A Distributed Accurate Average Consensus Algorithm and Its Application to State-of-Charge Balancing of Networked Battery Systems

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Jinshuo Xiao

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APPROVAL SHEET

This thesis is submitted in partial fulfillment of the
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Jinshuo Xiao, Author

This thesis has been read and approved by the examining committee:

Professor Zongli Lin, ECE, Thesis Advisor

Professor Nikhil Shukla, ECE, Chair

Professor Yan Wan, University of Texas at Arlington

Accepted for the School of Engineering and Applied Science:

Dean, School of Engineering and Applied Science

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Abstract

The dynamic average consensus problem, a group of agents, each associated with a time-varying signal, reaching consensus at the average of these signals by their own distributed estimators that interact with each other through the communication network among the agents, finds many applications such as distributed estimation, formation control and sensor fusion. Many distributed estimators have been constructed that achieve either consensus precisely at the average of the signals or around it depending on the properties of the signals. In this thesis, we revisit the dynamic average consensus problem in both the continuous-time and discrete-time settings. By utilizing the information on the frequency components of the signals, we construct distributed estimators that achieve accurate consensus at the average of the signals. We further establish that our distributed estimators are robust to the interruption of the network connectivity in the sense that connected subgroups of agents will continue to reach consensus around the average of all signals after an interruption of the network as along as the signals are bounded and the later the interruption occurs the more accurate the consensus will be. Numerical simulation is carried out to illustrate these theoretical conclusions. We apply our proposed distributed estimators to a networked battery system to achieve accurate state-of-charge balancing while delivering the desirable total power accurately. Simulation results also verify the robustness of the battery system when the communication is interrupted.

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Chapter 1

Introduction and Problem Statement

1.1 Introduction

The dynamic average consensus problem is the problem of all agents in a networked multi-agent system reaching consensus at the average of a set of time-varying signals, each associated with one agent, through distributed algorithms. This problem, which has been intensively studied over the past years, plays a vital role in numerous applications, such as battery management [1–3], distributed estimation [4–6], formation control [7–10], multi-robot coordination [11–13], distributed sensor fusion [14, 15], feature-based map merging [16, 17], distributed tracking [18, 19], and distributed mapping [20]. Various appealing features of dynamic average consensus have been explored in the literature, including its scalability, high rate of convergence, operation in the presence of constraints on the actuation capacity, and protection of the privacy of the information against competitors.

The dynamic average consensus problem has involved from the average consensus problem [21–23], where agents reach consensus at the average of their initial values. In [24], distributed proportional and proportional-integral algorithms were proposed to achieve dynamic average consensus with bounded steady-state errors. In [25], the authors designed distributed algorithms that achieve dynamic average consensus with steady-state errors that can be controlled by tuning design parameters. Conditions on the signals were also identified in [25] under which the designed distributed algorithms would achieve accurate dynamic average consensus, that is, with a zero steady-state error. Signals that satisfy these conditions include constant signals and time-varying signals that differ from one another by a constant value.

Separate efforts have also been made to develop distributed algorithms that achieve accurate dynamic average consensus. For example, based on the internal model principle, reference [26] designed filters to enhance the distributed algorithms proposed in [24] so as to achieve accurate dynamic average consensus. Explicit conditions on the denominator and the numerator polynomials of the filters were established. These conditions include that some denominator polynomials contains the frequency components of the time-varying signals. As such, knowledge of the frequency components of the signals is required. Explicit construction of the filters that satisfy the conditions established in [26] was given in [27]. On the other hand, in [28], authors introduced robust dynamic average consensus algorithms that achieve accurate averge consensus of time-varying signals with known bounded derivatives.

We note that a closely related problem is the leader-following consensus problem, where agents reach consensus at a signal associated with a leader and not all agents are directly connected to the leader. In particular, distributed observers were proposed in [29] to achieve accurate leader following consensus. These observers are in the state space form and all assume the knowledge of the frequency of the signal. In [30], only the leader itself has the knowledge of signal and all other agents uses an adaptive algorithm to learn the frequency information of the signal.

In this thesis, we revisit the problem of dynamic average consensus problem. We aim to develop distributed algorithms that will achieve accurate average consensus of time-varying signals with known frequency components. We will take the state space approach of [25, 29, 30] and explicitly construct distributed estimators in the state-space form that incorporate the frequency information of the signals. The incorporation of frequency information of the signals also makes our distributed estimator robust against the network connectivity interruption. In particular, we will establish that, after the interruption, connected subgroups of agents will continue to reach consensus around the average of all signals as along as the signals are bounded and the later the interruption occurs the more accurate the consensus will be.

1.2 Problem Statement

We consider a networked system of N agents, operating either in continuous-time or in discrete-time. Each agent is associated with a time-varying signal φ_i, generated by the exosystem

$$\dot{\varphi}_i = S\varphi_i,\ \varphi_i \in \mathbb{R}^M, \tag{1.1}$$

or

$$\varphi_i(k+1) = S\varphi_i(k),\ \varphi_i(k) \in \mathbb{R}^M. \tag{1.2}$$

We note that signals generated by the exosystem contain the frequency components determined by the eigenvalues of matrix S but their magnitudes and phases are freely determined by the initial conditions of the exosystem.

Our objective is to construct a distributed estimator for each agent such that, under an appropriate communication topology, its state will asymptotically converge to the average value of all signals, that is, $\frac{1}{N}\sum_{j=1}^{N}\varphi_i$. We would also like to establish the robustness property of our distributed estimators against the interruption of the network connectivity. To demonstrate its practical applicability, we apply our proposed distributed dynamic average consensus algorithm to a networked battery system to achieve accurate state-of-charge balancing while delivering the desirable total power accurately.

1.3 Preliminary

We will use an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ to represent the communication topology. Each agent is represented by a vertex from the set of vertices $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and the communication link between agent i and agent j is represented by an edge from the set of edges $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$.

The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ of the graph is defined as $a_{ij} = a_{ji} = 1$ if there is a bidirectional communication link between agent i and agent j, otherwise $a_{ij} = a_{ji} = 0$. In addition, we assume that $a_{ii} = 0$, $i = 1, 2, \cdots, N$. The Laplacian matrix associated with the graph is defined as $L = [l_{ij}] \in \mathbb{R}^{N\times N}$, where $l_{ij} = -a_{ij}$ if $i \neq j$ and $l_{ii} = \sum_{k=1,k\neq i}^{N} a_{ik}$. The set of neighbors of agent i is defined as $\mathcal{N} = \{j \in \{1, 2, \cdots, N\} : a_{ij} = 1\}$.

A path between v_{i_1} and v_{i_k} in $\mathcal{G}$ is defined as a sequence of edges $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \cdots, (v_{i_{k-1}}, v_{i_k})$, where $v_{i_1}, v_{i_2}, \cdots, v_{i_k}$ are distinct nodes. A graph is connected if there exists a path between any two distinct nodes.

We make the following assumption on the communication topology among the agents.

Assumption 1.1. *The graph associated with the communication topology is undirected and connected.*

Under Assumption 1.1, the graph Laplacian L is symmetric with eigenvalues $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$. In addition, 1_N is the eigenvector associated with $\lambda_1 = 0$, that is, $L1_N = 0$.

1.4 Organization and Notation

The remainder of the thesis is organized as follows. Chapter 2 proposes distributed estimators for accurate consensus of continuous-time signals, establishes their robustness against the network connectivity interruption and shows the properties with simulation. Chapter 3 studies the dynamic accurate average consensus problem in the discrete-time setting. Chapter 4 applies the proposed dynamic average consensus algorithms to the state-of-charge balancing problem of networked battery systems. Chapter 5 concludes the thesis.

Notation. Let $\mathbb{R}$ denote the real numbers. The vector 1_N is the vector of N ones, and I_N is the $N \times N$ identity matrix. For matrices A and B, $A \otimes B$ denotes their Kronecker product.

Chapter 2

Distributed Average Consensus of Continuous-time Signals

2.1 The Distributed Average Consensus Algorithm

In this chapter, we design, for each agent i, $i = 1, 2, \cdots, N$, the following distributed estimator in continuous-time,

$$\dot{\hat{y}}_i = -\mu \sum_{j=1}^{N} a_{ij} (\hat{y}_i - \hat{y}_j) - \mu (\hat{y}_i - \varphi_i) + S\hat{y}_i + \mu v_i, \tag{2.1a}$$

$$\dot{v}_i = -\mu \sum_{j=1}^{N} a_{ij} (\hat{y}_i - \hat{y}_j) + S v_i, \tag{2.1b}$$

where $\hat{y}_i, v_i \in \mathbb{R}^M$, $\mu > 0$ is a design parameter, whose value is to be determined, and the initial condition of v_i is chosen as $v_i(0) = 0$.

The following theorem establishes that each of the distributed estimators (2.1) asymptotically estimates the averaged signal $\frac{1}{N} \sum_{j=1}^{N} \varphi_j(t)$.

Theorem 2.1. *Consider a networked system of N agents, labeled as $1, 2, \cdots, N$.*

Each agent i is associated with a signal $\varphi_i(t)$ that is generated by an exosystem system (1.1)*, where, without loss of generality, all eigenvalues of S are assumed to be on the closed right-half plane. Let the communication topology among the agents satisfy Assumption 1.1. Let each agent be equipped with a distributed estimator as given in* (2.1)*. Then, there exists $\mu > 0$ such that*

$$\lim_{t\to\infty}\left(\hat{y}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\varphi_j(t)\right) = 0,\ i = 1, 2, \cdots, N. \tag{2.2}$$

Proof: Let $\hat{y} = \left[\hat{y}_1^{\mathrm{T}}\ \hat{y}_2^{\mathrm{T}} \cdots \hat{y}_N^{\mathrm{T}}\right]^{\mathrm{T}}$, $v = \left[v_1^{\mathrm{T}}\ v_2^{\mathrm{T}} \cdots v_N^{\mathrm{T}}\right]^{\mathrm{T}}$ and $\varphi = \left[\varphi_1^{\mathrm{T}}\ \varphi_2^{\mathrm{T}} \cdots \varphi_N^{\mathrm{T}}\right]^{\mathrm{T}}$. Then, by (2.1), we have

$$\dot{\hat{y}} = \Big(-\mu(L\otimes I_M) - \mu(I_N\otimes I_M) + I_N\otimes S\Big)\hat{y} + \mu\varphi + \mu v, \tag{2.3a}$$

$$\dot{v} = -\mu(L\otimes I_M)\hat{y} + (I_N\otimes S)v. \tag{2.3b}$$

Also, by (1.1), we have

$$\dot{\varphi} = (I_N\otimes S)\varphi. \tag{2.4}$$

Define the estimation error as

$$\tilde{y} = \hat{y} - 1_N\otimes\frac{1}{N}\sum_{j=1}^{N}\varphi_j.$$

Then, by (2.3a) and (2.4), we have

$$\begin{aligned}\dot{\tilde{y}} &= \dot{\hat{y}} - 1_N\otimes\frac{1}{N}\sum_{j=1}^{N}\dot{\varphi}_j\\ &= \Big(-\mu(L\otimes I_M) - \mu(I_N\otimes I_M) + I_N\otimes S\Big)\left(\tilde{y} + 1_N\otimes\frac{1}{N}\sum_{j=1}^{N}\varphi_j\right)\end{aligned}$$

$$
\begin{aligned}
&+\mu\varphi + \mu v - 1_N \otimes \frac{1}{N}\sum_{j=1}^{N}\dot{\varphi}_j \\
&= \Big(-\mu(L\otimes I_M) - \mu(I_N\otimes I_M) + I_N\otimes S\Big)\tilde{y} + \mu v + \mu\left(\varphi - 1_N\otimes\frac{1}{N}\sum_{j=1}^{N}\varphi_j\right) \\
&= \Big(-\mu(L\otimes I_M) - \mu(I_N\otimes I_M) + I_N\otimes S\Big)\tilde{y} + \mu\Big(v + (\Pi_N\otimes I_M)\varphi\Big), \qquad (2.5)
\end{aligned}
$$

where $\Pi_N = I_N - \frac{1}{N}1_N 1_N^{\mathrm{T}}$, and we have used

$$
\begin{aligned}
(I_N\otimes S)\left(1_N\otimes\frac{1}{N}\sum_{j=1}^{N}\varphi_j\right) &= (I_N 1_N)\otimes\left(S\frac{1}{N}\sum_{j=1}^{N}\varphi_j\right) \\
&= 1_N\otimes\frac{1}{N}\sum_{j=1}^{N}\dot{\varphi}_j,
\end{aligned}
$$

and the following identity resulting from the fact that $L1_N = 0$,

$$
(L\otimes I_M)\left(1_N\otimes\frac{1}{N}\sum_{j=1}^{N}\varphi_j\right) = (L1_N)\otimes\left(I_M\frac{1}{N}\sum_{j=1}^{N}\dot{\varphi}_j\right) = 0. \qquad (2.6)
$$

On the other hand, let

$$
\tilde{v} = v + (\Pi_N\otimes I_M)\,\varphi.
$$

Then, by (2.3b) and (2.4), we have

$$
\begin{aligned}
\dot{\tilde{v}} &= \dot{v} + (\Pi_N\otimes I_M)\dot{\varphi} \\
&= -\mu(L\otimes I_M)\left(\tilde{y} + \mathbf{1}_N\otimes\frac{1}{N}\sum_{j=1}^{N}\varphi_j\right) + (I_N\otimes S)\tilde{v} \\
&\quad -(I_N\otimes S)(\Pi_N\otimes I_M)\varphi + (\Pi_N\otimes I_M)\dot{\varphi} \\
&= -\mu(L\otimes I_M)\left(\tilde{y} + 1_N\otimes\frac{1}{N}\sum_{j=1}^{N}\varphi_j\right) + (I_N\otimes S)\tilde{v} \\
&\quad -(\Pi_N\otimes I_M)(I_N\otimes S)\varphi + (\Pi_N\otimes I_M)\,\dot{\varphi}
\end{aligned}
$$

$$
\begin{aligned}
&= -\mu(L \otimes I_M)\tilde{y} - \mu(L \otimes I_M)\left(1_N \otimes \frac{1}{N}\sum_{j=1}^{N}\varphi_j\right) \\
&\quad +(I_N \otimes S)\tilde{v} - (\Pi_N \otimes I_M)\Big((I_N \otimes S)\varphi - \dot{\varphi}\Big) \\
&= -\mu(L \otimes I_M)\tilde{y} + (I_N \otimes S)\tilde{v},
\end{aligned} \tag{2.7}
$$

where we have used (2.6) and the following identity,

$$
\begin{aligned}
(I_N \otimes S)(\Pi_N \otimes I_M) &= (I_N \Pi_N) \otimes (S I_M) \\
&= (\Pi_N I_N) \otimes (I_M S) \\
&= (\Pi_N \otimes I_M)(I_N \otimes S).
\end{aligned} \tag{2.8}
$$

Equations (2.5) and (2.7) can be written in the following compact form,

$$
\begin{bmatrix} \dot{\tilde{y}} \\ \dot{\tilde{v}} \end{bmatrix} = \widetilde{C} \begin{bmatrix} \tilde{y} \\ \tilde{v} \end{bmatrix}, \tag{2.9}
$$

where

$$
\widetilde{C} = \begin{bmatrix} I_N \otimes S & 0 \\ 0 & I_N \otimes S \end{bmatrix} + \mu \begin{bmatrix} -L \otimes I_M - I_N \otimes I_M & I_N \otimes I_M \\ -L \otimes I_M & 0 \end{bmatrix}.
$$

In what follows, we will analyze the asymptotic properties of the dynamical system (2.9). Under Assumption 1.1, that is, the graph representing the communication topology is undirected and connected, the Laplacian matrix L is symmetric with eigenvalues $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$ and 1_N is an eigenvector associated with $\lambda_1 = 0$. Let $h = \frac{1}{\sqrt{N}} 1_N$ and H be such that $h^{\mathrm{T}} H = 0$ and $H^{\mathrm{T}} H = I_{N-1}$.

Consider the following change of variables,

$$\begin{bmatrix} p \\ q \end{bmatrix} = T_1 T_2 \begin{bmatrix} \tilde{y} \\ \tilde{v} \end{bmatrix}, \tag{2.10}$$

where

$$T_1 = \begin{bmatrix} I_N \otimes I_M & 0 \\ -I_N \otimes I_M & I_N \otimes I_M \end{bmatrix}, \quad T_2 = \begin{bmatrix} T_3^{\mathrm{T}} & 0 \\ 0 & T_3^{\mathrm{T}} \end{bmatrix},$$

with $T_3 = [r \;\, R] = [h \otimes I_M \;\; H \otimes I_M]$. Then, by (2.9), we have

$$\begin{aligned} \begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} &= T_1 T_2 \widetilde{C} T_2^{-1} T_1^{-1} \begin{bmatrix} p \\ q \end{bmatrix} \\ &= \begin{bmatrix} T_3^{\mathrm{T}}\Big(-\mu(L \otimes I_M) + I_N \otimes S\Big) T_3 & T_3^{\mathrm{T}}\Big(\mu(I_N \otimes I_M)\Big) T_3 \\ 0 & T_3^{\mathrm{T}}\Big(-\mu(I_N \otimes I_M) + I_N \otimes S\Big) T_3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \end{aligned} \tag{2.11}$$

We next partition the state variables p and q as

$$p = \begin{bmatrix} p_1 \\ p_{2:N} \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ q_{2:N} \end{bmatrix},$$

where $p_1, q_1 \in \mathbb{R}^M$ and $p_{2:N}, q_{2:N} \in \mathbb{R}^{M(N-1)}$. Then, the dynamics (2.11) can be written as

$$\left[\begin{array}{c} \dot{p}_1 \\ \dot{q}_1 \\ \hline \dot{p}_{2:N} \\ \dot{q}_{2:N} \end{array}\right] = \left[\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_2 + \mu C_3 \end{array}\right] \left[\begin{array}{c} p_1 \\ q_1 \\ \hline p_{2:N} \\ q_{2:N} \end{array}\right], \tag{2.12}$$

where

$$C_1 = \begin{bmatrix} S & \mu I_M \\ 0 & -\mu I_M + S \end{bmatrix},$$

$$C_2 = \begin{bmatrix} R^{\mathrm{T}}(I_N \otimes S)R & 0 \\ 0 & R^{\mathrm{T}}(I_N \otimes S)R \end{bmatrix},$$

$$C_3 = \begin{bmatrix} -R^{\mathrm{T}}(L \otimes I_M)R & R^{\mathrm{T}}(I_N \otimes I_M)R \\ 0 & -R^{\mathrm{T}}(I_N \otimes I_M)R \end{bmatrix}.$$

The eigenvalues of C_2 are those of S, each with multiplicity $2(N-1)$. The eigenvalues of C_3 are $-\lambda_i, i = 2, 3, \cdots, N$, each with a multiplicity of M, and -1, with a multiplicity of $(N-1)M$. Recall that these λ_i's are eigenvalues of L, which, in view of Assumption 1.1, are all positive. Therefore, C_3 is Hurwitz. Let $\mu > 0$ be sufficiently large such that $S - \mu\lambda_i I_M,\ i = 2, 3, \cdots, N$, and $S - \mu I_M$ are all Hurwitz. Then, $C_2 + \mu C_3$ is Hurwitz, and, hence,

$$\lim_{t\to\infty} \begin{bmatrix} p_{2:N}(t) \\ q_{2:N}(t) \end{bmatrix} = 0.$$

The solution of the state equation (2.12) is given by

$$\begin{bmatrix} p_1(t) \\ q_1(t) \\ \hline p_{2:N}(t) \\ q_{2:N}(t) \end{bmatrix} = \Omega_1(t) \begin{bmatrix} p_1(0) \\ q_1(0) \\ \hline p_{2:N}(0) \\ q_{2:N}(0) \end{bmatrix},$$

where

$$\Omega_1(t) = \left[\begin{array}{cc|cc} \mathrm{e}^{St} & \mu\int_0^t \mathrm{e}^{(S-\mu I_M)(t-\tau)}\mathrm{e}^{S\tau}\mathrm{d}\tau & 0 & 0 \\ 0 & \mathrm{e}^{(S-\mu I_M)t} & 0 & 0 \\ \hline 0 & 0 & \Gamma_{11}(t) & \Gamma_{12}(t) \\ 0 & 0 & \Gamma_{21}(t) & \Gamma_{22}(t) \end{array}\right],$$

and

$$\Gamma(t) = \mathrm{e}^{(C_2+\mu C_3)t} := \begin{bmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ \Gamma_{21}(t) & \Gamma_{22}(t) \end{bmatrix},$$

$$\begin{aligned}
\Gamma_{11}(t) &= \mathrm{e}^{R^{\mathrm{T}}\left(I_N\otimes S-\mu(L\otimes I_M)\right)Rt}, \\
\Gamma_{12}(t) &= \mu\int_0^t \mathrm{e}^{R^{\mathrm{T}}\left(I_N\otimes S-\mu(L\otimes I_M)\right)R(t-\tau)} \\
&\quad \times \mathrm{e}^{R^{\mathrm{T}}\left(I_N\otimes S-\mu(I_N\otimes I_M)\right)R\tau}\mathrm{d}\tau, \\
\Gamma_{21}(t) &= 0, \\
\Gamma_{22}(t) &= \mathrm{e}^{R^{\mathrm{T}}\left(I_N\otimes S-\mu(I_N\otimes I_M)\right)Rt}.
\end{aligned}$$

Noting that

$$\mu\int_0^t \mathrm{e}^{(S-\mu I_M)(t-\tau)}\mathrm{e}^{S\tau}\mathrm{d}\tau = \mu\mathrm{e}^{(S-\mu I_M)t}\int_0^t \mathrm{e}^{\mu I_M\tau}\mathrm{d}\tau = \mathrm{e}^{St} - \mathrm{e}^{(S-\mu I_M)t},$$

we can simplify Ω_1 as

$$\Omega_1(t) = \left[\begin{array}{cc|cc} \mathrm{e}^{St} & \mathrm{e}^{St} - \mathrm{e}^{(S-\mu I_M)t} & 0 & 0 \\ 0 & \mathrm{e}^{(S-\mu I_M)t} & 0 & 0 \\ \hline 0 & 0 & \Gamma_{11}(t) & \Gamma_{12}(t) \\ 0 & 0 & \Gamma_{21}(t) & \Gamma_{22}(t) \end{array}\right].$$

Then, in view of (2.10), we have

$$\tilde{y}(t) = \Lambda_{11}(t)\tilde{y}(0) + \Lambda_{12}(t)\tilde{v}(0), \tag{2.13a}$$

$$\tilde{v}(t) = \Lambda_{21}(t)\tilde{y}(0) + \Lambda_{22}(t)\tilde{v}(0), \tag{2.13b}$$

where

$$\begin{aligned} \Lambda_{11}(t) &= r\mathrm{e}^{(S-\mu I_M)t}r^{\mathrm{T}} + R\Big(\Gamma_{11}(t) - \Gamma_{12}(t)\Big)R^{\mathrm{T}}, \\ \Lambda_{12}(t) &= r\left(\mathrm{e}^{St} - \mathrm{e}^{(S-\mu I_M)t}\right)r^{\mathrm{T}} + R\Gamma_{12}(t)R^{\mathrm{T}}, \\ \Lambda_{21}(t) &= R\Big(\Gamma_{11}(t) - \Gamma_{12}(t) + \Gamma_{21}(t) - \Gamma_{22}(t)\Big)R^{\mathrm{T}}, \\ \Lambda_{22}(t) &= r\mathrm{e}^{St}r^{\mathrm{T}} + R\Big(\Gamma_{12}(t) + \Gamma_{22}(t)\Big)R^{\mathrm{T}}. \end{aligned}$$

More explicitly, we have

$$\begin{aligned} \tilde{y}(t) &= r\mathrm{e}^{St}r^{\mathrm{T}}\tilde{v}(0) + \varepsilon(t) \\ &= 1_N \otimes \frac{1}{N}\mathrm{e}^{St}\sum_{j=1}^{N}\tilde{v}_j(0) + \varepsilon(t), \end{aligned} \tag{2.14}$$

where

$$\varepsilon(t) = \Bigg(r\mathrm{e}^{(S-\mu I_M)t} r^{\mathrm{T}} + R\Big(\Gamma_{11}(t) - \Gamma_{12}(t)\Big)R^{\mathrm{T}}\Bigg)\tilde{y}(0)$$
$$+ \left(R\Gamma_{12}(t)R^{\mathrm{T}} - r\mathrm{e}^{(S-\mu I_M)t} r^{\mathrm{T}}\right)\tilde{v}(0).$$

Recalling that $S - \mu I_M$ and $C_2 + \mu C_3$ are both Hurwitz, we have

$$\lim_{t\to\infty} \Gamma_{ij}(t) = 0,\ i, j = 1, 2, \tag{2.15}$$

and hence,

$$\lim_{t\to\infty} \varepsilon(t) = 0.$$

On the other hand, recalling that

$$\tilde{v}(t) = v(t) + (\Pi_N \otimes I_M)\varphi(t)$$

and $v_i(0) = 0, i = 1, 2, \cdots, N$, we have

$$\sum_{j=1}^{N} \tilde{v}_j(0) = \left(\sum_{j=1}^{N} \Pi_{N,j} \otimes I_M\right)\varphi(0) = 0,$$

where $\Pi_{N,j}$ is the j^{th} row of Π_N and $\sum_{j=1}^{N} \Pi_{N,j} = 0$.

Thus, it follows from (2.14) that

$$\lim_{t\to\infty} \tilde{y}(t) = 0,$$

and hence, for all $i = 1, 2, \cdots, N$,

$$\lim_{t\to\infty}\left(\hat{y}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\varphi_j(t)\right) = \lim_{t\to\infty}\tilde{y}_i(t) = 0.$$

This completes the proof. $\square$

2.2 Robustness Against Network Interruption

In what follows, we will establish that the distributed estimator (2.1) is robust against interruption of the network connectivity. We summarize this robustness result in the following theorem.

Theorem 2.2. *Consider a networked system of N agents, with the associated signals $\varphi_i(t)$ and the distributed estimators as described in Theorem 2.1. Let S be neutrally stable, that is, all its eigenvalues are on the $j\omega$-axis and are simple. Let the communication topology among the agents satisfy Assumption 1.1. Suppose that the communication is interrupted at time $t_{\mathrm{I}} > 0$, with individual subgroups of agents remaining connected. Then, there exists $\mu > 0$, independent of any subgroup, such that each agent i within a subgroup of $K < N$ agents continues to track the averaged signal $\frac{1}{N}\sum_{j=1}^{N}\varphi_j(t)$, with a bounded steady-state tracking error, that is,*

$$\hat{y}_i(t) = \frac{1}{N}\sum_{j=1}^{N}\varphi_j(t) + \varepsilon^{\mathrm{s}}(t, t_{\mathrm{I}}) + \delta(t, t_{\mathrm{I}}),$$

where $\|\delta(t, t_{\mathrm{I}})\| \leq \delta_0(t_{\mathrm{I}})$ and

$$\lim_{t\to\infty}\varepsilon^{\mathrm{s}}(t, t_{\mathrm{I}}) = 0,$$
$$\lim_{t_{\mathrm{I}}\to\infty}\delta_0(t_{\mathrm{I}}) = 0.$$

Proof: Recall from the proof of Theorem 2.1 that $\tilde{v}(t) = v(t) + (\Pi_N \otimes I_M)\varphi(t)$ and $\sum_{j=1}^{N} \tilde{v}_j(0) = 0$. Then, by (2.13b), we have

$$
\begin{aligned}
v(t_{\mathrm{I}}) &= \Lambda_{21}(t_{\mathrm{I}})\tilde{y}(0) + \Lambda_{22}(t_{\mathrm{I}})\tilde{v}(0) - (\Pi_N \otimes I_M)\varphi(t_{\mathrm{I}}) \\
&= \Lambda_{21}(t_{\mathrm{I}})\tilde{y}(0) + 1_N \otimes \frac{1}{N}\mathrm{e}^{St_{\mathrm{I}}} \sum_{j=1}^{N} \tilde{v}_j(0) \\
&\quad + R\Big(\Gamma_{12}(t_{\mathrm{I}}) + \Gamma_{22}(t_{\mathrm{I}})\Big)R^{\mathrm{T}}\tilde{v}(0) - (\Pi_N \otimes I_M)\varphi(t_{\mathrm{I}}) \\
&= \Lambda_{21}(t_{\mathrm{I}})\tilde{y}(0) + R\Big(\Gamma_{12}(t_{\mathrm{I}}) + \Gamma_{22}(t_{\mathrm{I}})\Big)R^{\mathrm{T}}\tilde{v}(0) - (\Pi_N \otimes I_M)\varphi(t_{\mathrm{I}}) \\
&= \chi(t_{\mathrm{I}}) - (\Pi_N \otimes I_M)\varphi(t_{\mathrm{I}}),
\end{aligned} \tag{2.16}
$$

where

$$
\chi(t_{\mathrm{I}}) = \Lambda_{21}(t_{\mathrm{I}})\tilde{y}(0) + R\Big(\Gamma_{12}(t_{\mathrm{I}}) + \Gamma_{22}(t_{\mathrm{I}})\Big)R^{\mathrm{T}}\tilde{v}(0),
$$

and by (2.15),

$$
\lim_{t_{\mathrm{I}}\to\infty} \chi(t_{\mathrm{I}}) = 0. \tag{2.17}
$$

Let us now consider the evolution for $t \geq t_{\mathrm{I}}$. Without loss of generality, let the subgroup of K agents be agents $1, 2, \cdots, K$. Denote the Laplacian matrix of the connected communication among these K agents as L^{s}. Let $\hat{y}^{\mathrm{s}} = \big[\hat{y}_1^{\mathrm{T}}\ \hat{y}_2^{\mathrm{T}} \cdots \hat{y}_K^{\mathrm{T}}\big]^{\mathrm{T}}$, $v^{\mathrm{s}} = \big[v_1^{\mathrm{T}}\ v_2^{\mathrm{T}} \cdots v_K^{\mathrm{T}}\big]^{\mathrm{T}}$, and $\varphi^{\mathrm{s}} = \big[\varphi_1^{\mathrm{T}}\ \varphi_2^{\mathrm{T}} \cdots \varphi_K^{\mathrm{T}}\big]^{\mathrm{T}}$. Then, we have

$$
\begin{aligned}
\dot{\hat{y}}^{\mathrm{s}} &= -\mu\Big((L \otimes I_M) + (I_K \otimes I_M) + I_K \otimes S\Big)\hat{y}^{\mathrm{s}} + \mu\varphi^{\mathrm{s}} + \mu v^{\mathrm{s}},\ t \geq t_{\mathrm{I}}, \\
\dot{v}^{\mathrm{s}} &= -\mu(L \otimes I_M)\hat{y}^{\mathrm{s}} + (I_K \otimes S)v^{\mathrm{s}},\ t \geq t_{\mathrm{I}},
\end{aligned}
$$

and

$$
\dot{\varphi}^{\mathrm{s}} = (I_K \otimes S)\varphi^{\mathrm{s}},\ t \geq t_{\mathrm{I}}.
$$

Following the proof of Theorem 2.1 for the subgroup of agents $1, 2, \cdots, K$, we define

$$\tilde{y}^{\mathrm{s}} = \hat{y}^{\mathrm{s}} - 1_K \otimes \frac{1}{K}\sum_{j=1}^{K} \varphi_j, \tag{2.18a}$$

$$\tilde{v}^{\mathrm{s}} = v^{\mathrm{s}} + (\Pi_K \otimes I_M)\varphi^{\mathrm{s}}. \tag{2.18b}$$

where $\Pi_K = I_K - \frac{1}{K}1_K 1_K^{\mathrm{T}}$.

Let $h^{\mathrm{s}} = \frac{1}{\sqrt{K}}1_K$ and H^{s} be such that $(h^{\mathrm{s}})^{\mathrm{T}} H^{\mathrm{s}} = 0$ and $(H^{\mathrm{s}})^{\mathrm{T}} H^{\mathrm{s}} = I_{K-1}$. Let $r^{\mathrm{s}} = h^{\mathrm{s}} \otimes I_M$, $R^{\mathrm{s}} = H^{\mathrm{s}} \otimes I_M$ and

$$C_2^{\mathrm{s}} = \begin{bmatrix} (R^{\mathrm{s}})^{\mathrm{T}}(I_K \otimes S)R^{\mathrm{s}} & 0 \\ 0 & (R^{\mathrm{s}})^{\mathrm{T}}(I_K \otimes S)R^{\mathrm{s}} \end{bmatrix},$$

$$C_3^{\mathrm{s}} = \begin{bmatrix} -(R^{\mathrm{s}})^{\mathrm{T}}(L^{\mathrm{s}} \otimes I_M)R^{\mathrm{s}} & (R^{\mathrm{s}})^{\mathrm{T}}(I_K \otimes I_M)R^{\mathrm{s}} \\ 0 & -(R^{\mathrm{s}})^{\mathrm{T}}(I_K \otimes I_M)R^{\mathrm{s}} \end{bmatrix}.$$

Let μ be such that $C_2^{\mathrm{s}} + \mu C_3^{\mathrm{s}}$ is Hurwtiz for any connected subgroup of agents.

Let

$$\Gamma^{\mathrm{s}}(t) = \mathrm{e}^{(C_2^{\mathrm{s}} + \mu C_3^{\mathrm{s}})t} := \begin{bmatrix} \Gamma_{11}^{\mathrm{s}}(t) & \Gamma_{12}^{\mathrm{s}}(t) \\ \Gamma_{21}^{\mathrm{s}}(t) & \Gamma_{22}^{\mathrm{s}}(t) \end{bmatrix}.$$

Then,

$$\lim_{t\to\infty} \Gamma_{ij}^{\mathrm{s}}(t) = 0,\ i, j = 1, 2. \tag{2.19}$$

Following the proof of Theorem 2.1, we can also obtain the following equation,

$$\tilde{y}^{\mathrm{s}}(t) = \Lambda_{11}^{\mathrm{s}}(t - t_{\mathrm{I}})\tilde{y}^{\mathrm{s}}(t_{\mathrm{I}}) + \Lambda_{12}^{\mathrm{s}}(t - t_{\mathrm{I}})\tilde{v}^{\mathrm{s}}(t_{\mathrm{I}}), \tag{2.20}$$

where

$$\Lambda_{11}^{\mathrm{s}}(t-t_{\mathrm{I}}) = r^{\mathrm{s}}\left(\mathrm{e}^{(S-\mu I_M)(t-t_{\mathrm{I}})}\right)(r^{\mathrm{s}})^{\mathrm{T}} + R^{\mathrm{s}}\Bigg(\Gamma_{11}^{\mathrm{s}}(t-t_{\mathrm{I}}) - \Gamma_{12}^{\mathrm{s}}(t-t_{\mathrm{I}})\Bigg)(R^{\mathrm{s}})^{\mathrm{T}},$$

$$\Lambda_{12}^{\mathrm{s}}(t-t_{\mathrm{I}}) = r^{\mathrm{s}}\left(\mathrm{e}^{S(t-t_{\mathrm{I}})} - \mathrm{e}^{(S-\mu I_M)(t-t_{\mathrm{I}})}\right)(r^{\mathrm{s}})^{\mathrm{T}} + R^{\mathrm{s}}\Bigg(\Gamma_{12}^{\mathrm{s}}(t-t_{\mathrm{I}})\Bigg)(R^{\mathrm{s}})^{\mathrm{T}}.$$

We can rewrite (2.20) as

$$\tilde{y}^{\mathrm{s}}(t) = 1_K \otimes \frac{1}{K}\mathrm{e}^{S(t-t_{\mathrm{I}})}\sum_{j=1}^{K}\tilde{v}_j(t_{\mathrm{I}}) + \varepsilon^s(t,t_{\mathrm{I}}), \tag{2.21}$$

where

$$\varepsilon^{\mathrm{s}}(t,t_{\mathrm{I}}) = \Bigg(R^{\mathrm{s}}\Gamma_{12}^{\mathrm{s}}(t-t_{\mathrm{I}})(R^{\mathrm{s}})^{\mathrm{T}} - r^{\mathrm{s}}\mathrm{e}^{(S-\mu I_M)(t-t_{\mathrm{I}})}(r^{\mathrm{s}})^{\mathrm{T}}\Bigg)\tilde{v}^{\mathrm{s}}(t_{\mathrm{I}}) + \Lambda_{11}^{\mathrm{s}}(t-t_{\mathrm{I}})\tilde{y}^{\mathrm{s}}(t_{\mathrm{I}}).$$

By (2.19), we have

$$\lim_{t\to\infty}\varepsilon^{\mathrm{s}}(t,t_{\mathrm{I}}) = 0. \tag{2.22}$$

Recalling that $\tilde{v}^{\mathrm{s}}(t) = v^{\mathrm{s}}(t) + (\Pi_K \otimes I_M)\varphi^{\mathrm{s}}(t)$ and $\sum_{j=1}^{K}\Pi_{K,j} = 0$ with $\Pi_{K,j}$ being the j^{th} row of Π_K, we have

$$\begin{aligned}\sum_{j=1}^{K}\tilde{v}_j(t_{\mathrm{I}}) = \sum_{j=1}^{K}v_j(t_{\mathrm{I}}) &= \sum_{j=1}^{K}\chi_j(t_{\mathrm{I}}) - \sum_{j=1}^{K}(\Pi_{N,j}\otimes I_M)\varphi(t_{\mathrm{I}})\\ &= \sum_{j=1}^{K}\chi_j(t_{\mathrm{I}}) - \Bigg(\sum_{j=1}^{K}\varphi_j(t_{\mathrm{I}}) - \frac{K}{N}\sum_{j=1}^{N}\varphi_j(t_{\mathrm{I}})\Bigg),\end{aligned} \tag{2.23}$$

where we have used (2.16), and $\chi_j(t_{\mathrm{I}})$ is the $\Bigg((j-1)M+1\Bigg)^{\mathrm{th}}$ to jM^{th} rows of $\chi(t_{\mathrm{I}})$ in (2.16).

It then follows from (2.21) and (2.23) that

$$\begin{aligned}
\hat{y}^{\mathrm{s}}(t) &= \tilde{y}^{\mathrm{s}}(t) + 1_K \otimes \frac{1}{K}\sum_{j=1}^{K} \varphi_j(t) \\
&= 1_K \otimes \left(\frac{1}{K}\sum_{j=1}^{K}\varphi_j(t) + \frac{1}{K}\mathrm{e}^{S(t-t_{\mathrm{I}})}\sum_{j=1}^{K}\tilde{v}_j(t_{\mathrm{I}})\right) + \varepsilon^{\mathrm{s}}(t,t_{\mathrm{I}}) \\
&= 1_K \otimes \left(\frac{1}{K}\sum_{j=1}^{K}\varphi_j(t) + \frac{1}{K}\mathrm{e}^{S(t-t_{\mathrm{I}})}\sum_{j=1}^{K}\chi_j(t_{\mathrm{I}})\right. \\
&\quad \left. -\frac{1}{K}\mathrm{e}^{S(t-t_{\mathrm{I}})}\left(\sum_{j=1}^{K}\varphi_j(t_{\mathrm{I}}) - \frac{K}{N}\sum_{j=1}^{N}\varphi_j(t_{\mathrm{I}})\right)\right) + \varepsilon^{\mathrm{s}}(t,t_{\mathrm{I}}) \\
&= 1_K \otimes \left(\frac{1}{K}\sum_{j=1}^{K}\varphi_j(t) + \frac{1}{K}\mathrm{e}^{S(t-t_{\mathrm{I}})}\sum_{j=1}^{K}\chi_j(t_{\mathrm{I}})\right. \\
&\quad \left. -\frac{1}{K}\left(\sum_{j=1}^{K}\varphi_j(t) - \frac{K}{N}\sum_{j=1}^{N}\varphi_j(t)\right)\right) + \varepsilon^{\mathrm{s}}(t,t_{\mathrm{I}}) \\
&= 1_K \otimes \frac{1}{N}\sum_{j=1}^{N}\varphi_j(t) + 1_K \otimes \left(\frac{1}{K}\mathrm{e}^{S(t-t_{\mathrm{I}})}\sum_{j=1}^{K}\chi_j(t_{\mathrm{I}})\right) + \varepsilon^{\mathrm{s}}(t,t_{\mathrm{I}}) \\
&= 1_K \otimes \frac{1}{N}\sum_{j=1}^{N}\varphi_j(t) + \varepsilon^{\mathrm{s}}(t,t_{\mathrm{I}}) + \delta(t,t_{\mathrm{I}}),
\end{aligned}$$

where

$$\delta(t,t_{\mathrm{I}}) = 1_K \otimes \left(\frac{1}{K}\mathrm{e}^{S(t-t_{\mathrm{I}})}\sum_{j=1}^{K}\chi_j(t_{\mathrm{I}})\right).$$

Since S is neutrally stable, we have $\left\|\mathrm{e}^{S(t-t_{\mathrm{I}})}\right\| \leq \sigma$ for some constant scalar $\sigma > 0$, and hence,

$$\|\delta(t,t_{\mathrm{I}})\| \leq \sigma K \|\chi(t_{\mathrm{I}})\| := \delta_0(t_{\mathrm{I}}).$$

In view of (2.22) and (2.17), it is clear that

$$\lim_{t\to\infty} \varepsilon^{\mathrm{s}}(t,t_{\mathrm{I}}) = 0,$$

$$\lim_{t_{\mathrm{I}} \to \infty} \delta_0(t_{\mathrm{I}}) = 0.$$

This completes the proof. $\square$

2.3 Simulation Results

In this section, we run simulation on a five agent system to verify our theoretical results both in the continuous-time settings. The communication topology among the agents is as shown in Fig. 2.1. We will also illustrate the robustness of our algorithms against interruptions of the network connectivity (see Fig. 2.2).

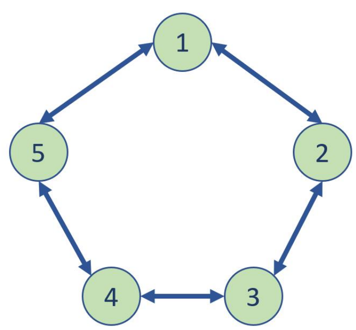

Figure 2.1: The communication topology.

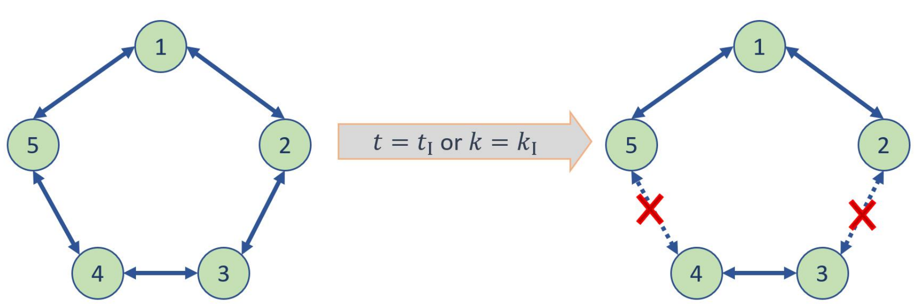

Figure 2.2: The communication topology after interruption at t_{I} or k_{I}.

Let the signals be generated by the exosystem (1.1) with

$$S = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the initial values $\varphi_i(0) = [4i-7, -3i+18, 4i-9, -2i+7]^{\mathrm{T}},\ i = 1, 2, \cdots, 5$.

For the given communication topology,

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

It can be verified that, with $\mu = 2$, all conditions of Theorem 2.1 are satisfied. Shown in Fig. 2.3 are the estimations of the average value of the five signals by the five agents.

To illustrate the robustness of our distributed algorithm against network connectivity interruption, we consider signals generated by a neutrally stable exosystem (1.1) with

$$S = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

and the initial conditions $\varphi_i(0) = [4i-7, -3i+18]^{\mathrm{T}}$, $i = 1, 2, \cdots, 5$. It can be verified that, with $\mu = 2$, all conditions of Theorem 2.2 are satisfied.

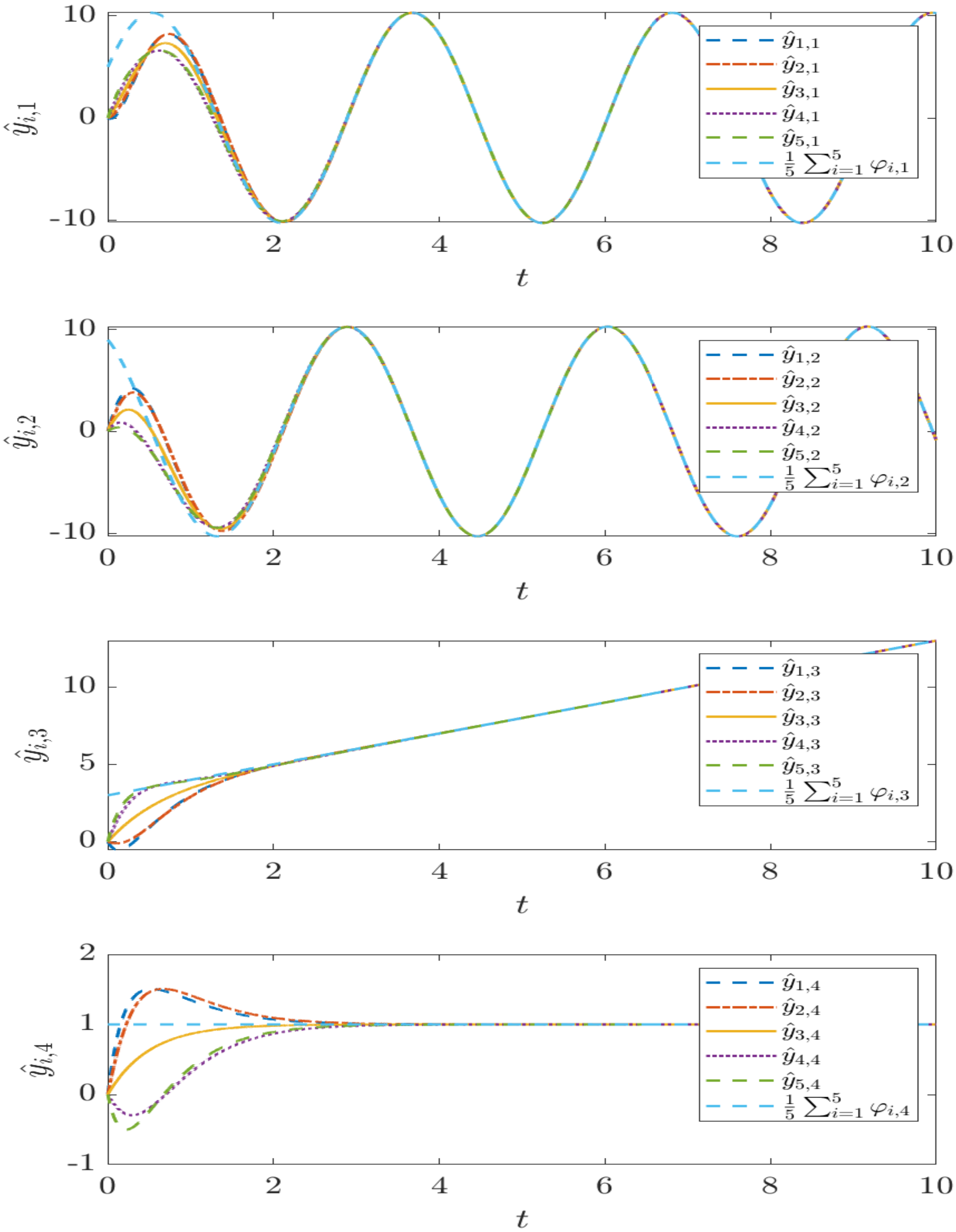

Figure 2.3: Estimations of the averaged signal by all agents.

Shown in Fig. 2.4 and Fig. 2.5 are the norms of the estimation errors of agents of both subgroups when the communication is interrupted at $t_{\mathrm{I}} = 1$ and $t_{\mathrm{I}} = 3$, respectively. As is obvious, the later the interruption occurs, the more accurate the estimations are.

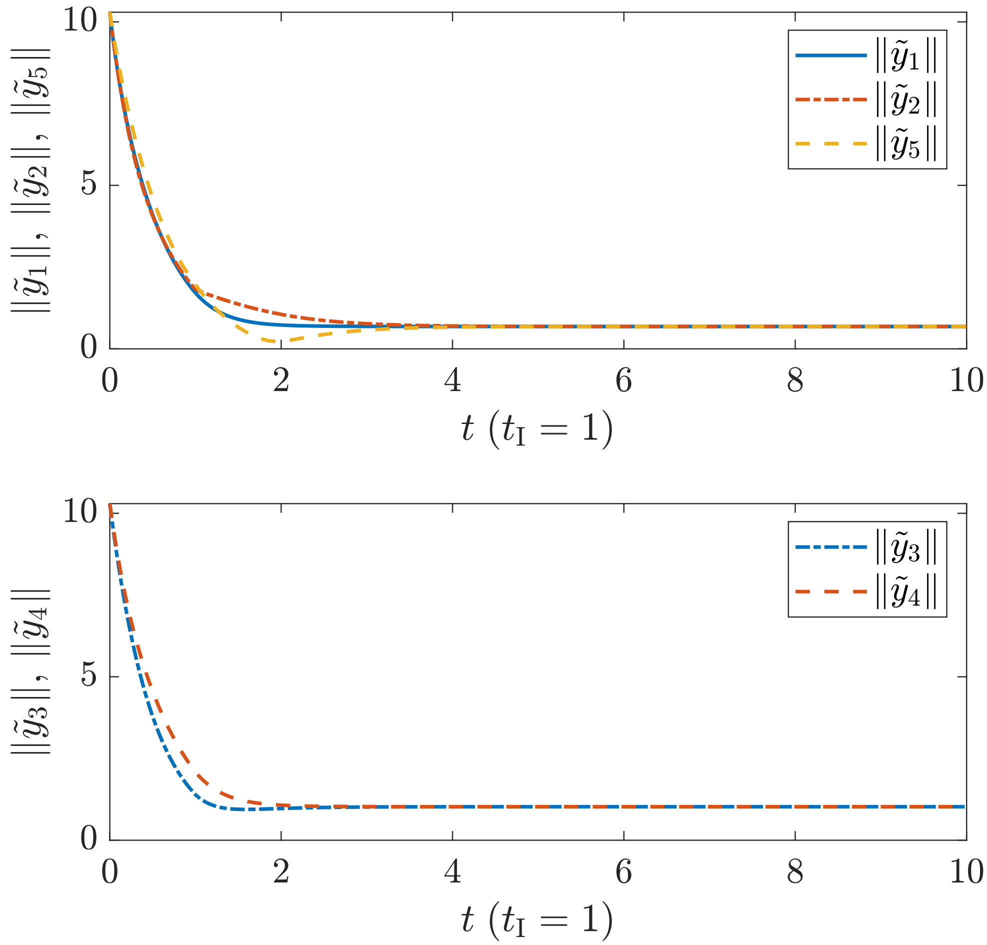

Figure 2.4: Estimation errors when the communication is interrupted at time $t_{\mathrm{I}} = 1$.

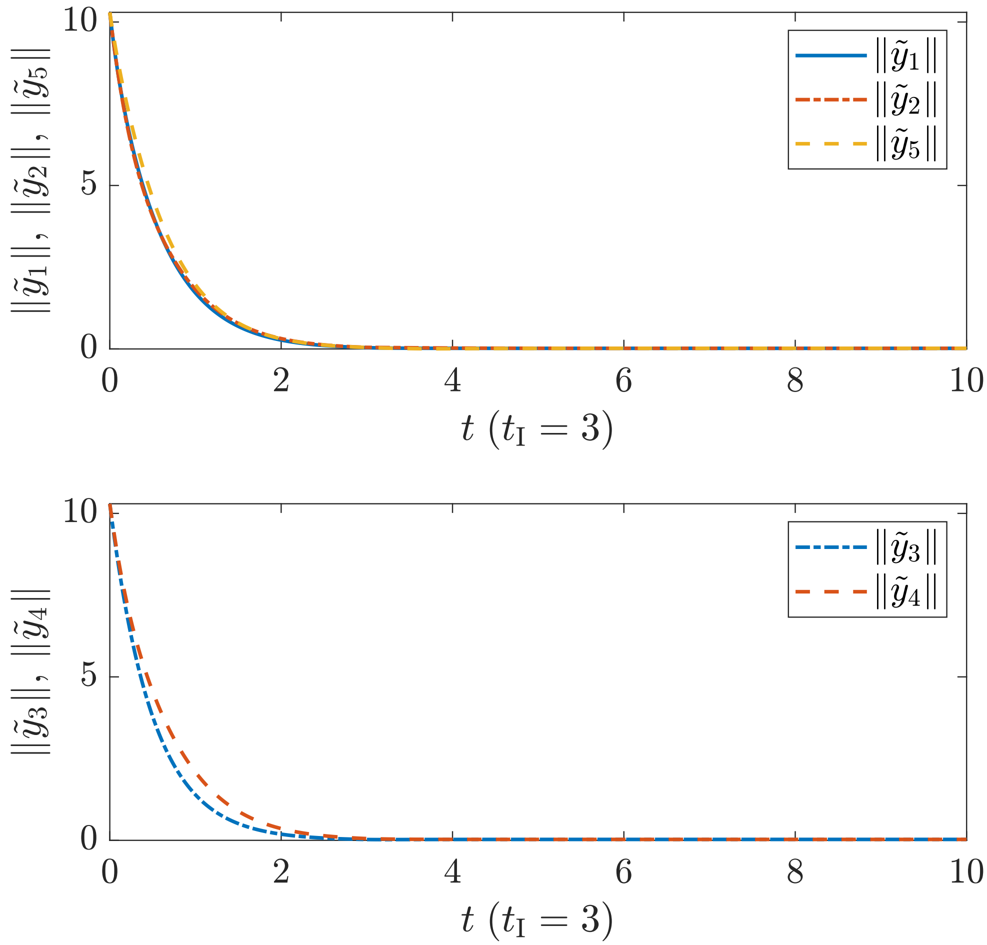

Figure 2.5: Estimation errors when the communication is interrupted at time $t_\mathrm{I} = 3$.

Chapter 3

Distributed Average Consensus of Discrete-time Signals

3.1 The Distributed Average Consensus Algorithm

In this chapter, we design, for each agent i, $i = 1, 2, \cdots, N$, the following distributed estimator in discrete-time,

$$\hat{y}_i(k+1) = -\alpha \sum_{j=1}^{N} a_{ij} \Big(\hat{y}_i(k) - \hat{y}_j(k) \Big) - \beta \Big(\hat{y}_i(k) - \varphi_i(k) \Big)$$
$$+ S\hat{y}_i(k) + \beta v_i(k), \tag{3.1a}$$
$$v_i(k+1) = -\alpha \sum_{j=1}^{N} a_{ij} \Big(\hat{y}_i(k) - \hat{y}_j(k) \Big) + S v_i(k), \tag{3.1b}$$

where $\hat{y}_i(k), v_i(k) \in \mathbb{R}^M$, α and β are design parameters, whose values are to be determined, and the initial condition of $v_i(k)$ is chosen as $v_i(0) = 0$.

The following theorem establishes that each of the distributed estimators (3.1) asymptotically estimates the averaged signal $\frac{1}{N} \sum_{j=1}^{N} \varphi_j(k)$.

Theorem 3.1. *Consider a networked system of N agents, labeled as $1, 2, \cdots, N$.*

Each agent i is associated with a discrete-time signal $\varphi_i(k)$ that is generated by an exosystem system (1.2)*, where, without loss of generality, all eigenvalues of S are assumed to be on or outside the unit circle. Let the communication topology among the agents satisfy Assumption 1.1. Let each agent be equipped with a distributed estimator as given in* (3.1)*. Then, if there exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that $S - \alpha\lambda_i I_M$, $i = 2, 3, \cdots, N$, and $S - \beta I_M$ are Schur. Then,*

$$\lim_{k\to\infty}\left(\hat{y}_i(k) - \frac{1}{N}\sum_{j=1}^{N}\varphi_j(k)\right) = 0,\ i = 1, 2, \cdots, N. \tag{3.2}$$

Remark 3.1. *Recall from Chapter 2 that, in the continuous-time setting, the average consensus can be realized for signals generated by any exosystem* (1.1)*. In the discrete-time setting, in the absence of high gain action, our ability to achieve average consensus is constrained by the properties of the communication topology. Conditions of Theorem 3.1 on parameters α and β reflect this constraint. Let λ_i^S, $i = 1, 2, \cdots, M$, be the eigenvalues of S. Assume that $|\mathrm{Im}(\lambda_i^S)| < 1$. Let $d_i^{\mathrm{r}} = \mathrm{Re}(\lambda_i^S) - \sqrt{1 - \mathrm{Im}^2(\lambda_i^S)}$ and $d_i^{\mathrm{l}} = \mathrm{Re}(\lambda_i^S) + \sqrt{1 - \mathrm{Im}^2(\lambda_i^S)}$. See Fig. 3.1 for an illustration. Then, there exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that $S - \alpha\lambda_i I_M, i = 2, 3, \cdots, N$, and $S - \beta I_M$ are Schur if*

$$\frac{\max_{i=2,3,\cdots,M} d_i^{\mathrm{r}}}{\lambda_2} < \frac{\min_{i=2,3,\cdots,M} d_i^{\mathrm{l}}}{\lambda_N},$$

which is always satisfied if eigenvalues of S are all on the left-half of the unit circle but not at $\pm j$ or they are all on the right-half of the unit circle but not at $\pm j$.

Proof of Theorem 3.1: Let $\hat{y}(k) = \left[\hat{y}_1^{\mathrm{T}}(k)\ \hat{y}_2^{\mathrm{T}}(k) \cdots \hat{y}_N^{\mathrm{T}}(k)\right]^{\mathrm{T}}$, $v(k) = \left[v_1^{\mathrm{T}}(k)\ v_2^{\mathrm{T}}(k) \cdots v_N^{\mathrm{T}}(k)\right]^{\mathrm{T}}$ and $\varphi = \left[\varphi_1^{\mathrm{T}}(k)\ \varphi_2^{\mathrm{T}}(k)\ \cdots \varphi_N^{\mathrm{T}}(k)\right]^{\mathrm{T}}$. Then, by (3.1), we have

$$\hat{y}(k+1) = \Big(-\alpha(L \otimes I_M) - \beta(I_N \otimes I_M) + I_N \otimes S\Big)\hat{y}(k)$$

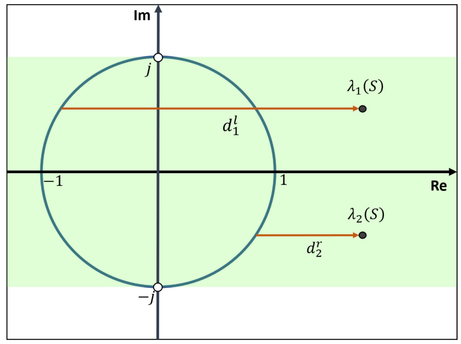

Figure 3.1: An illustration for Remark 3.1

$$+\beta\varphi(k) + \beta v(k), \tag{3.3a}$$

$$v(k+1) = -\alpha(L \otimes I_M)\hat{y}(k) + (I_N \otimes S)v(k). \tag{3.3b}$$

We also have

$$\varphi(k+1) = (I_N \otimes S)\,\varphi(k). \tag{3.4}$$

Define the estimation error as

$$\tilde{y}(k) = \hat{y}(k) - 1_N \otimes \frac{1}{N}\sum_{j=1}^{N}\varphi_j(k).$$

Then, by (3.3a) and (3.4), we have

$$\begin{aligned}
\tilde{y}(k+1) &= \hat{y}(k+1) - 1_N \otimes \frac{1}{N}\sum_{j=1}^{N}\varphi_j(k+1) \\
&= \Big(-\alpha(L \otimes I_M) - \beta(I_N \otimes I_M) + I_N \otimes S \Big)\hat{y}(k) \\
&\quad + \beta\varphi(k) + \beta v(k) - 1_N \otimes \frac{1}{N}\sum_{j=1}^{N} S\varphi_j(k) \\
&= \Big(-\alpha(L \otimes I_M) - \beta(I_N \otimes I_M) + I_N \otimes S \Big)\tilde{y}(k) \\
&\quad + \Big(-\alpha(L \otimes I_M) - \beta(I_N \otimes I_M) + I_N \otimes S \Big)\Big(1_N \otimes \frac{1}{N}\sum_{j=1}^{N}\varphi_j(k)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\beta\varphi(k) + \beta v(k) - 1_N \otimes \frac{1}{N}\sum_{j=1}^{N} S\varphi_j(k) \\
&= \Big(- \alpha(L \otimes I_M) - \beta(I_N \otimes I_M) + I_N \otimes S \Big)\tilde{y}(k) \\
&\quad +\beta \left(v(k) + (\Pi_N \otimes I_M)\, \varphi(k) \right),
\end{aligned} \tag{3.5}
$$

where $\Pi_N = I_N - \frac{1}{N} 1_N 1_N^{\mathrm{T}}$, which is the same as defined in the proof of Theorem 2.1, and we have used

$$
(I_N \otimes S)\left(1_N \otimes \frac{1}{N}\sum_{j=1}^{N}\varphi_j(k)\right) = (I_N 1_N) \otimes \left(S \frac{1}{N}\sum_{j=1}^{N} \varphi_j(k)\right),
$$

and (2.6) established in the proof of Theorem 2.1.

On the other hand, let

$$
\tilde{v}(k) = v(k) + (\Pi_N \otimes I_M)\, \varphi(k).
$$

Then, by (3.3b) and (3.4), we have

$$
\begin{aligned}
\tilde{v}(k+1) &= v(k+1) + (\Pi_N \otimes I_M)\varphi(k+1) \\
&= -\alpha(L \otimes I_M)\hat{y}(k) + (I_N \otimes S)\, v(k) + (\Pi_N \otimes I_M)\, (I_N \otimes S)\varphi(k) \\
&= -\alpha\, (L \otimes I_M) \left(\tilde{y}(k) + 1_N \otimes \frac{1}{N}\sum_{j=1}^{N} \varphi_j(k) \right) \\
&\quad + (I_N \otimes S)\, \tilde{v}(k) - (I_N \otimes S)\, (\Pi_N \otimes I_M)\, \varphi(k) \\
&\quad + (\Pi_N \otimes I_M)\, (I_N \otimes S)\, \varphi(k) \\
&= -\alpha\, (L \otimes I_M)\, \tilde{y}(k) + (I_N \otimes S)\, \tilde{v}(k),
\end{aligned} \tag{3.6}
$$

where we have used (2.6) and (2.8).

Equations (3.5) and (3.6) can be written in the following compact form,

$$\begin{bmatrix} \tilde{y}(k+1) \\ \tilde{v}(k+1) \end{bmatrix} = \widetilde{C} \begin{bmatrix} \tilde{y}(k) \\ \tilde{v}(k) \end{bmatrix}, \tag{3.7}$$

where

$$\widetilde{C} = \begin{bmatrix} I_N \otimes S & 0 \\ 0 & I_N \otimes S \end{bmatrix} + \begin{bmatrix} -\alpha(L \otimes I_M) - \beta(I_N \otimes I_M) & \beta(I_N \otimes I_M) \\ -\alpha(L \otimes I_M) & 0 \end{bmatrix}.$$

In what follows, we will analyze the asymptotic properties of the dynamical system (3.7). Let $h = \frac{1}{\sqrt{N}} 1_N$ and H be such that $h^{\mathrm{T}} H = 0$ and $H^{\mathrm{T}} H = I_{N-1}$. Consider the following change of variables,

$$\begin{bmatrix} p(k) \\ q(k) \end{bmatrix} = T_1 T_2 \begin{bmatrix} \tilde{y}(k) \\ \tilde{v}(k) \end{bmatrix}, \tag{3.8}$$

where

$$T_1 = \begin{bmatrix} I_N \otimes I_M & 0 \\ -I_N \otimes I_M & I_N \otimes I_M \end{bmatrix}, \quad T_2 = \begin{bmatrix} T_3^{\mathrm{T}} & 0 \\ 0 & T_3^{\mathrm{T}} \end{bmatrix},$$

with $T_3 = [r \ R] = [h \otimes I_M \ \ H \otimes I_M]$. Then, by (3.7), we have

$$\begin{bmatrix} p(k+1) \\ q(k+1) \end{bmatrix} = T_1 T_2 \widetilde{C} T_2^{-1} T_1^{-1} \begin{bmatrix} p(k) \\ q(k) \end{bmatrix}, \tag{3.9}$$

where

$$T_1 T_2 \widetilde{C} T_2^{-1} T_1^{-1} = \begin{bmatrix} T_3^{\mathrm{T}} (I_N \otimes S) T_3 & 0 \\ 0 & T_3^{\mathrm{T}} (I_N \otimes S) T_3 \end{bmatrix}$$

$$
+ \begin{bmatrix} -\alpha T_3^{\mathrm{T}}(L \otimes I_M)T_3 & \beta T_3^{\mathrm{T}}(I_N \otimes I_M)T_3 \\ 0 & -\beta T_3^{\mathrm{T}}(I_N \otimes I_M)T_3 \end{bmatrix}.
$$

We next partition the state variables p and q as

$$
p(k) = \begin{bmatrix} p_1(k) \\ p_{2:N}(k) \end{bmatrix}, \quad q(k) = \begin{bmatrix} q_1(k) \\ q_{2:N}(k) \end{bmatrix},
$$

where $p_1(k), q_1(k) \in \mathbb{R}^M$ and $p_{2:N}(k), q_{2:N}(k) \in \mathbb{R}^{M(N-1)}$. Then, the dynamics (3.9) can be written as

$$
\left[\begin{array}{c} p_1(k+1) \\ q_1(k+1) \\ \hline p_{2:N}(k+1) \\ q_{2:N}(k+1) \end{array}\right] = \left[\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_2 + C_3 \end{array}\right] \left[\begin{array}{c} p_1 \\ q_1 \\ \hline p_{2:N} \\ q_{2:N} \end{array}\right], \tag{3.10}
$$

where

$$
\begin{aligned}
C_1 &= \begin{bmatrix} S & \beta I_M \\ 0 & S - \beta I_M \end{bmatrix}, \\
C_2 &= \begin{bmatrix} R^{\mathrm{T}}(I_N \otimes S)R & 0 \\ 0 & R^{\mathrm{T}}(I_N \otimes S)R \end{bmatrix}, \\
C_3 &= \begin{bmatrix} -\alpha R^{\mathrm{T}}(L \otimes I_M)R & \beta R^{\mathrm{T}}(I_N \otimes I_M)R \\ 0 & -\beta R^{\mathrm{T}}(I_N \otimes I_M)R \end{bmatrix}.
\end{aligned}
$$

By the conditions of the Theorem 3.1, α and β are such that the eigenvalues of

$C_2 + C_3$ are all inside the unit circle. Hence, we have

$$\lim_{k\to\infty} \begin{bmatrix} p_{2:N}(k) \\ q_{2:N}(k) \end{bmatrix} = 0.$$

The solution of the state equation (3.10) is given by

$$\left[\begin{array}{c} p_1(k) \\ q_1(k) \\ \hline p_{2:N}(k) \\ q_{2:N}(k) \end{array}\right] = \Omega_1(k) \left[\begin{array}{c} p_1(0) \\ q_1(0) \\ \hline p_{2:N}(0) \\ q_{2:N}(0) \end{array}\right],$$

where

$$\Omega_1(k) = \left[\begin{array}{cc|cc} S^k & \beta \sum_{m=0}^{k-1} (S-\beta I_M)^{k-1-m} S^m & 0 & 0 \\ 0 & (S-\beta I_M)^k & 0 & 0 \\ \hline 0 & 0 & \Gamma_{11}(k) & \Gamma_{12}(k) \\ 0 & 0 & \Gamma_{21}(k) & \Gamma_{22}(k) \end{array}\right],$$

with

$$\begin{aligned} \Gamma(k) &= (C_2 + C_3)^k := \begin{bmatrix} \Gamma_{11}(k) & \Gamma_{12}(k) \\ \Gamma_{21}(k) & \Gamma_{22}(k) \end{bmatrix}, \\ \Gamma_{11}(k) &= \left(R^{\mathrm{T}} \Big(I_N \otimes S - \alpha (L \otimes I_M) \Big) R \right)^k, \\ \Gamma_{12}(k) &= \beta \sum_{m=0}^{k-1} \left(R^{\mathrm{T}} \Big(I_N \otimes S - \alpha (L \otimes I_M) \Big) R \right)^{k-1-m} \\ &\quad \times \left(R^{\mathrm{T}} \Big(I_N \otimes S - \beta (I_N \otimes I_M) \Big) R \right)^m, \\ \Gamma_{21}(k) &= 0, \end{aligned}$$

$$\Gamma_{22}(k) = \left(R^{\mathrm{T}}\left(I_N \otimes S - \beta(I_N \otimes I_M)\right)R\right)^k.$$

Noting that

$$\begin{aligned}
&\beta \sum_{m=0}^{k-1} (S - \beta I_M)^{k-1-m} S^m \\
&= \left(S - (S - \beta I_M)\right) \sum_{m=0}^{k-1} (S - \beta I_M)^{k-1-m} S^m \\
&= S^k - (S - \beta I_M)^k,
\end{aligned}$$

we can simplify Ω_1 as

$$\Omega_1(k) = \left[\begin{array}{cc|cc}
S^k & S^k - (S - \beta I_M)^k & 0 & 0 \\
0 & (S - \beta I_M)^k & 0 & 0 \\
\hline
0 & 0 & \Gamma_{11}(k) & \Gamma_{12}(k) \\
0 & 0 & \Gamma_{21}(k) & \Gamma_{22}(k)
\end{array}\right].$$

Then, in view of (3.8), we have

$$\tilde{y}(k) = \Lambda_{11}(k)\tilde{y}(0) + \Lambda_{12}(k)\tilde{v}(0), \tag{3.11a}$$

$$\tilde{v}(k) = \Lambda_{21}(k)\tilde{y}(0) + \Lambda_{22}(k)\tilde{v}(0), \tag{3.11b}$$

where

$$\begin{aligned}
\Lambda_{11}(k) &= r(S - \beta I_M)^k r^{\mathrm{T}} + R\left(\Gamma_{11}(k) - \Gamma_{12}(k)\right)R^{\mathrm{T}}, \\
\Lambda_{12}(k) &= r\left(S^k - (S - \beta I_M)^k\right) r^{\mathrm{T}} + R\Gamma_{12}(k)R^{\mathrm{T}}, \\
\Lambda_{21}(k) &= R\left(\Gamma_{11}(k) - \Gamma_{12}(k) + \Gamma_{21}(k) - \Gamma_{22}(k)\right)R^{\mathrm{T}},
\end{aligned}$$

$$\Lambda_{22}(k) = rS^k r^{\mathrm{T}} + R\Big(\Gamma_{12}(k) + \Gamma_{22}(k)\Big)R^{\mathrm{T}}.$$

More explicitly, we have

$$\begin{aligned}\tilde{y}(k) &= rS^k r^{\mathrm{T}}\tilde{v}(0) + \varepsilon(k) \\ &= 1_N \otimes \frac{1}{N} S^k \sum_{j=1}^{N} \tilde{v}_j(0) + \varepsilon(k),\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}\varepsilon(k) = &\Bigg(r(S-\beta I_M)^k r^{\mathrm{T}} + R\Big(\Gamma_{11}(k) - \Gamma_{12}(k)\Big)R^{\mathrm{T}}\Bigg)\tilde{y}(0) \\ &+\Big(-r(S-\beta I_M)^k r^{\mathrm{T}} + R\Gamma_{12}(k)R^{\mathrm{T}}\Big)\tilde{v}(0).\end{aligned}$$

Recalling that the eigenvalues of $C_2 + C_3$ and $S - \beta I_M$ are all inside the unit circle, we have

$$\lim_{k\to\infty} \Gamma_{ij}(k) = 0,\ i, j = 1, 2, \tag{3.13}$$

and hence,

$$\lim_{k\to\infty} \varepsilon(k) = 0.$$

On the other hand, recalling that

$$\tilde{v}(k) = v(k) + (\Pi_N \otimes I_M)\varphi(k),$$

and $v_i(0) = 0, i = 1, 2, \cdots, N$, we have

$$\sum_{j=1}^{N} \tilde{v}_j(0) = \left(\sum_{j=1}^{N} \Pi_{N,j} \otimes I_M\right)\varphi(0) = 0,$$

where $\Pi_{N,j}$ is the j^{th} row of Π_N and $\sum_{j=1}^{N} \Pi_{N,j} = 0$.

Thus, it follows from (3.12) that

$$\lim_{k\to\infty} \tilde{y}(k) = 0,$$

and hence, for all $i = 1, 2, \cdots, N$,

$$\lim_{k\to\infty} \left(\hat{y}_i(k) - \frac{1}{N} \sum_{j=1}^{N} \varphi_j(k) \right) = \lim_{k\to\infty} \tilde{y}_i(k) = 0.$$

This completes the proof. $\square$

3.2 Robustness Against Network Interruption

In what follows, we will establish that the distributed discrete-time estimator (3.1) is robust against interruption of the network connectivity. We summarize this robustness result in the following theorem.

Theorem 3.2. *Consider a networked system of N agents, with the associated discrete signal $\varphi_i(k)$ and the distributed estimators as described in Theorem 3.1. Let S be neutrally stable, that is, all its eigenvalues are all on the unit circle and are simple. Let the communication topology among the agents satisfy Assumption 1.1. Suppose that the communication is interrupted at time $k_{\mathrm{I}} > 0$, with individual subgroups of agents remaining connected. Then, without loss of generality, let the subgroup of K agents be agents $1, 2, \cdots, K$. Denote the Laplacian matrix of the connected communication among these K agents as L^{s} with eigenvalues $0 = \lambda_1^{\mathrm{s}} < \lambda_2^{\mathrm{s}} \leq \lambda_3^{\mathrm{s}} \leq ... \leq \lambda_K^{\mathrm{s}}$. If $S - \alpha\lambda_i^{\mathrm{s}}, i = 2, 3, \cdots, K$, are still Schur, then, each agent i within the subgroup continues to track the averaged signal $\frac{1}{N} \sum_{j=1}^{n} \varphi_j(k)$ with a bounded steady-state tracking error,* *that is,*

$$\hat{y}_i(k) = \frac{1}{N}\sum_{j=1}^{N}\varphi_j(k) + \varepsilon^{\mathrm{s}}(k,k_{\mathrm{I}}) + \delta(k,k_{\mathrm{I}}),$$

where $\|\delta(k,k_{\mathrm{I}})\| \le \delta_0(k_{\mathrm{I}})$ *and*

$$\lim_{k\to\infty}\varepsilon^{\mathrm{s}}(k,k_{\mathrm{I}}) = 0,$$
$$\lim_{k_{\mathrm{I}}\to\infty}\delta_0(k_{\mathrm{I}}) = 0.$$

Proof: Recall from the proof of Theorem 3.1 that $\tilde{v}(k) = v(k) + (\Pi_N \otimes I_M)\varphi(k)$ and $\sum_{j=1}^{N}\tilde{v}_j(0) = 0$. Then, by (3.11b), we have

$$\begin{aligned} v(k_{\mathrm{I}}) &= \Lambda_{21}(k_{\mathrm{I}})\tilde{y}(0) + \Lambda_{22}(k_{\mathrm{I}})\tilde{v}(0) - (\Pi_N \otimes I_M)\varphi(k_{\mathrm{I}}) \\ &= \Lambda_{21}(k_{\mathrm{I}})\tilde{y}(0) + 1_N \otimes \frac{1}{N}S^{k_{\mathrm{I}}}\sum_{j=1}^{N}\tilde{v}_j(0) \\ &\quad + R\Big(\Gamma_{12}(k_{\mathrm{I}}) + \Gamma_{22}(k_{\mathrm{I}})\Big)R^{\mathrm{T}}\tilde{v}(0) - (\Pi_N \otimes I_M)\varphi(k_{\mathrm{I}}) \\ &= \Lambda_{21}(k_{\mathrm{I}})\tilde{y}(0) + R\Big(\Gamma_{12}(k_{\mathrm{I}}) + \Gamma_{22}(k_{\mathrm{I}})\Big)R^{\mathrm{T}}\tilde{v}(0) - (\Pi_N \otimes I_M)\varphi(k_{\mathrm{I}}) \\ &= \chi(k_{\mathrm{I}}) - (\Pi_N \otimes I_M)\varphi(k_{\mathrm{I}}), \end{aligned} \tag{3.14}$$

where

$$\chi(k_{\mathrm{I}}) = \Lambda_{21}(k_{\mathrm{I}})\tilde{y}(0) + R\Big(\Gamma_{12}(k_{\mathrm{I}}) + \Gamma_{22}(k_{\mathrm{I}})\Big)R^{\mathrm{T}}\tilde{v}(0),$$

and by (3.13),

$$\lim_{k_{\mathrm{I}}\to\infty}\chi(k_{\mathrm{I}}) = 0. \tag{3.15}$$

Let us now consider the evolution for $k \ge k_{\mathrm{I}}$. Let $\hat{y}^{\mathrm{s}}(k) = \big[\hat{y}_1^{\mathrm{T}}(k)\ \hat{y}_2^{\mathrm{T}}(k)\cdots\hat{y}_K^{\mathrm{T}}(k)\big]^{\mathrm{T}}$, $v^{\mathrm{s}}(k) = \big[v_1^{\mathrm{T}}(k)\ v_2^{\mathrm{T}}(k)\cdots v_K^{\mathrm{T}}(k)\big]^{\mathrm{T}}$, and $\varphi^{\mathrm{s}}(k) = \big[\varphi_1^{\mathrm{T}}(k)\ \varphi_2^{\mathrm{T}}(k)\cdots\varphi_K^{\mathrm{T}}(k)\big]^{\mathrm{T}}$. Then, we have

$$\hat{y}^{\mathrm{s}}(k+1) = \Big(-\alpha(L\otimes I_M) - \beta(I_K\otimes I_M) + I_K\otimes S\Big)\hat{y}^{\mathrm{s}}(k) \\ +\beta\varphi^{\mathrm{s}}(k) + \beta v^{\mathrm{s}}(k),\ k\geq k_{\mathrm{I}},$$

$$v^{\mathrm{s}}(k+1) = -\alpha(L\otimes I_M)\hat{y}^{\mathrm{s}}(k) + (I_K\otimes S)v^{\mathrm{s}}(k),\ k\geq k_{\mathrm{I}},$$

and

$$\varphi^{\mathrm{s}}(k+1) = (I_K\otimes S)\varphi^{\mathrm{s}}(k),\ k\geq k_{\mathrm{I}}.$$

Following the proof of Theorem 3.1 for the subgroup of agents $1, 2, \cdots, K$, we define

$$\tilde{y}^{\mathrm{s}}(k) = \hat{y}^{\mathrm{s}}(k) - 1_K\otimes\frac{1}{K}\sum_{j=1}^{K}\varphi_j(k), \tag{3.16a}$$

$$\tilde{v}^{\mathrm{s}}(k) = v^{\mathrm{s}}(k) + (\Pi_K\otimes I_M)\varphi^{\mathrm{s}}(k). \tag{3.16b}$$

Let $h^{\mathrm{s}} = \frac{1}{\sqrt{K}}1_K$ and H^{s} be such that $(h^{\mathrm{s}})^{\mathrm{T}}H^{\mathrm{s}} = 0$ and $(H^{\mathrm{s}})^{\mathrm{T}}H^{\mathrm{s}} = I_{K-1}$. Let $r^{\mathrm{s}} = h^{\mathrm{s}}\otimes I_M$, $R^{\mathrm{s}} = H^{\mathrm{s}}\otimes I_M$ and

$$C_2^{\mathrm{s}} = \begin{bmatrix} (R^{\mathrm{s}})^{\mathrm{T}}(I_K\otimes S)R^{\mathrm{s}} & 0 \\ 0 & (R^{\mathrm{s}})^{\mathrm{T}}(I_K\otimes S)R^{\mathrm{s}} \end{bmatrix},$$

$$C_3^{\mathrm{s}} = \begin{bmatrix} -\alpha(R^{\mathrm{s}})^{\mathrm{T}}(L^{\mathrm{s}}\otimes I_M)R^{\mathrm{s}} & \beta(R^{\mathrm{s}})^{\mathrm{T}}(I_K\otimes I_M)R^{\mathrm{s}} \\ 0 & -\beta(R^{\mathrm{s}})^{\mathrm{T}}(I_K\otimes I_M)R^{\mathrm{s}} \end{bmatrix}.$$

By the conditions of the theorem, α and β are such that all eigenvalues of $C_2^{\mathrm{s}} + C_3^{\mathrm{s}}$ and $S - \beta I_M$ are inside the unit circle.

Let

$$\Gamma^{\mathrm{s}}(k) = (C_2^{\mathrm{s}} + C_3^{\mathrm{s}})^k := \begin{bmatrix} \Gamma_{11}^{\mathrm{s}}(k) & \Gamma_{12}^{\mathrm{s}}(k) \\ \Gamma_{21}^{\mathrm{s}}(k) & \Gamma_{22}^{\mathrm{s}}(k) \end{bmatrix}.$$

Then,

$$\lim_{k\to\infty} \Gamma_{ij}^{\mathrm{s}}(k) = 0,\ i,j = 1,2. \tag{3.17}$$

Following the proof of Theorem 3.1, we can also obtain the following equation,

$$\tilde{y}^{\mathrm{s}}(k) = \Lambda_{11}^{\mathrm{s}}(k-k_{\mathrm{I}})\tilde{y}^{\mathrm{s}}(k_{\mathrm{I}}) + \Lambda_{12}^{\mathrm{s}}(k-k_{\mathrm{I}})\tilde{v}^{\mathrm{s}}(k_{\mathrm{I}}), \tag{3.18}$$

where

$$\begin{aligned}\Lambda_{11}^{\mathrm{s}}(k-k_{\mathrm{I}}) &= r^{\mathrm{s}}(S-\beta I_M)^{k-k_{\mathrm{I}}}(r^{\mathrm{s}})^{\mathrm{T}} + R^{\mathrm{s}}\Big(\Gamma_{11}^{\mathrm{s}}(k-k_{\mathrm{I}}) - \Gamma_{12}^{\mathrm{s}}(k-k_{\mathrm{I}})\Big)(R^{\mathrm{s}})^{\mathrm{T}},\\ \Lambda_{12}^{\mathrm{s}}(k-k_{\mathrm{I}}) &= r^{\mathrm{s}}\left(S^{k-k_{\mathrm{I}}} - (S-\beta I_M)^{k-k_{\mathrm{I}}}\right)(r^{\mathrm{s}})^{\mathrm{T}} + R^{\mathrm{s}}\Big(\Gamma_{12}^{\mathrm{s}}(k-k_{\mathrm{I}})\Big)(R^{\mathrm{s}})^{\mathrm{T}}.\end{aligned}$$

We can rewrite (3.18) as

$$\tilde{y}^{\mathrm{s}}(k) = 1_K \otimes \frac{1}{K} S^{k-k_{\mathrm{I}}} \sum_{j=1}^{K} \tilde{v}_j(k_{\mathrm{I}}) + \varepsilon^{\mathrm{s}}(k, k_{\mathrm{I}}), \tag{3.19}$$

where

$$\begin{aligned}\varepsilon^{\mathrm{s}}(k,k_{\mathrm{I}}) = &\left(-r^{\mathrm{s}}(S-\beta I_M)^{k-k_{\mathrm{I}}}(r^{\mathrm{s}})^{\mathrm{T}} + R^{\mathrm{s}}\Big(\Gamma_{12}^{\mathrm{s}}(k-k_{\mathrm{I}})\Big)(R^{\mathrm{s}})^{\mathrm{T}}\right)\tilde{v}^{\mathrm{s}}(k_{\mathrm{I}})\\ &+\Lambda_{11}^{\mathrm{s}}(k-k_{\mathrm{I}})\tilde{y}^{\mathrm{s}}(k_{\mathrm{I}}).\end{aligned}$$

By (3.17), we have

$$\lim_{k\to\infty} \varepsilon^{\mathrm{s}}(k,k_{\mathrm{I}}) = 0. \tag{3.20}$$

Recalling that $\tilde{v}^{\mathrm{s}}(k) = v^{\mathrm{s}}(k) + (\Pi_K \otimes I_M)\varphi^{\mathrm{s}}(k)$ and $\sum_{j=1}^{K}\Pi_{K,j} = 0$ with $\Pi_{K,j}$ being the j^{th} row of Π_K, we have

$$\begin{aligned}\sum_{j=1}^{K}\tilde{v}_j(k_{\mathrm{I}}) =& \sum_{j=1}^{K} v_j(k_{\mathrm{I}}) = \sum_{j=1}^{K}\chi_j(k_{\mathrm{I}}) - \sum_{j=1}^{K}(\Pi_{N,j}\otimes I_M)\varphi(k_{\mathrm{I}}) \\ =& \sum_{j=1}^{K}\chi_j(k_{\mathrm{I}}) - \left(\sum_{j=1}^{K}\varphi_j(k_{\mathrm{I}}) - \frac{K}{N}\sum_{j=1}^{N}\varphi_j(k_{\mathrm{I}})\right),\end{aligned} \tag{3.21}$$

where we have used (3.14), and $\chi_j(k_{\mathrm{I}})$ is the $\Big((j-1)M+1\Big)^{\text{th}}$ to jM^{th} rows of $\chi(k_{\mathrm{I}})$ in (3.14).

It then follows from (3.19) and (3.21) that

$$\begin{aligned}\hat{y}^{\mathrm{s}}(k) =& \tilde{y}^{\mathrm{s}}(k) + 1_K \otimes \frac{1}{K}\sum_{j=1}^{K}\varphi_j(k) \\ =& 1_K \otimes \left(\frac{1}{K}\sum_{j=1}^{K}\varphi_j(k) + \frac{1}{K}\left(S^{k-k_{\mathrm{I}}}\right)\sum_{j=1}^{K}\tilde{v}_j(k_{\mathrm{I}})\right) + \varepsilon^{\mathrm{s}}(k,k_{\mathrm{I}}) \\ =& 1_K \otimes \Bigg(\frac{1}{K}\sum_{j=1}^{K}\varphi_j(k) + \frac{1}{K}S^{k-k_{\mathrm{I}}}\sum_{j=1}^{K}\chi_j(k_{\mathrm{I}}) \\ & -\frac{1}{K}S^{k-k_{\mathrm{I}}}\left(\sum_{j=1}^{K}\varphi_j(k_{\mathrm{I}}) - \frac{K}{N}\sum_{j=1}^{N}\varphi_j(k_{\mathrm{I}})\right)\Bigg) + \varepsilon^{\mathrm{s}}(k,k_{\mathrm{I}}) \\ =& 1_K \otimes \Bigg(\frac{1}{K}\sum_{j=1}^{K}\varphi_j(k) + \frac{1}{K}S^{k-k_{\mathrm{I}}}\sum_{j=1}^{K}\chi_j(k_{\mathrm{I}}) \\ & -\frac{1}{K}\left(\sum_{j=1}^{K}\varphi_j(k) - \frac{K}{N}\sum_{j=1}^{N}\varphi_j(k)\right)\Bigg) + \varepsilon^{\mathrm{s}}(k,k_{\mathrm{I}}) \\ =& 1_K \otimes \frac{1}{N}\sum_{j=1}^{N}\varphi_j(k) + 1_K \otimes \left(\frac{1}{K}S^{k-k_{\mathrm{I}}}\sum_{j=1}^{K}\chi_j(k_{\mathrm{I}})\right) + \varepsilon^{\mathrm{s}}(k,k_{\mathrm{I}}) \\ =& 1_K \otimes \frac{1}{N}\sum_{j=1}^{N}\varphi_j(k) + \varepsilon^{\mathrm{s}}(k,k_{\mathrm{I}}) + \delta(k,k_{\mathrm{I}}),\end{aligned}$$

where

$$\delta(k, k_{\mathrm{I}}) = 1_K \otimes \left(\frac{1}{K} S^{k-k_{\mathrm{I}}} \sum_{j=1}^{K} \chi_j(k_{\mathrm{I}}) \right).$$

Since S is neutrally stable, we have $\left\| S^{k-k_{\mathrm{I}}} \right\| \leq \sigma$ for some constant scalar $\sigma > 0$, and hence

$$\|\delta(k, k_{\mathrm{I}})\| \leq \sigma K \|\chi(k_{\mathrm{I}})\| := \delta_0(k_{\mathrm{I}}).$$

In view of (3.20) and (3.15), it is clear that

$$\lim_{k\to\infty} \varepsilon^{\mathrm{s}}(k, k_{\mathrm{I}}) = 0,$$
$$\lim_{k_{\mathrm{I}}\to\infty} \delta_0(k_{\mathrm{I}}) = 0.$$

This completes the proof. $\square$

3.3 Simulation Results

In this section, we run simulation on a five agent system to verify our theoretical results both in the discrete-time settings. The communication topology among the agents is as shown in Fig. 2.1. We will also illustrate the robustness of our algorithms against interruptions of the network connectivity (see Fig. 2.2).

Let the signals be generated by the exosystem (1.2) with

$$S = \begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} & 0 & 0 \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the initial values $\varphi_i(0) = [4i-7, -3i+18, 4i-9, -2i+7]^{\mathrm{T}},\ i = 1, 2, \cdots, 5.$

For the given communication topology,

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

It can be verified that, with $\alpha = 0.4$ and $\beta = 0.8$, all conditions of Theorem 3.1 are satisfied. Shown in Fig. 3.2 are the estimations of the average value of the five signals by the five agents.

To illustrate the robustness of our distributed algorithm against network connectivity interruption, we consider signals generated by a neutrally stable exosystem (1.2) with

$$S = \begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix}$$

and the initial conditions $\varphi_i(0) = [4i-7, -3i+18]^{\mathrm{T}}$, $i = 1, 2, \cdots, 5$. It can be verified that, with $\alpha = 0.4$ and $\beta = 0.8$, all conditions of Theorem 3.2 are satisfied.

Shown in Fig. 3.3 and Fig. 3.4 are the norms of the estimation errors of agents of both subgroups when the communication is interrupted at $k_{\mathrm{I}} = 5$ and $k_{\mathrm{I}} = 15$, respectively. As is obvious, the later the interruption occurs, the more accurate the estimations are.

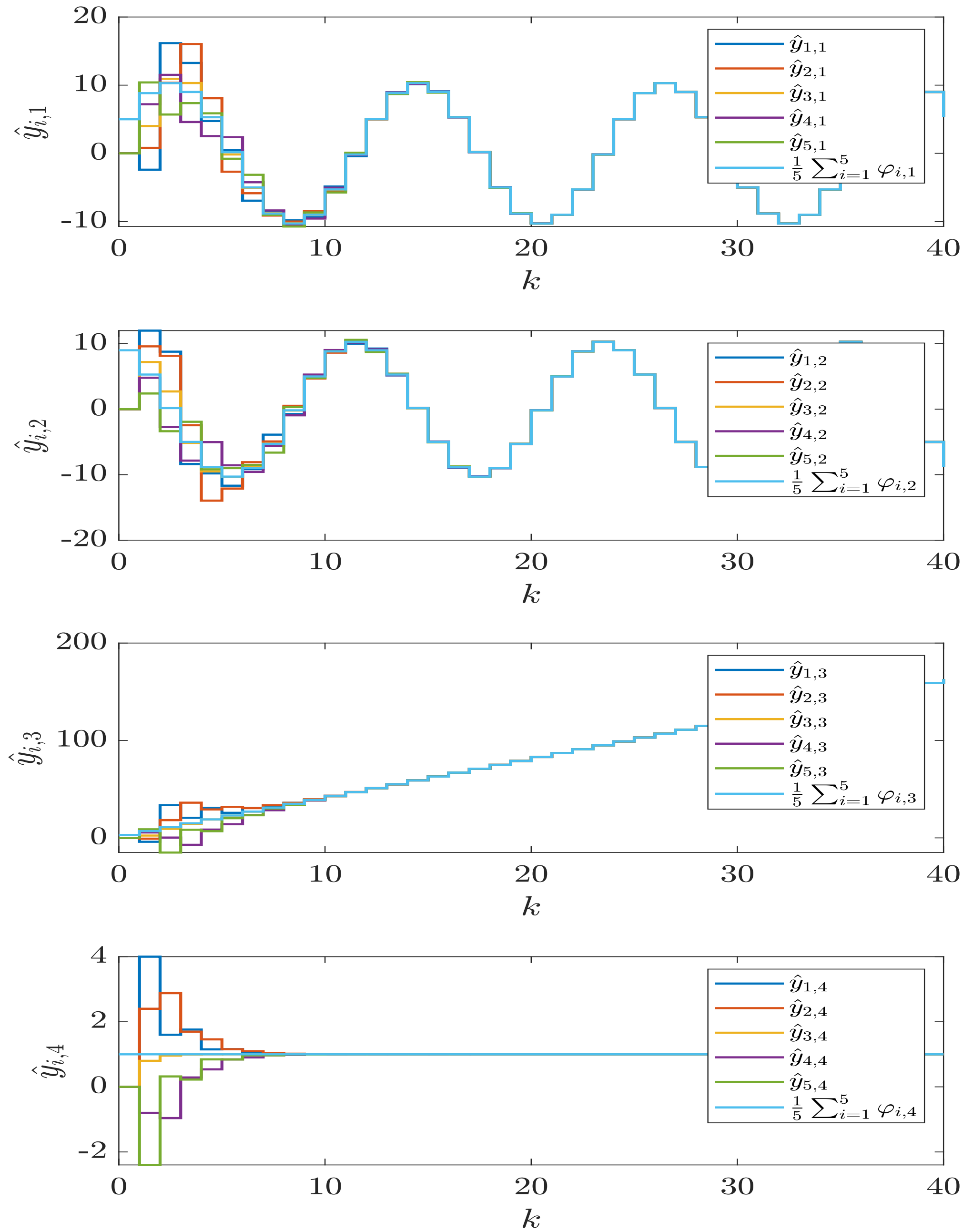

Figure 3.2: Estimations of the averaged signal by all agents.

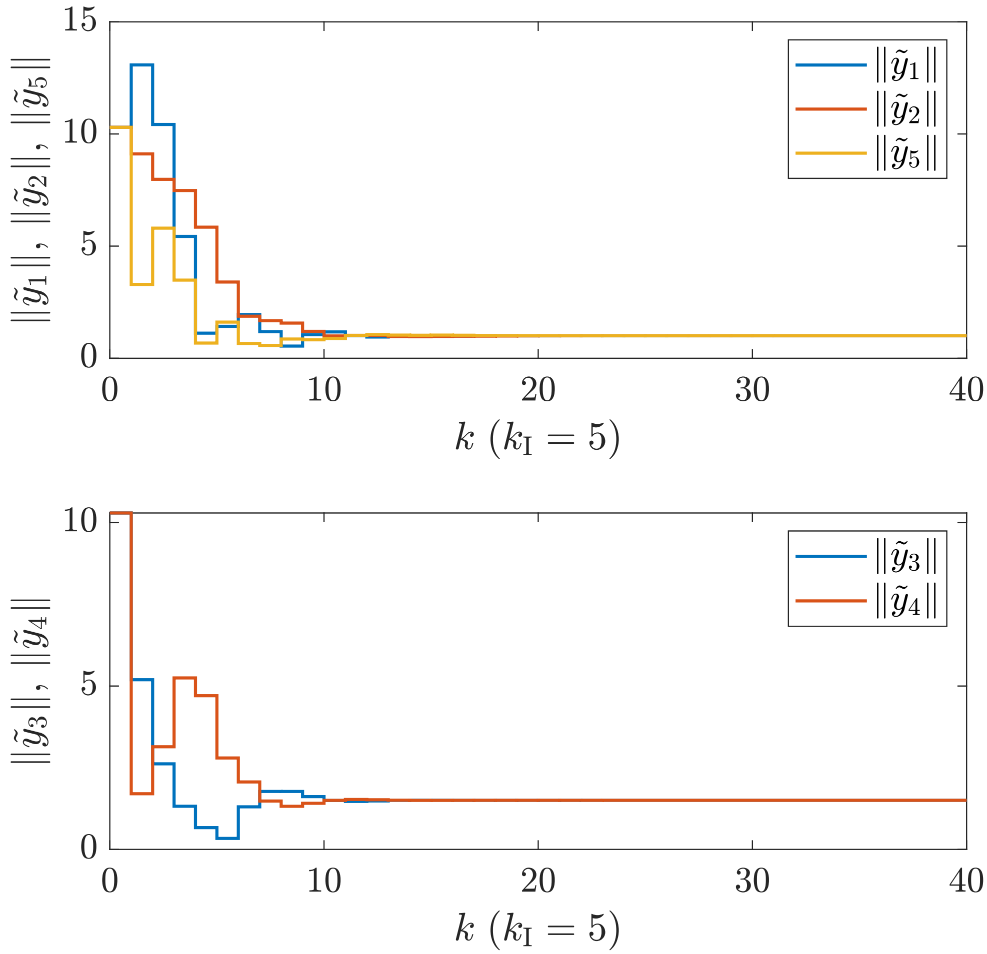

Figure 3.3: Estimation errors when the communication is interrupted at time $k_\mathrm{I} = 5$.

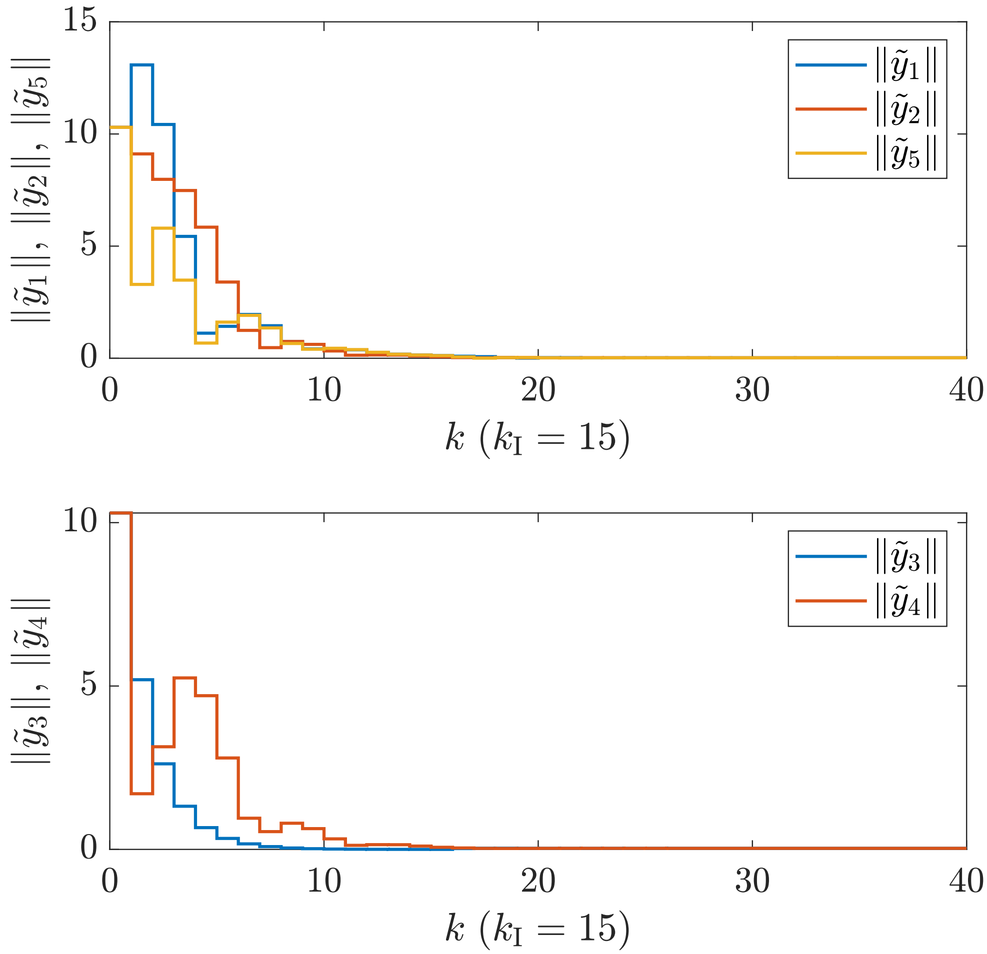

Figure 3.4: Estimation errors when the communication is interrupted at time $k_{\mathrm{I}} = 15$.

Chapter 4

State-of-Charge Balancing of a Networked Battery System

Energy storage systems are essential components in microgrids. They not only ensure the power quality and reliability but also reduce energy loss in microgrids. Among various energy storage technologies, battery energy storage systems have emerged as an appealing technology due to their versatility, rapid response, high energy density, and efficiency. By absorbing power from the grid during off-peak time and supplying power to the grid in peak time, battery systems enable the grid to have the ability of peak-shaving/shifting, power quality enhancement, and congestion relief. A fundamental control objective of a battery system is to satisfy the charging/discharging power desired by the grid while balancing the state-of-charge (SoC) of all its units [2]. In this chapter, we propose a practical application of our proposed dynamic average consensus algorithms to the SoC balancing problem of a networked battery system. Simulation results also show the robustness of the resulting battery system against interruption of the communication network.

4.1 Problem Statement

We consider a battery energy storage system consisting of N networked battery units. Each battery unit, with its own distributed control algorithm, is able to communicate with nearby battery units and exchange critical states such as their SoC and the desired power the grid needs. The desired power $p^*(t)$ is generated by an exosystem,

$$\begin{aligned} \dot{P}^*(t) &= SP^*(t),\ P^*(t) \in \mathbb{R}^M, \\ p^*(t) &= YP^*(t),\ p^*(t) \in \mathbb{R}. \end{aligned} \tag{4.1}$$

In order for the battery units to estimate the desired power $p^*(t)$, we make the following assumption.

Assumption 4.1. *There is at least one battery unit that has access to the desired power $p^*(t)$.*

We define the diagonal matrix $B = \mathrm{diag}\{b_1,\ b_2,\ \cdots,\ b_N\}$, where $b_i = 1$ if the i^{th} battery unit has access to the desired power $p^*(t)$ and $b_i = 0$ otherwise.

For each battery i, $i = 1, 2, \cdots, N$, the SoC dynamics of the i^{th} battery unit is given by

$$\dot{s}_i = -\frac{1}{C_i V_i} p_i, \quad p_i = V_i i_i \begin{cases} > 0, & \text{(discharging)}, \\ < 0, & \text{(charging)}, \end{cases} \tag{4.2}$$

where $s_i \in \mathbb{R}$ is the SoC of i^{th} battery, $C_i \in \mathbb{R}$ is the capacity of i^{th} battery, $V_i \in \mathbb{R}$ is the end voltage of i^{th} battery, $i_i \in \mathbb{R}$ is the current of i^{th} battery and $p_i \in \mathbb{R}$ is the power of i^{th} battery.

Define the state of the battery unit i as

$$x_i = \begin{cases} x_{d,i} = \ C_i V_i s_i, & \text{(in discharging mode)}, \\ x_{c,i} = C_i V_i (1 - s_i), & \text{(in charging mode)}. \end{cases} \tag{4.3}$$

Then, the dynamics of the battery unit i is described as

$$\dot{x}_i = \begin{cases} -p_i, & \text{(in discharging mode)}, \\ \ p_i, & \text{(in charging mode)}. \end{cases} \tag{4.4}$$

Based (4.4), the control algorithm is given by

$$p_i(t) = \frac{x_i(t)}{x_a(t)} p_a(t), \tag{4.5}$$

where $x_a(t) = \frac{1}{N} \sum_{j=1}^{N} x_j(t)$ is the average state for all the battery units and $p_a(t) = \frac{1}{N} p^*(t)$ is the average desired power among battery units. Since $x_a(t)$ and $p_a(t)$ are global information that might not be available to all battery units, the dynamic average consensus algorithms are needed to estimate $p_a(t)$ and $x_a(t)$.

By using consensus algorithms to estimate $x_a(t)$ and $p_a(t)$, (4.5) can be implemented as,

$$p_i(t) = \frac{x_i(t)}{\hat{x}_{a,i}(t)} \hat{p}_{a,i}(t), \ i = 1, 2, \cdots, N, \tag{4.6}$$

where $\hat{p}_{a,i}(t)$ and $\hat{x}_{a,i}(t)$ are respectively the estimated values for $p_a(t)$ and $x_a(t)$ by the i^{th} battery unit.

In [2], the dynamic leader-following consensus algorithm [31] and the dynamic average consensus algorithm [25] are adapted to implement (4.6). The dynamic leader-following consensus algorithm tracks the signal of the exosystem with a steady state error and the dynamic average consensus algorithm tracks the average value of all the agents also with a steady state error. As a result, the SoC balancing algorithm composed of the control algorithm (4.5), the dynamic leader-following consensus algorithm [31] and the dynamic average consensus algorithm [25] enables the distributed battery system to deliver the power desired by the grid with a steady state error and balance the SoC of all its units within a level of accuracy.

Our objective is to implement the control algorithm (4.6) by using the dynamic average consensus algorithm proposed in this thesis to accurately deliver the charging/discharging power while accurately balancing the SoC. In implementing (4.6) with our dynamic average consensus algorithm, we also relax the requirement of the knowledge of the number N of the battery units in the system.

4.2 Power Distribution Algorithm Design

Notice that only the exosystem knows the frequency components of the desired power $p^*(t)$. In this situation, we need to use the dynamic leader-following consensus algorithm to estimate the frequency components of the desired power $p^*(t)$ for each battery unit. In order to achieve this goal, here we upgrade the dynamic average consensus algorithm in Chapter 2 to

$$\dot{\hat{y}}_i = -\mu \sum_{j=1}^{N} a_{ij}(\hat{y}_i - \hat{y}_j) - \mu \left(\hat{y}_i - \varphi_i\right) + \hat{S}_i \hat{y}_i + \mu \sum_{j=1}^{N} a_{ij}(v_i - v_j)\,, \tag{4.7a}$$

$$\dot{v}_i = -\mu \sum_{j=1}^{N} a_{ij}\left(\hat{y}_i - \hat{y}_j\right) + \hat{S}_i v_i, \tag{4.7b}$$

$$\dot{\hat{S}}_i = -\mu \left(\sum_{j=1}^{N} a_{ij}\left(\hat{S}_i - \hat{S}_j\right) + b_i \left(\hat{S}_i - S\right) \right), \tag{4.7c}$$

where $\hat{y}_i(t), v_i(t) \in \mathbb{R}^M$, $\hat{S}_i(t) \in \mathbb{R}^{M\times M}$ and $\mu > 0$ is a design parameter, whose value is to be determined. Notice that, because of the presence of $v_j(t)$, $j \neq i$, in (4.7), this upgraded dynamic average consensus algorithm no longer possesses the robustness property of (2.1). That is, when the communication is interrupted at time $t_{\mathrm{I}} > 0$, with individual subgroups of agents remaining connected, each agent i within a subgroup of $K < N$ agents will no longer be guaranteed to continue to track the averaged signal $\frac{1}{N}\sum_{j=1}^{N}\varphi_j(t)$. However, it is precisely because of the communication of v_i among agents that the dynamic average consensus algorithm (4.7) possesses a different robustness feature. When the communication is interrupted at time $t_{\mathrm{I}} > 0$, with individual subgroups of agents remaining connected, each agent i within a subgroup of $K < N$ agents will track the averaged signal $\frac{1}{K}\sum_{j=1}^{K}\varphi_j(t)$. This robustness is an appealing feature for the distributed SoC balancing algorithm below.

In the following, we first apply the dynamic average consensus algorithm (4.7) to construct our dynamic average consensus algorithm to estimate the average desired power $p_a(t)$.

By (4.7c), the information S and Y in the exosystem (4.1) can be estimated by battery unit i with the following dynamic leader-following consensus algorithm,

$$\dot{\hat{S}}_i = -\mu\left(\sum_{j=1}^{N} a_{ij}\left(\hat{S}_i - \hat{S}_j\right) + b_i\left(\hat{S}_i - S\right)\right), \tag{4.8a}$$

$$\dot{\hat{Y}}_i = -\mu\left(\sum_{j=1}^{N} a_{ij}\left(\hat{Y}_i - \hat{Y}_j\right) + b_i\left(\hat{Y}_i - Y\right)\right), \tag{4.8b}$$

where $\hat{S}_i(t)$ and $\hat{Y}_i(t)$ are the estimation values of the i^{th} battery for S and Y, respectively. Notice that, even though $\sum_{j=i}^{N} a_{i,j}$ is defined for $j = 1$ to N, it can be computed over all $a_{i,j} \neq 0$. Hence, the information of N is not required.

For each battery unit i, $i = 1, 2, \cdots, N$, we use (4.7), (4.8) and (4.1) to design the following dynamic average consensus algorithm to estimate average desired power $p_a(t)$,

$$\dot{\hat{P}}_{a,i} = -\mu \sum_{j=1}^{N} a_{ij} \left(\hat{P}_{a,i} - \hat{P}_{a,j} \right) - \mu \left(\hat{P}_{a,i} - \frac{b_i}{\sum_{j=1}^{N} b_j} P^* \right) + \hat{S}_i \hat{P}_{a,i} + \mu \sum_{j=1}^{N} a_{ij} (V_i - V_j) \,, \tag{4.9a}$$

$$\dot{V}_i = -\mu \sum_{j=1}^{N} a_{ij} \left(\hat{P}_{a,i} - \hat{P}_{a,j} \right) + \hat{S}_i V_i, \tag{4.9b}$$

$$\hat{p}_{a,i} = \hat{Y} \hat{P}_{a,i}, \tag{4.9c}$$

where $\hat{P}_{a,i}(t)$ is the estimation values of the i^{th} battery for $\frac{1}{N} \sum_{i=1}^{N} \left(\frac{b_i}{\sum_{j=1}^{N} b_j} P^*(t) \right)$ and $\hat{p}_{a,i}(t)$ is the estimation values of the i^{th} battery for $\frac{1}{N} Y \sum_{i=1}^{N} \left(\frac{b_i}{\sum_{j=1}^{N} b_j} P^*(t) \right)$. Here, we notice that, even though $\sum_{j=i}^{N} a_{i,j}$ and $\sum_{j=i}^{N} b_{i,j}$ are defined for $j = 1$ to N, they can be computed over all $a_{i,j} \neq 0$ and $b_{i,j} \neq 0$, respectively. Hence, the information of N is not required. We also notice that

$$\frac{1}{N} \sum_{i=1}^{N} \left(\frac{b_i}{\sum_{j=1}^{N} b_j} P^*(t) \right) = \frac{1}{N} P^*(t) = P_a(t),$$
$$\frac{1}{N} Y \sum_{i=1}^{N} \left(\frac{b_i}{\sum_{j=1}^{N} b_j} P^*(t) \right) = Y P_a(t) = p_a(t),$$

which means that what $\hat{P}_{a,i}(t)$ and $\hat{p}_{a,i}(t)$ actually track are $P_a(t)$ and $p_a(t)$, respectively.

Next, we design, for each battery unit i, a dynamic average consensus algorithm to estimate the average unit state $x_a(t)$ for the discharging model by using (4.1), (4.7), (4.8) and (4.9),

$$\dot{\hat{X}}_{a,i} = -\mu \sum_{j=1}^{N} a_{ij} \left(\hat{X}_{a,i} - \hat{X}_{a,i} \right) - \mu \left(\hat{X}_{a,i} - P_i \right) + \hat{S}_i \hat{X}_{a,i}$$

$$+\mu \sum_{j=1}^{N} a_{ij} (W_i - W_j) \,, \tag{4.10a}$$

$$\dot{W}_i = -\mu \sum_{j=1}^{N} a_{ij} \left(\hat{X}_{a,i} - \hat{X}_{a,j} \right) + \hat{S}_i W_i, \tag{4.10b}$$

$$\dot{\hat{x}}_{a,i} = -\mu \sum_{j=1}^{N} a_{ij} (\hat{x}_{a,i} - \hat{x}_{a,i}) - \mu \left(\hat{x}_{a,i} - x_i \right) - \hat{Y}_i \hat{X}_{a,i} + \mu \sum_{j=1}^{N} a_{ij} (w_i - w_j) \,, \tag{4.10c}$$

$$\dot{w}_i = -\mu \sum_{j=1}^{N} a_{ij} \left(\hat{x}_i - \hat{x}_j \right) - \hat{Y}_i W_i, \tag{4.10d}$$

where $\hat{X}_{a,i}(t)$ and $\hat{x}_{a,i}(t)$ are the estimation values of the i^{th} battery for $\frac{1}{N} \sum_{i=1}^{N} P_i(t)$ and $\frac{1}{N} \sum_{i=1}^{N} x_i(t)$, respectively, and $P_i(t)$ can be given by

$$P_i(t) = \frac{x_i(t)}{\hat{x}_{a,i}(t)} \hat{P}_{a,i}(t).$$

Notice that, as with the implementation of (4.8) and (4.9), the information of N is not required in implementing (4.10). The dynamic average consensus algorithm to estimate the average unit state $x_a(t)$ for the charging model can be constructed by using similar way.

We can now implement the control algorithm (4.6) as follows,

$$p_i(t) = \frac{x_i(t)}{\hat{x}_{a,i}(t)} \hat{Y}_i(t) \hat{P}_{a,i}(t). \tag{4.11}$$

If the communication topology satisfies Assumptions 1.1 and 4.1, the battery dynamics satisfies (4.2) and (4.4), and the exosystem knows the information on the frequency components of the desired power, under the control algorithm (4.11) and the dynamic average consensus algorithms (4.9), (4.10) and (4.8), a battery energy storage system could accurately satisfy the charging/discharging power desired by the grid while accurately balancing the SoC of all its units without knowing the number of the battery units. Meanwhile, the resulting system possesses the robustness property when the communication is interrupted. When the communication is interrupted at time $t_{\mathrm{I}} > 0$, with subgroups of battery units remaining connected and containing at least one agent that has access to the information of the exosystem, the SoC of the battery units of each of these subgroups will reach balancing among battery units in the subgroup while all remaining battery units stop working. The total output power of the battery system still tracks the desired power accurately.

4.3 Simulation Results

In this section, we verify the effectiveness of the control algorithm (4.11) by simulation performed with a battery system consisting of six networked battery units.

The parameters of the battery units are $(C_1, C_2, C_3, C_4, C_5, C_6) = (180, 190, 200, 210, 220, 230)Ah$ and $(V_1, V_2, V_3, V_4, V_5, V_6) = (20, 20, 20, 20, 20, 20)\,V$. The initial SoC of the battery units are $(0.96, 0.89, 0.75, 0.8, 0.73, 0.88)$.

The communication topology of the system is shown in Fig. 4.1. In addition, only battery unit 1 has access to the desired charging/discharging power, i.e., $b_1 = 1$ and $b_i = 0, i = 2, 3, ..., 6$.

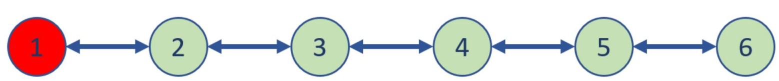

Figure 4.1: The communication topology of the battery system.

We will also illustrate the robustness of our algorithms against interruptions of the network connectivity (see Fig. 4.2).

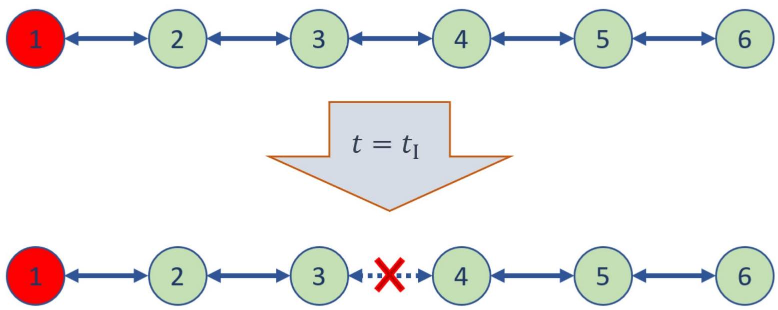

Figure 4.2: The communication topology of the battery system after interruption at $t = t_{\mathrm{I}}$.

We consider the discharge mode in the following simulation. The simulation for the charging mode can be carried out in a similar way.

Let matrices S and Y in the exosystem (4.1) be given by

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \; Y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.$$

Then the desired discharging power is given by

$$p^*(t) = (300\cos(t) + 300)\, W,$$

corresponding to $P^*(0) = [300,\ 300,\ 0]^{\mathrm{T}}$.

Shown in Fig. 4.3-4.6 are evolution of the SoC of all battery units during the discharging process, the total power of the battery system, the estimates of the average desired power by all battery units and the estimates of average units state by all battery units. It is observed that the SoC of all battery units reach balancing accurately and the total delivering power of the battery system tracks the desired power accurately.

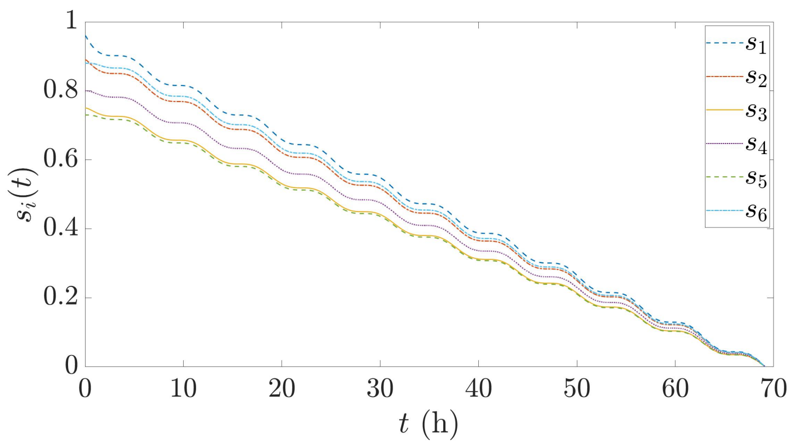

Figure 4.3: Evolution of SoC of all battery units.

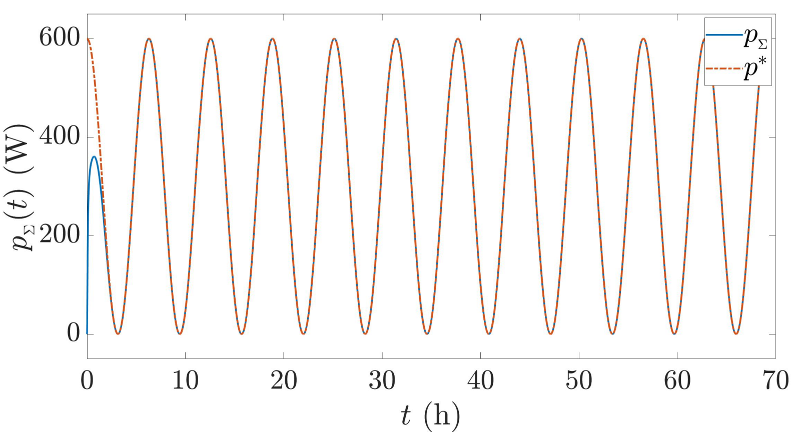

Figure 4.4: The total output power of all battery units and the desired power.

To illustrate the robustness of our distributed SoC balancing algorithm against network connectivity interruption, we consider the situation when the communication is interrupted at $t_{\mathrm{I}} = 25h$. After this interruption, a subgroup of battery units, units $1, 2$ and 3, remains connected with unit 1 having access to the information of the exosystem. Shown in Fig. 4.7-4.10 are evolution of the SoC of all battery units during the discharging process, the total power of the battery system, the estimates of the average desired power by all battery units and the estimates of average units state

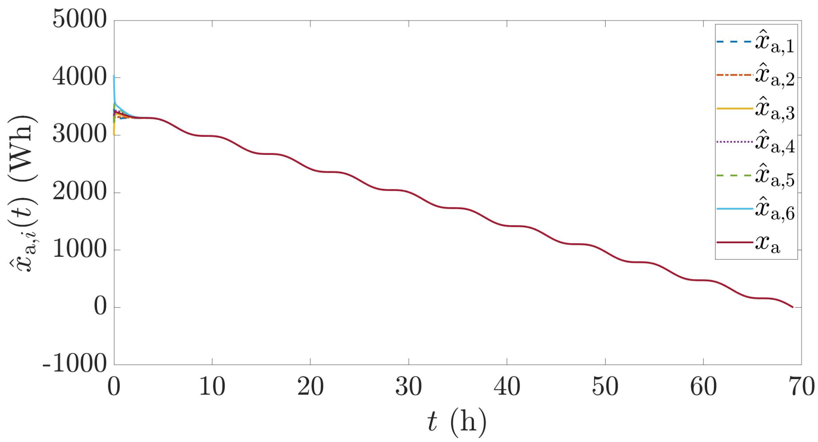

Figure 4.5: Estimates of the average unit state $\hat{x}_{a,i}$ by all battery units and the actual average unit state of x_a.

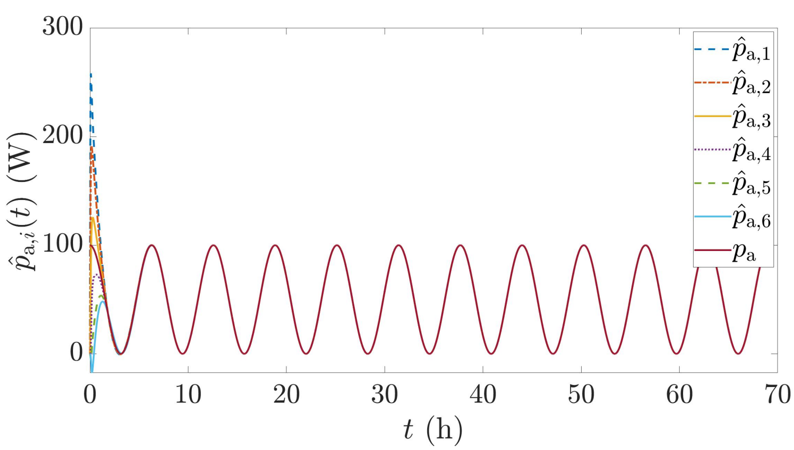

Figure 4.6: Estimates of the average desired power $\hat{p}_{a,i}$ by all battery units and the actual average desired power p_a.

by all battery units. It is observed that the SoC of the battery units in the subgroup of battery units $1, 2$ and 3 still reach balancing accurately and the remaining battery units stop working. The total output power of the battery system still tracks the desired power accurately.

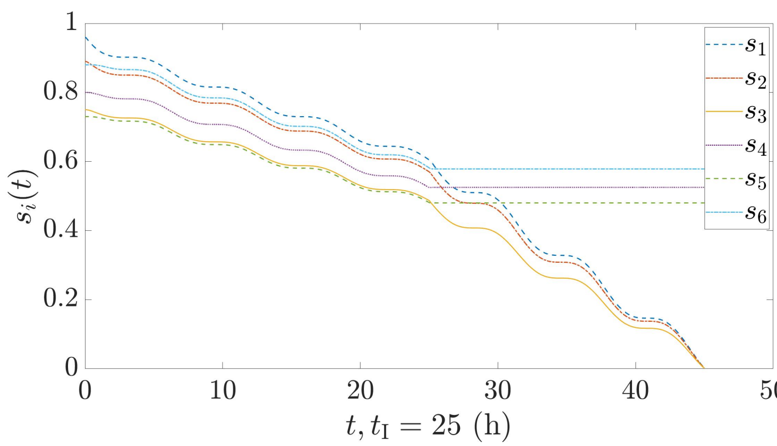

Figure 4.7: Evolution of SoC of all battery units.

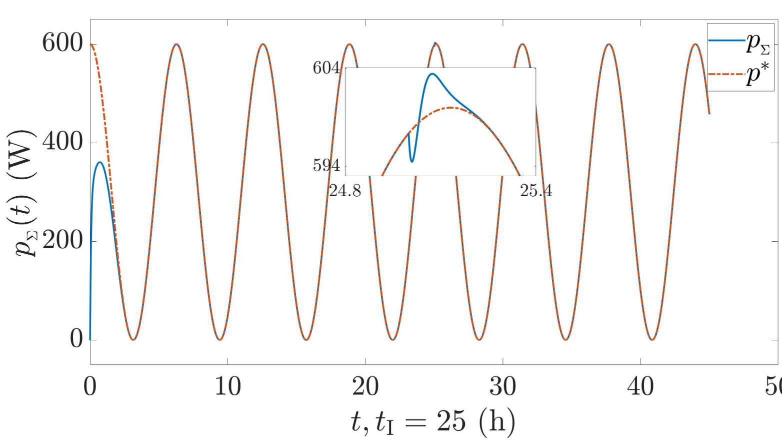

Figure 4.8: The total output power of all battery units and the desired power.

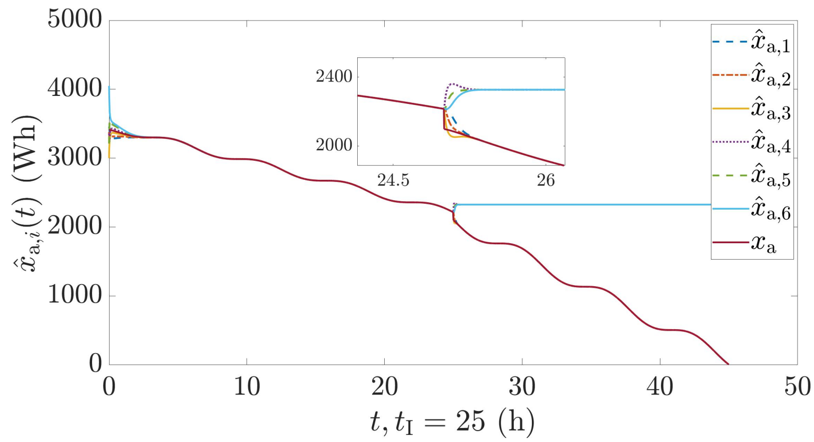

Figure 4.9: Estimates of the average unit state $\hat{x}_{a,i}$ by all battery units and the actual average unit state of x_a.

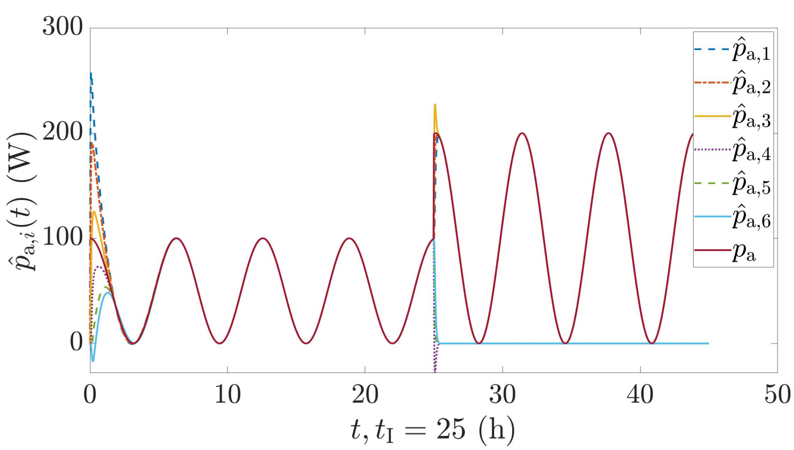

Figure 4.10: Estimates of the average desired power $\hat{p}_{a,i}$ by all battery units and the actual average desired power p_a.

Chapter 5

Conclusions

In this thesis, we have constructed distributed estimators for a group of agents operating on a connected directed communication network, in both the continuous-time and discrete-time settings. These distributed estimators reach consensus precisely at the average of the time-varying signals each associated with one agent. Such precise consensus is made possible by including the frequency information of time-varying signals in the estimators, which also makes the estimators robust to the interruption of the network connectivity. After the occurrence of an interruption, subgroups of connected agents that continue to reach consensus around the average of all signals as long as the signals are bounded and the later the interruption occurs the more accurate the consensus will be. To demonstrate their practical application, we apply our proposed distributed dynamic average consensus algorithms to a networked battery system to achieve accurate state-of-charge balancing while delivering the desirable total power accurately without knowing the number of the battery units, and to have robustness property when the communication is interrupted. When the communication is interrupted at time $t_\mathrm{I} > 0$, with subgroups of battery units remaining connected and containing at least one agent that has access to the information of the exosystem, the SoC of the battery units of each of these subgroups will reach balancing among battery units in the subgroup while all remaining battery units stop working. The total output power of the battery system still tracks the desired power accurately.

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