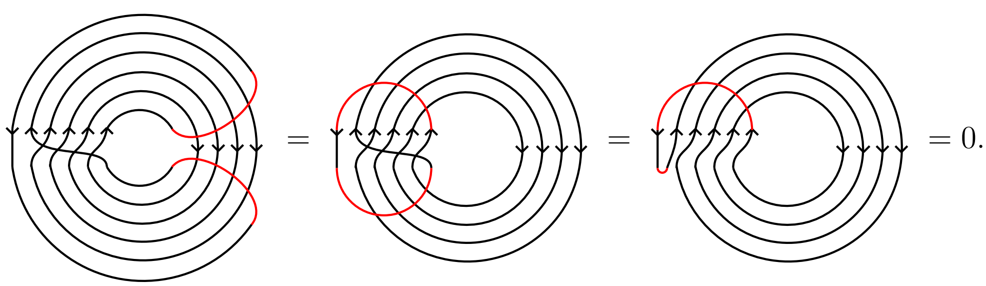
where the last equality follows from (4.24).

For the resolution term coming from the crossing of intermediate strands, consider a generic intermediate strand. We have

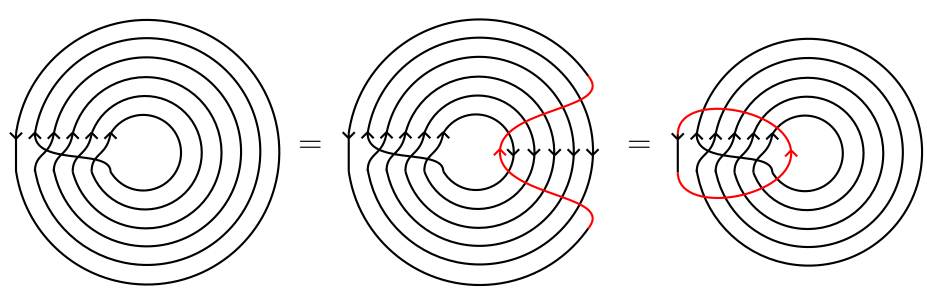


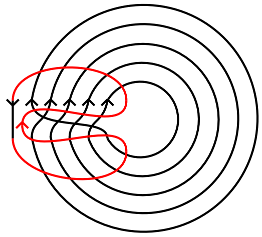
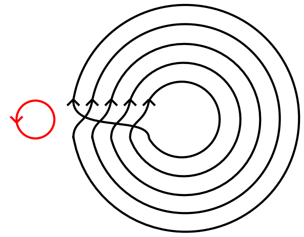
where the second and third equalities follow from a Reidemeister 3 move, and the fourth is a result of relation (4.22). Hence these resolution terms are zero as well. In general, for a resolution term coming from a crossing of intermediate strands, we can first pull the red string above the permutation using Reidemeister 3 moves, and then pull the red string into the permutation using relation (4.22) to get a left twist curl.

Finally, for the resolution term coming from the crossing of intermediate strands the situation is simpler:



Hence all the resolution terms are zero. This leaves us with



$=$  $=$ 

and a counter-clockwise oriented bubble is equal to 1 by the defining relation (4.24). These diagrammatic arguments clearly hold for arbitrary $k > 1$. Hence the action of $\omega_{(1,0)}$ on α_k for $k \neq 1$ is trivial.

This concludes the proof of the base case $\ell(\mu) = 1$. Now suppose $\omega_+ \cdot \alpha_\mu = \alpha_\mu$ for some $\mu \in \mathcal{OP}$ such that $\ell(\mu) = m - 1$, and let n be a positive integer. Then

$$\begin{aligned}
0 &= [\sqrt{2}\omega_{-(2n+1),0}, \omega_+] \cdot \alpha_\mu = \sqrt{2}\omega_{-(2n+1),0} \cdot (\omega_+ \cdot \alpha_\mu) - \omega_+ \cdot (\sqrt{2}\omega_{-(2n+1),0} \cdot \alpha_\mu) \\
&= \sqrt{2}\omega_{-(2n+1),0} \cdot \alpha_\mu - \omega_+ \cdot (\sqrt{2}\omega_{-(2n+1),0} \cdot \alpha_\mu) \\
&= \alpha_{(\mu,2n+1)} - \omega_+ \cdot \alpha_{(\mu,2n+1)}
\end{aligned}$$

and the result follows by induction.

Hence if μ doesn't contain any parts of size 1, then

$$\omega_+ \cdot \alpha_\mu = \alpha_\mu.$$

Now suppose γ is an odd partition without parts of size 1. We will prove that

$$\omega_+ \cdot \alpha_{(\gamma,1^k)} = \alpha_{(\gamma,1^k)} + 2k\alpha_{(\gamma,1^{k-1})}$$

by induction on k. The base case $k = 0$ was proved above. Suppose the formula holds for $\alpha_{(\gamma,1^k)}$.

$$\begin{aligned}
\omega_+ \cdot \alpha_{(\gamma,1^{k+1})} &= \omega_+ \cdot \left(\alpha_{(\gamma,1^k)}\alpha_1 - 2|(\gamma,1^k)|\alpha_{(\gamma,1^k)}\right) \quad \text{by Lemma 4.6.3} \\
&= \alpha_1\omega_+ \cdot \alpha_{(\gamma,1^k)} + 2\omega_+ \cdot \alpha_{(\gamma,1^k)} - 2|(\gamma,1^k)|\omega_+ \cdot \alpha_{(\gamma,1^k)} \\
&= (\alpha_1 + 2 - 2|(\gamma,1^k)|)\omega_+ \cdot \alpha_{(\gamma,1^k)} \\
&= (\alpha_1 + 2 - 2|(\gamma,1^k)|)(\alpha_{(\gamma,1^k)} + 2k\alpha_{(\gamma,1^{k-1})}) \quad \text{by the inductive hypothesis} \\
&= (\alpha_1 + 2 - 2|(\gamma,1^k)|)\alpha_{(\gamma,1^k)} + 2k(\alpha_1 + 2 - 2|(\gamma,1^k)|)\alpha_{(\gamma,1^{k-1})} \\
&= \alpha_{(\gamma,1^{k+1})} + 4\alpha_{(\gamma,1^k)} + 2k\alpha_{(\gamma,1^k)} \quad \text{by Lemma 4.6.3} \\
&= \alpha_{(\gamma,1^{k+1})} + 2(k+1)\alpha_{(\gamma,1^k)}
\end{aligned}$$

and the result follows after the identification $\alpha_\mu \to 2^{\ell(\mu)}\mathfrak{p}_\mu$.

For the action of $\omega_{0,3}$, note that this element acts on the center as multiplication by itself. Therefore

$$\omega_{0,3} \cdot \alpha_\mu = \alpha_3\alpha_\mu + \alpha_{(1,1)}\alpha_\mu$$

$$-2\omega_{0,3} \cdot 2^{\ell(\mu)}\mathfrak{p}_\mu = 2^{\ell(\mu)+1}\mathfrak{p}_3\mathfrak{p}_\mu + 2^{\ell(\mu)+2}\mathfrak{p}_{(1,1)}\mathfrak{p}_\mu$$

□

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