Small Seifert Fibered Zero-Surgery

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A Dissertation Presented to the Graduate Faculty

of the University of Virginia in Candidacy for the Degree of

Doctor of Philosophy

Department of Mathematics

University of Virginia

May, 2022

Abstract

This thesis provides new results regarding which small Seifert fibered spaces arise as 0-surgery on a knot in the 3-sphere. We generalize a 0-surgery obstruction of Ozsváth-Szabó to the setting of involutive Heegaard Floer homology, an extension of Heegaard Floer homology due to Hendricks-Manolescu. Using this obstruction, we find a new infinite family of small Seifert fibered spaces with first homology $\mathbb{Z}$ and weight 1 fundamental group that cannot be obtained by 0-surgery on a knot in the 3-sphere, generalizing a result of Hedden-Kim-Mark-Park. In fact, we show that these manifolds cannot even be the boundary of a negative semi-definite spin 4-manifold. On the opposite end of the spectrum, we also provide a new family of small Seifert fibered spaces that do arise as 0-surgery on a knot in the 3-sphere. This is a simple generalization of work of Ichihara-Motegi-Song. Additionally, we establish some constraints on the types of knots that can have small Seifert fibered 0-surgery.

Acknowledgments

I owe a great deal of credit and gratitude to many people for helping me on my journey to complete this thesis. I would like to thank all of my amazing mathematics teachers over the years, especially my advisor Tom Mark. Thank you for your constant support, encouragement, and valuable insight throughout my time as your student, and thank you for suggesting a great thesis project.

Special thanks also goes to my thesis committee members: Slava Krushkal, Sara Maloni, and Olivier Pfister. In addition, I would like to thank Ken Baker for helping me better understand the Montesinos trick and the Ichihara-Motegi-Song examples discussed in Chapter 4 of this thesis.

I would also like to thank my classmates at the University of Virginia for helping me become a better mathematician and making my time in Charlottesville a great experience. It would be impractical for me to list everyone since so many of you have become great friends, but I would like to specifically recognize Sarasij Maitra and Ross Akhmechet, who have been in the program with me since the beginning. I will miss hanging out in the basement of Kerchof with you and our frequent trips to Jack Brown's.

Last but not least, thank you to all of my friends and family, especially my parents Carl and Susan, who have supported me every step of the way.

Contents

Chapter 1

Introduction

1.1 Dehn surgery and the complexity of 3-manifolds

In the early 1900s, Max Dehn introduced a topological operation on 3-dimensional manifolds called *Dehn surgery* [10], the procedure of removing a tubular neighborhood of a knot (or link) from a 3-manifold and then gluing it back in a different way. In the 100+ years since its formulation, Dehn surgery has played a vital role in 3-manifold topology and geometry. In particular, via the following theorem of Lickorish and, independently, Wallace, Dehn surgery provides a framework for studying 3-manifolds from the perspective of knots and links in S^3 (the 3-sphere).

Theorem 1.1.1 (Lickorish [24], Wallace [48]). *Every closed oriented 3-manifold can be obtained by Dehn surgery on some, not necessarily unique, link in* S^3.

As a consequence of this theorem, one can gather a sense of the complexity of a 3-manifold Y in terms of the complexity of the links required to build Y from Dehn surgery. In [4], Auckly quantifies this notion of complexity with the following definition:

Definition 1.1.2. The *Dehn surgery number* of a closed oriented 3-manifold Y is $DS(Y) :=$ min{number of components of L | L is a link in S^3 on which Dehn surgery yields Y}.

Remark 1.1.3. $DS(S^3) = 0$ because we can regard S^3 as being obtained by Dehn surgery on the empty link.

Definition 1.1.2 leads to the following natural question:

Question 1.1.4. *For which 3-manifolds Y is $DS(Y) > 1$? In other words, which 3-manifolds cannot be obtained by Dehn surgery on a knot in S^3?*

The goal of this thesis is to provide new answers to Question 1.1.4 in the context of a particular class of 3-manifolds called small Seifert fibered spaces and a particular type of Dehn surgery called 0-surgery.

1.1.1 A brief background on Question 1.1.4

There are infinitely many different types of Dehn surgeries one can perform on a given knot in S^3. As described in detail in Section 2.1.1, these different types of Dehn surgeries can be parameterized by the extended rational numbers $\mathbb{Q}\cup\{\infty\}$. A choice of an extended rational number p/q specifies a *framing*, which describes how to glue back the tubular neighborhood of the knot that is removed during Dehn surgery.

Many interesting results regarding Question 1.1.4 have been achieved for non-zero framed Dehn surgery (see for example [4], [3], [17], [18], [19]), however much less is known about 0-surgery. That being said, there are some results known regarding which 3-manifolds cannot arise as 0-surgery on a knot in S^3, which we now outline.

At the most basic level, one can gain some knowledge by analyzing first homology and the fundamental group. If Y is the result 0-surgery on a knot in S^3, then $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ and $\pi_1(Y)$ has weight 1, meaning it is normally generated by a single element. At a much more advanced level, Gabai in [11] proved that if Y is obtained by 0-surgery on a knot K in S^3, then either $Y = S^1 \times S^2$ and K is the unknot, or Y is irreducible (i.e. every smoothly embedded $S^2 \subset Y$ bounds a 3-ball).

In [2], Aschenbrenner-Friedl-Wilton asked if the requirements described in the previous paragraph guarantee that Y can be obtained by 0-surgery on a knot in S^3. In other words, they asked if Y being a closed oriented irreducible 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ and weight 1 fundamental group implies that Y arises as 0-surgery on a knot in S^3. In 2018, Hedden-Kim-Mark-Park [14] showed that this is not true by producing infinitely many counterexamples.

One of their infinite families of counterexamples consists of small Seifert fibered spaces. Specifically, if for each positive integer j, we let $N_j = S^2\left(\frac{-2}{1}, \frac{-8j+1}{1}, \frac{16j-2}{8j+1}\right)$ (see Section 2.2 for the meaning of this notation), then by using an obstruction coming from the Rokhlin invariant, Hedden-Kim-Mark-Park prove the following theorem.

Theorem 1.1.5 (Hedden-Kim-Mark-Park [14])**.** *For $j \geq 1$, N_j is irreducible, has first homology $\mathbb{Z}$, weight 1 fundamental group, and if j is odd, then N_j is not homology cobordant to Dehn surgery on a knot in S^3. In particular, it does not arise as 0-surgery on a knot in S^3.*

1.2 Summary of results

In [21], we extend Theorem 1.1.5 of Hedden-Kim-Mark-Park to the case when j is even and also reprove the odd case by different means.

Theorem 1.2.1 (Johnson [21])**.** *For all positive integers j, N_j cannot be obtained by 0-surgery on a knot in S^3. In fact, N_j is not the oriented boundary of any smooth negative semi-definite spin 4-manifold.*

The method we use to prove Theorem 1.2.1 uses an extension of Heegaard Floer homology called involutive Heegaard Floer homology defined by Hendricks-Manolescu [16]. The idea is to extract a set of numerical invariants from involutive Heegaard Floer homology and show that for 3-manifolds obtained by 0-surgery on a knot in S^3 these invariants must satisfy certain properties that are not satisfied for the manifolds N_j.

The invariants that we extract are analogs of invariants called $\bar{d}$ and $\underline{d}$ defined for rational homology spheres by Hendricks-Manolescu in [16] and invariants called $d_{1/2}$, $d_{-1/2}$ defined for 3-manifolds Y with $H_1(Y;\mathbb{Z}) = \mathbb{Z}$ by Ozsváth-Szabó in [36]. Like $d_{1/2}$, $d_{-1/2}$, our invariants, denoted $\bar{d}_{1/2}, \bar{d}_{-1/2}, \underline{d}_{1/2}, \underline{d}_{-1/2}$, are also defined for 3-manifolds Y with $H_1(Y;\mathbb{Z}) = \mathbb{Z}$, the relevant condition for studying 0-surgery.

In [21], we proved the following theorems regarding the invariants $\bar{d}_{1/2}, \bar{d}_{-1/2}, \underline{d}_{1/2}, \underline{d}_{-1/2}$, which generalize [36, Theorem 9.11] and [36, Proposition 4.11] of Ozsváth-Szabó.

Theorem 1.2.2 (Johnson [21]). *Suppose X is a smooth oriented negative semi-definite spin 4-manifold with boundary a 3-manifold Y with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. Let $b_2(X) = \operatorname{rank}_{\mathbb{Z}} H_2(X;\mathbb{Z})$.*

1. *If the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is trivial, then*

$$b_2(X) - 3 \leq 4\underline{d}_{-1/2}(Y)$$

2. *If the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is non-trivial, then*

$$b_2(X) + 2 \leq 4\underline{d}_{1/2}(Y)$$

Remark 1.2.3. Hypothesis (1) implies $b_2(X) \geq 1$.

Theorem 1.2.4 (Johnson [21]). *Let M be an oriented integer homology 3-sphere and let Y and M' be the 3-manifolds obtained via 0 and +1 surgery respectively on a knot K in M. Then,*

1. $\underline{d}(M) - \frac{1}{2} \leq \underline{d}_{-1/2}(Y)$ *and* $\bar{d}(M) - \frac{1}{2} \leq \bar{d}_{-1/2}(Y)$
2. $\underline{d}_{1/2}(Y) - \frac{1}{2} \leq \underline{d}(M')$ *and* $\bar{d}_{1/2}(Y) - \frac{1}{2} \leq \bar{d}(M')$

As a consequence of these theorems, we obtain the following two corollaries:

Corollary 1.2.5 (Johnson [21]). *Suppose K is a knot in S^3 and Y is the result of 0-surgery on K. Then,*

$$-\frac{1}{2} \leq \underline{d}_{-1/2}(Y) \qquad \text{and} \qquad \bar{d}_{1/2}(Y) \leq \frac{1}{2}$$

Corollary 1.2.6 (Johnson [21]). *Suppose Y is a closed oriented 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. If*

$$\underline{d}_{-1/2}(Y) < -\frac{1}{2} \qquad \text{and} \qquad \underline{d}_{1/2}(Y) < \frac{1}{2}$$

then Y is not the boundary of any negative semi-definite spin 4-manifold.

We use these results, in combination with a calculation of $\underline{d}_{\pm 1/2}(N_j)$, to then prove Theorem 1.2.1. Specifically, we show that $\underline{d}_{-1/2}(N_j) = -2j - \frac{1}{2}$ and $\underline{d}_{1/2}(N_j) = -2j + \frac{1}{2}$ for all $j \geq 1$, and then apply Corollaries 1.2.5 and 1.2.6.

It is worth noting that the non-involutive version of Theorem 1.2.4, meaning [36, Proposition 4.11], which is the same statement as Theorem 1.2.4 except with the invariants $\bar{d}_{1/2}, \bar{d}_{-1/2}, \underline{d}_{1/2}, \underline{d}_{-1/2}$ replaced by their non-involutive counterparts $d, d_{1/2}, d_{-1/2}$, does not provide any useful information for the family of 3-manifolds $\{N_j\}_{j\geq 1}$. In fact, it is shown in [14, Theorem 5.4] that if Y is a Seifert fibered space with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ (or even just homology cobordant to such a manifold), then

$$-\frac{1}{2} \leq d_{1/2}(Y) \qquad \text{and} \qquad d_{1/2}(y) \leq \frac{1}{2}$$

Therefore, our results show that involutive d-invariants can detect Seifert fibered 0-surgery even though the non-involutive ones cannot.

In addition to providing new examples of small Seifert fibered spaces that cannot arise as 0-surgery on a knot S^3, we also provide new examples that can. In [20], Ichihara-Motegi-Song prove the following theorem.

Theorem 1.2.7 (Ichihara-Motegi-Song [20])**.** *For every integer n, the small Seifert fibered space*

$$S^2\left(\frac{2n+1}{n+1}, \frac{-(2n+3)}{n+1}, \frac{-(2n+1)(2n+3)}{2n+2}\right)$$

can be obtained by 0-surgery along a knot in S^3. Moreover, when $n \neq 0, -1, -2$, the space can be obtained by 0-surgery along a hyperbolic knot.

By analyzing the proof of Theorem 1.2.7 and making one simple observation, we are able to generalize Ichihara-Motegi-Song's result to obtain a 2-parameter family of small Seifert fibered spaces that arise as 0-surgery on a knot in S^3.

Theorem 1.2.8. *Let n and m be integers such that $(n-m)^2$ divides $1+n+m$. Then the small Seifert fibered space*

$$S^2\left(\frac{2n+1}{n+1}, \frac{-(2m+1)}{m}, \frac{-(2n+1)(2m+1)}{1+n+m}\right)$$

can be obtained by 0*-surgery along a knot in* S^3.

Ignoring the statement about hyperbolicty, setting $m = n+1$ recovers Theorem 1.2.7. Note that this 2-parameter family of small Seifert fibered spaces is indeed larger than the family in Theorem 1.2.7. For example, taking $n = 7$ and $m = 10$ gives

$$S^2\left(\frac{15}{8}, \frac{-21}{10}, \frac{-35}{2}\right)$$

which is not a member of the Ichihara-Motegi-Song family.

We also give a constraint on the type of knots that have Seifert fibered 0-surgery. Specifically, we prove the following theorem and corollary.

Theorem 1.2.9. *Let* $Y = S^2\left(\frac{pq}{\beta_1}, \frac{pr}{\beta_2}, \frac{rq}{\beta_3}\right)$ *with* p, q, r *positive pairwise coprime integers and* $\gcd(pq, \beta_1) = \gcd(pr, \beta_2) = \gcd(rq, \beta_3) = 1$. *Further suppose* $p, q, r \geq 2$. *Then the coefficient of the next to top degree term of the Alexander polynomial of any knot on which* 0*-surgery yields* Y *must be equal to* ± 2.

Remark 1.2.10. See Proposition 3.1.1 for the relevance of these conditions on Y.

Corollary 1.2.11. *If* Y *is as in the previous theorem, then* Y *cannot be obtained by* 0*-surgery on an L-space knot.*

1.3 Outline

In Chapter 2, we review relevant background about 3- and 4-manifolds. We define Dehn surgery, small Seifert fibered spaces, plumbed manifolds and discuss some basic properties. In Chapter 3, we discuss some basic small Seifert fibered 0-surgery obstructions. In particular, we prove Theorem 1.2.9 and Corollary 1.2.11. In Chapter 4, we review the Ichihara-Motegi-Song examples and show how to extend their result to prove Theorem 1.2.8. In Chapter 5, we review the definition of involutive Heegaard Floer homology and then prove Theorems 1.2.2 and 1.2.4. In Chapter 6, we develop the technique we use to compute the involutive Heegaard Floer homology of the manifolds $\{N_j\}_{j\geq 1}$. This involves graded roots and lattice cohomology, a combinatorially defined invariant of plumbed 3-manifolds due to Némethi [31], [32], which builds on earlier work of Ozsváth-Szabó [37] regarding the Heegaard Floer homology of plumbed 3-manifolds. We also utilize work of Rustamov [43] on the Heegaard Floer homology of plumbed 3-manifolds with rank 1 first homology and work of Dai-Manolescu [9] on computing the involutive Heegaard Floer homology of negative definite plumbed manifolds. Finally, in Chapter 7, we compute the involutive Heegaard Floer homology of the manifolds $\{N_j\}_{j\geq 1}$ and prove Theorem 1.2.1.

Chapter 2

Background on 3- and 4-manifolds

2.1 Dehn surgery

Definition 2.1.1. An *n-component link* in a 3-manifold Y is an embedding of a disjoint union of n circles into Y. A *knot* is a 1-component link. Throughout, we assume all links are smoothly embedded. We will often conflate links with the image of their embeddings.

Notation 2.1.2. For n a positive integer, let $D^n = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ and let $S^{n-1} = \partial D^n$.

Definition 2.1.3. If K is a knot contained in a 3-manifold Y, then *Dehn surgery* on K is the result of removing a tubular neighborhood $\nu(K)$ of K from Y and then gluing a solid torus $S^1 \times D^2$ to the complement $Y - \nu(K)$ via a diffeomorphism $\phi : \partial(S^1 \times D^2) \to \partial(\nu(K))$. The resulting manifold is $Y' = (Y - \nu(K)) \cup_\phi (S^1 \times D^2)$.

2.1.1 Framings

A priori, the diffeomorphism type of Y' depends on the choice of knot K, tubular neighborhood $\nu(K)$, and diffeomorphism ϕ. However, by standard results in differential topology, it follows that if A_t is an ambient isotopy of Y taking K to another knot K', then

$$(Y - \nu(K)) \cup_\phi (S^1 \times D^2) \cong_{diffeo} (Y - A_1(\nu(K))) \cup_{A_1 \circ \phi} (S^1 \times D^2)$$

In particular, if A fixes K, but changes $\nu(K)$, the two resulting Dehn surgered manifolds are diffeomorphic. Since any two tubular neighborhoods of a fixed knot K are ambiently isotopic, it therefore follows that the diffeomorphism type of Y' only depends on the isotopy class of K and the diffeomorphism ϕ. We now describe the dependence on ϕ.

First, choose a point $x \in S^1$ and consider the simple closed curve $\gamma = \phi(x \times \partial D^2) \subset \partial(\nu(K))$. It is well known that, given the choice of K and $\nu(K)$, the diffeomorphism type of Y' only depends on the isotopy class of γ in $\partial(\nu(K))$. Note that the choice of $x \in S^1$ is irrelevant because different choices yield isotopic γ. We call the isotopy class of a simple closed curve in $\partial(\nu(K))$ a *framing* for $K \subset Y$.

The set of framings is in bijection with the extended rational numbers $\mathbb{Q} \cup \{\infty\}$. To see this, first, let μ be the unique simple closed curve up to isotopy on $\partial(\nu(K))$ that bounds a disk in $\nu(K)$. We call μ the *meridian* of K. Let λ be any other simple closed curve on $\partial(\nu(K))$ that intersects μ in exactly one point. We call λ a *longitude*. Unlike μ, λ is not unique up to isotopy. We return to this ambiguity momentarily.

Next, choose an orientation on K. Then, orient μ according to the right-hand rule with respect to K. Also, orient λ so that it travels in the same direction as K. Locally, the orientations on K, μ, λ should look as in Figure 2-1.

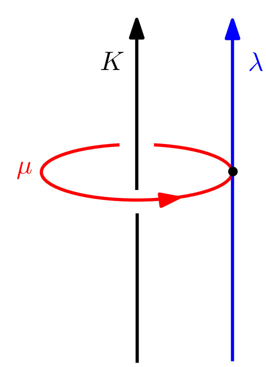


Figure 2-1: Dehn surgery orientations.

The homology classes of the oriented curves μ and λ form a basis for $H_1(\partial(\nu(K));\mathbb{Z})$, i.e., $H_1(\partial(\nu(K));\mathbb{Z}) = \mathbb{Z}[\mu] \oplus \mathbb{Z}[\lambda]$. In particular, if γ is an oriented simple closed curve on $\partial(\nu(K))$, then $[\gamma] = p[\mu] + q[\lambda]$ for a unique pair of coprime integers p, q. Conversely, given any pair of coprime integers p, q, there exists a unique oriented simple closed curve γ (up to isotopy) such that $[\gamma] = p[\mu] + q[\lambda]$. In other words, we have the following bijection:

$$\begin{array}{c} \{\text{isotopy classes of oriented simple closed curves on } \partial(\nu(K))\} \\ \updownarrow \\ \{\text{ordered pairs of coprime integers}\} \end{array}$$

Notice that if γ is associated to (p, q) under this bijection, then γ with the reverse orientation is associated to $(-p, -q)$. Therefore, since $\dfrac{p}{q} = \dfrac{-p}{-q}$ and p, q are coprime, the map defined by the following composition is an isomorphism:

$$\begin{array}{c} \text{framings} = \{\text{isotopy classes of unoriented simple closed curves on } \partial(\nu(K))\} \\ \Big\downarrow \text{Choose an arbitrary orientation} \\ \{\text{isotopy classes of oriented simple closed curves on } \partial(\nu(K))\} \\ \updownarrow \\ \{\text{ordered pairs of coprime integers}\} \\ \Big\downarrow (p,q)\mapsto p/q \\ \mathbb{Q} \cup \{\infty = \frac{\pm 1}{0}\} \end{array}$$

It is straightforward to check that the above identification of framings with $\mathbb{Q} \cup \{\infty\}$ only depends on the unoriented choice of longitude λ up to isotopy. Once we make such a choice, we define *p/q-surgery* on K to be Dehn surgery on K with respect to the framing associated to p/q in the above map. We denote the resulting manifold by $Y_{p/q}(K)$ and we call p/q the *framing coefficient* or *slope*.

Remark 2.1.4. ∞-surgery amounts to gluing $S^1 \times D^2$ to $Y - \nu(K)$ via a diffeomorphism that maps $x \times \partial D^2$ to the meridian μ. Since by definition, μ bounds a disk in $\nu(K)$, ∞-surgery is simply the operation of gluing back $\nu(K)$ in exactly the same way it originally sat. Hence, $Y_\infty(K) = Y$.

For an arbitrary knot in an arbitrary 3-manifold, there does not necessarily exist a canonical choice of longitude (i.e. 0-framing), and hence does not necessarily exist a canonical identification of framings with $\mathbb{Q} \cup \{\infty\}$. However, for S^3 there does. There are many ways to describe the canonical choice of longitude of a knot K in S^3. The route we take is via linking numbers.

Definition 2.1.5. Let K and K' be oriented knots in S^3 such that $K_1 \cap K_2 = \emptyset$. Let μ be a meridian for K, oriented by the right-hand rule. It is a standard result that $H_1(S^3 - K; \mathbb{Z}) = \mathbb{Z}[\mu]$. Define the *linking number* of K and K', $\ell k(K, K')$, to be $n \in \mathbb{Z}$ where $[K'] = n[\mu]$.

One can check that $\ell k(K, K')$ is an invariant of the isotopy class of the oriented link $K \cup K'$ and that $\ell k(K, K') = \ell k(K', K)$. The linking number can be computed diagrammatically as the signed number of times K crosses under K' in a planar diagram for $K \cup K'$, where the signs of crossings are specified in Figure 2-2.

(a) -1 crossing.

(b) $+1$ crossing.

Figure 2-2: Signs of crossings.

We now show how to specify a canonical choice of longitude using linking numbers. Given a knot $K \subset S^3$ with tubular neighborhood $\nu(K)$, put an arbitrary orientation on K. Let λ be an oriented simple closed curve on $\partial(\nu(K))$ such that $\ell k(K, \lambda) = 0$. It is straightforward to check that as an unoriented curve, λ is unique up to isotopy and is independent of the arbitrary choice of orientation on K. Define the *0-framed longitude* of K to be the unoriented isotopy class of λ.

2.1.2 Surgery diagrams and Rolfsen twists

One can generalize the notion of Dehn surgery on a knot to Dehn surgery on a link by performing Dehn surgery on each component of the link. The only added

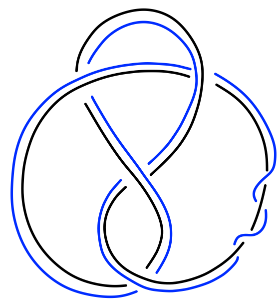


Figure 2-3: The blue curve is the 0-framed longitude of the figure-8 knot

requirement is that the tubular neighborhoods of each component must be small enough so as to not intersect each other. This requirement is superficial and does not have any substantive effect.

Just like with framings along knots in S^3, we would like to codify framings on links with numerical data. Given a link in S^3, one can specify a framing by assigning each component an extended rational number. However, for links with more than one component, there is a subtlety that must be handled to make this assignment unambiguous.

To describe this subtlety, consider Dehn surgery along a 2-component link $L = K \cup K'$ with framing coefficients r and r'. If one performs Dehn surgery on L by first doing r-framed Dehn surgery on K, then the ambient manifold in which K' sits before one completes the Dehn surgery on K' is $S^3_r(K)$. In $S^3_r(K)$, the framing coefficient of r' no longer has a well-defined meaning. For example, in $S^3_r(K)$, there might not even exist a canonical assignment of extended rational numbers to framings of K'. To fix this issue, we always interpret the framing coefficient of the component K_i of a link $L = K_1 \cup \cdots \cup K_n$ in S^3 as the framing coefficient of K_i considered as a knot in S^3, as if the other components don't exist.

Definition 2.1.6. A *surgery diagram* of a link L in S^3 is a planar diagram of L such that each component of L is decorated with an extended rational number. By the discussion above, a surgery diagram unambiguously specifies a Dehn surgery along a link in S^3. See Figure 2-4 for an example.

We now discuss some operations on surgery diagrams that preserve the diffeo-

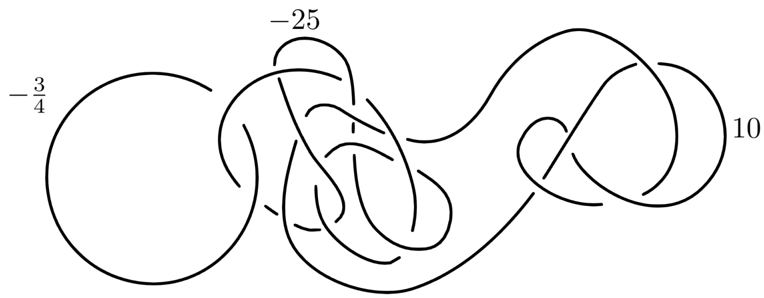
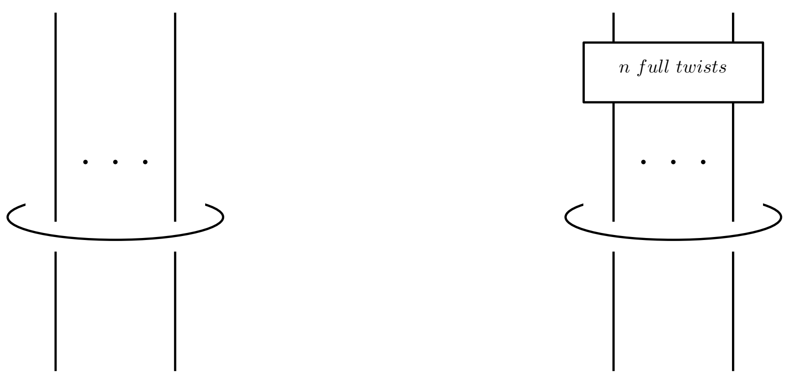


Figure 2-4: A surgery diagram on a 3-component link.

morphism type of the Dehn surgered 3-manifold that the diagram describes. If a component of a surgery diagram is labeled with ∞, then, by Remark 2.1.4, one may erase it. Conversely, one may introduce an arbitrary ∞-framed component to any diagram without affecting the 3-manifold. One may also isotope the diagram.

A more interesting move is the Rolfsen twist, [42, pp. 265-267], [12, p. 162]. For any integer n, this move allows us to locally replace Figure 2-5a with Figure 2-5b (or vice versa) in any surgery diagram.



(a) (b)

Figure 2-5: Rolfsen twist.

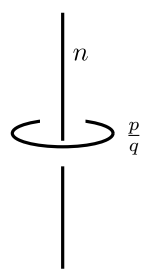
A few comments are necessary to fully explain this move. For one, there can be arbitrarily many strands running through the central unknot, independent of n. They can be strands of the same component or several different components. If $n \geq 0$, then the twist box means n full right-handed twists. If $n < 0$, then it means

$|n|$ full left-handed twists. We must also change the framings on the strands involved. Going from (a) to (b), the framing on the unknot changes from $\frac{p}{q}$ to $\frac{p}{q+np}$, and the framing of any component K which runs through the central unknot changes from r to $r + n\ell k(K, \text{central unknot}))^2$.

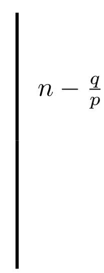
Remark 2.1.7. The Rolfsen twist in the case where the central unknot has framing ± 1 and $n = \mp 1$ is called a *blow-up/blow-down.*

Theorem 2.1.8 ([12, p.165])**.** *Two surgery diagrams describe the same 3-manifold if and only if one can be transformed formed into the other by a sequence of the moves described above, i.e. erasing or inserting ∞-framed components, isotopy, applying Rolfsen twists.*

We now describe one more move, called the *slam dunk.* The slam dunk is when we replace the local picture in Figure 2-6a with the local picture in Figure 2-6b (or vice versa).



(a)



(b)

Figure 2-6: Slam dunk. Note $n \in \mathbb{Z}$.

By Theorem 2.1.8, the slam dunk can be realized as a sequence of moves previously described, but it will be helpful to consider it as a separate move.

2.1.3 2-handlebodies and Kirby diagrams

If all of the framing coefficients of a surgery diagram are integers, then in addition to describing a 3-manifold Y, the surgery diagram also describes a particular type of 4-manifold X, called a *2*-handlebody, whose boundary is Y.

To build X from the data of a surgery diagram on a link L, first consider L as living in the boundary of a 4-ball, $L \subset \partial D^4 = S^3$. Then, for each link component K_i of L, attach a 2-handle, $D^2 \times D^2$, to D^4 via a diffeomorphism $\phi_i : \partial D^2 \times D^2 \to \nu(K_i)$ that maps $\partial D^2 \times x$ to $\gamma \subset \partial(\nu(K))$ where x is some point in the boundary of the second factor and γ is the framing (up to isotopy) determined by the framing coefficient of K_i. Like in Dehn surgery, the diffeomorphism type of Y is unambiguously specified by the framing coefficients of the K_i.

Remark 2.1.9. It is crucial in this construction, that all of the framing coefficients are integers. If the framing coefficient of K_i was a rational number that was not an integer, then no such ϕ_i exists because there are no diffeomorphisms: $S^1 \times D^2 \to S^1 \times D^2$ that map $S^1 \times x$ to a curve that winds around the S^1-factor of the target more than once. Similarly, the framing coeffcient cannot be equal to ∞ because there are no diffeomorphisms: $S^1 \times D^2 \to S^1 \times D^2$ that map $S^1 \times x$ to a curve that bounds a disk in the target.

When viewing a surgery diagram from the perspective of a 4-manifold, typically the surgery diagram is referred to as a *Kirby diagram*. In general, Kirby diagrams can involve 1- and 3-handles, in addition to 2-handles. They also come with their own set of moves, called *Kirby moves*. For details, we refer the reader to [23], [12], [1].

2.2 Small Seifert fibered spaces

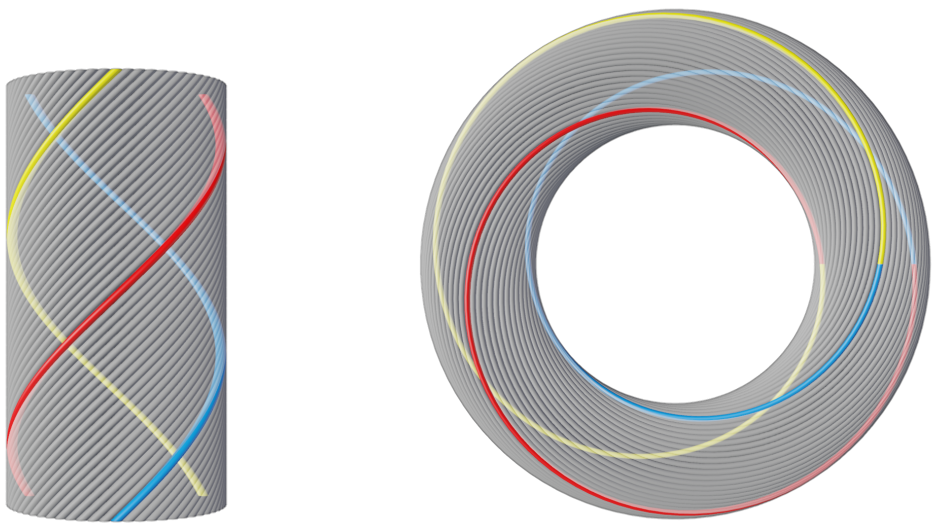
Seifert fibered spaces are an important class of 3-manifolds first introduced and studied in 1933 by Herbert Seifert [45]. To define Seifert fibered spaces, we first need to define their basic building blocks, *fibered solid tori*.

Definition 2.2.1. Consider the solid cylinder $D^2 \times [0,1]$ fibered as the trivial $[0,1]$-bundle over D^2 with fibers $x \times [0,1]$ for $x \in D^2$. Let p and q be a pair of coprime integers with $p > 0$. Define the *fibered solid torus* with *orbit type* p/q to be

$$T(p,q) = \frac{D^2 \times [0,1]}{(x,0) \sim (e^{2\pi i q/p}x, 1)}$$

together with the foliation by circles induced by the fibration on $D^2 \times [0,1]$.

If x is not the center of D^2, then the fiber containing $(x,0)$ in the above quotient is obtained by cyclically concatenating the line segments $x \times [0,1], e^{2\pi q/p}x \times [0,1], \ldots, e^{2\pi q(p-1)/p}x \times [0,1]$. If x is the center of D^2, then the fiber containing $(x,0)$ is obtained by gluing together the ends of the line segment $x \times [0,1]$. This fiber is called the *core* or *central fiber* of $T(p,q)$.



(a) The three line segments making up a non-core fiber before identifying the top and bottom.

(b) A non-core fiber.

Figure 2-7: $T(3,2)$.

Note that slicing $T(p,q)$ along a horizontal disk, then applying n full twists and regluing, produces a fiber and orientation preserving diffeomorphism from $T(p,q)$ to $T(p,q+np)$. Conversely, one can show that if there exists a fiber and orientation preserving diffeomorphism from $T(p,q)$ to $T(p',q')$, then $p' = p$ and $q' \equiv q \bmod p$. Note also that reflection through the horizontal disk $D^2 \times \frac{1}{2}$ yields an orientation reversing, fiber preserving diffeomorphism from $T(p,q)$ to $T(p,-q)$. Conversely, if there exists a fiber preserving diffeomorphism (which is not necessarily orientation preserving) from $T(p,q)$ to $T(p',q')$, then $p' = p$ and $q' \equiv \pm q \bmod p$.

From these observations we extract the following three facts:

1. The number p is an invariant of the fibered-diffeomorphism type of $T(p,q)$, independent of whether or not we allow orientation reversal.

2. The map

$$\{(p,q) \in \mathbb{Z}^2 \mid 0 \leq q < p, \gcd(p,q) = 1\} \to \frac{\text{solid fibered tori}}{\cong_{fo}}$$
$$(p,q) \mapsto T(p,q)$$

   is a bijection, where $\cong_{fo}$ denotes fiber and orientation preserving diffeomorphism.

3. The map

$$\{(p,q) \in \mathbb{Z}^2 \mid p > 0, 0 \leq q \leq p/2, \gcd(p,q) = 1\} \to \frac{\text{solid fibered tori}}{\cong_{f}}$$
$$(p,q) \mapsto T(p,q)$$

   is a bijection, where $\cong_f$ denotes fiber preserving diffeomorphism.

Definition 2.2.2. A *Seifert* fibered space is a 3-manifold Y together with a decomposition of Y into a disjoint union of circles called fibers such that:

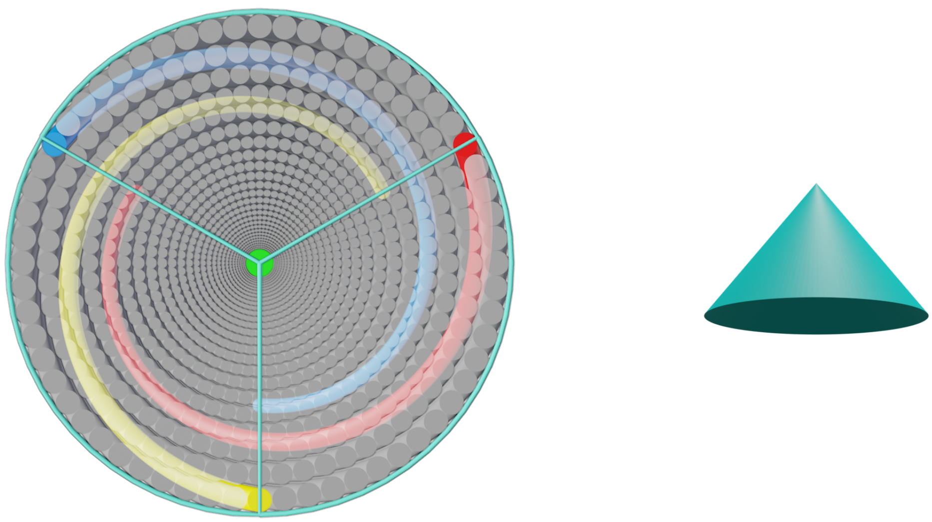
1. For each fiber F, there exists a tubular neighborhood T_F of F consisting of a union of fibers.

2. There exists a fiber preserving diffeomorphism from T_F to $T(p,q)$, for some p and q, such that F gets sent to the core of $T(p,q)$.

We call p the *multiplicity* of F. If $p > 1$, we say F is an *exceptional fiber*, otherwise we say F is an *ordinary fiber*.

Definition 2.2.3. Let Y be a Seifert fibered space, we define the *orbit space* or *base orbifold* of Y, Σ_Y, to be the quotient space of Y obtained by collapsing each fiber to a point.

As the name suggests, Σ_Y is indeed a surface that carries a natural orbifold structure. To see this, notice that the orbit surface of $T(p, q)$ is topologically a disk, which is flat if $p = 1$ and a cone if $p > 1$. In the case $p > 1$, the cone point corresponds to the quotient of the core of $T(p, q)$.



(a) Cross section view of $T(3, 2)$. The turquoise graph specifies three fundamental domains for the quotient.

(b) The base orbifold of $T(3, 2)$

Figure 2-8

Since, by definition, each fiber of a general Seifert fibered space Y has a fibered solid torus neighborhood, it follows from the above observations that Σ_Y is a surface with cone points at the image of the exceptional fibers under the quotient. For more details on orbifolds, including their formal definition, see [44].

We now focus our attention on a particular class of Seifert fibered spaces called *small Seifert fibered spaces.*

Definition 2.2.4. A *small Seifert fibered space* is a Seifert fibered space whose base orbifold is S^2 with at most three cone points.

Remark 2.2.5. In the literature, sometimes small Seifert fibered spaces are required to have exactly three cone points. However, to eliminate the need for added technicalities in later theorem statements, we allow them to have less than or equal to three cone points, i.e. less than or equal to three exceptional fibers.

Small Seifert fibered spaces can be parameterized by triples of non-zero extended rational numbers called *Seifert invariants*. Seifert invariants are essentially the same thing as the orbit types of the exceptional fibers, with a mild wrinkle, which we now describe.

Let Y be a small Seifert fibered space and fix an orientation on Y. Note, all small Seifert fibered spaces are closed orientable 3-manifolds. Let F be an exceptional fiber and let T_F be a fibered solid torus neighborhood of F. Let $\gamma \subset \partial T_F$ be a meridian of T_F, i.e. a simple closed curve on ∂T_F that bounds a disk in T_F. Choose an arbitrary orientation on F and then orient γ according to the right-hand rule with respect to F. Next, let μ be a simple closed curve on ∂T_F that intersects each fiber on ∂T_F exactly once and let λ be a fiber on ∂T_F. Orient μ and λ, so that λ travels in the same direction as F and the algebraic intersection number $\mu \cdot \lambda = 1$ with respect to the orientation on ∂T_F induced by the orientation on Y.

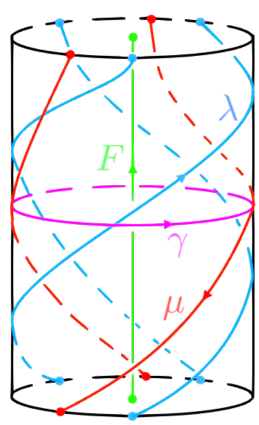


Figure 2-9: A model when $T_F \cong_{fo} T(3,2)$. Here, the top and bottom are identified.

Since μ and λ intersect exactly once, their homology classes form a basis for $H_1(\partial T_F; \mathbb{Z})$. Hence, we can express $[\gamma]$ uniquely as $[\gamma] = \alpha[\mu] + \beta[\lambda]$. We define $\dfrac{\alpha}{\beta}$ to be the *Seifert invariant* associated to the exceptional fiber F. We now make a couple of observations.

1. The terminology *Seifert invariant* is a little misleading. If we replace our choice of $[\mu]$ with $[\mu]+n[\lambda]$ for some integer n, then the Seifert invariant $\dfrac{\alpha}{\beta}$ transforms into $\dfrac{\alpha}{\beta+n\alpha}$. Hence, like orbit type, it is not actually an invariant until it is normalized. Since small Seifert fibered spaces are orientable, we will use the equivalence relation $\cong_{fo}$ rather than $\cong_f$. Under $\cong_{fo}$, one may normalize the Seifert invariants, so that $0 \leq \beta < \alpha$. However, we will not be concerned with normalization, and we will allow non-normalized invariants.

2. The relationship between the (unnormalized) orbit type p/q and the Seifert invariant α/β of T_F is: $p = |\alpha|$ and $\beta q \equiv 1 \bmod p$.

Using Seifert invariants and Dehn surgery, we can now concretely describe all small Seifert fibered spaces. Given $\dfrac{\alpha_1}{\beta_1}, \dfrac{\alpha_2}{\beta_2}, \dfrac{\alpha_3}{\beta_3} \in (\mathbb{Q}-\{0\}) \cup \{\infty\}$, let $S^2\left(\dfrac{\alpha_1}{\beta_1}, \dfrac{\alpha_2}{\beta_2}, \dfrac{\alpha_3}{\beta_3}\right)$ be the closed oriented 3-manifold described by the surgery diagram in Figure 2-10.

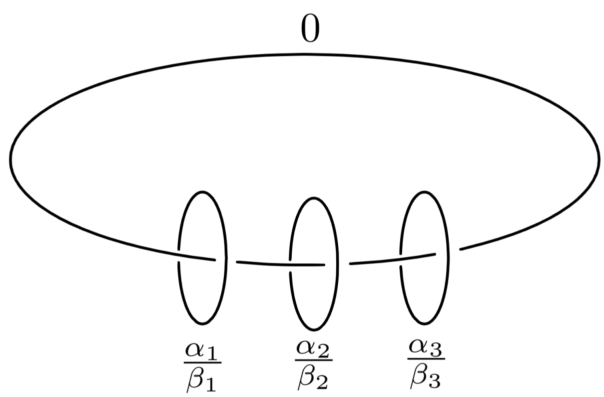


Figure 2-10: $S^2\left(\dfrac{\alpha_1}{\beta_1}, \dfrac{\alpha_2}{\beta_2}, \dfrac{\alpha_3}{\beta_3}\right)$

It is not difficult to show that $S^2\left(\dfrac{\alpha_1}{\beta_1}, \dfrac{\alpha_2}{\beta_2}, \dfrac{\alpha_3}{\beta_3}\right)$ is a small Seifert fibered space with (unnormalized) Seifert invariants $\dfrac{\alpha_i}{\beta_i}$. We briefly sketch this fact. First, note that 0-surgery on the central unknot in Figure 2-10 yields $S^1 \times S^2$, which can be regarded as the trivial small Seifert fibered space with no exceptional fibers. The other three components in the diagram, before doing surgery on them, are disjoint circle fibers of $S^1 \times S^2$. Therefore, performing $\frac{\alpha_i}{\beta_i}$-surgeries on these three components amounts to cutting out a trivially fibered solid torus neighborhood of each component and then gluing in a new fibered solid torus whose core has Seifert invariant $\frac{\alpha_i}{\beta_i}$. It is evident that the effect on the base orbifold of $S^1 \times S^2$, which is just S^2 with no-cone points, of these three surgeries is to replace three disjoint disks of S^2 with three cones, assuming $|\alpha_i| > 1$. If $|\alpha_i| = 1$, then the corresponding fiber is not exceptional, so rather than replacing a disk with a cone, one replaces a disk with another disk. In any case, it follows that the base orbifold of $S^2\left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \frac{\alpha_3}{\beta_3}\right)$ is S^2 with less than or equal to three cone points, and thus $S^2\left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \frac{\alpha_3}{\beta_3}\right)$ is a small Seifert fibered space. Moreover, every oriented small Seifert fibered space is orientation and fiber preservingly diffeomorphic to some $S^2\left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \frac{\alpha_3}{\beta_3}\right)$ and

Theorem 2.2.6 ([30, Theorem 1.1], [28])**.** *Let* $\frac{\alpha_i}{\beta_i}, \frac{\alpha_i'}{\beta_i'} \in (\mathbb{Q} - \{0\}) \cup \{\infty\}$ *for* $i = 1, 2, 3$. *WLOG, assume* $\alpha_i > 0$ *and* $\gcd(\alpha_i, \beta_i) = 1$ *for each* i. *Then,*

$$S^2\left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \frac{\alpha_3}{\beta_3}\right) \cong_{fo} S^2\left(\frac{\alpha_1'}{\beta_1'}, \frac{\alpha_2'}{\beta_2'}, \frac{\alpha_3'}{\beta_3'}\right)$$

if and only if, after possibly reindexing, there exists an integer k *with* $0 \leq k \leq 3$ *such that*

1. $\alpha_i = \alpha_i' \neq 1$ *for* $i = 1, \ldots, k$ *and* $\alpha_i = \alpha_i' = 1$ *for* $i > k$.

2. $\beta_i \equiv \beta_i' \bmod \alpha_i$ *for* $i = 1, \ldots, k$.

3. $\sum_{i=1}^{3} \frac{\beta_i}{\alpha_i} = \sum_{i=1}^{3} \frac{\beta_i'}{\alpha_i'}$

Definition 2.2.7. The quantity $-\sum_{i=1}^{3} \frac{\beta_i}{\alpha_i}$ from the previous theorem is called the *Euler number* of the Seifert fibered structure and is denoted by e.

2.3 Plumbed manifolds

Given a graph Γ, we denote the set of vertices of Γ by $\mathcal{V}(\Gamma)$ and the set of edges by $\mathcal{E}(\Gamma)$.

Definition 2.3.1. A *weighted graph* is a graph Γ together with a function $m : \mathcal{V}(\Gamma) \to \mathbb{Z}$, called a *weight function*. Given a vertex $v \in \mathcal{V}(\Gamma)$, we call $m(v)$ the weight of v. Usually we will refer to a weighted graph as Γ and do not explicitly write the weight function associated to it.

For the purposes of this thesis, we will use the term *plumbing graph* to mean a weighted graph Γ such that Γ is a forest (i.e. a disjoint union of trees) and $|\mathcal{V}(\Gamma)| < \infty$. Plumbing graphs in general can be more complicated, however for simplicity we only consider plumbing graphs of the type just described. As we explain in more detail below, our plumbed manifolds will be those obtained by plumbing disk bundles over 2-spheres according to such plumbing graphs.

Construction 2.3.2 (Plumbed manifolds)**.** Let Γ be a plumbing graph. First, suppose that Γ is connected. Then, the plumbed 4-manifold $X(\Gamma)$ is constructed in the following way:

1. To each vertex $v \in \mathcal{V}(\Gamma)$, associate the D^2-bundle $\pi_v : E(v) \to S^2$ over the 2-sphere with Euler number equal to $m(v)$. Here we are implicitly using the fact that the Euler number gives a bijection from bundle isomorphism classes of D^2-bundles over S^2 to $\mathbb{Z}$.

2. For each edge $[u, v]$ connecting vertices $u, v \in \mathcal{V}(\Gamma)$, we choose disks $D^2_{u,[u,v]}$ and $D^2_{v,[u,v]}$ in the base 2-spheres of the respective bundles $E(u)$ and $E(v)$. If a vertex u is adjacent to multiple edges $[u, v_1], \ldots, [u, v_\ell]$, then we choose the discs, $D^2_{u,[u,v_1]}, \ldots, D^2_{u,[u,v_\ell]}$, associated to u, to be pairwise disjoint in the base 2-sphere of the bundle $E(u)$.

3. Since $D^2_{u,[u,v]}$ and $D^2_{v,[u,v]}$ are contractible, the restrictions $\pi_u : \pi_u^{-1}(D^2_{u,[u,v]}) \to D^2_{u,[u,v]}$ and $\pi_v : \pi_v^{-1}(D^2_{v,[u,v]}) \to D^2_{v,[u,v]}$ are trivial D^2-bundles. Now, for each

edge $[u, v]$, we identify $\pi_u^{-1}(D^2_{u,[u,v]})$ with $\pi_v^{-1}(D^2_{v,[u,v]})$ by a diffeomorphism that swaps the two factors in the product structure, $D^2 \times D^2$, of these bundles. In other words, after choosing trivializations of the two restriction bundles, we send $(x, y) \in D^2 \times D^2$ to (y, x).

If Γ has multiple connected components, then apply the above construction to each component and then boundary connect sum the resulting 4-manifolds.

Definition 2.3.3. The 4-manifold $X(\Gamma)$ constructed from a plumbing graph Γ by the process described above is called the *plumbed 4-manifold* with plumbing graph Γ. The boundary of $X(\Gamma)$, denoted $Y(\Gamma)$, is called the *plumbed 3-manifold* with plumbing graph Γ.

Remark 2.3.4. In general, a given plumbed 3-manifold Y may bound many different plumbed 4-manifolds. In [29], Neumann describes a calculus for passing between different plumbing graphs that describe the same 3-manifold. This calculus is, in a sense, a specialized version of Kirby calculus for plumbed manifolds.

Given a plumbing graph Γ, a Kirby diagram for $X(\Gamma)$ (which is also a surgery diagram for $Y(\Gamma)$) is given by an $m(v)$-framed unknot for each $v \in \mathcal{V}(\Gamma)$ such that any pair of these unknots is either Hopf linked or unlinked depending on whether or not there is an edge between the vertices with which the unknots correspond.

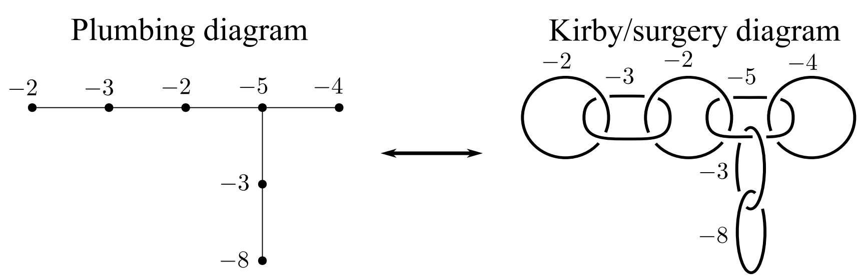


Figure 2-11: An example of how to pass between a plumbing graph and a Kirby/surgery diagram.

2.3.1 Algebraic topological properties of plumbings

Fix a plumbing graph Γ. Let $X = X(\Gamma)$ and $Y = Y(\Gamma)$ be the associated plumbed 4- and 3-manifolds. Label the vertices of Γ by $\mathcal{V}(\Gamma) = \{v_1, \ldots, v_s\}$ where $s = |\mathcal{V}(\Gamma)|$.

For each $v_j \in \mathcal{V}(\Gamma)$, let $[v_j] \in H_2(X;\mathbb{Z})$ be the homology class of the 2-sphere corresponding to the 0-section of the D^2-bundle associated to v_j. Equivalently, $[v_j]$ is represented by the capped-off core of the corresponding 2-handle. In particular, it is easy to see that $H_2(X;\mathbb{Z}) \cong \bigoplus_{j=1}^{s} \mathbb{Z}[v_j]$. Given $x = \sum a_j[v_j] \in H_2(X;\mathbb{Z})$, we write $x \geq 0$ if $a_j \geq 0$ for all j. If in addition, $x \neq 0$, we write $x > 0$. Given two elements $x, y \in H_2(X;\mathbb{Z})$, we write $x \geq y$ $(x > y)$ if $x - y \geq 0$ $(x - y > 0)$.

Denote the intersection form of X by

$$(\cdot,\cdot) : H_2(X,\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$$

By construction,

$$([v_i],[v_j]) = \begin{cases} m(v_i) & i = j \\ 1 & \text{if } i \neq j \text{ and there is an edge } [v_i, v_j] \text{ connecting } v_i \text{ and } v_j \\ 0 & \text{otherwise} \end{cases}$$

Let B be the matrix of the intersection form with respect to the ordered basis $([v_1],\ldots,[v_s])$. Notice, B is the incidence matrix of the graph Γ with the *ith*-diagonal entry equal to $m(v_i)$.

Definition 2.3.5. We define the *definiteness type* of a plumbing graph Γ to be the definiteness type of its associated intersection form $(\cdot,\cdot)$, or equivalently the definiteness type of B. For example, we say Γ is negative semi-definite if $(\cdot,\cdot)$ is negative semi-definite.

By an abuse of notation, we will also refer to the corresponding intersection pairing on cohomology as $(\cdot,\cdot) : H^2(X,Y;\mathbb{Z}) \times H^2(X,Y;\mathbb{Z}) \to \mathbb{Z}$. In addition, it will be useful to consider the slightly modified intersection pairing $(\cdot,\cdot)' : H^2(X;\mathbb{Z}) \times H^2(X,Y;\mathbb{Z}) \to \mathbb{Z}$ with a different domain, but still defined by the usual formula: $(\alpha,\beta)' = (\alpha \cup \beta)[X]$.

Recall, the set of characteristic vectors of X, denoted $\mathrm{Char}(X)$, is defined by

$$\begin{aligned}\mathrm{Char}(X) &= \{\alpha \in H^2(X;\mathbb{Z}) \mid (\alpha,\beta)' \equiv (\beta,\beta) \bmod 2, \forall \beta \in H^2(X,Y;\mathbb{Z})\} \\ &= \{\alpha \in H^2(X;\mathbb{Z}) \mid \alpha(x) \equiv (x,x) \bmod 2 \text{ for all } x \in H_2(X;\mathbb{Z})\}\end{aligned}$$

We now recall the relationship between the spin^c structures on X and Y and the characteristic vectors of X. The first observation is that we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{spin}^c(X) & \xrightarrow{|_Y} & \mathrm{spin}^c(Y) \\ \big\downarrow{\scriptstyle c_1} & & \big\downarrow{\scriptstyle c_1} \\ \mathrm{Char}(X) & \xrightarrow{\partial^*} & H^2(Y;\mathbb{Z}) \end{array}$$

Here, c_1 denotes the first Chern class of the determinant line bundle of the spin^c structure, the top horizontal map is restriction to Y, and the bottom horizontal map is the restriction of the map $\partial^* : H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ in the long exact sequence in cohomology of the pair (X,Y). The left vertical map is a bijection since $H_1(X,\mathbb{Z})$ has no 2-torsion (see [12, p. 56] for details). Therefore, c_1 provides a canonical identification of $\mathrm{spin}^c(X)$ with $\mathrm{Char}(X)$. Furthermore, since X is simply connected, we have the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^1(Y;\mathbb{Z}) & \xrightarrow{i^*} & H^2(X,Y;\mathbb{Z}) & \xrightarrow{j^*} & H^2(X;\mathbb{Z}) & \xrightarrow{\partial^*} & H^2(Y;\mathbb{Z}) & \longrightarrow & 0 \\ & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \\ 0 & \longrightarrow & H_2(Y;\mathbb{Z}) & \xrightarrow{i_*} & H_2(X;\mathbb{Z}) & \xrightarrow{j_*} & H_2(X,Y;\mathbb{Z}) & \xrightarrow{\partial_*} & H_1(Y;\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

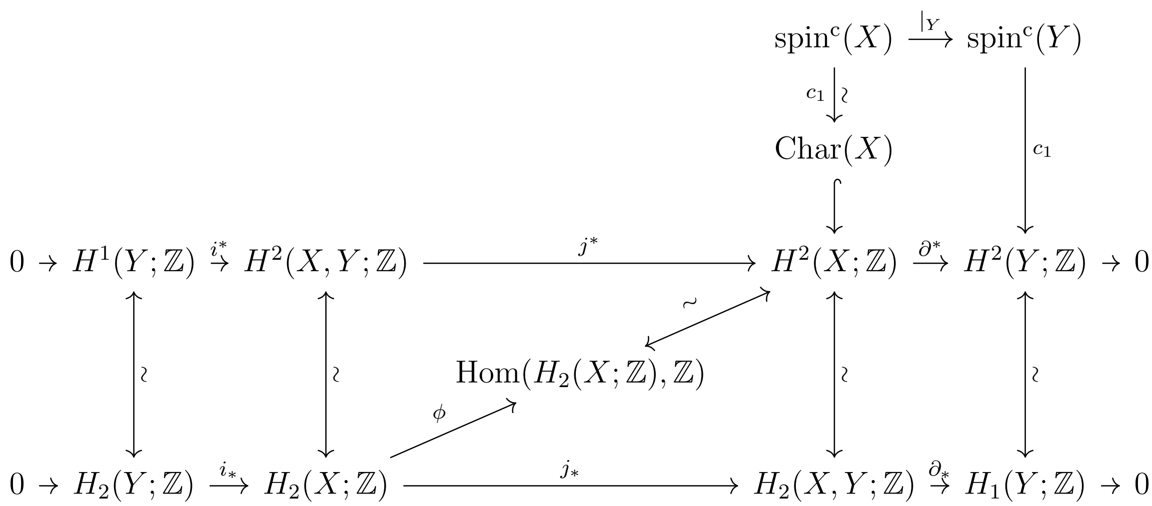
with exact rows coming from the long exact sequences in homology and cohomology of the pair (X,Y), and with vertical isomorphisms given by Poincaré/Lefschetz duality.

We have yet another commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}) & \cong & H^2(X;\mathbb{Z}) \\ {\scriptstyle \phi}\big\uparrow & & \|\wr \\ H_2(X;\mathbb{Z}) & \xrightarrow{j_*} & H_2(X,Y;\mathbb{Z}) \end{array}$$

Here, the top row is the isomorphism coming from the universal coefficient theorem, the right vertical map is the Lefschetz duality isomorphism, and the map ϕ is defined by $\phi(x) = (x,\cdot)$.

Combining the three previous diagrams we get the following commutative diagram:



In addition, there is a free and transitive action of $H^2(X;\mathbb{Z})$ on $\text{Char}(X)$, defined by $(\alpha, k) \mapsto k + 2\alpha$ for all $\alpha \in H^2(X;\mathbb{Z})$ and $k \in \text{Char}(X)$. Restricting this action to $j^*(H^2(X,Y;\mathbb{Z}))$, we get an action of $j^*(H^2(X,Y;\mathbb{Z}))$ on $\text{Char}(X)$. Let $\text{Char}(X)/2j^*(H^2(X,Y;Z))$ denote the set of orbits of this action and denote the orbit $k + 2j^*(H^2(X,Y;\mathbb{Z}))$ of an element k by $[k]$. We have the following standard fact.

Proposition 2.3.6. *The map* $\Psi : \text{Char}(X)/2j^*(H^2(X,Y;\mathbb{Z})) \to \text{spin}^c(Y)$ *given by*

$$\Psi([k]) = c_1^{-1}(k)|_Y$$

is well-defined and is a bijection.

Notation 2.3.7. Justified by the above proposition, we will use $[k]$ to denote both the orbit $k + 2j^*(H^2(X,Y;\mathbb{Z}))$ as well as the corresponding spin^c structure $\Psi([k])$.

Remark 2.3.8. From the above diagram, one can see that if k is a characteristic vector, then $[k]$ is a torsion spin^c structure on Y if and only if some integer multiple of k is in the image of j^*. Equivalently, $[k]$ is torsion if and only if there exists some $z_k \in H_2(X;\mathbb{Z}) \otimes \mathbb{Q}$ such that $k(x) = (z_k, x)$ for all $x \in H_2(X;\mathbb{Z})$.

2.3.2 Rationality and weight conditions

We now recall some terminology that will be useful in Chapter 6 when we discuss lattice cohomology and Heegaard Floer homology of plumbings.

If Γ is a negative definite plumbing tree, then there is a special characteristic vector K_{can} called the *canonical characteristic vector*. It is defined by the equation $K_{can}(v) = -m(v) - 2$ for all $v \in \mathcal{V}(\Gamma)$.

Definition 2.3.9. A plumbing graph Γ is called *rational* if it is a negative definite tree which satisfies the following condition: if $x \in H_2(X(\Gamma); \mathbb{Z})$ and $x > 0$, then

$$-\frac{K_{can}(x) - (x, x)}{2} \geq 1$$

In [31], Némethi introduces the following generalization of rational plumbings:

Definition 2.3.10 ([31, Definition 8.1])**.** A negative definite plumbing tree Γ is *almost rational* if there exists a vertex $v \in \mathcal{V}(\Gamma)$ and some integer $r \leq m(v)$ such that if you replace the weight of v with r, Γ becomes rational.

A further generalization of this notion is the following:

Definition 2.3.11 (See [35, Definition 2.1])**.** A plumbing tree Γ is *type n* if there exist n vertices of Γ such that if we reduce their weights sufficiently, the plumbing becomes rational.

Remark 2.3.12. A type n plumbing is not required to be negative definite.

The *degree*, denoted $\delta(v)$, of a vertex $v \in \mathcal{V}(\Gamma)$ is the number of edges adjacent to v. Following the terminology introduced in [37], we say a vertex is *bad* if $m(v) > -\delta(v)$. In particular, it can be shown that a negative definite plumbing with at most one bad vertex is almost rational.

Chapter 3

Basic small Seifert fibered 0-surgery obstructions

3.1 0-surgery homological constraints on Seifert invariants

As mentioned in the introduction, it is a standard result in the theory of Dehn surgery that for a knot $K \subset S^3$, $H_1(S_0(K);\mathbb{Z}) \cong \mathbb{Z}$. Since the focus of this thesis is to understand which small Seifert fibered spaces arise as 0-surgery on a knot in S^3, it will be useful to have a characterization of which small Seifert fibered spaces have first homology $\mathbb{Z}$.

Proposition 3.1.1. *Let* $Y = S^2\left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \frac{\alpha_3}{\beta_3}\right)$. *WLOG, assume* $(\alpha_i, \beta_i) = 1$ *and* $\alpha_i > 0$ *for* $i = 1, 2, 3$. *Then,* $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ *if and only if, after possibly reindexing, there exists pairwise coprime positive integers* p, q, r *such that:* $\alpha_1 = pq, \alpha_2 = pr, \alpha_3 = rq$, *and* $r\beta_1 + q\beta_2 + p\beta_3 = 0$.

Proof. There is a standard procedure for producing a presentation matrix A for the first homology of a 3-manifold from a surgery diagram [12, Proposition 5.3.11]. Applying this procedure to the surgery diagram for Y given in Figure 2-10, we see that

$$A = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ 0 & \alpha_2 & 0 & 1 \\ 0 & 0 & \alpha_3 & 1 \\ \beta_1 & \beta_2 & \beta_3 & 0 \end{pmatrix}$$

In other words, $H_1(Y;\mathbb{Z}) \cong \mathrm{coker}(A) = \mathbb{Z}^4 / A\,\mathbb{Z}^4$.

Next, we recall the notion of Fitting ideals. $\mathrm{Fitt}_i(A)$ is the ideal of $\mathbb{Z}$ generated by the $(n-i)$-dimensional minors of A. In particular, $\mathrm{Fitt}_0(A) = (\det(A))$, and, from a direct calculation, $\mathrm{Fitt}_1(A)$ is the ideal generated by the following elements:

$$\begin{array}{llll} \alpha_1\alpha_2\alpha_3 & \alpha_1\alpha_2\beta_3 & \alpha_1\alpha_3\beta_2 & \alpha_2\alpha_3\beta_1 \\ \alpha_1\alpha_2 & \alpha_2\beta_1 + \alpha_1\beta_2 & \alpha_1\beta_2 & \alpha_2\beta_1 \\ \alpha_1\alpha_3 & \alpha_1\beta_3 & \alpha_3\beta_1 + \alpha_1\beta_3 & \alpha_3\beta_1 \\ \alpha_2\alpha_3 & \alpha_2\beta_3 & \alpha_3\beta_2 & \alpha_3\beta_2 + \alpha_2\beta_3 \end{array}$$

We can greatly simplify this description of $\mathrm{Fitt}_1(A)$. Since $\gcd(\alpha_1, \beta_1) = 1$, there exists integers r, s such that $r\alpha_1 + s\beta_1 = 1$. In particular, $r\alpha_1\alpha_2 + s\alpha_2\beta_1 = \alpha_2$. Therefore, $\alpha_2 \in \mathrm{Fitt}_1(A)$. Similarly, $\alpha_1, \alpha_3 \in \mathrm{Fitt}_1(A)$. Hence, $\mathrm{Fitt}_1(A) = (\alpha_1, \alpha_2, \alpha_3)$.

It is a standard result [5, Proposition 1.4.9] that $\mathrm{coker}(A) \cong \mathbb{Z}$ if and only if $\mathrm{Fitt}_0(A) = 0$ and $\mathrm{Fitt}_1(A) = \mathbb{Z}$. In other words, $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ if and only if $\det(A) = 0$ and $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$. So suppose $\det(A) = 0$ and $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$. Then,

$$0 = \det(A) = \alpha_1\alpha_2\beta_3 + \alpha_1\alpha_3\beta_2 + \alpha_2\alpha_3\beta_1 \tag{3.1.2}$$

Thus, α_1 divides $\alpha_2\alpha_3\beta_1$. Since $\gcd(\alpha_1, \beta_1) = 1$, this implies α_1 divides $\alpha_2\alpha_3$. Similarly, α_2 divides $\alpha_1\alpha_3$ and α_3 divides $\alpha_1\alpha_2$.

Next, let $p = \gcd(\alpha_1, \alpha_2) \geq 1$. Then there exist coprime positive integers q and r such that $\alpha_1 = pq$ and $\alpha_2 = pr$. Since α_1 divides $\alpha_2\alpha_3$, we must have that q divides α_3. So we can write $\alpha_3 = nq$ for some integer n. Since α_2 divides $\alpha_1\alpha_3$, we must have that r divides n. Write $n = rm$ for some m, so $\alpha_3 = mrq$. Since α_3 divides $\alpha_1\alpha_2$, we must have that m divides p^2. Since p is a factor of α_1 and α_2 and $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$, we must have that $m = \pm 1$. Since we are assuming $\alpha_3 > 0$, we must have $m = 1$. Thus, $\alpha_1 = pq, \alpha_2 = pr, \alpha_3 = rq$. Also, it follows from the condition $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$ that p, q, r are pairwise coprime.

Now returning to equation 3.1.2, we see that:

$$(pqr)p\beta_3 + (pqr)q\beta_2 + (pqr)r\beta_1 = 0$$

Therefore, $p\beta_3 + q\beta_2 + r\beta_1 = 0$.

Conversely, it is immediate that if $\alpha_1 = pq, \alpha_2 = pr, \alpha_3 = rq$ for pairwise coprime positive integers p, q, r and $p\beta_3 + q\beta_2 + r\beta_1 = 0$, then $\det(A) = 0$ and $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$. □

Corollary 3.1.3. *The Euler number of a small Seifert fibered space with first homology $\mathbb{Z}$ is equal to* 0*.*

Proof. Let Y be a small Seifert fibered space with first homology $\mathbb{Z}$, i.e., $Y = S^3\left(\frac{pq}{\beta_1}, \frac{pr}{\beta_2}, \frac{rq}{\beta_3}\right)$ for $p, q, r, \beta_1, \beta_2, \beta_3$ as above in Proposition 3.1.1. First, assume p, q, r are non-zero. Then,

$$e(Y) = -\left(\frac{\beta_1}{pq} + \frac{\beta_2}{pr} + \frac{\beta_3}{rq}\right) = -\frac{\beta_1 r + q\beta_2 + p\beta_3}{pqr} = 0$$

Suppose instead that we are in the degenerate case where at least one of p, q, r are equal to 0. WLOG, assume $p = 0$. Since $\gcd(pq, \beta_1) = 1$ and $\gcd(pr, \beta_2) = 1$, we have $\beta_1, \beta_2 \in \{-1, 1\}$. Therefore, $r = \pm q$. But r and q are coprime and positive. Therefore, $rq = 1$. Since $\gcd(rq, \beta_3) = 1$, $\beta_3 = \pm 1$. In summary, $Y = S^2(0, 0, \pm 1)$, which, from the surgery description in Figure 2-10, is easily seen to be diffeomorphic to $S^1 \times S^2$. Hence, Y is an honest circle bundle over a surface and thus it's Euler number is defined in the usual fiber bundle sense. In particular, since Y is the trivial bundle, $e(Y) = 0$. □

3.2 Surface bundle structure and a constraint on the Alexander polynomial

Consider the small Seifert fibered space $Y = S^2\left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \frac{\alpha_3}{\beta_3}\right)$. One can show via a straightforward application of the Seifert-Van Kampen Theorem that

$$\pi_1(Y) = \langle x, y, z, h \mid xyz, x^{\alpha_1}h^{\beta_1}, y^{\alpha_2}h^{\beta_2}, z^{\alpha_3}h^{\beta_3}, [x,h], [y,h], [z,h]\rangle$$

where $x, y,$ and z are meridians of the 3 exceptional fibers and h is an ordinary fiber. In particular, the ordinary fiber is central. This fact together with Corollary 3.1.3 and [44, Theorem 5.4, (ii)] implies that all small Seifert fibered spaces with first homology $\mathbb{Z}$ are surface bundles over S^1 with periodic monodromy. Combining this observation with [11, Corollarly 8.19] yields the following proposition.

Proposition 3.2.1. *Let K be a knot in S^3. If $S^3_0(K)$ is a small Seifert fibered space, then K is a fibered knot.*

By using this proposition and analyzing the surface bundle structure of small Seifert fibered spaces with first homology $\mathbb{Z}$, we obtain the following constraint.

Theorem 3.2.2. *Let $Y = S^2\left(\frac{pq}{\beta_1}, \frac{pr}{\beta_2}, \frac{rq}{\beta_3}\right)$ with $p, q, r, \beta_1, \beta_2, \beta_3$ as in Proposition 3.1.1. Further suppose $p, q, r \geq 2$. If K is a knot in S^3 such that $S^3_0(K) = Y$, then the coefficient of the next to top degree term of the Alexander polynomial of K is equal to ± 2.*

Proof. From the observations above, we know that Y is a surface bundle over a circle with periodic monodromy. In other words,

$$Y = \frac{F \times [0,1]}{(x,0) \sim (\phi(x), 1)} \tag{3.2.3}$$

where F is a closed orientable surface and $\phi : F \to F$ is the periodic monodromy with order n. Moreover, as discussed in [44, p.442], the Seifert fibered structure of Y is induced by ϕ. Specifically, if $x \in F$ and $m_x \geq 1$ is the minimal integer such that $h^{m_x}(x) = x$, then the Seifert fiber containing $(x, 0) \in Y$ is the image of $x \times [0,1] \cup \cdots \cup h^{m_x - 1}(x) \times [0,1]$ in the quotient 3.2.3. From this, we see that the ordinary Seifert fibers are those with $m_x = n$ and the exceptional fibers are those with $m_x < n$, i.e., those with non-trivial stabilizer under the action of h. In particular, if the Seifert fiber passing through $(x, 0)$ is exceptional, then the multiplicity of that fiber is equal to the order of the stabilizier of x. So from the orbit-stabilizer theorem, the multiplicity of that fiber divides n. Since Y has 3 exceptional fibers of multiplicities pq, pr, rq and p, q, r are pairwise coprime, we must have that pqr divides n. By assumption $p, q, r \geq 2$, therefore it follows that n is strictly greater than the order of all stabilizers of the h action on F. Thus, ϕ has no fixed points.

Now by Proposition 3.2.1, if K is a knot in S^3 such that $S^3_0(K) = Y$, then K is a fibered knot, i.e.

$$S^3 - \nu(K) = \frac{\Sigma \times [0,1]}{(x,0) \sim (\psi(x), 1)}$$

where $\nu(K)$ is a tubular neighborhood of K, Σ is a closed oriented surface with a disk removed, and $\psi : \Sigma \to \Sigma$ is the monodromy of the knot. Performing 0-surgery on K has the effect of gluing a disk to each surface fiber of $S^3 - \nu(K)$, and hence we get a capped off monodromy $\widehat{\psi} : \Sigma \cup D^2 \to \Sigma \cup D^2$ on the closed surface $\Sigma \cup D^2$, which induces the structure of a surface bundle over a circle on Y. Since closed oriented 3-manifolds with rank 1 2nd homology admit at most one surface bundle structure up to isotopy, we see that $\Sigma \cup D^2 \cong F$ and $\widehat{\psi}$ is isotopic to ϕ.

Next, it is well-known that the Alexander polynomial of K is equal to the characteristic polynomial of the induced map $\psi_* : H_1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^{2g} \to \mathbb{Z}^{2g} \cong H_1(\Sigma; \mathbb{Z})$ where g is the genus of Σ. Since extending ψ to the capped off monodromy $\widehat{\psi}$ does not effect the induced map on H_1 and since ϕ is isotopic to $\widehat{\psi}$, it follows that the Alexander polynomial of K is equal to the characteristic polynomial of the map $\phi_* : H_1(F; \mathbb{Z}) \cong \mathbb{Z}^{2g} \to \mathbb{Z}^{2g} \cong H_1(F; \mathbb{Z})$. But we know ϕ has no fixed points. Therefore, by the Lefschetz fixed point theorem

$$0 = trace(\phi_*|H_0(F;\mathbb{Z})) - trace(\phi_*|H_1(F;\mathbb{Z})) + trace(\phi_*|H_2(F;\mathbb{Z}))$$

Since ϕ is an orientation preserving diffeomorphism, $trace(\phi_*|H_0(F;\mathbb{Z})) = 1$ and $trace(\phi_*|H_2(F;\mathbb{Z})) = 1$. Thus, $trace(\phi_*|H_1(F;\mathbb{Z})) = 2$, which is (-1)-times the coefficient of the next to top degree term of of the characteristic polynomial of $\phi_* : H_2(F;\mathbb{Z}) \to H_2(F;\mathbb{Z})$. Since the Alexander polynomial is only defined up multiplication by units, this implies that the coefficient of the next to top degree term of the Alexander polynomial of K is equal to ± 2. □

Corollary 3.2.4. *If Y is as in Theorem 3.2.2. Then, Y cannot be obtained by* 0-*surgery on an L-space knot.*

Proof. This follows immediately from Theorem 3.2.2 and [34, Corollary 1.3]. □

Chapter 4

A 2-parameter generalization of the Ichihara-Motegi-Song examples

The goal of this brief chapter is to point out a simple observation which generalizes the construction in [20] to produce a new set of examples of small Seifert fibered spaces that are 0-surgery on a knot in S^3. Let n be an integer and consider the surgery diagram in Figure 4-1 with one unframed component labeled K_n.

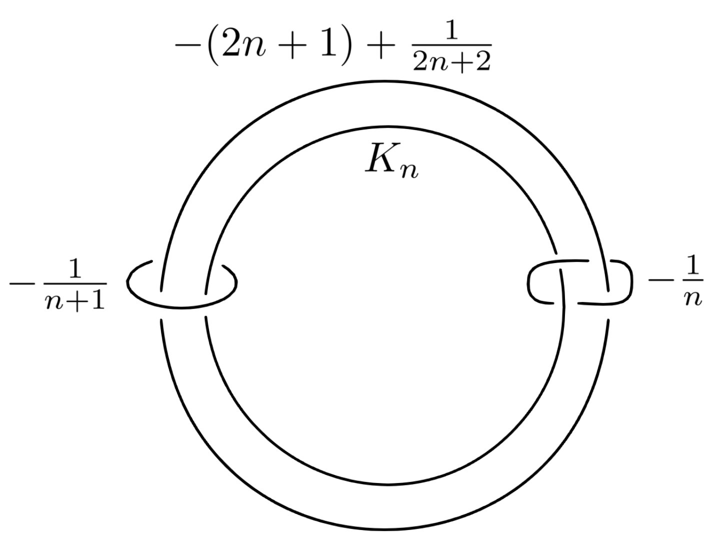
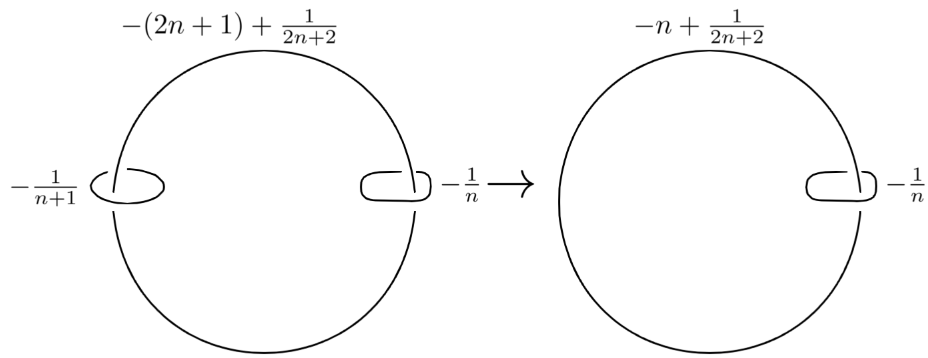
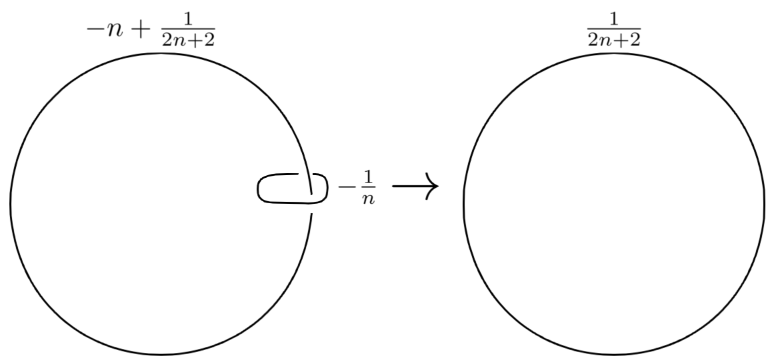


Figure 4-1: Ichihara-Motegi-Song knot.

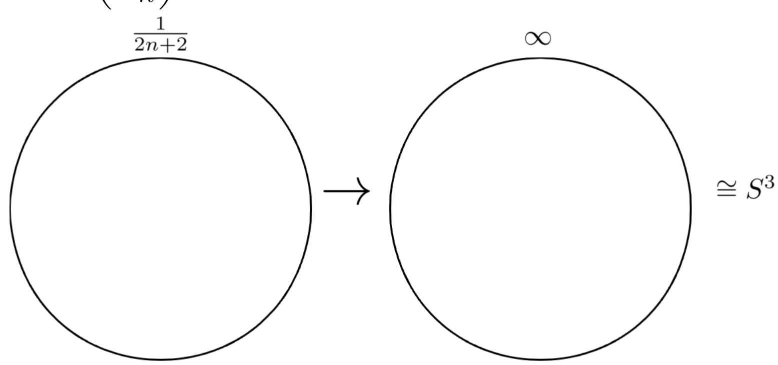
If one ignores, the unframed component, then surgery on the remaining three components yields S^3 as shown in Figure 4-2.



(a) $(n+1)$-Rolfsen twist the $\left(-\dfrac{1}{n+1}\right)$-framed unknot and delete resulting ∞-framed component.



(b) n-Rolfsen twist the $\left(-\dfrac{1}{n}\right)$-framed unknot and delete resulting ∞-framed component.



(c) $-(2n+2)$-Rolfsen twist the $\left(\dfrac{1}{2n+2}\right)$-framed unknot and delete resulting ∞-framed component.

Figure 4-2

Therefore, K_n is actually a knot in S^3. For example, if one tracks what happens to K_1 as the Rolfsen twists in Figure 4-2a are applied, one sees that K_1 is the knot given in Figure 4-3.

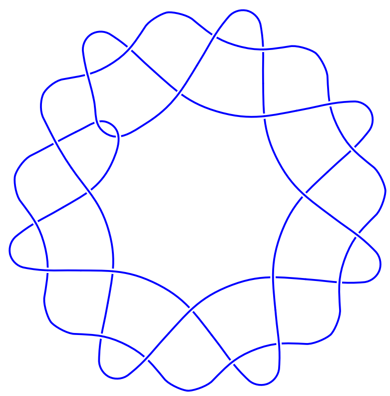
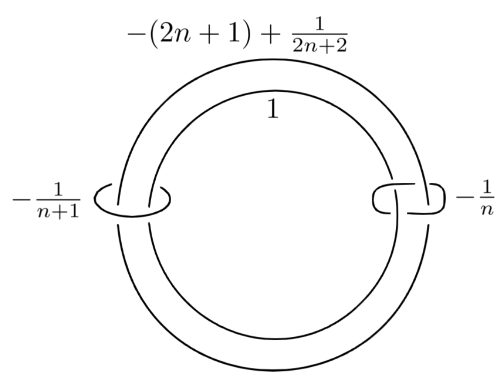


Figure 4-3: Ichihara-Motegi-Song knot K_1. Computed with KLO [46].

Returning to the general K_n, if one assigns the framing $+1$ to K_n in Figure 4-1, then after applying the Rolfsen twists on the other three components, one sees that the framing on the component K_n transforms into 0. In other words, 0-surgery on $K_n \subset S^3$ is the same thing as surgery on the 4-component link in Figure 4-1 with framing $+1$ assigned to K_n.

Theorem 4.0.1 (Ichihara-Motegi-Song [20]).

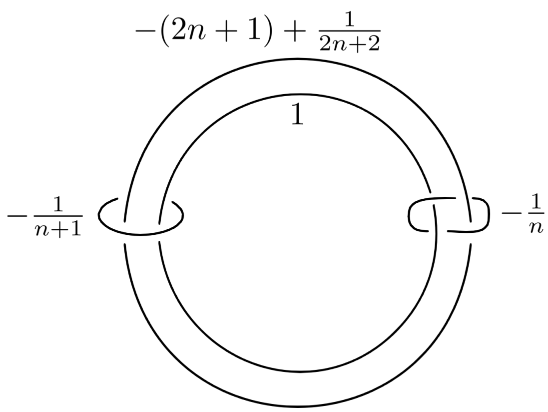
$$S^2\left(\frac{2n+1}{n+1}, \frac{-(2n+3)}{n+1}, \frac{-(2n+1)(2n+3)}{2n+2}\right) \cong$$



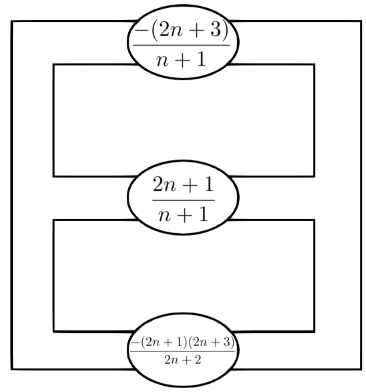
$$\cong S^3_0(K_n)$$

To prove this theorem, Ichihara-Motegi-Song use the Montesinos trick [26], [25], a method to transform a double branched cover description of a 3-manifold into a surgery description and vice versa by using the language of rational tangles [6].

Specifically, they show that the manifold given by the surgery diagram in Figure 4-4a is diffeomorphic to the double branch cover over the link in Figure 4-4b.



(a) Surgery diagram.



(b) Double branch cover over link.

Figure 4-4

In Figure 4-4b, the circles with numbers in them represent rational tangles (see [6] for details). It is result of Montesinos [25] that the double branch cover over the link in Figure 4-4b is the small Seifert fibered space $S^2\left(\frac{2n+1}{n+1}, \frac{-(2n+3)}{n+1}, \frac{-(2n+1)(2n+3)}{2n+2}\right)$ and hence the Theorem 4.0.1 follows.

We now generalize Theorem 4.0.1. Let n and m be integers such that $(n-m)^2$ divides $1+n+m$ and let $p = \frac{1+n+m}{(n-m)^2} \in \mathbb{Z}$. Consider the surgery diagram given in Figure 4-5 with one unframed component $K_{n,m}$.

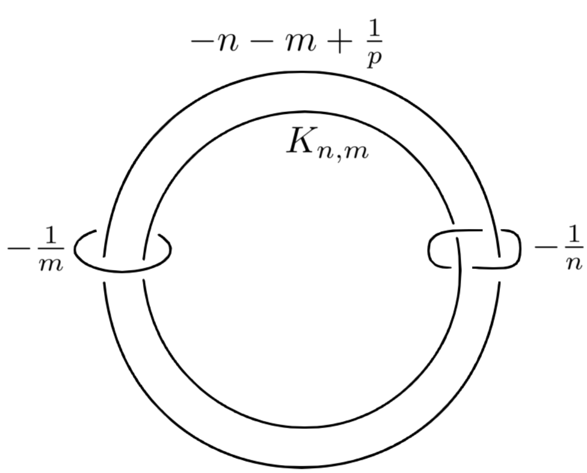
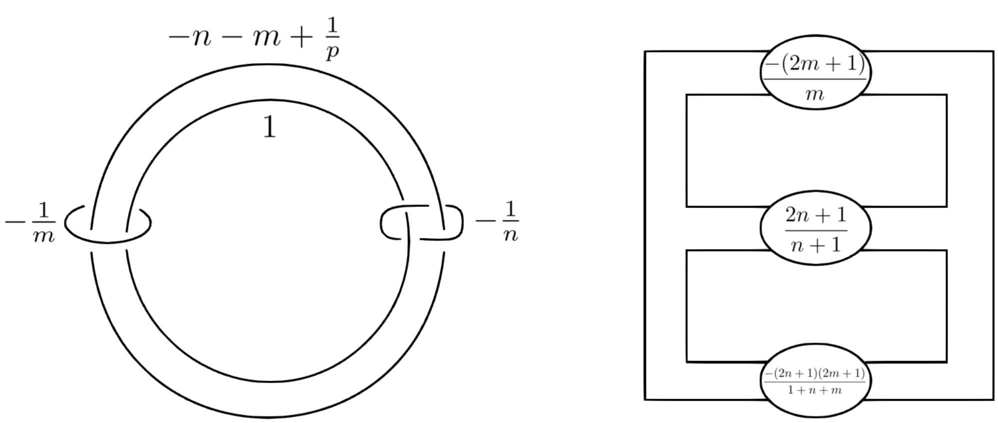


Figure 4-5

As with the surgery diagram in Figure 4-1, if one ignores the unframed component, then surgery on the remaining three yields S^3. One can verify this by applying the moves analogous to those given in Figure 4-2. From this sequence of moves, one can see why we impose the condition $(n-m)^2$ divides $1+n+m$. If we did not have this condition, then p would not be an integer and therefore we would not be able to apply the move analogous to (c) from 4-2 to erase the $\frac{1}{p}$-framed unknot. In other words, if p were not an integer, then we would end up with a lens space rather than S^3.

As a consequence of this, we see that $K_{n,m}$ is a knot in S^3. Moreover, like in the Ichihara-Motegi-Song construction, one can check that if we assign the framing $+1$ to $K_{n,m}$ in Figure 4-5, then after performing surgery on the other three components, the framing on K_n transforms into 0.

Next, if we use exactly the same moves that are in Ichihara-Motegi-Song's implementation of the Montesinos trick (see [20, p. 23, Figures 3 and 4]), except with $\frac{-1}{n+1}$ replaced with $\frac{-1}{m}$ and $-(2n+1)+\frac{1}{2n+2}$ replaced with $-n-m+\frac{1}{p}$, then we get that the manifold given by the surgery diagram in Figure 4-4a is diffeomorphic to the double branch cover over the link in Figure 4-4b.



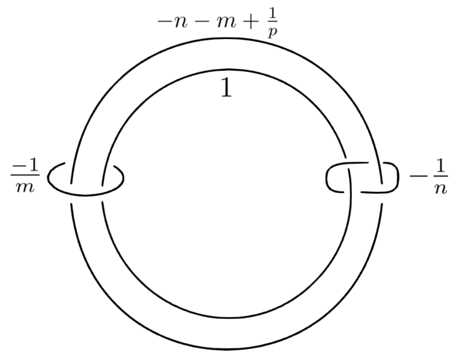
(a) Surgery diagram. (b) Double branched cover over link.

Figure 4-6

Finally, once again invoking Montesinos's result [25] that the double branch cover over the link in 4-6 is diffeomorphic to the small Seifert fibered space with the corresponding invariants, we get the following theorem:

Theorem 4.0.2. *Let n and m be integers such that $(n-m)^2$ divides $1+n+m$ and let $p = \dfrac{1+n+m}{(n-m)^2}$. Then,*

$$S^2\left(\frac{2n+1}{n+1}, \frac{-(2m+1)}{m}, \frac{-(2n+1)(2m+1)}{1+n+m}\right) \cong$$



$$\cong S^3_0(K_{n,m})$$

Example 4.0.3.

$$S^2\left(\frac{15}{8}, \frac{-21}{10}, \frac{-35}{2}\right) \cong$$

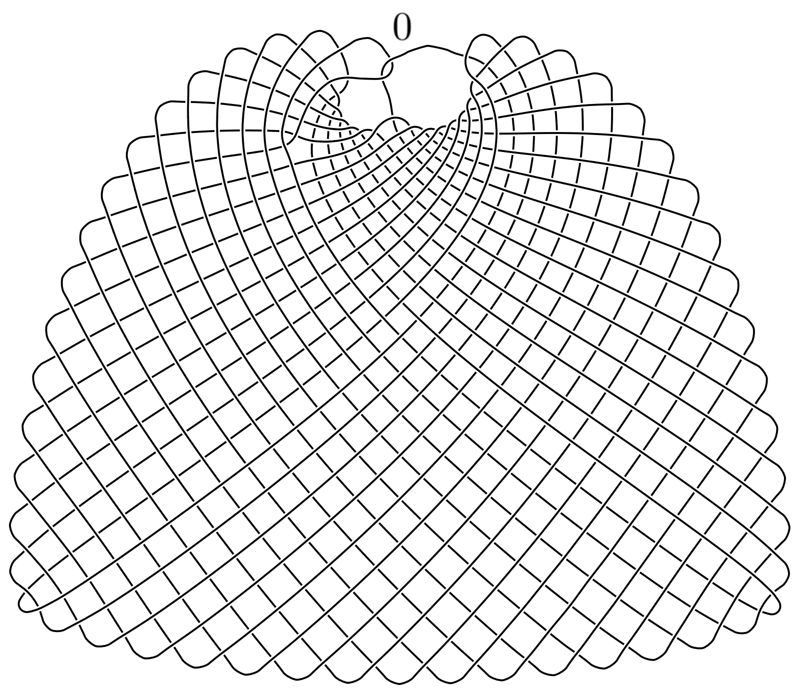


Figure 4-7: 0-surgery on $K_{7,10}$, computed with KLO [46].

This small Seifert fibered space has exceptional fiber multiplicities of the form pq, pr, rq where $p, q, r \geq 2$ and are pairwise coprime. Therefore, by Theorem 3.2.2 and Corollary 3.2.4 $K_{7,10}$ is not an L-space and the next to top degree term of its Alexander polynomial is ± 2. A direct computation using SnapPy [7] shows that the

Alexander polynomial of $K_{7,10}$ is:

$$t^{92} - \boxed{2t^{91}} + t^{90} + t^{89} - 2t^{88} + 2t^{87} - t^{86} + t^{82} - 2t^{81} + 2t^{80} - t^{79}$$
$$+ t^{77} - 2t^{76} + 2t^{75} - t^{74} + t^{72} - t^{71} + t^{68} - t^{67} + t^{65} - t^{64} + t^{62}$$
$$- t^{61} + t^{59} - t^{58} + t^{57} - t^{56} + t^{54} - t^{53} + t^{52} - t^{51} + t^{50} - t^{49}$$
$$+ 2t^{47} - 3t^{46} + 2t^{45} - t^{43} + t^{42} - t^{41} + t^{40} - t^{39} + t^{38} - t^{36} + t^{35}$$
$$- t^{34} + t^{33} - t^{31} + t^{30} - t^{28} + t^{27} - t^{25} + t^{24} - t^{21} + t^{20} - t^{18}$$
$$+ 2t^{17} - 2t^{16} + t^{15} - t^{13} + 2t^{12} - 2t^{11} + t^{10} - t^{6} + 2t^{5} - 2t^{4} + t^{3}$$
$$+ t^{2} - 2t + 1$$

Chapter 5

Spin filling and 0-surgery obstructions from involutive Heegaard Floer homology

In this chapter, we briefly review the construction of involutive Heegaard Floer homology. We then recall the involutive d invariants, $\underline{d}$ and $\bar{d}$, defined by Hendricks-Manolescu for rational homology spheres and define analogous invariants, $\underline{d}_{\pm 1/2}$ and $\bar{d}_{\pm 1/2}$, for closed oriented 3-manifolds with first homology $\mathbb{Z}$. We show that $\underline{d}_{\pm 1/2}$ and $\bar{d}_{\pm 1/2}$ are spin integer homology cobordism invariants and use them to establish constraints on the intersection forms of negative semi-definite spin 4-manifolds whose boundary is a 3-manifold with first homology $\mathbb{Z}$. Furthermore, we establish new obstructions to a 3-manifold being realized as 0-surgery on a knot in an integer homology sphere.

We assume the reader is familiar with Heegaard Floer homology (see for example: [39], [38], [40], [41]).

Notation 5.0.1.

- We use $\mathbb{F} = \mathbb{Z}_2$ coefficients for all Heegaard Floer and involutive Heegaard Floer homology groups.
- Given a graded $\mathbb{F}[U]$-module $\mathcal{A}$, we let $\mathcal{A}[r]$ be the graded $\mathbb{F}[U]$-module defined by $\mathcal{A}[r]_k = \mathcal{A}_{k+r}$. The subscripts denote the homogeneous elements of the

corresponding grading.

- We let $\mathcal{T}^+ = \mathbb{F}[U, U^{-1}]/(U \cdot \mathbb{F}[U])$ be the graded $\mathbb{F}[U]$-module where $\mathrm{gr}(U^n) = -2n$.

- We let $\mathcal{T}_d^+ \coloneqq \mathcal{T}^+[-d]$. In other words, $\mathcal{T}_d^+$ is the $\mathbb{F}[U]$-module $\mathcal{T}^+$ with grading shifted so that the minimal non-zero grading level is d.

5.1 Review of involutive Heegaard Floer homology

For complete details of the construction of involutive Heegaard Floer homology see [16].

Let Y be any closed, connected, oriented 3-manifold. Fix a spin^c structure $\mathfrak{s}$ on Y and let $\overline{\omega} = \{\mathfrak{s}, \bar{\mathfrak{s}}\}$ be the orbit of $\mathfrak{s}$ under the conjugation action. Let $\mathcal{H} = (H, J)$ be a Heegaard pair, i.e., $H = (\Sigma, \alpha, \beta, z)$ is a pointed Heegaard diagram for Y admissible with respect to $\mathfrak{s}$ and J is a generic family of almost complex structures on $\mathrm{Sym}^g(\Sigma)$. Given this setup, define

$$CF^\circ(\mathcal{H}, \overline{\omega}) = \bigoplus_{\mathfrak{t} \in \overline{\omega}} CF^\circ(\mathcal{H}, \mathfrak{t})$$

where $CF^\circ(\mathcal{H}, \mathfrak{t})$ is the usual Heegaard Floer chain complex associated to $(\mathcal{H}, \mathfrak{t})$.

We call $\overline{\mathcal{H}} = (\overline{H}, \bar{J})$ the conjugate Heegaard pair where $\overline{H} = (-\Sigma, \beta, \alpha, z)$ and where $\bar{J}$ is the corresponding conjugate family of almost complex structures. As shown by Ozsváth-Szabó [38, Theorem 2.4], there is a canonical isomorphism of chain complexes:

$$\eta : CF^\circ(\mathcal{H}, \mathfrak{s}) \to CF^\circ(\overline{\mathcal{H}}, \bar{\mathfrak{s}})$$

Moreover, H and $\overline{H}$ both represent the same 3-manifold Y; swapping the order of the α and β curves and reversing the orientation of Σ both have the effect of reversing the orientation on Y and thus cancel each other out. One may think of $\overline{H}$ as being obtained from H by flipping the handle decomposition corresponding to H upside down.

Using naturality results of Juhász, Thurston, and Zemke [22], it is observed by Hendricks-Manolescu in [16, Proposition 2.3] that given two Heegaard pairs representing the same 3-manifold there is a chain homotopy equivalence between their respective Heegaard Floer chain complexes. Furthermore, these chain homotopy equivalences form a transitive system. In particular, since $\mathcal{H}$ and $\overline{\mathcal{H}}$ both represent Y, we get a chain homotopy equivalence:

$$\Phi(\overline{\mathcal{H}}, \mathcal{H}) : CF^\circ(\overline{\mathcal{H}}, \bar{\mathfrak{s}}) \to CF^\circ(\mathcal{H}, \bar{\mathfrak{s}})$$

Taking the composition of η and Φ, we obtain a map:

$$\iota = \Phi(\overline{\mathcal{H}}, \mathcal{H}) \circ \eta : CF^\circ(\mathcal{H}, \mathfrak{s}) \to CF^\circ(\mathcal{H}, \bar{\mathfrak{s}})$$

which is uniquely determined up to chain homotopy. By swapping the roles of $\mathfrak{s}$ and $\bar{\mathfrak{s}}$ in the above discussion, we get a second map going in the opposite direction which, by an abuse of notation, we again call ι.

$$\iota : CF^\circ(\mathcal{H}, \bar{\mathfrak{s}}) \to CF^\circ(\mathcal{H}, \mathfrak{s})$$

It is shown in [16] that $\iota^2 : CF^\circ(\mathcal{H}, \mathfrak{s}) \to CF^\circ(\mathcal{H}, \mathfrak{s})$ is chain homotopic to the identity.

By a further abuse of notation, we let ι also denote the direct sum of the two ι maps above, i.e.,

$$\iota : CF^\circ(\mathcal{H}, \overline{\omega}) \to CF^\circ(\mathcal{H}, \overline{\omega})$$

We then define the involutive Heegaard Floer complex, $CFI^\circ(\mathcal{H}, \overline{\omega})$, to be the mapping cone complex:

$$CF^\circ(\mathcal{H}, \overline{\omega}) \xrightarrow{Q(1+\iota)} Q \cdot CF^\circ(\mathcal{H}, \overline{\omega})[-1]$$

Here, Q is a formal variable that shifts the grading down by 1. Therefore, as graded $\mathbb{F}[U]$-modules, $Q \cdot CF^\circ(\mathcal{H}, \overline{\omega})[-1] \cong CF^\circ(\mathcal{H}, \overline{\omega})$ (strictly, these are $\mathbb{Z}_2$-graded modules; there is only an absolute $\mathbb{Q}$-grading lifting the $\mathbb{Z}_2$-grading when $\mathfrak{s}$ is torsion, for example when $\mathfrak{s}$ is self-conjugate). Introducing the formal variable Q gives $CFI^\circ(\mathcal{H}, \overline{\omega})$ the extra structure of a $\mathbb{F}[Q, U]/(Q^2)$-module rather than just an $\mathbb{F}[U]$-module. The involutive Heegaard Floer homology, $HFI^\circ(\mathcal{H}, \overline{\omega})$, is then defined to be the homology of $CFI^\circ(\mathcal{H}, \overline{\omega})$. It turns out that the isomorphism class of $HFI^\circ(\mathcal{H}, \overline{\omega})$ as a graded $\mathbb{F}[Q, U]/(Q^2)$-module is independent of the choice of auxiliary data $\mathcal{H}$. Therefore, we will write $HFI^\circ(Y, \overline{\omega})$ rather than $HFI^\circ(\mathcal{H}, \overline{\omega})$. If $\mathfrak{s}$ is self-conjugate ($\mathfrak{s} = \bar{\mathfrak{s}}$), we write $HFI^\circ(Y, \mathfrak{s})$.

Remark 5.1.1. When involutive Heegaard Floer homology was first introduced, the module $HFI^\circ(Y, \overline{\omega})$ was only known to be well-defined up to isomorphism. Therefore, to discuss elements of, or maps on, HFI°, one needed to make a choice of auxiliary data. However, recently it was shown by Hendricks-Hom-Stoffregen-Zemke [15] that one can associate to $HFI^\circ(Y, \overline{\omega})$ a specific module rather than just an isomorphism class. Moreover, they construct cobordism maps that are independent of an initial choice auxiliary data.

5.2 Involutive d invariants

In [16, Section 5], Hendricks-Manolescu define involutive d invariants, denoted $\bar{d}$ and $\underline{d}$, for self-conjugate spin^c structures of rational homology spheres. Before recalling their definitions and generalizing them to 3-manifolds with $H_1 = \mathbb{Z}$, we need to review a few basic properties.

Proposition 5.2.1 (See [16, Proposition 4.6]). *Suppose Y is a closed, connected, oriented 3-manifold and $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ with $\mathfrak{s} = \bar{\mathfrak{s}}$. Then, there exists an exact triangle of $\mathbb{F}[U]$-modules:*

$$\begin{array}{ccc} HF^\circ(Y, \mathfrak{s}) & \xrightarrow{\;Q(1+\iota_*)\;} & Q \cdot HF^\circ(Y, \mathfrak{s})[-1] \\ & {\scriptstyle h}\nwarrow \quad \swarrow{\scriptstyle g} & \\ & HFI^\circ(Y, \mathfrak{s}) & \end{array}$$

where h decreases grading by 1 and the maps $Q(1+\iota_*)$ and g preserve grading.

Corollary 5.2.2. *With $(Y,\mathfrak{s})$ as in the previous proposition, if $HF^\circ_r(Y,\mathfrak{s}) \cong 0$ or $\mathbb{F}$, then the map $Q(1+\iota_*) : HF^\circ_r(Y,\mathfrak{s}) \to Q \cdot HF^\circ_r(Y,\mathfrak{s})[-1]$ is trivial.*

Proof. Since ι^2 is chain homotopic to the identity, the induced map $\iota_*^2 = 1$. In particular, ι_* is an automorphism. Since the only automorphisms of $\mathbb{F}$ or 0 are the identity, if r is a grading for which $HF^\circ_r(Y,\mathfrak{s}) \cong 0$ or $\mathbb{F}$, then ι_* is the identity. Thus, $Q(1+\iota_*) = Q(1+1) = 0$. □

Next we recall a structure result for the ∞-flavor of Heegaard Floer homology. To be consistent with the reference [38], we phrase the next theorem in terms of $\mathbb{Z}$-coefficients. However, we will only be concerned with the mod 2 reduction of this result.

Theorem 5.2.3 (See [38, section 10])**.** *Let Y be a closed, connected, oriented 3-manifold. If $b_1(Y) \leq 2$, then there exists an equivalence class of orientation system over Y such that for any torsion spin^c structure $\mathfrak{s}$, we have:*

$$HF^\infty(Y,\mathfrak{s}) \cong \mathbb{Z}[U,U^{-1}] \otimes_{\mathbb{Z}} \Lambda^* H^1(Y;\mathbb{Z})$$

as $\mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^*(H_1(Y;\mathbb{Z})/Tors)$-modules.

In Heegaard Floer terminology, HF^∞ is said to be *standard* if it satisfies the conclusion of the above theorem. In other words, Theorem 5.2.3 says that if $b_1(Y) \in \{0,1,2\}$, then $HF^\infty(Y,\mathfrak{s})$ is automatically standard. In particular, if $b_1(Y) = 0$, i.e., if Y is a rational homology sphere, then $HF^\infty(Y,\mathfrak{s}) \cong \mathbb{F}[U,U^{-1}]$. In this case, as graded $\mathbb{F}[U]$-modules, we have the following (non-canonical) splitting:

$$HF^+(Y,\mathfrak{s}) \cong \mathcal{T}^+_d \oplus HF^+_{\text{red}}(Y,\mathfrak{s})$$

where $d = d(Y,\mathfrak{s})$ is the usual d invariant of $(Y,\mathfrak{s})$ and $\mathcal{T}^+_d$ corresponds to the image of $\pi_* : HF^\infty(Y,\mathfrak{s}) \to HF^+(Y,\mathfrak{s})$.

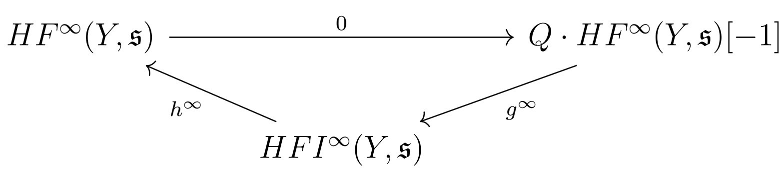
Similarly, if $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ and $\mathfrak{s}_0$ is the unique torsion spin^c structure on Y, then we have that $HF^\infty(Y,\mathfrak{s}_0) \cong \mathbb{F}[U,U^{-1}] \oplus \mathbb{F}[U,U^{-1}]$ and we get the following (non-canonical) splitting:

$$HF^+(Y,\mathfrak{s}_0) \cong \mathcal{T}^+_{d_{-1/2}} \oplus \mathcal{T}^+_{d_{1/2}} \oplus HF^+_{\mathrm{red}}(Y,\mathfrak{s}_0)$$

where $d_{-1/2} = d_{-1/2}(Y,\mathfrak{s}_0)$ and $d_{1/2} = d_{1/2}(Y,\mathfrak{s}_0)$ are the two d invariants for $(Y,\mathfrak{s}_0)$ and $\mathcal{T}^+_{d_{-1/2}} \oplus \mathcal{T}^+_{d_{1/2}}$ corresponds to the $\mathrm{Im}(\pi_*)$. Recall, $d_{\pm 1/2} \equiv \pm 1/2 \bmod 2$.

Remark 5.2.4. The previous paragraph applies more generally to Y with $b_1(Y) = 1$, not just $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$, but to simplify the exposition we will restrict to the case $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. Ultimately, we are concerned with 0-surgery applications, so this restriction suffices for our purposes.

Proposition 5.2.5. *Let Y be a closed, connected oriented 3-manifold with $b_1(Y) = 0$ or $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. If $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ with $\mathfrak{s} = \bar{\mathfrak{s}}$, then we get an exact triangle of $\mathbb{F}[U]$-modules:*



Proof. By the above discussion, if r is a grading for which $HF^\infty_r(Y,\mathfrak{s}) \neq 0$, then $HF^\infty_r(Y,\mathfrak{s}) \cong \mathbb{F}$. The proposition then follows immediately from Corollary 5.2.2 and Proposition 5.2.1. □

We now analyze the conclusion of Proposition 5.2.5 in the case $b_1 = 0$ and recall the definition of the involutive d invariants $\bar{d}$ and $\underline{d}$. After this, we consider the case $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. To minimize confusion, for the rest of this section we use the letter M to denote rational homology spheres and the letter Y to denote 3-manifolds with $b_1 = 1$.

Consider a rational homology sphere M equipped with a self-conjugate spin^c structure $\mathfrak{s}$. Then the exact triangle of Proposition 5.2.5 decomposes into exact sequences:

$$0 \longrightarrow Q \cdot HF_r^\infty(M,\mathfrak{s})[-1] \xrightarrow{\sim} HFI_r^\infty(M,\mathfrak{s}) \longrightarrow 0 \text{ (if } r \equiv d(M,\mathfrak{s}) \bmod 2)$$

$$0 \longrightarrow HFI_r^\infty(M,\mathfrak{s}) \xrightarrow{\sim} HF_{r-1}^\infty(M,\mathfrak{s}) \longrightarrow 0 \text{ (if } r \equiv d(M,\mathfrak{s}) + 1 \bmod 2)$$

Since the maps in the exact triangle are U-equivariant, we further get that HFI^∞ splits as a graded $\mathbb{F}[U]$-module as follows:

$$HFI^\infty(M,\mathfrak{s}) \cong Q \cdot HF^\infty(M,\mathfrak{s})[-1] \oplus HF^\infty(M,\mathfrak{s})[-1]$$

This splitting is canonical since HF_r^∞ is supported in alternating degrees. Moreover, as graded $\mathbb{F}[Q,U]/(Q^2)$-modules (up to possibly an overall grading shift) one can check that

$$HFI^\infty(M,\mathfrak{s}) \cong \mathbb{F}[Q,U,U^{-1}]/(Q^2)$$

Therefore, we may think of $HFI^\infty(M,\mathfrak{s})$ as the direct sum of two doubly infinite towers: one which is not in the image of Q, and the other which is the image of the first under multiplication by Q. Both towers have involutive grading congruent to $d(M,\mathfrak{s}) \bmod 2\,\mathbb{Z}$.

We now recall the definition of the involutive d invariants introduced by Hendricks-Manolescu. To make sense of the definition, it is useful to recall that:

$$\mathrm{Im}(\pi_* : HFI^\infty(M,\mathfrak{s}) \to HFI^+(M,\mathfrak{s})) = \mathrm{Im}(U^n)$$

for $n \gg 0$ [39, Lemma 4.6].

Definitions 5.2.6 ([16, 5.1 Definitions])**.** Let M be an oriented rational homology 3-sphere and $\mathfrak{s} \in \mathrm{Spin}^c(M)$ with $\mathfrak{s} = \bar{\mathfrak{s}}$. Define the lower and upper involutive correction terms of $(M,\mathfrak{s})$ to be $\underline{d}(M,\mathfrak{s})$ and $\bar{d}(M,\mathfrak{s})$, respectively, where

$$\underline{d}(M,\mathfrak{s}) = \min\{r \mid \exists\, x \in HFI_r^+(M,\mathfrak{s}), x \in \mathrm{Im}(U^n), x \notin \mathrm{Im}(U^n Q) \text{ for } n \gg 0\} - 1$$

$$\bar{d}(M,\mathfrak{s}) = \min\{r \mid \exists\, x \in HFI_r^+(M,\mathfrak{s}), x \neq 0, x \in \mathrm{Im}(U^n Q) \text{ for } n \gg 0\}$$

It is conceptually useful to think of $\bar{d}$ and $\underline{d}$ in terms of a splitting of HFI^+ into towers and reducible elements as follows:

Corollary 5.2.7. *Suppose M is an oriented rational homology 3-sphere and $\mathfrak{s} \in \mathrm{Spin}^c(M)$ with $\mathfrak{s} = \bar{\mathfrak{s}}$. Then, we get a (non-canonical) splitting as graded $\mathbb{F}[U]$-modules:*

$$HFI^+(M, \mathfrak{s}) \cong \mathcal{T}^+_{\bar{d}} \oplus \mathcal{T}^+_{\underline{d}+1} \oplus HFI^+_{red}(M, \mathfrak{s})$$

Here, $\mathcal{T}^+_{\bar{d}} \oplus \mathcal{T}^+_{\underline{d}+1}$ corresponds to $\mathrm{Im}(\pi_*)$, with $\mathcal{T}^+_{\bar{d}}$ in the image of Q and $\mathcal{T}^+_{\underline{d}+1}$ not in the image of Q.

The invariants $\underline{d}$ and $\bar{d}$ satisfy the following basic properties:

Proposition 5.2.8 ([16, Propositions 5.1, 5.2])**.** *With M and $\mathfrak{s}$ as in Definitions 5.2.6,*

1. $\underline{d}(M, \mathfrak{s}) \leq d(M, \mathfrak{s}) \leq \bar{d}(M, \mathfrak{s})$
2. $\underline{d}(M, \mathfrak{s}) = -\bar{d}(-M, \mathfrak{s})$

Additionally, Hendricks-Manolescu generalize [36, Theorem 9.6] to the involutive setting to obtain:

Theorem 5.2.9 ([16, Theorem 1.2])**.** *With M and $\mathfrak{s}$ as in Definitions 5.2.6, if X is a smooth negative definite 4-manifold with boundary M and $\mathfrak{t}$ is a spin structure on X such that $\mathfrak{t}|_M = \mathfrak{s}$, then*

$$\mathrm{rank}(H^2(X; \mathbb{Z})) \leq 4\underline{d}(M, \mathfrak{s})$$

The method of proof of Theorem 5.2.9 is used to further show that $\underline{d}$ and $\bar{d}$ are spin rational homology cobordism invariants.

Now suppose Y is a closed oriented 3-manifold with $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}$ and let $\mathfrak{s}_0$ be the unique torsion spin^c-structure on Y. Then the exact triangle of Proposition 5.2.5 decomposes into short exact sequences:

$$0 \longrightarrow Q \cdot HF^\infty_r(Y, \mathfrak{s}_0)[-1] \xrightarrow{g^\infty} HFI^\infty_r(Y, \mathfrak{s}_0) \xrightarrow{h^\infty} HF^\infty_{r-1}(Y, \mathfrak{s}_0) \longrightarrow 0$$

These short exact sequences are of the form:

$$0 \longrightarrow \mathbb{F} \longrightarrow \mathbb{F} \oplus \mathbb{F} \longrightarrow \mathbb{F} \longrightarrow 0$$

Therefore, as vector spaces, we get a splitting:

$$HFI^\infty_r(Y, \mathfrak{s}_0) \cong Q \cdot HF^\infty_r(Y, \mathfrak{s}_0)[-1] \oplus HF^\infty_{r-1}(Y, \mathfrak{s}_0)$$

where each summand is one dimensional. Unlike in the $b_1 = 0$ case, this splitting is not canonical. However, we are still able to get the following structure result:

Proposition 5.2.10. *Suppose Y is a closed connected oriented 3-manifold with $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}$ and $\mathfrak{s}_0 \in \mathrm{Spin}^c(Y)$ is the unique spin^c structure with $\mathfrak{s}_0 = \bar{\mathfrak{s}}_0$. Then, as graded $\mathbb{F}[Q, U]/(Q^2)$-modules,*

$$HFI^\infty(Y, \mathfrak{s}_0) \cong \mathbb{F}[Q, U, U^{-1}]/(Q^2) \oplus \mathbb{F}[Q, U, U^{-1}]/(Q^2)$$

where, on the right side of the equation, the first factor has gradings congruent to $1/2 \bmod 2$ and the second factor has gradings congruent to $-1/2 \bmod 2$.

Proof. Fix a Heegaard pair $\mathcal{H} = (H, J)$ representing Y and admissible with respect to $\mathfrak{s}_0$. Let ∂^I be the boundary map on the involutive chain complex. We can compactly write ∂^I as $\partial^I = \partial + Q(1 + \iota)$ where ∂ is the usual boundary map on the Heegaard Floer chain complex extended by Q-linearity.

By Theorem 5.2.3, $HF^\infty_{1/2}(\mathcal{H}, \mathfrak{s}_0) \cong \mathbb{F}$ and $HF^\infty_{-1/2}(\mathcal{H}, \mathfrak{s}_0) \cong \mathbb{F}$. Let $\alpha \in HF^\infty_{1/2}(\mathcal{H}, \mathfrak{s}_0)$ and $\beta \in HF^\infty_{-1/2}(\mathcal{H}, \mathfrak{s}_0)$ be the unique non-zero generators. Let $a, b \in CF^\infty(\mathcal{H}, \mathfrak{s}_0)$ be representatives of α and β respectively. Then, the unique non-zero element in the image of

$$g^\infty : Q \cdot HF^\infty_{1/2}(\mathcal{H}, \mathfrak{s}_0)[-1] \to HFI^\infty_{1/2}(\mathcal{H}, \mathfrak{s}_0)$$

is $[Qa]$. Similarly, $[Qb]$ is the unique non-zero element in the image of

$$g^\infty : Q \cdot HF^\infty_{-1/2}(\mathcal{H}, \mathfrak{s}_0)[-1] \to HFI^\infty_{-1/2}(\mathcal{H}, \mathfrak{s}_0)$$

As we have observed above, $1 + \iota_*$ is the zero map on homology. Therefore, there exists some $x, y \in CF^\infty(\mathcal{H}, \mathfrak{s}_0)$ such that $(1 + \iota)a = \partial x$ and $(1 + \iota)b = \partial y$. Thus, $\partial^I(a+Qx) = 0$ and $\partial^I(b+Qy) = 0$. Furthermore, we have that $Q[a+Qx] = [Qa]$ and $Q[b+Qy] = [Qb]$. Therefore, the first summand in the decomposition can be taken to be $(\mathbb{F}[Q, U, U^{-1}]/(Q^2))\,[b+Qy]$ and the second to be $(\mathbb{F}[Q, U, U^{-1}]/(Q^2))\,[a+Qx]$. $\square$

The isomorphism in Proposition 5.2.10 is not canonical with respect to a given Heegaard pair $\mathcal{H} = (H, J)$ because the elements $[a + Qx]$ and $[b + Qy]$ depend on our choice of representatives a, b, x, y. Despite this, we can still define involutive d invariants in this situation. We only need to know the $\mathbb{F}[Q, U]/(Q^2)$-module structure of HFI^∞, regardless of a canonical isomorphism.

Definitions 5.2.11. Let Y be a closed oriented 3-manifold with $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}$. Let $\mathfrak{s}_0$ be the unique spin^c structure on Y with $\mathfrak{s}_0 = \bar{\mathfrak{s}}_0$. Define:

$$\underline{d}_{1/2}(Y, \mathfrak{s}_0) = \min\{r \mid r \equiv -1/2 \bmod 2, \exists x \in HFI^+_r(Y, \mathfrak{s}_0), x \in \mathrm{Im}(U^n), x \notin \mathrm{Im}(U^n Q) \text{ for } n \gg 0\} - 1$$
$$\underline{d}_{-1/2}(Y, \mathfrak{s}_0) = \min\{r \mid r \equiv 1/2 \bmod 2, \exists x \in HFI^+_r(Y, \mathfrak{s}_0), x \in \mathrm{Im}(U^n), x \notin \mathrm{Im}(U^n Q) \text{ for } n \gg 0\} - 1$$
$$\bar{d}_{1/2}(Y, \mathfrak{s}_0) = \min\{r \mid r \equiv 1/2 \bmod 2, \exists x \in HFI^+_r(Y, \mathfrak{s}_0), x \neq 0, x \in \mathrm{Im}(U^n Q) \text{ for } n \gg 0\}$$
$$\bar{d}_{-1/2}(Y, \mathfrak{s}_0) = \min\{r \mid r \equiv -1/2 \bmod 2, \exists x \in HFI^+_r(Y, \mathfrak{s}_0), x \neq 0, x \in \mathrm{Im}(U^n Q) \text{ for } n \gg 0\}$$

Remark 5.2.12. Since $\mathfrak{s}_0$ is unique, we will often just write $\underline{d}_{\pm 1/2}(Y)$ and $\bar{d}_{\pm 1/2}(Y)$, or $\underline{d}_{\pm 1/2}$ and $\bar{d}_{\pm 1/2}$ if Y is clear from context.

As in the $b_1 = 0$ case, it is again useful to think of these invariants in terms of a splitting of HFI^+.

Corollary 5.2.13. *Suppose Y is a closed, connected, oriented 3-manifold with $H_1(Y, \mathbb{Z}) \cong \mathbb{Z}$ and $\mathfrak{s}_0 \in \mathrm{Spin}^c(Y)$ is the unique Spin^c structure with $\mathfrak{s}_0 = \bar{\mathfrak{s}}_0$. Then, there exists a* *(non-canonical) splitting:*

$$HFI^+(Y, \mathfrak{s}_0) \cong \mathcal{T}^+_{\bar{d}_{1/2}} \oplus \mathcal{T}^+_{\bar{d}_{-1/2}} \oplus \mathcal{T}^+_{\underline{d}_{1/2}+1} \oplus \mathcal{T}^+_{\underline{d}_{-1/2}+1} \oplus HFI^+_{red}(Y, \mathfrak{s}_0)$$

where $\mathcal{T}^+_{\bar{d}_{1/2}} \oplus \mathcal{T}^+_{\bar{d}_{-1/2}} \oplus \mathcal{T}^+_{\underline{d}_{1/2}+1} \oplus \mathcal{T}^+_{\underline{d}_{-1/2}+1}$ *corresponds to* $\mathrm{Im}(\pi_*)$ *and* $\mathcal{T}^+_{\bar{d}_{1/2}} \oplus \mathcal{T}^+_{\bar{d}_{-1/2}}$ *is contained in the image of multiplication by* Q.

Proposition 5.2.14. *The involutive correction terms* $\underline{d}_{\pm 1/2}$ *and* $\bar{d}_{\pm 1/2}$ *satisfy the following basic properties:*

1. $\underline{d}_{\pm 1/2}(Y) \leq d_{\pm 1/2}(Y) \leq \bar{d}_{\pm 1/2}(Y)$
2. $\underline{d}_{\pm 1/2}(Y) = -\bar{d}_{\mp 1/2}(-Y)$

Proof. The proof of (1) follows from the same arguments as the proof of [16, Proposition 5.1]. The proof of (2) follows from [16, Proposition 4.4] and the same arguments as in the proof of [16, Proposition 5.2]. □

5.3 Spin filling constraints, homology cobordism invariance, and 0-surgery obstruction.

In [36, Theorem 9.11], Ozsváth-Szabó establish constraints in terms of $d_{\pm 1/2}$ on the intersection form of a negative semi-definite 4-manifold with boundary a given 3-manifold Y with $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}$. Furthermore, Ozsváth-Szabó establish 0-surgery obstructions in terms of $d_{\pm 1/2}$ (see [36, Corollary 9.14, Proposition 4.11]). In this section, we establish the analogous results in the involutive setting.

Theorem 5.3.1. *Suppose* X *is a smooth oriented negative semi-definite spin 4-manifold with boundary a 3-manifold* Y *with* $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}$.

1. *If the restriction* $H^1(X; \mathbb{Z}) \to H^1(Y; \mathbb{Z})$ *is trivial, then*

$$b_2(X) - 3 \leq 4\underline{d}_{-1/2}(Y)$$

2. If the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ *is non-trivial, then*

$$b_2(X) + 2 \leq 4\underline{d}_{1/2}(Y)$$

Proof. Let $\mathfrak{s}$ be a spin structure on X. In particular, $c_1^2(\mathfrak{s}) = 0$. We follow the proof strategy of [36, Theorem 9.11].

(1) Suppose the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is trivial. First, surger out all of $b_1(X)$ without changing the non-degenerate part of the intersection form of X. Then, remove a ball from X to obtain W which we regard as a cobordism $W : S^3 \to Y$. As observed in the proof of [36, Theorem 9.11], the map induced from the cobordism W

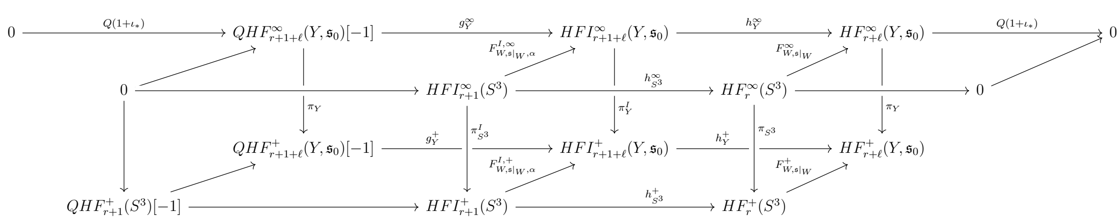
$$F^{\infty}_{W,\mathfrak{s}|_W} : HF^{\infty}(S^3) \to HF^{\infty}(Y, \mathfrak{s}|_Y)$$

is injective with image equal to the doubly infinite tower with degrees congruent to $-1/2$ mod 2 and shifts degree by $\ell = \frac{b_2(X)-3}{4}$. Also, by [16, Section 4.5] there exists an induced map

$$F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha} : HFI^{\infty}(S^3) \to HFI^{\infty}(Y, \mathfrak{s}_0)$$

which also shifts degree by $\ell = \frac{b_2(X)-3}{4}$. Note that the involutive cobordism map $F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha}$ depends on an additional choice of auxiliary data α.

Combining the results in [16, Section 4.5] with Proposition 5.2.5, we see that for every even integer r, we have the following commutative diagram with exact horizontal rows:



By definition of $\underline{d}_{-1/2}(Y)$, there exists some y^+ in $HFI^+_{\underline{d}_{-1/2}+1}(Y, \mathfrak{s}_0)$ such that $y^+ \in$

$\mathrm{Im}(U^n)$ for $n \gg 0$ and $y^+ \notin \mathrm{Im}(U^n Q)$ for $n \gg 0$. The condition [$y^+ \in \mathrm{Im}(U^n)$ for $n \gg 0$] is equivalent to the condition [$y^+ \in \mathrm{Im}(\pi^I_Y)$]. Therefore, there exists some $y^\infty \in HFI^\infty_{\underline{d}_{-1/2}+1}(Y, \mathfrak{s}_0)$ such that $\pi^I_Y(y^\infty) = y^+$. The condition [$y^+ \notin \mathrm{Im}(U^n Q)$ for $n \gg 0$] implies that $y^\infty \notin \mathrm{Im}(g^\infty_Y)$. Therefore, by exactness, $h^\infty_Y(y^\infty) \neq 0 \in HF^\infty_{\underline{d}_{-1/2}}(Y, \mathfrak{s}_0)$. By assumption, the map $F^\infty_{W,\mathfrak{s}|_W} : HF^\infty_{\underline{d}_{-1/2}-\ell}(S^3) \to HF^\infty_{\underline{d}_{-1/2}}(Y, \mathfrak{s}|_Y)$ is an isomorphism. Moreover, by exactness, the map $h^\infty_{S^3} : HFI^\infty_{\underline{d}_{-1/2}+1-\ell}(S^3) \to HF^\infty_{\underline{d}_{-1/2}-\ell}(S^3)$ is also an isomorphism. Therefore, there exists some $x^\infty \in HFI^\infty_{\underline{d}_{-1/2}+1-\ell}(S^3)$ such that $(F^\infty_{W,\mathfrak{s}|_W} \circ h^\infty_{S^3})(x^\infty) = h^\infty_Y(y^\infty)$. Let $z^\infty = F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha}(x^\infty) \in HFI^\infty_{\underline{d}_{-1/2}+1}(Y, \mathfrak{s}_0)$. By commutativity, $h^\infty_Y(z^\infty) = h^\infty_Y(y^\infty)$. Therefore, $z^\infty + y^\infty \in \ker(h^\infty_Y)$. So, by exactness, there exists some $w^\infty \in Q \cdot HF^\infty_{\underline{d}_{-1/2}+1}(Y, \mathfrak{s}_0)[-1]$ such that $g^\infty_Y(w^\infty) = z^\infty + y^\infty$. If $\pi^I_Y(z^\infty) = 0$, then that would imply $\pi^I_Y(g^\infty(w^\infty)) = y^+$. But this would be a contradiction because that would imply $y^+ \in \mathrm{Im}(U^n Q)$ for $n \gg 0$. Therefore, $\pi^I_Y(z^\infty) \neq 0$. Thus, $(\pi^I_Y \circ F^{I,\infty}_{W,\mathfrak{s}|_W,\alpha})(x^\infty) \neq 0$. So, by commutativity, $(F^{I,+}_{W,\mathfrak{s}|_W,\alpha} \circ \pi^I_{S^3})(x^\infty) \neq 0$. In particular, $\pi^I_{S^3}(x^\infty) \neq 0$. Therefore, the element $x^+ = \pi^I_{S^3}(x^\infty) \in HFI^+_{\underline{d}_{-1/2}+1-\ell}(S^3)$ has the property that $x^+ \in \mathrm{Im}(U^n)$ for $n \gg 0$ and $x^+ \notin \mathrm{Im}(U^n Q)$ for $n \gg 0$. It follows that:

$$\underline{d}(S^3) + 1 \leq \underline{d}_{-1/2}(Y) + 1 - \ell \tag{5.3.2}$$

Observing that $\underline{d}(S^3) = 0$ and rearranging/canceling the terms, we get:

$$b_2(X) - 3 \leq 4\underline{d}_{-1/2}(Y)$$

(2) Now suppose the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is non-trivial. Surger out the 1 dimensional homology of X until $b_1(X) = 1$ and so that the map $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is still non-trivial. Again, remove a ball from X to obtain a cobordism $W : S^3 \to Y$. In this case, the induced map

$$F^\infty_{W,\mathfrak{s}|_W} : HF^\infty(S^3) \to HF^\infty(Y, \mathfrak{s}|_Y)$$

is injective with image equal to the doubly infinite tower with degrees congruent to $+1/2 \bmod 2$. The degree shift of this map is now $\frac{b_2(X)+2}{4}$. We then repeat the analogous diagram chase to establish the inequality. We leave the details to the reader. □

Corollary 5.3.3. *Suppose Y is a closed oriented 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. If*

$$\underline{d}_{-1/2}(Y) < -1/2 \qquad \textit{and} \qquad \underline{d}_{1/2}(Y) < 1/2$$

then Y is not the boundary of any negative semi-definite spin manifold.

Proof. Suppose X is a smooth negative semi-definite spin 4-manifold with boundary Y. If the restriction $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is trivial, then the map $H^1(Y;\mathbb{Z}) \to H^2(X,Y;\mathbb{Z})$ is injective. Since $H^1(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$ and $H^2(X,Y;\mathbb{Z}) \cong H_2(X;\mathbb{Z})$, it follows that $b_2(X) \geq 1$. Hence, by Theorem 5.3.1, $-1/2 \leq \underline{d}_{-1/2}(Y)$. If instead $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is non-trivial, then all we can say about $b_2(X)$ is that $b_2(X) \geq 0$. Theorem 5.3.1 therefore implies $1/2 \leq \underline{d}_{1/2}(Y)$. The conclusion now follows. □

Proposition 5.3.4. *Suppose Y_1 and Y_2 are closed oriented 3-manifolds with $H_1(Y_i;\mathbb{Z}) \cong \mathbb{Z}$ for $i \in \{1,2\}$. If there exists a spin integer homology cobordism $(W,\mathfrak{s}) : Y_1 \to Y_2$, then $\underline{d}_{\pm 1/2}(Y_1) = \underline{d}_{\pm 1/2}(Y_2)$ and $\bar{d}_{\pm 1/2}(Y_1) = \bar{d}_{\pm 1/2}(Y_2)$.*

Proof. The argument is the same as in the proof of [16, Proposition 5.4], using the fact that W induces an isomorphism

$$F^{\infty}_{W,\mathfrak{s},\alpha} : HFI^{\infty}(Y_1, \mathfrak{s}|_{Y_1}) \to HFI^{\infty}(Y_2, \mathfrak{s}|_{Y_2})$$

□

Theorem 5.3.5. *Let M be an oriented integer homology 3-sphere and let Y and M' be the 3-manifolds obtained via 0 and $+1$ surgery respectively on a knot K in M. Then,*

1. $\underline{d}(M) - \frac{1}{2} \leq \underline{d}_{-1/2}(Y)$ *and* $\bar{d}(M) - \frac{1}{2} \leq \bar{d}_{-1/2}(Y)$
2. $\underline{d}_{1/2}(Y) - \frac{1}{2} \leq \underline{d}(M')$ *and* $\bar{d}_{1/2}(Y) - \frac{1}{2} \leq \bar{d}(M')$

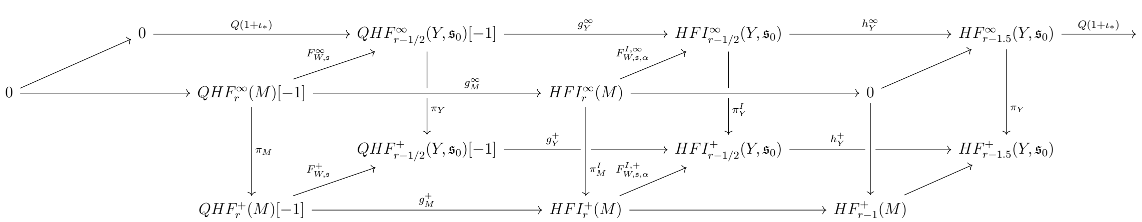
Proof. First, we prove the inequalities in (1).

Let $(W, \mathfrak{s})$ be the spin cobordism from M to Y obtained by attaching a 0-framed 2-handle along K and let $\mathfrak{s}_0$ be the trivial spin^c structure on Y. Then, then by [36, Proposition 9.3], the induced map

$$F^{\infty}_{W,\mathfrak{s}} : HF^{\infty}(M) \to HF^{\infty}(Y, \mathfrak{s}_0)$$

shifts grading by $-1/2$ and is injective with image equal to the doubly infinite tower with gradings congruent to $-1/2 \bmod 2$. The first inequality of (1) now follows by repeating exactly the same argument as in the proof of Theorem 5.3.1 where now M assumes the role of S^3 and $\ell = -1/2$ (see inequality 5.3.2).

To establish the second inequality in (1), we consider the rightward continuation of the commutative diagram used in the proof of Theorem 5.3.1 again replacing S^3 with M. Specifically, for r even, we have the following commutative diagram with exact horizontal rows:



Now we get that g^{∞}_M is an isomorphism, and we again know that $F^{\infty}_{W,\mathfrak{s}}$ is an isomorphism. Furthermore, g^{∞}_Y is injective with $\mathrm{Im}(g^{\infty}_Y) = \ker(h^{\infty}_Y)$. Thus, $F^{I,\infty}_{W,\mathfrak{s},\alpha}$ maps $HFI^{\infty}_r(M)$ isomorphically onto $\mathrm{Im}(g^{\infty}_Y)$.

By definition of the value $\bar{d}_{-1/2}$, there exists some non-zero $y^+ \in HFI^+(Y, \mathfrak{s}_0)$ such that $\mathrm{gr}(y^+) = \bar{d}_{-1/2}$ and $y^+ \in \mathrm{Im}(U^n Q)$ for $n \gg 0$. This implies that there exists some element $y^\infty \in \mathrm{Im}(g_Y^\infty) \subset HFI^\infty_{\bar{d}_{-1/2}}(Y, \mathfrak{s}_0)$ such that $\pi_Y^I(y^\infty) = y^+$. Therefore, the unique non-zero element of $HFI^\infty_{\bar{d}_{-1/2}+1/2}(M)$, which we will call x^∞, maps to y^∞ under $F^{I,\infty}_{W,\mathfrak{s},\alpha}$. Since $(\pi_Y^I \circ F^{I,\infty}_{W,\mathfrak{s},\alpha})(x^\infty) = y^+ \neq 0$, the commutativity of the diagram implies $\pi_M^I(x^\infty) \neq 0$. Additionally, $\pi_M^I(x^\infty) \in \mathrm{Im}(U^n Q)$ for $n \gg 0$. Therefore,

$$\bar{d}(M) \leq \bar{d}_{-1/2}(Y) + \frac{1}{2}$$

The proofs of the inequalities in (2) follow the same arguments as the proofs of (1), except that now we consider the maps:

$$F^\circ_{W',\mathfrak{s}'} : HF^\circ(Y, \mathfrak{s}_0) \to HF^\circ(M')$$

and

$$F^{I,\circ}_{W',\mathfrak{s}',\alpha'} : HFI^\circ(Y, \mathfrak{s}_0) \to HFI^\circ(M')$$

induced by the spin cobordism $(W', \mathfrak{s}') : Y \to M'$ obtained by attaching a 2-handle to the dual of K in Y with framing so that the resulting space is M'. Analyzing the corresponding commutative diagrams and using the fact that for all r even,

$$F^\infty_{W',\mathfrak{s}'} : HF^\infty_{r+1/2}(Y, \mathfrak{s}_0) \to HF^\infty_r(M')$$

is an isomorphism, we get statement (2). We leave the details to the reader. □

Corollary 5.3.6. *Suppose K is a knot in S^3 and Y is the result of 0-surgery on K. Then,*

$$-\frac{1}{2} \leq \underline{d}_{-1/2}(Y) \qquad \textit{and} \qquad \bar{d}_{1/2}(Y) \leq \frac{1}{2}$$

Proof. $0 = d(S^3) = \underline{d}(S^3) = \bar{d}(S^3)$. Therefore, (1) follows immediately from Theorem

5.3.5. For (2), let $\bar{K}$ be the mirror of K. Then, 0-surgery on $\bar{K}$ is $-Y$. Thus, we have $-\frac{1}{2} \leq \underline{d}_{-1/2}(-Y, \mathfrak{s}_0)$. Now by Proposition 5.2.14, $\underline{d}_{-1/2}(-Y, \mathfrak{s}_0) = -\bar{d}_{1/2}(Y, \mathfrak{s}_0)$. Therefore, $\bar{d}_{1/2}(Y, \mathfrak{s}_0) \leq \frac{1}{2}$. □

Chapter 6

Lattice cohomology and involutive calculation techniques for negative semi-definite plumbings

6.1 Heegaard Floer homology and lattice cohomology of plumbings

In this section, we review some of the key developments in the Heegaard Floer homology and lattice cohomology of plumbed 3-manifolds. We then present a modified version of lattice cohomology that involves passing to a quotient lattice. This presentation enables us to readily adapt and combine the work of Rustamov in [43] and the work of Dai-Manolescu in [9] to compute HFI^+ of certain negative semi-definite plumbed 3-manifolds with $b_1 = 1$ and at most one bad vertex.

6.1.1 O-S description of HF^+ of negative definite plumbed 3-manifolds with at most one bad vertex

In an early paper on Heegaard Floer homology [37], Ozsváth-Szabó provide a combinatorial description of the Heegaard Floer homology of 3-manifolds plumbed along negative definite forests with at most one bad vertex. We briefly review their description.

Given a plumbing presentation Γ of a 3-manifold Y, there is a naturally associated cobordism from S^3 to Y via attaching two handles to $S^3 \times [0,1]$ according to the plumbing graph Γ. One can turn this cobordism around and use the fact that there is an orientation preserving diffeomorphism from $-S^3$ to S^3 to yield a cobordism $W_\Gamma : -Y \to S^3$. For each spin^c structure $\mathfrak{s}$ on W_Γ, we get a U-equivariant map

$$F^+_{W_\Gamma,\mathfrak{s}} : HF^+(-Y, \mathfrak{s}|_Y) \to HF^+(S^3)$$

It is easy to see that the spin^c structures on W_Γ correspond in a direct way to spin^c structures on the plumbed 4-manifold $X(\Gamma)$ since W_Γ is diffeomorphic to $X(\Gamma) - D^4$. Because of this we will work with spin^c structures on $X(\Gamma)$ rather than on W_Γ.

Now by the basic facts about spin^c structures and characteristic vectors described in the Section 2.3 and the fact that $HF^+(S^3) \cong \mathcal{T}^+$ as a graded $\mathbb{F}[U]$-module, we can define a map $T^+ : HF^+(-Y) \to \mathrm{Map}(\mathrm{Char}(X(\Gamma), \mathcal{T}^+)$ via the formula:

$$T^+(\xi)(c_1(\mathfrak{s})) = F^+_{W_\Gamma,\mathfrak{s}}(\xi)$$

Here $\mathrm{Map}(\mathrm{Char}(X(\Gamma), \mathcal{T}^+)$ simply denotes the set of functions from $\mathrm{Char}(X(\Gamma))$ to $\mathcal{T}^+$.

Let $\mathrm{H}^+(\Gamma) \subset \mathrm{Map}(\mathrm{Char}(X(\Gamma), \mathcal{T}^+)$ be the functions ϕ of finite support which satisfy the following adjunction relations: For each $k \in \mathrm{Char}(X(\Gamma))$ and $v_i \in \mathcal{V}(\Gamma)$, let $2n_i = k([v_i]) + ([v_i],[v_i])$. Then,

1. if $n_i \geq 0$, we require $U^{n_i}\phi(k + 2PDj_*[v_i]) = \phi(k)$
2. if $n_i < 0$, we require $U^{-n_i}\phi(k) = \phi(k + 2PDj_*[v_i])$

The set $\mathrm{H}^+(\Gamma)$ naturally inherits an $\mathbb{F}[U]$-module structure from $\mathcal{T}^+$. One can also introduce a grading on $\mathrm{H}^+(\Gamma)$ by defining $\phi \in \mathrm{H}^+(\Gamma)$ to be a homogeneous element of degree d if $\phi(k) \in \mathcal{T}^+$ is a homogeneous element of degree $d + \dfrac{k^2 + |\mathcal{V}(\Gamma)|}{4}$ for all $k \in \mathrm{Char}(X(\Gamma))$. Furthermore, we can decompose $\mathrm{H}^+(\Gamma)$ into a direct sum over spin^c structures of Y by defining $\mathrm{H}^+(\Gamma, [k])$ to be the elements of $\mathrm{H}^+(\Gamma)$ which are supported on the *set* $[k]$. Recall $[k]$ denotes both a spin^c structure on Y as well as a subset of $\mathrm{Char}(X(\Gamma))$ (see Notation 2.3.7).

Remark 6.1.1. In [37], $\mathrm{H}^+(\Gamma)$ is instead denoted by $\mathbb{H}^+(\Gamma)$. We have changed the notation in this paper to $\mathrm{H}^+(\Gamma)$ to avoid confusion with lattice cohomology which is denoted by $\mathbb{H}^*(\Gamma)$.

The main result (Theorem 1.2) in [37] states that if Γ is a negative definite plumbing with at most one bad vertex, then $T^+ : HF^+(-Y(\Gamma), [k]) \to \mathrm{H}^+(\Gamma, [k])$ is an isomorphism of graded $\mathbb{F}[U]$-modules for all spin^c structures $[k]$ on $Y(\Gamma)$. Moreover, $\mathrm{H}^+(\Gamma, [k])$ can be computed combinatorially from the data encoded by the plumbing graph. Therefore, this result enables one to compute $HF^+(-Y(\Gamma), [k])$ without having to count holomorphic disks. In particular, Ozsváth-Szabó provide a relatively simple algorithm to compute $\ker(U) \subset \mathrm{H}^+(\Gamma, [k])$.

6.1.2 Némethi's graded roots and lattice cohomology

Building upon the work of Oszváth-Szabó, Némethi in [31] provides an algorithm to compute the entire $\mathbb{F}[U]$-module H^+ for almost rational plumbings by adapting methods of computation sequences used in the study of normal surface singularities. On the way to computing H^+, Némethi's algorithm first computes an intermediate object called a graded root whose definition we review below (see Definitions 6.1.21). For now, we will just mention that a graded root is weighted graph associated to $Y(\Gamma)$ from which one can easily calculate H^+ and therefore HF^+. Furthermore, by using the language of graded roots, Némethi shows that [37, Theorem 1.2] holds for almost rational plumbed manifolds, a strictly larger class of plumbed 3-manifolds than the class of negative definite trees with at most one bad vertex.

Remark 6.1.2. We say trees in the previous sentence because strictly speaking almost rational plumbings are typically assumed to be connected. This assumption, however, is not important. The same methods apply to yield the isomorphism if you drop the connectedness assumption in the definition of almost rational.

Motivated by questions involving complex analytic normal surface singularities and the Seiberg-Witten invariant, Némethi further generalizes his work on negative definite plumbed 3-manifolds by introducing the broader framework of lattice cohomology in [32]. Lattice cohomology assigns to any negative definite plumbed 3-manifold and spin^c structure a graded $\mathbb{F}[U]$-module, which we denote $\mathbb{H}^*$.

Némethi's original definition provides two different, but equivalent, realizations of lattice cohomology. One realization is constructed by first decomposing Euclidean space $\mathbb{R}^s = \mathbb{R} \otimes H_2(X(\Gamma); \mathbb{Z})$ into cubes using the $\mathbb{Z}$-lattice $H_2(X(\Gamma); \mathbb{Z})$ with basis $[v_1], \ldots, [v_s]$. Then, one considers the usual cellular cohomology of $\mathbb{R}^s$, except with the differential modified by a set of weight functions which encode information about the intersection form of $X(\Gamma)$. The other realization is built by taking the cellular cohomology of certain sublevel sets of these weight functions on cubes.

Lattice cohomology also comes equipped with an extra $\mathbb{Z}$-grading. Namely $\mathbb{H}^*$ decomposes as $\mathbb{H}^* = \bigoplus_{q=0}^{\infty} \mathbb{H}^q$ such that each $\mathbb{H}^q$ is itself a $\mathbb{Z}$-graded $\mathbb{F}[U]$-module. In particular, together with his work in [31], Némethi shows that for a negative definite almost rational plumbed 3-manifold, $Y(\Gamma)$, and $\mathfrak{s} \in \text{spin}^c(Y(\Gamma))$, $\mathbb{H}^0(Y(\Gamma), \mathfrak{s})$ is isomorphic to $HF^+(-Y(\Gamma), \mathfrak{s})$ as graded $\mathbb{F}[U]$-modules (up to an overall grading shift), and, moreover, $\mathbb{H}^q(Y, \mathfrak{s}) \cong 0$ for $q \geq 1$. In general, however, it is not the case that for arbitrary negative definite plumbed 3-manifolds $\mathbb{H}^q \cong 0$ for all $q \geq 1$. For example, Némethi shows the existence of a negative definite plumbed rational homology sphere with non-trivial $\mathbb{H}^1$ (see [32, Example 4.4.1]). Of course though, this plumbing is not almost rational.

Very recently, Zemke [49] proved that, in fact, the lattice (co)homology of any plumbing tree is isomorphic to its Heegaard Floer homology.

6.1.3 Modified formulation of lattice cohomology

In this subsection, we construct a modified version of lattice cohomology in order to deal with negative semi-definite plumbings rather than just negative definite plumbings. Before defining this modified version, it is important to point out that subsequent to Némethi's original definition of lattice cohomology several other variants/generalizations have been defined which apply to much more general plumbings including negative semi-definite plumbings (see for example: [13], [33], [35]). The modified construction we provide is very similar to these formulations in many regards; the main difference is that we handle degenerate plumbings by passing to a certain quotient lattice. As in [32], we begin by giving the constructions in general terms, without reference to plumbings.

Construction 1

Let A be a free finitely generated $\mathbb{Z}$-module with a specified ordered basis $(e_1, \ldots, e_n)$. Let $\bar{A}$ be a quotient of A with the property that $\bar{A}$ is itself a free finitely generated $\mathbb{Z}$-module. Given $a \in A$, we write $\bar{a}$ for the corresponding element of $\bar{A}$.

We define a chain complex as follows. For each $0 \leq q \leq n$, let C_q be the free $\mathbb{F}$-module generated by the set $\mathcal{Q}_q = \bar{A} \times \{I \subseteq \{1, \ldots n\} \mid |I| = q\}$. Because later we will want to think of these generators as cubes in a cube complex (see Construction 2), we denote the generator of C_q and the element of $\mathcal{Q}_q$ corresponding to $(\bar{a}, I)$ by $\square(\bar{a}, I)$. We define a differential $\partial : C_q \to C_{q-1}$ by the following formula,

$$\partial \square(\bar{a}, I) = \sum_{i \in I} \Big[\square(\bar{a}, I - \{i\}) + \square(\bar{a} + \bar{e}_i, I - \{i\}) \Big]$$

Remark 6.1.3. Intuitively, it may be helpful to think of this differential as a cellular boundary map on cubes. We make this point of view precise in Construction 2.

Proposition 6.1.4. $\partial^2 = 0$

Proof.

$$\begin{aligned} \partial^2 \square(\bar{a}, I) = &\sum_{i \in I} \sum_{j \in I - \{i\}} \Big[\square(\bar{a}, I - \{i,j\}) + \square(\bar{a} + \bar{e}_j, I - \{i,j\}) \Big] \\ &+ \sum_{i \in I} \sum_{j \in I - \{i\}} \Big[\square(\bar{a} + \bar{e}_i, I - \{i,j\}) + \square(\bar{a} + \bar{e}_i + \bar{e}_j, I - \{i,j\}) \Big] \end{aligned}$$

Now observe that the terms of the form $\square(\bar{a}, I - \{i,j\})$ cancel in pairs as i and j

vary, as do the terms of the form $\square(\bar{a} + \bar{e}_i + \bar{e}_j, I - \{i, j\})$. Finally, the cross terms also cancel. Therefore, $\partial^2 = 0$. $\square$

Remark 6.1.5. If one wanted to work over the coefficient ring $\mathbb{Z}$ instead of $\mathbb{F}$, then signs could be introduced as follows: Given a non-empty subset I of $\{1, \ldots, n\}$ with $|I| = q$, let $g_I : I \to \{1, \ldots, q\}$ be the unique order preserving bijection. Define the differential via the formula:

$$\partial\square(\bar{a}, I) = \sum_{i \in I} (-1)^{g_I(i)} \Big[\square(\bar{a}, I - \{i\}) - \square(\bar{a} + \bar{e}_i, I - \{i\})\Big]$$

One can check that we still have $\partial^2 = 0$. For the purposes of this paper, we will stick with the coefficient ring $\mathbb{F}$.

For each $0 \le q \le s$, define $\mathcal{F}^q = \mathrm{Hom}_{\mathbb{F}}(C_q, \mathcal{T}^+)$. We endow $\mathcal{F}^q$ with a $\mathbb{F}[U]$-module structure by the following formula: $(U^n \cdot \phi)(\square_q) = U^n \phi(\square_q)$ for all $\square_q \in \mathcal{Q}_q$. Our goal now is to define a differential, δ_w, on our cochain modules $\mathcal{F}^q$ by modifying the usual coboundary map by a set of weight functions w.

Definition 6.1.6 (See [32, 3.1.4. Definition])**.** A set of functions $w_q : \mathcal{Q}_q \to \mathbb{Z}$, $0 \le q \le n$ is called a set of compatible weight functions if the following hold:

1. For any integer $k \in \mathbb{Z}$, the set $w_0^{-1}((-\infty, k])$ is finite.
2. For any $\square(\bar{a}, I) \in \mathcal{Q}_q$ and any $i \in I$, $w_q(\square(\bar{a}, I)) \ge w_{q-1}(\square(\bar{a}, I - \{i\}))$ and $w_q(\square(\bar{a}, I)) \ge w_{q-1}(\square(\bar{a} + \bar{e}_i, I - \{i\}))$.

Fix a set of compatible weight functions w (we drop the subscript for simplicity). By using w, we are able to define a $\mathbb{Z}$-grading on our cochain modules $\mathcal{F}^q$. Specifically, we say that $\phi \in \mathcal{F}^q$ is homogeneous of degree $d \in \mathbb{Z}$ if $\phi(\square_q)$ is a homogeneous element of $\mathcal{T}^+$ of degree $d - 2w(\square_q)$ whenever $\phi(\square_q) \neq 0$.

The differential.

Mimicking the formula for the differential given in [32, 3.1.4 Definition], we define $\delta_w : \mathcal{F}^q \to \mathcal{F}^{q+1}$ as follows:

- Let $\square_{q+1} \in \mathcal{Q}_{q+1}$ and write $\partial \square_{q+1} = \sum_k \square_q^k$.
- Given $\phi \in \mathcal{F}^q$, let

$$(\delta_w \phi)(\square_{q+1}) = \sum_k U^{w(\square_{q+1}) - w(\square_q^k)} \phi(\square_q^k)$$

Proposition 6.1.7. $\delta_w^2 = 0$

Proof. This follows directly from the definition and the fact that $\partial^2 = 0$. $\square$

Definition 6.1.8. The homology of the cochain complex $(\mathcal{F}^*, \delta_w)$ is called the *lattice cohomology* of the triple $(\bar{A}, (e_1, \ldots, e_n), w)$ and is denoted by $\mathbb{H}^*(\bar{A}, (e_1, \ldots, e_n), w)$.

Remarks 6.1.9.

1. For each q, the $\mathbb{Z}$-grading on $\mathcal{F}^q$ induces a $\mathbb{Z}$-grading on $\mathbb{H}^q$. Therefore, $\mathbb{H}^q$ is a $\mathbb{Z}$-graded $\mathbb{F}[U]$-module.

2. If $\bar{A} = A$, then we recover the usual lattice cohomology defined by Némethi in [32].

Construction 2

We now give a more geometric, but equivalent formulation of the lattice cohomology theory we defined in Construction 1. This is analogous to [32, 3.1.11 Definitions].

First, we give a geometric realization of the chain complex C_q. For each $1 \leq q \leq s$, let $\mathbf{c}_q$ be denote the q-dimensional cube $[0,1]^q$ oriented in the standard way. Additionally, let $\mathbf{c}_0$ be a fixed 0-dimensional cube (i.e. point) oriented positively. To each $\square(\bar{a}, I) \in \mathcal{Q}_q$ we associate a distinct copy of $\mathbf{c}_q$. By an abuse of notation, from now on we will regard each $\square(\bar{a}, I) \in \mathcal{Q}_q$ as both a distinct copy of $\mathbf{c}_q$ and a generator of C_q depending on which point of view is more convenient in a given context.

We now construct a cube complex $\mathcal{C}$ whose q-dimensional cubes are precisely the elements of $\mathcal{Q}_q$ with attaching maps defined as follows:

- First, we prescribe a method for identifying each $(q-1)$-dimensional face of $\mathbf{c}_q$ with $\mathbf{c}_{q-1}$. Let $\{x_j\}_{j=1}^q$ be the standard coordinate functions on $\mathbf{c}_q = [0,1]^q$. Each $(q-1)$-dimensional face of $\mathbf{c}_q$ is defined by an equation $x_i = \epsilon$ for some $\epsilon \in \{0,1\}$. Denote this face by $f_{i,\epsilon}$. For $q \geq 2$, we identify $f_{i,\epsilon}$ with $\mathbf{c}_{q-1}$ via the map $(x_1, \ldots, x_q) \mapsto (x_1, \ldots, \hat{x}_i, \ldots, x_q)$. For $q = 1$, we send the point $f_{i,\epsilon}$ to the point $\mathbf{c}_0$.

- Given $\square(\bar{a}, I) \in \mathcal{Q}_q$, the face $f_{i,\epsilon}$ of $\square(\bar{a}, I)$ gets glued to the cube $\square(\bar{a} + \epsilon\bar{e}_i, I - \{i\})$ via the map defined in the first bullet point.

By construction the q-dimensional cellular chain group of the cube complex $\mathcal{C}$ is equal to C_q and the cellular boundary map is equal to the differential $\partial : C_q \to C_{q-1}$ defined in Construction 1.

Again, fix a set of compatible weight functions w. For every integer $n \geq 1$, let S_n be the subcomplex of $\mathcal{C}$ consisting of all cubes $\square$ such that $w(\square_q) \leq n$ where q ranges over all dimensions. Let $m_w = \min\{w(\square_q) \mid \square_q \in \mathcal{Q}_q,\ 0 \leq q \leq n\}$. Define

$$\mathbb{S}^q(\bar{A}, (e_1, \ldots, e_n), w) = \bigoplus_{n \geq m_w} H^q(S_n; \mathbb{F})$$

where H^q denotes the qth-cellular cohomology. For each fixed q, we give $\mathbb{S}^q(\bar{A}, (e_1, \ldots, e_n), w)$ the structure of an $\mathbb{F}[U]$-module by defining the U action to be the restriction map

$$U : H^q(S_{n+1}; \mathbb{Z}) \to H^q(S_n; \mathbb{Z})$$

We additionally put a $\mathbb{Z}$-grading on $\mathbb{S}^q(\bar{A}, (e_1, \ldots, e_n), w)$ by declaring the elements of $H^q(S_n, \mathbb{Z})$ to be homogeneous of degree $2n$.

Proposition 6.1.10. *As graded* $\mathbb{F}[U]$*-modules,* $\mathbb{H}^*(\bar{A}, (e_1, \ldots, e_n), w) \cong \mathbb{S}^*(\bar{A}, (e_1, \ldots, e_n), w)$.

Proof. This is proved in exactly the same way as [32, 3.1.12 Theorem (a)]. $\square$

Notation 6.1.11. From now on we will denote lattice cohomology by $\mathbb{H}^*$ regardless of which construction we are using.

Lattice cohomology associated to negative semi-definite plumbings

Fix a negative semi-definite plumbing graph Γ and let k be a characteristic vector of $X(\Gamma)$ such that $[k]$ is a torsion spin^c structure on $Y(\Gamma)$.

We now show how to associate a lattice cohomology module to the pair (Γ, k). Let $L = H_2(X(\Gamma); \mathbb{Z})$ and $\bar{L} = H_2(X(\Gamma); \mathbb{Z})/\ker(j_*)$. By the long exact sequence in homology, $\bar{L}$ is isomorphic to a submodule of the free finitely generated $\mathbb{Z}$-module $H_2(X, Y; \mathbb{Z})$ and therefore is itself free and finitely generated. As in section 2.3, let $s = \text{rank}(H_2(X; \mathbb{Z}))$. Also, let $\sigma = s - b_1(Y)$. With this notation, we have that $\bar{L} \cong \mathbb{Z}^\sigma$. Furthermore, after choosing an ordering on the vertices, the plumbing gives us an ordered basis $([v_1], \ldots, [v_s])$ of L.

We now have almost all the data we need in order to get lattice cohomology. It remains to define a set of weight functions. To do this, we rely on our choice of characteristic vector k.

Weight functions

Let $\chi_k : L \to \mathbb{Z}$ be the function defined by $\chi_k(x) = -\dfrac{k(x) + (x, x)}{2}$.

Proposition 6.1.12. *$\chi_k : L \to \mathbb{Z}$ descends to a well-defined function $\bar{\chi}_k : \bar{L} \to \mathbb{Z}$.*

Proof. Since $[k]$ is assumed to be a torsion spin^c structure on Y there exists, by Remark 2.3.8, some $z_k \in L \otimes \mathbb{Q}$ such that $k(x) = (z_k, x)$ for all $x \in L$. Now suppose $x \in L$ and $x' \in \ker(j_*)$. Then,

$$\begin{aligned}
\chi_k(x + x') = -\frac{k(x + x') + (x + x', x + x')}{2} &= \chi_k(x) - \frac{k(x') + 2(x, x') + (x', x')}{2} \\
&= \chi_k(x) - \frac{1}{2}(z_k + 2x + x', x') \\
&= \chi_k(x) - \frac{1}{2}PD[j_*(x')](z_k + 2x + x') \\
&= \chi_k(x)
\end{aligned}$$

□

To make it easier to state some qualitative properties of $\bar{\chi}_k$, we now consider the extension of $\bar{\chi}_k$ by scalars to the function $\bar{\chi}_k^{\mathbb{R}} : \bar{L} \otimes \mathbb{R} \to \mathbb{R}$. Notice that the negative semi-definite intersection form $(\cdot, \cdot) : L \times L \to \mathbb{Z}$ descends to a negative definite symmetric bilinear pairing on $\bar{L}$ which we denote by $(\cdot, \cdot)_{\bar{L}}$. Extending by scalars, we get a negative definite intersection form $(\cdot, \cdot)_{\bar{L}\otimes\mathbb{R}} : (\bar{L} \otimes \mathbb{R}) \times (\bar{L} \otimes \mathbb{R}) \to \mathbb{R}$. Therefore, we have

$$\bar{\chi}_k^{\mathbb{R}}(\bar{x}) = -\frac{k(x) + (\bar{x}, \bar{x})_{\bar{L}\otimes\mathbb{R}}}{2} = -\frac{1}{2}(\bar{z}_k + \bar{x}, \bar{x})_{\bar{L}\otimes\mathbb{R}}$$

In particular, we see that $\bar{\chi}_k^{\mathbb{R}}$ is a positive definite quadratic form plus a linear shift. Putting these observations together yields the following proposition.

Proposition 6.1.13.

1. *$\bar{\chi}_k^{\mathbb{R}}$ is bounded below.*
2. *Let $\{\bar{x}_1, \ldots, \bar{x}_\sigma\}$ be any $\mathbb{R}$-basis of $\bar{L} \otimes \mathbb{R}$. Identify $\bar{L} \otimes \mathbb{R}$ with $\mathbb{R}^\sigma$ via $\bar{L} \otimes \mathbb{R} = \bigoplus_{j=1}^{\sigma} \mathbb{R}\,\bar{x}_j$. Then, the level sets of $\bar{\chi}_k^{\mathbb{R}} : \mathbb{R}^\sigma \to \mathbb{R}$ are $(\sigma - 1)$-dimensional ellipsoids and the sublevel sets are σ-dimensional balls bounded by these ellipsoids.*

Corollary 6.1.14. *$\bar{\chi}_k : \bar{L} \to \mathbb{Z}$ is bounded below and its sublevel sets are finite.*

Definition 6.1.15. Define $w_q : \mathcal{Q}_q \to \mathbb{Z}$ by

$$w(\square(\bar{l}, I)) = \max\{\bar{\chi}_k(\bar{x}) \mid \bar{x} = \bar{l} + \sum_{j\in J} \overline{[v_j]}, J \subseteq I\}$$

Note, $w : \mathcal{Q}_0 \to \mathbb{Z}$ is simply $\bar{\chi}_k$.

By Corollary 6.1.14, w is a valid set of weight functions.

Definition 6.1.16. Define $\mathbb{H}^*(\Gamma, k) = \mathbb{H}(\bar{L}, ([v_1], \ldots, [v_s]), w)$

As in the case with negative definite plumbings, different choices of representatives for $[k]$ yield isomorphic lattice cohomology up to an overall grading shift. More specifically,

Lemma 6.1.17 (See [31, 3.3.2 Lemma])**.** *If $k' = k + 2PD[j_*(l)]$ for some $l \in L$, then $\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, k')[2\bar{\chi}_k(\bar{l})]$.*

Remark 6.1.18. Némethi uses the opposite convention for grading shifts. Hence, [31, 3.3.2 Lemma] is stated as: $\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, k')[-2\bar{\chi}_k(\bar{l})]$.

Examples 6.1.19.

1. (Compare [35, Examples 3.11]) Consider the plumbing graph Γ consisting of a single 0-framed vertex v_1. Γ is a negative semi-definite plumbing whose corresponding plumbed 3-manifold, $Y(\Gamma)$, is diffeomorphic to $S^1 \times S^2$. In this case, $\bar{L} = \langle \overline{[v_1]} \rangle = \{0\}$, $\mathcal{Q}_0 = \{\square(0, \emptyset)\}$, and $\mathcal{Q}_1 = \{\square(0, \{1\})\}$. Therefore, $\mathcal{F}^0 \cong \mathcal{T}^+$ and $\mathcal{F}^1 \cong \mathcal{T}^+$. Let $k = 0 \in H_2(X(\Gamma); \mathbb{Z})$. Then, $[k]$ is the unique torsion spin^c structure on $Y(\Gamma)$ and $\bar{\chi}_k \equiv 0$. Therefore, the weight functions w are identically zero. So in this very simple case, the lattice cohomology coboundary map δ_w is literally the dual of ∂. But,

$$\begin{aligned}\partial\square(0, \{1\}) &= -\square(0, \{\emptyset\}) + \square(0 + \overline{[v_1]}, \emptyset)\\ &= -\square(0, \{\emptyset\}) + \square(0, \{\emptyset\})\\ &= 0\end{aligned}$$

   Thus, $\delta_w = 0$. It follows that

$$\mathbb{H}(\Gamma, k) \cong \mathbb{H}^0(\Gamma, k) \oplus \mathbb{H}^1(\Gamma, k) \cong \mathcal{T}^+ \oplus \mathcal{T}^+$$

   In particular, up to the appropriate grading shifts, $\mathbb{H}^*(\Gamma, k) \cong HF^+(-(S^1 \times S^2), [k])$.

2. Even though our main focus is when $b_1 = 1$, we think it is instructive to generalize the previous example. Specifically, let Γ be the plumbing graph consisting of s disjoint vertices $v_1, \ldots, v_s$ all with weight 0 and no edges. Then, $Y(\Gamma) = \#_s S^1 \times S^2$. Again, let $k = 0 \in H_2(X; \mathbb{Z})$. One can check that the associated cube complex $\mathcal{C}$ is $T^s = \underbrace{S^1 \times \cdots \times S^1}_{s \text{ times}}$. The singular cohomology ring of T^s is $H^*(T^s; \mathbb{F}) \cong \Lambda(\mathbb{F}^s)$, where Λ denotes the exterior algebra. Since in this case the weight functions are all identically zero, Construction 2 tells us

that $\mathbb{H}^*(\Gamma, k) \cong \Lambda(\mathbb{F}^s) \otimes \mathcal{T}^+$. In particular, up to the appropriate grading shifts, $\mathbb{H}^*(\Gamma, k) \cong HF^+(-T^s, [k])$.

Remark 6.1.20. In the above examples, there are of course many other plumbing descriptions of the same 3-manifolds. Therefore, it is worth noting that at this stage we have not yet shown that this modified version of lattice cohomology is independent of the plumbing description (i.e. that it is a topological invariant). However, these examples do suggest that up to grading shifts (at least for "nice enough" negative semi-definite plumbings) lattice cohomology agrees with HF^+, which of course is a topological invariant. We would like to point out though that in [35] Oszváth-Stipsicz-Szabó construct a spectral sequence relating a completed version of HF^+ to their version of lattice cohomology and show that these two objects coincide for plumbing trees of type 2. In particular, their isomorphism holds for negative semi-definite plumbings of type 2.

6.1.4 Graded roots associated to negative semi-definite plumbings

Definitions 6.1.21 (See [31, 3.2 Definitions])**.**

1. Let R be an infinite tree with vertices $\mathcal{V}$ and edges $\mathcal{E}$. We denote by $[u, v]$ the edge with end-points u and v. We say that R is a graded root with grading $\chi : \mathcal{V} \to \mathbb{Z}$ if

   (a) $\chi(u) - \chi(v) = \pm 1$ for any $[u, v] \in \mathcal{E}$

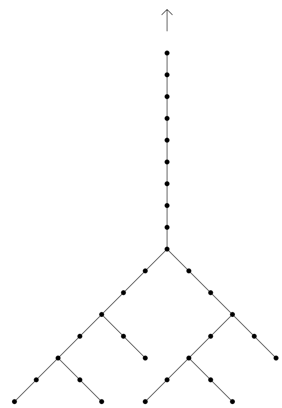
   (b) $\chi(u) > \min\{\chi(u), \chi(w)\}$ for any $[u, v], [u, w] \in \mathcal{E}, v \neq w$

   (c) χ is bounded below, $\chi^{-1}(k)$ is finite for any $k \in \mathbb{Z}$, and $\#\chi^{-1}(k) = 1$ if k is sufficiently large.

2. We say that $v \in \mathcal{V}$ is a local minimum point of the graded root (R, χ) if $\chi(v) < \chi(w)$ for any edge $[v, w]$.

3. If (R, χ) is a graded root, and $r \in \mathbb{Z}$, then we denote by $(R, \chi)[r]$ the same R with the new grading $\chi[r](v) := \chi(v) + r$. (This can be generalized for any $r \in \mathbb{Q}$ as well.)

Example 6.1.22.



We now show how to associate a graded root to a pair (Γ, k) where γ is a negative semi-definite plumbing and k is a characteristic vector of $X(\Gamma)$ such that $[k]$ is a torsion spin^c structure on $Y(\Gamma)$. For each $n \in \mathbb{Z}$, let $\bar{L}_{k,\leq n}$ be the graph whose vertex set is $\mathcal{V}(\bar{L}_{k,\leq n}) = \{\bar{x} \in \bar{L} : \bar{\chi}_k(\bar{x}) \leq n\}$ and such that there is an edge between two vertices $\bar{x}_1, \bar{x}_2$ if and only if $\bar{x}_1 - \bar{x}_2 = \pm\overline{[v_j]}$ where the v_j are as in subsection 2.3.1. Now let $\pi_0(\bar{L}_{k,\leq n})$ denote the set of connected components of the graph $\bar{L}_{k,\leq n}$.

The graded root $(\bar{R}_k, \bar{\chi}_k)$ associated to Γ and k is constructed as follows:

- The vertex set is $\mathcal{V}(\bar{R}_k) = \bigsqcup_{n \in \mathbb{Z}} \pi_0(\bar{L}_{k,\leq n})$. By an abuse of notation, we denote the grading $\mathcal{V}(\bar{R}_k) \to \mathbb{Z}$ by $\bar{\chi}_k$ where now $\bar{\chi}_k|_{\pi_0(\bar{L}_{k,\leq n})} = n$.

- There is an edge between two vertices $v, v' \in \mathcal{V}(\bar{R}_k)$, which correspond to connected components C_v and $C_{v'}$, if and only if after possibly reordering v and v', we have $\bar{\chi}_k(v') = \bar{\chi}_k(v) + 1$ and $C_v \subset C_{v'}$.

Remark 6.1.23. When Γ is negative definite, $(\bar{R}_k, \bar{\chi}_k)$ is precisely the graded root, (R_k, χ_k), defined by Némethi in [31, Section 4].

Remark 6.1.24. The graph $\bar{L}_{k,\leq n}$ is the 1-skeleton of the space S_n considered above in Construction 2 of lattice cohomology. In particular, we can think of $\pi_0(\bar{L}_{k,\leq n})$ equivalently as $\pi_0(S_n)$.

Proposition 6.1.25 (See [31, 4.3 Proposition]). *$(\bar{R}_k, \bar{\chi}_k)$ is a graded root.*

Proof. This proof is essentially identical to the proof of [31, 4.3 Proposition]. Condition (a) of Definition 6.1.21 (1) follows immediately from the construction of $(R_k, \bar{\chi}_k)$. The proof of condition (b) is the same as in [31, 4.3 Proposition]. The first two conditions of (c) follow from Corollary 6.1.14. The last condition of (c) follows the same argument as Némethi's proof, with mild modification. Essentially just replace the function χ_k in Némethi's proof with $\bar{\chi}_k$ and use that $\bar{\chi}_k$ has a (not necessarily unique) global minimum and that $(\cdot,\cdot)_{\bar{L}}$ is negative definite. □

Again, as in the case with negative definite plumbings, the graded roots, $(\bar{R}_k, \bar{\chi}_k)$ and $(\bar{R}_{k'}, \bar{\chi}_{k'})$ corresponding to two characteristic vectors k and k', which restrict to the same torsion spin^c structure on Y, are equal up to an overall grading shift. More specifically,

Proposition 6.1.26 (See [31, 4.4 Proposition]). *If $k' = k + 2PD[j_*(l)]$ for some $l \in L$ and $k \in \mathrm{Char}(X(\Gamma))$ with $[k]$ torsion, then*

$$(\bar{R}_{k'}, \bar{\chi}_{k'}) = (\bar{R}_k, \bar{\chi}_k)[\bar{\chi}_k(\bar{l})]$$

6.1.5 The relationship between lattice cohomology, H^+, and graded roots

In subsection 6.1.1, we recalled the definition of the $\mathbb{F}[U]$-module $\mathrm{H}^+(\Gamma, [k])$ introduced by Ozsváth-Szabó where Γ is a negative definite plumbing and $[k]$ is a spin^c structure on $Y(\Gamma)$. The same definition makes sense for negative semi-definite plumbings and $[k]$ torsion except that we adjust the grading as follows: we say $\phi \in \mathrm{H}^+(\Gamma, [k])$ is a homogeneous element of degree d if for each $k' \in [k]$ with $\phi(k') \neq 0$, we have that $\phi(k') \in \mathcal{T}^+$ is a homogeneous element of degree

$$d + \frac{(k')^2 + |\mathcal{V}(\Gamma)| - 3b_1(Y(\Gamma))}{4}$$

Proposition 6.1.27. *As graded* $\mathbb{F}[U]$*-modules,*

$$\mathrm{H}^+(\Gamma, [k]) \cong \mathbb{H}^0(\Gamma, k)\left[\frac{k^2 + |V(\Gamma)| - 3b_1(Y)}{4}\right]$$

Proof. The isomorphism is induced by the map $Z : \mathrm{H}^+(\Gamma, [k]) \to \mathcal{F}^0$ defined by

$$Z(\phi)(\square(\bar{l}, \emptyset)) = \phi(k + 2PDj_*(l))$$

We leave the details to the reader. □

As described in [37] and [43], for calculation purposes it is convenient to consider the "dual space" of $\mathrm{H}^+(\Gamma, [k])$, which we denote by $\mathrm{K}^+(\Gamma, [k])$. To recall their definition of $\mathrm{K}^+(\Gamma, [k])$, first consider the set $\mathbb{Z}_{\geq 0} \times [k]$. Write elements $(m, k') \in \mathbb{Z}_{\geq 0} \times [k]$ as $U^m \otimes k'$. Define an equivalence relation $\sim$ on $\mathbb{Z}_{\geq 0} \times [k]$ in the following way: for each $k' \in [k]$ and $v_i \in \mathcal{V}(\Gamma)$, let $2n_i = k'([v_i]) + ([v_i], [v_i])$. Then,

1. if $n_i \geq 0$, we require $U^{n_i+m} \otimes (k' + 2PDj_*[v_i]) \sim U^m \otimes k'$

2. if $n_i < 0$, we require $U^m \otimes (k' + 2PDj_*[v_i]) \sim U^{m-n_i} \otimes k'$

In other words, two elements $U^m \otimes k'$ and $U^n \otimes k''$ are equivalent if and only if there exists a finite sequence of elements $U^{m_0} \otimes k_1, \ldots, U^{m_\ell} \otimes k_\ell$ such that $U^{m_0} \otimes k_1 = U^m \otimes k'$, $U^{m_\ell} \otimes k_\ell = U^n \otimes k''$ and each adjacent pair in the sequence is related by a relation of type (1) or (2) as given above. We call such a sequence a *path* connecting $U^m \otimes k'$ and $U^n \otimes k''$.

Remark 6.1.28. In general, there are many different paths connecting a given pair of elements $U^m \otimes k'$ and $U^n \otimes k''$.

Write the equivalence class containing $U^m \otimes k'$ as $\underline{U^m \otimes k'}$ and define $\mathrm{K}^+(\Gamma, [k])$ to be the set of these equivalence classes. $\mathrm{K}^+(\Gamma, [k])$ is the dual of $\mathrm{H}^+(\Gamma, [k])$ (or maybe more naturally $\mathrm{H}^+(\Gamma, [k])$ is the dual of $\mathrm{K}^+(\Gamma, [k])$) in the following sense:

- Define $\mathrm{K}^+(\Gamma, [k])^*$ to be the set of finitely supported functions $\phi : \mathrm{K}^+(\Gamma, [k]) \to \mathcal{T}^+$ such that $\phi(\underline{U^{n+m} \otimes k'}) = U^n \phi(\underline{U^m \otimes k'})$ for all $n, m \geq 0$ and $k' \in [k]$. Endow $(\mathrm{K}^+)^*$ with an $\mathbb{F}[U]$-module structure by inheriting that of $\mathcal{T}^+$.

- Define a map $F : \mathrm{H}^+(\Gamma, [k]) \to \mathrm{K}^+(\Gamma, [k])^*$ by

$$F(\phi)(\underline{U^m \otimes k'}) = U^m \phi(k')$$

  It is straightforward to check that F is a well-defined $\mathbb{F}[U]$-module isomorphism.

We can put more structure on $\mathrm{K}^+(\Gamma, [k])$ by thinking of it as a graph. Specifically, define $g\,\mathrm{K}^+(\Gamma, [k])$ to be the graph whose vertices are the elements of $\mathrm{K}^+(\Gamma, [k])$ and such that there is an edge between to vertices $\underline{U^m \otimes k'}$ and $\underline{U^n \otimes k''}$ if and only if either $\underline{U^{m+1} \otimes k'} = \underline{U^n \otimes k''}$ or $\underline{U^m \otimes k'} = \underline{U^{n+1} \otimes k''}$.

Proposition 6.1.29. *As graphs, $g\,\mathrm{K}^+(\Gamma, [k])$ is isomorphic to the graded root $(\bar{R}_k, \bar{\chi}_k)$.*

Proof. This proof is essentially the same as Némethi's proof of [31, Proposition 4.7]. For completeness, we provide the details here.

By definition each element $k' \in [k]$ can be written as $k' = k + 2PD[j_*(l)]$ for some $l \in L$. Let $\bar{l}_{k'} := \bar{l} \in \bar{L}$. Define a map $p : \mathrm{K}^+(\Gamma, [k]) \to \mathcal{V}(\bar{R}_k)$ as follows:

$$p(\underline{U^m \otimes k'}) = \text{ the connected component of } \bar{L}_{k, \leq \bar{\chi}_k(\bar{l}_{k'})+m} \text{ containing } \bar{l}_{k'}$$

To show that p is well-defined, let $2n_i = k'([v_i]) + ([v_i], [v_i])$. Suppose first that $n_i \geq 0$ so that we have $U^{n_i+m} \otimes (k' + 2PDj_*[v_i]) \sim U^m \otimes k'$. Let $k'' = k' + 2PDj_*[v_i]$. Then, $\bar{l}_{k''} = \bar{l}_{k'} + \overline{[v_i]}$. Thus,

$$\begin{aligned} \bar{\chi}_k(\bar{l}_{k''}) + n_i + m &= \bar{\chi}_k(\bar{l}_{k'}) + \bar{\chi}_{k'}(\overline{[v_i]}) + n_i + m \\ &= \bar{\chi}_k(\bar{l}_{k'}) - n_i + n_i + m \\ &= \bar{\chi}_k(\bar{l}_{k'}) + m \end{aligned}$$

Therefore, $\bar{L}_{k, \leq \bar{\chi}_k(\bar{l}_{k'})+m} = \bar{L}_{k, \leq \bar{\chi}_k(\bar{l}_{k''})+n_i+m}$ and $\bar{l}_{k'}$ and $\bar{l}_{k''}$ are in the same connected component since they differ by $\overline{[v_i]}$. The case when $n_i < 0$ is similar. This establishes that p is well-defined.

Next we define a map $q : \mathcal{V}(\bar{R}_k) \to \mathrm{K}^+(\Gamma, [k])$ which we will show is the inverse of p. Suppose $v \in \mathcal{V}(\bar{R}_k)$. Let C_v be the corresponding connected component in $\bar{L}_{k,\leq \bar{\chi}_k(v)}$ and let $\bar{l}_v$ be some element in $\bar{L} \cap C_v$. Define

$$q(v) = \underline{U^{\bar{\chi}_k(v)-\bar{\chi}_k(\bar{l}_v)} \otimes (k + 2PDj_*(l_v))}$$

To show q is well-defined, suppose $\bar{l}'$ is some other element in $\bar{L} \cap C_v$. It suffices to consider the case that $\bar{l}' = \bar{l}_v + \overline{[v_i]}$ for some i. First note,

$$\bar{\chi}_k(\bar{l}') = \bar{\chi}_k(\bar{l}_v + \overline{[v_i]}) = \bar{\chi}_k(\bar{l}_v) + \bar{\chi}_k(\overline{[v_i]}) - ([v_i], l_v)$$

Also,

$$\begin{aligned}(k + 2PDj_*(l_v))(v_i) + (v_i, v_i) &= k(v_i) + (v_i, v_i) + 2(v_i, l_v) \\ &= -2[\bar{\chi}_k(\overline{[v_i]}) - (v_i, l_v)]\end{aligned}$$

Hence, if $-[\bar{\chi}_k(\overline{[v_i]}) - (v_i, l_v)] \geq 0$, then

$$\begin{aligned}U^{\bar{\chi}_k(v)-\bar{\chi}_k(\bar{l}_v)} \otimes (k + 2PDj_*(l_v)) &\sim U^{\bar{\chi}_k(v)-\bar{\chi}_k(\bar{l}_v)-\bar{\chi}_k(\overline{[v_i]})+(v_i,l_v)} \otimes (k + 2PDj_*(l_v + [v_i])) \\ &= U^{\bar{\chi}_k(v)-\bar{\chi}_k(\bar{l}')} \otimes (k + 2PDj_*(l'))\end{aligned}$$

Similarly, if $-[\bar{\chi}_k(\overline{[v_i]}) - (v_i, l_v)] < 0$, then

$$\begin{aligned}U^{\bar{\chi}_k(v)-\bar{\chi}_k(\bar{l}')} \otimes (k + 2PDj_*(l')) &\sim U^{\bar{\chi}_k(v)-\bar{\chi}_k(\bar{l}')+\bar{\chi}_k(\overline{[v_i]})-(v_i,l_v)} \otimes (k + 2PDj_*(l_v)) \\ &= U^{\bar{\chi}_k(v)-\bar{\chi}_k(\bar{l}_v)} \otimes (k + 2PDj_*(l_v))\end{aligned}$$

Therefore, q is well-defined.

Now consider $qp(\underline{U^m \otimes k'})$ where $k' = k + 2PDj_*(\bar{l}_k)$. Let $v = p(\underline{U^m \otimes k'})$ and C_v

be the connected component of $\bar{L}_{k,\leq\bar{\chi}_k(\bar{l}_{k'})+m}$ containing $\bar{l}_{k'}$. Then, by definition

$$\begin{aligned} q(v) &= \underline{U^{\bar{\chi}_k(\bar{l}_{k'})+m-\bar{\chi}_k(\bar{l}_{k'})} \otimes k + 2PDj_*(\bar{l}_{k'})} \\ &= \underline{U^m \otimes k'} \end{aligned}$$

Hence, $qp = Id$. The other direction, i.e. that $pq = Id$, is tautological. Therefore, p is a bijection. To see that p takes edges to edges bijectively, let $v_1 = p(\underline{U^m \otimes k'})$ and $v_2 = p(\underline{U^{m+1} \otimes k'})$. It follows directly from the definition that $C_{v_1} \subset C_{v_2}$ and $\bar{\chi}_k(v_2) - \bar{\chi}_k(v_1) = 1$. □

Remark 6.1.30. It is useful to point out that under the isomorphism p constructed in the above proof, we have that

$$\mathrm{gr}(p(\underline{U^m \otimes k'})) = m - \frac{(k')^2 - k^2}{8}$$

6.1.6 A quick review of Rustamov's results on negative semi-definite plumbings with $\mathbf{b_1 = 1}$

In [43], Rustamov generalizes the setting in which the isomorphism T^+, described in Subsection 6.1.1, holds. In particular, Rustamov proves the following theorem:

Theorem 6.1.31 ([43, Theorem 1.2])**.** *Let Γ be a negative semi-definite plumbing with at most one bad vertex and with $b_1(Y(\Gamma)) = 1$. Further, let $[k]$ be a torsion* spin^c *structure. Then,*

1. $T^+ : HF^+_{odd}(-Y(\Gamma), [k]) \to \mathrm{H}^+(\Gamma, [k])$ *is an isomorphism of graded* $\mathbb{F}[U]$*-modules.*

2. $HF^+_{even}(-Y(\Gamma), [k]) \cong \mathcal{T}^+_d$ *where* $d = d_{-1/2}(-Y(\Gamma), [k])$.

Here $HF^+_{odd}(-Y(\Gamma), [k])$ and $HF^+_{even}(-Y(\Gamma), [k])$ refer to the submodules generated by elements of $HF^+(-Y(\Gamma), [k])$ of degrees congruent to $1/2 \bmod 2$ *and* $-1/2 \bmod 2$ *respectively.*

Combining Rustamov's result with the observations of the previous subsection, we get:

Corollary 6.1.32. *With* Γ *as above,* $HF^+_{odd}(-Y(\Gamma),[k]) \cong \mathbb{H}^0(\Gamma,k)\left[\frac{k^2+|V(\Gamma)|-3}{4}\right]$ *as graded* $\mathbb{F}[U]$*-modules. In particular, up to an overall grading shift,* $\mathbb{H}^0(\Gamma,k)$ *is a topological invariant of* $Y(\Gamma)$*.*

Remark 6.1.33. It is likely the case that one can prove $\mathbb{H}^0(\Gamma,k)\left[\frac{k^2+|V(\Gamma)|-3}{4}\right]$ is a topological invariant without appealing to Heegaard Floer homology, by showing invariance under Neumann moves as in the proof of [31, Proposition 4.6].

6.2 Calculation method

Throughout this section, fix a negative semi-definite plumbing Γ with at most one bad vertex and such that $b_1(Y(\Gamma)) = 1$. Let $[k]$ be a self-conjugate spin^c structure on $Y(\Gamma)$. In other words, $[k] = [-k]$ or, equivalently, $k = PD[j_*(l)]$ for some $l \in L$. Note that by identifying $\bar{l}$ with k, we can think of k as an element of $\bar{L}$.

6.2.1 Involutions on lattice cohomology and Heegaard Floer homology

As in [9, Section 2], define $J_0 : \bar{L} \to \bar{L}$ by $J_0(\bar{x}) = -\bar{x} - \bar{l}$. Clearly, $J_0^2 = Id$. We can extend J_0 to a cubical involution on the cube complex $\mathcal{C}$ considered in Construction 2 of lattice cohomology via the formula,

$$J_0\square(\bar{a}, I) = \square(J_0(\bar{a} + \sum_{i\in I}\overline{[v_i]}), I)$$

It is straightforward to check that J_0 is compatible with the gluing of the cells. Moreover, since $\bar{\chi}_k(J_0(\bar{x})) = \bar{\chi}_k(\bar{x})$ for all $\bar{x} \in \bar{L}$, J_0 maps the subcomplex S_n of $\mathcal{C}$ to itself. Therefore, J_0 induces an involution on $H^q(S_n;\mathbb{Z})$ or each n, q and hence on lattice cohomology. By an abuse of notation, we denote the involution on lattice cohomology again by J_0. In a similar manner, one could alternatively define J_0 by using Construction 1, but we leave the details to the reader.

Focusing our attention on the $0th$-level of lattice cohomology, we can think of the action of J_0 on $\mathbb{H}^0$ from the dual perspective by realizing an involution on the associated graded root. More specifically, since J_0 acts continuously on S_n, J_0 also induces an involution on the connected components of S_n. Hence, J_0 induces an involution on the graded root $(\bar{R}_k, \bar{\chi}_k)$. From another perspective, under the identification of $(\bar{R}_k, \bar{\chi}_k)$ with $g\,\mathrm{K}^+(\Gamma, [k])$ given in Proposition 6.1.29, the involution J_0 sends $\underline{U^m \otimes k'}$ to $\underline{U^m \otimes -k'}$.

There is a fundamental difference between the action of J_0 on the graded root in the negative definite and negative semi-definite cases. Before describing this difference, we need to recall the following definition:

Definition 6.2.1 (See [9, Definition 2.11])**.** A *symmetric graded root* is a graded root (R, χ) together with an involution $J : \mathcal{V}(R) \to \mathcal{V}(R)$ such that

- $\chi(v) = \chi(Jv)$ for any vertex v
- $[v, w]$ is an edge in R if an only if $[Jv, Jw]$ is an edge in R
- for every $r \in \mathbb{Q}$, there is at most one J invariant vertex v with $\chi(v) = r$

We call such a J a *symmetric involution.*

In [8, Lemma 2.1] (see also [9, Section 2.1]) it is shown that the graded root (R_k, χ_k) of a negative definite almost rational plumbing with $[k]$ self-conjugate is symmetric and J_0 is a symmetric involution; in particular, this holds if the plumbing has at most one bad vertex. However, if the plumbing is negative semi-definite and has at most one bad vertex, then the proof of [8, Lemma 2.1] no longer works and, as we show in section 7.1.2, J_0 need not be a symmetric involution.

The proof of [8, Lemma 2.1] uses the classical Lefshetz fixed-point theorem and relies crucially on the fact that for Γ negative definite and almost rational, $\mathbb{H}^q(\Gamma, k) = 0$ for $q > 0$. However, as we have seen in Examples 6.1.19, when Γ is negative semi-definite, it is not necessarily true that $\mathbb{H}^q(\Gamma, k) = 0$ for $q > 0$. Hence, the proof that

J_0 is symmetric fails in this case. As we will demonstrate in Chapter 7, the possibility that J_0 is not symmetric has important implications on the involutive d invariants and hence on properties regarding spin cobordism and 0-surgery.

Despite the difference in behavior of the involution J_0 in the negative definite and negative semi-definite cases, the proof of [9, Theorem 3.1] still holds in the negative semi-definite setting to give an identification of J_0 on $\mathbb{H}^0(\Gamma, k)$ with the involution ι_* on $HF^+(-Y(\Gamma), [k])$. More precisely,

Theorem 6.2.2 (See [9, Theorem 3.1]). *Let Γ be a negative semi-definite plumbing with at most one bad vertex and such that $b_1(Y(\Gamma)) = 1$. If $[k]$ is a self-conjugate* spin^c *structure, then under the isomorphism given in Corollary 6.1.32 the maps J_0 and the restriction of ι_* to $HF^+_{odd}(-Y(\Gamma), [k])$ are identified.*

The action of ι_* on the even part of HF^+ is less interesting. Since $HF^+_{even}(-Y(\Gamma), [k]) \cong \mathcal{T}_d$ and ι_* is U-equivariant, the restriction of ι_* to the even part must be the identity. Moreover, if one knows HF^+ and ι_*, then by using the mapping cone exact triangle in Proposition 5.2.1, one can completely determine HFI^+ as a graded $\mathbb{F}$-vector space.

In the context of negative definite almost rational plumbings, Dai-Manolescu show that one can actually determine the entire $\mathbb{F}[U, Q]/(Q^2)$-module structure of HFI^+ just from knowing J_0 (see [9, Sections 4-5]). However, one encounters issues when trying to extrapolate their methods to the case of negative semi-definite plumbings with at most one bad vertex. The main difficulty is that in the negative definite almost rational case, HF^+ is supported in even gradings, whereas in the negative semi-definite case, HF^+ has gradings in both even and odd dimensions which allows for the possibility of a more complicated action of ι at the chain level. Despite this issue, for negative semi-definite plumbings with at most one bad vertex whose HF^+ and ι_* are sufficiently simple, it is still possible to compute much, if not all, of the $\mathbb{F}[U, Q]/(Q^2)$-module structure of HFI^+ as well as some of the involutive d invariants just from the mapping cone exact triangle. We illustrate this via the examples in Chapter 7.

6.2.2 Computation of $HFI^+(-Y(\Gamma), [k])$ as a graded $\mathbb{F}$-vector space

We summarize the strategy we use to compute $HFI^+(-Y(\Gamma), [k])$ as a graded graded $\mathbb{F}$-vector space in the following 3-step process and then elaborate on each individual step.

1. Compute $HF^+(-Y(\Gamma), [k])$ using the methods from section 6.1.

2. Use Theorem 6.2.2 to compute the involution

$$\iota_* : HF^+(-Y(\Gamma), [k]) \to HF^+(-Y(\Gamma), [k])$$

3. Apply the exact triangle relating HF^+ and HFI^+ from Proposition 5.2.1.

Step (1): To compute $HF^+(-Y(\Gamma), [k])$, the first and main step is to determine the set

$$\mathcal{L}(\Gamma, [k]) := \{x \in \mathrm{K}^+(\Gamma, [k]) \mid x \text{ has no representative of the form } U^n \otimes k' \text{ for } n > 0\}$$

It is easy to see that the elements of $\mathcal{L}(\Gamma, [k])$ correspond to the leaves of the graded root $(\bar{R}_k, \bar{\chi}_k)$ under the isomorphism in Proposition 6.1.29. Moreover, from the results in section 6.1.6, it follows that the leaves of $(\bar{R}_k, \bar{\chi}_k)$ correspond to a basis of the $\mathbb{F}$-vector space:

$$\ker(U) \cap HF^+_{odd}(-Y(\Gamma), [k])$$

In [43, Section 3], Rustamov provides an algorithm to compute $\mathcal{L}(\Gamma, [k])$ which builds on the Ozsváth-Szabó algorithm in [37, Section 3] for negative definite plumbings. For our computations in Chapter 7, rather than use Rustamov's algorithm directly, we instead will use a simple criterion (see Proposition 6.2.3 below) which characterizes the elements of $\mathcal{L}(\Gamma, [k])$.

To explain this criterion, first recall from section 6.1.5 that two elements $U^m \otimes k'$ and $U^n \otimes k''$ are equivalent (i.e. represent the same element of $\mathrm{K}^+(\Gamma, [k])$) if and only if there is a path between them. In particular, every element of $\mathcal{L}(\Gamma, [k])$ is represented by an element of the form $U^0 \otimes k'$ and every element of a path connecting $U^0 \otimes k'$ to another representative must also have 0 as the exponent on the U term. Therefore, when discussing representatives or paths for elements in $\mathcal{L}(\Gamma, [k])$, we can drop the U^0 term and instead think of a representative as an element $k' \in [k]$ and a path as a sequence of vectors $k_1, \ldots, k_j \in [k]$. Furthermore, the relations defining such a path imply that for adjacent elements k_i, k_{i+1} we have that $k_{i+1} = k_i \pm 2PD[v]$ for some $v \in \mathcal{V}(\Gamma)$ with $k_i(v) = \mp m(v)$. Additionally, it follows from the definition that a representative k' of an element in $\mathcal{L}(\Gamma, [k])$ must satisfy the following property:

$$m(v) \leq k'(v) \leq -m(v)$$

for all $v \in \mathcal{V}(\Gamma)$. We refer to this property as $\star$ and we let $\star[k] = \{k' \in [k] : k' \text{ satisfies } \star\}$.

Combining these observations, we get the following proposition:

Proposition 6.2.3. *An element $k' \in [k]$ represents an element of $\mathcal{L}(\Gamma, [k])$ if and only if k' satisfies $\star$ and every element on every path containing k' also satisfies $\star$.*

After using Proposition 6.2.3 to find elements $k_1, \ldots, k_n \in [k]$ which represent the distinct elements of $\mathcal{L}(\Gamma, [k])$, it then follows that every other vertex of $(\bar{R}_k, \bar{\chi}_k)$ corresponds to an element of the form $\underline{U^m \otimes k_i}$ for some m and i. Of course, there could be relations of the form $\underline{U^{m_1} \otimes k_i} = \underline{U^{m_2} \otimes k_j}$. To determine these relations, in principle, one can write down the elements of the equivalence classes $\underline{U^{m_1} \otimes k_i}$ and $\underline{U^{m_2} \otimes k_j}$ and see whether they are equal. However, this can be quite tedious to do by hand and, in simple enough situations, there are shortcuts one can take by leveraging properties of HF^+. For example, we will use the relationship between Turaev torsion and HF^+ established in [38, Theorem 10.17] to complete the computation of $(\bar{R}_k, \bar{\chi}_k)$ for the manifolds N_j.

By sections 6.1.5 and 6.1.6, once we have computed $(\bar{R}_k, \bar{\chi}_k)$, we know $HF^+_{odd}(-Y(\Gamma), [k])$.

Furthermore, by Rustamov, we know that $HF^+_{even}(-Y(\Gamma),[k]) = \mathcal{T}^+_{d_{-1/2}}$. So to complete the computation of $HF^+(-Y(\Gamma),[k])$ it suffices to compute $d_{-1/2}(-Y(\Gamma),[k])$. As noted in Rustamov, one strategy to compute $d_{-1/2}(-Y(\Gamma),[k])$ is to first notice that $d_{-1/2}(-Y(\Gamma),[k]) = d_{1/2}(Y(\Gamma),[k])$. Then, if we can find a negative semi-definite plumbing with one bad vertex representing $-Y(\Gamma)$, we can repeat the above steps to compute $HF^+_{even}(Y(\Gamma),[k])$ which then gives us $d_{1/2}(Y(\Gamma),[k])$ and hence $d_{-1/2}(-Y(\Gamma),[k])$. This is the approach we take.

Step (2): Having done the computations in step (1), it is now easy to complete step (2). By Theorem 6.2.2, to compute ι_*, we just need to compute J_0. As noted in section 6.2.1, J_0 simply maps $\underline{U^m \otimes k'}$ to $\underline{U^m \otimes -k'}$. J_0 is also U-equivariant. Thus, to compute J_0, we just need to determine for each leaf representative k_i, which representative k_j corresponds to $-k_i$. This amounts to finding a path from $-k_i$ to one of the k_j.

Step (3): It follows from Proposition 5.2.1 and basic homological algebra, that as a graded $\mathbb{F}$-vector space:

$$HFI^+_r(-Y(\Gamma),[k]) \cong \ker Q(1+\iota_*)_{r-1} \oplus \operatorname{coker} Q(1+\iota_*)_r$$

where

$$\ker Q(1+\iota_*)_{r-1} = \ker[Q(1+\iota_*) : HF^+_{r-1}(-Y(\Gamma),[k]) \to Q \cdot HF^+_{r-1}(-Y(\Gamma),[k])]$$

and

$$\operatorname{coker} Q(1+\iota_*)_r = \operatorname{coker}[Q(1+\iota_*) : HF^+_r(-Y(\Gamma),[k]) \to Q \cdot HF^+_r(-Y(\Gamma),[k])]$$

Furthermore, steps (1) and (2) give us all of the ingredients to compute $\ker Q(1+\iota_*)_{r-1}$ and $\operatorname{coker} Q(1+\iota_*)_r$ for each r.

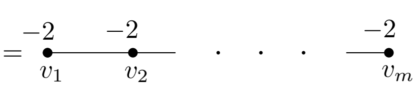
Chapter 7

A new infinite family of small Seifert fibered spaces that cannot be obtained by 0-surgery on a knot in the 3-sphere

In this chapter, we compute $HFI^+(-N_j, \mathfrak{s}_0)$ for the infinite family of small Seifert fiber spaces $\{N_j\}_{j\in\mathbb{N}}$ described in the introduction. As an application, we prove Theorem 1.2.1. We also compute $HFI^+(-S^3_0(K_1), \mathfrak{s}_0)$ where $S^3_0(K_1)$ is the manifold obtained by 0-surgery on the Ichihara-Motegi-Song knot K_1 from [20]. We then compare $HFI^+(-S^3_0(K_1), \mathfrak{s}_0)$ and $HFI^+(-N_1, \mathfrak{s}_0)$.

7.1 Moves between equivalent vectors

Let Γ be a negative semi-definite plumbing with at most one bad vertex and with $b_1 = 1$. Suppose Γ contains a linear subgraph Λ with framing -2 at each vertex:

$\Lambda =$ 

Let $[k]$ be a self-conjugate spin^c structure on $Y(\Gamma)$. Given a characteristic vector $k' \in [k]$, let $k'_\Lambda = (a_1, \ldots, a_m)$ be the subvector corresponding to the vertices

$v_1, \ldots, v_m$. We call k'_Λ the Λ-subvector of k'.

Note, if $k' \in [k]$ and satisfies $\star$, then we must have $a_i \in \{-2, 0, 2\}$ for each $1 \leq i \leq m$. If there exists some i such that $a_i = \pm 2$, then $k'' = k' \pm 2PD[v_i]$ is an equivalent vector. In particular, $k''_\Lambda = (a_1, \ldots, a_{i-1} \pm 2, \mp 2, a_{i+1} \pm 2, \ldots, a_m)$. Of course, other entries of k'' not contained in k''_Λ may also differ from those of k'. Specifically, any entry a of k' corresponding to a vertex adjacent to v_i will change from a to $a \pm 2$. We call the replacement of k' with $k'' = k' \pm 2PD[v_i]$ where $k'(v_i) = \pm 2$ a move of **type** ± 2.

Next suppose $k'_\Lambda = (a_1, \ldots, a_i, 0, \ldots, 0, 2, -2, a_j, \ldots, a_m)$. Then, by iteratively applying type $+2$ moves to the $+2$-entry, we can convert k' into an equivalent vector k'' with: $k''_\Lambda = (a_1, \ldots, a_i, 2, -2, 0, \ldots, 0, a_j, \ldots, a_m)$. We call the replacement of k' with k'' or k'' with k' a $(2, -2)$-**slide**. We define a $(-2, 2)$-**slide** analogously.

Lemma 7.1.1. *Let $k' \in [k]$ be a vector with $k'_\Lambda = (a_1, \ldots, a_i, 0, \pm 2, 0, \ldots, 0, \mp 2, a_j, \ldots, a_m)$. Then, k' is equivalent to a vector k'' with $k''_\Lambda = (a_1, \ldots, a_i, \pm 2, 0, \ldots, 0, \mp 2, 0, a_j, \ldots, a_m)$.*

Proof. Apply a type ± 2 move to the ± 2-entry to get an equivalent vector h' with $h'_\Lambda = (a_1, \ldots, a_i, \pm 2, \mp 2, \pm 2, 0, \ldots, 0, \mp 2, a_j, \ldots, a_m)$. Now do a rightward $(\mp 2, \pm 2)$-slide to h' to convert h' into an equivalent vector h'' with:

$$h''_\Lambda = (a_1, \ldots, a_i, \pm 2, 0, \ldots, 0, \mp 2, \pm 2, \mp 2, a_j, \ldots, a_m)$$

Finally apply a type ± 2 move to the rightmost ± 2-entry to get an equivalent vector k'' with $k''_\Lambda = (a_1, \ldots, a_i, \pm 2, 0, \ldots, 0, \mp 2, 0, a_j, \ldots, a_m)$. □

By iterating the sequence of moves described in the above proof, we can now convert any vector $k' \in [k]$ with $k'_\Lambda = (a_1, \ldots, a_i, 0, \ldots, 0, \pm 2, 0, \ldots, 0, \mp 2, a_j, \ldots, a_m)$ into an equivalent vector k'' with:

$$k''_\Lambda = (a_1, \ldots, a_i, \pm 2, 0, \ldots, 0, \mp 2, 0, \ldots, 0, a_j, \ldots, a_m)$$

By an abuse of notation, we also call the replacement of k' with k'' or k'' with k' via the above sequence of moves a $(\pm 2, \mp 2)$-**slide**.

Lemma 7.1.2. *Suppose $k' \in [k]$ represents an element of $\mathcal{L}(\Gamma, [k])$, then either k'_Λ is the zero vector or it has entries which alternate between 2 and -2 with possibly 0s inbetween.*

Proof. Suppose k' represents an element of $\mathcal{L}(\Gamma, [k])$ and k'_Λ contains a subvector of the form $(2, \underbrace{0, \ldots, 0}_{j}, 2)$ where $j \geq 0$. Then, by doing a type +2 move on the leftmost +2-entry, k' is equivalent to a vector whose corresponding subvector is $(-2, 2, \underbrace{0, \ldots, 0}_{j-1}, 2)$ if $j \geq 1$ or $(-2, 4)$ if $j = 0$. In the latter case, the vector fails to satisfy $\star$ and thus we get a contradiction by Proposition 6.2.3. So we can assume the subvector is $(-2, 2, \underbrace{0, \ldots, 0}_{j-1}, 2)$ with $j \geq 1$. Now do a rightward $(-2, 2)$-slide to produce an equivalent vector whose corresponding subvector is $(\underbrace{0, \ldots, 0}_{j-1}, -2, 2, 2)$. Next apply a type +2 move to get an equivalent vector whose corresponding subvector is $(\underbrace{0, \ldots, 0}_{j}, -2, 4)$. We again get a contradiction for the same reason as before. Therefore, k'_Λ cannot contain a subvector of the form $(2, \underbrace{0, \ldots, 0}_{j}, 2), j \geq 0$. By an analogous argument, k'_Λ also cannot contain a subvector of the form $(-2, \underbrace{0, \ldots, 0}_{j}, -2), j \geq 0$. This completes the proof. □

Lemma 7.1.3. *Suppose $k' \in [k]$ represents an element of $\mathcal{L}(\Gamma, [k])$. Then, k' is equivalent to a vector k'' such that k''_Λ is the zero vector except for possibly one non-zero entry equal to ± 2.*

Proof. We induct on the number of non-zero entries of k'_Λ. Obviously the statement is true if k'_Λ is the zero vector or has only one-nonzero entry. So suppose k'_Λ has $n \geq 2$ non-zero entries. Let a_i and a_{i+j} be the leftmost non-zero entries. Then by the Lemma 7.1.2, $a_i = \pm 2$ and $a_{i+j} = \mp 2$. For simplicity, assume $a_i = 2$. (The argument when $a_i = -2$ is identical up to sign changes.) We can write k'_Λ as:

$$k'_\Lambda = (0, \ldots, 0, 2, 0, \ldots, 0, -2, a_{i+j+1}, \ldots, a_m)$$

where there are possibly no initial 0 entries and no 0 entries between a_i and a_{i+j}. If there are initial 0 entries, then by doing a leftward $(2,-2)$-slide, k' is equivalent to a vector whose Λ-subvector is $(2,0,\ldots,0,-2,0,\ldots,0,a_{i+j+1},\ldots,a_m)$. Now apply a type $+2$ move to the left most $+2$-entry to get an equivalent vector whose Λ-subvector is: $(-2,2,0,\ldots,0,-2,0,\ldots,0,a_{i+j+1},\ldots,a_m)$ if $j>1$ or $(-2,0,\ldots,0,a_{i+2},\ldots,a_m)$ if $j=1$. In the latter case, we have reduced the number of non-zero entries in the Λ-subvector by 1. Hence, we can assume $j>1$. In this case, if we do a rightward $(-2,2)$-slide on leftmost $(-2,2)$-pair, we get an equivalent vector whose Λ-subvector is $(0,\ldots,0,-2,2,-2,0,\ldots,0,a_{i+j+1},\ldots,a_m)$. Finally apply a type $+2$ move to produce an equivalent vector whose Λ-subvector is $(0,\ldots,0,0,-2,0,0,\ldots,0,a_{i+j+1},\ldots,a_m)$. We have reduced the number of non-zero entries by 1. Therefore, by induction the result follows. □

Lemma 7.1.4. *Suppose $k'\in[k]$ with:*

$$k'_\Lambda = (\underbrace{0,\ldots,0}_{j\ copies},2,\underbrace{0,\ldots,0}_{m-j-1\ copies})$$

Then k' is equivalent to a vector k'' with:

$$k''_\Lambda = (\underbrace{0,\ldots,0}_{m-j-1\ copies},-2,\underbrace{0,\ldots,0}_{j\ copies})$$

Proof. We list the sequence of moves needed to obtain the relevant vector. In each move, we only write the resulting Λ-subvector.

1. Type $+2$ move: $(\underbrace{0,\ldots,0}_{j-1\ copies},2,-2,2\ \underbrace{0,\ldots,0}_{m-j-2\ copies})$
2. Leftward $(2,-2)$-slide: $(2,-2,\underbrace{0,\ldots,0}_{j-1\ copies},2,\underbrace{0,\ldots,0}_{m-j-2\ copies})$
3. Type $+2$ move: $(-2,\underbrace{0,\ldots,0}_{j\ copies},2,\underbrace{0,\ldots,0}_{m-j-2\ copies})$
4. Rightward $(-2,2)$-slide: $(\underbrace{0,\ldots,0}_{m-j-2\ copies},-2,\underbrace{0,\ldots,0}_{j\ copies},2)$
5. Type $+2$ move: $(\underbrace{0,\ldots,0}_{m-j-2\ copies},-2,\underbrace{0,\ldots,0}_{j-1\ copies},2,-2)$

6. Leftward $(2,-2)$-slide: $(\underbrace{0,\ldots,0}_{m-j-2\ copies}, -2, 2, -2\, \underbrace{0,\ldots,0}_{j-1\ copies})$

7. Type $+2$ move: $(\underbrace{0,\ldots,0}_{m-j-1\ copies}, -2, \underbrace{0,\ldots,0}_{j\ copies})$

□

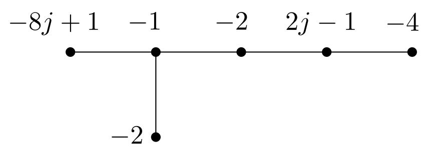
Remark 7.1.5. If one traces through the above sequence of moves, it is easy to see that if v is a vertex not in Λ, but is adjacent to the initial vertex v_1 or terminal vertex v_m of Λ, then $k''(v) = k'(v) + 2$.

7.1.1 Computation of $HFI^+(-N_j, \mathfrak{s}_0)$

Step 1

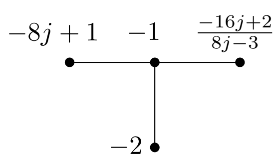
Recall from the Section 1.2, for $j \geq 1$, $N_j = S^2\left(\frac{-2}{1}, \frac{-8j+1}{1}, \frac{16j-2}{8j+1}\right)$. In [14, Section 7], it is shown that N_j can be represented as a plumbing as follows:

$N_j =$



By performing two slam dunks on the rightward stem, we get:

$N_j =$



One can further check that:

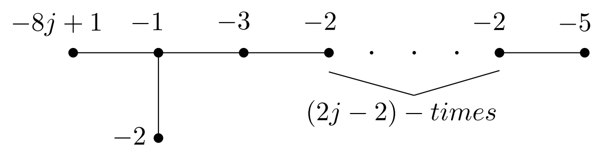
$$\frac{-16j+2}{8j-3} = -3 - \cfrac{1}{-2 - \cfrac{1}{\ddots - 2 - \cfrac{1}{\frac{-8j+3+4r}{8j-7-4r}}}}$$

where there are r copies of -2 along the diagonal. In particular, setting $r = 2j-2$, the last term becomes:

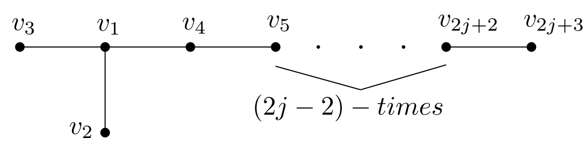
$$\frac{-8j+3+4(2j-2)}{8j-7-4(2j-2)} = -5$$

Hence, by performing the corresponding slam dunks, we get:

$N_j =$



Let Γ_j be the above plumbing graph with vertices labeled as follows:



With respect to the ordered basis $([v_1], \ldots, [v_{2j+3}])$, the matrix for the intersection form of $X(\Gamma_j)$ is:

$$B_j = \begin{pmatrix} -1 & 1 & 1 & 1 & & & & & \\ 1 & -2 & & & & & & & \\ 1 & & -8j+1 & & & & & & \\ 1 & & & -3 & 1 & & & & \\ & & & 1 & -2 & 1 & & & \\ & & & & 1 & -2 & 1 & & \\ & & & & & & \ddots & & \\ & & & & & & 1 & -2 & 1 \\ & & & & & & & 1 & -5 \end{pmatrix}$$

It is straightforward to check that B_j is negative semi-definite and $H_1(N_j;\mathbb{Z}) \cong \mathbb{Z}$. We leave this to the reader.

Note, the $\mathbb{Z}$-kernel of B_j is generated by the vector:

$$x = (16j-2, 8j-1, 2, 8j-3, 8j-7, 8j-11, \ldots, 1)$$

Therefore, the unique self-conjugate spin^c-structure $\mathfrak{s}_0$ on N_j can be thought of as:

$$[k] = \{k' \in \text{Char}(X(\Gamma_j)) \mid k' \cdot x = 0\}$$

Let Λ_j be the linear subgraph of Γ_j given by:

$$\Lambda_j = \overset{-2}{\underset{v_5}{\bullet}} \!\!-\!\!- \quad \cdot \quad \cdot \quad \cdot \quad -\!\!\!- \overset{-2}{\underset{v_{2j+2}}{\bullet}}$$

We write vectors $k' \in [k]$ as: $k' = (a_1, a_2, a_3, a_4, b_5, \dots, b_{2j+2}, c_{2j+3})$, where $k'_{\Lambda_j} = (b_5, \dots, b_{2j+2})$.

Lemma 7.1.6. *If $k' \in [k]$ represents an element of $\mathcal{L}(\Gamma_j, [k])$, then k' is equivalent to a vector whose Λ_j-subvector is not equal to the zero vector.*

Proof. Suppose $k' \in [k]$ represents an element of $\mathcal{L}(\Gamma_j, [k])$. For the purpose of contradiction, suppose the Λ_j-subvector of every representative of every element of $\mathcal{L}(\Gamma_j, [k])$ is zero. Then, in particular, $k'_{\Lambda_j} = 0$. Also, since k' represents an element of $\mathcal{L}(\Gamma_j, [k])$, it must satisfy $\star$. So we must have $a_4 \in \{-3, -1, 1, 3\}$. If $a_4 = \pm 3$, then by adding $\pm 2PD[v_4]$ to k' we would obtain an equivalent vector with a non-zero Λ_j-subvector. Thus, $a_4 \in \{-1, 1\}$.

Since k' must satisfy $\star$, we also have $a_1 = \pm 1$. If $a_1 = 1$ and $a_4 = 1$, then by adding $2PD[v_1]$ to k', a_4 becomes 3. But we just showed that a_4 cannot be equal to 3. Similarly, if $a_1 = -1$ and $a_4 = -1$, then by adding $-2PD[v_1]$ to k', a_4 becomes -3, which is again a contradiction. Hence, $a_1 = \pm 1$ and $a_4 = \mp 1$. By adding $-2PD[v_1]$ if necessary, we may assume $a_1 = 1$ and $a_4 = -1$. Again, by $\star$, we must have $a_2 \in \{-2, 0, 2\}$. If $a_2 = 2$, then by adding $2PD[v_1]$ to k', we get an equivalent vector with $a_2 = 4$, which contradicts Proposition 6.2.3. Therefore, $a_2 \in \{0, -2\}$. If $a_2 = -2$, then by adding $-2PD[v_2]$ to k' we obtain an equivalent vector with $a_1 = -1$ and $a_4 = -1$, which we already determined cannot happen. Therefore, $a_2 = 0$. Now add $2PD[v_1]$ to k'. The result is an equivalent vector with $a_1 = -1$, $a_2 = 2$, and $a_4 = 1$. Since $a_2 = 2$, we can add $2PD[v_2]$ to get an equivalent vector with $a_1 = 1, a_2 = -2$, and $a_4 = 1$, but we have already shown that we cannot have both $a_1 = 1$ and $a_4 = 1$. Therefore, we get a contradiction and hence k' must be equivalent to some vector whose Λ_j-subvector is not equal to the zero vector. $\square$

Somewhat counter-intuitively, we are now going to use the previous lemma to find a small finite set of possible representatives of $\mathcal{L}(\Gamma_j, [k])$, all of whose Λ_j-subvectors are all equal to the zero vector.

Lemma 7.1.7. *If $k' \in [k]$ represents an element of $\mathcal{L}(\Gamma_j, [k])$, then k' is equivalent to a vector of the form $k'' = (-1, 0, a_3, 3, 0, \ldots, 0, c_{2k+3})$ where $a_3 \in \{-8k+1, -8k+3, \ldots, 8k+1\}$ and $c_{2j+3} \in \{-5, -3, -1, 1, 3\}$.*

Proof. Suppose k' represents an element of $\mathcal{L}(\Gamma_j, [k])$. Then, by combining Lemmas 7.1.3, 7.1.4, and 7.1.6, we may assume:

$$k'_{\Lambda_j} = (\underbrace{0, \ldots, 0}_{\ell \text{ copies}}, 2, \underbrace{0, \ldots, 0}_{2j-3-\ell \text{ copies}})$$

for some $0 \leq \ell \leq 2j-3$. By $\star$, $a_1 = \pm 1$. If $a_1 = -1$, then we can add $-2PD[v_1]$ from k' to get an equivalent vector with $a_1 = 1$. This addition does not effect any of the entries in k'_{Λ_j}. Thus, we may assume $a_1 = 1$.

Next, by $\star$, $a_2 \in \{-2, 0, 2\}$. If $a_2 = 2$, then adding $2PD[v_1]$ to k' yields an equivalent vector with $a_2 = 4$, which violates $\star$. Therefore, $a_2 \in \{-2, 0\}$. Suppose $a_2 = -2$. Then, by adding $-2PD[v_2]$, we get an equivalent vector with $a_2 = 2$ and $a_1 = -1$. k'_{Λ_j} is unaffected by this move. If we then add $-2PD[v_1]$, we get an equivalent vector with $a_1 = 1$ and $a_2 = 0$. Again k'_{Λ_j} is unaffected. Therefore, we may assume $a_2 = 0$.

Next, with $k' = (1, 0, a_3, a_4, \underbrace{0, \ldots, 0}_{\ell \text{ copies}}, 2, \underbrace{0, \ldots, 0}_{2j-3-\ell \text{ copies}}, c_{2j+3})$, add $2PD[v_1]$ to k' to get the equivalent vector:

$$(-1, 2, a_3 + 2, a_4 + 2, \underbrace{0, \ldots, 0}_{\ell \text{ copies}}, 2, \underbrace{0, \ldots, 0}_{2j-3-\ell \text{ copies}}, c_{2j+3})$$

Next, add $2PD[v_2]$ to get $(1, -2, a_3 + 2, a_4 + 2, \underbrace{0, \ldots, 0}_{\ell \text{ copies}}, 2, \underbrace{0, \ldots, 0}_{2j-3-\ell \text{ copies}}, c_{2j+3})$. Then, add another $2PD[v_1]$, to get $(-1, 0, a_3+4, a_4+4, \underbrace{0,\ldots,0}_{\ell \text{ copies}}, 2, \underbrace{0,\ldots,0}_{2j-3-\ell \text{ copies}}, c_{2j+3})$.

Now, if we apply the move in Lemma 7.1.4 and take into account Remark 7.1.5, one can check that we get an equivalent vector whose $4th$-entry is a_4+6. Since we assumed k' represents an element of $\mathcal{L}(\Gamma_j, [k])$, we must therefore have that $a_4 \in \{-3,-1,1,3\}$ and $a_4+6 \in \{-3,-1,1,3\}$. Hence, we must have had $a_4 = -3$. To summarize, we have now shown that we can assume: $k' = (1, 0, a_3, -3, \underbrace{0,\ldots,0}_{\ell \text{ copies}}, 2, \underbrace{0,\ldots,0}_{2j-3-\ell \text{ copies}}, c_{2j+3})$.

Next, by $\star$, $c_{2j+3} \in \{-5,-3,-1,1,3,5\}$. If $c_{2j+3} = 5$, then again by applying the move from Lemma 7.1.4, one can check that we transform c_{2j+3} into 7, which violates $\star$. Therefore, we must have had $c_{2j+3} \in \{-5,-3,-1,1,3\}$.

Now add $-2PD[v_4]$ to get an equivalent vector (which we again call k') with $a_1 = -1$, $a_2 = 0$, $a_4 = 3$ and k'_{Λ_j} unchanged except for the first entry which decreases by 2. Also, c_{2k+3} remains unchanged. If $\ell = 0$, then k'_{Λ_j} is now the zero vector, so we are done. Thus, suppose $\ell > 0$. Then, $k' = (-1, 0, a_3, 3, -2, \underbrace{0,\ldots,0}_{\ell-1 \text{ copies}}, 2, \underbrace{0,\ldots,0}_{2j-3-\ell \text{ copies}}, c_{2j+3})$.

Now consider the following sequence of moves:

1. Rightward $(-2,2)$-slide: $(-1, 0, a_3, 3, \underbrace{0,\ldots,0}_{2j-3-\ell \text{ copies}}, -2, \underbrace{0,\ldots,0}_{\ell-1 \text{ copies}}, 2, c_{2j+3})$
2. Type $+2$ move: $(-1, 0, a_3, 3, \underbrace{0,\ldots,0}_{2j-3-\ell \text{ copies}}, -2, \underbrace{0,\ldots,0}_{\ell-2 \text{ copies}}, 2, -2, c_{2j+3}+2)$
3. Leftward $(2,-2)$-slide: $(-1, 0, a_3, 3, \underbrace{0,\ldots,0}_{2j-3-\ell \text{ copies}}, -2, 2, -2\ \underbrace{0,\ldots,0}_{\ell-2 \text{ copies}}, c_{2j+3}+2)$
4. Type $+2$ move: $(-1, 0, a_3, 3, \underbrace{0,\ldots,0}_{2j-2-\ell \text{ copies}}, -2, \underbrace{0,\ldots,0}_{\ell-1 \text{ copies}}, c_{2j+3}+2)$
5. Apply Lemma 7.1.4 and Remark 7.1.5: $(-1, 0, a_3, 1, \underbrace{0,\ldots,0}_{\ell-1 \text{ copies}}, 2, \underbrace{0,\ldots,0}_{2j-2-\ell \text{ copies}}, c_{2j+3})$
6. Add $-2PD[v_1]$: $(1, -2, a_3-2, -1, \underbrace{0,\ldots,0}_{\ell-1 \text{ copies}}, 2, \underbrace{0,\ldots,0}_{2j-2-\ell \text{ copies}}, c_{2j+3})$
7. Add $-2PD[v_2]$: $(-1, 2, a_3-2, -1, \underbrace{0,\ldots,0}_{\ell-1 \text{ copies}}, 2, \underbrace{0,\ldots,0}_{2j-2-\ell \text{ copies}}, c_{2j+3})$
8. Add $-2PD[v_1]$: $(1, 0, a_3-4, -3, \underbrace{0,\ldots,0}_{\ell-1 \text{ copies}}, 2, \underbrace{0,\ldots,0}_{2j-2-\ell \text{ copies}}, c_{2j+3})$

The net effect of this sequence of moves is that the +2-entry in k'_{Λ_j} shifts one space to the left while every other entry, excluding a_3, remains the same. So now we can repeat the above process until +2-entry is in the first position of k'_{Λ_j}. Then add $-2PD[v_4]$ to get $(-1, 0, a'_3, 3, 0, \ldots, 0, c_{2j+3})$ with $c_{jk+3} \in \{-5, -3, -1, 1, 3\}$ and, by $\star$, $a'_3 \in \{-8j + 1, -8j + 3, \ldots, 8j + 1\}$. □

Proposition 7.1.8. *If k' represents an element of $\mathcal{L}(\Gamma_j, [k])$, then k' is equivalent to*

$$k_1 = (-1, 0, 5 - 4j, 3, 0, \ldots, 0, -3) \text{ or } k_2 = (-1, 0, 3 - 4j, 3, 0, \ldots, 0, 1)$$

In particular, $|\mathcal{L}(\Gamma_j, [k])| \leq 2$.

Proof. Up to this point, we have not used the fact that $k' \cdot x = 0$ where x is a generator of $\ker_{\mathbb{Z}}(B_j)$ as above. So assume k' is of the form in the previous lemma. Then, we get the following equation:

$$0 = k' \cdot x = 8j - 7 + 2a_3 + c_{2j+3}$$

where $a_3 \in \{-8j + 1, -8k + 3, \ldots, 8j + 1\}$ and $c_{2j+3} \in \{-5, -3, -1, 1, 3\}$. The only solutions to this equation with the given constraints are: $(a_3, c_{2j+3}) = (5 - 4j, -3)$ and $(3 - 4j, 1)$, corresponding to k_1 and k_2, respectively. □

We have not yet proved that k_1 and k_2 represent different elements of $\mathcal{L}(\Gamma_j, [k])$. To do this we will do a similar analysis for $-N_j$ and then use Turaev torsion. However, before we undertake this task, we first compute the HF^+ grading associated to the vectors k_1 and k_2.

Corollary 7.1.9. $d_{1/2}(-N_j; \mathfrak{s}_0) = \frac{1}{2}$

Proof. Let

$$\alpha_1 = (-12j + 6, -6j + 3, -1, -6j + 3, -6j, -6j - 3, \ldots, 2, 1, 0)$$

$$\alpha_2 = (4j + 2, 2j + 1, 1, 2j - 1, 2j - 2, 2j - 3, \ldots, 2, 1, 0)$$

Then, $\alpha_1 B_j = k_1$ and $\alpha_2 B_j = k_2$. Thus,

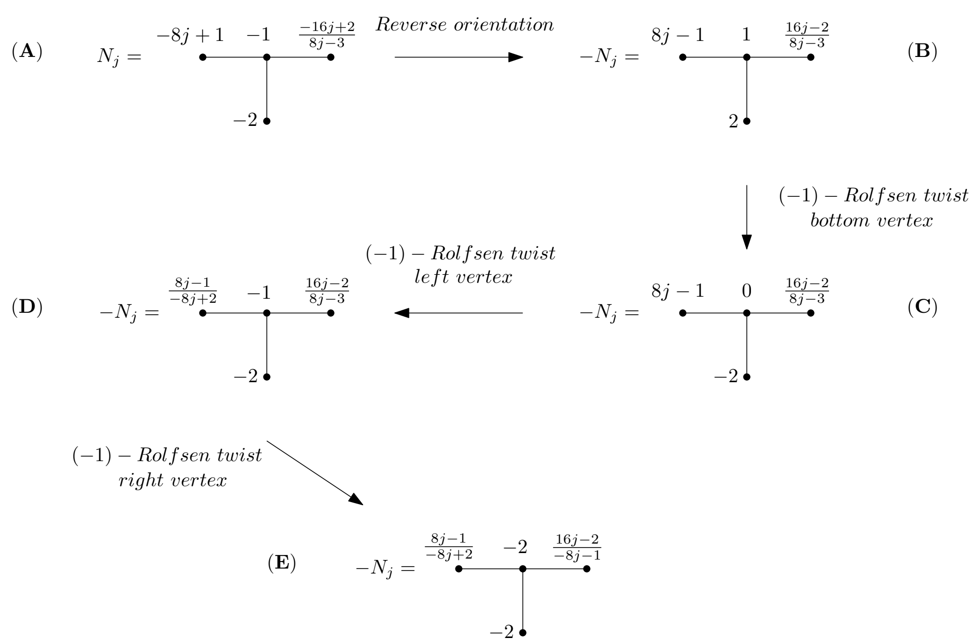
$$k_1^2 = k_1 \cdot \alpha_1 = -2j - 2$$
$$k_2^2 = k_2 \cdot \alpha_2 = -2j - 2$$

Hence, under the isomorphism from Corollary 6.1.32, the elements of $HF^+(-N_j, \mathfrak{s}_0)$ corresponding to k_1 and k_2 have gradings:

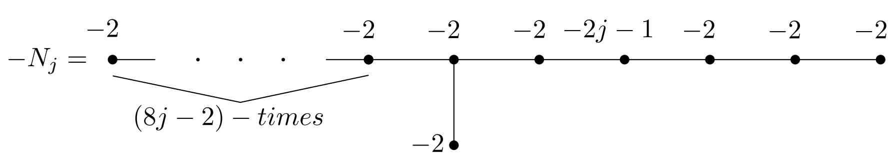
$$\mathrm{gr}(k_1) = \mathrm{gr}(k_2) = -\frac{k_2^2 + |\mathcal{V}(\Gamma_j)| - 3}{4} = -\frac{-2j - 2 + 2j + 3 - 3}{4} = \frac{1}{2}$$

□

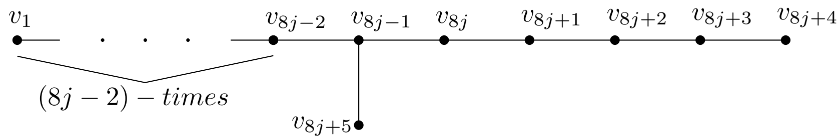
We now find a negative semi-definite plumbing representation of $-N_j$. First,



Next, do slam dunks on the left and right vertices to get:



Let Γ_j' be the above plumbing graph with vertices labeled as follows:



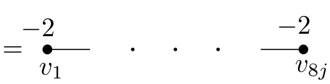
With respect to the ordered basis $([v_1], \ldots, [v_{8j+5}])$, the matrix for the intersection form of $X(\Gamma'_j)$ is:

$$B'_j = \begin{pmatrix}
-2 & 1 & & & & & & & & & & & \\
1 & -2 & 1 & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & \\
 & & & & 1 & -2 & 1 & & & & & & \\
 & & & & & 1 & -2 & 1 & & & & & 1 \\
 & & & & & & 1 & -2 & 1 & & & & \\
 & & & & & & & 1 & -2j-1 & 1 & & & \\
 & & & & & & & & 1 & -2 & 1 & & \\
 & & & & & & & & & 1 & -2 & 1 & \\
 & & & & & & & & & & 1 & -2 & \\
 & & & & & & 1 & & & & & & -2
\end{pmatrix}$$

Again, it is straightforward to check that B'_j is negative semi-definite. Also, the $\mathbb{Z}$-kernel of B'_j is generated by the vector $x' = (2, 4, 6, \ldots, 16j-2, 8j+1, 4, 3, 2, 1, 8j-1)$.

Let t denote a characteristic vector representing the trivial spin^c structure $\mathfrak{s}_0$. Then again, we can think of $\mathfrak{s}_0$ as $[t] = \{t' \in \mathrm{Char}(X(\Gamma'_j)) \mid t' \cdot x' = 0\}$.

Let Λ'_j be the linear subgraph of Γ'_j given by:

$\Lambda'_j =$ 

We write vectors $t' \in [t]$ as: $t' = (a_1, a_2, \ldots, a_{8j}, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, d_{8j+5})$ where $t'_{\Lambda'_j} = (a_1, a_2, \ldots, a_{8j})$.

Lemma 7.1.10. *If $t' \in [t]$ represents an element of $\mathcal{L}(\Gamma'_j, [t])$, then t' is equivalent to a vector whose Λ'_j-subvector is of the form $(0, \ldots, 0, a_{8j})$ where $a_{8j} \in \{0, 2\}$.*

Proof. Suppose t' represents and element of $\mathcal{L}(\Gamma'_j, [t])$. By Lemmas 7.1.3 and 7.1.4, it suffices to consider the case when:

$$t'_{\Lambda'_j} = (\underbrace{0,\ldots,0}_{\ell \textit{ copies}}, 2, \underbrace{0,\ldots,0}_{8j-1-\ell \textit{ copies}})$$

for some $0 \le \ell \le 8j-2$. Furthermore, by considering the linear subgraph of Γ'_j whose endpoints are $v_{\ell+1}$ and v_{8j+5}, it follows from Lemma 7.1.2 that $d_{8j+5} \in \{0, -2\}$.

Case 1: Suppose $d_{8j+5} = -2$ and $\ell = 8j-2$. If we add $-2PD[v_{8j+5}]$, then the Λ'_j-subvector of the resulting vector is zero, so we are done.

Case 2: Suppose $d_{8j+5} = -2$ and $\ell \le 8j-3$. Consider the following sequence of moves:

1. Add $-2PD[v_{8j+5}]$: $(\underbrace{0,\ldots,0}_{\ell \textit{ copies}}, 2, \underbrace{0,\ldots,0}_{8j-3-\ell \textit{ copies}}, -2, 0, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, 2)$

2. Rightward $(2,-2)$-slide: $(\underbrace{0,\ldots,0}_{\ell+1 \textit{ copies}}, 2, \underbrace{0,\ldots,0}_{8j-3-\ell \textit{ copies}}, -2, b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}, 0)$.
   Note, the rightmost entry of the vector changes from 2 to 0.

3. Type -2 move on the leftmost -2:

$$\begin{cases} (\underbrace{0,\ldots,0}_{8j-1 \textit{ copies}}, 2, b_{8j+1}-2, c_{8j+2}, c_{8j+3}, c_{8j+4}, 0) & \text{if } \ell = 8j-3 \\ (\underbrace{0,\ldots,0}_{\ell+1 \textit{ copies}}, 2, \underbrace{0,\ldots,0}_{8j-4-\ell \textit{ copies}}, -2, 2, b_{8j+1}-2, c_{8j+2}, c_{8j+3}, c_{8j+4}, 0) & \text{if } \ell \le 8j-4 \end{cases}$$

   If $\ell = 8j-3$ we are done. If $\ell = 8j-4$, then by applying a type -2 move on the leftmost -2, we get $(\underbrace{0,\ldots,0}_{8j-2 \textit{ copies}}, 2, 0, b_{8j+1}-2, c_{8j+2}, c_{8j+3}, c_{8j+4}, -2)$. Hence, we are back to case 1. Therefore, we may assume $\ell \le 8j-5$. We now continue as follows:

4. Leftward $(-2,2)$-slide: $(\underbrace{0,\ldots,0}_{\ell+1 \textit{ copies}}, 2, -2, 2\ \underbrace{0,\ldots,0}_{8j-4-\ell \textit{ copies}}, b_{8j+1}-2, c_{8j+2}, c_{8j+3}, c_{8j+4}, -2)$.
   Note, the rightmost entry of the vector now changes back to -2.

5. Type -2 move on the leftmost -2:

$$(\underbrace{0,\ldots,0}_{\ell+2\ copies},2,\underbrace{0,\ldots,0}_{8j-3-\ell\ copies},b_{8j+1}-2,c_{8j+2},c_{8j+3},c_{8j+4},-2)$$

We are now back to the vector we started with at the beginning of case 2, except that the $+2$ entry of the Λ'_j-subvector has shifted two positions to the right. Therefore, we can iterate this process until $\ell = 8j-3$ or $8j-4$, and we have already dealt with both of those cases.

Case 3: Suppose $d_{8j+5} = 0$ and $\ell = 8j-2$. Add $2PD[v_{8j-1}]$ to get the equivalent vector: $(\underbrace{0,\ldots,0}_{8j-3\ copies},2,-2,2,b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},2)$.

Now add $2PD[v_{8j+5}]$ to get: $(\underbrace{0,\ldots,0}_{8j-3\ copies},2,0,2,b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},-2)$. This vector violates Lemma 7.1.2 and hence cannot be a representative of $\mathcal{L}(\Gamma'_j,[t])$.

Case 4: Suppose $d_{8j+5} = 0$ and $\ell \leq 8j-3$, so that we start with a vector of the form $(\underbrace{0,\ldots,0}_{\ell\ copies},2,\underbrace{0,\ldots,0}_{8j-1-\ell\ copies},b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},0)$. Now consider the following sequence of moves:

1. Type $+2$ move: $(\underbrace{0,\ldots,0}_{\ell-1\ copies},2,-2,2\ \underbrace{0,\ldots,0}_{8j-2-\ell\ copies},b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},0)$
2. Rightward $(-2,2)$-slide: $(\underbrace{0,\ldots,0}_{\ell-1\ copies},2,\underbrace{0,\ldots,0}_{8j-3-\ell\ copies},-2,2,0,b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},0)$
3. Add $2PD[v_{8j-1}]$: $(\underbrace{0,\ldots,0}_{\ell-1\ copies},2,\underbrace{0,\ldots,0}_{8j-2-\ell\ copies},-2,2,c_{8j+2},c_{8j+3},c_{8j+4},2)$
4. Add $2PD[v_{8j+5}]$: $(\underbrace{0,\ldots,0}_{\ell-1\ copies},2,\underbrace{0,\ldots,0}_{8j-1-\ell\ copies},2,c_{8j+2},c_{8j+3},c_{8j+4},-2)$

Again, this vector violates Lemma 7.1.2 and hence cannot be a representative of $\mathcal{L}(\Gamma'_j,[t])$. □

Proposition 7.1.11. *If $t' \in [t]$ represents an element of $\mathcal{L}(\Gamma'_j,[t])$, then t' is equivalent to $t_1 = (0,\ldots,0,-1,0,2,0,0)$ or $t_2 = (0,\ldots,0,2,-1,0,0,0,-2)$.*

Proof. Suppose $t' \in [k]$ represents an element of $\mathcal{L}(\Gamma'_j,[t])$. By the previous lemma, we can assume $t' = (0,\ldots,0,a_{8j},b_{8j+1},c_{8j+2},c_{8j+3},c_{8j+4},d_{8j+5})$, where $a_{8j} \in \{0,2\}$,

$b_{8j+1} \in \{-2j-1, -2j+1, \ldots, 2j-1, 2j+1\}$, $c_{8j+2}, c_{8j+3}, c_{8j+4} \in \{-2, 0, 2\}$, $d_{8j+5} \in \{-2, 0, 2\}$.

Since we are assuming t' represents an element of $\mathcal{L}(\Gamma'_j, [t])$, we must have:

$$0 = t' \cdot x' = (8j+1)a_{8j} + (8j-1)d_{8j+5} + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \quad (7.1.12)$$

By Lemmas 7.1.3 and 7.1.4, we can assume $(c_{8j+2}, c_{8j+3}, c_{8j+4})$ is the zero vector or has exactly one non-zero entry equal to $+2$. In particular, we can assume $3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \in \{0, 2, 4, 6\}$. Note the moves required to put the subvector $(c_{8j+2}, c_{8j+3}, c_{8j+4})$ into this form only effect the entry b_{8j+1} and leave all of the others unchanged.

Now suppose $a_{8j} = 2$ and $b_{8j+1} = 2j+1$. Then by adding $2PD[v_{8j+1}]$ we would obtain an equivalent vector with $a_{8j} = 4$. But this violates $\star$. Hence, if $a_{8j} = 2$, we can assume $b_{8j+1} \leq 2j-1$.

Now suppose $a_{8j} = 0$ and $b_{8j+1} = 2j+1$. If $(c_{8j+2}, c_{8j+3}, c_{8j+4})$ is not the zero vector, but rather a vector with precisely one non-zero entry equal to $+2$, then by applying the move in Lemma 7.1.4 and taking into account Remark 7.1.5, we would obtain an equivalent vector with $b_{8j+1} = 2j+3$, which violates $\star$. Therefore, if $a_{8j} = 0$ and $b_{8j+1} = 2j+1$, we must have that $(c_{8j+2}, c_{8j+3}, c_{8j+4})$ is the zero vector. Plugging this into equation 7.1.12 yields $(8j-1)d_{8j+5} = -8j-4$. This clearly has no solutions with the given constraints. Therefore, we can assume $b_{8j+1} \leq 2j-1$, regardless of whether $a_{8j} = 0$ or 2. In particular,

$$-8j - 4 \leq 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \leq 8j + 2$$

<u>Case 1:</u> Suppose $a_{8j} = 0$ and $d_{8j+5} = -2$. Then:

$$0 = t' \cdot x' = -16j + 2 + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \leq -8j + 4 < 0$$

which is a contradiction.

Case 2: Suppose $a_{8j} = 0$ and $d_{8j+5} = 0$. Then:

$$0 = t' \cdot x' = 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4}$$

The only solution to this equation given the constraints we have established is

$$(b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}) = (-1, 0, 2, 0)$$

which corresponds to t_1.

Case 3: Suppose $a_{8j} = 0$ and $d_{8j+5} = 2$. Then:

$$0 = t' \cdot x' = 16j - 2 + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \geq 8j - 6 > 0$$

which again is a contradiction.

Case 4: Suppose $a_{8j} = 2$ and $d_{8j+5} = -2$. Then:

$$0 = t' \cdot x' = 4 + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4}$$

The only solution to this equation given the constraints we have established is

$$(b_{8j+1}, c_{8j+2}, c_{8j+3}, c_{8j+4}) = (-1, 0, 0, 0)$$

which corresponds to t_2.

Case 5: Suppose $a_{8j} = 2$ and $d_{8j+5} = 0$. Then:

$$0 = t' \cdot x' = 16j + 2 + 4b_{8j+1} + 3c_{8j+2} + 2c_{8j+3} + c_{8j+4} \geq 8j - 2 > 0$$

which again is a contradiction. Finally, Case 6: Suppose $a_{8j} = 2$ and $d_{8j+5} = 2$. This case is ruled out by Lemma 7.1.2. $\square$

Again, we have not yet proved that t_1 and t_2 represent different elements of $\mathcal{L}(\Gamma'_j, [t])$, however, we do have:

Corollary 7.1.13. $d_{1/2}(N_j, \mathfrak{s}_0) = -2j + \frac{1}{2}$

Proof. Let

$$\beta_1 = (2, 4, 6, \ldots, 16j - 2, 8j + 1, 4, 2, 0, 0, 8j - 1)$$
$$\beta_2 = (0, \ldots, 0, -1, 0, 0, 0, 0, 1)$$

Then, $\beta_1 B'_j = t_1$ and $\beta_2 B'_j = t_2$. Thus,

$$t_1^2 = t_1 \cdot \beta_1 = -4 \qquad \text{and} \qquad t_2^2 = t_2 \cdot \beta_2 = -4$$

Hence, under the isomorphism in Corollary 6.1.32, the elements of $HF^+(N_j, \mathfrak{s}_0)$ corresponding to t_1 and t_2 have gradings:

$$\mathrm{gr}(t_1) = \mathrm{gr}(t_2) = -\frac{t_2^2 + |\mathcal{V}(\Gamma'_j)| - 3}{4} = -\frac{-4 + 8j + 5 - 3}{4} = -2j + \frac{1}{2}$$

$\square$

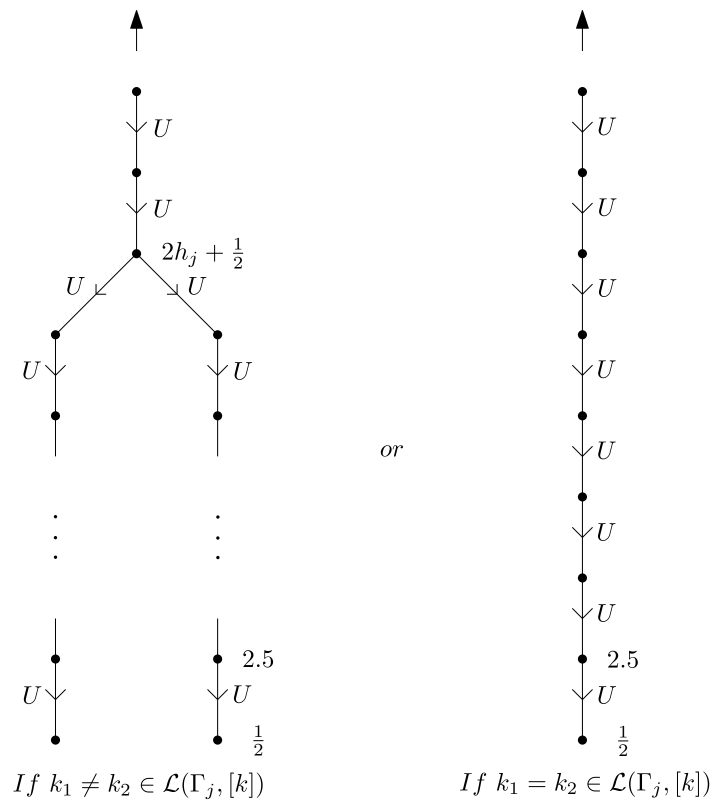
Now combining Corollaries 7.1.9, 7.1.13, and the basic fact that $d_{\pm 1/2}(-Y) = -d_{\mp 1/2}(Y)$ (see [36, Proposition 4.10]), we have:

$$d_{1/2}(-N_j) = \frac{1}{2} \qquad \text{and} \qquad d_{-1/2}(-N_j) = 2j - \frac{1}{2}$$

In particular, by Theorem 6.1.31, $HF^+_{even}(-N_j, \mathfrak{s}_0) = \mathcal{T}^+_{2j-1/2}$.

We have yet to completely determine $HF^+_{odd}(-N_j, \mathfrak{s}_0)$. So far, from Proposition 7.1.8, we know that $\dim_{\mathbb{F}}[\ker(U) \cap HF^+_{odd}(-N_j, \mathfrak{s}_0)] = 1$ or 2 depending on whether k_1 and k_2 represent the same element or not in $\mathcal{L}(\Gamma_j, [k])$. Therefore, as graded $\mathbb{F}[U]$-modules, we have: (see next page)

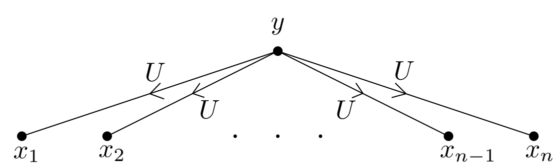
$HF^+_{odd}(-N_j, \mathfrak{s}_0) \cong$



Here, h_j is some positive integer depending on j which we have not yet determined.

A word of explanation is in order since on the left side of the above isomorphism we have an $\mathbb{F}[U]$-module and on the right we have one of two possible graphs. The right side is to be interpreted as follows:

- Each vertex at grading r corresponds to a basis element of the $\mathbb{F}$-vector space $HF^+_r(-N_j, \mathfrak{s}_0)$.
- If the edges emanating from a vertex y are of the form:



  then $Uy = x_1 + x_2 + \cdots + x_{n-1} + x_n$. In particular, if there are no edges emanating from y, then $Uy = 0$.

We now utilize Turaev Torsion to complete step (1). Combining our computations thus far with [38, Theorem 10.17], we see that

$$T_{N_j}(\mathfrak{s}_0) = \begin{cases} h_j + j & \text{if } k_1 \neq k_2 \in \mathcal{L}(\Gamma_j, [k]) \\ j & \text{otherwise} \end{cases}$$

where T_{N_j} is the Turaev torsion function associated to N_j (see [47, p. 119]). Therefore, to precisely determine $HF^+_{odd}(-N_j, \mathfrak{s}_0)$, it suffices to compute $T_{N_j}(\mathfrak{s}_0)$.

There are many standard ways to compute $T_{N_j}(\mathfrak{s}_0)$. For example, in [47], Turaev provides a formula in terms of a surgery description. We will now give a brief outline of how to carry out the calculation using this method, but we leave the details to the reader.

1. Let $H = H_1(N_j; \mathbb{Z})$. Consider the group ring $\mathbb{Z}[H]$. Since $H \cong \mathbb{Z}$, we can think of $\mathbb{Z}[H]$ as a $\mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials in the indeterminate t. Let $Q(H)$ denote the field of fractions of $\mathbb{Z}[H]$. The first step is to compute the Turaev torsion $\tau(N_j, \mathfrak{s}_0) \in Q(H)$. For this, we use the formula given in [47, VII.2, Theorem 2.2]. To apply this formula, we need to choose a surgery diagram for N_j and orient the underlying link. We have a surgery description of N_j from its plumbing representation and also from its surgery description in Figure 2-10 as a small Seifert fibered space. However, it will be easier to instead use the surgery description provided in [14], as surgery on the 2-component link as shown in Figure 7-1. We call the underlying link L_j and orient it as indicated by the arrows in the surgery diagram.

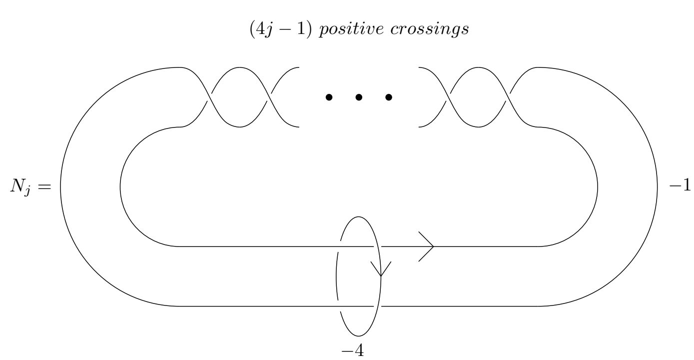


Figure 7-1

The bulk of the work in computing $\tau(N_j, \mathfrak{s}_0)$ using [47, VII.2, Theorem 2.2] is calculating the multivariable Alexander-Conway function $\nabla(L_j)$. Again, there are various approaches to computing $\nabla(L_j)$. For example, in [27] Murakami provides a skein formula for ∇. Using this formula, we find that

$$\nabla(L_j) = yx^{4j-1} + y^{-1}x^{-4j+1}$$

where the variable x corresponds to the torus knot component and the variable y corresponds to the unknot component. Plugging this into the formula for $\tau(N_j, \mathfrak{s}_0)$, we get:

$$\tau(N_j, \mathfrak{s}_0) = \frac{t^{8j-1}+1}{t^{4j-2}(t-1)^2(t+1)}$$

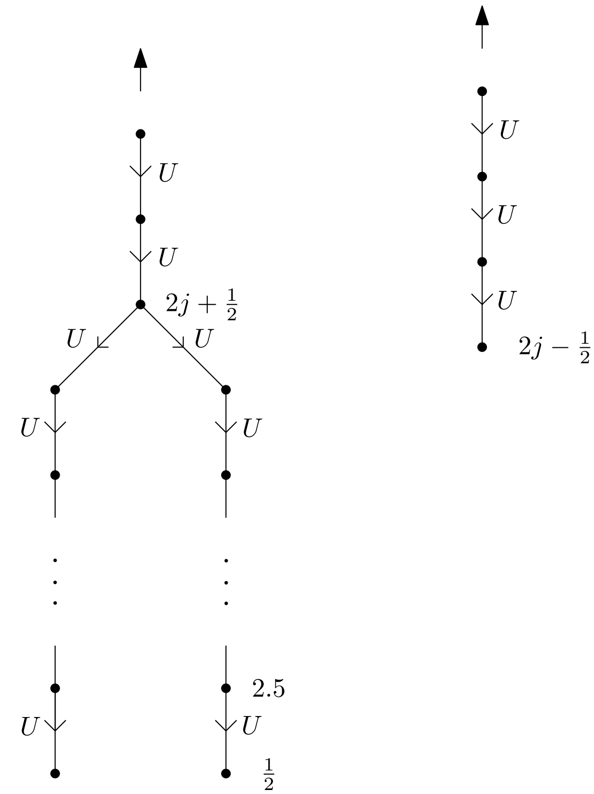
2. Next, we compute $[\tau(N_j, \mathfrak{s}_0)]$ which is a Laurent polynomial obtained by truncating $\tau(N_j, \mathfrak{s}_0)$ in a certain way (see [47, p.22]). We find that:

$$\begin{aligned}
[\tau(N_j, \mathfrak{s}_0)] &= \frac{t^{8j-1}+1}{t^{4j-2}(t-1)^2(t+1)} - \frac{t}{(t-1)^2} \\
&= \left(\sum_{i=0}^{4j-4} t^{i-4j+2}\right)\left(\sum_{i=1}^{2j-1} t^{2i}\right) + \sum_{i=0}^{8j-4} t^{i-4j+2} \\
&= 2j + \text{non-constant terms}
\end{aligned}$$

3. By definition, $T_{N_j}(\mathfrak{s}_0)$ is the constant term of $[\tau(N_j, \mathfrak{s}_0)]$. Hence, $T_{N_j}(\mathfrak{s}_0) = 2j$.

Thus, we have the following isomorphism of graded $\mathbb{F}[U]$-modules: (see next page)

$HF^+(-N_j, \mathfrak{s}_0) \cong$



Step 2

We now compute the involution ι_* on homology. This amounts to determining whether $-k_1$ is equivalent to k_1 or k_2. If $-k_1$ is equivalent to k_2, then the involution swaps the two legs of the left-hand graph of the above figure and leaves the right-hand graph fixed. If $-k_1$ is equivalent to k_1, then ι_* is the identity. We know show that, in fact, $-k_1$ is equivalent to k_2.

Recall, $-k_1 = (1, 0, -5 + 4j, -3, 0, \ldots, 0, 3)$ and $k_2 = (-1, 0, 3 - 4j, 3, 0, \ldots, 0, 1)$. Consider the following sequence of moves from $-k_1$ to k_2:

1. Add $2PD[v_4]$:

$$\begin{cases} (-1, 0, -1, 3, 1) = k_2 & \text{if } j = 1 \\ (-1, 0, -5 + 4j, 3, -2, \underbrace{0, \ldots, 0}_{2j-3}, 3) & \text{if } j \geq 2 \end{cases}$$

So we can assume for the subsequent moves that $j \geq 2$.

2. Apply Lemma 7.1.4 and Remark 7.1.5: $(-1, 0, -5+4j, 1, \underbrace{0, \dots, 0}_{2j-3}, 2, 1)$

3. Add $-2PD[v_1]$: $(1, -2, -5+4j-2, -1, \underbrace{0, \dots, 0}_{2j-3}, 2, 1)$

4. Add $-2PD[v_2]$: $(-1, 2, -5+4j-2, -1, \underbrace{0, \dots, 0}_{2j-3}, 2, 1)$

5. Add $2PD[v_1]$: $(1, 0, -5+4j-4, -3, \underbrace{0, \dots, 0}_{2j-3}, 2, 1)$

6. Add $-2PD[v_4]$: $(-1, 0, -5+4j-4, 3, -2, \underbrace{0, \dots, 0}_{2j-4}, 2, 1)$

7. Add $-2PD[v_5]$: $(-1, 0, -5+4j-4, 1, 2, -2\underbrace{0, \dots, 0}_{2j-5}, 2, 1)$

8. Rightward $(2, -2)$-slide: $(-1, 0, -5+4j-4, 1, \underbrace{0, \dots, 0}_{2j-5}, 2, -22, 1)$

9. Type -2 move: $(-1, 0, -5+4j-4, 1, \underbrace{0, \dots, 0}_{2j-4}, 2, 0, 1)$

Now notice that we are back to the same vector as in (2), except we have decreased the 3rd entry by 4 and shifted the $+2$ entry one slot to the left. Therefore, if we iterate this sequence of moves $(2j-4)$-more times, we get the vector:

$$(-1, 0, 7-4j, 1, 2, \underbrace{0, \dots, 0}_{2j-3}, 1)$$

Now consider the sequence of moves:

1. Add $-2PD[v_1]$: $(1, -2, 5-4j, -1, 2, \underbrace{0, \dots, 0}_{2j-3}, 1)$

2. Add $-2PD[v_2]$: $(-1, 2, 5-4j, -1, 2, \underbrace{0, \dots, 0}_{2j-3}, 1)$

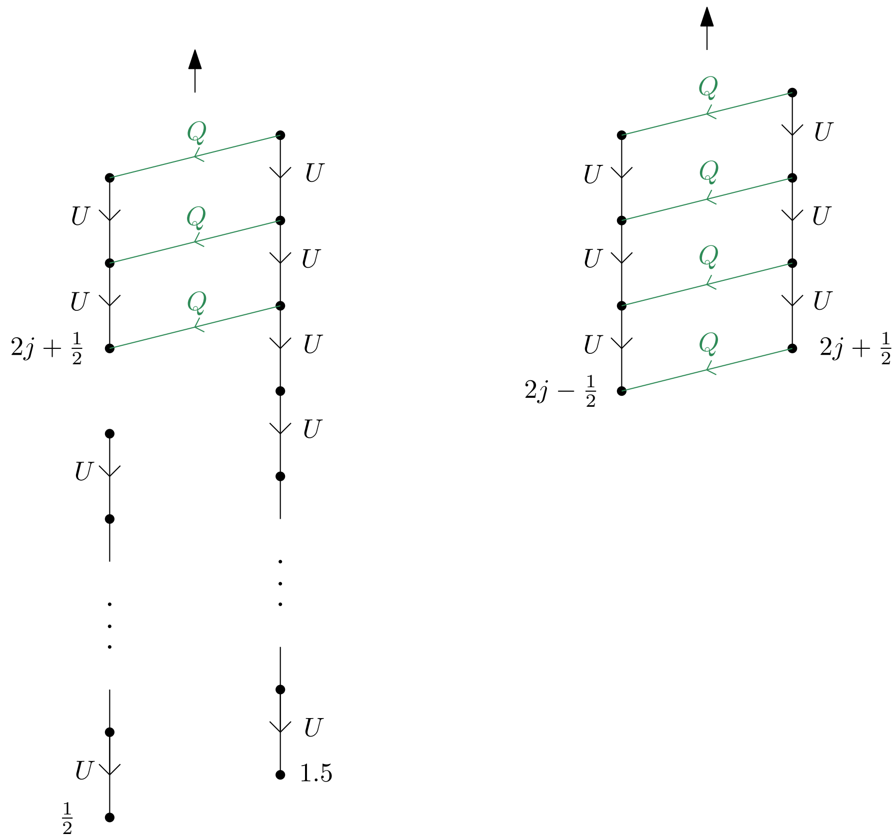
3. Add $-2PD[v_1]$: $(1, 0, 3-4j, -3, 2, \underbrace{0, \dots, 0}_{2j-3}, 1)$

4. Add $-2PD[v_4]$: $(-1, 0, 3-4j, 3, 0, \dots, 0, 1) = k_2$

Step 3

Theorem 7.1.14. *We have the following isomorphism of graded* $\mathbb{F}[U,Q]/(Q^2)$*-modules:*

$HFI^+(-N_j, \mathfrak{s}_0) \cong$



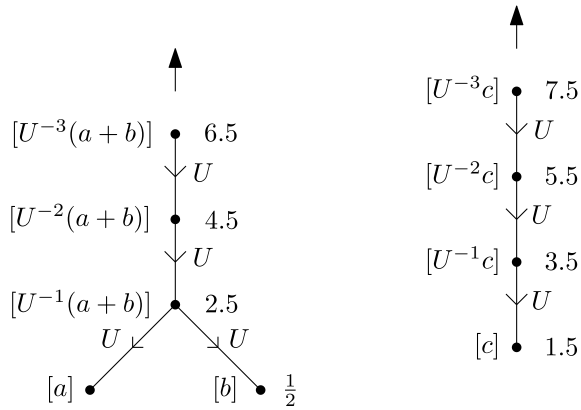
Remark 7.1.15. The graph on the right-hand side of the above isomorphism should be interpreted as a graded $\mathbb{F}[U,Q]/(Q^2)$-module in a manner similar to what was described earlier in the context of $\mathbb{F}[U]$-modules, except now there are additional arrows labeled with Q to indicate the action of Q.

Proof. For simplicity of exposition, we prove the statement for $j = 1$. The proof for $j \geq 2$ is completely analogous and is left to the reader.

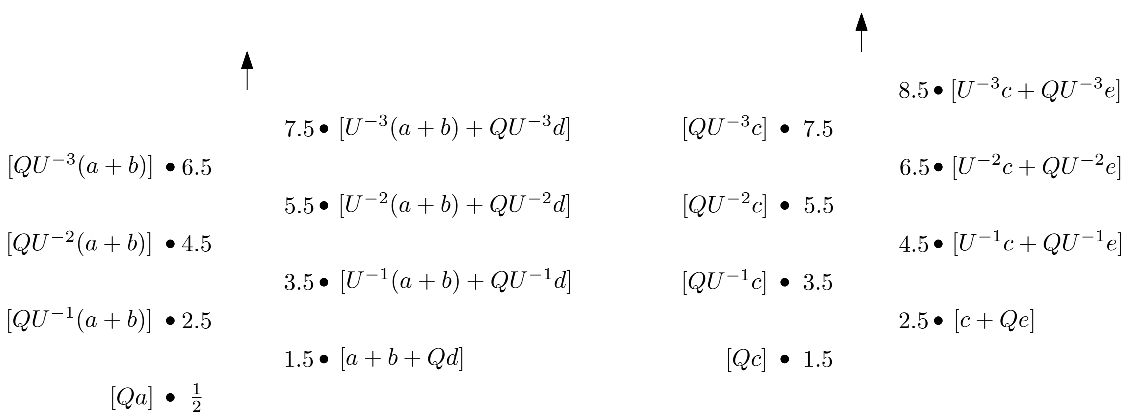
Fix an admissible Heegaard pair $\mathcal{H} = (H, J)$ for $(-N_1, \mathfrak{s}_0)$. We can choose representative cycles $a, b, c \in CF^+(\mathcal{H}, \mathfrak{s}_0)$ such that:

$$[a+b], [c] \in \mathrm{Im}[\pi_* : HF^\infty(\mathcal{H}, \mathfrak{s}_0) \to HF^+(\mathcal{H}, \mathfrak{s}_0)]$$

and the corresponding HF^+ homology generators are:



Since $\iota_*([a]) = \iota_*([b])$, we have that $(1+\iota_*)([a+b]) = 0$. Therefore, there exists some $d \in CF^+(\mathcal{H}, \mathfrak{s}_0)$ such that $\partial d = a + b + \iota(a+b)$. Similarly, since $(1+\iota_*)([c]) = 0$, there exists some $e \in CF^+(\mathcal{H}, \mathfrak{s}_0)$ such that $\partial e = c + \iota(c)$. It then follows from Proposition 5.2.1 and step 3 of section 6.2.2, that as graded $\mathbb{F}$-vector spaces we have $HFI^+(-N_1; \mathfrak{s}_0) \cong$



From this explicit description of generators, we see that for $n \geq 2$:

$$U \cdot [QU^{-n}(a+b)] = [QU^{-n+1}(a+b)]$$

and for $n \geq 1$:

$$U \cdot [U^{-n}(a+b) + QU^{-n}d] = [U^{-n+1}(a+b) + QU^{-n+1}d]$$
$$U \cdot [QU^{-n}c] = [QU^{-n+1}c]$$
$$U \cdot [U^{-n}c + QU^{-n}e] = [U^{-n+1}c + QU^{-n+1}e]$$

Next, we have:

$$U \cdot [QU^{-1}(a+b)] = [Q(a+b)] = [\partial^I a] = 0$$

Moreover, by grading considerations, we must have:

$$U \cdot [Qa] = 0 \qquad \text{and} \qquad U \cdot [a+b+Qd] = 0 \qquad \text{and} \qquad U \cdot [Qc] = 0$$

Also, either $U \cdot [c+Qe] = 0$ or $U \cdot [c+Qe] = [Qa]$. In the former case, we would have:

$$\dim_{\mathbb{F}}[\ker(U : HFI^+(-N_j, \mathfrak{s}_0) \to HFI^+(-N_j, \mathfrak{s}_0))] = 5$$
$$\dim_{\mathbb{F}}[\operatorname{coker}(U : HFI^+(-N_j, \mathfrak{s}_0) \to HFI^+(-N_j, \mathfrak{s}_0))] = 1$$

whereas in latter we would have:

$$\dim_{\mathbb{F}}[\ker(U : HFI^+(-N_j, \mathfrak{s}_0) \to HFI^+(-N_j, \mathfrak{s}_0)] = 4$$
$$\dim_{\mathbb{F}}[\operatorname{coker}(U : HFI^+(-N_j, \mathfrak{s}_0) \to HFI^+(-N_j, \mathfrak{s}_0))] = 0$$

Thus, by [16, Proposition 4.1], we would have either $\dim_{\mathbb{F}}(\widehat{HFI}(-N_j, \mathfrak{s}_0)) = 6$ or 4. But by [16, Corollary 4.7] we see that:

$$\begin{aligned}\dim_{\mathbb{F}}(\widehat{HFI}(-N_j, \mathfrak{s}_0)) &= \dim_{\mathbb{F}}[\ker(Q(1+\iota_*) : \widehat{HF}(-N_j, \mathfrak{s}_0) \to Q \cdot \widehat{HF}(-N_j, \mathfrak{s}_0))] \\ &\quad + \dim_{\mathbb{F}}[\operatorname{coker}(Q(1+\iota_*) : \widehat{HF}(-N_j, \mathfrak{s}_0) \to Q \cdot \widehat{HF}(-N_j, \mathfrak{s}_0))] \\ &= 3+3 = 6\end{aligned}$$

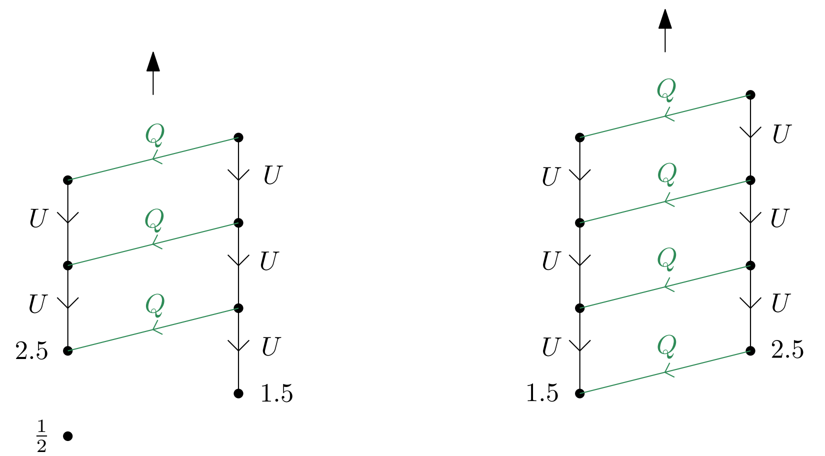
Hence, we must have had $U \cdot [c+Qe] = 0$. We have now completely determined the U-action on $HFI^+(-N_j, \mathfrak{s}_0)$.

Next, for the Q-action, it follows from the explicit description of the generators that for $n \geq 1$ $Q \cdot [U^{-n}(a+b) + QU^{-1}d] = [QU^{-n}(a+b)]$ and for $n \geq 0$:

$$Q \cdot [U^{-n}c + QU^{-n}e] = [QU^{-n}c]$$

Also, $Q \cdot [a + b + Qd] = [Q(a + b)] = [\partial^I a] = 0$. It is clear that the action of Q on all of the other generators is zero. Thus, we have:

$HFI^+(-N_1, \mathfrak{s}_0) \cong$



□

Theorem 7.1.16. *For all positive integers j, N_j cannot be obtained by 0-surgery on a knot in S^3. In fact, N_j is not the oriented boundary of any smooth negative semi-definite spin 4-manifold.*

Proof. From previous theorem, we have:

$$\bar{d}_{1/2}(-N_j) = 2j + \frac{1}{2} \qquad \bar{d}_{-1/2}(-N_j) = 2j - \frac{1}{2}$$
$$\underline{d}_{1/2}(-N_j) = \frac{1}{2} \qquad \underline{d}_{-1/2}(-N_j) = 2j - \frac{1}{2}$$

Equivalently,

$$\underline{d}_{-1/2}(N_j) = -2j - 1/2 \qquad \underline{d}_{1/2}(N_j) = -2j + \frac{1}{2}$$
$$\bar{d}_{-1/2}(N_j) = -\frac{1}{2} \qquad \bar{d}_{1/2}(N_j) = -2j + \frac{1}{2}$$

The conclusion now follows immediately from Corollaries 5.3.3 and 5.3.6.

□

7.1.2 $HFI^+(-S^3_0(K_1), \mathfrak{s}_0)$

Consider the Ichihara-Motegi-Song knot K_1 from Chapter 4 (see Figure 4-3). We know that

$$S^3_0(K_1) = S^2\left(\frac{3}{2}, -\frac{5}{2}, -\frac{15}{4}\right)$$

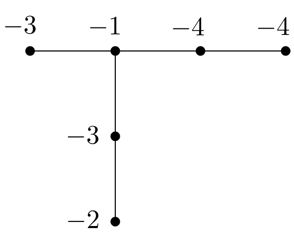
Since, by definition, $S^3_0(K_1)$ is 0-surgery on a knot in S^3, we know from Corollary 5.3.6 that:

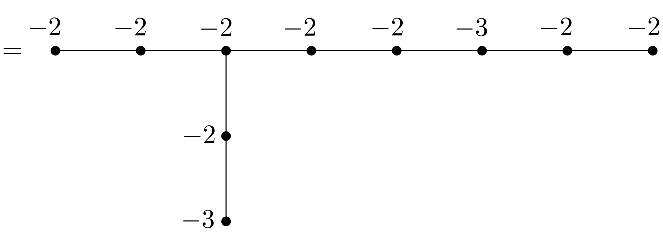
$$-\frac{1}{2} \leq \underline{d}_{-1/2}(S^3_0(K_1)) \text{ and } \bar{d}_{1/2}(S^3_0(K_1)) \leq \frac{1}{2}$$

We now verify these bounds directly by computing $HFI^+(-S^3_0(K_1), \mathfrak{s}_0)$ and then we compare this to $HFI^+(-N_1, \mathfrak{s}_0)$. In the interest of brevity, we are only going to give an outline of the calculation and leave the details to the reader.

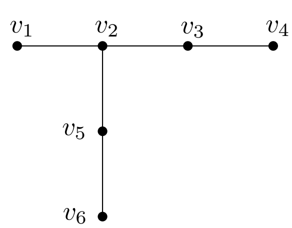
Step 1

We use Rolfsen twists, slam dunks, and the fact that $S^3_0(K_1)$ is a small Seifert fibered space to find negative-semi definite plumbing representations of $S^3_0(K_1)$ and $-S^3_0(K_1)$:

$S^3_0(K_1) =$ 

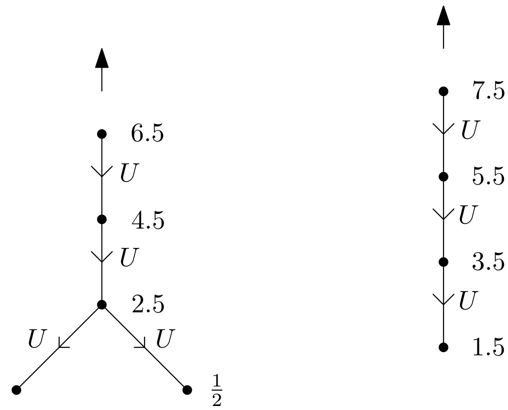
$-S^3_0(K_1) =$ 

Label the vertices of the above left plumbing graph as:



Using the methods of the previous section, one can show that as graded $\mathbb{F}[U]$-modules:

$HF^+(-S^3_0(K_1), \mathfrak{s}_0) \cong$



where the two leaves on the left graph correspond to the representative vectors:

$$z_1 = (-1, -1, 4, -2, 1, 0) \text{ and } z_2 = (1, -1, 0, 4, 1, 0)$$

Note that $HF^+(-S_0(K_1), \mathfrak{s}_0) \cong HF^+(-N_1, \mathfrak{s}_0)$.

Step 2

To determine ι_*, consider the following sequence of moves starting with the vector $-z_1 = (1, 1, -4, 2, -1, 0)$:

1. Add $-2PD[v_3]$: $(1, -1, 4, 0, -1, 0)$
2. Add $-2PD[v_2]$: $(-1, 1, 2, 0, -3, 0)$
3. Add $-2PD[v_5]$: $(-1, -1, 2, 0, 3, -2)$
4. Add $-2PD[v_6]$: $(-1, -1, 2, 0, 1, 2)$
5. Add $-2PD[v_2]$: $(-3, 1, 0, 0, -1, 2)$
6. Add $-2PD[v_1]$: $(3, -1, 0, 0, -1, 2)$
7. Add $-2PD[v_2]$: $(1, 1, -2, 0, -3, 2)$
8. Add $-2PD[v_5]$: $(1, -1, -2, 0, 3, 0)$
9. Add $-2PD[v_2]$: $(-1, 1, -4, 0, 1, 0)$

10. Add $-2PD[v_3]$: $(-1,-1,4,-2,1,0) = z_1$

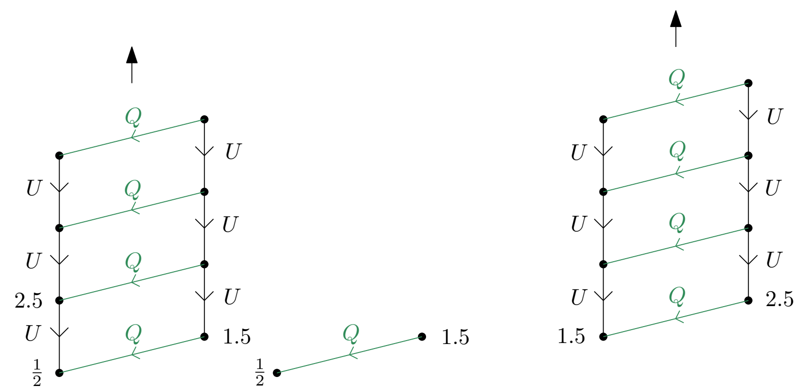
Therefore, ι_* is the identity. In particular, unlike for $HF^+(-N_1, \mathfrak{s}_0)$, ι_* is not a symmetric involution as defined in Definition 6.2.1.

Step 3

Applying the same methods as in the proof of Theorem 7.1.14, we get:

Theorem 7.1.17. *As graded* $\mathbb{F}[U,Q]/(Q^2)$*-modules:*

$HFI^+(-S^3_0(K_1), \mathfrak{s}_0) \cong$



In particular,

$$\bar{d}_{1/2}(-S^3_0(K_1)) = \frac{1}{2} \qquad \bar{d}_{-1/2}(-S^3_0(K_1)) = 1.5$$
$$\underline{d}_{1/2}(-S^3_0(K_1)) = \frac{1}{2} \qquad \underline{d}_{-1/2}(-S^3_0(K_1)) = 1.5$$

In summary, even though

$$HF^+(-N_1, \mathfrak{s}_0) \cong HF^+(-S^3_0(K_1), \mathfrak{s}_0)$$

we see that

$$HFI^+(-N_1, \mathfrak{s}_0) \not\cong HFI^+(-S^3_0(K_1), \mathfrak{s}_0)$$

Bibliography

[1] Selman Akbulut. *4-manifolds*, volume 25 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2016.

[2] Matthias Aschenbrenner, Stefan Friedl, and Henry Wilton. *3-manifold groups*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2015.

[3] David Auckly. An irreducible homology sphere which is not dehn surgery on a knot. *Preprint*. Available at https://www.math.ksu.edu/~dav/irr.pdf.

[4] David Auckly. Surgery numbers of 3-manifolds: a hyperbolic example. In *Geometric topology (Athens, GA, 1993)*, volume 2 of *AMS/IP Stud. Adv. Math.*, pages 21–34. Amer. Math. Soc., Providence, RI, 1997.

[5] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[6] J. H. Conway. An enumeration of knots and links, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*, pages 329–358. Pergamon, Oxford, 1970.

[7] Marc Culler, Nathan M. Dunfield, Matthias Goerner, and Jeffrey R. Weeks. SnapPy, a computer program for studying the geometry and topology of 3-manifolds. Available at http://snappy.computop.org (4/10/2022).

[8] Irving Dai. On the Pin(2)-equivariant monopole Floer homology of plumbed 3-manifolds. *Michigan Math. J.*, 67(2):423–447, 2018.

[9] Irving Dai and Ciprian Manolescu. Involutive Heegaard Floer homology and plumbed three-manifolds. *J. Inst. Math. Jussieu*, 18(6):1115–1155, 2019.

[10] M. Dehn. Über die Topologie des dreidimensionalen Raumes. *Math. Ann.*, 69(1):137–168, 1910.

[11] David Gabai. Foliations and the topology of 3-manifolds. III. *J. Differential Geom.*, 26(3):479–536, 1987.

[12] Robert E. Gompf and András I. Stipsicz. *4-manifolds and Kirby calculus*, volume 20 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1999.

[13] Joshua Evan Greene. A surgery triangle for lattice cohomology. *Algebr. Geom. Topol.*, 13(1):441–451, 2013.

[14] Matthew Hedden, Min Hoon Kim, Thomas E. Mark, and Kyungbae Park. Irreducible 3-manifolds that cannot be obtained by 0-surgery on a knot. *Trans. Amer. Math. Soc.*, 372(11):7619–7638, 2019.

[15] Kristen Hendricks, Jennifer Hom, Matthew Stoffregen, and Ian Zemke. Naturality and functoriality in involutive heegaard floer homology. *arXiv preprint arXiv:2201.12906*, 2022.

[16] Kristen Hendricks and Ciprian Manolescu. Involutive Heegaard Floer homology. *Duke Math. J.*, 166(7):1211–1299, 2017.

[17] Neil R. Hoffman and Genevieve S. Walsh. The big Dehn surgery graph and the link of S^3. *Proc. Amer. Math. Soc. Ser. B*, 2:17–34, 2015.

[18] Jennifer Hom, Çağrı Karakurt, and Tye Lidman. Surgery obstructions and Heegaard Floer homology. *Geom. Topol.*, 20(4):2219–2251, 2016.

[19] Jennifer Hom and Tye Lidman. A note on surgery obstructions and hyperbolic integer homology spheres. *Proc. Amer. Math. Soc.*, 146(3):1363–1365, 2018.

[20] Kazuhiro Ichihara, Kimihiko Motegi, and Hyun-Jong Song. Seifert fibered slopes and boundary slopes on small hyperbolic knots. *Bull. Nara Univ. Ed. Natur. Sci.*, 57(2):21–25, 2008.

[21] Peter K. Johnson. On the involutive heegaard floer homology of negative semi-definite plumbed 3-manifolds with b_{1}= 1. *arXiv preprint arXiv:2108.13548*, 2021.

[22] András Juhász, Dylan P Thurston, and Ian Zemke. Naturality and mapping class groups in heegaard floer homology. *arXiv preprint arXiv:1210.4996*, 2012.

[23] Robion Kirby. A calculus for framed links in S^3. *Invent. Math.*, 45(1):35–56, 1978.

[24] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. of Math. (2)*, 76:531–540, 1962.

[25] José M. Montesinos. Seifert manifolds that are ramified two-sheeted cyclic coverings. *Bol. Soc. Mat. Mexicana (2)*, 18:1–32, 1973.

[26] José M. Montesinos. Surgery on links and double branched covers of S^3. In *Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox)*, pages 227–259. Ann. of Math. Studies, No. 84. 1975.

[27] Jun Murakami. A state model for the multivariable Alexander polynomial. *Pacific J. Math.*, 157(1):109–135, 1993.

[28] Walter D. Neumann. S^1-actions and the α-invariant of their involutions. *Bonn. Math. Schr.*, (44):iv+82, 1970.

[29] Walter D. Neumann. A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves. *Trans. Amer. Math. Soc.*, 268(2):299–344, 1981.

[30] Walter D. Neumann and Frank Raymond. Seifert manifolds, plumbing, μ-invariant and orientation reversing maps. In *Algebraic and geometric topology (Proc. Sympos., Univ. California, Santa Barbara, Calif., 1977)*, volume 664 of *Lecture Notes in Math.*, pages 163–196. Springer, Berlin, 1978.

[31] András Némethi. On the Ozsváth-Szabó invariant of negative definite plumbed 3-manifolds. *Geom. Topol.*, 9:991–1042, 2005.

[32] András Némethi. Lattice cohomology of normal surface singularities. *Publ. Res. Inst. Math. Sci.*, 44(2):507–543, 2008.

[33] András Némethi. Two exact sequences for lattice cohomology. In *Noncommutative geometry and global analysis*, volume 546 of *Contemp. Math.*, pages 249–269. Amer. Math. Soc., Providence, RI, 2011.

[34] Peter Ozsváth and Zoltán Szabó. On knot Floer homology and lens space surgeries. *Topology*, 44(6):1281–1300, 2005.

[35] Peter Ozsváth, András I. Stipsicz, and Zoltán Szabó. A spectral sequence on lattice homology. *Quantum Topol.*, 5(4):487–521, 2014.

[36] Peter Ozsváth and Zoltán Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Adv. Math.*, 173(2):179–261, 2003.

[37] Peter Ozsváth and Zoltán Szabó. On the Floer homology of plumbed three-manifolds. *Geom. Topol.*, 7:185–224, 2003.

[38] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and three-manifold invariants: properties and applications. *Ann. of Math. (2)*, 159(3):1159–1245, 2004.

[39] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math. (2)*, 159(3):1027–1158, 2004.

[40] Peter Ozsváth and Zoltán Szabó. An introduction to Heegaard Floer homology. In *Floer homology, gauge theory, and low-dimensional topology*, volume 5 of *Clay Math. Proc.*, pages 3–27. Amer. Math. Soc., Providence, RI, 2006.

[41] Peter Ozsváth and Zoltán Szabó. Lectures on Heegaard Floer homology. In *Floer homology, gauge theory, and low-dimensional topology*, volume 5 of *Clay Math. Proc.*, pages 29–70. Amer. Math. Soc., Providence, RI, 2006.

[42] Dale Rolfsen. *Knots and links*. Mathematics Lecture Series, No. 7. Publish or Perish, Inc., Berkeley, Calif., 1976.

[43] Raif Rustamov. On heegaard floer homology of plumbed three-manifolds with b_1= 1. *arXiv preprint math/0405118*, 2004.

[44] Peter Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.

[45] H. Seifert. Topologie Dreidimensionaler Gefaserter Räume. *Acta Math.*, 60(1):147–238, 1933.

[46] Frank Swenton. KLO (Knot-Like Objects) software (version 0.969 alpha). Available at https://klo-software.blogspot.com/.

[47] Vladimir Turaev. *Torsions of 3-dimensional manifolds*, volume 208 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2002.

[48] Andrew H. Wallace. Modifications and cobounding manifolds. *Canadian J. Math.*, 12:503–528, 1960.

[49] Ian Zemke. The equivalence of lattice and Heegaard Floer homology. *arXiv preprint arXiv:2111.14962*, 2021.