Fluid limits and the batched processor sharing model

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Abstract

We consider a sequence of single-server queueing models operating under a service policy that incorporates batches into processor sharing. As a processor sharing model is serving all jobs present simultaneously, the rate that it serves each job at is dependent on the number of jobs present in the system. For this reason, keeping track of the residual service times of each job is essential in a processor sharing model in order to be informed of significant impending changes in queue length, and therefore processing rate. We require a tool that will not only allow the recovery of characteristics such as the queue length process but also encodes the residual service time of each job. Each model is described by a measure-valued process that evolves according to a family of dynamic equations. This measure-valued process is defined by placing a unit mass at the residual service time for each job in the system, thereby encapsulating the characteristic that analyzing a processor sharing system requires.

Under mild conditions and a law-of-large-numbers scaling, we prove that the sequence of measure-valued processes converges in distribution to an essentially deterministic limit process. This result heralds back to the consequence of the Law of Large Numbers, where letting n tend to infinity in a sum of n random variables scaled by $1/n$, we obtain a constant. Thus we can approximate the complicated sum by a simple number. In our setting, we show that the limit process obeys periodic dynamics that are easy to describe as a function of the initial condition.

Contents

Chapter 1

Introduction

Queueing theory is the study of waiting lines and networks of such lines. The basic queueing model consists of the following: a server that processes jobs from a buffer; a sequence of jobs, each with an associated service time, which indicates the amount of work that needs to be done to that particular job, or the amount of attention a job requires from the system; an arrival process that describes when each job arrives to the buffer; a service policy, which is the rule that dictates how the server will process the jobs present. Typical quantities one wishes to study for such a model are the queue length, which is the number of jobs in the system, the workload, or sojourn times.

The first paper in queueing theory is attributed to A. K. Erlang in 1909. Erlang worked for the Copenhagen Telephone Exchange and was to determine the number of switchboards necessary to handle all incoming calls in a timely fashion. The Copenhagen Telephone Exchange realized that while one switchboard was insufficient, each additional switchboard increased overhead cost. From their perspective, it was important to handle calls in an efficient manner while reducing overhead costs. Erlang studied a model under which each incoming call took a fixed, deterministic amount of time, and calls came into the Copenhagen Telephone Exchange at random times. While the arrival times were random, for simplicity's sake, he placed heavy restrictions on the characteristics that describe the time between each call. He published his findings from this analysis in 1917.

In 1930 F. Pollaczek built on Erlang's model, removing the restrictions that Erlang had placed on the length of each call. Pollaczek, maintaining the restrictive behavior of time between calls, now allowed each call to take some random length of time. This advancement made the model much more realistic. Pollaczek's paper focused on providing details about queue length and the amount of time a job can expect to spend in the system, both waiting to be processed and actually receiving service. A. Khinchin reproduced Pollaczek's work a few years later, taking an approach that incorporated probability theory.

Beginning in 1940 queueing theory became incorporated as a branch of probability theory. In 1953 G. Kendall emerged as a pioneer in the field, studying systems where the restrictions Erlang, Pollczek, and Khinchin had placed on time between jobs arriving were removed. Kendall instead placed heavy restrictions on the service times of each job.

Queueing theory has continued to develop over the last sixty-five years. It is viewed these days as an area intersecting mathematics, operations research, and industrial engineering. Over the years, queueing models have become more complicated, allowing for example for multiple jobs to be serviced simultaneously, jobs that abandon the system early if service is taking too long, interconnected networks of queues, or randomly assigned priorities to jobs, allowing them to jump ahead in the line.

1.1 Fluid limits and batched processor sharing

We consider a single server queue operating under a service policy that is a variant of processor sharing. In typical processor sharing, every job in the system is worked on simultaneously with the system devoting an equal share of attention to each. Each job remains in the system until it has received the necessary amount of attention. One aspect of processor sharing that has been criticized is that it can take a long time for large jobs to receive full service and exit the system if there are many small jobs present that will slow the system down. Queues using a service policy such as first in first out (FIFO) don't encounter this problem. One way to mitigate such slowdown is to incorporate the notion of a gate into processor sharing which has the effect of creating batches of jobs that are sequenced in a FIFO manner. We require that when the system turns on it begins working on the jobs present using the typical processor sharing approach. These jobs are referred to as the initial batch. If a new job enters while the initial batch is being processed the system does not slow down to begin working on this job too. Instead the system completely ignores any jobs that arrive after it has started processing a set of jobs. Once every job in the initial batch has departed, the system will turn to pick up all jobs that have arrived while the initial batch was being processed. These jobs form the next batch. Again the system treats those jobs using processor sharing and while the system is processing those jobs it will ignore all incoming jobs. The system repeats this cycle indefinitely.

The way the batches are formed is sometimes referred to as using a gated service policy. Gated service policies have been studied in many different settings. In some cases a gated service policy will specify that each batch can only be of some maximal size. In our setting we allow that size to be arbitary so that all jobs present when a batch ends can be processed with the next batch. In [11], [1], and [10] a gated processor sharing system allowing m customers per batch (where $m \leq \infty$) is considered. In [11] they study the distribution of queue length, mean time in system, and distribution of busy periods for a system where jobs arrive according to a Poisson process with exponential service times. In [1] the sojourn time and response time distributions are studied for a system where jobs arrive according to a Poisson process and have arbitrary service time distributions. In [10] a system with bulk Poisson arrivals and arbitrary service time distribution is considered. In particular, they study the wait time, queue length, and batch size distribution for this setting. More recently, gated service policies have been studied in polling models. A polling model is a system of queues with one server. The server rotates cyclically through the queues providing service as the policy dictates at each queue. The system we consider could be viewed as an idealization of a gated polling model. In [3] the queue length and workload distribution is considered for a system with a Poisson arrival process. In [7] they derive the first and second moments of the number of customers at each queue and the expected customer wait time where jobs arrive according to a Poisson process.

A different service policy that combines FIFO and processor sharing protocols is head-of-the-line proportional processor sharing. Under this service discipline, there are jobs of different classes. The server uses processor sharing to simultaneously serve one job from each class. Within each class, the jobs are served FIFO. The diffusion limits for this model under heavy traffic were studied by Bramson [4] and Williams [12].

Another method that has been studied in an attempt to compensate for system slowdown is referred to as limited processor sharing. In limited processor sharing the system uses processor sharing to serve maximally k jobs. If there are less than k jobs in service, an incoming job will be served immediately upon entering. If there are k jobs in service any incoming job waits to begin service. Once a job leaves, the job that has been waiting the longest begins service. The fluid limit was derived in by Dai, Zhang, and Zwart in [5] for a system with jobs arriving according to a general distribution with a general service time distribution.

We record the jobs in the system via a measure-valued process that places a unit mass at the remaining service time for each job present. This measure-valued process, $\{\mu(t) : t \geq 0\}$, was used by Grishechkin [8] and by Gromoll, Puha, and Williams [9] in their analysis of the fluid limit for a processor sharing queue. This measure-valued approach is particularly well-suited to analyzing queues that are operating under a service policy that is similar to processor sharing in nature because in those situations knowing the remaining service time for each job is crucial for understanding how the system is functioning. In this paper, we study a sequence of queues operating under our previously defined service policy. We are primarily concerned with deriving the fluid limit approximation of the measure-valued processes we obtain from the sequence of queues. As with the law of large numbers the fluid limit will provide a simple approximation for the dynamics of our system.

When analyzing our system we will split the measure valued process $\mu(t)$ into two parts: $\mu_1(t)$ and $\mu_2(t)$. We define $\mu_1(t)$ as the process that keeps track of the jobs currently being processed. That is, $\mu_1(t)$ places a unit mass at the residual service time for each job in the batch currently being processed. We define $\mu_2(t)$ to keep track of the jobs that are forming the next batch. That is to say, $\mu_2(t)$ places a unit mass at the remaining service time for each job that enters the system while a batch is being processed. Notice that the residual service times that $\mu_2(t)$ are tracking will remain unchanged because those jobs are being ignored by the system. This approach is necessary because of the batched nature of the service policy.

We consider a critical model where the limiting interarrival rate is denoted by α and the limiting service time distribution is given by ν. Here we assume ν is a Borel probability measure on $\mathbb{R}_+$ with a finite first moment. We further assume $\nu(0) = 0$. Both the interarrival times and the service time of jobs arriving are given by i.i.d. sequences and by definition a critical model has the property that

$$\alpha^{-1} = \mathbb{E}[\nu].$$

When considering this sequence of queues, each queue in the sequence has an associated measure-valued process $\mu^r(\cdot)$. The specific object we study is the fluid scaled process $\frac{1}{r}\mu^r(r\cdot)$. The limiting measure-valued process as $r \to \infty$ will be an element of the Skorohod space $D(\mathcal{M}_F)$, where $\mathcal{M}_F$ is the space of finite, nonnegative Borel measures on $\mathbb{R}_+$ with the weak topology. Notationally, for any function g and measure $\zeta \in \mathcal{M}_F$, we define

$$\langle g, \zeta \rangle = \int g(x)\zeta(dx).$$

Let

$$\mathcal{C} = \{g \in C_b^1(\mathbb{R}_+) \, : g(0) = g'(0) = 0\}$$

where $C_b^1(\mathbb{R}_+)$ is the set of continuous, once differentiable functions on $\mathbb{R}_+$ whose infinity norm and that of their first derivatives are bounded.

The fluid limit solution we obtain is a continuous function $\bar{\mu} : [0, \infty) \to \mathcal{M}_F$. As with the pre-limit process, we split $\bar{\mu}(t)$ into two pieces: $\bar{\mu}_1(t)$ and $\bar{\mu}_2(t)$. As before, $\bar{\mu}_1(t)$ describes the behavior of the jobs in the system that are currently being processed and $\bar{\mu}_2(t)$ describes the jobs in the system that are waiting to receive attention. The description for $\bar{\mu}_1(t)$ that we obtain is

$$\begin{aligned}\langle g, \bar{\mu}_1(t)\rangle &= \langle g, \bar{\mu}(0)\rangle 1_{\{\lfloor t/\langle \chi, \bar{\mu}(0)\rangle \rfloor = 0\}} \\ &\quad + \alpha \langle \chi, \bar{\mu}(0)\rangle \langle g, \nu\rangle 1_{\{\lfloor t/\langle \chi, \bar{\mu}(0)\rangle \rfloor \neq 0\}} - \int_{\lfloor t/\langle \chi, \bar{\mu}(0)\rangle \rfloor \langle \chi, \bar{\mu}(0)\rangle}^{t} \frac{\langle g', \bar{\mu}_1(s)\rangle}{\langle 1, \bar{\mu}_1(s)\rangle} ds.\end{aligned}$$

Here we use χ to denote the identity function (that is, $\chi(x) = x$). Notice we need the indicator function on the first two terms because the system will behave differently depending on whether it is processing the initial batch or any subsequent batch. The description for $\bar{\mu}_2(t)$ is given by

$$\langle g, \bar{\mu}_2(t)\rangle = \alpha \left(t - \left\lfloor \frac{t}{\langle \chi, \bar{\mu}(0)\rangle} \right\rfloor \langle \chi, \bar{\mu}(0)\rangle \right) \langle g, \nu\rangle.$$

The main result of this paper is that under mild assumptions the fluid scaled measure-valued processes $\frac{1}{r}\mu_1^r(\cdot)$ and $\frac{1}{r}\mu_2^r(\cdot)$ converge to limits satisfying the equations above and that the solutions to those equations are unique. We will refer to the solutions as the fluid limit for the batched processor sharing queue.

1.2 Notation

Throughout the paper we adhere to the convention that $\sum_{i=a}^{b} s_i = 0$ if $b < a$. We will also use the following notation:

$$\delta_x^+ = \begin{cases} \delta_x, & x > 0, \\ 0, & x = 0, \end{cases}$$

where δ_x is the standard Dirac measure. We let $\mathbb{Z}_+ = \{0, 1, 2, ...\}$. We use $\Rightarrow$ for convergence in distribution and $\overset{w}{\to}$ to indicate weak convergence of measures. We use χ to denote the identity function, that is to say for all x, $\chi(x) = x$. Given two random variables, we write $X \sim Y$ if X and Y have the same distribution.

Before we define our model rigorously we review some pertinent concepts in analysis and probability theory.

Chapter 2

Background

2.1 Methods of convergence

There are several notions of convergence that we will be concerned with understanding and manipulating. Determining convergence of a sequence of measures will be of primary interest. We will use the Prohorov metric as defined in [6] to determine distance between probability measures defined on the same space.

Definition 2.1.1. *The distance between two elements $P, Q \in \mathcal{M}_F$ as given by the* ***Prohorov metric*** *is*

$$d(P, Q) = \inf\{\epsilon > 0 \,:\, P(F) \leq Q(F^\epsilon) + \epsilon \text{ for all closed } F \subset \mathbb{R}_+\}$$

where

$$F^\epsilon = \{y \in \mathbb{R}_+ \,:\, \inf_{x \in F} r(y, x) < \epsilon\}.$$

One notion of convergence that we will use repeatedly is weak convergence of measures.

Definition 2.1.2. *Let* $\{\mu_n\}$ *be a sequence of finite measures and* μ *be a finite measure defined on* $(S, \mathcal{S})$*. Then* μ_n ***converges weakly to*** μ*, written* $\mu_n \xrightarrow{w} \mu$*, if*

$$\lim_{n\to\infty} \mu_n(A) = \mu(A)$$

for any measurable A *with* $\mu(\bar{A}\backslash A) = 0$.

When proving that a sequence of measures converges weakly to another measure, the Portmanteau Theorem is often very powerful. Let $C_b(S)$ denote the set of real-valued, continuous, bounded functions on S.

Theorem 2.1.3. *Let* $\{\mu_n\}$ *be a sequence of finite measures and* μ *be a finite measure defined on* $(S, \mathcal{S})$*. The following conditions are equivalent:*

1. $\lim_{n\to\infty} d(\mu_n, \mu) = 0$,
2. $\mu_n \xrightarrow{w} \mu$,
3. $\lim_{n\to\infty} \langle f, \mu_n\rangle = \langle f, \mu\rangle$ *for all* $f \in C_b(S)$,
4. $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$ *for all closed* F *in* $\mathcal{S}$,
5. $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$ *for all open* G *in* $\mathcal{S}$*, and*
6. $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ *for* $A \in \mathcal{S}$ *with* $\mu(\bar{A}\backslash A) = 0$.

Given a sequence of measures, we can develop other criteria for determining when a subsequence may converge weakly to some measure by using Prohorov's Theorem. We require the following definitions:

Definition 2.1.4. *A probability measure P is **tight** if for every $\epsilon > 0$ there exists some compact K with $P(K) > 1 - \epsilon$.*

If P is defined on a separable, complete space then P is tight. We can extend the notion of tightness to a collection of probability measures.

Definition 2.1.5. *A collection of probability measures Π is **tight** if for every $\epsilon > 0$ there exists some compact K such that for all $P \in \Pi$, $P(K) > 1 - \epsilon$.*

Definition 2.1.6. *A collection of probability measures Π is **relatively compact** if every sequence of elements from Π contains a weakly convergent subsequence.*

Prohorov's Theorem links the notion of tightness and relative compactness.

Theorem 2.1.7. *Let Π be a collection of probability measures on $(S, \mathcal{S})$ where S is separable and complete. Then Π is relatively compact if and only if it is tight.*

This theorem will prove important as we can more easily show that a collection of measures is tight and then we may work along a weakly convergent subsequence. Since each random variable has an associated distribution, which is a measure, we can extend the notion of weak convergence to random elements.

Definition 2.1.8. *Let $\{X_n\}$ be a sequence of random elements with associated distributions given by $\{P_n\}$. Let X be a random element with associated distribution P. Then X_n* ***converges in distribution to*** *X, written $X_n \Rightarrow X$, if $P_n \xrightarrow{w} P$.*

It is interesting to note here that while each X_n and X must take values in the same space for $P_n \xrightarrow{w} P$ to have meaning, we do not require that the sample space that each X_n and X are defined on be the same. We can now restate the Portmanteau Theorem through the lens of convergence in distribution, as is done in Billingsley [2] and Ethier and Kurtz [6].

Theorem 2.1.9. *Let $\{X_n\}$ be a sequence of random elements of S, and let X be a random element of S. The following conditions are equivalent:*

1. *$X_n \Rightarrow X$,*
2. *$\lim_{n\to\infty} E[f(X_n)] = E[f(X)]$ for all $f \in C_b(S)$,*
3. *$\limsup_n P(X_n \in F) \leq P(X \in F)$ for all closed F,*
4. *$\liminf_n P(X_n \in G) \geq P(X \in G)$ for all open G, and*
5. *$\lim_{n\to\infty} P(X_n \in A) = P(X \in A)$ for all sets A such that $P(X \in \bar{A}\backslash A) = 0$.*

In addition to convergence in distribution, we have the notion of convergence in probability.

Definition 2.1.10. *Let $\{X_n\}$ be a sequence of random elements mapping to the metric space (S, d), and let X be a random element mapping to (S, d). If for every $\epsilon > 0$*

$$\lim_{n\to\infty} P(d(X_n, X) \geq \epsilon) = 0$$

we say X_n ***converges in probability to*** *X and we write $X_n \xrightarrow{P} X$.*

Convergence in probability implies convergence in distribution. When X is constant, the two notions are equivalent provided the sequence of random elements and X are defined on the same space. If we have a sequence of random elements converging in distribution and a second sequence converging in distribution to a constant, we can obtain joint convergence using the theorem below.

Theorem 2.1.11. *If $X_n \Rightarrow X$ and $Y_n \Rightarrow a$ then*

$$(X_n, Y_n) \Rightarrow (X, a).$$

Another type of convergence is almost sure convergence. We first need to define an almost sure event.

Definition 2.1.12. *We say an event A is* ***almost sure*** *if $P(A) = 1$.*

Definition 2.1.13. *Let $\{X_n\}$ be a sequence of random elements, and let X be a random element with $X_n, X : \Omega \to S$. Then X_n* ***converges almost surely to*** *X, written $X_n \xrightarrow{a.s.} X$, if*

$$P\left(\lim_{n\to\infty} X_n = X\right) = 1.$$

Notice that almost sure convergence will imply both convergence in probability and distribution. However the requirement that each X_n and X have the same sample space and state space is sometimes too restrictive. In these situations, we can use the Skorohod Representation Theorem to move between convergence in distribution and almost sure convergence.

Theorem 2.1.14. *Let $\{X_n\}$ be a sequence of random elements and let X be a random element whose state space is separable. If $X_n \Rightarrow X$ then there is some sequence of random elements $\{\tilde{X}_n\}$, and a random element $\tilde{X}$ defined on a common probability space $(\Omega, \mathcal{F}, P)$ with*

$$\tilde{X}_n \sim X_n \text{ for each } n,$$

$$\tilde{X} \sim X \text{ and,}$$

$$\tilde{X}_n \stackrel{a.s.}{\to} \tilde{X}.$$

Often times we wish to study how a sequence of random elements interacts but even the simplest operations can result in a complicated distribution function. Consider for example the distribution function for the sum of a sequence of random elements. While it is easy to see why we might want to study this example, to understand this distribution function requires computing a messy convolution of measures. The Law of Large Numbers is an incredible theorem which allows us to better understand this sum and use a simple number to approximate it. To state the theorem we need to define independent random variables.

Definition 2.1.15. *Two random variables X, and Y are **independent**, written $X \perp Y$ if their joint law $\mathcal{L}_{X,Y}$ is equivalent to $\mathcal{L}_X \times \mathcal{L}_Y$.*

We then say two random variables are independent, and identically distributed (i.i.d.) if they are independent and have the same distribution.

Theorem 2.1.16. *Let $\{X_n\}$ be an i.i.d. sequence of random elements with expected value μ. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} X_i = \mu \text{ a.s.}$$

2.2 Stochastic processes

We often want to consider a system of random variables that evolve over time. To do this we make the following definition.

Definition 2.2.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let S be some measurable space and let $\mathcal{A}$ be some collection such that for each $\alpha \in \mathcal{A}$, $X(\alpha) : \Omega \to S$ is a random variable. The collection $\{X(\alpha) : \alpha \in \mathcal{A}\}$ is a **stochastic process**.*

In order to allow our stochastic processes to develop over time, we take $\mathcal{A} = [0, \infty)$. We consider a **sample path** of a stochastic process by fixing some $\omega \in \Omega$ and studying

$$X(\cdot, \omega) : \mathbb{R}_+ \to S.$$

We often suppress the ω-notation and simply refer to the sample path by $X(\cdot)$. Under certain conditions the stochastic process will be a member of càdlàg space. That is to say, $X(\cdot)$ will be right continuous with left limits (RCLL). In general we will let

$$D_E(\mathbb{R}_+) = \{X(\cdot) : \mathbb{R}_+ \to E \ \text{ RCLL}\}.$$

We will commonly be concerned with the situation where $E = \mathcal{M}_{\mathcal{F}}$. Here $\mathcal{M}_{\mathcal{F}}$ refers to the set of finite, non-negative Borel measures on $\mathbb{R}_+$. We use the Skorohod J_1-topology on $D_E(\mathbb{R}_+)$. In the J_1-topology we define a metric that allows two elements to be close if applying a slight time change (or horizontal shift) will allow the outputs to be close together. Formally, given a sequence of elements $\{x_n\}$ in $D_E(\mathbb{R}_+)$ and $x \in D_E(\mathbb{R}_+)$, we say x_n converges to x if and only if there exists continuous functions $\{\lambda_n\}$ such that λ_n converges to e uniformly on compacts, where $e(t) = t$ and $x_n(\lambda_n)$ converges to x uniformly on compacts. Alternatively, for any $T > 0$, $x(\cdot), y(\cdot) \in D_{\mathbb{R}_+}(\mathbb{R}_+)$ we define the Skorohod distance restricted to $[0,T]$ by

$$r_T(x,y) = \inf\{\epsilon > 0 \, : \, \|\lambda(t) - t\|_T \le \epsilon, \, \|x(t) - y(\lambda(t))\|_T \le \epsilon\}$$

where $\lambda(\cdot)$ is a continuous, strictly increasing function and $\|\cdot\|_T$ denotes the infinity norm on $[0,T]$. We also have a notion of relative compactness for stochastic processes. Within the definition for a relative compact family of stochastic processes we need the notion of the modulus of continuity. Intuitively, the modulus of continuity is a measure of how close a function is to being uniformly continuous.

Definition 2.2.2. *For $x \in D_E([0,\infty))$, $\delta > 0$, and $T > 0$, define the* ***modulus of continuity*** *to be*

$$w'(x,\delta,T) = \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} d(x(s),x(t))$$

where $\{t_i\}$ ranges over all partitions of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ with $\min_{1\le i\le n} t_i - t_{i-1} > \delta$.

Definition 2.2.3. *Let $\{X_\alpha\}$ be a family of stochastic processes whose sample paths lie in $D_E([0,\infty))$ for a separable space E. Then $\{X_\alpha\}$* ***is relatively compact*** *if the associated family of probability distributions $\{P_\alpha\}$ is.*

Note that $\{P_\alpha\}$ will be relatively compact in $\mathcal{P}(D_E([0,\infty)))$ if it has compact closure. Alternatively, we can determine if a family of stochastic processes is relatively compact by employing the following theorem stated in [6]:

Theorem 2.2.4. *Let (E,d) be a separable metric space and let $\{X_\alpha\}$ be a family of stochastic processes taking sample paths in $D_E([0,\infty))$. Then $\{X_\alpha\}$ is relatively compact if and only if:*

1. *For every $\eta > 0$ and rational $t \geq 0$, there exists a compact set $\Gamma_{\eta,t} \subset E$ with*

$$\inf_\alpha P(X_\alpha(t) \in \Gamma_{\eta,t}) \geq 1-\eta, \text{ and}$$

2. *for every $\eta > 0$ and $T > 0$, there exists some $\delta > 0$ such that*

$$\sup_{\alpha} P(w'(X_\alpha, \delta, T) \geq \eta) \leq \eta.$$

Notice to show the first condition above, it is more than sufficient to show that for every $\eta > 0$ and $T > 0$ there is a compact $\Gamma_{\eta,T} \subset E$ with

$$\inf_{\alpha} P(X_\alpha(t) \in \Gamma_{\eta,T} \text{ for all } 0 \leq t \leq T) \geq 1 - \eta.$$

Additionally, notice that the second condition above is automatically satisfied if each stochastic process in the family is monotone in t.

Chapter 3

Model and main Result

3.1 Model definition

Let $\{u_i\}$ be independent, identically distributed inter-arrival times. Define $U_j = \sum_{i=1}^{j} u_i$ and $E(t)$ to be the number of jobs that have arrived to the system by time t. That is to say,

$$E(t) = \max\left\{ j \,:\, \sum_{i=1}^{j} u_i \leq t \right\}.$$

Let $\{v_i\}$ be independent, identically distributed service times where these service times are distributed like ν, a Borel probability measure on $\mathbb{R}_+$ with $\nu(\{0\}) = 0$. Define $V_j = \sum_{i=1}^{j} v_i$. Let Z_0 be a non-negative integer valued random variable. Let $\{\tilde{v}_j\}$ be a sequence of positive random variables, and assume $\mathbb{E}\left[\sum_{j=1}^{Z_0} \tilde{v}_j\right] < \infty$. The random variable Z_0 represents the initial queue length and $\{\tilde{v}_j \,:\, j = 1, 2, ...Z_0\}$ are the initial service times. Define the initial workload W_0 and initial batch size B_0 by

$$W_0 = B_0 = \sum_{j=1}^{Z_0} \tilde{v}_j.$$

The initial condition can be encoded using the measure $\mu_1(0) = \sum_{j=1}^{Z_0} \delta_{\tilde{v}_j}$ where $W_0 = \langle \chi, \mu_1(0) \rangle$.

Let $I(t)$ be the amount of time the system has spent idle on $[0, t]$, that is

$$I(t) = \sup\{(W_0 + V(E(s)) - s)^-,\ 0 \leq s \leq t\}, t \geq 0.$$

Define the workload process

$$W(t) = W_0 + \sum_{i=1}^{E(t)} v_i - t + I(t). \tag{3.1.1}$$

Next, define the batch start times and sizes. Let $\beta_0 = 0$ and for positive $k \in \mathbb{Z}$ define β_k and B_k in an alternating, inductive fashion where

$$\begin{aligned}\beta_k &= (\beta_{k-1} + B_{k-1})1_{\{W(\beta_{k-1}+B_{k-1}) \neq 0\}} \\ &\quad + \inf\{s \geq \beta_{k-1} + B_{k-1} \ :\ W(s) > 0\}1_{\{W(\beta_{k-1}+B_{k-1})=0\}},\end{aligned}$$

and

$$B_k = \sum_{j=E(\beta_{k-1})+1}^{E(\beta_k)} v_j.$$

Then for any positive integer k, β_k gives the start time for the kth batch and B_k gives the workload of the kth batch. Notice for any s,

$$\begin{aligned} W(s) - W(s-) &= B_0 + \sum_{i=1}^{E(s)} v_i - s + I(s) - B_0 - \lim_{t \uparrow s} \sum_{i=1}^{E(t)} v_i + s - I(s) \\ &= \lim_{t \uparrow s} \sum_{i=E(t)+1}^{E(s)} v_i \end{aligned}$$

Which due to our convention for sums equals the amount of work arriving at s. We also want to keep track of the number of jobs in the system, or queue length. Define the index of the most recently started batch

$$\ell(t) = \max\{j \,:\, \beta_j \leq t\}.$$

Denote the residual service time a t of initial job j by $\tilde{R}_j(t)$ and of job j by $R_j(t)$. This is the remaining amount of processing time at time t required to fulfill the job's service requirement. The state descriptor at time t is given by two measures $\mu_1(t)$, representing the batch of jobs currently in service, and $\mu_2(t)$, representing the jobs currently waiting to begin service. These are defined in terms of one another and an additional process $S(t)$, called the cumulative service per job process. Specifically, the above processes are defined by the equations

$$\mu_1(t) = \sum_{j=1}^{Z_0} \delta^+_{\tilde{R}_j(t)} + \sum_{j=1}^{E(\beta_{\ell(t)})} \delta^+_{R_j(t)} \tag{3.1.2}$$

$$\mu_2(t) = \sum_{j=E(\beta_{\ell(t)})+1}^{E(t)} \delta^+_{R_j(t)} \tag{3.1.3}$$

$$R_j(t) = v_j - \left[S(t) - S(\beta_{\ell(U_j-)+1})\right]^+ \tag{3.1.4}$$

$$\tilde{R}_j(t) = \tilde{v}_j - S(t) \tag{3.1.5}$$

$$S(t) = \int_0^t \phi\left(\langle 1, \mu_1(s)\rangle\right) ds, \tag{3.1.6}$$

where $\phi(x) = 1/x$, $x > 0$, $\phi(0) = 0$. In particular, given the primitive processes $E(\cdot)$ and V., and the initial condition Z_0 and $\{\tilde{v}_j\}$, equations (3.1.2)-(3.1.6) determine the state descriptor $(\mu_1(\cdot), \mu_2(\cdot))$, the service process $S(\cdot)$, and the residual service times $\{\tilde{R}_j(\cdot)\}$, $\{R_j(\cdot)\}$. Since the above describes a discrete event system, this fact is not difficult although somewhat tedious to show. Define

$$S_{t,s} = S(s) - S(t) = \int_t^s \phi\left(\langle 1, \mu_1(u)\rangle\right) du.$$

Then

$$\mu(t) = \mu_1(t) + \mu_2(t)$$

describes the evolution of the system.

Observe that for all $t \geq 0$, we have the valuation

$$\beta_{\ell(t)} = \sup\{s \leq t \, : \, \mu_1(s-) < \mu_1(s)\} \tag{3.1.7}$$

Using induction, we will now give a more versatile expression for β_k.

Lemma 3.1.1. *For $k \in \mathbb{Z}_+$,*

$$\beta_k = \sum_{i=0}^{k-1} B_i + I(\beta_k). \tag{3.1.8}$$

Proof First notice that $I(0) = 0 = I(B_0)$ since

$$I(B_0) = \sup\{(B_0 + \sum_{i=1}^{E(s)} v_i - s)^- \, : \, 0 \leq s \leq B_0\}$$

and for any $0 \leq s \leq B_0$

$$(B_0 + \sum_{i=1}^{E(s)} v_i - s)^- \leq (B_0 - s)^- = 0.$$

Therefore, if $W(B_0) \neq 0$, $\beta_1 = B_0$ and

$$\begin{aligned} W(\beta_1) &= B_0 + \sum_{i=1}^{E(\beta_1)} v_i - \beta_1 + I(\beta_1) \\ &= B_0 + B_1 - B_0 + 0 = B_1. \end{aligned}$$

Solving equation (3.1.1) for $t = \beta_1$ we have

$$\beta_1 = B_0 + \sum_{i=1}^{E(\beta_1)} v_i - W(\beta_1) + I(\beta_1) = B_0 + I(\beta_1).$$

If $W(B_0) = 0$, then $B_1 = W(\beta_1)$. To see this we rewrite (3.1.1) to obtain

$$B_0 = B_0 + \sum_{i=1}^{E(B_0)} v_i + I(B_0)$$

which implies $\sum_{i=1}^{E(B_0)} v_i = 0$. Suppose there is some t with $B_0 < t < \beta_1$ with $\sum_{E(B_0)+1}^{E(t)} v_i > 0$. Then there is some s, $B_0 < s < t$ with $\lim_{u\uparrow s} \sum_{E(u)+1}^{E(s)} v_i > 0$ but this implies

$$W(s) - W(s-) = W(s) > 0$$

contradicting the infimum definition of β_1. Thus we conclude that

$$\sum_{1}^{E(s)} v_i = 0$$

for any $0 \leq s < \beta_1$ and

$$W(\beta_1) = W(\beta_1) - W(\beta_1-) = \lim_{s\uparrow\beta_1} \sum_{E(s)+1}^{E(\beta_1)} v_i = \sum_{1}^{E(\beta_1)} v_i = B_1.$$

We can now rewrite (3.1.1) to give

$$\beta_1 = B_0 + \sum_{i=1}^{E(\beta_1)} v_i - W(\beta_1) + I(\beta_1) = B_0 + I(\beta_1).$$

Suppose we know $\beta_k = \sum_{i=0}^{k-1} B_i + I(\beta_k)$ and suppose $W(\beta_k + B_k) \neq 0$. Then by definition,

$$\beta_{k+1} = \beta_k + B_k.$$

Notice for $\beta_k < s \leq \beta_{k+1}$ we have

$$\begin{aligned} s \leq \beta_k + B_k &= \sum_{i=0}^{k-1} B_i + I(\beta_k) + B_k = \sum_{i=0}^{k} B_i + I(\beta_k) \\ &= B_0 + \sum_{i=1}^{E(\beta_k)} v_i + I(\beta_k) \\ &\leq B_0 + \sum_{i=1}^{E(s)} v_i + I(\beta_k). \end{aligned}$$

Rearranging terms this yields

$$s - B_0 - \sum_{i=1}^{E(s)} v_i \leq I(\beta_k)$$

which implies

$$(B_0 + \sum_{i=1}^{E(s)} v_i - s)^- \leq I(\beta_k).$$

It follows from this and the definition of $I(\cdot)$ that $I(\beta_{k+1}) = I(\beta_k)$ and we can write

$$\begin{aligned} W(\beta_{k+1}) &= B_0 + \sum_{i=1}^{E(\beta_{k+1})} v_i - \beta_{k+1} + I(\beta_{k+1}) \\ &= B_0 + \sum_{i=1}^{k+1} B_i - \beta_k - B_k + I(\beta_k) \\ &= \sum_{i=0}^{k+1} B_i - \left(\sum_{i=0}^{k-1} B_i + I(\beta_k) \right) - B_k + I(\beta_k) = B_{k+1}. \end{aligned}$$

It follows from (3.1.1) that

$$\beta_{k+1} = B_0 + \sum_{i=1}^{E(\beta_{k+1})} v_i - W(\beta_{k+1}) + I(\beta_{k+1}) = \sum_{i=0}^{k} B_i + I(\beta_{k+1}).$$

If $W(\beta_k + B_k) = 0$ then $B_{k+1} = W(\beta_{k+1})$. To see this we rewrite the workload equation to obtain

$$\beta_k + B_k = B_0 + \sum_{i=1}^{E(\beta_k + B_k)} v_i + I(\beta_k + B_k).$$

This yields

$$\sum_{i=0}^{k-1} B_i + I(\beta_k) + B_k = \sum_{i=0}^{k} B_i + \sum_{i=E(\beta_k)+1}^{E(\beta_k+B_k)} v_i + I(\beta_k),$$

which implies $\sum_{i=E(\beta_k)+1}^{E(\beta_k+B_k)} v_i = 0$. Suppose there is some t with $\beta_k + B_k < t < \beta_{k+1}$ with $\sum_{E(\beta_k+B_k)+1}^{E(t)} v_i > 0$. Then there is some s, $\beta_k + B_k < s < t$ with $\lim_{u \uparrow s} \sum_{E(u)+1}^{E(s)} v_i > 0$ but this implies

$$W(s) - W(s-) = W(s) > 0$$

contradicting the infimum definition of β_{k+1}. Thus we conclude that

$$\sum_{E(\beta_k+B_k)+1}^{E(s)} v_i = 0$$

for any $\beta_k + B_k \le s < \beta_{k+1}$ and

$$W(\beta_{k+1}) = W(\beta_{k+1}) - W(\beta_{k+1}-) = \lim_{s\uparrow\beta_{k+1}} \sum_{E(s)+1}^{E(\beta_{k+1})} v_i = \sum_{E(\beta_k)+1}^{E(\beta_{k+1})} v_i = B_{k+1}.$$

Now by (3.1.1),

$$\begin{aligned}
\beta_{k+1} &= B_0 + \sum_{i=1}^{E(\beta_{k+1})} v_i - W(\beta_{k+1}) + I(\beta_{k+1}) \\
&= \sum_{i=0}^{k+1} B_i - B_{k+1} + I(\beta_{k+1}) \\
&= \sum_{i=0}^{k} B_i + I(\beta_{k+1}),
\end{aligned}$$

as desired. ■

3.2 Fluid model

We now define the object that will appear as the limit in our fluid limit theorem.

Definition 3.2.1. *Given α, ν, and ξ where $\xi \in \mathcal{M}_F$ has no atoms, we say that (μ_1, μ_2) are fluid model solutions for (α, ν) and initial condition ξ if $\mu_1(0) = \xi$ and for all $g \in \mathcal{C}$ we have*

$$\langle g, \mu_1(t)\rangle = \langle g, \xi\rangle 1_{\left\{\left\lfloor \frac{t}{w_0}\right\rfloor = 0\right\}} + \alpha w_0 \langle g, \nu\rangle 1_{\left\{\left\lfloor \frac{t}{w_0}\right\rfloor \neq 0\right\}} - \int_{\left\lfloor \frac{t}{w_0}\right\rfloor w_0}^{t} \frac{\langle g', \mu_1(s)\rangle}{\langle 1, \mu_1(s)\rangle} ds, \quad (3.2.1)$$

and

$$\langle g, \mu_2(t)\rangle = \alpha \left(t - \left\lfloor \frac{t}{w_0} \right\rfloor w_0\right) \langle g, \nu\rangle \quad (3.2.2)$$

for all $t \geq 0$, *where* $w_0 = \langle \chi, \xi\rangle$. *We refer to* $\mu_1(\cdot)$ *as the shifting solution and* $\mu_2(\cdot)$ *as the growing solution.*

3.3 Sequence of models and main result

Consider a sequence of models indexed by r. For each model in the sequence we can define corresponding processes as in (3.1.2)-(3.1.6). We study this sequence under fluid or law of large numbers scaling using the following definition:

Definition 3.3.1. *Consider a sequence of queues indexed by* r *and the corresponding process* $\mu^r(\cdot)$. *The resulting fluid scaled process is defined by* $\bar{\mu}^r(\cdot) = \frac{1}{r}\mu^r(r\cdot)$.

In this sequence of queues we assume the following asymptotic properties

1. $\left(\bar{\mu}^r(0), \frac{1}{r}B_0^r\right) \Rightarrow (\theta_0, W_0)$
2. $\mathbb{E}[u_1^r] = \alpha^r \to \alpha$
3. $\nu^r \stackrel{w}{\to} \nu$,

4. $\langle \chi, \nu^r \rangle \to \langle \chi, \nu \rangle$,

5. $\mathbb{E}[\nu] = \frac{1}{\alpha}$,

6. $\mathbb{E}[u_1^r; u_1^r > r] \to 0$.

Theorem 3.3.2. *Consider a sequence of models as defined in Section 3.1 and suppose there are $\alpha > 0$, a non-atomic probability measure ν on $\mathbb{R}_+$, and a random measure θ_0 such that $\theta_0 \neq 0$ almost surely and such that the asymptotic assumptions in Definition 3.3.1 hold. Then as $r \to \infty$ the sequence of fluid scaled state descriptors $\{(\bar{\mu}_1^r(\cdot), \bar{\mu}_2^r(\cdot))\}$ converges in distribution on $D \times D$ to a limit $(\theta_1(\cdot), \theta_2(\cdot))$ that is almost surely a fluid model solution for α, ν, θ_0.*

Chapter 4

Uniqueness of fluid model solutions

Theorem 4.0.1. *If* (μ_1, μ_2) *is a fluid model solution for* (α, ν) *and initial condition* ξ *then* (μ_1, μ_2) *is unique. Moreover, it is periodic in that* $\mu_1(t) = \mu_1(t + kw_0)$ *for all* $t \geq w_0$ *and* $k \in \mathbb{Z}_+$*, and* $\mu_2(t) = \mu_2(t + kw_0)$ *for all* $t \geq 0$ *and* $k \in \mathbb{Z}_+$*.*

We build off of the uniqueness arguments developed in [9]. We begin with the following technical lemmas.

Lemma 4.0.2. *Let* $\mu_1 : [kw_0, (k+1)w_0) \to \mathcal{M}_F$ *be continuous. For each* $f \in \boldsymbol{C}_b^1([kw_0, (k+1)w_0) \times \mathbb{R}_+)$*,*

$$t \to \langle f(t, \cdot), \mu_1(t) \rangle$$

is a continuous function of $t \in [kw_0, (k+1)w_0)$*.*

Proof Fix $t \in [kw_0, (k+)w_0)$, let $t_n \to t$. Then $\mu_1(t_n) \stackrel{w}{\to} \mu_1(t)$ and $f(t_n, \cdot) \to f(t, \cdot)$ pointwise. So for any k and w

$$
\begin{aligned}
|\langle f(t_n,\cdot),\mu_1(t_n)\rangle - \langle f(t,\cdot),\mu_1(t)\rangle| \\
&\le |\langle f(t_n,\cdot),\mu_1(t_n)\rangle - \langle f(t_k,\cdot),\mu_1(t_n)\rangle| \\
&+ |\langle f(t_k,\cdot),\mu_1(t_n)\rangle - \langle f(t_k,\cdot),\mu_1(t)\rangle| \\
&+ |\langle f(t_k,\cdot),\mu_1(t)\rangle - \langle f(t,\cdot),\mu_1(t)\rangle|. \quad (4.0.1)
\end{aligned}
$$

The first right-hand term is bounded above for any $M < \infty$ by

$$
\|f(t_n,\cdot) - f(t_k,\cdot)\|_M \sup_n \langle 1,\mu_1(t_n)\rangle + 2\|f\|_\infty \sup_n \langle 1_{(M,\infty)},\mu_1(t_n)\rangle.
$$

Since $\{\mu_1(t_n)\}$ is tight, we can make this arbitrarily small by first choosing M large enough, and then n, k large enough, since $f(t_n,\cdot)$, $f(t_k,\cdot)$ converge uniformly on $[0, M]$. Finally the latter two terms in (4.0.1) can be made small by first choosing k sufficiently large and applying bounded convergence, and then choosing n sufficiently large. ■

Lemma 4.0.3. *Suppose* μ_1 *is a solution to* (3.2.1)
then for all $k \in \mathbb{Z}_+$, $t \in [kw_0,(k+1)w_0)$, *and* $f \in \boldsymbol{C}_b^1([kw_0,(k+1)w_0) \times \mathbb{R}_+)$ *such that* $f(\cdot,0) \equiv 0$ *and* $f_x(\cdot,0) \equiv 0$, $\mu_1(\cdot)$ *satisfies*

$$
\begin{aligned}
\langle f(t,\cdot),\mu_1(t)\rangle &= \langle f(0,\cdot),\xi\rangle\, 1_{\{k=0\}} + \alpha w_0 \,\langle f(kw_0,\cdot),\nu\rangle\, 1_{\{k\neq 0\}} \\
&+ \int_{\lfloor \frac{t}{w_0}\rfloor w_0}^{t} \langle f_s(s,\cdot),\mu_1(s)\rangle ds - \int_{\lfloor \frac{t}{w_0}\rfloor w_0}^{t} \frac{\langle f_x(s,\cdot),\mu_1(s)\rangle}{\langle 1,\mu_1(s)\rangle} ds.
\end{aligned}
\tag{4.0.2}
$$

Proof Fix $k \in \mathbb{Z}_+$, $t \in [kw_0, (k+1)w_0)$, and $f \in \mathbf{C}_b^1([kw_0, (k+1)w_0) \times \mathbb{R}_+)$ with $f(\cdot, 0) \equiv 0$ and $f_x(\cdot, 0) \equiv 0$. Take h small enough so that $t + h \in [kw_0, (k+1)w_0)$. Then we have

$$\begin{aligned}
&\langle f(t+h,\cdot), \mu_1(t+h)\rangle - \langle f(t,\cdot), \mu_1(t)\rangle \\
&\qquad = \langle f(t+h,\cdot), \mu_1(t+h)\rangle - \langle f(t,\cdot), \mu_1(t+h)\rangle \\
&\qquad + \langle f(t,\cdot), \mu_1(t+h)\rangle - \langle f(t,\cdot), \mu_1(t)\rangle.
\end{aligned}$$

Since

$$f(t+h,\cdot) - f(t,\cdot) = \int_t^{t+h} f_s(s,\cdot)ds,$$

we obtain

$$\begin{aligned}
\langle f(t+h,\cdot), \mu_1(t+h)\rangle &- \langle f(t,\cdot), \mu_1(t+h)\rangle \\
&= \langle f(t+h,\cdot) - f(t,\cdot), \mu_1(t+h)\rangle \\
&= \left\langle \int_t^{t+h} f_s(s,\cdot)ds, \mu_1(t+h) \right\rangle.
\end{aligned} \tag{4.0.3}$$

Performing a substitution with $s = t + hv$ we obtain

$$\left\langle \int_t^{t+h} f_s(s,\cdot)ds, \mu_1(t+h) \right\rangle = \left\langle \int_0^1 f_s(t+hv,\cdot)hdv, \mu_1(t+h) \right\rangle. \tag{4.0.4}$$

Since $f \in \mathbf{C}_b^1([kw_0,(k+1)w_0) \times \mathbb{R}_+)$ there exists some M such that $\|f_s(\cdot,\cdot)\|_\infty \leq M$. Therefore, since $\mu_1(t+h) \in \mathcal{M}_F$ we have

$$\left\langle \int_t^{t+h} f_s(s,\cdot)ds, \mu_1(t+h) \right\rangle \leq \left\langle \int_t^{t+h} M ds, \mu_1(t+h) \right\rangle \leq \langle hM, \mu_1(t+h)\rangle < \infty.$$

Therefore, we can interchange the integrals to write

$$\left\langle \int_0^1 f_s(t+hv,\cdot)hdv, \mu_1(t+h) \right\rangle = h\int_0^1 \langle f_s(t+hv,\cdot), \mu_1(t+h)\rangle dv. \tag{4.0.5}$$

For every $v \in [0,1]$ we define $f^v : [kw_0,(k+1)w_0) \times \mathbb{R}_+) \to \mathbb{R}$ by

$$f^v(u,t) = f_s(t+(u-t)v, x).$$

By definition, taking $u = t+h$ we have $f^v(t+h,\cdot) = f_s(t+hv,\cdot)$. So for any $v \in [0,1]$ we have

$$\lim_{h\to 0}\langle f_s(t+hv,\cdot), \mu_1(t+h)\rangle = \lim_{h\to 0}\langle f^v(t+h,\cdot), \mu_1(t+h)\rangle. \tag{4.0.6}$$

Notice that $f^v \in \mathbf{C}_b([kw_0,(k+1)w_0) \times \mathbb{R}_+)$, and so by Lemma 4.0.2, it follows that the map defined by $u \mapsto \langle f^v(u,\cdot), \mu_1(u)\rangle$ is continuous for any $u \in [kw_0,(k+1)w_0)$. Therefore, we can pass the limit inside in the equation above to obtain

$$\lim_{h\to 0}\langle f^v(t+h,\cdot), \mu_1(t+h)\rangle = \langle f^v(t,\cdot), \mu_1(t)\rangle = \langle f_s(t,\cdot), \mu_1(t)\rangle. \tag{4.0.7}$$

Combining this with (4.0.3)-(4.0.5) we have

$$\begin{aligned}
&\lim_{h\to 0}\frac{\langle f(t+h,\cdot),\mu_1(t+h)\rangle-\langle f(t,\cdot),\mu_1(t+h)\rangle}{h}\\
&=\lim_{h\to 0}\frac{h\int_0^1\langle f_s(t+hv,\cdot),\mu_1(t+h)\rangle dv}{h}\\
&=\lim_{h\to 0}\int_0^1\langle f_s(t+hv,\cdot),\mu_1(t+h)\rangle dv.
\end{aligned}$$

Notice that since $f_s(\cdot,\cdot)\in C_b([kw_0,(k+1)w_0),\times\mathbb{R}_+)$ and $\mu_1(\cdot)\in\mathcal{M}_F$, by the bounded convergence theorem, (4.0.6), and (4.0.7) we have

$$\begin{aligned}
&\lim_{h\to 0}\int_0^1\langle f_s(t+hv,\cdot),\mu_1(t+h)\rangle dv\\
&=\int_0^1\lim_{h\to 0}\langle f_s(t+hv,\cdot),\mu_1(t+h)\rangle dv\\
&=\int_0^1\langle f_s(t,\cdot),\mu_1(t)\rangle dv\\
&=\langle f_s(t,\cdot),\mu_1(t)\rangle.
\end{aligned}$$

Next, to consider $\langle f(t,\cdot),\mu_1(t+h)\rangle-\langle f(t,\cdot),\mu_1(t)\rangle$ we use the assumption that on $[kw_0,(k+1)w_0)$, $\mu_1(\cdot)$ is a solution to

$$\langle g,\mu_1(t)\rangle=\langle g,\xi\rangle 1_{\{k=0\}}+\alpha w_0\langle g,\nu\rangle 1_{\{k\neq 0\}}-\int_{\lfloor\frac{t}{w_0}\rfloor w_0}^{t}\frac{\langle g',\mu_1(s)\rangle}{\langle 1,\mu_1(s)\rangle}ds.$$

Define $g(\cdot)=f(t,\cdot)$ and note that $g\in\mathcal{C}$. Since $t+h\in[kw_0,(k+1)w_0)$ it follows that

$$\begin{aligned}
&\langle f(t,\cdot),\mu_1(t+h)\rangle - \langle f(t,\cdot),\mu_1(t)\rangle \\
&= \langle f(t+h,\cdot),\xi\rangle 1_{\{k=0\}} + \alpha w_0 \langle f(t+h,\cdot),\nu\rangle 1_{\{k\neq 0\}} \\
&- \int_{\left\lfloor \frac{t+h}{w_0}\right\rfloor w_0}^{t+h} \frac{\langle f_x(t,\cdot),\mu_1(s)\rangle}{\langle 1,\mu_1(s)\rangle} ds - \\
&\left(\langle f(t,\cdot),\xi\rangle 1_{\{k=0\}} + \alpha w_0 \langle f(t,\cdot),\nu\rangle 1_{\{k\neq 0\}} - \int_{\left\lfloor \frac{t}{w_0}\right\rfloor w_0}^{t} \frac{\langle f_x(t,\cdot),\mu_1(s)\rangle}{\langle 1,\mu_1(s)\rangle} ds \right).
\end{aligned}$$

But since $t+h \in [kw_0,(k+1)w_0)$ we have $\left\lfloor \frac{t+h}{w_0}\right\rfloor = \left\lfloor \frac{t}{w_0}\right\rfloor$. Therefore,

$$\begin{aligned}
&- \int_{\left\lfloor \frac{t+h}{w_0}\right\rfloor w_0}^{t+h} \frac{\langle f_x(t,\cdot),\mu_1(s)\rangle}{\langle 1,\mu_1(s)\rangle} ds + \int_{\left\lfloor \frac{t}{w_0}\right\rfloor w_0}^{t} \frac{\langle f_x(t,\cdot),\mu_1(s)\rangle}{\langle 1,\mu_1(s)\rangle} ds \\
&= -\int_t^{t+h} \frac{\langle f_x(t,\cdot),\mu_1(s)\rangle}{\langle 1,\mu_1(s)\rangle} ds.
\end{aligned}$$

By the continuity of $\mu_1(\cdot)$ on $[kw_0,(k+1)w_0)$ and the fact that on $[kw_0,(k+1)w_0)$, $\langle 1,\mu_1(\cdot)\rangle > 0$ we know that the function given by

$$s \mapsto \frac{\langle f_x(s,\cdot),\mu_1(s)\rangle}{\langle 1,\mu_1(s)\rangle}$$

is continuous for $s \in [kw_0,(k+1)w_0)$. Consider

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \frac{\langle f_x(s,\cdot),\mu_1(s)\rangle}{\langle 1,\mu_1(s)\rangle} ds.$$

By the continuity of the integrand and the mean value theorem for integrals we have

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \frac{\langle f_x(s,\cdot),\mu_1(s)\rangle}{\langle 1,\mu_1(s)\rangle} ds = \frac{\langle f_x(t,\cdot),\mu_1(t)\rangle}{\langle 1,\mu_1(t)\rangle}.$$

Combining the results above we have for $t \in [kw_0, (k+1)w_0)$,

$$\begin{aligned}\frac{d}{dt}\langle f(t,\cdot), \mu_1(t)\rangle &= \lim_{h\to 0}\frac{1}{h}\langle f(t+h,\cdot), \mu_1(t+h)\rangle - \langle f(t,\cdot), \mu_1(t)\rangle \\ &= \langle f_s(t,\cdot), \mu_1(t)\rangle - \frac{\langle f_x(t,\cdot), \mu_1(t)\rangle}{\langle 1, \mu_1(t)\rangle}.\end{aligned}$$

By Lemma 4.0.2, we know that each term on the right hand side of this expression is continuous on $[kw_0, (k+1)w_0)$ since f, f_s, and f_x, and $\frac{1}{\langle 1, \mu_1(\cdot)\rangle}$ are continuous on $[kw_0, (k+1)w_0)$. The desired result follows by integrating both side from kw_0 to t and noting that

$$\begin{aligned}&\langle f\left(\left(kw_0\right),\cdot\right), \mu_1\left(kw_0\right)\rangle \\ &= \langle f\left(\left(kw_0\right),\cdot\right), \xi\rangle\, 1_{\{k=0\}} + \alpha w_0 \langle f(kw_0,\cdot), \nu\rangle 1_{\{k\neq 0\}}.\end{aligned}$$

■

Lemma 4.0.4. *Suppose that $(\mu_1(\cdot), \mu_2(\cdot))$ is a fluid model solution for α, ν, ξ. Then for all $w \in [0,\infty]$, $k \in \mathbb{Z}_+$, and $t \in [kw_0, (k+1)w_0)$, we have*

$$\begin{aligned}\langle 1_{(0,w)}, \mu_1(t)\rangle &= \langle 1_{(0,w)}(\cdot - S(t)), \xi\rangle 1_{\left\{\left\lfloor \frac{t}{w_0}\right\rfloor = 0\right\}} \\ &+ \alpha w_0 \langle 1_{(0,w)}(\cdot - S(t) + S(kw_0), \nu\rangle 1_{\left\{\left\lfloor \frac{t}{w_0}\right\rfloor \neq 0\right\}}.\end{aligned} \tag{4.0.8}$$

Proof Fix $w \in [0,\infty]$, $k \in \mathbb{Z}_+$, and $t \in [kw_0,(k+1)w_0)$. Notice that if $t = kw_0$ (4.0.8) is satisfied by considering a sequence of $\{g_n\} \subset \mathcal{C}$ that increase up to $1_{(0,w)}$, applying the dominated convergence theorem, and appealing to (3.2.1). Now suppose $t \in (kw_0,(k+1)w_0)$, and let $g \in \mathbf{C}_b^1(\mathbb{R})$ such that $g \equiv 0$ on $(-\infty, 0]$. Define

$$f(s,x) = \begin{cases} g(x - S(t) + S(s)), & (s,x) \in [kw_0, t] \times \mathbb{R}_+, \\ \\ 0, & (s,x) \in (t,(k+1)w_0) \times \mathbb{R}_+. \end{cases}$$

Since $S(\cdot)$ is differentiable on $[kw_0,(k+1)w_0)$ we know $f \in \mathbf{C}_b^1([kw_0,t] \times \mathbb{R}_+)$ and since $S'(\cdot) = 1/Z(\cdot)$, on $[kw_0,t]$,

$$f_s(s,x) = \frac{g'(x - S(t) + S(s))}{Z(s)}, \text{ and}$$

$$f_x(s,x) = g'(x - S(t) + S(s)).$$

Let $0 < \epsilon < t - kw_0$ be given. Define

$$h^\epsilon(s) = \begin{cases} 1, & s \in [kw_0, t-\epsilon], \\ \\ 0, & s \in [t - \epsilon/2, (k+1)w_0). \end{cases}$$

Take $f^\epsilon : [kw_0,(k+1)w_0) \times \mathbb{R}_+ \to \mathbb{R}$ to be the map taking $(s,x) \mapsto f(s,x)h^\epsilon(s)$. Then $f^\epsilon \in \mathbf{C}_b^1([kw_0,(k+1)w_0) \times \mathbb{R}_+)$ and $f^\epsilon(\cdot,0) \equiv 0$ and $f_x^\epsilon(\cdot,0) \equiv 0$. Then $f|_{[kw_0,t-\epsilon]} \equiv f^\epsilon$ and by Lemma 4.0.3 applied to f^ϵ, for any $kw_0 \le s \le t - \epsilon$,

$$\begin{aligned} \langle f(s,\cdot), \mu_1(s) \rangle &= \langle f(kw_0,\cdot), \xi \rangle 1_{\{k=0\}} + \alpha w_0 \langle f(kw_0,\cdot), \nu \rangle 1_{\{k \neq 0\}} \\ &\quad + \int_{kw_0}^s \langle f_u(u,\cdot), \mu_1(u) \rangle du - \int_{kw_0}^s \frac{\langle f_x(u,\cdot), \mu_1(u) \rangle}{\langle 1, \mu_1(u) \rangle} du. \end{aligned}$$

But, by definition we know $f_u(u,x) = \frac{g'(x-S(t)+S(u))}{Z(u)}$ and $f_x(u,x) = g'(x-S(t)+S(u))$. Therefore, the above equality gives

$$\begin{aligned}
&\langle f(s,\cdot), \mu_1(s)\rangle \\
&\qquad = \langle g(\cdot - S(t), \xi\rangle 1_{\{k=0\}} + \alpha w_0 \langle g(\cdot - S(t) + S(kw_0), \nu\rangle 1_{\{k\neq 0\}} \\
&\qquad + \int_{kw_0}^{s} \left\langle \frac{g'(\cdot - S(t) + S(u))}{Z(u)}, \mu_1(u) \right\rangle du \\
&\qquad - \int_{kw_0}^{s} \frac{\langle g'(\cdot - S(t) + S(u)), \mu_1(u)\rangle}{\langle 1, \mu_1(u)\rangle} du \\
&\qquad = \langle g(\cdot - S(t), \xi\rangle 1_{\{k=0\}} + \alpha w_0 \langle g(\cdot - S(t) + S(kw_0)), \nu\rangle 1_{\{k\neq 0\}} \\
&\qquad + \int_{kw_0}^{s} \frac{\langle g'(\cdot - S(t) + S(u)), \mu_1(u)\rangle}{Z(u)} du \\
&\qquad - \int_{kw_0}^{s} \frac{\langle g'(\cdot - S(t) + S(u)), \mu_1(u)\rangle}{Z(u)} du \\
&\qquad = \langle g(\cdot - S(t), \xi\rangle 1_{\{k=0\}} + \alpha w_0 \langle g(\cdot - S(t) + S(kw_0), \nu\rangle 1_{\{k\neq 0\}}.
\end{aligned}$$

Since $0 < \epsilon < t - kw_0$ was arbitrary, it follows that for any $s \in [kw_0, t)$, we have

$$\langle f(s,\cdot), \mu_1(s)\rangle = \langle g(\cdot - S(t), \xi\rangle 1_{\{k=0\}} + \alpha w_0 \langle g(\cdot - S(t) + S(kw_0), \nu\rangle 1_{\{k\neq 0\}}.$$

Now letting $s \to t$ above, as a consequence of Lemma 4.0.2 we obtain

$$\langle f(t,\cdot), \mu_1(t)\rangle = \langle g(\cdot - S(t), \xi\rangle 1_{\{k=0\}} + \alpha w_0 \langle g(\cdot - S(t) + S(kw_0), \nu\rangle 1_{\{k\neq 0\}}.$$

But $f(t,\cdot) = g(\cdot)$, so in fact we have

$$\langle g, \mu_1(t)\rangle = \langle g(\cdot - S(t), \xi\rangle 1_{\{k=0\}} + \alpha w_0 \langle g(\cdot - S(t) + S(kw_0), \nu\rangle 1_{\{k\neq 0\}}.$$

We obtain (4.0.8) from this by applying the monotone convergence theorem to a sequence $\{g_n\} \subset \mathbf{C}_b^1(\mathbb{R})$ which increases pointwise to $1_{(0,w)}$. ■

Proof of Theorem 4.0.1 Uniqueness of the growing solution $\mu_2(\cdot)$ is immediate because $\mathcal{C}$ is a separating class on the subspace of $\mathcal{M}_F$ that does not charge the origin, so any solution to (3.2.2) for all $g \in \mathcal{C}$ is uniquely determined by α, ν, ξ. For the shifting solution, define

$$H_\xi(x) = \int_0^x \langle 1_{(y,\infty)}, \xi\rangle dy, \text{ and}$$

$$H_\nu(x) = \int_0^x \langle 1_{(y,\infty)}, \nu\rangle dy.$$

Recall we have defined

$$S(t) = \int_0^t \frac{1}{\langle 1, \mu_1(s)\rangle} ds = \int_0^t \frac{1}{Z(s)} ds.$$

So we have $S'(t) = \frac{1}{Z(t)}$. Next define for each $k \in \mathbb{Z}_+$

$$S_k : [kw_0, (k+1)w_0) \to \mathbb{R} : t \mapsto S(t) - S(kw_0).$$

Notice that S_k is increasing on its domain and $S'|_{[kw_0,(k+1)w_0)}(\cdot) = S_k'(\cdot)$. Let

$$T_k : [S_k(kw_0), S_k(k+1)w_0) \to \mathbb{R}$$

be the inverse of S_k and take $w = \infty$ in (4.0.8). Then by Lemma 4.0.4, for all $t \in [kw_0, (k+1)w_0)$

$$\begin{aligned} Z(t) &= \langle 1_{(0,\infty)}, \mu_1(t)\rangle \\ &= \langle 1_{(0,\infty)}(\cdot - S(t)), \xi\rangle 1_{\{k=0\}} + \alpha w_0 \langle 1_{(0,\infty)}(\cdot - S(t) + S(kw_0)), \nu\rangle 1_{\{k\neq 0\}} \\ &= \langle 1_{(S(t),\infty)}, \xi\rangle 1_{\{k=0\}} + \alpha w_0 \langle 1_{(0,\infty)}(\cdot - S(t) + S(kw_0)), \nu\rangle 1_{\{k\neq 0\}} \\ &= H'_\xi(S(t)) 1_{\{k=0\}} + \alpha w_0 H'_\nu(S(t) - S(kw_0)) 1_{\{k\neq 0\}}. \end{aligned}$$

Let $u = S(t) - S(kw_0)$. Then

$$Z(T_k(u)) = Z(t) = H'_\xi(u) 1_{\{k=0\}} + \alpha w_0 H'_\nu(u) 1_{\{k\neq 0\}}.$$

In addition, we know

$$T'_k(u) = \frac{1}{S'_k(T_k(u))} = \frac{1}{S'(T_k(u))} = \frac{1}{S'(t)} = Z(t).$$

Therefore, we have $T'_k(u) = H'_\xi(u) 1_{\{k=0\}} + \alpha w_0 H'_\nu(u) 1_{\{k\neq 0\}}$ and

$$\begin{aligned}
T_k(u) &= \int T_k'(u)du \\
&= \int H_\xi'(u)1_{\{k=0\}} + \alpha w_0 H_\nu'(u)1_{\{k\neq 0\}}du \\
&= H_\xi(u)1_{\{k=0\}} + \alpha w_0 H_\nu(u)1_{\{k\neq 0\}}.
\end{aligned}$$

So each T_k is uniquely determined by ξ, α, and ν. This implies S_k is also uniquely determined by ξ, α, and ν. Since this holds for all k, we have $S(t) - S(kw_0)$ is uniquely determined by ξ, α, and ν. It follows that

$$\langle g, \mu_1(t)\rangle = \langle g(\cdot - S(t)), \xi\rangle 1_{\{k=0\}} + \alpha w_0 \langle g(\cdot - S(t) + S(kw_0)), \nu\rangle 1_{\{k\neq 0\}}$$

is uniquely determined by ξ, α, and ν. So if $\mu_1(\cdot)$ satisfies (3.2.1), it is uniquely determined by ξ, α, and ν. ■

Chapter 5

Some general convergence considerations

Lemma 5.0.1. *If $X^r \Rightarrow X$ where $X^r, X > 0$ then for any $M > 0$,*

$$X^r 1_{\{X^r < M\}} \Rightarrow X 1_{\{X < M\}}.$$

Proof Let $b > a > 0$. We consider two cases. First suppose $b < M$. Then

$$\{X^r 1_{\{X^r < M\}} \in (a, b)\} = \{X^r \in (a, b)\}, \text{and}$$

$$\{X 1_{\{X < M\}} \in (a, b)\} = \{X \in (a, b)\}.$$

Since $X^r \Rightarrow X$, by the Portmanteau Theorem we have

$$\liminf_{r \to \infty} P(X^r \in (a, b)) \geq P(X \in (a, b)).$$

So we have

$$\liminf_{r\to\infty} P(X^r 1_{\{X^r<M\}} \in (a,b)) = \liminf_{r\to\infty} P(X^r \in (a,b))$$
$$\geq P(X \in (a,b)) = P(X1_{\{X<M\}} \in (a,b)).$$

If $b \geq M$ we have

$$\{X^r 1_{\{X^r<M\}} \in (a,b)\} = \{X^r \in (a,M)\}, \text{and}$$

$$\{X1_{\{X<M\}} \in (a,b)\} = \{X \in (a,M)\}.$$

Since $X^r \Rightarrow X$, by the Portmanteau Theorem we have

$$\liminf_{r\to\infty} P(X^r \in (a,b)) \geq P(X \in (a,b)).$$

So we have

$$\liminf_{r\to\infty} P(X^r 1_{\{X^r<M\}} \in (a,b)) = \liminf_{r\to\infty} P(X^r \in (a,M))$$
$$\geq P(X \in (a,M)) = P(X1_{\{X<M\}} \in (a,b)).$$

Since open intervals form a generating set, by Portmanteau Theorem we know

$$X^r 1_{\{X^r<M\}} \Rightarrow X1_{\{X<M\}}.$$

■

Lemma 5.0.2. *Let X^r be a sequence of random variables such that for any $\eta > 0$ there exists some Ω^r with $P(\Omega^r) \geq 1 - \eta$. Suppose further $X^r 1_{\Omega^r} \Rightarrow 0$. Then*

$$X^r \Rightarrow 0.$$

Proof Let $\epsilon > 0$, $f \in C_b(\mathbb{R}_+)$ be given. Take $\eta < \frac{\epsilon}{4\|f\|_\infty}$. Let Ω^r_η be such that $P(\Omega^r) \geq 1 - \eta$ and $X^r 1_{\Omega^r} \Rightarrow 0$. Then

$$\begin{aligned}
|E[f(X^r)] - E[f(0)]| &= \left|E[f(X^r)1_{\Omega^r}] + E[f(X^r)1_{(\Omega^r)^c}] - E[f(0)]\right| \\
&= \left|E[f(X^r 1_{\Omega^r})] - E[f(0)1_{(\Omega^r)^c}] + E[f(X^r)1_{(\Omega^r)^c}] - E[f(0)]\right| \\
&\leq |E[f(X^r 1_{\Omega^r})] - E[f(0)]| + \left|E[f(0)1_{(\Omega^r)^c}]\right| + \left|E[f(X^r)1_{(\Omega^r)^c}]\right| \\
&\leq |E[f(X^r 1_{\Omega^r})] - E[f(0)]| + 2\eta\|f\|_\infty.
\end{aligned}$$

Taking r large enough so $|E[f(X^r 1_{\Omega^r})] - E[f(0)]| < \epsilon/2$. It follows that

$$|E[f(X^r)] - E[f(0)]| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have the desired convergence. ■

Throughout this paper we will use the following lemma repeatedly.

Lemma 5.0.3. *Let $\{X_n\}, \{Y_n\}, \{Z_n\}$ be a sequence of random elements and X, Y, Z be random elements where $\{X_n\} \subset S_X, X \in S_X$, $\{Y_n\} \subset S_Y, Y \in S_Y$, $\{Z_n\} \subset S_Z, Z \in S_Z$ for some topological spaces S_X, S_Y, S_Z. Then*

$$((X_n, Y_n), Z_n) \Rightarrow ((X, Y), Z)$$

if and only if

$$(X_n, Y_n, Z_n) \Rightarrow (X, Y, Z).$$

Proof First assume $((X_n, Y_n), Z_n) \Rightarrow ((X, Y), Z)$. Define

$$g : (S_X \times S_Y) \times S_Z \to S_X \times S_Y \times S_Z$$

by g((X, Y), Z)=(X, Y, Z). Then g is continuous with respect to the product topology which implies

$$(X_n, Y_n, Z_n) = g(((X_n, Y_n), Z_n)) \Rightarrow g(((X, Y), Z)) = (X, Y, Z).$$

Since g is invertible the same argument applied with g^{-1} proves the opposite direction.

■

Lemma 5.0.4. *Let X, Y, V be random elements and let $f : W \to Z$ be a measurable function where W, Z are any topological spaces. If*

$$(X, f(X)) \sim (Y, V)$$

then $V = f(Y)$ almost surely.

Proof Suppose not. Then there exists some measurable set A, $A \subset G(f)^c$, and $P((Y,V) \in A) > 0$. Since $(X, f(X)) \sim (Y, V)$,

$$P((Y,V) \in A) = P((X, f(X)) \in A) > 0,$$

a contradiction since $P((X, f(X)) \in G(f)) = 1$.

■

Lemma 5.0.5. *If $X(\cdot) \sim \tilde{X}(\cdot)$ and f is measurable then*

$$(X(\cdot), f(X(\cdot))) \sim (\tilde{X}(\cdot), f(\tilde{X}(\cdot))).$$

Proof Let $\mu(\cdot)(\tilde{\mu}(\cdot))$ be the distribution of $(X(\cdot), f(X(\cdot)))$ (respectively, $(\tilde{X}(\cdot), f(\tilde{X}(\cdot)))$). We aim to show that $\mu(\cdot) = \tilde{\mu}(\cdot)$. We know the marginal distributions of $(X(\cdot), f(X(\cdot)))$ and $(\tilde{X}(\cdot), f(\tilde{X}(\cdot)))$ agree. Let $R = A \times B$ be any rectangle in $D_{\mathbb{R}_+} \times D_{\mathbb{R}_+}$. It suffices to show $\mu(\cdot)(A \times B) = \tilde{\mu}(\cdot)(A \times B)$. Notice that

$$\{\omega \,:\, (X(\cdot)(\omega), f(X(\cdot)(\omega))) \in A \times B\} = \{\omega \,:\, X(\cdot)(\omega) \in A \cap f^{-1}(B)\}$$

so

$$\begin{aligned}\mu(\cdot)(A\times B) =& P((X(\cdot),f(X(\cdot)))\in A\times B)\\ =& P(X(\cdot)\in A\cap f^{-1}(B))\\ =& P(\tilde{X}(\cdot)\in A\cap f^{-1}(B))\\ =& P((\tilde{X}(\cdot),f(\tilde{X}(\cdot)))\in A\times B)=\tilde{\mu}(\cdot)(A\times B).\end{aligned}$$

So $\mu(\cdot)=\tilde{\mu}(\cdot)$ implying $(X(\cdot),f(X(\cdot)))\sim(\tilde{X}(\cdot),f(\tilde{X}(\cdot)))$. ■

Given any measurable f define the continuity set of f to be

$$C_f=\{x\,:\,f(x_n)\to f(x)\ \text{ whenever }\ x_n\to x\}.$$

Lemma 5.0.6. *Suppose $X^r(\cdot)\Rightarrow X(\cdot)$ and let f be a measurable, real valued function such that $P(X\in C_f)=1$. Then*

$$(X^r(\cdot),f(X^r(\cdot)))\Rightarrow(X(\cdot),f(X(\cdot))).$$

Proof By the continuous mapping theorem we know $f(X^r(\cdot))\Rightarrow f(X(\cdot))$. This implies that $(X^r(\cdot),f(X^r(\cdot)))$ is tight and so for some subsequence

$$(X^{r_k}(\cdot),f(X^{r_k}(\cdot)))\Rightarrow(U(\cdot),V(\cdot))$$

where $U(\cdot)\sim X(\cdot)$, $V(\cdot)\sim f(X(\cdot))$. For convenience sake, we drop subsequence notation in all that follows. By the Skorohod representation theorem, there is some $(Y^r(\cdot), W^r(\cdot)) \sim (X^r(\cdot), f(X^r(\cdot)))$, $(Y(\cdot), W(\cdot)) \sim (U(\cdot), V(\cdot))$ with

$$(Y^r(\cdot), W^r(\cdot)) \to (Y(\cdot), W(\cdot)) \text{ almost surely.}$$

By Lemma 5.0.4 we know $W^r(\cdot) = f(Y^r(\cdot))$ almost surely. In fact, we have

$$(Y^r(\cdot), f(Y^r(\cdot))) \to (Y(\cdot), W(\cdot)) \text{ almost surely.}$$

This implies $Y^r(\cdot) \to Y(\cdot)$ and $f(Y^r(\cdot)) \to W(\cdot)$, almost surely. Since

$$P(Y(\cdot) \in C_f) = P(X(\cdot) \in C_f) = 1,$$

it follows that $f(Y^r(\cdot)) \to f(Y(\cdot))$ almost surely. We can then conclude

$$(Y^r(\cdot), f(Y^r(\cdot))) \to (Y(\cdot), f(Y(\cdot))).$$

This implies $(Y(\cdot), f(Y(\cdot))) \sim (U(\cdot), V(\cdot))$. By Lemma 5.0.4 we know

$$(U(\cdot), V(\cdot)) \sim (U(\cdot), f(U(\cdot))).$$

Therefore we have

$$(X^r(\cdot), f(X(\cdot))^r) \Rightarrow (U(\cdot), f(U(\cdot))).$$

Since $X^r(\cdot) \Rightarrow X(\cdot)$ and $U(\cdot) \sim X(\cdot)$ it follows by Lemma 5.0.5 that

$$(X^r(\cdot), f(X^r(\cdot))) \Rightarrow (X(\cdot), f(X(\cdot))).$$

Since any convergent subsequence must behave in this manner, we conclude the entire sequence converges jointly, as desired.■

Chapter 6

Joint convergence of batch lengths and start times

As in [9] our asymptotic assumptions imply that

$$\bar{I}^r(\cdot) = \frac{1}{r} I^r(r\cdot) \Rightarrow 0,$$

$$\bar{E}^r(\cdot) \Rightarrow \alpha(\cdot), \text{ and}$$

$$X_g^r(\cdot) = \frac{1}{r} \sum_{i=1}^{r\bar{E}^r(\cdot)} g(v_i^r) \Rightarrow \alpha(\cdot)\langle g, \nu \rangle$$

for any fixed ν-a.s. continuous Borel measurable $g : \mathbb{R}_+ \to \mathbb{R}$ that is both ν^r- and ν-integrable (here $\alpha(t) = \alpha t$); see Lemma A.2 in [9]. In particular, if we define $Id(t) = t$,

$$X_\chi^r(\cdot) = \frac{1}{r} \sum_{i=1}^{r\bar{E}^r(\cdot)} v_i^r \Rightarrow Id(\cdot).$$

Let $\eta > 0$ and take $M > 0$ such that

$$P(W_0 < M) \geq 1 - \eta. \tag{6.0.1}$$

Proposition 6.0.1. *Suppose* $\left(\bar{\mu}^r(0), \frac{1}{r}B_0^r\right) \Rightarrow (\theta_0, W_0)$. *For any fixed* $n \in \mathbb{N}$ *we have the following joint convergence*

$$\left(\bar{\mu}^r(0), \frac{1}{r}B_0^r, \frac{1}{r}\beta_1^r, \frac{1}{r}B_1^r, \frac{1}{r}\beta_2^r, \frac{1}{r}B_2^r, \cdots, \frac{1}{r}\beta_n^r, \frac{1}{r}B_n^r\right)$$
$$\Rightarrow (\theta_0, W_0, 1W_0, W_0, 2W_0, W_0, \cdots, nW_0, W_0).$$

It is well known that if $\rho^r \uparrow 1$ then $\bar{W}^r(\cdot) \Rightarrow \bar{W}(\cdot)$ where $\bar{W}(\cdot) \equiv W(0)$. Since $W(0) \sim W_0 > 0$ almost surely by assumption, it follows that for any $\eta, M > 0$ there exists some $\epsilon > 0$ such that

$$\liminf_{r\to\infty} P\left(\inf_{t\in[0,M]} \bar{W}^r(t) > \epsilon\right) \geq 1 - \eta. \tag{6.0.2}$$

Proof of Proposition 6.0.1 Notice that

$$\frac{1}{r}\beta_1^r = \frac{1}{r}B_0^r 1_{\{\bar{W}^r\left(\frac{1}{r}B_0^r\right)\neq 0\}} + \frac{1}{r}\inf\{s \geq B_0^r \,:\, \bar{W}^r(s) > 0\}1_{\{\bar{W}^r\left(\frac{1}{r}B_0^r\right)=0\}}.$$

Let $\eta > 0$ be given. Since $\frac{1}{r}B_0^r \Rightarrow W_0$ there exists some $M_0 > 0$ such that if

$$C_0^r = \left\{\frac{1}{r}B_0^r \leq M_0\right\}$$

then

$$\liminf_{r\to\infty} P(C_0^r) \geq 1-\eta.$$

By (6.0.2) we know there exists an $\epsilon > 0$ such that if

$$A_0^r = \left\{ \inf_{t\in[0,2M_0]} \bar{W}^r(t) > \epsilon \right\}$$

then

$$\liminf_{r\to\infty} P(A_0^r) \geq 1-\eta.$$

It follows that on $A_0^r \cap C_0^r$, $\frac{1}{r}\beta_1^r = \frac{1}{r}B_0^r \leq M_0$ and for large enough r we have

$$P(A_0^r \cap C_0^r) \geq 1-2\eta.$$

It follows from the fact that $\bar{I}^r(\cdot) \Rightarrow 0$ that

$$\bar{I}^r\left(\frac{1}{r}\beta_1^r\right) 1_{A_0^r\cap C_0^r} \Rightarrow 0.$$

Since $\eta > 0$ was arbitrary, by Lemma 5.0.2

$$\bar{I}^r\left(\frac{1}{r}\beta_1^r\right) \Rightarrow 0.$$

Since the first two components converge jointly by assumption and the remaining components have deterministic limits, it follows that

$$Y_0^r(\cdot) = \left(\bar{\mu}^r(0), \frac{1}{r}B_0^r, X_g^r(\cdot), \bar{E}^r(\cdot), \bar{I}^r(\cdot), X_\chi^r(\cdot), \bar{I}^r\left(\frac{1}{r}\beta_1^r\right)\right)$$
$$\Rightarrow (\theta_0, W_0, \alpha(\cdot)\langle g, \nu\rangle, \alpha(\cdot), 0, Id(\cdot), 0) = Y_0(\cdot).$$

So $\frac{1}{r}\beta_1^r = f_1(Y_0^r(\cdot))$ where $f_1 : \mathcal{M} \times \mathbb{R} \times D_{\mathbb{R}_+}(\mathbb{R}_+)^4 \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f_1((\nu, x_1, y_1(\cdot), y_2(\cdot), y_3(\cdot), y_4(\cdot), x_2)) = x_1 + x_2.$$

This function is continuous. It follows by Lemma 5.0.6 that since $f_1(Y_0(\cdot)) = W_0$, we have

$$\left(Y_0^r(\cdot), \frac{1}{r}\beta_1^r\right) \Rightarrow (Y_0(\cdot), W_0).$$

Recall

$$\frac{1}{r}B_1^r = \frac{1}{r}\sum_{i=1}^{r\bar{E}^r\left(\frac{\beta_1^r}{r}\right)} v_i^r = X_\chi^r\left(\frac{\beta_1^r}{r}\right).$$

Define $f_2 : (\mathcal{M} \times \mathbb{R} \times D_{\mathbb{R}_+}(\mathbb{R}_+)^4 \times \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ by

$$f_2(((\nu, x_1, y_1(\cdot), y_2(\cdot), y_3(\cdot), y_4(\cdot), x_2), x_3)) = y_4(x_3).$$

Then $\frac{1}{r}B_1^r = f_2\left(\left(Y_0^r, \frac{1}{r}\beta_1^r\right)\right)$. Since f_2 is continuous at points such that $y_4(\cdot)$ is continuous, and since $X_\chi^r(\cdot) \Rightarrow Id(\cdot)$,we have $P\left((Y_0, 1W_0) \in C_{f_2}\right) = 1$. So by Lemma 5.0.6 and Lemma 5.0.3, since $f_2((Y_0(\cdot), W_0)) = W_0$,

$$Y_1^r(\cdot) = \left(Y_0^r(\cdot), \frac{1}{r}\beta_1^r, \frac{1}{r}B_1^r\right) \Rightarrow (Y_0(\cdot), W_0, W_0) = Y_1(\cdot).$$

We now proceed by induction. Suppose that

$$Y_n^r(\cdot) = \left(Y_1^r(\cdot), \bar{I}^r\left(\frac{1}{r}\beta_2^r\right), \frac{1}{r}\beta_2^r, \frac{1}{r}B_2^r, \cdots, \bar{I}^r\left(\frac{1}{r}\beta_n^r\right), \frac{1}{r}\beta_n^r, \frac{1}{r}B_n^r\right)$$
$$\Rightarrow (Y_1(\cdot), 0, 2W_0, W_0, \cdots, 0, nW_0, W_0) = Y_n(\cdot).$$

We aim to show

$$Y_{n+1}^r(\cdot) = \left(Y_1^r(\cdot), \bar{I}^r\left(\frac{1}{r}\beta_2^r\right), \frac{1}{r}\beta_2^r, \frac{1}{r}B_2^r, \cdots, \bar{I}^r\left(\frac{1}{r}\beta_n^r\right),\right.$$
$$\left.\frac{1}{r}\beta_n^r, \frac{1}{r}B_n^r, \bar{I}^r\left(\frac{1}{r}\beta_{n+1}^r\right), \frac{1}{r}\beta_{n+1}^r, \frac{1}{r}B_{n+1}^r\right)$$
$$\Rightarrow (Y_1(\cdot), 0, 2W_0, W_0, \cdots, 0, nW_0, W_0, 0, (n+1)W_0, W_0) = Y_{n+1}(\cdot).$$

Notice that

$$\frac{1}{r}\beta_{n+1}^r = \left(\frac{\beta_n^r}{r} + \frac{B_n^r}{r}\right)1_{\{\bar{W}^r\left(\frac{\beta_n^r}{r}+\frac{B_n^r}{r}\right)\neq 0\}} + \frac{1}{r}\inf\{s \geq \beta_n^r + B_n^r : \bar{W}^r(s) > 0\}1_{\{\bar{W}^r\left(\frac{\beta_n^r}{r}+\frac{B_n^r}{r}\right)=0\}}.$$

Let $\eta > 0$ be given. Since $\frac{1}{r}\beta_n^r \Rightarrow nW_0$ and $\frac{1}{r}B_n^r \Rightarrow W_0$ we know there exists some $M_n > 0$ so that if $C_n^r = \left\{\frac{1}{r}\beta_n^r + \frac{1}{r}B_n^r \leq M_n\right\}$ then

$$\liminf_{r\to\infty} P(C_n^r) \geq 1 - \eta.$$

By (6.0.2) we know there exists an ϵ such that if

$$A_n^r = \left\{ \inf_{t \in [0, 2M_n]} \bar{W}^r(t) > \epsilon \right\}$$

then

$$\liminf_{r \to \infty} P(A_n^r) \geq 1 - \eta.$$

It follows that on $A_n^r \cap C_n^r$, $\frac{1}{r}\beta_{n+1}^r = \frac{1}{r}\beta_n^r + \frac{1}{r}B_n^r \leq M_n$ and for large enough r we have

$$P(A_n^r \cap C_n^r) \geq 1 - 2\eta.$$

It follows from the fact that $\bar{I}^r(\cdot) \Rightarrow 0$ that

$$\bar{I}^r \left(\frac{1}{r}\beta_{n+1}^r \right) 1_{A_n^r \cap C_n^r} \Rightarrow 0.$$

Since $\eta > 0$ was arbitrary, by Lemma 5.0.2

$$\bar{I}^r \left(\frac{1}{r}\beta_{n+1}^r \right) \Rightarrow 0.$$

Since the previous limit is deterministic, we have

$$\left(Y_n^r(\cdot), \bar{I}^r \left(\frac{1}{r}\beta_{n+1}^r \right) \right) \Rightarrow (Y_n(\cdot), 0).$$

Since

$$\begin{aligned}\frac{1}{r}\beta^r_{n+1} &= \frac{1}{r}\beta^r_n + \frac{1}{r}B^r_n + \bar{I}^r\left(\frac{1}{r}\beta^r_{n+1}\right) - \bar{I}^r\left(\frac{1}{r}\beta^r_n\right)\\ &= f_{2n-1}\left(\left(Y^r_n(\cdot), \bar{I}^r\left(\frac{1}{r}\beta^r_{n+1}\right)\right)\right),\end{aligned}$$

where

$$f_{2n-1} : \mathcal{M} \times \mathbb{R} \times D_{\mathbb{R}_+}(\mathbb{R}_+)^4 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{R}$$

is defined by

$$\begin{aligned}&f_{2n-1}((\nu, x_1, y_1(\cdot), y_2(\cdot), y_3(\cdot), y_4(\cdot), x_2, x_3, \cdots, x_{3n-3}, x_{3n-2}, x_{3n-1}))\\ &\quad = x_{3n-3} + x_{3n-2} + (x_{3n-1} - x_{3n-4}).\end{aligned}$$

Notice f_{2n-1} is continuous and since $f_{2n-1}\left((Y_n, 0)\right) = (n+1)W_0$, it follows by Lemma 5.0.6 that

$$\left(\left(Y^r_n(\cdot), \bar{I}^r\left(\frac{1}{r}\beta^r_{n+1}\right)\right), \frac{1}{r}\beta^r_{n+1}\right) \Rightarrow ((Y_n(\cdot), 0), (n+1)W_0).$$

Notice also

$$\frac{1}{r}B^r_{n+1} = X^r_\chi\left(\frac{\beta^r_{n+1}}{r}\right) - \sum_{i=1}^{n}\frac{1}{r}B^r_i = f_{2n}\left(\left(Y^r_n(\cdot), \bar{I}^r\left(\frac{1}{r}\beta^r_{n+1}\right), \frac{1}{r}\beta^r_{n+1}\right)\right)$$

where

$$f_{2n} : \mathcal{M} \times \mathbb{R} \times D_{\mathbb{R}_+}(\mathbb{R}_+)^4 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{R}$$

is defined by

$$f_{2n}((\nu, x_1, y_1(\cdot), y_2(\cdot), y_3(\cdot), y_4(\cdot), x_2, x_3, \cdots, x_{3n-3}, x_{3n-2}, x_{3n-1}, x_{3n}))$$
$$= y_4(x_{3n}) - \sum_{i=1}^{n-1} x_{3i+1}.$$

Since $X^r_\chi(\cdot) \Rightarrow Id(\cdot)$, a continuous function and

$$f_{2n}(((Y_n(\cdot), 0, (n+1)W_0)) = W_0$$

it follows by the continuity of f_{2n} at points such that $y_4(\cdot)$ is continuous and Lemmas 5.0.6 and 5.0.3 that

$$\left(Y^r_n(\cdot), \bar{I}^r\left(\frac{1}{r}\beta^r_{n+1}\right), \frac{1}{r}\beta^r_{n+1}, \frac{1}{r}B^r_{n+1}\right) \Rightarrow (Y_n(\cdot), 0, (n+1)W_0, W_0).$$

Ignoring extraneous terms, the desired convergence follows. ■

Lemma 6.0.2. *If*

$$(\theta'_0, W'_0, V_1, V_2, V_3, V_4, \cdots, V_{2n-1}, V_{2n})$$
$$\sim (\theta_0, W_0, 1W_0, W_0, 2W_0, W_0, \cdots, nW_0, W_0)$$

then almost surely

$$(\theta_0', W_0', V_1, V_2, V_3, V_4, \cdots, V_{2n-1}, V_{2n})$$
$$= (\theta_0', W_0', 1W_0', W_0', 2W_0', W_0', \cdots, nW_0', W_0').$$

Proof Let $U_0 = (\theta_0, W_0)$. Notice that

$$((\theta_0, W_0), 1W_0, W_0, 2W_0, W_0, \cdots nW_0, W_0) = (U_0, f(U_0))$$

where $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}^{2n} : (x, y) \mapsto (1y, y, 2y, y, \cdots, ny, y)$. So the statement follows immediately from Lemma 5.0.4. ■

Chapter 7

Tightness of the state descriptors

Recall $\ell(t) = \max\{j : \beta_j \leq t\}$. Notice that for $t = \beta_m^r/r$,

$$\bar{\mu}^r(t) = \bar{\mu}_1^r(t) \tag{7.0.1}$$

since

$$\ell^r(rt) = \ell^r(\beta_m^r) = \max\{j : \beta_j^r \leq \beta_m^r\} = m,$$

and so

$$\bar{\mu}_2^r(t) = \frac{1}{r} \sum_{j=E^r(\beta_m^r)+1}^{E^r(\beta_m^r)} \delta_{R_j(t)}^+ = 0.$$

Before we show that $\{\bar{\mu}_1^r(\cdot)\}$ and $\{\bar{\mu}_2^r(\cdot)\}$ are individually tight we prove the following technical result.

Theorem 7.0.1. *Given $T, \eta, \epsilon > 0$, there exist $\delta, M, \kappa > 0$ (without loss of generality we take $\epsilon, \kappa < 1$), and a sequence of events $\{\Omega^r\}$ such that $P(\Omega^r) \geq 1 - \eta$ and on Ω^r the following hold (provided r sufficiently large):*

1. $\ell^r(rT) \leq \lceil \frac{T}{\delta} \rceil$

2. $\sup_{x\in\mathbb{R}_+} \bar{\mu}_1^r \left(\frac{\beta_i}{r}\right)([x, x+\kappa]) < \frac{\epsilon}{2}$ *for any* $0 \leq i \leq \lfloor \frac{T}{\delta} \rfloor$

3. $\|\langle \chi, \bar{\mu}^r(t)\rangle - \langle \chi, \bar{\mu}^r(0)\|_T < \epsilon$

4. $\langle \chi, \bar{\mu}^r(0)\rangle \vee \langle 1, \bar{\mu}^r(0)\rangle < M$

5. $\bar{E}^r(t+h) - \bar{E}^r(t) \leq 2\alpha h$ *for all* $t, t+h \in [0, T]$

6. $\bar{E}^r(T) \leq 2\alpha T$

Lemma 7.0.2. *Let* $T, \eta > 0$ *be given. There exists* $\delta > 0$ *and* R *such that for* $r > R$ *there is* Ω_1^r *where* $P(\Omega_1^r) \geq 1 - \eta$ *and on* Ω_1^r

$$\ell^r(rT) \leq \left\lceil \frac{T}{\delta} \right\rceil.$$

Proof Take δ such that $P(W_0 > \delta) \geq 1 - \eta$ and denote $W_\delta = \{W_0 > \delta\}$. By Proposition 6.0.1 we know for any n

$$\frac{1}{r}\beta_n^r \Rightarrow nW_0.$$

In particular, take $n = \lceil \frac{T}{\delta} \rceil$. On W_δ

$$\left\lceil \frac{T}{\delta} \right\rceil W_0 > \frac{T}{\delta}\delta = T.$$

By the Portmanteau Theorem it follows that

$$\liminf_{r\to\infty} P\left(\frac{1}{r}\beta^r_{\lceil \frac{T}{\delta}\rceil} > T\right) \geq P\left(\left\lceil \frac{T}{\delta}\right\rceil W_0 > T\right) \geq 1-\eta.$$

By definition of $\ell^r(rT)$ it follows that

$$\liminf_{r\to\infty} P\left(\ell^r(rT) \leq \left\lceil \frac{T}{\delta}\right\rceil\right) \geq 1-\eta.$$

■

Lemma 7.0.3. *Given $T, \delta > 0$, for any $1 \leq i \leq \lceil \frac{T}{\delta}\rceil$, $\bar{\mu}^r\left(\frac{\beta^r_i}{r}\right) \Rightarrow \phi$ where for any $g \in C_b(\mathbb{R}_+)$*

$$\langle g, \phi\rangle = \alpha W_0 \langle g, \nu\rangle.$$

Proof Let $g \in C_b(\mathbb{R}_+)$. We know from Proposition 6.0.1 that for any i we have

$$Z^r_i(\cdot) = \left(X^r_g(\cdot), \frac{1}{r}\beta^r_{i-1}, \frac{1}{r}\beta^r_i\right) \Rightarrow (\alpha(\cdot)\langle g, \nu\rangle, (i-1)W_0, iW_0) = Z_i(\cdot).$$

Notice that

$$\left\langle g, \bar{\mu}^r\left(\frac{1}{r}\beta^r_i\right)\right\rangle = \sum_{r\bar{E}^r\left(\frac{\beta^r_{i-1}}{r}\right)+1}^{r\bar{E}^r\left(\frac{\beta^r_i}{r}\right)} g(v^r_i) = X^r_g\left(\frac{\beta^r_i}{r}\right) - X^r_g\left(\frac{\beta^r_{i-1}}{r}\right).$$

So

$$\left\langle g, \bar{\mu}^r\left(\frac{1}{r}\beta^r_i\right)\right\rangle = f\left(\left(X^r_g(\cdot), \frac{1}{r}\beta^r_{i-1}, \frac{1}{r}\beta^r_i\right)\right)$$

where $f : D_{\mathbb{R}_+}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f((y(\cdot), x_1, x_2)) = y(x_2) - y(x_1).$$

and is continuous at points $(y(\cdot), x_1, x_2)$ such that $y(\cdot)$ is continuous.

Therefore $P(Z_i(\cdot) \in C_f) = 1$ and by the continuous mapping theorem, since $f(Z(\cdot)) = \alpha W_0\langle g, \nu\rangle$, it follows that

$$\left\langle g, \bar{\mu}^r\left(\frac{1}{r}\beta_i^r\right)\right\rangle \Rightarrow \alpha W_0\langle g, \nu\rangle.$$

Since this description holds for any $g \in C_b(\mathbb{R}_+)$ we have for every i,

$$\bar{\mu}^r\left(\frac{1}{r}\beta_i^r\right) \Rightarrow \phi.$$

■

Lemma 7.0.4. *Let $\eta, \epsilon > 0$ be given. Then there exists some $\tilde{\kappa} > 0$ such that*

$$P\left(\sup_{x\in\mathbb{R}_+}\langle 1_{[x,x+\tilde{\kappa}]}, \theta_0\rangle < \epsilon\right) \geq 1 - \eta.$$

Proof This follows directly from the assumption that θ_0 has no atoms. ■

Lemma 7.0.5. *Let $\epsilon, \kappa > 0$ be given. Then*

$$A = \{\theta \in \mathcal{M}_F \,:\, \sup_{x\in\mathbb{R}_+}\langle 1_{[x,x+\kappa]}, \theta\rangle < \epsilon\}$$

is open.

Proof Consider a sequence $\{\zeta_n\} \subset \mathcal{M}_F$ such that $\zeta_n \xrightarrow{w} \zeta \in A$. Then

$$a = \sup_{x \in \mathbb{R}_+} \langle 1_{[x,x+\kappa]}, \zeta \rangle < \epsilon \tag{7.0.2}$$

Suppose that

$$\limsup_{n \to \infty} \sup_{x \in \mathbb{R}_+} \langle 1_{[x,x+\kappa]}, \zeta_n \rangle \geq \epsilon. \tag{7.0.3}$$

Then on a subsequence $\{k\} \subset \{n\}$, $\sup_{x \in \mathbb{R}_+} \langle 1_{[x,x+\kappa]}, \zeta_k \rangle > b$ for $b \in (a, \epsilon)$ and all k. So for each k, there exists x_k such that $\langle 1_{[x_k, x_k+\kappa]}, \zeta_k \rangle > b$. Since $M < \infty$ can be chosen so that $\langle 1_{[M,\infty)}, \zeta \rangle \leq b/2$ and so

$$\limsup_{k \to \infty} \langle 1_{[M,\infty)}, \zeta_k \rangle \leq b/2$$

by the Portmanteau Theorem, all but finitely many x_k must be bounded by M. So on a further subsequence $\{j\} \subset \{k\}$ $x_j \to x$. For each $\delta > 0$, all but finitely many $[x_j, x_j + \kappa]$ are subsets of the interval $I^\delta = [x - \delta, x + \kappa + \delta]$, which implies $\liminf_{j \to \infty} \zeta_j(I^\delta) \geq b$. By the Portmanteau Theorem since I^δ is closed, $\zeta(I^\delta) \geq \liminf_{j \to \infty} \zeta_j(I^\delta) \geq b$. But this implies $\zeta([x, x + \kappa]) = \lim_{\delta \to 0} \zeta(I^\delta) \geq b > a$ contradicting (7.0.2). We conclude that (7.0.3) is false and thus

$$\limsup_{n \to \infty} \sup_{x \in \mathbb{R}_+} \langle 1_{[x,x+\kappa]}, \zeta_n \rangle < \epsilon$$

which implies that $\zeta_n \in A$ for sufficiently large n and A is open. ■

Lemma 7.0.6. *Let $T, \delta, \eta, \epsilon > 0$ be given. Then there exists some $\kappa > 0$ and R such that if $r > R$*

$$P\left(\sup_{x\in\mathbb{R}_+} \bar{\mu}_1^r\left(\frac{\beta_i^r}{r}\right)([x, x+\kappa]) < \epsilon\right) \geq 1-\eta$$

for any $0 \leq i \leq \left\lceil \frac{T}{\delta} \right\rceil$.

Proof Recall $\bar{\mu}_1^r\left(\frac{\beta_i^r}{r}\right) = \bar{\mu}^r\left(\frac{\beta_i^r}{r}\right)$. By Lemma 7.0.4 we know there exists some $\tilde{\kappa}$ such that

$$P(\sup_{x\in\mathbb{R}_+} \langle 1_{[x,x+\tilde{\kappa}]}, \theta_0\rangle < \epsilon) \geq 1-\eta.$$

Define

$$\tilde{A} = \{\zeta \in \mathcal{M}_F \ : \ \sup_{x\in\mathbb{R}_+} \langle 1_{[x,x+\tilde{\kappa}]}, \zeta\rangle < \epsilon\}.$$

Then $\tilde{A}$ is open by Lemma 7.0.5. Since $\bar{\mu}^r(0) \Rightarrow \theta_0$

$$\liminf_{r\to\infty} P(\bar{\mu}^r(0) \in A) \geq P(\theta_0 \in A) \geq 1-\eta.$$

We also know by Lemma 7.0.3 that for $i \in \mathbb{Z}_{\geq 1}$, $\bar{\mu}^r\left(\frac{\beta_i^r}{r}\right) \Rightarrow \phi$ where ϕ is non-atomic. It follows that there exists some κ' such that

$$P(\sup_{x\in\mathbb{R}_+} \langle 1_{[x,x+\kappa']}, \phi\rangle < \epsilon) \geq 1-\eta.$$

Define

$$A' = \{\zeta \in \mathcal{M}_F \ : \ \sup_{x \in \mathbb{R}_+} \langle 1_{[x,x+\kappa']}, \zeta \rangle < \epsilon\}.$$

Then A' is open by Lemma 7.0.5. It follows that for each i,

$$\liminf_{r\to\infty} P\left(\bar{\mu}^r\left(\frac{\beta_i^r}{r}\right) \in A'\right) \geq P(\phi \in A') \geq 1 - \eta.$$

So for each i, there exists some R_i such that for $r > R_i$

$$\bar{\mu}^r\left(\frac{\beta_i^r}{r}\right) \in A'.$$

Take $\kappa = \tilde{\kappa} \wedge \kappa'$ and define $R = \max_{0\leq i \leq \lceil \frac{T}{\delta} \rceil}$. Then the desired result holds. ■

Lemma 7.0.7. $\{\langle \chi, \bar{\mu}_1^r(\cdot) + \bar{\mu}_2^r(\cdot)\rangle\}$ *converges in distribution to a process that is equal to* W_0 *a.s. for all time.*

Proof Recall $\bar{\mu}^r(\cdot) = \bar{\mu}_1^r(\cdot) + \bar{\mu}_2^r(\cdot)$ and using the definitions (3.1.2)-(3.1.6) $\langle \chi, \bar{\mu}^r(\cdot)\rangle$ is the workload process $\bar{W}^r(\cdot)$ defined in (3.1.1) which is the same process for any single server queue operating under a work conserving service protocol. The statement is thus the well-known result of our asymptotic assumptions in Definition 3.3.1. ■

Lemma 7.0.8. *Let* $T, \epsilon, \eta > 0$ *be given. Then there exists some* R *such that if* $r > R$

$$P\left(\sup_{t\in[0,T]} |\langle \chi, \bar{\mu}^r(t)\rangle - \langle \chi, \bar{\mu}^r(0)\rangle| < \epsilon\right) \geq 1 - \eta.$$

Proof This follows immediately since $\langle \chi, \bar{\mu}^r(\cdot)\rangle$ converges in distribution to its initial value. ■

Lemma 7.0.9. *Let $\eta > 0$ be given. Then there exists some M such that*

$$\liminf_{r\to\infty} P(\langle 1, \bar{\mu}^r(0)\rangle \vee \langle \chi, \bar{\mu}^r(0)\rangle < M) \geq 1 - \eta.$$

Proof This follows since $(\bar{\mu}^r(0), \langle \chi, \bar{\mu}^r(0)\rangle) \Rightarrow (\theta_0, \langle \chi, \theta_0\rangle)$ where $\mathbb{E}[\langle 1, \theta_0\rangle] < \infty$, $\mathbb{E}[\langle \chi, \theta_0\rangle] < \infty$. ■

Lemma 7.0.10. *Let $T > 0$ and $g \in C_b(\mathbb{R}_+)$ and let $0 < h \leq T$. Then*

$$\lim_{r\to\infty} P\left(\sup_{t\in[0,T-h]} \frac{1}{r}\sum_{i=r\bar{E}^r(t)+1}^{r\bar{E}^r(t+h)} g(v_i^r) \leq 2\alpha h\langle g, \nu\rangle\right) = 1.$$

Proof Since $g \in C_b(\mathbb{R}_+)$ we know $\langle g, \nu^r\rangle \to \langle g, \nu\rangle$. From the appendix of [9] we have

$$X_g^r(\cdot) = \frac{1}{r}\sum_{i=1}^{r\bar{E}^r(t)} g(v_i^r) \Rightarrow \alpha(\cdot)\langle g, \nu\rangle.$$

The statement follows. ■

Corollary 7.0.11. *Let $T, \eta > 0$ be given. Then for $h \in [0, T]$ there exists some R such that if $r > R$*

$$P\left(\sup_{t\in[0,T-h]} \bar{E}^r(t+h) - \bar{E}^r(t) \leq 2\alpha h\right) \geq 1 - \eta.$$

In particular using $h = T$,

$$P(\bar{E}^r(T) \leq 2\alpha T) \geq 1 - \eta.$$

Proof This follows immediately by applying Lemma 7.0.10 with $g \equiv 1$. ■

Proof of Theorem 7.0.1 We make the following definitions:

1. $\Omega_1^r = \left\{\ell^r(rT) \leq \left\lceil \frac{T}{\delta} \right\rceil\right\}$

2. $\Omega_2^r = \left\{\sup_{x \in \mathbb{R}_+} \langle 1_{[x,x+\kappa]}, \theta_0 \rangle < \frac{\epsilon}{2}\right\}$

3. $\Omega_3^r = \left\{\sup_{x \in \mathbb{R}_+} \bar{\mu}^r \left(\frac{\beta_i^r}{r}\right)([x, x+\kappa]) < \frac{\epsilon}{2},\ 0 \leq i \leq \left\lceil \frac{T}{\delta} \right\rceil\right\}$

4. $\Omega_4^r = \left\{\sup_{t \in [0,T]} |\langle \chi, \bar{\mu}^r(t) \rangle - \langle \chi, \bar{\mu}^r(0) \rangle| < \epsilon\right\}$

5. $\Omega_5^r = \{\langle 1, \bar{\mu}^r(0) \rangle \vee \langle \chi, \bar{\mu}^r(0) \rangle < M\}$

6. $\Omega_6^r = \left\{\sup_{t \in [0,T-h]} \bar{E}^r(t+h) - \bar{E}^r(t) \leq 2\alpha h\right\}$

7. $\Omega_7^r = \{\bar{E}^r(T) \leq 2\alpha T\}$

Let $\Omega^r = \cap_{i=1}^7 \Omega_i^r$. By the proceeding lemmas given $T, \eta, \epsilon > 0$, we may pick δ, κ, M, and R, in that order, such that if $r > R$ we have

$$\liminf_{r \to \infty} P(\Omega^r) \geq 1 - \eta.$$

■

Equipped with the above result, we now show that $\{\bar{\mu}_1(\cdot)\}$ and $\{\bar{\mu}_2^r(\cdot)\}$ are individually tight. Recall that

$$d[\mu,\nu] = \inf\{\epsilon > 0 \,:\, \mu(B) \le \nu(B^\epsilon) + \epsilon, \nu(B) \le \mu(B^\epsilon) + \epsilon \text{ for all closed } B \subset \mathbb{R}_+\}.$$

To show tightness, it is sufficient to verify the following two conditions

1. For every $\eta > 0$ and $T > 0$ there is a compact set $\Gamma_{\eta,T} \subset \mathcal{M}_F$ such that

$$\liminf_{r\to\infty} P(\bar{\mu}_1^r(t) \in \Gamma_{\eta,T} \text{ for } 0 \le t \le T) \ge 1 - \eta.$$

2. For every $\eta > 0$ and $T > 0$ there exists some $\gamma > 0$ such that

$$\limsup_{r\to\infty} P(w'(\bar{\mu}_1^r(t),\gamma,T) \ge \eta) \le \eta$$

   where

$$w'(\bar{\mu}_1^r,\gamma,T) = \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} d[\bar{\mu}_1^r(s),\bar{\mu}_1^r(t)]$$

   and $\{t_i\}$ are partitions of $[0,T]$ with $t_i - t_{i-1} > \gamma$, $0 = t_0$, and $t_n = T$.

Theorem 7.0.12. *The sequence $\{\bar{\mu}_1^r(\cdot)\}$ is tight in $D = D([0,\infty),\mathcal{M}_F)$.*

Proof We begin by verifying property 2 above. Fix ϵ, η, T and let δ, κ, M and Ω^r be given by Theorem 7.0.1. For each r consider a partition $\{t_i^r\}$ of $[0,T]$ such that

$\frac{\beta_j^r}{r} = t_i^r$ for some i, $0 \le j \le \ell(rT)$. We also require that $t_i - t_{i-1} < \frac{\kappa\epsilon^2}{2}$ for all i. We can set $\gamma = \frac{\min_i |t_i - t_{i-1}|}{2}$. Let $B \subset \mathbb{R}_+$ be closed. Given any $t < s \in [t_{i-1}, t_i)$

$$\begin{aligned}
\bar{\mu}_1^r(t)(B) &\le \bar{\mu}_1^r(t)([0,\kappa]) + \bar{\mu}_1^r(t)(B \cap (\kappa,\infty)) \\
&\le \bar{\mu}_1^r\left(\frac{\beta_{\ell(rt)}}{r}\right)([\bar{S}^r_{\beta^r_{\ell^r(rt)}/r,t}, \bar{S}^r_{\beta^r_{\ell^r(rt)}/r,t} + \kappa]) + \bar{\mu}_1^r(t)(B \cap (\kappa,\infty)) \\
&\le \sup_{x\in\mathbb{R}_+} \left\langle 1_{[x,x+\kappa]}, \bar{\mu}_1^r\left(\frac{\beta^r_{\ell^r(rt)}}{r}\right)\right\rangle + \bar{\mu}_1^r(t)(B \cap (\kappa,\infty)) \\
&\le \frac{\epsilon}{2} + \bar{\mu}_1^r(t)(B \cap (\kappa,\infty)),
\end{aligned}$$

where we have used (3.1.2)-(3.1.6).

Let $I = \{u \in [t,s] : \langle 1, \bar{\mu}_1^r(u)\rangle < \epsilon/2\}$. Suppose $I = \emptyset$. Then $\langle 1, \bar{\mu}_1^r(u)\rangle \ge \epsilon/2$ on $[t,s]$. So

$$\bar{S}^r_{t,s} = \int_t^s \frac{1}{\langle 1, \bar{\mu}_1^r(u)\rangle} du \le \frac{(s-t)}{\epsilon/2} \le \frac{2(t_i - t_{i-1})}{\epsilon} < \frac{\kappa\epsilon^2}{\epsilon} = \kappa\epsilon < \kappa \wedge \epsilon.$$

If $x \in B \cap (\kappa,\infty)$ then $x - \bar{S}^r_{t,s} \in B^\epsilon$, so

$$\bar{\mu}_1^r(t)(B \cap (\kappa,\infty)) \le \bar{\mu}_1^r(t)(B^\epsilon + \bar{S}^r_{t,s}) = \bar{\mu}_1^r(s)(B^\epsilon).$$

In this case, we have

$$\bar{\mu}_1^r(t)(B) \le \epsilon + \bar{\mu}_1^r(s)(B^\epsilon).$$

If on the other hand $I \neq \emptyset$, let $\tau = \inf I$. So $\langle 1, \bar{\mu}_1^r(\tau)\rangle \leq \epsilon/2$ by right continuity. If $\tau = t$, then

$$\bar{\mu}_1^r(t)(B) \leq \langle 1, \bar{\mu}_1^r(t)\rangle \leq \frac{\epsilon}{2}.$$

If $\tau > t$ then $\langle 1, \bar{\mu}_1^r(u)\rangle \geq \epsilon/2$ for all $u \in [t, \tau)$. So

$$\bar{S}_{t,\tau}^r = \int_t^\tau \frac{1}{\langle 1, \bar{\mu}_1^r(u)\rangle} du < \frac{\tau - t}{\epsilon/2} \leq \frac{2(t_i - t_{i-1})}{\epsilon} < \frac{\kappa\epsilon^2}{\epsilon} = \kappa\epsilon < \epsilon \wedge \kappa.$$

Since

$$\begin{aligned}
\bar{\mu}_1^r(t)(B \cap (\kappa, \infty)) &\leq \bar{\mu}_1^r(t)(B \cap (\bar{S}_{t,\tau}^r, \infty)) \\
&\leq \bar{\mu}_1^r(t)((\bar{S}_{t,\tau}^r, \infty)) \\
&= \bar{\mu}_1^r(\tau)(\mathbb{R}_+) \\
&= \langle 1, \bar{\mu}^r(\tau)\rangle \leq \frac{\epsilon}{2}.
\end{aligned}$$

In this case we have

$$\bar{\mu}_1^r(t)(B) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon + \bar{\mu}_1^r(s)(B^\epsilon).$$

So in each case we have the desired inequality and it remains to show the symmetric inequality. Again let $I = \{u \in [t, s] : \langle 1, \bar{\mu}_1^r(u)\rangle < \epsilon/2\}$. Suppose $I = \emptyset$. Then on $[t, s]$ we have $\langle 1, \bar{\mu}_1^r(u)\rangle \geq \epsilon/2$. Exactly as before, we find $\bar{S}_{t,s}^r < \kappa \wedge \epsilon < \epsilon$. Let $x \in B + \bar{S}_{t,s}^r$, then there exists some $y \in B$ such that $x = y + \bar{S}_{t,s}^r < y + \epsilon$, so $x \in B^\epsilon$. So if $I = \emptyset$

$$\bar{\mu}_1^r(s)(B) = \bar{\mu}_1^r(t)(B + \bar{S}_{t,s}^r) \leq \bar{\mu}_1^r(t)(B^\epsilon) + \epsilon.$$

Now suppose $I \neq \emptyset$. Again, as before, let $\tau = \inf I$. Then

$$\bar{\mu}_1^r(s)(B) = \bar{\mu}_1^r(\tau)(B + \bar{S}_{\tau,s}^r) \leq \bar{\mu}_1^r(\tau)(\mathbb{R}_+) = \langle 1, \bar{\mu}_1^r(\tau)\rangle \leq \frac{\epsilon}{2}.$$

Therefore if $I \neq \emptyset$

$$\bar{\mu}_1^r(s)(B) \leq \frac{\epsilon}{2} \leq \bar{\mu}_1^r(t)(B^\epsilon) + \epsilon.$$

By definition it follows that $d[\bar{\mu}_1^r(s), \bar{\mu}_1^r(t)] \leq \epsilon$ on Ω^r. Therefore $w'(\bar{\mu}_1^r, \delta, T) \leq \epsilon$ on Ω^r. Since $P(\Omega^r) \geq 1 - \eta$, condition 2 is satisfied.

To show tightness of $\{\bar{\mu}_1^r(\cdot)\}$ it remains to show the compact containment condition. Define

$$M' = (M + 2\alpha T) \vee (M + \epsilon)$$

and let

$$C = \{\zeta \in \mathcal{M}_F \,:\, (\langle 1, \zeta\rangle \vee \langle \chi, \zeta\rangle \leq M'\}.$$

Notice that on Ω^r, $\bar{\mu}_1^r(\cdot) \in C$ for $0 \leq t \leq T$ because

$$\langle 1, \bar{\mu}_1^r(t)\rangle \leq \langle 1, \bar{\mu}_1^r(0)\rangle + \bar{E}^r(T)$$

$$\leq M + 2\alpha T$$

and

$$\langle \chi, \bar{\mu}_1^r(t)\rangle \leq \langle \chi, \bar{\mu}_1^r(0)\rangle + \epsilon$$

$$\leq M + \epsilon.$$

Notice that C is pre-compact since

$$\begin{aligned} M' &\geq \langle \chi, \zeta\rangle \\ &= \langle \chi 1_{[0,K)}, \zeta\rangle + \langle \chi 1_{[K,\infty)}, \zeta\rangle \\ &\geq \langle \chi 1_{[K,\infty)}, \zeta\rangle \\ &\geq K\langle 1_{[K,\infty)}, \zeta\rangle. \end{aligned}$$

This implies that

$$\langle 1_{[K,\infty)}, \zeta\rangle \leq \frac{M'}{K} \to 0$$

as K tends to infinity. Therefore we have

$$\sup_{\zeta\in C}\langle 1_{[K,\infty)}, \zeta\rangle \to 0$$

and $\sup_{\zeta \in C} \zeta(\mathbb{R}_+) \leq M' < \infty$. Therefore

$$\liminf_{r \to \infty} P(\bar{\mu}_1^r(\cdot) \in \overline{C}) \geq 1 - \eta$$

and $\{\bar{\mu}_1^r(\cdot)\}$ is tight. ■

Theorem 7.0.13. *The sequence $\{\bar{\mu}_2^r(\cdot)\}$ is tight in $D = D([0, \infty), \mathcal{M}_F)$.*

Proof Again it suffices to verify conditions 1 and 2 as stated in Theorem 7.0.12. We begin by verifying that 2 holds. Take Ω^r as previously defined. For each r consider a partition $\{t_i^r\}$ such that $\frac{\beta_j^r}{r} = t_i^r$ for some i, $0 \leq j \leq \ell(rT)$ and $t_i - t_{i-1} < \frac{\epsilon}{2\alpha}$ for all i. We can take $\gamma = \frac{\min_i |t_i - t_{i-1}|}{2}$. Let $B \subset \mathbb{R}_+$ be closed. Given $t < s \in [t_{i-1}, t_i)$ we immediately have

$$\bar{\mu}_2^r(t)(B) \leq \bar{\mu}_2^r(s)(B) \leq \bar{\mu}_2^r(s)(B^\epsilon) + \epsilon.$$

We now consider the symmetric inequality.

$$
\begin{aligned}
\bar{\mu}_2^r(s)(B) &\le \bar{\mu}_2^r(t)(B) + \bar{\mu}_2^r(s)(\mathbb{R}_+) - \bar{\mu}_2^r(t)(\mathbb{R}_+) \\
&= \bar{\mu}_2^r(t)(B) + \bar{E}^r(s) - \bar{E}^r(t) \\
&\le \bar{\mu}_2^r(t)(B) + 2\alpha(s-t) \\
&\le \bar{\mu}_2^r(t)(B) + 2\alpha(t_i - t_{i-1}) \\
&\le \bar{\mu}_2^r(t)(B) + 2\alpha\frac{\epsilon}{2\alpha} \\
&\le \bar{\mu}_2^r(t)(B) + \epsilon \\
&\le \bar{\mu}_2^r(t)(B^\epsilon) + \epsilon.
\end{aligned}
$$

Therefore we have $d[\bar{\mu}_2^r(s), \bar{\mu}_2^r(t)] \le \epsilon$ on Ω^r and it follows that $w'(\bar{\mu}_2^r, \gamma, T) \le \epsilon$ on Ω^r. Since $P(\Omega^r) \ge 1 - \eta$, condition 2 is satisfied. To show tightness of $\{\bar{\mu}_2^r(\cdot)\}$ it remains to show the compact containment. Take

$$M' = 2\alpha T \vee (M + \epsilon)$$

and define

$$C = \{\zeta \in \mathcal{M}_F \,:\, \langle 1, \zeta\rangle \vee \langle \chi, \zeta\rangle \le M'\}.$$

Notice that on Ω^r, $\bar{\mu}_2^r(\cdot) \in C$ for $0 \le t \le T$ because

$$\langle 1, \bar{\mu}_2^r(t)\rangle \le \bar{E}^r(T) \le 2\alpha T, \text{ and}$$

$$\langle \chi, \bar{\mu}_2^r(t) \rangle \leq \langle \chi, \bar{\mu}^r(t) \rangle \leq \langle \chi, \bar{\mu}^r(0) \rangle + \epsilon < M + \epsilon.$$

By the same argument presented in Theorem 7.0.12, C is relatively compact. Therefore

$$\liminf P(\bar{\mu}_2^r(\cdot) \in \overline{C}) \geq 1 - \eta$$

and $\{\bar{\mu}_2^r(\cdot)\}$ is tight. ■

Corollary 7.0.14. *The sequence* $\{(\bar{\mu}_1^r(\cdot), \bar{\mu}_2^r(\cdot))\}$ *is jointly tight on* $D \times D$ *equipped with the product topology.*

Chapter 8

Dynamic equation for $\langle g, \bar{\mu}_1^r(t)\rangle$ and $\langle g, \bar{\mu}_2^r(t)\rangle$

Let

$$\tilde{\mathcal{C}} = \{g \in \mathcal{C} \,:\, supp(g') \subset K \;\text{ for some compact set} K\}.$$

Lemma 8.0.1. *Let $g \in \mathcal{C}$ and $t \geq 0$ be fixed. If*

$$m^r = \sup\left\{\langle 1, \bar{\mu}_1^r(s)\rangle^{-1} \,:\, \frac{1}{r}\beta^r_{\ell^r(rt)} \leq s \leq t\right\} < \infty,$$

then

$$\begin{aligned} \langle g, \bar{\mu}_1^r(t)\rangle &= \langle g, \bar{\mu}_1^r(0)\rangle 1_{\{\ell^r(rt)=0\}} \\ &+ \left(X_g^r\left(\frac{1}{r}\beta^r_{\ell^r(rt)}\right) - X_g^r\left(\frac{1}{r}\beta^r_{\ell^r(rt)-1}\right)\right) 1_{\{\ell^r(rt)>0\}} - \int_{\beta_{\ell^r(rt)/r}}^t \frac{\langle g', \bar{\mu}_1^r(s)\rangle}{\langle 1, \bar{\mu}_1^r(s)\rangle} ds, \end{aligned} \tag{8.0.1}$$

and

$$\langle g, \bar{\mu}_2^r(t)\rangle = X_g^r(t) - X_g^r\left(\frac{1}{r}\beta^r_{\ell^r(rt)}\right). \tag{8.0.2}$$

Proof We first consider $g \in \tilde{\mathcal{C}}$. For any $n = 1, 2, ...$ and $j = 0, 1, ..., n-1$ define

$$t_j^r = \frac{\beta_{\ell^r(rt)}}{r} + \frac{j\left(t - \frac{\beta^r_{\ell^r(rt)}}{r}\right)}{n}, \text{ and } t^{j,r} = t^r_{j+1}.$$

Then we have

$$\begin{aligned}\langle g, \bar{\mu}_1^r(t)\rangle - \left\langle g, \bar{\mu}_1^r\left(\frac{\beta_{\ell^r(rt)}}{r}\right)\right\rangle &= \sum_{j=0}^{n-1}\left(\langle g, \bar{\mu}_1^r(t^{j,r})\rangle - \langle g, \bar{\mu}_1^r(t_j^r)\rangle\right) \\ &= \sum_{j=0}^{n-1}\left(\langle g(\cdot - \bar{S}^r_{t_j^r, t^{j,r}}), \bar{\mu}_1^r(t_j^r)\rangle - \langle g, \bar{\mu}_1^r(t_j^r)\rangle\right) \\ &= \sum_{j=0}^{n-1}\langle g(\cdot - \bar{S}^r_{t_j^r, t^{j,r}}) - g, \bar{\mu}_1^r(t_j^r)\rangle. \quad (8.0.3)\end{aligned}$$

For each $j = 0, 1, ..., n-1$ and each $x \in \mathbb{R}_+$

$$g(x - \bar{S}^r_{t_j^r, t^{j,r}}) - g(x) = g'(w_j^{x,r})h_j^r$$

where $h_j^r = -\bar{S}^r_{t_j{}^r, t^{j,r}}$ and $w_j^{x,r} \in \mathbb{R}$ is in the interval $[x - \bar{S}^r_{t_j^r, t^{j,r}}, x]$. Note that

$$\max_{j<n}|h_j^r| = \max_{j<n}|\bar{S}^r_{t_j^r, t^{j,r}}| \leq \frac{t}{n}\|\langle 1, \bar{\mu}^r(\cdot)\rangle^{-1}\|_{\left[\frac{1}{r}\beta^r_{\ell^r(rt)}, t\right]} = \frac{t}{n}m^r.$$

In our application in Chapter 9, we can ensure $m^r < \infty$ by taking r large enough because the system is asymptotically critical and so idleness converges to zero. For each $j \in \{0, 1, ..., n-1\}$, let $z_j^r = \sup_{s \in [t_j^r, t^{j,r}]}\langle 1, \bar{\mu}_1^r(s)\rangle^{-1}$ and define $\tilde{h}_j^r = -z_j^r\left(\frac{t - \frac{\beta^r_{\ell^r(rt)}}{r}}{n}\right)$.

Then

$$\sum_{j=0}^{n-1}|h_j^r-\tilde h_j^r| = \sum_{j=0}^{n-1}\left|z_j^r\left(\frac{t-\frac{\beta^r_{\ell^r(rt)}}{r}}{n}\right)-\bar S^r_{t_j^r,t^{j,r}}\right|$$
$$=\sum_{j=0}^{n-1}\left(z_j^r\left(\frac{t-\frac{\beta^r_{\ell^r(rt)}}{r}}{n}\right)-\bar S^r_{t_j^r,t^{j,r}}\right)$$
$$=\sum_{j=0}^{n-1}\left(z_j^r\left(\frac{t-\frac{\beta^r_{\ell^r(rt)}}{r}}{n}\right)\right)-\bar S^r_{\frac{\beta^r_{\ell^r(rt)}}{r},t}.$$

For each $n=1,2,...$ and $s\in\left[\frac{1}{r}\beta^r_{\ell^r(rt)},t\right)$ let $k_r^n(s)=\sum_{j=0}^{n-1}z_j^r 1_{[t_j^r,t^{r,j})}(s)$ and define $k^n(t)=0$. Then

$$\left|\sum_{j=0}^{n-1}\langle g(\cdot-\bar S^r_{t_j^r,t^{j,r}})-g(\cdot),\bar\mu_1^r(t_j^r)\rangle-\sum_{j=0}^{n-1}\langle g'(\cdot)\tilde h_j^r,\bar\mu_1^r(t_j^r)\rangle\right|$$
$$\le\sum_{j=0}^{n-1}\sup_{x\in\mathbb{R}_+}|g(x-\bar S^r_{t_j^r,t^{j,r}})-g(x)-g'(x)\tilde h_j^r|\langle 1,\bar\mu_1^r(t_j^r)\rangle$$
$$=\sum_{j=0}^{n-1}\sup_{x\in\mathbb{R}_+}|g'(w_j^{x,r})h_j^r-g'(x)\tilde h_j^r|\langle 1,\bar\mu_1^r(t_j^r)\rangle$$
$$\le\|\langle 1,\bar\mu_1^r(\cdot)\rangle\|_{\left[\frac{1}{r}\beta^r_{\ell^r(rt)},t\right]}\sum_{j=0}^{n-1}\sup_{x\in\mathbb{R}_+}\left(|g'(w_j^{x,r})-g'(x)|\right)|h_j^r|+|g'(x)||h_j^r-\tilde h_j^r|$$
$$\le M^r\left(n\psi_g\left(\frac{tm^r}{n}\right)\frac{tm^r}{n}+\|g'\|_\infty\sum_{j=0}^{n-1}\left(z_j^r\left(\frac{t-\frac{\beta^r_{\ell^r(rt)}}{r}}{n}\right)\right)-\bar S^r_{\frac{\beta^r_{\ell^r(rt)}}{r},t}\right)$$

where $M^r=\|\langle 1,\bar\mu_1^r(\cdot)\rangle\|_{\left[\frac{1}{r}\beta^r_{\ell^r(rt)},t\right]}$ and ψ_g is a continuous nondecreasing function with $\psi_g(0)=0$ and

$$\sup_{x\in\mathbb{R}_+}\|g'(x+h)-g'(x)\|\le\psi_g(|h|)$$

which exists since $g'\in\tilde{\mathcal{C}}$ is uniformly continuous. Notice that

$$\sum_{j=0}^{n-1}\left(z_j^r\left(\frac{t-\frac{\beta^r_{\ell^r(rt)}}{r}}{n}\right)\right)=\int_{\beta^r_{\ell^r(rt)/r}}^t k^n(s)ds, \text{ and}$$

$$\bar{S}^r_{\frac{\beta^r_{\ell^r(rt)}}{r},t}=\int_{\beta^r_{\ell^r(rt)/r}}^t \langle 1,\bar{\mu}_1^r(s)\rangle^{-1}ds.$$

So we have

$$\begin{aligned}&M^r\left(n\psi_g\left(\frac{tm^r}{n}\right)\frac{tm^r}{n}+\|g'\|_\infty\sum_{j=0}^{n-1}\left(z_j^r\left(\frac{t-\frac{\beta^r_{\ell^r(rt)}}{r}}{n}\right)\right)-\bar{S}^r_{\frac{\beta^r_{\ell^r(rt)}}{r},t}\right)\\ =&M^r\left(\psi_g\left(\frac{tm^r}{n}\right)tm^r+\|g'\|_\infty\left(\int_{\beta^r_{\ell^r(rt)/r}}^t k^n(s)ds-\int_{\beta^r_{\ell^r(rt)/r}}^t\langle 1,\bar{\mu}_1^r(s)\rangle^{-1}ds\right)\right).\end{aligned}$$

Notice that

$$\lim_{n\to\infty}M^r\left(\psi_g\left(\frac{tm^r}{n}\right)tm^r+\|g'\|_\infty\left(\int_{\beta^r_{\ell^r(rt)/r}}^t k^n(s)ds-\int_{\beta^r_{\ell^r(rt)/r}}^t\langle 1,\bar{\mu}_1^r(s)\rangle^{-1}ds\right)\right)=0.$$

It follows that

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}\langle g(\cdot-\bar{S}^r_{t_j^r,t^{j,r}})-g(\cdot),\bar{\mu}_1^r(t_j^r)\rangle=\lim_{n\to\infty}\sum_{j=0}^{n-1}\langle g'(\cdot)\tilde{h}_j^r,\bar{\mu}_1^r(t_j^r)\rangle.$$

Since

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}\langle g'(\cdot)\tilde{h}_j^r, \bar{\mu}_1^r(t_j^r)\rangle$$
$$= \lim_{n\to\infty} -\sum_{j=0}^{n-1}\langle g', \bar{\mu}_1^r(t_j^r)\rangle z_j^r \left(\frac{t-\frac{\beta^r_{\ell^r(rt)}}{r}}{n}\right)$$
$$= -\int_{\beta^r_{\ell^r(rt)/r}}^{t} \frac{\langle g', \bar{\mu}_1^r(s)\rangle}{\langle 1, \bar{\mu}_1^r(s)\rangle} ds,$$

we have

$$\langle g, \bar{\mu}_1^r(t)\rangle = \left\langle g, \bar{\mu}_1^r\left(\frac{\beta^r_{\ell^r(rt)}}{r}\right)\right\rangle - \int_{\beta^r_{\ell^r(rt)/r}}^{t} \frac{\langle g', \bar{\mu}_1^r(s)\rangle}{\langle 1, \bar{\mu}_1^r(s)\rangle} ds.$$

Since a new batch has started at $\beta^r_{\ell^r(rt)}$, if $\ell^r(rt) > 0$, the jobs present at this time are any jobs that arrived after the start of the $\ell^r(rt)-1$st batch, which began at $\beta^r_{\ell^r(rt)-1}$. In this case, we may write the following expression

$$\left\langle g, \bar{\mu}_1^r\left(\frac{\beta_{\ell^r(rt)}}{r}\right)\right\rangle = \frac{1}{r}\sum_{i=r\bar{E}^r\left(\frac{1}{r}\beta^r_{\ell^r(rt)-1}\right)+1}^{r\bar{E}^r\left(\frac{1}{r}\beta^r_{\ell^r(rt)}\right)} g(v_i^r)$$
$$= X_g^r\left(\frac{1}{r}\beta^r_{\ell^r(rt)}\right) - X_g^r\left(\frac{1}{r}\beta^r_{\ell^r(rt)-1}\right).$$

Therefore, we have

$$\langle g, \bar{\mu}_1^r(t)\rangle = \langle g, \bar{\mu}_1^r(0)\rangle 1_{\{\ell^r(rt)=0\}} +$$
$$\left(X_g^r\left(\frac{1}{r}\beta^r_{\ell^r(rt)}\right) - X_g^r\left(\frac{1}{r}\beta^r_{\ell^r(rt)-1}\right)\right) 1_{\{\ell^r(rt)>0\}} - \int_{\beta_{\ell^r(rt)/r}}^{t} \frac{\langle g', \bar{\mu}_1^r(s)\rangle}{\langle 1, \bar{\mu}_1^r(s)\rangle} ds.$$

The extension from $g \in \tilde{\mathcal{C}}$ to $g \in \mathcal{C}$ follows in the same way as in the proof of Property (3) on p. 855 in [9].

For $\bar{\mu}_2^r(\cdot)$, note that since

$$\mu_2(t) = \sum_{i=E(\beta_{\ell(t)})+1}^{E(t)} \delta_{v_i},$$

for any $g \in \mathcal{C}$, we can express

$$\langle g, \bar{\mu}_2^r(t) \rangle = \sum_{i=r\bar{E}^r\left(\frac{1}{r}\beta^r_{\ell^r(rt)}\right)+1}^{r\bar{E}^r(t)} g(v_i^r) = X_g^r(t) - X_g^r\left(\frac{1}{r}\beta^r_{\ell^r(rt)}\right).$$

for all $t \geq 0$. ■

Chapter 9

Convergence to fluid limit solutions

In this chapter we complete the proof of our limit theorem. Since $\{(\bar{\mu}_1^r(\cdot), \bar{\mu}_2^r(\cdot))\}$ is tight, it has a subsequence converging to some limit $(\theta_1(\cdot), \theta_2(\cdot))$, which we now show is almost surely a fluid model solution for α, ν, θ_0. By Theorem 4.0.1, the limit is thus unique and Theorem 3.3.2 is proved. To ease notation we index the subsequence by r. Let $T, \eta > 0$ be given and let $g \in \mathcal{C}$. We will show that $(\theta_1(\cdot), \theta_2(\cdot))$ satisfies Definition 3.2.1 for all $t \in [0, T]$ with probability at least $1 - \eta$. To that end, take $n = \lceil \frac{T}{\delta} \rceil$ where $P(W_0 > \delta) \geq 1 - \eta$. Then by Proposition 6.0.1,

$$\left\{ \left(\bar{\mu}^r(0), X_g^r(\cdot), \frac{1}{r} B_0^r, \frac{1}{r}\beta_1^r, \frac{1}{r} B_1^r, \frac{1}{r}\beta_2^r, \frac{1}{r} B_2^r, \cdots, \frac{1}{r}\beta_n^r, \frac{1}{r} B_n^r, \bar{\mu}_1^r(\cdot), \bar{\mu}_2^r(\cdot) \right) \right\}$$

are jointly tight. So by passing to a further subsequence if necessary,

$$\begin{aligned} V_n^r(\cdot) = \Big(& \bar{\mu}^r(0), X_g^r(\cdot), \frac{1}{r} B_0^r, \frac{1}{r}\beta_1^r, \frac{1}{r} B_1^r, \frac{1}{r}\beta_2^r, \frac{1}{r} B_2^r, \\ & \cdots, \frac{1}{r}\beta_n^r, \frac{1}{r} B_n^r, \bar{\mu}_1^r(\cdot), \bar{\mu}_2^r(\cdot) \Big) \\ & \Rightarrow (\theta_0, \alpha(\cdot)\langle g, \nu \rangle, W_0, 1W_0, W_0, 2W_0, W_0, \cdots, nW_0, W_0, \theta_2(\cdot), \theta_2(\cdot)) = V_n(\cdot). \end{aligned}$$

By the Skorohod representation theorem there exist

$$\tilde{V}_n^r(\cdot) = \left(\tilde{\bar{\mu}}^r(0), \tilde{X}_g^r(\cdot), \frac{1}{r}\tilde{B}_0^r, \frac{1}{r}\tilde{\beta}_1^r, \frac{1}{r}\tilde{B}_1^r, \frac{1}{r}\tilde{\beta}_2^r, \frac{1}{r}\tilde{B}_2^r, \cdots, \frac{1}{r}\tilde{\beta}_n^r, \frac{1}{r}\tilde{B}_n^r, \tilde{\bar{\mu}}_1^r(\cdot), \tilde{\bar{\mu}}_2^r(\cdot)\right)$$

such that $\tilde{V}_n^r(\cdot) \sim V_n^r(\cdot)$ and

$$\tilde{V}(\cdot) = (\tilde{\theta}(0), \alpha(\cdot)\langle g, \nu\rangle, \tilde{W}_0, 1\tilde{W}_0, \tilde{W}_0, 2\tilde{W}_0, \tilde{W}_0, \cdots, n\tilde{W}_0, \tilde{W}_0, \tilde{\theta}_1(\cdot), \tilde{\theta}_2(\cdot))$$

such that $\tilde{V}(\cdot) \sim V(\cdot)$, defined on a common probability space such that almost surely,

$$\left(\tilde{\bar{\mu}}^r(0), \tilde{X}_g^r(\cdot), \frac{1}{r}\tilde{B}_0^r, \frac{1}{r}\tilde{\beta}_1^r, \frac{1}{r}\tilde{B}_1^r, \frac{1}{r}\tilde{\beta}_2^r, \frac{1}{r}\tilde{B}_2^r, \cdots, \frac{1}{r}\tilde{\beta}_n^r, \frac{1}{r}\tilde{B}_n^r, \tilde{\bar{\mu}}_1^r(\cdot), \tilde{\bar{\mu}}_2^r(\cdot)\right)$$
$$\to (\tilde{\theta}_0, \alpha(\cdot)\langle g, \nu\rangle, \tilde{W}_0, 1\tilde{W}_0, \tilde{W}_0, 2\tilde{W}_0, \tilde{W}_0, \cdots, n\tilde{W}_0, \tilde{W}_0, \tilde{\theta}_1(\cdot), \tilde{\theta}_2(\cdot)).$$

Note that $\tilde{V}(\cdot)$ has the specified form in terms of a common random variable $\tilde{W}_0$ by Lemma 6.0.2. Let Ω denote the event where the above convergence holds.

Define $\tilde{\ell}^r(rt) = \max\{j : \tilde{\beta}_j^r \leq rt\}$. By Theorem 7.0.1, $P\left(\ell^r(rt) \leq \lceil \frac{T}{\delta} \rceil\right) \geq 1 - \eta$ for all r. This implies that on this event, for all $t \in [0, T]$

$$\frac{1}{r}\beta_{\ell^r(rt)}^r = \max\left\{\frac{1}{r}\beta_j^r : \frac{1}{r}\beta_j^r \leq t, j = 1, \cdots, n\right\} = \sup\{s \leq t : \bar{\mu}_1^r(s-) < \bar{\mu}_1^r(s)\}$$

and so $\frac{1}{r}\beta^r_{\ell^r(rt)}$ is a function of $V_n^r(\cdot)$ such that (8.0.1) and (8.0.2) hold. Consequently there is an event $\tilde{W}_\delta$ with $P(\tilde{W}_\delta) \geq 1-\eta$ on which for all $t \in [0,T]$

$$\frac{1}{r}\tilde{\beta}^r_{\tilde{\ell}^r(rt)} = \max_{j=1,\cdots,n}\left\{\frac{1}{r}\tilde{\beta}^r_j \,:\, \frac{1}{r}\tilde{\beta}^r_j \leq t\right\} = \sup\{s \leq t \,:\, \tilde{\tilde{\mu}}^r_1(s-) < \tilde{\tilde{\mu}}^r_1(s)\}$$

is the same function of $\tilde{V}_n^r(\cdot)$ and (8.0.1) and (8.0.2) hold for $\tilde{\tilde{\mu}}^r_1(\cdot)$. That is

$$\langle g, \tilde{\tilde{\mu}}^r_2(t)\rangle = \tilde{X}^r_g(t) - \tilde{X}^r_g\left(\frac{\tilde{\beta}^r_{\tilde{\ell}^r(rt)}}{r}\right), \tag{9.0.1}$$

and

$$\langle g, \tilde{\tilde{\mu}}^r_1(t)\rangle = \langle g, \tilde{\tilde{\mu}}^r_1(0)\rangle 1_{\{\tilde{\ell}^r(rt)=0\}} + \left(\tilde{X}^r_g\left(\frac{1}{r}\tilde{\beta}^r_{\tilde{\ell}^r(rt)}\right) - \tilde{X}^r_g\left(\frac{1}{r}\tilde{\beta}^r_{\tilde{\ell}^r(rt)-1}\right)\right) 1_{\{\tilde{\ell}^r(rt)>0\}} - \int_{\tilde{\beta}_{\tilde{\ell}^r(rt)/r}}^t \frac{\langle g', \tilde{\tilde{\mu}}^r_1(s)\rangle}{\langle 1, \tilde{\tilde{\mu}}^r_1(s)\rangle} ds.$$

It suffices to show that $(\tilde{\theta}_1(\cdot), \tilde{\theta}_2(\cdot))$ satisfy (3.2.1) and (3.2.2) on $\Omega \cap \tilde{W}_\delta$ for all $t \in [0,T]$.

Theorem 9.0.1. *On $\Omega \cap \tilde{W}_\delta$ for all $t \in [0,T]$*

$$\langle g, \tilde{\theta}_1(t)\rangle = \langle g, \tilde{\theta}_0\rangle \boldsymbol{1}_{\{\lfloor t/\tilde{W}_0\rfloor = 0\}} + \alpha\tilde{W}_0\langle g, \nu\rangle \boldsymbol{1}_{\{\lfloor t/\tilde{W}_0\rfloor > 0\}} - \int_{\lfloor t/\tilde{W}_0\rfloor \tilde{W}_0}^t \frac{\langle g', \tilde{\theta}_1(s)\rangle}{\langle 1, \tilde{\theta}_1(s)\rangle} ds \tag{9.0.2}$$

and

$$\langle g, \tilde{\theta}_2(t)\rangle = \alpha\left(t - \left\lfloor \frac{t}{\tilde{W}_0}\right\rfloor \tilde{W}_0\right)\langle g, \nu\rangle. \tag{9.0.3}$$

Proof Fix some $\omega \in \Omega \cap \tilde{W}_\delta$. Since $\tilde{\bar{\mu}}_1^r(\cdot) \to \tilde{\theta}_1(\cdot)$ in J_1, there exist $\lambda^r(\cdot) \to Id(\cdot)$ uniformly on compact sets such that $\tilde{\bar{\mu}}_1^r(\lambda^r(\cdot)) \to \tilde{\theta}_1(\cdot)$ uniformly on compact sets. We first restrict to times that are not integer multiples of $\tilde{W}_0$. For each r, and $t \in [0,T] \backslash \tilde{W}_0 \mathbb{Z}$,

$$\langle g, \tilde{\bar{\mu}}_1^r(\lambda^r(t))\rangle = \langle g, \tilde{\bar{\mu}}_1^r(0)\rangle 1_{\{\tilde{\ell}^r(r\lambda^r(t))=0\}} +$$
$$\left(\tilde{X}_g^r\left(\frac{1}{r}\tilde{\beta}^r_{\tilde{\ell}^r(r\lambda^r(t))}\right) - \tilde{X}_g^r\left(\frac{1}{r}\tilde{\beta}^r_{\tilde{\ell}^r(r\lambda^r(t))-1}\right)\right) 1_{\{\tilde{\ell}^r(r\lambda^r(t))>0\}} - \int_{\tilde{\beta}_{\tilde{\ell}^r(r\lambda^r(t))/r}}^{\lambda^r(t)} \frac{\langle g', \tilde{\bar{\mu}}_1^r(s)\rangle}{\langle 1, \tilde{\bar{\mu}}_1^r(s)\rangle} ds.$$

Take the limit of both sides as $r \to \infty$. The left hand side becomes

$$\lim_{r\to\infty} \langle g, \tilde{\bar{\mu}}_1^r(\lambda^r(t))\rangle = \langle g, \tilde{\theta}_1(t)\rangle,$$

and the right hand side converges to

$$\langle g, \tilde{\theta}_0\rangle 1_{\{\lfloor t/\tilde{W}_0\rfloor = 0\}} + \left(\alpha \left\lfloor \frac{t}{\tilde{W}_0}\right\rfloor \tilde{W}_0 \langle g, \nu\rangle - \alpha\left(\left\lfloor \frac{t}{\tilde{W}_0}\right\rfloor - 1\right)\tilde{W}_0\langle g,\nu\rangle\right) 1_{\{\lfloor t/\tilde{W}_0\rfloor > 0\}}$$
$$- \lim_{r\to\infty} \int \frac{\langle g', \tilde{\bar{\mu}}_1^r(s)\rangle}{\langle 1, \tilde{\bar{\mu}}_1^r(s)\rangle} 1_{\left\{\frac{\tilde{\beta}_{\ell^r(r\lambda^r(t))}}{r} \le s \le \lambda^r(t)\right\}} ds.$$

To see this, note that $\lambda^r(t) \to t$, so for sufficiently large r, since $t \in \tilde{W}_0\mathbb{Z}$,

$$\frac{1}{r}\tilde{\beta}^r_{\tilde{\ell}^r(r\lambda^r(t))} = \max\left\{\frac{1}{r}\tilde{\beta}_j^r \,:\, \frac{1}{r}\tilde{\beta}_j^r \le \lambda^r(t)\right\} = \max\left\{\frac{1}{r}\tilde{\beta}_j^r \,:\, \frac{1}{r}\tilde{\beta}_j^r \le t\right\}$$
$$\to \max\{j\tilde{W}_0 \,:\, j\tilde{W}_0 \le t\} = \left\lfloor \frac{t}{\tilde{W}_0}\right\rfloor \tilde{W}_0.$$

Notice

$$\left| \frac{\langle g', \tilde{\bar{\mu}}_1^r(s)\rangle}{\langle 1, \tilde{\bar{\mu}}_1^r(s)\rangle} 1_{\left\{\frac{\tilde{\beta^r}_{\ell^r(r\lambda^r(t))}}{r} \le s \le \lambda^r(t)\right\}} \right| \le \|g'\|_\infty \left| \frac{\langle 1, \tilde{\bar{\mu}}_1^r(s)\rangle}{\langle 1, \tilde{\bar{\mu}}_1^r(s)\rangle} 1_{\{0\le s\le T\}} \right| \in L^1(\mathbb{R}).$$

By the dominated convergence theorem we have

$$\begin{aligned}\lim_{r\to\infty} \int \frac{\langle g', \tilde{\bar{\mu}}_1^r(s)\rangle}{\langle 1, \tilde{\bar{\mu}}_1^r(s)\rangle} 1_{\left\{\frac{\tilde{\beta^r}_{\ell^r(r\lambda^r(t))}}{r} \le s \le \lambda^r(t)\right\}} ds \\ = \int \lim_{r\to\infty} \frac{\langle g', \tilde{\bar{\mu}}_1^r(s)\rangle}{\langle 1, \tilde{\bar{\mu}}_1^r(s)\rangle} 1_{\left\{\frac{\tilde{\beta^r}_{\ell^r(r\lambda^r(t))}}{r} \le s \le \lambda^r(t)\right\}} ds \\ = \int_{\lfloor t/\tilde{W}_0\rfloor \tilde{W}_0}^t \frac{\langle g', \tilde{\theta}_1(s)\rangle}{\langle 1, \tilde{\theta}_1(s)\rangle} ds.\end{aligned}$$

It follows that

$$\langle g, \tilde{\theta}_1(t)\rangle = \langle g, \tilde{\theta}_0\rangle \mathbf{1}_{\{\lfloor t/\tilde{W}_0\rfloor = 0\}} + \alpha \tilde{W}_0 \langle g, \nu\rangle \mathbf{1}_{\{\lfloor t/\tilde{W}_0\rfloor > 0\}} - \int_{\lfloor t/\tilde{W}_0\rfloor \tilde{W}_0}^t \frac{\langle g', \tilde{\theta}_1(s)\rangle}{\langle 1, \tilde{\theta}_1(s)\rangle} ds.$$

It remains to show (3.2.2) for the growing measure. Since $\tilde{\bar{\mu}}_2^r(\cdot) \to \tilde{\theta}_2(\cdot)$ in J_1, there exists $\lambda^r(\cdot) \to Id(\cdot)$ uniformly on compact sets such that $\tilde{\bar{\mu}}_2^r(\lambda^r(\cdot)) \to \tilde{\theta}_2(\cdot)$ uniformly on compact sets. For $t \in [0,T]\backslash \tilde{W}_0\mathbb{Z}$, and using (9.0.1) at the time $\lambda^r(t)$, we see that

$$\lim_{r\to\infty} \langle g, \tilde{\bar{\mu}}_2^r(\lambda^r(t))\rangle = \langle g, \tilde{\theta}_2(t)\rangle,$$

and for the right hand side,

$$\lim_{r\to\infty}\left(\tilde{X}^r_g(\lambda^r(t)) - \tilde{X}^r_g\left(\frac{\tilde{\beta}^r_{\tilde{\ell}^r(r\lambda^r(t))}}{r}\right)\right) = \alpha t\langle g,\nu\rangle - \alpha\left\lfloor\frac{t}{\tilde{W}_0}\right\rfloor\tilde{W}_0\langle g,\nu\rangle,$$

using the same reasoning as above to conclude that $\frac{1}{r}\tilde{\beta}^r_{\tilde{\ell}^r(r\lambda^r(t))} \to \left\lfloor\frac{t}{\tilde{W}_0}\right\rfloor\tilde{W}_0$.

Therefore we have

$$\langle g,\tilde{\theta}_2(t)\rangle = \alpha t\langle g,\nu\rangle - \alpha\left\lfloor\frac{t}{\tilde{W}_0}\right\rfloor\tilde{W}_0\langle g,\nu\rangle.$$

It remains to extend the result to all $t \in [0,T]$. But this follows immediately from the fact that $\tilde{\theta}_1(\cdot)$ and $\tilde{\theta}_2(\cdot)$ are elements of D and thus are right-continuous. Indeed if $t = j\tilde{W}_0$ then taking $t_n \downarrow t$ in (9.0.2) and (9.0.3) completes the proof. ■

Bibliography

[1] Avi-Itzhak, B., and Halfin, S. Response times in gated M/G/1 queues: The processo- sharing case. *Queueing Systems. 4* (1989), 263–279.

[2] Billingsley, P. *Convergence of Probability Measures.* John Wiley & Sons, Inc., 1999.

[3] Boxma, O. J., Kella, O., and Kosiński, K. M. Queue lengths and workoads in polling systems. *Operations Research Letters 39* (2011), 401–405.

[4] Bramson, M. State space collapse with application to heavy traffic limits for multiclass queueuing networks. *Queueing Systems: Theory and Applications 30* (1998), 89–148.

[5] Dai, J. G., Zhang, J., and Zwart, B. Law of large number limits of limited processor-sharing queues. *Mathematics of Operations Research 34* (2009), 937–970.

[6] Ethier, S. N., and Kurtz, T. G. *Markov Processes: Characterization and Convergence.* John Wiley & Sons, Inc., 1985.

[7] Fuhrmann, S., Levy, H., and Sidi, M. A queueing network with a single cyclically roving server. *Queueing Systems 11* (1992), 121–144.

[8] Grishechkin, S. GI/G/1 processor sharing queue in heavy traffic. *Adv. in Appl. Probab. 26*, 539–555.

[9] Gromoll, H. C., Puha, A. L., and Williams, R. J. The fluid limit of a heavily loaded processor sharing queue. *Ann. Appl. Probab. 12* (2002), 797–859.

[10] Jagerman, D. L., and Sengupta, B. A functional equation arising in a queue with a gating mechanism. *Probability in the Engineering and Information Sciences 3* (1989), 417–433.

[11] Rege, K. M., and Sengupta, B. A single server queue with gated processor-sharing discipline. *Queueing Systems 4* (1989), 249–261.

[12] Williams, R. J. Diffusion approximations for open multiclass queueing networks: Sufficient conditions involving state space collapse. *Queueing Systems: Theory and Applications 30* (1998), 27–88.