INTERPRETING AND ENACTING THE DANISH MATHEMATICS COMMUNICATIONS COMPETENCY: THE RELATIONSHIP BETWEEN POLICY, PEDAGOGY, CURRICULUM, AND UNDERSTANDING

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Abstract

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The purpose of this study is to examine how Danish teachers interpret the mathematics communications competency and how those interpretations are enacted in classroom practice. Denmark implemented mathematics process standards in 2003 and teachers and students in Denmark have had over a decade of working with those standards. This study provides insight into factors influencing how teachers interpret and implement oral and written mathematical communication in their classrooms. The results of this study can be used to inform mathematics communication instructional practice in the United States.

A grounded theory methodology was used to investigate two research questions: a) How do teachers interpret the Danish communications competency? and b) In what ways are those interpretations enacted in classroom practice? Data sources include observations, interviews with teachers and pupils, and classroom artifacts. Five themes emerged from the analysis of the data: understanding, communications, pedagogy, curriculum, and policy.

Two forms of mathematics understanding are described: procedural understanding – which includes understanding what to do to solve a mathematics problem, and understanding how to solve a mathematics problem, and connectional understanding – understanding why a problem is solved in a certain way. These two forms of understanding correspond directly with two levels of mathematics communications -

procedural and connectional. The corresponding levels of mathematics communication influence not only the types of mathematical understanding a pupil develops but also how a teacher assesses a pupil's degree of that understanding.

Education policies and the circumstances of teaching such as available instructional time and the structure of national assessments account for another aspect of how and why teachers interpret the mathematics communications competency. Two areas of pedagogy that relate to teachers' views of mathematical understanding and communications are beliefs about communicating in mathematics, and beliefs about the role of the teacher and student. A key factor in the enactment of classroom mathematics curriculum is not the specific curriculum materials that are used, but how they are used, and how they are used depends on a teacher's views of understanding.

The answer to the first research question is that teachers interpret the mathematics communications competency in a way that correspond with their beliefs and views of mathematics understanding as being either procedural or connectional. The answer to the second research question is that teachers enact the mathematics communications competency in classroom practice in ways that are largely consistent with their views of mathematics understanding as being either procedural or connectional. Mathematical communications is used in the classroom as a tool for both supporting and assessing different forms of procedural and connectional understanding. Implications of this study include reframing the discussion regarding classroom mathematics instruction as a continuum of mathematics understanding rather than one that emphasizes rote memorization and algorithms versus an expectation to *teach for understanding*.

DEDICATION

To my parents, Waverly and Cheryl Reames. You are my first and most important teachers.

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CHAPTER 1

INTRODUCTION

Mathematics teaching and learning is a topic of much attention among many people, including legislators, educators, and parents. News articles focus on teacher incentives and test scores (Chang, 2014) as well as changes in mathematics curriculum (Tucker, 2014). Social media sites such as Facebook and Twitter have numerous posts devoted to, and generally aimed against, standards initiatives such as Common Core (Lynch, 2014). These articles and social media posts often cite specific examples of classroom practice, such as worksheets or test questions (Torres, 2014) as examples meant to prove the writer's point that standards, such as Common Core, are fundamentally flawed.

Consider a video posted on You Tube of an Arkansas mother addressing the state Board of Education ("Arkansas mother obliterates Common Core in 4 minutes," 2013). In this video, Karen Lamoreaux states she represents 1,110 "parents, educators, and taxpayers in our state who have some very serious reservations about the Common Core initiative." Lamoreaux claims that, while parents and legislators were told Common Core was a set of "rigorous, college-ready, internationally-benchmarked standards that prepare our kids to compete in a global economy," Common Core is nothing more than "an empty sales pitch for corporations and government agencies to profit from our kids and sell them downriver in the name of saving education."

To illustrate her argument, Lamoreaux ("Arkansas mother obliterates Common Core in 4 minutes," 2013) ask the Board members, "Are you smarter than a Common Core fourth-grader?" and reads the following math problem: "Mr. Yamata's class has 18 students. If the class counts around by a number and ends with 90, what number did they count by?" She asks Board members to solve the problem and share how they solved the problem. An off-camera Board member gives an answer and, though viewers are unable to hear her answer or method of solving the problem, Lamoreaux responds by saying, "You know why? Because that makes sense, right? That's the way we were taught to do it." She holds up a photocopied paper and continues,

> This, however, is what Common Core expects our fourth-graders to do. If they solve it in those two steps, they get it marked wrong. They are expected to draw 18 circles, with 90 hash marks, solving this problem in exactly 108 steps. ... This is not rigorous. This is not college-ready. This is not preparing our children to compete in a global economy. Skipping rote memorization of multiplication tables is hindering their ability to master long-division and fractions later on in the semester, and now our children, who were testing in the 80th or higher percentile in math last year, are now coming home with Cs, D, and Fs on their report cards.

Lamoreaux's ("Arkansas mother obliterates Common Core in 4 minutes," 2013) main point centers around what the Common Core State Standards for Mathematics require of students. In her example, she describes a student's work being marked as incorrect unless a child draws a representation using circles and hash marks, and she says Common Core requires students to illustrate a division problem using this specific diagram, otherwise the problem must be incorrect, regardless of how a student solved the problem. Lamoreaux's central premise is that the Common Core State Standards require specific instructional and assessment practices.

A look at the Common Core State Standards, however, shows a different story. The grade 4 overview gives this description of division:

> Students apply their understanding of models for division, place value, properties of operations, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multi-digit dividends. They select and accurately apply appropriate methods to estimate and mentally calculate quotients, and interpret remainders based upon the context (Common Core State Standards for Mathematics, 2014, p. 27).

The specific standards state students should be able to:

> Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. For example, recognize that 700 ÷ 70 = 10 by applying concepts of place value and division (p. 29), and
>
> Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models (p. 30).

Not only do the Common Core standards not require students to draw a diagram using circles and hash marks, they actually state students should use "efficient procedures to find quotients" (p. 27) and "explain calculations using equations" (p. 30). These statements are not substantially different from the previous Arkansas mathematics standards that apply to this problem: "Demonstrate fluency with combinations for multiplication and division facts (12 x 12) and use these combinations to mentally compute related problems (30 x 50)" (K-8 Mathematics Curriculum Framework, 2004, p. 14), and "Solve simple problems using operations involving addition, subtraction, and multiplication using a variety of methods and tools (e.g., objects, mental computation, paper and pencil and with and without appropriate technology)" (p. 16). Both sets of curriculum standards say students should use a variety of methods and tools to solve division problems.

As an additional concern, Lamoreaux ("Arkansas mother obliterates Common Core in 4 minutes," 2013) states children will be unprepared for long division or fractions. Long division is not mentioned at all in the previous Arkansas mathematics standards but is in the Common Core standards, though not until grade 7. The Common Core standards also state that fourth-graders should "build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers" (Common Core State Standards for Mathematics, 2014, p. 30). Rather than separating student's understanding of division from their understanding of fractions, Common Core specifically links fractions with division. The previous Arkansas standards made no mention of linking fractions to any mathematical operations.

Lamoreaux ("Arkansas mother obliterates Common Core in 4 minutes," 2013) makes one particular statement that seems to sum up not only her main concern, but that of many of those posting comments about Common Core on social media: "Because that makes sense, right? That's the way we were taught to do it." At its center, the Common Core State Standards for Mathematics are based on a set of process standards that are likely quite different from what many parents and teachers experienced when they were in school. As a result, there is a tension point on the continuum between algorithms and rote memorization at one extreme and depth and understanding on the other.

Much of the current conversations in the United States about the tension between memorization and understanding are a result of the newly-implemented Common Core State Standards. The history of mathematics education in the twentieth century indicates that negotiation of the memorization/understanding continuum is not just a recent trend, but has been the focus of several major curriculum movements.

A Brief History of Initiatives in Mathematics Education

In the early part of the twentieth century, educators recognized that children's mathematical knowledge was often limited to reproducing rote mathematical procedures rather than any sort of true understanding (Thorndike, 1922). Thorndike notes this lack of understanding or even any sense of deductive thinking extends to textbooks as well as classroom practice: "one seems to sense in the better textbooks a recognition of the futility of the attempt to secure deductive derivations of those manipulations" (p. 67). The Progressive Movement in the 1920s developed in part as a reaction to the practice of rote teaching (Ellis & Berry, 2005). This movement emphasized the connection of learning to students' experiences and that student interest should be the guiding motivation for curriculum. Many educators saw the Progressive Movement as too radical a change, both in the way it removed both authority from teachers and organization from the curriculum.

The Progressive Movement's emphasis on student experience and interest eventually led to tracking, with the majority of students placed in vocational or other perceived lower-level tracks, and a decline in the number of students taking algebra (Ellis & Berry, 2005). The decline in numbers of students taking advanced mathematics courses was, by the middle of the century, seen as a national security issue, particularly after the Soviet Union launched *Sputnik I* in 1957. To counter this perceived threat to the security of the United States, the National Science Foundation (NSF) funded a series of *New Math* initiatives and textbooks. Many of the New Math initiatives of the mid-twentieth-century were seen as too abstract and unrelated to real-world contexts. This poor reception led to calls to return to basic skills. This *back-to-basics* movement in the

1970s emphasized decontextualized skills-based mathematics. The basic-skills approach, however, was soon criticized for many of the same reasons Thorndike criticized mathematics education in the 1920s: with rote learning and de-contextualized content, students fail to understand the mathematics they were being taught.

To counter the back-to-basics movement, the National Council of Teachers of Mathematics (NCTM) laid the foundation for standards-based mathematics teaching and learning. In 1989, the NCTM released *Curriculum and Evaluation Standards for School Mathematics* which endorsed the idea that students should make sense of mathematical concepts and connections (NCTM, 1989). These ideas were further amplified by the NCTM in 1991 when they published *Professional Standards for Teaching Mathematics*, and again in 1995 with the release of *Assessment Standards for School Mathematics* ("Standards Overview," 2014). These three documents provide focus and coherence to mathematics education in the United States. In 1998, the NCTM published an updated resource, *Principles and Standards for School Mathematics*, combining and expanding upon the three earlier standards documents.

The national movement towards standards-based instruction has led to the development of the Common Core State Standards for mathematics and their implementation in the majority of the states. States not participating in the Common Core movement also have their own sets of curricular standards. For example, Virginia Department of Education has developed the Standards of Learning (Board of Education, Commonwealth of Virginia, 2009).

The development of standards-based mathematics instruction is not limited to the United States, but is an international development. In 1988, the Education Reform Act introduced a national curriculum to schools in England and Wales (Ball & Bowe, 1992). Similar national standards for mathematics were developed in Australia and New Zealand in the early 1990s (Priestly, 2008). In 2003, Denmark introduced the Common Objectives, a set of standards for all school subjects (Ramboll, 2011), which have been revised and clarified in 2013 ("Agreement between…," 2013).

Each of these sets of curricular standards includes not only content standards, but also process standards. One of the first examples of mathematics process standards are those appearing in the *Principles and Standards for School Mathematics* ("Process Standards," 2014). These five standards - problem solving, reasoning and proof, communication, connections, and representation - are methods by which students learn and understand mathematical content. The NCTM Process Standards are reflected in the *Standards for Mathematical Practice* found in the Common Core State Standards for Mathematics ("Common Core State Standards - Standards for mathematical practice," 2014). Internationally, process standards appear in both the National Curriculum Framework in England (Department for Education, 2013, p. 88), as well as in the form of competencies in the Danish national standards ("Matematik: EMU," 2014). These examples clearly indicate a common movement towards using mathematics process standards in the teaching and learning of mathematics.

Caught at the Tension Point

When Lamoreaux ("Arkansas mother obliterates Common Core in 4 minutes," 2013) stated, "Because that makes sense, right? That's the way we were taught to do it," she was describing the tension between *what parents are used to seeing in math* and *what* *it looks like to use process standards to learn math*. On one end of the continuum is a focus on algorithms and memorization and on the other end, a focus on understanding.

An example of memorization and algorithms-focused teaching is how many students learn long division. By memorizing a mnemonic such as “Dracula Must Suck Blood” or “Dead Monkeys Smell Bad,” (“Mnemonic,” 2014), children are told to remember the steps to use: divide, multiply, subtract, bring down. Other examples of these mnemonics are “Everybody's Daddy, Mother, Sister, Brother, Reunion, Celebration,” to remember “Estimate, Divide, Multiply, Subtract, Bring Down, Redo (all steps), Check,” or “Does McDonald's Sell CheeseBurgers and Shakes?” for “Divide, Multiply, Subtract, Check that the divisor is larger than your remainder, Bring down the next number, and Start all over again.” In many cases, students are taught to memorize the algorithm rather than understand the concept of division.

An understanding-based approach to long division uses long division as a tool for efficient calculation, but in combination with an understanding of place value and properties of operations, and students will have explored a range or representations, such as manipulatives and arrays, and the relationships between these things. Students are also encouraged to explain their reasoning. Teaching and learning for understanding looks far different from learning a mnemonic phrase in isolation.

Though teaching is contextual and situational, mathematics standards are not. Standards are sets of policy documents that exist outside the classroom context. As teachers work to implement the standards, they must make frequent, daily interpretations and decisions as they negotiate the tension point on the continuum between memorization and understanding. These decisions are situational and contextual and based on a multitude of factors specific to their individual classroom. Simply following textbooks and other curriculum materials is not sufficient because they are generally planned and written for hypothetical students (Goodlad, Klein, & Tye, 1979) rather than for a specific set of individuals.

Examining a Process Standard

There are several common themes in the process standards from both the United States and other countries. One theme focuses on specific practices such as using accurate and precise mathematical language, expressing oneself clearly to others, justifying conclusions, analyzing and evaluating the mathematical thinking of others, and being able to do these practices orally, visually, and in written form. These specific practices are all part of mathematical communication.

The NCTM Process Standards include communication as one of the five standards ("Process Standards," 2014). As described by NCTM, teachers at all levels, from prekindergarten to twelfth grade, should enable students to communicate mathematically by:

- Organizing and consolidating their mathematical thinking through communication;
- Communicating their mathematical thinking coherently and clearly to peers, teachers, and others;
- Analyzing and evaluating the mathematical thinking and strategies of others; and
- Using the language of mathematics to express mathematical ideas precisely.

These same processes are reflected in the Common Core State Standards for Mathematics in the set of Standards for Mathematical Practice. These process standards state:

> Mathematically proficient students are expected to understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments (Standards for Mathematical Practice, 2014).

Communication as a process standard also appears in international mathematics curriculum standards. Though not described using the terms *practice standards* or *goals*, the National Curriculum Framework in England includes similar language about the aims of the curriculum and how teachers should support students' learning of mathematical content,

> "The national curriculum for mathematics aims to ensure that all pupils: … reason mathematically by following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language…." (Department for Education, 2013, p. 88).

In Denmark, the Mathematics Competencies include a communications standard: "The communication competence is about students being able to express themselves and understand others' communication about mathematical topics, including oral, written and visual forms of communication," ("Matematik: EMU," 2014).

Though each of these four examples of state or national curriculum process standards include language about mathematical communication, in many cases it is up to the individual teacher to interpret these standards for herself or himself. There are a number of possible ways of interpreting a phrase such as *students should be able to express themselves mathematically*. As described earlier, these interpretations are made at the tension point between different ends of the procedure/understanding continuum. The way a teacher interprets this specific standard likely impacts a range of classroom practice and, ultimately, student achievement.

Though the process standards that form part of the Common Core State Standards have resulted in increased attention in the United States to the tension between memorization and understanding in mathematics, process standards have been part of the curriculum in Denmark since 2003 (Ramboll, 2011). Results from the 2012 PISA test indicate that the mean performance of 15-year-olds in Denmark is higher than that of their counterparts in the United States (Organisation for Economic Co-operation and Development, 2014). Additionally, data from the 2012 PISA tests show that students who seek mathematical explanations, can connect ideas, and solve complex problems – behaviors closely linked to processes and process standards - score higher in mathematics than students who do not exhibit these behaviors. For these reasons, it is reasonable to look at mathematics practices in Denmark to inform practice in the United States.

The Danish Mathematics Competency for Communication

The Danish communication process standard, called a competency, is given below. As many of the examples in this study are in Danish, where appropriate I will give the original Danish accompanied by an English translation. Each English translation is not necessarily a literal, word-for-word translation, but instead a translation intended to give the sense and meaning of the original, while fitting English sentence structure.

Kommunikation	**Communication**
Kommunikationskompetence handler om at kunne udtrykke sig og forstå andres kommunikation om matematikholdige emner, herunder mundtlige, skriftlige og visuelle kommunikationsformer.	The communication competence is about students being able to express themselves and understand others' communication about mathematical topics, including oral, written and visual forms of communication.
I indskolingen er der specielt fokus på mundtlige og visuelle kommunikationsformer med brug af enkle fagord og begreber. På mellemtrinnet kommer der fokus på skriftsproget også, og eleverne arbejder med at forstå og udtrykke sig med et mere præcist fagsprog. I udskolingen øges denne grad af præcision yderligere, samtidig med at der kommer øget fokus på brugen af det matematiske fagsprogs begreber og notation, såvel skriftligt som mundtlig.	The early school years are especially focused on oral and visual forms of communication with the use of simple mathematics terms and concepts. By middle school, students will focus on written language also, and students are working to understand and express themselves in a more precise terminology. In early adolescence, students increase the degree of precision further, while there will be an increased focus on the use of the mathematical concepts and notation of technical terms, both written and oral.

It is quite possible for teachers to interpret the communications competency in different ways. A result of differing interpretations is different types of classroom learning activities, and quire likely, different learning outcomes.

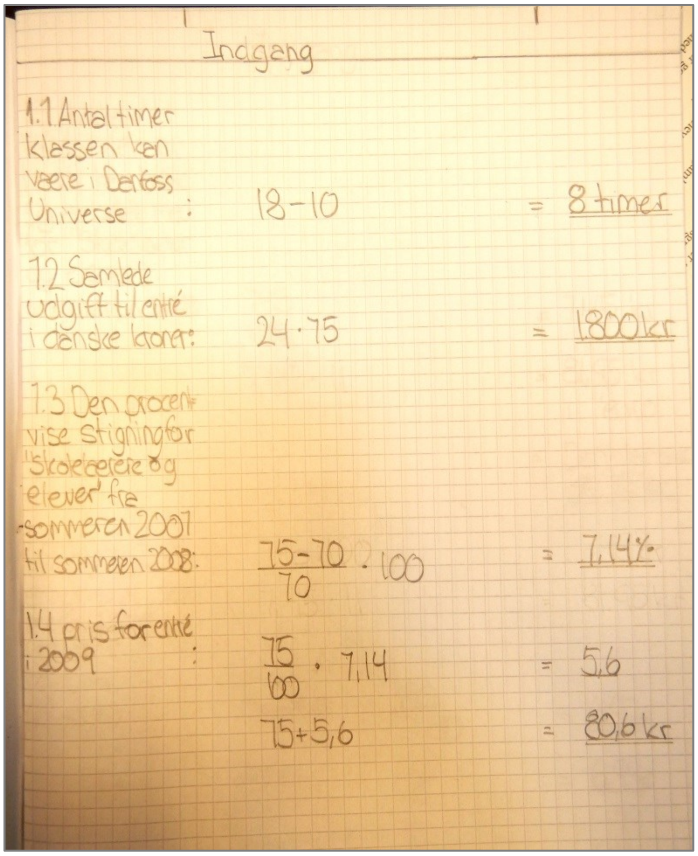
Consider two different seventh-grade teachers, Laura and Birgit. Laura views written language as a tool for student expression and asks her students to explain and justify their problem-solving process. As a way of promoting connections between mathematical ideas and concepts, Laura gives students mathematical tasks that have more than one possible solution method in order to push students into making their own decisions about what mathematics to use to find a solution.   Their solutions include diagrams and other mathematical representations. To help organize their thoughts and ideas, students discuss mathematics problems in small groups and as a whole class. Laura models precise terminology and expects her students to use precise mathematical language as a means to conveying their ideas accurately to other people. Laura's students regularly write paragraphs in which they describe not only how they arrived at a solution, but why they made the mathematical decisions they made. Their paragraphs often include graphs and diagrams to accompany their explanations. The students frequently share these paragraphs with each other and have an opportunity to ask questions about their classmates' ideas solutions.

Birgit encourages her students to be as precise as possible. In her mind, precision is closely related to order and structure. To help her students structure their solutions clearly, Birgit models a problem-solving framework in which students first identify the key terms in a problem, then they show the mathematical procedure for solving the problem, and finally they present the answer.   To make sure students use the correct terminology, Birgit emphasizes to her class that they must make certain to include the proper units with their solution. Birgit gives students feedback regarding the structure of their solution to help them place each type of information in the proper location. Students learn to express themselves mathematically through repetition of this process. To Birgit, this method of problem-solving allows the mathematics to speak for itself.

These two hypothetical teachers, Laura and Birgit, interpret the same communications competency quite differently, based on a number of different factors.

As a result, students' learning opportunities in each classroom are quite different. Laura's students have opportunities to explore different ways of using and connecting mathematical ideas, while Birgit's students have opportunities to apply repeated procedures as they practice individual concepts. These different opportunities to learn are likely to result in quite different outcomes for their students. For example, where Birgit's students have learned highly-structured methods for efficiently solving specific types of problems, Laura's students are possibly better prepared to solve problems that they have not seem before. Laura and Birgit interpret the communications competency at very different points along the procedure/understanding continuum.

Now consider Martin, an actual seventh-grade teacher in Denmark. Martin says that many of his students have problems solving word problems: "It's the text that gives them problems." To help solve the problem of using text in mathematics, Martin teaches his students a three-column method of solving problems as shown in figure 1.1. He describes this problem format as "a good example of writing mathematics." In the left-most column, students' "text has to have something to do with the answer." The middle column is "for the working" and the last column is for the answer: "Text, working, and then the results." Martin says it often takes students until ninth grade before he considers they are successful at this method, but that, once students understand it well, it shows "how the assignment communicates with you – so you actually could read the assignment" without looking at the problem in the textbook.



Input

1.1 Number of hours the class may be in Danfoss Universe	18-10	= 8 hours
1.2 Total cost of admission in Danish kroner	24 x 75	= 1,800 Kr
1.3 The percentage increase for "teachers and students" from summer 2007 to summer 2008.	$\frac{75-70}{70} \times 100$	= 7.14 %
1.4 Price for admission in 2009	$\frac{75}{100} \times 7.14$	= 5.6
	75 + 5.5	= 80.6 Kr

Figure 1.1. Martin's three-column method of solving problems

Purpose of Study

These examples show how the same communications competency can be interpreted in different ways. Differing interpretations of the competence have a potential impact on wider learning activities and, quite likely, student outcomes. This study considers the following questions: How do teachers interpret the Danish communications competency? and In what ways are those interpretations enacted in classroom practice?

Significance

This study will provide insight into how curricular content, educational policy, and classroom pedagogy influence how teachers interpret and implement oral and written mathematical communication in their classrooms. It will also look at the ways in which these different interpretations of the same communications competency can impact classroom instruction. International rankings such as PISA indicate that students in Denmark outperform students in the United States (Organisation for Economic Co-operation and Development, 2014). Denmark implemented mathematics process standards in 2003 and teachers and students in Denmark have had over a decade of working with those standards. Therefore, it is likely that the results of this study can be used to inform mathematics instructional practice in the United States.

Definition of Terms

Competence. The term competence is "someone's insightful readiness to act in response to the challenges of a given situation" (Højgaard, 2009, p. 226).

Concept. At varying psychological levels, concept development involves a process of naming, classifying, using abstractions, utilizing examples and non-examples, definitions, and shared cultural experiences (Skemp, 1987). In this dissertation, recognizing both the epistemological sense of the term as well as more commonly-used senses of the term, I use the *concept* in a general sense that means *an idea of something*.

Curriculum. This term has different meanings, depending on the context. I describe several of these meanings and contexts in Chapter 2. In general, unless otherwise specified, curriculum will encompass textbooks, additional learning resources

– including online, teacher support materials, learning activities, and classroom assessments.

Mathematical competence. This is “the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role” (Niss, 2003, pp. 6-7). It can also be defined as a person’s “insightful readiness to act in response to a mathematical challenge of a given situation (Højgaard, 2009, p. 226).

Mathematical competency. This is defined as “a clearly recognisable and distinct, major constituent of mathematical competence,” (Niss, 2003, p. 7).

Pedagogy. The term pedagogy means “the process through which knowledge is produced” (Lusted, 1986, p. 2).

Policy. For this dissertation, the term policy is used to encompass the circumstances of teaching imposed on teachers by government agencies. Examples of educational policies are national curriculum standards, regulations regarding teacher licensure, and government mandates about instructional time.

Problem. As used in this dissertation, the term *problem* refers to a mathematics assignment set for pupils by a teacher. These mathematics problems might range from basic computational mathematics such as *56 ÷ 7* to more complex assignments such as *How much does is cost when you use toothpaste on a toothbrush?* The term *mathematical task* is sometimes used in the literature to refer to “a problem or set of problems that focuses students’ attention on a particular mathematical idea (content standards) and/or provides an opportunity to develop or use a particular mathematical habit of mind” (McCallum, 2011, slide 30) or mathematics process standard. The distinction between *mathematical problem* and *mathematical task* is often found in the context or intent of the specific assignment and therefore the same assignment might be used in an isolated manner or in a context designed to develop specific process skills or competencies.

Process standards. Mathematical process standards are the mathematical processes students use to acquire and apply mathematical content knowledge. These may alternately be referred to as goals, practice standards, or competencies.

Standards-based curriculum. A standards-based curriculum is one in which instruction and assessment are based on a set of knowledge and skills students are expected to learn and understand ("Standards-Based Definition," 2014).

Organization of Study

In this chapter, I presented two different ways teachers interpret the same curricular policy statement as an introduction to the problem. I outlined the purpose and significance of the study and explained key terms. In Chapter 2, I provide an overview of the Danish educational system as well as a comprehensive review of the literature on three main areas and their impact in the mathematics classroom: policy, content, and pedagogy. This literature review provides a framework for my study, and my research questions are presented at the end of Chapter 2. In Chapter 3, I describe the methodology I used for collecting my data as well as the proposed methods of analysis. Chapter 4 presents the major findings of the study in the context of class profiles. Finally, Chapter 5 provides a summary of the study, implications, recommendations for future research, limitations of the study, and conclusions.

CHAPTER 2

REVIEW OF THE LITERATURE

In this chapter, I present a review of the literature and an outline of the need for this study. I provide a brief history and overview of the Danish education system in order to provide context for my study. I explain my theoretical framework and present a comprehensive review of the literature on three main areas and their potential impact in the mathematics classroom: policy, content, and pedagogy.

A Brief History of Education in Denmark

Education in a society is shaped by a range of factors, including history and cultural influences, as well as political, social, and economic change (Hoffman, 2000). Danish education has been strongly influenced by the Nordic ideas of democracy. Democracy in the Nordic countries, which include Denmark, Iceland, Norway, Finland, and Sweden, "is not only a form of government but also comprises social and economic democracy as well as the democratic principles underlying justice, education, and culture, etc." (Andersen, 1981). These democratic ideals are deeply rooted in Nordic culture and references to democratic decision-making assemblies as early as the ninth century, with the earliest reference to such an assembly in Denmark being a treaty from 811 (Líndal, 1981). In schools, democracy is both an aim of education as well as a means of achieving other educational objectives (Sysiharju, 1981). The principles of democracy are reflected in the history and development of formal education in Denmark, particularly in the reforms made in the mid-20th century.

The history of schools in Denmark begins in the Middle Ages. Much of the development of the education system in Denmark parallels similar movements in the other Nordic countries (Sysiharju, 1981). The first school is said to have been established by the monk Ansgar for twelve serf boys in 826 (Boje, Borup, & Rützebeck, 1932). The Catholic Church later established grammar schools near its cathedrals, with the primary purpose of educating future priests (Sysiharju, 1981). After completing grammar school, the top pupils often moved to universities in central Europe to continue their studies. The Catholic Church founded first Nordic universities in Uppsala in 1477 and Copenhagen in 1479, which allowed Nordic students the option of continuing their studies closer to home. Though only a very small part of the population attended school, the grammar schools and universities provided one of the only paths for upward social mobility.

In the early part of the 16th century, the Reformation helped lay the foundation for general literacy skills the entire population, both men and women (Sysiharju, 1981). The kingdom of Denmark-Norway-Iceland broke from the Catholic Church and became Evangelical Lutheran. This national conversion to Protestantism brought with it an emphasis on reading the Bible (Munck, 2004), and with it, the need to teach people how to read. Though formal schools were not yet established, the Lutheran Church encouraged its clergy and other literate members to provide instruction to the population (Sysiharju, 1981), this instruction centered mainly on reading and catechism (Boje et al., 1932).

During the 17th and 18th centuries, elementary schools were established by local governments in towns and cities and provided instruction in reading, writing, arithmetic, and religion (Sysiharju, 1981). These elementary schools were not a replacement to the grammar schools, but were essentially a separate educational path entirely. Slowly, elementary schools were established in rural areas as well. Though the legislation took time to enact across the entire country, in 1814, Denmark enacted a seven-year system of compulsory education for children between ages seven and 14 (Boje et al., 1932; Sysiharju, 1981). This education, however, was not required to take place in schools and parents were allowed to home-school their children (Patrinos, 2001). Education past the elementary and grammar school levels was formalized into the *gymnasium* or high school in an Act of 1809 (Boje et al., 1932).

By the late 19th century, both elementary schools and grammar schools had become well-established in Denmark (Sysiharju, 1981). Girls and boys both had access to education by way of compulsory elementary schools, though grammar schools remained mainly to prepare students for further university study. This two-track elementary/grammar school structure reflected the social divisions in wider Danish society: although extremely bright lower-class children had the opportunity to move from elementary schools into grammar schools and further education, in general the grammar schools were for the upper class, and the elementary schools were for the lower class.

In the early 20th century, the elementary and grammar school systems became somewhat more linked and, in principle, children could spend their first four to six years in elementary school before entering grammar school for their remaining years of compulsory schooling (Sysiharju, 1981). In 1903, further links were made between elementary schools at one end and the secondary grammar schools and gymnasiums at the other by the formation of *Enhedsskole* or middle schools. (Boje et al., 1932). By the mid-20th century, increasing numbers of students were attending grammar schools as a result of higher overall standards of living throughout Denmark. At the same time, industrialization of many trades meant the existing elementary education was no longer sufficient (Sysiharju, 1981). Throughout the Nordic countries, national governments began a process of school reform.

In Denmark, major school reform was legislated in 1975 (Sysiharju, 1981). Instead of two systems of schools, there was now a single nine-year system of compulsory comprehensive schools, called the *Folkeskole* ("The Folkeskole," 2008). Children began school at age seven and would remain in the same school for nine years. This system of basic, primary schooling was provided by the government at no expense to families. Primary schools offered a standard curriculum that varied little across the country and made specific efforts to meet the needs of children with special needs within the same schools and classes as other children. This focus on providing approximately the same instruction to all children was specifically in place as a way of attempting to avoid a school that reflected the inequality in the local community.

There are several general aims of the 1975 system of compulsory comprehensive education in Denmark and Nordic education in general (Sysiharju, 1981). These aims include student self-realization, an emphasis on equality of educational opportunities, and preparation of students for participation in a democratic society. These main aims are described in a set of principles in 1977:

- A pupil should be able to remain in the same "unstreamed" class throughout the whole of the compulsory school course.

- The pupils' self-realization should be achieved by the individualization of teaching within the class framework and by a freedom to choose between subjects.
- Some differentiation within individual subjects will continue, but the pupils' choice of various syllabuses must not have in itself any influence on their later choice of further education or training (as cited in Sysiharju, 1981).

Of particular note in these principles are the ideas that students are in unstreamed, heterogeneous class groupings, and that any differentiation should take place within those classes rather than in separate, streamed classes. Further, in Denmark, all subjects are compulsory until eighth-grade, at which point students could begin to have a certain amount of streamed subjects as well as elective subjects.

The Danish educational reforms of the 20th century continued and added to secondary school options. The non-compulsory *gymnasium* continued to additional education beyond the compulsory nine years (Sysiharju, 1981). In 1966, a more flexible secondary option, the HF *(Højere forberedelseseksamen)* was introduced, followed in 1972 by the eight EFG *(Erhvervsfaglige grunduddannelser)* vocational tracks.

Further changes to the Education Act have made slight modifications to the *Folkeskole* system. Currently, the *Folkeskole* consists of a pre-school class (one year), primary and lower secondary education (nine years) and a non-compulsory 10th form year, ("The *Folkeskole*," 2008). Education between the ages of 6-7 and 16 is compulsory but may take place in the *Folkeskole*, in a private school, or at home. The aims of the *Folkeskole* remain very much as they were in 1975: student self-realization, an emphasis on equality of educational opportunities, and preparation of students for participation in a democratic society.

1. The Folkeskole is, in cooperation with the parents, to provide students with the knowledge and skills that will prepare them for further education

and training and instill in them the desire to learn more; familiarise them with Danish culture and history; give them an understanding of other countries and cultures; contribute to their understanding of the interrelationship between human beings and the environment; and promote the well-rounded development of the individual student.

2. The Folkeskole is to endeavour to develop the working methods and create a framework that provides opportunities for experience, in-depth study and allows for initiative so that students develop awareness and imagination and a confidence in their own possibilities and backgrounds such that they are able to commit themselves and are willing to take action.

3. The Folkeskole is to prepare the students to be able to participate, demonstrate mutual responsibility and understand their rights and duties in a free and democratic society. The daily activities of the school must, therefore, be conducted in a spirit of intellectual freedom, equality and democracy ("*The Folkeskole*," 2008).

Students in *Folkskole* have classes in three areas: humanities subjects (including Danish, English, Christian studies, and history and social studies), practical/creative subjects (including physical education, music, visual arts, design, and home economics), and science subjects (including mathematics, natural sciences and technology, geography, biology, and physics/chemistry) ("The *Folkskole,*" 2008). Additional topics such a road safety, health and sexual education, and educational, vocational, and labor-market orientation are also required. The Ministry of Education establishes regulations, including objectives for each subject area. These subject-area objectives are published for each subject as part of the *Fælles Mål* or Common Goals.

Students in the *Folkskole* are grouped into classes based on age, not ability ("The *Folkskole,*" 2008). Classes typically range in size from 20 to a maximum of 28 or 30. The concept of "school failure is an almost non-existing phenomenon" (p. 3). In this context, *school failure* refers to pupil-retention; only in very rare situations are students not moved ahead to the next grade with the rest of their class. In principle, differentiation is emphasized, but is to be done within a student's class or team consisting of students from different classes of levels, though students must be taught in their own class for the majority of the time. Each class has a class teacher who is responsible for both the academic and social welfare of students in the class. The class teacher teaches at least one academic subject to that class, and other teachers teach additional academic subjects to that class. Not only does the class remain together for several years of school, the class teacher remains with that class as well (Morrill, 2003). In some cases, the class and their class teacher remain together for the entire nine years of primary and lower secondary education. In other cases, schools make the decision to keep classes and teachers together for shorter periods of time, for example grades 1 through 4 and 5 through 9, or grades 1 to 3, 4 to 6, and 7 to 9.

Evaluation of student progress is generally done by subject teachers and the results are expected to inform further lesson planning and teaching ("The *Folkskole,*" 2008). Each *Folkeskole* student must have a written plan in place that details on-going evaluations in each subject as well as sets the course for further action based on those evaluations ("Evaluation, tests, and student plans," 2014). These written plans are to be updated by schools at least once each year. Schools are required to keep parents informed of student progress. National testing takes place at several levels. For example, Danish tests are given in grades 2, 4, 6, and 8, while mathematics testing takes place in grades 3 and 6. These tests are adaptive and computer-based. School-leaving examinations are required in seven subjects at the end of ninth grade. These examinations are to ensure students leaving the *Folkskole* have sufficient foundation for further education. Students who have not achieved their desired levels by the end of ninth-grade or who feel the need for additional qualifications may undertake a further, voluntary tenth-grade year of education, with examinations at the end of that year. Individual test results are kept confidential ("Evaluation, tests, and student plans," 2014; Patrinos, 2001), but school averages for Danish, English, physics/chemistry, and mathematics are available to the public.

Students who have completed their basic education in the *Folkeskole* may continue their education in one of four upper secondary education programs ("Four upper secondary education programmes in Denmark," 2014). The HF or Higher Preparatory Examination program takes two years and is open to students who have completed grade 10. The HF program emphasizes both theory and practice in natural science, social science and humanities. The remaining three programs each take three years and are open to students who have completed ninth grade. The STX or Gymnasium program is considered a general preparation in the natural science, social science and humanities and is intended to lead to higher education. The HHX or Higher Commercial Examination Program is focused on business economics, socioeconomics and foreign languages. The HTX or Higher Technical Examination program focuses on technical and natural sciences. Students also have the option of entering other types of vocational training ("Overview of the Danish education system," 2014). A graphic describing the Danish education system is provided in figure 2.1.

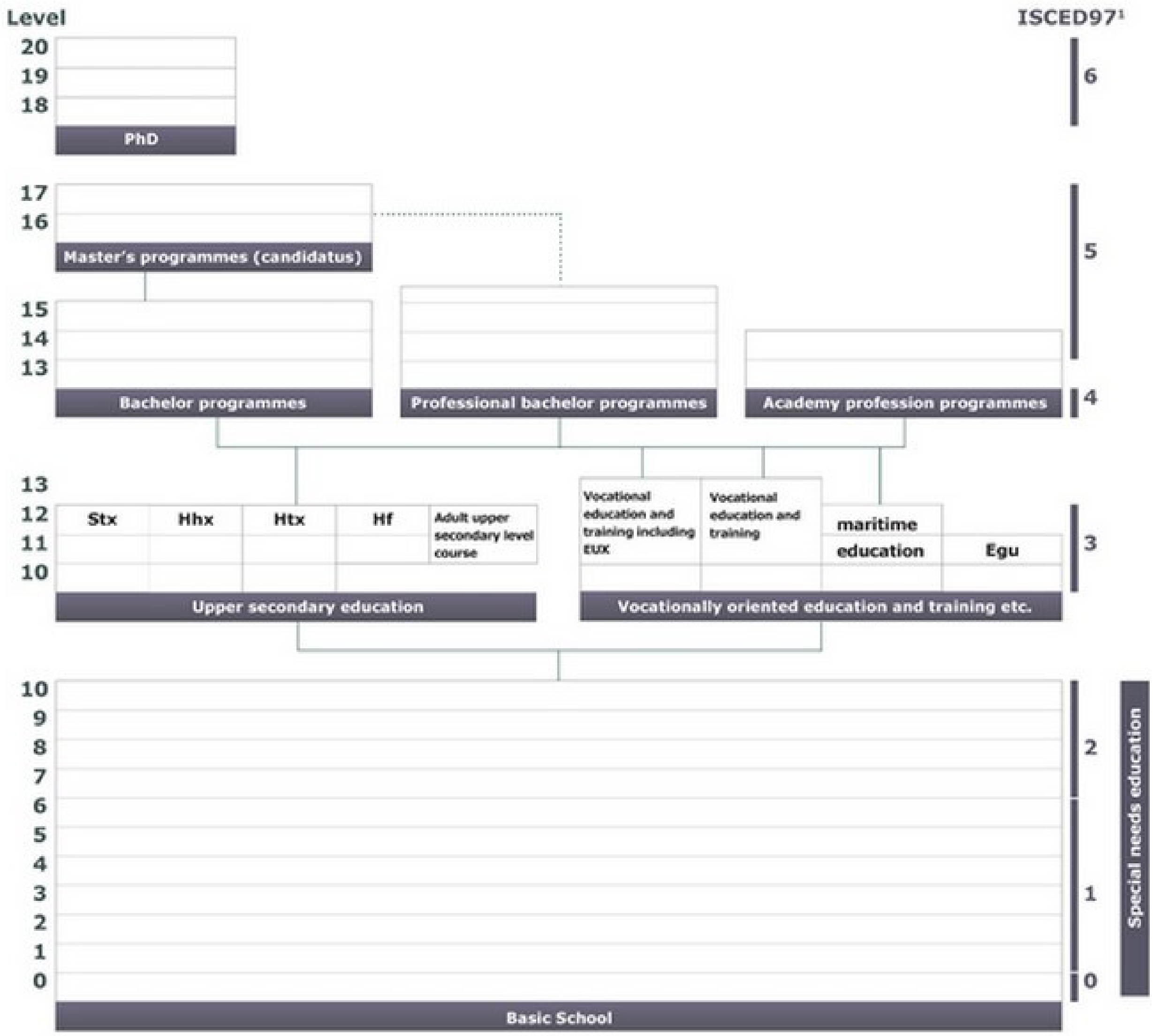


Figure 2.1. Overview of the Danish Education System
From "Overview of the Danish education system: Ministry of Education," (2014).

An additional type of Danish school is called the *free school* – a private school outside the direct control of the government (Patrinos, 2001; Wiborg, 2010). The 1915 Constitution of Denmark required compulsory education rather than compulsory school attendance and parents have three options for providing this education: at public primary and secondary schools, at private schools, or at home ("Private Schools," 2014). The right to open free schools was legislated in 1855, and beginning in 1899, Danish free schools received government funding (Boje et al., 1932; Patrinos, 2001). In the 2008-2009 school year, 14.2% of all Danish children (95,931 out of 675,588 students) at the primary-school level and about 6% of secondary school pupils attended private schools ("Private Schools," 2014). Current legislation requires not only that the educational content of private education be consistent with that of public education, but that private schools also receive government funding that matches funding given to public schools, subtracting school fees paid by parents. Private schools must be self-governing and run as non-profit organizations. Though not part of the *Folkeskole* system, many private schools nonetheless follow the *Folkeskole* model, particularly in regards to curriculum and class organization.

An agreement between Denmark's five main political parties in 2013 made several changes to the *Folkeskole* system that are being implemented starting in the 2014-2015 school year ("Agreement between…," 2013). Several key aspects of this agreement are a longer school day for students and increased numbers of lessons per week in certain grades and subjects, including Danish and mathematics. In grades 4 through 9, students will receive one additional 45-minute mathematics lesson a week, for a total of five 45-minute weekly mathematics lessons. This time, however, can be allocated flexibly throughout the week, for example as two 90-minute lessons and one 45-minute lesson, or as three 75-minute lessons. The *Fælles Mål* (Common Goals) have been clarified and simplified to provide additional support for teachers as they enact the standards in their classrooms. The agreement also clarifies provisions regarding class formation (in which students are taught primarily in their class groups).

In a number of ways, the history of teacher preparation in Denmark parallels that of the development of the school system, from relatively informal teaching and learning to a much more structured system (European Commission, 2006/07). Until relatively recently in Danish history, few specific requirements existed specifying how teachers should be prepared. The Teacher Training Act of 1818 stated,

> There is no doubt that those who are themselves born of country folk are best fitted to be village school teachers, for not only are they more readily accepted by country people but also they are better able to put up with the primitive conditions which country teachers have to accept (p. 100).

Village schools and their village teachers were the focus of Danish teacher training until the Teacher Training Act of 1954. This act formally established teacher-training college programs and stated primary teachers should be able to teach at all levels of the basic school (grades 1 to 10). A further Teacher Training Act of 1966 revised the teacher-training college program to make it more academic, and in 1985 the length of the training program was increased from 3.5 to 4 years. From August 1998, people preparing to become *Folkskole* teachers were required to specialize in four subject areas, one of which must be mathematics or Danish. Teachers only teach the subjects in which they have specialized.

Folkeskole teachers in Denmark generally must hold a Bachelor of Education degree from one of seven university colleges ("Teacher training," 2014). As part of their degree, pre-service teachers generally select three subject areas in which to specialize. Danish and mathematics subjects are now *age-specialized*, meaning pre-service teachers receive specific training for either beginner and intermediate grades (grades 1 to 6), or intermediate and final grades (grades 4 to 10). Before age-specialization was implemented, teachers of mathematics and Danish were trained to teach all grades, 1 to

10. Therefore, many teachers currently working in the *Folkeskole* system have been trained in teaching mathematics or Danish to grades 1 to 10 rather than a only a subset of those grades.

The fact that Danish pre-service teachers were being trained to teach not only mathematics in grades 1 to 10, but also in several other subjects was a growing concern to the Ministry of Education (Niss, 2003). A minority of pre-service teachers chose to specialize in mathematics and, for most of those pre-service teachers, their specialization was lacking depth. Though pre-service teachers received training in pedagogical techniques, they lacked the university mathematics degrees held by upper secondary mathematics teachers.

In addition to teacher training concerns, there were also concerns regarding methods of characterizing and measuring students' mathematical progression as well as overall ways of assessing students' mathematics learning (Niss, 2003). Not only were there questions regarding how components of mathematical understanding were interpreted on assessments, but also a misalignment between assessment techniques and the ways teaching and learning were enacted in classrooms. Additionally, the Ministry of Education noted problems with the transition between levels of mathematics, particularly between lower and upper secondary schools.

To help address these concerns, a committee of twelve mathematicians, mathematics teachers, mathematics education researchers and others outside of mathematics was appointed by the Ministry of Education in 2000 to determine ways to improve mathematics teaching and learning in primary and lower secondary schools (Niss, 2003). The committee developed the *KOM* Project, which represents the Danish words for *Competencies and Learning of Mathematics*. The purpose of the project was to address several questions:

- To what extent is there a need for innovation of the prevalent forms of mathematics education?
- Which mathematical competencies need to be developed with students at different stages of the education system?
- How do we ensure progression and coherence in mathematics teaching and learning throughout the education system?
- How do we measure mathematical competence?
- What should be the content of up-to-date mathematics curricula?
- How do we ensure the ongoing development of mathematics as an education subject as well as of its teaching?
- What does society demand and expect of mathematics teaching and learning?
- What will mathematical teaching materials look like in the future?
- How can we, in Denmark, make use of international experiences with mathematics teaching?
- How should mathematics teaching be organised in the future? (pp. 5-6)

The members of the KOM Project settled on the following definition of mathematical competence: "the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role" (Niss, 2003, pp. 6-7). Content knowledge and understanding were recognized as necessary but not sufficient for mathematical competence. A more complete mathematical competence required additional components other than content knowledge and understanding. The committee decided upon this definition of a mathematical competency: "a clearly recognisable and distinct, major constituent of mathematical competence," (p. 7). Eight additional mathematical competencies in two separate groups were adopted by the committee as necessary constituents of mathematical competence:

Asking and answering questions in and with mathematics:

1. Thinking mathematically
2. Posing and solving mathematical problems

3. Modelling mathematically
4. Reasoning mathematically

Managing mathematical language and tools:

5. Representing mathematical entities (objects and situations)
6. Handling mathematical symbols and formalisms
7. Communicating in, with, and about mathematics
8. Making use of aids and tools

The committee described these eight competencies as closely related to one another and as behavioral in the sense they are related to mental and physical processes, activities, and behaviors. The intent is that these eight competencies be understood and interpreted in a strict mathematical sense. For example, the sixth competency related specifically to mathematical symbols, not other types of symbols such as literary symbols or chemical symbols, for example.

Each mathematical competency has three dimensions (Niss, 2003). The *degree of coverage* relates to the extent an individual masters the competency characteristics given by the description of each competency. The *radius of action* describes the range of contexts in which an individual can use and apply a particular competency. The *technical level* of a mathematical competency is the level of how conceptually and technically advanced the skills and tools are with which an individual is able to use that competency. Though not measured on a quantitative scale, the committee used a volume metaphor to describe these dimensions: the product of the three dimensions gives an indication of the level of mastery of a competency. If one dimension has a measure of zero, then the level of mastery of that competence is also zero.

The most recent revision of the *Fælles Mål* (Common Goals) for *Folkeskolen* has six mathematics competencies: problem-treatment (posing and solving mathematical problems), modeling, reasoning and thinking, representation and symbolic processing, communications, and tools (“Læseplan for faget matematik,” 2014). This updated set of competencies includes all of the aspects of the original set of eight competencies, however the reasoning and thinking competencies have been combined, and the representation and symbols competencies have been combined. At levels of the educational system after *Folkeskole*, the competencies are not combined and eight competencies remains as originally presented.

These mathematical competencies can be used in three different ways in mathematics education (Niss, 2003). When used *normatively*, the competencies are a tool for clarifying how mathematics education should happen. In a *descriptive* context, the competencies are used to characterize not only teaching practice but also assessment and student outcomes. The competencies may also be used *meta-cognitively* to help teachers and students monitor their own teaching and learning in the classroom.

Danish mathematical competencies exist in contrast to the Danish tem *færdighed,* or procedural skills (Højgaard, 2009). A procedural skill is the “ability to carry out a given act with unambiguous characteristics,” (p. 227). These procedural skills can form part of mathematical content, for example, identifying an obtuse angle or demonstrating the correct technique for long division, but mathematical competence is more than simply accumulating procedural skills.

The contrast between mathematical competencies and procedural skills is an important distinction, particularly when assessing students’ work. While assessing procedural skills such as a student’s ability to identify an obtuse angle or divide two numbers correctly is generally straightforward, assessing a student’s use of a mathematical competency, such as communication, is more difficult. When considering mathematical competencies, Højgaard (2009) suggests an assessment process that includes three sub-processes:

- Characterizing what you are looking for.
- Identifying the extent to which what you are looking for is present in the situations involved in the assessment.
- Judging the identified (p. 228).

The mathematical competency assessment process has implications for mathematics instruction. Højgaard's (2009) three sub-processes for assessment indicate that without very specific guidelines, a certain amount of interpretation takes place within and between these sub-processes. Differing interpretations of the sub-processes have specific implications for instruction. Højgaard suggests teachers consider a series of questions when planning mathematics instruction:

- Which (competency) *learning aims* exist for the unit of teaching I am about to assess?
- How do I *understand* these aims – especially if they are not initially chosen and formulated be me?
- Which kind of *presentations to guide student activity* – tasks, presentation of cases, oral and/or written stories, questions for discussion, etc. – can I find or construct that I believe will be well suited to help the students in developing toward the established aims?
- What *signs* in terms of certain kinds of student activity should I pay special attention to in order to *identify* the extent to which our aims are present in the situations assessed?
- How do I *judge* what I have identified? (p. 230)

Theoretical Framework

The way teachers interpret the competencies has an impact on classroom practice and what students do in their mathematics classes. To examine how teachers interpret a specific mathematics competency and how teachers enact that competency in their classrooms, I will use the theoretical framework set forth by Kennedy (2005) in her studies of the gap between reform ideals and everyday teaching practice. Kennedy states teachers' interpretations of their teaching situation influence their practice. Further, Kennedy suggests teachers' beliefs about the nature of the subject matter, about how students learn, and about the role of the teacher shape their interpretations. It is difficult, however, to separate teachers' beliefs from the circumstances of teaching. Kennedy's framework can be summarized into three major areas: subject matter - the *curriculum*, beliefs about how students learn and the role of the teacher – the *pedagogy*, and the circumstances of teaching – the *policies*.

Kennedy's (2005) framework aligns well with Cohen & Ball's (1990) three areas of change in mathematics education teaching and learning. In their call for a new approach to primary mathematics, Cohen and Ball describe three areas of change: new goals for learning, new conceptions of mathematical content, and new pedagogy. *Goals for learning* are described as a need for students to reason mathematically, to apply what they are learning, to understand mathematical concepts, and to evaluate mathematical arguments. These components of *goals for learning* are major aspects of mathematical process standards and competencies. In *conceptions of mathematical content*, Cohen and Ball suggest ways for teachers to make changes in what they teach. For example, they call for teachers to provide students with contextualized problems and to encourage students to find alternative solutions to problems. Cohen and Ball also describe new and different ways teachers should think about how students should learn and what the teacher should be doing during lessons. They suggest teachers provide students with more opportunities for reasoning, explaining, justifying, and writing in mathematics so that students develop connections and depth of understanding. These three areas map directly on to those described by Kennedy: (a) goals for learning are described in national curriculums and standards, (b) conceptions of mathematical content are exemplified in mathematical content, and (c) ideas about how student learn and how teachers guide that process is pedagogy.

For my theoretical framework, I will explore how three areas - curriculum, pedagogy, and policies, each influence what happens in a mathematics classroom. A diagram of my framework is given in figure 2.2. Aspects of curriculum, pedagogy, and policies exist outside of classroom practice. For example, in many cases, educational policies are created with little consideration for specific individuals or classrooms. Similarly, curriculum materials are often created by publishing companies for hypothetical students or for ideal classroom situations rather than for specific teachers and their students. Teachers' ideas about pedagogy develop over time based on a number of factors, including their own experiences in school. It is in the classroom, however, that these three areas meet and influence how a teacher enacts mathematics teaching and learning for her or his students. Teachers make decisions about their curriculum when they select tasks, resources, materials, and activities to use with their students. Teachers' ideas about pedagogy in turn influence their curriculum choices. These decisions are often made within the structures of educational policies covering required content and mandated lesson time. The next three sections will address the place of each area – curriculum, policy, and pedagogy – in the mathematics classroom.

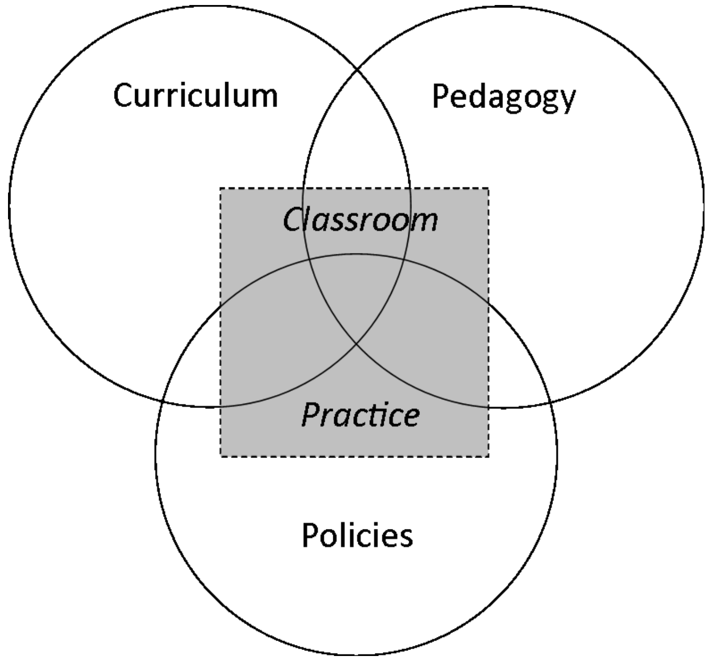


Figure 2.2. Theoretical Framework

The Place of Curriculum in the Mathematics Classroom

Although Stein, Remillard, & Smith (2007) state the term *curriculum* is generally used to refer "to the content of teaching and learning – the *what* of teaching and learning (as distinguished from the *how* of teaching)" (p. 321), the choices teachers make and the ways teachers use learning materials, the *hows* of teaching, contribute substantially to the classroom experience. Given the many different uses of the same term, it is understandable that there can be confusion about the word *curriculum*. In considering Helen Simons' explanation of curriculum, "Curriculum is not an abstract entity, but a lived experience that has relevance for particular students and particular teachers in a particular context" (Simons, 1998, p. 367), I will consider the various definitions of curriculum and then address three main questions: What is the role of curriculum in the mathematics classroom? How do teachers make curricular decisions? What is necessary for teachers to implement an adequate, effective curriculum in their classrooms?

Defining Curriculum

The term *curriculum* is used in a variety of ways, and its meaning depends on the context. Though generally taken to mean something relating to teaching and learning (Stein et al., 2007), it is necessary to explore these multiple meanings in order to have a clearer understanding of the issues involved. Some definitions or usages are rather narrow while others are much wider in their scope.

Curriculum can be used to refer to frameworks that set out content expectations (Stein et al., 2007). In England, for example, the term *national curriculum* is used to describe the programs of study and attainment targets for twelve different subject areas ("National Curriculum – GOV.UK," 2014). The National Curriculum does not specify textbooks or other learning materials, teaching strategies, or classroom assessments. These specific decisions have generally been left up to individual schools (Wragg, Bennett, & Carre, 1989). The National Curriculum document states what teachers and pupils should do in each subject area, for example,

> Teachers should develop pupils' numeracy and mathematical reasoning in all subjects so that they understand and appreciate the importance of mathematics. Pupils should be taught to apply arithmetic fluently to problems, understand and use measures, make estimates and sense check their work. Pupils should apply their geometric and algebraic understanding, and relate their understanding of probability to the notions of risk and uncertainty. They should also understand the cycle of collecting, presenting and analysing data. They should be taught to apply

> their mathematics to both routine and non-routine problems, including breaking down more complex problems into a series of simpler steps. (Department for Education, 2013, p. 9)

The document also lists the statutory requirements for each topic area (those things schools are legally obligated to teach), for example:

> Pupils should be taught to:
>
> - solve problems with addition and subtraction:
>   - using concrete objects and pictorial representations, including those involving numbers, quantities and measures
>   - applying their increasing knowledge of mental and written methods
> - recall and use addition and subtraction facts to 20 fluently, and derive and use related facts up to 100 (Department for Education, 2013, p. 108).

The Virginia Standards of Learning ("Virginia Standards of Learning," 2012) and the Common Core State Standards ("Mathematics Standards," 2014) are also examples of standards and content expectations, as are the set of Danish *Fælles Mål* or Common Goals. For my dissertation, however, despite sometimes having the term *curriculum* in their name, these sets of mandated standards are considered aspects of educational policy and are discussed further in the policy section. When I refer to these sets of mandated curriculum expectations, I will generally use the terms *frameworks, standards*, or the Danish term *Common Goals*.

Curriculum can also be used to refer to materials designed for classroom use (Stein et al., 2007). In contrast to England's Department for Education usage of the term, textbook publishers often use the term *curriculum* to encompass a range of learning materials. For example, Houghton Mifflin Harcourt markets, among other products, Saxon Math curriculum ("Saxon Math," 2014). This product includes textbooks, online learning resources, teacher support materials, and classroom assessments, all of which are developed around a specific pedagogy or method of teaching. I will generally use the term *curriculum materials* to refer to this collective set of materials intended for classroom use.

Curriculum Theories

Although *curriculum* is often used to refer "to the content of teaching and learning – the 'what' of teaching and learning (as distinguished from the 'how' of teaching)" (Stein et al., 2007, p. 321), there are often considerable differences between curriculum as it is designed or expected to be implemented, and curriculum as it is implemented in classrooms. Curriculum researchers use a range of terms to describe these differences.

Curriculum before it gets to the classroom.

There is no single term researchers use to describe curriculum before teachers use it. Doyle (1992) describes an *institutional curriculum* as "a tacitly understood and shared conception or paradigm of schooling" (p. 487). He then describes a *formal* or *written curriculum* as a document that attempts to capture much of the institutional curriculum. The set of learning goals and activities described by textbooks or school policies are variously called the *planned curriculum* (Gehrke, Knapp, & Sirotnik, 1992), the *overt curriculum* (McCutcheon, 1988), or the *intended* (Eisner, 1979) or *institutional curriculum* (Brophy, 1982; Stein et al., 2007). These correspond to the different uses of *curriculum* by both the Department for Education in England and the Saxon Math textbook publishers as described earlier. The actual substance of these uses of the term *curriculum* may vary, and could include things like state or national standards, learning goals, textbooks, supplemental learning activities, and scope and sequence documents.

Curriculum in the classroom.

Once the pre-classroom curriculum gets into a classroom, there are again a number of terms used to describe the form the curriculum takes, what happens in the classroom, and how students experience a specific curriculum. *Enacted curriculum* can used to describe what happens in the classroom (Gehrke et al., 1992), while *attained curriculum* (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002) and *experienced curriculum* (Gehrke et al., 1992) might be used to describe how students experience and interact with the curriculum.

These different forms of curriculum are important to consider. Teachers should be aware that there is potential for misalignment between the content and processes of what is intended and what is actually enacted in order to be reflective about what is actually being taught. There are also potential assessment issues as a result of misaligned curriculum. In many cases, teachers may create their own formative and summative mandated assessments based on the curriculum as it is enacted or experienced in their classrooms, while mandated assessments or other assessments, such as those accompanying a textbook series, are based on the curriculum as it is envisioned by the standards or frameworks.

Curriculum Models

A number of researchers have described models that account for pre-classroom and in-classroom components of curriculum, several of which will be presented here. In his model of school learning, Carroll (1963) states a student “will succeed in learning a given task to the extent that he spends the amount of time that he needs to learn the task” (p. 725). He further elaborates on this model by describing five factors that influence a student’s learning. Aptitude or the amount of time a student needs to learn the task, a student’s ability to understand classroom instruction, and quality of instruction all help determine the time needed to be allocated for learning. Time devoted for learning and a student’s perseverance or the time that student is willing to spend learning help to determine the amount of time spend in learning. Though not specifically a curricular model, Carroll’s model of school learning is important in this context for two main reasons. First, time allocated to specific topics, and the time actually spent learning those topics is an important aspect of curriculum. Second, Carroll described the time available and devoted to learning as *opportunity* or *opportunity to learn* (Carroll, 1989). Though the term *opportunity to learn* is closely linked to Carroll’s model, other researchers also use the term *opportunity* when discussing curriculum.

The Tripartite Curriculum Model (Valverde et al., 2002) has three curricular dimensions: intended (the intent and goals), implemented (the strategies and activities) and attained (the knowledge and ideas). These three dimensions together form a model for the educational opportunity or the “social, political and pedagogical conditions to provide pupils chances to acquire knowledge, to develop skills and to form attitudes concerning school subjects” (p. 6).

Stein et al. (2007) illustrate a dynamic process of curriculum from the pre-classroom stages to the student experience. Their model consists of four phases: written curriculum (including school guidelines and textbook materials), intended curriculum (what teachers plan for their instruction), enacted curriculum (how the teachers' plans are implemented in the classroom, all of which lead to student learning. Both between and within the phases are opportunities for transformation, both in how material is interpreted by teachers and in the interaction between students and teachers (Stein & Smith, 2007).

Goodlad, Klein, & Tye (1979) describe five curriculum domains: *ideal, formal, perceived, operational*, and *experienced*. The ideal curriculum is that which is done for mainly hypothetical students: it is planned and developed but, other than in trial classrooms, not yet implemented by teachers in specific classrooms. Many commercially-developed curriculum materials are in the ideal curriculum category. When curriculum materials are adopted or approved by organizations, be they national, state or local organizations, they become the formal curriculum. The formal curriculum is generally an ideal curriculum onto which are added additional decisions and ideas specific to the organization. The perceived curriculum is the curriculum people envision when they think of the curriculum. Parents, for example, often have very specific things in mind when they think of a mathematics curriculum. These ideas and perceptions can vary widely from the formal curriculum. Teachers also can have certain percepts in mind about curriculum and these perceptions can alter the formal curriculum as it becomes the operational curriculum. The operational curriculum is curriculum as it is implemented in the classroom by teachers. Finally, each student experiences the curriculum differently depending on their own prior experiences, opinions, and perceptions. The curriculum in each domain changes to some degree as it moves from domain to domain.

Perhaps the model that most comprehensively describes the factors that influence the curriculum teachers implement is described by Tarr et al. (2008) and reproduced in figure 2.3. In this model, the intended curriculum – the framework teachers use - is impacted by factors such as societal needs, advances in mathematics and technology, outside policies, and values and beliefs about mathematics and education in general. This intended curriculum influences both the textbook curriculum and the assessed curriculum, each of which have their own influencing factors. Each of these three types of curriculum influence the teacher's implemented curriculum, which is also influenced by the teacher's own subject knowledge and beliefs about learning, as well as students' motivation, effort, and prior knowledge. Finally, each of these curricula and outside forces all influence the curriculum students learn.

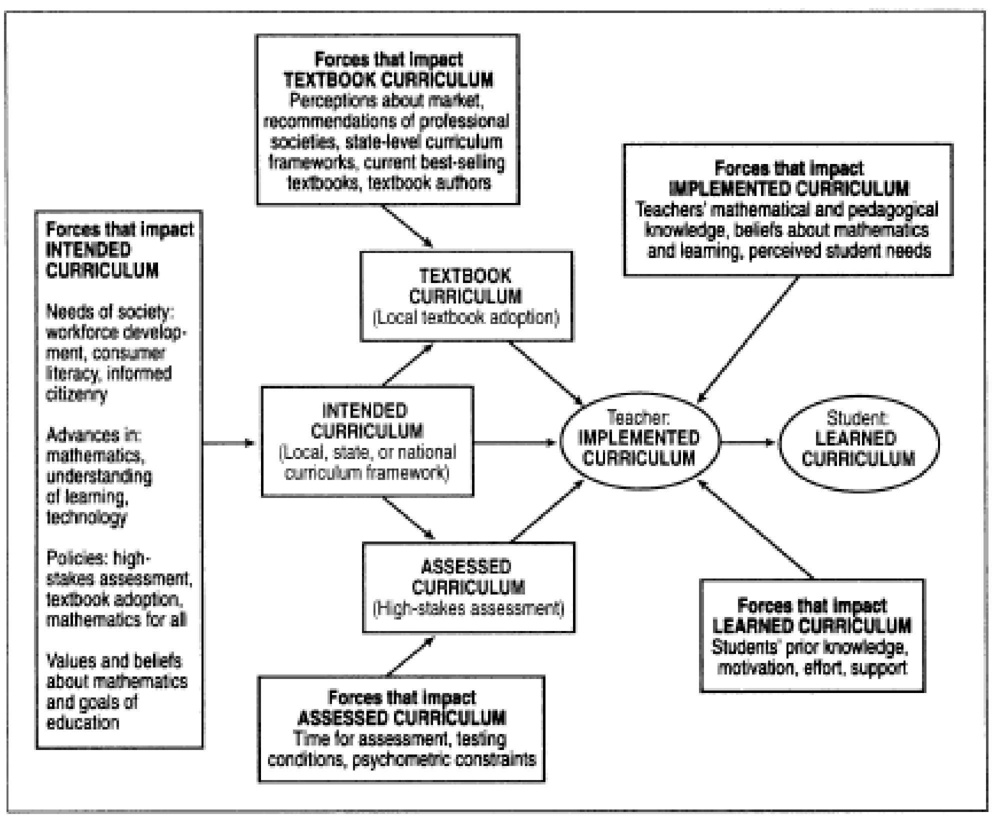


Figure 2.3 Model depicting the relationship of various types of curriculum and the forces that influence the content of those curricula.
From: Tarr et al., 2008, p. 251. Reprinted with permission from The Journal for Mathematics Education Research, copyright 2008, by the National Council of Teachers of Mathematics.

The Hidden Curriculum.

In the classroom, however, there is another curriculum present: the *hidden curriculum*. This hidden curriculum is not intended, but is a by-product of the school environment and policies, teacher actions, or other learning materials. It can develop through the interactions between teachers and their students and can be thought of as the values students develop about learning and one another as learners. Consider a school policy in which students are required to earn 80 percent or higher on proficiency tests, and are allowed to retake these tests if they score lower than 80 percent. In this example, the hidden curriculum could be encouraging students not to take the initial test seriously. The teachers may fear that this school policy conveys to students there will always a second chance, not only in school but in life.

Teachers often make decisions regarding what tasks they assign to students, how much time is spent on particular topics, and what is to be evaluated (Romberg, 1983). Teachers are often limited, however, by a range of outside factors, including school boards and administrators, textbook publishers and curriculum developers, as well as curricular traditions. Romberg states several curricular traditions often enter into decisions about what is taught and how, and these traditions can provide teachers with regularity and predictability. These curricular traditions also help form parts of the *hidden curriculum* in schools (Anyon, 1980; Romberg, 1983). In the *discipline tradition*, curriculum development is done by breaking the subject into topics, studies, and then into lessons, with specific facts for each lesson. This is often seen in mathematics, which, for example, might be broken-down into Algebra I, linear equations, slope, and then positive and negative slopes. The *psychological engineering* tradition focuses on teaching methods from an education psychology background. An example of this tradition would be having children begin with a concrete learning experience and then move to a more abstract experience. The third curricular tradition is *critical sociology.* In this tradition, the emphasis is on what is considered legitimate knowledge in certain classes and groups of people; education focuses on ideology. In describing these three traditions, Romberg emphasizes that, while teachers may be placed to make curricular decisions, these decisions are often influenced, knowingly or not, on certain curricular decisions.

In some cases, the different social classes of school can have different educational philosophies which can influence a school's hidden curriculum (Anyon, 1980). In her observations in schools of different social classes, Anyon noted wide differences in how two-digit division was taught in each type of school. In the predominantly *working class* schools, those with most family incomes at or near the poverty level, the work was very procedural and students were expected to follow the steps they were given: "Divide, Multiply, Subtract, Bring Down" (Anyon, 1980, p. 69). In the *middle class* school, with a mixture of social classes and family incomes in the mid-40% of the population, instruction went beyond the procedural but the focus was on getting correct answers: ""I want to make sure you understand what you're doing - so you get it right" (Anyon, 1980, p. 72). The *affluent professional* school, whose families had incomes in the top7% of the population, children were expected to apply independent thought and students were able to expected to apply their knowledge of division to solve problems about averages. The final school, the *executive elite* school, with family incomes in the top 1% of the population, the focus was on developing students' reasoning and mathematical thinking. Though Anyon's study used only a small sample of schools, these examples are illustrative of how, even when the content is the same, additional curricular factors are quite different.

The Null Curriculum.

A further curriculum to consider is the *null curriculum*: the things students do not have a chance to learn (McCutcheon, 1988). This category includes material that has been consciously chosen for exclusion from the curriculum for a variety of reasons, including time, lack of necessary equipment or materials, or a topic's potential to cause controversy. The null curriculum can vary for different students, and might apply to certain students but not others. For example, in the mid-twentieth century, it was not uncommon for girls to receive little or no useful educational counseling in schools (Lewis, 1965). In many cases, classes would be routinely offered to boys, but not girls. Precisely defining the null curriculum is virtually impossible given its near limitlessness, however curriculum designers should be aware of what is being excluded and the reasons for exclusion (Flinders, Noddings, & Thornton, 1986).

Eisner (1979) says that these three domains form the curriculum that every school teaches: the *explicit* curriculum – that which is stated and written down, the *implicit* curriculum – that which is unstated but impacts student learning, and the *null* curriculum – that which is not taught. These three domains, in whatever form they take in a specific school or classroom, are important factors to consider when exploring curriculum as enacted in the classroom.

The Role of Curriculum in the Mathematics Classroom

With the many and varied uses of the term *curriculum,* it is reasonable to ask the role of curriculum in the mathematics classroom: what purpose does it serve and how do the diverse meanings of *curriculum* relate to the mathematics classroom?

One role of curriculum is to provide a detailed list of what should be taught (Stein & Smith, 2010). In the United States, this is often specified by state standards such as the Virginia Standards of Learning ("Virginia Standards of Learning," 2012) or the Common Core State Standards ("Mathematics Standards," 2014). These standards and frameworks remove much of the larger content decisions from teachers, in an attempt to make sure students throughout the jurisdiction are exposed to the same content.

Curriculum materials are the primary mathematics teaching tool for the majority of teachers (Grouws, Smith, & Sztajn, 2000). Additionally, "students do not learn content to which they are not exposed" (Stein & Smith, 2010, p. 327) and teachers generally do not cover topics that do not appear in their curriculum materials. Though the implementation of national or state standards and frameworks has meant that topic gaps in textbooks are often supplemented by other materials, comprehensive curriculum materials are important in making sure students have an opportunity to learn specific topics in mathematics.

One concern often expressed about curriculum, however, whether it is frameworks or materials, is that it is rarely designed for the children for whom it is meant (Simons, 1998). In other words, it is designed for hypothetical children or "other people's children" (Grumet, 1988, p. 164), rather than specific children in a specific classroom. In some cases, material for standards-based curricula are seen to be conceptually weak and neglect or ignore mathematical problem-solving (Stein & Smith, 2010).

In some cases, curriculum materials are designed to address not only student learning but also development of teacher knowledge and understanding (Stein & Smith, 2010). This helps to address concerns such as limited teacher content knowledge, varying interpretation of content and materials, and teacher difficulties implementing the materials. Often, however, a teacher's own mindset towards a specific set of curriculum materials influences what a teacher learns from that material (Remillard & Bryans 2004). A teacher who views specific material as being unaligned with his or her own philosophies of education is unlikely to make use of teacher educational opportunities in that material, even if they are available.

Though this paper is primarily about the role of curriculum in the mathematics classroom, it is necessary to examine the range of meanings and domains of curriculum to understand what happens before curriculum gets to a mathematics classroom. Providing teachers with a set of content to be taught and curricular materials covering those topics is not sufficient to ensure students learn those topics (Stein & Smith, 2010). Standards, frameworks, and comprehensive curriculum materials must be combined with effective instructional strategies and teachers decisions.

Implementation Matters

What happens in classrooms makes a difference. In their study of more than 2500 students in 10 middle schools, Tarr et al. (2008) concluded that textbook type was not a significant predictor of student achievement. A number of factors impacted the textbook curriculum implementation, including how often teachers used the textbook materials and, when teachers supplemented the textbook with other materials, what types of additional materials were used. Even when textbook materials were used for specific lessons, they were used differently in different classrooms. These findings are supported by Kilpatrick (2003) who stated,

> Two classrooms in which the same curriculum is supposedly being 'implemented' may look very different; the activities of teacher and students in each room may be quite dissimilar, with different learning opportunities available, different mathematical ideas under consideration, and different outcomes achieved (p. 473).

One factor that impacts student learning is teacher knowledge (Grouws et al., 2000). A teacher's knowledge of mathematics and instructional practices impacts the organization and management of a lesson which in turn influences what students learn and to what extent they learn it. Elementary- and middle-school teachers in the United States often have limited mathematical knowledge. Teachers also need mathematical pedagogical knowledge. This category includes:

> the ability to select and enact mathematical tasks that are appropriate for students, flexibly represent mathematical concepts and procedures, facilitate discourse among students so as to make foundational mathematical ideas salient, and assess what students know and understand through a variety of means (p. 231).

Implementing Curriculum and the Curricular Decisions Teachers Make

While frameworks, standards, textbooks, and other curricular materials provide teachers with guidance about what to teach, and in some cases when and how to teach particular topics, it is the teacher who orchestrates the transfer between the pre-classroom curriculum and in-classroom curriculum. Though some organizations provide far more scripted curriculum materials for teachers, other organizations, such as the Dutch National Institute for Curriculum Development, expect teachers to adapt materials for their students and supplement and extend the materials as needed (van den Akker, 1988). In some cases, teachers consciously make decisions about curriculum implementation, but other less-conscious factors can also substantially influence this implementation.

Though many areas specific curriculum standards and objectives, these objectives are not necessarily the first thing teachers consider when planning (McCutcheon, 1980). For many teachers, the first curriculum decision is the learning activity or the content, rather than what they hope students will learn. In moving between pre-classroom curriculum, be it standards or materials, to implementing curriculum in their classrooms, teachers read and interpret the material they are given (Stein & Smith, 2010). This process of interpretation is based on teachers' knowledge and experience. In this way, teachers play a key role in transforming the pre-classroom curriculum into the classroom curriculum. In some situations, teachers have difficulty implementing curriculum because of limited content knowledge and the varying ways they interpret and implement the material (Stein & Smith, 2010). Some curriculum materials are substantially different from the materials they are replacing (Remillard & Bryans 2004). In these situations, teachers may need guidance in considering their own philosophies about teaching and mathematics, thinking about how curriculum materials might be used and adapted, and how to critically analyze supplemental curriculum materials.

While researching implementation of the 1998 English national curriculum, Wragg, Bennett, and Carre (1998) noted several areas of teachers' concerns, including teachers' perceived competence in the subject area, short implementation timescale, shortages or feared shortages of resources, personal stress, and apprehension regarding national assessments. Teachers identified a need for help with assessments and testing as their greatest in-service training need. Several of these concerns are echoed in a study of science curriculum in the Netherlands. Van den Akker (1988) recorded four categories of difficulties teachers have with new curriculum: a change in the role of the teacher in the new curriculum, a lack of adequate subject knowledge, the complexities of lesson preparation, and lack of understanding of how to measure and assess student learning.

Teachers also implement curriculum materials in a range of ways. In their study of classroom teachers and their use of new mathematics curriculum, Remillard & Bryans (2004) describe the construct of *orientation toward curriculum* as:

> a set of perspectives and dispositions about mathematics, teaching, learning, and curriculum that together influence how a teacher engages and interacts with a particular set of curriculum materials and consequently the curriculum enacted in the classroom and the subsequent opportunities for student and teacher learning (p. 364).

The degree to which teachers' views of a specific curriculum match the teachers' own views of mathematics influence teachers' use of curriculum resources. This interaction between teacher beliefs, the curriculum materials, how the materials are used in the classroom, and student opportunity to learn is a complex and dynamic one.

Teacher orientation towards curriculum can be described in three broad categories (Remillard & Bryans 2004). Some teachers are in the *thorough piloting* category; they carefully read and make use of each part of the curriculum guides. These teachers use the new curriculum as their primary teaching resource. Other teachers are considered to be in the *adopting and adapting* category. Though they rely on the new curriculum materials for the scope and sequence of topics, they use their own ideas and strategies in the classroom. The final group of teachers is the *intermittent and narrow use* category. If they use the new curriculum materials at all, they rely more on their own ideas and activities familiar to them. In some cases, it takes time for teachers to more fully embrace new curriculum. Remillard & Bryans (2004) note that some changes in teacher attitudes and implementation of new curriculum did not happen until after more than a year of using the curriculum.

In his study of how teachers utilize science curriculum materials, Brown (2002) notes three similar categories of use: offloading – placing all of the curriculum design responsibilities on the new curriculum, adapting – using some of the material in the new curriculum, and improvising – moving away from the new curriculum materials to a considerable extent. These categories, however, are not as closely aligned to teacher beliefs and philosophy about the new curriculum as the categories of Remillard & Bryans (2004), but may be more aligned with teacher ability and competence in the subject. In this case, a more novice or less-knowledgeable teacher might utilize substantially more offloading than a more experienced teachers. Brown also notes the approach can change from lesson to lesson and unit to unit based on a variety of factors, including varying extents of teacher knowledge and familiarity about specific topics, and a teacher's desire to focus on other instructional needs in the classroom. He also notes situations in which delegating curricular responsibilities to pre-made materials did not provide adequate support for teachers, suggesting that even structured curricular materials require adaptation to specific students.

Despite these reasons why teachers may not implement curriculum as envisioned and despite some of the difficulties teachers have, teachers also make conscious decisions about their classroom curriculum. The decisions teachers make regarding the content of their mathematics instruction are complex and varied, including the time needed and available to devote to a topic, the intended audience, and to what extent the topics are to be learned (Brophy, 1982). When planning and teaching a specific topic, some teachers rely on their past experiences teaching that particular topic. Teachers who feel they have sufficient background knowledge about a topic may feel less reliant on textbooks than teachers who are less knowledgeable. Additionally, teachers often hold certain beliefs about specific content areas and their importance in the curriculum. Brophy suggests teachers take these factors into account in a "benefits and costs" assessment when planning. In some cases, teachers make curriculum decisions because they think the decisions are helping student learning, but make those decisions with little or no pre-or post-assessment to verify the teacher's thoughts (Klein, 1979).

Brophy (1982) describes teachers as policy brokers who work within existing frameworks and structures to adapt curriculum to what they see are the needs of their students. In this way, "the content actually taught to students is likely to be a compromise between the officially adopted content and the needs of the students" (p. 3). Other reasons for this compromise between what is intended and what is accomplished can result from issues arising during the course of the lesson, such as certain aspects taking longer than expected and students who have difficulty understanding the material. In some cases, the material is taught incorrectly or incompletely.

Student factors also influence teachers' curriculum decisions (Brophy, 1982). Students' prior knowledge, or lack of prior knowledge, can influence instructional decisions. For example, sometimes teachers must teach topics that are not specifically in that year's curriculum because students had an incomplete or incorrect mastery of the material in previous years (Brophy, 1982). In other cases, teachers fail to assess students' prior knowledge and address material students already know.

In some cases, teachers make instructional decisions based on their personal attitudes to a subject or topic. Depending on the organization, a teacher might be asked or expected to create a timeline of teaching: what will be taught when, and how long will be devoted to each topic (Eisner, 1979). This timeline might be for an entire school year or could be on a shorter, week-by-week basis. Some teachers allocate instructional time on subjects according to their feelings and attitudes about those subjects (Brophy, 1982). For example, a study of elementary-school teachers showed who enjoyed mathematics allocated more than 50 percent more time to teaching mathematics than teachers who did not enjoy mathematics (Buchmann, & Schmidt, 1981). Teachers sometimes make changes to units or lessons because of personal preference rather than learning objectives. It is reasonable to suggest that mathematics teachers could follow a similar pattern with specific mathematics topics they enjoy compared with those they do not.

Thus far, I have looked at reasons for or against specific curriculum implementation and some of the decisions teachers might make. Eisner (1979) describes seven areas in which curriculum decisions can be made. Depending on the specific situation, some of these decisions might be made at national, state, or local levels, and an individual teacher may be allowed or expected to make decisions in different areas. The first area is curriculum goals, aims, and objectives. Aims – the educational values, goals – the educational purposes, and objectives – statements of what students should be able to do, are generally decided on a national or state level. The second area, content, is in many cases, particularly in localities with curriculum standards and frameworks, generally decided for teachers, though teachers often make decisions about extent of content coverage. Teachers quite often make decisions about the third area: types of learning opportunities. Eisner describes this area as one in which the "educational imagination" (p. 138) is necessary as the objectives and content are transformed for the classroom. The next area is organization of learning opportunities: how learning activities are planned over time, whether they are sequential or more interlinked and connectional. The organization of content areas is another decision-making area. Though these decisions might be made for teachers, teachers still must be aware of how their subject area is or could be organized and what connections are possible between these areas. The sixth area of curriculum decision-making is in regards to how material is presented and how students will respond to that material. For example, lectures or reading textbooks are two modes of presentation that are quite different from hands-on, experiential learning. Finally, decisions are made regarding the evaluation of student learning. While formal end-of-course testing is often a decision made by a state or local authority, teachers make frequent decisions regarding in-class assessments.

Implementing an Adequate, Effective Curriculum in the Classroom

When making curricular decisions in the classroom, teachers should keep several factors in mind. Teachers often focus on what content they are teaching and how to teach it rather than on the objectives underlying that content (Brophy, 1982). Instead, when planning mathematics instruction, teachers should begin with the learning objectives and then structure the lesson to meet those objectives. Teachers should not overly rely on packaged curriculum or textbooks when planning (Brophy, 1982). Teachers must know how to choose and modify materials for their own students. Additionally, teachers must understand the material they are teaching in order to present it to students. Teachers should also be aware of student misconceptions and be prepared to correct them when they arise. For teachers without a strong mathematics background, a mathematics specialist can provide support for teachers as they develop their mathematics content and pedagogical knowledge (Grouws, Smith, & Sztajn, 2000). Teachers require support through focused mathematics professional development opportunities as well as adequate time to prepare lessons and then reflect on them afterwards. In some cases, teachers benefit from interacting with other teachers who are using the same curriculum (Remillard & Bryans 2004). This process allows teachers to discuss ideas, consider different interpretations, and consider how to implement the curriculum in their own classrooms.

Grouws et al. (2000) seem to summarize these factors as they identify four factors needed for learning: "Increasing United States students' knowledge of mathematics requires well-qualified teachers teaching high-quality lessons focused on important mathematics in a context that supports students' opportunity to learn" (p. 263).

What Teachers Do vs. What Teachers Should Do

The previous two sections examine some of the ways teachers make their curricular decisions as well as some of the things teachers should do when making curricular decisions. In some cases, these two domains align, such as factors regarding teachers' mathematical content knowledge. In other areas, such as a focus on learning objectives, seem to happen far less often. Figure 2.4 gives a comparison of factors that help teachers implement effective classroom curriculum compared with factors that generally do impact teachers' curriculum decisions.

	Factors that help teachers implement an effective classroom curriculum					**Other Factors**
	Aligned with learning objectives	**Tailored to their own students**	**Mathematical subject knowledge**	**Professional development**	**Time**	
Factors impacting teachers' curriculum decisions		Knowledge of students, student interest and questions	Mathematical knowledge and perceived competence in mathematics		Time available for teaching/ learning, time available for planning	Teachers' philosophy of education, attitudes, feelings; Available resources; Tests and assessment; interpretation of frameworks

Figure 2.4. Comparison of factors that help teachers implement effective classroom curriculum compared with factors impacting teachers' curriculum decisions

In a previous section, I explored some reasons for varying levels of curriculum planning and implementation by teachers and why teachers often use supplemental materials. What seems to be missing from the literature, however, is a clearer examination into why teachers make decisions about specific curriculum material: why a teacher chooses worksheet A over worksheet B, or why a teacher uses technique C compared to technique D. These types of curriculum decisions are often made on an informal basis in the form of *curriculum deliberation* rather than in a more formal or elaborate lesson plan (Eisner, 1979). Anecdotally, I fear that many teachers rely on the "it looks cute" method of resource selection, particularly at the elementary-school level, rather than on many of the factors shown to help teachers implement effective curriculum. A Google search for the term *"this looks cute" math lesson planning* gives over 13 million results, suggesting that for at least some teachers, the appearance of the curriculum resource is a crucial decision-making factor.

It seems reasonable to state that additional research is needed in the specific factors that teachers use in their curriculum deliberation: Why does a teacher choose resource E instead of resource F? Additionally, as certain factors have been shown to help teachers implement an effective classroom curriculum, how do those two factors align and how *could* those two factors align?

For this dissertation, I will be considering how teachers' interpretations of the communications competency are enacted in their classroom curriculum. This lens on classroom curriculum could help determine some possible ways teachers make curricular decisions.

The Place of Policy in the Mathematics Classroom

The previous section is a discussion of the place of curriculum in the mathematics classroom. In many cases, teachers are responsible for making decisions about how to enact curriculum in their own classrooms. Teachers base these decisions on a number of factors as described earlier, but in increasing numbers of states and countries, teachers must also make these decisions within the structures of policies mandated by government agencies.

Government agencies have long had an influence on educational policy. In recent decades, however those policies have extended to many areas, including student assessments, teacher evaluation programs, teacher certification or licensing, curriculum frameworks (Cohen & Ball, 1990a), and other areas such as graduation rates, academic requirements for graduation, and summer school instruction (Regulations Establishing Standards for Accrediting Public Schools in Virginia, 2013). These policies have an impact on schools and on students in a variety of ways, but some of these policies have a direct impact on what happens in school mathematics classrooms.

This section examines several areas of educational policy and their potential impact on mathematics instruction in the classroom, specifically policy regarding curriculum frameworks, assessment, instructional time, and teacher licensure and certification. The specific focus of this section is on policies made at the state and national level rather than those made by school districts or individual schools.

In the United States, we are generally familiar with the public school model in which each state makes its own educational policies – often influenced by federal policies - in which states are divided into school districts, and in which each school district is run by a superintendent and overseen by a school board or similar authority. State departments of education establish a set of content standards, and sometimes process standards, for public schools in that state. The Common Core State Standards are an example of content and process standards that are being adopted as policy by a number of states for use by public schools.

In other countries, the model is not necessarily the same. In England, the government sets the National Curriculum and individual schools are responsible for following those guidelines. There is little, if any, involvement by a Local Education Authorities, the equivalent of local school district administration, on curriculum or assessment matters. As described earlier, the Ministry of Education sets education policies for *folkeskole* in Denmark.

What and how mathematics is taught

The early 1980s saw a change in the educational policy landscape as government authorities began to exert increasing authority over school curriculums (Cohen & Hill, 2000). In some cases, such as California in the mid-1980s, this authority was through state instructional frameworks, and in other cases, the authority was in the form of national curriculums and frameworks, such as England and Wales in the late 1980s and New Zealand in the early 1990s (Priestley, 2002). The 1985 Mathematics Framework in California not only gives content guidance by grade cluster, for example kindergarten through grade three, but also describes *characteristics of instruction*, including "teaching for understanding" (California State Department of Education, 1985, p. 12), "problem solving" (p. 13), and "cooperative learning groups" (p. 16). The Framework also provided standards for mathematics textbooks and descriptions of two content courses for pre-service mathematics teachers. Though the 1985 Mathematics Framework was only an advisory document for local districts, the California State Board of Education used the Framework to drive textbook adoption (Cohen & Hill, 2000). Adoption of approved textbooks was required for localities to qualify for state aid. In this way, the advisory framework became much more of a driving force in determining the content and methods of instruction in California schools.

The emergence of state and national frameworks is important because it represents not only increasing governmental control over education, but also it represents the emergence of a structure to what is to be taught and what is to be learned. Before governmental curriculum frameworks, content decisions often rested at the district or school level. Government frameworks, however, provide organization and shape to content. The National Curriculum in England and Wales in the late 1980s divided subjects into strands, each of which was further divided into learning outcomes. (Priestley, 2002). The New Zealand Framework of the early 1990s divided subjects into strands, each of which was subdivided into eight levels, each of which had three or four achievement objectives. For example, currently the National Curriculum in England gives several strands for mathematics: number, measurement, geometry, statistics, ratio and proportion, and algebra (Department for Education, 2013).

Current state and national frameworks have built upon the frameworks of the 1980s and 1990s. The Common Core State Standards for Mathematics, for example, provide a set of curriculum standards for what students should be able to do by the end of each grade level or course ("Math Standards," 2014). Similarly to how the 1985 Mathematics Framework of California (California State Department of Education, 1985) specified not only content but also characteristics of instruction, the Common Core includes a set of eight Standards for Mathematical Practice that "describe varieties of expertise that mathematics educators at all levels should seek to develop in their students" ("Standards for Mathematical Practice," 2014). These eight standards are:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

The 2014 National Curriculum for England has a similar structure, providing content requirements for each year group as well as a set of curricular aims that include fluency in mathematical fundamentals, reasoning mathematically, and solving problems by applying mathematics ("National Curriculum – GOV.UK," 2014). In Denmark, the Ministry of Education provides the *Fælles Mål* (*Common Objectives*) for several subjects, including mathematics. The set of mathematics Common Objectives includes content for grade level clusters (grades 1-3, 4-6, and 7-9), as well as a set of six mathematical *competencies* that describe types of mathematical knowledge and skills: problem solving, modeling, reasoning and thinking, representation and symbolic processing, communication, and mathematical tools.

By specifying both the content and processes (sometimes called competencies) of a subject, governmental organizations specify what should be taught as well as, to an extent, how it should be taught. Content standards give information about specific content, for example, "students should measure the perimeter of simple 2D shapes," (Department for Education, 2013, p. 117), while process standards are ways of learning mathematics and are tools for clarifying how mathematics education should happen (Niss, 2003). In this case, the *how* is often in broad terms. For example, a curriculum framework might state students should use mathematical tools appropriately, and accompany this statement with guidance that a spreadsheet is an example of a mathematical tool, but the framework generally does not specify a fifth-grade student should use a spreadsheet to collect and organize data from a survey. These curriculum decisions are left to be made elsewhere, be it in curriculum materials provided by a publisher, curriculum guidelines from a school system, or by a teacher when making lesson plans.

The existence of education policy in the form of curriculum frameworks and standards do not, however, guarantee uniform implementation in the classroom. Though many researchers and policymakers seem to assume that practice follows policy, the reality is often quite different (Cohen & Hill, 2000, Darling-Hammond, 1990). New policies can actually increase the variability of mathematics instruction (Cohen & Ball, 1990b). Policies about mathematics education often fail to provide teachers with opportunities to change their beliefs about mathematics and teaching or opportunities to develop new methods of teaching mathematics. Without these opportunities, teachers are left to interpret educational policies on their own, often based on how teachers understand the meaning of key terms (Cohen & Ball, 1990b), and what teachers are used to doing (Darling-Hammond, 1990), not so much changing their practice as just adding new parts to what they already do (Cohen & Hill, 2000).

When teachers interpret curriculum policy, they do so for several reasons (Darling-Hammond, 1990). The meanings of terms such as *problem-solving* and *understanding* can be interpreted in different ways by different people (Cohen & Ball, 1990b). Often policies lack clarity or teachers do not fully understand the policy or framework and are therefore unable to determine how the framework should influence their teaching. In other cases, local administrators present the policy to teachers, and as a result teachers receive information as "through a filter, with most of the contextual clues filtered out" (p. 342). Teachers turn to textbooks and other supplemental materials in order to help them interpret curriculum policy. In this case, however, teachers will often add new content to their lessons, but may not change the nature of their teaching. Some teachers base their interpretation of curriculum frameworks on their own beliefs and attitudes about mathematics and learning, as well as on their prior learning and experiences. Other teaching decisions are based on selective interpretation of policy based on other factors such as classroom-management, grading, and lesson-planning (Reynolds & Saunders, 1987).

Teachers need the opportunity to learn about the specific curriculum materials related to the frameworks as well as an opportunity to learn about special topics in the framework. Cohen and Hill (2000) describe teacher opportunity to learn as the key dimension to bridge the gap between policy and practice in their "instructional model for instructional policy" (p. 6). Using the example of the Common Core State Standards ("Standards for Mathematical Practice," 2014), a special topic might be *modeling with mathematics*. Teachers need the opportunity to learn specifically about what this topic means and how it relates to the curriculum standards. Finally, teachers need to participate in learning opportunities and related professional development, however these opportunities should be well-focused. These specific opportunities to learn, those that are aimed directly at the curriculum, are more likely to impact teacher practice than other opportunities to learn. Generic professional development, that which is not focused specifically on the curriculum framework, has less of an impact on classroom practice. A focus on mathematical content is crucial to improving teachers' knowledge and understanding for mathematics.

Teacher opportunity to learn is one of the factors Darling-Hammond (1990) describes as necessary for effective curriculum policy implementation. Many teachers struggle with new curriculum frameworks because of a lack of professional development. Darling-Hammond also notes the cumulative effects of policies: while a new policy might completely replace a previous policy, the way a teacher implements a new curriculum policy is influenced by the previous policy (or policies).

Though policy specifying curriculum frameworks provides structure and guidance for classroom teachers, the existence of the policy is not sufficient. In order for curriculum to have the desired classroom impact, provision must be in place for teachers to understand the curriculum content and objectives, as well as provision for teachers to adapt their practice to the expected processes.

How mathematics is assessed

The rise of state and national curriculum frameworks in the 1980s and 1990s was accompanied by a rise in frameworks of assessment (Priestley, 2002). In many cases, the main focus of educational policy has gone from the inputs - the curriculum and how it is enacted in the classroom, to the outputs - the results of testing (Hannaway & Hamilton, 2008). In addition to being a measure of student achievement, in many localities test performance is used as a measure of teacher accountability, often with the aim of improving the lowest-performing schools and students. Localities set achievement targets as well as provide incentives for schools and teachers whose students meet, or fail to meet, those targets. These incentives include such things as performance-pay for teachers whose students reach certain thresholds (Springer et al., 2011), or state-mandated academic reviews of schools where students fail to meet specific achievement targets (Regulations Establishing Standards for Accrediting Public Schools in Virginia, 2013).

It is important to distinguish the assessments referred to in this section from other methods of assessment. In this section, the term *assessment* refers to policy-mandated assessments, such as the SOL (Standards of Learning) tests in Virginia ("VDOE – Standards of Learning (SOL) and Testing," 2012). These SOL tests are a major component in school accreditation (Regulations Establishing Standards for Accrediting Public Schools in Virginia, 2013). Other methods of classroom assessment include formative assessment methods that take place during instruction (Kurz, 2011), such as student interviews, observations, journals, and portfolios (Vásquez-Levy, D., Garofalo, J., Timmerman, M. A. & Drier, H. S., 2001), as well as summative measures such as unit tests. These types of classroom assessments are important as well, but are not the focus of this section.

Traditional mandated assessments are often in multiple-choice format, however many researchers have noted limitations with this format (Linn, Baker, & Dunbar, 1991). One main limitation of the multiple-choice format is the difficulty in assessing process standards such as those in the Common Core State Standards or in the Danish mathematics goals. In a movement to transform assessment policies, researchers and government education agencies are pushing for assessments that incorporate more open-ended problems, essays, hands-on problem-solving, computer simulations, and portfolios. These measures are often termed *authentic* or *performance-based assessments*. With this push for more performance-based assessments, there are issues of consistency, reliability and comparability of assessments, particularly from year to year.

In 1992, Vermont implemented a system of mathematics portfolios as a major part of their statewide assessment of student achievement (Klein, McCaffrey, Stetcher, & Koretz, 1995). Teachers assigned several exercises they believed promoted problem-solving and mathematical communication. At the end of each school year, students selected between five and seven pieces to include in these portfolios. Each piece in the portfolio was rated on seven dimensions of up to four points each. Portfolio readers participated in training before rating the portfolios. During the first two years of the portfolio program, there was little agreement between readers in regards to a portfolio's score, and scores often varied widely across the items within a portfolio. The problems with reliability were so great that Vermont was unable to allow student-level portfolio results. Though not part of the statewide assessment program any longer, Vermont schools still use portfolios as part of the local assessment process ("Portfolio/Problem Solving Resources," 2014).

Currently, two consortia are developing and implementing statewide assessments to measure students' attainment of Common Core State Standards (Herman & Linn, 2013). Both consortia, Smarter Balanced Assessment Consortium and Partnership for Assessment of Readiness and Careers (PARCC), were formed through state partnerships in order to pool resources to develop the assessments. The 2014-2015 school year is the first year of full implementation of these statewide tests. Though the main focus of each consortia is on the summative, end-of-year assessments, each is also developing formative assessments for use during the school year. Assessments from each consortia are technology-based, though the Smarter Balanced assessments are adaptive and respond to student ability levels, the PARCC assessments are standard and fixed. Compared to previous statewide assessments, the consortia assessments appear to test students' deeper learning of each subject's assessment targets – approximately 29 mathematics targets at each grade, though full results are not yet available. Depth of learning will be measured using Webb's Depth of Knowledge (DOK) classification (Webb, Alt, Ely, & Vesperman, 2005). The four mathematics DOK levels are:

> Level 1 – Recall
> Level 2 – Skill/Concept
> Level 3 – Strategic Thinking
> Level 4 – Extended thinking (pp. 45-46).

One challenge for states as they transition from their previous statewide assessments to those developed by the consortia will be to help schools and teachers effectively interpret and drive instruction based on the large amount of information provided by the assessments.

The implementation of mandated performance-based assessments can lead to changes in the amount of instructional time for each subject. For tested subjects, such as mathematics, instructional time often increases, which sometimes results in time taken away from non-tested subjects (Hannaway & Hamilton, 2008). In some cases, instructional time for tested subjects increases only in those grades being tested. These shifts in instructional time may happen because of teacher decisions, as in the case of elementary teachers who often have more control over their class schedule, or from wider school or district decisions. As curriculum frameworks call for a focus on conceptual understanding in mathematics, teachers are expected to respond with classroom techniques and activities that help students develop that understanding. This type of teaching often requires more instructional time than merely teaching for knowledge (Darling-Hammond, 1990).

Though increased accountability through mandated testing can lead to increased instructional time on those subjects being tested, often some of that time, and sometimes a substantial amount of that time, is spent teaching test-taking skills rather than content (Hannaway & Hamilton, 2008; Hoffman, Assaf, & Paris, 2001, Supovitz, 2009). Teaching curriculum is different from teaching test items and, though focusing on items from previous tests may improve test scores, it is far less likely to indicate content learning (Popham, 2001). One way to reduce item-teaching is to focus instruction on the content represented by test items rather than on the specific test items themselves.

The format of the assessment also has an impact on what is emphasized in classroom instruction (Darling-Hammond, 1990; Hannaway & Hamilton, 2008; Supovitz, 2009). For example, a locality that uses a test that emphasizes problem-solving or writing is likely to see an increase in instructional emphasis in those areas. Likewise, a test which stresses basic facts and recall will likely lead to instruction focusing on facts and recall. Additionally, teachers sometimes decide to de-emphasize curriculum topics that are not assessed and focus more instructional time on topics that are assessed (Hannaway & Hamilton, 2008). Because of the alignment between instruction and assessment, alignment between curriculum expectations and assessment is crucial. Critics of standardized testing often cite a misalignment between curriculum and assessment. In some cases, the type of knowledge and performance measured by certain tests is quite different from the conceptual understanding called for in curriculum frameworks (Darling-Hammond, 1990). Even when content is aligned, teachers often stress the types of problems student will encounter on assessments (Stecher & Mitchell, 1995). For example, if an assessment format is multiple-choice, teachers may emphasize those types of problems over other formats.

Assessments can also have a differential impact on students in a specific classroom (Hannaway & Hamilton, 2008). In order to achieve the highest possible number of passing test scores in a class, a teacher might focus greater instructional effort on students who are at or near the passing threshold. While this strategy has the potential to influence test scores, it does little to help students who are far below the passing threshold nor does it help advance students who are far above the threshold. In fact, resources are often diverted away from the students who are struggling the most (Hamilton, Stecher, Marsh, McCombs, & Robyn, 2007). This practice is at odds with one major aim of many assessment-based accountability policies, specifically raising achievement of the lowest performing students (Hannaway & Hamilton, 2008).

A further criticism is that, despite their widespread use as indicators of school progress and teacher effectiveness, these state and national tests have limited use in instructional guidance (Supovitz, 2009). Not only are the assessments generally summative rather than formative, they also give little insight into student thinking, understanding or misconceptions. The assessments can provide information about student starting points, but not information about effective instructional techniques. For example, an fourth-grade end-of-year assessment can provide a student's fifth-grade teacher with information about things that student knows or does not know, but gives little, if any, guidance on how to move forward instructionally. As a response to the lack of instructional information provided by such summative assessments, many localities have implemented regular benchmark assessments (Herman & Baker, 2005). These benchmark assessments are designed to help teachers get useful information to use when planning instruction.

As with curriculum policies, the existence of assessment policies is not enough to make sure that classroom practice changes in the desired ways. Teachers must be aware that factors such as increased instructional time do not always lead to increased time on content instruction, but can often be eroded away by test preparation. It is crucial to focus on content and skills represented by tests instead of focusing on individual test items. Additionally, teachers have the responsibility to focus on the needs of all students, not only those at or near certain test-score thresholds. While teachers may not be able to change testing policy, they have the ability to monitor and adapt their own responses to testing policy.

How much time is spent teaching and learning

Instructional time can contribute strongly to student achievement (Kurz, 2011), and as noted earlier, one side effect of assessment policy can be additional instructional time for tested subjects. Policymakers also attempt to determine how much educational time there should be and, in many cases, how that time is spent (Benavot, 2007). As such, this can also directly impact what happens in mathematics classrooms. Some instructional time policies specify the exact amount of time to be spent on mathematics instruction, while other policies give less specific guidelines. As noted earlier, agreements at the national level in Denmark establish the minimum instructional time for grades in the *Folkeskole* ("Agreement between," 2013), while still allowing individual schools to decide how to allocate that time through the instructional week. Other policies, such as the accreditation standards for Virginia public schools, specifically mention instructional hours in mathematics. The Virginia standards mandate an average of 5.5 hours of instructional time for students in grades 1 through 12 and at least 180 teaching days for a total of at least 990 instructional hours (Regulations Establishing Standards for Accrediting Public Schools in Virginia, 2013). Elementary-school students are expected to spend 75% of their 990 instructional hours (742.5 hours) in English, mathematics, science, and social studies or history, though specific times for each subject are not given. Middle-school students must receive 140 hours of mathematics instruction each year (for reference, approximately 46 minutes per day for 180 days). High-school students must earn 4 credits of mathematics in order to meet graduation requirements, with 1 unit of credit being 140 hours of instruction in a specific course. Other state policies are much less specific. State law in Minnesota, for example, requires 935 hours of instruction for students in grades 1 through 6, and 1,020 hours in grades 7 through 12 (Minnesota Statutes, 2012). The law does not specify how those instructional hours are to be distributed.

Though instructional time policies exist, they often leave room for a great deal of variance. For example, though Virginia middle-school students must receive 140 hours of mathematics instruction, there is no similar requirement for elementary students. The 742.5 hours is likely to be unevenly distributed, with large parts devoted to English instruction (Benavot, 2007). Schools in England are required to teach students at least 190 days but there are no specific requirements for how those days or time within those days is to be distributed ("School day and school year," 2013). Schools that are under policies with specific instructional time expectations have a greater chance of uniform amounts of instruction time for mathematics than those schools whose policies are open to greater interpretation.

Teacher Preparation and Licensing/Certification

The terms *license* and *certification* are both commonly used in the United States and Canada to refer to approval from a governmental body to teach in a public school (Youngs et al., 2003). For example, teachers in Virginia earn a teaching license ("VDOE – Licensure," 2012), while teachers in Ontario, Canada, must qualify for a teaching certificate ("The Teaching Profession," 2014). In other countries, different terms may be used. For example, in England and Wales, teachers must obtain *qualified teacher status* (QTS) in order to teach in certain schools ("Qualified teacher status," 2014). *Folkeskole* teachers in Denmark generally must hold a Bachelor of Education degree from one of seven university colleges ("Teacher training," 2014), and pre-service teachers generally select three subject areas in which to specialize. Though the terms and accompanying processes vary somewhat from jurisdiction to jurisdiction, for the purposes of this paper, terms such as *license, certification,* and *qualification* refer to the means policymakers use to give approval for individuals to teach in schools.

Concerns about the quality of teachers can lead to teacher certification standards (McNergney, Medley, & Caldwell, 1988). Teacher content and pedagogical knowledge is crucially important to student achievement (Darling-Hammond, 1990), so policymakers use teacher licensure as a means of ensuring teachers have a basic level or skills or knowledge (Goldhaber & Brewer, 2000). While the specific policies vary widely from jurisdiction to jurisdiction, these licensure requirements often include factors such as completion of - and grade point averages in - certain university course work, scores on standardized tests, assessments of basic skills, content knowledge, pedagogical knowledge, and performance assessments (Youngs, Odden, & Porter, 2003). Many jurisdictions set standards for colleges and university teacher preparation programs, though these requirements also vary substantially (Goldhaber & Brewer, 2000). Teacher education programs with an emphasis on courses in teaching and learning that are also closely aligned with well-supervised clinical experience tend to produce more effective teachers (Darling-Hammond, 2000).

In some classrooms, teachers are prepared and licensed, but not in the subject area being taught. Mathematics teachers who have subject-specific preparation are more effective teachers than those teachers lacking a mathematics background or license (Goldhaber & Brewer, 2000). The mathematics students of teachers who are not licensed for mathematics do not achieve as well as students who have licensed mathematics teachers. Not only does teacher preparation and licensure matter, but subject-specific preparation and licensure also matters.

Some states are facing, or have faced, shortages of qualified teachers. As a result, some states have issued emergency or alternative teaching licenses to individuals with little or no formal teacher training (Darling-Hammond, 1990). In other cases, jurisdictions have established alternative licensure routes such as the National Teacher Corps in the 1960s and 1970s (McNergney et al., 1988) and the more recent Teach for America program (Eckert, 2011). Critics of these alternate certification routes note that the relatively minimal training these teachers receive, often only several weeks in the summer before they enter the classroom, leads to poor teaching performance and lower student achievement (Darling-Hammond, 2000). These teachers also tend to have less subject-specific training, including subject-specific pedagogy, than traditionally licensed teachers. These alternately-certified individuals are often the least-prepared for teaching but tend to teach in schools that are high-minority, low-income, and have the highest need of well-prepared teachers. In a study of school districts in Texas, Ferguson (1991) notes that teacher performance on a statewide recertification exam required for all teachers in the state accounted for more variance in student reading and mathematics achievement that student socioeconomic status. Clearly, teacher knowledge, experience, and preparation are crucial to student achievement (Fetler, 1999).

Strong models of teacher preparation have several specific aspects (Darling-Hammond, 2006). First, these teacher preparation programs focus on the *what* of teaching: knowledge of curriculum content and goals, a thorough understanding of the pedagogical skills necessary for teaching, and knowledge of how students learn and develop. The second aspect of a successful teacher preparation program is that it focuses on *how* to teach. This aspect has several features, including helping pre-service teachers learn and understand ways to teaching that are often different from their own experiences as a student, learning how to not only think as a teacher but to act as a teacher, and to learn how to handle the many simultaneous demands of teaching. These *whats* and *hows* of teaching can only be achieved through coherence and integration, both among the courses pre-service teachers take, and between their courses and their clinical experiences, and through extensive clinical work what is well-supervised and offers access to a diverse range of students.

While policies regarding teacher preparation and licensure help to address a range of aspects before teachers take full responsibility for a class of students, other teacher qualifications are also important to student achievement. Aspects such as scores on teacher licensing exams, type of teaching license, and whether a teacher has a graduate degree or not are all have an impact on student achievement (Clotfelter, Ladd, & Vigdor, 2007) and are often considered as part of licensure policies, but initial, policies regarding basic teacher qualifications are not sufficient. Additional teacher credentials such as years of teaching experience and National Board Certification also impact student achievement. Policies could and should exist that not only encourage teacher retention but also enable teachers to obtain additional academic degrees and National Board Certification.

Conclusions

Educational policy contributes a great deal to what happens in the mathematics classroom. Some of these policies can have a positive impact. Providing curriculum structure and guidance, assessing student learning, providing for adequate instructional time, and making sure teachers meet minimum qualifications are all encouraging policy goals. But policymakers should not neglect the unintended impacts of polices. Vague or unclear curriculum polices can be interpreted numerous ways. High-stakes testing can result in reduced instructional time for content, the educational neglect of students who are most in need of academic support, and data that does not provide for effective instructional decisions. Teacher preparation and licensing policies can give a misleading sense of teaching ability: having a teaching license does not necessarily imply an individual is prepared to teach mathematics. It is how policy is interpreted and enacted at the classroom level that has the potential for the greatest impact on students.

The Place of Pedagogy in the Mathematics Classroom

The previous section is a discussion of the place of policy in the mathematics classroom. Teachers make instructional decisions based on curriculum factors as described earlier and they make these decisions within the structures of policies mandated by government agencies. The beliefs teachers have about how students learn and the role of the teacher also influence classroom practice. This section explores the place of pedagogy in the mathematics classroom.

The term *pedagogy* literally means “the art and science of teaching children” (Knowles, 1973, p. 42). In practice, pedagogy describes “the process through which knowledge is produced” (Lusted, 1986, p. 2). Whereas educational policies often help answer the question, *What content?*, and curriculum addresses *What materials?*, pedagogy focuses on *What method?*

The methods of teaching and learning mathematics can vary depending on teachers’ attitudes about mathematics (Ernest, 1989; Wilkins, 2008). A teacher who enjoys mathematics and is interested in the subject is likely to use quite different teaching methods than a teacher who dislikes or fears mathematics. The curriculum materials teachers select, and the learning activities teachers plan can be influenced to a strong degree by their attitudes towards mathematics. Teachers who have more positive attitudes towards mathematics are more likely to use inquiry-based instruction in their classrooms (Wilkins, 2008).

Teacher beliefs about mathematics also influence classroom practice. Ernest (1989) describes teachers’ beliefs about mathematics as fitting into one or more of three mathematical philosophies. The first is the problem-solving view: that mathematics is a dynamic subject, open to inquiry. The second is a static view: mathematics is an unchanging body of interconnected body of truths. The third philosophy is the instrumentalist view: mathematics is an unrelated collection of rules, skills, and facts. Ernest notes teachers may draw from more than one of these philosophies. Each of these beliefs or combinations of beliefs about mathematics can lead to different classroom environments. A teacher with an instrumentalist view of mathematics might insist students learn the rules and algorithms for long division by doing pages of problems, for example, while a teacher with a problem-solving perspective is far more likely to include investigations and discussions about real-life contexts that involve long-division in conjunction with other skills and concepts.

Teachers' beliefs about the nature of mathematics greatly influence their beliefs about teaching and learning mathematics. Ernest (1989) sets forth a set of six models of mathematics teaching:

- the pure investigational, problem posing and solving model
- the conceptual understanding enriched with problem-solving model
- the conceptual understanding model
- the mastery of skills and facts with conceptual understanding model
- the mastery of skills and facts model
- the day to day survival model (p. 22)

This set of models aligns with Ernest's ideas about teachers' beliefs about the nature of mathematics. A teacher with a very instrumentalist view of mathematics will likely view mathematics teaching as a set of facts necessary for day-to-day survival of mastery of basic skills, compared with a teacher who view mathematics as a dynamic subject and encourages his or her students to investigate and problem-solve. These beliefs about mathematics and mathematics teaching will be reflected in the curriculum materials a teacher creates, selects, or adapts.

In presenting his models of mathematics teaching, Ernest (1989) also presents a set of models for learning mathematics. These models incorporate not only teachers' beliefs about the nature of mathematics, but also beliefs about the role of the student, about appropriate behaviors and activities, and what makes appropriate learning activities. The six models of student learning of mathematics are:

- child's exploration and autonomous pursuit of own interests model
- child's constructed understanding and interest driven model
- child's constructed understanding driven model
- child's mastery of skills model
- child's linear progress through curricular scheme model
- child's compliant behavior model (p. 23)

Similarly to the models for teaching mathematics, it is clear that a teacher's beliefs about learning mathematics will greatly influence a how a student experiences mathematics in the classroom.

Teachers' beliefs about the nature of mathematics and about mathematics teaching and learning come from a variety of sources (Raymond, 1997). Beliefs about the nature of mathematics often come from teachers' own experiences as students, while beliefs about teaching and learning are also influenced by teachers' own teaching practice. Other influencing factors are teacher education programs, societal and school teaching norms, as well as personality traits of individual teachers. Despite their beliefs, however, teachers' teaching practices can also be influenced by additional factors, including limitations of time and resources, mandated assessments, as well as classroom management issues. In many cases, managing the immediate classroom situation can take priority over a teacher's beliefs about teaching and learning.

A search for *pedagogy* will turn up a number of pedagogical models, each focused on teaching and learning methods relevant to specific groups. For example, *critical* *pedagogy* focuses on the relationship between schooling and the social conditions of disaffected students (McLaren, 1998). *Pedagogy of the oppressed* is a pedagogy that proposes changes to the relationship traditional colonizer/colonized between teachers, students, and society (Freire, 2000). Each pedagogical model is a view of teaching that applies to students in a particular group.

Of particular relevance to mathematics education is *culturally relevant pedagogy.* This specific pedagogical approach describes “effective teaching in culturally diverse classrooms” (Irvine, 2010, p. 57). It is based on the premise that learning varies across cultures and student success can be increased when teachers use this understanding in their teaching practice. In mathematics education, culturally relevant pedagogy sometimes specifically refers to teaching methods to help African American students be more successful in mathematics classes (Tate, 1995). These methods draw on cultural traditions such as children’s oral expression, and active learning, rather than remaining seated and working problems from a textbook all lesson. In many cases, culturally relevant pedagogy emphasizes not rote memorization, but open-ended problems and problem-solving strategies, mathematical persuasion and justification, and connecting mathematics to other areas of students’ lives.

In chapter 1, I discussed a continuum with algorithms and rote memorization at one extreme and depth and understanding on the other, and ways in which teachers negotiate tension points on this continuum. In view of Ernest’s (1989) ideas about models of mathematics teaching and learning, many of the discussions and conflicts surrounding Common Core State Standards for Mathematics actually center around differences in pedagogy. Though the mathematics itself may not be different from when parents were in school, the ways of teaching and learning mathematics, in many cases, are quite different.

The introduction of process skills – whether they are called practices, goals, or competencies – has the potential to imply certain mathematics pedagogy. For example, look again at the eight Common Core Standards for Mathematical Practice:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning ("Standards for Mathematical Practice," 2014)

The existence of these standards for mathematical practice imply that mathematics is not, as described by Ernest's (1989) instrumentalist view of mathematics, an unrelated collection of rules, skills, and facts. The standards also imply certain beliefs about mathematics teaching and learning: if nothing else, teaching and learning mathematics is about more than just mastery of skills and facts.

The existence of process skills is not sufficient, however, in defining a specific mathematics pedagogy. In chapter 1, I described two teachers, Laura and Birgit, and showed how their interpretation of the same competency could lead to quite different classroom experiences and student outcomes. Even within the framework of process standards, teachers still make interpretations and pedagogical decisions based on their own attitudes and beliefs about mathematics and mathematics teaching and learning.

Pedagogical Content Knowledge

In chapter 2, I have presented my theoretical framework in which curriculum, pedagogy, and policies, each influence what happens in a mathematics classroom. I have discussed each area separately, and in some cases described how the intersection of two areas. For example, curriculum materials are selected, in part, based on the content set by education policy in the form of standards. Curriculum materials are also selected based on a teacher's pedagogical ideas. These intersections between two areas – such as policy and curriculum, or curriculum and pedagogy, will be explored further in my analysis.

The intersection of pedagogy and one aspect of policy – specifically content – deserves greater attention at this point. There is often an acute distinction between mathematical content and pedagogy (Shulman, 1986). On one hand, policymakers often cite the need for teachers to have specific subject-area content knowledge, while on the other hand people call for teachers to have skills in classroom management (Ball, 2000). In actuality, neither content knowledge nor pedagogical knowledge alone are sufficient for effective mathematics teaching. Pedagogical content is distinctly different form either mathematical content knowledge or general pedagogical content knowledge, but requires a "special amalgam" (Shulman, 1987, p. 8) of both. The intersection of and interplay between these two types of knowledge is called *pedagogical content knowledge* (Ball, 2000).

Pedagogical content knowledge (PCK) in mathematics is the specific knowledge about how to teach mathematics content to students (Shulman, 1986). This includes knowing the most effective representations and examples for conveying ideas and knowing alternate representations when necessary. In short, it is being able to make mathematics understandable to others. PCK is also knowing what particular aspects of a topic students often find difficult, what are the common preconceptions or misconceptions students have, and how to help them overcome any misconceptions. It is also being able to read a student's work and not only recognize any errors, but to understand how and why a student made that mistake, and to know how the next step in correcting that error. For Shulman (1987), pedagogical content knowledge is the ability to take desired student outcomes and transform those outcomes into pedagogical representations and activities. This blending of content and pedagogy is "an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners and presented for instruction" (p. 8).

Shulman's (1986) initial ideas about pedagogical content knowledge have been further refined by Ball, Thames, and Phelps (2008). The Ball et al. Mathematical Knowledge of Teaching (MKT) model draws heavily on Shulman's work. In addition to three types of subject matter knowledge, the MKT describes three types of pedagogical content knowledge: knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. These three areas encompass Shulman's original ideas, but help to more clearly describe the ways teachers must understand content and pedagogy. Knowledge of content and teaching incorporates knowing what examples are generally effective in conveying concepts and ideas, and helping to address students' errors. Knowledge of content and students refers to knowing how best to help individual students learn specific mathematics content. The final area, knowledge of content and curriculum, means being able to select the appropriate curriculum materials to help students learn specific content. These three areas together describe the varying yet inter-related ways content and pedagogy intersect.

Research Questions

The review of the literature indicates three areas - curriculum, policy, and pedagogy – and their intersections that each have an influence on how teachers' classroom practice. The way teachers interpret a mathematics standards has an impact on classroom practice and what students do in their mathematics classes. Absent from the literature is an examination of how the interplay between these three areas influences classroom practice. The research questions guiding this study are:

1. How do teachers interpret the Danish communications competency?
2. In what ways are those interpretations enacted in classroom practice?

CHAPTER 3

METHODOLOGY

The review of literature in the previous chapter provides insight into three areas impacting mathematics classroom practice – curriculum, policy, and pedagogy - as well as insight into the history of education in Denmark. This study builds on previous research by investigating a specific part of mathematics classroom practice, namely how teachers implement the writing portion of the communication competency. The primary methods of data collection are instructional artifacts, classroom observations and interviews with teachers and students. This chapter presents the research design, participants, methods for data collection, and methods for data analysis.

The Research Design

A grounded theory research design can be used when studying a process (Creswell, 2012). This type of research, first described by Glaser and Strauss (1967), is a specific research design defined by its focus on allowing a theory to emerge from the data rather than using pre-determined coding strategies or theories. It stresses ongoing comparative coding that compares "incident to incident, incident to category, and category to category" (Creswell, 2012, p. 429). The rationale for a grounded theory research design is that the specific factors involved in a process are not yet known and the lack of a predetermined theory allows for greater in-depth exploration of a topic. While grounded theory methods are used to develop a theory, grounded theory analysis is not atheoretical (Elliott & Higgins, 2012). A critical review of relevant literature is useful in formulating the research questions and helping the researcher to develop a conceptual map of the subject (Hart, 1988). Glaser and Strauss (1967) state the researcher "does not approach reality as a *tabula rasa*" (p. 3).

Participants

The participants in this study were classroom teachers in a large city in Denmark, their mathematics students, and a National Mathematics Advisor for primary school mathematics. I selected teacher participants through a process similar to that described by Michelle Foster (1997) as *community nomination*. In this study, the process of community nomination is an attempt to obtain an insider's view, an emic perspective (Etter-Lewis & Foster, 1996), of mathematics teachers in Denmark. For my study, community nomination involved selecting teachers through contact with Danish mathematics educators. Two Danish mathematics education professors provided the names of three teachers based on their own prior work and experiences with those teachers. The professors have worked with these teachers in the teaching-college classroom as pre-service and in-service teachers, and continue their work with these teachers and their primary-school mathematics classes. In some cases, the professors take their own teaching-college students to these teacher's primary mathematics classes as a field experience. From their work with these teachers, the professors selected teachers who are interested in sharing their mathematics practice with others.

This study had a total of seven teacher participants. Three of the teachers were nominated because they were described by mathematics education faculty at a teaching university in the city as being individuals who welcome visitors into their classrooms and are interested in sharing their mathematics teaching practice. A fourth teacher was selected because she is a National Mathematics Advisor for the secondary-school HTX program. Three of these four teachers then each selected a colleague from their own school, choosing someone who would be willing to be both observed and interviewed about their own mathematics teaching practice. This additional nomination step allowed for extension of the community nomination process to a wider pool of potential participants. As the interviews took place in English rather than Danish, the selection of teachers who felt comfortable conversing in English was also a consideration. The seven teachers taught a range of grades, from third graders to students in their last year of high school, and represent both one private school and three public schools.

Data Collection

The data collection of this study consisted of three main sections: instructional artifacts, lesson observations, and interviews.

Instructional Artifacts

One primary source of data for this study was instructional artifacts. Samples of student work in the form of photographs provided a vital insight into how these students are using mathematical writing in their classes. In some cases, these photos were taken as students were working on the particular assignment in class, and in other cases, after the work was handed in. Other instructional artifacts included photos of textbook and workbook pages and class handouts or photocopies of the handouts themselves. Where possible, I also took photographs of things teachers and students wrote on the board, as well as related classroom displays or bulletin boards. Another set of instructional artifacts were in the form of blog postings on a variety of mathematics topics created by students in two classes. Finally, from each school's website I also collected documents relating to weekly class timetables and other instructional data where available.

Lesson Observations

One teacher was observed teaching seven mathematics lessons, six of which were to the same class of third-grade students and one lesson to a class of tenth-grade students. These six observations took place over a period of two weeks and were interspersed with observations in other classrooms. I purposefully selected this teacher to observe in depth because of the specific types of mathematics activities taking place in her classroom as well as because she taught mathematics in both third and tenth grades.

Of the remaining teachers, one teacher was observed twice, once with each of two different classes. The remaining five teachers were observed teaching one mathematics lesson each. Lesson observations took place during regularly-scheduled lessons and teachers were asked to teach the topic and in the style they would have done had I not been present. For much of the lessons, I was a passive observer, though while students were working, teachers encouraged me to move around the classroom and ask students questions if I chose.   During each lesson, I took field notes by hand and then wrote-up full versions of my field notes, in most cases the same day, but in all cases within 24 hours of the observation. The field notes and typed write-ups consist of my own observations and impressions, thoughts and questions, classroom diagrams, examples of teacher and student discourse, and examples of what was written on the board. In several cases, the field notes, and therefore the write-ups, are supplemented with photographs of student work, textbook pages, things written on the board, and general classroom appearance.

Interviews

Five of the teachers were formally interviewed about their mathematics teaching and their use of writing in mathematics classes. Each formal interview was semi-structured based on the protocol in Appendix 1. Two of the interviews took place after I observed those teachers' mathematics lessons, two took place before the observations, and the final interview, of the teacher who was observed multiple times, took place after one lesson observation, but before the remaining observations. The interviews took place in the teacher's school, either in the classroom after the lesson, or in the school teachers' room. Each interview was conducted in English, was audio recorded, and I took written notes throughout. The teacher interviews ranged in duration from 12 minutes (though this interview was supplemented with additional informal conversations) to 63 minutes. I later transcribed each interview into a typewritten document. The remaining two teachers were not interviewed formally, but instead took part in informal conversations addressing topics similar to those in the interview protocol. These conversations were not recorded but I took written notes and included these teachers' thoughts and ideas in the typed observation records.

I conducted 14 student interviews with 16 different students. In all but 12 interviews, students chose to be interviewed individually. Four students asked to be interviewed in pairs rather than individually. In nine cases, the interviews were one-on-one with just myself and the participant. The remaining five cases three third-grade students for whom their teacher, Maria, assisted with the interviews because of the low English proficiency of three students, and the two interviews done with pairs of students. In these two paired interviews, the students' teacher, Anna, was also present to help with translations is necessary. Students were selected in most cases by their teachers based on a range of characteristics, including student performance in mathematics, student English language proficiency, and student willingness to be interviewed. In the case of the three third-grade students, Maria conferred with another of the students' teachers about which students should be selected. These three students were selected because of their overall success in all lessons. In two other cases, with teachers Jacob and Martin, I specifically asked these two teachers if they would each ask a specific individual to agree to an interview. These requests were based on observations I made during the lessons, and hoped to follow-up on further during the interviews. Each interview was semi-structured based on the protocol in Appendix 1. In each case, the interviews took place after I observed those students' mathematics lessons. Each interview was conducted in English, though as noted above, in some cases teachers helped to translate portions of the interview between Danish and English. The interviews took place at the students' schools, either in their classrooms after the lesson, or in a nearby room. The student interviews ranged in length from 7 to 25 minutes. Each interview was audio recorded, and I took written notes throughout. I later transcribed each interview into a typewritten document. In one case, a student was unsure of a mathematics term in English so he gave the Danish term. During transcription, I extracted the sentence in which the student used this term and sent the 17-second audio clip to a native Danish speaker who is a professor of mathematics education for translation.

The final participant in the study was the National Mathematics Advisor for primary mathematics. This advisor was invited to participate in the study because of his role in advising the Danish Ministry of Education on primary mathematics education policy, as well as his role in helping teacher implement these policies in their mathematics classrooms. This interview lasted for 67 minutes and was conducted in English at the Ministry of Education in Copenhagen. The interview was audio recorded, and I took written notes throughout. I later transcribed the interview into a typewritten document. A summary of teachers, schools, observations, and interviews is given in Figure 3.1.

Participant Consent and Notification

All participants were given information about the study as approved by the University of Virginia Institutional Review Board in IRB 2014-0059. The Board approved consent forms and notification letters in their English-language form, which were then translated into Danish by a native Danish speaker who is also a professor of mathematics education. These Danish translations were further checked by a second native Danish speaker who is also a mathematics teacher and National Mathematics Advisor for secondary mathematics. Teachers each received an Informed Consent Agreement for the lesson observations and interviews. Teachers each signed and returned a copy of the consent form to me and kept a copy for their own records. I provided notification letters to teachers who gave them to students and their parents. The primary mathematics advisor also received a notification letter. These consent and notification documents are provided in Appendix 2. IRB approval was granted until February 20, 2015, with a continuation granted for an additional year until February 20, 2016, to allow for analysis and follow-up data collection as needed. All of the data collection took place during twelve school days in March, 2014. The data collection was funded in part by a Graduate Research Grant from the University of Virginia Center for International Studies.

School	Teacher	Gender, years of math teaching experience	Class (number indicates grade)	Lesson observations	Pupil Interviews	Teacher Interviews
A – Primary (private)	Maria	Female 20	3B	6	3	1
			10	1	2	
	Jacob	Male 8 (11 total)	7B	1	4	1
B – Primary (public)	Charlotte	Female 15 Matematik-vejleder*	4C	1	0	1
	Martin	Male 15 Matematik-vejleder*	7A	1	3	1
C – Primary (public)	Anna	Female 17	6	1	2 (4 pupils)	1
D – High School – HTX (public)	Henrik	Male 2	B2 (2nd year)	1		Informal conversation
	Pernille	Female 24 National Advisor for HTX program	A2 (2nd year)	1		Informal conversation
			A3 (3rd year)	1		
Totals: 4 schools	7 teachers		8 grades 9 classes	14 observations	14 interviews with 16 pupils	5 teacher interviews; 2 informal conversations; 1 interview with National Mathematics Advisor for Primary Mathematics

* Matematikvejleder is an additional Danish qualification as a mathematics specialist. This qualification is undertaken after several years of teaching and is earned by completing six additional semester-long courses over a period of three years.

Figure 3.1. Summary of teachers, schools, observations, and interviews

Data Analysis

Analytic Approach

I used Creswell's (2012) systematic approach to grounded theory to analyze the data. This approach has three main components: (a) open coding, (b) axial coding, and (c) selective coding. The open coding phase involves creating initial, broad categories of information relating to the process being studied. Then, in the axial coding phase, one specific category is selected as the main phenomenon and the remaining categories are related to the main category. In the selective coding stage, the categories in the axial coding stage are used to form a theory of how the categories are interrelated.

Trustworthiness

Lincoln and Guba (1985) describe trustworthiness as those things necessary to persuade an audience that a researcher's findings are worthy of attention. In quantitative research designs, trustworthiness often refers to internal and external validity, reliability, and objectivity. In qualitative research, Lincoln and Guba suggest several necessary aspects that should be considered in the research design.

Credibility

In order for findings and interpretations to be credible, a term comparable to internal validity (Lincoln & Guba, 1985), several techniques were used during data collection and were used in the analysis and interpretation phases.

Triangulation. Triangulation refers to the use of multiple sources and methods of gathering information (Lincoln & Guba, 1985). While collecting data, I used classroom observations, participant interviews, instructional artifacts, and informal conversations. I used multiple sources for data collection, including four different schools and, in three cases, two teachers within each school, and, in classes in which student interviews took place, two to four student interviews per class. During my data collection phase, I was also able to discuss certain observations and ideas with a local professor of mathematics education. These discussions helped to provide additional cultural and educational context to my data collection.

Peer debriefing. Peer debriefing allows the researcher to discuss observations and interpretations with a peer who has no stake in the study itself (Lincoln & Guba, 1985). I had two peer debriefers for my study. Both individuals were fellow PhD students who had extensive experience as classroom teachers before beginning their doctoral programs in education. Though each individual had experience in the classroom, neither individual was a mathematics teacher nor was working on a PhD in mathematics education. These two individuals were chosen specifically because they were not specialists in mathematics education. Having such individuals as peer debriefers was valuable during my analysis and interpretation phase. Their experience in research methods and analysis provided me with important insights, while the fact they were not mathematics specialists forced me to consider my observations and interpretations from the perspective of non-mathematics educators.

Referential adequacy. This term refers to the ability to refer to data when making critiques of research findings in order to decide if these critiques are valid and sufficient. Though my classroom observations were not videotaped, my field notes were written-up within 24 hours of each observation and provide a record of my observations and impressions at the time. Additionally, these observations are accompanied by photos of instructional artifacts. Each interview was transcribed and the transcription serves as data which to refer.

Member checking. Member checking refers to the formal or informal process whereby data, analytic categories, interpretations, and conclusions are tested with those from whom the data was originally collected (Lincoln & Guba, 1985). During data collection, I often asked teachers, either in the interview or informal discussions, about certain aspects of the lesson or about things they had told me earlier in order to assess my own understanding and interpretation of what was said or done. During interviews, I often summarized participants' thoughts and ideas and asked participants if my summary was correct and adequate. I was also able to do informal member checking through regular, informal conversations with two of the teachers as well as with the mathematics education professor who helped to identify the initial group of teachers. Additionally, I used email to do additional further member checking with participants during my analysis and interpretation phase.

Transferability

Though qualitative research often attempts to make generalizations across wider populations, the purpose of qualitative research is to provide rich description about smaller populations. The qualitative researcher provides the rich description in order for other researchers to determine the transferability of the data (Lincoln, & Guba, 1985). By researching a range of examples of and factors influencing mathematical writing in classrooms, I have helped to identify variables upon which other research can be designed.

Dependability

In quantitative research, reliability is critical to ensure that the findings are consistent. The corresponding term in qualitative research is dependability. In qualitative research, this is done by examining the process through which the data were kept and examining the product for accuracy (Lincoln & Guba, 1985). Careful records of data collection and analysis helped ensure dependability.

Confirmability

The degree to which research findings can be verified is called confirmability (Lincoln & Guba, 1985). One main way of ensuring confirmability is through the use of an audit trail which is maintained throughout the data collection and analysis. A further audit process at each stage helped to make certain I drew accurate conclusions from the data.

Researcher as Instrument Statement

To a great extent in qualitative research, the researcher is an instrument in that research. The researcher's own experiences, beliefs, and preconceptions have an impact at each stage in the research. It is necessary for the researcher to reflect on those factors in order to be aware and mindful of them during the research study, but also for readers to have an idea of the researcher's own background. Several main factors have helped to shape my beliefs about mathematics and mathematics education.

First, I have been involved with mathematics professionally for sixteen years, as a teacher, department chair, teacher-trainer, writer of mathematics curriculum, author of practitioner articles, and researcher. These six different roles have allowed me to develop ideas and understandings about mathematics education that are more robust that I would have had otherwise. Therefore, my observations and experiences in other teachers' mathematics classrooms are filtered through my experiences in a number of different roles.

Second, I was a middle school mathematics and science teacher in a Virginia public school for several years, and then I was a mathematics teacher at two state schools and one independent school in England for six years. Not only do my classroom experiences influence how I see and experience other teachers' classrooms, but my experiences living and teaching in another country give me a lens through which to experience education in a third country. Additionally, as a teacher in England, I was able to participate in programs that allowed me to travel and observe mathematics lessons in Taiwan, Slovenia, Bulgaria, and Poland. These international experiences have given me additional insight into how the culture of education can vary from country to country. My experiences, both in the classroom and in international visits, as well as background work in preparation for my observations and interviews in Denmark have helped me to become well-prepared and situated to collect and analyze this set of data.

Third, as a non-Danish speaker, all of my experiences in Denmark were filtered through a sieve of language ability of others. Though many Danes are fluent or nearly fluent in English, the nuances of language vary depending on whether one is a native speaker or not. My lack of Danish language ability no doubt restricted my ability to ask and answer questions with a high degree of precision. In addition, my position as an outsider in Danish culture forced me to examine my observations and ask additional questions, such as whether or not what I was seeing was a common occurrence in Danish schools or was more specific to an individual class or teacher.

During my data analysis, I returned to this statement periodically and updated it as necessary. This process helped me to check for any effects I might bring to the research.

CHAPTER 4

FINDINGS

The previous chapter provides details about the methodology of my study, including participants, data collection, and data analysis. A grounded theory research design (Glaser & Strauss, 1967) was used to analyze interviews with teachers and pupils, classroom observations, and instructional artifacts in order to address the two research questions:

1. How do teachers interpret the Danish communications competency?
2. In what ways are those interpretations enacted in classroom practice?

In this chapter I present my findings of my analysis. These findings are focused on teachers, teaching, and interactions with students. First, I present profiles of four Danish mathematics classes in order to provide context for a discussion of the main themes. Next, I describe and examine each of the main themes emerging from the data analysis in regards to the two research questions. Finally, I present a reconceptualization of my theoretical framework in light of the emerging themes.

I began with three themes as informed by my theoretical framework: policy, pedagogy, and curriculum. *Policy* refers to the circumstances of teaching, including curriculum standards such as the Danish *Fælles Mål* (*Common Objectives*), government policies about national exams for pupils, use of instructional time, and teacher education and preparation, as well as other circumstances beyond teachers' control such as learning profiles of individual pupils in a specific classroom. In my theoretical framework, I described *pedagogy* as beliefs about teaching and learning and *curriculum* as a widely encompassing term including textbooks, additional learning resources – including online, teacher support materials, learning activities, and classroom assessments. It was also clear in my analysis that two additional themes emerged as being both distinct from, yet closely related to, the first three themes: communications – both oral and written, and understanding.

Understanding emerged as a theme that, while closely related to the three areas of policy, curriculum, and pedagogy, was actually a much more encompassing idea. The term *understand* appeared in many instances within these other themes. The communications competency itself refers to understanding:

> The communication competence is about pupils being able to express themselves and *understand* others' communication about mathematical topics, including oral, written and visual forms of communication.
>
> The early school years are especially focused on oral and visual forms of communication with the use of simple mathematics terms and concepts. By middle school, pupils will focus on written language also, and pupils are working to *understand* and express themselves in a more precise terminology. In early adolescence, pupils increase the degree of precision further, while there will be an increased focus on the use of the mathematical concepts and notation of technical terms, both written and oral.

A number of study participants referred to understanding in one form or another, whether it was explaining beliefs about why understanding is important in mathematics, describing how a teacher or pupil assesses understanding, or sharing details of curriculum activities designed to help pupils gain understanding of a certain topic.

Although communications also emerged as a distinct theme, pupils' written and oral communications seems to be the primary way teachers and pupils measure understanding of mathematics. This assessment of understanding takes several forms, from a simple, cursory check for understanding to a more in-depth level of understanding. The relationship between these five main themes is illustrated in Figure 4.1. The ways in which teachers interpret and enact the Danish mathematics communications competency is influenced by policy, pedagogy, and curriculum, and teachers use the resulting mathematical communication to assess pupils' understanding of mathematical topics and concepts.

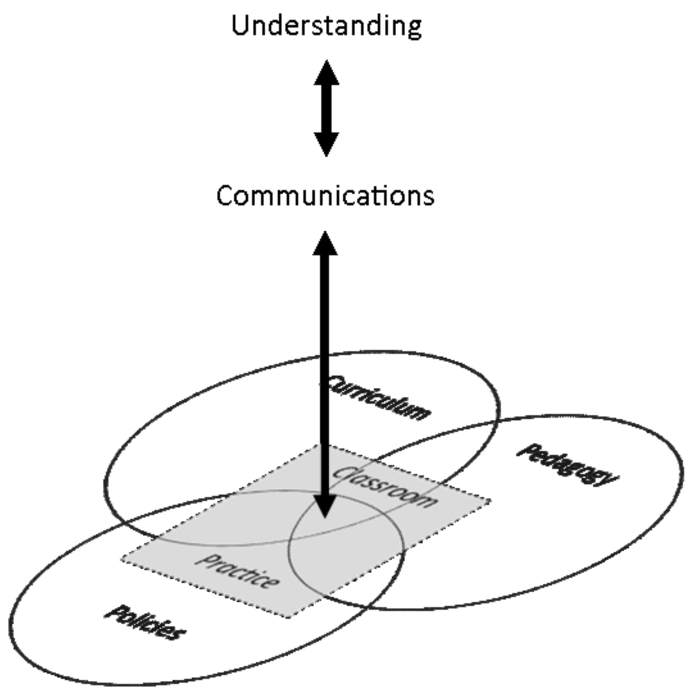


Figure 4.1. Relationship between the five main themes

Before discussing each of the five main themes, I will first provide context for my findings by describing four Danish mathematics classes. After the four profiles, I will examine each of the main themes emerging from the data analysis in regards to the two research questions before giving a reconceptualization of my theoretical framework in light of the emerging themes.

Context for Findings – Class Profiles

To provide context for these findings, I will first describe four different mathematics classes. I will use the term *class* to refer to specific groups of pupils and the term *lesson* to refer to the time in which mathematics was taught to a specific class. For the schools I visited, the class forms the core social structure for pupils and therefore profiles will focus on classes rather than schools. Though I observed nine different classes, each of the four classes and their teachers profiled here was specifically selected. Figure 4.2 provides an extract of Figure 3.1, showing the profiled classes and teachers. Although only four classes and their teachers are profiled here, my analysis will include data from each class and teacher.

School	Teacher	Gender, years of math teaching experience	Class (number indicates grade)	Lesson observations	Pupil Interviews	Teacher Interviews
A – Primary (private)	Maria	Female 20	3B	6	3	1
B – Primary (public)	Charlotte	Female 15 Matematik-vejleder*	4C	1	0	1
	Martin	Male 15 Matematik-vejleder*	7B	1	3	1
D – High School – HTX (public)	Henrik	Male 2	B2 (2nd year of high school)	1		Informal conversation

* Matematikvejleder is an additional Danish qualification as a mathematics specialist. This qualification is undertaken after several years of teaching and is earned by completing six additional semester-long courses over a period of three years.

Figure 4.2. Summary of profiled teachers, classes, observations, and interviews

The first profile describes class 3B and their teacher, Maria. During the data collection phase of this project, I observed six mathematics lessons with class 3B during a 12-day period. In between observations of this third-grade class, I observed eight additional mathematics classes, and conducted interviews with other teachers and pupils. Thus, class 3B became an anchor or reference point in my study as, in many cases, things I observed and ideas I developed during my time with class 3B influenced and gave additional focus to the other lesson observations, discussions, and interviews.

The second profile describes class 7B and their teacher, Martin. Martin's ideas and enactment of mathematical communications provide a contrast to the ideas and enactment found in Maria's lessons. Class 4C and their teacher, Charlotte, are the focus of the third profile. Charlotte teaches at the same school as Martin and they are both *matematikvejledere*, or mathematics specialists. Although Charlotte and Martin took part in the three-year matematikvejleder training together, their ideas about mathematical communication and pupil understanding are quite different. In contrast to the first three class profiles, class B2 is not a primary school class. The pupils in class B2 are in their second year of the three-year high school HTX B-level mathematics program (one of the four upper-secondary programs), after which they will be going into communications-related fields. In contrast to the teaching experience of the other teachers profiled, class B2's teacher, Henrik, is in his second year of teaching. In conversations with Henrik, he explained how his teaching style differs from the teaching he experienced in school.

The name of each participant has been changed as required by the terms of my IRB. In the classes I visited in Denmark, pupils call their teachers by their first names, therefore I have followed this convention throughout. As noted in chapter 3, English is not the first language of any of the participants in this study. In quotations I have removed linguistic fillers such as hesitations and repetitions (such as umms and ahhs) but have kept the rest of the quotation as the speaker originally spoke.

Class 3B

During the mathematics lessons

Class 3B's room is on the second-floor of the main building of a private school. I was told by a teacher in the school that they have approximately 400 pupils. Along one wall of the classroom are three large windows facing a church just across the driveway. In this third-grade class, teacher Maria has 20 pupils, an equal number of girls and boys. Maria has been the class teacher for these pupils since they were in first grade and says she hopes to stay with them through the end of ninth grade. The term *class teacher* describes the teacher who is responsible for both the academic and social welfare of pupils in the class. As the class teacher, Maria has a weekly 40-minute *klassens time* with her pupils. This is a time devoted to developing the social structures within the class group and can be used for things such as class excursions. In conversations with teachers during my visit, I learned that in Denmark it is not uncommon for one teacher to stay with the same class for nine years, although it is becoming less common that it once was. Maria tells me she has done this more than once in her 20-year teaching career. In addition to being responsible for the overall academic and social welfare of her pupils, each week Maria teaches three 80-minute math lessons and one 80-minute science lesson to her third-graders. This year, Maria also teaches eighth-grade math and tenth-grade math.

When asked to talk about their mathematics class, two pupils in class 3B, Sofie and Sebastian, both described their class as *hyggeligt.* This term has no direct English equivalent although the terms *cozy, welcoming*, and *comfortable* are often used to describe *hyggeligt.* A common example given by adults to describe *hyggeligt* includes features such as sitting on a couch in front of a warm fire with a drink and some friends. Clearly, however, these pupils have no couch and no fireplace in their classroom, yet they still described their mathematics class as *hyggeligt*. Sebastian and Sofie both explained that working with their friends and helping one another in class were two things that helped create a *hyggeligt* environment.

When the bell rings, Maria and I walk to the classroom. She explains to me that in Denmark, the pupils remain in the same classroom and the teachers move from room to room. The pupils are seated at individual desks arranged in pairs. There are three columns of desks, but they are staggered so that the rows are not evenly aligned. On the walls of the classroom are samples of pupils' projects and a row of colorfully-decorated paper fish hang from a string stretched across the middle of the room (Figure 4.3).

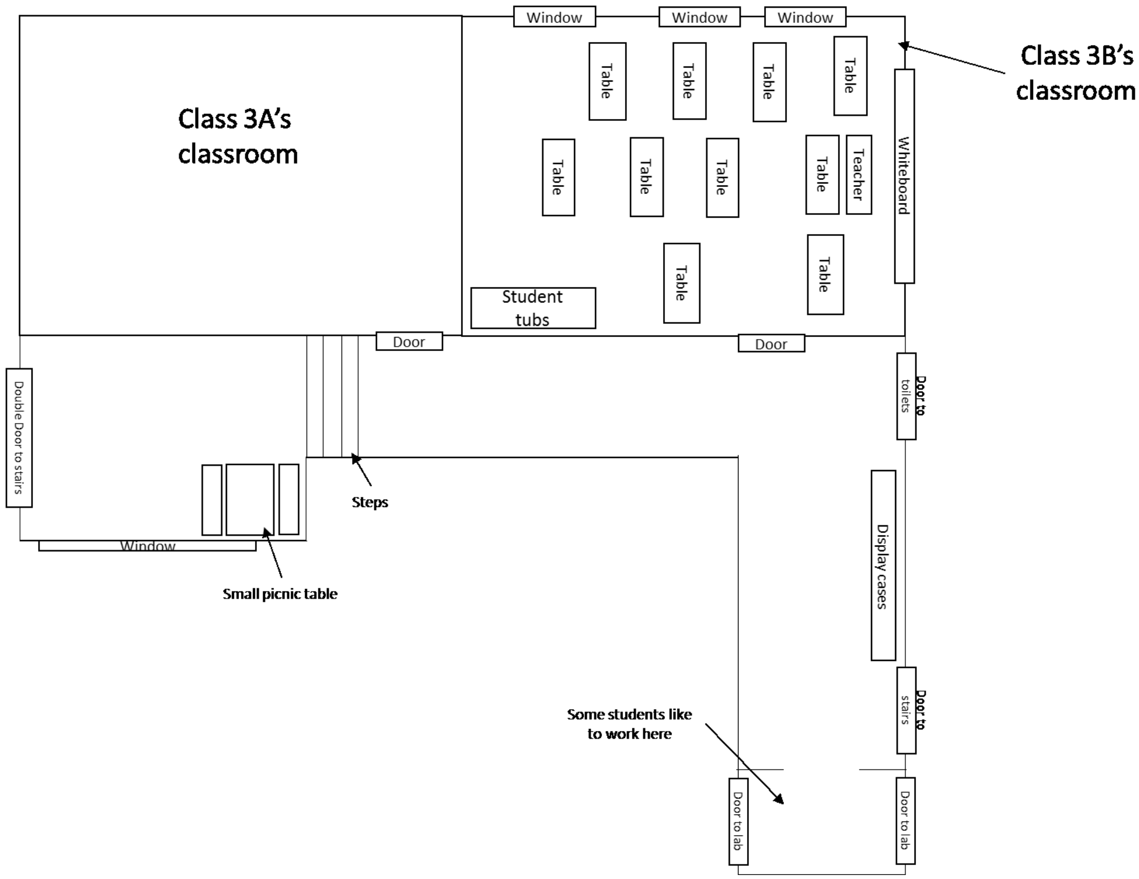


Figure 4.3. Diagram of 3B's classroom and nearby areas

At the beginning of the first lesson I observed with 3B, as the pupils worked in small notebooks with squared-paper, Maria talked briefly with her pupils about the topic of the lesson: strategies. After a few minutes, Maria went over the answers with the pupils, asking questions and giving pupils time to explain their answers. When pupils took out their textbooks, Maria said to me, "this [textbook] is from a very popular and good textbook series in Denmark," and explains it is popular "because it goes deep and emphasizes understanding, not just learning facts." Pupils turn to a probability activity that spreads across the top half of two pages. Along the top, there are three photos with a blank space at the far right end. In the middle of the page, below each photo, are statements and questions above spaces for pupils to write. Below each of these are the words, *Hvad er en god strategi?* (*What is a good strategy?)* followed by three lines for pupils to write (textbook and translations shown in Figure 4.4).

Shoe landing correctly?	How many sixes are there?	Rock, scissors and paper	Create an experiment
"Remember to repeat a number of times!"		"Remember to repeat a number of times!"	
Observed number ___ out of 10?	Observed number ___ out of 10?	Observed number ___ out of 10?	Observed number ___ out of 10?
What is a good strategy?	What is a good strategy?	What is a good strategy?	What is a good strategy?
To throw well	Shake them well and throw	Change when he gets rock then I get paper	

Figure 4.4. Textbook pages and translations for strategy activity.
Workbook pages from *Matematrix 3b*. Copenhagen, Denmark: Alinea. Reprinted with permission.

Maria projected a copy of the textbook pages on the board and asked the class to say some ideas about strategies. As pupils shared their ideas with Maria, she wrote on the board in Danish (for convenience, the English translation is given throughout even though only Danish was used in classes):

Strategi	Strategy
- plan for at gætte	- plan for making a guess
- det smartest at gætte på	- the smartest way to guess
- viden	- knowledge
- tænking	- thinking
- erfaringer	- experience

Maria explained to pupils that, "We know something and then we make a strategy from our knowledge." She highlighted three of the terms as being key to effective strategies: *viden, tænking,* and *erfaringer* (knowledge, thinking, and experience). For some things, she explained, a strategy is more difficult to create than for other things. As an example, she demonstrated by gently throwing a pencil case in the air and letting it fall to the floor. She explained it is possible to develop a strategy to make the pencil case fall a certain way most of the time.

After this short introduction, Maria wrote directions on the board:

1. gæt	1. guess
2. lav strategi	2. make a strategy
3. udfør forsøget	3. do the experiment

She divided pupils into partners and they began the activity. During this activity, pupils were trying three different experiments and were asked to make predictions and consider strategies for each: throwing a shoe and making it land on its sole, making six dice each land on 6, and winning at the game Rock, Paper, Scissors. For the final part of the activity, each pupil had to create his or her own experiment, make predictions, and then create a strategy for that experiment.

Several pairs of pupils moved into the hallway to work while others remained in the room. For the first experiment, I saw a pupil remove one of his shoes, flip it in the air and watch it fall. He made a note in his book and repeated the action. Other pupils began to do similar things, dropping or flipping their shoes in different ways. Pupils wearing a single shoe moved around the classroom and hallway. Maria seemed unconcerned by the noise or activity but moved to different groups to talk with them. As pupils worked, each pair worked at their own pace. Maria spent several minutes each with two of the groups and seemed unhurried and very willing to spend all the time necessary to answer questions, discuss the activity with pupils, and ask questions of each pupil. At two points, she was talking to different pairs of pupils, and several other pupils came to hear what she was saying. She also helped two groups keep track of when each partner won at Rock, Paper, Scissors. At times, two or three pupils seemed to be off-task for a minute or two, but Maria did not seem concerned about monitoring this – the expectation seemed to be that the pupils know what they are meant to be doing and will do it at the pace that is right for them.

After about 30 minutes, Maria called the pupils back together to discuss their findings. Several pupils discussed how they had different strategies for making their shoes land a specific way because the shoes are designed a specific way. One girl shared that her prediction for the number of dice landing on a six was 37 because 37 is her lucky number. Maria used this opportunity to lead the class in a discussion about this type of guess compared to a guess based on the three aspects of strategy (knowledge, thinking, and experience of rolling dice).

Talking with Maria and her pupils

The next day, I had a chance to talk with Maria before observing class 3B again. When asked to describe her mathematics class, she said (T: teacher; I: interviewer):

> I –Tell me some about your class.
> T – Yes. It often starts with some explanations about the stuff [topic] and some talking about understanding the stuff. And we often start with talking about what they know until now about a new stuff, because I have to know where are they, what can I build on, do you know what I mean?
> I – Oh, yes, I do.
> T – So we talk about new stuff, and what they know about the stuff, and what they are going to read in this subject. And sometimes we make some, like yesterday, some groups, and they try to…they make small exercises together. And sometimes they make it on their own. Sometimes they move around in the whole school to make some exercises and sometimes they are in the classroom. It's different.

In Maria's class, she thinks pupils should enjoy mathematics:

> I think it's good when they … I think it's good when they like what they do. So I try to find some exercises where it can be funny and where it can be, you use your body or you use some instruments, but also where you can use writing and talking.

Mathematical communication, both oral and written, is something mentioned both in interviews with Maria and her pupils as well as a part of the learning activities during the lesson. Maria explained why she works to incorporate talking and writing into her lessons:

> T – Because if you have to understand a subject, you have to talk it out and you have to write it down and maybe you have to try it on your body if it's about, if you have to know how long this table is, you have to try to [Danish]…
> I – To measure?
> T – Yes. Dimensions, yes.
> I – Ok.
> T – I think it's good when you can talk, write, think, try it in real practice and all around, use your body.
> I – Ok.

T – I think it's good when you can talk to your partner about it, so when you hear your partner's talking, you can use it in your own thinking and your own talking.

I also spoke with three of Maria's pupils. Although the pupils have begun to learn English in school, each pupil preferred to have Maria translate my questions in to Danish and their own responses into English. Therefore, each pupil's response to my questions was in Danish and Maria's translated responses would use third-person pronouns such as "they talked about," or "she says." In the following quotations from Maria's pupils, I have changed the third-person pronouns to first-person pronouns in order to provide for clarity of pupil voice rather than use wording such as, *Maria said Sofie said, "They talked about…"*. An example of these pronoun changes is given below in Figure 4.5.

Original interview transcription:	Quotation as modified for clarity of pupil voice (underlines pronouns indicate changes):
Interviewer: Did you talk about what you were going to write down? [exchange in Danish] Maria [translating for Sofie]: They talked about what to write down and Sofie gave her partner some suggestions for strategies to write in her book.	Interviewer: Did you talk about what you were going to write down? Sofie: <u>We</u> talked about what to write down and <u>I</u> gave <u>my</u> partner some suggestions for strategies to write in her book.

Figure 4.5. Example of pronoun modification to quotations

One of Maria's pupils, Sofie, described how she and her partner worked together on the strategy activity in class. "We talked about what to write down and I gave my partner some suggestions for strategies to write in her book. We didn't write the same [thing]. We had different results." Sofie's classmate, Sebastian, says they often write in math lessons.

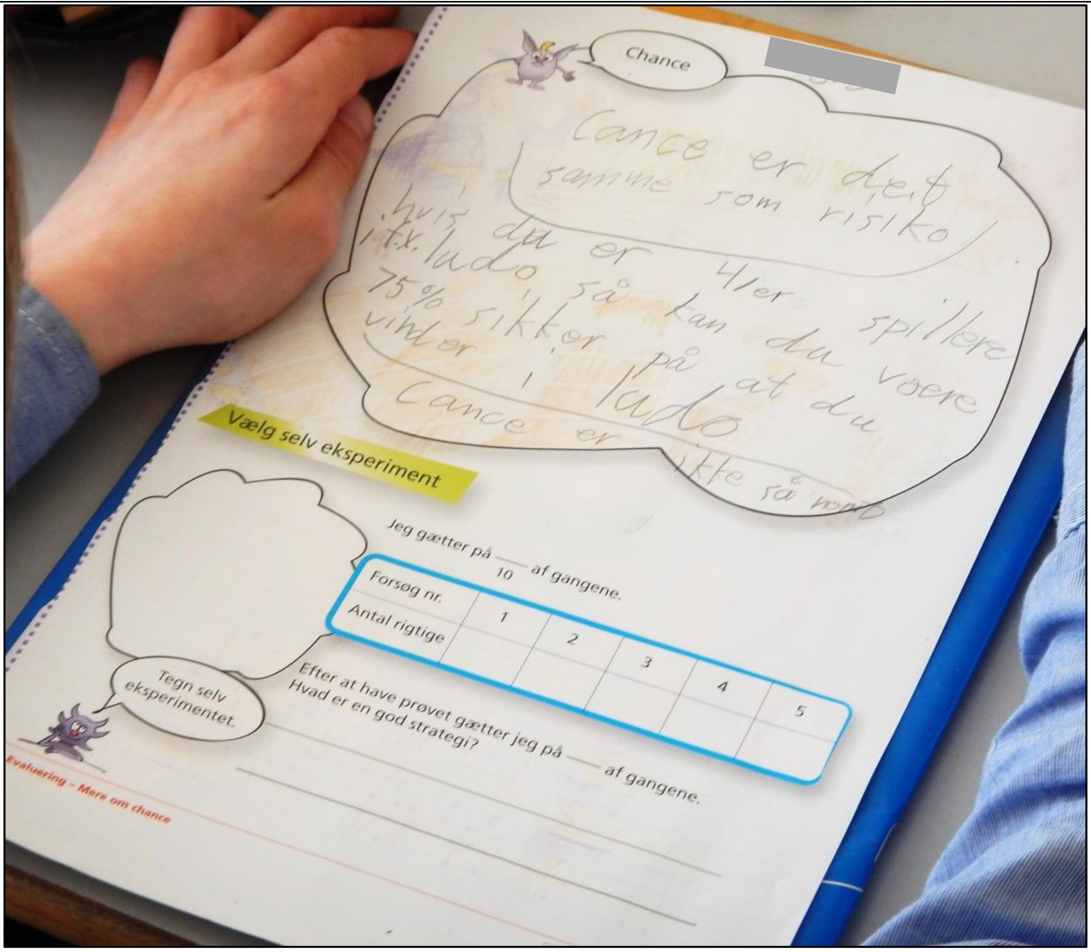
I – So how do you know what you're supposed to write down?
[exchange in Danish]
P – It's different. It's from the book. I know it from the book or the teacher.

After he has written something in class, Sebastian explained that his teacher Maria reads what he has written. A third pupil, Emma, describes her class as a place in which "I talk mathematics. … We often make experiences like yesterday. … And we get some questions that we have to answer in groups. Sometimes we make exercises [math problems] in the books." Emma describes classroom activities in which she and a partner each write word-problems and then exchange them with one another. Emma says they "sometimes we write text, not [just] exercises but words … and math problems in a story."

For Maria, using pupils' oral and written communication in mathematics lessons is an important part of knowing whether pupils understand a mathematical concept or not. She explains to me her beliefs about the role of the teacher:

> The teacher has to make situations where the pupils are going to think and try and try and try and work with the stuff in different ways. And the teacher has to be good at evaluation because you can build your new [Danish], when you're going to make plans for the rest of the year, you have all your evaluation and then you can build it up in a better way, so we make testing every month in our class, you can see today how we do it, it's an open form, it's not an old testing form. They sit on their place and they have their own test and it's quiet in the classroom but we can talk about the exercise and I take them home and I give it back to one of the pupils if I don't understand what he or she had made, and we talk about it again until I know that the pupil understands his subject.

In class activities, Maria uses pupils' written work and verbal conversations as a measure of understanding: "I looked at groups and saw if they made the right exercises and I listened to their talking in the group and if they didn't understand it, I helped." She explains that the monthly tests also help her know if a pupil understands the subject. An example of a test to which she refers is shown in Figure 4.6. As she described, the test is in open-response form: pupils must use written descriptions and explanations to show their understanding.



Chance

Chance is the same as risk.
If there are 4 players in Ludo, for example, you can be 75% certain that you win in Ludo.
Chance is not easy.

Choose an experiment

I guess _____of the times
10

Trial number:	1	2	3	4	5
Correct answers					

After my guess, I had _____ correct answers.
What is a good strategy?
"Draw your own experiment."

Figure 4.6. Probability open-response test and translation.
Textbook page from *Matematrix 3b*. Copenhagen, Denmark: Alinea. Reprinted with permission.

Before pupils begin the test, Maria gives a short introduction. On the board, she writes three words:

Chance	Chance
Risiko	Risk
Sandsynlighed	Probability

Maria asks pupils to write about these three words in the thought bubble at the top of the test. She acknowledges that these words can be difficult, but encourages pupils to write words, draw pictures, or answer however they like – the thought bubble is for their thoughts.

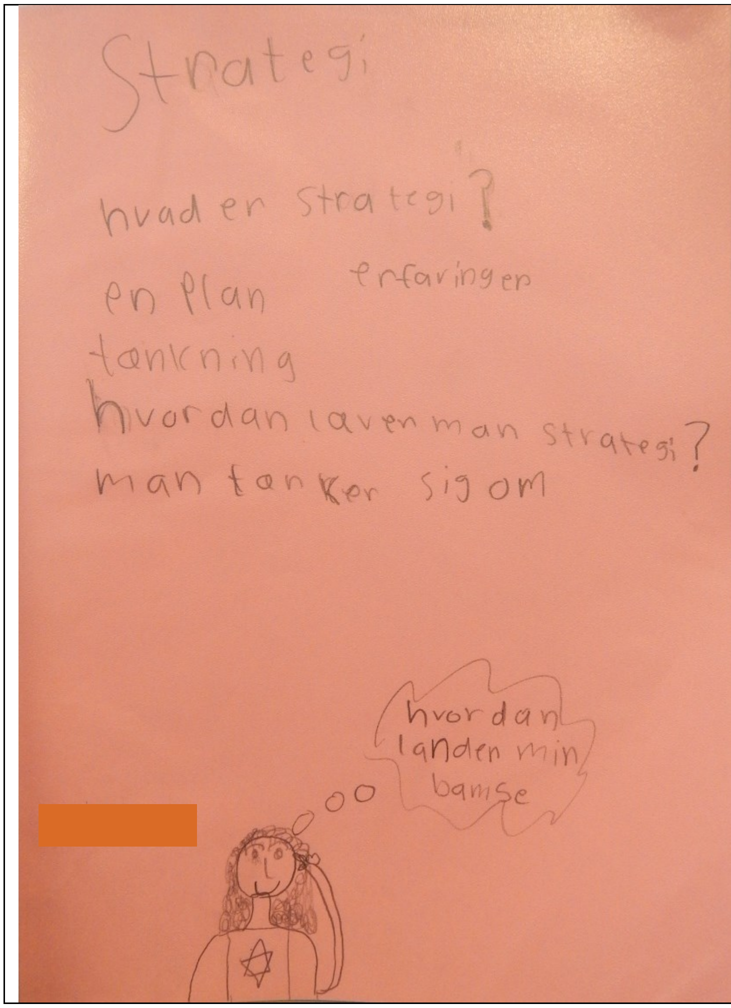
Maria also describes other forms of mathematical communications she includes in her lessons:

> When they [were] small kids, it's exercises they write down in books or on paper. It's often exercises or it's a kind of evaluation when we make our own books, book about multiplication, for example, or book about another subject, so writing is trying and evaluation and writing is talking to your partner, for example, when you make exercises for another partner. And writing can also be trying to write the numbers in the correct for, but it was more last year…they were smaller so that… Sometimes in this class we talk about the right way to use the [Danish] how do you make a 7 or an 8 or…

The books are another way Maria assesses pupil understanding, and also shares that understanding with parents. When pupils have completed the open-response test in which they describe chance, probability, and risk, Maria wrote on the board:

Mappe med STRATEGI	Book with STRATEGY
Hvad?	What?
Hvordan?	How?

Maria explained to the class they will be making a book about strategies: what a strategy is and how to make a strategy. Each pupil will create one page for the book. Figure 4.7shows an example page from this book. A pupil asks if they have to use a situation from yesterday's lesson or if they can use a different one. Maria tells them "it is free," meaning they are free to choose. Maria will use this book both to assess whether pupils understand about strategies, but also as a way to communicate with parents so they can see what the pupils are learning. Each night, a different pupil will take the book home and share it with his or her family. After a month, Maria says, each pupil will have taken the book home. This is also a method of eliciting mathematical conversations at home. For example, one of Maria's pupils, Sofie, describes taking these books home and, "Sometimes I tell my mother what I did in school, I show it to my mother."



Strategy

What is a strategy?

a plan experiences

thinking

how does one make a strategy?
you think [about the problem]

how will my
teddy bear
land

Figure 4.7. Strategy book page and translation

For Maria, there is a progression in mathematical communication. The mathematical communications pupils use when they are younger help to build a foundation for effective communications later on (T: teacher; I: interviewer):

> I – Is that something they do, writing sentences and things like that, as opposed to just numbers and exercises?
> T – Yesterday they wrote some sentence but it's not so often.
> I – Not so often? Ok. Is that important for them, do you think? And maybe not this year, but maybe later on? Is it important then?
> T – Yes, it's very important later on.
> I – Yes? Tell me about that.
> T – Because this use of language is helping the understanding of math when they write and when they talk, and speak, make some speeches about some [unintelligible], easier to remember or easier to understand the subject.
> I – So, something, it's using the language that's important?
> T – Yeah.
> I – Ok.
> T – Also, when they get older, it's good to take a book and read what you wrote about this stuff a month ago, or…it gets more and more important when they get older.

Class 7A

During the mathematics lesson

I first meet Martin in the teachers' room of his school. To get to class 7A's room, we walked down two long hallways in this large single-level school. Martin greets several pupils as we near the classroom. This is Martin's first year with this seventh-grade class. He explained that at this school, at the start of grade 7, all of the classes are mixed up so that pupils are with different people than before. This caused some concern among pupils and parents when it was started several years before, but it is now accepted as, "just the way it works." According to Martin, this school policy gives a fresh start to pupils who may have had problems in earlier grades. In addition to being their class teacher, Martin also teaches these pupils mathematics (one 60-minute lesson, and one 90-minute lesson each week), history (one 90-minute lesson each week), biology (one 45-

minute lesson each week) and geography (one 45-minute lesson each week). He plans to remain the class teacher these pupils until the end of grade 9.

Martin is also a *matematikvejleder.* A Matematikvejleder is an additional Danish qualification as a mathematics specialist. This qualification is undertaken after several years of teaching and is earned by completing six additional semester-long courses over a period of three years. Martin describes the role of the *matematikvejleder* is "to give advice to other math teachers when they have any source of trouble in their classroom." Despite this additional qualification, Martin says he acts in this role only in limited situations when other teachers have a specific question or classroom need:

> We're not using it [*matematikvejleder*] that much because it's, we would like to, but teachers also, it's also a limit for them to, they don't probably feel too confident with having me watching their teaching and should point out what could be developed and, so, yeah. That can be, so we are not using them much, as much as we, this is more like special, that's not quite the plan with having the education, but now, since no one is using the [*matematikvejleder*], and I said yes to do something about it.

As we arrive at the door to class 7A, two pupils greet us and shake my hand. The classroom is roughly square with a row of windows along the wall opposite the door. Pupils are seated in pairs at tables arranged roughly into inverted F-shapes. Class 7A has 27 pupils, but one pupil is absent. Today, there are an equal number of girls and boys in the class. The ceiling is high and sloped upwards to a peak. On the walls are some examples of pupils' work from math and other classes. Martin tells me there is usually more on the walls, but pupils have recently had to opportunity to take things home. A diagram of 7A's class and nearby areas is shown in Figure 4.8.

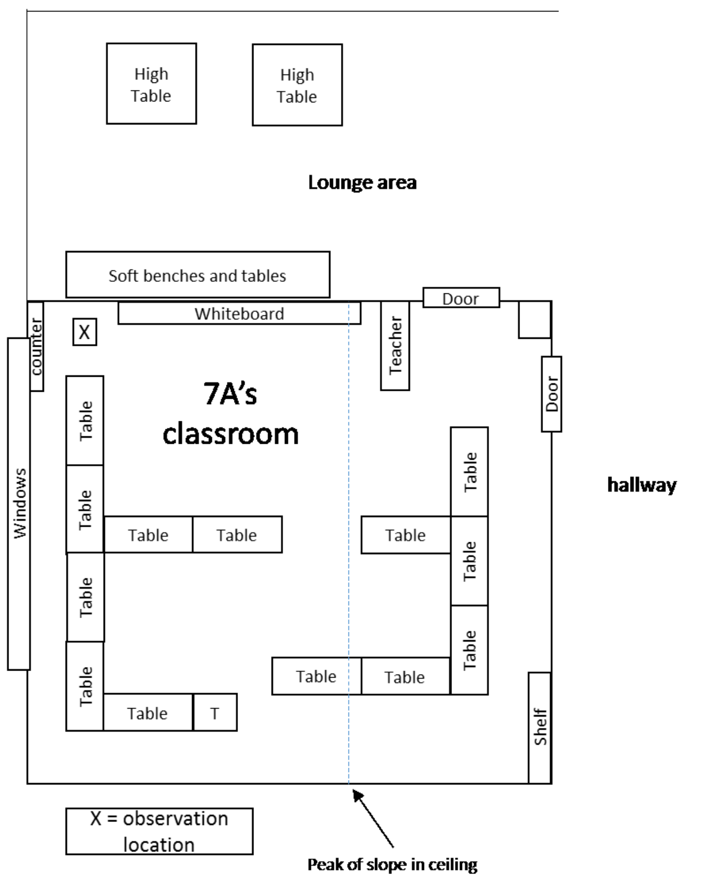
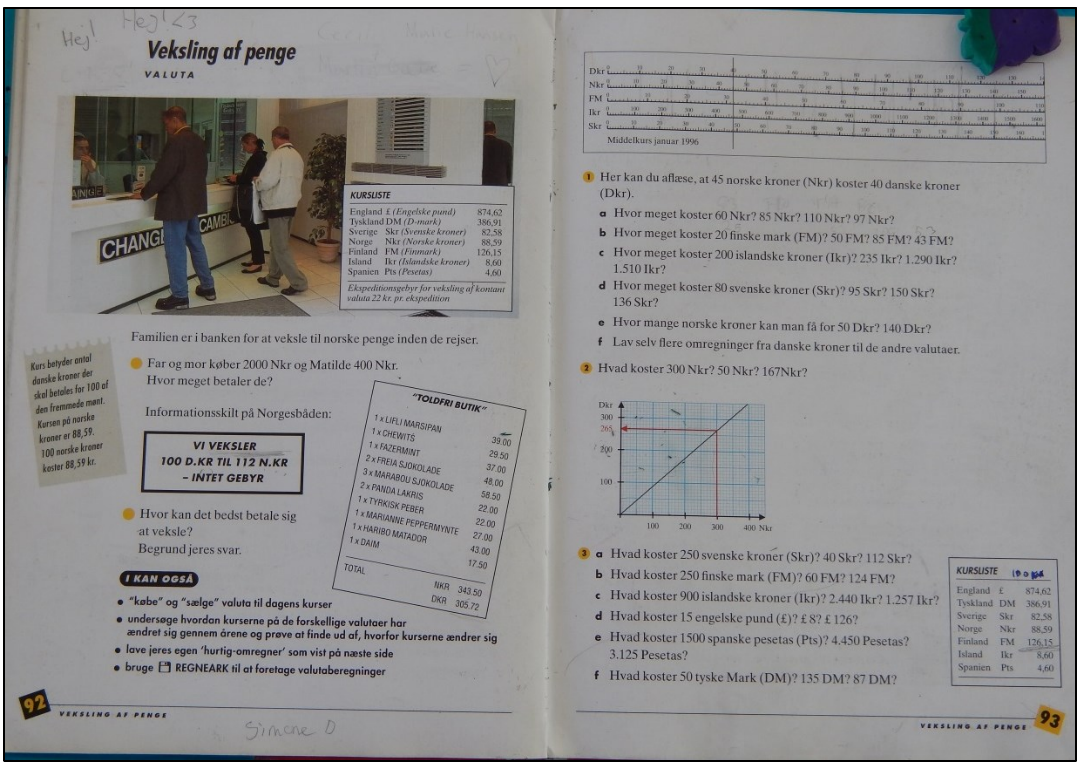


Figure 4.8. Diagram of 7A's classroom and nearby areas

After introducing me, I sit in the front corner of the room, near the windows. As pupils move to get their textbooks, Martin says to me, "There is a very good atmosphere with this class." One girl throws her pencil case in a high arc across the room and Martin walks over and speaks quietly with her. Later in the lesson, as pupils are working in groups, Martin talks to me about *menneskesyn*, which he describes as *a way of looking at people,* that he thinks is very important. He explains to me how important it is for pupils to feel appreciated and that it is more important than completing all the mathematics problems. He refers specifically to the pupil who threw the pencil case and explains she recently came from another school where she was having problems and he is trying to work with her in a very kind, careful way that respects who she is.

Once pupils have their books, Martin introduces the lesson by asking pupils to tell him about different currencies. He writes pupil responses on the board. Among the currencies listed are Danish kroner, Swedish kroner, Euros, British pounds, US dollars and Turkish lira. For several of the currencies, Martin writes their value in Danish kroner. While he discusses different currencies, pupils refer to their textbook (see Figure 4.9) and raise their hands to answer Martin's questions. He writes an example problem on the board:

$$\frac{2400\ n\,Kr}{100} = 24\ \cdot 88{,}59 = 2126{,}16\ d\,Kr$$



Hej! Hej! <3

Veksling af penge

VALUTA

KURSLISTE

England	£ (Engelske pund)	874,62
Tyskland	DM (D-mark)	386,91
Sverige	Skr (Svenske kroner)	82,58
Norge	Nkr (Norske kroner)	88,59
Finland	FM (Finmark)	126,15
Island	Ikr (Islandske kroner)	8,60
Spanien	Pts (Pesetas)	4,60

Ekspeditionsgebyr for veksling af kontant valuta 22 kr. pr. ekspedition

Kurs betyder antal danske kroner der skal betales for 100 af den fremmede mønt. Kursen på norske kroner er 88,59. 100 norske kroner koster 88,59 kr.

Familien er i banken for at veksle til norske penge inden de rejser.

- Far og mor køber 2000 Nkr og Matilde 400 Nkr. Hvor meget betaler de?

Informationsskilt på Norgesbåden:

VI VEKSLER
100 D.KR TIL 112 N.KR
– INTET GEBYR

- Hvor kan det bedst betale sig at veksle? Begrund jeres svar.

"TOLDFRI BUTIK"

1 x LIFLI MARSIPAN	39.00
1 x CHEWITS	29.50
1 x FAZERMINT	37.00
2 x FREIA SJOKOLADE	48.00
3 x MARABOU SJOKOLADE	58.50
2 x PANDA LAKRIS	22.00
1 x TYRKISK PEBER	22.00
1 x MARIANNE PEPPERMYNTE	27.00
1 x HARIBO MATADOR	43.00
1 x DAIM	17.50
TOTAL NKR	343.50
DKR	305.72

I KAN OGSÅ

- "købe" og "sælge" valuta til dagens kurser
- undersøge hvordan kurserne på de forskellige valutaer har ændret sig gennem årene og prøve at finde ud af, hvorfor kurserne ændrer sig
- lave jeres egen 'hurtig-omregner' som vist på næste side
- bruge REGNEARK til at foretage valutaberegninger

92 VEKSLING AF PENGE

Simone D

Dkr
Nkr
FM
Ikr
Skr
Middelkurs januar 1996

1 Her kan du aflæse, at 45 norske kroner (Nkr) koster 40 danske kroner (Dkr).

a Hvor meget koster 60 Nkr? 85 Nkr? 110 Nkr? 97 Nkr?
b Hvor meget koster 20 finske mark (FM)? 50 FM? 85 FM? 43 FM?
c Hvor meget koster 200 islandske kroner (Ikr)? 235 Ikr? 1.290 Ikr? 1.510 Ikr?
d Hvor meget koster 80 svenske kroner (Skr)? 95 Skr? 150 Skr? 136 Skr?
e Hvor mange norske kroner kan man få for 50 Dkr? 140 Dkr?
f Lav selv flere omregninger fra danske kroner til de andre valutaer.

2 Hvad koster 300 Nkr? 50 Nkr? 167Nkr?

Dkr
300
265
200
100
100 200 300 400 Nkr

3 a Hvad koster 250 svenske kroner (Skr)? 40 Skr? 112 Skr?
b Hvad koster 250 finske mark (FM)? 60 FM? 124 FM?
c Hvad koster 900 islandske kroner (Ikr)? 2.440 Ikr? 1.257 Ikr?
d Hvad koster 15 engelske pund (£)? £ 8? £ 126?
e Hvad koster 1500 spanske pesetas (Pts)? 4.450 Pesetas? 3.125 Pesetas?
f Hvad koster 50 tyske Mark (DM)? 135 DM? 87 DM?

KURSLISTE

England	£	874,62
Tyskland	DM	386,91
Sverige	Skr	82,58
Norge	Nkr	88,59
Finland	FM	126,15
Island	Ikr	8,60
Spanien	Pts	4,60

VEKSLING AF PENGE 93

Figure 4.9. Currency conversion textbook pages.
Textbook pages from *Matematik-tak 7*. Copenhagen, Denmark: Alinea. Reprinted with permission.

After this 15-minute introduction to the topic, Martin divides pupils into partners. Several pairs of pupils move into the hallway or adjoining lounge area to work. Martin tells me pupils are allowed to choose where they work, but if they go to the library they must return to Martin if they have a question, because the library is too far for Martin to walk to check on them. As pupils work, Martin describes what he prefers pupils to do if they have questions. If pupils have trouble, they should ask their partner first, otherwise Martin feels that if he has many pupils waiting to ask him something, he feels he quickly gives the answer in order to help people quickly and get to other pupils' questions. He feels the ask-a-partner-first method allows him to spend more time with pupils who have more in-depth questions.

Martin structures his teaching around his beliefs about how pupils learn mathematics (T: teacher; I: interviewer):

> T – I teach, fundamentally I would like the pupils to work a lot by themselves, working together, solving problems together, discussing why they came to find the right answer, or discussing different ways to get an answer, that is what I would like to achieve with my teaching. If I start something new like I did today, I would often give them a little instruction, just to make sure they have a way of getting started with the things. I experience that my pupils prefer to work instead of listen, instead of….isn't to sit quietly in the classroom and listen to other pupils or me, they would like to do something themselves. So, I try to use as little time as possible with I am talking in the classroom, and as much time as possible when they are working and doing different problems, assignments. So, that's how I try to, I would like to make different ways of doing it, smaller groups, bigger groups, sometimes they present what they…at the blackboard, they come up and explain to the others and write what they do. So, they can speak about what they are doing. Like I told you earlier, I think it's first when they are capable of speaking, talking about what they did, I think they learned it. Some pupils just learn a method, do you understand?
>
> I –Yeah.
>
> T – Just a method and then they can do it again, and again, and again, probably without knowing why they do it. So, in order to know why they do like they do, I would like them to explain it.

For Martin, listening to the teacher is not how he thinks his pupils learn. Instead, he thinks pupils should spend time working on problems and discussing the problems with other pupils. He uses pupils' explanations as a means of determining how well pupils understand what they are learning.

> If they don't understand, I don't think they are capable of explaining it, so probably they can explain it [partially] without understanding deeply, but then they have a picture of what it's all about.

Martin uses his assessments of pupils' understanding to plan for additional instruction.

> Then I try to push it to the next level, find something that is right above the level they have now. I say, "What can you do about it? Could you create your own assignments or problems? So you see the math in a different way." So now I have to create a problem instead. Make something that challenges them a little bit above where they are now.

He thinks pupils should each have mathematics work on their own appropriate levels, however there are limits to the time available for planning and preparation.

> Today it was just an example of where we could work together, but also there I have to give different things to different pupils because they can't do the same. So that's probably the thing about preparing your teaching that takes the longest time, the most time. That's preparing for different levels.

Martin sometimes uses the website matematikfessor.dk to assign differentiated homework to his pupils. "Differentiating, in the homework, because I don't think two pupils should have the same thing. It's also a big preparation work to say you have to do this."

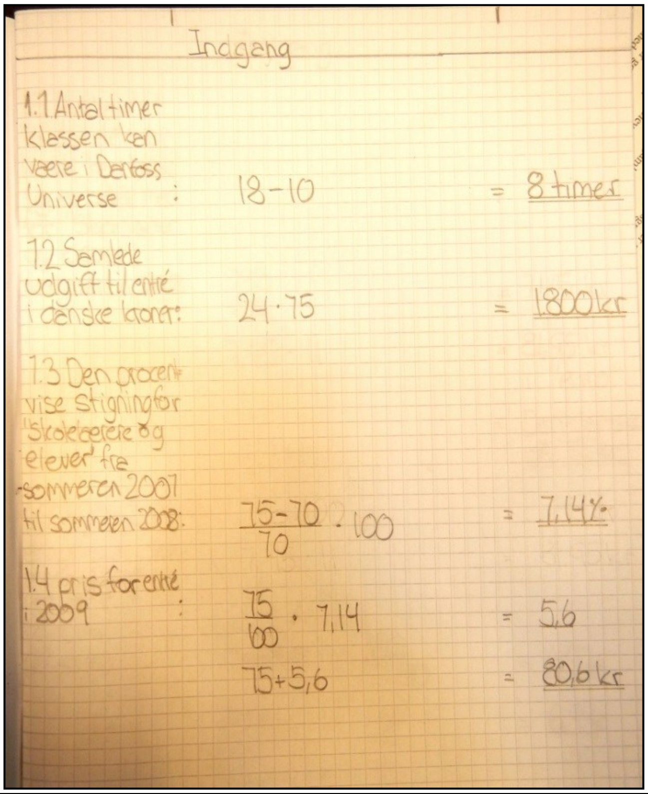
In today's lesson, pupils are all working on the same topic from a textbook – converting currency. Sometimes, Martin explains, the textbook "has only problems and not many examples of math in context" and he tries to use other material to supplement the textbook. In some cases, pupils are working far above or below the levels of the other pupils. He describes a pupil in another teacher's eighth-grade class. As a *matematikvejleder*, Martin has been asked by this teacher to help a pupil who is working far below grade level. In his own eight-grade math class, Martin also has a pupil who is working far below grade level.

> I'm going, five or six years back in curriculum and finding stuff that will challenge him, that he can work with, because the things we work with in the rest of the class, he can't do. Not even close.

In the same eighth-grade class, Martin has begun using writing assignments with one of his pupils because she is working so far above the level of the other pupils.

> And we talked about the girl I have in eighth grade that is so far ahead of the others, and I have to, you know…she's doing writing assignments, writing in words what she does and combine it with exams with numbers and calculations and then make arguments based on the numbers she used.

Martin describes for me the writing assignments he is using with his eighth-grade pupil. He typically would only use one or two of these in seventh- and eighth-grades and not focus on these writing assignments until ninth-grade. In math, his pupils write "mostly pictures, writing how they come to the right answer, and the right answer. They use a little text and I'm teaching them to divide the page in to three columns." Martin explains that in the left-most column, pupils' "text has to have something to do with the answer." The middle column is "for the working" and the last column is for the answer: "Text, working, and then the results." He describes this problem format as "a good example of writing mathematics." Figure 4.10 shows an example of this format.



Indgang

1.1 Antal timer klassen kan være i Danfoss Universe : 18-10 = 8 timer

1.2 Samlede udgift til entré i danske kroner: 24·75 = 1800kr

1.3 Den procentvise stigning for 'Skolelærere og elever' fra sommeren 2007 til sommeren 2008: $\frac{75-70}{70} \cdot 100$ = 7,14%

1.4 pris for entré i 2009 : $\frac{75}{100} \cdot 7{,}14$ = 5,6

75+5,6 = 80,6 kr

Input		
1.1 Number of hours the class may be in Danfoss Universe	18-10	= 8 hours
1.2 Total cost of admission in Danish kroner	24 x 75	= 1,800 Kr
1.3 The percentage increase for "teachers and pupils" from summer 2007 to summer 2008.	$\frac{75-70}{70} \times 100$	= 7.14 %
1.4 Price for admission in 2009	$\frac{75}{100} \times 7.14$	= 5.6
	75 + 5.5	= 80.6 Kr

Figure 4.10. Martin's three-column method of solving problems

Martin explains that this three-column writing framework is commonly-used in his school and throughout Denmark (I: interviewer; T: teacher):

> I – Now is this something you came up with or is this commonly used?
> T – It's commonly used. I was teached this when I was in school.
> I – It's something you learned.
> T – And when I started here and everything, the teachers did that, so it's done this way so we are talking about the way the assignment communicates with you, so you should actually could read the assignment without knowing this one [points to textbook problem].
> I –Oh, so without actually having the book or the test in front of you, you should be able to look at the pupil's work and know what the problem is?
> T – Yeah, that's how we read it here at the school, when we discuss it, that's something we discuss a lot because it can interfere with your final result. If you are on the edge between A and B, so if you have a poor way of putting it, poor communication value, you can go to the low side.

For Martin, communicating clearly in mathematics is important, and the way to communicate clearly is to have well-organized, well-structured solutions.

Although Martin's long-term plan is for pupils to learn and use the three-column framework for their mathematics problems, today his pupils focused more on answering questions about currency conversion rather than on showing the problem or the steps to their solutions. Figure 4.11 shows two pupils' work for today's lesson. Rather than using the example problem Martin wrote on the board as a model for what they needed to write in their books, pupils seemed to be using the problem as a procedural example for the numbers and operations they entered into their calculators.

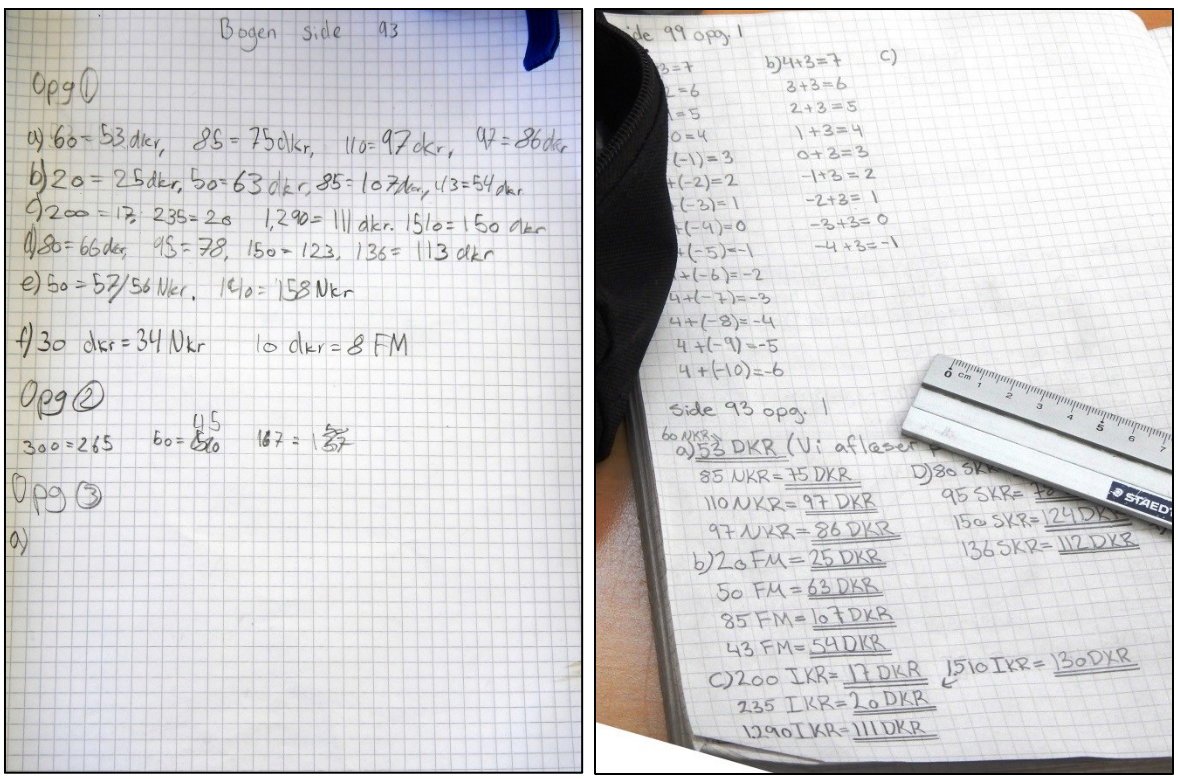


Figure 4.11. Two examples of pupils' currency conversion work

As pupils work, Martin moves around from pair to pair, both in the classroom and to pupils working in nearby areas. After speaking to one group, Martin shares part of the conversation with me.

> They were talking about the value of the Turkish Lira – when one pupil visited Turkey recently it was 4 Turkish lira to the Danish kroner, but now it is 2.5 which indicates there are "problems for Turkey." The pupil looked up the current rate on his phone. It is good to use what pupils already know and to learn how to find out what they need.

As the lesson draws to a close, Martin calls pupils back to the classroom before they are dismissed.

Talking with pupils in 7A

Later I spoke with several pupils about mathematics and about their mathematics class. Isabella described today's lesson as "sort of what we do every time." She explains, "We sit in there and then he sort of tells us what we have to do and then we start doing it." Isabella's classmate Marcus agrees their lessons usually begin with Martin giving instructions:

> P – Yeah, we have like…some pages we make about a thing, like negative numbers or…yes, other stuff. It's different. We make it. And when we are all done, we start on something new.
> I – Ok. So today, what I saw in there, is that like it is most of the time?
> P – What?
> I – Where Martin tells you a little about it at the beginning…and then you do the problems?
> P – Yes, yes, and then we do it, yes.
> I – So that's like it is usually.
> P – Yes.

Isabella says that, although normally the class works from the textbook, occasionally Martin "shows us things on the board and then we have to do them." As an example, Isabella describes a recent activity: "Right now they're doing like a game. A horse race where they have to use some dices and then you have to see who gets their horse in first by rolling the dice."

In the lessons where pupils work from textbooks, Isabella explains they are often allowed to choose whether they work alone or with a partner. Marcus tells me, "usually I like to sit by myself so I can concentrate better, but sometimes I like also to sit with people and if there's something I don't understand, I can just ask." Isabella explains what Martin does as pupils work (P: pupil, I: interviewer):

> P – Well, usually he's always got somebody that asks about something and he goes around and looks at what we're doing and usually there's somebody not doing anything then he tells them to do something.
> I – Ok, so he's making sure people are doing what they're supposed to when he's answering questions?
> P – Yeah.
> I – Ok. Does he ever look at what somebody's done and say, "Oh, that's not correct"?
> P – Ah…not really, he sort of looks in the book. Usually we ask if it's right.

Marcus explains that "if there's something [Martin] wants to tell us, he does it on the board."

Isabella says she thinks Martin feels it is more important to understand what you are doing than to complete all of the problems on a given page.

> P – And, we…he…I think the most important thing for him is that we learn how or why it's not right and not, "That's not right," he'll say why it's not right and explain how we can do it other, another way.
> I – Ok, so is that important to know why it's not right?
> P – Yeah.
> I – Why do you think that's important?
> P – Because otherwise you won't learn anything, because that's why you make mistakes then you learn something from that.
> I – So is it more important to understand what you're doing or to finish all the problems on that page?
> P – To understand what you're doing.

In some cases, pupils have to write the steps showing how they found their answer, Isabella says; sometimes "if we don't know how to explain it in math, then we'll do it with words instead." Generally, however, "We usually write which question it is, but we don't write anything else." Isabella later adds, "I don't really like writing, I would rather just multiply and divide and all that kind of stuff." Marcus also prefers numbers over text: "If there is something we have to write, I do that, but usually I just write the numbers and how I have calculated." Alexander prefers doing his mathematics work on his laptop computer. "It's actually easier than writing by hand." In some cases, Alexander just writes the answer to a problem, although sometimes, he says, "when we go through it, we need sometimes a description." Sometimes, Isabella tells me, Martin will show the whole class how an individual or pair of pupils solved a particular problem. Isabella finds it useful to see how others have solved problems.

> It shows that there isn't only one way to do it, because people think differently so maybe somebody…if you're partners with somebody and they maybe will find [solve] it another way then you'll maybe do that one differently but you'll then do it together afterwards and see, "Ok, we've actually got the same answer, but it's still different ways we're trying to find it," then we'll ask him [Martin] if that's okay and he'll say, "That's okay," because we think differently.

To assess pupils' understanding, Isabella says. "He [Martin] asks us and he uses questions in different ways than it maybe says in the book so that we maybe understand it a bit more and he changes the words a bit and stuff like that." When pupils check their work in class, Isabella says Martin goes over the answers with pupils and gives them an opportunity to ask questions if their answer is incorrect. Alexander says, "He trusts us that we can do it on our own, he doesn't need to check it. We just ask him, 'Is this right?' or, 'Have I done it right?'" Marcus explains further that Martin "doesn't really look at our things. If there's something we don't understand or if we want to have it checked, we just say it." This is in contrast, Marcus says, to his teacher the previous year who would collect pupils' work and check each problem.

When I ask Marcus if he would consider his mathematics class to be *hyggeligt,* he smiles and says, "No. Not really." Isabella is more direct: "No, no. I don't think it's possible. … I don't think that most kids really like, "Yay, now we're having math! I'm gonna sit and work things out! That's just the best thing I'm gonna do all day." Alexander agrees with Isabella's feelings about their mathematics class not being *hyggeligt*. In his opinion, a mathematics class would be more *hyggeligt* if "instead of being just straight from the book, it would be, it would be nice to be a little more creative about it, so we don't just sit there like [here Alexander lets out a loud anguished sigh]." Marcus compares his mathematics class to an out-of-school enrichment program he took part in for several years. When asked if the enrichment program has aspects that were *hyggeligt*, Marcus was very clear:

> Yes, because it was [*hyggeligt]*. We were about fifteen [pupils] and we all had something different [we could contribute], we were good at school and it's nice to be with someone who really understands the same thing as I do, because sometimes when I work with people in math, in the school, I understand the things better and I feel like I know a lot of things that they don't because I'm very fast to learn. I'm very, very fast learner. And it's sometimes when I have learned something it takes long, long time before other pupils learn it. And then it's good to have someone who can do the same.

One problem about his mathematics class, Marcus says, is that "I work better with people that understand and that can make the things fast than with people I have to help all the time." A consequence of understanding the material quickly is that he often has to teach material to other pupils. "Sometimes I get a little annoyed, because if I tell them something and they just sit and look at me like I'm talking in another language they don't understand, it sometime it, I can be a little, 'Arrrghhh!'"

Class 4C

During the mathematics lesson

When I first meet Charlotte, Martin's colleague and class 4C's mathematics teacher, she explains to me today's lesson is about fractions and adds about her fourth-graders, "they like… they LOVE fractions! I don't know why!" Throughout my time with Charlotte, she frequently tells me how much she enjoys teaching mathematics and hopes to pass along that excitement to her pupils. This is the first year Charlotte has taught these pupils, but she will stay with them through the end of sixth grade. In addition to teaching mathematics to class 4C (two 90-minute lessons – each counting as two lesson period - each week), she is also their class teacher and teachers them PE, science, history and religion for a total of 12 lessons a week with 4C. Charlotte has taught at this school for 10 years and was at another school for 5 years before that.

We enter the room and the pupils in 4C are sitting at tables facing the whiteboard. There are 19 pupils: 12 girls and 7 boys. The classroom arrangement is shown in Figure 4.12. With the exception of three boys who are sitting together, pupils are sitting in same-gender pairs at the tables. Each pupil is eating a lollypop. I later learn one of the pupils has a birthday today and has brought the lollypops for her classmates to celebrate.

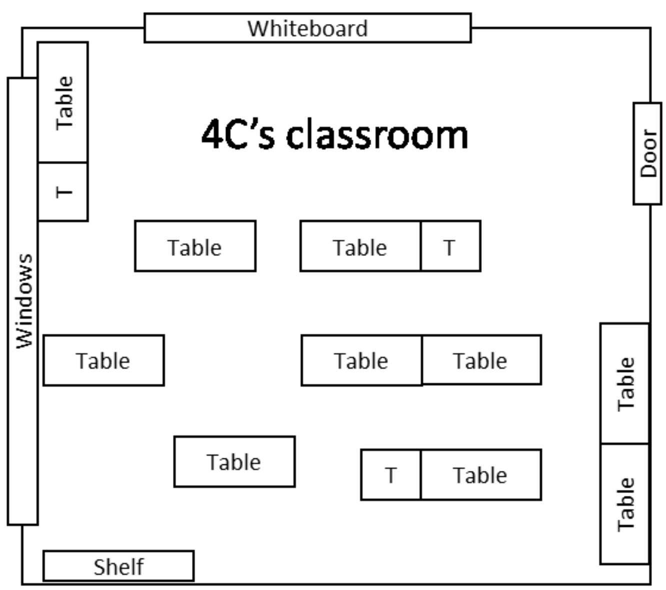


Figure 4.12. Diagram of 4C's classroom

After introducing me, Charlotte starts writing on the board. In green marker, she writes the number 4 and draws 4 smiley faces, then she writes the number 17 and, using pupils' lollipops as an example, draws 17 lollipops (see Figure 4.13.)

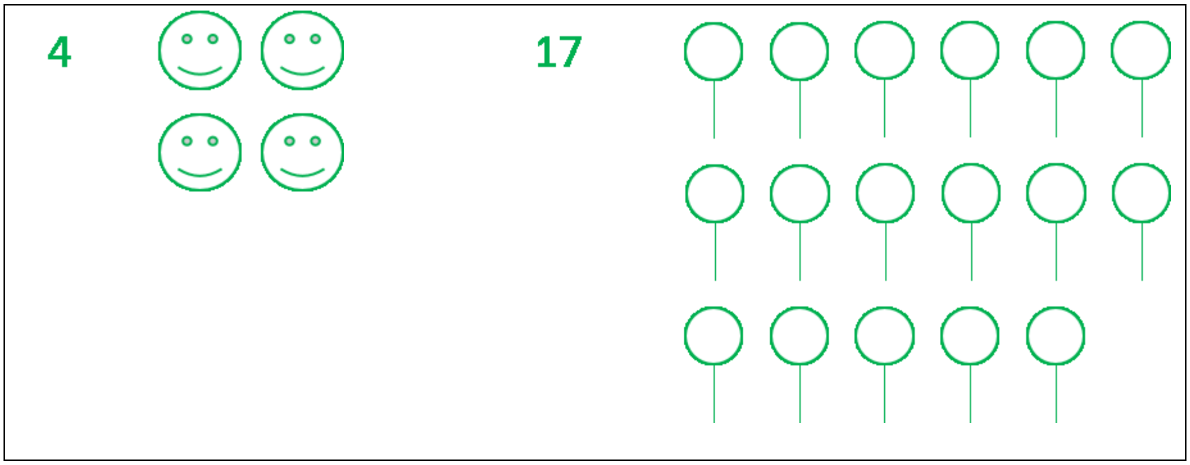


Figure 4.13. Smiley faces and lollipops

Charlotte speaks to the pupils in Danish and asks pupils questions. The pupils are responding individually, some by raising their hands, some by being called on. Their responses are, in most cases, sentences or explanation rather than just numbers or short answers. When one boy on the front row is initially unsure of a response, Charlotte waits patiently for him to consider what to say and how to answer. She does not rush him or go to another pupil for the answer. Charlotte continues talking and illustrating on the board what she is saying. She draws ovals around groups of 4 figures and writes $4\frac{1}{4}$ as shown in Figure 4.14. After discussing this example with pupils, Charlotte draws and discusses a similar problem in which pupils share 11 circles among three people, and then sharing 19 Xs among five people.

After Charlotte and her pupils work through and discuss each problem, Charlotte draws a number line starting at zero and going past 5 on the board. As shown in Figure

4.15, Charlotte divides the interval between 4 and 5 into four equal sections and uses an arrow to indicate where $4\frac{1}{4}$ is located.

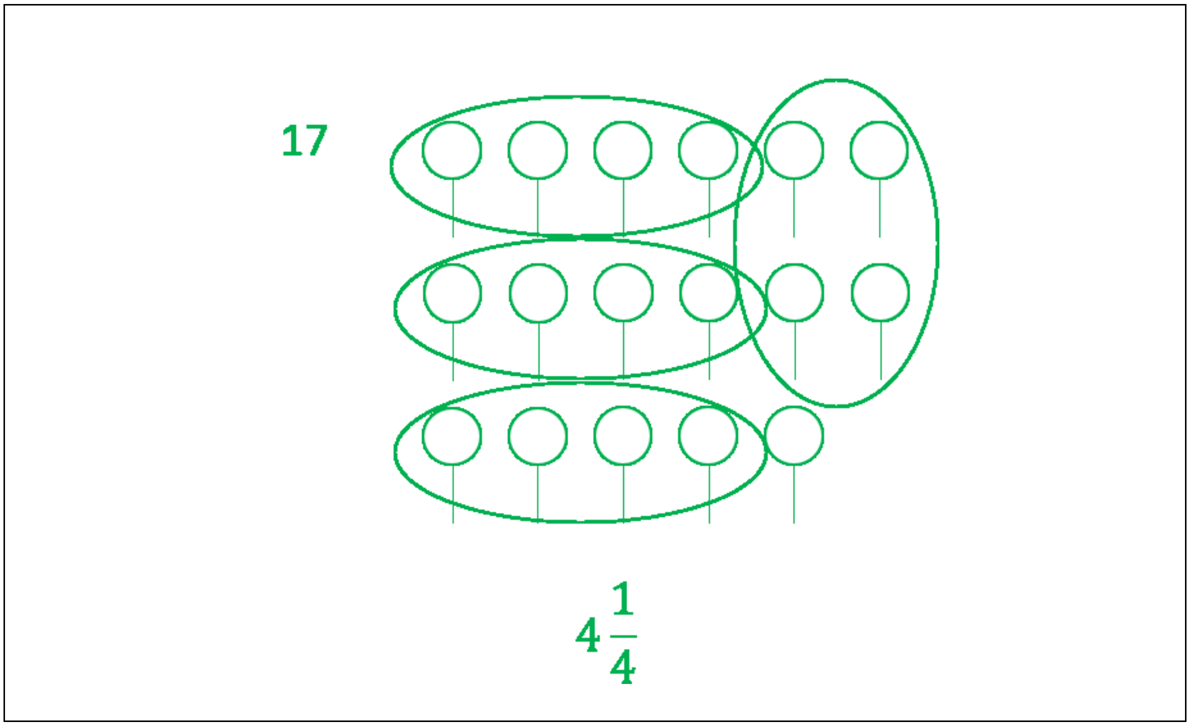


Figure 4.14. Ovals around lollipops

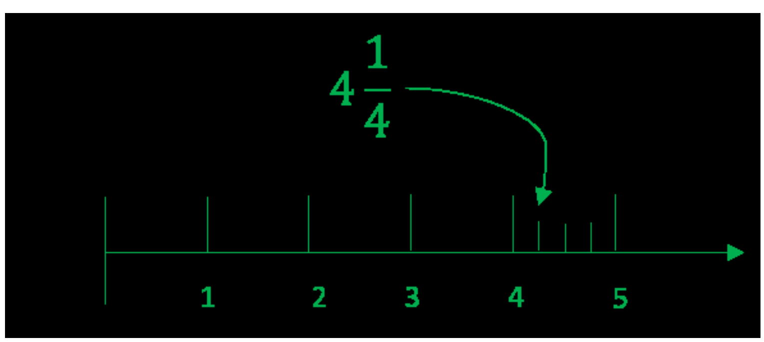


Figure 4.15. Number line showing 4 ¼

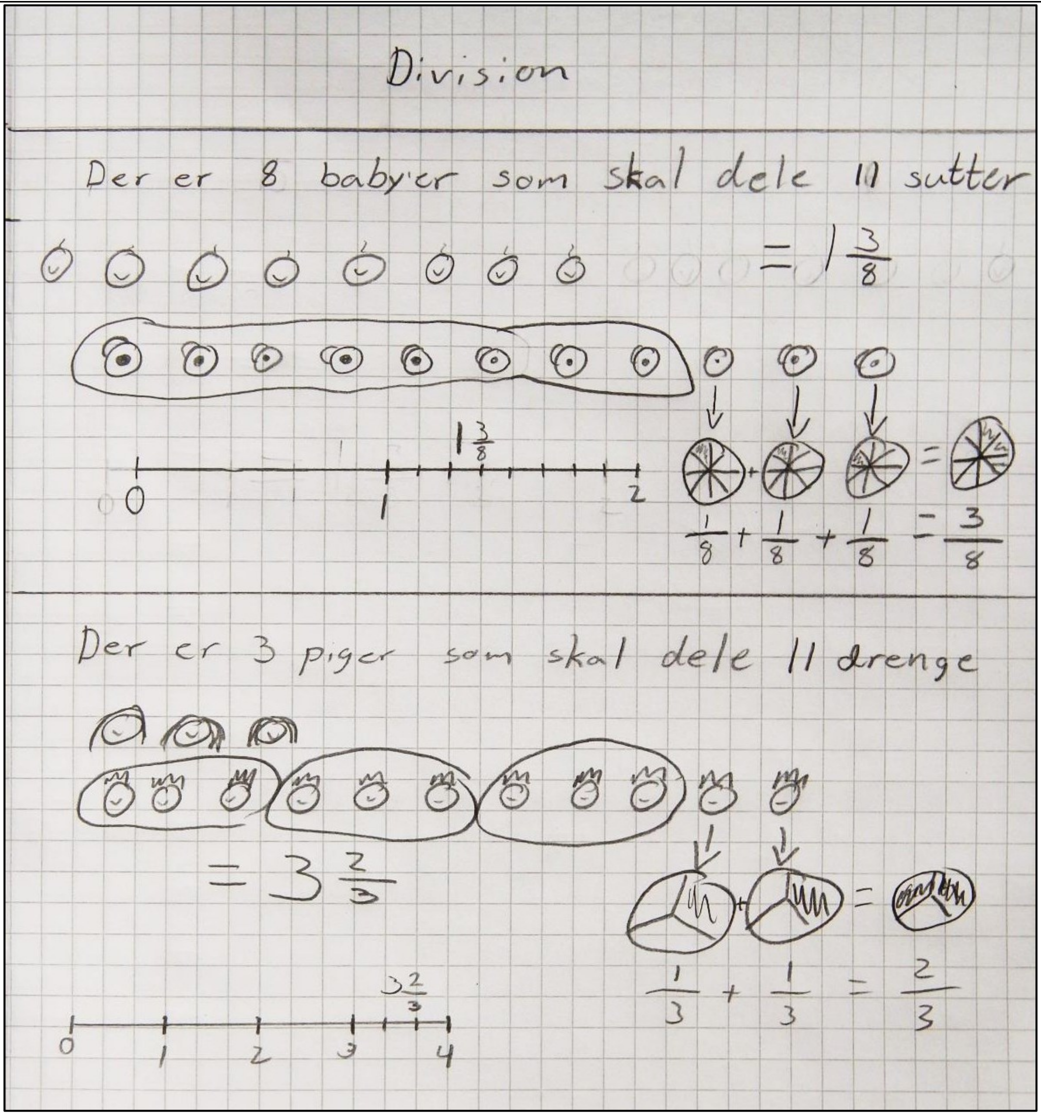
Charlotte repeats this process for the second example. For the third example. Charlotte asks pupils to explain to her where to locate $3\frac{4}{5}$ on the number line. This time, there is some discussion between pupils about how many sections to divide the interval into and how many lines are needed to divide the interval. After discussing the number lines, Charlotte also discussed with pupils which whole numbers each fraction is between. Though this is illustrated on the number lines, she took the opportunity to emphasize it further. On either side of $4\frac{1}{4}$, Charlotte wrote 4 and 5, and on either side of $3\frac{2}{3}$, she wrote 3 and 4. For the last example, she has pupils tell her what whole numbers $3\frac{4}{5}$ is between. There was some discussion before pupils settled on 3 and 4.

At this point, after spending 12 minutes on the initial discussion of division and fractions, Charlotte makes a transition from having pupils seated to a shorter five-minute activity requiring pupils to move about. She holds up a stack of laminated cards and explains the next exercise. The purple cards are about 4 inches by 6 inches. On each card is a small rectangle of pale yellow paper glued in the center. The yellow rectangles have either a fraction (such as $\frac{3}{5}$ or $\frac{4}{7}$) or a circle divided into sections with some of the sections shaded. Pupils line up across the front of the room to get two cards from Charlotte. As pupils get their cards, they go into the hallway. Charlotte explains to pupils they must work in pairs to say the fraction and then decide who has the larger fraction. They use the picture to compare and check their answer. She tells me each pupil has one card with the fraction in numbers and a card with a corresponding picture. In the hallway, pupils are working in pairs to talk about which fraction is larger and then show each other their pictures to compare. As one pair finishes, the pupils individually go to find a new partner.

After the pupils return from discussing relative sizes of fractions, they sit at their tables as Charlotte explains the next activity. In addition to their notebooks, several pupils also have their textbook or their workbook open on their desks. Pupils are to write word problems, and for each problem, write the fraction, draw a picture, and show the fraction on a number line. (See Figure 4.16 for an example of how one pupil completed this activity.) In blue ink, Charlotte writes on the board:

Der er __ børn som skal dele __ ting	There are __ pupils to share __ things
tegning	drawing
tallinje	number line

Charlotte explains pupils have 10 minutes to create and solve as many of these tasks as they can. She tells pupils their partners and each pair moves to work together.



Division

There are 8 babies who need to share 11 pacifiers.

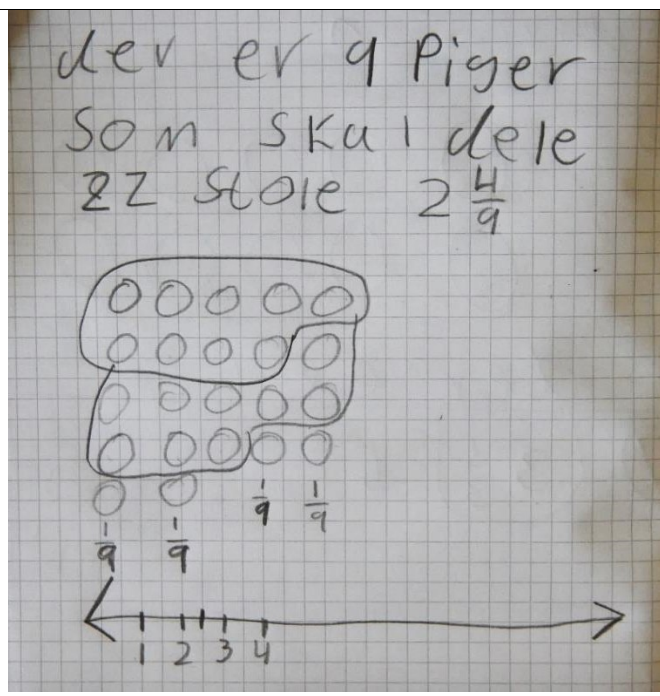
There are 3 girls who must work with 11 boys.

Figure 4.16. Examples of pupil division problems

As the pupils begin working, Charlotte explains to me the differences in fraction notation between third and fourth grades. She refers to something she wrote at the start of the lesson:

3. kl: 4 og 1 rest — third grade: 4 and 1 left over
4. kl: $4\frac{1}{4}$ — fourth grade: $4\frac{1}{4}$

Part of Charlotte's goal in this lesson is for the pupils to correctly use the fraction notation. While pupils are working, Charlotte moves around the room, talking to partners if she sees something she needs to discuss with them or if a group has a question. At one point, Charlotte goes over to Jonathan and his partner. Jonathan has drawn 22 circles and he is using 9 as his whole. He has circled two groups of 9 as shown but he has written, incorrectly, $2\frac{4}{2}$. Charlotte talks with them about the mistake and has Jonathan label each remaining circle as one-ninth. (The conversation seems to include what each remaining part is – not a half, but a ninth.) Jonathan seems to understand and changes his answer to $2\frac{4}{9}$. Figure 4.17 shows Jonathan's work after correcting this fraction to $2\frac{4}{9}$. At the end of the lesson, Charlotte tells each pupil to write one additional problem for homework.



there are 9 girls who must share 22 chairs

Figure 4.17. Jonathan's word problem

Talking with Charlotte

After the lesson, Charlotte and I have an opportunity to talk further. She describes how she feels it is very important for pupils to be discussing, writing, and connecting their ideas.

> They're going to talk [about] it, it's very important that they just…they're not just looking at things, and doing it because the teacher said, "Do this, do that," but they can think it, they can talk it, and [do] the next activity where they had to write down the story, is also a way of getting them to connect what they already know with a picture and try to combine all of the knowledge.

Despite the importance of these types of communications activities, Charlotte acknowledges she is not able to include as many of these activities as she would like because they take more instructional time than more routine types of problems.

> Because the activity I did today, with the fractions and the picture of the fractions, it takes so much time! But I think I have to do it, because it's a way to get them to think about, "Oh, one-seventh, one-eighth, what is the larger, the bigger thing?"

She also discusses how she tries to balance the activities involving multiple types of mathematical communication with the more routine mathematics problems.

> But I do that [the communication activities], because of that, there's not much time for exercises all the time, so I try to find out which one is the important one, and a little of it is going to be the week study at home, the not-difficult parts where you don't have to think that much, that's mostly routine work, and then the difficult parts, we do it up here [in class].

Although Charlotte makes clear she feels that talking and writing are critical to learning mathematics, she is also clear that the instructional time available to her causes her to include fewer in-depth, communications-rich learning activities than she would like.

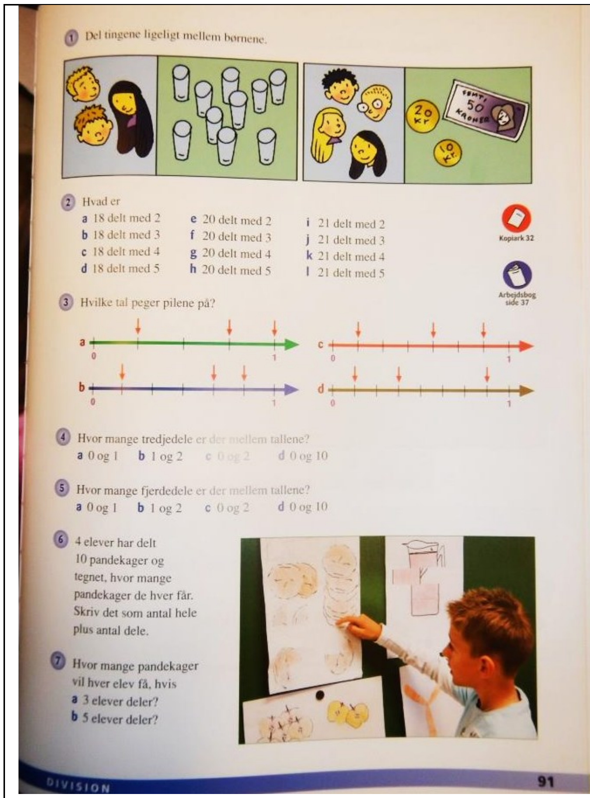
Like her colleague Martin, Charlotte is also a *matematikvejleder* – a mathematics specialist. She and Martin took their courses together several years before. When describing the *matematikvejleder* qualification, Charlotte says, "we were taught how to see math in different ways and how to tell [this to] our colleagues, to give them ideas for their math [lessons] and their way of teaching math." She describes the content of what she learned not only as mathematics content but also how to teach specific mathematics topics as well as "which programs for computers work well and how do you analyze a math book to see if they're a good math book or not."

When Charlotte refers to "seeing math in different ways," part of what she means is knowing when and how to extend a problem beyond what is printed in a textbook. Referring to the textbook page about today's lesson topic, she explains:

> To open up the exercise like this, by just using another word, or also just like this, it says [reading from book], "Put the fractions on the line." It's a very dull [problem]. The next one, I could say, "Well, now you have done this, now make your own fractions and your own lines. Make some that [are] easy, make some that are on the same place at the line, but is a different [fraction]."

She points to the textbook page as she talks (see Figure 4.18). She explains how she could make additional modifications to the problems in the textbook.

> Some of the brighter ones…they have to do a different kind of explaining. If they had number 6, it wouldn't be enough just to draw the picture, they would have to do something more like the [number] line today or [explain] in words, "then I do, then I do, then I do." That would be an exercise for the skilled ones.



1 Del tingene ligeligt mellem børnene.

2 Hvad er

a 18 delt med 2
b 18 delt med 3
c 18 delt med 4
d 18 delt med 5

e 20 delt med 2
f 20 delt med 3
g 20 delt med 4
h 20 delt med 5

i 21 delt med 2
j 21 delt med 3
k 21 delt med 4
l 21 delt med 5

Kopiark 32

Arbejdsbog side 37

3 Hvilke tal peger pilene på?

4 Hvor mange tredjedele er der mellem tallene?
a 0 og 1 b 1 og 2 c 0 og 2 d 0 og 10

5 Hvor mange fjerdedele er der mellem tallene?
a 0 og 1 b 1 og 2 c 0 og 2 d 0 og 10

6 4 elever har delt 10 pandekager og tegnet, hvor mange pandekager de hver får. Skriv det som antal hele plus antal dele.

7 Hvor mange pandekager vil hver elev få, hvis
a 3 elever deler?
b 5 elever deler?

DIVISION 91

1. Share things equally between the pupils.

2. What is

18 divided by 2, etc.

3. What number is the arrow pointing to?

4. How many thirds are between the numbers?

5. How many quarters are between the numbers?

6. 4 pupils have shared 10 pancakes. How many pancakes do they each get? Write it as the number of whole plus number of parts.

7. How many pancakes will each pupil get if
a. 3 pupils share?
b. 5 pupils share?

Figure 4.18. Division textbook page.
From *Matematrix 4*. Copenhagen, Denmark: Alinea. Reprinted with permission.

Although Charlotte often modifies the problems from how they are presented in the textbook, she feels it important that pupils have the structure of the workbook that accompanies the textbook.

> In the fourth grade, [it is] very difficult for the boys to get the organization of the notebook, to get, "Oh, all this writing!" Some of them still write very big and it's very slow for them to write. So they need a kind of a workbook where they can write the answer.

Charlotte also stresses to her pupils the importance of using the textbook as a source of reference, not just a source of problems for the day's lesson. When describing an activity involving graphing and fractions earlier in the year, Charlotte explains how she expected her pupils to decide what information they needed and then find and use that information (T: teacher; I: interviewer):

T - Graphing and fractions, they were divided into groups, about 2 or 3 pupils in each group and they had to make a small movie where they had some drawings in the bag, they were the teacher and they had to explain, "Then we do, and then we do, and then we do...and here's an example, here you can see we have to go out this way and then up, and so forth," and we put them online so the parents could see it and they were so good, and perhaps that's why they love fractions because it was a very good exercise…took AGES! I think we used four or six lessons just rehearsing. I wouldn't help them. It sounds stupid, but it was their problem, they had to find the examples they want to use, they had to find out what to do in planning it, and so forth, and of course I was going around saying, "This is not right, you have to change…how is it, let's look at it again…" trying to get it the right level.

I – So what do the pupils think when you don't even tell them the answer, it was their problem to do, so I could imagine that for some of them, that's a struggle.

T – It was.

I – And how…what …how do they do with that?

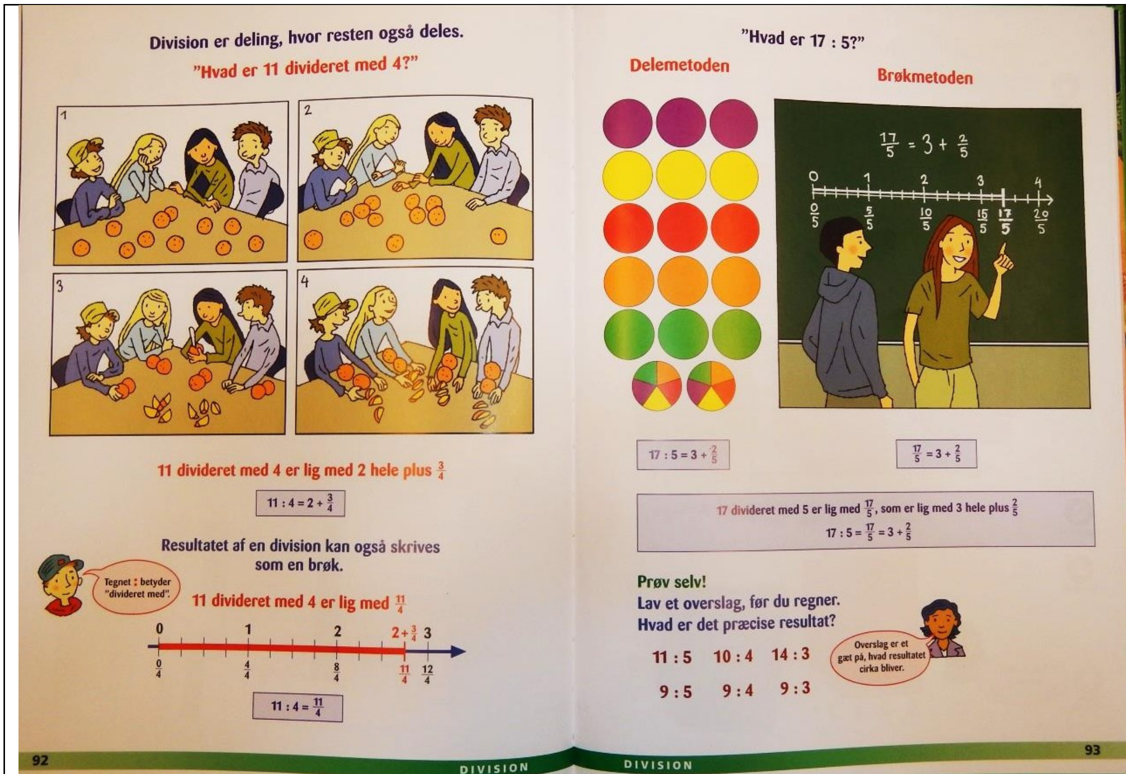
T – Every chapter has this kind of pages. (As an example, Charlotte indicates the pages shown in Figure 4.19.)

I – Oh, like a review page.

T – Yes, so I would say, "Find these instruction pages and get some ideas from it. But you can't use these examples, you have to find your own. But you can do similar."

Much of this emphasis on effective use of resources stems from Charlotte's desire to help her pupils become more reliant on themselves as problem-solvers rather than always expect Charlotte to show them the way to an answer.

> I want to do that because if I always tell them what to do, they never learn it. It will just be (the) teacher saying…but when they stand in the shop, and have to buy something, I'm not there to tell them what to do, and they need a strategy to do [it] fast. So they have to do it by themselves.



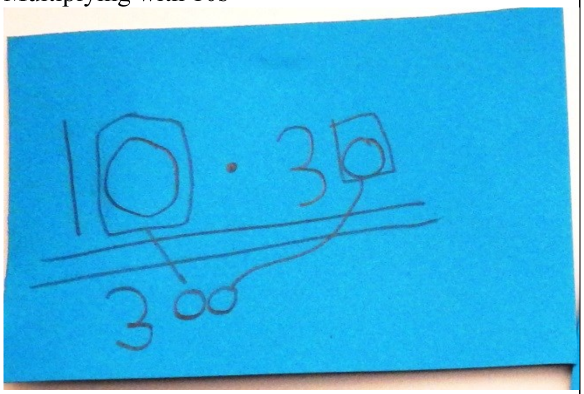
Division where the remainder is also shared. "What is 11 divided by 4?" 11 divided by 4 is equal to 2 wholes and $\frac{3}{4}$ The result of a division can also be written as a fraction. 11 divided by 4 is $\frac{11}{4}$	"What is 17 ÷ 5?" Parts method Fraction method 17 divided by 5 is equal to $\frac{17}{5}$, also equal to 3 wholes plus $\frac{2}{5}$ Try it yourself! Make an estimate before you solve. What is the exact answer? An estimate is a guess on what the answer will be.

Figure 4.19. Example of instruction pages.
From *Matematrix 4*. Copenhagen, Denmark: Alinea. Reprinted with permission.

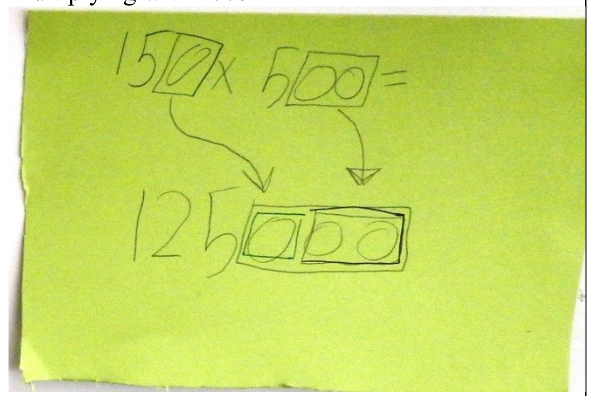
For Charlotte, another aspect of seeing math in different ways is revealed by a set of posters and cards on the back wall of 4C's classroom. This display, shown in Figure 4.20, has at its center, the words *Gange Begrebskort (Multiplication Concept Map)*. Around the title are seven headings (their English translations, from the upper left just above the center): Different multiplication methods, Multiplying with 10s, Word problems, Multiplying with 100s, The different multiplication tables, How can you use multiplication in your life?, and Written explanation showing how to use different methods of multiplication. Each heading is color-coded and pupils have a set of hand-written cards to accompany each heading (examples are shown in Figure 4.21 and Figure 4.22). Above and to the left of the concept map, Charlotte has created a poster illustrating and annotating seven different multiplication algorithms, including The Traditional Method, The Disaggregation Method, and The Egyptian Method.

Figure 4.20. Multiplication wall display

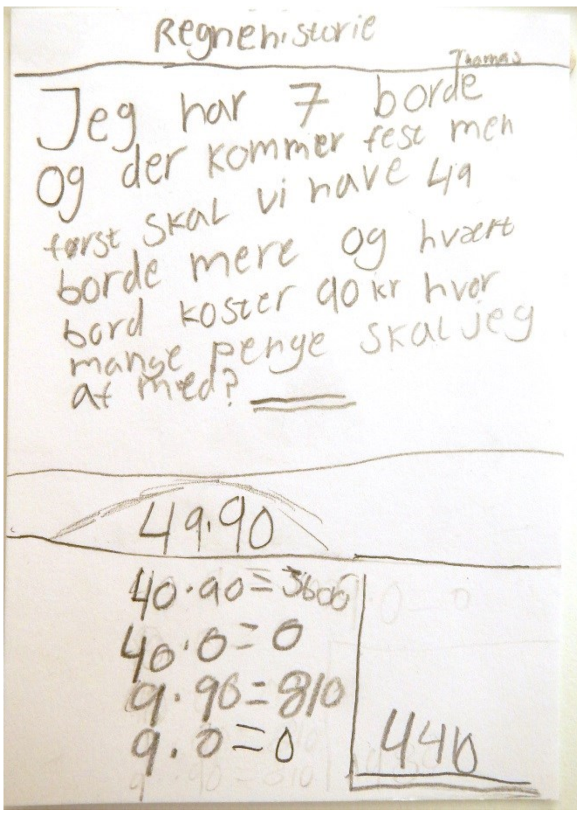
Multiplying with 10s



Multiplying with 100s

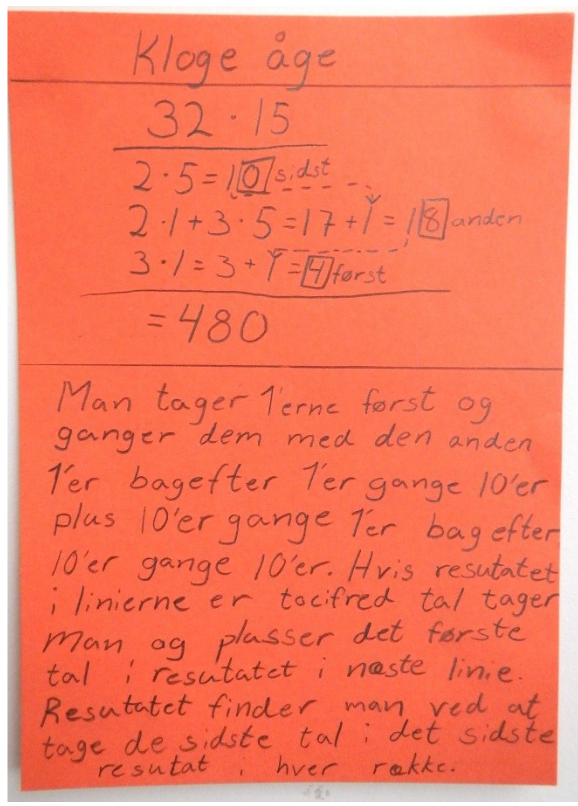


Word problems



I have 7 tables for the coming feast but first we need 49 more tables and each table costs 90 kr how much money do I need to pay with?

Written explanation showing how to use different methods of multiplication

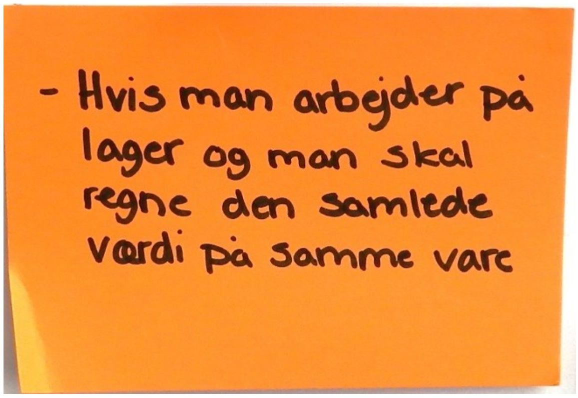


Wise age

Take 1's first and multiply them with the bottom 1s, afterwards 1s times 10s plus 10s times, afterwards 10s times 10s. If the result of the lines are two-digit figures take the sums and the first digit of the result in the next line. The result is found by taking the last digit of the last result each row.

Figure 4.21. Pupil examples from multiplication concept map

How can you use multiplication in your life?



If you work in stock and you have to calculate the total value of the same product.

The different multiplication tables

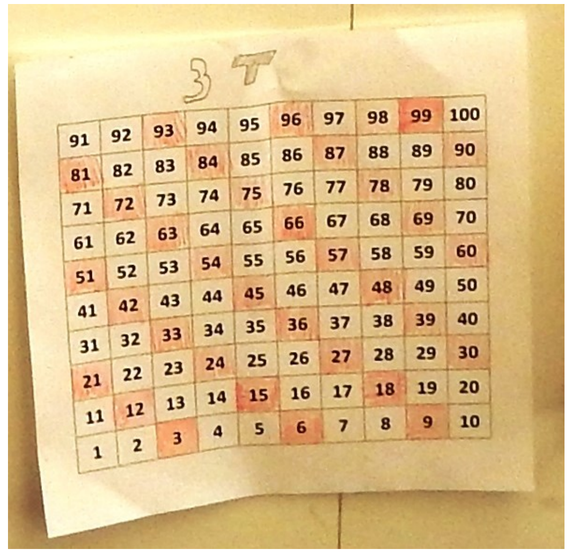


Figure 4.22. Additional pupil examples from multiplication concept map

Charlotte summarizes her ideas about teaching learning mathematics when I ask if she would describe her mathematics class as *hyggeligt*.

> Nope. Not cozy. In my math class, I want the pupils to be active, they need to be appreciated for what they are and what they can do, mostly for who they are. I want them to be safe, feel safe, feel secure in trying to do the exercises. It's not important if they do it right the first time, they just have to try it. So the safety part is very important. Cozy isn't for me important. It needs to be fun! Math for me is fun, I want the pupils to feel excited when they are to solve this kind of exercises. They loved the one today because they could do drawings, and talking to each other, so for them it was much fun today. But for me as a teacher, I just went around checking what is their background? How are they thinking right now? What can I do next? So, by doing a fun exercise, where they do drawings and talking to each other and…there wasn't anyone talking about something else. All of them were talking about the topic. That's also important to find exercises that get them excited and get their energy this way. But cozy, no.

For Charlotte, it is important that pupils actively participate in mathematics, rather than just solving problems from a textbook. She feels a large part of active participation in mathematics is communicating orally about mathematics:

> I would like pupils to do the talking, so I always try to open up the questions so they give me the answers. I give them a problem, but they have to do the answering, they have to put in all the words, it's not just an answer, it's an explanation.

It is through these pupil communications that Charlotte assesses how well pupils have understood what they have learned.

> I think it is important that the teacher takes time to listen to the explanations, more than just checking if the answers is correct, but step up and say, "Ok, can you explain this one for me?" To take the time because it's behind the explanations you find out if the pupils have learned anything or just can do it automatically.

Class B2

During the mathematics lesson

In contrast to the previous class profiles, B2 is not a primary school class. The pupils in class B2 are in their second year of the high school HTX B-level mathematics program. Although this is the last year of mathematics for these pupils, they have one additional year in the THX program, after which they will be going into communications-related fields. After completing their first nine years of school, Danish pupils have a choice of four upper secondary-school programs. The Higher Technical Examination (HTX) program that focuses on technical and natural sciences. Class B2's teacher, Henrik is in his second year of teaching. Before becoming a mathematics teacher, he worked for five years as a robotics engineer. In addition to teaching, he is also taking a course designed to help him learn more about how to teach – he recognizes that having the mathematical knowledge is not sufficient to be a good mathematics teacher. When talking about his style of teaching, Henrik says,

> [My] way of teaching math is much different from how I learned it – there is now more group work and a focus on understanding and mathematics in context, in integrating other subjects. This is more of what it actually is to work in math-related fields such as engineering.

Henrik refers to the eight mathematics competencies and notes the competencies require this type of teaching rather than just memorization of algorithms.

We enter the classroom where class B2 is already assembled. There are 26 boys and 1 girl in the class. Pupils are seated roughly in two long, shallow U-shaped arrangements of desks (illustrated in Figure 4.23). Henrik explains today's task to the pupils: in preparation for the upcoming exams, pupils will be divided into small groups to work on a sample exam problem. In this problem, pupils are given information about a go-kart track and they have to use the math they are learning to devise a solution. The problem pupils are given is shown in Figure 4.24.

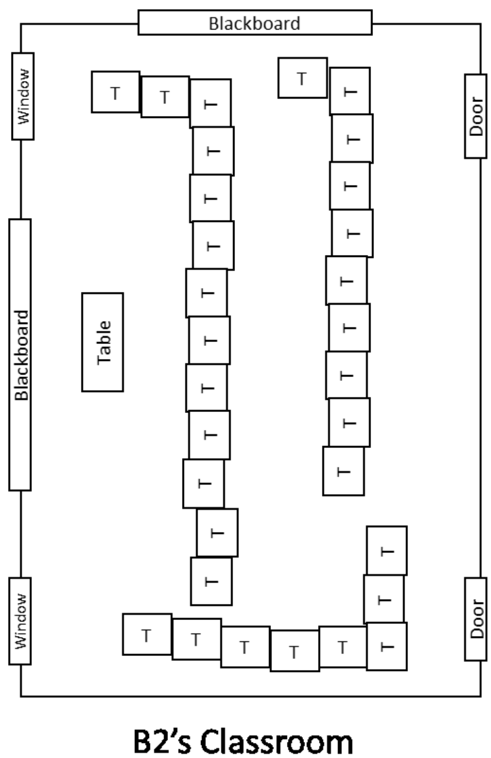
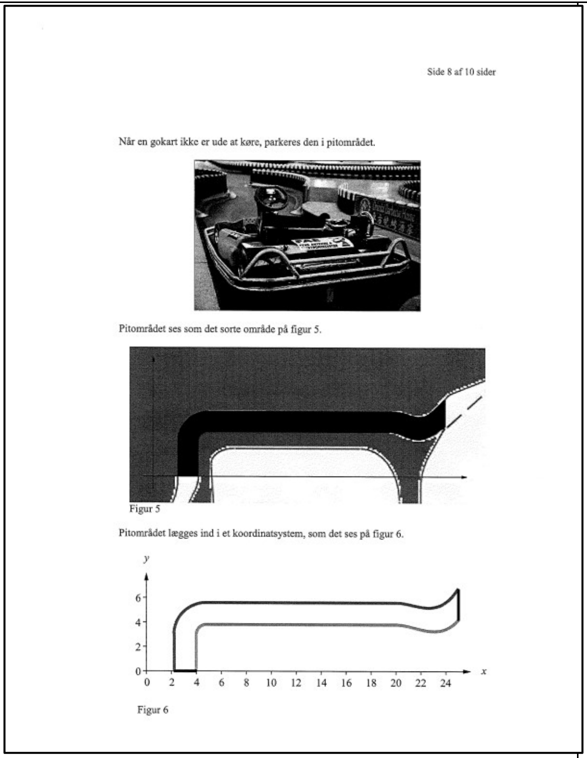


Figure 4.23. Diagram of B2's classroom



Side 8 af 10 sider

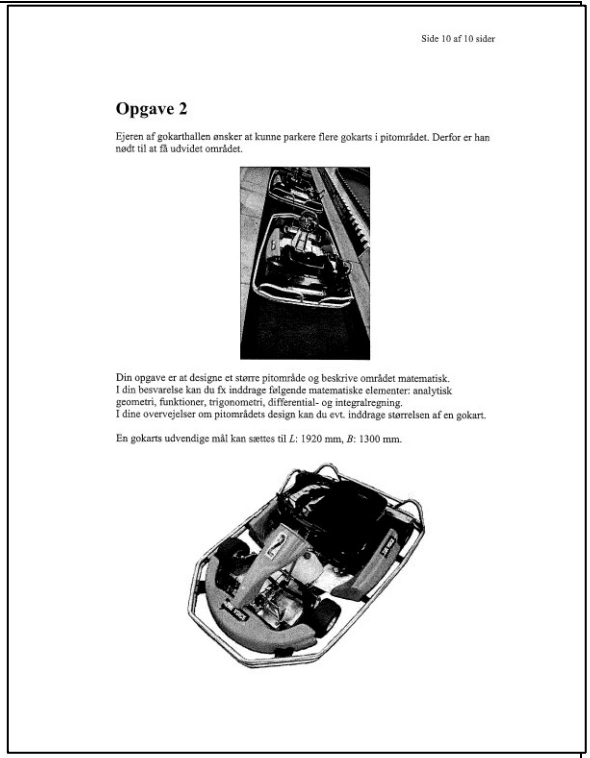
Når en gokart ikke er ude at køre, parkeres den i pitområdet.

Pitområdet ses som det sorte område på figur 5.

Figur 5

Pitområdet lægges ind i et koordinatsystem, som det ses på figur 6.

Figur 6



Side 10 af 10 sider

Opgave 2

Ejeren af gokarthallen ønsker at kunne parkere flere gokarts i pitområdet. Derfor er han nødt til at få udvidet området.

Din opgave er at designe et større pitområde og beskrive området matematisk.
I din besvarelse kan du fx inddrage følgende matematiske elementer: analytisk geometri, funktioner, trigonometri, differential- og integralregning.
I dine overvejelser om pitområdets design kan du evt. inddrage størrelsen af en gokart.

En gokarts udvendige mål kan sættes til *L*: 1920 mm, *B*: 1300 mm.

When a kart is not put on the track, it is parked in the pit. The pit is marked by the black area in figure 5. The pit is placed in a coordinate system as seen in figure 6.	The owner of the kart track needs to be able to park more karts in the pit area. Hence, he needs to expand the current area. It is your task to design a larger pit area and describe it in mathematical terms. Your answer may include the following mathematical elements: analytical geometry, trigonometry, differential and integral calculus. In your considerations of the design you may need to use the size of a kart. The outside measurements of a cart can be assumed to equal length: L = 1920 mm and width: B = 1300 mm

Figure 4.24. HTX sample exam paper and translations
HTX Examination. Copenhagen, Denmark: Ministry of Education. Reprinted with permission

After receiving their instructions, the pupils break into seven groups – six groups of four pupils and a group of three pupils. Henrik has created the groups for this activity, making sure to have "a strong, medium, and weak pupil" in each group. As pupils work, they discuss the problem with their partners.

> We know this part is difficult for pupils – defining the problem and deciding what mathematics to use. It is possible for each group to solve the problem using completely different mathematics and one challenge is using the higher-level mathematics they have been learning. They are learning what questions we can ask in math and what we can answer with math. … Is often difficult to get pupils to raise the bar and use the new mathematical things from this class instead of the simpler things they are more comfortable with. They are not familiar with the types of questions you can answer with differential calculus.

Pupils have laptop computers and some of the screens show Maple 17, Geogebra, or a calculator. Some pupils are also using laptops to look at reference material. One group has used Google to find photographs of existing race tracks. As pupils work, Henrik moves around the room from group to group, observing and asking questions. From time to time during the lesson, he uses the board to explain something to a group of pupils. Pupils seem to be using various method of solving the problem, including sketching by hand, using software to define track shapes, and formulating equations (see Figure 4.25 for examples).

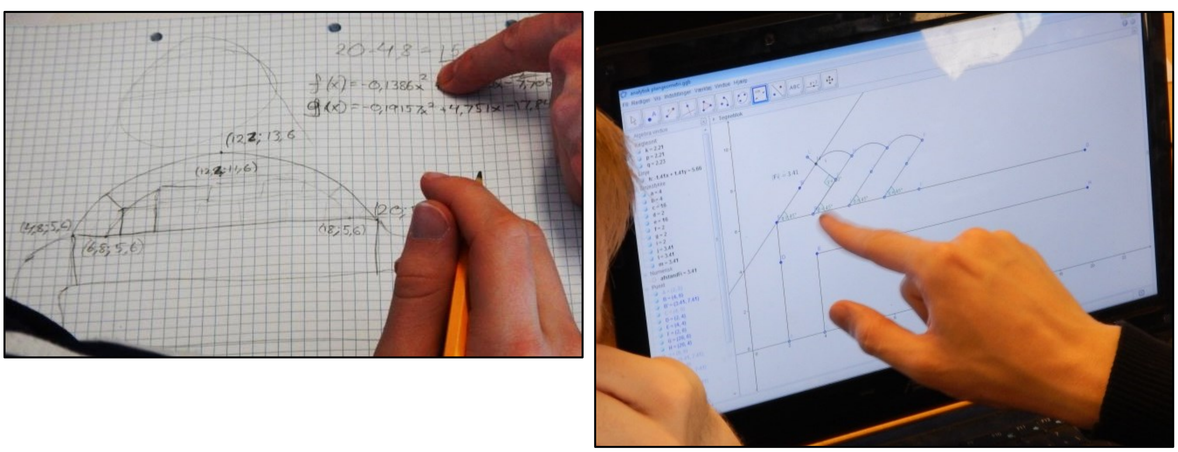


Figure 4.25. Examples of pupil work on the go-kart problem

As I observe the groups, I see one pupil is using Google and has typed "1920 mm to m" in order to convert the units. Midway through the 105-minute lesson, Henrik gives pupils a short break. During the break, Henrik talks about today's lesson with me.

“Today is more a counselling session than teaching.” After the break his pupils return and continue working on the problem. At the end of the lesson, Henrik tells me, “Two hours has not been enough for the pupils to solve the problem, though I suggest two hours is the time they spend on this part of the paper.”

Talking with Henrik

Later, Henrik tells me sometimes his pupils have trouble with number sense. One pupil calculated the length of the track as 3.2 km. Henrik says he tries to teach the pupils to ask themselves if their answer makes sense. On many assignments, he requires an additional sentence after the answer, in which pupils write about the answer in the context of the problem and have to use necessary rounding or correct units. Henrik says his pupils find things like fractions difficult: “they have little understanding of the concepts and have not even memorized the algorithm for solving such problems” so he often has to teach some of this content in his class. He also tells me that the transition from grade 9 to this school can be difficult for some of the pupils because they are not used to explaining their thinking and telling why.

One technique Henrik uses with his pupils throughout their time in the HTX program is a class wiki – a website in which pupils can create webpages for particular mathematics topics. He explains that pupils were assigned topics to include in the wiki. For each topic, the pupils had to decide the important information to include in order to clearly and accurately explain the topic. In this way, the class wiki serves both as a means of generating authentic pupil written mathematical communication but also as a reference for later. An example of a group’s blog entry on the intersection of two circles is shown in Figure 4.26 and Figure 4.27.

mat1h-2012

Skæring mellem to cirkler

Edit 0 16 ...

Hvis man kender det basale omkring en cirkel og substitutionsmetoden, så er skæringen mellem to cirkler let.

Læs evt.:
Cirklens ligning
To ligninger med to ubekendte

Så bruger man bare substitutionsmetoden, altså sæt dem lig med hinanden, og det giver en linjens ligning hvis man isolere y.
Derefter bruger du igen substitutionsmetoden og indsætter ligningen på y's plads i en af cirklernes ligninger. Det giver en andengradsligning.
Den kan så enten have to løsninger, altså to skæringspunkter, diskriminanten vil i dette tilfælde være positiv.
Der kan også være 1 løsning, så rammer cirklerne kun lige hinanden i et punkt. Diskriminanten vil være 0.
Der kan også være ingen løsninger, altså at cirklerne ikke skære hinanden. Diskriminanten vil være negativ.
De løsninger som denne andengradsligning giver, kan man nu sætte ind i den linjes ligning som man har fået fra tidligere.

Intersection of two circles

If you know the basics about a circle and the substitution method, the intersection of two circles is easy.

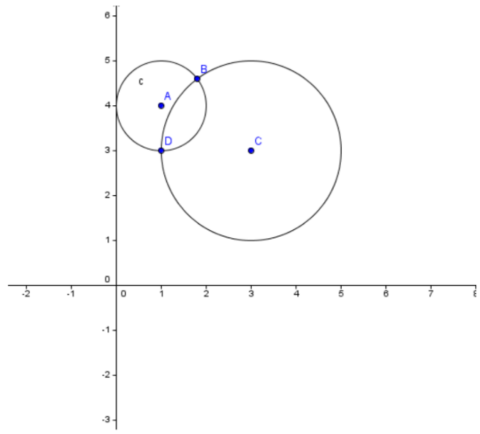
If necessary:
Circle equation
Two equations with two unknowns

So you just use the substitution method, then set them equal to each other, and it provides a one line equation to isolate y. Then use the back substitution method and inserts the equation y value in one of the circle's equations. It gives a quadratic equation. It can then either have two solutions, thus two intersections discriminant in this case will be positive. There may also be one solution, then the circles meet in just one point. The discriminant will be 0. There may also be no solutions, that is, the circles do not intersect each other. The discriminant will be negative. The solutions that this quadratic equation gives, you can now put into the equation of the line you got earlier.

Figure 4.26. Class wiki example and translation

Eksempel:

- $c : (x-1)^2 + (y-4)^2 = 1$
- $d : (x-3)^2 + (y-3)^2 = 4$



I dette eksempel bruger vi disse ligninger:
c er den lille cirkel og d er den store cirkel.

$c : (x-1)^2 + (y-4)^2 = 1$
$d : (x-3)^2 + (y-3)^2 = 4$

Vi sætter dem begge til at være lig med 0, altså sådan her:

$c : (x-1)^2 + (y-4)^2 - 1 = 0$
$d : (x-3)^2 + (y-3)^2 - 4 = 0$

Så sætter vi dem lig med hinanden:

$(x-1)^2 + (y-4)^2 - 1 = (x-3)^2 + (y-3)^2 - 4$
$\Updownarrow$
$y = 2x + 1$

I denne ligning er y isoleret, dvs. at vi kan sætte denne y værdi ind i den anden ligning.
Dette giver den andengradsligning som giver os to løsninger.
Her regner vi altså en linjes skæringspunkter med en cirkel.
Læs evt. mere her:
Skæring mellem linie og cirkel

Her fortsætter vi eksemplet ved at indsætte y-værdien i d ligningen:

$(x-3)^2 + ((2x+1)-3)^2 - 4 = 0$
$\Updownarrow$
$5x^2 - 14x + 9 = 0$
Det giver disse to svar:
$x = 1$
&
$x = \frac{9}{5}$

Så indsætter vi x værdierne i vores "linjes" ligning: y = 2x + 1

$y = 2 * 1 + 1$
$\Updownarrow$
$y = 3$

$y = 2 * \frac{9}{5} + 1$
$\Updownarrow$
$y = \frac{23}{5}$

Nu har vi x og y værdierne:
$(1; 3) og (\frac{9}{5}; \frac{23}{5})$

Example:

In this example, we use these equations: c is the small circle and d is the great circle.

We put them both to be equal to 0, that is like this:

So we set them equal to each other:

In this equation, y is isolated, that is, that we can put this y value into the second equation.
This gives the quadratic equation that gives us two solutions. Here we are therefore dealing with a line intersecting with a circle.
More here: Intersection between line and circle

Here we continue the example by inserting the y value into the equation:

It gives these two answers:

Then we insert x values in our "line to its maximum" equation: y = 2x + 1

Now we have x and y values:

Figure 4.27. Class wiki example and translation continued

Despite the difficulties his pupils sometimes encounter, Henrik tells me he likes problems like the one pupils worked on today, "because pupils have to think. I like to train [teach] the competencies and not the curriculum." I say that often I see pupils who face a new problem and they say, "I can't do it," but that did not seem the case for his class. He laughs and says, "This is not the first time we have done something like this!" Pupils have had previous practice problems where they were required to struggle to define the problem. "Sometimes it is enough for pupils to make a start even if they don't finish."

The Five Main Themes

Five main themes influence teachers' classroom practice: types of understanding, understanding, mathematical communication, policy, pedagogy, and curriculum. The ways in which teachers interpret and enact the Danish mathematics communications competency is influenced by policy, pedagogy, and curriculum. In turn, students develop mathematical understanding through use of mathematical communications and teachers use the resulting mathematical communication to assess pupils' understanding of mathematical topics and concepts. Although there is a great deal of overlap among these five themes, I will present each theme individually and discuss how each is a distinct theme.

Understanding

The term *understanding* is used in a range of ways in the mathematics education literature. Schroeder (1987) describes three broad categories of mathematics understanding found in the literature: epistemological and constructivist models, Piagetian and neo-Piagetian models, and cognitive science models. In each category, Schroeder recognized differing levels or types of understanding and noted that, though the conceptualizations of understanding differed from category to category, the ideas described are related. The ideas described by each category of models are similar to those explicitly stated by Pirie and Schwarzenberger (1988): "We consider understanding to encompass the comprehension of concepts, the relationships between these concepts and ordinary language or physical objects. Such comprehension must also include the procedural and process skills which depend upon familiarity with these relationships" (p. 461). In this dissertation, the term *understanding* will follow the use and definitions of the participants. Further, the definition and use of *understanding* is interpreted through the lens of the participants.

Three tiers of understanding emerged from this theme: understanding what to do to solve a mathematics problem, understanding how to solve specific mathematics problem, and understanding why specific problems are solved the way they are. The first two tiers are closely related and I will discuss these two together, followed by a discussion of the third tier of understanding. After discussing each tier of understanding, I will discuss the three tiers together and present a framework for how understanding influences mathematical communications at each tier of understanding as well as how understanding is assessed at each tier.

Procedural Understanding: Understanding what to do and how to do it

For some teachers and pupils, understanding is important because it lays the foundation for future work, whether that future work is a related mathematics topic later in the school year or an examination at the end of grade nine. Isabella, a pupil in class

7A, describes what she thinks her teacher, Martin, feels is important about understanding (P: pupil; I: interviewer):

> P – And, we…he…I think the most important thing for him is that we learn how or why it's not right and not, "That's not right," he'll say why it's not right and explain how we can do it other, another way.
> I – Ok, so is that important to know why it's not right?
> P – Yeah.
> I – Why do you think that's important?
> P – Because otherwise you won't learn anything, because that's why you make mistakes then you learn something from that.
> I – So is it more important to understand what you're doing or to finish all the problems on that page?
> P – To understand what you're doing.

According to Isabella, knowing why an answer is incorrect leads to understanding how to solve the problem correctly. For Isabella, this is important because she uses her notes and classwork to help her review a topic when she encounters that topic again and when she is preparing for exams.

> I – Ok. In the things that people did today…
> P – Mmmhmm.
> I - …the stuff they wrote down, what happens next with that?
> P – It's usually in our books and then we can either save the books or throw them out, but I usually save them, because then we've got them for when we're going to exams and stuff so we know what…so we can look back at things and see.
> I – Ok. So sometimes you go back and look at what you've done before?
> P – Yeah, if I'm not sure how we do it.
> I – So it can be useful?
> P – Yeah.

Isabella describes a two-phase cycle of understanding, shown in Figure 4.28 that is somewhat self-perpetuating: It is important that the pupil understand a topic because the pupil will see this topic again, therefore it is important that the pupil understand this topic. Two key aspects of this two-phase cycle of understanding a specific mathematics problem are a pupil knowing what to do and knowing how to do it.

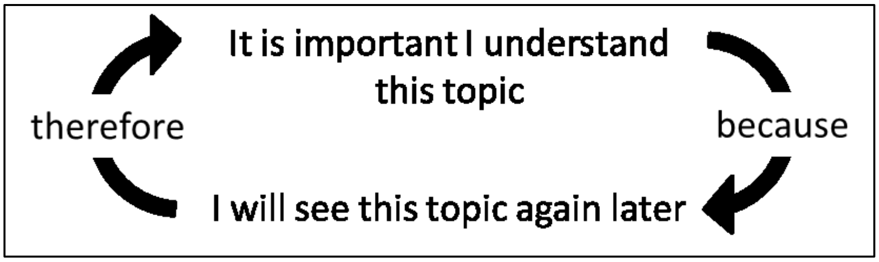


Figure 4.28. Two-phase cycle of understanding

The two procedural indicators of this two-phase cycle of understanding - understanding what to do and understanding how to do it - appeared during other conversations as well. Charlotte, class 4C's teacher, echoes Isabella's ideas about the importance of understanding when creating notes and writing classwork to refer to later (T: teacher; I: interviewer):

> T – Yes, others. Because I always tell the children, yes, you can say this is…and the answer…but if you don't show me what you did, I can't give you credit if you have the wrong answer. Perhaps you did it the right way…if there's only the answer and not the exercise, the way you did it, I can't give you any credit, it's just, "Oh, you couldn't do it." So…and it's very important that they, in their mind, can make the exercise, "I do it like this and this and this," and write it down. It makes it easier for them, they get help from it when they do it, the writing, and also it helps me to understand how they think.
> I – Do you think it's also helpful for them later?
> T – Yes, I think so. It's a good habit.
> I – Also, if they see, if they see fractions in two more months…
> T – Yes, they can go back. And look in their notebook at what to do.
> I- Because the answer itself is 3, but how did I get 3?
> T – Yes, and what exercise is it? It's very important I think.

Anna, a sixth grade teacher, at a different school from Charlotte, agrees:

> T – I use writing to see what they can and not only the answer. I'll have a description up, how they come to the result, and sometimes when we are finished with a thing, I'll have, they write some notes so that when we have the same theme in one or two years, we can go back and see how, what did we learn last?
> I – Oh, so they save their work from year to year?
> T – Yes.
> I- And they go back?
> T – Yes.

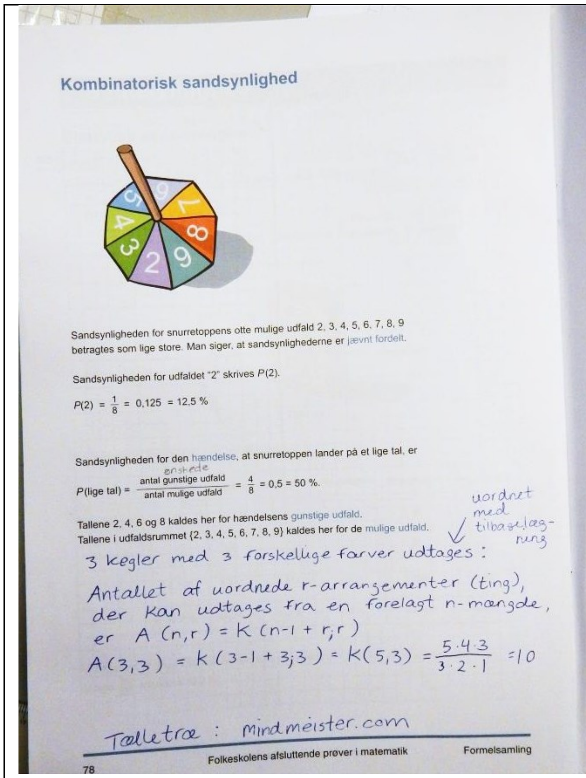
Martin provides an example of the emphasis on a clear structure to demonstrate understanding of what to do and how to solve mathematics problems is shown in the three-column method he teaches his pupils. His pupils write "mostly pictures, writing how they come to the right answer, and the right answer. They use a little text and I'm teaching them to divide the page in to three columns." Martin explains that in the left-most column, pupils' "text has to have something to do with the answer." The middle column is "for the working" and the last column is for the answer: "Text, working, and then the results." He describes this problem format as "a good example of writing mathematics." This format provides structure for pupils to show they know what they need to do in order to solve a mathematics problem and how to solve that problem, although there are limits to this format. The National Advisor for Primary Mathematics discussed this format: "It's not good for reasoning, not good for geometry." He described how the three-column format limits how pupils present their mathematical reasoning because there is only a narrow space. A better format, he explained, is one that has three horizontal rows which allow pupils to work across the entire width of the page.

Martin describes the three-column framework (shown in Figure 4.10) as "a good example of writing mathematics." This framework is introduced gradually, in many cases once pupils already understand many mathematics topics and concepts. Rather than being an integral part of how pupils develop their mathematical communication skills, Martin seems to view the framework as a separate mathematical concept that focuses on format and structure, almost as an algorithm in its own right.

In some ways, the two indicators of procedural understanding are encouraged by the national examination format. At the end of ninth grade, pupils take a four-hour mathematics examination. The first part of the exam, lasting one hour, contains 50 questions and focuses on basic facts. The second part of the exam, lasting three hours, focuses on problem solving. For this problem-solving portion, pupils may bring to the exam any of the curriculum material they used in class, including their textbook and something called a *Formelsamling* or *Formulary*. Produced by the Ministry of Education, the *Formelsamling* is a place for pupils to write formulas and explanations for each topic they encounter throughout seventh, eighth, and ninth grades.

In Maria's tenth-grade class, I observed how her older pupils use their written mathematics communications not only as a means for making sure they understand the current topic but also as a source of reference for the future. In Denmark, pupils in seventh grade receive a booklet called the *Formelsamling*. As described in chapter 2, tenth grade is an additional voluntary year of schooling for pupils who have not achieved the desired levels by the end of ninth grade or who feel the need for additional qualifications. For this reason, Maria's tenth-grade pupils are using the *Formelsamling* to prepare for their end-of-year examinations.

The structure of the *Formelsamling* is such that two pages are devoted to each topic. When opened to a specific topic, the left page has information about the topic and the right page has space for pupils to write additional notes, explanations, and sample problems. Figure 4.29 shows the *Formelsamling* pages and pupil notes for the topic titled *Combinatoric probability.*



Kombinatorisk sandsynlighed

Sandsynligheden for snurretoppens otte mulige udfald 2, 3, 4, 5, 6, 7, 8, 9 betragtes som lige store. Man siger, at sandsynlighederne er jævnt fordelt.

Sandsynligheden for udfaldet "2" skrives $P(2)$.

$P(2) = \frac{1}{8} = 0,125 = 12,5\ \%$

Sandsynligheden for den hændelse, at snurretoppen lander på et lige tal, er

$P(\text{lige tal}) = \frac{\text{antal gunstige udfald}}{\text{antal mulige udfald}} = \frac{4}{8} = 0,5 = 50\ \%.$

Tallene 2, 4, 6 og 8 kaldes her for hændelsens gunstige udfald.
Tallene i udfaldsrummet {2, 3, 4, 5, 6, 7, 8, 9} kaldes her for de mulige udfald.

uordnet med tilbagelægning

3 kegler med 3 forskellige farver udtages:

Antallet af uordnede r-arrangementer (ting), der kan udtages fra en forelagt n-mængde, er $A(n,r) = K(n-1+r, r)$

$A(3,3) = K(3-1+3;3) = K(5,3) = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$

Tælletræ : mindmeister.com

78 Folkeskolens afsluttende prøver i matematik Formelsamling

(Selected translations)

The probability of each of the 8 possible outcomes 2, 3, 4, 5, 6, 7, 8, 9 of the spinner are considered equally big. One says that the probability is equally distributed.

The probability of the outcome 2 is written p(2)

The probability of the event: The spinner yields an even number is

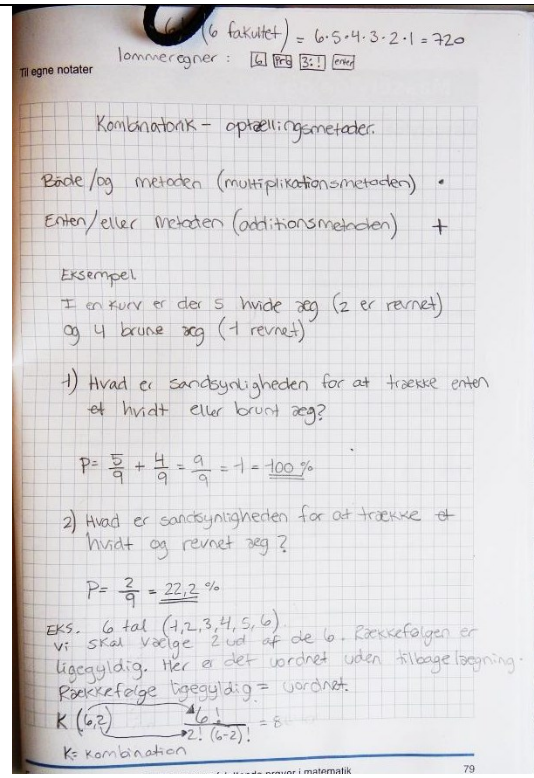
P(even number) = $\frac{\text{number of “good” outcomes}}{\text{total number of outcomes}}$

The numbers 2, 4, 6 and 8 are referred to as the "good" outcomes of the event

The numbers {2, 3, 4, 5, 6, 7, 8, 9} are referred to as the total number of outcomes

3 cones with 3 different colors are selected. The number of non-ordered r-arrangements (things) which can be chosen from a given n-set is A(n,r) = C(n-1+r, r)

Counting trees: mindmeister.com



$(6 \text{ fakultet}) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

lommeregner : 6 shift 3:! enter

Til egne notater

Kombinatorik – optællingsmetoder

Både/og metoden (multiplikationsmetoden) ·

Enten/eller metoden (additionsmetoden) +

Eksempel

I en kurv er der 5 hvide æg (2 er revnet) og 4 brune æg (1 revnet)

1) Hvad er sandsynligheden for at trække enten et hvidt eller brunt æg?

$P = \frac{5}{9} + \frac{4}{9} = \frac{9}{9} = 1 = 100\ \%$

2) Hvad er sandsynligheden for at trække et hvidt og revnet æg?

$P = \frac{2}{9} = 22,2\ \%$

Eks. 6 tal (1,2,3,4,5,6)
Vi skal vælge 2 ud af de 6. Rækkefølgen er ligegyldig. Her er det uordnet uden tilbagelægning.
Rækkefølge ligegyldig = uordnet.

$K(6,2) = \frac{6!}{2!\,(6-2)!} = 8$

K = kombination

79

Combinatorics – methods for counting
Both/and method (multiplication principle)
Either/or method (addition principle)

2) What is the probability of selecting a white and cracked egg?

Example: 6 numbers (1, 2, 3, 4, 5, 6)

We must pick 2 out of the 6. The order is of no importance. Here it is non-ordered without repetition.

The order of no importance = non-ordered
C(6,2)

C = combinations

Figure 4.29. Combinatoric probability pages and notes from the Formelsamling

One of Maria’s tenth-graders, Christian, described how he uses the *Formelsamling* in mathematics lessons (I: interviewer; P: pupil):

I - Ok. And so when you write, what do you put on the paper? Do you just write the problem or...?
P - Em, well, it's kinda like different, different, but you're sure you write down the entre problem so we know what to do next time.
I - So it's important that you don't just put the answer?
P - Yeah, yeah, we need to know what to do.
I - So sometimes you go back and look to see how you've done problems before?
P - Yeah.
I - Is that helpful for when you get ready for the exam?
P - Yeah, because like at the exams, you can bring tons of notes and whatever.
I - So I've seen the book, *Formelsamling*, I don't know how you say it in Danish, the book where you write...
P - Yeah, yeah, yeah. *Formelsamling*.
I - Do you write in there how to do the problems?
P - Yeah, yeah, and I write an example each time.
I - That, you said, is very helpful when you, when you're preparing for the exam or when you go to the exam?
P - Yeah.

Christian's classmate, David, describes how he uses the *Formelsamling* in a similar manner:

P - Yeah, it's a book where you have everything about math, and you can show, see [Danish - *geometri*], yeah, triangles...
I - Geometry.
P - Yeah, Geometry. And if we have to, to find something about...triangles and how you Pythagorize and stuff like that, then you can look in our F*ormelsamling* and then see what we done earlier.
I - Ok.
P - And there's a lot of forms about how you do it and it's, it's good to have when you don't know what to do.

When asked if he used sentences to explain his reasoning, David described what he usually wrote:

I - Ok. Do you ever write sentences? Like, "This is how I solved it," or something like that? Do you write lots of words? Or just mainly like this (gestures to paper).
P - Only when we have homework to do, when we get a problem with the person who maybe have to find the, find and answer about how many apples he have to buy or how many money he have to get back from the shop. Then we have to write, "He have, he has maybe 100 dollars and he have to pay 50 for the thing he's going to buy, and then we have to write how many money does he have when he's done in the shop. Then we have to write,

> "He have....maybe....50 dollars when he coming out of the shop," and then we have to write the solution and how we solved the problems.

The teachers and pupils describe this two-phase cycle of procedural understanding, it is important for pupils to understand the mathematics they are learning because they will see these topics again. These two levels of procedural understanding are limited to knowing what the correct answer is and how to solve the problems correctly, and therefore teachers assess these tiers of procedural understanding by first checking if an answer is correct and then, often only if the answer is incorrect, by checking the steps to the answer.

To assess Martin's pupils' understanding, Isabella says. "He [Martin] asks us and he uses questions in different ways than it maybe says in the book so that we maybe understand it a bit more and he changes the words a bit and stuff like that." When pupils check their work in class, Isabella says Martin goes over the answers with pupils and gives them an opportunity to ask questions if their answer is incorrect. Alexander says, "He trusts us that we can do it on our own, he doesn't need to check it. We just ask him, 'Is this right?' or, 'Have I done it right?'" Marcus explains further that Martin "doesn't really look at our things. If there's something we don't understand or if we want to have it checked, we just say it." A strong emphasis seems to be placed on pupils taking the initiative to ask Martin questions if they are unsure of their understanding. This is in contrast, Marcus says, to his teacher the previous year who would collect pupils' work and check each answer as correct or not.

Thomas**,** a seventh grader in Jacob's class, describes a classroom routine in which his teacher checks daily classwork (P: pupil, I: interviewer).

P - Yes. "This row come up here and I'll check your answers."

I - Ok. And then if you get something wrong?
P - Yeah. So he says, "What? Can you see what you [did] wrong there?"
I - So you explain how you got it. .
P - Yeah. And how did you make that answer?
I - And then you explain it to him?
P - Yeah.
I - And then what? Then does he say, "Oh, here is where your mistake is"?
P - Yeah.
I - And then you learn how to do it right?
P - But if they're not any...mistakes, so we just [gestures making check marks].
I - Just checks?
P - Yeah.

In Thomas' description of this routine, a correct answer represents understanding: if an answer is correct, there is no further action with that problem. If an answer is incorrect, however, it initiates a short conversation between teacher and pupil about the problem: identifying the mistake or explaining how the pupil solved the problem.

Jacob, Thomas' mathematics teacher and a colleague of Maria's, describes how he assesses a pupil's understanding in mathematics (I: interviewer; t: teacher):

I – So how do you know if a pupil understands something? Or doesn't understand something, I guess.
T – First and foremost, I can tell if they solve the problems correctly, is probably one thing.
I – So I guess the way you set things up, they have to show you that they understand.
T – Yeah, yeah. And I would ask them check.
I – Check your work.
T – Check. I would ask them questions to make sure…
I – Oh, right, to prove, you want them to prove.
T – Yeah. Say, "Ok, you can solve this, but could you solve this, too, then." To think a little bit different now to solve this one. A bit the same, but I've made it a little bit more difficult for you.
I – So if a pupil shows…
T – If they could multiply a two two-digit numbers, I could probably ask them to multiply three digit numbers, to make sure they know what's going on.

Jacob assesses pupil understanding by whether or not they have solved a problem correctly. As a next step, Jacob uses this assessment of a pupil's understanding to ask them to solve a similar but more difficult problem. By correctly solving the more difficult problem, the pupil demonstrates his or her understanding to Jacob of how to solve the problem.

According to teachers and pupils, the types of understanding described in this section – understanding what to do and how to do it - are gained through repetition, asking questions of a partner or the teacher, and seeing how other pupils have solved a particular problem. Jacob says that for some pupils, repetition is important for understanding.

> Also for some of the not-so-skilled pupils, there's the, there's a lot of repetition in doing additions, and they can, they can get into kind of a rhythm, this, this this…this, this, this, and they can learn to multiply or divide or so, even though they're probably not very skilled at it. Some of those who have really big problems in math, I think it can help them a lot.

I asked Thomas, a seventh grader, whether he preferred a more open-ended type of project or a page of problems from the textbook (described in our conversation by Thomas as "a page of plus, plus, plus").

> Maybe, I think I like...I like the plus, plus, plus, because you learn really how you do it, if there really, then you can maybe have problems with the start...later after that then it become easy and you know how you can do it and...yeah.

Marcus, a pupil in Martin's seventh-grade class, describes how he can find out how to solve a problem if he has questions (I: interviewer; P: pupil).

> I – So if there's something, for examples, that you don't understand, you said you can ask your partner? If you have a partner.
> P – Yes, yes, and if they don't know, I ask Jacob.

Sofie, a pupil in Maria's third-grade class, explains that she is sometimes the one to explain to other pupils:

> P – I often help the other pupils.

I – Oh, do you? Do you help them by answering their questions, or do you show them examples?
P – I often tells the other pupil what to do and sometimes I finds another example like the one they are having problems with.

Isabella, Marcus' classmate, explains that seeing how another pupil has solved a problem can be useful to her understanding of how to solve a problem.

I – Ok. So you said sometimes people will find the answer in different ways?
P – Mmmhmm.
I – Is that okay?
P – Yeah.
I – Is it?
P – Yeah, because he…
I – Because you're like, "Yeah, yeah…that's REALLY good." So why is that really good?
P – Because then it shows that there isn't only one way to do it, so, because people think differently so maybe somebody…if you're partners with somebody and they maybe will find it another way then you'll maybe do that one differently but you'll then do it together afterwards and see, "Ok, we've actually got the same answer, but it's still different ways we're trying to find it," then we'll ask him if that's okay and he'll say, "That's okay," because we think differently so…
I – Ok. Do you ever have a chance to see what other people, how other people have solved it?
P – Ah, yeah.
I – You said you could talk to your partner but what if maybe these people over here have done it another way?
P – Well, if we don't understand it, then we'll ask them how they done it, and then we'll say, "Okay, you could also do it like that."

The two tiers of procedural understanding – understanding what to do and how to do it – align well with the two-phase cycle of understanding. The cycle describes why pupils need to understand a specific topic and the tiers describe the understanding necessary to meet that need. At this level, mathematical communications, be they oral or written, remain procedural in nature. Gaining these types of understanding is generally limited to solving repeated examples of these problems and, if necessary, asking others about the steps to solve a particular problem. When assessing understanding is limited to checking for a correct answer and correct solution steps, understanding is reduced to an *either a pupil understands it or not* dichotomy with little room for partial understanding. There is little data thus far that suggest this two-phase cycle of understanding - and therefore the tiers of understanding what to do and how to do it - includes many written explanations or discussions, or much at all beyond definitions, example problems and steps to a solution.

Connectional Understanding: Understanding why

Although many of the observations and interviews touched on the importance of procedural mathematics - understanding what to do and how to do it - when faced with a mathematics problem, it is clear from the class profiles that neither mathematical understanding nor mathematical communications were limited to simply understanding what to do and how to do it. For example, although Jacob talked about the importance of pupils' repetition in learning how to solve mathematics problems, he later alluded to a deeper form of understanding than just procedural understanding: "That might be okay that they don't actually quite understand what they do in the first place, but after some time, they will hopefully learn what's actually going on." When asked how important is it that a pupil understands what is actually going on, he answered,

> Oh, that's very important. I mean, they could possibly end up passing their exam in the ninth grade, but it would be worthless knowledge as I see it, because they wouldn't be able to apply it on any other, how do you say it, in another situation, they wouldn't [?] those skills, in another setting, in another…yeah.

Seen from this perspective, mathematical understanding involves more than learning something just because it will be on an exam. Martin echoes this view about the importance of understanding more than just a method of solving a problem when he describes how he structures his teaching (T: teacher; I: interviewer):

T – I teach, fundamentally I would like the pupils to work a lot by themselves, working together, solving problems together, discussing why they came to find the right answer, or discussing different ways to get an answer, that is what I would like to achieve with my teaching. If I start something new like I did today, I would often give them a little instruction, just to make sure they have a way of getting started with the things. I experience that my pupils prefer to work instead of listen, instead of….isn't to sit quietly in the classroom and listen to other pupils or me, they would like to do something themselves. So, I try to use as little time as possible with I am talking in the classroom, and as much time as possible when they are working and doing different problems, assignments. So, that's how I try to, I would like to make different ways of doing it, smaller groups, bigger groups, sometimes they present what they…at the blackboard, they come up and explain to the others and write what they do. So, they can speak about what they are doing. Like I told you earlier, I think it's first when they are capable of speaking, talking about what they did, I think they learned it. Some pupils just learn a method, do you understand?

I – Yeah.

T – Just a method and then they can do it again, and again, and again, probably without knowing why they do it. So, in order to know why they do like they do, I would like them to explain it.

This mention of knowing why pupils do what they do to solve problems represents a shift in the meaning of the word *understand*. Rather than referring only to pupils understanding what to do to solve a problem and understanding how to solve a problem, Martin's statement uses pupil explanations to describe another type of understanding: understanding why a problem is solved a certain way.

Others described using oral and written communications to assess deeper levels of understanding. For Charlotte, using a pupil's written work is also part of assessing understanding.

> Because I always tell the pupils, yes, you can say this is the answer…but if you don't show me what you did, I can't give you credit if you have the wrong answer. Perhaps you did it the right way…if there's only the answer and not the exercise, the way you did it, I can't give you any credit, it's just, "Oh, you couldn't do it." So…and it's very important that they, in their mind, can make the exercise, "I do it like this and this and this," and write it down.

Charlotte continues, however, by saying a pupil's explanation is the crucial way of determining not just understanding but depth of understanding. "Their explanations of how they do things tells me how complex are they thinking." Charlotte also structures many of her learning activities to include pupil discussion and explanations. She jokingly told me, "They can't do my kind of exercises if they don't understand!"

When Charlotte refers to "my kind of exercises," she means exercises in which she has extended textbook problems in order to include pupil discussion and explanations as well as exercises she has created on her own such as the activity in which her pupils had to explain to one another why the fraction on one card was larger than the fraction on the other card and how they determined which was larger.

Anna, a sixth-grade mathematics teacher, described to me how she also uses activities that require pupils to understand why they are solving problems in a specific way. Rather than always using textbook problems, Anna tries to being in real-world examples (T: teacher; I: interviewer):

> T – I would like that they can meet problems in the book that they can meet in, outside the school. It's not always that I can find something that they can meet outside the school, but if I can, I try to take the outside in to the school.
> I – And you said if it's not in the book, sometimes, then what do you do?
> T – Yes. Maybe I can give them a problem, "What does it cost when you use toothpaste on a toothbrush?"
> I – Ok.
> T – "What does one…, one of…."
> I – Oh, ok. One of….yes. [gestures to indicate one glob of toothpaste on a toothbrush]
> T – Yes, yes.
> I – So you, sometimes you make the problems up yourself?
> T – Yes, and they don't have it on a paper.
> I – Ah, you tell them?
> T – Yes. And then they have to make it, they have to decide if it, how they want…how they can…
> I – How to solve it or how they do it?

T – Yes, how they solve it, yes.

This is an example of a problem in which pupils are not told what they need to do in order to solve the problem. Instead, pupils must work together to discuss what information is necessary to solve the problem and then explain and justify their mathematical decisions to each other and to Anna.

The profile of Maria's third-grade class provides examples of how Maria incorporates this tier of understanding into her lessons. When pupils were conducting the strategy experiments, they had to write their strategies for each experiment. During the class discussion, pupils discussed how they had different strategies for making their some shoes land a specific way because the shoes are designed a specific way. Later, when pupils wrote about chance (as in Figure 4.6), pupils were writing sentences describing and explaining what they know about the topic. These types of discussions and open-response opportunities required students to use different types of mathematical communications than just writing an answer and showing their steps.

Similarly, Maria was using these forms of mathematical communication to assess a different type of understanding than just understanding what to do and how to do it. When pupils discussed that they had different strategies for making their shoes land a specific way because the shoes are designed a specific way and when a pupil shared that her prediction for the number of dice landing on a six was 37 because 37 is her lucky number, Maria used these discussions and explanations to explore and develop pupils' understanding of why predictions and strategies function as they do.

For Maria, lesson activities that require pupils to discuss and explain are not extension activities to be added on later after pupils have already learned and understood the mathematics concepts. Instead, these types of lesson activities are the way in which pupils develop an understanding of mathematics. (I: interviewer; T: teacher):

> I – So tell me about using writing and talking. You think that's very important?
> T – Yes.
> I – Why?
> T – Because if you have to understand a subject, you have to talk it out and you have to write it down and maybe you have to try it on your body if it's about, if you have to know how long this table is, you have to try to [Danish]…
> I – To measure.
> T – Yes. Dimensions, yes.
> I – Ok.
> T – I think it's good when you can talk, write, think, try it in real practice and all around, use your body.
> I – Ok.
> T – I think it's good when you can talk to your partner about it, so when you hear your partner's talking, you can use it in your own thinking and your own talking.

The understanding to which Maria refers goes well beyond getting the correct answer to a mathematics problem. When she says it is good for students to "try it in real practice," Maria is referring to mathematics as something more than just problems in textbooks and work in notebooks. This is similar to what Jacob referred to when he described that he wanted his pupils to be able to use their mathematics skills in another setting:

> Oh, that's very important. I mean, they could possibly end up passing their exam in the ninth grade, but it would be worthless knowledge as I see it, because they wouldn't be able to apply it on any other, how do you say it, in another situation, they wouldn't [?] those skills, in another setting, in another…yeah.

Anna also referred to this idea when she said:

> I would like that they can meet problems in the book that they can meet in, outside the school. It's not always that I can find something that they can meet outside the school, but if I can, I try to take the outside in to the school.

There is common thread running through what teachers say when they refer to deeper types of understanding such as understanding why a problem is solved the way it is. This common thread is the idea of mathematical connections: teachers would like for students to connect what they are currently learning in mathematics both to other mathematics topics and to contexts outside of the textbook. These mathematical connections are formed when pupils discuss not just methods of solving problems, but also include reasoning and justifications for their solution methods. Only giving a correct answer and showing correct solution steps is not sufficient for understanding why.

For Henrik, communicating mathematically forms part of the core of his HTX mathematics lessons. There is "a focus on understanding and mathematics in context" and in order to assess how well his pupils understand a problem or topic, pupils must be able to communicate this to each other and to Henrik. Different from the procedural way Martin uses mathematical communication - almost as something to be learned later after the other topics have been covered – Henrik recognizes that communicating effectively in mathematics is a tool by which pupils learn and understand mathematics.

In contrast to the tiers of procedural understanding in which a correct answer and correct solution steps indicate an *understand it or not* dichotomy with little room for partial understanding, connectional understanding – understanding why – allows for a wide range of understanding. Rather than *Does a student understand?* a better question might be *How much does a student understand?*

Charlotte describes this range of connectional understanding when she explains why listening to what pupils say is so important (I: interviewer, T: teacher):

I – Why do you think it's so important to listen to the children?

T – Their explanations of how they do things tells me how complex are they thinking. If they can only say, “I do this and do the line,” it’s a division problem again, “if I have to do the coloring and counting,” then they are not very abstract in the way they’re thinking, but if they can say, “I just use, this is…” what is this one, I can’t see it (looking at book), if must be 5, 5 and 5. They do the 5, 10, 15 and then they get one out of five and one out of five and one out of five. If they can say it that way, they are more complex in the way of thinking, they are at higher level. And then some of them just say, “Oh, of course it must be 3 and some fraction.” They are at even higher level, so the explanation tells me what is the next step? If a student need the coloring like they are doing here, then I can’t just say, “well, you need to do, what is it called? What is called 5, 10, 15?

I – Oh, counting by fives.

T – Yeah. Then I won’t say that to them. I would still say you need to group and group and group and say, “Here is five, one for each, here is five, one for each, here is five, one for each.” So it still needs to be almost like, “Here’s marbles, give out marbles to the students.” Still needs to be hands-on. If they can do five by…then they are thinking in their head, they don’t need hands-on materials as much anymore. So the lower, the more hands-on, the higher, the more thinking skills, the more problem-solving.

I - And I like how you said that how complex their thinking tells you what the next step is. It sounds like to me that there is always a next step.

T – Always.

I – Even if they’re here, there’s always a next step.

T – Perhaps I need to get some expert in, but there’s always a next step.

For Charlotte, a pupil getting the correct answer is less important than a pupil being able to explain why he or she solved the problem a certain way. She describes not only a range of understanding why but also that a pupil’s level of understanding why leads to the next step. As Charlotte describes it, “there’s always a next step.” Without using a pupil’s written and oral communications such as explanations, descriptions, reasoning, and justification, Charlotte would have no guidance on *the next step*.

The understanding to which Charlotte and other teachers refer – understanding why – requires pupils to make connections between and within mathematical topics. It also situates learning and understanding in a different context from the two-phase cycle of understanding illustrated in Figure 4.28. This two-phase cycle of understanding uses two indicators of understanding - understanding what to do and understanding how to do it – to assess whether or not a pupil understands mathematics. In addition, the two-phase cycle of understanding situates mathematical understanding in a rather narrow context: I need to understand it because I will see it later in math class or on an exam.

Charlotte describes a connectional cycle of mathematical understanding that uses a wider range of mathematical communications, not only the answer and solution steps, but also explanations, descriptions, reasoning, and justification in order to assess not if a pupil understands why a problem is solved the way it is, but to what extent a pupil understands why a problem is solved the way it is. This cycle, shown in Figure 4.30, also implies that pupils not only connect previous mathematical topics to what they are currently learning but also that pupils connect the current topic to future topics. For Charlotte's pupils, *the next step* flows naturally from the current step.

A characteristic of the connectional cycle of understanding is the recognition that pupils will have different extents of understanding why. There is a continuum of understanding and each pupil might be at a different point on the continuum. To use a technology metaphor, connectional understanding is a slider: though the slider may be labeled from 0 to 10, it is possible to locate the slider at any position between 0 and 10. In contrast, procedural understanding focuses on either a pupil understands it or not. Instead of a slider, it is a switch: it is either on or off.

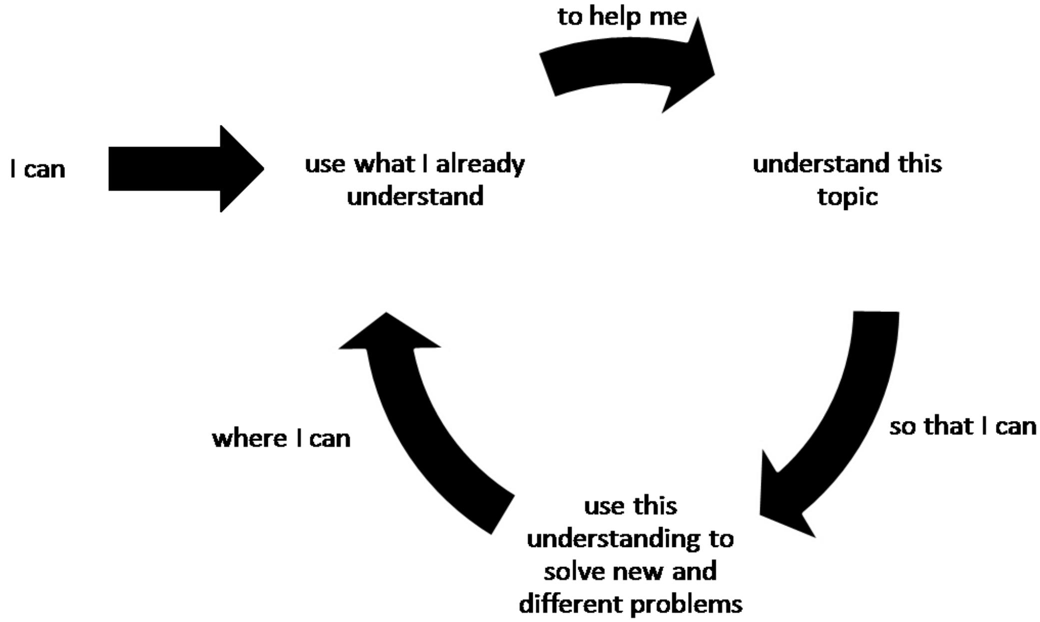


Figure 4.30. Connectional cycle of understanding

When asked to describe her mathematics lessons, Maria explained how she often begins her lessons:

> It often starts with some explanations about the stuff and some talking about understanding the stuff. And we often start with talking about what they know until now about a new stuff, because I have to know where are they, what can I build on, do you know what I mean?

For Maria, assessing what pupils already understand – to what extent they know understand the topic – and then using the understanding pupils already have is a crucial step in understanding the current lesson. This use of pupils' prior understanding is an entry point in the connectional cycle of understanding.

Anna describes how she uses a range of mathematical communications depending on the extent of individual pupils' understanding of a topic (I: interviewer; T: teacher):

> I – Is that difficult for you to have the range in the same class?
> T – No.

I – No? Ok, so how do you do with the high ones and for the other ones?
T – I try to, I want that they, the good one do more than the middle. I want them to explain more than the other. The other could maybe write the letter…no, not the letters, the numbers…and the good one have to make a description of what they have done. Yeah.
I – So in the same problem, they have to do more, in different ways.
T – Different ways. Yes.
I – Not just, these people you give three problems to, and these people you give five problems to?
T – No.
I – Ok, so not more number of problems…
T – No.
I - …but…
T – Sometimes, but that's not what I want to.
I – Right, you want them to go deeper…
T – Yes.
I – In the problem?
T – Yes. And maybe do other reflections after, maybe they can, I don't know what it's called in English [DANISH].
I – Oh, generalize?!
T – Yeah, yes, yes.

Recognizing understanding why as a continuum rather than an *understands it or not* dichotomy leads some teachers to use mathematical communications as a type of differentiation. Anna continues the connectional cycle of understanding by using what pupils are able to do, and encourages pupils to solve new and different problems by using that understanding.

The relationship between understanding and communication

One theory to emerge from the data relates to understanding and how mathematical communications, both oral and written, are used to assess that understanding. As described above, there are three tiers of mathematical understanding: understanding what to do, understanding how to do something, and understanding why something is done. Each tier of understanding represents a progressively deeper understanding of a topic or concept, and each tier of understanding uses different types of mathematical communication.

To illustrate the idea of how mathematical communications is used to assess understanding, let us imagine a hypothetical fourth-grade mathematics class in which the pupils are working on division of fractions. Consider five pupils seated along the front row of the class and we will assume that each pupil is focused on the lesson and not making attempts to be intentionally uncooperative. Each pupil is working on the same problem: 2/3 ÷ 3/4. Figure 4.31 describes what each pupil is writing on his or her paper.

$\frac{2}{3} \div \frac{3}{4} =$	$\frac{2}{3} \div \frac{3}{4} =$	$\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$	$\frac{2}{3} \div \frac{3}{4} =$ $\frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$	$\frac{2}{3} \div \frac{3}{4} =$ $\frac{2}{3}(\frac{4}{3}) \div \frac{3}{4}(\frac{4}{3}) =$ $\frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$
Pupil A	Pupil B	Pupil C	Pupil D	Pupil E

Figure 4.31. Five pupils and their papers

In this example, a teacher looking at just the answer as a measure of understanding can conclude that pupils A and B do not understand the problem while pupils C, D, and E each understand the problem. A teacher looking at pupil written work as a measure of understanding can conclude that pupils D and E understand the problem and pupil C does not (because pupil C did not show the second step of the problem).

Figure 4.32 shows not only what the hypothetical pupils wrote on their papers, but also how they responded to the prompt, “describe your work.” This figure shows the limitations on relying solely on a pupil’s answer to assess understanding. At best, the answer to a problem can only be used to assess whether or not a pupil understands what to do. Using a pupil's work as a measure of understanding has limitations as well. While looking at a pupil's work can determine if a pupil knows how to solve a problem (such as the difference between pupils C and D), it is unable to determine the differences between pupils D and E.

$\frac{2}{3} \div \frac{3}{4} =$	$\frac{2}{3} \div \frac{3}{4} =$	$\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$	$\frac{2}{3} \div \frac{3}{4} =$ $\frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$	$\frac{2}{3} \div \frac{3}{4} =$ $\frac{2}{3}\left(\frac{4}{3}\right) \div \frac{3}{4}\left(\frac{4}{3}\right) =$ $\frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$
I don't know what I am supposed to do here.	I know it's dividing fractions, but I don't know how to do that.	I wrote 8/9 because it's dividing fractions and that's what I heard my friend say as the answer.	The answer is 8/9 because when you divide fractions, you flip the second one and multiply.	The answer is 8/9 because when you divide, you flip and multiply. This works because of reciprocal fractions.
Pupil A	Pupil B	Pupil C	Pupil D	Pupil E

Figure 4.32. Five pupils and their written responses

In order to determine if a pupil understands why something is done in mathematics – depth and understanding as opposed to algorithms and memorization – a teacher must rely on a pupil's explanations and reasoning. In chapter 1, I described a continuum between algorithms and rote memorization at one extreme and depth and understanding on the other. By relying only on a pupil's written work, it is not possible to determine the difference between pupils D and E. It is only by using a pupil's explanation that a teacher can determine pupil D is actually relying on memorization of an algorithm while pupil E shows signs of greater depth of understanding of division of fractions. Figure 4.33 shows this relationship.

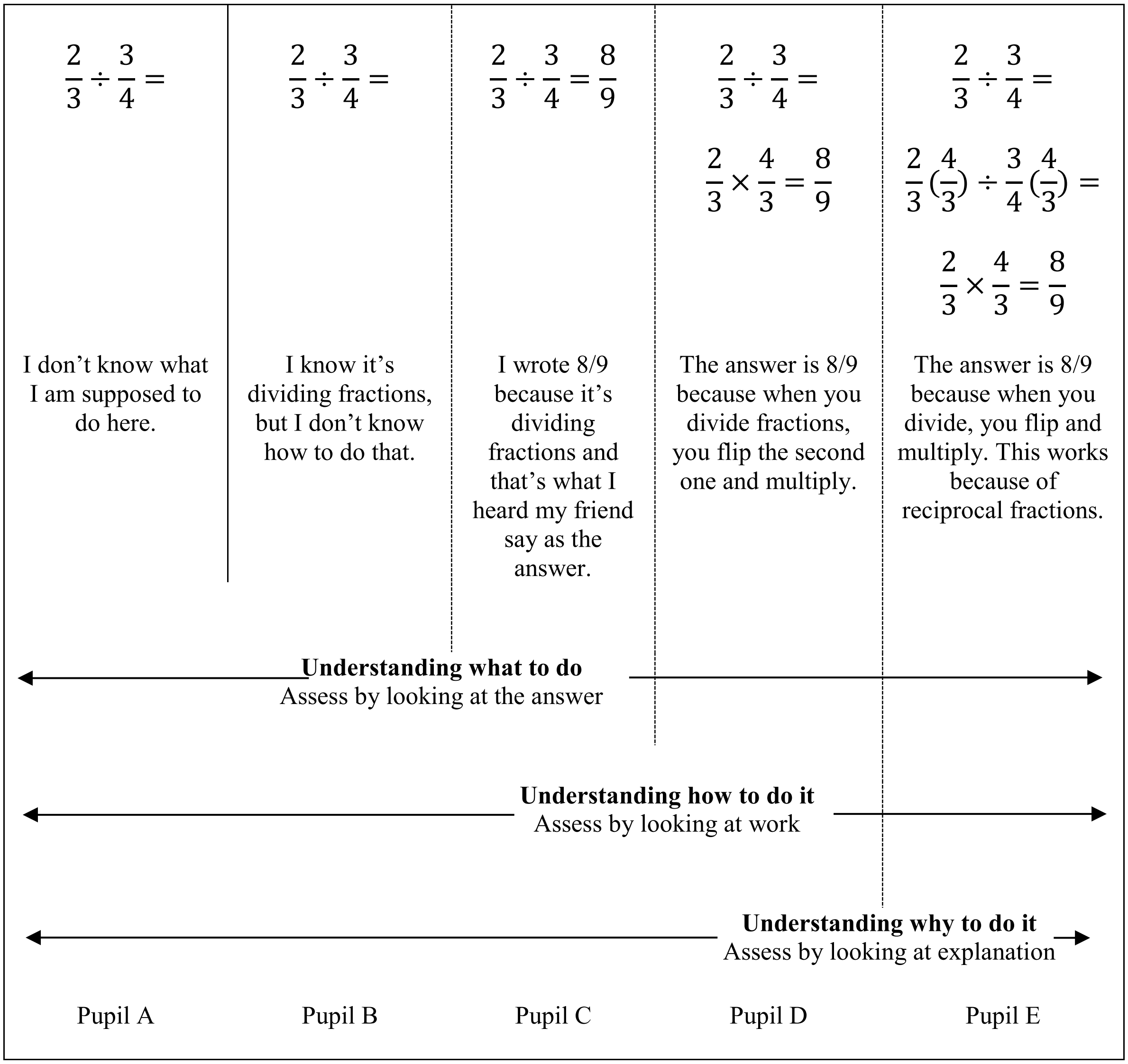


Figure 4.33. Assessing understanding using mathematical communication

The existing data can be applied to this framework. Alexander, a seventh-grader in Martin's mathematics class describes two questions students use to assess their understanding: "He trusts us that we can do it on our own, he doesn't need to check it. We just ask him, 'Is this right?' or, 'Have I done it right?'" By asking, "Is this right?"

Alexander is assessing his understanding of what to do. By asking, “Have I done it right?” he is assessing his understanding of how to solve the problem. Alexander’s classmate, Marcus, gave an example of assessing understanding what to do when he described his teacher the previous year who would collect pupil notebooks and mark each answer as correct or not. As Marcus described it, a correct answer signified pupil understanding.

Thomas**,** a seventh grader, described how his teacher, Jacob, uses assesses both understanding what to do and understanding how to do it to solve mathematics problems: (P: pupil, I: interviewer).

> P - Yes. “This row come up here and I'll check your answers.”
> I - Ok. And then if you get something wrong?
> P - Yeah. So he says, "What? Can you see what you [did] wrong there?"
> I - So you explain how you got it. .
> P - Yeah. And how did you make that answer?
> I - And then you explain it to him?
> P - Yeah.
> I - And then what? Then does he say, "Oh, here is where your mistake is"?
> P - Yeah.
> I - And then you learn how to do it right?
> P - But if they're not any...mistakes, so we just [gestures making check marks].
> I - Just checks?
> P - Yeah.

Although Jacob does use pupil work as a way of assessing whether or not his pupils understand how to do a specific mathematics problem, it seems the primary indicator of pupil understanding is the correct answer. According to Thomas, it is only if a student has a lack of understanding what to do as evidenced by an incorrect answer, that Jacob assesses whether a student understood how to solve the problem. It is entirely possible that a pupil has a correct answer but has used an incorrect method to solve the problem.

These examples illustrate the dichotomy described previously – either a pupil understands it or not. If an answer I correct, there is little need to look at the solution steps. If an answer is not correct, the solution steps reveal exactly what a student does or does not understand. In Figure 4.33, the arrows to the left of *Understanding what to do* and *Understanding how to do it* could be labelled *yes* and the arrows to the right labelled *no*.

The third tier of understanding, *Understanding why to do it*, however, is not easily answered *yes* or *no*, but is instead a continuum of understanding: *To what extent does the pupil understand why to solve a problem this way?* As Charlotte described, "Their explanations of how they do things tells me how complex are they thinking," and no matter the extent of a pupil's understanding, there's always a next step." In order to determine the extent to which her pupils understand a topic, Maria assesses her pupils using open-response tests in which pupils write sentences giving explanations and describe their reasoning.

The types of mathematical communication a teacher uses in his or her mathematics lessons seems to relate closely to how a teacher views understanding in mathematics. In turn, a teacher's views on understanding in mathematics seem related to possible beliefs about mathematics in general. As described in chapter 2, Ernest (1989) describes teachers' beliefs about mathematics as fitting into one or more of three mathematical philosophies. The first is the problem-solving view: that mathematics is a dynamic subject, open to inquiry. The second is a static view: mathematics is an unchanging body of interconnected body of truths. The third philosophy is the instrumentalist view: mathematics is an unrelated collection of rules, skills, and facts.

Ernest notes teachers may draw from more than one of these philosophies. A mathematics teacher who views mathematics as dynamic and open to inquiry is likely to view it as important for pupils to understanding why will likely incorporate more mathematical communication such as explanations, reasoning, and justifications in addition to answers and solution steps. Someone who views mathematics as an unrelated collection of rules, however, is likely to view mathematical understanding as procedural and use mathematical communications that focus only on the correct answer and solution steps.

Understanding Understanding: A Teacher/Pupil Disconnect

In conversations with Martin, he describes pupils' explanations as important in determining whether or not a pupil understands a topic. When talking with three of his pupils, they describe their answers as being the primary way of determining understanding, and it is only when a pupil has a question or has the wrong answer that Martin uses pupil explanations. Much of the responsibility for asking questions relies on pupils assessing their own understanding and taking the initiative to ask questions.

There seems to be a disconnect between how Martin describes the importance of oral and written mathematical communications and how his pupils understand and interpret the importance of these forms of communication. Martin spoke about the importance of understanding more than just what to do and how to do it (T: teacher; I: interviewer):

> T – … Like I told you earlier, I think it's first when they are capable of speaking, talking about what they did, I think they learned it. Some pupils just learn a method, do you understand?
>
> I –Yeah.

> T – Just a method and then they can do it again, and again, and again, probably without knowing why they do it. So, in order to know why they do like they do, I would like them to explain it.

Martin's pupil Isabella, however, says Martin goes over the answers with pupils and gives them an opportunity to ask questions if their answer is incorrect and discussion is generally limited to how to find the correct answer.

In part, this disconnect is echoed in how Martin described the alignment between the mathematical competencies and the national exams:

> I think the oral exam is way better at testing the competencies. This [written test] tests skills, fundamental skills and so on. You don't get to test, that is a problematic version in the way we test and the way we teach. So, if you want to have the best possible test and the pupils want that and their parents want it, and they want the best grades, so we have probably a little bit too much teaching for the test, instead of teaching for what we should, the curriculum.

Despite the importance Martin said he places on explanations and descriptions, the correct answer seemed to be the primary indicator of pupils' understanding during the observation of his teaching.

A similar disconnect appears between conversations with Jacob and conversations with his pupils. Jacob, a seventh-grade mathematics teacher, discussed both using pupils' solutions as a measure of understanding but also the importance of being able to use and apply what they know in other situations, not just the textbook problems.

> Oh, that's very important. I mean, they could possibly end up passing their exam in the ninth grade, but it would be worthless knowledge as I see it, because they wouldn't be able to apply it on any other, how do you say it, in another situation, they wouldn't [?] those skills, in another setting, in another…yeah.

Jacob's pupils, however, described more limited uses of mathematical understanding. Daniel is in Jacob's mathematics class and described for me what he typically writes in mathematics lessons and how that writing is used later (I: interviewer; P: pupil):

I - Do you just write the answer? Do you have to write x equals 7 or do you have to tell more?
P - I ever...I...sometimes I write more. But in some of these questions of...most of these question are just write the answer.
I - Ok. And then, then what happens when you finish them? Do you give them to Jacob?
P - Yeah.
I - And then what?
P - He would check it and he would correct and not. Sometimes he don't but we talk it...about it in the class...at the start of it.

While Jacob sometimes assigns larger, project-style mathematics assignments to his pupils in order to help them apply the mathematics they have learned to other contexts, pupils seem to interpret the discussion about those projects as focused on the correct answer. Sofia, another of Jacob's pupils, describes that discussion:

I - Ok. When you finish something like this, [the project] that you were working on yesterday, and you give it to Jacob, then what happens next? Does he look at it, or...?
P - He looks at is. I'm sending it on the internet then he is correcting it and we are talking about it in the class when everyone has got it back and then if we couldn't find out, we are talking about how to do it.

There is a clear disconnect between what some teachers say is important about mathematical understanding and how pupils interpret those beliefs. There are several places where this disconnect might occur. For example, teachers might say one thing but they actually enact something different. Another possible point of disconnect is that teachers think they are enacting one thing but the actual enactment is different. A third possibility is that teachers are actually enacting learning activities that focus on deeper levels of understanding but this is not clear to pupils.

This third possibility, enactment that focuses on deeper levels of understanding without this being clear to pupils, could happen in a number of ways. One way is that pupils are assigned a learning activity such as the assignment Jacob gave about calculating how much a person earns at an after-school job. Although there is a clearly correct answer, there are multiple correct steps to the answer. As pupils work together and discuss the problem, Jacob moves around the room listening to discussions and looking at the things pupils are writing. At times, he stops and asks questions of certain groups or point out a step that is not correct. It is possible that Jacob is listening for specific aspects of pupils' discussions about why they are solving the problem in certain ways and then offering input when he hears pupils say or sees pupils write something that does not support a deeper understanding of why. These communications about deeper levels of understanding might be present and encouraged by Jacob, without Jacob overtly specifying types of communication he is looking for and without pupils being actively aware of it.

As noted earlier, there is a great deal of overlap among the five themes. Although this section has focused on types of mathematical understanding, it is clear from the examples that different types of mathematical understanding utilize different types of mathematical communication. The next section will focus on describing mathematical communications in greater detail

Communications

As discussed in chapter 1, a number of mathematics curriculum standards refer to communication in mathematics. Though these curriculum standards recognize the importance of communicating in mathematics, there is often very little guidance on what mathematical communication actually is. In some sets of standards, mathematical communication is defined or described using a form of the word *communicate*. The English translation of the Danish mathematics communications competency describes itself in terms of *communication* without giving specific examples: “The communication competence is about students being able to express themselves and understand others' communication about mathematical topics, including oral, written and visual forms of communication,” (“Matematik: EMU,” 2014).

Given the rather vague description of mathematical communications in the Danish mathematics communications competency, there are a number of possible ways of interpreting a phrase such as *students should be able to express themselves mathematically*. Therefore it should be no surprise that there is a wide range of mathematical communication within the data set. During the interview with Jacob, we discussed writing in mathematics. When asked why he uses writing in his mathematics lessons, Jacob replied,

> I tell them that, like in any other subject, it’s a form of communication, if the recipient can’t either read or understand what you’re writing, it’s not communication. So, I tell them to put themselves in the recipient’s place, in a few words, read this, could you, would it make any sense?

Jacob’s description of communications – if someone can understand what you are writing – is easily expanded to oral communications. Whether speaking or writing, someone needs to understand what you are sharing orally or in writing. The previous section about mathematical understanding describes ways in which mathematical communications are used to support and assess mathematics different levels of mathematical understanding. This section focuses on the various forms of mathematics communication and the structure of these communications, as well as who is creating and using these communications.

Procedural mathematics communication

As noted in the discussion about understanding, one basic form of mathematical communication is procedural communication: the answer to a mathematics problem and the solution steps. In some cases, pupils write only the answer to a specific problem such as the example shown in the pupil work from Martin's class in Figure 4.11. Daniel, a pupil in Jacob's seventh-grade mathematics class explains, "Sometimes I write more. But in some of these questions of, most of these question are just write the answer." Daniel's classmate Andrea, however, says she writes the problem also (I: interviewer; P: pupil):

> I - Is that important to write not just the answer but to write the problem also?
> P - It's very important because you can go back and look. If it's just the answer, you can't really do anything with it. It's just the answer. So if it's something you have been working on a time ago, and we just started getting it up again, you can look...go back and look at the projects and problems you did before.

Thomas, a third classmate, explains that, although each student decides how much procedural information to write, it is important to show how a problem was solved (P: pupil; I: interviewer):

> P - Yeah, it's...it's maybe...it's different…it's different for person to person how they do that....they do that their work, but we have to write, we have to write the [Danish word]- I don't know how to say that.
> I - That's ok.
> P – [Danish. Asks a classmate] Yeah...and you have to make...if you have to write… 61 plus 94...
> I - Right.
> P - Then you have, you don't have to write the answer, but you also have to write the...(gestures to paper)
> I - Oh, how, how you did it.
> P - How. How, yeah.
> I - So Jacob is interested in how you did it, also and the answer, but not just the answer?
> P - Yeah. Because the person write just the answer could look at another. Another one.

I - Ah, ok, so it that important to write how?
P - Yeah.
I - Yes, why?
P - Because, if we only write the answer, then Jacob don't know how we are working with some stuff, and he don't know what other, he don't know… how we do it. Yeah.

In some cases, this type of procedural mathematics writing includes formulas. For David, a tenth-grade pupil, formulas are a way of implying steps to a solution without writing down each step (I: interviewer; P: pupil – note that David uses the word *formals* for *formula*):

I - What kind of things do you write in math?
P – If I, if it's a long formal, do you have some... [gestures for paper]
I - Oh, yeah.
P - It's a big....big numbers, or some formals we have to write down, on my calculator, I'm just writing down the formal as you can see, it was just I'm writing down the formal we just made and write into our calculator, then I write it down so I can show my teacher and the other person I'm working with what I actually do instead of just showing him my calculator and next time we have to do it the same, the same...
I - The same kind of problem? The same thing?
P - Yes, exactly. Then I just can, I can look back in my notebook and see what I have done but also when it's big numbers and I don't want to use my calculator, then I'm just, just [writing] 1000 plus 1000, then I'm just writing, maybe [writing] then I'm writing it down here, then I can see which, what I have to do, if I have to plus, it's easier, easy problem you can see like then I just say, 8 plus 7, that's 15, then I doing this [writing on paper].

Whether using a formula or not, numbers and mathematical notation such as symbols for multiplication and division form the majority of procedural communications.

Some problems, however, require additional information other than numerical answers. Jacob explains how additional information is sometimes needed: "Information in the problem that's important to do the calculation should be included, for instance. It's very important that they remember to put in units, if it's Kroner or dollars." For Jacob, including units is a way of making sure an answer is understood correctly.

Written procedural communications – the problem, solution steps, and answer – have similar oral forms. Caroline and Sara, two sixth-grade pupils in Anna's mathematics class, explained how sometimes in class, they will say just the answer, but sometimes they will explain how they solved a problem (I: interviewer; P: pupils who were interviewed together and are not individually distinguishable in the interview).

I – Do you speak a lot of answers?
P – Yeah.
I – Ok, why?
P – Because we know many thing about math and, yeah…I talk a lot, so…
I – What kind of things do you talk about? Do you just say the answer or do you say more?
P – We also say how do you find the answer sometimes, and sometimes we just say the answer if…if we want to know how you do it.
I – If you don't know how to do it?
P – Yeah.

Pupils also describe writing referential details when working in their notebooks. In the interview with Andrea, when asked what she writes in mathematics, she described details such as page numbers and problem numbers.

I - Yeah. I'm just trying to make sure I understand. So what kind of things do you write in math?
P - What kind of things?
I - Yeah. Or what do you write in math?
P - We write the page we're on...and the, called, it's called....
I - Do you want to show me an example?
P - You know...this. (Writes a problem number)
I - Oh, the problem number?
P - Yes. Or the, like this, the problem...
I - Right, the problem, which problem it is.
P - Yes. And there's some...
I - Number 1, number 2, number 3...
P - Yes. And then there's, you know, if this is the, title, it's...
I - Oh, you write the title?
P - And it's a, b, c, d....
I - Ok. And then you write, what else do you write?
P – We, we, [thinking] and these, like "a". We write how to solve it.
I - Ah, so you write how to? Ok. Could you show me an example?
P - Let's say it's page, I'm going to write it in Danish.

I - That's fine.
P - Page 18. And 1, and then you have “a”. And if it's, let's say [thinking] just write, it's 10.

Accompanying these forms of procedural communications are explanations and examples of how to solve problems and, as well as questions and classroom conversations. Martin provided this type of procedural explanation when he gave his class an example problem about converting currency:

$$\frac{2400\ n\ Kr}{100} = 24\ \cdot 88{,}59 = 2126{,}16\ d\ Kr$$

This type of explanation serves as a guide for students in how to solve specific types of problems.

In many cases, teachers purposefully have pupils work with a partner or small group so that if a pupil has difficulty solving a problem, he or she can ask a teacher or classmate for help. Marcus describes how pupils get help in Martin’s mathematics class (I: interviewer; P: pupil):

I – I saw Martin going around and checking on different partners? Is that what he usually does?
P – Yes, and helps people if there’s something they don’t understand.
I – So if there’s something, for examples, that you don’t understand, you said you can ask your partner? If you have a partner.
P – Yes, yes, and if they don’t know, I ask Martin.

Pupils also ask questions to assess their own understanding of what to do and how to solve a problem. Caroline and Sara explain how they know if an answer is correct or not.

I – How do you know if what you’ve done is correct?
P – Hmmm. We ask Anna.
I – You ask her if it’s correct? Ok, and what does Anna tell you?
P – She help us if it’s the wrong answer, and then we get the right answer and we learn more.

Later in lessons, when teachers go over work with pupils, there is often another opportunity to ask questions about how to solve problems. Andrea explains such opportunities in Jacob's class:

> P - Yeah, we go through…when we're done.
> I - You mean you go through it with the class? P - Yes. Together. It's like if we don't finish the pages we're on, we have to do it at home. And when we get back to school, we go through the pages and the things we…
> I - And if somebody gets something wrong, then what happens?
> P - Well, we just have to write the...
> I - So you have a chance to ask questions or say, "I didn't understand how to do it"?
> P - Yeah.
> I - "Tell me how to do it," or something?
> P - That's why we do go through it.

While these types of student-initiated questions are common, similar teacher-initiated procedural questioning also takes place. Clara and Victoria describe how their teacher Anna sometimes goes over answers in lessons (I: interviewer; P: as with Caroline and Sara, these students were interviewed together and are not individually distinguishable in the interview but are identified here are Pupil 1 and Pupil 2):

> P1 – Our teacher saying that our, the corr…
> P2 – The correct answer is.
> P1 - …or Anna says, "Victoria, what is the answer with page 21, [Danish]…"
> P2 – "…and B?" I say, "It's 12," and if the class say , "Yes, it's fine," so it's right or they say, "No," we…Anna is writing on the table, and we see what the right answer is.

Thomas**,** a pupil in Jacob's class, describes a similar classroom routine in which his Jacob checks daily classwork (P: pupil, I: interviewer).

> P - Yes. "This row come up here and I'll check your answers."
> I - Ok. And then if you get something wrong?
> P - Yeah. So he says, "What? Can you see what you [did] wrong there?"
> I - So you explain how you got it. .
> P - Yeah. And how did you make that answer?
> I - And then you explain it to him?
> P - Yeah.

I - And then what? Then does he say, "Oh, here is where your mistake is"?
P - Yeah.
I - And then you learn how to do it right?
P - But if they're not any...mistakes, so we just [gestures making check marks].
I - Just checks?
P - Yeah.

In Thomas' description of this routine, an incorrect answer initiates a short conversation between teacher and pupil about the problem: identifying the mistake or explaining how the pupil solved the problem. In some cases this teacher-initiated conversation takes place after pupils have done their work, but in other cases it takes place as pupils are working.

As described earlier, some pupils learn a particular format for presenting their procedural communications in mathematics. The *Formelsamling* (shown in Figure 4.29) is one example. Though it's two-page-per-topic format with one page of information and one blank page for notes is relatively open to whatever types of writing students care to include, the nature of the *Formelsamling* as an exam-preparation tool seems to limit pupils to procedural mathematics. The three-column format Martin teaches his pupils (see Figure 4.10) is another example of a procedural format. According to Rasmus, the National Advisor for Primary Mathematics, this format, however, "It's not good for reasoning, not good for geometry." The three-column vertical format limits how pupils present their mathematical reasoning because there is only a narrow space. A less limiting format is one that has three horizontal rows which allow pupils to work across the entire width of the page.

Connectional mathematics communication

The forms of communication described thus far include procedural communications such as the problem, solution steps or a formula, and the answer;

supporting text such as units; referential details such as page numbers and problem numbers; and explanations, examples, questioning and class discussion that focuses on procedural mathematics. These forms of written and oral mathematical communications support the early tiers of understanding: understanding what to do and how to solve a specific problem. Two students, Isabella and Andrea, both in seventh grade but from different schools, explained that sometimes these forms of communication are not sufficient. For Isabella, sometimes she does not yet know how to “do it with math” (I: interviewer; P: pupil):

> I – Ok, but you don’t write sentences or something to explain it?
> P – Sometimes.
> I – Sometimes? So what do you…what kind of sentences do you write?
> P – Maybe if we don’t know how to explain it in math, then we’ll do it with words instead.
> I – Oh, ok.
> P – And then he’ll sort of show us how we do it with math.
> I – Like if you know what you’re supposed to do, but you don’t know the math way of writing it?
> P – Yeah.

Andrea describes how explanations require more than just numbers:

> P - Well, sometimes you need to because there are some things you can't write with numbers.
> I - Ok, like what?
> P - Like how...um...how…(thinking)…trying to think of...(thinking) ...how is this different from this? And then you have to explain.
> I - Ok, and you can't just do that with numbers, you have to use words to explain?
> P - Yes.

In these two examples, Isabella and Andrea begin to describe something more than just procedural mathematical communications. Just as how, in the previous section, Jacob and Martin described the importance of understanding more than just what to do and how to solve a mathematic problem, Isabella and Andrea refer to the need for mathematical communications that do more than just tell what or how; they are using communications that help explain why.

If connections within and between mathematical ideas are a characteristic of understanding why mathematics problems are solved certain ways, mathematical communications that support this type of understanding can be considered connectional communications. Charlotte gives an example of connectional communications when she describes why she feels it is important for pupils to talk in mathematics:

> They're going to talk [about] it, it's very important that they just…they're not just looking at things, and doing it because the teacher said, "Do this, do that," but they can think it, they can talk it, and [do] the next activity where they had to write down the story, is also a way of getting them to connect what they already know with a picture and try to combine all of the knowledge.

In this description, Charlotte touches on two important types of connections: connections between this topic and other topics by drawing on what pupils already know, and connections within the current topic by using a story and a picture.

One form of connectional mathematics communication is a guess or a prediction. One of Maria's pupils, Sebastian, described what he wrote for the strategy experiments (I: interviewer; P: pupil):

> I – Ok. Yesterday when I was in your class, and you were doing the shoes and the dice, I saw that people were writing things in their books. So, what kinds of things were you writing?
> P – My guess. [Danish] What the real answer was. [Danish] And we had to find our own…exercise…
> I – Oh, your own, your own… experiments?
> P – Experiments. Experiments. And we wrote also our strategy.

When Maria asked her pupils to write down their guess about the outcome of the experiment, she was having pupils connect their background knowledge and understanding of strategies to the day's experiments. Later, when pupils write their actual results (Sebastian's "real answer") and when the pupils discussed their results with the class, Maria helped pupils make additional connections between their prior knowledge and the day's lesson.

Rasmus, the National Primary Mathematics Advisor, describes how it is necessary to connect a pupil's thinking and understanding to the current mathematics topic instead of teaching a standard algorithm in isolation. He also describes how algorithms are not all that is necessary for pupil understanding (A: advisor; I: interviewer):

> A – In the beginning, informal writing, just like I do here [taps on paper], nothing…
> I – Not sentences you mean?
> A – Not sentences at all, standard algorithms. There is some hard research from the United States, I think it was neurology, that when the pupil had problems in math, and you go on try to learn them one standard algorithm for addition, or so, they never learn anything. So you have to be more flexible in using algorithms, building on the pupil's own thinking and understanding, so, I start with informal writing and then when you come up, you have to do more writing, but there comes something before, as I think also, that the pupils in the beginning are also not writing but tell, they tell what they are thinking, for instance to a computer. There's a lot of small programs with some figures that they can do some dialogue with themselves, they can talk to the [computer], and the teachers or the other pupils can see how they are doing the work, if they are doing something in geometry, in Geogebra, we use this Geogebra, you know it, they can have a screencast in the computer and they can speak what they are thinking, what they are reasoning, their reflections, and show it for the teacher, or show it for the pupils. When they grow older, maybe instead of writing about, they have been outside in the woods, and should measure the height of a tree, and what they have to do the next day, produce a film, two minutes, how did you do this, so when you are doing this, you are filming it with your mobile phone, and then you speak and it will only take two minutes and you are using trigonometry, for instance.

Another aspect of connectional communication in mathematics is giving explanations. Where explanations in procedural mathematics focus on the steps of a solution, connectional explanations help tell why a problem is solved the way it is. Jacob gives an example (T: teacher; I: interviewer):

T – I don't know what it's called in English, but when you…you need to make a calculation like [writing on paper] minus 3…7…378…you could, you do this, then you do this…what do you call it? When you round up?
I – Oh, I guess you're rounding up? Yeah.
T – Yeah, I might ask them to write down why is that a good idea to sometimes, and what does it mean to round up and they have to explain it in words, what does it mean to round up and down.
I – Right, yes. Ok, so not just writing out the problem 598 minus 378, but explain…
T – Also, explain what does it actually mean to round up.
I – So sentences and things like that?
T – Yeah.

Rather than only assessing whether or not a pupil has rounded the answer correctly, Jacob describes looking for a pupil's explanation of what it means to round and uses this explanation as an indicator of how much a pupil understands about why he or she has solved the problem in a specific manner.

One characteristic of connectional communications in mathematics is what Rasmus, the National Primary Mathematics Advisor, refers to as "multimodal." When I asked him about pupil's using words in their mathematical explanations, he agreed but said words were not all that was possible (I: interviewer, A: advisor):

I – So the children are explaining, they're telling, they're using words to tell what they did and why they did it.
A – Yes. So this is a different oral work and then they of course have to write and do, I think, maybe fourth grade, that they have to explain themselves in writing.
I – In words and sentences?
A – Yes. But what we call multimodal, you know? What do you call it? Multimodal? We call it (writes)…multi…modal…text with pictures and diagrams and…what do you call it?
I – Diagrams, yes.
A – No, no, when you…
I – Representations, I think. Different representations, different ways…yes.
A – Yes, yes, but we start(?) talk about when you have a text, you have pictures…
I – Oh, illustrate. Or to…
A – Sometimes more…
I – To explain? I know what you mean, multi…

> A – Multimodal. Texts with, and we will work very much with that so when you are writing you have some texts, then you have a screen dump from Geogebra, some show you and you come and you have some reasoning, so on…

This multimodal form of mathematics communication was alluded to above when Charlotte talked about combining a mathematical story with a picture. When I first met Charlotte, she described learning about "seeing math in different ways." Part of what she means by this is using different ways of representing mathematics ideas. She uses a textbook page (see Figure 4.18) as she explains how she could make modifications to the problems in the textbook in order to incorporate additional modes of communicating:

> Some of the brighter ones…they have to do a different kind of explaining. If they had number 6, it wouldn't be enough just to draw the picture, they would have to do something more like the [number] line today or [explain] in words, "then I do, then I do, then I do." That would be an exercise for the skilled ones.

In the wall of 4C's classroom, Charlotte's pupils have created a display that illustrates seven different aspects of multiplication (see Figure 4.20). For Charlotte, knowing how to correctly use an algorithm to multiply two numbers is not sufficient to understanding multiplication. Charlotte wants pupils to be able to connect a range of multiplication-related ideas, including visual representations of multiplication tables, examples of different multiplication algorithms, word problems involving multiplication and ways of using multiplication in pupils' lives.

Similarly, Henrik encourages his pupils to multiple modes of representations in their work. When his pupils were working in small groups on the sample exam problem, pupils were sketching pictures on paper and using Maple 17 and Geogebra software to create diagrams. Some pupils are also using laptops to look at reference material. For pupils in some classes, working in a multimodal manner is something they do regularly.

Multimodal forms of connectional mathematics communication refer not only to how pupils present their understanding but also to how teachers present mathematics topics. When asked what he likes to see in mathematics classrooms, Rasmus, the National Primary Mathematics Advisor, begins by describing what is on the walls, but then extends this to describing multimodal ways of communicating mathematics ideas to pupils (I: interviewer; A: advisor):

> I – So, you said that the curriculum reform is going to be focusing on what children are learning. So when you think about a math class, when you think about, if you go into a math class, what do you like to see? Or what are the things you walk in and go, "Ah, this is a good math class." Or should I give you an example?
>
> A – You know we don't have special places for math in Denmark, you have a classroom for the class, and teacher come. Normally when you look at the walls in the classroom, it's the Danish language teacher that have decided…
>
> I – Oh, that puts posters and things…
>
> A – The math teachers are not very good at show THEIR language in posters and so on. Their subject, but it is coming. So, what I like to see is, what are the goals for this day, or this…
>
> I – Written on the board?
>
> A – Written somewhere, if it's a longer period, you can have it as a poster, especially in lower secondary. I would like to see pictures; I would like to see art that has something to do with what we are doing now. If we are having geometry, have some, it would be from Alhambra, it could be, we have…yes. Then I would like to see maybe the pupils own pictures that we using in the teaching. I like to go with children to make some pictures then when we come home, analyze geometry. Have you been to the Central [Train] Station [in Copenhagen]?
>
> I – I have, yes.
>
> A – Just have a look on one side of the wall, there is a lot of mosaic. I use them very much for geometry in fourth or fifth grade, and I have them to draw this and sketch, and when we are coming home, use Geogebra or something like that to analyze and they can make their own patterns and mosaics.

As Rasmus describes it, pupils must be in a learning environment that uses multimodal forms of mathematical communication for teaching and learning in order for a pupil to use multimodal forms of mathematical communication to explain their understanding of why problems are solved the way they are.

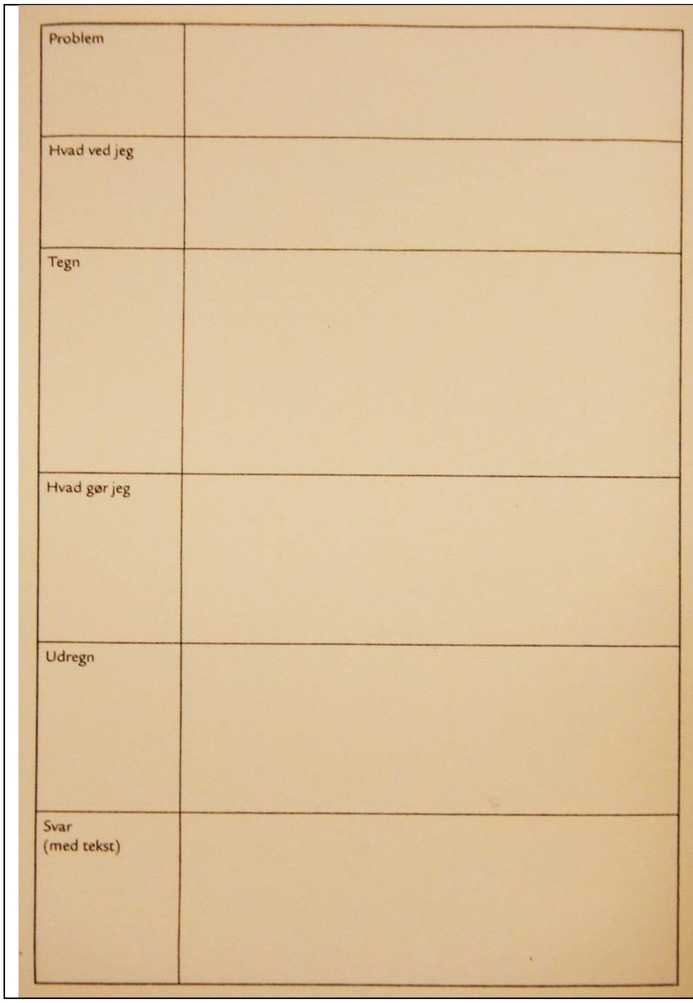
In an effort to encourage his high-school pupils to use connectional communications rather than only procedural communications, Henrik has created a wiki for his class. A wiki is a website in which pupils can create webpages for particular mathematics topics. He explains that pupils were assigned topics to include in the wiki, and for each topic, pupils had to decide the important information to include in order to clearly and accurately explain the topic using not only text but images as well. Part of each entry should be explaining why problems in the particular topic are solved the way they are. An example of a group's blog entry on the intersection of two circles is shown in Figure 4.26 and Figure 4.27.

Anna uses a similar method with her sixth-grade mathematics class. Her pupils have created blog posts that give both procedural information on a topic, such as how to calculate the circumference of a circle, but also connectional information, such as the relationship between a circle's radius and diameter. Anna's pupils have used a mixture of all text, text and images, and explanatory videos in their blog posts. Clara and Victoria, two of Anna's pupils, say that Anna sometimes uses certain blog posts as a way of beginning class discussion the next day. Anna also explains that pupils are able to post comments for each other on the blog posts and these comments can help pupils revise and refine their explanations.

In addition to multimodal representations, and computer-based wikis and blogs, teachers also use classroom discussions and conversations to help their pupils develop their understanding of why problems are solved the way they are. Though classroom discussions are also a feature of procedural communications, those discussions focus on procedural mathematics: what to do and how to solve a specific problem. Discussions focused on connectional mathematics communications focus on understanding why a particular problem is solved the way it is.

The fraction activity in Charlotte's lesson is an example of how pupils use connectional communications in discussions. Each pupil received two cards: one card with a fraction in numbers and one card with a corresponding picture. Pupils are working in pairs to discuss which fraction is larger and then use the picture card to prove and justify their answers. Another example is Anna's toothpaste problem: "What does it cost when you use toothpaste on a toothbrush?" This is a problem in which pupils are not told what they need to do in order to solve the problem. Instead, pupils must discuss what information they need to solve the problem and then explain and justify their mathematical decisions to each other and to Anna.

Just as some pupils learn procedural formats for presenting their work, other pupils learn more connectional formats for presenting their work. In Anna's mathematics lessons, pupils learn a six-row horizontal problem-solving framework (see Figure 4.34). Anna explains she found this format in a book for "two-language students" and thought it was useful. She began using it with her pupils last year (in grade 5) and at first she had them complete one chart for each problem, but she tells me that now pupils know the format and do not have to use the framework for each problem. Also, Anna explains pupils may not need all the steps, particularly the drawings, for each problem. Although this framework includes procedural forms of communication such as the problem, an explanation of what to do, calculations, and the solution, it also includes connectional forms of communication as well: links to what a pupil already knows, and a representation of the problem in the form of a drawing. This horizontal format provides more space for pupils to show their reasoning, particularly if pupils write their own framework on their paper – an aspect Rasmus specifically noted was missing from the vertical column format.



Problem

What do I know

Drawings

What to do

Calculate

Solution (with text)

Figure 4.34. Anna's six-row horizontal problem-solving framework

Teachers who incorporate connectional communications into their mathematics lessons do so throughout their pupils' experience in mathematics. For the pupils in Maria's third-grade class, for example, using a range of mathematical communications is an integral component of their classroom experience. Because of her views about teaching and learning mathematics and through her conscious selection of and planning for learning activities, Maria incorporates a range of oral and written mathematical discussion, explanation, and problem solving into her lessons. For Maria, both how a pupil has solved a problem as well as the detailed explanations are crucial components in assessing whether or not, or to what extent, a pupil has understood a mathematical topic, and she uses that information to inform her planning.

The development of connectional mathematics communication is not a sudden process, but rather one that begins in the early years of school and develops over time using both oral and written components. Rasmus, the National Advisor for Primary Mathematics, explains (I: interviewer; A: advisor):

> I – Can you tell me about what you think about writing and mathematics, I guess is a good place to start.
> A – Yes. But I will start where it starts because I think it's very important to be a good writer that you have oral working with math, that the pupils from the start are telling what they are thinking, what they are doing…
> I – From very small, from first grade?
> A – First grade. From the kindergarten.
> I – As soon as they are able to, they should be doing that?
> A – Yes. So the language is very important and it should be central for the working with math all the time.

Even before children are able to write, they are still able to talk and use the language of mathematics. Anna describes using mathematical language with young children – "the small classes":

> T – In the small classes, they don't have an explicit speak, yes, they can point at the things and I try to give them a language, a language so they can explain very clear in mathematic problems.
> I – Alright, so you help the younger ones, you help them learn the language, the things to say and the words to use?
> T – Yes.

Charlotte describes how she uses oral language, drawings, and "children's spelling" as part of "the whole process" of using connected mathematics communication with young children:

> T – Yes. Exactly. But also in second grade it could be just, How do I add numbers? Or in first grade, just counting, How many apples in the tree? In the beginning of first grade. Or…
> I – Would you do that with sentences? With words also?
> T – Yes. No, no, sorry. Not words, just words speaking it.
> I – Oh, telling it.
> T – Because in first grade they can't write it.
> I – So when is it, when do children start, or when do you start with children, the writing by words?
> T – It depends on how good they are at it. Not the math part, but the writing part. As soon as they can do just a little. In Danish we call it "children's spelling" – they do it by what they can hear of the words. Sometimes I can't read it, but they know what it says, so they write it.
> I – So then they tell you what it says.
> T – Yes.
> M - So even when they're very young, it's not, "you must wait until grade 4 or grade 7," it's as soon as possible.
> T – No, no, no, it starts as soon as possible.
> I – As soon as they're ready. Do you also, do you have them explain, tell you in speaking, before even, before they can write?
> T – Yes. As soon as they have, as soon as I start math classes, we do a lot of drawings. One apple tree, another apple tree, here are five apples, here's three apples. The problem is, how many apples are there at the two trees? They solve, the answer is 5 plus 3. Not 8. No, 5 plus 3.
> I – So really showing the whole process…
> T – The whole process.

Procedural and Connectional Mathematical Communications

Procedural and connectional mathematical communications have distinct characteristics as shown in Figure 4.35. Procedural communications seem limited to those forms of communication shown. Teachers who encourage connectional forms of communications also encourage procedural communications, but their emphasis is on connectional communications. For example, although Charlotte places a strong emphasis on her pupils learning and using connectional communications, they also learn procedural communications. Teachers focused on procedural understanding in mathematics emphasize procedural communications in their classrooms. Teachers focused on connectional understanding in mathematics use procedural communications, although there is a strong emphasis on connectional communications.

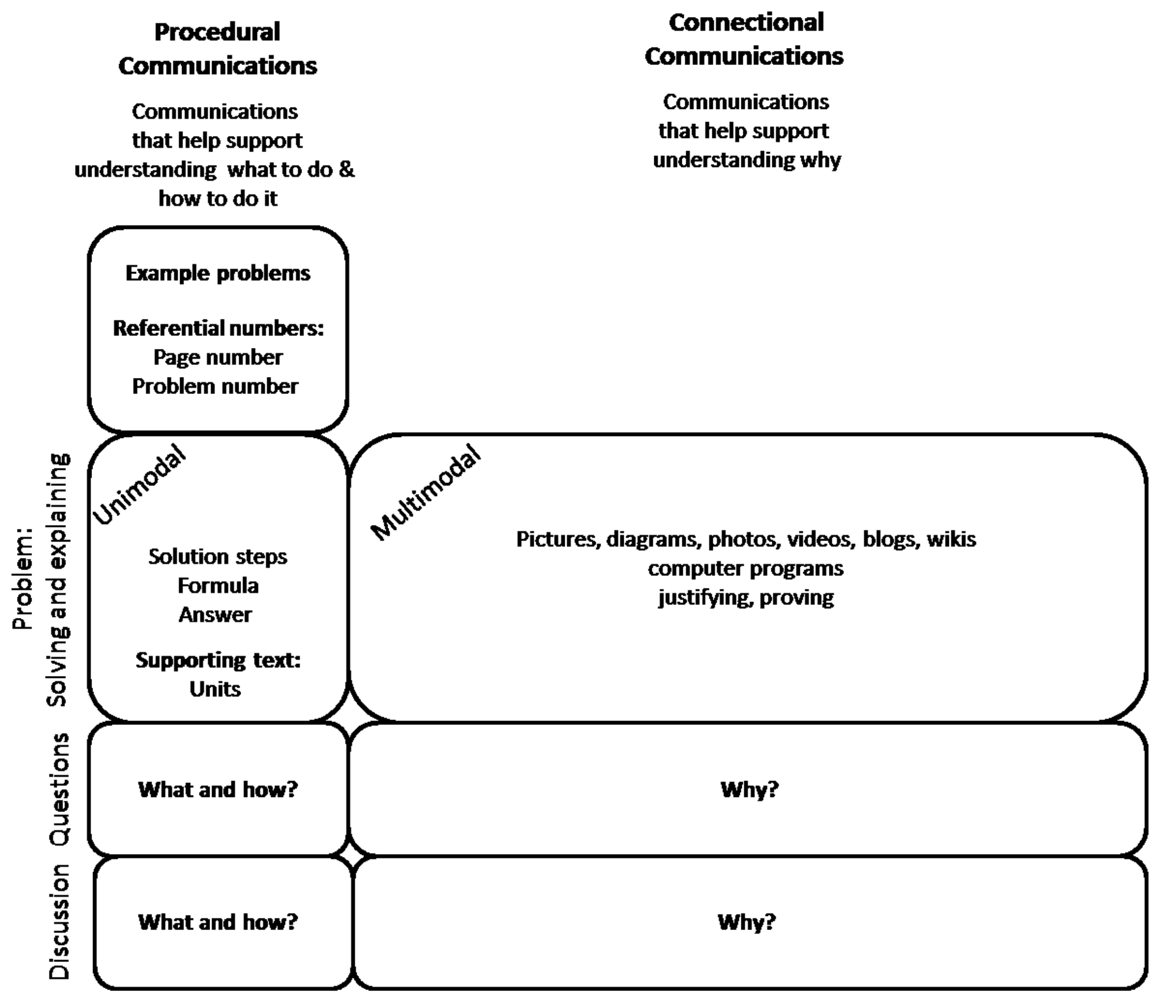


Figure 4.35. Procedural and Connectional Mathematical Communications

Policy

Policy refers to the circumstances of teaching. These circumstances can be curriculum standards such as the Danish *Fælles Mål* (*Common Objectives*) that set out government expectations for both content and process standards in Danish schools. These circumstances of teaching can also relate to government policies about national exams for pupils, and how instructional time is to be used in schools.

The Mathematics Communications Competency

In Denmark, the *Fælles Mål* (*Common Goals*) is a set of content and process standards for each subject area. Teachers in both public and private schools are required to follow these standards in their teaching. The mathematics communication competency is a part of these expectations. As described earlier, this competency is somewhat vague and open to interpretation: "The communication competence is about students being able to express themselves and understand others' communication about mathematical topics, including oral, written and visual forms of communication," ("Matematik: EMU," 2014).

When specifically asked to talk about the mathematics communications competency and whether their classroom use of talking and writing was informed by the communications competency, teachers spoke of the competency but indicated their own thoughts and views of mathematics communication guided how they used speaking and writing. In some cases, the communications competency itself was mentioned almost as an after-thought. For Jacob, using correct mathematics terms and expressing thoughts and solutions is important:

> I believe it's very important, that the students are able to communicate math using the right terms. … I personally believe it's incredibly important the students learn to verbalize their thoughts and solutions to math problems. It also happens to be one of the competencies from the Ministry of education.

Martin indicates he is aware of the competencies and tries to teach towards them. His own beliefs about mathematical understanding, however, seem to guide his interpretation and enactment of mathematical speaking and writing in the classroom:

> Like you said, I use a lot of talking in my teaching. Primarily because of my own beliefs. I believe that the student have a better understanding, when they are capable of communicating it to me or each other. It gives them room to make arguments about what they are thinking when they are solving mathematical problems. But I am also aware of the competences, that is described in the law. I usually try to teach towards two of the competences, but often more of the competences is in present.

Maria's beliefs about mathematics communication and multi-modal representations also seem to guide her interpretation of the communications competency:

> We have to be focused on mathematics communications competency. It is written in the law. I have to do it and I also believe in saying, writing, drawing especially with small children. I think it is a good way of understanding. It is also a way to search for solutions.

For Anna, the communications competency is even more of an after-thought.

> The pupils have to understand mathematics communication when they meet it in for example a newspaper and they have to understand something other people explain in instruction books and tell them - not written. And they have to express mathematics both written and spoken, so that they can take part in professional/mathematic discussions. (In the curriculum from 2009 there were 8 competences but in 2014 they have put some of them together, so now there are 6).

From these examples, it is clear that teachers are aware of the mathematics communications competency but do not look to it for guidance in how to enact mathematical communication in their lessons. Instead, teachers interpret the curriculum policy, at least in part, according to what each individual teacher expresses are his or her beliefs about what mathematics communications are important. As seen from the class profiles and previous findings, however, interpretation and enactment of the mathematical communications competency is more complex than stated beliefs.

Instructional Time

Another circumstance of teaching that has an impact on how some teachers enact parts of the mathematics communication competency is instructional time. As noted in chapter 2, the Danish government mandates the minimum instructional times for certain subjects, including mathematics. While it is up to individual schools and classroom teachers to determine how to organize this time, in some cases teachers feel limited by the instructional time available for mathematics. Charlotte typically has two 90-minute mathematics lessons each week with her class of fourth graders. She describes how she feels it is very important for pupils to be discussing, writing, and connecting their ideas.

> They're going to talk [about] it, it's very important that they just…they're not just looking at things, and doing it because the teacher said, "Do this, do that," but they can think it, they can talk it, and [do] the next activity where they had to write down the story, is also a way of getting them to connect what they already know with a picture and try to combine all of the knowledge.

Despite the important of these types of communications activities, Charlotte acknowledges she is not able to include as many of these activities as she would like because they take more instructional time than more routine types of problems.

> Because the activity I did today, with the fractions and the picture of the fractions, it takes so much time! But I think I have to do it, because it's a way to get them to think about, "Oh, one-seventh, one-eighth, what is the larger, the bigger thing?"

She also discusses how she tries to balance the activities involving multiple types of mathematical communication with the more routine mathematics problems.

> But I do that [the communication activities], because of that, there's not much time for exercises all the time, so I try to find out which one is the important one, and a little of it is going to be the week study at home, the not-difficult parts where you don't have to think that much, that's mostly routine work, and then the difficult parts, we do it up here [in class].

Although Charlotte makes clear she feels talking and writing are critical to learning mathematics, she is also clear that the instructional time available to her causes her to include fewer in-depth, communications-rich learning activities than she would like.

Assessment

Another circumstance of teaching is government policy regarding assessment. In Denmark, while most assessment is done in classrooms by teachers, national testing in mathematics takes place in grades 3 and 6 with school-leaving examinations at the end of grade 9. As described in chapter 2, after completing ninth grade, Danish pupils have several options for continuing their education into upper secondary school. Each option has its own mathematics examinations at the end.

Few of the primary-school teachers in this study indicated that the national mathematics assessments had much, if any, influence on their interpretations or enactments of the communications competency. One teacher, however, briefly discussed the alignment between the examinations and the curriculum. During our discussion, Martin mentioned the examinations at the end of grade 9 and provided me with several samples of previous written exam papers. Martin noted the written exams focus less on the competencies, particularly the communication competency, than on the mathematics content, while the oral exams – taken by only a small fraction of pupils – is more aligned with the competencies. In other words, Martin views the written exams as focused on procedural understanding and the oral exams as focused on connectional understanding.

> I think the oral exam is way better at testing the competencies. This [written test] tests skills, fundamental skills and so on. You don't get to test, that is a problematic version in the way we test and the way we teach. So, if you want to have the best possible test and the pupils want that and their parents want it, and

> they want the best grades, so we have probably a little bit too much teaching for the test, instead of teaching for what we should, the curriculum.

The difficulties of assessing the mathematical competencies in the national tests is affirmed by the National Advisor for Primary Mathematics. With the tests in grades three, and six, "you can't evaluate [pupils'] competencies," he notes. In regards to the communications competency specifically, "it is clear that they have to do this writing, but not in which way." A lack of alignment with or capacity to adequately assess the mathematics competencies helps move the focus from competencies towards content alone. Additionally, the format of the *Formelsamling* discussed earlier, in which pupils make notes in grades 7, 8 and 9 to use on the national examinations, focuses on how to solve individual content topics rather than competencies or process skills. David, a pupil preparing to take the national exams, explained to me:

> In grammar, there's a lot of things that you have to, that have to be correct, as a comma and how you spell words, but in math it's not so important because it's only the final answer you have to, have to be correct, all the solutions final.

This focus on content rather than competencies, or procedure rather than connections, in grades 1-9 is echoed by one of the upper-secondary teachers I observed. Henrik teaches mathematics in the HTX program – the Higher Technical Examination program that focuses on technical and natural sciences. He noted the transition from grade 9 to his high school can be difficult for some of the pupils because they are not used to explaining their thinking and telling why. In the HTX program, Henrik says there is more group work in high school and a focus on understanding and mathematics in context and in integrating other subjects with mathematics. This, he says, is more of what it actually is to work in math-related fields such as engineering. Henrik attributes this style of teaching directly to the Danish competencies for mathematics and notes the competencies require this type of teaching.

One reason for differing interpretations and enactment of the communications competency in grades 1-9 compared with high school is that the national examination format is different. While the national tests and exams in grades 3, 6, and 9 focus less on competencies, the examinations at the end of the HTX program requires pupils to clearly express themselves mathematically in both a written paper and an oral presentation.

Observing a mathematics lesson lead by Henrik's colleague, Pernille, gives an additional insight into the competency-focused oral presentation. Pernille's mathematics lesson was a review of differential calculus with her A-level HTX pupils. These pupils were in their final year of high school and were nearing their final exams. In the first part of the lesson, Pernille gave pupils problems to solve such as:

$$(f \circ g)'x = f'(x)g(x) + f(x)g'(x)$$

Rather than focusing solely on procedural mathematics such as the answer and solution steps, for each problem, Pernille invited a different pupil to the board to work through the problem and explain the steps of his or her solution in a manner similar to the oral exam format. At various points during the solution, Pernille and the pupil would discuss different aspects of the solution such as why a pupil was solving the problem in a certain way. Other pupils would contribute suggestions or alternate methods and explain why they suggested those alternatives. Figure 4.36 shows how the pupil solved the problem. Small stars and arrows at various points in the solution indicate where he made reference to specific steps during his explanation.

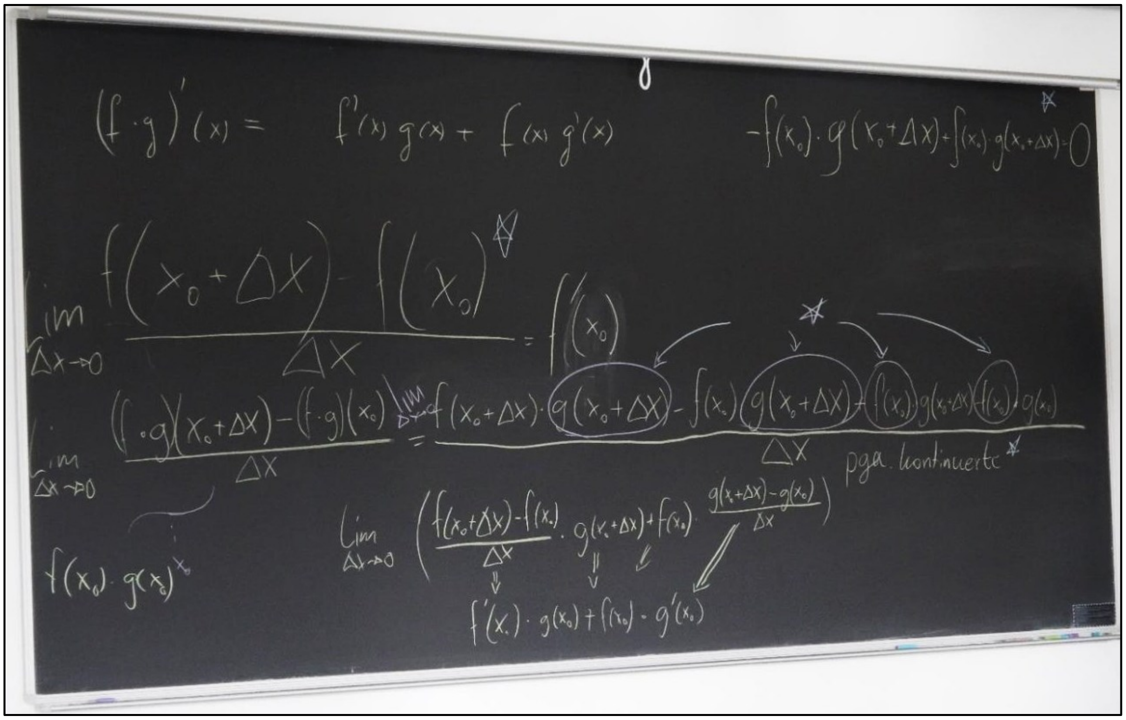


Figure 4.36. One pupil's solution to differential calculus problem

Education policies and the circumstances of teaching account for one aspect of how and why teachers interpret the mathematics communications competency. Instructional time, or lack thereof, can limit the types of communication-rich learning activities teachers feel are important. The structure of national tests and the alignment of those tests with the curriculum content and competency standards can also influence the extent to which teacher enact the communications competency.

Pedagogy

Pedagogy is defined as beliefs about teaching and learning, and curriculum is defined as a widely encompassing term including textbooks, additional learning resources – including online, teacher support materials, learning activities, and classroom assessments. In many cases, separating pedagogy from curriculum is difficult because a teacher's beliefs about teaching and learning often have such a strong influence on what is done in class. For my analysis of pedagogy and curriculum, I used a lens that considered pedagogy to be a set of beliefs and curriculum to be the enactment of those beliefs. In other words, when a teacher talked about what he or she believed about teaching and learning, I considered that as an example of pedagogy. When a teacher referred to specific classroom actions or when I observed things in the classroom, I considered that an example of curriculum.

Teachers' beliefs about mathematics teaching and learning influence their classroom practice, including how they interpret and enact the mathematics communications competency. From my data, two main areas of beliefs emerged from the data: beliefs about communicating in mathematics, and beliefs about the roles of teachers and pupils. In this section, I will discuss each of these beliefs and how they relate to the two research questions.

Beliefs about communicating in mathematics

As discussed earlier in the policy section, it is difficult to separate what a teacher says about the mathematics communications competency from that teacher's beliefs about mathematics communication. These stated beliefs, however, give insight to the types of mathematical understanding and communications teachers believe are important and, therefore, are likely to enact in their mathematics lessons.

As described earlier, Jacob uses pupils' answers and solution steps as a primary indicator of understanding, although he says learning something just for the exam is less important than being able to use mathematics skills in other contexts:

> I mean, they could possibly end up passing their exam in the ninth grade, but it would be worthless knowledge as I see it, because they wouldn't be able to apply

> it on any other, how do you say it, in another situation, they wouldn't [?] those skills, in another setting, in another…yeah.

Jacob also described the value of repetition in gaining mathematical understanding.

> Also for some of the not-so-skilled pupils, there's the, there's a lot of repetition in doing additions, and they can, they can get into kind of a rhythm, this, this this…this, this, this, and they can learn to multiply or divide or so, even though they're probably not very skilled at it. Some of those who have really big problems in math, I think it can help them a lot.

The ways in which Jacob uses mathematical communications and assesses pupil understanding aligns with his stated beliefs about communications in mathematics:

> I believe it's very important, that the students are able to communicate math using the right terms. … I personally believe it's incredibly important the students learn to verbalize their thoughts and solutions to math problems.

Jacob's beliefs about what is important in mathematics communication seem to drive the focus on procedural forms of understanding and communications in his mathematics lessons.

Maria's beliefs about understanding and communicating in mathematics also drive how she uses mathematical communications in her lessons. Maria's pupils use a range of oral and written mathematical discussion, as well as pictures and diagrams in their lessons. For Maria, both how a pupil has solved a problem as well as the detailed explanations are crucial components in assessing whether or not, or to what extent, a pupil has understood a mathematical topic, and she uses that information to inform her planning. This use of connectional mathematics in her lessons is in direct alignment with how Maria describes her beliefs about communicating in mathematics:

> We have to be focused on mathematics communications competency. It is written in the law. I have to do it and I also believe in saying, writing, drawing especially with small children. I think it is a good way of understanding. It is also a way to search for solutions.

Anna's description of understanding and communication in mathematics are even more connectional in nature. Rather than understanding mathematics for an exam, Anna situates mathematics communication and understanding in a much wider context:

> The pupils have to understand mathematics communication when they meet it in for example a newspaper and they have to understand something other people explain in instruction books and tell them - not written. And they have to express mathematics both written and spoken, so that they can take part in professional/mathematic discussions.

For Anna, pupils will encounter mathematics in many contexts in their lives – not just in school – and mathematical communications are the means in which pupils learn to participate in mathematics. This belief in participating, in being able to "take part" in mathematics is a very connectional perspective.

Beliefs about the roles of teachers and pupils

In a similar manner to how teachers' beliefs about communicating in mathematics is reflected in the forms of mathematical communication and understanding enacted in their classrooms, teachers' beliefs about the roles of teachers and students is also reflected in what is enacted in the classroom. The beliefs about these roles align with levels of mathematical understanding and forms of mathematical communication.

Jacob, for example, says teachers should help pupils know there is often more than one solution to a problem and that mathematics can be applied in contexts outside of school (I: interviewer; T – teacher):

> I – What do you think teachers should be doing in general, not necessarily your class specifically?
> T – They should…I don't quite know how to phrase it…they should try to see if they can get their pupils to approach a problem from many angles, that there's not just one solution. Learn them, teach them to think out of the box, so to speak. To get them excited about math, and let them know that it's not…that you actually use math, it's not just for solving math problems in class, but you can use it in

> your everyday life, and it's a skill that you need throughout your life. It's applicable in many different situations.

Jacob's beliefs about the role of the teacher are fairly procedural. Though he believes in making connections to real-life contexts, it is in a procedural-focused manner: there may be more than one way to get to the answer, but the answer is how understanding is assessed. Jacob's use of a specific real-world set of problems will be discussed later in the curriculum section.

In contrast to Jacob's focus on procedural mathematics, Anna believes one role of the teacher is to create an environment in which pupils feel comfortable using the language of mathematics.

> I – Ok. So what should the teacher be doing in a class?
> T – Have a good relation to the pupils. Then, then I think it's easier for them to ask when they doesn't understand, and if they use speak very often, it's easier for them to explain for me what they can and what they doesn't can and then I can see, Ok, it's that what you know, now I can, I know what they know and I can, I know a way to the target.
> I – So, speaking a lot is not just for you…
> T – No.
> I - …but it also helps them, because they can say, "Oh, Anna, I don't know how to do this…"
> T – Yes.
> I – …and it helps you be able to help them.
> T – Yes

For Anna, another role of the teacher is to use what a pupil says in order to assess how much that pupil understands about a topic. A teacher should then use what that pupil understands to help move towards the learning target.

In the section about connectional understanding, I described the recognition that each pupil might have a different extent of connectional understanding. Pedagogical beliefs that support the development of connectional understanding in mathematics are those that recognize what a pupil already understands and uses that understanding to help move that pupil towards, as Charlotte said, "the next step."

Maria describes beliefs that support the development of connectional understanding when she talks about what a teacher should do in a math lesson. Knowing pupils well helps Maria to better assess their understanding.

> I – What do you think teachers should do in a math class?
> T – They have to know their students very well. Know how they work, and if you can, know how they are thinking so you can use it when, when you are going to help. And I think they has to be the expert.
> I – And then if the teacher's the expert, then how or what does the teacher do then to share that or to help give the information to the students?
> T – The teacher has to make situations where the students are going to think and try and try and try and work with the stuff in different ways. And the teacher has to be good at evaluation because you can build your new [Danish], when you're going to make plans for the rest of the year, you have all your evaluation and then you can build it up in a better way, so we make testing every month in our class, you can see today how we do it, it's an open form, it's not, not an old testing form. They sit on their place and they have their own test and it's quiet in the classroom but we can talk about the exercise and I take them home and I give it back to one of the students if I don't understand what he or she had made, and we talk about it again until I know that the student understands his subject

Teacher's enactment of mathematical understanding and communications is reflected in their beliefs about the role of pupils in a mathematics class. Jacob provides a procedural view when he describes how he expects pupils to participate in mathematics lessons:

> I – What is the students' role in the math class? Does that make sense?
> T – Ah, yeah. Well, I expect them to contribute to the lesson with either questions, "I do not understand this, would you please help," or with possible solutions to a problem, so that's my expectations.
> I – Some people, you really want them to participate, even if they say, "I don't know."
> T – Yeah, I have a rule, that you have to put up your hand five times each lesson, and it's not to be excused or anything like that. [laughs] But you have to put up your hand five times each day.
> I – Whether it's to ask a question, or say, "I don't understand what you just said"?
> T – Yeah.

> I – Or, "Maybe we can solve it this way," or "Have you thought of that?"
> T – Yeah, yeah. That's a general rule I have.

While encouraging pupils to ask questions in class, the types of questions he encourages focus on procedural understanding.

A more connectional view of pupils' role in a mathematics lesson is given by Charlotte:

> I – So, what do you think students should be doing in math class?
> T – Thinking. Explaining. Consider what is the right one, which way to go. I like open topic, where they can find the way, maybe different kind of answering. In my experience, the students are very good at making exercises at their own level. So, they actually, if I say, like I did today with the stories, today was just a mix, good and skilled ones and less skilled and, just a match.

As Charlotte describes the role of pupils in mathematics lessons, pupils should be participating in connectional ways such as explaining and finding different types of answers. She recognizes pupils are working at different extents of understanding and provides support for each pupil.

As noted earlier, separating teachers' beliefs about teaching and learning from their enactments of curriculum and learning is often difficult. This section has explained two main categories of beliefs – beliefs about communicating in mathematics and beliefs about the role of the teacher and student – and described how those beliefs relate to teachers' views of mathematical understanding and communications. The following section will describe specific curriculum findings related to mathematics communication.

Curriculum

C*urriculum* can have a variety of meanings. For this analysis, however, *curriculum* refers to the materials that are used and the activities that take place within the classroom. These can include textbooks, supplemental workbooks, online resources, teacher support materials, learning activities, and classroom assessments. Some of these curriculum materials, such as the third-grade open-response test and the *Formelsamling* exam preparation book, have been discussed previously in great detail and there is no need to explore those further here. Other curriculum materials, specifically textbooks, use of real-world contexts in mathematics, and differentiation, are curriculum topics related to how teachers interpret and enact the mathematics communication competency and will be discussed in detail.

Textbooks

Each primary mathematics class I observed used a textbook at some point. I will include consumable workbooks such as those used by class 3B and shown in Figure 4.4 as a textbook for purposes of this discussion. For the primary schools in which I observed lessons, mathematics textbook selection was a school-based decision. Maria explained that the textbook she uses with her pupils is the same as that used in the other third-grade class.

> The textbook I used with 3.b (Matematrix) is a textbook series that we use from 0. to 5. grade. We have chosen the book some years ago. From 6. to 9. we use another book (Kontext).

As the school's only tenth-grade teacher that school year, Maria was able to select the textbook she uses with her pupils:

> For the 10. class we use the red book you saw (matama10k). I chose the book last year. Before me the other teacher in 10. used another book. I made the choice. I still use the book [this year] and I think it fits the students.

Jacob teaches at the same school as Maria and provides additional information about textbook selection at their school:

> Regarding the textbooks...I did not chose the books myself. The math teachers at [school name] met two years ago and had a variety of books we had ordered from various publishers. We decided on one book series for 1st to 5th grade, and another system for 6th to 9th grade. The other grade 7 class uses the same textbooks we [use in] my class. I'm still free though to use any sort of additional textbook material I find necessary (paper copies and such). I'm obviously not free to buy new textbooks by myself.

A similar school-based textbook selection process was in place at the other two primary schools I visited. Martin explained that textbooks at his school were chosen before the current *Fælles Mål* (Common Goals) were in place:

> The textbook we use in 7. grade is used by all the 7. graders on our school. The student have no influence on what textbook we use. This textbook we have chosen a number of years ago. Today we would like to change in to new textbook-system, that is built on the demands from the ministry of education. The textbook we have today is very old, therefore I/we often choose to bring in other materials to make a better teaching.

Although at a different school from the other teachers, Anna also uses textbooks she feels do not align with the current curriculum. She, like Jacob and Martin, describes supplementing textbooks with other materials:

> The textbook I used in my 6 class is from the beginning of the 2000, and at that time I was with in the decision to buy that "book-system". We bought the system from 0. to 9. class. Since that we have got new a new curriculum in mathematics in Denmark (both in 2009 and in 2014), and I don't think the booksystem comes up to the targets, but it's expensive to change to a new system. - The school haven't money for that. Sometimes I supplement with other materials I have made by myself or found in other books, so that I think the target can be succeed. (Not all teachers do that;-)). The two other 6 class on [school name] use the same textbook, and when I use other materials I share them with my collegeagues and they share with me;-) On the neighbour-school (where my own children are) they use another system that are closer to the new targets. If I had the money I would use another system, and if I had the time I would make it all by myself!

These teachers describe school textbook selection, with the exception of Maria's tenth-grade class, as a school-based process involving multiple teachers. The factors involved in textbook selection might include teacher beliefs about the nature of mathematical understanding and communications in mathematics, but an investigation of those factors is beyond the scope of this study. What is clear is that these teachers are permitted to supplement textbooks with other materials as they see fit.

One factor that emerged about textbook use in classrooms relates to how those textbooks were used in mathematics lessons. In some lessons, such as Martin's lesson on currency conversion, the textbook (shown in Figure 4.9) served primarily as a source of procedural problems for pupils to solve. After Martin led a short introduction to the lesson and used the examples in the textbook as the basis of this introduction, pupils spent the remainder of the lesson working on problems from the textbook.

Jacob described how he uses textbooks to provide additional extension work for more advanced pupils. The first book is the regular textbook for the class that includes mainly pages of procedural mathematical problems. The second is a supplemental workbook from the same series as the textbook. It is filled mainly with additional procedural exercises and Jacob describes them as slightly more difficult. When pupils finish problems in the first two books, Jacob uses a third book to supplement the first two. The third book is from a different publisher and contains pages of mainly more exercises, but not necessarily aligned with the content in the first two. Jacob explained "the third book, I use for advanced students, is a book I personally ordered online." I ask Jacob if the students need to complete all of the work in the other two books before moving to the third, and Jacob said, "Yes, they do."

Maria's pupils also used a textbook (shown in Figure 4.4) but in quite a different way. While pupils were doing strategy experiments as described in their textbooks, the textbook served as more of a logbook - a place for pupils to write the results of their experiments, rather than as a source of procedural problems. This particular textbook was something pupils used periodically throughout their lessons rather than something pupils used as their main learning material. It provides a structure for mathematics communication for younger pupils. Charlotte describes how a workbook or consumable textbook can help pupils who struggle with organization:

> Also, in the fourth grade, very difficult for the boys to get the organization of the notebook. To get, "Oh, all this writing!" Some of them still write very big and it's very slow for them to write. So they need a kind of a workbook where they can write the answer, and just chill.

Part of how a textbook is used to support mathematical understanding depends on the format of the textbook. The book used by Maria's pupils was clearly designed to support different learning activities than the textbook used by Martin's pupils. Textbook format, particularly one that is focused on procedural mathematics, does not, however, necessary limit how a teacher uses that textbook with his or her class.

Although Charlotte views certain problems in her grade 4 textbooks as "dull," she describes how she uses the textbook problems (an example is shown in Figure 4.18) not at something that limits her, but as a starting point to "open up the exercises":

> T – Most of it is because of my experience, but some of it is because I had a good teacher, too, (laughs) so some examples of, Oh, you can do this, and you can do this, to open up the exercise like this, by just using another word, or also just like this, it says (reading from book) …"put the fractions on the line," …it's a very dull one. The next one, I could say, "Well, now you done this, now make your own fractions and your own lines. Make some that (are) easy, make some that is the same place at the line, but is a different kind of name."
>
> I – So ways to extend the problem beyond what is just on the paper. Do you think that is important for the student?
>
> T – Very much. This is dull [points at book problem].

Despite having a textbook with relatively procedural problems, Charlotte often uses the textbook problems, but changes and adapts these problems to fit her views on mathematics understanding and communications.

Charlotte also encourages her pupils to use mathematics textbooks as a reference source:

> I – And I think it's also interesting the way you have them use the textbook, because sometimes you said you have them do the problems…
> T – Yes.
> I - …but there's more to it than that. You have them use it as a reference book.
> T – Yes, very much.
> I – And that's something that I don't see a lot.
> T – Oh?
> I – Often when I go to classrooms, the book is for doing the problems.
> T – Yeah.
> I – So I really like how you're showing them that it's something else also.
> T – But also this is a very good system, that's part of it. I wouldn't…because as I said, it ALWAYS had this information pages, ideas pages with different strategies, and help and…so it IS a very good system, and it always gets the idea that, "Oh, we have to look at this." And, "Oh, if we don't remember, how is it about the point," they can always look backwards. And, "Oh! This one! I have to do this first." That's why it's called the first one. "Ah, the second one, oh I can use this as a help."

She uses the pages shown in Figure 4.19 as an example. Charlotte's pupils learn to use textbooks as a source of information to help when they are working on problems that are not in the textbook.

The existing data provides incomplete insight into textbook formats and no information about author intentions. Nonetheless, it begins to provide an idea of different ways teachers use textbooks to support mathematics understanding and communications. While textbooks can be used in procedural ways in lessons, it seems that a procedural textbook does not always have to be used solely for procedural purposes. As shown in how Charlotte adapts and changes procedural problems, there is evidence that teachers who have connectional beliefs about mathematics are not limited by procedural mathematics in their textbooks.

Supplementing Textbooks Using Real-world Contexts

In several cases, teachers spoke to me about the importance of a real-life context to the mathematics done in their classrooms. Jacob described mathematics as, “not just for solving math problems in class, but you can use it in your everyday life, and it’s a skill that you need throughout your life. It’s applicable in many different situations.” Jacob enacts this belief through the curriculum choices he makes, including mathematics projects that require his pupils to apply their understanding of mathematics to real-life situations.

In a project he described to me, Jacob seventh-grade pupils had to design houses, calculate the area of each room, and then build models of those houses while remaining within a budget. In another project, Jacob asked his pupils to rate a set of music videos from the 1980s and then explain and discuss the results of the survey. In the lesson I observed, Jacob’s pupils were working to calculate someone’s weekly salary from a part-time job. Pupils were given basic information about the number of hours worked on certain days and rates of pay including weekend and overtime pay. Using this information, pupils not only had to calculate how much the person earned that week, but also had to present their calculations in a clear and organized format. Throughout the lesson, Jacob stressed to his pupils the importance of expressing their work clearly so that others could understand what they were writing. His belief about the importance of real-life examples of mathematics is enacted in the curriculum choices he makes, including opportunities for pupils to communicate their work to others. In this way, Jacob interprets the communications competency as a means providing focus to pupil work: in addition to considering the mathematics content, there is an additional focus on the clarity and structure. Though Jacob recognized there are multiple ways to arrive at the correct answer, the emphasis was still on procedural mathematics and procedural communication.

Anna also views real-world problem such at the toothpaste problem described earlier. Pupils were not told what they needed to do in order to solve the problem, but instead had to work together to discuss what information is necessary to solve the problem and then explain and justify their mathematical decisions to each other and to Anna. In this example, however, although procedural mathematics has a role in pupils' work, the focus is clearly on connectional communications and connectional understanding.

These two examples illustrate the limits of using real-world problems in a mathematics lesson. Simply including problems in a real-world context, while possibly making a connection to pupils' lives, can still have a procedural focus.

Differentiation

Pupils in Denmark are, by policy, heterogeneously grouped in unstreamed classes and differentiation must take place within these classes. Martin explained that, although sometimes pupils work on the same things in a mathematics lesson, teachers are expected to meet different levels of pupils' needs:

> T- Today it was just an example of where we could work together, but also there I have to give different things to different students because they can't do the same. So that's probably the thing about preparing your teaching that takes the longest time, the most time. That's preparing for different levels.
> I – Right, it's not just one lesson, it's several lessons in the same group.
> T – Yeah, you are obligated to teach in different levels.

One way Martin works to address the different levels at which pupils are working is by creating in advance individualized assignments using an online mathematics site.

> T – We do as well, yeah. We using internet site called matematikfessor.dk. I can show you. It's a site that, where you can, that's just to do the problems so you don't have to copy from this book, and this book, and this book, and this book. Oh, my computer's so… [getting out his computer] That was not it. [typing] See, that's the site we have bought license to, matematikfessor.dk…course it takes a little while [to load]. The main idea is you have all the topics, you have the levels, and I can say, ok, I want to work with the percentage, third grade. And then I can select the problems I want, and I can put together an assignment for him.
> I – Ah, and the student goes on the computer and does those.
> T – Yeah. And work it from there. That's a fairly simple way to, still if you have to do that five or six ways, that's still a big preparation. So, that's probably the biggest.
> I – Because I was wondering where you find all the, because I know you said you supplement with other things, but I wasn't sure. Just finding all of the things, I think can be difficult.
> T – Yeah, it can, it is. And like in, you say in here we have the same problems. Sometimes you haven't prepared for all and then you think that they can work with you and then, oh, goodness. Then you are standing there, you have to find out how to, but a lot of problems can be worked in different levels as well, so you have to…I also, in Denmark it's called [Danish], to make different.
> I – Differentiating. Yes.
> T – Differentiating, in the homework, because I don't think two students should have the same thing. It's also a big preparation work to say you have to do this and you have to do, I think it's more like a half an hour's work is better than five problems because, [name], the clever girl is doing ten minutes and [name – a boy who is struggling in mathematics] could be three hours which, I talked to you about motivation, which I think it's very important to learn. If he has to work three hours every time, and experience that he can't make it…
> I – Right, if he's already having, if he's already struggling, yeah. So then for the homework, do you do it on here [gestures to the website] also? Do you set assignments on the website for homework?
> T – No. Sometimes I do, but no.

Although Martin gives this website as an example of differentiating the work pupils are assigned, the types of differentiation provided by the problems on this website seem similar to how Jacob described using different textbooks and wording extension problems for pupils: “If they could multiply a two two-digit numbers, I could probably ask them to multiply three digit numbers, to make sure they know what’s going on.”

Martin later described how he provides differentiated work for one of his grade 8 pupils:

> We talked about the girl I have in eighth grade that is so far ahead of the others, and I have to, you know…she’s doing writing assignments, writing in words what she does and combines it with examples with numbers and calculations and then makes arguments based on the numbers she used.

The writing assignment to which Martin refers is the three-column format described earlier and shown in Figure 4.10. These types of differentiation, although seemingly addressing needs of pupils who are working on different levels, remains quite procedural. The writing assignment is only accessible to students who have successfully completed earlier procedural mathematics.

In contrast to procedural types of differentiation, Charlotte describes how she uses connectional understanding and communications to provide differentiation.

> I – You told me a lot of what students should be doing, and you’ve told me some of the next part, but I don’t know if there’s anything you wanted, anything else you wanted to tell me, what do you think teachers should be doing in a math class?
> T – Teachers should not give the answers, but the question. I think it’s important that the teacher take time to listen to the explanations, more than just checking if the answers is correct, but step up and say, “Ok, can you explain this one for me?” To take the time because it’s behind the explanations you find out if the students have learned anything or just can do it automatically. … But also if the teachers, the teacher have to be very…have to think a lot about how do I want the students to work? It takes a lot of planning. You have to think a lot of scenarios through, from the beginning and also ALL the way to the end, and perhaps you never get to the end, the end can be over here in the end, but you have to think it all through and say, “Ok, for starter, I want this to end up here, I want to go through this and this and this and this and perhaps that’s 10 lessons,” but on the way you have to stop and say, “Ok, now we’re here, but the students are much better than I thought so I need this kind of exercises for the next lesson,” so you have to make changes all the time, so your plan, you have to do the planning because that’s the way of

> organizing your own thoughts and the way to teach, but you have to do, yeah, to stop and think, "Ok, what changes are needed?" and do the changes.
> I – So it sounds like to me you don't just say, "Ok…"
> T – We start here and end up here…
> I - …Here's the page for today, tomorrow we will be on the next page…
> T – No.
> I – It sounds like you don't do that at all.
> T – [laughs] No, I don't do that. I say, "I want this and that and then this one makes sense and then I want to do this and, oh! Perhaps I like this as well but this one I don't like," and so I skip a lot and I choose a lot and I say, "This one is too difficult for these three students but this one is very good for the ten others, and here's for the seven, and this one is only for the three." I do a lot of that.

For Charlotte, part of the process of differentiation is being flexible with expectations of pupils. This does not mean lowering expectations, but it means being aware that what she expected pupils to already understand might need to be adjusted and being aware that pupils might make progress at differing rates that she expected. In some cases, Charlotte makes adjustments to her lesson plans during a lesson in response to pupil needs. Differentiated classwork and homework are often selected in immediate response to what individual pupils did or said earlier in the same lesson.

While Charlotte is aware that pupils are working at different levels and works to provide connectional differentiation, another feature of the type of differentiation she provides is that children working at a perceived lower level still have access to the mathematics that children working at a perceived higher level have:

> T –In my experience, the students are very good at making exercises at their own level. So, they actually, if I say, like I did today with the stories, today was just a mix, good and skilled ones and less skilled and, just a match.
> I – Oh, you mean the groups?
> T – The groups, yeah. Sometimes I sit them together deliberately, two skilled ones, the next one, and the low one, but today it was just a mix.
> I – Just kind of, you two and you two…
> T – Yes, exactly. And they start working and they find out, "Oh, you can do this, but I'm very good at this," and they sit together and they find a level that is just in between, but never the low one, that is the funny part. It's never the low level that

> gets, it's always almost as much as they can do, sometimes even more than you would be able to do alone.
> I – So it brings at least one of them up further than they'd be alone?
> T – Yes. Yes.

By working together with other pupils using connectional communications, Charlotte feels children make more progress than they would by working on their own. Providing access to mathematics at a higher level than a pupil is currently working is another contrast to differentiation described by Jacob. For a pupil in Jacob's mathematics lesson, access to higher-level problems is only gained by successfully completing all of the problems at a lower level.

These examples provide insight into how teachers use mathematics understanding and communications as a tool of differentiation. Some teachers interpret and enact the communications competency as a procedural, ability-focused competency: smarter, more able pupils should be focusing on certain aspects mathematical communication or only smarter, more able pupils need to focus on certain aspects of mathematical communications. A more connectional view of the communications competency is that students are provided with work at their own level that helps them develop mathematical understanding. Additionally, and importantly, however, all pupils, regardless of ability, have access to mathematics that helps them work towards increasing all levels of mathematics understanding and these increasing levels of understanding are gained by using connectional mathematics communication.

The evidence in the data supports the idea that teachers enact classroom mathematics curriculum based on their view of mathematics understanding and communications. Rather than being limited by procedural textbooks, teachers are able to use these textbooks in connectional ways. Conversely, supplementing textbooks with real-world contexts and providing pupils with differentiation in mathematics lessons does not directly equate to connectional understanding. Differentiation and use of supplemental real-world problems can be done at a very procedural level or done on a more connectional level. What makes the difference in how teachers enact mathematics communications in their classrooms is not the specific curriculum materials that are used, but how they are used, and how they are used depends on a teacher's views of understanding.

Reconceptualizing the Theoretical Framework

The initial theoretical framework suggested three themes – pedagogy, curriculum, and policies that influence how teachers interpreted and enacted the mathematics communications competency. An analysis of the data sources revealed two additional themes as factors in how teachers interpret and enact the communications competency: types of understanding and types of mathematics communication. Figure 4.37 shows the relationship between the five themes in comparison with the original theoretical framework. The diagram on the left is the original framework. The diagram on the right is the revised theoretical framework that shows how the emerging themes of understanding and communications are related to pedagogy, curriculum, and policies. The three themes of pedagogy, curriculum, and policies each influence communications and understanding as used in classroom practice and, conversely, communications and understanding have an influence on pedagogy, curriculum, and policies. The bidirectional nature of this relationship needs further exploration.

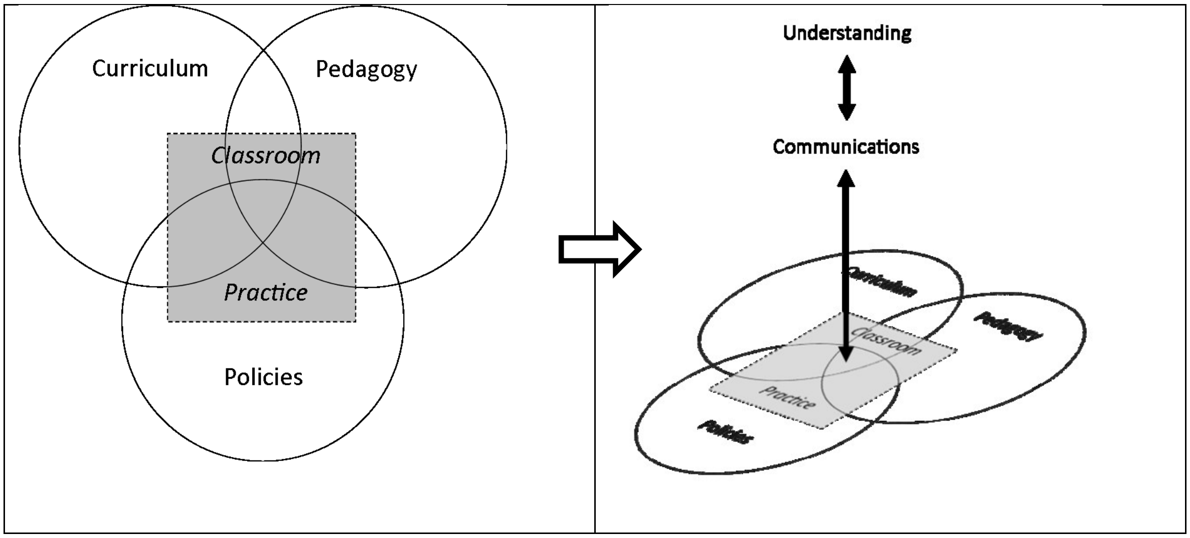


Figure 4.37. Relationship between the five main themes

Curriculum, pedagogy, and policies each influence classroom practice, although the influence of each is not equal. Beliefs about pedagogy impact the classroom decisions teachers make regarding curriculum and learning activities. Curriculum issues such as textbook choice, supplemental materials, and styles of differentiation influence a pupil's classroom experience. Policy issues such as the existence of mathematics standards and competencies impact the content taught and, to an extent, the processes used in teaching. Rules regarding mixed-ability classrooms impact the need for differentiation. Available instructional time can limit the curriculum activities a teacher uses in lessons. The format of national assessments impacts the types of mathematics learning activities teachers select.

Each of these three themes – curriculum, pedagogy, and polices – certainly have an impact on classroom practice, although the way in which these three areas intersect as classroom practice are to a large degree influenced by mathematical understanding and communications. Pupils use mathematics communications to gain mathematics understanding and teachers use pupils' mathematics communications to assess pupils' level of understanding of mathematics.

Two levels of mathematics understanding correspond directly with two levels of mathematics communications. These two levels of understanding, procedural and connectional, describe different ways of understanding mathematics. The corresponding procedural and connectional levels of mathematics communication influence not only the types of mathematical understanding a pupil develops but also how a teacher assesses the degree of that understanding. Classroom practice that supports and drives mainly procedural communications and understanding is different from classroom practice that supports and drives connectional communications and understanding.

CHAPTER 5

CONCLUSIONS

In this chapter, I present a summary of my study, implications, recommendations for future research, limitations of the study, and my conclusions. The study summary provides a review of the purpose, methodology, and findings before addressing the research questions and relating those questions to the results of the study.

Summary of Study

The purpose of this study was to examine how Danish teachers interpret the mathematics communications competency and how those interpretations are enacted in classroom practice. Denmark implemented mathematics process standards in 2003 and teachers and students in Denmark have had over a decade of working with those standards. This study provides insight into factors influencing how teachers interpret and implement oral and written mathematical communication in their classrooms. The results of this study can be used to inform mathematics communication instructional practice in the United States.

A grounded theory methodology was used to investigate two research questions: a) How do teachers interpret the Danish communications competency? and b) In what ways are those interpretations enacted in classroom practice? Data sources include observations, interviews with teachers and pupils, and classroom artifacts. Five themes emerged from the analysis of the data: understanding, communications, pedagogy, curriculum, and policy.

Two forms of mathematics understanding are described: procedural understanding – which includes understanding what to do to solve a mathematics problem, and understanding how to solve a mathematics problem, and connectional understanding – understanding why a problem is solved in a certain way. These two forms of understanding correspond directly with two levels of mathematics communications - procedural and connectional. The corresponding levels of mathematics communication influence not only the types of mathematical understanding a pupil develops but also how a teacher assesses a pupil's degree of that understanding.

Classroom practice that supports and drives mainly procedural communications and understanding is different from classroom practice that supports and drives connectional communications and understanding. Teachers who hold views consistent with a mainly procedural understanding of mathematics interpret and enact the communications competency in a way that emphasizes clarity of procedural communications. Teachers who hold views consistent with a more connectional understanding of mathematics interpret and enact the communications competency in a way that includes procedural communications but emphasizes connectional communications.

Education policies and the circumstances of teaching such as available instructional time and the structure of national assessments account for another aspect of how and why teachers interpret the mathematics communications competency. Two areas of pedagogy that relate to teachers' views of mathematical understanding and communications are beliefs about communicating in mathematics, and beliefs about the role of the teacher and student. A key factor in the enactment of classroom mathematics curriculum is not the specific curriculum materials that are used, but how they are used, and how they are used depends on a teacher's views of understanding.

The answer to the first research question is that teachers interpret the mathematics communications competency in a way that correspond with their beliefs and views of mathematics understanding as being either procedural or connectional. The answer to the second research question is that teachers enact the mathematics communications competency in classroom practice in ways that are largely consistent with their views of mathematics understanding as being either procedural or connectional. Mathematical communication is used in the classroom as a tool for both supporting and assessing different forms of procedural and connectional understanding. Implications of this study include reframing the discussion regarding classroom mathematics instruction as a continuum of mathematics understanding rather than one that emphasizes rote memorization and algorithms versus an expectation to *teach for understanding*.

Existing literature supports a recognition between mathematics in the form of "how to do it" compared with "why a piece of mathematics works" (Pirie & Schwarzenberger, 1988, p. 461). In his 1976 article, Skemp described two types of mathematical understanding: procedural and relational. He describes procedural understanding in a similar manner to how the term is used in this study: as knowing the rules for solving problems without knowing the reasons for solving them a certain way. He describes relational understanding as not only knowing what to do to solve a problem but why it is solved in a certain way. (It is worth noting that Skemp credits Stieg Mellin-

Olsen with first sharing these two terms with Skemp, however Mellin-Olsen is not cited in the article.) Though two distinct types of understanding, Skemp describes procedural understanding as a necessary part of relational understanding. Skemp notes four main situational factors that contribute to why teachers focus on procedural understanding compared with relational understanding: the format of examinations, course syllabi that too ambitious, difficulty in assessing relational understanding, and the difficulty for teachers in restructuring their existing schemas.

Skemp (1976) describes these four situational factors as, in part, choices teachers make in their teaching. Indeed at least two of these factors are reflected in the current study: the differing format of examinations and the relative ease of assessing procedural understanding can each influence how a teacher enacts mathematics understanding in lessons. As each teacher was responsible for following a national curriculum, the content of the course syllabi was beyond the scope of this study. Skemp's final factor, however, relates more directly to the findings of this study. As Skemp describes it, even when teachers are aware of and want to change their teaching from procedural to relational, it is a very difficult process and teachers are unable or unwilling to change how they teach. For other teachers, they lack relational understanding of mathematics itself and are therefore unable to make the choice between procedural or relational understanding in their teaching.

Hiebert describes two similar types of knowledge: procedural knowledge and conceptual knowledge (1986). Procedural knowledge focuses on understanding mathematical symbols and the rules and procedures of mathematics problem solving. Conceptual knowledge is "rich in relationships" (p. 406). These two types of knowledge are considered as being distinctly separate from one another, although procedural knowledge is a necessary component of conceptual knowledge. More recently, Star (2005) has built upon the knowledge described by Hiebert to suggest that both procedural and conceptual knowledge can be separated into two forms: superficial and deep. Superficial procedural knowledge fits with Hiebert's original definition, while an example of deep procedural knowledge is procedural flexibility: the ability to consider multiple ways to find a solution to a particular problem and to identify the more efficient methods. Baroody, Feil, and Johnson (2007) suggest that some degree of conceptual knowledge is necessary for deep procedural knowledge. They also state an additional type of knowledge – a well-structured knowledge – is necessary to allow a useful problem representation.

This study builds on this previous work in several ways. Whereas the previous work describes different types of mathematical understanding, there is limited focus on the foundational of how these types of understandings are enacted in mathematics classrooms. The findings of this study indicate that, although policy and curriculum factors have an influence on the enactment of mathematical understanding, rather than teacher choice of one type compared with another, it is a teacher's own views and beliefs about mathematics understanding that correspond with the enactment of that understanding. The findings of this study also show how mathematical communications is used as a tool for supporting and assessing different forms of mathematical understanding.

Additionally, the previous literature has portrayed the differences in mathematical understanding (or *knowledge*, depending on the author's choice of terminology) as distinctly separate. Though Star goes somewhat further when he describes procedural and conceptual knowledge as each having additional sublevels of superficial and deep knowledge, he nonetheless described these in a 2x2 matrix. The following section will discuss the implications of this study and present a revised mathematics learning continuum that considers procedural and connectional understanding not as separate types of understanding, but as part of the same continuum.

Implications

In Chapter 1, I described a mathematics learning continuum with algorithms and rote memorization at one extreme and depth and understanding on the other. I noted that there is often a conflict of tension point on this continuum. For example, the Common Core State Standards for Mathematics are based on a set of process standards that are likely quite different from what many parents and teachers experienced when they were in school. In other cases, this tension results from teachers being told to *teach for understanding* when that teacher thinks he or she *is* teaching for understanding. An illustration of this continuum is given in Figure 5.1.

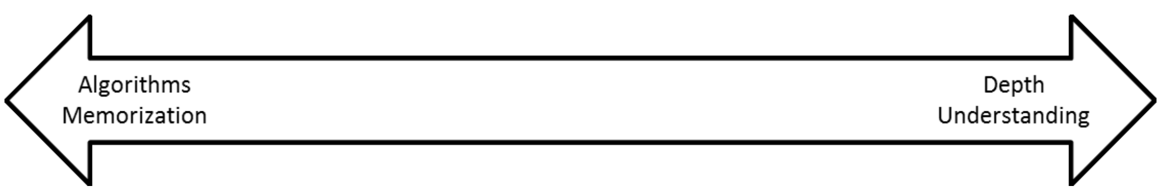


Figure 5.1. Initial mathematics learning continuum

The findings of this study, however, can reframe how we convey mathematics teaching and learning expectations to teachers, pupils, and parents. Figure 5.2 shows the same mathematics learning continuum, but relabeled in light of the findings related to mathematical understanding. Relabeling the continuum to focus on the emphasis of procedural or connectional understanding helps acknowledge the confusion a teacher might have when told to *teach for understanding*. The revised mathematics learning continuum provides a much clearer picture of what is possible in mathematics as well as where the emphasis is placed in classroom learning activities. This continuum recognizes algorithms and memorization as a type of mathematical understanding that is necessary but also recognizes the importance of helping pupils make mathematical connections. The continuum can help reframe discussions with teachers: from "You are teaching incorrectly," to "You are very good at teaching for procedural understanding, and now we would like you to incorporate more connectional understanding into your lessons." It changes the conversation from, "The things you learned in math as a child aren't good enough anymore," to "The ways you learned to solve problems were likely quite procedural. We would like your child to learn not only procedures, but also how those ideas fit with other areas of mathematics."

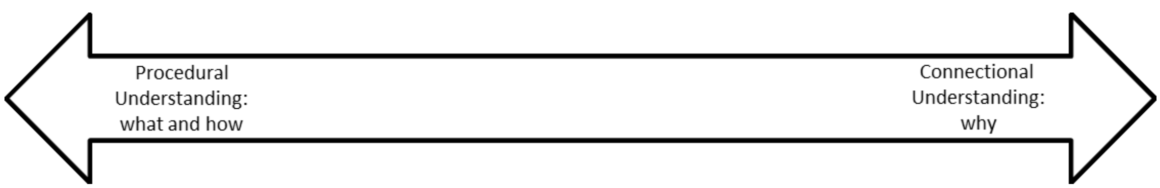


Figure 5.2. Revised mathematics learning continuum

In addition to reframing how we convey mathematics teaching and learning expectations to teachers, pupils, and parents, the findings have teacher education implications as well. In order for pupils are to develop connectional mathematics understanding, they must be taught by teachers who have a firm connectional understanding of mathematics. If, as Skemp suggested, individuals who lack a relational -

or in this sense, connectional – understanding of mathematical are unable to teach in a connectional manner, this becomes a matter of how to effectively develop connectional pedagogies in teachers. This has particularly crucial implications in a system such as Denmark where pupils have the same mathematics teacher for several years in a row.

A issue closely-related to development of connectional understanding in teachers, is how schools (and others) assess the quality of mathematic lessons. This study indicates that connectional understanding uses specific forms of communication but also incorporates procedural mathematics. In order to more accurately assess the quality of mathematics lessons, observations need to focus on not just what is taking place in lessons but how those activities are being used. For example, as described earlier, the presence of real-world contexts in a mathematics lesson does not on its own indicate it is being used to support connectional mathematics. Similarly, the presence of learning activities that help develop computational fluency, such as reviews of multiplication tables, does not indicate connectional mathematics is not taking place. Assessments of lesson quality must be based on more than snapshot judgments of individual lesson activities, but how those activities fit together. There are existing lesson observation instruments that take into account aspects of teaching such as *demonstrating an understanding of the subject area* and *selecting appropriate resources to meet individual needs* (Fairfax County Public Schools, 2010), but there is no guidance to the lesson observer about determining whether or not those things support connectional understanding. Additionally, this raises the question that if someone does not have connectional mathematics understanding for themselves, can they adequately assess and make judgments about connectional understanding in lessons they observe?

The findings of this study indicate that mathematical communications are used to both support the development of and assess the extent of pupils' mathematics understanding. This finding helps provide evidence for reexamining how the mathematics competencies or process skills are presented to teachers. The Danish communications competency as it is currently written does not provide sufficient guidance for teachers in order to effectively enact this competency in their mathematics lessons. In the mathematics curriculum reforms implemented in the 2014-2015 school year, Danish teachers now have a much more specific set of guidance on each of the mathematics competencies. Each of the mathematical competencies are broken down by grade phase. Within each grade phase, each competence has a set of three skill objectives and three content objectives designed to help teachers better understand each competency and therefore help more effectively assess pupils' learning of each competency.

The Common Core State Standards for Mathematics (CCSSM) have a set of eight Standards for Mathematical Practice (SMPs) that are presented in a similar fashion to the Danish mathematics competencies. The SMPs are a part of the CCSSM that describe the types of mathematical process skills mathematics teachers should help their pupils to develop. Two of these standards relate specifically to mathematics communication as being procedural or connectional. Standard MP.2 refers to the ability to reason abstractly and quantitatively:

> Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent

> representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. ("Common Core State Standards - Standards for mathematical practice," 2014).

A strength of this standard is that is recognizes both the procedural emphasis of mathematics understanding and communication – the ability to decontextualize – and the connectional emphasis of mathematics understanding and communication – the ability to contextualize. What would make this standard stronger is guidance for teachers on how to help pupils develop and move between these complementary abilities at different grade levels.

One SMP focuses specifically on connectional understanding. Standard MP.3 refers to a pupil's ability to construct viable arguments and critique the reasoning of others:

> Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. ("Common Core State Standards - Standards for mathematical practice," 2014).

This standard provides teachers with clear examples of what it means to communicate connectionally in mathematics: things such as analyzing, justifying, reasoning, making arguments, and using drawings. It still leaves room, however, for teachers to interpret what aspects of this standard are relevant for specific problems. Without more specific examples, a teacher could still interpret some of these statements in very procedural ways.

In the most recent revision of the *Fælles Mål* (Common Goals) for *Folkeskolen,* there are six mathematics competencies: problem-treatment (posing and solving mathematical problems), modeling, reasoning and thinking, representation and symbolic processing, communications, and tools ("Læseplan for faget matematik," 2014). As no specific ranking of importance is provided, communications is presented on equal footing with the other five competencies. Connectional mathematics communications, however, includes elements of modeling, reasoning and thinking, and representation and symbolic processing in order for pupils to solve mathematical problems. In the Common Core State Standards for Mathematics, the set of eight Standards for Mathematical Practice are presented in a similar fashion to the Danish mathematics competencies. Though numbered for ease of reference, the Standards for Mathematical Practice are presented on equal footing with one another ("Common Core State Standards - Standards for mathematical practice," 2014). For these reasons, mathematical communications must take precedence over the competencies for it is through a communications competency or standard that the other competencies and standards are gained and assessed.

Recommendations for future research

The findings of this study indicate that teachers interpret the Danish communications competency in a way that corresponds to their individual views of mathematics understanding. Further, teachers enact their interpretations of the Danish communications competency in their classroom practice in ways that are largely consistent with their views of mathematics understanding as being either procedural or connectional. In light of these findings, current reforms to the Danish mathematics curriculum policy are particularly interesting and relevant.

Recommendations in Light of Danish Curriculum Reforms

The 2014-2015 school year brings with it a number of reforms to Danish mathematics curriculum policy. Previously, the mathematics curriculum focused on what content teachers should teach. Under the reform, the focus is shifting to what pupils should be learning and how teachers know their pupils are learning. Each content standard now has examples of learning the specific standard, two or three evaluation levels, and an example of a challenging task. An English translation of an example of this guidance is given in Figure 5.3. These evaluation levels are designed to help assist in assessment of learning both formatively by giving classroom teachers a tool to determine the level of pupil understanding, as well as summatively by providing pupil levels on national tests in first, third, and sixth grade.

According to the National Advisor for Primary Mathematics, while other subject areas in Denmark were just beginning to introduce competencies in their subject areas during the current curriculum reform, competencies have existed in mathematics for some time and the reforms in mathematics will not be as extreme. Though the Danish mathematics standards have long included both content and process standards (competencies), the current reforms to the standards are introducing a *Model for* *planlægningen af undervisningsforløb i matematik* (*Model for the planning of teaching mathematics)* in the 2014-2015 school year (Læseplan for faget matematik, 2014).

Number		
Phase 2: 4th - 6th grade The pupil can use negative integers The pupil has knowledge about negative integers Mandatory ✓		
Indicative examples of learning, evidence of learning and challenging tasks		
Learning *Examples of learning objectives for a course.* Pupils can put negative integers in order of size. Pupils can describe everyday situations where negative integers used. Pupils can explain the difference between a sign and a rain characters.	**Signs of Learning** *Examples of character and learning of selected area.* Pupils can give describe everyday situations where negative integers used. The pupil must write a letter to his cousin, who does not know much about math, where the pupil explains what a "negative number" is and gives some examples of the use of negative numbers. **Level 1** - refers to at least one example of the use of negative numbers. **Level 2** - cites several examples of the use of negative numbers. **Level 3** - comparing negative for known representations and other terms (eg "The positive numbers belong to this part of the number line and the negative numbers belong to this part of the number line.")	**Challenging Task** *Example of a task that can challenge the academically talented pupil.* Four brothers agree to take on the casino and play. Afterwards they will share what they win or lose. When they come home, two of them each lost 200 crowns, one has won 300 dollars, and the last has not won or lost. You have to make a calculation using negative and positive numbers to fit a calculation story. You must formulate a new calculation story which includes subtraction of negative numbers.

EMU Danmarks læringsportal (2015)

Figure 5.3. Guidance example for grades 4 to 6, negative numbers

In this planning model, the curriculum standards for mathematics are divided into grade phases: phase 1: grades 1-3, phase 2: grades 4-6, and phase 3: grades 7-9. Within each phase, the content standards are divided into three categories: numbers and algebra, geometry and measurement, and statistics and probability. When planning for their lessons, teachers must include not only the relevant content standard(s) but also the competency (process standard) by which they will help pupils learn the content. An example of the planning grid is given in Figure 5.4. According to the National Advisor for Primary Mathematics, this specific emphasis on planning for both mathematical content and competency is meant to help pupils "build up competency in math".

		Competencies (Process standards)					
		problem solving	modeling	reasoning and thinking	representation and symbols	communication	mathematical tools
Content standards	numbers and algebra						
	geometry and measurement						
	statistics and probability						

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Figure 5.4. Model for the planning of teaching mathematics

Another part of the current curriculum reform is the addition of an entirely new set of guidance regarding the mathematical competencies. The National Advisor previously made available to teachers some examples of mathematical communications but they were limited in scope. There were no set of examples in the national mathematics curriculum and, as a result, teachers were generally left to interpret the communications competency on their own. Now, however, each of the mathematical competencies are broken down by grade phase. Within each grade phase, each competence has a set of three skill objectives and three content objectives designed to help teachers better understand each competency and therefore help more effectively assess pupils' learning of each competency. An example of the communication competency guidelines from the grades 4-6 phase is shown in Figure 5.5.

Grades 4-6 - The pupil can act with an overview of complex situations with mathematics.

Communications relates to pupils expressing themselves with and about mathematics and familiarizing themselves with and interpreting others' expressions and mathematics.

In grades 4-6, there is emphasis on pupils being able to apply mathematics academic texts.

From the beginning of grades 4-6, pupils should develop skills to decode and understand texts about and with mathematics. Pupils' understanding of texts including knowledge of mathematical texts' purpose and structure, including informational, instructional or argumentative texts is developed through a variety of modes.

Pupils must be able to decode and read texts of authentic character, in which mathematics is used as a tool for communication and texts to support their mathematics learning. In connection with the latter type of texts, pupils should include develop the skills to decode and read mathematical problems. This includes pupils' skills in finding and reading the relevant information.

The other work with communication in mathematics at the intermediate level, is aimed both at pupils' oral and written communication. Pupils should be able to communicate with varied types of mathematics, including the use of digital tools. They must be able to express their ideas, actions and reasoning in mathematics and to use specialist terminology and concepts.

It is essential that pupils develop skills and knowledge of mathematical communication in meaningful contexts.

Attention point: Pupils can extract relevant information from simple mathematical texts.

Phase 1

Skill objectives
The pupil can read and write simple texts and mathematics

Concept objectives
The pupil has knowledge about the purpose and structure of texts and mathematics

Phase 2

Skill objectives
The pupil can orally and in writing communicate with varied and mathematics

Concept objectives
The pupil has knowledge of oral and written forms of communication with and about mathematics, including digital media

Phase 3

Skill objectives
The pupil can use the technical terms and concepts orally and in writing

Concept objectives
The pupil has knowledge of specialist terminology and concepts

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Figure 5.5. Grades 4-6 Communications Competency Guidelines

These new policy guidelines form a potentially useful resource for teachers as they interpret the Danish mathematics communications competency. Whereas previously teachers were left to interpret and enact the communications competency based on a vague guideline, the new competency provides much more clarity and in-depth guidance. Additionally, the Danish government is mandating additional instructional time in mathematics for upper primary pupils. Rasmus, the National Primary Mathematics Advisor, states the additional instructional time comes with no additional content expectations and his hope is that the additional instructional time will provide teachers more time for learning activities that support deeper mathematical understanding.

The new Danish curriculum reforms provide an excellent opportunity to compare the types of mathematical communications used before the reforms (and described in this study) to those used under the new curriculum guidance. Future research could investigate how teachers describe a change in their views of mathematical communications in light of the new guidance.

Additional Recommendations for Future Research

Further research is needed to study the role mathematics understanding and communications plays in other areas of mathematics education. The findings of this study indicate that the types of mathematics communications and understandings enacted in classroom practice are highly connected to a teacher's own beliefs and views about mathematics. What is not clear is the specific directionality of these views and enactments: whether or not teachers' views of mathematical understanding help form their mathematical pedagogy or if their pedagogy helps form their views of mathematical understanding. What is also not clear from the existing data is how mathematics understandings and communications interact with curriculum and policy. For example, what is the role of mathematics understanding in the development of educational policy? What is the role of mathematics communication in the development and selection of mathematics textbooks? Figure 5.6 shows how these unknowns relate to the revised theoretical framework.

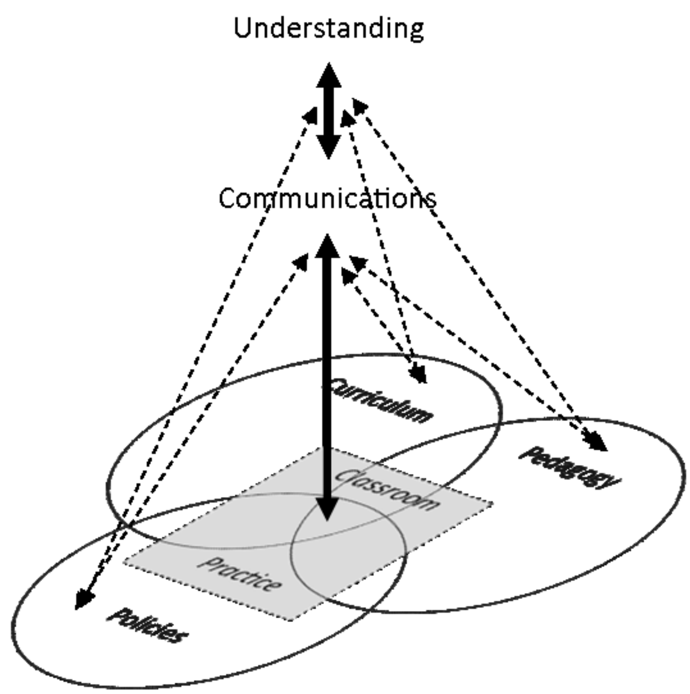

Figure 5.6. What is not clear: Dashed lines indicate unknowns

As described in chapter 2, *pedagogical content knowledge* is intersection of and interplay between mathematical content knowledge or general pedagogical content knowledge (Ball, 2000). The findings of this study suggest that mathematics teaching and learning requires something at the intersection of connectional understanding of mathematics and pedagogical views of mathematics teaching and learning: a sort of connectional pedagogical understanding for mathematics teaching. Although the types of mathematics communications and understandings enacted in classroom practice are highly related to a teacher's own beliefs and views about mathematics, we lack an understanding of this connectional pedagogical understanding. Future research could focus first on defining the necessary components of connectional pedagogical understanding and then devising an effective instrument for measuring or assessing the extent of a teachers' connectional pedagogical understanding.

Finally, if the continuum of mathematics learning can be described as having procedural forms of mathematics at one end and deep connectional mathematics on the other, what other mathematical landmarks might be on this continuum? For example, Star (2005) described computational flexibility as a specific example of deep procedural knowledge. Several examples described in chapter 4 provide evidence of how teachers encourage pupils to look for and share alternate solution steps when solving mathematics problems. It may be the case that computational flexibility is a useful or even necessary point on the continuum as pupils move further towards connectional understanding. Further research could explore this topic and others as possible landmarks of understanding.

Limitations

The goal of this study was to formulate a theory about how teachers interpret and enact the Danish communications competence in their classrooms with a view that the results of this study can be used to inform further study about mathematics instructional practice. Though the findings of this study support a theory that teachers interpret and enact the Danish communications competency in a way that corresponds to their individual views of mathematics understanding, the sample size is small. One limitation of the study is that the small sample size. Another crucial limitation is the fact that with all but one of the classes I visited, I observed only a single lesson and it is possible those individual lessons were not representative of the wider range of types of lessons. The purpose of the study, however, was to inform a theory rather than make generalizations about all teachers in all locations. This theory can now form the basis of further research as described in the previous section.

Additionally, my role as an English-only speaker in a Danish context created challenges. Relying on the English-language proficiency of native speakers of Danish meant that all of my conversations were filtered through someone else's knowledge of English. Nuances of language, such as those described by *understanding what, understanding how,* and *understanding why,* made my work more challenging and made my reliance on collecting a range of instructional artifacts and detailed lesson observations more crucial.

Conclusions

This study confirms previous work about types mathematical understanding (see Hiebert, 1986; Skemp, 1976, & Starr, 2005) and builds on that framework to show that teachers' beliefs and views of mathematics understanding correspond with their interpretation and enactment of mathematics communications in the classroom. The findings of this study also show how mathematical communications is used as a tool for supporting and assessing different forms of mathematical understanding. Although this study was conducted in Denmark, there is no substantial reason not to expect similar findings among mathematics teachers in the United States.

The existence of procedural and connectional forms of mathematics understanding, and the corresponding types of mathematics communication that are used to support and assess these types of understanding, form an important basis for reframing the discussion regarding rote memorization and algorithms compared with an expectation to *teach for understanding*. This reframed discussion of mathematical learning as a continuum of understanding has implications not only for classroom practice, but also classroom lesson observations and teacher education. Additional research is necessary to determine the precise nature of the relationship between a teacher's beliefs about mathematics and their view of understanding mathematics. Further research is also needed to define the necessary components of connectional pedagogical understanding in mathematics and to devise an effective instrument for measuring or assessing the extent of a teachers' connectional pedagogical understanding. Finally, specific research is needed to further investigate the continuum of mathematics learning to determine what landmarks of understanding, if any, are between procedural forms of mathematics at one end and deep connectional mathematics on the other.

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APPENDIX 1

INTERVIEW PROTOCOLS

Interviews will all be semi-structured with the focus of interviews on to role of writing in mathematics. The prompts below are designed to facilitate discussion about mathematics and the role of writing in math. Where possible, examples from a participant's own class will be used to supplement these prompts. For example, "When I visited your math class today, I noticed that you were writing about how you solved that problem. How did you decide what to include in your paragraph?"

Also note that the term "writing" can include graphs, sketches, charts, tables, etc., when used in mathematics. It can also include typed writing or computer-based written products.

Protocol for younger children – primary school

- When I visit math classes, I sometimes see different things. For example, in some classes, the students are all sitting in rows and the teacher stands in front of the room and does most of the talking. In other classes, students are moving around and working together in groups to solve problems. What would I see if I came to your math class?
    - What does your teacher usually do in your math class?
    - What do the students usually do in your math class?
    - What do you usually do in your math class?
- In some math classes, students work on problems from a textbook or worksheet and the problems often have only one correct answer. In other classes, students have problems that that are more open; problems that can be solved in more than

one way or have more than one answer. What kinds of problems do you work on in your math class?

    - What kinds of things do you like to work on in math?
    - Why do you like those kinds of things?
- When you have solved a problem, what happens next?
    - How do you know your solution is correct?
- How do you know if you have understood what you did?
    - Are there things your classmates do to help you understand? Things your teacher does?
- Do you write things down in your math class?
    - Why do you write things down?
    - What do you write down?
    - How do you decide what is important to write down?
    - What kind of things do you NOT write down?
    - What do you do with the things you write down?
        - Do you ever go back and read the things you wrote?
        - Do you ever read anything anyone else has written?
        - What does your teacher do with the things you write?
- If you were the teacher, what things might you do to help students understand things in math?
- What is a subject you enjoy in school?
    - Why?
    - Do you write in this class?
    - How is this the same/different from writing in mathematics?
    - Does it help you with writing in mathematics?

Protocol for older children – secondary school

- When I visit math classes, I sometimes see different things. For example, in some classes, the students are all sitting in rows and the teacher stands in front of the room and does most of the talking. In other classes, students are moving around and working together in groups to solve problems. What would I see if I came to your math class?
    - What does each person usually do in your math class? The teacher? The students?
- In some math classes, students work on problems from a textbook or worksheet and the problems often have only one correct answer. In other classes, students have problems that that are more open; problems that can be solved in more than one way or have more than one answer. What kinds of problems do you work on in your math class?

- What kinds of things do you like to work on in math?
- Why do you like those kinds of things?

- In math class, how do you know if you have understood something you have done?
    - Are there things your classmates do to help you understand? Things your teacher does?
    - Are there some things that might help you even more to understand things in math?
- Do you write things down in your math class?
    - Why do you write things down?
    - What do you write down?
    - How do you decide what is important to write down?
    - What kind of things do you NOT write down?
    - What do you do with the things you write down?
        - Do you ever go back and read the things you wrote?
        - Do you ever read anything anyone else has written?
        - What does your teacher do with the things you write?
- Part of the HTX mathematics examination is a paper in which you have to solve a set of problems and explain your approach and your thinking. What work do you do in math class that prepares you for this paper?
    - In what ways is this work useful?
    - Are there other things that might also be useful to prepare you?
- In some countries, students have exams in which they solve a number of problems and are graded only on their answers, and not on their methods or thinking. How do you think an exam like this might change the way your math class is taught?
- If you were the teacher, what things might you do to help students understand things in math?
- What is a subject you enjoy in school?
    - Why?
    - Do you write in this class?
    - How is this the same/different from writing in mathematics?
    - Does it help you with writing in mathematics?

Interview protocol for teachers

- When I visit math classes, I sometimes see different things. For example, in some classes, the students are all sitting in rows and the teacher stands in front of the room and does most of the talking. In other classes, students are moving around and working together in groups to solve problems. How would you describe your math class?

    - What should students be doing in math class?
    - What should teachers be doing?
- When I talk to teachers in other countries, sometimes teachers tell me that it is very important for students to have a deep understanding of the math they are learning. Other times, teachers have told me that they are not interested in whether a student understands, but only on whether a student can answer questions. How important do you think it is that students understand the mathematics they are learning?
    - How do you know if a student understands something?
    - If a student shows a good understanding of something in math, what happens next?
    - What happens is a student doesn't understand something very well?
    - What are some things that happen in your classroom that help students understand math?
- What does writing in mathematics look like in a model classroom or lesson?
    - Have you observed a model classroom or lesson?
    - For classrooms, that do not meet the model classroom or lesson standard, what do they typically look like?
    - Where does the typically Danish classroom fall within the description of a model classroom?
- What does writing in mathematics look like in your classroom?
    - How often do students write things in math?
        - What do they write?
        - Why do they write things?
        - How do students decide what is important to write down?
        - What kind of things do students NOT write down?
        - Do students ever go back and read the things they wrote?
        - Do students ever read anything anyone else has written?
        - What do you do with the things students write?
- What role does writing have in helping students understand math?
- (Secondary teachers) Part of the HTX mathematics examination is a paper in which students have to solve a set of problems and explain their approach and thinking. What work do you do in math class that prepares students for this paper?
    - In what ways is this work useful?
- How does what students write in math compare to what students write in other classes?
    - How does writing fit into the curriculum in general?
- Are there other things you would like to say about writing in mathematics?

Interview Protocol - National Mathematics Advisor

- When I visit math classes, I sometimes see different things. For example, in some classes, the students are all sitting in rows and the teacher stands in front of the room and does most of the talking. In other classes, students are moving around and working together in groups to solve problems. What would you like to see when you visit a math class?
    - What should students be doing in math class?
    - What should teachers be doing?
- When I talk to teachers in other countries, sometimes teachers tell me that it is very important for students to have a deep understanding of the math they are learning. Other times, teachers have told me that they are not interested in whether a student understands, but only on whether a student can answer questions. How important do you think it is that students understand the mathematics they are learning?
    - What can teachers do to know if a student understands something?
- What does writing in mathematics look like in a model classroom or lesson?
    - Have you observed a model classroom or lesson?
    - For classrooms, that do not meet the model classroom or lesson standard, what do they typically look like?
        - How does the Ministry support or help these classrooms?
    - Where does the typically Danish classroom fall within the description of a model classroom?
- How does writing fit into mathematics? What are some examples?
- How does mathematics fit into the mathematics curriculum?
- How does writing in mathematics change as student get older?
    - What is the progression of writing in mathematics?
- What role does writing have in helping students understand math?
- As a National Mathematics Advisor, what can you do (what do you do) to incorporate writing into mathematics?
    - Into curriculum?
    - Into instruction?
    - Into assessment?
- Are there other things you would like to say about writing in mathematics?

APPENDIX 2

CONSENT AND NOTIFICATION DOCUMENTS

Informed Consent Agreement

Please read this consent agreement carefully before you decide to participate in the study.

Purpose of the research study: The purpose of the study is to investigate how teachers and students use writing in mathematics.

What you will do in the study: I would like to visit your class to observe mathematics lessons. During my observation, you will be asked to conduct your lesson as usual. Additionally, I would like to interview you to find out more about how you and your students use writing in mathematics. This interview will be a semi-structured time for me to ask questions about classroom practice and student understanding. It will not be an evaluation of any sort. You can skip any question that makes you uncomfortable and you can stop the interview/survey at any time. With your permission, I would like to take written notes and audio-record the conversation. The audio recording will be only to help me supplement my written notes and I will delete all audio recordings no later than July 31, 2014.

The data I plan to collect during the study will include the following:

- Your name, mathematics education background, and number of years of mathematics teaching experience.
- Students' first names and grade in school.
- Notes I make during classroom observations.
- Audio recordings of interviews.
- Photographs of classrooms and schools as allowed by individual teachers and schools.
- Field notes made by the researcher during classroom observations and interviews.
- Related communication between you and myself such as emails.
- School artifacts such as samples of lesson plans, workbook pages, teacher pages and work products as agreed by the teachers and students directly concerned. These may be physical samples or photographs.

Time required: The study will require about 30-45 minutes of your time for the interview. Observations will take place during your scheduled classes and will not require any additional time outside of what you spend in that class. I plan to observe your class at least one time, and possibly additional times if invited by you.

Risks: There are no anticipated risks in this study.

Benefits: There are no direct benefits to you for participating in this research study. The study may help us understand how teachers and students in Denmark use writing in mathematics.

Confidentiality: I will not allow access to the data by anyone unrelated to the project. Names of teachers and students will be collected only for the purpose of the interviews with students and teachers while on-site. Names of these participants will not be used or reported after the analysis phase of the project. I will use audio recordings and photographs to supplement written field notes. This is particularly important as I may need to review written work or interview recordings in which certain aspects are in Danish. If necessary, I will use a translator (or translators) to assist with translating Danish into English. Emails and files (such as Word files, Excel files, audio files, or images of student work) will be saved in secure, password-protected files. If in physical form, these files will be stored in a locked file cabinet. Audio recordings will be deleted no later than July 31, 2014.

Voluntary participation: Your participation in the study is completely voluntary. You may choose to participate in an observation, an interview, both, or neither.

Right to withdraw from the study: You have the right to withdraw from the study at any time without penalty. Any audio recordings will be deleted if you decide to withdraw.

How to withdraw from the study: If you wish to withdraw from the study, please contact me. There is no penalty for withdrawing. If you would like to withdraw after I have observed your class or after the interview, please contact me at mr7c@virginia.edu or 1-434-924-3182.

If you want to withdraw from the study, tell me to stop the interview. There is no penalty for withdrawing.

Payment: You will receive no payment for participating in the study.

If you have questions about the study, contact:
Matthew Reames
Curry School of Education
University of Virginia, Charlottesville, VA 22903
United States of America
Telephone: +1-434-924-3182
mr7c@virginia.edu

Robert Berry, PhD
Curry School of Education
University of Virginia, Charlottesville, VA 22903
United States of America
Telephone: +1-434-924-0767
rqb3e@virginia.edu

If you have questions about your rights in the study, contact:
Tonya R. Moon, Ph.D.
Chair, Institutional Review Board for the Social and Behavioral Sciences
One Morton Dr Suite 500
University of Virginia, P.O. Box 800392
Charlottesville, VA 22908-0392
Telephone: +1-434- 924-5999
Email: irbsbshelp@virginia.edu
Website: www.virginia.edu/vpr/irb/sbs

Agreement:
I agree to participate in the research study described above.

Signature: ____ **Date:** _____________

You will receive a copy of this form for your records.

Curry School of Education
405 Emmet Street South
P.O. Box 400260
Charlottesville, VA 22904-4260
www.curry.virginia.edu

Dear Family,

I am an education researcher from the University of Virginia in the United States of America. I am a former mathematics teacher and am currently working on a PhD in mathematics education. As part of my research, I will be visiting your child's class to observe mathematics lessons. I will be looking primarily at how students use writing in mathematics.

The data I plan to collect during the study will include the following:

- Your child's first name and grade in school.
- Notes I make during classroom observations.
- Samples of student work related to writing in mathematics. These may be physical samples or photographs.

Additionally, I would like to interview students to find out more about how children use writing in mathematics. This interview will take place at school and will be a semi-structured time for me to ask questions about writing and understanding. Your child may skip any questions that make him or her feel uncomfortable, and your child may stop the interview at any time by telling me. It will not be an evaluation of any sort. With your permission, I would like to take written notes and audio-record the conversation. The audio recording will be only to help me supplement my written notes and I will delete all audio recordings no later than July 31, 2014. If you would NOT like for me to consider interviewing your child, please tell your child's teacher. Also, if you would prefer I not audio-record the interview, please tell your child's teacher.

I will not allow access to the data by anyone unrelated to the project. All participant identities will be removed from all data during the analysis phase.

Your participation in this research study is completely voluntary. You do not have to participate in the research study. Additionally, you have the right to withdraw yourself from the study at any time. If you wish to withdraw from the study, please contact me. There is no penalty for withdrawing. If you would like to withdraw after I have observed your class or after the interview, please contact me at mr7c@virginia.edu or +1-434-924-3182.

If you have questions about your rights as a research participant, please contact:

Tonya R. Moon, Ph.D.,
Chair, Institutional Review Board for the Social and Behavioral Sciences
One Morton Dr., Suite 500
University of Virginia, P.O. Box 800392
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Telephone: 1-434-924-5999
Email: irbsbshelp@virginia.edu
Website: www.virginia.edu/vpr/irb/sbs

Best regards,

Matthew Reames

Curry School of Education
405 Emmet Street South
P.O. Box 400260
Charlottesville, VA 22904-4260
www.curry.virginia.edu

Dear National Mathematics Advisor,

I am an education researcher from the University of Virginia in the United States of America. I am a former mathematics teacher and am currently working on a PhD in mathematics education. As part of my research, I would like to talk with you to get a better understanding of how writing fits into the Danish mathematics curriculum.

The data I plan to collect during the study will include the following:

- Your name, mathematics background, and number of years of mathematics teaching and advising experience.
- Notes I make during our conversation.
- Related communication between you and myself such as emails.

This conversation will be a semi-structured time for me to ask questions about classroom practice and student understanding in relation to the national curriculum. It will not be an evaluation of any sort. With your permission, I would like to take written notes and audio-record the conversation. The audio recording will be only to help me supplement my written notes and I will delete all audio recordings no later than July 31, 2014.

I will not allow access to the data by anyone unrelated to the project. All participant identities will be removed from all data during the analysis phase. In the event I would like to attribute a specific statement to you in your professional role as a National Mathematics Advisor, I will obtain written permission from you before doing so.

Your participation in this research study is completely voluntary. You do not have to participate in the research study. Additionally, you have the right to withdraw yourself from the study at any time. If you wish to withdraw from the study, please contact me. There is no penalty for withdrawing. If you would like to withdraw after I have observed your class or after the interview, please contact me at mr7c@virginia.edu or 1-434-924-3182.

If you have questions about your rights as a research participant, please contact:
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Website: www.virginia.edu/vpr/irb/sbs

Best regards,

Matthew Reames