The G_r $(r<5)$ of n-Dimensional Space

and

Their Differential Invariants for $n = r = 3$.

INTRODUCTION.

According to Prof. Sophus Lie, the "theory of Differential Equations is the most important branch of modern mathematics." During the last century, this branch of mathematical science has been developed in a number of different directions, one of the most important of which is that based on the theory of transformation groups. As is well known, this whole method was originated by Lie in 1869–70, when he showed that most of the older theories of integration owe their origin to a common source and at the same time introduced new theories of integration, based on the theory of groups.

In order to apply Lie's method to the problem of integration, it is necessary to know what group, if any, a given differential equation admits of. In his "Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen" and various other publications, particularly in Vol. XXXII of the *Mathematische Annalen*, Lie established in complete detail all differential invariants of every group in two variables, and showed how to reduce as far as possible the problem of integrating the differential equations invariant under such groups. It would seem most desirable to do the same thing, as far as possible, for groups in x, y, z. This great problem has been solved only for a few special cases: by Lie, for example, for the group of Euclidean movements and a few other special groups (see "Continuierliche Gruppen," Kap. 22); by Tresse for the G_{10} of conform transformations (see *Comptes Rendus*, 1892, Tom. 114); by Dr. G. Noth in a Leipzig thesis on the differential invariants of a certain G_{10}. One object of the present paper is to begin the solution of this general problem in a systematic manner, by establishing the desired results for all G_3's in x, y, z.

The problem before us divides itself naturally into three parts :

I. The establishment of the normal forms of the G_r $(r < 5)$ in n-variables.

II. The establishment of the differential invariants of the G_3's in x, y, z: (i) when y and z are each functions of x; (ii) when z is a function of x and y.

III. Applications of, and remarks on, the results obtained.

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CHAPTER I.

The Normal Forms of the $G_r(r < 5)$.

In Vol. III of the " Theorie der Transformationsgruppen " by Lie and Engel, two methods of establishing these groups in x, y, z are indicated ; but the groups of this class — namely, those imprimitive ones which leave invariant one curve family

$$\phi(x, y, z) = \text{const.} \quad \text{and} \quad \psi(x, y, z) = \text{const.},$$

and at the same time at least one surface family

$$\Omega(\phi, \psi) = \text{const.}$$

— are exceedingly numerous, and, so far as the present writer is able to discover, have never been established and tabulated, as was done for the other classes. We shall make direct use of neither of the methods given by Lie in the reference mentioned above, but shall follow other methods frequently used by him in other connections for establishing the normal forms of groups. It is believed that the G_2's, given on pp. 6–7, and the tables of G_3's, G_4's, given on pp. 15–16 and pp. 29–35 of this paper are accurate and exhaustive so that they can be confidently referred to by any one desiring to use these groups.

§ 1. Normal Forms of the G_2's of Space.

In his " Theorie der Berührungstransformationen," Lie established in all essentials the normal forms of the G_2's in x, y, z. In deducing the G_2's of n-dimensional space, we shall follow the method used by him in the above mentioned case, and at the same time show that the four types obtained by Lie are exhaustive also for n-dimensional space. We shall use p_i to represent $\partial f / \partial x_i$.

If

$$U_k f \equiv \sum_{i=1}^{n} \xi_{k, i}(x_1, \cdots, x_n) p_i \qquad (k = 1, 2)$$

are two representative transformations of the group, it is well known

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that they can be assumed to satisfy one of the structures (Zusammensetzungen)

$$(U_1 U_2) = 0, \quad (U_1 U_2) = U_1.$$

Hence
$$U_1 f = 0, \quad U_2 f = 0$$

will in each case define a complete system of at most two members in the n-variables $x_1, \cdots, x_n$. Hence this complete system will have at least $n - 2$ common solutions, which are independent functions of the variables $x_1, \cdots, x_n$. If $\psi_3(x_1, \cdots, x_n), \cdots, \psi_n(x_1, \cdots, x_n)$ are the solutions of the complete system, we may without loss of generality assume the ψ_i to be independent functions with regard to the $n - 2$ variables $x_3, \cdots, x_n$; hence we may introduce as a new system of independent variables the $\bar{x}_j$ defined by

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = x_2, \quad \bar{x}_s = \psi_s(x_1, \cdots, x_n) \quad (s = 3, \cdots, n).$$

In these new variables, we have

$$\bar{U}_k f = \bar{\xi}_{k,1}(\bar{x}_1, \cdots, \bar{x}_n)\bar{p}_1 + \bar{\xi}_{k,2}(\bar{x}_1, \cdots, \bar{x}_n)\bar{p}_2 \quad (k = 1, 2),$$

where the $\bar{x}_3, \cdots, \bar{x}_n$ play the rôle of constants. One of these transformations $\bar{U}_1 f$ (say) can, by the introduction of new variables

$$\bar{\bar{x}}_1 = \phi_1(\bar{x}_1, \cdots, \bar{x}_n), \quad \bar{\bar{x}}_2 = \phi_2(\bar{x}_1, \cdots, \bar{x}_n), \quad \bar{\bar{x}}_3 = \bar{x}_3, \cdots, \bar{\bar{x}}_n = \bar{x}_n,$$

always be thrown into the form of a translation $\bar{p}_1$, or let us say simply p_1. Suppose that by the introduction of the new variables the other transformation becomes

$$U_2 f = \xi_1(x_1, \cdots, x_n)p_1 + \xi_2(x_1, \cdots, x_n)p_2.$$

Now there are two cases according as the two transformations have the same path-curves or not. We consider these in order.

I. When the path-curves are the same,

$$U_2 f = \rho(x_1, \cdots, x_n)p_1.$$

(i) $(U_1 U_2) = 0$: $\therefore \left(p_1, \rho(x_1, \cdots, x_n)p_1\right) \equiv \rho_{x_1} \cdot p_1 = 0,$

Hence ρ is free of x_1. Also ρ cannot be a constant, since then there would be only one transformation. As ρ must contain at least one

of the variables $x_2, \cdots, x_n$, suppose that it contains x_j. Then we may introduce ρ as a new $\bar{x}_j$, so that

$$\overline{U}_2 f \equiv \bar{x}_j p_1 \qquad\qquad (j > 1).$$

Hence for this case we have the group

$$\boxed{p_1, \ x_j p_1 \quad (j > 1)}.$$

(ii) $(U_1, U_2) = U_1$: $\quad \therefore \left(p_1, \rho(x_1, \cdots, x_n)p_1\right) \equiv \rho_{x_1} \cdot p_1 = p_1.$

Hence $\rho \equiv x_1 + \alpha(x_2, \cdots, x_n).$

Putting $\bar{x}_1 = x_1 + \alpha(x_2, \cdots, x_n)$, we have $\overline{U}_2 f = \bar{x}_1 \bar{p}_1$, while $U_1 f \equiv \bar{p}_1.$ Hence for this case we have the group

$$\boxed{p_1, \ x_1 p_1}.$$

II. When the path-curves of the transformations are different,

$$U_2 f \not\equiv \rho(x_1, \cdots, x_n)\, U_1 f.$$

For this case $U_1 f \equiv p_1$, $U_2 f \equiv \xi_1(x_1, \cdots, x_n)p_1 + \xi_2(x_1, \cdots, x_n)p_2.$

(i) $(U_1 U_2) = 0$: $\quad \therefore (p_1, \xi_1 p_1 + \xi_2 p_2) \equiv \xi_{1_{x_1}} p_1 + \xi_{2_{x_1}} p_2 = 0.$

Hence ξ_1 and ξ_2 are free of x_1, so that

$$U_2 f \equiv \alpha(x_2, \cdots, x_n)p_1 + \beta(x_2, \cdots, x_n)p_2 \qquad (\beta \neq 0).$$

Introducing as new variables

$$\bar{x}_1 = x_1 - \int \frac{\alpha}{\beta} dx_2, \quad \bar{x}_2 = \int \frac{dx_2}{\beta},$$

we have

$$\overline{U}_1 f = \bar{p}_1, \quad \overline{U}_2 f = \bar{p}_2.$$

Hence for this case, we obtain the group

$$\boxed{p_1, \ p_2}.$$

(ii) $(U_1 U_2) = U_1$: $\quad \therefore (p_1, \xi_1 p + \xi_2 p_2) \equiv \xi_{1_{x_1}} p_1 + \xi_{2_{x_1}} p_2 = p_1.$

Hence

$$\xi_1 \equiv x_1 + \alpha(x_2, \cdots, x_n), \quad \xi_2 \equiv \beta(x_2, \cdots, x_n) \qquad (\beta \neq 0).$$

Introducing as new variables

$$\bar{x}_1 = x_1 + \phi(x_2, \cdots, x_n), \quad \bar{x}_2 = \epsilon^{\int \frac{dx_2}{\beta}},$$

and choosing ϕ so as to make the arbitrary function in ξ_1 disappear, we obtain

$$\overline{U}_1 f = \bar{p}_1, \quad \overline{U}_2 f = \bar{x}_1 \bar{p}_1 + \bar{x}_2 \bar{p}_2.$$

Without changing the first transformation, the second can be thrown into the symmetrical form

$$U_2 f \equiv \sum_{i=1}^{n} x_i p_i$$

by introducing $x_1 = \bar{x}_1$, $x_2 = \bar{x}_2$, $x_j = \bar{x}_2 \bar{x}_j$ $(j = 3, \cdots, n)$. Hence for this case we obtain the group

$$\boxed{p_1, \; \sum_{i=1}^{n} x_i p_i}.$$

As all possible cases have been considered these four types must be exhaustive.

§ 2. *Normal Forms of the G_3's of Space.*

Lie has shown that the representative transformations $U_1 f$, $U_2 f$, $U_3 f$, of a G_3 in n variables can always be chosen so that the G_3 has one of the structures (Zusammensetzungen) I, $\cdots$, V, below (see "Theorie der Transformationsgruppen," Dritter Abschnitt, Kap. 28). Under each one of these structures we must consider the various cases which arise when the transformations satisfy two independent linear relations, one such relation, or no such relation. These will be taken up in order.

I. Let the transformations satisfy the structure

$$(U_1 U_2) \equiv U_1, \quad (U_1 U_3) \equiv 2U_2, \quad (U_2 U_3) \equiv U_3.$$

(1) Suppose there are two independent linear relations connecting the transformations, and let them be

$$\rho_1 U_1 + \rho_2 U_2 + \rho_3 U_3 = 0, \quad \sigma_1 U_1 + \sigma_2 U_2 + \sigma_3 U_3 = 0,$$

ρ_i and σ_i being functions of $x_1, \cdots, x_n$.

Solving these two equations for U_2 and U_3, we obtain

$$U_2 f \equiv \rho(x_1, \cdots, x_n) U_1 f, \quad U_3 f \equiv \sigma(x_1, \cdots, x_n) U_1 f.$$

Hence if we suppose, as can be done, that $U_1 f$ has been thrown into the form of a translation p_1, we have

$$U_2 f \equiv \rho(x_1, \cdots, x_n) p_1, \quad U_3 f \equiv \sigma(x_1, \cdots, x_n) p_1.$$

Since the path curves of U_1 and U_2, which form a G_2, are the same, we can, by the preceding section, put

$$U_2 f \equiv x_1 p_1.$$

Substituting in the structure, we find.

$$(U_1 U_3) \equiv (p_1, \sigma p_1) \equiv \sigma_{x_1} p_1 = 2 x_1 p_1, \quad \text{or} \quad \sigma_{x_1} = 2 x_1$$

$$(U_2 U_3) \equiv (x_1 p_1, \sigma p_1) \equiv 2 x_1^2 p_1 - \sigma p_1 = \sigma p_1,$$

whence

$$\sigma = x_1^2 \quad \text{and} \quad U_3 f \equiv x_1^2 p_1.$$

Hence for this case we have the group

$$\boxed{p_1, \ x_1 p_1, \ x_1^2 p_1} \ .$$

(2) Suppose there is only one linear relation connecting the transformations, namely

$$\rho_1 U_1 + \rho_2 U_2 + \rho_3 U_3 = 0,$$

ρ_i being functions of $x_1, \cdots, x_n$. Since the $U_i f$ do not occur symmetrically in the structure of the G_3, we shall further subdivide this case according as:

(i) $\rho_1 \equiv 0, \rho_2 \cdot \rho_3 \not\equiv 0$;   (ii) $\rho_2 \equiv 0, \rho_1 \cdot \rho_3 \not\equiv 0$;

(iii) $\rho_3 \equiv 0, \rho_1 \cdot \rho_2 \not\equiv 0$;   (iv) $\rho_1 \cdot \rho_2 \cdot \rho_3 \not\equiv 0$.

(i) $\rho_1 \equiv 0, \quad \rho_2 \cdot \rho_3 \not\equiv 0.$

Then $U_3 f \equiv \rho(x_1, \cdots, x_n) U_2 f$, while $U_1 f$ and $U_2 f$ are not connected by any linear relation, i. e.,

$$U_2 f \not\equiv \sigma(x_1, \cdots, x_n) U_1 f.$$

If $U_1 f \equiv p_1$, we must have by the preceding section

$$U_2 f \equiv x_1 p_1 + x_2 p_2.$$

Hence by the structure, we obtain

$$\big(p_1, \rho(x_1 p_1 + x_2 p_2)\big) \equiv (\rho + x_1 \rho_{x_1}) p_1 + x_2 \rho_{x_1} p_2 \equiv 2(x_1 p_1 + x_2 p_2).$$

From this relation

$$\rho + x_1 \rho_{x_1} = 2x_1, \quad x_2 \rho_{x_1} = 2x_2,$$

i. e., $\rho = 0$, consequently there is no G_3 for this case. In the same way it is shown that the subdivisions (ii) and (iii) above give no groups.

Let us consider the last of these, i. e.,

(iv) $$\rho_1 \cdot \rho_2 \cdot \rho_3 \not\equiv 0.$$

For this case we may take $U_1 f = p_1$, $U_2 f = x_1 p_1 + x_2 p_2$, since no linear relation connects any two of the transformations. Also we can take

$$U_3 f \equiv \rho U_1 f + \sigma U_2 f \qquad\qquad \rho \cdot \sigma \not\equiv 0,$$

ρ and σ being functions of $x_1, \cdots, x_n$.

Thus we have

$$U_3 f \equiv (\rho + x_1 \sigma) p_1 + x_2 \sigma p_2.$$

Substituting in the structure

$$\big(p_1, (\rho + x_1 \sigma) p_1 + x_2 \sigma p_2 \big) \equiv (\rho_{x_1} + x_1 \sigma_{x_1} + \sigma) p_1 + x_2 \sigma_{x_1} p_2 \equiv 2(x_1 p_1 + x_2 p_2),$$

$$\big(x_1 p_1 + x_2 p_2, (\rho + x_1 \sigma) p_1 + x_2 \sigma p_2 \big) \equiv 2x_1 (x_1 p_1 + x_2 p_2)$$
$$+ x_2 \{ (\rho_{x_2} + x_1 \sigma_{x_2}) p_1 + (x_2 \sigma_{x_2} + \sigma) p_2 \} - (\rho + x_1 \sigma) p_1 - x_2 \sigma p_2$$
$$= (\rho + x_1 \sigma) p_1 + x_2 \sigma p_2.$$

Equating coefficients in these two relations, we have

$$\rho_{x_1} + x_1 \sigma_{x_1} + \sigma = 2x_1, \qquad\qquad 2x_1 x_2 + x_2^2 \sigma_{x_2} = x_2 \sigma,$$

$$2x_1^2 + x_2 \rho_{x_2} + x_1 x_2 \sigma_{x_2} = 2\rho + 2x_1 \sigma, \qquad\qquad x_2 \sigma_{x_1} = 2x_2.$$

From these relations we obtain immediately

$$\sigma \equiv 2x_1 + x_2 \cdot \lambda, \quad \rho \equiv -x_1^2 - x_1 x_2 \lambda + x_2^2 \mu,$$

λ and μ being functions of $x_3, \cdots, x_n$.

Hence

$$U_3 f \equiv (x_1^2 + x_2^2 \mu) p_1 + (2x_1 x_2 + x_2^2 \lambda) p_2.$$

Introducing as new x_1, $\bar{x}_1 = x_1 + x_2 \phi(x_3, \cdots, x_n)$, and choosing ϕ so that $\phi^2 - \phi \lambda - \mu = 0$, we obtain

$$\overline{U}_3 f \equiv \overline{x}_1^2 \overline{p}_1 + \lceil 2\overline{x}_1 x_2 + x_2^2 \lambda_1(x_3, \cdots, x_n) \rceil p_2,$$

$$\lambda_1(x_3, \cdots, x_n) \equiv \lambda(x_3, \cdots, x_n) - 2\phi(x_3, \cdots, x_n).$$

Now if $\lambda_1 = 0$, then $U_3 f \equiv x_1^2 p_1 + 2x_1 x_2 p_2$, and we have the G_3

$$\boxed{p_1, \ x_1 p_1 + x_2 p_2, \ x_1^2 p_1 + 2x_1 x_2 p_2} .$$

If $\lambda_1 \neq 0$, put $\overline{x}_2 = x_2 \lambda_1$, and we have the G_3

$$\boxed{p_1, \ x_1 p_1 + x_2 p_2, \ x_1^2 p_1 + (2x_1 x_2 + x_2^2) p_2} .$$

(3) Suppose there is no linear relation connecting the transformations. Then we may take

$$U_1 f \equiv p_1, \quad U_2 f \equiv x_1 p_1 + x_2 p_2,$$

and as $U_i f = 0$ $(i = 1, 2, 3)$ form a complete system of three members in n-variables, we may assume the variables so chosen that $U_3 f$ has the form

$$U_3 f \equiv \xi_1(x_1, \cdots, x_n) p_1 + \xi_2(x_1, \cdots, x_n) p_2 + \xi_3(x_1, \cdots, x_n) p_3.$$

Substituting in the structure

$$(U_1 U_3) \equiv (p_1, \ \xi_1 p_1 + \xi_2 p_2 + \xi_3 p_3) \equiv \xi_{1_{x_1}} p_1 + \xi_{2_{x_1}} p_2 + \xi_{3_{x_1}} p_3$$

$$= 2(x_1 p_1 + x_2 p_2).$$

Whence

$$\xi_1 \equiv x_1^2 + \alpha(x_2, \cdots, x_n), \quad \xi_2 \equiv 2x_1 x_2 + \beta(x_2, \cdots, x_n),$$

$$\xi_3 \equiv \gamma(x_2, \cdots, x_n) \qquad\qquad (\gamma \neq 0).$$

So that

$$U_3 f \equiv (x_1^2 + \alpha) p_1 + (2x_1 x_2 + \beta) p_2 + \gamma \cdot p_3.$$

Next we have

$$(U_2 U_3) \equiv x_1(2x_1 p_1 + 2x_2 p_2) + x_2 \{ \alpha_{x_2} p_1 + (2x_1 + \beta_{x_2}) p_2 + \gamma_{x_2} p_3 \}$$

$$- (x_1^2 + \alpha) p_1 - (2x_1 x_2 + \beta) p_2 = (x_1^2 + \alpha) p_1 + (2x_1 x_2 + \beta) p_2 + \gamma p_3.$$

From this relation

$$2x_1^2 + x_2 \cdot \alpha_{x_2} = 2x_1^2 + 2\alpha,$$

$$2x_1 x_2 + 2x_1 x_2 + x_2 \beta_{x_2} = 4x_1 x_2 + 2\beta,$$

$$x_2 \gamma_{x_2} = \gamma,$$

so that

$$\alpha \equiv x_2^2 \lambda(x_3, \cdots, x_n), \quad \beta \equiv x_2^2 \mu(x_3, \cdots, x_n), \quad \gamma \equiv x_2 \nu(x_3, \cdots, x_n),$$

and

$$U_3 f \equiv (x_1^2 + x_2^2 \lambda) p_1 + (2x_1 x_2 + x_2^2 \mu) p_2 + x_2 \nu p_3 \quad (\nu \neq 0).$$

Introduce $\bar{x}_3 = \int dx_3 / \nu$ and denote the new arbitrary functions in the coefficients of p_1, p_2, by λ_1, μ_1, respectively.   Then

$$\overline{U}_3 f \equiv (x_1^2 + x_2^2 \lambda_1) p_1 + (2x_1 x_2 + x_2^2 \mu_1) p_2 + x_2 \bar{p}_3.$$

Introduce again

$$\bar{x}_1 = x_1 + x_2 \phi(x_3, \cdots, x_n),$$

and choose ϕ to satisfy

$$\lambda_1 + \mu_1 \phi - \phi^2 + \phi_{x_3} = 0.$$

Then

$$U_3 f \equiv \bar{x}_1^2 \bar{p}_1 + [2\bar{x}_1 x_2 + x_2^2 \rho(x_3, \cdots, x_n)] p_2 + x_2 p_3,$$

where, for brevity, we substitute

$$\rho(x_3, \cdots, x_n) \equiv \mu_1 - 2\phi.$$

Now introduce

$$\bar{x}_2 = x_2 \psi(x_3, \cdots, x_n), \quad \bar{x}_3 = \int \psi dx_3,$$

in which ψ must be different from zero.   Leaving off the bars, we have finally

$$U_3 f \equiv x_1^2 p_1 + 2x_1 x_2 p_2 + x_2 p_3,$$

and thus the G_3

$$\boxed{p_1, \quad x_1 p_1 + x_2 p_2, \quad x_1^2 p_1 + 2x_1 x_2 p_2 + x_2 p_3}.$$

II. Let the three transformations satisfy the structure

$$(U_1 U_2) = 0, \quad (U_1 U_3) = U_1, \quad (U_2 U_3) = cU_2.$$

(1) If two linear independent relations exist among the transformations, then as in the preceding case we have

$$U_2 f \equiv \rho(x_1, \cdots, x_n) p_1, \quad U_3 f \equiv \sigma(x_1, \cdots, x_n) p_1,$$

if, as usual, we assume that $U_1 f \equiv p_1$.

Since the path curves for U_1f and U_2f are the same, we must by the preceding section take

$$U_2f = x_kp_1 \qquad\qquad (k > 1).$$

Substituting in the structure

$$(p_1, \sigma p_1) \equiv \sigma_{x_1}p_1 \equiv p_1, \quad (x_kp_1, \sigma p_1) \equiv x_kp_1 \equiv cx_kp_1.$$

Hence for this case to exist $c = 1$ and

$$\sigma \equiv x_1 + \alpha(x_2, \cdots, x_n).$$

Introducing $\bar{x}_1 = x_1 + \alpha$, we have $U_3f \equiv x_1p_1$. Hence the G_3

$$\boxed{p_1, \ x_kp_1, \ x_1p_1 \qquad (k > 1)}.$$

(2) If one linear relation exists, we have four cases as under I, page 8.

(i) $$\rho_1 = 0, \quad \rho_2 \cdot \rho_3 \neq 0,$$
so that

$$U_3f \equiv \rho(x_1, \cdots, x_n)U_2f, \quad U_2f \neq \sigma(x_1, \cdots, x_n)U_1f.$$

Hence for $(U_1 U_2) = 0$, we must by the preceding section have

$$U_1f \equiv p_1, \qquad U_2f \equiv p_2, \qquad U_3f \equiv \rho p_2.$$

Evidently this case gives a contradiction under the above structure, so that there is no G_3 for it.

(ii) $$\rho_2 = 0, \quad \rho_1 \cdot \rho_3 \neq 0.$$
In this case

$$U_3f \equiv \rho(x_1, \cdots, x_n)U_1f, \quad U_2f \not\equiv \sigma(x_1, \cdots, x_n)U_1f.$$

Hence by the preceding section we must take

$$U_1f \equiv p_1, \quad U_2f \equiv p_2, \quad U_3f \equiv \rho(x_1, \cdots, x_n)p_1.$$

Applying the structure we see that this case holds only when $c = 0$. Then $\rho_{x_1} = 1, \rho_{x_2} = 0$. Whence $\rho \equiv x_1 + \alpha(x_3, \cdots, x_n)$. Putting $x_1 \equiv x_1 + \alpha$, we have $U_3f \equiv \bar{x}_1 p_1$ and thus the G_3

$$\boxed{p_1, \ p_2, \ x_1p_1}.$$

(iii) $$\rho_3 = 0, \quad \rho_1 \cdot \rho_2 \neq 0.$$

Hence $U_2 f \equiv \rho(x_1, \cdots, x_n) U_1 f$, and by the preceding section, we have

$$U_1 f \equiv p_1, \quad U_2 f \equiv x_2 p_1.$$

It remains to determine $U_3 f$, for which there will be two cases and which according to hypothesis is not linearly connected with $U_1 f$ and $U_2 f$. Now $U_i f = 0 \ (i = 1, 2, 3)$ form a complete system, of which, since two of the equations are independent, there are $n - 2$ independent solutions. Now we make two cases according as ξ_2 in

$$U_3 f \equiv \sum_{i=1}^{n} \xi_i p_i$$

(a) is not zero; or (b) is zero.

(a) $\xi_2(x_1, \cdots, x_n) \neq 0$.

x_1 and x_2 are not solutions of the complete system and can therefore be introduced as new variables. Introducing also $\bar{x}_j = \psi_j(x_1, \cdots, x_n)$, $(j = 3, \cdots, n)$ where the ψ_j are the $n - 2$ independent solutions, we have

$$\overline{U}_3 f \equiv \bar{\xi}_1(\bar{x}_1, \cdots, \bar{x}_n) \bar{p}_1 + \bar{\xi}_2(\bar{x}_1, \cdots, \bar{x}_n) \bar{p}_2.$$

Leaving off the bars and applying the structure, we have immediately $\xi_1 \equiv x_1 + \alpha(x_2, \cdots, x_n)$, $\xi_2 \equiv (1 - c) x_2$. Introducing again $\bar{x}_1 = x_1 + \phi(x_2, \cdots, x_n)$ and choosing ϕ so that

$$\alpha - \phi + (1 - c) x_2 \phi_{x_2} = 0,$$

we get

$$\overline{U}_3 f \equiv \bar{x}_1 \bar{p} + (1 - c) x_2 p_2 \qquad\qquad (c \neq 1)$$

Thus we have the group

$$\boxed{p_1, \ x_2 p_1, \ x_1 p_1 + (1 - c) x_2 p_2}.$$

(b) $\xi_2(x_1, \cdots, x_n) = 0$.

In this case x_2 will be a solution of the system, but at least one of the variables, x_3 (say), will not be a solution. We can introduce as new variables $\bar{x}_1 = x_1$, $\bar{x}_3 = x_3$ and the $n - 2$ solutions

$$\bar{x}_2 = x_2, \quad \bar{x}_j = \psi_j(x_1, \cdots, x_n) \qquad (j = 4, \cdots, n).$$

Then

$$\overline{U}_3 f \equiv \bar{\xi}_1(\bar{x}_1, \cdots, \bar{x}_n) \bar{p}_1 + \bar{\xi}_3(\bar{x}_1, \cdots, \bar{x}_n) \bar{p}_3.$$

Leaving off the bars, and applying the structure we find

$$\bar{x}_1 = x_1 + \alpha(x_2, \cdots, x_n), \quad \bar{x}_3 = \beta(x_2, \cdots, x_n).$$

Introducing again

$$\bar{x}_1 = x_1 + \phi(x_2, \cdots, x_n), \quad \bar{x}_3 = \int \frac{dx_3}{\beta},$$

and determining ϕ from $\alpha - \phi + \beta \cdot \phi_{x_3} = 0$, we have

$$U_3 f \equiv x_1 p_1 + p_3.$$

Thus we get the G_3

$$\boxed{p_1, \ x_2 p_1, \ x_1 p_1 + p_3}.$$

(iv) $$\rho_1 \cdot \rho_2 \cdot \rho_3 \neq 0.$$

   By § 1, we must have in this case,

$$U_1 f \equiv p_1, \quad U_2 f \equiv p_2.$$

Then

$$U_3 f \equiv \rho(x_1, \cdots, x_n) p_1 + \sigma(x_1, \cdots, x_n) p_2.$$

The structure gives

$$(p_1, \rho p_1 + \sigma p_2) \equiv \rho_{x_1} p_1 + \sigma_{x_1} p_2 = p_1,$$

$$(p_2, \rho p_1 + \sigma p_2) \equiv \rho_{x_2} p_1 + \sigma_{x_2} p_2 = c p_2,$$

whence

$$\rho \equiv x_1 + \alpha(x_3, \cdots, x_n), \quad \sigma \equiv c x_2 + \beta(x_3, \cdots, x_n).$$

If $c \neq 0$, we can put

$$\bar{x}_1 = x_1 + \alpha, \ \bar{x}_2 = x_2 + \frac{\beta}{c}.$$

Then $\bar{U}_3 f \equiv x_1 p_1 + c x_2 p_2$ and we have the G_3

$$\boxed{p_1, \ p_2, \ x_1 p_1 + c x_2 p_2 \quad (c \neq 0)}.$$

   If $c = 0$, put $\bar{x}_3 \equiv \beta(x_3, \cdots, x_n)$ and we have the G_3

$$\boxed{p_1, \ p_2, \ x_1 p_1 + x_3 p_2}.$$

   (3) Finally let us suppose that no linear relation exists among the transformations. Then by § 1, we must have

$$U_1 f \equiv p_1, \quad U_2 f \equiv p_2.$$

Now, since $U_i f = 0$ form a complete system of three independent

equations, there are $n - 3$ common solutions $\psi_4(x_1, \cdots, x_n), \cdots,$ $\psi_n(x_1, \cdots, x_n)$, (say).

Introducing as new variables

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = x_2, \quad \bar{x}_3 = x_3, \quad \bar{x}_j = \psi_j(x_1, \cdots, x_n) \quad (j = 4, \cdots, n),$$

we have $\overline{U}_3 f \equiv \xi_1(\bar{x}_1, \cdots, \bar{x}_n)\bar{p}_1 + \xi_2(\bar{x}_1, \cdots, \bar{x}_n)\bar{p}_2 + \xi_3(\bar{x}_1, \cdots, \bar{x}_n)\bar{p}_3.$ By the structure

$$(p_1, \xi_1 p_1 + \xi_2 p_2 + \xi_3 p_3) \equiv \xi_{1_{x_1}} p_1 + \xi_{2_{x_1}} p_2 + \xi_{3_{x_1}} p_3 = p_1,$$

$$(p_2, \xi_1 p_1 + \xi_2 p_2 + \xi_3 p_3) \equiv \xi_{1_{x_2}} p_1 + \xi_{2_{x_2}} p_2 + \xi_{3_{x_2}} p_3 = c p_2,$$

whence

$$\xi_1 = x_1 + \alpha(x_3, \cdots, x_n), \quad \xi_2 = c x_2 + \beta(x_3, \cdots, x_n), \quad \xi_3 = \gamma(x_3, \cdots, x_n).$$

By hypothesis γ cannot be zero. Introducing

$$\bar{x}_1 = x_1 + \phi(x_3, \cdots, x_n), \quad \bar{x}_2 = x_2 + \psi(x_3, \cdots, x_n), \quad \bar{x}_3 = \int \frac{dx_3}{\gamma},$$

and choosing ϕ and ψ so as to make the functions in ξ_1 and ξ_2 disappear, we have

$$\overline{U}_3 f \equiv x_1 p_1 + c x_2 p_2 + p_3,$$

and thus the group

$$\boxed{p_1, \quad p_2, \quad x_1 p_1 + c x_2 p_2 + p_3}.$$

The treatment for the remaining structures is quite similar to the preceding; we shall therefore not give the calculations in detail, but shall, in concluding this section, give the results in a complete table of all G_3's classified according to the various structures.

I. $\quad (U_1 U_2) = U_1, \quad (U_1 U_3) = 2 U_2, \quad (U_2 U_3) = U_3.$

$$\boxed{p_1, \ x_1 p_1, \ x_1^2 p_1} \qquad \boxed{p_1, \ x_1 p_1 + x_2 p_2, \ x_1^2 p_1 + 2 x_1 x_2 p_2 + x_2 p_3}$$

$$\boxed{p_1, \ x_1 p_1 + x_2 p_2, \ x_1^2 p_1 + 2 x_1 x_2 p_2}$$

$$\boxed{p_1, \ x_1 p_1 + x_2 p_2, \ x_1^2 p_1 + (2 x_1 x_2 + x_2^2) p_2}$$

II. $\quad (U_1 U_2) = 0, \quad (U_1 U_3) = U_1, \quad (U_2 U_3) = c U_2.$

$$\boxed{p_1, \; x_k p_1, \; x_1 p_1} \qquad \boxed{p_1, \; p_2, \; x_1 p_1} \qquad \boxed{p_1, \; x_2 p_1, \; x_1 p_1 + (1-c) x_2 p_2}$$

$$\boxed{p_1, \; x_2 p_1, \; x_1 p_1 + p_3} \qquad \boxed{p_1, \; p_2, \; x_1 p_1 + c x_2 p_2} \qquad \boxed{p_1, \; p_2, \; x_1 p_1 + x_3 p_2}$$

$$\boxed{p_1, \; p_2, \; x_1 p_1 + c x_2 p_2 + p_3}$$

III. $\quad (U_1 U_2) = 0, \quad (U_1 U_3) = U_1, \quad (U_2 U_3) = U_1 + U_2.$

$$\boxed{p_1, \; x_2 p_1, \; x_1 p_1 - p_2} \qquad \boxed{p_1, \; p_2, \; (x_1 + x_2) p_1 + x_2 p_2}$$

$$\boxed{p_1, \; p_2, \; (x_1 + x_2) p_1 + x_2 p_2 + p_3}$$

IV. $\quad (U_1 U_2) = 0, \quad (U_1 U_3) = 0, \quad (U_2 U_3) = U_1.$

$$\boxed{p_1, p_2, x_2 p_1} \qquad \boxed{p_1, p_2, x_2 p_1 + x_j p_2 \quad (j > 2)} \qquad \boxed{p_1, \; p_2, \; x_2 p_1 + p_3}$$

V. $\quad (U_1 U_2) = 0, \quad (U_1 U_3) = 0, \quad (U_2 U_3) = 0.$

$$\boxed{p_1, \; p_2, \; p_3} \qquad \boxed{p_1, \; p_2, \; x_3 p_1 + F(x_3, \cdots, x_n) p_2}$$

$$\boxed{p_1, \; x_2 p_1, \; F(x_2, \cdots, x_n) p_1} \qquad \boxed{p_1, \; p_2, \; x_j p_2 \quad (j > 2)}$$

§ 3. *Normal Forms of the G_4's of Space.*

It is known (see Lie's "Theorie der Transformationsgruppen," Dritter Abschnitt, Kap. 28) that the G_4's of n-dimensional space satisfy one of the structures I, $\cdots$, XII below. In each case, the G_4 contains a sub-G_3 for which we can assume one of the normal forms established in the preceding section. In the following the fourth transformation of the G_4 is determined from the fact that it must either be connected linearly with the transformations of the sub-G_3, or not so connected.

I. Suppose the transformations of the G_4 satisfy the structure

$$(U_1 U_2) = U_1, \quad (U_1 U_3) = 2 U_2, \quad (U_2 U_3) = U_3, \quad (U_i U_4) = 0$$
$$(i = 1, 2, 3).$$

For the sub-G_3 U_1, U_2, U_3 of this structure, we have found in the preceding section the following cases:

(i) $\qquad\qquad\qquad \boxed{p_1, \; x_1 p_1, \; x_1^2 p_1}$

(ii) $\boxed{p_1, \; x_1 p_1 + x_2 p_2, \; x_1^2 p_1 + 2 x_1 x_2 p_2}$

(iii) $\boxed{p_1, \; x_1 p_1 + x_2 p_2, \; x_1^2 p_1 + (2 x_1 x_2 + x_2^2) p_2}$

(iv) $\boxed{p_1, \; x_1 p_1 + x_2 p_2, \; x_1^2 p_1 + 2 x_1 x_2 p_2 + x_2 p_3}$.

Further, we have for each of these cases the subdivisions

(1) $\qquad U_4 f \equiv \sum_{i=1}^{3} \rho_i U_i f;$ or (2) $U_4 f \not\equiv \sum_{i=1}^{3} \rho_i U_i f,$

the ρ_i being functions of $x_1, \cdots, x_n$. We shall proceed to consider the cases and sub-cases for this structure in order.

(i) $\qquad U_1 f = p_1, \quad U_2 f = x_1 p_1, \quad U_3 f = x_1^2 p_1.$

(1) For this case

$$U_4 f \equiv \sum_{i=1}^{3} \rho_i U_i f \equiv \xi(x_1, \cdots, x_n) p_1.$$

The structure shows that $\xi = 0$, hence there is no G_4.

(2) $\qquad U_4 f \not\equiv \sum_{i=1}^{3} \rho_i U_i f \equiv \sum_{i=1}^{n} \xi_i(x_1, \cdots, x_n) p_i.$

$U_k f = 0$ $(k = 1, \cdots, 4)$ form a complete system of two members; hence there will be $n - 2$ solutions independent of each other and of x_1. Since $U_4 f$ is not connected linearly with $U_1 f, \cdots, U_3 f$, at least one other variable, x_2 (say), is not a solution. Suppose the common solutions are $\psi_3(x_1, \cdots, x_n), \cdots, \psi_n(x_1, \cdots, x_n)$. We introduce as new variables

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = x_2, \quad \bar{x}_j = \psi_j(x_1, \cdots, x_n) \quad (j = 3, \cdots, n),$$

and obtain

$$\bar{U}_1 f = \bar{p}_1, \quad \bar{U}_2 f = \bar{x}_1 \bar{p}_1, \quad \bar{U}_3 f = \bar{x}_1^2 \bar{p}_1, \quad \bar{U}_4 f = \bar{\xi}_1 \bar{p}_1 + \bar{\xi}_2 \bar{p}_2,$$

$\bar{\xi}_1$ and $\bar{\xi}_2$ being functions of $\bar{x}_1, \cdots, \bar{x}_n$. Leaving off the bars and applying the structure we find

$$\xi_1 \equiv 0, \quad \xi_2 \equiv \beta(x_2, \cdots, x_n) \not\equiv 0.$$

Putting

$$\bar{x}_2 = \int \frac{dx_2}{\beta},$$

we have $\overline{U}_4 f \equiv \overline{p}_2$, and thus the G_4

$$\boxed{p_1, \; x_1 p_1, \; x_1^2 p_1, \; p_2}.$$

(ii) $U_1 f = p_1, \quad U_2 f = x_1 p_1 + x_2 p_2, \quad U_3 f = x_1^2 p_1 + 2 x_1 x_2 p_2.$

(1) $U_4 f \equiv \sum\limits_{i=1}^{3} \rho_i \, U_i f \equiv \xi_1(x_1, \cdots, x_n) p_1 + \xi_2(x_1, \cdots, x_n) p_2.$

By the structure

$$(U_1 U_4) \equiv \xi_{1_{x_1}} p_1 + \xi_{2_{\overline{x_1}}} p_2 = 0,$$

$$(U_2 U_4) \equiv x_2(\xi_{1_{x_2}} p_1 + \xi_{2_{x_2}} p_2) - \xi_1 p_1 - \xi_2 p_2 = 0,$$

$$(U_3 U_4) \equiv 2 x_1(\xi_1 p_1 + \xi_2 p_2) - 2 \xi_1 (x_1 p_1 + x_2 p_2) - 2 x_1 \xi_2 p_2 = 0.$$

Whence

$$\xi_1 = 0, \quad \xi_2 = x_2 \beta(x_3, \cdots, x_n) \neq 0.$$

Putting $\overline{x}_3 = \beta(x_3, \cdots, x_n) \neq$ const., we have the G_4

$$\boxed{p_1, \; x_1 p_1 + x_2 p_2, \; x_1^2 p_1 + 2 x_1 x_2 p_2, \; x_2 x_3 p_2}.$$

If $\beta = $ const., the G_4 has the form

$$\boxed{p_1, \; x_1 p_1 + x_2 p_2, \; x_1^2 p_1 + 2 x_1 x_2 p_2, \; x_2 p_2}.$$

(2) $$U_4 f \neq \sum\limits_{i=1}^{3} U_i f.$$

As under (i), we can introduce new variables so that

$$U_4 f \equiv \xi_1(x_1, \cdots, x_n) p_1 + \xi_2(x_1, \cdots, x_n) p_2 + \xi_3(x_1, \cdots, x_n) p_3.$$

By the structure

$$\xi_1 \equiv 0, \quad \xi_2 \equiv x_2 \beta(x_3, \cdots, x_n), \quad \xi_3 \equiv \gamma(x_3, \cdots, x_n) \neq 0.$$

Introducing

$$\overline{x}_2 = x_2 \phi(x_3, \cdots, x_n), \quad \overline{x}_3 = \int \frac{dx_3}{\gamma},$$

and choosing ϕ so that $\beta\phi + \gamma\phi_{x_3} = 0$, we have

$$\overline{U}_4 f = \overline{p}_3.$$

Thus for this case we get the group

$$\boxed{p_1, \; x_1 p_1 + x_2 p_2, \; x_1^2 p_1 + 2 x_1 x_2 p_2, \; p_3}.$$

(iii) $U_1 f = p_1,$ $U_2 f = x_1 p_1 + x_2 p_2,$ $U_3 f = x_1^2 p_1 + (2x_1 x_2 + x_2^2) p_2.$

(1) $U_4 f \equiv \sum_{i=1}^{3} \rho_i U_i f \equiv \xi_1(x_1, \cdots, x_n) p_1 + \xi_2(x_1, \cdots, x_n) p_2.$

The structure gives the conditions

$(U_1 U_4) \equiv \xi_{1_{x_1}} p_1 + \xi_{2_{x_1}} p_2 = 0,$

$(U_2 U_4) \equiv x_2 (\xi_{1_{x_2}} p_1 + \xi_{2_{x_2}} p_2) - \xi_1 p_1 - \xi_2 p_2 = 0,$

$(U_3 U_4) \equiv (2x_1 + x_2)(\xi_1 p_1 + \xi_2 p_2) - 2\xi_1(x_1 p_1 + x_2 p_2)$
$$- 2x_1 \xi_2 p_2 - 2x_2 \xi_2 p_2 = 0.$$

From the third relation $\xi_1 = \xi_2 = 0$; hence there is no G_4 for this case.

(2) $$U_4 f \not\equiv \sum_{i=1}^{3} \rho_i U_i f.$$

As before, we can introduce new variables so that

$U_4 f \equiv \xi_1(x_1, \cdots, x_n) p_1 + \xi_2(x_1, \cdots, x_n) p_2 + \xi_3(x_1, \cdots, x_n) p_3.$

Hence by the structure,

$(U_1 U_4) \equiv \xi_{1_{x_1}} p_1 + \xi_{2_{x_1}} p_2 + \xi_{3_{x_1}} p_3 = 0,$

$(U_2 U_4) \equiv x_2 (\xi_{1_{x_2}} p_1 + \xi_{2_{x_2}} p_2 + \xi_{3_{x_2}} p_3) - \xi_1 p_1 - \xi_2 p_2 = 0,$

$(U_3 U_4) \equiv (2x_1 + x_2)(\xi_1 p_1 + \xi_2 p_2) - 2\xi_1(x_1 p_1 + x_2 p_2) - 2(x_1 + x_2)\xi_2 p_2 = 0.$

From these relations

$$\xi_1 = \xi_2 = 0, \quad \xi_3 \equiv \gamma(x_3, \cdots, x_n) \neq 0.$$

Introducing

$$\bar{x}_3 = \int \frac{dx_3}{\gamma},$$

we have $\overline{U}_4 f \equiv \bar{p}_3.$

Hence we find the G_4:

$$\boxed{p_1, \; x_1 p_1 + x_2 p_2, \; x_1^2 p_1 + (2x_1 x_2 + x_2^2) p_2, \; p_3}.$$

(iv) $U_1 f = p_1,$ $U_2 f = x_1 p_1 + x_2 p_2,$ $U_3 f = x_1^2 p_1 + 2x_1 x_2 p_2 + x_2 p_3.$

(1) $U_4 f \equiv \sum_{i=1}^{3} \rho_i U_i f \equiv \sum_{i=1}^{3} \xi_i(x_1, \cdots, x_n) p_i.$

The structure gives the conditions

$$(U_1 U_4) \equiv \xi_{1_{x_1}} p_1 + \xi_{2_{x_1}} p_2 + \xi_{3_{x_1}} p_3 = 0,$$

$$(U_2 U_4) \equiv x_2(\xi_{1_{x_2}} p_1 + \xi_{2_{x_2}} \cdot p_2 + \xi_{3_{x_2}} \cdot p_3) - \xi_1 p_1 - \xi_2 p_2 = 0,$$

$$(U_3 U_4) \equiv 2x_1(\xi_1 p_1 + \xi_2 p_2) + x_2(\xi_{1_{x_3}} p_1 + \xi_{2_{x_3}} p_2 + \xi_{3_{x_3}} p_3)$$
$$- 2\xi_1(x_1 p_1 + x_2 p_2) - \xi_2(2x_1 p_2 + p_3) = 0.$$

Whence

$$\xi_1 \equiv x_2 \alpha(x_4, \cdots, x_n), \quad \xi_2 \equiv 2x_2 x_3 \alpha(x_4, \cdots, x_n) + x_2 \beta(x_4, \cdots, x_n),$$

$$\xi_3 \equiv x_3^2 \alpha(x_4, \cdots, x_n) + x_3 \beta(x_4, \cdots, x_n) + \gamma(x_4, \cdots, x_n),$$

so that

$$U_4 f \equiv x_2 \alpha p_1 + (2x_2 x_3 \alpha + \beta) p_2 + (x_3^2 \alpha + x_3 \beta + \gamma) p_3.$$

(a) $\alpha(x_4, \cdots, x_n) \neq 0.$

Introduce

$$\bar{x}_2 = x_2 \alpha(x_4, \cdots, x_n), \quad \bar{x}_3 = x_3 \alpha(x_4, \cdots, x_n).$$

Then (omitting the bars)

$$U_4 f \equiv x_2 p_1 + 2x_2 \left(x_3 + \frac{\beta}{2}\right) p_2 + (x_3^2 + x_3 \beta + \alpha \gamma) p_3.$$

Introduce again

$$\bar{x}_3 = x_3 + \frac{\beta}{2}, \quad \bar{x}_4 = \alpha \cdot \gamma - \frac{\beta^2}{4} \neq \text{const.}$$

Hence

$$\boxed{p_1, \; x_1 p_1 + x_2 p_2, \; x_1^2 p_1 + 2x_1 x_2 p_2 + x_2 p_3, \; x_2 p_1 + 2x_2 x_3 p_2 + (x_3^2 + x_4) p_3}.$$

If $\alpha\gamma - \beta^2/4 = \text{const.}$, we have

$$\boxed{p_1, \; x_1 p_1 + x_2 p_2, \; x_1^2 p_1 + 2x_1 x_2 p_2 + x_2 p_3, \; x_2 p_1 + 2x_2 x_3 p_2 + x_3^2 p_3}.$$

(b) $\alpha(x_4, \cdots, x_n) = 0, \quad \beta(x_4, \cdots, x_n) \neq 0.$

Introduce

$$\bar{x}_2 = x_2 \beta, \quad \bar{x}_3 = x_3 \beta + \gamma, \quad \bar{x}_4 = \beta \neq \text{const.}$$

Then

$$\bar{U}_4 f \equiv \bar{x}_2 \bar{x}_4 \bar{p}_2 + \bar{x}_3 \bar{x}_4 \bar{p}_3,$$

and we get the G_4

$$\boxed{p_1, \; x_1 p_1 + x_2 p_2, \; x_1^2 p_1 + 2x_1 x_2 p_2 + x_2 p_3, \; x_4(x_2 p_2 + x_3 p_3)}.$$

If $\beta = $ const. we have

$$\boxed{p_1, \quad x_1 p_1 + x_2 p_2, \quad x_1^2 p_1 + 2x_1 x_2 p_2 + x_2 p_3, \quad x_2 p_2 + x_3 p_3}.$$

(c) If β is also zero, $\gamma \neq 0$. Then introduce $\bar{x}_2 = x_2/\gamma$, $\bar{x}_3 = x_3/\gamma$, and we have the G_4

$$\boxed{p_1, \quad x_1 p_1 + x_2 p_2, \quad x_1^2 p_1 + 2x_1 x_2 p_2 + x_2 p_3, \quad p_3}.$$

(2)
$$U_4 f \equiv \sum_{i=1}^{3} \rho_i U_i f.$$

As before, we can introduce new variables so that

$$U_4 f \equiv \sum_{i=1}^{4} \xi_i (x_1, \cdots, x_n) p_i.$$

By the structure we get, as in the preceding case,

$$U_4 f \equiv x_2 \alpha p_1 + 2x_2 \left(x_3 \alpha + \frac{\beta}{2} \right) p_2 + (x_3^2 \alpha + x_3 \beta + \gamma) p_3 + \delta p_4,$$

$\alpha, \beta, \gamma, \delta$ being functions of $x_4, \cdots, x_n$.

δ cannot be zero, so that we can assume that the coefficient of p_4 has been made unity by the substitution

$$\bar{x}_4 = \int \frac{dx_4}{\delta}.$$

Then

$$U_4 f \equiv x_2 \alpha p_1 + 2x_2 \left(x_3 \alpha + \frac{\beta}{2} \right) p_2 + (x_3^2 \alpha + x_3 \beta + \gamma) p_3 + p_4.$$

Now by assuming $\alpha = 0$, which is certainly one value the arbitrary function α may have, and by introducing as new variables

$$\bar{x}_2 = x_2 \epsilon^{-\int \beta \, dx_4}, \quad \bar{x}_3 = x_3 \epsilon^{-\int \beta \, dx_4} - \int \gamma \epsilon^{-\int \beta \, dx_4} \cdot dx_4,$$

we find

$$\bar{U}_4 f = p_4.$$

Thus we get the group

$$\boxed{p_1, \quad x_1 p_1 + x_2 p_2, \quad x_1^2 p_1 + 2x_1 x_2 p_2 + x_2 p_3, \quad p_4}.$$

This group is simply transitive in the four variables $x_1, \cdots, x_4$. Hence by Lie's "Theorie der Transformationsgruppen" (Vol. I,

page 340), all types under this structure and subdivision that arise when $\alpha \neq 0$ and $n = 4$ can be transformed into the above given form. Moreover when $n > 4$, the variables $x_5, \cdots, x_n$ play the rôle of constants. Hence all G_4's belonging to this type — i. e., which have this structure and are simply transitive with regard to $x_1, \cdots, x_4$ — can be transformed into the above G_4 by a proper choice of variables.

II. Let us consider next the groups satisfying the structure

$$(U_1 U_2) = (U_1 U_3) = (U_2 U_4) = (U_3 U_4) = 0,$$

$$(U_2 U_3) = U_2, \quad (U_1 U_4) = U_1.$$

For the sub-G_3, U_1, U_2, U_3, we having the following cases:

(i)
$$\boxed{p_1, \ p_2, \ x_2 p_2}$$

(ii)
$$\boxed{p_1, \ x_2 p_1, \ -x_2 p_2}$$

(iii)
$$\boxed{p_1, \ p_2, \ x_3 p_1 + x_2 p_2}$$

(iv)
$$\boxed{p_1, \ p_2, \ x_2 p_2 + p_3}.$$

(1) We will consider first all the cases for which

$$U_4 f \equiv \sum_{i=1}^{3} \rho_i(x_1, \cdots, x_n) U_i f.$$

(i) For the case (i) above,

$$U_4 f \equiv \xi_1(x_1, \cdots, x_n) p_1 + \xi_2(x_1, \cdots, x_n) p_2.$$

By the structure, we find immediately

$$\xi_1 \equiv x_1 + \alpha(x_3, \cdots, x_n), \quad \xi_3 \equiv 0.$$

Putting $\bar{x}_1 = x_1 + \alpha$, we obtain the G_4,

$$\boxed{p_1, \ p_2, \ x_2 p_2, \ x_1 p_1}.$$

(ii) Likewise here

$$U_4 f \equiv \xi_1(x_1, \cdots, x_n) p_1 + \xi_2(x_1, \cdots, x_n) p_2.$$

The structure gives $\xi_1 \equiv x_1 + \alpha(x_3, \cdots, x_n)$, $\xi_2 \equiv x_2$. Hence putting $\bar{x}_1 = x_1 + \alpha$, we get the G_4,

$$\boxed{p_1, \ x_2 p_1, \ -x_2 p_2, \ x_1 p_1 + x_2 p_2}.$$

(iii) The structure gives no new G_4 for this case.

(iv) For this case

$$U_4 f \equiv \sum_{i=1}^{3} \xi_i (x_1, \cdots, x_n) p_i.$$

By the structure we find

$$\xi_1 \equiv x_1 + \alpha(x_4, \cdots, x_n), \quad \xi_2 \equiv \epsilon^{x_3 + \beta(x_4, \cdots, x_n)}, \quad \xi_3 \equiv \gamma(x_4, \cdots, x_n).$$

If α and β are not zero, they can be made to disappear by the new variables

$$\bar{x}_1 = x_1 + \alpha, \quad \bar{x}_3 = x_3 + \beta, \quad \bar{x}_4 = \gamma(x_4, \cdots, x_n) \neq \text{const.}$$

Thus we obtain the G_4

$$\boxed{p_1, \ p_2, \ x_2 p_2 + p_3, \ x_1 p_1 + \epsilon^{x_3} p_2 + x_4 p_3}.$$

If $\gamma = \text{const.}$, we have

$$\boxed{p_1, \ p_2, \ x_2 p_2 + p_3, \ x_1 p_1 + \epsilon^{x_3} p_2 + c p_3}.$$

(2) We will consider next the cases for which

$$U_4 f \not\equiv \sum_{i=1}^{3} \rho_i U_i f \equiv \sum_{i=1}^{n} \xi_i (x_1, \cdots, x_n) p_i.$$

(i) In this case $U_k f = 0$ $(k = 1, \cdots, 4)$ form a complete system of 3 independent members; hence there will be $n - 3$ common solutions, $\psi_4(x_1, \cdots, x_n), \cdots, \psi_n(x_1, \cdots, x_n)$, independent of each other and of x_1, x_2, and at least one of the remaining variables, x_3 (say). Then we can introduce as a new system of variables

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = x_2, \quad \bar{x}_3 = x_3, \quad \bar{x}_j = \psi_j(x_1, \cdots, x_n), \quad (j = 4, \cdots, n).$$

By these U_1, U_2, U_3 remain unchanged, and

$$\bar{U}_4 f \equiv \bar{\xi}_1 \bar{p}_1 + \bar{\xi}_2 \bar{p}_2 + \bar{\xi}_3 \bar{p}_3,$$

ξ_i being functions of the new variables $\bar{x}_1, \cdots, \bar{x}_n$. From the structure we find immediately

$$U_4 f \equiv x_1 p_1 + p_3,$$

and thus we have the G_4

$$\boxed{p_1, \; p_2, \; x_2 p_2, \; x_1 p_1 + p_3}.$$

(ii) As in the preceding case we have

$$U_4 f \equiv \sum_{i=1}^{3} \xi_i(x_1, \cdots, x_n) p_i,$$

and by the structure we find

$$U_4 f \equiv x_1 p_1 + x_2 p_2 + p_3.$$

Hence the G_4

$$\boxed{p_1, \; x_2 p_1, \; x_2 p_2, \; x_1 p_1 + x_2 p_2 + p_3}.$$

(iii) For this case we have to make two subdivisions, according as $\xi_3 = 0$ or $\xi_3 \neq 0$.

(a) $\xi_3 \neq 0$. Then x_3 is not a common solution of the complete system, and we can introduce, as above,

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = x_2, \quad \bar{x}_3 = x_3, \quad \bar{x}_j = \psi_j \quad (j = 4, \cdots, n).$$

Then

$$\overline{U}_4 f \equiv \sum_{i=1}^{3} \xi_i(x_1, \cdots, x_n) p_i,$$

and we obtain the G_4

$$\boxed{p_1, \; p_2, \; x_3 p_1 + x_2 p_2, \; x_1 p_1 + x_3 p_3}.$$

(b) $\xi_3 = 0$. For this case x_3 is a common solution of the complete system, and at least one of the other variables $x_4, \cdots, x_n$ is not a solution (say, x_4). Then we can introduce as a new system of variables

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = x_2, \quad \bar{x}_4 = x_4, \quad \bar{x}_3 = x_3, \quad \bar{x}_j = \psi_j \quad (j = 5, \cdots, n),$$

the first three because they are not solutions, and the last $n - 3$ because they are independent common solutions. Thus we get

$$\overline{U}_4 f \equiv \bar{\xi}_1(\bar{x}_1, \cdots, \bar{x}_n) \bar{p}_1 + \bar{\xi}_2(\bar{x}_1, \cdots, \bar{x}_n) \bar{p}_2 + \bar{\xi}_4(\bar{x}_1, \cdots, x) \bar{p}_4.$$

Applying the structure, we find

$$(U_1 U_4) \equiv \xi_{1_{x_1}} p_1 + \xi_{2_{x_1}} p_2 + \xi_{4_{x_1}} p_4 = p_1,$$

$$(U_2 U_4) \equiv \xi_{1_{x_2}} p_1 + \xi_{2_{x_2}} p_2 + \xi_{4_{x_2}} p_4 = 0,$$

$$(U_3 U_4) \equiv x_3 p_1 - \xi_2 p_2 = 0.$$

Hence this case holds only when $x_3 = 0$, which is impossible.

(iv) For this case

$$U_4 f \equiv \sum_{i=1}^{4} \xi_i(x_1, \cdots, x_n) p_i.$$

The structure gives the conditions

$$(U_1 U_4) \equiv \sum_{i=1}^{4} \xi_{i x_1} p_i = p_1,$$

$$(U_2 U_4) \equiv \sum_{i=1}^{4} \xi_{i x_2} p_i = 0,$$

$$(U_3 U_4) \equiv \sum_{i=1}^{4} \xi_{i x_3} p_i - \xi_2 \cdot p_2 = 0.$$

Whence

$$\xi_1 \equiv x_1 + \alpha(x_4, \cdots, x_n), \quad \xi_2 \equiv \beta(x_4, \cdots, x_n) \epsilon^{x_3},$$

$$\xi_3 \equiv \gamma(x_4, \cdots, x_n), \quad\quad \xi_4 \equiv \delta(x_4, \cdots, x_n).$$

ξ_3 can be made zero and ξ_4 can be made unity by the introduction of new variables. Also if we assume $\beta \equiv 1$, we get a G_4, to which, being simply transitive in four variables, all groups under this case will be similar, as on page 22. Hence we obtain the G_4

$$\boxed{p_1, \; p_2, \; x_2 p_2 + p_3, \; x_1 p_1 + \epsilon^{x_3} p_2 + p_4}.$$

III. Suppose the four transformations satisfy the structure

$$(U_1 U_2) = 0, \quad (U_1 U_3) = 0, \quad (U_2 U_3) = U_1,$$

$$(U_1 U_4) = c U_1, \; (U_2 U_4) = U_2, \; (U_3 U_4) = (c-1) U_3, \; (c \neq 1).$$

For the sub-G_3, U_1, U_2, U_3, we found in § 2 these cases:

(i) $$\boxed{p_1, \; p_2, \; x_2 p_1}.$$

(ii) $$\boxed{p_1, \; p_2, \; x_2 p_1 + x_3 p_2}$$

(iii) $$\boxed{p_1, \; p_2, \; x_2 p_1 + p_3}.$$

To these will be added

(iv) $$\boxed{p_1, \; x_2 p_1, \; -p_2},$$

which as a G_3 is the same as (i), but these two give rise to different G_4's.

(1) As before we will consider first all the cases for which

$$U_4 f \equiv \sum_{i=1}^{3} \rho_i U_i f.$$

(i) For this case

$$U_4 f \equiv \xi_1(x_1, \cdots, x_n) p_1 + \xi_2(x_1, \cdots, x_n) p_2.$$

From the structure we find immediately

$$\xi_1 \equiv c x_1 + \alpha(x_3, \cdots, x_n), \quad \xi_2 \equiv x_2.$$

(a) If $c \neq 0$, put $\bar{x}_1 = c x_1 + \alpha$, and we have the group

$$\boxed{p_1, \quad p_2, \quad x_2 p_1, \quad c x_1 p_1 + x_2 p_2} .$$

(b) If $c = 0$, put $\bar{x}_3 = \alpha$, and we have

$$\boxed{p_1, \quad p_2, \quad x_2 p_1, \quad x_3 p_1 + x_2 p_2} .$$

(ii) Here also

$$U_4 f \equiv \xi_1(x_1, \cdots, x_n) p_1 + \xi_2(x_1, \cdots, x_n) p_2.$$

We find from the structure that $c = 2$ for this case to hold.
Then follows that

$$\xi_1 \equiv 2 x_1 + \alpha(x_3, \cdots, x_n), \quad \xi_2 \equiv x_2.$$

Putting $\bar{x}_1 = x_1 + \alpha/2$, we obtain the group

$$\boxed{p_1, \quad p_2, \quad x_2 p_1 + x_3 p_2, \quad 2 x_1 p_1 + x_2 p_2} .$$

(iii) For this case

$$U_4 f \equiv \xi_1(x_1, \cdots, x_n) p_1 + \xi_2(x_1, \cdots, x_n) p_2 + \xi_3(x_1, \cdots, x_n) p_3.$$

Substituting in the structure, we find

$$\xi_1 \equiv c x_1 + \alpha(x_3, \cdots, x_n), \quad \xi_3 \equiv (c-1) x_3 + \gamma(x_4, \cdots, x_n),$$

$$\xi_2 \equiv x_2 + \alpha_{x_3} \equiv x_2 + \beta(x_4, \cdots, x_n).$$

Whence

$$\alpha \equiv x_3 \beta + \lambda(x_4, \cdots, x_n).$$

λ and γ can be made to disappear in the usual way.

Then introduce as new variables

$$\bar{x}_1 = x_1 + \beta(x_4, \cdots, x_n) \cdot x_3, \quad \bar{x}_2 = x_2 + \beta(x_4, \cdots, x_n).$$

Thus we obtain the group

$$\boxed{p_1, \ p_2, \ x_2 p_1 + p_3, \ c x_1 p_1 + x_2 p_2 + (c-1) x_3 p_3}.$$

(iv) For this case

$$U_4 f \equiv \xi_1(x_1, \cdots, x_n) p_1 + \xi_2(x_1, \cdots, x_n) p_2.$$

By the structure we find immediately

$$\xi_1 \equiv c x_1 + \alpha(x_3, \cdots, x_n), \quad \xi_2 \equiv (c-1) x_2.$$

(a) $c \neq 0$. Introduce $c \bar{x}_1 = c x_1 + \alpha(x_3, \cdots, x_n)$. Then we have

$$\boxed{p_1, \ x_2 p_1, \ -p_2, \ c x_1 p_1 + (c-1) x_2 p_2}.$$

(b) $c = 0$. Put $\bar{x}_3 = \alpha(x_3, \cdots, x_n) \neq$ const. Thus

$$\boxed{p_1, \ x_2 p_1, \ -p_2, \ x_3 p_1 - x_2 p_2}.$$

(c) $c = 0$, $\alpha =$ const. Then we have

$$\boxed{p_1, \ x_2 p_1, \ p_2, \ p_1 - x_2 p_2}.$$

(2) We will next consider all the cases for which

$$U_4 f \neq \sum_{i=1}^{3} \rho_i U_i f \equiv \sum_{i=1}^{n} \xi_i(x_1, \cdots, x_n) p_i.$$

(i) For this case we can introduce new variables so that

$$U_4 f \equiv \sum_{i=1}^{3} \xi_i(x_1, \cdots, x_n) p_i.$$

Applying the structure and reducing in the usual way, we obtain the group

$$\boxed{p_1, \ p_2, \ x_2 p_1, \ c x_1 p_1 + x_2 p_2 + p_3}.$$

(ii) Here we have to make two cases according as

$$\xi_3 = 0 \quad \text{or} \quad \xi_3 \neq 0.$$

(a) $\xi_3 \neq 0$. Then since there are three independent members of the complete system, obtained by writing $U_i f = 0$, there will be $n-3$ common solutions, independent of x_1, x_2, x_3, and of each other. Suppose these solutions are $\psi_4(x_1, \cdots, x_n), \cdots, \psi_n(x_1, \cdots, x_n)$.

Then we can introduce as new variables

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = x_2, \quad \bar{x}_3 = x_3, \quad \bar{x}_j = \psi_j(x_1, \cdots, x_n) \quad (j = 4, \cdots, n),$$

and thus obtain

$$\bar{U}_4 f \equiv \xi_1(\bar{x}_1, \cdots, \bar{x}_n)\bar{p}_1 + \xi_2(\bar{x}_1, \cdots, \bar{x}_n)\bar{p}_2 + \xi_3(\bar{x}_1, \cdots, \bar{x}_n)\bar{p}_3.$$

Applying the structure we find

$$U_4 f \equiv c x_1 p_1 + x_2 p_2 + (2 - c) x_3 p_3,$$

and thus we have the G_4

$$\boxed{p_1, \quad p_2, \quad x_2 p_1 + x_3 p_2, \quad c x_1 p_1 + x_2 p_2 + (2 - c) x_3 p_3}.$$

(b) $\xi_3 = 0$. In this case x_3 will be a solution, and there will be only $n - 4$ others, $\psi_5(x_1, \cdots, x_n), \cdots, \psi_n(x_1, \cdots, x_n)$. At least one of the remaining variables, x_4(say), must not be a solution for this case to hold. Hence we can introduce as new variables

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = x_2, \quad \bar{x}_4 = x_4, \quad \bar{x}_3 = x_3, \quad \bar{x}_j = \psi_j(x_1, \cdots, x_n) \quad (j = 5, \cdots, n).$$

Then

$$\bar{U}_4 f \equiv \xi_1(\bar{x}_1, \cdots, \bar{x}_n)\bar{p}_1 + \xi_2(\bar{x}_1, \cdots, \bar{x}_n)\bar{p}_2 + \xi_4(\bar{x}_1, \cdots, \bar{x}_n)\bar{p}_4.$$

From the structure we find, leaving off the bars,

$$\xi_1 \equiv 2x_1 + \alpha(x_3, \cdots, x_n), \quad \xi_2 \equiv x_2, \quad \xi_4 \equiv \gamma(x_3, \cdots, x_n) \neq 0.$$

This shows $c = 2$ for this case.

Now introduce

$$\bar{x}_1 = x_1 + \phi(x_3, \cdots, x_n), \quad \bar{x}_4 = \int \frac{dx_4}{\gamma},$$

and choosing ϕ so as to make the arbitrary function in ξ_1 disappear, we find

$$U_4 f \equiv 2x_1 p_1 + x_2 p_2 + p_4.$$

Thus the G_4

$$\boxed{p_1, \quad p_2, \quad x_2 p_1 + x_3 p_2, \quad 2x_1 p_1 + x_2 p_2 + p_4}.$$

(iii) We can introduce new variables directly so that

$$U_4 f \equiv \sum_{i=1}^{4} \xi_i(x_1, \cdots, x_n) p_i.$$

From the structure, we find immediately

$$\xi_1 \equiv cx_1 + \alpha(x_3, \cdots, x_n), \qquad \xi_2 \equiv x_2 + \beta(x_4, \cdots, x_n),$$

$$\xi_3 \equiv (c-1)x_3 + \gamma(x_4, \cdots, x_n), \quad \xi_4 \equiv \delta(x_4, \cdots, x_n) \neq 0.$$

ξ_4 can be made unity by the substitution

$$\bar{x}_4 \equiv \int \frac{dx_4}{\delta},$$

and γ can be made to vanish in the usual way. Now if α and β, which are arbitrary, have the value zero, we obtain the group, which is simply transitive in four variables

$$\boxed{p_1, \ p_2, \ x_2 p_1 + p_3, \ cx_1 p_1 + x_2 p_2 + (c-1)x_3 p_3 + p_4}.$$

To this group, being simply transitive, every group under this case, when $n = 4$, must be similar, whatever be the forms of the functions α and β, according to Lie's "Theorie der Transformationsgruppen," (Vol. I, page 340). The remaining variables $x_5, \cdots, x_n$ play the rôle of constants in the arbitrary functions, and do not affect the equations of transformations in four variables, that would transform any type into the one above. Hence all G_4's belonging to this type can be transformed into the above-given G_4.

(iv) Finally for this case, we can introduce new variables so that

$$U_4 f \equiv \sum_{i=1}^{3} \rho_i U_i f.$$

Applying the structure and reducing in the usual way, we obtain the group

$$\boxed{p_1, \ x_2 p_1, \ p_2, \ cx_1 p_1 + (c-1)x_2 p_2 + p_3}.$$

The treatment for the remaining structures is very similar to the preceding; we will therefore not give the calculations in full, but will in concluding this section give the results in a complete table of all G_4's, classified according to their structures.

I. $(U_1 U_2) = U_1, \quad (U_1 U_3) = 2U_2, \quad (U_2 U_3) = U_3, \quad (U_i U_k) = 0$

$$(i, k, = 1, 2, 3).$$

$$p_1, \ x_1p_1, \ x_1^2p_1, \ p_2 \qquad p_1, \ x_1p_1+x_2p_2, \ x_1^2p_1+2x_1x_2p_2, \ x_2x_3p_2$$

$$p_1, \ x_1p_1+x_2p_2, \ x_1^2p_1+2x_1x_2p_2, \ x_2p_2$$

$$p_1, \ x_1p_1+x_2p_2, \ x_1^2p_1+2x_1x_2p_2, \ p_3$$

$$p_1, \ x_1p_1+x_2p_2, \ x_1^2p_1+(2x_1x_2+x_2^2)p_2, \ p_3$$

$$p_1, \ x_1p_1+x_2p_2, \ x_1^2p_1+2x_1x_2p_2+x_2p_3, \ x_2p_1+2x_2x_3p_2+(x_3^2+x_4)p_3$$

$$p_1, \ x_1p_1+x_2p_2, \ x_1^2p_1+2x_1x_2p_2+x_2p_3, \ x_2p_1+2x_2x_3p_2+x_3^2p_3$$

$$p_1, \ x_1p_1+x_2p_2, \ x_1^2p_1+2x_1x_2p_2+x_2p_3, \ x_4(x_2p_2+x_3p_3)$$

$$p_1, \ x_1p_1+x_2p_2, \ x_1^2p_1+2x_1x_2p_2+x_2p_3, \ x_2p_2+x_3p_3$$

$$p_1, \ x_1p_1+x_2p_2, \ x_1^2p_1+2x_1x_2p_2+x_2p_3, \ p_3$$

$$p_1, \ x_1p_1+x_2p_2, \ x_1^2p_1+2x_1x_2p_2+x_2p_3, \ p_4$$

II. $(U_1U_2)=(U_1U_3)=(U_2U_4)=(U_3U_4)=0, \ (U_2U_3)=U_2,$
$$(U_1U_4)=U_1.$$

$$p_1, \ p_2, \ x_2p_2, \ x_1p_1 \qquad p_1, \ x_2p_1, \ x_2p_2, \ x_1p_1+x_2p_2$$

$$p_1, \ p_2, \ x_2p_2+p_3, \ x_1p_1+\epsilon^{x_3}p_2+x_4p_3$$

$$p_1, \ p_2, \ x_2p_2, \ x_1p_1+p_3$$

$$p_1, \ p_2, \ x_2p_2+p_3, \ x_1p_1+\epsilon^{x_3}p_2+cp_3$$

$$p_1, \ x_2p_1, \ x_2p_2, \ x_1p_1+x_2p_2+p_3$$

$$p_1, \ p_2, \ x_3p_1+x_2p_2, \ x_1p_1+x_3p_3$$

$$p_1, \ p_2, \ x_2p_2+p_3, \ x_1p_1+\epsilon^{x_3}p_2+p_4$$

III. $(U_1 U_2) = (U_1 U_3) = 0$, $(U_2 U_3) = U_1$, $(U_1 U_4) = c U_1$,

$$(U_2 U_4) = U_2, \quad (U_3 U_4) = (c-1) U_3 \qquad (c \neq 1).$$

$$\boxed{p_1, \; p_2, \; x_2 p_1, \; c x_1 p_1 + x_2 p_2}$$

$$\boxed{p_1, \; p_2, \; x_2 p_1, \; x_3 p_1 + x_2 p_2}$$

$$\boxed{p_1, \; p_2, \; x_2 p_1 + x_3 p_2, \; 2 x_1 p_1 + x_2 p_2}$$

$$\boxed{p_1, \; p_2, \; x_2 p_1 + p_3, \; c x_1 p_1 + x_2 p_2 + (c-1) x_3 p_3}$$

$$\boxed{p_1, \; x_2 p_1, \; p_2, \; c x_1 p_1 + (c-1) x_2 p_2}$$

$$\boxed{p_1, \; x_2 p_1, \; p_2, \; x_3 p_1 - x_2 p_2} \qquad (c = 0).$$

$$\boxed{p_1, \; p_2, \; x_2 p_1, \; c x_1 p_1 + x_2 p_2 + p_3}$$

$$\boxed{p_1, \; p_2, \; x_2 p_1 + x_3 p_2, \; c x_1 p_1 + x_2 p_2 + (2-c) x_3 p_3}$$

$$\boxed{p_1, \; p_2, \; x_2 p_1 + x_3 p_2, \; 2 x_1 p_1 + x_2 p_2 + p_4}$$

$$\boxed{p_1, \; p_2, \; x_2 p_1 + p_3, \; c x_1 p_1 + x_2 p_2 + (c-1) x_3 p_3 + p_4}$$

$$\boxed{p_1, \; x_2 p_1, \; p_2, \; c x_1 p_1 + (c-1) x_2 p_2 + p_3}$$

$$\boxed{p_1, \; x_2 p_1, \; p_2, \; p_1 - x_2 p_2}$$

IV. $(U_1 U_2) = (U_1 U_3) = 0$, $(U_2 U_3) = U_1$, $(U_1 U_4) = 2 U_1$,

$$(U_2 U_4) = U_2, \quad (U_3 U_4) = U_2 + U_3.$$

$$\boxed{p_1, \; x_2 p_1, \; p_2, \; \left(2 x_1 - \tfrac{1}{2} x_2^2\right) p_1 + x_2 p_2}$$

$$\boxed{p_1, \; p_2, \; x_2 p_1 + p_3, \; \left(2 x_1 + \tfrac{1}{2} x_3^2\right) p_1 + (x_2 + x_3) p_2 + (x_3 + x_4) p_3}$$

$$\boxed{p_1, \; x_2 p_1, \; p_2, \; \left(2 x_1 - \tfrac{1}{2} x_2^2\right) p_1 + x_2 p_2 + p_3}$$

$$\boxed{p_1, \; p_2, \; x_2 p_1 + x_3 p_2, \; 2 x_1 p_1 + x_2 p_2 - p_3}$$

$$\boxed{p_1,\ p_2,\ x_2 p_1 + p_3,\ \left(2x_1 + \tfrac{1}{2}x_3^2\right)p_1 + (x_2 + x_3)p_2 + x_3 p_3 + p_4}$$

V. $(U_1 U_2) = (U_1 U_3) = 0,\quad (U_2 U_3) = U_1,\quad (U_1 U_4) = U_1,$
$$(U_2 U_4) = U_2,\quad (U_3 U_4) = 0.$$

$$\boxed{p_1,\ p_2,\ x_2 p_1,\ x_1 p_1 + x_2 p_2}\qquad \boxed{p_1,\ x_2 p_1,\ p_2,\ x_1 p_1}$$

$$\boxed{p_1,\ p_2,\ x_2 p_1 + p_3,\ x_1 p_1 + x_2 p_2 + F(x_4,\ \cdots,\ x_n)p_3}$$

$$\boxed{p_1,\ p_2,\ x_2 p_1,\ x_1 p_1 + x_2 p_2 + p_3}\qquad \boxed{p_1,\ x_2 p_1,\ p_2,\ x_1 p_1 + p_3}$$

$$\boxed{p_1,\ p_2,\ x_2 p_1 + x_3 p_2,\ x_1 p_1 + x_2 p_2 + x_3 p_3}$$

$$\boxed{p_1,\ p_2,\ x_2 p_1 + p_3,\ x_1 p_1 + x_2 p_2 + p_4}$$

VI. $(U_1 U_4) = U_1,\quad (U_2 U_4) = a U_2,\quad (U_3 U_4) = c U_3,$
$$(U_i U_k) = 0 \qquad\qquad (i,\ k = 1,\ 2,\ 3)$$

$$\boxed{p_1,\ x_2 p_1,\ F(x_2,\ \cdots,\ x_n)p_1,\ x_1 p_1}\qquad (a = c = 1)$$

$$\boxed{p_1,\ p_2,\ x_3 p_2,\ x_1 p_1 + a x_2 p_2}\qquad (a = c)$$

$$\boxed{p_1,\ x_3 p_1,\ p_2,\ x_1 p_1 + c x_2 p_2}\qquad (a = 1)$$

$$\boxed{p_1,\ p_2,\ x_3 p_1 + F(x_3,\ \cdots,\ x_n)p_2,\ x_1 p_1 + x_2 p_2}\quad (a = c = 1)$$

$$\boxed{p_1,\ p_2,\ p_3,\ x_1 p_1 + a x_2 p_2 + c x_3 p_3}$$

$$\boxed{p_1,\ x_2 p_1,\ F_1(x_2,\ \cdots,\ x_n)p_1,\ x_1 p_1 + F_2(x_2,\ \cdots,\ x_n)p_4}\quad (a = 1)$$

$$\boxed{p_1,\ x_2 p_1,\ x_3 p_1,\ p_1 + (1 - c)x_3 p_3}\qquad (a = 1)$$

$$\boxed{p_1,\ x_2 p_1,\ x_3 p_1,\ p_1 + (1 - a)x_2 p_2 + (1 - c)x_3 p_3}$$

$$\boxed{p_1,\ x_2 p_1,\ x_2^{(1-c)/(1-a)}p_1,\ x_1 p_1 + (1 - a)x_2 p_2}$$

$$\boxed{p_1,\ p_2,\ x_3 p_2,\ x_1 p_1 + a x_2 p_2 + p_4}\qquad (a = c)$$

$$\boxed{p_1,\ p_2,\ x_3 p_2,\ x_1 p_1 + a x_2 p_2 + F(x_3) p_3}$$

$$\boxed{p_1,\ x_3 p_1,\ p_2,\ x_1 p_1 + c x_2 p_2 + (1 - a) x_3 p_3}$$

$$\boxed{p_1,\ p_2,\ x_3 p_1 + x_4 p_2,\ x_1 p_1 + a x_2 p_2 + (1 - a) x_4 p_4}$$

$$\boxed{p_1,\ p_2,\ x_3 p_1 + F_1(x_3, \cdots, x_n) p_2,\ x_1 p_1 + x_2 p_2 + F_2(x_3, \cdots, x_n) p_5}$$
$$(a = c = 1)$$

$$\boxed{p_1,\ p_2,\ x_3 p_1 + x_4 p_2,\ x_1 p_1 + a x_2 p_2 + (1 - c) x_3 p_3 + (a - c) x_4 p_4}$$

$$\boxed{p_1,\ p_2,\ p_3,\ x_1 p_1 + a x_2 p_2 + c x_3 p_3 + p_4}$$

VII. $(U_1 U_4) \equiv c U_1,\quad (U_2 U_4) \equiv (1 + c) U_2,\quad (U_3 U_4) \equiv U_1 + c U_3,$
$$(U_i U_k) = 0 \qquad (i, k, = 1, 2, 3).$$

$$\boxed{p_1,\ p_2,\ p_3,\ (c x_1 + x_3) p_1 + (1 + c) x_2 p_2 + c x_3 p_3}\quad (c \neq 0)$$

$$\boxed{p_1,\ p_2,\ p_3,\ x_3 p_1 + x_2 p_2 + F(x_4, \cdots, x_n) p_3}\quad (c = 0)$$

$$\boxed{p_1,\ x_2 p_1,\ x_3 p_1,\ c x_1 p_1 - x_2 p_2 - p_3}$$

$$\boxed{p_1,\ x_2 p_1,\ \log x_2 p_1,\ c x_1 p_1 - x_2 p_2}$$

$$\boxed{p_1,\ x_3 p_1,\ p_2,\ (c x_1 + x_2) p_1 + c x_2 p_2 + x_3 p_3}$$

$$\boxed{p_1,\ p_2,\ x_3 p_1 + x_4 p_2,\ c x_1 p_1 + (1 + c) x_2 p_2 - p_3 + x_4 p_4}$$

$$\boxed{p_1,\ p_2,\ x_3 p_1 + \epsilon^{-x_3} p_2,\ c x_1 p_1 + (1 + c) x_2 p_2 - p_3}$$

$$\boxed{p_1,\ p_2,\ p_3,\ (c x_1 + x_3) p_1 + (1 + c) x_2 p_2 + c x_3 p_3 + p_4}$$

VIII. $(U_1 U_4) = U_2 f,\quad (U_2 U_4) = 0,\quad (U_3 U_4) = U_1,$
$$(U_i U_k) = 0 \qquad (i, k, = 1, 2, 3).$$

$$\boxed{p_1,\ p_2,\ p_3,\ x_3 p_1 + x_1 p_2 + F(x_4, \cdots, x_n) p_3}$$

$$\boxed{p_1,\ x_2 p_1,\ x_3 p_1,\ x_1 x_2 p_1 + x_2^2 p_2 + (x_2 x_3 - 1) p_3}$$

$$\boxed{p_1, \ x_2 p_1, \ x_2^{-1} p_1, \ x_1 x_2 p_1 + x_2^2 p_2}$$

$$\boxed{p_1, \ x_3 p_1, \ p_2, \ (x_1 x_3 + x_2) p_1 + x_3^2 p_3}$$

$$\boxed{p_1, \ p_2, \ x_3 p_1 + x_4 p_2, \ x_1 p_2 - p_3 + x_3 p_4}$$

$$\boxed{p_1, \ p_2, \ 2x_3 p_1 - x_3^2 p_2, \ x_1 p_2 - p_3}$$

$$\boxed{p_1, \ p_2, \ p_3, \ x_3 p_1 + x_1 p_2 + p_4}$$

IX. $(U_1 U_4) = U_1 f + U_2 f, \ (U_2 U_4) = U_2 f, \ (U_3 U_4) = U_1 f + U_3 f,$

$$(U_i, \ U_k) = 0 \qquad (i, k = 1, 2, 3).$$

$$\boxed{p_1, \ p_2, \ p_3, \ (x_1 + x_3) p_1 + (x_1 + x_2) p_2 + x_3 p_3}$$

$$\boxed{p_1, \ x_2 p_1, \ x_3 p_1, \ (x_1 + x_1 x_2) p_1 + x_2^2 p_2 + (x_2 x_3 - 1) p_3}$$

$$\boxed{p_1, \ x_2 p_1, \ x_2^{-1} p_1, \ (x_1 + x_1 x_2) p_1 + x_2^2 p_2}$$

$$\boxed{p_1, \ x_3 p_1, \ p_2, \ (x_1 + x_1 x_3 + x_2) p_1 + x_2 p_2 + x_3^2 p_3}$$

$$\boxed{p_1, \ p_2, \ x_3 p_1 + x_4 p_2, \ x_1 p_1 + (x_1 + x_2) p_2 - p_3 + x_3 p_4}$$

$$\boxed{p_1, \ p_2, \ 2x_3 p_1 - x_3^2 p_2, \ x_1 p_1 + (x_1 + x_2) p_2 - p_3}$$

$$\boxed{p_1, \ p_2, \ p_3, \ (x_1 + x_3) p_1 + (x_1 + x_2) p_2 + x_3 p_3 + p_4}$$

X. $(U_3 U_4) = U_2 f, \ (U_1 U_4) = (U_2 U_4) = (U_i, \ U_k) = 0 \quad (i, k = 1, 2, 3)$

$$\boxed{p_1, \ x_3 p_1, \ p_2, \ x_2 x_3 p_1 + F(x_3, \cdots, x_n) p_2}$$

$$\boxed{p_1, \ p_2, \ p_3, \ x_4 p_1 + x_3 p_2 + F(x_4, \cdots, x_n) p_3}$$

$$\boxed{p_1, \ x_2 p_1, \ x_3 p_1, \ x_2 p_3}$$

$$\boxed{p_1, \ p_2, \ x_3 p_2, \ p_3}$$

$$\boxed{p_1, \ x_3 p_1, \ p_2, \ x_2 x_3 p_1 + F(x_3, \cdots, x_n) p_4}$$

$$\boxed{p_1,\ p_2,\ x_3 p_1 + x_4 p_2,\ p_4}$$

$$\boxed{p_1,\ p_2,\ p_3,\ x_3 p_2 + p_4}$$

XI. $(U_1 U_4) = U_1 f,$ $(U_2 U_4) = U_2 f,$ $(U_3 U_4) = U_2 f + U_3 f,$

$$(U_i U_k) = 0 \qquad\qquad (i,\, k = 1,\, 2,\, 3).$$

$$\boxed{p_1,\ x_3 p_1,\ p_2,\ (x_1 + x_2 x_3) p_1 + x_2 p_2}$$

$$\boxed{p_1,\ p_2,\ p_3,\ x_1 p_1 + (x_2 + x_3) p_2 + x_3 p_3}$$

$$\boxed{p_1,\ x_2 p_1,\ x_3 p_1,\ x_1 p_1 - x_2 p_3}$$

$$\boxed{p_1,\ p_2,\ x_3 p_2,\ x_1 p_1 + x_2 p_2 - p_3}$$

$$\boxed{p_1,\ x_3 p_1,\ p_2,\ (x_1 + x_2 x_3) p_1 + x_2 p_2 + F(x_3, \cdots, x_n) p_4}$$

$$\boxed{p_1,\ p_2,\ x_3 p_1 + x_4 p_2,\ x_1 p_1 + x_2 p_2 - p_4}$$

$$\boxed{p_1,\ p_2,\ p_3,\ x_1 p_1 + (x_2 + x_3) p_2 + x_3 p_3 + p_4}$$

XII. $\qquad\qquad (U_i,\, U_k) = 0 \qquad (i,\, k = 1,\, 2,\, 3,\, 4).$

$$\boxed{p_1,\ x_2 p_1,\ F_1(x_2, \cdots, x_n) p_1,\ F_2(x_2, \cdots, x_n) p_1}$$

$$\boxed{p_1,\ p_2,\ x_3 p_2,\ F_1(x_3, \cdots, x_n) p_1 + F_2(x_3, \cdots, x_n) p_2}$$

$$\boxed{p_1,\ p_2,\ x_3 p_1 + F_1(x_3, \cdots, x_n) p_2,\ F_2(x_3, \cdots, x_n) p_1 + F_3(x_3, \cdots, x_n) p_2}$$

$$\boxed{p_1,\ p_2,\ p_3,\ x_4 p_1 + x_5 p_2 + x_6 p_3}$$

$$\boxed{p_1,\ x_2 p_1,\ x_3 p_1,\ p_4} \qquad \boxed{p_1,\ x_2 p_1,\ F_1(x_2) p_1,\ F_2(x_2, \cdots, x_n) p_3}$$

$$\boxed{p_1,\ p_2,\ x_3 p_2,\ F(x_3, \cdots, x_n) p_4}$$

$$\boxed{p_1,\ p_2,\ x_3 p_1 + x_4 p_2,\ F(x_3, \cdots, x_n) p_j \qquad (j > 4)}$$

$$\boxed{p_1,\ p_2,\ x_3 p_1 + F_1(x_3) p_2,\ F_2(x_3, \cdots, x_n) p_j \qquad (j > 3)}$$

$$\boxed{p_1,\ p_2,\ p_3,\ p_4}$$

CHAPTER II.

INVARIANTS OF THE G_r $(r < 4)$.

Having established all the groups of two, three, and four parameters in n variables, we are next concerned with the finding of their differential invariants when $n < 4$. In this problem a division is to be made according as y and z are functions of x; or z is a function of x and y. The calculations for finding the invariants for these two cases are different and will be given in different sections.

§ 1. *Invariants When y and z are Functions of x.*

In this section we shall give the detailed work for several of the more difficult types, and tabulate the remaining results at the end of the paper. Lie's general method consists in equating to zero the transformations after they have been extended, and finding the solutions of the resulting complete systems.

As is customary, we shall use y_1, z_1 to represent the total first derivatives of y and z respectively with respect to x; y_2, z_2, the total second derivatives, etc. The expressions by means of which the increments of the higher derivatives are obtained were deduced by Lie, and for this case they have the forms

$$\eta_{(n)} \equiv \frac{d\eta_{(n-1)}}{dx} - y_n \frac{d\xi}{dx}, \quad \zeta_{(n)} \equiv \frac{d\zeta_{(n-1)}}{dx} - z_n \frac{d\xi}{dx},$$

where $\eta_{(n)}$, $\zeta_{(n)}$ represent the increments, which y_n, z_n, respectively receive.

1. Let us find first the differential invariants of the group,

$$\boxed{p, \ xp + yq, \ x^2 p + 2xyq}.$$

Calculating the increments of y_1, z_1 for each of the transformations by means of the expressions given above, we have for the once-extended transformations:

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$$U_1'f \equiv p = 0,$$

$$U_2'f \equiv xp + yq + 0r + 0q_1 - z_1r_1 = 0,$$

$$U_3'f \equiv x^2p + 2xyq + 0r + 2yq_1 - 2xz_1r_1 = 0,$$

$$\left(q_1 \equiv \frac{\partial f}{\partial y_1}, \quad r_1 \equiv \frac{\partial f}{\partial z_1}, \quad \text{etc.}\right).$$

The first equation shows that x does not occur in any of the invariants. The second linear partial differential equation is equivalent to

$$\frac{dy}{y} = \frac{dz}{0} = \frac{dy_1}{0} = -\frac{dz_1}{z_1},$$

of which the integrals are z, y_1, $yz_1 \equiv u$.

Introducing these as new variables into $U_3'f = 0$, the coefficients in the new equation can be expressed as functions of the new variables. Performing the operation, we get

$$\overline{U}_3'f \equiv 0r + q_1 + 0\frac{\partial f}{\partial u} = 0.$$

The solutions of this linear partial differential equation are directly z, $u \equiv yz_1$.

To find the invariants of the second order, the transformations must be extended twice. They then have the forms:

$$U_1''f \equiv p = 0,$$

$$U_2''f \equiv xp + yq + 0r + 0q_1 - z_1r_1 - y_2q_2 - 2z_2r_2 = 0,$$

$$U_3''f \equiv x^2p + 2xyq + 0r + 2yq_1 - 2xz_1r_1$$
$$+ 2(y_1 - xy_2)q_2 - 2(z_1 + 2xz_2)r_2 = 0.$$

The third equation can be considerably simplified by replacing it by $V_3''f \equiv U_3''f - 2xU_2''f + x^2U_1''f$, giving

$$V_3''f \equiv 0r + 2yq_1 + 2y_1q_2 - 2z_1r_2 = 0 \cdot$$

As before, $U_1''f$ merely shows that the invariants are independent of x. Then the solutions of $U_2''f$ are directly

$$z, y_1, yz_1 \equiv u, yy_2 \equiv v, y^2z_2 \equiv w.$$

Introducing these solutions as new variables into $V_3''f$, it becomes

$$\overline{V}_3''f \equiv 0\frac{\partial f}{\partial z} + \frac{\partial f}{\partial y_1} + 0\frac{\partial f}{\partial u} + 2y_1\frac{\partial f}{\partial v} - u\frac{\partial f}{\partial w},$$

of which the solutions are: z, u, $y_1^2 - 2v$, $y_1 + w/u$. Hence the invariants of the second order are

$$y_1^2 - 2yy_2, \quad y_1 + \frac{yz_2}{z_1}.$$

By continuing to extend the transformations in this way, the differential invariants of any order can be found; but the work becomes very complicated after a few extensions and a simpler method is at hand.

This method makes use of the differential parameter to find the invariants of higher orders.   If ϕ is an invariant of lowest order, we seek a function

$$\Omega\left(x, y, z, y_1, z_1, \cdots, \phi, \frac{d\phi}{dx}\right)$$

which shall be a differential invariant whenever ϕ is.

Indicating by δ the increment received by means of an infinitesimal transformation of the group, we have, since $d\phi \equiv \phi' dx$,

$$\delta d\phi = \delta\phi' dx + \phi' \delta dx;$$

or since the operations denoted by d and δ are interchangeable,

$$\delta\phi' \equiv \frac{d\delta\phi}{dx} - \phi' \frac{d\delta x}{dx}.$$

Now if ϕ is an invariant, it receives no increment, i. e., $\delta\phi = 0$, so that

$$\delta\phi' \equiv -\phi' \frac{d\delta x}{dx}.$$

This is the expression to determine the increment, which ϕ' receives by means of a transformation.

Applying to the transformations of the group under consideration, we have the following complete system:

$$U_1 f \equiv \frac{\partial\Omega}{\partial x} = 0,$$

$$U_2 f \equiv x\frac{\partial\Omega}{\partial x} + y\frac{\partial\Omega}{\partial y} - \phi'\frac{\partial\Omega}{\partial\phi'} = 0,$$

$$U_3 f \equiv x^2\frac{\partial\Omega}{\partial x} + 2xy\frac{\partial\Omega}{\partial y} - 2x\phi'\frac{\partial\Omega}{\partial\phi'} = 0.$$

One common solution of this system is found immediately to be $y\phi'$. Hence if ϕ is an invariant under the transformations of the group $\Delta\phi \equiv y\phi'$, obtained by multiplying the first derivative of ϕ with respect to x by y, is also an invariant.

Thus, knowing the two invariants above of second order, we can by this process find those of the third order; then applying the parameter to them, find those of the fourth order, and so on.

(2) Let us consider next the group

$$\boxed{p, \; q, \; yp + zq}.$$

The invariants are obtained more easily if, by the introduction of the new variables

$$\bar{x} = \frac{x}{z}, \quad \bar{y} = \frac{y}{z}, \quad \bar{z} = \frac{1}{z},$$

we change the group into the similar one

$$\boxed{zp, \; zq, \; yp + q}.$$

The twice-extended forms of these last transformations are:

$$U_1''f \equiv zp - y_1 z_1 q_1 - z_1^2 r_1 - (y_1 z_2 + 2y_2 z_1)q_2 - 3z_1 z_2 r_2 = 0,$$

$$U_2''f \equiv zq + z_1 q_1 + z_2 q_2 = 0,$$

$$U_3''f \equiv yp + q - y_1^2 q_1 - y_1 z_1 r_1 - 3y_1 y_2 q_2 - (2y_1 z_2 + y_2 z_1)r_2 = 0.$$

The solutions of the second equation are

$$x, \; z, \; z_1, \; z_2, \; zy_1 - z_1 y \equiv u, \; zy_2 - z_2 y \equiv v.$$

Introducing these as new variables into the first equation, it becomes

$$\bar{U}_1''f \equiv zp + 0r - z_1^2 r_1 - 3z_1 z_2 r_2 - uz_2 \frac{\partial f}{\partial u} - (uz_2 + 2vz_1)\frac{\partial f}{\partial v} = 0.$$

The integrals of the corresponding simultaneous system are

$$z, \; \frac{x}{z} - \frac{1}{z_1} \equiv \rho_1, \; \frac{z_1^3}{z_2} \equiv \rho_2, \; \frac{u}{z_1} \equiv \rho_3, \; \frac{v}{z_1^2} - \frac{uz_2}{z_1^3} \equiv \rho_4.$$

Introducing these as new variables into the third equation, we have

$$\bar{U}_3''f \equiv 0r - \frac{\rho_3}{z}\frac{\partial f}{\partial \rho_1} + \frac{\rho_4 \rho_2^2}{z} \cdot \frac{\partial f}{\partial \rho_2} - \frac{\partial f}{\partial \rho_3} + 0\frac{\partial f}{\partial \rho_4} = 0,$$

of which the solutions are

$$z, \quad 2z\rho_1 - \rho_3^2, \quad \rho_4, \quad \frac{z}{\rho_4\rho_2} - \rho_3,$$

or in the original variables

$$z, \quad 2x - \frac{2z}{z_1} - \frac{(zy_1 - z_1y)^2}{z_1^2} \equiv I_1; \quad \frac{z(z_1y_2 - z_2y_1)}{z_1^3} \equiv I_2;$$

$$\frac{z_2}{z_1y_2 - z_2y_1} - \frac{zy_1 - z_1y}{z_1} \equiv J_2.$$

By the method of the preceding paragraph, we find

$$\Delta\phi \equiv \frac{\phi'}{z_1}.$$

Applying this to the invariants I_2, J_2, above, we find I_3, J_3 of the third order, etc.

We will now tabulate all the groups of two and three parameters and their total differential invariants. The classification is made according to the order of the first derived groups. The invariants are found up to the second order and we will use I_1, J_1, to represent those of the first order, I_2, J_2, those of the second order. The invariants of higher orders can easily be obtained by the use of the differential parameter for each group, as explained on pages 38–39. From these invariants, the invariant differential equations can easily be set up by equating an arbitrary function of the invariants to zero.

G_2's.

I. When the first-derived group is one-membered:

[1] $\boxed{p, \ xp}$: $I_1 \equiv \dfrac{z_1}{y_1};$ $I_2 \equiv \dfrac{y_2}{y_1^2};$ $J_2 \equiv \dfrac{z_2}{z_1^2};$ $\Delta\phi \equiv \dfrac{\phi'}{z_1}.$

[2] $\boxed{p, \ xp + yq + zr}$: $I_1 \equiv y_1;$ $J_1 \equiv z_1;$ $I_2 \equiv zy_2;$ $J_2 \equiv zz_2;$

$\quad \Delta\phi \equiv z\phi'.$

II. When the first-derived group is zero-membered:

[3] $\boxed{p, \ zp}$: $I_1 \equiv \dfrac{y_1}{z_1};$ $I_2 \equiv \dfrac{z_2}{z_1^3};$ $J_2 \equiv \dfrac{y_1z_2 - y_2z_1}{z_1^3};$ $\Delta\phi \equiv \dfrac{\phi'}{z_1}.$

[4] $\boxed{p, \ q}$: $I_1 \equiv y_1;$ $J_1 \equiv z_1;$ $I_2 \equiv y_2;$ $J_2 \equiv z_2;$ $\Delta\phi \equiv \phi'.$

G_3's.

I. When the first-derived group is three-membered.

[1] $\boxed{p,\ xp,\ x^2 p}$: $I_1 \equiv \dfrac{y_1}{z_1}$;　$I_2 \equiv \dfrac{y_1 z_2 - y_2 z_1}{y_1^3}$;　$\Delta\phi \equiv \dfrac{\phi'}{z_1}$.

[2] $\boxed{p,\ xp + yq,\ x^2 p + 2xyq}$: $I_1 \equiv yz_1$;　$I_2 \equiv y_1^2 - 2yy_2$;

$\qquad J_2 \equiv \dfrac{y_1 z_1 + yz_2}{z_1}$;　$\Delta\phi \equiv y\phi'$.

[3] $\boxed{p,\ xp + yq,\ x^2 p + (2xy + y^2)q}$: $I_1 \equiv \dfrac{y^2 z_1^2}{1 + y_1}$;

$\qquad I_2 \equiv y^2 z_2 - 2yz_1$;　$J_2 \equiv yy_2 - (1 + y_1)^3 \displaystyle\int \dfrac{y_1\, dy_1}{(1 + y_1)^3}$;

$\qquad \Delta\phi \equiv \dfrac{\phi'}{z_1}$.

[4] $\boxed{p,\ xp + yq,\ x^2 p + 2xyq + yr}$: $I_1 \equiv 2z - y_1$;　$J_1 \equiv y_1^2 - 4yz_1$;

$\qquad I_2 \equiv yy_2 - 2yz_1$;　$J_2 \equiv yz(y_2 - 2z_1) - y^2 z_2$;　$\Delta\phi \equiv y\phi'$.

II. When the first-derived group is two-membered:

[5] $\boxed{r,\ xr,\ zr}$: $I_1 \equiv y_1$;　$I_2 \equiv y_2$;　$\Delta\phi \equiv \phi'$.

[6] $\boxed{r,\ xr,\ q + zr}$: $I_1 \equiv y_1$;　$I_2 \equiv y_2$;　$J_2 \equiv \epsilon^{-y} \cdot z_2$;　$\Delta\phi \equiv \phi'$.

[7] $\boxed{r,\ xr,\ (1 - c)xp + zr}$: $I_1 \equiv xy_1$;　$I_2 \equiv x^2 y_2$;　$J_2 \equiv x^{1-2c} \cdot z_2^{1-c}$;

$\qquad \Delta\phi \equiv \dfrac{\phi'}{y_1}$.

[8] $\boxed{p,\ q,\ xp + zq}$: $I_1 \equiv \dfrac{y_1}{z_1} + \log z_1$;　$I_2 \equiv z_1^{-2} \cdot z_2$;

$\qquad J_2 \equiv \dfrac{2y_2}{z_2} + \log z_2$;　$\Delta\phi \equiv \dfrac{\phi'}{z_1}$.

[9] $\boxed{p,\ q,\ xp + cyq}$: $I_1 \equiv y_1 z_1^{c-1}$;　$I_2 \equiv y_1^{c-2} \cdot y_2^{1-c}$;　$J_2 \equiv z_1^{-2} z_2$;

$\qquad \Delta\phi \equiv \dfrac{\phi'}{z_1}$.

[10] $\boxed{p,\ q,\ xp + cyq + r}$: $I_1 \equiv z + \log z_1$; $J_1 \equiv z_1 \cdot y_1^{1/(c-1)}$;

$$I_2 \equiv z_1^{c-2}\, y_2;\quad J_2 \equiv z_1^{-2} z_2;\quad \Delta\phi \equiv \frac{\phi'}{z_1}.$$

[11] $\boxed{r,\ xr,\ p - zr}$: $I_1 \equiv y_1$; $I_2 \equiv y_2$; $J_2 \equiv \epsilon^x \cdot z_2$; $\Delta\phi \equiv \phi'$

[12] $\boxed{p,\ q,\ (x+y)p + yq}$: $I_1 \equiv \dfrac{z_1}{y_1}\,\epsilon^{\frac{1}{y_1}}$; $I_2 \equiv \dfrac{y^2}{y_1^3}\,\epsilon^{\frac{1}{y_1}}$;

$$J_2 \equiv \epsilon^{\frac{2}{y_1}}\left(\frac{y_1 z_2 - y_2 z_1}{y_1^3}\right);\quad \Delta\phi \equiv \frac{\phi'}{z_1}.$$

[13] $\boxed{p,\ q,\ (x+y)p + yq + r}$: $I_1 \equiv z - \dfrac{1}{y_1}$; $J_1 \equiv \dfrac{z_1}{y_1}\,\epsilon^{\frac{1}{y_1}}$;

$$I_2 \equiv \frac{y_2}{y_1^3}\,\epsilon^{\frac{1}{y_1}};\quad J_2 \equiv \epsilon^{\frac{2}{y_1}}\left(\frac{y_1 z_2 - y_2 z_1}{y_1^3}\right);\quad \Delta\phi \equiv \frac{\phi'}{z_1}.$$

III. When the first-derived group is one-membered :

[14] $\boxed{p,\ q,\ xp}$: $I_1 \equiv \dfrac{y_1}{z_1}$; $I_2 \equiv \dfrac{y_2}{z_1^2}$; $J_2 \equiv \dfrac{z_2}{z_1^2}$; $\Delta\phi \equiv \dfrac{\phi'}{z_1}$.

[15] $\boxed{p,\ q,\ yp + zq} \equiv \boxed{zp,\ zq,\ yp + q}$: $I_1 \equiv \dfrac{x}{z} - \dfrac{1}{z_1}$; $I_2 \equiv \dfrac{z_1^3}{z_2}$;

$$J_1 \equiv \frac{yz_1 - zy_1}{z_1};\quad J_2 \equiv \frac{z(y_1 z_2 - y_2 z_1)}{z_1^3};\quad \Delta\phi \equiv \frac{\phi'}{z_1}.$$

[16] $\boxed{p,\ q,\ yp}$: $I_1 \equiv \dfrac{z_1}{y_1}$; $I_2 \equiv \dfrac{y_1^3}{y_2}$; $J_2 \equiv \dfrac{y_2 z_1 - y_1 z_2}{y_1^3}$; $\Delta\phi \equiv \dfrac{\phi'}{z_1}$.

[17] $\boxed{p,\ q,\ yp + r}$: $I_1 \equiv z - \dfrac{1}{y_1}$; $J_1 \equiv \dfrac{z_1}{y_1}$; $I_2 \equiv \dfrac{y_1^3}{y_2}$;

$$J_2 \equiv \frac{y_1 z_2 - y_2 z_1}{y_1^3};\quad \Delta\phi \equiv \frac{\phi'}{z_1}.$$

[18] $\boxed{r,\ xr,\ F(x,y)r}$: $I_1 \equiv y_1$; $I_2 \equiv y_2$; $\Delta\phi \equiv \phi'$.

[19] $\boxed{p,\ q,\ zp}$: $I_1 \equiv \dfrac{y_1}{z_1}$; $I_2 \equiv \dfrac{z_2}{z_1^3}$; $J_2 \equiv \dfrac{y_1 z_2 - y_2 z_1}{z_1^3}$; $\Delta\phi \equiv \dfrac{\phi'}{z_1}$.

[20] $\boxed{p,\ q,\ zp + \eta(z)q}$: $I_1 \equiv \dfrac{y_1 - \eta'(z)}{z_1}$; $I_2 \equiv \dfrac{z_2}{z_1^3}$;

$$J_2 \equiv \frac{y_2 z_1 - y_1 z_2 + \eta'(z) \cdot z_2 - \eta''(z) \cdot z_1^2}{z_1^3}; \quad \Delta\phi \equiv \frac{\phi'}{z_1}.$$

[21] $\boxed{p,\ q,\ r}$: $I_1 \equiv y_1$; $J_1 \equiv z_1$; $I_2 \equiv y_2$; $J_2 \equiv z_2$; $\Delta\phi \equiv \phi'$.

§ 2. *The Invariants when z is a Function of x and y.*

In this section the invariants when z is a function of x and y are found for the groups of two and three parameters. For the four-membered groups the results are not given, though they can be deduced, by a theorem due to Lie (see, "Continuierliche Gruppen," page 761), from the total invariants of the G_4's, which have been obtained by the writer, though not given in this paper.

For these invariants, we shall make use of the following relations to determine the increments of the partial derivatives:

$$d\delta z = \delta p\, dx + \delta q\, dy + p\, d\delta x + q\, d\delta y,$$
$$d\delta p = \delta r\, dx + \delta s\, dy + r\, d\delta x + s\, d\delta y,$$
$$d\delta q = \delta s\, dx + \delta t\, dy + s\, d\delta x + t\, d\delta y,$$

$p,\ q,\ r,\ s,\ t$ standing respectively for

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}, \quad \frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial x \cdot \partial y}, \quad \frac{\partial^2 z}{\partial y^2}.$$

In the same way the corresponding expression for finding the differential parameters is

$$d\delta\phi = \delta\phi_x \cdot dx + \delta\phi_y \cdot dy + \phi_x d\delta x + \phi_y \cdot d\delta y.$$

I. Let us consider first the groups of two parameters, which were found in Chapter I.

(1) $\boxed{\dfrac{\partial f}{\partial z},\ x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z}}$.

The twice extended forms of these transformations, when z is a function of x and y, are

$$\frac{\partial f}{\partial z} = 0,$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} + 0\frac{\partial f}{\partial p} + 0\frac{\partial f}{\partial q} - r\frac{\partial f}{\partial r} - s\frac{\partial f}{\partial s} - t\frac{\partial f}{\partial t} = 0.$$

The first equation shows merely that the invariants do not contain z. Then the solutions of the second are directly

$$p, \quad q, \quad xr, \quad xs, \quad xt.$$

By using the above-given expression for the differential parameters, they are found in the same way to be

$$x\phi_x, \quad x\phi_y.$$

(2) For the group

$$\boxed{\frac{\partial f}{\partial x}, \quad z\frac{\partial f}{\partial x}}$$

the invariants are found in the same way to be

$$\frac{q}{p}, \quad \frac{r}{p^3}, \quad \frac{ps - qr}{p^3}, \quad \frac{p^2 t - 2pqs + q^2 r}{p^3},$$

and the differential parameters are

$$\frac{\phi_x}{p}, \quad \frac{q\phi_x - p\phi_y}{q}.$$

(3) For

$$\boxed{\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}},$$

the invariants are p, q, r, s, t, and the differential parameters ϕ_x, ϕ_y.

(4) For

$$\boxed{\frac{\partial f}{\partial x}, \quad x\frac{\partial f}{\partial x}}$$

the invariants are $q, t, r/p^2, s/p$ and the parameters $\phi_y, \phi_x/p$.

II. We will now give in detail the finding of the invariants for some of the G_3's, and tabulate the results of the rest.

(1) Let us consider the group

$$\boxed{\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad z\frac{\partial f}{\partial x} + \eta(z)\frac{\partial f}{\partial y}}.$$

The twice extended forms of these transformations are

$$\frac{\partial f}{\partial x} = 0,$$

$$\frac{\partial f}{\partial y} = 0,$$

$$z\frac{\partial f}{\partial x} + \eta(z)\frac{\partial f}{\partial y} - p(p+q\eta')\frac{\partial f}{\partial p} - q(p+q\eta')\frac{\partial f}{\partial q} - (3pr + qr\eta$$

$$+ 2ps\eta' + p^2 q\eta'')\frac{\partial f}{\partial r} - (2ps + 2qs\eta' + pt\eta' + pq^2\eta'' + qr)\frac{\partial f}{\partial s}$$

$$- (2qs + pt + 3qt\eta' + q^3\eta'')\frac{\partial f}{\partial t} = 0.$$

The first two equations show merely that the invariants are free of x and y. The simultaneous system corresponding to the third is then

$$\frac{dz}{0} = \frac{dp}{p(p+q\eta')} = \frac{dq}{q(p+q\eta')} = \frac{dr}{3pr + qr\eta' + 2ps\eta' + p^2 q\eta''}$$

$$= \frac{ds}{2ps + 2qs\eta' + pt\eta' + p^2 q\eta'' + qr} = \frac{dt}{2qs + pt + 3qt\eta' + q^3\eta'''} \cdot \quad (A)$$

Two integrals are immediately z, p/q, and to find the remaining, the differential parameters have to be used. They are found from the system

$$\frac{dp}{p(p+q\eta')} = \frac{dq}{q(p+q\eta')} = \frac{d\phi_x}{p(\phi_x + \phi_y \cdot \eta')} = \frac{d\phi}{q(\phi_x + \phi_y \cdot \eta')}, \quad (B)$$

of which the integrals are $(q\phi_x - p\phi_y)/q$, $(\phi_x + \phi_y \cdot \eta')/p$. Operating with these on the solution p/q, two solutions of the second order are found to be

$$\frac{q^2 r - 2pqs + p^2 t}{q^3} \quad \text{and} \quad \frac{qr - ps + (qs - pt)\eta'}{pq^2}.$$

Equating p/q and these two solutions of the second order to constants, using the second ratio of (A) with the fourth, or fifth, or sixth, and eliminating, by means of the solutions equated to constants, from the denominators the variables that do not occur in the numerators, we have left ordinary differential equations of the first order, which are linear in r, or s, or t, respectively. These can be integrated by quadratures and give three solutions of the second order, of which the two above are functions.

2. We will next find the invariants of the group

$$\boxed{\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad y\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial y}}\,.$$

As in the last section, we will use the similar group, obtained by the introduction of the new variables

$$\bar{x} = \frac{x}{z}, \quad \bar{y} = \frac{y}{z}, \quad \bar{z} = \frac{1}{z},$$

$$\boxed{z\frac{\partial f}{\partial x}, \quad z\frac{\partial f}{\partial y}, \quad y\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}}\,.$$

These transformations, twice extended, have the forms

$$z\frac{\partial f}{\partial x} - p^2\frac{\partial f}{\partial p} - pq\frac{\partial f}{\partial q} - 3pr\frac{\partial f}{\partial r} - (2ps+qr)\frac{\partial f}{\partial s} - (pt+2qs)\frac{\partial f}{\partial t} = 0,$$

$$z\frac{\partial f}{\partial y} - pq\frac{\partial f}{\partial p} - q^2\frac{\partial f}{\partial q} - (2ps+qr)\frac{\partial f}{\partial r} - (2qs+pt)\frac{\partial f}{\partial s} - 3qt\frac{\partial f}{\partial t} = 0,$$

$$y\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + 0\frac{\partial f}{\partial z} + 0\frac{\partial f}{\partial p} - p\frac{\partial f}{\partial q} + 0\frac{\partial f}{\partial r} - r\frac{\partial f}{\partial s} - 2s\frac{\partial f}{\partial t} = 0.$$

The solutions of the third equation are immediately

$$z, \ p, \ r, \ y^2 - 2x \equiv \rho_1, \ q + py \equiv \rho_2, \ s + ry \equiv \rho_3, \ s^2 - rt \equiv \rho_4.$$

Introducing these as new variables, the first equation becomes

$$0\frac{\partial f}{\partial z} + p^2\frac{\partial f}{\partial p} + 3pr\frac{\partial f}{\partial r} + 2z\frac{\partial f}{\partial \rho_1} + p\rho_2\frac{\partial f}{\partial \rho_2} + (2p\rho_3 + r\rho_2)\frac{\partial f}{\partial \rho_3} + 4p\rho_4\frac{\partial f}{\partial \rho_4} = 0,$$

of which the integrals are, in the original variables,

$$z, \ \frac{p^3}{r} \equiv u, \ \frac{y^2-2x}{2z} + \frac{1}{p} \equiv v, \ \frac{q+py}{p} \equiv w, \ \frac{s^2-rt}{p^4} \equiv m, \ \frac{sp-qr}{p^3} \equiv n.$$

Introducing these as new variables into the second equation, it becomes

$$0\frac{\partial f}{\partial z} + 2u^2 n\frac{\partial f}{\partial u} + w\frac{\partial f}{\partial v} + z\frac{\partial f}{\partial w} + 0\frac{\partial f}{\partial m} - (un^2 + um)\frac{\partial f}{\partial n} = 0.$$

The solutions are $z, \ m, \ w^2 - 2vz, \ un^2 + um, \ w(un^2 + um) + zn$, or in the original variables:

$$z, \ \frac{s^2 - rt}{p^4}, \ \frac{(q+py)^2}{p^2} - 2z\left(\frac{y^2-2x}{2z} + \frac{1}{p}\right), \ \frac{p^3}{r}\left[\left(\frac{ps-qr}{p^3}\right)^2 + \frac{s^2-rt}{p^4}\right],$$

$$\frac{p^2(q+py)}{r}\left\{\left(\frac{sp-qr}{p^3}\right)^2 + \frac{s^2-rt}{p^4}\right\} + z\left(\frac{sp-qr}{p^3}\right).$$

In the usual way, the differential parameters are found to be

$$\frac{p\phi_y - q\phi_x}{p}, \ \frac{pz\phi_x + (q+py)(p\phi_x - q\phi_y)}{p^2}.$$

(3) We will next consider the groups:

(i)
$$\boxed{\frac{\partial f}{\partial x}, \ x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \ x^2\frac{\partial f}{\partial x} + 2xy\frac{\partial f}{\partial y}} \ ;$$

(ii)
$$\boxed{\frac{\partial f}{\partial x}, \ x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \ x^2\frac{\partial f}{\partial x} + (2xy + y^2)\frac{\partial f}{\partial y}} \ ;$$

(iii)
$$\boxed{\frac{\partial f}{\partial x}, \ x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \ x^2\frac{\partial f}{\partial x} + 2xy\frac{\partial f}{\partial y} + y\frac{\partial f}{\partial z}} \ .$$

The first two transformations of these three groups are the same, and the partial differential invariants are easily found to be:

$$z, \ yp \equiv \rho_1, \ yq \equiv \rho_2, \ y^2 r \equiv \rho_3, \ y^2 s \equiv \rho_4, \ y^2 t \equiv \rho_5.$$

(i) Introducing these into the twice-extended form of the third transformation, it becomes

$$0\frac{\partial f}{\partial z} - 2\rho_2\frac{\partial f}{\partial \rho_1} + 0\frac{\partial f}{\partial \rho_2} - 2(\rho_1 + 2\rho_4)\frac{\partial f}{\partial \rho_3} - 2(\rho_2 + \rho_5)\frac{\partial f}{\partial \rho_4} + 0\frac{\partial f}{\partial \rho_5} = 0 \ ;$$

the solutions of which are

$$z, \ \rho_2, \ \rho_5, \ \rho_1\rho_2 + \rho_1\rho_5 - \rho_4\rho_2, \ (\rho_2 + \rho_5)^2\rho_3 - \tfrac{3}{2}\rho_4^2\rho_2 - \rho_4^2\rho_5$$
$$- \rho_4(\rho_1\rho_2 + \rho_1\rho_5 - \rho_4\rho_2).$$

Expressed in the original variables, these are:

$$z, \ yq, \ y^2 t, \ y^2(pq + pty - qsy), \ y^4(q + ty)^2 r - \tfrac{1}{2}y^5 qs - y^6 s^2 t$$
$$- y^4 s(pq + pty).$$

The differential parameters for this group are:

$$y\phi_y, \quad y^2q\phi_x - y^2p\phi_y.$$

(ii) By the new variables, the third member of the second group is

$$0\,\frac{\partial f}{\partial z} + (\rho_1 - 2\rho_2)\frac{\partial f}{\partial \rho_1} - \rho_2\frac{\partial f}{\partial \rho_2} + (2\rho_3 - 2\rho_1 - 4\rho_4)\frac{\partial f}{\partial \rho_3}$$

$$- 2(\rho_2 + \rho_5)\frac{\partial f}{\partial \rho_4} - 2(\rho_2 + \rho_5)\frac{\partial f}{\partial \rho_5} = 0.$$

The solutions are:

$$z, \quad \rho_4 - \rho_5, \quad \rho_1\rho_2 - \rho_2^2, \quad \frac{\rho_5 + 2\rho_2}{\rho_2^2}, \quad \rho_2^2 \cdot \rho_3 - 2\rho_2^2(\rho_4 - \rho_5) - \tfrac{3}{2}\rho_2^4\left(\frac{\rho_5}{\rho_2^2} + \frac{2}{\rho_2}\right) - 4\rho_2^3.$$

These can easily be expressed in the old variables. The differential parameters are found to be

$$\frac{\phi_y}{q}, \quad y^2q(\phi_x - \phi_y).$$

(iii) In the same way, the invariants for the third of the above groups are:

$$z - yq, \quad y^2t, \quad yp + y^2q^2, \quad y^2s + y^2q^2 + 2y^3t\cdot q,$$

$$y^2r + 2y^2pq + 4y^3qs + 4y^2q^2 + 4y^4q^2t,$$

and the differential parameters are:

$$y\phi_y, \quad y\phi_x + 2zy\phi_y.$$

(4) As the finding of the differential invariants of the remaining G_3's offers no special difficulties, we will merely state the results. We use I_1, J_1, for the invariants of the first order, and I_2, I_2, K_2, for those of the second order. The differential parameters can be easily computed for each group.

Group.	Invariants.
(i) $\boxed{\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}}$	: $I_1 \equiv p;$ $J_1 \equiv q;$ $I_2 \equiv r;$ $J_2 \equiv s;$ $K_2 \equiv t.$
(ii) $\boxed{\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, z\dfrac{\partial f}{\partial x}}$	: $I_1 \equiv \dfrac{p}{q};$ $I_2 \equiv \dfrac{r}{p^3};$ $J_2 \equiv \dfrac{ps - qr}{p^3};$

$$K_2 \equiv \frac{p^2t - 2pqs + q^2r}{p^3}.$$

(iii) $\boxed{\dfrac{\partial f}{\partial z}, \ x\dfrac{\partial f}{\partial z}, \ y\dfrac{\partial f}{\partial z}}$: $I_2 \equiv r$; $J_2 \equiv s$; $K_2 \equiv t$.

(iv) $\boxed{\dfrac{\partial f}{\partial z}, \ x\dfrac{\partial f}{\partial z}, \ F(x)\dfrac{\partial f}{\partial z}}$: $I_1 \equiv q$; $I_2 \equiv s$; $J_2 \equiv t$.

(v) $\boxed{\dfrac{\partial f}{\partial x}, \ \dfrac{\partial f}{\partial y}, \ y\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial z}}$: $I_1 \equiv p$; $J_1 \equiv z+\dfrac{q}{p}$; $I_2 \equiv r$;

$$J_2 \equiv z+\frac{s}{r}; \quad K_2 \equiv rz^2 + 2sz + t.$$

(vi) $\boxed{\dfrac{\partial f}{\partial x}, \ \dfrac{\partial f}{\partial y}, \ y\dfrac{\partial f}{\partial x}}$: $I_1 \equiv p$; $I_2 \equiv r$; $J_2 \equiv qr-ps$;

$$K_2 \equiv q^2 r + 2p^2 t - 2pqs.$$

(vii) $\boxed{\dfrac{\partial f}{\partial x}, \ \dfrac{\partial f}{\partial y}, \ (x \pm y)\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}+\dfrac{\partial f}{\partial z}}$: $I_1 \equiv z + \log p$;

$$J_1 \equiv \frac{q}{p} - \log p; \quad I_2 \equiv \frac{r}{p^2}; \quad J_2 \equiv \frac{s - r\log p}{p^2};$$

$$K_2 \equiv \frac{pt - pr\,[\log p]^2 - 2s\log p}{p^3}.$$

(viii) $\boxed{\dfrac{\partial f}{\partial x}, \ \dfrac{\partial f}{\partial y}, \ (x + y)\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}}$: $I_1 \equiv \dfrac{q}{p} - \log p$; $I_2 \equiv \dfrac{r}{p_2}$;

$$J_2 \equiv \frac{s - r\log p}{p^2}; \quad K_2 \equiv \frac{pt - pr\,[\log p]^2 - 2s\log p}{p^3}.$$

(ix) $\boxed{\dfrac{\partial f}{\partial z}, \ x\dfrac{\partial f}{\partial z}, \ \dfrac{\partial f}{\partial x}-z\dfrac{\partial f}{\partial z}}$: $I_1 \equiv x + \log q$; $I_2 \equiv x + \log r$;

$$J_2 \equiv x + \log s; \quad K_2 \equiv x + \log t.$$

(x) $\boxed{\dfrac{\partial f}{\partial x}, \ \dfrac{\partial f}{\partial y}, \ x\dfrac{\partial f}{\partial x}+cy\dfrac{\partial f}{\partial y}+\dfrac{\partial f}{\partial z}}$: $I_1 \equiv z + \log p$; $J_1 \equiv \dfrac{p^c}{q}$;

$$I_2 \equiv \frac{p^2}{r}; \quad J_2 \equiv \frac{p^{1+c}}{s}; \quad K_2 \equiv \frac{p^{2c}}{t}.$$

(xi) $\boxed{\dfrac{\partial f}{\partial x}, \quad \dfrac{\partial f}{\partial y}, \quad x\dfrac{\partial f}{\partial x} + cy\dfrac{\partial f}{\partial y}}$: $I_1 \equiv \dfrac{p^c}{q}$; $\quad I_2 \equiv \dfrac{p^2}{r}$; $\quad J_2 \equiv \dfrac{p^{1+c}}{s}$;

$$K_2 \equiv \dfrac{p^{2c}}{t}.$$

(xii) $\boxed{\dfrac{\partial f}{\partial z}, \quad x\dfrac{\partial f}{\partial z}, \quad (1-c)x\dfrac{\partial f}{\partial x} + z\dfrac{\partial f}{\partial z}}$: $I_1 \equiv \dfrac{q^{1-c}}{x}$; $\quad I_2 \equiv \dfrac{q^{2c-1}}{r}$;

$$J_2 \equiv \dfrac{q^c}{s}; \quad K_2 \equiv \dfrac{q}{t}.$$

(xiii) $\boxed{\dfrac{\partial f}{\partial x}, \quad x\dfrac{\partial f}{\partial x}, \quad x^2\dfrac{\partial f}{\partial x}}$: $I_1 \equiv q$; $\quad I_2 \equiv \dfrac{s}{p}$; $\quad J_2 \equiv t$.

(xiv) $\boxed{\dfrac{\partial f}{\partial x}, \quad \dfrac{\partial f}{\partial y}, \quad x\dfrac{\partial f}{\partial x} + z\dfrac{\partial f}{\partial y}}$: $I_1 \equiv \log\dfrac{p}{q} + \dfrac{1}{q}$; $\quad I_2 \equiv \dfrac{t}{q}$;

and for this group, the remaining two invariants of the second order are found by quadratures from

$$\frac{ds}{dq} = \frac{2q+1}{q^2}\,s + b\epsilon^{a-1/q}, \quad \frac{dr}{dq} = \frac{2+q}{q^2}\,r + \frac{2s\epsilon^{a-1/q}}{q},$$

where for s in the second equation must be substituted its value in terms of q from the first equation. In these two equations

$$a \equiv \log\frac{p}{q} + \frac{1}{q}, \quad b \equiv \frac{t}{q}.$$

(xv) $\boxed{\dfrac{\partial f}{\partial x}, \quad \dfrac{\partial f}{\partial y}, \quad x\dfrac{\partial f}{\partial x}}$: $I_1 \equiv q$; $\quad I_2 \equiv t$; $\quad J_2 \equiv \dfrac{p^2}{r}$; $\quad K_2 \equiv \dfrac{p}{s}$.

(xvi) $\boxed{\dfrac{\partial f}{\partial z}, \quad x\dfrac{\partial f}{\partial z}, \quad z\dfrac{\partial f}{\partial z}}$: $I_2 \equiv \dfrac{r}{q}$; $\quad J_2 \equiv \dfrac{s}{q}$; $\quad K_2 \equiv \dfrac{t}{q}$.

(xvii) $\boxed{\dfrac{\partial f}{\partial z}, \quad x\dfrac{\partial f}{\partial z}, \quad \dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z}}$: $I_1 \equiv \epsilon^{-y}q$; $\quad I_2 \equiv \dfrac{r}{q}$; $\quad J_2 \equiv \dfrac{s}{q}$;

$$K_2 \equiv \dfrac{t}{q}.$$

To these we will tabulate, for the sake of reference, the partial differential invariants obtained on pp. 44–48.

(xviii) $\boxed{\dfrac{\partial f}{\partial x},\ \dfrac{\partial f}{\partial y},\ z\dfrac{\partial f}{\partial x} + \eta(z)\dfrac{\partial f}{\partial y}}$: $I_1 \equiv \dfrac{p}{q}$; $I_2 \equiv \dfrac{q^2 r - 2pqs + p^2 t}{q^3}$;

$$J_2 \equiv \frac{qr - ps + (qs - pt)\eta'(z)}{pq^2};\ \ \text{[see page 45].}$$

(xix) $\boxed{z\dfrac{\partial f}{\partial x},\ z\dfrac{\partial f}{\partial y},\ y\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}}$: $I_1 \equiv \left(\dfrac{q + py}{p}\right)^2 - 2z\left(\dfrac{y^2 - 2x}{2z} + \dfrac{1}{p}\right)$;

$$I_2 \equiv \frac{s^2 - rt}{p^4};\ \ \ J_2 \equiv \frac{p^3}{r}\left[\left(\frac{ps - qr}{p^3}\right)^2 + \frac{s^2 - rt}{p^4}\right];$$

$$K_2 \equiv \frac{p^2}{r}(q + py)\left[\left(\frac{ps - qr}{p^3}\right)^3 + \frac{s^2 - rt}{p^4}\right] + z\left(\frac{sp - qr}{p^3}\right).$$

(xx) $\boxed{\dfrac{\partial f}{\partial x},\ x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y},\ x^2\dfrac{\partial f}{\partial x} + 2xy\dfrac{\partial f}{\partial y}}$: $I_1 \equiv yq$; $I_2 \equiv y^2 t$;

$$J_2 \equiv y^2(pq + pty - qsy);\ \ \ K_2 \equiv y^4(q + ty)^2 r$$
$$- \tfrac{1}{2}y^5 qs - y^6 s^2 t - y^4 s(pq + pty).$$

(xxi) $\boxed{\dfrac{\partial f}{\partial x},\ x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y},\ x^2\dfrac{\partial f}{\partial x} + 2xy\dfrac{\partial f}{\partial y} + y\dfrac{\partial f}{\partial z}}$: $I_1 \equiv z - yq$;

$$J_1 \equiv yp + y^2 q^2;\ \ \ I_2 \equiv y^2 t;\ \ \ J_2 \equiv y^2 s + y^2 q^2 + 2y^3 tq;$$
$$K_2 \equiv y^2 r + 2y^2 pq + 4y^3 qs + 4y^2 q^2 + 4y^4 q^2 t.$$

(xxii) $\boxed{\dfrac{\partial f}{\partial x},\ x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y},\ x^2\dfrac{\partial f}{\partial x} + (2xy + y^2)\dfrac{\partial f}{\partial y}}$: $I_1 \equiv y^2 pq - y^2 q^2$;

$$I_2 \equiv y^2(s - t);\ \ \ J_2 \equiv \frac{t}{q^2} + \frac{2}{yq};$$

$$K_2 \equiv y^4 q^2 r - 2y^4 q^2(s - t) - \tfrac{3}{2}y^4 q^4\left[\frac{t}{q^2} + \frac{2}{yq}\right] - 4y^3 q^2.$$

CHAPTER III.

OBSERVATIONS AND APPLICATIONS.

Many interesting investigations are suggested by the knowledge of the groups which have been found. We shall conclude this paper with a few remarks that have an important bearing on the general problem of integration.

The Operations Necessary for Reducing the $G_r (r < 4)$ to Normal Forms.

1. As is well known if the G_1

$$U_1 f \equiv \xi_1 p + \eta_1 q + \zeta_1 r$$

is given, it can be thrown into the form of a translation, $\bar{p}$, only by the integration of the simultaneous system

(1)
$$\frac{dx}{\xi_1} = \frac{dy}{\eta_1} = \frac{dz}{\zeta_1},$$

unless the path-curves of $U_1 f$ are known.

2. If a given G_2, $U_1 f$, $U_2 f$, belongs to the type p, xp, it is easily seen that the reduction to normal form requires, in general, the integration of the simultaneous system (1). If the G_2 belongs to either of the other three types, the reduction can be accomplished by the integration of an ordinary differential equation of first order in two variables, together with certain quadratures.

3. Let us next consider the G_3's.

I. When no linear relation exists.

If the G_3 belongs to the type ([4], table of G_3, page 41),

[4] $\qquad p_1, \; x_1 p_1 + y_1 q_1, \; x_1^2 p_1 + 2 x_1 y_1 q_1 + y_1 r_1,$

the reduction is accomplished as follows:

If the three transformations are given in the form

$$U_i f \equiv \xi_i p + \eta_i q + \zeta_i r \qquad (i = 1, 2, 3),$$

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we know that there must exist three functions of x, y, z — $x_1(x, y, z)$, $y_1(x, y, z)$, $z_1(x, y, z)$ — such that

$$U_1 f \equiv \xi_1 p + \eta_1 q + \zeta_1 r = p_1,$$

(1) $\qquad U_2 f \equiv \xi_2 p + \eta_2 q + \zeta_2 r = x_1 p_1 + y_1 q_1,$

$$U_3 f \equiv \xi_3 p + \eta_3 q + \zeta_3 r = x_1^2 p_1 + 2 x_1 y_1 q_1 + y_1 r_1.$$

Hence x_1 is determined by

$$U_1(x_1) \equiv \xi_1 \frac{\partial x_1}{\partial x} + \eta_1 \frac{\partial x_1}{\partial y} + \zeta_1 \frac{\partial x_1}{\partial z} = 1,$$

(2) $\qquad U_2(x_1) \equiv \xi_2 \frac{\partial x_1}{\partial x} + \eta_2 \frac{\partial x_1}{\partial y} + \zeta_2 \frac{\partial x_1}{\partial z} = x_1,$

$$U_3(x_1) \equiv \xi_3 \frac{\partial x_1}{\partial x} + \eta_3 \frac{\partial x_1}{\partial y} + \zeta_3 \frac{\partial x_1}{\partial z} = x_1^2.$$

By DuBois-Raymond's method (see Lie's "Differentialgleichungen," page 553), x_1 can be determined from (2) by the integration of a Riccati equation.

Now considering that x_1 has been found and introduced as a new variable in place of x, y_1 is given by

$$U_1(y_1) \equiv \frac{\partial y_1}{\partial x_1} + \eta_1 \frac{\partial y_1}{\partial y} + \zeta_1 \frac{\partial y_1}{\partial z} = 0,$$

$$U_2(y_1) \equiv x_1 \frac{\partial y_1}{\partial x_1} + \eta_2 \frac{\partial y_1}{\partial y} + \zeta_2 \frac{\partial y_1}{\partial z} = y_1,$$

$$U_3(y_1) \equiv x_1^2 \frac{\partial y_1}{\partial x_1} + \eta_3 \frac{\partial y_1}{\partial y} + \zeta_3 \frac{\partial y_1}{\partial z} = 2 x_1 y_1.$$

Here $x_1 \equiv x_1(x, y, z)$; and the third equation is not the same as the second. Dividing the third equation by the known function x_1, and solving for the partial derivatives, we get

$$\frac{\partial y_1}{\partial x_1} = \rho_1 + \sigma_1 y_1, \qquad \frac{\partial y_1}{\partial y} = \rho_2 + \sigma_2 y_1, \qquad \frac{\partial y_1}{\partial z} = \rho_3 + \sigma_3 y_1.$$

Hence y_1 is found by quadrature.

In the same way z_1 is given by

$$U_1(z_1) \equiv \frac{\partial z_1}{\partial x_1} + \eta_1 \frac{\partial z_1}{\partial y} + \zeta_1 \frac{\partial z_1}{\partial z} = 0,$$

$$U_2(z_1) \equiv x_1 \frac{\partial z_1}{\partial x_1} + \eta_2 \frac{\partial z_1}{\partial y} + \zeta_2 \frac{\partial z_1}{\partial z} = 0,$$

$$U_3(z_1) \equiv x_1^2 \frac{\partial z_1}{\partial x_1} + \eta_3 \frac{\partial z_1}{\partial y} + \zeta_3 \frac{\partial z_1}{\partial z} = y_1,$$

where y_1 is a known function. Hence the partial derivatives of z_1 are known, and z_1 is given by a quadrature.

In this case, therefore, the integration of a *Riccati* equation and certain quadratures are demanded. If the G_3 belongs to any one of the other four types ([10], [13], [17], [21], pages 42–43) of this class, the reduction is seen to require only quadratures.

II. When one linear relation exists.

If the G_3 belongs to one of the types ([8], [15], [19], [20], pages 41–43), the reduction to normal form requires only quadratures. In the other cases of this class, the integration of an ordinary differential equation of first order in two variables is necessary.

III. When two linear relations exist.

If the G_3 belongs to the type [1], the reduction can be accomplished only by integrating a simultaneous system

$$\frac{dx}{\xi_1} = \frac{dy}{\eta_1} = \frac{dz}{\zeta_1}.$$

If the G_3 belongs to either one of the other two types of this class ([5], [18]), the reduction requires in the most unfavorable case the integration of an ordinary differential equation of the first order in two variables.

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In the *Leipziger Berichte*, Vol. 47, pp. 494–499, Lie shows that if with the transformations of the G_3 of rotations

$$X_1 f \equiv yp - xq$$

$$X_2 f \equiv zq - yr$$

$$X_3 f \equiv xr - zp$$

we form the functions $X_3 \phi(X_1/X_3, X_2/X_3)$ where ϕ is arbitrary, these functions can be considered to be the characteristic functions of contact transformations. From this, it follows that this G_3 is a finite

sub-group of an infinite group of contact transformations, and several interesting properties are obtained from this fact. As we now have all the G_3's, we can use each one as the starting-point for theories similar to those deduced by Lie in the case of rotations.

In his "Berührungstransformationen" [Kap. 13], Lie has discussed the general problem of the integration of a non-linear partial differential equation of the first order, which is invariant under two transformations. As we can obtain a G_2, under which the equation is invariant when two transformations are known under which it is invariant, we can obtain Lie's results in an exceedingly simple manner, by using the typical forms of the G_2's, established above.

I. The Partial Differential Equation admits of a G_1. In this case, by the introduction of new variables, the transformation can be brought to the form of a translation $\partial f / \partial x$. Hence this shows that, in the new variables, the equation will be free of x and will have the form

$$F(y, z, p, q) = 0,$$

as is well known.

II. The equation admits of a G_2.

(1) If the path-curves are the same, then by Theorem 5, p. 615, Lie's "Berührungstransformationen," the equation is linear in p, q. We see the same here for the two G_2's, which have the same path-curves.

(i)
$$\boxed{\frac{\partial f}{\partial x}, \quad z \frac{\partial f}{\partial x}}$$

By page 44, the typical invariant equation is

$$F\left(y, z, \frac{p}{q}\right) = 0.$$

This is equivalent to

$$\alpha(y, z)\frac{\partial z}{\partial x} + \beta(y, z)\frac{\partial z}{\partial y} = 0,$$

and since $z = \text{const.}$, is a solution, the problem can be solved completely by a quadrature.

(ii)

$$\boxed{\frac{\partial f}{\partial x}, \quad x\frac{\partial f}{\partial x}}$$

By page 44, the typical invariant equation is

$$F(y, z, q) = 0,$$

which is equivalent to

$$\frac{\partial z}{\partial y} = f(y, z), \quad \text{or} \quad \frac{dy}{1} = \frac{dz}{f(y, z)}.$$

This is an ordinary differential equation of the first order, and thus far only, will the knowledge of the G_2 be of service in the integration in this case.

(2). The path-curves are different.

(i)

$$\boxed{\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}}$$

By page 44, the typical invariant equation is

$$F(z, p, q) = 0.$$

This equation is immediately integrable by quadrature.

(ii)

$$\boxed{\frac{\partial f}{\partial z}, \quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}}$$

The invariant equation is

$$F\left(\frac{x}{y}, p, q\right) = 0.$$

The solution of this equation demands at most the integration of an ordinary differential equation of the first order.

III. Suppose the equation admits of a G_3, $U_i f$, $(i = 1, 2, 3)$. In cases (i), (ii) under (1), pages 55–56, we have $U_2 = \rho U_1$; so that the invariant partial differential equation of first order, $\Omega = 0$, is *linear* and ρ is one solution. Then if $U_3 = \sigma \cdot U_1$, where σ is not a function of ρ, σ is the other solution. Or if no linear relation connects U_3 and U_1, the other solution of $\Omega = 0$ can be found by quadrature. But if $\rho = F(\sigma)$, above, the integration of an ordinary differential equation of first order is still necessary to integrate $\Omega = 0$.

In case (2), (ii), Lie has shown that without reducing the G_2, U_1, U_2 to normal form, the integration of the invariant differential equation $\Omega(x, y, z, p, q) = 0$ may be made to depend upon the integration of an integrable total equation

$$Pdx + Qdy + Rdz = 0.$$

Then if $P\xi_3 + Q \cdot \eta_3 + R\zeta_3 \neq 0$, this equation is integrable by quadratures. [See Page's "Note on the Invariant Total Differential Equation $Pdx + Qdy + Rdz = 0$," *Annals of Mathematics*, Vol. 12, No. 6.]

If the G_3 can be reduced to normal form without an integration, as is often possible, the invariant partial differential equation of first order must take the form (see pages 48–51],

(i) $$yq = \Omega(z),$$

or

(ii) $$yq(yp - yq) = \Omega(z) \quad \text{or} \quad yq = \text{const.},$$

or

(iii) $$yq - z = \text{const.}$$

Each of these equations is integrable by quadrature.

If I_1, J_1, are the total differential invariants, first order, of any given G_3, the equation

$$\Omega(I_1, J_1) = 0$$

will be an invariant Monge equation. Its integration depends, (see Lie, "Berührungstransformationen," page 553), upon that of the corresponding non-linear partial differential equation of the first order

$$F(x, y, z, p, q) = 0,$$

which of course is also invariant and is handled by the method above.

It is interesting to note that G_3's belonging to the types [1], [2], [3], [5], [6], [7], [11], [14], [15], [16], [18], [19], [20],

leave no Monge equation, but only integrable total equations of first order of the form

(1) $Pdx + Qdy + Rdz = 0$

invariant.   This equation is integrable by quadrature if

$$P\xi_i + Q\eta_i + R\zeta_i \neq 0$$

for any value of $i = 1, 2, 3$.   These groups consequently leave no non-linear partial differential equation of the first order invariant. The other types may leave both Monge equations and equations of form (1) invariant.

From the partial differential invariants, we see that every G_3 in normal form leaves a partial differential equation of first order invariant except

$$\boxed{r, \; xr, \; yr}$$

In this, the invariant partial differential equation is of at least the second order.

We saw above that certain G_3's leave no Monge equations invariant, and hence no non-linear partial differential equations of first order.   This was evident, *a priori*, if any transformation of the G_3 was connected with any other of the G_3 by a linear relation, (see Lie, " Berührungstransformationen," Kap. 13).   But the G_3's

$$\boxed{p, \; q, \; zp + \eta(z)q} \qquad\qquad (\eta \neq c \cdot z)$$

and

$$\boxed{p, \; xp + yq, \; x^2p + 2xyq}$$

are exceptional, as no such linear relation exists; and still only a linear partial differential equation of first order can be invariant under each of them.

By the method developed by Lie, (*Leipziger Berichte*, Oct. 10, 1893, pages 98, ff.), if a partial differential equation of order m, $\Omega^{(m)} = 0$, is invariant under a known G_3, we can find, by integrating an ordinary differential equation of mth order in two variables,

∞^{m+2} integral surfaces of $\Omega^{(m)} = 0$, which are composed of path curves of the general transformation of the G_3,

$$Uf \equiv \sum_{i=1}^{3} a_i U_i f.$$

Hence if the form of $\Omega^{(m)} = 0$ shows that it is invariant under a known G_3, it is frequently possible to find ∞^{m+2} integral surfaces of $\Omega^{(m)} = 0$ by Lie's method. Of course the operations are much simplified if the G_3 has been thrown into normal form, and the path curves of Uf have been determined.

It is easily seen that the most general transformation of space that leaves directions invariant belongs to one of the G_4's. The coordinates of a line element in space are x, y, z, y_1, z_1, and if the directions are to be invariant, the increments of y_1 and z_1 must be zero. Thus we have

$$\eta_1 \equiv \frac{d\eta}{dx} - y_1 \frac{d\xi}{dx} \equiv \eta_x + \eta_y \cdot y_1 + \eta_z \cdot z_1 - y_1(\xi_x + \xi_y \cdot y_1 + \xi_z \cdot z_1) = 0,$$

$$\zeta_1 \equiv \frac{d\zeta}{dx} - z_1 \frac{d\xi}{dx} \equiv \zeta_x + \zeta_y \cdot y_1 + \zeta_z \cdot z_1 - z_1(\xi_x + \xi_y \cdot y_1 + \xi_z \cdot z_1) = 0.$$

Since these relations are independent, we have

$$\xi_y = \xi_z = \eta_z = \eta_x = \zeta_x = \zeta_y = 0, \quad \xi_x = \eta_y = \zeta_z.$$

Whence, immediately $\xi \equiv mx + \alpha$, $\eta \equiv my + \beta$, $\zeta \equiv mz + \gamma$; m, α, β, γ, being arbitrary constants. Hence the transformation is

$$(mx + \alpha)\frac{\partial f}{\partial x} + (my + \beta)\frac{\partial f}{\partial y} + (mz + \gamma)\frac{\partial f}{\partial z}.$$

This is recognized at once as the most general transformation of the fifth group under VI, (see table G_4's, p. 32), obtained by putting the arbitrary constants a and c equal to unity.

This G_4 is of importance in the problem of spherical representation of curves.

In the same way, it can be shown that the transformation obtained above is the most general one that leaves the direction of surface elements invariant.

We notice that there is only one G_2 in one variable; that there are four in two variables, and four in n-variables. There is one G_3 in one variable, eleven in two variables, twenty-one in three variables, and twenty-one in n-variables. There are 16 G_4's in two variables, 71 in three variables, 105 in four variables, and 105 in n-variables. That for these three cases there is the same number of G_r's in r-variables as in n-variables is to be expected according to Theorem 84, page 458, of the first volume of Lie's "Theorie der Transformationsgruppen." It will readily be seen that more is shown in the above-given result than is stated in the theorem referred to.