Categorification of Tensor Products of Representations for Current Algebras and Quantum Groups

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Abstract

Motivated by the categorified quantum group of Khovanov-Lauda and Rouquier, Webster defined diagrammatic categories whose split Grothendieck groups are isomorphic to tensor products of integrable highest weight modules for the quantum group. Losev and Webster have proposed an axiomatic definition for a tensor product categorification (TPC) of integrable highest weight modules and shown that these TPCs are unique up to a strong form of equivalence.

In this dissertation, we study the categorification of tensor products of different classes of modules. We show that in ADE type, Webster's category can be regarded as a categorification of a tensor product of Weyl modules for the current algebra by considering the trace decategorification functor.

We also establish a new uniqueness theorem for TPCs of modules over $\mathfrak{sl}_{\mathbb{Z}}$ motivated by work of Brundan, Losev, and Webster. Using this, we lift the super duality equivalence between infinite-rank parabolic BGG categories of general linear Lie (super) algebras of Cheng, Lam, and Wang to a graded equivalence between Koszul graded lifts.

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Chapter 1

Introduction

1.1 Categorification

Categorification is the process of enriching algebraic objects by increasing their categorical dimension by one, for example by passing from a set to a category. For instance, we can regard the category of finite-dimensional $\Bbbk$-vector spaces as a categorification of the integers since its Grothendieck group is isomorphic to $\mathbb{Z}$. Passing from $\mathbb{Z}$ to the category of vector spaces lifts the algebraic structure - addition and multiplication become direct sum and tensor product - and reveals additional structure - linear maps between vector spaces. *Higher representation theory* is concerned with categorifying Lie algebras (and related algebras) and their representation theory.

Categorified quantum groups

Any semisimple Lie algebra $\mathfrak{g}$ has an associated *quantum group* $\dot{U}_q^{\mathbb{Z}} = \dot{U}_q^{\mathbb{Z}}(\mathfrak{g})$. This is a $\mathbb{Z}[q^{\pm 1}]$-algebra which specializes to the universal enveloping algebra of $\mathfrak{g}$ at $q = 1$. The study of quantum groups was initiated independently by Drinfel'd [Dri87] and Jimbo [Jim86]. An important step in their development was the construction of canonical bases for $\dot{U}_q^{\mathbb{Z}}$ and its integrable modules made independently by Lusztig [Lus90] and Kashiwara [Kas91]. These bases have strong compatibility with the algebraic structure. In particular, the non-zero images of basis elements in $\dot{U}_q^{\mathbb{Z}}$ acting on an integrable highest weight module $V_q^{\mathbb{Z}}(\lambda)$ of dominant weight λ form the canonical basis of $V_q^{\mathbb{Z}}(\lambda)$, and canonical bases behave well under tensor products (especially at the combinatorial $q \to 0$ limit). Moreover, in ADE type the product of two canonical basis elements in $\dot{U}_q^{\mathbb{Z}}$ can be written as a $\mathbb{Z}_{\geq 0}[q^{\pm 1}]$-linear combination of other basis elements. This "positivity" phenomenon is a sign that $\dot{U}_q^{\mathbb{Z}}$ is the shadow of a higher categorical structure.

Rouquier [Rou08] and Khovanov-Lauda [KL10] independently constructed a graded 2-category, the *categorified quantum group* $\dot{\mathcal{U}} = \dot{\mathcal{U}}(\mathfrak{g})$, whose split Grothendieck group $K_0(\dot{\mathcal{U}})$ is isomorphic to $\dot{U}_q^{\mathbb{Z}}$. The action of the parameter q is induced by the grading shift on $\dot{\mathcal{U}}$. Khovanov-Lauda's presentation of $\dot{\mathcal{U}}$ is explicit and diagrammatic; the 2-morphism spaces are spanned by oriented string diagrams with strings labelled by simple roots and decorated by dots. The two constructions were shown to be equivalent in [Bru16].

We expect representations of $\dot{U}_q^{\mathbb{Z}}$ to lift to graded 2-representations of $\dot{\mathcal{U}}$. Roughly speaking, linear maps on a vector space should lift to functors on a $\Bbbk$-linear category and string diagrams should correspond to natural transformations between these functors. For a dominant weight λ, there is an associated graded 2-representation $\mathcal{U}^\lambda$ of $\dot{\mathcal{U}}$ called a *cyclotomic quotient*. Kang-Kashiwara [KK12] and Webster [Web17] independently showed that $K_0(\mathcal{U}^\lambda)$ is isomorphic to the highest weight module $V_q^{\mathbb{Z}}(\lambda)$ of $\dot{U}_q^{\mathbb{Z}}$.

As hinted above, higher representation theory can be motivated by connections with canonical bases. Indeed, in [VV11] Varagnolo-Vasserot showed that in ADE type (and over characteristic zero) the classes of indecomposables in $K_0(\mathcal{U}^\lambda)$ coincide with the canonical basis of $V_q^{\mathbb{Z}}(\underline{\lambda})$ and in [Web15] Webster proved the analogous statement for the categorified quantum group $\dot{\mathcal{U}}$ (and the categories $\mathcal{X}^{\underline{\lambda}}$ discussed below). From this perspective positivity of canonical bases is obvious; it reflects the decomposition of a tensor product of objects in $\dot{\mathcal{U}}$ into indecomposables.

Tensor product algebras

In [Web17], Webster constructed a graded 2-representation $\mathcal{X}^{\underline{\lambda}}$ of $\dot{\mathcal{U}}$ for any sequence $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ of dominant weights. The split Grothendieck group of $\mathcal{X}^{\underline{\lambda}}$ is isomorphic to the tensor product $V_q^{\mathbb{Z}}(\underline{\lambda}) = V_q^{\mathbb{Z}}(\lambda^{(1)}) \otimes \cdots \otimes V_q^{\mathbb{Z}}(\lambda^{(n)})$ (actually Webster initially worked with the *tensor product algebra* $T^{\underline{\lambda}}$, but it will be convenient for us to work instead with the category $\mathcal{X}^{\underline{\lambda}} \cong \text{pmod-}T^{\underline{\lambda}}$ given by generators and relations). Morphism spaces in $\mathcal{X}^{\underline{\lambda}}$ are spanned by string diagrams containing red strings that separate the tensor factors. If $\underline{\lambda} = (\lambda)$ then $\mathcal{X}^{\underline{\lambda}}$ is equivalent to the corresponding cyclotomic quotient $\mathcal{U}^{\lambda}$.

These categories provide an illustrative example of the connection between higher representation theory and (quantum) topology. In [RT91], Reshetikhin and Turaev associated an element of $\mathbb{Z}[q^{\pm 1}]$ to any link (a collection of disjoint knots in $\mathbb{R}^3$) with each knot labelled by an integral highest weight representation $V_q^{\mathbb{Z}}(\lambda)$. They defined these "quantum link polynomials" by projecting links to a plane and associating homomorphisms between tensor products $V_q^{\mathbb{Z}}(\underline{\lambda})$ associated to crossings, cups, and caps in the link projection, with the sequence $\underline{\lambda}$ determined by the labels of strands. Composing the homomorphisms yields an element of $\mathbb{Z}[q^{\pm 1}]$ which is an invariant of the link (the most famous of these is the Jones polynomial obtained by taking $\mathfrak{g} = \mathfrak{sl}_2$ and labelling each knot with the defining representation).

Webster was able to categorify this construction by replacing homomorphisms between modules $V_q^{\mathbb{Z}}(\underline{\lambda})$ with functors between categories $\mathcal{X}^{\underline{\lambda}}$ and obtained a bigraded vector space for any link and choice of labels. This yields a richer invariant, analogous to lifting the Euler characteristic of a topological space to its homology. Indeed, taking the graded Euler characteristic of one of these bigraded vector spaces yields the quantum link polynomial of Reshetikhin and Turaev (for the example of the Jones polynomial Webster's construction yields Khovanov homology [Kho00]). This contributes to the long-term goal of constructing 4-dimensional TQFTs in fulfillment of predictions made by Crane and Frenkel [CF94].

Uniqueness of categorification

Another application of categorification comes from uniqueness theorems. For any dominant weight λ, Rouquier defined a natural deformation $\check{\mathcal{U}}^\lambda$ of the cyclotomic quotient $\mathcal{U}^\lambda$. This is also a graded 2-representation of $\dot{\mathcal{U}}$ with Grothendieck group isomorphic to $V_q^{\mathbb{Z}}(\lambda)$, but is *universal* with these properties. That is, any graded 2-representation of $\dot{\mathcal{U}}$ whose split Grothendieck group is isomorphic to $V_q^{\mathbb{Z}}(\lambda)$ is equivalent to a base change of $\check{\mathcal{U}}^\lambda$ (by "equivalent" we mean that there is an equivalence of categories compatible with the 2-representations of $\dot{\mathcal{U}}$).

In [LW15] the authors proved a similar property for $\mathcal{X}^{\underline{\lambda}}$. They gave an axiomatic definition of a *tensor product categorification* (TPC) of the tensor product $V_q^{\mathbb{Z}}(\underline{\lambda})$ and built on Rouquier's work to show that $V_q^{\mathbb{Z}}(\underline{\lambda})$ has a unique TPC. In particular, any TPC is equivalent to $\mathcal{X}^{\underline{\lambda}}$ for some sequence $\underline{\lambda}$. The reader should note that the axioms for a TPC are much stronger than just an isomorphism on the level of the Grothendieck group.

Up to now we have only discussed diagrammatic categories which carry 2-representations of $\dot{\mathcal{U}}$ by design. However, we can also define 2-representations of $\dot{\mathcal{U}}$ on categories found "in nature". Showing such actions are well defined is often difficult, but doing so allows us to apply the powerful machinery of higher representation theory. In particular, the uniqueness results above can be used to show that two categories are equivalent even when constructing an explicit equivalence is infeasible.

1.2 Trace decategorification of tensor product algebras

As described in the previous section, many categories have been constructed whose split Grothendieck groups are, by design, isomorphic to important objects in Lie theory. Recently there has been interest in applying alternative decategorification functors to these categories, in particular the trace decategorification functor defined in [BGHL14]. For example, the traces of the Heisenberg category, diagrammatic Hecke category, and categorified quantum group have been identified with the W-algebra $W_{1+\infty}$ ([CLLS18]), the semidirect product of the Weyl group and a polynomial algebra ([EL16]), and the corresponding current algebra ([BHLŽ17] and [BHLW17]), respectively.

Trace decategorification

The *trace decategorification* of a $\Bbbk$-linear category $\mathcal{C}$, denoted $\mathrm{Tr}(\mathcal{C})$, is the $\Bbbk$-vector space

$$\mathrm{Tr}(\mathcal{C}) = \Big(\bigoplus_{x\in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(x,x)\Big) / \operatorname{span}_{\Bbbk}\{fg - gf\},$$

where $\mathcal{C}(x,x)$ denotes the endomorphism algebra of x and the span is over all $f : x \to y$ and $g : y \to x$ with $x, y \in \mathrm{Ob}(\mathcal{C})$. The trace of a category is invariant under idempotent completion, which makes it particularly convenient for diagrammatic categories.

The trace and split Grothendieck group are related by the *Chern character map* $h_{\mathcal{C}}$ from $K_0(\mathcal{C})$ to $\mathrm{Tr}(\mathcal{C})$ which sends the isomorphism class of an object to the trace class of the identity morphism on that object. It is often injective but not surjective. In particular, if $\mathcal{C}$ is a graded category and $\mathcal{C}^*$ is the category obtained from $\mathcal{C}$ by enlarging morphism spaces to include morphisms of non-zero degree, then $\mathrm{Tr}(\mathcal{C}^*)$ is a graded vector space and the image of $h_{\mathcal{C}^*}$ is concentrated in degree zero, so the trace of $\mathcal{C}^*$ is considerably richer than its Grothendieck group.

Trace and categorified quantum groups

The trace of the starred categorified quantum group $\dot{\mathcal{U}}^*$ is a graded algebra and graded 2-representations of $\dot{\mathcal{U}}^*$ induce graded representations of $\mathrm{Tr}(\dot{\mathcal{U}}^*)$. In [BHLŽ17] the authors showed that $\mathrm{Tr}(\dot{\mathcal{U}}^*(\mathfrak{sl}_2))$ is isomorphic to the *current algebra* $\dot{U}(\mathfrak{sl}_2[t])$ with the indeterminate t in degree 2. The analogous statement for $\mathfrak{sl}_3$ was proved in [Živ14] and this was extended to any $\mathfrak{g}$ of ADE type in [BHLW17], where the authors also identified the traces of deformed and undeformed starred cyclotomic quotients $\mathrm{Tr}(\check{\mathcal{U}}^{\lambda,*})$ and $\mathrm{Tr}(\mathcal{U}^{\lambda,*})$ with the *global Weyl module*

$\mathbb{W}(\lambda)$ and *local Weyl module* $W(\lambda)$ for $\dot{U}(\mathfrak{g}[t])$, respectively.

Recall Webster's category $\mathcal{X}^{\underline{\lambda}}$ from §1.1. In Part I of this dissertation we extend the results of [BHLW17] to the starred category $\mathcal{X}^{\underline{\lambda},*}$ and its deformed counterpart $\check{\mathcal{X}}^{\underline{\lambda},*}$ as introduced in [Web12]. More precisely, we prove the following:

Theorem A. *Let $\mathfrak{g}$ be of ADE type and take a sequence of dominant weights $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)})$. There are isomorphisms of graded $\dot{U}(\mathfrak{g}[t])$-algebras*

$$\begin{aligned} \mathbb{W}(\underline{\lambda}) := \mathbb{W}(\lambda^{(1)}) \otimes \cdots \otimes \mathbb{W}(\lambda^{(n)}) &\longrightarrow \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}) \\ W(\underline{\lambda}) := W(\lambda^{(1)}) \otimes \cdots \otimes W(\lambda^{(n)}) &\longrightarrow \mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*}). \end{aligned} \tag{1.1}$$

These maps are uniquely characterized by the fact that they commute with the action of $\mathrm{Tr}(\dot{\mathcal{U}}^*) \cong \dot{U}(\mathfrak{g}[t])$ and that adding an additional tensor factor in $\mathbb{W}(\underline{\lambda})$ or $W(\underline{\lambda})$ and taking its cyclic vector corresponds to drawing an additional red string on the left of a diagram.

This establishes a categorification of a tensor product of Weyl modules. Moreover, it gives a new perspective on the categories $\mathcal{X}^{\underline{\lambda}}$ and $\mathcal{X}^{\underline{\lambda},*}$ which are interesting algebraic objects in their own right.

Symmetric functions and bubbles

The algebra Sym of symmetric functions is central to how we relate $\dot{U}(\mathfrak{g}[t])$ and $\dot{\mathcal{U}}^*$ and their (2-)representations. Let $\Pi = \bigotimes_{i\in I} \mathrm{Sym}$, where I is the indexing set for simple roots of $\mathfrak{g}$. This is a graded Hopf algebra with $p_{i,r}$, the rth power-sum symmetric function in the i^{th} copy of Sym, in degree $2r$ and coproduct δ given by

$$\delta(p_{i,r}) = p_{i,r} \otimes 1 + 1 \otimes p_{i,r}. \tag{1.2}$$

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ with standard basis $\{\xi_i \mid i \in I\}$. For an integral weight μ, there is a commutative diagram of graded algebras:

$$\begin{array}{ccc} & \Pi & \\ \swarrow & & \searrow^{b_\mu} \\ U(\mathfrak{h}[t]) & \longrightarrow & \dot{\mathcal{U}}^*(1_\mu, 1_\mu) \end{array} \tag{1.3}$$

The left diagonal map (a map of Hopf algebras) sends $p_{i,r} \mapsto \xi_i \otimes t^r$ and the map b_μ sends complete homogeneous (resp. elementary) symmetric functions to clockwise (resp. counter-clockwise) bubbles in $\dot{\mathcal{U}}^*$. The horizontal map agrees with the isomorphism $\dot{U}(\mathfrak{g}[t]) \cong \mathrm{Tr}(\dot{\mathcal{U}}^*)$ after passing to the trace.

The algebra $\mathbb{A}_{\underline{\lambda}}$

In [CFK10] the authors showed that each global Weyl module $\mathbb{W}(\lambda^{(k)})$ carries a graded right action of Π and this action factors through the surjection

$$a_k : \Pi \longrightarrow \mathbb{A}_{\lambda^{(k)}} = \bigotimes_{i \in I} \mathrm{Sym}_{\langle i, \lambda^{(k)} \rangle}, \tag{1.4}$$

where Sym_m denotes the algebra of symmetric polynomials in m variables. So the tensor product $\mathbb{W}(\underline{\lambda})$ is naturally a graded $(\dot{U}(\mathfrak{g}[t]), \mathbb{A}_{\underline{\lambda}})$-bimodule, where

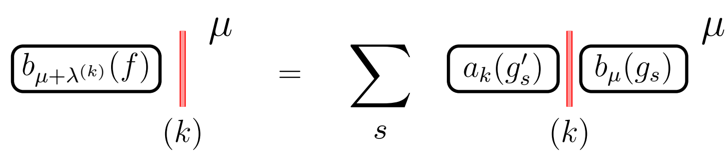
$$\mathbb{A}_{\underline{\lambda}} = \bigotimes_{k=1}^{n} \mathbb{A}_{\lambda^{(k)}}. \tag{1.5}$$

If we consider $\mathbb{A}_{\underline{\lambda}}$ as a subalgebra of a polynomial algebra $\Bbbk[Z]$ in a set of indeterminates Z then $W(\underline{\lambda})$ is obtained from $\mathbb{W}(\underline{\lambda})$ by tensoring with the left $\mathbb{A}_{\underline{\lambda}}$-module $\Bbbk$ on which all $z \in Z$ act as zero.

The category $\check{\mathcal{X}}^{\underline{\lambda},*}$ is obtained by deforming the defining relations in $\mathcal{X}^{\underline{\lambda},*}$ over $\mathbb{A}_{\underline{\lambda}}$ so that a dot on a string, rather than being nilpotent, has spectrum contained in the set of indeterminates Z. It is enriched over graded right $\mathbb{A}_{\underline{\lambda}}$-modules and the undeformed category

$\mathcal{X}^{\underline{\lambda},*}$ is obtained from $\check{\mathcal{X}}^{\underline{\lambda},*}$ by tensoring morphism spaces with the left $\mathbb{A}_{\underline{\lambda}}$-module $\Bbbk$ as above. The spectrum of a dot is determined by the action of bubbles, so it is important for us to understand how these interact with red strings in $\check{\mathcal{X}}^{\underline{\lambda},*}$. In particular we show the following:

Proposition B. *Let μ be an integral weight and take $1 \leq k \leq n$. Recall the coproduct δ on Π, the map b_μ sending elements of Π to bubbles, and the projection $a_k : \Pi \to \mathbb{A}_{\lambda^{(k)}} \leq \mathbb{A}_{\underline{\lambda}}$. If $f \in \Pi$ and $\delta(f) = \sum_s g_s \otimes g'_s$ then in $\check{\mathcal{X}}^{\underline{\lambda},*}$:*



(1.6)

The label (k) on the red string indicates that this corresponds to the dominant weight $\lambda^{(k)}$ and the weight μ indicates that we are in the μ weight space.

Outline of the proof of Theorem A

To prove Theorem A we first show that $\operatorname{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ and $\operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*})$ are spanned by the classes of diagrams with no crossings between red and black strings, so the maps in (1.1) are surjective if they are well defined. Combining this with a filtration of $\operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*})$, the results of [BHLW17] allow us to derive an upper bound for the dimension of the trace:

$$\dim_{\Bbbk} \operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*}) \leq \dim_{\Bbbk} W(\underline{\lambda}). \tag{1.7}$$

It remains to show that these dimensions are equal and the maps are well defined. We do this by showing the maps are well defined at the "generic point" using Webster's technique of unfurling 2-representations and apply upper semi-continuity of dimension under deformation (see Lemma 6.1.1).

Let $\mathbb{K} = \overline{\Bbbk(Z)}$, the algebraic closure of the field of rational polynomials in Z. In [Web16] Webster showed that the idempotent completion of the extension of scalars $\check{\mathcal{X}}^{\underline{\lambda},*} \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K}$ carries a

2-representation of the categorified quantum group $\dot{\mathcal{U}}^*(\tilde{\mathfrak{g}})$ for a larger Lie algebra $\tilde{\mathfrak{g}} = \bigoplus_{z \in Z} \mathfrak{g}$, called an *unfurling* of $\mathfrak{g}$, and moreover it is equivalent to a cyclotomic quotient of $\dot{\mathcal{U}}^*(\tilde{\mathfrak{g}})$. The corresponding statement for Weyl modules was proved in [CFK10]: $\mathbb{W}(\underline{\lambda}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K}$ is isomorphic to a tensor product of local Weyl modules for $\dot{U}(\mathfrak{g}[t])$ indexed by Z; that is, to a single local Weyl module for $\dot{U}(\tilde{\mathfrak{g}}[t])$ (actually the module structures on these local Weyl modules are "twisted" according to the parameter z).

Combining these two pictures with [BHLW17] allows us to construct an isomorphism

$$\mathbb{W}(\underline{\lambda}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K} \longrightarrow \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K}. \tag{1.8}$$

In particular their dimensions are equal, so the upper bound (1.7) and upper semi-continuity of dimension under deformation implies that $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ is flat over $\mathbb{A}_{\underline{\lambda}}$. Moreover, Proposition B implies that the right actions of $\mathbb{A}_{\underline{\lambda}}$ on $\mathbb{W}(\underline{\lambda})$ and $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ are compatible and so flatness allows us to lift (1.8) to a bimodule isomorphism $\mathbb{W}(\underline{\lambda}) \to \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$. Tensoring over $\mathbb{A}_{\underline{\lambda}}$ with $\Bbbk$ gives the isomorphism $W(\underline{\lambda}) \to \mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})$.

1.3 Graded super duality

As described in §1.1, constructing 2-representations and applying uniqueness theorems can show that two categories are equivalent without having to construct an explicit equivalence. In Part II of this dissertation we apply this idea to certain BGG categories $\mathcal{O}$ for general linear Lie (super)algebras.

Brundan-Kazhdan-Lusztig conjecture

The classical Kazhdan-Lusztig (KL) conjecture [KL79] states that characters in the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$ for a complex semisimple Lie algebras are controlled by the Weyl group, or more precisely by canonical bases for the corresponding Hecke algebra.

This statement fails for Lie superalgebras. For example, the Weyl group of the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ is the product $S_m \times S_n$ of symmetric groups, so in particular the Weyl group of $\mathfrak{gl}_{1|1}$ is trivial. But category $\mathcal{O}$ for $\mathfrak{gl}_{1|1}$ is far from semisimple (in fact it has blocks containing infinitely many simples).

However, for the general linear Lie algebra the KL conjecture has a well-known reformulation. Recall that Schur-Weyl duality describes commuting actions of $\mathfrak{sl}_{\mathbb{Z}}$ and S_m on $V^{\otimes m}$, where $V = \mathbb{C}^{\mathbb{Z}}$ is the natural $\mathfrak{sl}_{\mathbb{Z}}$-module and S_m acts by permuting factors. This has a natural quantization called Schur-Jimbo duality (see [Jim86]), where $\mathfrak{sl}_{\mathbb{Z}}$ is replaced by its quantum group $U_q(\mathfrak{sl}_{\mathbb{Z}})$ and S_m by the corresponding Hecke algebra. Moreover, the canonical bases on this module coming from the quantum group and from the Hecke algebra coincide, so the KL conjecture can be rephrased entirely in terms of canonical bases in the $\mathfrak{sl}_{\mathbb{Z}}$-module $V^{\otimes m}$. In particular, it can be phrased without reference to the Weyl group.

In [Bru03] Brundan conjectured that characters in the BGG category $\mathcal{O}$ for $\mathfrak{gl}_{m|n}$ are controlled by canonical bases in the $\mathfrak{sl}_{\mathbb{Z}}$-module $V^{\otimes m} \otimes W^{\otimes n}$, where $W = V^*$ is the restricted dual of V. This is known as the *Brundan-Kazhdan-Lusztig (BKL) conjecture*. It was first proved in full generality in [CLW15] and was later proved in [BLW17] using different techniques (see below).

Super duality

As in the non-super case, the BKL conjecture admits natural parabolic variants (see e.g. [CW08]). The corresponding $\mathfrak{sl}_{\mathbb{Z}}$-modules are of the form

$$\bigwedge^{\underline{m}} V \otimes \bigwedge^{\underline{n}} W := \bigwedge^{m_1} V \otimes \cdots \otimes \bigwedge^{m_s} V \otimes \bigwedge^{n_1} W \otimes \cdots \otimes \bigwedge^{n_t} W \tag{1.9}$$

where $\underline{m} = (m_1, \ldots, m_s)$, $\underline{n} = (n_1, \ldots, n_t)$, $m = \sum m_i$, and $n = \sum n_j$ (in fact the wedges of Vs and Ws can be in any order, but we restrict our attention to this case for the introduction).

There is a natural $\mathfrak{sl}_{\mathbb{Z}}$-module isomorphism $\bigwedge^{\infty} V \cong \bigwedge^{\infty} W$ between semi-infinite wedge spaces and in [CW08] the authors considered the induced isomorphism

$$\bigwedge^m V \otimes \bigwedge^n W \otimes \bigwedge^\infty V \cong \bigwedge^m V \otimes \bigwedge^n W \otimes \bigwedge^\infty W. \tag{1.10}$$

Motivated by this, they defined infinite-rank parabolic categories $\mathcal{O}$, denoted $\mathcal{O}_0^{++}$ and $\mathcal{O}_1^{++}$ respectively, for the corresponding infinite-rank general linear Lie superalgebras and conjectured an equivalence $\mathcal{O}_0^{++} \cong \mathcal{O}_1^{++}$. This is super duality (the special case $s = 1$, $t = 0$ was first conjectured in [CWZ08]).

Super duality was proved in [CL10] (see also [CLW11] where an analogous statement was proved for ortho-symplectic Lie superalgebras). It can be regarded as a bridge between categories $\mathcal{O}$ for general linear Lie algebras and superalgebras. Its utility stems from "truncation functors" from the categories $\mathcal{O}_\epsilon^{++}$ ($\epsilon \in \{0, 1\}$) to finite-rank categories $\mathcal{O}$. These allow us to read off information about the finite-rank categories from their infinite-rank counterparts. For example, the BKL conjecture for the special case $t = 1$ is an immediate corollary of super duality, as observed in [CW08]. Super duality was also a key component in the first general proof of the BKL conjecture in [CLW15].

Tensor product categorification

In a highly non-trivial extension of the uniqueness result of [LW15] (see §1.1), Brundan, Losev, and Webster proved in [BLW17] that tensor product categorifications (TPCs) of $\mathfrak{sl}_\mathbb{Z}$-modules of the form (1.9) are unique up to equivalence. Note that since wedges of V and W are not highest weight, Losev and Webster's original result doesn't apply, and there is no diagrammatic category $\mathcal{X}^{\underline{\lambda}}$ that we might regard as a "canonical" TPC.

For $r \in \mathbb{N}$, let $I_r = \{i \in \mathbb{Z} \mid 1 - r \leq i \leq r - 1\}$ and let $\mathfrak{sl}_{I_r}$ be the special linear Lie algebra of traceless complex matrices with rows and columns indexed by $I_r^+ := I_r \cup (I_r + 1)$. Let V_r be the natural $\mathfrak{sl}_{I_r}$-module and $W_r = V_r^*$ its dual.

Roughly, the uniqueness theorem in [BLW17] was proved by regarding the $\mathfrak{sl}_\mathbb{Z}$-module as a union of highest weight $\mathfrak{sl}_{I_r}$-modules

$$\bigwedge^m V \otimes \bigwedge^n W = \bigcup_{r=1}^{\infty} \bigwedge^m V_r \otimes \bigwedge^n W_r \tag{1.11}$$

and showing that the data in Losev and Webster's uniqueness argument for these highest weight modules lifts appropriately. Moreover, they showed that translation functors on a finite-rank parabolic category $\mathcal{O}$ for $\mathfrak{gl}_{m|n}$ make the category a TPC of the corresponding $\mathfrak{sl}_{\mathbb{Z}}$-module and used this to give another proof of the BKL conjecture.

We examine TPCs of modules of the form (1.10) and applications to the infinite-rank categories $\mathcal{O}_\epsilon^{++}$. We prove the following:

Theorem C. *For $\epsilon \in \{0, 1\}$, the infinite-rank parabolic category $\mathcal{O}_\epsilon^{++}$ is a tensor product categorification of the $\mathfrak{sl}_{\mathbb{Z}}$-module* (1.10)*. Moreover, any two TPCs of this module are equivalent. In particular, there is an equivalence of categories*

$$\mathbb{S} : \mathcal{O}_0^{++} \longrightarrow \mathcal{O}_1^{++}. \tag{1.12}$$

Outline of proof of Theorem C

We prove the uniqueness part of the theorem by regarding semi-infinite wedges as a direct limit of finite wedges. So the decomposition (1.11) is replaced by the direct limit

$$\bigwedge^m V \otimes \bigwedge^n W \otimes \bigwedge^\infty V = \varinjlim \bigwedge^m V_r \otimes \bigwedge^n W_r \otimes \bigwedge^r V_r, \tag{1.13}$$

or the analogous limit for $\bigwedge^\infty W$.

Our main tools for showing that the categories $\mathcal{O}_\epsilon^{++}$ are TPCs are the truncation functors already mentioned. These allow us to consider a module in $\mathcal{O}_\epsilon^{++}$ as a direct limit of modules for finite-rank general linear Lie superalgebras as the rank goes to infinity. In particular, we show that $\mathcal{O}_\epsilon^{++}$ has enough projectives by constructing projective covers as the direct limit of projective covers in certain subcategories of finite-rank categories $\mathcal{O}$. The key property is that the Verma flags of these projective covers stabilize when the rank is sufficiently large.

An unexpected difficulty occurs in defining the functors $\mathcal{E}_j$ on $\mathcal{O}_\epsilon^{++}$ lifting the action of the Chevalley generators e_j. The corresponding translation functors on finite-rank categories $\mathcal{O}$ are not well behaved with respect to truncation, so we cannot define connecting maps for the direct limit in the obvious way. We actually define two sets of connecting maps, leading to isomorphic functors $\mathcal{E}_j^{\mathtt{L}}$ and $\mathcal{E}_j^{\mathtt{R}}$ which are naturally left and right adjoint respectively to the functor $\mathcal{F}_j$ lifting the action of f_j. Again, these only give well-defined functors on $\mathcal{O}_\epsilon^{++}$ because the associated composition multiplicities eventually stabilize.

Gradings

Observe that so far we have been working with $\mathfrak{sl}_{\mathbb{Z}}$-modules rather than modules over the quantum group $U_q(\mathfrak{sl}_{\mathbb{Z}})$. This is because there is no obvious grading on the BGG categories which would induce an action of q on the Grothendieck group. However, in [BLW17] the authors showed that any TPC of the $\mathfrak{sl}_{\mathbb{Z}}$-module (1.9) lifts uniquely to a Koszul graded category which is a TPC of the corresponding $U_q(\mathfrak{sl}_{\mathbb{Z}})$-module. We prove the analogous statement in our setting and deduce the following "graded super duality":

Theorem D. *For $\epsilon \in \{0,1\}$, the category $\mathcal{O}_\epsilon^{++}$ has a unique Koszul graded lift ${}^{gr}\mathcal{O}_\epsilon^{++}$ and the super duality equivalence $\mathbb{S}$ lifts to an equivalence of graded categories*

$$ {}^{gr}\mathbb{S} : {}^{gr}\mathcal{O}_0^{++} \longrightarrow {}^{gr}\mathcal{O}_1^{++}. \tag{1.14}$$

After a preprint of this work was submitted to the `arXiv`, Chih-Whi Chen and Ngau Lam posted a paper [CL18] independently showing that the categories $\mathcal{O}_\varepsilon^{++}$ have enough projectives and possess Koszul graded lifts and extending these results to other classical types. Their work uses a different approach.

1.4 Organization

This dissertation is separated into two largely independent parts.

Part I

Part I consists of Chapters 2-8 and treats trace decategorification of the categories $\mathcal{X}^{\underline{\lambda},*}$ and $\check{\mathcal{X}}^{\underline{\lambda},*}$. This work originally appeared in [LR19] and is joint with Michael Reeks.

In Chapter 2 we recall preliminaries on quantum groups, current algebras, and Weyl modules. We describe the connection between the current algebra and symmetric functions leading to a right action of an algebra $\mathbb{A}_{\underline{\lambda}}$ on a tensor product of global Weyl modules.

In Chapter 3 we recall the diagrammatic presentation of the categorified quantum group $\dot{\mathcal{U}}(\mathfrak{g})$. We state isomorphisms relating symmetric functions and bubbles in $\dot{\mathcal{U}}(\mathfrak{g})$ and use generating functions for symmetric functions to concisely state bubble slide equations. We recall the notion of a 2-representation and define the deformed and undeformed cyclotomic quotients of $\dot{\mathcal{U}}(\mathfrak{g})$.

In Chapter 4 we define the category $\mathcal{X}^{\underline{\lambda}}$ whose split Grothendieck group is a tensor product of integrable highest weight modules. We also define a category $\check{\mathcal{X}}^{\underline{\lambda}}$ by deforming the relations of $\mathcal{X}^{\underline{\lambda}}$ over the algebra $\mathbb{A}_{\underline{\lambda}}$. We prove Proposition B yielding equations for sliding bubbles past red strings in $\check{\mathcal{X}}^{\underline{\lambda}}$.

In Chapter 5 we consider the category $\mathcal{G}^{\underline{\lambda}}$ obtained by specializing $\check{\mathcal{X}}^{\underline{\lambda},*}$ at the generic point of the deformation. We determine the spectrum of a dot acting on a black string in $\mathcal{G}^{\underline{\lambda}}$ using the relations in $\dot{\mathcal{U}}(\mathfrak{g})$ and $\check{\mathcal{X}}^{\underline{\lambda}}$. Following [Web16], we describe how the decomposition of the categorified Chevalley generators $\mathcal{E}_i$ and $\mathcal{F}_i$ into generalized eigenspaces for the actions of dots induces a 2-representation of the categorified quantum group $\dot{\mathcal{U}}(\tilde{\mathfrak{g}})$ on $\mathcal{G}^{\underline{\lambda}}$, where $\tilde{\mathfrak{g}}$ is a larger Lie algebra called an unfurling of $\mathfrak{g}$. We compare the 2-representations of $\dot{\mathcal{U}}(\mathfrak{g})$ and $\dot{\mathcal{U}}(\tilde{\mathfrak{g}})$ on $\mathcal{G}^{\underline{\lambda}}$ and give an equivalence of categories between $\mathcal{G}^{\underline{\lambda}}$ and a cyclotomic quotient of $\dot{\mathcal{U}}(\tilde{\mathfrak{g}})$.

In Chapter 6 we compare dimensions of morphism spaces in $\mathcal{X}^{\underline{\lambda},*}$ and $\mathcal{G}^{\underline{\lambda}}$ using the unfurled 2-representation on $\mathcal{G}^{\underline{\lambda}}$ and apply upper semi-continuity of dimension to show that $\check{\mathcal{X}}^{\underline{\lambda},*}$ is a flat deformation of $\mathcal{X}^{\underline{\lambda},*}$ over $\mathbb{A}_{\underline{\lambda}}$. We use flatness to show that the split Grothendieck group of $\check{\mathcal{X}}^{\underline{\lambda}}$ is a tensor product of integrable highest weight modules and that $\check{\mathcal{X}}^{\underline{\lambda}}$ is equivalent to a deformed cyclotomic quotient of $\dot{\mathcal{U}}(\mathfrak{g})$ if the sequence $\underline{\lambda}$ consists of a single weight.

In Chapter 7 we recall the process of trace decategorification and the results of [BHLW17]. By considering a careful chosen spanning set for morphism spaces in $\mathcal{X}^{\underline{\lambda},*}$ and $\check{\mathcal{X}}^{\underline{\lambda},*}$ we show that the traces $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})$ and $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ are spanned by classes of diagrams with no crossings between red and black strings. We deduce that the Chern character maps for the unstarred categories $\mathcal{X}^{\underline{\lambda}}$ and $\check{\mathcal{X}}^{\underline{\lambda}}$ are isomorphisms. Using standardization functors from [Web17] we construct a filtration of $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})$ whose associated graded space admits a surjection from a tensor product of local Weyl modules. This gives an upper bound on the dimension of $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})$.

Finally in Chapter 8 we prove Theorem A, the main result of Part I. First we construct a homomorphism to $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ from a tensor product of Verma-like modules for the current algebra. The spanning set for $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ shows this is surjective and the bubble slides in $\check{\mathcal{X}}^{\underline{\lambda}}$ show that it is compatible with the right $\mathbb{A}_{\underline{\lambda}}$-action. We show that this descends to an isomorphism from a tensor product of global Weyl modules at the generic point using the results of [BHLW17] and the unfurled 2-representation on $\mathcal{G}^{\underline{\lambda}}$ from Chapter 5. Combining this with the upper bound on the dimension of $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})$, Theorem A follows by upper semi-continuity of dimension.

Part II

Part II consists of Chapters 9-12 and treats uniqueness of tensor product categorifications and applications to super duality. This work originally appeared in [Leo17].

In Chapter 9 we describe the combinatorics of wedge spaces. We fix a tensor product $\bigwedge^{\infty,\underline{\epsilon}} V$ of finite and semi-infinite wedges of the $\mathfrak{sl}_{\mathbb{Z}}$-modules V and W and describe how to regard it as a direct limit over r of tensor products $\bigwedge^{\underline{r},\epsilon} V_r$ of finite wedges over special linear Lie algebras $\mathfrak{sl}_{I_r}$. We state the combinatorial shadow of super duality.

In Chapter 10 we define the notion of tensor product categorification (TPC) of $\bigwedge^{\underline{r},\epsilon} V_r$ for $r \in \mathbb{N} \cup \{\infty\}$. We show that for r sufficiently large, a TPC $\mathcal{C}$ of $\bigwedge^{\underline{\infty},\epsilon} V$ has a subquotient $\mathcal{C}_r$ that is a TPC of $\bigwedge^{\underline{r},\epsilon} V_r$. Following [BLW17] we define a category of stable modules and use this to lift the uniqueness apparatus for the TPCs $\mathcal{C}_r$ up to $\mathcal{C}$ resulting in the proof that TPCs of $\bigwedge^{\underline{\infty},\epsilon} V$ are unique.

In Chapter 11 we define parabolic BGG categories $\mathcal{O}^{++}$ for infinite-rank general linear Lie superalgebras. We recall truncation functors which allow us to regard modules in $\mathcal{O}^{++}$ as direct limits of modules in categories $\mathcal{O}_r^{++}$ of modules for finite-rank general linear Lie superalgebras. We construct projective covers in $\mathcal{O}^{++}$ as direct limits of projective covers in $\mathcal{O}_r^{++}$ and deduce that $\mathcal{O}^{++}$ is a highest weight category. We define functors $\mathcal{F}_j$ on $\mathcal{O}^{++}$ lifting the Chevalley generators f_j and construct left and right adjoints $\mathcal{E}_j^{\mathtt{L}}$ and $\mathcal{E}_j^{\mathtt{R}}$ to $\mathcal{F}_j$ as direct limits of translation functors on the finite-rank categories $\mathcal{O}_r^{++}$. We show that $\mathcal{E}_j^{\mathtt{L}}$ and $\mathcal{E}_j^{\mathtt{R}}$ are naturally isomorphic and deduce that $\mathcal{O}^{++}$ is a TPC of a module of the form $\bigwedge^{\underline{\infty},\epsilon} V$. Super duality follows immediately.

Finally in Chapter 12 we consider graded lifts. Following [BHLW17] we show that any TPC of $\bigwedge^{\underline{\infty},\epsilon} V$ lifts uniquely to a Koszul graded category which is a TPC of the analogous module for the quantum group $U_q(\mathfrak{sl}_{\mathbb{Z}})$. We deduce that the categories $\mathcal{O}^{++}$ have Koszul graded lifts and prove graded super duality.

1.5 Conventions and notation

Throughout this dissertation $\Bbbk$ denotes a fixed field of characteristic zero. For an integer m, $[1, m]$ denotes the set of integers k such that $1 \leq k \leq m$.

Let $\mathcal{C}$ be a small $\Bbbk$-linear category (we will generally just say "category" and "functor" and the reader can assume that everything is small and linear). We write $\mathrm{Ob}(\mathcal{C})$ for the set of objects in $\mathcal{C}$, $\mathcal{C}(x,y)$ for the $\Bbbk$-vector space of morphisms from x to y, and $1_x \in \mathcal{C}(x,x)$ for the identity morphism on x.

If $\mathcal{C}$ is a graded category with grading shift $\langle 1 \rangle$ we let $\mathcal{C}^*$ denote the category with the same objects as $\mathcal{C}$ and morphism spaces given by

$$\mathcal{C}^*(x,y) = \bigoplus_{t \in \mathbb{Z}} \mathcal{C}(x, y\langle t \rangle). \tag{1.15}$$

Morphism spaces in $\mathcal{C}^*$ are $\mathbb{Z}$-graded with $\mathcal{C}(x, y\langle t \rangle)$ in degree t.

Let $K_0(\mathcal{C})$ denote the split Grothendick group of $\mathcal{C}$ and write $[x] \in K_0(\mathcal{C})$ for the class of $x \in \mathrm{Ob}(\mathcal{C})$. Write $K_0^{\Bbbk}(\mathcal{C}) = K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \Bbbk$. If $\mathcal{C}$ is graded then grading shift induces a $\mathbb{Z}[q^{\pm 1}]$ action on $K_0(\mathcal{C})$ and $K_0(\mathcal{C}^*) \cong K_0(\mathcal{C}) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{Z}$ where q acts on $\mathbb{Z}$ as 1.

By a *Schurian category* we mean a *abelian* category $\mathcal{C}$ such that all objects are of finite length, there are enough projectives and injectives, and the endomorphism algebras of irreducible objects are one-dimensional. Write $G_0(\mathcal{C})$ for the Grothendieck group of $\mathcal{C}$ and $G_0^{\mathbb{C}}(\mathcal{C}) = G_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ for the complexified Grothendieck group. We write proj-$\mathcal{C}$ for the additive subcategory of projective objects in $\mathcal{C}$.

If A is an associative algebra we denote the category of finite-dimensional right A-modules by mod-A. When we have a collection of objects X_r indexed by $\mathbb{N} \cup \{\infty\}$ (as it often the case in Part II) we will drop the subscript in the $r = \infty$ case and write $X = X_\infty$. If $\mathfrak{a}$ is a Lie superalgebra we will say '$\mathfrak{a}$-module' to mean $\mathfrak{a}$-supermodule.

Part I

Trace decategorification of Webster's tensor product algebras

Chapter 2

Quantum groups and current algebras

We fix notation and recall some standard results about quantum groups, current algebras, and their modules. The main reference for current algebras is [CFK10].

2.1 Quantum groups

Cartan datum

Fix a symmetric simply-laced Cartan datum consisting of:

- a free $\mathbb{Z}$-module X, the weight lattice;
- a finite indexing set I, simple roots $\alpha_i \in X$ and fundamental weights $\Lambda_i \in X$ for $i \in I$;
- simple coroots $\alpha_i^\vee \in X^\vee := \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ for $i \in I$;
- a symmetric bilinear form $(-,-)$ on X.

Write $\langle -,- \rangle : X^\vee \times X \to \mathbb{Z}$ for the canonical pairing. Assume that

- $(\alpha_i, \alpha_i) = 2$ for all $i \in I$;
- $\langle i, \lambda \rangle := \langle \alpha_i^\vee, \lambda \rangle = (\alpha_i, \lambda)$ for $i \in I$ and $\lambda \in X$;

- $(\alpha_i, \alpha_j) \in \{0, -1\}$ for $i, j \in I$ with $i \neq j$;
- $\langle i, \Lambda_j \rangle = \delta_{ij}$ for $i, j \in I$.

For $i, j \in I$ we will write $\langle j, i \rangle := \langle \alpha_j^\vee, \alpha_i \rangle$. Let $X^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i$ be the set of dominant weights in X. We sometimes write $+I$ for I and define a set of formal negatives $-I = \{-i \mid i \in I\}$. Define the "absolute value" $|\pm i| = i$. For $i, j \in I$ let $\alpha_{-i} = -\alpha_i \in X$ and $\langle j, -i \rangle = -\langle j, i \rangle$.

We assume that the Cartan datum is of finite type, so the Cartan matrix $(\langle i, j \rangle)_{i,j \in I}$ is invertible. Let Γ denote the corresponding graph without loops or multiple edges. It has vertex set I and an edge between i and j if any only if $\langle i, j \rangle = -1$. For the rest of Part II we fix an orientation on Γ.

Definition of quantum group

Let q be an indeterminate and let $U_q = U_q(\mathfrak{g})$ denote the quantum group associated to the Cartan datum above. This is the $\mathbb{Q}(q)$-algebra generated by E_i, F_i, and K_μ for $i \in I$ and $\mu \in X^\vee$ and subject to familiar relations (we use the same conventions as in [BHLW17, §3.1].

The quantum group is a Hopf algebra with coproduct Δ_q given by

$$\Delta_q(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta_q(K_\mu) = K_\mu \otimes K_\mu, \quad \Delta_q(F_i) = F_i \otimes K_{-i} + 1 \otimes F_i$$

for $i \in I$ and $\mu \in X^\vee$. Let $U_q^{\mathbb{Z}}$ denote the integral form of U_q; the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of U_q generated by the K_μ for $\mu \in X^\vee$ and the divided powers $E_i^{(a)}$ and $F_i^{(a)}$ for $i \in I$, $a \in \mathbb{N}$.

Let $\dot{U}_q = \dot{U}_q(\mathfrak{g})$ denote the idempotent form of U_q; the locally unital $\mathbb{Q}(q)$-algebra obtained from U_q by adjoining mutually orthogonal idempotents 1_λ for $\lambda \in X$ satisfying

$$E_i 1_\lambda = 1_{\lambda + \alpha_i} E_i, \quad K_\mu 1_\lambda = 1_\lambda K_\mu = q^{\langle \mu, \lambda \rangle} 1_\lambda, \quad F_i 1_\lambda = 1_{\lambda - \alpha_i} F_i. \tag{2.1}$$

The algebra $\dot{U}_q$ decomposes as a direct sum

$$\dot{U}_q = \bigoplus_{\lambda,\mu \in X} 1_\mu U_q 1_\lambda. \tag{2.2}$$

Let $\dot{U}_q^{\mathbb{Z}} = \bigoplus 1_\mu U_q^{\mathbb{Z}} 1_\lambda$ denote the integral form of $\dot{U}_q$.

For $\lambda \in X^+$, let $V_q(\lambda)$ denote the irreducible $\dot{U}_q$-module generated by a highest weight vector v_λ of weight λ and let $V_q^{\mathbb{Z}}(\lambda) = \dot{U}_q^{\mathbb{Z}} v_\lambda$ be its integral form. For a sequence $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ of dominant weights, let

$$V_q^{\mathbb{Z}}(\underline{\lambda}) = V_q^{\mathbb{Z}}(\lambda^{(1)}) \otimes_{\mathbb{Z}[q^{\pm 1}]} \cdots \otimes_{\mathbb{Z}[q^{\pm 1}]} V_q^{\mathbb{Z}}(\lambda^{(n)}), \tag{2.3}$$

regarded as a $\dot{U}_q^{\mathbb{Z}}$-module via the coproduct Δ_q.

2.2 Current algebras

Let $\mathfrak{g}$ be the semisimple Lie algebra over $\mathbb{Q}$ determined by the Cartan matrix. For $i \in I$, we write e_i and f_i for the root vectors of weights α_i and $-\alpha_i$ respectively, and write $\xi_i = [e_i, f_i]$. We will sometimes also write e_{-i} for f_i. Let $\mathfrak{h} = \mathrm{span}_{\mathbb{Q}}\{\xi_i \mid i \in I\}$ be the canonical Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{n} = \langle e_i \mid i \in I \rangle$ the sum of the positive root spaces.

Definition of current algebra

Let t be an indeterminate and set $\mathfrak{g}[t] := \mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{Q}[t]$ with Lie bracket given by

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} \tag{2.4}$$

for $x, y \in \mathfrak{g}$ and $r, s \in \mathbb{N}$.

Definition 2.2.1. The *current algebra* $U(\mathfrak{g}[t]) = U_{\Bbbk}(\mathfrak{g}[t])$ of $\mathfrak{g}$ over $\Bbbk$ is the universal enveloping algebra of $\mathfrak{g}[t]$ over the field $\Bbbk$. It is $\mathbb{Z}$-graded with $x \otimes t^r$ in degree $2r$ for $x \in \mathfrak{g}$ and $r \in \mathbb{N}$.

This is a Hopf algebra with coproduct sending x to $x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}[t]$.

The current algebra has an idempotent form $\dot{U}(\mathfrak{g}[t])$; the locally unital $\Bbbk$-algebra obtained from $U(\mathfrak{g}[t])$ by adjoining mutually orthogonal idempotents 1_λ for $\lambda \in X$ satisfying

$$(e_i \otimes t^r)1_\lambda = 1_{\lambda+\alpha_i}(e_i \otimes t^r), \quad (\xi_i \otimes t^r)1_\lambda = 1_\lambda(\xi_i \otimes t^r), \quad (f_i \otimes t^r)1_\lambda = 1_{\lambda-\alpha_i}(f_i \otimes t^r)$$

and

$$(\xi_i \otimes 1)1_\lambda = 1_\lambda(\xi_i \otimes 1) = \langle i, \lambda\rangle 1_\lambda \tag{2.5}$$

for $i \in I$ and $r \in \mathbb{N}$. The idempotent form is also $\mathbb{Z}$-graded. We will pass freely between weight modules for $U(\mathfrak{g}[t])$ with integral weights and $\dot{U}(\mathfrak{g}[t])$-modules.

Symmetric functions

Let Sym denote the ring of symmetric functions over $\Bbbk$ and define $\Pi := \bigotimes_{i\in I} \mathrm{Sym}$. We denote power sum, elementary, and complete homogeneous symmetric functions in the i^{th} component of Π by $\{p_{i,r}\}_{r\geq 1}$, $\{e_{i,r}\}_{r\geq 1}$, and $\{h_{i,r}\}_{r\geq 1}$ respectively. We consider Π as a $\mathbb{Z}$-graded algebra with $p_{i,r}$, $e_{i,r}$, and $h_{i,r}$ in degree $2r$.

It will be useful for us to consider the generating functions

$$p_i(x) = \sum_{r=1}^{\infty} p_{i,r}x^r, \quad e_i(x) = \sum_{r=0}^{\infty} e_{i,r}x^r, \quad h_i(x) = \sum_{r=0}^{\infty} h_{i,r}x^r, \tag{2.6}$$

where x is a formal indeterminate (by convention $h_{i,0} = e_{i,0} = 1$). If we regard the ith copy of Sym in Π as consisting of symmetric functions in countably many variables $y_1, y_2, \ldots$ then

$$p_i(x) = \sum_{j=1}^{\infty} \frac{y_j x}{1 - y_j x}, \quad e_i(x) = \prod_{j=1}^{\infty}(1 + y_j x), \quad h_i(x) = \prod_{j=1}^{\infty}(1 - y_j x)^{-1}. \tag{2.7}$$

For $n \in \mathbb{N}$, let Sym_n denote the polynomial algebra in n variables over $\Bbbk$, also $\mathbb{Z}$-graded with each variable having degree 2. For $m, n \in \mathbb{N}$, there is an inclusion

$$\mathrm{Sym}_{m+n} \longrightarrow \mathrm{Sym}_m \otimes \mathrm{Sym}_n. \tag{2.8}$$

Taking the direct limit over m and n this gives a map from Sym to Sym$\otimes$Sym, and tensoring over I copies yields a coproduct $\delta : \Pi \to \Pi \otimes \Pi$. In terms of the generating functions:

$$\delta(p_i(x)) = p_i(x) \otimes 1 + 1 \otimes p_i(x), \quad \delta(e_i(x)) = e_i(x) \otimes e_i(x), \quad \delta(h_i(x)) = h_i(x) \otimes h_i(x).$$

For any $\lambda \in X$, there is an isomorphism of graded Hopf algebras

$$\begin{aligned} \Pi &\longrightarrow 1_\lambda U(\mathfrak{h}[t]) 1_\lambda \\ p_{i,r} &\longmapsto 1_\lambda (\xi_i \otimes t^r) 1_\lambda \end{aligned} \tag{2.9}$$

for $i \in I$ and $r \geq 1$. Note in particular that this intertwines δ with the coproduct on $U(\mathfrak{h}[t])$.

For $\lambda \in X^+$ define a $\mathbb{Z}$-graded algebra

$$\mathbb{A}_\lambda := \bigotimes_{i \in I} \mathrm{Sym}_{\langle i, \lambda \rangle}. \tag{2.10}$$

There is a surjective homomorphism of graded algebras $\Pi \to \mathbb{A}_\lambda$ sending $p_{i,r}$, $e_{i,r}$, and $h_{i,r}$ to the corresponding symmetric polynomials.

2.3 Weyl modules

Let $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}[t] \subseteq \mathfrak{g}[t]$. For $\lambda \in X^+$, let $M(\lambda)$ denote the Verma-like module

$$M(\lambda) = U(\mathfrak{g}[t]) \otimes_{U(\mathfrak{p})} \mathbb{k}_\lambda, \tag{2.11}$$

where $\mathbb{k}_\lambda$ is the one-dimensional $\mathfrak{h}$-module on which each ξ_i acts by $\langle i, \lambda \rangle$, induced up to $\mathfrak{p}$.

Let $m_\lambda := 1 \otimes 1 \in M(\lambda)$. Then $M(\lambda)$ is generated by m_λ subject to the following relations:

$$\mathfrak{n}[t] \cdot m_\lambda = 0, \quad (\xi_i \otimes 1) \cdot m_\lambda = \langle i, \lambda \rangle m_\lambda \tag{2.12}$$

for $i \in I$. It is a graded $U(\mathfrak{g}[t])$-module with m_λ in degree 0. There is a right action of Π on $M(\lambda)$ given by

$$(um_\lambda) \cdot p_{i,r} = u(\xi_i \otimes t^r)m_\lambda \tag{2.13}$$

for $i \in I$, $r \geq 1$, and $u \in U(\mathfrak{g}[t])$. This implicitly uses the identification in (2.9) of Π with $1_\lambda U(\mathfrak{h}[t])1_\lambda$, and makes $M(\lambda)$ a graded $(U(\mathfrak{g}[t]), \Pi)$-bimodule.

The *global Weyl module* $\mathbb{W}(\lambda)$ is the $U(\mathfrak{g}[t])$-module quotient of $M(\lambda)$ by the relation

$$(f_i \otimes 1)^{\langle i, \lambda \rangle + 1} \cdot m_\lambda = 0 \tag{2.14}$$

for $i \in I$. It is also $\mathbb{Z}$-graded. We write w_λ for the image of m_λ in $\mathbb{W}(\lambda)$. The right action of Π on $M(\lambda)$ descends to an action on $\mathbb{W}(\lambda)$. In fact we have the following:

Theorem 2.3.1. *[CFK10] The action of* Π *on* $\mathbb{W}(\lambda)$ *factors through a faithful action of* $\mathbb{A}_\lambda$*. Moreover,* $M(\lambda)$ *and* $\mathbb{W}(\lambda)$ *are free right* Π*- and* $\mathbb{A}_\lambda$*-modules respectively.*

Proof. The first claim is [CFK10, Theorem 4]. The fact that $\mathbb{W}(\lambda)$ is a free $\mathbb{A}_\lambda$-module follows from the work of [CP01], [CL06], [FL07], and [BN04] (see [CFK10, §7.2] for a more detailed discussion). Finally $M(\lambda)$ is free by the PBW theorem. □

In particular, $\mathbb{W}(\lambda)$ is a graded $(U(\mathfrak{g}[t]), \mathbb{A}_\lambda)$-bimodule.

The *local Weyl module* $W(\lambda)$ associated to $\lambda \in X^+$ is

$$W(\lambda) := \mathbb{W}(\lambda) \otimes_{\mathbb{A}_\lambda} \Bbbk, \tag{2.15}$$

where $\Bbbk$ is the unique simple graded $\mathbb{A}_\lambda$-module. Equivalently, $W(\lambda)$ is the $U(\mathfrak{g}[t])$-module quotient of $\mathbb{W}(\lambda)$ by the relation

$$\mathfrak{h}t[t] \cdot w_\lambda = 0. \tag{2.16}$$

It is a graded $U(\mathfrak{g}[t])$-module.

Notation for tensor products

For the rest of the paper we fix $n \in \mathbb{N}$ and a sequence $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ of dominant weights. For $k \in [1, n]$ and $i \in I$ let $\lambda_i^{(k)} := \langle i, \lambda^{(k)} \rangle$. Define

$$M(\underline{\lambda}) = \bigotimes_{k=1}^{n} M(\lambda^{(k)}), \quad \mathbb{W}(\underline{\lambda}) = \bigotimes_{k=1}^{n} \mathbb{W}(\lambda^{(k)}), \quad W(\underline{\lambda}) = \bigotimes_{k=1}^{n} W(\lambda^{(k)}). \tag{2.17}$$

Each of these is a graded $U(\mathfrak{g}[t])$-module via the coproduct.

Define also

$$\mathbb{A}_{\underline{\lambda}} = \bigotimes_{k=1}^{n} \mathbb{A}_{\lambda^{(k)}} = \bigotimes_{i \in I} \bigotimes_{k=1}^{n} \mathrm{Sym}_{\lambda_i^{(k)}} \tag{2.18}$$

and write a_k for the projection $\Pi \to \mathbb{A}_{\lambda^{(k)}} \subseteq \mathbb{A}_{\underline{\lambda}}$. We regard elements of $\mathrm{Sym}_{\lambda_i^{(k)}}$ as symmetric polynomials in a set $Z_i^{(k)}$ of $\lambda_i^{(k)}$ indeterminates. Define disjoint unions

$$Z^{(k)} := \coprod_{i \in I} Z_i^{(k)}, \qquad Z_i := \coprod_{k \in [1,n]} Z_i^{(k)} \qquad Z := \coprod_{k \in [1,n]} Z^{(k)}. \tag{2.19}$$

The algebra $\mathbb{A}_{\underline{\lambda}}$ is graded local ring. Let $\Bbbk$ denote the unique simple graded $\mathbb{A}_{\underline{\lambda}}$-module (on which all $z \in Z$ act as zero) and let $\mathbb{K} := \overline{\Bbbk(Z)}$ (the algebraic closure of the field of rational functions in the indeterminates Z). We consider $\mathbb{A}_{\underline{\lambda}}$ as embedded in $\mathbb{K}$.

There is a graded $(U(\mathfrak{g}[t]), \Pi^{\otimes n})$-bimodule structure on $M(\underline{\lambda})$ where $\Pi^{\otimes n}$ acts component-wise (note that the right actions are defined component-wise, not using the coproduct δ). This induces a graded $(U(\mathfrak{g}[t]), \mathbb{A}_{\underline{\lambda}})$-bimodule structure on $\mathbb{W}(\underline{\lambda})$ from which we obtain $W(\underline{\lambda}) = \mathbb{W}(\underline{\lambda}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \Bbbk$.

Chapter 3

Categorified quantum groups

In this chapter we recall the definition of the categorified quantum group $\dot{\mathcal{U}} = \dot{\mathcal{U}}_{\Bbbk}(\mathfrak{g})$. We state an isomorphism between the tensor product $\Pi = \bigotimes_{i \in I} \mathrm{Sym}$ of symmetric functions and bubbles in $\dot{\mathcal{U}}$ and formulate bubble slides in $\dot{\mathcal{U}}$. Finally we recall the notion of a (graded) 2-representation of $\dot{\mathcal{U}}$ and the definition of the deformed and undeformed cyclotomic quotients of $\dot{\mathcal{U}}$, denoted $\check{\mathcal{U}}^{\lambda} = \check{\mathcal{U}}^{\lambda}_{\Bbbk}(\mathfrak{g})$ and $\mathcal{U}^{\lambda} = \mathcal{U}^{\lambda}_{\Bbbk}(\mathfrak{g})$ respectively.

3.1 Definition

We use the "cyclic" formulation of the categorified quantum group defined in [BHLW16], where it is denoted $\mathcal{U}_Q^{cyc}(\mathfrak{g})$. It is shown in Theorem 2.1 in loc. cit. that $\mathcal{U}_Q^{cyc}(\mathfrak{g})$ is equivalent to the 2-category defined in [CL15].

Note that we read our diagrams from *right to left* and bottom to top following [BHLW16], [CL15], [KL10], and [Lau10]. In contrast, Webster ([Web15], [Web16], [Web17]) reads his diagrams from left to right.

Choice of parameters

The definition of $\dot{\mathcal{U}}$ depends on two additional pieces of information: a *choice of scalars* $t_{ij} \in \Bbbk$ for $i, j \in I$ and a choice of *bubble parameters* $c_{i,\lambda} \in \Bbbk$ for $i \in I$ and $\lambda \in X$.

Recall from §2.1 that we have fixed an orientation on the graph Γ associated the Cartan datum. For the choice of scalars we set

$$t_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } \langle i, j \rangle = 0; \\ 1 & \text{if there is an arrow } j \to i; \\ -1 & \text{if there is an arrow } i \to j. \end{cases} \tag{3.1}$$

In [Web16] this is called the *geometric* choice of scalars.

For the bubble parameters we allow any choice of $c_{i,\lambda} \in \Bbbk \setminus \{0\}$ for $i \in I$ and $\lambda \in X$ consistent with the conditions

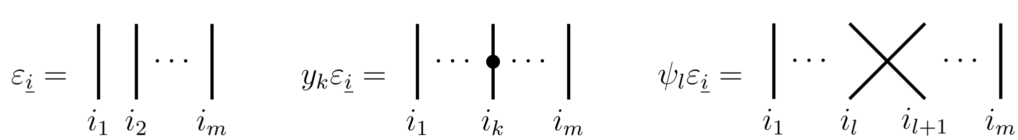
$$c_{i,\lambda+\alpha_j} = t_{ij} c_{i,\lambda} \tag{3.2}$$

for $i, j \in I$ and $\lambda \in X$.

KLR algebras

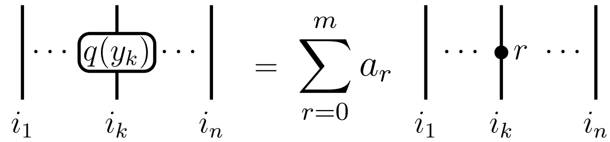
For $m \in \mathbb{N}$, the Khovanov-Lauda-Rouquier algebra, or *KLR algebra*, R_m is a $\Bbbk$-algebra with generators $\{\varepsilon_{\underline{i}} \mid \underline{i} \in I^m\}$, $\{y_k \mid k \in [1, m]\}$, and $\{\psi_l \mid l \in [1, m-1]\}$.

We will often represent elements of R_m diagrammatically: write



The product $a \cdot b$ is represented by placing the diagram for a above that for b and attempting to connect the strings. If the labels do not match then we get zero (i.e. the $\varepsilon_{\underline{i}}$ are mutually orthogonal).

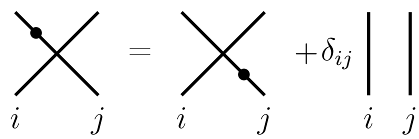
If $q(y) = \sum_{r=0}^{m} a_r y^r \in \mathbb{k}[y]$ is a polynomial, we will denote the element $q(y_k)\varepsilon_{\underline{i}}$ diagrammatically by



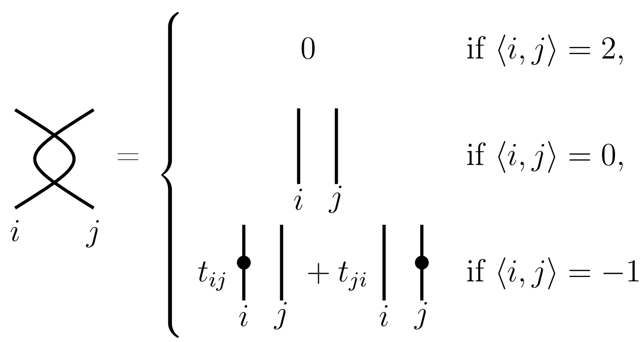
(3.3)

where r next to a dot indicates r dots.

Isotopic diagrams are equal and subject to well-known local relations. In particular:



(3.4)



(3.5)

For the remaining relations the reader is referred to [CL15, (2.8)-(2.14)] (we have $r_i = 1$, $d_{i,j} = 1$, and $s_{ij}^{pq} = 0$ for all i, j, p, q).

Definition of $\mathcal{U}$

We define a graded $\mathbb{k}$-linear 2-category $\mathcal{U} = \mathcal{U}_{\mathbb{k}}(\mathfrak{g})$ (so morphism spaces $\mathcal{U}(\lambda, \mu)$ in $\mathcal{U}$ are graded $\mathbb{k}$-linear categories). Objects in $\mathcal{U}$ are weights $\lambda \in X$. The 1-morphisms $\lambda \to \mu$ are formal direct sums of grading shifts of symbols $1_\mu \mathcal{E}_{\underline{i}} 1_\lambda$, where $\underline{i} = (i_1, \ldots, i_m) \in (\pm I)^m$ for some $m \in \mathbb{N}$ such that

$$\lambda + \sum_{j=1}^{m} \alpha_{i_j} = \mu, \tag{3.6}$$

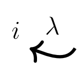
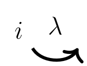
(recall that we write $\alpha_{-i} = -\alpha_i$). Since μ is uniquely determined we often drop it from our notation and write $\mathcal{E}_{\underline{i}} 1_\lambda$ for $1_\mu \mathcal{E}_{\underline{i}} 1_\lambda$.

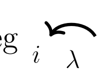
The 2-morphisms in $\mathcal{U}$ are $\Bbbk$-linear combinations of (grading shifts of) *Khovanov-Lauda (KL) diagrams.* A KL diagram consists of finitely many oriented black strings in $\mathbb{R} \times [0,1]$, labelled by elements of I and decorated with finitely many dots, with the regions between strings labelled by weights. Diagrams have no triple points or tangencies and any open end of a string must meet one of the lines $y = 0$ or $y = 1$ at a distinct point from all other strings. The labelling of regions must be consistent with the local rules below:

(3.7)

Since the labelling of all regions is uniquely determined by that of a single one, we will often only label one region.

Take a KL diagram. Let λ and μ be the weights of the right- and left-most regions respectively. Reading along the bottom ($y = 0$) of the diagram yields a signed sequence $\underline{i} = (i_1, \ldots, i_m) \in (\pm I)^m$ where $|i_k|$ is the label of the kth string from the left, and $i_k \in +I$ (resp. $-I$) if the string is oriented upward (resp. downward). Similarly, reading along the top ($y = 1$) of the diagram yields a signed sequence $\underline{j}$. We assign the diagram a degree d by taking the sum of the degrees of the elementary diagrams of which it is composed: a dot has degree 2, a crossing between and i-string and a j-string has degree $-\langle i, j\rangle$, and cups and caps have the following degrees:

deg  $= \langle i, \lambda\rangle - 1$ deg  $= -\langle i, \lambda\rangle - 1$ (3.8)

deg  $= \langle i, \lambda\rangle - 1$ deg  $= -\langle i, \lambda\rangle - 1.$ (3.9)

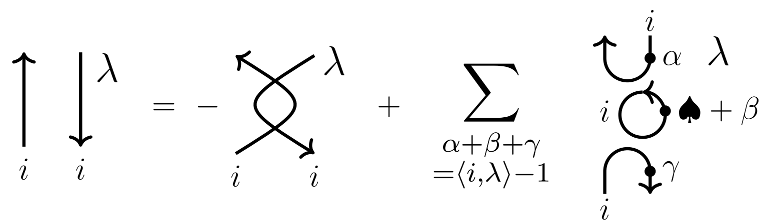
Then this diagram is a 2-morphism $1_\mu \mathcal{E}_{\underline{i}} 1_\lambda \Rightarrow 1_\mu \mathcal{E}_{\underline{j}} 1_\lambda \langle d\rangle$.

The *horizontal composition* $D \circ D'$ of KL diagrams D and D' is given by placing D to the left of D' if the weights of the corresponding regions match. Their *vertical composition* $D \cdot D'$ is given by placing D on top of D' and connecting strings if the corresponding 1-

morphisms match. These definitions extend to all 2-morphisms. So "$\cdot$" denotes composition in the category $\mathcal{U}(\lambda, \mu)$ and "$\circ$" denotes composition

$$\mathcal{U}(\nu, \mu) \times \mathcal{U}(\lambda, \mu) \longrightarrow \mathcal{U}(\lambda, \nu). \tag{3.10}$$

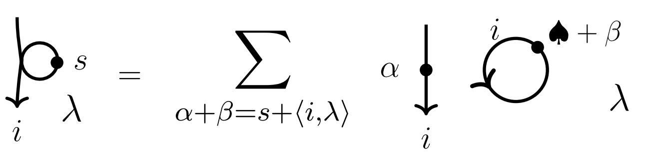
The 2-morphisms in $\mathcal{U}$ are subject to the additional local relations listed in [BHLW16, Definition 1.3]. In particular, if we interpret diagrams in the KLR algebra as having all strings oriented *upward* then 2-morphisms locally satisfy the KLR relations, bubbles are subject to the relations described in §3.2, and we have the following for any $i \in I$ and $\lambda \in X$:



(3.11)

(see 3.2) for an explanation of the ♠-notation).

The cyclic duality in $\mathcal{U}$ (see [BHLW16, §1.2]) means that rotating any relation by 180 degrees yields another valid relation in $\mathcal{U}$. In particular, rotating the second relation in [BHLW16, Proposition 3.3] yields the following for any $i \in I$, $\lambda \in X$, and $s \geq 0$:



(3.12)

Categorified quantum group

The categorified quantum group $\dot{\mathcal{U}} = \dot{\mathcal{U}}_{\Bbbk}(\mathfrak{g})$ is the idempotent completion of $\mathcal{U}$. More precisely, the morphism space $\dot{\mathcal{U}}(\lambda, \mu)$ is the idempotent completion of $\mathcal{U}(\lambda, \mu)$. It is a graded $\Bbbk$-linear 2-category. The starred variants $\mathcal{U}^*$ and $\dot{\mathcal{U}}^*$ are defined by adding stars to the morphisms categories.

The split Grothendieck group $K_0(\dot{\mathcal{U}})$ of $\dot{\mathcal{U}}$ is a locally unital $\mathbb{Z}$-algebra:

$$K_0(\dot{\mathcal{U}}) = \bigoplus_{\lambda,\mu \in \mathrm{Ob}(\mathcal{U})} 1_\mu K_0(\dot{\mathcal{U}}(\lambda,\mu))1_\lambda \tag{3.13}$$

with multiplication induced by horizontal composition. There is an isomorphism of locally unital $\mathbb{Z}$-algebras

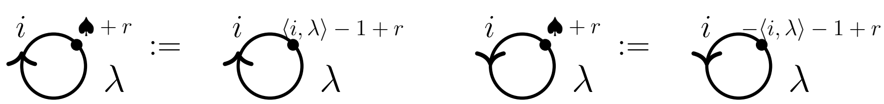
$$\dot{U}_q^{\mathbb{Z}} \longrightarrow K_0(\dot{\mathcal{U}}). \tag{3.14}$$

where $\dot{U}_q^{\mathbb{Z}}$ is the integral idempotent quantum group from §2.1.

This map exists and is surjective by [KL10, Theorem 1.1] and injectivity is equivalent to non-degeneracy of the graphical calculus by Theorem 1.2 in loc. cit. In finite-type non-degeneracy follows from the decategorification of cyclotomic quotients proved by [KK12] and [Web17] (c.f. §3.3).

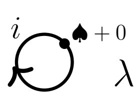
3.2 Bubbles and symmetric functions

We will use the following shorthand for bubbles:

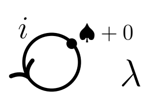


(3.15)

where $\lambda \in X$, $i \in I$, and $r \in \mathbb{Z}$. When $r > |\langle i,\lambda\rangle|$ these bubbles have degree $2r$. When $r \leq |\langle i,\lambda\rangle|$ the number of dots is negative and so this doesn't make sense as a 2-morphism. We resolve this by adding this "fake bubble" as a new generator of degree $2r$ and impose the following relations: bubbles of negative degree are zero, degree zero bubbles satisfy

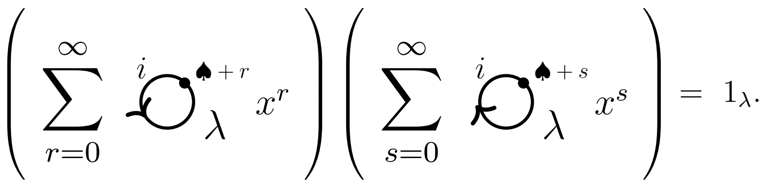


$= c_{i,\lambda}1_\lambda,$



$= c_{i,\lambda}^{-1}1_\lambda,$ (3.16)

and higher degree bubbles satisfy the equations arising from the homogeneous terms in x of the infinite Grassmannian equation:



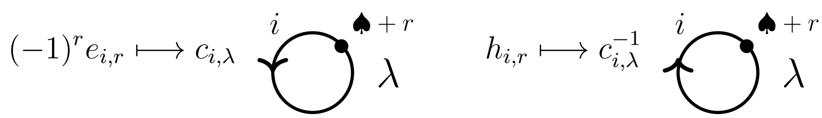
(3.17)

From this it follows that all fake bubbles can actually be written in terms of real bubbles.

Recall the graded algebra $\Pi = \bigotimes_{i\in I} \mathrm{Sym}$ from §2.2. For any $\lambda \in X$, there is a homomorphism of graded algebras

$$b_\lambda : \Pi \longrightarrow \dot{\mathcal{U}}^*(1_\lambda, 1_\lambda) \tag{3.18}$$

sending



(3.19)

By non-degeneracy, this is an isomorphism.

Remark 3.2.1. In [Lau12, §3.4.4] and [CL15, §5.1] the authors identify elementary symmetric fuctions with *clockwise bubbles*. Our homomorphism differs from theirs by the automorphism of Π interchanging $(-1)^r e_{i,r}$ and $h_{i,r}$ and fixing $p_{i,r}$. We believe our choice is more natural given the relationship between bubbles and the deformed cyclotomic relation (3.26).

Remark 3.2.2. Observe that we now have isomorphisms relating symmetric functions, bubbles, and the Cartan subalgebra of $\mathfrak{g}$. More precisely, for any $\lambda \in X$ there are isomorphisms of graded algebras:

$$\begin{array}{ccc} & \Pi & \\ \swarrow & & \searrow \\ 1_\lambda U(\mathfrak{h}[t])1_\lambda & \longrightarrow & \dot{\mathcal{U}}^*(1_\lambda, 1_\lambda) \end{array} \tag{3.20}$$

where the left diagonal map was defined in 2.2. The horizontal map descends to the restriction of the isomorphism $\dot{U}(\mathfrak{g}[t]) \cong \mathrm{Tr}(\mathcal{U}^*)$ in the trace.

It will be convenient for us to consider the image under b_λ of the generating functions $e_i(x)$, $h_i(x)$, and $p_i(x)$ from §2.2. For example, the relation (3.17) is just the image under b_λ

of $h_i(x)e_i(-x) = 1$.

Generating functions also greatly simplify the statement of bubble slide equations. To save space, we do not state these diagrammatically. Instead for $i \in \pm I$ we use $1_{\mathcal{E}_i}$ to denote a string labelled by $|i|$, oriented upward if $i \in +I$ and downward if $i \in -I$, and let y denote a dot on that string. Recall that "$\circ$" denotes horizontal composition in $\dot{\mathcal{U}}$.

The reader is invited to compare these with the equations (2.7).

Lemma 3.2.3. *Take $i \in \pm I$, $j \in I$, and $\lambda \in X$. Then*

$$\begin{aligned} b_{\lambda+\alpha_i}(e_j(x)) \circ 1_{\mathcal{E}_i} &= (1+yx)^{\langle j,i\rangle} \circ b_\lambda(e_j(x)), \\ b_{\lambda+\alpha_i}(h_j(x)) \circ 1_{\mathcal{E}_i} &= (1-yx)^{-\langle j,i\rangle} \circ b_\lambda(h_j(x)), \\ b_{\lambda+\alpha_i}(p_j(x)) \circ 1_{\mathcal{E}_i} &= 1_{\mathcal{E}_i} \circ b_\lambda(p_j(x)) + \langle j,i\rangle \frac{yx}{1-yx}. \end{aligned} \tag{3.21}$$

Proof. The first two follow directly from [BHLW16, §3.2]. The third follows from these, the coproduct δ, and (2.7). □

3.3 2-representations and cyclotomic quotients

In this section we recall the definition of a 2-representation of $\dot{\mathcal{U}}$ and 2-natural transformations between them. We also recall the undeformed and deformed cyclotomic quotients of $\dot{\mathcal{U}}$ - 2-representations of $\dot{\mathcal{U}}$ that categorify integrable highest weight modules over the quantum group.

2-representations

A 2-representation of $\dot{\mathcal{U}}$ on a category $\mathcal{M}$ consists of a weight decomposition $\mathcal{M} = \bigoplus_{\lambda\in X} \mathcal{M}(\lambda)$ into subcategories and compatible functors from $\dot{\mathcal{U}}(\lambda,\mu)$ to the category of functors $\mathcal{M}(\lambda) \to \mathcal{M}(\mu)$. A 2-representation is graded if $\mathcal{M}$ is graded and all the functors are graded. This induces a 2-representation of $\dot{\mathcal{U}}^*$ on $\mathcal{M}^*$.

A graded 2-representation of $\dot{\mathcal{U}}$ on $\mathcal{M}$ induces a $K_0(\dot{\mathcal{U}}) \cong \dot{U}_q^{\mathbb{Z}}$-module structure on

$$K_0(\mathcal{M}) = \bigoplus_{\lambda \in \mathrm{Ob}(\mathcal{U})} 1_\lambda K_0(\mathcal{M}(\lambda)) \tag{3.22}$$

compatible with the action of $\mathbb{Z}[q^{\pm 1}]$.

A 2-natural transformation η between 2-representations on $\mathcal{M}$ and $\mathcal{N}$ consists of functors

$$\eta(\lambda) : \mathcal{M}(\lambda) \longrightarrow \mathcal{N}(\lambda) \tag{3.23}$$

for all $\lambda \in X$ together with compatible natural isomorphisms

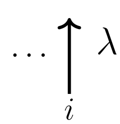
$$\eta(x) : x_{\mathcal{N}} \circ \eta(\lambda) \Rightarrow \eta(\mu) \circ x_{\mathcal{M}} \tag{3.24}$$

for any 1-morphism $x \in \mathrm{Ob}(\mathcal{U}(\lambda, \mu))$, where $x_{\mathcal{M}}$ (resp. $x_{\mathcal{N}}$) denotes the functor $\mathcal{M}(\lambda) \to \mathcal{M}(\mu)$ (resp. $\mathcal{N}(\lambda) \to \mathcal{N}(\mu)$) associated to x. We call η a 2-natural isomorphism if the $\eta(\lambda)$ are all equivalences.

If $\mathcal{M}$, $\mathcal{N}$ and all $\eta(\lambda)$ and $\eta(x)$ are graded we call η a graded 2-representation. This induces a 2-representation from $\mathcal{M}^*$ to $\mathcal{N}^*$. A graded 2-representation of $\dot{\mathcal{U}}$ induces $\dot{U}_q^{\mathbb{Z}}$-module homomorphism $K_0(\mathcal{M}) \to K_0(\mathcal{N})$. These are isomorphisms if η is a 2-natural isomorphism.

Cyclotomic quotients

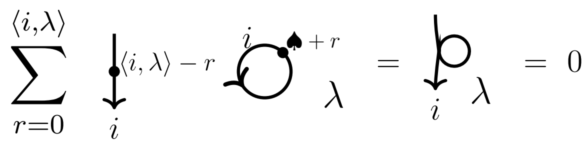
If $\lambda \in X^+$, the corresponding *deformed cyclotomic quotient* $\check{\mathcal{U}}^\lambda = \check{\mathcal{U}}_{\Bbbk}^\lambda(\mathfrak{g})$ of $\dot{\mathcal{U}}$ is the graded category obtained from $\bigoplus_{\mu \in X} \dot{\mathcal{U}}(\lambda, \mu)$ by imposing the following global relation for any $i \in I$:



$$= 0; \tag{3.25}$$

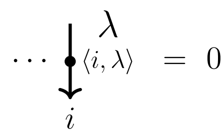
that is, any diagram with an upward string at the far right is equal to zero. This preserves the direct sum decomposition $\check{\mathcal{U}}^\lambda = \bigoplus_{\mu \in X} \check{\mathcal{U}}^\lambda(\mu)$ according to the left-most weight, and horizontal composition in $\dot{\mathcal{U}}$ induces an action of $\dot{\mathcal{U}}$ on $\check{\mathcal{U}}^\lambda$ by placing a diagram on the left. So $\check{\mathcal{U}}^\lambda$ is a graded 2-representation of $\dot{\mathcal{U}}$.

Taking $s = 0$ in (3.12) yields the deformed cyclotomic relation in $\check{\mathcal{U}}^\lambda$:



(3.26)

The *undeformed cyclotomic quotient* $\mathcal{U}^\lambda = \mathcal{U}^\lambda_{\Bbbk}(\mathfrak{g})$ of $\dot{\mathcal{U}}$ is obtained from $\check{\mathcal{U}}^\lambda$ by setting any diagram with a positive degree bubble at the far right equal to zero. In $\mathcal{U}^\lambda$ we have the undeformed cyclotomic relation:



(3.27)

at the far right of any diagram. The weight decomposition and action of $\dot{\mathcal{U}}$ on $\check{\mathcal{U}}^\lambda$ are preserved, so $\mathcal{U}^\lambda$ is also a graded 2-representation of $\dot{\mathcal{U}}$.

It was proved independently by [Web17, Corollary 3.22] and [KK12, Theorem 6.2] that for any $\lambda \in X^+$ there are isomorphisms of $\dot{U}_q^{\mathbb{Z}}$-modules:

$$K_0(\check{\mathcal{U}}^\lambda) \cong K_0(\mathcal{U}^\lambda) \cong V_q^{\mathbb{Z}}(\lambda), \tag{3.28}$$

where $V_q^{\mathbb{Z}}(\lambda)$ is the integrable highest weight $\dot{U}_q^{\mathbb{Z}}$-module from §2.1.

Chapter 4

Bubble slides in tensor product algebras

Recall from §2.3 that we have fixed $n \in \mathbb{N}$ and a sequence $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ of dominant weights and that if $k \in [1, n]$ and $i \in I$ then $\lambda_i^{(k)} = \langle i, \lambda^{(k)} \rangle$. The corresponding graded algebra $\mathbb{A}_{\underline{\lambda}}$ consists of certain symmetric polynomials in a set Z of indeterminates.

In this chapter we recall Webster's categorification $\mathcal{X}^{\underline{\lambda}}$ of a tensor product $V_q^{\mathbb{Z}}(\underline{\lambda})$ of integral highest weight modules for $\dot{U}_q^{\mathbb{Z}}$ and its deformation $\check{\mathcal{X}}^{\underline{\lambda}}$, and prove equations for passing bubbles through red strings in $\check{\mathcal{X}}^{\underline{\lambda}}$ analogous to the bubble slide equations in $\dot{\mathcal{U}}$ (see §3.2). Note that to match with the conventions of [BHLW16] our categories differ from Webster's by a reflection in a vertical line. Also, since we work with a fixed sequence $\underline{\lambda}$ of dominant weights we will label red strings in $\mathcal{X}^{\underline{\lambda}}$ by (k) with $k \in [1, n]$ rather than by the actual weights $\lambda^{(k)}$.

The category $\mathcal{X}^{\underline{\lambda}}$ is equivalent to the category of graded projective modules over the tensor product algebra $T^{\underline{\lambda}}$ from [Web17], but it is more convenient for us to define $\mathcal{X}^{\underline{\lambda}}$ by generators and relations as in [Web15]. Morphisms spaces in $\mathcal{X}^{\underline{\lambda}}$ are spanned by string diagrams containing red strings which separate tensor factors and there is a graded 2-representation of $\dot{\mathcal{U}}$ on $\mathcal{X}^{\underline{\lambda}}$ by placing diagrams on the left and composing.

The category $\check{\mathcal{X}}^{\underline{\lambda}}$ is equivalent to the category of graded projective modules over the algebra in [Web12, §3.2]. It is obtained by deforming the defining relations for $\mathcal{X}^{\underline{\lambda}}$ over $\mathbb{A}_{\underline{\lambda}}$ such that setting all $z \in Z$ equal to zero recovers $\mathcal{X}^{\underline{\lambda}}$. The effect of this is that dots in $\check{\mathcal{X}}^{\underline{\lambda}}$, rather than being nilpotent, can have any $z \in Z$ as a generalized eigenvalue (see Chapter 5). The relationship between $\mathcal{X}^{\underline{\lambda}}$ and $\check{\mathcal{X}}^{\underline{\lambda}}$ is analogous to that between the undeformed and deformed $\check{\mathcal{U}}^{\lambda}$ cyclotomic quotients $\mathcal{U}^{\lambda}$ and $\check{\mathcal{U}}^{\lambda}$ of $\dot{\mathcal{U}}$. There is a graded 2-representation of $\dot{\mathcal{U}}$ on $\check{\mathcal{X}}^{\underline{\lambda}}$ and morphism spaces in the starred category $\check{\mathcal{X}}^{\underline{\lambda},*}$ are graded right $\mathbb{A}_{\underline{\lambda}}$-modules.

In §4.2 we prove new equations for passing bubbles through red strings in $\check{\mathcal{X}}^{\underline{\lambda}}$. We state these in terms of the coproduct δ on $U(\mathfrak{h}[t])$ which shows that the actions of $\mathbb{A}_{\underline{\lambda}}$ on $\mathbb{W}(\underline{\lambda})$ and $\check{\mathcal{X}}^{\underline{\lambda},*}$ are compatible. We also state the bubble slides in terms of the generating functions $e_i(x)$, $h_i(x)$, and $p_i(x)$ for symmetric functions. This allows us to study the spectrum of a dot in $\mathcal{X}^{\underline{\lambda},*}$ in Chapter 5.

4.1 Tensor product algebras

An auxiliary category

We begin by defining an auxiliary graded $\Bbbk$-linear category $\widetilde{\mathcal{X}}^{\underline{\lambda}}$ from which both $\mathcal{X}^{\underline{\lambda}}$ and $\check{\mathcal{X}}^{\underline{\lambda}}$ can be obtained. We will not need $\widetilde{\mathcal{X}}^{\underline{\lambda}}$ outside this section.

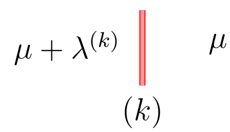
Objects in $\widetilde{\mathcal{X}}^{\underline{\lambda}}$ are formal direct sums of grading shifts of *Stendhal pairs* $(\underline{i}, \kappa)$, where $\underline{i} \in (\pm I)^m$ for some $m \in \mathbb{N}$ and κ is a weakly increasing function from $[1, 2, \ldots, n+1]$ to $[0, 1, \ldots, m]$ with $\kappa(n+1) = m$ (the equivalent objects in [Web15] and [Web17] are called tricolore quadruples and double Stendhal triples respectively).

Morphisms in $\widetilde{\mathcal{X}}^{\underline{\lambda}}$ are $\Bbbk$-linear combinations of grading shifts of *Stendhal diagrams* (double Stendhal diagrams in [Web17]). A Stendhal diagram consists of finitely many strings in $\mathbb{R} \times [0, 1]$. Each string is either:

- coloured black, given an orientation, labelled with an element $i \in I$, and decorated with finitely many dots; or
- coloured red and labelled with (k) for $k \in [1, n]$.

Diagrams have no triple points or tangencies and any open end of a string must meet one of the lines $y = 0$ or $y = 1$ at a distinct point from all other strings. Red strings have no critical points (that is, they never turn back on themselves) and two red strings cannot cross. Reading the labels of red strings from left to right as the intersect a horizontal line $y = c$ must give the sequence $(n), (n-1), \ldots, (1)$.

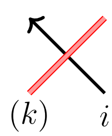
Regions between strings are labelled by weights. The right-most region is labelled by 0 and the labels of the other regions are determined by the consistency rules in (3.7) and the additional condition:

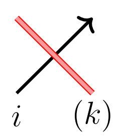


(4.1)

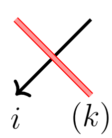
Since the labelling of regions is uniquely determined we will often not record it.

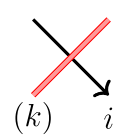
Take a Stendhal diagram. As with 2-morphisms in $\mathcal{U}$, recording the labels and orientations of the black strings along the bottom of the diagram yields a signed sequence $\underline{i} = (i_1, \ldots, i_m) \in (\pm I)^m$. For $k \in [1, n]$, let $\kappa(k) \in \mathbb{N}$ be the number of black strings to the right of the red (k)-string reading along the line $y = 0$, and let $\kappa(n+1) = m$. This defines a weakly increasing function from $[1, \ldots, n+1]$ to $[0, \ldots, m]$, so $(\underline{i}, \kappa)$ is a Stendhal pair. Similarly, reading the top of the diagram we get an Stendhal pair $(\underline{i}', \kappa')$. We assign the diagram a degree d by taking the sum of the degrees of elementary diagrams: in addition to those set in $\mathcal{U}$ such as ((3.9)) we set

deg 

= deg 

$= 0,$ (4.2)

deg 

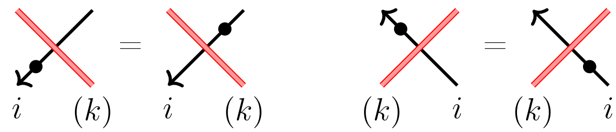
= deg 

$= \lambda_i^{(k)}.$ (4.3)

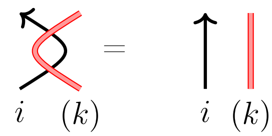
This Stendhal diagram is a morphism from $(\underline{i}, \kappa)$ to $(\underline{i}', \kappa')\langle d \rangle$. Composition of morphisms is induced by vertical composition of diagrams as in §3.1.

We sometimes conflate a Stendhal pair S and the identity 1_S on S and refer to, for example, the red string in S labelled by (k).

Morphisms in $\widetilde{\mathcal{X}}^{\underline{\lambda}}$ are subject to local relations: black strings satisfy the relations of $\mathcal{U}$ (see §3.1) and for any $i \in I$ and $k \in [1, n]$ the following hold as well as their reflections in a vertical line:



(4.4)



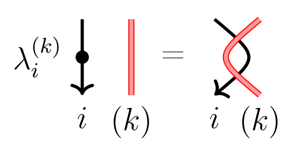
(4.5)

We also impose the relations [Web17, (4.4a)-(4.4c)] and their reflections in a vertical line for any labelling of red and black strings (the reader should ignore the orientation on red strings). Finally, we also set to zero any *violated* diagrams; that is, diagrams which at some horizontal slice $y = c$ have a black string to the right of all red strings.

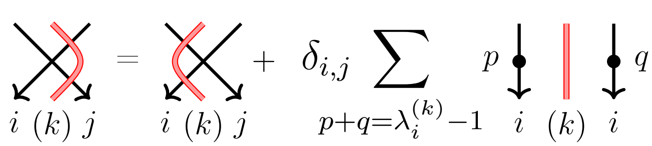
Tensor product algebras

The following category was introduced in [Web17, §4], where it is presented as the category of graded projective modules over an algebra:

Definition 4.1.1. Let $\mathcal{X}^{\underline{\lambda}}$ be the idempotent completion of the category obtained from $\widetilde{\mathcal{X}}^{\underline{\lambda}}$ by imposing the following additional local relations as well as the reflection of (4.6) in a vertical line:



(4.6)



(4.7)

It is a graded $\Bbbk$-linear category. There is a weight decomposition $\mathcal{X}^{\underline{\lambda}} = \bigoplus \mathcal{X}^{\underline{\lambda}}(\mu)$ according to the left-most weight $\mu \in X$ of a diagram, and placing diagrams in $\dot{\mathcal{U}}$ on the left of those in

$\mathcal{X}^{\underline{\lambda}}$ defines a graded 2-representation of $\dot{\mathcal{U}}$ on $\mathcal{X}^{\underline{\lambda}}$.

The 2-representation of $\dot{\mathcal{U}}$ on $\mathcal{X}^{\underline{\lambda}}$ induces a $\dot{U}_q^{\mathbb{Z}}$-action on the split Grothendieck group $K_0(\mathcal{X}^{\underline{\lambda}})$. By [Web17, Theorems 4.38] there is an isomorphism of $\dot{U}_q^{\mathbb{Z}}$-modules

$$K_0(\mathcal{X}^{\underline{\lambda}}) \longrightarrow V_q^{\mathbb{Z}}(\underline{\lambda}), \tag{4.8}$$

where $V_q^{\mathbb{Z}}(\underline{\lambda})$ is the product of integrable highest weight modules from §2.1.

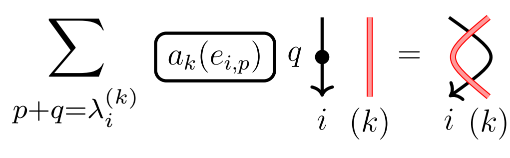
Deformed tensor product algebras

Recall the graded algebras $\Pi = \bigotimes_{i\in I}$ and $\mathbb{A}_{\underline{\lambda}}$ and the projections $a_k : \Pi \to \mathbb{A}_{\lambda^{(k)}} \subseteq \mathbb{A}_{\underline{\lambda}}$ from §4.1. Let $\widetilde{\mathcal{X}}^{\underline{\lambda}} \otimes \mathbb{A}_{\underline{\lambda}}$ denote the extension of scalars of $\widetilde{\mathcal{X}}^{\underline{\lambda}}$ to $\mathbb{A}_{\underline{\lambda}}$. This is a graded category and $(\widetilde{\mathcal{X}}^{\underline{\lambda}} \otimes \mathbb{A}_{\underline{\lambda}})^*$ is enriched over graded right $\mathbb{A}_{\underline{\lambda}}$-modules.

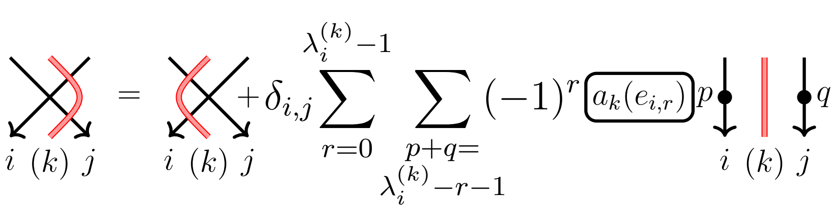
Morphisms in $\widetilde{\mathcal{X}}^{\underline{\lambda}} \otimes \mathbb{A}_{\underline{\lambda}}$ are $\mathbb{A}_{\underline{\lambda}}$-linear combinations of Stendhal diagrams. If D is a Stendhal diagram and $a \in \mathbb{A}_{\underline{\lambda}}$ then we will represent the morphism $D \cdot a$ in $\widetilde{\mathcal{X}}^{\underline{\lambda}} \otimes \mathbb{A}_{\underline{\lambda}}$ diagrammatically by placing a in a box somewhere in D. The placement of the box does not change the morphism.

The following was introduced in [Web12, §3.2] where it was presented as a category of graded projective modules over an algebra:

Definition 4.1.2. Let $\check{\mathcal{X}}^{\underline{\lambda}}$ be the idempotent completion of the category obtained from $\widetilde{\mathcal{X}}^{\underline{\lambda}} \otimes \mathbb{A}_{\underline{\lambda}}$ by imposing the following additional local relations as well as the reflection of (4.9) in a vertical line:



(4.9)



(4.10)

This a graded $\Bbbk$-linear category and $\check{\mathcal{X}}^{\underline{\lambda},*}$ is enriched over graded right $\mathbb{A}_{\underline{\lambda}}$-modules.

Observe that relations (4.9)-(4.10) reduce to the relations (4.6)-(4.7) in the undeformed category $\mathcal{X}^{\underline{\lambda}}$ if we specialize $z = 0$ for all $z \in Z$. So if $\Bbbk$ denotes the unique simple graded $\mathbb{A}_{\underline{\lambda}}$-module (on which all $z \in Z$ act as zero) then tensoring morphism spaces over $\mathbb{A}_{\underline{\lambda}}$ with $\Bbbk$ gives a functor from $\check{\mathcal{X}}^{\underline{\lambda},*}$ to $\mathcal{X}^{\underline{\lambda},*}$. Restricting to degree zero morphisms gives a functor from $\check{\mathcal{X}}^{\underline{\lambda}}$ to $\mathcal{X}^{\underline{\lambda}}$.

As with $\mathcal{X}^{\underline{\lambda}}$, there is a weight decomposition $\check{\mathcal{X}}^{\underline{\lambda}} = \bigoplus \check{\mathcal{X}}^{\underline{\lambda}}(\mu)$ according to the left-most weight $\mu \in X$ of a diagram and placing diagrams in $\dot{\mathcal{U}}$ on the left gives a graded 2-representation of $\dot{\mathcal{U}}$ on $\check{\mathcal{X}}^{\underline{\lambda}}$.

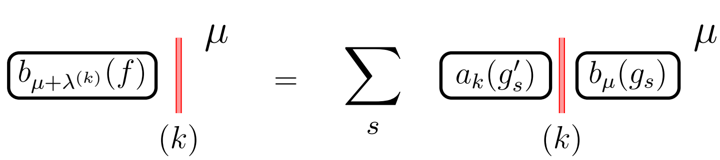
In Chapter 6 we will show that the split Grothendieck group of $\check{\mathcal{X}}^{\underline{\lambda}}$ is isomorphic to $V_q^{\mathbb{Z}}(\underline{\lambda})$ as a $\dot{U}_q^{\mathbb{Z}}$-module.

4.2 Bubble slides

Recall the algebra $\Pi = \bigotimes_{i \in I} \mathrm{Sym}$ from §2.2. We can interpret symmetric functions in $\check{\mathcal{X}}^{\underline{\lambda}}$ either through the isomorphism with bubbles $b_\mu : \Pi \to \mathcal{U}^*(1_\mu, 1_\mu)$ from §3.2 or through the projections $a_k : \Pi \to \mathbb{A}_{\lambda^{(k)}} \subseteq \mathbb{A}_{\underline{\lambda}}$. Our description of how the two of these interact takes the form of bubble slides through red strings and mirrors the relationship between the actions of $U(\mathfrak{h}[t])$ and $\mathbb{A}_{\underline{\lambda}}$ on the tensor product $\mathbb{W}(\underline{\lambda})$ of global Weyl modules.

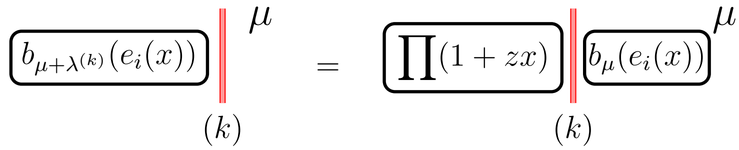
Recall the coproduct $\delta : \Pi \to \Pi \otimes \Pi$ from §2.2.

Proposition 4.2.1. *Take $\mu \in X$ and $k \in [1, n]$. If $f \in \Pi$ and $\delta(f) = \sum_s g_s \otimes g'_s$ then*

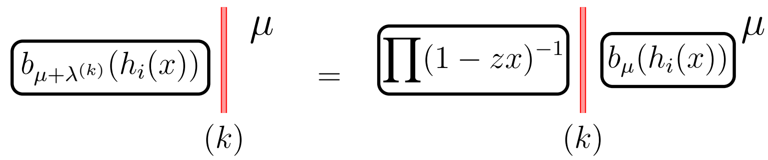


(4.11)

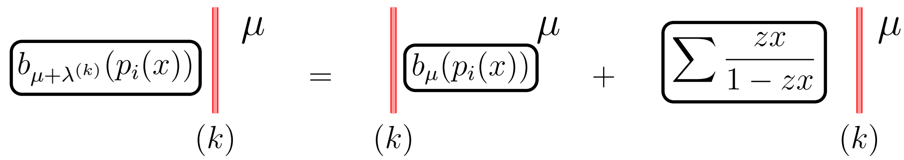
Take $i \in I$ and recall the generating functions $e_i(x)$, $h_i(x)$, and $p_i(x)$ from §2.2 and the set $Z_i^{(k)}$ of indeterminates from §2.3. In Lemma 3.2.3 we stated bubble slides in $\dot{\mathcal{U}}^*$ for $e_i(x)$, $h_i(x)$, and $p_i(x)$. The proposition yields the analogous equations in $\check{\mathcal{X}}^{\underline{\lambda},*}$ (the products and sums below are taken over all $z \in Z_i^{(k)}$):



(4.12)



(4.13)

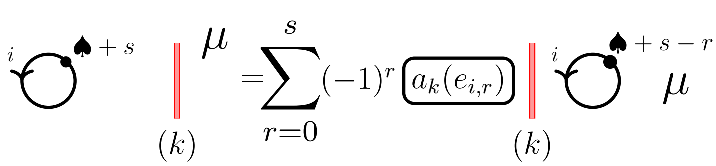


(4.14)

The idea behind the proof of the proposition is simple, but the reality is a little fiddly. Through explicit diagrammatic calculations based on the deformed relations in Definition 4.1.2, we will show that (4.11) holds for all f in a generating set for Π. The difficulty is that a priori the deformed relations only hold when the bubbles are *real*, so the generating set we choose depends on the weight μ.

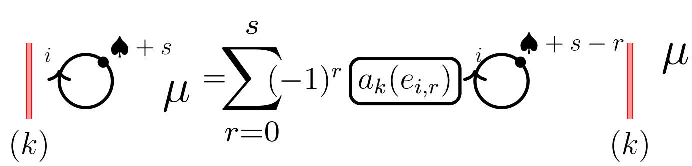
Lemma 4.2.2. *Take $\mu \in X$, $i \in I$, $k \in [1, n]$, and $s \geq 1$.*

(i) If $s > \lambda_i^{(k)} + \langle i, \mu \rangle$ then



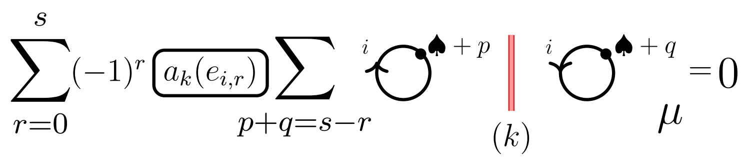
(4.15)

(ii) If $s > -\langle i, \mu \rangle$ then



(4.16)

(iii) If $-\lambda_i^{(k)} < \langle i, \mu\rangle < 0$ *and* $s < \lambda_i^{(k)}$ *then*



(4.17)

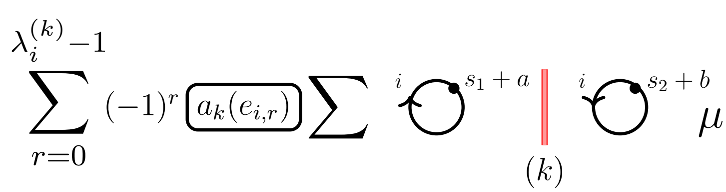
Proof. (i) By the assumption on s, the bubble on the left is real, so we can pull the right edge through the red string and use the mirror image of deformed relation (4.9) in a vertical line to pull the rest of the bubble through. This yields the right hand side of the equation, but with the sum running to $\lambda_i^{(k)}$. The difference between these two sums either involves negative degree bubbles, which are zero, or symmetric polynomials $a_k(e_{i,r})$ with $r > \lambda_i^{(k)}$, which are also zero. So the equation holds.

(ii) This is similar to (i), but we pull the bubble left through the red string.

(iii) This is more involved. Set $s_1 := \max\{0, s + \langle i, \mu\rangle\}$ and $s_2 := \max\{s-1, -\langle i, \mu\rangle\}$. Observe that $s_1 + s_2 = s - 1$ and

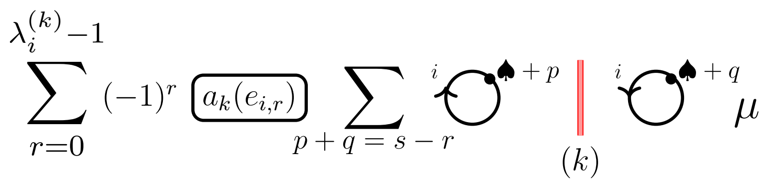
$$0 \leq s_1 < \lambda_i^{(k)} + \langle i, \mu\rangle, \qquad 0 \leq s_2 < -\langle i, \mu\rangle. \tag{4.18}$$

Consider the following diagram:



(4.19)

where the second sum is over all $a, b \geq 0$ with $a + b = \lambda_i^{(k)} - r - 1$. We can rewrite this in “spade notation” as



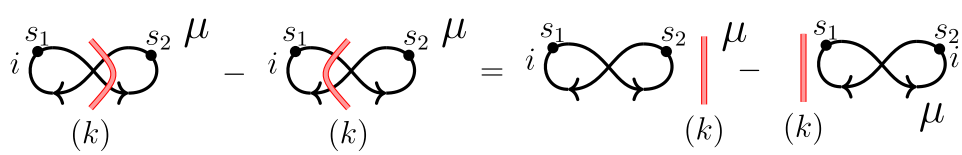
(4.20)

where p and q in the second sum must satisfy

$$p \geq s_1 - \lambda_i^{(k)} - \langle i, \mu \rangle + 1, \qquad q \geq s_2 - \langle i, \mu \rangle + 1. \tag{4.21}$$

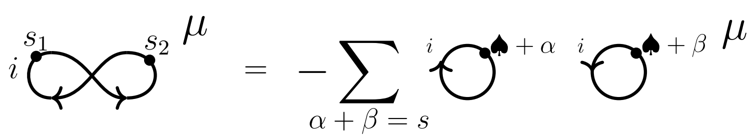
By (4.18) the expressions on the right-hand side of both inequalities are less than or equal to zero and bubbles of negative degree are zero, so we can change the second summation to be over $p, q \geq 0$ without changing the value of the expression. Now the first sum is empty for $s < r < \lambda_i^{(k)}$ so we change the upper limit to s, which yields the left hand side (4.17).

Now we claim that (4.19) is equal to zero. Since $s_1, s_2 \geq 0$, all the bubbles in the diagram are real, so we can apply the deformed relation (4.10) to get



We claim that both of these “infinity” diagrams are zero.

For the rightmost one, consider (3.12) with $m = s_2$ and $\lambda = \mu$. Adding s_1 dots to the free string and closing it with a loop on the left yields



(4.22)

with the sum over $\alpha \geq 0$ and $\beta \geq s_2 + \langle i, \mu \rangle + 1$. Again, since $s_2 + \langle i, \mu \rangle + 1 \leq 0$ and bubbles of negative degree are zero we can just take $\beta \geq 0$. Now this is the homogeneous component of the infinite Grassmannian (3.17) of degree $s > 0$ so is zero.

The calculation for the other “infinity" diagram is similar using the second identity in [BHLW16, Proposition 3.3]. The claim follows. □

Proof of Proposition 4.2.1. We fix $i \in I$ and show the claim for the copy of Sym in Π indexed by i. We will prove (4.11) for all f in a generating set of Sym. Since we have only proved the bubble slides in Lemma 4.2.2 for real bubbles, the generating set we choose depends on $\lambda_i^{(k)}$. There are three cases.

1. If $\langle i, \mu \rangle \leq -\lambda_i^{(k)}$ then (4.15) holds for all $s > 0$. Since

$$\delta(e_{i,s}) = \sum_{p+q=s} e_{i,p} \otimes e_{i,q}, \tag{4.23}$$

   the bubble slides hold for all $f = e_{i,s}$, $s \geq 1$.

2. If $\langle i, \mu \rangle \geq 0$ then (4.16) holds for all $s > 0$. Since

$$h_{i,s} \otimes 1 = \sum_{p+q=s} (-1)^p (1 \otimes e_{i,p}) \delta(h_{i,q}), \tag{4.24}$$

   by induction on s the bubble slides hold for all $f = h_{i,s}$, $s \geq 1$.

3. If $-\lambda_i^{(k)} < \langle i, \mu \rangle < 0$ then (4.17) holds for $0 < s < \lambda_i^{(k)}$ and (4.15) holds for $s \geq \lambda_i^{(k)}$. Since

$$0 = \sum_{p+q+r=s} (-1)^{p+q} (e_{i,p} \otimes e_{i,q}) \delta(h_{i,r}), \tag{4.25}$$

   by induction the bubble slides hold for $f = h_{i,s}$ for $0 < s < \lambda_i^{(k)}$ and they hold for $f = e_{i,s}$ for $s \geq \lambda_i^{(k)}$ as in (1).

□

Chapter 5

Unfurling 2-representations

Recall from §4.1 that $\check{\mathcal{X}}^{\underline{\lambda}}$ is a deformation of the tensor product categorification $\mathcal{X}^{\underline{\lambda}}$ over the algebra $\mathbb{A}_{\underline{\lambda}}$ which consists of certain symmetric polynomials in a set Z of indeterminates and is regarded as a subset of $\mathbb{K} = \overline{\Bbbk(Z)}$ (see §2.3). In this chapter we study $\check{\mathcal{X}}^{\underline{\lambda}}$ at the generic point of the deformation; that is, we study the idempotent completion $\mathcal{G}^{\underline{\lambda}}$ of $\check{\mathcal{X}}^{\underline{\lambda},*} \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K}$.

Define a new symmetric simply-laced Cartan datum with indexing set $\tilde{I} = I \times Z$, weight lattice $\tilde{X} = X \times Z$, and symmetric bilinear form obtained from that on X via

$$((\mu, z), (\mu', z')) = \delta_{z,z'}(\mu, \mu'). \tag{5.1}$$

for $(\mu, z), (\mu', z') \in \tilde{X}$. We call $\tilde{I}$ an *unfurling* of I and we identify the corresponding Lie algebra $\tilde{\mathfrak{g}}$ with a direct sum $\mathfrak{g}^{\oplus Z}$ of copies of $\mathfrak{g}$ indexed by $z \in Z$.

Morphism spaces in $\mathcal{G}^{\underline{\lambda}}$ are (ungraded) $\mathbb{K}$-vector spaces and the deformed relations (4.9)-(4.10) in $\check{\mathcal{X}}^{\underline{\lambda}}$ imply that a dot acting on a black string in a Stendhal pair $S \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}})$ has spectrum contained in Z. In [Web16], Webster showed that the decomposition of S into generalized eigenspaces indexed by Z induces a 2-representation of the categorified quantum group $\dot{\mathcal{U}}^*_{\mathbb{K}}(\tilde{\mathfrak{g}})$ for $\tilde{\mathfrak{g}}$ on $\mathcal{G}^{\underline{\lambda}}$ where a dot in $\dot{\mathcal{U}}^*_{\mathbb{K}}(\tilde{\mathfrak{g}})$ acts locally nilpotently. Moreover, he showed that there is a 2-natural isomorphism

$$\eta : \mathcal{U}_{\mathbb{K}}^{\tilde{\lambda},*}(\tilde{\mathfrak{g}}) \longrightarrow \mathcal{G}^{\underline{\lambda}}, \tag{5.2}$$

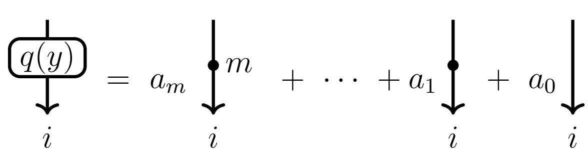
between $\mathcal{G}^{\underline{\lambda}}$ and a cyclotomic quotient of $\dot{\mathcal{U}}_{\mathbb{K}}^{*}(\tilde{\mathfrak{g}})$. Since the trace decategorification of cyclotomic quotients is known from [BHLW17], this allows us to determine the structure of $\mathrm{Tr}(\mathcal{G}^{\underline{\lambda}})$ in Chapter 8.

Actually, the setting considered in [Web16] is slightly different; Webster considers a categorification of a tensor product of highest *and lowest* weight modules with a less generic deformation. The main ideas are the same, but since [Web16] is a preprint and we need to understand the structure of the 2-representation of $\dot{\mathcal{U}}_{\mathbb{K}}^{*}(\tilde{\mathfrak{g}})$ on $\mathcal{G}^{\underline{\lambda}}$ in some detail for Chapters 6 and §8, we give a relatively complete description of its construction.

Remark 5.0.3. Since the underlying field and Lie algebra vary in this chapter, we will take care to include them in our notation. This applies particularly to the categorified quantum group $\dot{\mathcal{U}}_{\Bbbk}(\mathfrak{g})$, cyclotomic quotients $\mathcal{U}_{\Bbbk}^{\lambda,*}(\mathfrak{g})$, current algebra $U_{\Bbbk}(\mathfrak{g}[t])$, and Weyl modules $W_{\Bbbk}(\underline{\lambda})$. Since we only ever consider (deformed) tensor product categorifications over $\Bbbk$ with respect to $\mathfrak{g}$, we still write these as $\mathcal{X}^{\underline{\lambda}}$ and $\check{\mathcal{X}}^{\underline{\lambda}}$.

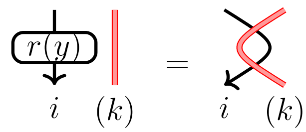
5.1 Spectrum of dots

Recall that if $q(w) = \sum_{r=0}^{m} a_r w^r$ is a polynomial then we write



(5.3)

Recall that $\mathcal{G}^{\underline{\lambda}}$ is the idempotent completion of $\check{\mathcal{X}}^{\underline{\lambda},*} \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K}$. Since the elements of Z are scalars in $\mathbb{K}$, the deformed relation (4.9) factors in $\mathcal{G}^{\underline{\lambda}}$ to give



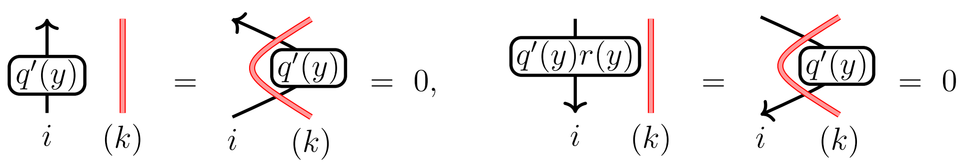
(5.4)

where $r(w)$ is the product of $(w-z)$ over $z \in Z_i^{(k)}$. This allows us to determine the spectrum of a dot on a black string in $\mathcal{G}^{\underline{\lambda}}$.

Lemma 5.1.1. *Take a Stendhal pair $S \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}})$ and let B denote a black string in S with label $i \in I$. Suppose that the first red string R to the right of B is labelled by (k). Then a dot y acting on B satisfies a polynomial with roots in $Z^{(\leq k)} = \bigcup_{l \leq k} Z^{(l)}$.*

Proof. We proceed by induction on the total number of strings (red or black) to the right of B.

First suppose there are no black strings between B and R. Let S' denote the Stendhal pair obtained from S by swapping B and R and denote their images in S' by B' and R', respectively. By the inductive hypothesis, a dot on B' satisfies a polynomial $q'(w)$ with roots in $Z^{(\leq k-1)}$. The relations (4.5) and (5.4) imply that in $\mathcal{G}^{\underline{\lambda}}(S,S)$,

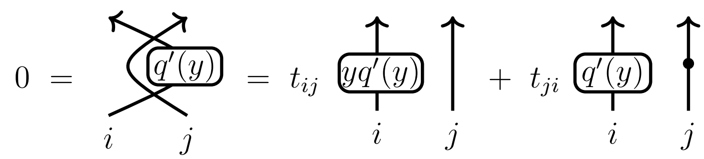


where $r(w)$ is the product of $(w-z)$ taken over all $z \in Z_i^{(k)}$. The claim follows.

Now suppose that the string C immediately to the right of B in S is black and labelled by $j \in I$. Let S' denote the Stendhal pair obtained by swapping B and C and denote their images in S' by B' and C' respectively. By the inductive hypothesis, a dot acting on C in S (resp. B' in S') satisfies a polynomial $q(w)$ (resp. $q'(w)$) with roots in $Z^{(\leq k)}$.

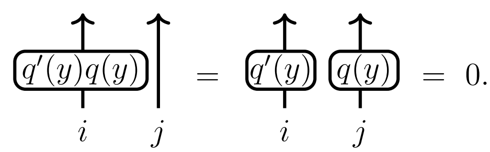
First suppose that B and C have the same orientation. Without loss of generality we may assume they are both oriented down. Recall that the KLR relations (3.4) and (3.5) hold on upward-oriented black strings.

If $\langle i, j\rangle = 0$ then we can pull B through C and dots commute with the crossing, so $q'(y)$ acts as zero on B. If $\langle i, j\rangle = -1$ then



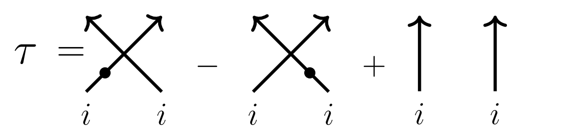
(5.5)

Since $t_{ij} = -t_{ji}$ by (3.1), repeated applications of this show that



(5.6)

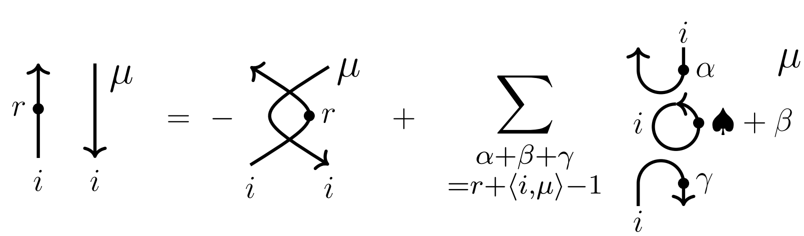
If $i = j$ then let



(5.7)

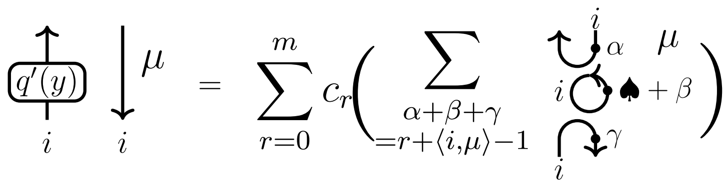
As in [BK09, Lemma 2.1], this squares to the identity on S, and conjugating by τ sends a dot on C to a dot on B. So $q(y)$ acts on B as zero.

Now suppose that B and C have opposite orientations. Assume that B (resp. C) is oriented up (resp. down). The other case is similar. If $i \neq j$ then we can pull B through C and dots commute with the crossing so $q'(y)$ acts as zero on B. If $i = j$ then for $r \in \mathbb{N}$, the relation (3.11) and KLR relations imply that



(5.8)

so if $q'(w) = c_m w^m + \cdots + c_0$ then



(5.9)

Apply $q(y)$ to the top of the left string on both sides of the equation. On the right hand side we can slide these new dots through the cup and so the whole expression is equal to zero. So $q'(y)q(y)$ acts on B as zero and the claim holds. $\square$

5.2 2-representations on $\mathcal{G}^{\underline{\lambda}}$

Recall that the unfurled Cartan datum has indexing set $\tilde{I} = I \times Z$ and weight lattice $\tilde{X} = X \times Z$. The associated graph $\tilde{\Gamma} \cong \Gamma \times Z$ inherits an orientation from Γ. We write p for the projection $\tilde{X} \to X$. For notational convenience we identify $(\pm\tilde{I})^m = (\pm I)^m \times Z^m$ for $m \in \mathbb{N}$, so, for example, $\mathcal{E}_{(-i,z)}$ means $\mathcal{E}_{-(i,z)}$ for $i \in I$ and $z \in Z$.

The definition of the corresponding categorified quantum group $\dot{\mathcal{U}}_{\mathbb{K}}(\tilde{\mathfrak{g}})$ depends on a choice of parameters - see §3.1. We use the geometric choice of scalars:

$$t_{(i,z),(j,z')} = \begin{cases} t_{i,j} & \text{if } z = z' \\ 1 & \text{if } z \neq z' \end{cases} \tag{5.10}$$

and the following bubble parameters:

$$c_{(i,z),\tilde{\mu}} = c_{i,\tilde{\mu}_z}, \tag{5.11}$$

where $(i,z),(j,z') \in \tilde{I}$ and $\tilde{\mu}_z$ is the z-component of $\tilde{\mu} \in \tilde{X}$.

There is an ungraded 2-representation of $\dot{\mathcal{U}}^*_{\mathbb{K}}(\mathfrak{g})$ on $\mathcal{G}^{\underline{\lambda}}$ arising from the 2-representation of

$\dot{\mathcal{U}}^*_{\mathbb{K}}(\mathfrak{g})$ on $\check{\mathcal{X}}^{\underline{\lambda},*}$. We have the following:

Theorem 5.2.1. *[Web16, Theorem 3.13] There is a 2-representation of $\dot{\mathcal{U}}^*_{\mathbb{K}}(\tilde{\mathfrak{g}})$ on $\mathcal{G}^{\underline{\lambda}}$ such that dots in $\dot{\mathcal{U}}^*_{\mathbb{K}}(\tilde{\mathfrak{g}})$ act locally nilpotently.*

In this section we describe how to construct this 2-representation. The reader is referred to [Web16] for the proof that it is well defined.

From the 2-representation of $\dot{\mathcal{U}}^*_{\mathbb{K}}(\mathfrak{g})$ there is a decomposition $\mathcal{G}^{\underline{\lambda}} = \bigoplus \mathcal{G}^{\underline{\lambda}}(\mu)$ according to the weight $\mu \in X$ of the left-hand region of a diagram and compatible functors from $\dot{\mathcal{U}}^*_{\mathbb{K}}(\mathfrak{g})(\mu,\nu)$ to the category of functors $\mathcal{G}^{\underline{\lambda}}(\mu) \to \mathcal{G}^{\underline{\lambda}}(\nu)$ for any $\mu, \nu \in X$.

In particular there are functors $\mathcal{E}_i 1_\mu$ from $\mathcal{G}^{\underline{\lambda}}(\mu)$ to $\mathcal{G}^{\underline{\lambda}}(\mu+\alpha_i)$ for all $i \in \pm I$ (recall that $\alpha_{-i} = -\alpha_i$). Define

$$\mathcal{E}_i = \bigoplus_{\mu \in X} \mathcal{E}_i 1_\mu, \qquad \mathcal{E} = \bigoplus_{i \in I} \mathcal{E}_i, \tag{5.12}$$

regarded as endofunctors of $\mathcal{G}^{\underline{\lambda}}$. For $(i,z) \in \pm\tilde{I}$, let $\mathcal{E}_{(i,z)}$ denote the functor sending $A \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}})$ to the generalized z-eigenspace of a dot acting on $\mathcal{E}_i A$ and given by restriction on morphisms. By Lemma 5.1.1 we can decompose $\mathcal{E}_i = \bigoplus_{z \in Z} \mathcal{E}_{(i,z)}$.

By [Rou08, Theorem 5.25], to define a 2-representation of $\dot{\mathcal{U}}^*_{\mathbb{K}}(\tilde{\mathfrak{g}})$ it remains to give a refined weight decomposition

$$\mathcal{G}^{\underline{\lambda}}(\mu) = \bigoplus_{\tilde{\mu} \in p^{-1}(\mu)} \mathcal{G}^{\underline{\lambda}}(\tilde{\mu}) \tag{5.13}$$

such that for any $(i,z) \in \pm\tilde{I}$, the restriction $\mathcal{E}_{(i,z)} 1_{\tilde{\mu}}$ of $\mathcal{E}_{(i,z)}$ sends

$$\mathcal{G}^{\underline{\lambda}}(\tilde{\mu}) \longrightarrow \mathcal{G}^{\underline{\lambda}}(\tilde{\mu} + \alpha_{(i,z)}), \tag{5.14}$$

and to give a suitable action of the unfurled KLR algebra $\tilde{R}_m$ on $\mathcal{E}^m$.

Weight decomposition of $\mathcal{G}^{\underline{\lambda}}$

The definition of the categories $\mathcal{G}^{\underline{\lambda}}(\tilde{\mu})$ below is equivalent to that in [Web16, §3.2].

Take a weight $\mu \in X$ and a Stendhal pair $S = (\underline{i}, \kappa) \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}}(\mu))$ with $\underline{i} \in (\pm I)^m$ and for $k \in [1, m]$ let y_k denote a dot acting on the black string in S labelled by $|i_k|$. By Lemma 5.1.1, y_k has generalised eigenvalues in Z. For a sequence $\underline{z} = (z_1, \dots, z_m) \in Z^m$ let $S_{\underline{z}} \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}})$ denote the simultaneous generalised eigenspace of S where y_k has eigenvalue z_k (note that the y_k commute). Let $\mathcal{I}_{\underline{z}}$ denote the 2-sided ideal of $\mathcal{G}^{\underline{\lambda}}(S_{\underline{z}}, S_{\underline{z}})$ generated by $y_k - z_k$ for $1 \leq k \leq m$.

Recall the generating functions $e_j(x)$ for $j \in I$ from §2.2 and the isomorphism b_μ with bubbles from §3.2. The dot y_k acts as z_k in $\mathcal{G}^{\underline{\lambda}}(S_{\underline{z}}, S_{\underline{z}})/\mathcal{I}_{\underline{z}}$ so by the bubble slides in Lemma 3.2.3 and (4.12), $b_\mu(e_j(x))$ acts in $\mathcal{G}^{\underline{\lambda}}(S_{\underline{z}}, S_{\underline{z}})/\mathcal{I}_{\underline{z}}$ as a rational function in x. We define the weight $\tilde{\mu} \in \tilde{X}$ of $S_{\underline{z}}$ as the unique weight such that

$$b_\mu(e_j(x)) \circ 1_{S_{\underline{z}}} \equiv \prod_{z \in Z} (1 + zx)^{\langle (j,z), \tilde{\mu} \rangle} 1_{S_{\underline{z}}} \qquad (\mathrm{mod}\ \mathcal{I}_{\underline{z}}) \tag{5.15}$$

for all $j \in I$. From the bubble slides and induction, $\sum_z \langle (j, z), \tilde{\mu} \rangle = \langle j, \mu \rangle$ and so $\tilde{\mu} \in p^{-1}(\mu)$.

Let $\mathcal{G}^{\underline{\lambda}}(\tilde{\mu})$ denote the full, additive subcategory of $\mathcal{G}^{\underline{\lambda}}$ generated by objects $S_{\underline{z}}$ with weight $\tilde{\mu}$ defined as above, and closed under taking direct summands. Since Stendhal pairs generate $\check{\mathcal{X}}^{\underline{\lambda},*}$, there is an induced decomposition $\mathcal{G}^{\underline{\lambda}}(\mu) = \bigoplus \mathcal{G}^{\underline{\lambda}}(\tilde{\mu})$ taken over all $\tilde{\mu} \in p^{-1}(\mu)$. If $(i, z) \in \pm \tilde{I}$, then $\mathcal{E}_{(i,z)}$ denotes the generalized z-eigenspace of a dot on $\mathcal{E}_i$, so by Lemma 3.2.3 acting by $\mathcal{E}_{(i,z)}$ contributes a factor of $(1 + zx)^{\langle j,i \rangle}$ to (5.15). Thus

$$\mathcal{E}_{(i,z)} 1_{\tilde{\mu}} : \mathcal{G}^{\underline{\lambda}}(\tilde{\mu}) \longrightarrow \mathcal{G}^{\underline{\lambda}}(\tilde{\mu} + \alpha_{(i,z)}) \tag{5.16}$$

as required.

Actions of KLR algebras

Take $m \in \mathbb{N}$. Let R_m denote the KLR algebra associated to I defined over the field $\mathbb{K}$ (see §3.1). Since the KLR relations hold on upward oriented strings in $\dot{\mathcal{U}}_{\mathbb{k}}(\mathfrak{g})$, the 2-representation of $\dot{\mathcal{U}}^*_{\Bbbk}(\mathfrak{g})$ on $\mathcal{G}^{\underline{\lambda}}$ leads to algebra homomorphisms

$$R_m \longrightarrow \mathcal{G}^{\underline{\lambda}}(\mathcal{E}^m A, \mathcal{E}^m A) \tag{5.17}$$

which are natural in $A \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}})$. The idempotent $\varepsilon_{\underline{i}} \in R_m$ acts as projection onto $\mathcal{E}_{\underline{i}}A$ and by Lemma 5.1.1 a dot $y_k\varepsilon_{\underline{i}}$ acts with generalized eigenvalues in Z. We can package such actions in a completion $\hat{R}_m$ in which we can formally separate the spectrum of the dot into components indexed by $z \in Z$.

For $N \in \mathbb{N}$, let $J_m^{(N)}$ be the two-sided ideal of R_m generated by elements

$$\left(\prod_{z\in Z}(y_k - z)^N\right)\varepsilon_{\underline{i}} \tag{5.18}$$

as $\underline{i}$ ranges over I^m and k ranges over $[1,m]$. These form a decreasing chain $J_m^{(1)} \supseteq J_m^{(2)} \supseteq J_m^{(3)} \supseteq \cdots$ so we can form the completion:

$$\hat{R}_m = \varprojlim R_m/J_m^{(N)}. \tag{5.19}$$

If $A \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}})$ then each $y_k\varepsilon_{\underline{i}}$ acts on $\mathcal{E}^m A$ with generalised eigenvalues in Z, so there is an induced homomorphism from $\hat{R}_m$ to $\mathcal{G}^{\underline{\lambda}}(\mathcal{E}^m A, \mathcal{E}^m A)$. By abstract Jordan decomposition there is a unique decomposition $\varepsilon_{\underline{i}} = \sum_{\underline{z}\in Z^m} \varepsilon_{(\underline{i},\underline{z})}$ into mutually orthogonal idempotents such that if $\underline{z} = (z_1, \ldots, z_m) \in Z^m$ then $\varepsilon_{(\underline{i},\underline{z})}$ projects onto the simultaneous generalized $(z_1, \ldots, z_m)$-eigenspace for $(y_1\varepsilon_{\underline{i}}, \ldots, y_m\varepsilon_{\underline{i}})$ on $\mathcal{E}_{\underline{i}}A$ and this operation is natural in A.

Let $\tilde{R}_m$ be the KLR algebra for $\tilde{I}$ defined over $\mathbb{K}$. To distinguish them from elements R_m, we write the generators of $\tilde{R}_m$ as $\{\tilde{\varepsilon}_{(\underline{i},\underline{z})} \mid (\underline{i},\underline{z}) \in \tilde{I}^m\}$, $\{\tilde{y}_k \mid 1 \le k \le m\}$, and $\{\tilde{\psi}_l \mid 1 \le l \le m-1\}$. Note in particular that $\tilde{\varepsilon}_{(\underline{i},\underline{z})}$ denotes a generator of $\tilde{R}_m$, whereas $\varepsilon_{(\underline{i},\underline{z})}$ denotes the element of $\hat{R}_m$ projecting onto a generalized $\underline{z}$-eigenspace for dots.

We can package actions of $\tilde{R}_m$ in which dots act *nilpotently* into a completion. For $N \in \mathbb{N}$, let $\tilde{J}_m^{(N)}$ be the two-sided ideal of $\tilde{R}_m$ generated by elements $\tilde{y}_k^N\tilde{\varepsilon}_{(\underline{i},\underline{z})}$ as $(\underline{i},\underline{z})$ ranges over $\tilde{I}^m$ and k ranges over $[1,m]$. Let $\hat{\tilde{R}}_m$ denote the completion of $\tilde{R}_m$ along these ideals.

Proposition 5.2.2. *[Web16, Proposition 3.3] There is an algebra isomorphism* $\hat{\tilde{R}}_m \to \hat{R}_m$ *sending*

$$\tilde{\varepsilon}_{(\underline{i},\underline{z})} \longmapsto \varepsilon_{(\underline{i},\underline{z})} \qquad \tilde{y}_k\tilde{\varepsilon}_{(\underline{i},\underline{z})} \longmapsto (y_k - z_k)\varepsilon_{(\underline{i},\underline{z})} \tag{5.20}$$

for $(\underline{i}, \underline{z}) \in \tilde{I}^m$ *and* $k \in [1, m]$*.*

The formula for the image of $\tilde{\psi}_l\tilde{\varepsilon}_{(\underline{i},\underline{z})}$ is more involved and we will not need it explicitly.

Since dots in R_m act on $\mathcal{G}^{\underline{\lambda}}$ with generalized eigenvalues in Z, the action of R_m on $\mathcal{E}^m A$ for an object A induces an action of the completion $\hat{R}_m$. The action of $\tilde{R}_m$ comes from the compositions

$$\tilde{R}_m \longrightarrow \hat{\tilde{R}}_m \longrightarrow \hat{R}_m \longrightarrow \mathcal{G}^{\underline{\lambda}}(\mathcal{E}^m A, \mathcal{E}^m A) \tag{5.21}$$

which are natural in $A \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}})$. For $(\underline{i}, \underline{z}) \in \tilde{I}^m$, $\tilde{\varepsilon}_{(\underline{i},\underline{z})}$ acts as projection onto

$$\mathcal{E}_{(\underline{i},\underline{z})}A = \mathcal{E}_{(i_1,z_1)} \cdots \mathcal{E}_{(i_m,z_m)}A, \tag{5.22}$$

and each $\tilde{y}_k\varepsilon_{(\underline{i},\underline{z})}$ acts nilpotently. This completes our description of the construction of the 2-representation of $\dot{\mathcal{U}}^*_{\mathbb{K}}(\tilde{\mathfrak{g}})$ on $\mathcal{G}^{\underline{\lambda}}$.

5.3 Equivalence with cyclotomic quotient

In this section we analyze the structure of $\mathcal{G}^{\underline{\lambda}}$ as a 2-representation of $\dot{\mathcal{U}}^*_{\mathbb{K}}(\tilde{\mathfrak{g}})$.

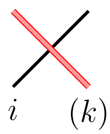
Let $S \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}})$ be a Stendhal pair. Take $k \in [1, n]$, $i \in I$, and $z \in Z_i^{(k)}$. Let S_1 be the Stendhal pair obtained from S by placing a black string B_1 labelled by i immediately to the left of the red (k)-string. Define S_2 by placing a black string B_2 labelled by i and of the same orientation as B_1 immediately to the *right* of the red string. For $j = 1, 2$, let $T_j \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}})$ denote the generalized z-eigenspace for a dot on B_j - a direct summand of S_j.

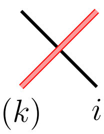
Lemma 5.3.1. *There is an isomorphism* $T_1 \cong T_2$ *and this intertwines a dot acting on* B_1 *with a dot on* B_2*.*

Proof. For $j = 1, 2$, we can consider dots acting on B_j in S_j as coming from an action of the KLR algebra R_m with $m = 1$. Recall the completion $\hat{R}_1$ from §5.2 and the element $\varepsilon_{(i,z)} \in \hat{R}_1$. This acts as projection onto T_j and is independent of j. We can identify

$$\mathcal{G}^{\underline{\lambda}}(T_j, T_j) \cong \varepsilon_{(i,z)} \mathcal{G}^{\underline{\lambda}}(S_j, S_j) \varepsilon_{(i,z)}. \tag{5.23}$$

Define Stendhal diagrams in $D_1 \in \mathcal{G}^{\underline{\lambda}}(S_1, S_2)$ and $D_2 \in \mathcal{G}^{\underline{\lambda}}(S_2, S_1)$ by

$D_1 =$ 

and $D_2 =$ 

(5.24)

where the black string is given the same orientation as B_1 and B_2. Note that since dots can pass through red strings by (4.4), D_1 and D_2 intertwine the actions of $\hat{R}_1$ on S_1 and S_2 so we can project them to morphisms between T_1 and T_2 by multiplying by $\varepsilon_{(i,z)}$ on either the top or bottom.

If B_1 and B_2 are both oriented upward then by relation (4.5) in $\check{\mathcal{X}}^{\underline{\lambda}}$, $D_1 \cdot D_2 = 1_{S_2}$ and $D_2 \cdot D_1 = 1_{S_1}$ so $\varepsilon_{(i,z)} D_1$ and $\varepsilon_{(i,z)} D_2$ are mutually inverse isomorphisms $T_1 \cong T_2$. Suppose B_1 and B_2 are oriented downward. Let $r(w) = \prod (w - z')$ where the product is taken over $z' \in Z_i^{(k)}$. By (5.4), separating the red and black strings in $D_1 \cdot D_2$ and $D_2 \cdot D_1$ introduces a factor of $r(y)$. Since $z \notin Z_i^{(k)}$, $r(w)$ and $(w - z)$ are coprime and so there exists $s \in \hat{R}_1$ such that $r(y) s \varepsilon_{(i,z)} = \varepsilon_{(i,z)}$. Thus $s\varepsilon_{(i,z)} D_1$ and $\varepsilon_{(i,z)} D_2$ give mutually inverse isomorphisms $T_1 \cong T_2$. □

Corollary 5.3.2. *Let $S \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}})$ be a Stendhal pair and take $k \in [1, n]$ and $i \in \pm I$. Suppose that there are no black strings to the left of the red $(k + 1)$-string in S, and let S^+ denote the Stendhal pair obtained from S by placing a black string labelled by $|i|$ and oriented according* *to the sign of i immediately to the right of this red string. Then there is an isomorphism*

$$\bigoplus_{z \in Z^{(\leq k)}} \mathcal{E}_{(i,z)} S \longrightarrow S^+ \tag{5.25}$$

intertwining a dot on the left-most black string in $\mathcal{E}_i S$ with a dot on the new string in S^+.

Proof. If $z \in Z^{(\leq k)}$ then by repeated applications of Lemma 5.3.1, the generalized z-eigenspace of a dot on the new string in S^+ is isomorphic to $\mathcal{E}_{(i,z)} S$. But by Lemma 5.1.1 the eigenvalues of the dot in S^+ are contained in $Z^{(\leq k)}$, so the claim holds. □

Let $\emptyset \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}})$ denote the trivial Stendhal pair with no black strings. Define a dominant weight for $\tilde{\mathfrak{g}}$ by

$$\tilde{\lambda} = \sum_{i \in I} \sum_{z \in Z_i} \Lambda_{(i,z)}, \tag{5.26}$$

where $Z_i = \bigcup_k Z_i^{(k)}$. By the construction of the weight decomposition §5.2 and the bubble slides (4.12) $\emptyset$ has weight $\tilde{\lambda}$ in $\mathcal{G}^{\underline{\lambda}}$. Repeated applications of Corollary 5.3.2 show that $\emptyset$ generates $\mathcal{G}^{\underline{\lambda}}$ under the action of $\dot{\mathcal{U}}^*_{\mathbb{K}}(\tilde{\mathfrak{g}})$. Moreover, $\mathcal{E}_{(i,z)}\emptyset = 0$ for all $(i,z) \in \tilde{I}$ and dots in $\dot{\mathcal{U}}^*_{\mathbb{K}}(\tilde{\mathfrak{g}})$ act locally nilpotently on $\mathcal{G}^{\underline{\lambda}}$ by Theorem 5.2.1, so $\emptyset$ is a highest weight object in $\mathcal{G}^{\underline{\lambda}}$.

Recall the definition of cyclotomic quotients from §3.3. By [Rou12, Theorem 4.25], the undeformed cyclotomic quotient $\mathcal{U}^{\tilde{\lambda},*}_{\mathbb{K}}(\tilde{\mathfrak{g}})$ of $\dot{\mathcal{U}}^*_{\mathbb{K}}(\tilde{\mathfrak{g}})$ of weight $\tilde{\lambda}$ is the universal highest weight categorification on which dots act nilpotently, so there is a 2-natural transformation

$$\eta : \mathcal{U}^{\tilde{\lambda},*}_{\mathbb{K}}(\tilde{\mathfrak{g}}) \longrightarrow \mathcal{G}^{\underline{\lambda}} \tag{5.27}$$

sending $1_{\tilde{\lambda}}$ to $\emptyset$ and this is an isomorphism if and only if $\emptyset \ncong 0$ in $\mathcal{G}^{\underline{\lambda}}$. Webster shows this by explicitly constructing an inverse to η:

Theorem 5.3.3. *[Web16, Lemma 4.8] There is a 2-natural isomorphism sending $1_{\tilde{\lambda}}$ to $\emptyset$:*

$$\eta : \mathcal{U}^{\tilde{\lambda},*}_{\mathbb{K}}(\tilde{\mathfrak{g}}) \longrightarrow \mathcal{G}^{\underline{\lambda}}. \tag{5.28}$$

Chapter 6

Flatness of $\check{\mathcal{X}}^{\underline{\lambda},*}$

In this chapter we use the results of chapter 5 to show that $\check{\mathcal{X}}^{\underline{\lambda},*}$ is a flat deformation of $\mathcal{X}^{\underline{\lambda},*}$ over $\mathbb{A}_{\underline{\lambda}}$ and deduce that $K_0(\check{\mathcal{X}}^{\underline{\lambda}}) \cong K_0(\mathcal{X}^{\underline{\lambda}})$ and $\check{\mathcal{X}}^{\underline{\lambda}}$ is equivalent to a deformed cyclotomic quotient if $\underline{\lambda} = (\lambda)$.

6.1 Dimensions of morphism spaces

Recall from §2.3 that $\mathbb{A}_{\underline{\lambda}} \leq \mathbb{K} = \overline{\Bbbk(Z)}$ is a graded local ring with unique simple graded module $\Bbbk$ on which all $z \in Z$ act as zero. The following is well-known, but we state it precisely for sake of clarity:

Lemma 6.1.1 (Upper semi-continuity of dimension)**.** *Let M be a finitely-generated graded right $\mathbb{A}_{\underline{\lambda}}$-module. Then*

$$\dim_{\Bbbk} M \otimes_{\mathbb{A}_{\underline{\lambda}}} \Bbbk \geq \dim_{\mathbb{K}} M \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K} \tag{6.1}$$

with equality if and only if M is a free graded $\mathbb{A}_{\underline{\lambda}}$-module.

Note that since $\mathbb{A}_{\underline{\lambda}}$ is a graded local ring, M is free if and only if it is flat.

So to establish that morphism spaces in $\check{\mathcal{X}}^{\underline{\lambda},*}$ are flat $\mathbb{A}_{\underline{\lambda}}$-modules it suffices to compare the dimensions of morphism spaces in $\mathcal{X}^{\underline{\lambda},*} = \check{\mathcal{X}}^{\underline{\lambda},*} \otimes_{\mathbb{A}_{\underline{\lambda}}} \Bbbk$ and $\check{\mathcal{X}}^{\underline{\lambda},*} \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K}$, or its idempotent completion $\mathcal{G}^{\underline{\lambda}}$. Dimensions of morphism spaces are encoded in the *Euler form* on the Grothendieck group, so the problem essentially reduces to one of bilinear forms on $\mathfrak{g}$-modules.

Proposition 6.1.2. *Morphism spaces in $\check{\mathcal{X}}^{\underline{\lambda},*}$ are free over $\mathbb{A}_{\underline{\lambda}}$.*

Proof. Recall that $K_0(\mathcal{X}^{\underline{\lambda},*})$ is obtained from $K_0(\mathcal{X}^{\underline{\lambda}})$ by tensoring over $\mathbb{Z}[q^{\pm 1}]$ with the module $\mathbb{Z}$ on which q acts as 1, so if $\dot{U}^{\mathbb{Z}}(\mathfrak{g})$ is the idempotent form of the integral universal enveloping algebra for $\mathfrak{g}$ and $V^{\mathbb{Z}}(\underline{\lambda})$ is a tensor product of integral highest weight modules for $\dot{U}^{\mathbb{Z}}(\mathfrak{g})$ then there is an isomorphism $K_0(\mathcal{X}^{\underline{\lambda},*}) \cong V^{\mathbb{Z}}(\underline{\lambda})$ of $\dot{U}^{\mathbb{Z}}(\mathfrak{g})$-modules sending the class of a Stendhal pair $S \in \mathrm{Ob}(\mathcal{X}^{\underline{\lambda},*})$ to a vector $v_S \in V^{\mathbb{Z}}(\underline{\lambda})$.

The set $\tilde{I} = I \otimes Z$ indexes simple roots for the unfurled Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}^{\oplus Z}$. By Theorem 5.3.3 there is a 2-natural isomorphism between $\mathcal{G}^{\underline{\lambda}}$ and $\mathcal{U}_{\mathbb{K}}^{\tilde{\lambda},*}(\tilde{\mathfrak{g}})$ - the undeformed cyclotomic quotient of $\dot{\mathcal{U}}_{\mathbb{K}}^{*}(\tilde{\mathfrak{g}})$ of weight

$$\tilde{\lambda} = \sum_{i \in I} \sum_{z \in Z_i} \Lambda_{(i,z)}. \tag{6.2}$$

So there is an isomorphism $K_0(\mathcal{G}^{\underline{\lambda}}) \cong V^{\mathbb{Z}}(\tilde{\lambda})$ of $\dot{U}^{\mathbb{Z}}(\tilde{\mathfrak{g}})$-modules sending the class of a Stendhal pair $S \in \mathrm{Ob}(\mathcal{G}^{\underline{\lambda}})$ to a vector $\tilde{v}_S \in V^{\mathbb{Z}}(\tilde{\lambda})$.

For $k \in [1, n]$ there is an inclusion of $\dot{U}^{\mathbb{Z}}(\mathfrak{g})$-modules

$$V^{\mathbb{Z}}(\lambda^{(k)}) \longrightarrow \bigotimes_{i \in I} V^{\mathbb{Z}}(\Lambda_i)^{\otimes \lambda_i^{(k)}} \tag{6.3}$$

sending the cyclic vector $v_{\lambda^{(k)}}$ to the tensor product of the cyclic vectors. Taking the tensor product over all k we get an embedding $V^{\mathbb{Z}}(\underline{\lambda}) \hookrightarrow V^{\mathbb{Z}}(\tilde{\lambda})$, where we use the identification $\tilde{\mathfrak{g}} = \mathfrak{g}^{\oplus Z}$. We claim this sends $v_S \mapsto \tilde{v}_S$ for any Stendhal pair S.

This is clear for the trivial Stendhal pair $\emptyset$ with no black strings. Take $k \in [1, n]$ and assume the claim holds for some S with no black strings to the left of the red $(k+1)$-string. Take $i \in \pm I$ and let S^+ denote the Stendhal pair obtained from S as in Corollary 5.3.2; that is, by placing a black string labelled by $|i|$ and oriented according to the sign of i immediately to the right of this red string. We prove the claim for S^+. It then follows in general by induction.

By [Web17, Theorem 4.38], v_{S^+} is obtained from $v_S \in V^{\mathbb{Z}}(\underline{\lambda})$ by applying e_i to the first k tensor factors. So by the construction of the map (6.3) and the inductive hypothesis, v_{S^+} is mapped to

$$\sum_{z \in Z^{(\leq k)}} e_{(i,z)} \tilde{v}_S \tag{6.4}$$

in $V^{\mathbb{Z}}(\tilde{\lambda})$, where $Z^{(\leq k)} = \bigcup_{l \leq k} Z^{(l)}$. But by Corollary 5.3.2,

$$S^+ \cong \bigoplus_{z \in Z^{(\leq k)}} \mathcal{E}_{(i,z)} S \tag{6.5}$$

in $\mathcal{G}^{\underline{\lambda}}$, so their classes are equal in the Grothendieck group and the claim follows.

Recall that for a $\Bbbk$-linear category $\mathcal{C}$, the Euler form is the bilinear form on the Grothendieck group $K_0(\mathcal{C})$ defined by

$$\langle [x], [y] \rangle = \dim_{\Bbbk} \mathcal{C}(x, y) \tag{6.6}$$

for $x, y \in \mathrm{Ob}(\mathcal{C})$. By [Web17, Theorem 4.38], the isomorphisms $K_0(\mathcal{G}^{\underline{\lambda}}) \cong V^{\mathbb{Z}}(\tilde{\lambda})$ and $K_0(\mathcal{X}^{\underline{\lambda},*}) \cong V^{\mathbb{Z}}(\underline{\lambda})$ intertwine the Euler forms with the Shapovalov form and factorwise Shapovalov form, respectively. By the uniqueness of contravariant forms on highest weight modules, the inclusions (6.3), and consequently the embedding $V^{\mathbb{Z}}(\underline{\lambda}) \hookrightarrow V^{\mathbb{Z}}(\tilde{\lambda})$, are isometries.

Now take Stendhal pairs S and S'. Then

$$\dim_{\Bbbk} \check{\mathcal{X}}^{\underline{\lambda},*}(S, S') = \langle v_S, v_{S'} \rangle = \langle \tilde{v}_S, \tilde{v}_{S'} \rangle = \dim_{\mathbb{K}} \mathcal{G}^{\underline{\lambda}}(S, S'), \tag{6.7}$$

so $\check{\mathcal{X}}^{\underline{\lambda},*}(S, S')$ is a free $\mathbb{A}_{\underline{\lambda}}$-module by upper semi-continuity of dimension - Lemma 6.1.1. Since flatness is preserved under taking direct sums and direct summands, this shows that every morphism space in $\check{\mathcal{X}}^{\underline{\lambda},*}$ is free. □

6.2 Applications of flatness

Recall that there is a functor from $\check{\mathcal{X}}^{\underline{\lambda}}$ to $\mathcal{X}^{\underline{\lambda}}$ given by taking the degree zero component of the projection $\check{\mathcal{X}}^{\underline{\lambda},*} \to \mathcal{X}^{\underline{\lambda},*}$ defined by tensoring over $\mathbb{A}_{\underline{\lambda}}$ with $\Bbbk$.

Corollary 6.2.1. *The functor $\check{\mathcal{X}}^{\underline{\lambda}} \to \mathcal{X}^{\underline{\lambda}}$ induces an isomorphism of $\dot{U}_q^{\mathbb{Z}}$-modules*

$$K_0\left(\check{\mathcal{X}}^{\underline{\lambda}}\right) \longrightarrow K_0\left(\mathcal{X}^{\underline{\lambda}}\right) \tag{6.8}$$

and so $K_0(\check{\mathcal{X}}^{\underline{\lambda}}) \cong V_q^{\mathbb{Z}}(\underline{\lambda})$.

Proof. It suffices to show that the functor realises a bijection between isomorphism classes of indecomposables. Since every indecomposable object is a direct summand of (a grading shift of) a Stendhal pair, it suffices to show that if S is a Stendhal pair then idempotents (and isomorphisms between them) lift uniquely under the algebra homomorphism

$$\check{\mathcal{X}}^{\underline{\lambda}}(S,S) \longrightarrow \mathcal{X}^{\underline{\lambda}}(S,S) \tag{6.9}$$

The kernel K of this map is the degree zero component of $\check{\mathcal{X}}^{\underline{\lambda},*}(S,S) \cdot (\mathbb{A}_{\underline{\lambda}})_+$, where $(\mathbb{A}_{\underline{\lambda}})_+$ is the augmentation ideal of $\mathbb{A}_{\underline{\lambda}}$. So K^N is contained in the degree zero component of $\check{\mathcal{X}}^{\underline{\lambda},*}(S,S) \cdot (\mathbb{A}_{\underline{\lambda}})_+^N$. Degrees of morphisms between Stendhal pairs are bounded below (this follows from Lemma 7.2.1 which is independent of this proposition) so K is nilpotent ideal of $\check{\mathcal{X}}^{\underline{\lambda}}(S,S)$. Since idempotents can be lifted uniquely modulo nilpotent ideals, the claim holds. □

If $\underline{\lambda} = (\lambda)$ consists of a single dominant weight then we write $\check{\mathcal{X}}^{\lambda}$ for $\check{\mathcal{X}}^{\underline{\lambda}}$. Recall the deformed cyclotomic quotient $\check{\mathcal{U}}^{\lambda}$ from §3.3.

Corollary 6.2.2. *If $\lambda \in X^+$ is a dominant weight then there is a graded 2-natural isomorphism*

$$\xi : \check{\mathcal{U}}^{\lambda} \longrightarrow \check{\mathcal{X}}^{\lambda} \tag{6.10}$$

sending 1_λ *to the trivial Stendhal pair* $\emptyset$.

Proof. Horizontal composition induces a graded right action of $\dot{\mathcal{U}}^*(1_\lambda, 1_\lambda)$ on morphism spaces in $\check{\mathcal{U}}^{\lambda,*}$. Since an upward string at the far right is zero in $\check{\mathcal{U}}^{\lambda,*}$, real clockwise bubbles act as zero. So the bubble isomorphism b_λ from §3.2 induces a right action of Π such that elementary symmetric functions $e_{i,r}$ with $r > \langle i, \lambda\rangle$ act as zero. These generate the kernel of the projection $\Pi \to \mathbb{A}_\lambda$, so the action of Π factors through $\mathbb{A}_\lambda$.

Let $\emptyset \in \mathrm{Ob}(\check{\mathcal{X}}^{\underline{\lambda}})$ denote the trivial Stendhal pair with no black strings. By relation (4.5), $\mathcal{E}_i\emptyset = 0$ for any $i \in I$, so $\emptyset$ is a highest weight object of $\check{\mathcal{X}}^\lambda$ of weight λ. Moreover, $\emptyset$ generates $\check{\mathcal{X}}^\lambda$ under the action of $\dot{\mathcal{U}}$ so by [Rou12, Theorem 4.25] there is an essentially surjective 2-natural transformation

$$\xi : \check{\mathcal{U}}^\lambda \longrightarrow \check{\mathcal{X}}^\lambda \tag{6.11}$$

sending 1_λ to $\emptyset$, and ξ is an isomorphism if and only if the induced map

$$\check{\mathcal{U}}^{\lambda,*}(1_\lambda, 1_\lambda) \longrightarrow \check{\mathcal{X}}^{\lambda,*}(\emptyset, \emptyset) \tag{6.12}$$

is an isomorphism. But the bubble slides (4.12)-(4.14) in $\check{\mathcal{X}}^\lambda$ imply that ξ respects the $\mathbb{A}_\lambda$-actions and both $\check{\mathcal{U}}^{\lambda,*}(1_\lambda, 1_\lambda)$ and $\check{\mathcal{X}}^{\lambda,*}(\emptyset, \emptyset)$ are generated by the identity morphism under $\mathbb{A}_\lambda$, so in particular $\check{\mathcal{X}}^{\lambda,*}(\emptyset, \emptyset) \cong \mathbb{A}_\lambda$. The claim follows. □

Chapter 7

Trace of $\mathcal{X}^{\underline{\lambda},*}$

In this chapter we begin discussing the trace of $\mathcal{X}^{\underline{\lambda},*}$ and $\check{\mathcal{X}}^{\underline{\lambda},*}$. After recalling basic properties of the trace and the results of [BHLW17] we show that $\operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*})$ and $\operatorname{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ are spanned by the classes of Stendhal diagrams with no crossings between red and black strings. We use this to determine the trace of the unstarred categories $\mathcal{X}^{\underline{\lambda}}$ and $\check{\mathcal{X}}^{\underline{\lambda}}$ and find an upper bound for $\dim_{\Bbbk} \operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*})$.

7.1 Trace decategorification

First we recall the necessary background on trace decategorification and the relevant results from [BHLW17] on the trace of the categorified quantum group and its cyclotomic quotients.

Definition

The *trace* or zeroth Hochschild homology of a $\Bbbk$-linear category $\mathcal{C}$, denoted $\operatorname{Tr}(\mathcal{C})$, is the $\Bbbk$-vector space defined by:

$$\operatorname{Tr}(\mathcal{C}) = \left(\bigoplus_{x \in \operatorname{Ob}(\mathcal{C})} \mathcal{C}(x,x) \right) \Big/ \operatorname{span}_{\Bbbk}\{fg - gf\},$$

where the span is over all $f \in \mathcal{C}(x,y)$ and $g \in \mathcal{C}(y,x)$ for $x, y \in \operatorname{Ob}(\mathcal{C})$. If $f \in \mathcal{C}(x,x)$ then write $[f]$ for the class of f in $\mathrm{Tr}(\mathcal{C})$. In diagrammatic categories, applying the trace relation $fg = gf$ can be thought of as cutting a diagram at a horizontal line and swapping the top and bottom parts of the diagram.

The trace and the split Grothendieck group of $\mathcal{C}$ are related by the $\Bbbk$-linear *Chern character map*:

$$\begin{aligned} h_{\mathcal{C}} : K_0^{\Bbbk}(\mathcal{C}) &\longrightarrow \mathrm{Tr}(\mathcal{C}) \\ [x] &\longrightarrow [1_x]. \end{aligned} \tag{7.1}$$

This is injective under relatively weak hypotheses (see [BHLW17, Proposition 2.4]), but often fails to be surjective. Unlike the split Grothendieck group, the trace is invariant under taking the idempotent completion (c.f. [BHLŽ17, Proposition 3.2]).

If $\mathcal{C}$ is a graded category then grading shift $\langle 1 \rangle$ induces an automorphism q on the trace. This gives $\mathrm{Tr}(\mathcal{C})$ a $\Bbbk[q^{\pm 1}]$-module structure with respect to which $h_{\mathcal{C}}$ is a homomorphism of $\Bbbk[q^{\pm 1}]$-modules.

If $\mathcal{C}^*$ is the corresponding starred category then the q-action on $\mathrm{Tr}(\mathcal{C}^*)$ is trivial, but since $\mathcal{C}^*$ is enriched over graded vector spaces, the trace $\mathrm{Tr}(\mathcal{C}^*)$ is a graded vector space also. The corresponding Chern character map $h_{\mathcal{C}^*}$ is a homomorphism of graded vector spaces where $K_0^{\Bbbk}(\mathcal{C}^*)$ is concentrated in degree zero.

Since the morphism spaces in $\mathcal{C}^*$ are larger than in $\mathcal{C}$, we should expect $\mathrm{Tr}(\mathcal{C}^*)$ to be richer than $\mathrm{Tr}(\mathcal{C})$. In fact in this paper the Chern character maps $h_{\mathcal{C}}$ for unstarred categories are always isomorphisms, so we focus our attention almost entirely on the traces of starred categories $\mathrm{Tr}(\mathcal{C}^*)$.

If $\mathcal{C}$ is enriched over (graded) right A-modules for some $\Bbbk$-algebra A then $\mathrm{Tr}(\mathcal{C})$ is a (graded) right A-module. Moreover, the trace commutes with base change; if $\mathcal{C}$ is a $\Bbbk$-linear category and A is a $\Bbbk$-algebra then

$$\mathrm{Tr}(\mathcal{C} \otimes A) \cong \mathrm{Tr}(\mathcal{C}) \otimes A \tag{7.2}$$

as right A-modules.

Trace decategorification and 2-representations

The trace $\mathrm{Tr}(\mathcal{U}^*) \cong \mathrm{Tr}(\dot{\mathcal{U}}^*)$ of the (starred) categorified quantum group is a locally unital graded $\Bbbk$-algebra:

$$\mathrm{Tr}(\mathcal{U}^*) = \bigoplus_{\mu,\nu \in X} 1_\nu \, \mathrm{Tr}(\mathcal{U}^*(\mu,\nu)) 1_\mu \tag{7.3}$$

with multiplication given by horizontal composition. A (graded) 2-representation of $\dot{\mathcal{U}}^*$ on $\mathcal{M} = \bigoplus_{\mu \in X} \mathcal{M}(\mu)$ induces a locally unital (graded) $\mathrm{Tr}(\mathcal{U}^*)$-module structure on

$$\mathrm{Tr}(\mathcal{M}) = \bigoplus_{\mu \in X} 1_\mu \, \mathrm{Tr}(\mathcal{M}(\mu)). \tag{7.4}$$

A 2-natural transformation η between 2-representations on $\mathcal{M}$ and $\mathcal{N}$ induces a homomorphism of locally unital $\mathrm{Tr}(\mathcal{U}^*)$-modules $\mathrm{Tr}(\eta) : \mathrm{Tr}(\mathcal{M}) \to \mathrm{Tr}(\mathcal{N})$. If η is a 2-natural isomorphism then $\mathrm{Tr}(\eta)$ is an isomophism. If $\mathcal{M}$, $\mathcal{N}$, and η are graded then $\mathrm{Tr}(\eta)$ respects this structure.

In particular, $\mathrm{Tr}(\check{\mathcal{U}}^{\lambda,*})$, $\mathrm{Tr}(\mathcal{U}^{\lambda,*})$, $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$, and $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})$ are locally unital graded $\mathrm{Tr}(\mathcal{U}^*)$-modules. The action is given by placing a diagram on the left if weights match, and taking the class in the trace. Moreover, since morphism spaces in $\check{\mathcal{X}}^{\underline{\lambda},*}$ are enriched over graded right $\mathbb{A}_{\underline{\lambda}}$-modules, $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ has the structure of a graded $(\mathrm{Tr}(\mathcal{U}^*), \mathbb{A}_{\underline{\lambda}})$-bimodule.

Results of [BHLW17]

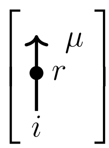
Recall from §2.2 that the current algebra $\dot{U}(\mathfrak{g}[t])$ is a locally unital graded $\Bbbk$-algebra and for $\lambda \in X^+$ it has (locally unital) graded modules $\mathbb{W}(\lambda)$ and $W(\lambda)$ called global and local Weyl modules respectively. Recall, also for $\lambda \in X^+$, the deformed and undeformed cyclotomic quotients $\check{\mathcal{U}}^\lambda$ and $\mathcal{U}^\lambda$ - graded 2-representations of $\dot{\mathcal{U}}$. Finally recall the isomorphisms b_μ ($\mu \in X$) between symmetric functions and bubbles from §3.2.

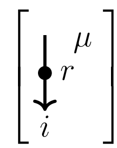
The following theorem comprises [BHLW17, Theorems 7.4, 7.5, and 8.4]:

Theorem 7.1.1. *There is an isomorphism of locally unital graded* $\Bbbk$*-algebras*

$$\rho : \dot{U}(\mathfrak{g}[t]) \longrightarrow \mathrm{Tr}(\mathcal{U}^*) \tag{7.5}$$

sending

$(e_i \otimes t^r)1_\mu \longmapsto$ 

, $\quad (\xi_i \otimes t^r)1_\mu \longrightarrow [b_\mu(p_{i,r})], \quad (f_i \otimes t^r)1_\mu \longmapsto$ 

for any $\mu \in X$, $i \in I$, *and* $r \in \mathbb{N}$. *Moreover, for any* $\lambda \in X^+$ *there are isomorphisms of graded modules*

$$\mathbb{W}(\lambda) \longrightarrow \mathrm{Tr}(\check{\mathcal{U}}^{\lambda,*}(\mathfrak{g})) \quad \textit{and} \quad W(\lambda) \longrightarrow \mathrm{Tr}(\mathcal{U}^{\lambda,*}) \tag{7.6}$$

intertwining ρ *and sending the distinguished generator* w_λ *to the class* $[1_\lambda]$ *of the empty diagram.*

So $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})$ is a graded $\dot{U}(\mathfrak{g}[t])$-module and $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ is a graded $(\dot{U}(\mathfrak{g}[t]), \mathbb{A}_{\underline{\lambda}})$-bimodule.

In [BHLW17, Theorem 8.1] the authors also showed that the Chern character maps $h_{\dot{\mathcal{U}}}$, $h_{\check{\mathcal{U}}^\lambda}$, and $h_{\mathcal{U}^\lambda}$ for the unstarred categories are isomorphisms, thereby determining the traces of these categories. Our proof that $h_{\check{\mathcal{X}}^{\underline{\lambda},*}}$ and $h_{\mathcal{X}^{\underline{\lambda},*}}$ are isomorphisms follows theirs (see Proposition 7.2.3).

7.2 Spanning set

In this section we show that $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ and $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})$ are spanned by diagrams with no crossings between red and black strings and deduce that the Chern character maps $h_{\mathcal{X}^{\underline{\lambda}}}$ and $h_{\check{\mathcal{X}}^{\underline{\lambda}}}$ for the unstarred categories are isomorphisms.

We begin by constructing a spanning set for morphism spaces in $\check{\mathcal{X}}^{\underline{\lambda},*}$ and $\mathcal{X}^{\underline{\lambda},*}$ following [KL10, §3.2]. Choosing this set carefully allows us to use an inductive argument in the proof of Lemma 7.2.1.

Let C denote the set of compositions $\nu = (\nu(1), \ldots, \nu(n))$ of length n. Write $|\nu| = \sum \nu(j)$ and let $C(m)$ denote the set of $\nu \in C$ with $|\nu| = m$. Each $C(m)$ is a poset under the reverse dominance order: $\nu \leq \nu'$ if

$$\sum_{j=1}^{k} \nu(j) \geq \sum_{j=1}^{k} \nu'(j) \tag{7.7}$$

for all $k \in [1, n]$. If $S = (\underline{i}, \kappa)$ is a Stendhal pair then define a composition

$$\nu_S = (\kappa(2) - \kappa(1), \ldots, \kappa(n+1) - \kappa(n)), \tag{7.8}$$

so $\nu_S(k)$ is the number of black strings between the red strings labelled by (k) and $(k+1)$. Observe that "right" crossings move us down in the partial order and "left" crossings move us up:

"left" crossing "right" crossing

Take Stendhal pairs $S = (\underline{i}, \kappa)$ and $S' = (\underline{i}', \kappa')$. Any Stendhal diagram with S as its bottom and S' as its top induces a matching on the disjoint union $\underline{i} \sqcup \underline{i}'$ by pairing elements of $\pm I$ that are connected by strings. Any such matching either connects an occurrence of $\pm i \in \pm I$ in $\underline{i}$ with one in $\underline{i}'$, or occurrences of i and $-i$ that lie either both in $\underline{i}$ or both in $\underline{i}'$.

For each such matching we fix a diagram D that attaches matched elements. We require that:

1. D is minimal (no two strings of any color cross more than once);

2. D has no closed loops (so no bubbles);

3. there are no dots on any of the strings of D;

4. on a given red string, all "right" crossings with black strings occur below all "left" crossings.

Fix a point on each black string of D away from intersections. Let $B_{S,S'}$ denote the union over all matchings of diagrams obtained from D by placing an arbitrary number of dots at the chosen points on D.

Lemma 7.2.1. *The set $B_{S,S'}$ generates the morphism space $\mathcal{X}^{\underline{\lambda},*}(S,S')$ (resp. $\check{\mathcal{X}}^{\underline{\lambda},*}(S,S')$) as a $\Bbbk$-vector space (resp. $\mathbb{A}_{\underline{\lambda}}$-module).*

Proof. This follows from the spanning set argument in [KL10, §3.2] together with the new bubble slides from Proposition 4.2.1. □

Lemma 7.2.2. *The trace $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})$ (resp. $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$) is generated as a $\Bbbk$-vector space (resp. $\mathbb{A}_{\underline{\lambda}}$-module) by the classes of Stendhal diagrams with no crossings between red and black strings.*

Proof. We show the statement for the undeformed category $\mathcal{X}^{\underline{\lambda},*}$. The argument for $\check{\mathcal{X}}^{\underline{\lambda},*}$ is identical. It suffices to show that $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})$ is generated as an vector space by the classes of diagrams in $B_{S,S}$ with no red-black crossings, taken over all Stendhal pairs S.

Take a Stendhal pair S and suppose $D \in B_{S,S}$ has a red-black crossing. Condition 4 in the definition of $B_{S,S}$ implies that D factors through a Stendhal pair S' with $\nu_{S'} < \nu_S$. Let $D = D_1 \cdot D_2$ be the corresponding factorization of D and set $D' = D_2 \cdot D_1 \in \mathcal{X}^{\underline{\lambda},*}(S',S')$. Then $[D] = [D']$ in the trace, D' lies in the span of $B_{S',S'}$, and $\nu_{S'} < \nu_S$. The claim follows by induction on C. □

Together with the results of [BHLW17] this allows us to determine the structure of the trace of the unstarred categories $\check{\mathcal{X}}^{\underline{\lambda}}$ and $\mathcal{X}^{\underline{\lambda}}$.

Corollary 7.2.3. *The Chern character maps*

$$h_{\mathcal{X}^{\underline{\lambda}}} : K_0^{\Bbbk}(\mathcal{X}^{\underline{\lambda}}) \longrightarrow \mathrm{Tr}(\mathcal{X}^{\underline{\lambda}}), \qquad h_{\check{\mathcal{X}}^{\underline{\lambda}}} : K_0^{\Bbbk}(\check{\mathcal{X}}^{\underline{\lambda}}) \longrightarrow \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda}}) \tag{7.9}$$

are isomorphisms, so by Corollary 6.2.1 both $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda}})$ *and* $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda}})$ *are isomorphic to* $V_q^{\mathbb{Z}}(\underline{\lambda})$ *with scalars extended to* $\Bbbk$*.*

Proof. We prove the claim for the undeformed tensor product category $\mathcal{X}^{\underline{\lambda}}$; the argument for $\check{\mathcal{X}}^{\underline{\lambda}}$ is identical. The field $\Bbbk$ is perfect since it has characteristic zero, and morphism spaces in $\mathcal{X}^{\underline{\lambda}}$ are finite-dimensional vector spaces by Lemma 7.2.1 so $h_{\mathcal{X}^{\underline{\lambda}}}$ is injective by [BHLW17, Proposition 2.4]. The same argument as Lemma 7.2.2 shows that $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda}})$ is spanned by classes of Stendhal diagrams of degree zero with no red-black crossings, so by [BHLW17, Corollary 6.3] $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda}})$ is spanned by concatenations of idempotents projecting onto divided powers, separated by red strings. Since $\mathcal{X}^{\underline{\lambda}}$ is idempotent complete, any idempotent lies in the image of $h_{\mathcal{X}^{\underline{\lambda}}}$ and so it is surjective. □

7.3 Upper bound on dimension

In this section we show that the dimension of $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})$ is bounded above by the dimension of the tensor product $W(\underline{\lambda})$ of local Weyl modules. This will allow us to show that $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ is free over $\mathbb{A}_{\underline{\lambda}}$ in Corollary 8.2.2 using upper semi-continuity of dimension.

Recall the cyclotomic quotients from §3.3. We wish to relate $\mathcal{X}^{\underline{\lambda}}$ and the undeformed cyclotomic quotients $\mathcal{U}^{\lambda^{(k)}}$ and their traces. Consider the product category

$$\mathcal{U}^{\underline{\lambda}} = \mathcal{U}^{\lambda^{(1)}} \times \cdots \times \mathcal{U}^{\lambda^{(n)}}. \tag{7.10}$$

Objects (resp. morphisms) in $\mathcal{U}^{\underline{\lambda}}$ are tuples of objects (resp. morphisms) with composition defined component-wise. The trace of a product category is isomorphic to the product of the traces.

We would like to define a functor sending a tuple of objects in $\mathcal{U}^{\underline{\lambda}}$ to the obvious horizontal composition in $\mathcal{X}^{\underline{\lambda}}$ with different components separated by red strings. But in $\mathcal{X}^{\underline{\lambda}}$ the cyclotomic-type relations (4.5) and (4.6) only allow us to pull black strings through red strings, while in cyclotomic quotients they give zero, so we would need to pass to a filtration (or stratification) of $\mathcal{X}^{\underline{\lambda}}$ for this functor to be well defined. In [Web17, §6] Webster constructed such a functor. To deal with taking quotients of objects in $\mathcal{X}^{\underline{\lambda}}$, he defined this as a functor from $\mathcal{U}^{\underline{\lambda}}$ to the category of representations of $\mathcal{X}^{\underline{\lambda}}$ - an abelian category in which $\mathcal{X}^{\underline{\lambda}}$ embeds fully-faithfully.

To avoid this complication we ignore objects; instead of defining a functor we construct homomorphisms from morphism spaces in $\mathcal{U}^{\underline{\lambda},*}$ to quotients of morphism spaces in $\mathcal{X}^{\underline{\lambda},*}$. In the trace this gives surjections

$$\operatorname{Tr}(\mathcal{U}^{\underline{\lambda},*})_\nu \longrightarrow \operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*})_{\leq\nu} / \operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*})_{<\nu}, \tag{7.11}$$

where the $\operatorname{Tr}(\mathcal{U}^{\underline{\lambda},*})_\nu$ and $\operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*})_{\leq\nu}$ form a grading and a filtration of $\operatorname{Tr}(\mathcal{U}^{\underline{\lambda},*})$ and $\operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*})$ respectively, both indexed by the poset C of compositions ν of length n (see §7.2). The upper bound on dimension follows from the trace decategorification of cyclotomic quotients.

Corollary 7.3.1. *The dimension of* $\operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*})$ *is bounded above:*

$$\dim_{\Bbbk} \operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*}) \leq \dim_{\Bbbk} W(\underline{\lambda}). \tag{7.12}$$

Proof. For $m \geq 0$ let $Q_m \in \operatorname{Ob}(\mathcal{X}^{\underline{\lambda}})$ be the direct sum of all Stendhal pairs $(\underline{i}, \kappa)$ with $\underline{i} \in (-I)^m$ (so there are m black strings and they are all oriented downward). By the categorified commutation relation for $\mathcal{E}_{+i}$ and $\mathcal{E}_{-i}$ and the fact that by (4.5) upward black strings commute with red strings, the objects Q_m additively generate $\mathcal{X}^{\underline{\lambda}}$. So [BHLW17, Lemma 2.1] implies that $\operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*})$ is isomorphic to the trace of the full subcategory of $\mathcal{X}^{\underline{\lambda},*}$ with objects Q_m. Since there are no morphisms between Q_m and $Q_{m'}$ for $m \neq m'$, this implies that the trace decomposes as a direct sum:

$$\operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*}) = \bigoplus_{m\geq 0} \operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*})_m, \tag{7.13}$$

where $\operatorname{Tr}(\mathcal{X}^{\underline{\lambda},*})_m$ is the image of the morphism space $\mathcal{X}^{\underline{\lambda},*}(Q_m, Q_m)$ in the trace.

We can refine this by a filtration indexed by the poset $C(m)$ of compositions of m (c.f. §3.3). For $\nu \in C(m)$ let $\mathcal{X}^{\underline{\lambda},*}(Q_m, Q_m)_{\leq \nu}$ and $\mathcal{X}^{\underline{\lambda},*}(Q_m, Q_m)_{<\nu}$ denote the subalgebras of $\mathcal{X}^{\underline{\lambda},*}(Q_m, Q_m)$ spanned by diagrams that factor through a Stendhal pair S with $\nu_S \leq \nu$ and $\nu_S < \nu$ respectively. Let $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})_{\leq \nu}$ and $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})_{<\nu}$ denote their images in the trace.

The spaces $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})_{\leq \nu}$ with $|\nu| = m$ form a filtration of $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})_m$. The components of the associated graded space are defined by

$$\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})_{\nu} = \mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})_{\leq \nu} \big/ \mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})_{<\nu} \tag{7.14}$$

for $\nu \in C(m)$. In particular, this implies that

$$\dim_{\Bbbk} \mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})_m \geq \sum_{\nu \in C(m)} \dim_{\Bbbk} \mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})_{\nu}. \tag{7.15}$$

Now consider the undeformed cyclotomic quotients $\mathcal{U}^{\lambda^{(k)}}$. For $k \in [1, n]$ and $m \geq 0$ let $P_m^{(k)} \in \mathrm{Ob}(\mathcal{U}^{\lambda^{(k)}})$ be the direct sum of all $\mathcal{E}_{\underline{i}} 1_{\lambda^{(k)}}$ with $\underline{i} \in (-I)^m$. For a composition $\nu \in C$, the tuple $P_\nu = (P^{(1)}_{\nu(1)}, \cdots, P^{(n)}_{\nu(n)})$ is an object in the product category $\mathcal{U}^{\underline{\lambda}}$.

As with $\mathcal{X}^{\underline{\lambda}}$, the objects P_ν additively generate $\mathcal{U}^{\underline{\lambda}}$ and there are no non-zero morphisms between P_ν and $P_{\nu'}$ for $\nu \neq \nu'$, so there is a direct sum decomposition

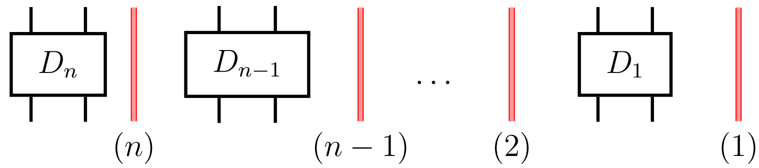
$$\mathrm{Tr}(\mathcal{U}^{\underline{\lambda},*}) = \bigoplus_{\nu} \mathrm{Tr}(\mathcal{U}^{\underline{\lambda},*})_{\nu}, \tag{7.16}$$

where $\mathrm{Tr}(\mathcal{U}^{\underline{\lambda},*})_\nu$ is the image of $\mathcal{U}^{\underline{\lambda},*}(P_\nu, P_\nu)$ in the trace.

For a composition ν with $|\nu| = m$, define an algebra homomorphism

$$\mathcal{U}^{\underline{\lambda},*}(P_\nu, P_\nu) \longrightarrow \mathcal{X}^{\underline{\lambda},*}(Q_m, Q_m)_{\leq \nu} \big/ \mathcal{X}^{\underline{\lambda},*}(Q_m, Q_m)_{<\nu} \tag{7.17}$$

by sending a tuple $(D_1, \ldots, D_n)$ of diagrams to the coset of



(7.18)

If D_k has a downward string with $\lambda_i^{(k)}$ dots (or an upward string) at the far right then by (4.6) (resp. (4.5)) we can pass that string through the red (k)-string in $\mathcal{X}^{\underline{\lambda}}$. So we have an element of $\mathcal{X}^{\underline{\lambda},*}(Q_m, Q_m)_{<\nu}$ and this map respects the cyclotomic relations (3.27).

It isn't obvious that this homomorphism is well defined since it doesn't respect the weights of regions and the defining relations of $\mathcal{U}^{\lambda}$ and $\mathcal{X}^{\underline{\lambda}}$ depend on these weights. However, by [Web17, Proposition 3.13] the algebra $\mathcal{U}^{\underline{\lambda},*}(P_\nu, P_\nu)$ is generated by diagrams whose black strings have no critical points (that is they never turn back on themselves) subject only to KLR relations and the cyclotomic relation. Since these relations are independent of the weight of the region, the map is well defined.

Passing to the trace we get a linear map

$$\mathrm{Tr}(\mathcal{U}^{\underline{\lambda},*})_\nu \longrightarrow \mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})_\nu \tag{7.19}$$

for any composition ν. By Lemma 7.2.2, $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})$ is spanned by diagrams with no red-black crossings, so this is surjective. Since the trace commutes with products, Theorem 7.1.1 implies that

$$\begin{aligned}\dim_{\Bbbk} W(\underline{\lambda}) &= \sum_{\nu \in C} \dim_{\Bbbk} \mathrm{Tr}(\mathcal{U}^{\underline{\lambda},*})_{\nu} \\ &\geq \sum_{\nu \in C} \dim_{\Bbbk} \mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})_{\nu} \\ &\geq \sum_{m \geq 0} \dim_{\Bbbk} \mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})_{m} \\ &= \dim_{\Bbbk} \mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*})\end{aligned} \tag{7.20}$$

as required. □

Chapter 8

Proof of Theorem A

In this chapter we prove the main theorem of Part II on the trace decategorification of $\mathcal{X}^{\underline{\lambda},*}$ and $\check{\mathcal{X}}^{\underline{\lambda},*}$. First we introduce some notation to make the statement more precise: for $u_1, \ldots, u_n \in \dot{U}(\mathfrak{g}[t])$, recursively define elements

$$w(u_1, \ldots, u_k) \in \mathbb{W}(\lambda^{(1)}) \otimes \cdots \mathbb{W}(\lambda^{(k)}) \tag{8.1}$$

for $k \in [1, n]$ by setting $w(u_1) := u_1 w_{\lambda^{(1)}}$ and

$$w(u_1, \ldots, u_k) := u_k(w(u_1, \ldots, u_{k-1}) \otimes w_{\lambda^{(k)}}). \tag{8.2}$$

So $w(u_1, \ldots, u_n)$ is a well-defined element of $\mathbb{W}(\underline{\lambda})$.

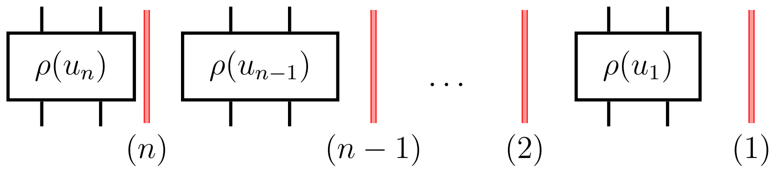
Recall the isomorphism $\rho : \dot{U}(\mathfrak{g}[t]) \to \mathrm{Tr}(\mathcal{U}^*)$ from Theorem 7.1.1.

Theorem 8.0.2. *There are isomorphisms*

$$\mathbb{W}(\underline{\lambda}) \longrightarrow \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}), \qquad W(\underline{\lambda}) \longrightarrow \mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*}) \tag{8.3}$$

of graded $(\dot{U}(\mathfrak{g}[t]), \mathbb{A}_{\underline{\lambda}})$-bimodules and graded $\dot{U}(\mathfrak{g}[t])$-modules respectively, sending $w(u_1, \ldots, u_n)$

to the class of the diagram



(8.4)

for any $u_1, \ldots, u_n \in \dot{U}(\mathfrak{g}[t])$.

In §8.1 we construct a $\dot{U}(\mathfrak{g}[t])$-module homomorphism from the tensor product of Verma-like modules $M(\underline{\lambda})$ to $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ and show it is surjective and compatible with the right actions by symmetric functions. Then in §8.2 we use the results of Chapter 5 to show that this descends to an isomorphism at the generic point:

$$\mathbb{W}(\underline{\lambda}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K} \longrightarrow \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K}. \tag{8.5}$$

Finally we use the upper bound on dimension from Proposition 7.3.1 and upper semi-continuity of dimension to establish the theorem.

8.1 Homomorphism from $M(\underline{\lambda})$

Recall the Verma-like modules $M(\lambda^{(k)})$ from §2.3. For $u_1, \ldots, u_n \in \dot{U}(\mathfrak{g}[t])$, recursively define elements

$$m(u_1, \ldots, u_k) \in M(\lambda^{(1)}) \otimes \cdots \otimes M(\lambda^{(k)}) \tag{8.6}$$

by analogy with (8.2), so $m(u_1, \ldots, u_n) \in M(\underline{\lambda})$.

Lemma 8.1.1. *There is a unique homomorphism of graded* $\dot{U}(\mathfrak{g}[t])$*-modules*

$$\varphi : M(\underline{\lambda}) \longrightarrow \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}) \tag{8.7}$$

sending $m(u_1, \ldots, u_n)$ *to the class of the diagram* (8.4) *for any* $u_1, \ldots, u_n \in \dot{U}(\mathfrak{g}[t])$.

Proof. We proceed by induction on n; the number of tensor factors. If $n = 1$ then by Corollary 6.2.2 there is a 2-natural isomorphism between $\check{\mathcal{X}}^{\lambda,*}$ and the deformed cyclotomic quotient $\check{\mathcal{U}}^{\lambda,*}$, so the claim follows from Theorem 7.1.1.

Assume the claim holds for $n \in \mathbb{N}$ and take $\mu \in X^+$. Recall the Lie algebra $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}[t]$ and the one dimensional $\mathfrak{p}$-module $\mathbb{k}_\mu$ from §2.3. The $\dot{U}(\mathfrak{g}[t])$-module map from the inductive hypothesis induces a $\mathfrak{p}$-module map $M(\underline{\lambda}) \to \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$. There is a map of graded $\mathbb{k}$-vector spaces from $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ to $\mathrm{Tr}(\check{\mathcal{X}}^{(\underline{\lambda},\mu),*})$ induced by placing the new red string at the far left of the diagram. By (4.1) this increases the weight by μ and so there is an induced $\mathfrak{h}$-module map from $M(\underline{\lambda}) \otimes \mathbb{k}_\mu$ to $\mathrm{Tr}(\check{\mathcal{X}}^{(\underline{\lambda},\mu),*})$. In fact this is a $\mathfrak{p}$-module map by (4.4) and (4.5). By Frobenius reciprocity and the tensor identity this yields a $\mathfrak{g}[t]$-module homomorphism $M(\underline{\lambda}, \mu) \to \mathrm{Tr}(\check{\mathcal{X}}^{(\underline{\lambda},\mu),*})$ of the desired form. Uniqueness is clear. □

Recall that $M(\underline{\lambda})$ carries a right action by $\Pi^{\otimes n}$. We can consider $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ as a right $\Pi^{\otimes n}$-module via projection $\bigotimes a_k : \Pi^{\otimes n} \to \mathbb{A}_{\underline{\lambda}}$.

Proposition 8.1.2. *The map φ above is a surjective homomorphism of $(\dot{U}(\mathfrak{g}[t]), \Pi^{\otimes n})$-bimodules.*

Proof. Take $u_1, \ldots, u_n \in \dot{U}(\mathfrak{g}[t])$. By the definition of the action of $\Pi^{\otimes n}$ and the coproduct on $U(\mathfrak{g}[t])$, a power sum symmetric function $p_{i,r}$ in the kth copy of Π in $\Pi^{\otimes n}$ sends $m(u_1, \ldots, u_n)$ to

$$m(u_1, \ldots, u_k(\xi_i \otimes t^r), \ldots, u_n) - m(u_1, \ldots, (\xi_i \otimes t^r)u_{k-1}, \ldots, u_n) \tag{8.8}$$

By Theorem 7.1.1 and the bubbles slides in Proposition 4.2.1, this is mapped under φ to

$$\varphi(m(u_1, \ldots, u_n)) \cdot a_k(p_{i,r}). \tag{8.9}$$

Surjectivity follows from the spanning set Lemma 7.2.1 and surjectivity of ρ. □

8.2 Unfurling and the trace

In this section we apply the trace decategorification results from [BHLW17] to the unfurled 2-representation on $\mathcal{G}^{\underline{\lambda}}$ from Chapter 5. We use this to determine the structure of $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ at the generic point and apply upper semi-continuity of dimension to show that φ descends to an isomorphism from $\mathbb{W}_{\Bbbk}(\underline{\lambda})$. Since the underlying field and Cartan datum vary in this section we take care to include them in notation.

The trace of $\mathcal{G}^{\underline{\lambda}}$

Recall from Chapter 5 that the set $\tilde{I} = I \times Z$ indexes simple roots for the unfurled Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}^{\oplus Z}$ and there is a 2-representation of the corresponding categorified quantum group $\dot{\mathcal{U}}^*_{\mathbb{K}}(\tilde{\mathfrak{g}})$ on the idempotent completion $\mathcal{G}^{\underline{\lambda}}$ of $\check{\mathcal{X}}^{\underline{\lambda},*} \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K}$, where $\mathbb{K} = \overline{\Bbbk(Z)}$. By Theorem 7.1.1 there is an induced $\dot{U}_{\mathbb{K}}(\tilde{\mathfrak{g}}[t])$-action on $\mathrm{Tr}(\mathcal{G}^{\underline{\lambda}})$.

We need to be a little careful with interpreting this action: the isomorphism ρ is expressed diagrammatically, but the diagrammatics in $\dot{\mathcal{U}}^*_{\mathbb{K}}(\tilde{\mathfrak{g}})$ doesn't match the diagrammatics of $\mathcal{G}^{\underline{\lambda}}$. In particular, for $(i,z) \in \pm\tilde{I}$ and $r \in \mathbb{N}$, $e_{(i,z)} \otimes t^r$ acts on the class of a morphism in $\mathrm{Tr}(\mathcal{G}^{\underline{\lambda}})$ by applying the functor $\mathcal{E}_{(i,z)}$ and acting by $\tilde{y}\tilde{\varepsilon}_{(i,z)} \in \tilde{R}_1$ - the KLR algebra for $\tilde{\mathfrak{g}}$. By Proposition 5.2.2 this is the same as applying $\mathcal{E}_i$ and acting by $(y-z)^r\varepsilon_{(i,z)} \in R_1$. So the action of $\dot{U}_{\mathbb{K}}(\tilde{\mathfrak{g}}[t])$ on $\mathrm{Tr}(\mathcal{G}^{\underline{\lambda}})$ is "twisted" according to $z \in Z$.

We wish to modify the action of $\dot{U}_{\mathbb{K}}(\tilde{\mathfrak{g}}[t])$ on $\mathrm{Tr}(\mathcal{G}^{\underline{\lambda}})$ to remove this twist. For $z \in Z$, the current algebra $U_{\mathbb{K}}(\mathfrak{g}[t])$ of $\mathfrak{g}$ has an automorphism σ_z given by

$$\sigma_z(x \otimes t^r) = x \otimes (t+z)^r \tag{8.10}$$

for $x \in \mathfrak{g}$ and $r \geq 0$. For any $U_{\mathbb{K}}(\mathfrak{g}[t])$-module M we define a *z-twisted action* on M by

$$u * m = \sigma_z(u)m \tag{8.11}$$

for $u \in U_{\mathbb{K}}(\mathfrak{g}[t])$ and $m \in M$. Let σ_Z denote the automorphism of $U_{\mathbb{K}}(\tilde{\mathfrak{g}}[t])$ which restricts to σ_z on the copy of $\mathfrak{g}[t]$ indexed by z under the identification $\tilde{\mathfrak{g}}[t] = \mathfrak{g}[t]^{\oplus Z}$. We define the *$Z$-twisted action* on a $U_{\mathbb{K}}(\tilde{\mathfrak{g}}[t])$-module as above.

Let $\mathrm{Tr}^Z(\mathcal{G}^{\underline{\lambda}})$ denote the trace of $\mathcal{G}^{\underline{\lambda}}$ under the Z-twisted action of $\dot{U}_{\mathbb{K}}(\tilde{\mathfrak{g}}[t])$. Now $e_{(i,z)} \otimes t^r$ acts on the class of a morphism by applying $\mathcal{E}_i$ and acting by $y^r \varepsilon_{(i,z)}$ or, in diagrammatic terms, by adding a black i-string at the left, projecting to the generalized z-eigenspace of a dot, and applying r dots.

Recall from Theorem 5.3.3 that there is a 2-natural isomorphism

$$\eta : \mathcal{U}_{\mathbb{K}}^{\tilde{\lambda},*}(\tilde{\mathfrak{g}}) \longrightarrow \mathcal{G}^{\underline{\lambda}} \tag{8.12}$$

where $\mathcal{U}_{\mathbb{K}}^{\tilde{\lambda},*}(\tilde{\mathfrak{g}})$ is the cyclotomic quotient of $\dot{\mathcal{U}}_{\mathbb{K}}^{*}(\tilde{\mathfrak{g}})$ of weight

$$\tilde{\lambda} = \sum_{i \in I} \sum_{z \in Z_i} \Lambda_{(i,z)}. \tag{8.13}$$

By Theorem 7.1.1, the trace of $\mathcal{U}_{\mathbb{K}}^{\tilde{\lambda},*}(\tilde{\mathfrak{g}})$ is isomorphic to the local Weyl module for $\dot{U}_{\mathbb{K}}(\tilde{\mathfrak{g}}[t])$ of weight $\tilde{\lambda}$, so taking the trace of η and twisting by Z yields an isomorphim of $\dot{U}_{\mathbb{K}}(\tilde{\mathfrak{g}}[t])$-modules:

$$\mathrm{Tr}^Z(\eta) : W_{\mathbb{K}}^{Z}(\tilde{\lambda}) \longrightarrow \mathrm{Tr}^Z(\mathcal{G}^{\underline{\lambda}}), \tag{8.14}$$

where $W_{\mathbb{K}}^{Z}(\tilde{\lambda})$ denotes the local Weyl module under the Z-twisted action.

Application to $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$

By [CFK10, Corollary 6], there is an isomorphism of $\dot{U}_{\mathbb{K}}(\mathfrak{g}[t])$-modules

$$\mathbb{W}_{\Bbbk}(\underline{\lambda}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K} \longrightarrow \bigotimes_{i \in I} \bigotimes_{z \in Z_i} W_{\mathbb{K}}^{z}(\Lambda_i), \tag{8.15}$$

where $W_{\mathbb{K}}^{z}(\Lambda_i)$ denotes the z-twisted $U_{\mathbb{K}}(\mathfrak{g}[t])$-module structure on the local Weyl module

$W_{\mathbb{K}}(\Lambda_i)$. The module on the right coincides with $W^Z_{\mathbb{K}}(\tilde{\lambda})$ using the identification $\tilde{\mathfrak{g}} = \mathfrak{g}^{\oplus Z}$. Moreover, since the trace is invariant under idempotent completion, there are $\mathbb{K}$-linear isomorphisms

$$\mathrm{Tr}^Z(\mathcal{G}^{\underline{\lambda}}) \longrightarrow \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K} \longrightarrow \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K}. \tag{8.16}$$

Now we can relate $\mathbb{W}_{\Bbbk}(\underline{\lambda})$ and $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ at the generic point:

Proposition 8.2.1. *The composition*

$$\mathbb{W}_{\Bbbk}(\underline{\lambda}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K} \longrightarrow W^Z_{\mathbb{K}}(\tilde{\lambda}) \longrightarrow \mathrm{Tr}^Z(\mathcal{G}^{\underline{\lambda}}) \longrightarrow \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K} \tag{8.17}$$

of the $\mathbb{K}$*-linear isomorphisms* (8.15)*,* (8.14)*, and* (8.16) *sends* $w(u_1, \ldots, u_n)$ *to the class of the diagram* (8.4) *for any* $u_1, \ldots, u_n \in \dot{U}_{\Bbbk}(\mathfrak{g}[t])$*.*

Proof. The claim is immediate if all $u_k = 1$. Assume the statement holds for an element $w := w(u_1, \ldots, u_k, 1, \ldots, 1)$ for some $k \in [1, n]$ and $u_1, \ldots, u_k \in \dot{U}_{\Bbbk}(\mathfrak{g}[t])$ and take $i \in \pm I$ and $r \in \mathbb{N}$. We will deduce the claim for $w^+ := (u_1, \ldots, (e_i \otimes t^r)u_k, 1, \ldots, 1)$. The proposition follows by induction.

By assumption, the composition (8.17) sends w to the class of an endomorphism D of a Stendhal pair S with no black strings to the left of the red $(k+1)$-string. Let S^+ be the Stendhal pair obtained from S as in Corollary 5.3.2; that is, by placing a black string labelled by $|i|$ and oriented according to the sign of i immediately to the right of this red string. Let D^+ be the endomorphism of S^+ obtained from D by inserting this new string and acting on it with r dots. We need to show that $w^+ \mapsto [D^+]$.

We obtain w^+ from $w \in \mathbb{W}_{\Bbbk}(\underline{\lambda})$ by applying $e_i \otimes t^r$ to the first k tensor factors. So in $W^Z_{\mathbb{K}}(\tilde{\lambda})$,

$$w^+ = \sum_{z \in Z^{(\leq k)}} (e_{(i,z)} \otimes t^r) * w \tag{8.18}$$

under the Z-twisted action, where $Z^{(\leq k)} = \bigcup_{l \leq k} Z^{(l)}$.

Since $\mathrm{Tr}^Z(\eta)$ is an isomorphism of $U_{\mathbb{K}}(\tilde{\mathfrak{g}}[t])$-modules, it sends w^+ to

$$\sum_{z \in Z^{(\leq k)}} (e_{(i,z)} \otimes t^r) * [D]. \tag{8.19}$$

By the discussion in §8.2 this is the class of r dots acting on the left-most string in

$$\sum_{z \in Z^{(\leq k)}} \mathcal{E}_{(i,z)} D. \tag{8.20}$$

The isomorphism in Corollary 5.3.2 sends this to D^+ and so they are equal in the trace. The claim follows. ☐

Corollary 8.2.2. *The trace* $\mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*})$ *is a free graded right* $\mathbb{A}_{\underline{\lambda}}$*-module.*

Proof. By Proposition 8.2.1,

$$\dim_{\mathbb{K}} \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K} = \dim_{\mathbb{K}} \mathbb{W}_{\Bbbk}(\underline{\lambda}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K}. \tag{8.21}$$

If $\Bbbk$ denotes the unique simple graded $\mathbb{A}_{\underline{\lambda}}$-module then we have $W_{\Bbbk}(\underline{\lambda}) \cong \mathbb{W}_{\Bbbk}(\underline{\lambda}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \Bbbk$ and $\mathrm{Tr}(\mathcal{X}^{\underline{\lambda},*}) \cong \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \Bbbk$, so Corollary 7.3.1 implies that

$$\dim_{\Bbbk} \mathrm{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \Bbbk \leq \dim_{\Bbbk} \mathbb{W}_{\Bbbk}(\underline{\lambda}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \Bbbk. \tag{8.22}$$

By Theorem 2.3.1, $\mathbb{W}_{\Bbbk}(\underline{\lambda})$ is a free graded right $\mathbb{A}_{\underline{\lambda}}$-module so the right hand terms in the above equations are equal by upper semi-continuity of dimension (see Lemma 6.1.1). Now the claim follows by appealing to upper semi-continuity again. ☐

Now we can prove the main theorem.

Proof of Theorem 8.0.2. It suffices to construct the isomorphism from $\mathbb{W}(\underline{\lambda})$. There are surjective homomorphisms of graded $(\dot{U}(\mathfrak{g}[t]), \mathbb{A}_{\underline{\lambda}})$-bimodules

$$\mathbb{W}(\underline{\lambda}) \xleftarrow{\psi} M(\underline{\lambda}) \otimes_{\Pi^{\otimes n}} \mathbb{A}_{\underline{\lambda}} \xrightarrow{\varphi} \operatorname{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}) \tag{8.23}$$

coming from the natural projection from $M(\underline{\lambda})$ onto $\mathbb{W}(\underline{\lambda})$ and the map in Proposition 8.1.2 respectively. By Theorem 2.3.1 and Corollary 8.2.2, all three of these $\mathbb{A}_{\underline{\lambda}}$-modules are graded free, so $\operatorname{Ker}(\varphi)$ and $\operatorname{Ker}(\psi)$ are also free (they are flat and so free since $\mathbb{A}_{\underline{\lambda}}$ is graded local).

By Proposition 8.2.1, these factor through an isomorphism at the generic point:

$$\begin{array}{ccc} & M(\underline{\lambda}) \otimes_{\Pi^{\otimes n}} \mathbb{K} & \\ {\scriptstyle \psi\otimes\mathbb{K}} \swarrow & & \searrow {\scriptstyle \varphi\otimes\mathbb{K}} \\ \mathbb{W}(\underline{\lambda}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K} & \longrightarrow & \operatorname{Tr}(\check{\mathcal{X}}^{\underline{\lambda},*}) \otimes_{\mathbb{A}_{\underline{\lambda}}} \mathbb{K} \end{array} \tag{8.24}$$

So $\operatorname{Ker}(\varphi\otimes\mathbb{K}) = \operatorname{Ker}(\psi\otimes\mathbb{K})$. Freeness implies that $\operatorname{Ker}(\varphi) = \operatorname{Ker}(\psi)$ and so φ factors through an isomorphism from $\mathbb{W}(\underline{\lambda})$ as claimed. □

Part II

Graded super duality

Chapter 9

Combinatorics

For $r \in \mathbb{N} \cup \{\infty\}$, let $I_r = [1-r, r-1]$ and let $\mathfrak{sl}_{I_r}$ be the special linear Lie algebra of traceless complex matrices with rows and columns indexed by $I_r^+ := I_r \cup (I_r + 1)$ (for $r = \infty$ we restrict to matrices with finitely many non-zero entries in each row and column). Let $f_i := e_{i,i+1}$ (resp. $e_i := e_{i+1,i}$) denote the matrix unit with 1 in position $(i,\, i+1)$ (resp. $(i+1,\, i)$) and 0s elsewhere.

Define the weight and root lattices of $\mathfrak{sl}_{I_r}$ by

$$P_r := \bigoplus_{i \in I_r} \mathbb{Z}\varpi_i \qquad \text{and} \qquad Q_r := \bigoplus_{i \in I_r} \mathbb{Z}\alpha_i \tag{9.1}$$

respectively, where $\alpha_i := 2\varpi_i - \varpi_{i-1} - \varpi_{i+1}$ and we interpret ϖ_i as 0 if $i \notin I_r$. We will denote the non-degenerate pairing $P_r \times Q_r \to \mathbb{Z}$ by $(\varpi, \alpha) \mapsto \varpi \cdot \alpha$.

Let $P_r^+ \subseteq P_r$ (resp. $Q_r^+ \subseteq Q_r$) denote the $\mathbb{Z}_{\geq 0}$-span of the ϖ_i (resp. α_i). Define the dominance ordering on P_r by declaring that $\beta \geq \gamma$ if $\beta - \gamma \in Q_r^+$. For $i \in I_r^+$ let $\varepsilon_i := \varpi_i - \varpi_{i-1}$ where again we interpret ϖ_i as 0 if $i \notin I_r$.

9.1 Exterior powers

Let V_r be the natural $\mathfrak{sl}_{I_r}$-module with standard basis $\{v_i \mid i \in I_r^+\}$. Let W_r be the dual of V_r with dual basis $\{w_i \mid i \in I_r^+\}$. We will denote the exterior powers $\bigwedge^n V_r$ and $\bigwedge^n W_r$ by $\bigwedge^{n,0} V_r$ and $\bigwedge^{n,1} V_r$, respectively. For $n \in \mathbb{N}$ and $\epsilon \in \{0,1\}$ let $\Xi_{r;n,\epsilon}$ be the set of 01-tuples $\lambda = (\lambda_i)_{i \in I_r^+}$ such that

$$\left|\left\{ i \in I_r^+ \,\middle|\, \lambda_i \neq \epsilon \right\}\right| = n. \tag{9.2}$$

This set indexes the monomial basis of $\bigwedge^{n,\epsilon} V_r$: if $\lambda \in \Xi_{r;n,\epsilon}$ and $i_1 > \cdots > i_n$ with $\lambda_{i_1}, \ldots, \lambda_{i_n} \neq \epsilon$ then set

$$v_\lambda := \begin{cases} v_{i_1} \wedge \cdots \wedge v_{i_n} & \text{if } \epsilon = 0 \\ w_{i_n} \wedge \cdots \wedge w_{i_1} & \text{if } \epsilon = 1. \end{cases} \tag{9.3}$$

The weight of v_λ is

$$|\lambda| := \sum_{\substack{i \in I_r^+ \\ \lambda_i \neq \epsilon}} (-1)^\epsilon \varepsilon_i \in P_r. \tag{9.4}$$

We wish to consider spaces of semi-infinite wedges. We will construct these as direct limits of the modules above. For $r < s < \infty$ there is an $\mathfrak{sl}_{I_r}$-module embedding $\bigwedge^{r,\epsilon} V_r \hookrightarrow \bigwedge^{s,\epsilon} V_s$ given by linearly extending the assignment

$$v_\lambda \longmapsto \begin{cases} v_\lambda \wedge v_{-r} \wedge v_{-r-1} \wedge \cdots \wedge v_{1-s} & \text{if } \epsilon = 0 \\ v_\lambda \wedge w_{r+1} \wedge w_{r+2} \wedge \cdots \wedge w_s & \text{if } \epsilon = 1. \end{cases} \tag{9.5}$$

The corresponding embedding $\Xi_{r;r,\epsilon} \hookrightarrow \Xi_{s;s,\epsilon}$ is given by extending a 01-tuple $\lambda = (\lambda_j)_{j \in I_r^+}$ by setting

$$\lambda_j = \begin{cases} 0 & \text{if } r < j \le s \\ 1 & \text{if } 1-s \le j < 1-r. \end{cases} \tag{9.6}$$

Let $\Xi_{\infty,\epsilon} := \varinjlim \Xi_{r;r,\epsilon}$ and let $\bigwedge^{\infty,\epsilon} V := \varinjlim \bigwedge^{r,\epsilon} V_r$, an $\mathfrak{sl}_{\mathbb{Z}}$-module. Then $\Xi_{\infty,\epsilon}$ parameterizes the natural monomial basis of $\bigwedge^{\infty,\epsilon} V$, the union of the monomial bases of the $\bigwedge^{r,\epsilon} V_r$. It is natural to think of an element $\lambda \in \Xi_{\infty,\epsilon}$ as a 01-tuple $\lambda = (\lambda_i)_{i\in\mathbb{Z}}$ and of the corresponding element $v_\lambda \in \bigwedge^{\infty,\epsilon} V$ as a semi-infinite wedge.

9.2 Tensor products of wedges

Once and for all fix $l \in \mathbb{N}$, $n_1, \dots, n_l \in \mathbb{N}$, and $c_1, \dots, c_l \in \{0, 1\}$.

We will often suppress dependence on this data in our notation. For $r \in \mathbb{N} \cup \{\infty\}$ and $\epsilon \in \{0,1\}$ write $\underline{r} = (n_1, \dots, n_l, r)$ and $\underline{\epsilon} = (c_1, \dots, c_l, \epsilon)$. Our main objects of study will be modules of the form

$$\bigwedge^{\underline{r},\underline{\epsilon}} V_r := \bigwedge^{n_1,c_1} V_r \otimes \cdots \otimes \bigwedge^{n_l,c_l} V_r \otimes \bigwedge^{r,\epsilon} V_r. \tag{9.7}$$

This has a monomial basis indexed by the set

$$\Xi_{\underline{r},\underline{\epsilon}} := \Xi_{r;n_1,c_1} \times \cdots \times \Xi_{r;n_l,c_l} \times \Xi_{r;r,\epsilon}, \tag{9.8}$$

where for $\lambda = (\lambda^i)_{1\le i\le l+1} \in \Xi_{\underline{r},\underline{\epsilon}}$ we set

$$v_\lambda := v_{\lambda^1} \otimes \cdots \otimes v_{\lambda^{l+1}}. \tag{9.9}$$

It will sometimes be convenient to think of λ as a 01-matrix $\lambda = (\lambda^i_j)_{1\le i\le l+1, j\in I_r^+}$ whose i^{th} row is λ^i.

For $r < \infty$ and $\lambda \in \Xi_{\underline{r},\underline{\epsilon}}$, v_λ has weight

$$|\lambda| = |\lambda^1| + \cdots + |\lambda^{l+1}| \in P_r \tag{9.10}$$

Define a poset structure on $\Xi_{\underline{r},\underline{\epsilon}}$ by declaring that if $\lambda, \mu \in \Xi_{\underline{r},\underline{\epsilon}}$ then $\lambda \le \mu$ if and only if $|\lambda| = |\mu|$ and $|\lambda^1| + \cdots + |\lambda^i| \ge |\mu^1| + \cdots + |\mu^i|$ for each $1 \le i \le l+1$.

For $r < s < \infty$ there are maps $\bigwedge^{\underline{r},\underline{\epsilon}} V_r \hookrightarrow \bigwedge^{\underline{s},\underline{\epsilon}} V_s$ induced by inclusion on the first l factors and (9.5) on the last. There is a corresponding map $\Xi_{\underline{r},\underline{\epsilon}} \hookrightarrow \Xi_{\underline{s},\underline{\epsilon}}$ given by setting 'new' λ^i_j equal to c_i if $1 \le i \le l$ and by (9.6) on the last factor. So we have

$$\bigwedge^{\underline{\infty},\underline{\epsilon}} V = \varinjlim \bigwedge^{\underline{r},\underline{\epsilon}} V_r, \qquad\qquad \Xi_{\underline{\infty},\underline{\epsilon}} = \varinjlim \Xi_{\underline{r},\underline{\epsilon}}. \tag{9.11}$$

We freely identify elements of $\Xi_{\underline{r},\underline{\epsilon}}$ with their image in $\Xi_{\underline{\infty},\underline{\epsilon}}$ and regard $\Xi_{\underline{\infty},\underline{\epsilon}}$ as the union $\bigcup_{r=1}^{\infty} \Xi_{\underline{r},\underline{\epsilon}}$. Similarly we will consider the $\bigwedge^{\underline{r},\underline{\epsilon}} V_r$ as $\mathfrak{sl}_{I_r}$-submodules of $\bigwedge^{\underline{\infty},\underline{\epsilon}} V$.

The maps (9.11) are order preserving so there is an induced partial order on $\Xi_{\underline{\infty},\underline{\epsilon}}$.

Remark 9.2.1. The maps (9.11) are order preserving but they are *not* weight preserving. If $\lambda \in \Xi_{\underline{r},\underline{\epsilon}} \subseteq \Xi_{\underline{\infty},\underline{\epsilon}}$ we will write $|\lambda|_s$ ($s \ge r$) to denote the weight of v_λ when considering λ as an element of $\Xi_{\underline{s},\underline{\epsilon}}$.

Remark 9.2.2. Most of our notations match those of [BLW17, §2.2]. The only notable differences are that our V_r and W_r correspond to V_{I_r} and W_{I_r} in *loc. cit.* and, when $r < \infty$, our $\Xi_{r;n,\epsilon}$ and $\Xi_{\underline{r},\underline{\epsilon}}$ correspond to their $\Lambda_{I_r;n,\epsilon}$ and $\Lambda_{I_r;\underline{r},\underline{\epsilon}}$ respectively. They do not consider semi-infinite wedges so do not have notation for these indexing sets when $r = \infty$.

The following straightforward proposition is the basis for super-duality.

Proposition 9.2.3. *We have equality of posets* $\Xi_{\underline{\infty},\underline{0}} = \Xi_{\underline{\infty},\underline{1}}$ *and this induces an* $\mathfrak{sl}_{\mathbb{Z}}$*-module* *isomorphism*

$$\begin{aligned} \bigwedge\nolimits^{\frac{\infty}{2},0} V &\longrightarrow \bigwedge\nolimits^{\frac{\infty}{2},1} V \\ v_\lambda &\longmapsto v_\lambda \end{aligned} \tag{9.12}$$

Chapter 10

Tensor product categorifications

In [CR08] Chuang and Rouquier defined the notion of a categorical $\mathfrak{sl}_2$-action on an abelian category. This definition was extended to any Lie algebra of type A in [Rou08]. Roughly, a categorical action on an abelian category $\mathcal{C}$ consists of pairs of adjoint endofunctors $(\mathcal{E}_i, \mathcal{F}_i)$ on $\mathcal{C}$ indexed by simple roots such that powers of $\mathcal{F} = \bigoplus \mathcal{F}_i$ carry an action of the degenerate affine Hecke algebra and $K_0^{\mathbb{C}}(\text{proj-}\mathcal{C})$ has the structure of an integrable representation of the Lie algebra.

Affine Hecke algebras are closely related to type A KLR algebras. In [Rou08] proved an isomorphism between localizations of affine Hecke algebras and KLR algebras and used this to show that the data of a categorical action is equivalent to an integrable 2-representation of the categorified quantum group on $\mathcal{C}$ (see also [BK09] where the authors constructed isomorphisms between cyclotomic quotients of the two algebras).

In the remaining chapters we use the language of categorical actions in place of that of 2-representations. In particular, the KLR algebra is replaced by the affine Hecke algebra and 2-natural transformations (resp. isomorphisms) between 2-representations are replaced by *strongly equivariant functors* (resp. isomorphisms).

10.1 Recollection of definitions

We recall some important definitions. See [BLW17, §2.3-2.6] for more details.

Definition 10.1.1. The (degenerate) *affine Hecke algebra* AH_k is the $\mathbb{C}$-algebra with generators $x_1, \dots, x_k, t_1, \dots t_{k-1}$ such that $x_1, \dots, x_k$ generate a polynomial algebra $\mathbb{C}[x_1, \dots, x_k]$, $t_1, \dots, t_{k-1}$ generate a copy of the symmetric group S_k with t_j corresponding to the transposition $(j\ j+1)$, and

$$t_i x_j - x_{t_i(j)} t_i = \begin{cases} 1 & \text{if } j = i+1 \\ -1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \tag{10.1}$$

For $\varpi \in P_r^+$, the *cyclotomic affine Hecke algebra* AH_k^{ϖ} is the quotient of AH_k by the two-sided ideal generated by $\prod_{i \in I_r} (x_1 - i)^{\varpi \cdot \alpha_i}$.

The image of $\mathbb{C}[x_1, \dots, x_k]$ in AH_k^{ϖ} is a finite-dimensional commutative algebra, so it contains mutually orthogonal idempotents $\left\{ 1_{\boldsymbol{i}} \,\middle|\, \boldsymbol{i} = (i_1, \dots, i_k) \in \mathbb{C}^k \right\}$ such that $1_{\boldsymbol{i}}$ acts on any finite-dimensional AH_k^{ϖ}-module M as projection onto

$$M_{\boldsymbol{i}} = \left\{ v \in M \,\middle|\, (x_j - i_j)^N v = 0 \text{ for } 1 \leq j \leq k \text{ and } N \gg 0 \right\}. \tag{10.2}$$

Define

$$AH_{r,k}^{\varpi} := \bigoplus_{\boldsymbol{i}, \boldsymbol{j} \in I_r^k} 1_{\boldsymbol{i}} AH_k^{\varpi} 1_{\boldsymbol{j}} = AH_k^{\varpi} \;/\; \langle 1_{\boldsymbol{i}} \,|\, \boldsymbol{i} \notin I_r^k \rangle. \tag{10.3}$$

Definition 10.1.2. Let $\mathcal{C}$ be a Schurian category with an endofunctor $\mathcal{F} \in \operatorname{End}(\mathcal{C})$, a right adjoint $\mathcal{E}$ to $\mathcal{F}$ (with a fixed adjunction), and natural transformations $x \in \operatorname{End}(\mathcal{F})$ and $t \in \operatorname{End}(\mathcal{F}^2)$. Via the adjunction there are induced natural transformations $x \in \operatorname{End}(\mathcal{E})$ and

$t \in \mathrm{End}\,(\mathcal{E}^2)$. For $i \in I_r$ let $\mathcal{F}_i$ (resp. $\mathcal{E}_i$) be the endofunctor of $\mathcal{C}$ defined by setting $\mathcal{F}_i M$ (resp. $\mathcal{E}_i M$) equal to the generalized i-eigenspace of x on $\mathcal{F}M$ (resp. $\mathcal{E}M$). We say this data defines a *categorical* $\mathfrak{sl}_{I_r}$*-action* on $\mathcal{C}$ if:

(CA1) $\mathcal{F}$ decomposes as a direct sum $\mathcal{F} = \bigoplus_{i \in I_r} \mathcal{F}_i$;

(CA2) the endomorphisms $x_j := \mathcal{F}^{k-j} x \mathcal{F}^{j-1}$ and $t_j := \mathcal{F}^{k-j-1} t \mathcal{F}^{j-1}$ of $\mathcal{F}^k$ satisfy the relations of the affine Hecke algebra AH_k for all $k \geq 0$;

(CA3) $\mathcal{F}$ is isomorphic to a right adjoint of $\mathcal{E}$;

(CA4) the endomorphisms f_i and e_i of $K_0^{\mathbb{C}}(\text{proj-}\mathcal{C})$ induced by $\mathcal{E}_i$ and $\mathcal{F}_i$ make it into an integrable $\mathfrak{sl}_{I_r}$-module such that the classes of indecomposable projectives are weight vectors.

As a consequence of these axioms, $\mathcal{E} = \bigoplus_{i \in I_r} \mathcal{E}_i$ and $\mathcal{E}_i$ and $\mathcal{F}_i$ are biadjoint for all $i \in I_r$.

Recall the definition of a (Schurian) highest weight category $\mathcal{C}$ in the sense of [CPS88]. As in [BLW17, Definition 2.8] we allow the associated poset $(\Xi, \leq)$ to be infinite as long as it is interval-finite: for $\lambda, \mu \in \Xi$ the set $\{\, \nu \in \Xi \mid \lambda \leq \nu \leq \mu \,\}$ is finite. We write L, Δ, and P for the irreducibles, standards, and projectives in $\mathcal{C}$ respectively. Let $\mathcal{C}^\Delta$ be the full subcategory of $\mathcal{C}$ consisting of objects with a Δ-flag and let $G_0^{\mathbb{C}}(\mathcal{C}^\Delta)$ denote its complexified Grothendieck group.

Definition 10.1.3. Take $r \in \mathbb{N} \cup \{\infty\}$ and $\epsilon \in \{0, 1\}$. An $\mathfrak{sl}_{I_r}$*-tensor product categorification* (TPC) of type $(\underline{r}, \underline{\epsilon})$ is a Schurian highest weight category with poset $\Xi_{\underline{r},\underline{\epsilon}}$ and a categorical $\mathfrak{sl}_{I_r}$-action such that

(TPC1) $\mathcal{F}_i$ and $\mathcal{E}_i$ restrict to endofunctors of $\mathcal{C}^\Delta$;

(TPC2) there is a linear isomorphism

$$\begin{aligned} G_0^{\mathbb{C}}(\mathcal{C}^\Delta) &\longrightarrow \textstyle\bigwedge^{\underline{r},\underline{\epsilon}} V_r, \\ [\Delta(\lambda)] &\longmapsto v_\lambda \end{aligned} \tag{10.4}$$

intertwining the endomorphisms induced by $\mathcal{E}_i$ and $\mathcal{F}_i$ and the Chevalley generators $e_i, f_i \in \mathfrak{sl}_{I_r}$.

As $K_0^{\mathbb{C}}(\text{proj-}\mathcal{C}) \hookrightarrow G_0^{\mathbb{C}}(\mathcal{C}^{\Delta})$, the axiom (CA4) is actually a consequence of (TPC2).

The following proposition immediately follows from the definitions and Proposition 9.2.3.

Proposition 10.1.4. *Every $\mathfrak{sl}_{\mathbb{Z}}$-TPC of type $(\underline{\infty}, \underline{0})$ is also a TPC of type $(\underline{\infty}, \underline{1})$ and vice-versa.*

10.2 Strategy for uniqueness

The rest of this chapter will be dedicated to proving the following theorem (see [LW15] or [BLW17] for the definition of a strongly equivariant equivalence):

Theorem 10.2.1. *Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are $\mathfrak{sl}_{I_r}$-TPCs of type $(\underline{r}, \underline{\epsilon})$. Then there is a strongly equivariant equivalence $\mathbb{G} : \mathcal{C} \to \mathcal{C}'$ with $\mathbb{G}L(\lambda) = L'(\lambda)$ for all $\lambda \in \Xi_{\underline{r},\underline{\epsilon}}$.*

For $r < \infty$ this is a special case of the main result in [LW15] so we will only actually provide a proof for the case $r = \infty$.

To establish the theorem, it suffices to show that for a given type $(\underline{r}, \underline{\epsilon})$ there exists a Schurian category $\mathcal{D}$ with a categorical $\mathfrak{sl}_{I_r}$-action such that for any $\mathfrak{sl}_{I_r}$-TPC $\mathcal{C}$ of type $(\underline{r}, \underline{\epsilon})$ there is an exact functor $\mathbb{U} : \mathcal{C} \to \mathcal{D}$ satisfying the following:

(U1) $\mathbb{U}$ is strongly equivariant;

(U2) $\mathbb{U}$ is fully faithful on projectives;

(U3) for each $\lambda \in \Xi_{\underline{r},\underline{\epsilon}}$, $Y(\lambda) := \mathbb{U}P(\lambda)$ is independent (up to isomorphism) of the choice of $\mathcal{C}$.

See [BLW17, §2.7] for an explanation of why this is sufficient.

For the rest of this chapter we fix an $\mathfrak{sl}_{\mathbb{Z}}$-TPC $\mathcal{C}$ of type $(\underline{\infty}, \underline{\epsilon})$. In §10.3 below we define a subquotient $\mathcal{C}_r$ of $\mathcal{C}$ that is an $\mathfrak{sl}_{I_r}$-TPC of type $(\underline{r}, \underline{\epsilon})$ and construct the corresponding functors

$\mathbb{U}_r$ satisfying (U1)-(U3). Then in §10.4 we construct a functor $\mathbb{U}$ from $\mathcal{C}$ to a category of "stable modules" using the functors $\mathbb{U}_r$ and show that $\mathbb{U}$ satisfies (U1)-(U3), thus establishing the theorem. Our proof of Theorem 10.2.1 for $r = \infty$ closely follows the argument in [BLW17, §4]. The reader is referred there for many of the proofs in this chapter as they pass over with minimal change. Our $\Xi_{\underline{r},\underline{\epsilon}}$ corresponds to Λ_r in their notation and we have $I = \mathbb{Z}$.

10.3 Reduction to finite intervals

Let $r_0 := \max\{n_1, \ldots, n_l\}$ and take r such that $r_0 \leq r < \infty$. Define subsets $\Xi^{\leq}_{\underline{r},\underline{\epsilon}}$ and $\Xi^{<}_{\underline{r},\underline{\epsilon}}$ of $\Xi_{\underline{\infty},\underline{\epsilon}}$ as follows. Take $\lambda \in \Xi_{\underline{\infty},\underline{\epsilon}}$ and choose $s \geq r$ such that $\lambda \in \Xi_{\underline{s},\underline{\epsilon}}$. Then $\lambda \in \Xi^{\leq}_{\underline{r},\underline{\epsilon}}$ if it satisfies the following conditions:

$$\begin{cases} \displaystyle\sum_{i=1}^{k} \sum_{\substack{j \in I_s \\ j \leq h \\ \lambda^i_j \neq c_i}} (-1)^{c_i} \geq 0 \text{ for all } h < \min(I_r^+) \text{ and } 1 \leq k \leq l+1 \\ \displaystyle\sum_{i=1}^{k} \sum_{\substack{j \in I_s \\ j \geq h \\ \lambda^i_j \neq c_i}} (-1)^{c_i} \leq 0 \text{ for all } h > \max(I_r^+) \text{ and } 1 \leq k \leq l+1 \end{cases} \tag{10.5}$$

(we write ϵ as c_{l+1} for convenience) and $\lambda \in \Xi^{<}_{\underline{r},\underline{\epsilon}}$ if at least one of the above inequalities is strict. This definition is independent of the choice of s. These are ideals of $\Xi_{\underline{\infty},\underline{\epsilon}}$ and $\Xi_{\underline{r},\underline{\epsilon}} = \Xi^{\leq}_{\underline{r},\underline{\epsilon}} \setminus \Xi^{<}_{\underline{r},\underline{\epsilon}}$. Moreover the vector subspaces of $\bigwedge^{\infty,\epsilon} V$ spanned by the corresponding v_λ are $\mathfrak{sl}_{I_r}$-submodules.

Let $\mathcal{C}_{\leq r}$ (resp. $\mathcal{C}_{<r}$) be the Serre subcategory of $\mathcal{C}$ generated by those $L(\lambda)$ with $\lambda \in \Xi^{\leq}_{\underline{r},\underline{\epsilon}}$ (resp. $\Xi^{<}_{\underline{r},\underline{\epsilon}}$) and let $\mathcal{C}_r$ be the Serre quotient category

$$\mathcal{C}_r = \mathcal{C}_{\leq r} \,/\, \mathcal{C}_{<r}. \tag{10.6}$$

All three of these are highest weight categories. The functors $\mathcal{F}_i$ and $\mathcal{E}_i$ with $i \in I_r$ preserve the subcategories $\mathcal{C}_{\leq r}$ and $\mathcal{C}_{<r}$ and thus they have induced actions on $\mathcal{C}_r$. The following proposition is easily established by checking the necessary axioms:

Proposition 10.3.1. *The category $\mathcal{C}_r$ is an $\mathfrak{sl}_{I_r}$-tensor product categorification of type $(\underline{r}, \underline{\epsilon})$ under the induced action of $\mathcal{F}_{I_r} := \bigoplus_{i \in I_r} \mathcal{F}_i$, together with the restrictions of x and t to $\mathcal{F}_{I_r}$ and $\mathcal{F}_{I_r}^2$ respectively.*

Define an equivalence relation '$\sim$' on $\Xi_{\underline{\infty},\underline{\epsilon}}$ as follows. Take $\lambda, \mu \in \Xi_{\underline{\infty},\underline{\epsilon}}$. Take $r < \infty$ such that $\lambda, \mu \in \Xi_{\underline{r},\underline{\epsilon}}$ and write $\lambda \sim \mu$ if $|\lambda|_r = |\mu|_r$ (see Remark 9.2.1). This definition is independent of the choice of r. For $\lambda \in \Xi_{\underline{\infty},\underline{\epsilon}}$, let $\mathcal{C}_{[\lambda]}$ be the Serre subcategory of $\mathcal{C}$ generated by those $L(\mu)$ with $\mu \sim \lambda$. Then we can decompose

$$\mathcal{C} = \bigoplus_{[\lambda] \in \Xi_{\underline{\infty},\underline{\epsilon}}/\sim} \mathcal{C}_{[\lambda]} \tag{10.7}$$

If $r_0 \leq r < \infty$ then $\Xi_{\underline{r},\underline{\epsilon}}$ has a unique maximal element κ_r. It is the 01-matrix with the 1s in each row as far left as possible. The corresponding vector v_{κ_r} is the highest weight vector in $\bigwedge^{\underline{r},\underline{\epsilon}} V_r$.

Lemma 10.3.2. *$L(\kappa_r) \in \mathcal{C}$ is both projective and injective.*

Proof. This follows from the proof of [BLW17, Lemma 2.20], using (10.7) above in place of [BLW17, (2.16)]. □

For $r_0 \leq r < \infty$, define $T_r := \bigoplus_{k \geq 0} F_{I_r}^k L(\kappa_r)$ and $H_r := \bigoplus_{k \geq 0} AH_{r,k}^{|\kappa_r|}$.

Proposition 10.3.3. *[BLW17, Theorem 4.1] The action of the affine Hecke algebras on T_r induces a canonical algebra isomorphism $H_r \cong \mathcal{C}(T_r, T_r)$.*

In particular, considering T_r as a left H_r-module we can define a functor:

$$\mathbb{U}_r := \mathcal{C}(T_r, -) : \mathcal{C} \longrightarrow \text{mod-}H_r \tag{10.8}$$

By Lemma 10.3.2, T_r is both projective and injective and so $\mathbb{U}_r$ is exact. In the finite interval case, mod-H_r fills the role of the category $\mathcal{D}$ mentioned in §10.2; that is, there exists a categorical $\mathfrak{sl}_{I_r}$-action on mod-H_r such that the functor $\mathcal{C}_r \to$ mod-H_r induced by $\mathbb{U}_r$ satisfies (U1)-(U3) (see [BLW17, Theorem 2.14 and Lemma 2.16]).

10.4 Stable modules

Take $r_0 \leq r < \infty$. Then there exist $a \geq 0$, $p_1, \dots, p_a \in \mathbb{N}$, and $s_1, \dots, s_a \in I_r$ such that

$$f_{s_1}^{(p_1)} f_{s_2}^{(p_2)} \cdots f_{s_a}^{(p_a)} v_{\kappa_{r+1}} = v_{\kappa_r} \tag{10.9}$$

in $\bigwedge^{\infty,\epsilon} V$. Let $p = p_1 + \cdots + p_a$. For $i \in \mathbb{Z}$ and $m \geq 1$, let $\mathcal{F}_i^{(m)}$ denote the summand of $\mathcal{F}_i^m$ that induces $f_i^{(m)}$ on the level of the Grothendieck group.

Lemma 10.4.1. *There is an algebra embedding $\phi_r : H_r \to H_{r+1}$, independent of the choice of $\mathcal{C}$, such that $e_r := \phi_r(1_{H_r})$ acts as projection*

$$\mathcal{F}_{I_{r+1}}^p L(\kappa_{r+1}) \longrightarrow \mathcal{F}_{s_1}^{(p_1)} \mathcal{F}_{s_2}^{(p_2)} \cdots \mathcal{F}_{s_a}^{(p_a)} L(\kappa_{r+1}) \cong L(\kappa_r) \tag{10.10}$$

in $\mathcal{C}$. Moreover there is an isomorphism $\theta_r : T_r \xrightarrow{\sim} e_r T_{r+1}$ intertwining the action of H_r on T_r with its action on T_{r+1} via $\phi_r : H_r \xrightarrow{\sim} e_r H_{r+1} e_r \subseteq H_{r+1}$.

Proof. The assumption $r \geq r_0$ ensures that an identity of the form (10.9) holds. With this, the proof is the same as that of [BLW17, Lemma 4.2]. □

Remark 10.4.2. It is more natural to think of the above map in terms of cyclotomic KLR algebras as in [BLW17, §4.1]. With this perspective, the map between cyclotomic quotients is induced by tensoring on the left with the diagram that realizes projection $\mathcal{F}_{I_{r+1}}^p \to \mathcal{F}_{s_1}^{(p_1)} \mathcal{F}_{s_2}^{(p_2)} \cdots \mathcal{F}_{s_a}^{(p_a)}$ in the categorified quantum group.

Since $\phi_r : H_r \xrightarrow{\sim} e_r H_{r+1} e_r$, we can induce and restrict modules between the H_r for varying r. Given a right (resp. left) H_{r+1}-module M, we consider Me_r (resp. $e_r M$) as a right (resp. left) H_r-module via ϕ_r. For $M \in \text{mod-}H_r$ and $N \in \text{mod-}H_{r+1}$, define

$$\begin{aligned} \operatorname{Ind}_r^{r+1} M &:= M \otimes_{H_r} e_r H_{r+1} \in \text{mod-}H_{r+1} \\ \operatorname{Res}_r^{r+1} N &:= N e_r \in \text{mod-}H_r \end{aligned} \tag{10.11}$$

More generally, we define functors

$$\begin{aligned} \operatorname{Ind}_r^s &:= \operatorname{Ind}_{s-1}^s \circ \cdots \circ \operatorname{Ind}_r^{r+1} \\ \operatorname{Res}_r^s &:= \operatorname{Res}_r^{r+1} \circ \cdots \circ \operatorname{Res}_{s-1}^s \end{aligned} \tag{10.12}$$

for any $s > r$.

Definition 10.4.3. Define a category $\text{mod-}H_\infty$ as follows. Objects in $\text{mod-}H_\infty$ are sequences $M = (M_r, \iota_r)_{r \geq r_0}$ such that $M_r \in \text{mod-}H_r$ and

$$\iota_r : M_r \longrightarrow \operatorname{Res}_r^{r+1} M_{r+1} \subseteq M_{r+1} \tag{10.13}$$

is an H_r-module isomorphism for each r. Morphisms $f : M \to N$ in $\text{mod-}H_\infty$ are sequences $f = (f_r)_{r \geq r_0}$ with $f_r \in \operatorname{Hom}_{H_r}(M_r, N_r)$ such that the following diagram commutes for all $r \geq r_0$:

$$\begin{array}{ccc} M_r & \xrightarrow{\iota_r} & M_{r+1} \\ {\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle f_{r+1}} \\ N_r & \xrightarrow{\iota_r} & N_{r+1} \end{array} \tag{10.14}$$

For $r \geq r_0$, define a functor $\operatorname{st}_r : \text{mod-}H_r \to \text{mod-}H_\infty$ by sending $M \in \text{mod-}H_r$ to the sequence $(M_s, \iota_s)_{s\geq r_0}$ with

$$M_s := \begin{cases} \operatorname{Ind}_r^s M & \text{if } s > r \\ M & \text{if } s = r \\ \operatorname{Res}_s^r M & \text{if } r_0 \leq s < r. \end{cases} \tag{10.15}$$

For $r_0 \leq s < r$, $M_s = \operatorname{Res}_s^{s+1} M_{s+1}$ so we can take $\iota_s = 1_{M_s}$, and for $s \geq r$, ι_s is defined via the obvious natural isomorphism $1 \xrightarrow{\sim} \operatorname{Res}_s^{s+1} \operatorname{Ind}_s^{s+1}$.

The functor st_r corresponds to $\mathrm{pr}_r^!$ in [BLW17, §4.2].

Definition 10.4.4. An object in mod-H_∞ is *r-stable* if it is in the essential image of st_r. It is *stable* if it is r-stable for some $r \geq r_0$. Let mod-H denote the full subcategory of mod-H_∞ consisting of stable modules.

Recall the functors $\mathbb{U}_r$ from §10.3. Define a functor

$$\begin{aligned} \mathbb{U} : \mathcal{C} &\longrightarrow \text{mod-}H_\infty \\ M &\longmapsto (\mathbb{U}_r M, \iota_r)_{r\geq r_0} \end{aligned} \tag{10.16}$$

where

$$\begin{aligned} \iota_r : \operatorname{Hom}_{H_r}(T_r, M) &\xrightarrow{\sim} \operatorname{Hom}_{H_{r+1}}(e_r T_{r+1}, M) \\ \phi &\longmapsto \phi \circ (\theta_r)^{-1} \end{aligned} \tag{10.17}$$

It is defined on morphisms by setting $\mathbb{U} f = (\mathbb{U}_r f)_{r\geq r_0}$.

Theorem 10.4.5. *The category mod-H is Schurian and $\mathbb{U}M \in$ mod-H for every $M \in \mathcal{C}$. Moreover, there exists a categorical $\mathfrak{sl}_\mathbb{Z}$-action on mod-H such that $\mathbb{U}$ satisfies (U1)-(U3).*

Proof. The analogous results in [BLW17] comprise Theorems 4.7, 4.9, and 4.10, and §4.3. The proof is formally identical: our Lemma 10.3.2 takes the place of [BLW17, Lemma 2.20], our

Lemma 10.4.1 takes the place of [BLW17, Lemma 4.2], and [BLW17, Theorem 2.24] and its proof go through unchanged. □

By the discussion in §10.2 we have established Theorem 10.2.1.

Remark 10.4.6. As mentioned in the proof above, [BLW17, Theorem 2.24] holds in our setting. This gives a classification of projective-injective objects in $\mathcal{C}$. Together with Theorem 11.3.9 this yields a classification on projective-injective modules in the infinite-rank categories $\mathcal{O}_{\epsilon}^{++}$ described in the next chapter.

Chapter 11

Categories $\mathcal{O}$

Fix $\epsilon \in \{0,1\}$. The aim of this chapter is to show that a certain infinite-rank limit $\mathcal{O}_\epsilon^{++}$ of parabolic BGG categories of general linear Lie superalgebras is an $\mathfrak{sl}_\mathbb{Z}$-tensor product categorification of type $(\underline{\infty}, \underline{\epsilon})$. In §11.1 we set up the necessary notation and define the category $\mathcal{O}_\epsilon^{++}$. In §11.2 we show that it has enough projectives and conclude that it is a highest weight category. Finally in §11.3 we describe the actions of $\mathcal{E}$ and $\mathcal{F}$ on $\mathcal{O}_\epsilon^{++}$ and show that it is an $\mathfrak{sl}_\mathbb{Z}$-TPC. This yields the super duality equivalence $\mathcal{O}_0^{++} \cong \mathcal{O}_1^{++}$.

Since ϵ will be fixed until the discussion of super duality at the end of this chapter, we will write $\mathcal{O}^{++}$ for $\mathcal{O}_\epsilon^{++}$ up to that point. Our constructions also depend on the sequences $n_1, \dots, n_l \in \mathbb{N}$ and $c_1, \dots, c_l \in \{0,1\}$ which we fixed in Chapter 9. For cleanness we will always drop reference to these from our notation.

11.1 Set up

Let $m = \sum_{c_i=1} n_i$ and $n = \sum_{c_i=0} n_i$. For notational convenience we will sometimes write $n_{l+1} = r$ and $c_{l+1} = \epsilon$.

For $1 \leq j \leq m+n+r$, define $p_j = c_i \in \{0,1\}$, where $1 \leq i \leq l+1$ is maximal such that $n_1 + \cdots + n_{i-1} < j$. For $r < \infty$ let U_r be the vector superspace with basis $u_1, \dots, u_{m+n+r}$, where u_j has degree p_j. Let $u_1^* \ldots, u_{m+n+r}^* \in U_r^*$ be the dual basis. Let $\mathfrak{g}_r := \mathfrak{gl}(U_r)$ be the Lie superalgebra of endomorphisms of U_r under the supercommutator bracket. Let $U = U_\infty = \bigcup_{r=1}^\infty U_r$ and let $\mathfrak{g} = \mathfrak{g}_\infty = \varinjlim \mathfrak{g}_r$ be the Lie superalgebra of endomorphisms of U that vanish on all but finitely many of the u_j.

For $r < \infty$, let $\{ e_{ij} \,|\, 1 \le i, j \le m+n+r \}$ be the basis of matrix units for $\mathfrak{g}_r$. Denote by $\mathfrak{b}_r$ the Borel subalgebra of $\mathfrak{g}_r$ consisting of upper triangular matrices. Let $\mathfrak{h}_r$ be the Cartan subalgebra of $\mathfrak{g}_r$ with basis $\{ e_{ii} \,|\, 1 \le i \le m+n+r \}$ and let $\{\delta_i \,|\, 1 \le i \le m+n+r\} \subseteq \mathfrak{h}_r^*$ be the dual basis. Define a bilinear form $(-,-)$ on $\mathfrak{h}_r^*$ by declaring that

$$(\delta_i, \delta_j) = \begin{cases} (-1)^{p_i} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{11.1}$$

Let $\Phi_r = \{ \delta_i - \delta_j \,|\, 1 \le i, j \le m+n+r,\ i \ne j \}$ be the root system for $\mathfrak{g}_r$. A root $\delta_i - \delta_j$ is even if $p_i = p_j$ and is odd otherwise. It is positive (with respect to $\mathfrak{b}_r$) if $i < j$ and is negative otherwise. Let $\Phi_{r,\bar{0}}$, $\Phi_{r,\bar{1}}$, Φ_r^+, and Φ_r^- be the sets of even, odd, positive, and negative roots respectively and decompose $\Phi_r^+ = \Phi_{r,\bar{0}}^+ \sqcup \Phi_{r,\bar{1}}^+$ and $\Phi_r^- = \Phi_{r,\bar{0}}^- \sqcup \Phi_{r,\bar{1}}^-$ in the obvious way.

For $r < \infty$, define the Weyl vector

$$\overline{\rho}_r = \frac{1}{2} \sum_{\alpha \in \Phi_{r,\bar{0}}^+} \alpha - \frac{1}{2} \sum_{\beta \in \Phi_{r,\bar{1}}^+} \beta \in \mathfrak{h}_r^* \tag{11.2}$$

and the normalized version

$$\rho_r = \overline{\rho}_r + \left(\frac{n - m + 1 - (-1)^\epsilon r}{2} \right) \sum_{i=1}^{m+n+r} (-1)^{p_i} \delta_i. \tag{11.3}$$

This has the following properties:

$$(\rho_r, \delta_i - \delta_{i+1}) = \begin{cases} (-1)^{p_i} & \text{if } p_i = p_{i+1} \\ 0 & \text{otherwise} \end{cases} \qquad (\rho_r, \delta_{m+n+r}) = \begin{cases} 1-r & \text{if } \epsilon = 0 \\ r & \text{if } \epsilon = 1 \end{cases} \tag{11.4}$$

and if $r < s$ then ρ_r is the restriction of ρ_s to $\mathfrak{h}_r$.

For $r < \infty$, let

$$X_r = \bigoplus_{j=1}^{m+n+r} \mathbb{Z}\delta_j \tag{11.5}$$

be the integral weight lattice for $\mathfrak{g}_r$ and define

$$X_r^+ = \{\, \lambda \in X_r \,|\, (-1)^{p_j}(\lambda + \rho_r, \delta_j - \delta_{j+1}) > 0 \text{ unless } j = n_1 + \cdots + n_i \text{ for some } i \},$$

$$X_r^{++} = \left\{ \lambda \in X_r^+ \,\middle|\, (-1)^{\epsilon}(\lambda, \delta_{m+n+r}) \geq 0 \right\}.$$

Using the natural embeddings $X_r^{++} \subseteq X_{r+1}^{++}$, define $X^{++} = X_\infty^{++} = \bigcup X_r^{++}$.

We will identify these sets with certain indexing sets for wedges from Chapter 9. Take $r < \infty$. Take $\lambda \in X_r$ and define a corresponding 01-matrix $(\lambda_j^i)_{1 \leq i \leq l+1, j \in \mathbb{Z}}$ as follows. Fix $1 \leq i \leq l+1$ and let $k = n_1 + \cdots + n_{i-1}$. Set

$$\lambda_j^i = \begin{cases} 1 - c_i & \text{for } j = (\lambda + \rho_r, \delta_{k+1}), \ldots, (\lambda + \rho_r, \delta_{k+n_i}) \\ c_i & \text{otherwise} \end{cases} \tag{11.6}$$

This provides an identification between X_r^+ and the indexing set for the basis of the module

$$\bigwedge^{n_1, c_1} V \otimes \cdots \bigwedge^{n_l, c_l} V \otimes \bigwedge^{r, \epsilon} V. \tag{11.7}$$

We will freely identify these sets and use the inherited partial order on X_r^+. For $r < s$, the linear map

$$\operatorname{span}\{\, v_\lambda \,|\, \lambda \in X_r^{++} \} \longrightarrow \operatorname{span}\{\, v_\lambda \,|\, \lambda \in X_s^{++} \} \tag{11.8}$$

induced by the inclusion $X_r^{++} \subseteq X_s^{++}$ coincides with the map induced by the assignment (9.5) and the direct limit along these maps is $\bigwedge^{\underline{\infty},\epsilon} V$. Thus we can identify X^{++} and $\Xi_{\underline{\infty},\epsilon}$. This induces a partial order on X^{++} compatible with the partial orders on the X_r^{++}.

Remark 11.1.1. For finite r, the indexing set $\Xi_{\underline{r},\epsilon}$ corresponds to a **proper subset** of X_r^{++}. The former indexes tensor products of wedges of V_rs and W_rs and the latter indexes tensor products of wedges of $V = V_\infty$s and $W = W_\infty$s with the wedges in the final tensor factor restricted just enough that the assignment (9.5) is well behaved.

For $r \leq \infty$, define a Levi subalgebra $\mathfrak{l}_r$ of $\mathfrak{g}_r$ by

$$\mathfrak{l}_r = \mathfrak{gl}_{n_1} \oplus \cdots \mathfrak{gl}_{n_l} \oplus \mathfrak{gl}_r \tag{11.9}$$

and let $\mathfrak{p}_r = \mathfrak{l}_r + \mathfrak{b}_r$ be the corresponding parabolic subalgebra. For $\lambda \in X_r^+$ (or $\lambda \in X^{++}$ if $r = \infty$), let $L_r^0(\lambda)$ be the irreducible $\mathfrak{l}_r$-module of highest weight λ. The corresponding parabolic Verma module is

$$\Delta_r(\lambda) = U(\mathfrak{g}_r) \otimes_{U(\mathfrak{p}_r)} L_r^0(\lambda) \tag{11.10}$$

It has a unique irreducible quotient $L_r(\lambda)$.

For $r < \infty$, let $\mathcal{O}_r$ be the integral BGG category $\mathcal{O}$ for $\mathfrak{g}_r$; the category of finitely-generated, $\mathfrak{h}_r$-semisimple, integral weight $\mathfrak{g}_r$-modules that are locally $\mathfrak{b}_r$-finite. Morphisms are all (not necessarily even) homomorphisms of $\mathfrak{g}_r$-modules. Let $\mathcal{O}_r^+$ be the parabolic subcategory of $\mathcal{O}_r^+$ associated to $\mathfrak{p}_r$; the full subcategory of $\mathcal{O}_r$ consisting of modules that are $\mathfrak{l}_r$-semisimple and locally $\mathfrak{p}_r$-finite. Equivalently, $\mathcal{O}_r^+$ is the Serre subcategory of $\mathcal{O}_r$ generated by the $L_r(\lambda)$ with $\lambda \in X_r^+$. It contains the parabolic Verma modules $\Delta_r(\lambda)$ for $\lambda \in X_r^+$. Let $\mathcal{O}_r^{++}$ be the Serre subcategory of $\mathcal{O}_r^+$ generated by $\{L_r(\lambda) \,|\, \lambda \in X_r^{++}\}$. Finally let $\mathcal{O}^{++} = \mathcal{O}_\infty^{++}$ be the category of finitely generated, finite length, $\mathfrak{h}$-semisimple $\mathfrak{g}$-modules that are locally $\mathfrak{p}_r$-finite for each $r < \infty$ and whose composition factors are of the form $L(\lambda) = L_\infty(\lambda)$ with $\lambda \in X^{++}$. It contains the parabolic Verma modules $\Delta(\lambda) = \Delta_\infty(\lambda)$ for $\lambda \in X^{++}$.

Remark 11.1.2. The analogous categories in [BLW17, Definition 3.1] are constructed slightly differently. The authors add a parity assumption to the weight spaces that ensures all morphisms are even. This makes it clear that the categories are abelian. The different constructions yield equivalent categories and so $\mathcal{O}_r$ and $\mathcal{O}^{++}$ are also abelian, see e.g. [CL10, §2.5].

Take $r \leq \infty$. The supertranspose x^{st} of a matrix $x = (x_{ij}) \in \mathfrak{g}_r$ is the matrix with (i,j)-entry $(-1)^{p_i(p_i+p_j)}x_{ji}$. The assignment $x \mapsto x^{\mathrm{st}}$ defines an anti-automorphism of $\mathfrak{g}_r$. If $M \in \mathcal{O}_r$, define its dual by

$$M^\vee = \bigoplus_{\lambda \in X_r} M_\lambda^* \tag{11.11}$$

with $\mathfrak{g}_r$-action given by

$$(x \cdot f)(m) := (-1)^{|x|\cdot|f|} f(x^{st} \cdot m), \tag{11.12}$$

where x and f are homogeneous and $|\cdot|$ denotes their parity. This defines exact, contravariant, self-equivalences on $\mathcal{O}_r$, $\mathcal{O}_r^+$, and $\mathcal{O}_r^{++}$. The dual Verma module corresponding to $\lambda \in X_r$ is

$$\nabla_r(\lambda) := \Delta_r(\lambda)^\vee. \tag{11.13}$$

The following important functors were first introduced in [CWZ08, Definition 3.4].

Definition 11.1.3. For $r < s \leq \infty$ and $M = \bigoplus_{\lambda \in X_s} M_\lambda \in \mathcal{O}_s^{++}$, define

$$\mathrm{tr}_r^s(M) = \bigoplus_{\lambda \in X_r} M_\lambda \in \mathcal{O}_r^{++}, \tag{11.14}$$

where we regard $X_r \subseteq X_s$. This defines an exact *truncation functor* tr_r^s from $\mathcal{O}_s^{++}$ to $\mathcal{O}_r^{++}$.

We will sometimes drop the sub/superscripts when they are clear from context.

Proposition 11.1.4. *If $r < s \leq \infty$, $\lambda \in X_s^{++}$, and $Z = L,\ \Delta,$ or ∇, then*

$$\operatorname{tr}_r^s Z_s(\lambda) = \begin{cases} Z_r(\lambda) & \textit{if } \lambda \in X_r^{++} \\ 0 & \textit{otherwise.} \end{cases} \tag{11.15}$$

Proof. For $Z = L$ or Δ this is [CLW15, Proposition 7.5]. For $Z = \nabla$ this follows from the $Z = \Delta$ case and the observation that duality commutes with truncation. □

Take $M \in \mathcal{O}^{++}$ and $r' \in \mathbb{N}$ such that if $\lambda \in X^{++}$ with $[M : L(\lambda)] \neq 0$ then $\lambda \in X_{r'}^{++}$. For $r \in \mathbb{N}$, let $M_r := \operatorname{tr}_r^\infty(M)$. The proposition implies that the composition multiplicities of M_r are independent of $r \geq r'$ in the sense that if $\infty \geq s > r \geq r'$ and $\lambda \in X_s^{++}$ then

$$[M_s : L_s(\lambda)] = \begin{cases} [M_r : L_r(\lambda)] & \text{if } \lambda \in X_r^{++} \\ 0 & \text{otherwise.} \end{cases} \tag{11.16}$$

The following lemma gives a converse to this statement.

Lemma 11.1.5. *Take $r' \in \mathbb{N}$. Suppose we have modules $M_r \in \mathcal{O}_r^{++}$ and injective $\mathfrak{g}_r$-module maps $f_r : M_r \to \operatorname{tr}_r^{r+1}(M_{r+1}) \subseteq M_{r+1}$ for all $r \geq r'$. Suppose further that the composition multiplicities of the M_r are independent of $r \geq r'$ as above. Then the f_r are isomorphisms and the direct limit $M := \varinjlim M_r$ along the maps f_r is a module in $\mathcal{O}^{++}$.*

Proof. By assumption, M_r and $\operatorname{tr}_r^{r+1}(M_{r+1})$ have the same composition multiplicites when $r \geq r'$. This implies f_r is an isomorphism and $f_r^{-1} \circ \operatorname{tr}_r^{r+1}$ sends a composition series of M_{r+1} to a composition series of M_r with the same ordered sequence of weights. Now since taking direct limits is exact and $L(\lambda) = \varinjlim L_r(\lambda)$ for any $\lambda \in X^{++}$ by Proposition 11.1.4, M has a finite composition series with the same ordered sequence of weights as M_r for any $r \geq r'$. This implies M is finitely generated. Truncation to M_r shows that M is $\mathfrak{h}$-semisimple and locally $\mathfrak{p}_r$-finite for any r. So $M \in \mathcal{O}^{++}$. □

Remark 11.1.6. The same conclusion holds if $M_r \in (\mathcal{O}_r^{++})^\Delta$ for all $r \geq r'$ and we replace composition multiplicities with Δ-multiplicities. Moreover, in this situation we can conclude that $M \in (\mathcal{O}^{++})^\Delta$.

11.2 Highest weight structure

Proposition 11.2.1. *If $r < \infty$ then $\mathcal{O}_r^{++}$ is a highest weight category with weight poset $(X_r^{++}, \leq)$, standard objects $\Delta_r(\lambda)$, and costandard objects $\nabla_r(\lambda)$.*

Proof. The parabolic category $\mathcal{O}_r^+$ is a highest weight category with weight poset $(X_r^+, \leq)$ and standard objects $\{\, \Delta_r(\lambda) \mid \lambda \in X_r^+ \,\}$ (see e.g. [BLW17, Theorem 3.8]). Since X_r^{++} is an ideal in X_r^+ (an easy generalisation of [CW08, Lemma 3.4]), the proposition follows from the general theory of highest weight categories. □

Remark 11.2.2. We will write $P_r(\lambda)$ for the projective cover of $L_r(\lambda)$ in $\mathcal{O}_r^{++}$. This will generally be a proper quotient of the projective cover of $L_r(\lambda)$ in the larger categories $\mathcal{O}_r^+$ and $\mathcal{O}_r$.

We wish to extend this to $r = \infty$. Most of the necessary ingredients are already in the literature, it only remains to establish that $\mathcal{O}^{++}$ has enough projectives. Our main tool will be the truncation functors from Definition 11.1.3. We will show that these send $P_s(\lambda)$ to $P_r(\lambda)$ for $\lambda \in X_r^{++}$ and use them to construct projective covers in $\mathcal{O}^{++}$ direct limits of the $P_r(\lambda)$ as in Lemma 11.1.5.

We will need a left adjoints $(\mathrm{tr}_r^s)^!$ to the tr_r^s. Let $i_r^! : \mathcal{O}_r \to \mathcal{O}_r^{++}$ be the functor that sends a module to its largest quotient in $\mathcal{O}_r^{++}$. It is left adjoint to the inclusion $i_r : \mathcal{O}_r^{++} \to \mathcal{O}_r$. Let $\mathfrak{p}_{r,1}$ be the parabolic subalgebra of $\mathfrak{g}_{r+1}$ corresponding to the Levi subalgebra $\mathfrak{g}_{r,1} := \mathfrak{g}_r + \mathfrak{h}_{r+1}$. Take $M \in \mathcal{O}_r^{++}$ and trivially extend the $\mathfrak{g}_r$-action on M to an action of $\mathfrak{p}_{r,1}$. Then $U(\mathfrak{g}_{r+1}) \otimes_{U(\mathfrak{p}_{r,1})} M \in \mathcal{O}_{r+1}$. Define

$$(\mathrm{tr}_r^{r+1})^!(M) = i_{r+1}^! \left(U(\mathfrak{g}_{r+1}) \otimes_{U(\mathfrak{p}_{r,1})} M \right). \tag{11.17}$$

Now set

$$(\mathrm{tr}_r^s)^! = (\mathrm{tr}_{s-1}^s)^! \circ \cdots (\mathrm{tr}_r^{r+1})^! : \mathcal{O}_r^{++} \to \mathcal{O}_s^{++} \tag{11.18}$$

for $r < s < \infty$.

Lemma 11.2.3. *If $r < s < \infty$ then $(\mathrm{tr}_r^s)^!$ is left adjoint to tr_r^s.*

Proof. It suffices to prove this in the case $s = r + 1$. Take $M \in \mathcal{O}_r^{++}$ and $N \in \mathcal{O}_{r+1}^{++}$ and take a $\mathfrak{g}_{r,1}$-module homomorphism $f : M \to N$. We claim this is a homomorphism of $\mathfrak{p}_{r,1}$-modules. Indeed, since all composition factors of N are of the form $L_{r+1}(\lambda)$ with $\lambda \in X_{r+1}^{++}$, all weights of N must have non-negative $\delta_{m+n+r+1}$-component. As f preserves weight spaces and the weight of any root vector in the nilradical of $\mathfrak{p}_{r,1} \subseteq \mathfrak{g}_{r+1}$ has $\delta_{m+n+r+1}$-component -1, the nilradical must act trivially on im $f$and therefore f is a homomorphism of $\mathfrak{p}_{r,1}$-modules. Now we have a chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}_r}(M, \mathrm{tr}_r^{r+1} N) &= \mathrm{Hom}_{\mathfrak{g}_{r,1}}(M, N) \\
&= \mathrm{Hom}_{\mathfrak{p}_{r,1}}(M, N) \\
&\cong \mathrm{Hom}_{\mathfrak{g}_{r+1}}(U(\mathfrak{g}_{r+1}) \otimes_{U(\mathfrak{p}_{r,1})} M, N) \\
&= \mathrm{Hom}_{\mathfrak{g}_{r+1}}((\mathrm{tr}_r^{r+1})^! M, N),
\end{aligned} \tag{11.19}$$

where the penultimate isomorphism comes from the usual adjunction between induction and restriction. □

Proposition 11.2.4. *If $r < s < \infty$ and $\lambda \in X_r^{++}$ then*

$$\mathrm{tr}_r^s\, P_s(\lambda) = P_r(\lambda). \tag{11.20}$$

Proof. Without loss of generality assume $s = r + 1$. Throughout the proof we write tr for tr_r^{r+1} and $\mathrm{tr}^!$ for $(\mathrm{tr}_r^{r+1})^!$. First we claim $\mathrm{tr}^! \,\mathrm{tr}\, \mathrm{tr}^! = \mathrm{tr}^!$. Take $M \in \mathcal{O}_r^{++}$. The weight of any root vector in the complement to $\mathfrak{p}_{r,1}$ in $\mathfrak{g}_{r+1}$ has $\delta_{m+n+r+1}$-component 1. So

$$\operatorname{tr}\left(U(\mathfrak{g}_{r+1}) \otimes_{U(\mathfrak{p}_{r,1})} M\right) = M. \tag{11.21}$$

Hence, since $\operatorname{tr}^!(M)$ is a quotient of $U(\mathfrak{g}_{r+1}) \otimes_{U(\mathfrak{p}_{r,1})} M$, by exactness of tr there is a surjection $M \twoheadrightarrow \operatorname{tr}\operatorname{tr}^!(M)$. Left adjoints are right exact so this induces a surjection $\operatorname{tr}^!(M) \twoheadrightarrow \operatorname{tr}^! \operatorname{tr}\operatorname{tr}^!(M)$. But by the counit-unit equations, $\operatorname{tr}^!(M)$ is a direct summand of $\operatorname{tr}^! \operatorname{tr}\operatorname{tr}^!(M)$. Thus they are equal.

Take $M = P_r(\lambda)$. A left adjoint to an exact functor sends projectives to projectives, so $\operatorname{tr}^! P_r(\lambda)$ is projective. If $\mu \in X^{++}_{r+1}$, the multiplicity of $P_{r+1}(\mu)$ as a direct summand of $\operatorname{tr}^! P_r(\lambda)$ equals

$$\dim \operatorname{Hom}_{\mathfrak{g}_{r+1}}(\operatorname{tr}^! P_r(\lambda), L_{r+1}(\mu)) = \dim \operatorname{Hom}_{\mathfrak{g}_r}(P_r(\lambda), \operatorname{tr} L_{r+1}(\mu)) = \delta_{\lambda\mu} \tag{11.22}$$

by Proposition 11.1.4. So $\operatorname{tr}^! P_r(\lambda) = P_{r+1}(\lambda)$. This implies $\operatorname{tr}^! \operatorname{tr} P_{r+1}(\lambda) = P_{r+1}(\lambda)$ and so there are isomorphisms of functors on $\mathcal{O}^{++}_{r+1}$:

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{g}_{r+1}}(P_{r+1}(\lambda), -) &\cong \operatorname{Hom}_{\mathfrak{g}_{r+1}}(\operatorname{tr}^! \operatorname{tr} P_{r+1}(\lambda), -) \\ &\cong \operatorname{Hom}_{\mathfrak{g}_r}(\operatorname{tr} P_{r+1}(\lambda), \operatorname{tr}(-)). \end{aligned} \tag{11.23}$$

In particular the last functor is exact. One can show that the composition of these isomorphisms is just the map induced by tr.

Now take $\mu \in X^{++}_r \subseteq X^{++}_{r+1}$. There is a short exact sequence

$$0 \longrightarrow L_{r+1}(\mu) \longrightarrow \nabla_{r+1}(\mu) \longrightarrow C \longrightarrow 0 \tag{11.24}$$

and applying tr yields

$$0 \longrightarrow L_r(\mu) \longrightarrow \nabla_r(\mu) \longrightarrow \mathrm{tr}(C) \longrightarrow 0. \tag{11.25}$$

Consider the long exact sequence induced by $\mathrm{Hom}_{\mathfrak{g}_r}(\mathrm{tr}\, P_{r+1}(\lambda), -)$. By (11.23) the Hom terms form a short exact sequence and so there is an injection

$$\mathrm{Ext}^1_{g_r}(\mathrm{tr}\, P_{r+1}(\lambda), L_r(\mu)) \hookrightarrow \mathrm{Ext}^1_{g_r}(\mathrm{tr}\, P_{r+1}(\lambda), \nabla_r(\mu)). \tag{11.26}$$

But $P_{r+1}(\lambda)$ has a Δ-flag, so $\mathrm{tr}\, P_{r+1}(\lambda)$ does also, and therefore $\mathrm{Ext}^1_{\mathfrak{g}_r}(\mathrm{tr}\, P_{r+1}(\lambda), \nabla_r(\mu)) = 0$. So $\mathrm{Ext}^1_{\mathfrak{g}_r}(\mathrm{tr}\, P_{r+1}(\lambda), L_r(\mu)) = 0$ and now induction on length shows $\mathrm{Ext}^1_{g_r}(\mathrm{tr}\, P_{r+1}(\lambda), N)$ =0 for any $N \in \mathcal{O}_r^{++}$. So $\mathrm{tr}\, P_{r+1}(\lambda)$ is projective. Arguing as in (11.22) shows $\mathrm{tr}\, P_{r+1}(\lambda) = P_r(\lambda)$. □

Definition 11.2.5. Take $\lambda \in X^{++}$. If $s > r \gg 0$ then $\lambda \in X_r^{++}$ so by Proposition 11.2.4 there is an inclusion of $\mathfrak{g}_r$-modules $P_r(\lambda) = \mathrm{tr}_r^s P_s(\lambda) \hookrightarrow P_s(\lambda)$. Define a $\mathfrak{g} = \mathfrak{g}_\infty$-module $P(\lambda)$ by

$$P(\lambda) = \varinjlim P_r(\lambda). \tag{11.27}$$

Theorem 11.2.6. *Take $\lambda \in X^{++}$ and let $r_\lambda \in \mathbb{N}$ be minimal such that $\lambda \in X_{r_\lambda}^{++}$ and $1 - r_\lambda \leq (\lambda + \rho_{r_\lambda}, \delta_i) \leq r_\lambda$ for all i. Then*

(i) the Δ-multiplicities of $P_r(\lambda)$ are independent of $r \geq r_\lambda$ in the sense of (11.16)*, so $P(\lambda) \in \mathcal{O}^{++}$ by Remark 11.1.6;*

(ii) $P(\lambda)$ is a projective cover of $L(\lambda)$ in $\mathcal{O}^{++}$.

In particular, $\mathcal{O}^{++}$ has enough projectives.

Proof. (i) Take $s > r \geq r_\lambda$. Applying tr_r^s to a Δ-flag of $P_s(\lambda)$ yields a Δ-flag of $P_r(\lambda)$. In light of Proposition 11.1.4 it suffices to show that if $\mu \in X_s^{++}$ and $(P_s(\lambda) : \Delta_s(\mu)) \neq 0$ then $\mu \in X_r^{++}$. But if $(P_s(\lambda) : \Delta_s(\mu)) \neq 0$ then $\mu \geq \lambda$. Thus it suffices to show that if $\mu \in X^{++}$ with $\mu \geq \lambda$ then $\mu \in X_{r_\lambda}^{++}$.

Assume $\epsilon = 0$ (the case $\epsilon = 1$ is similar). Take r minimal such that $\mu \in X_r^{++}$. Suppose for contradiction that $r > r_\lambda$. We have $r, r_\lambda \in X_r^{++}$ so by [BLW17, Lemma 3.9],

$$\sum_{\substack{1\leq i\leq m+n+r \\ (\lambda+\rho_r,\delta_i)\leq 1-r}} (-1)^{p_i} = \sum_{\substack{1\leq i\leq m+n+r \\ (\mu+\rho_r,\delta_i)\leq 1-r}} (-1)^{p_i}. \tag{11.28}$$

Since $r > r_\lambda$, $(\lambda+\rho_r, \delta_i) > 1-r$ unless $i = m+n+r$ so the left hand side is equal to $(-1)^\epsilon = 1$. Similarly, $\mu \in X_r^{++} \setminus X_{r-1}^{++}$ implies $(\mu + \rho_r, \delta_i) > 1 - r$ for $i > m+n$. So the right hand side of (11.28) does not change if we restrict the sum to be over $1 \leq i \leq m+n$. But

$$\sum_{\substack{1\leq i\leq m+n \\ (\mu+\rho_r,\delta_i)\leq 1-r}} (-1)^{p_i} \leq \sum_{\substack{1\leq i\leq m+n \\ (\lambda+\rho_r,\delta_i)\leq 1-r}} (-1)^{p_i}, \tag{11.29}$$

again by [BLW17, Lemma 3.9] and the right hand side is equal to 0, a contradiction. So $\mu \in X_{r_\lambda}^{++}$ and (i) holds.

(ii) We claim that $P(\lambda)$ is projective. Take a diagram

$$\begin{array}{ccccc} & & P(\lambda) & & \\ & & \big\downarrow f & & \\ M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array} \tag{11.30}$$

Without loss of generality assume $f \neq 0$. Let $\mu_1, \ldots, \mu_k \in X_{r_\lambda}^{++}$ be the ordered sequence of weights in a Δ-flag of $P(\lambda)$ and take weight vectors $v_1, \ldots, v_k \in P(\lambda)$ such that v_i projects onto the highest weight vector of weight μ_i in the subquotient $\Delta(\mu_i)$ of $P(\lambda)$. Then $v_1, \ldots, v_k$ lie in $P_r(\lambda)$ for any $r \geq r_\lambda$ by part (i).

If $r \geq r_\lambda$, then applying tr_r^∞ to (11.30) and using projectivity of $P_r(\lambda)$ yields a commutative diagram:

$$\begin{array}{ccccc} & & P_r(\lambda) & & \\ & \swarrow{\scriptstyle h_r} & \big\downarrow f_r & & \\ M_r & \xrightarrow[g_r]{} & N_r & \longrightarrow & 0 \end{array} \tag{11.31}$$

where $M_r = \mathrm{tr}_r^\infty(M)$, $N_r = \mathrm{tr}_r^\infty(N)$, and f_r and g_r denote the restrictions of f and g respectively. Write $\underline{v} = (v_1, \dots, v_k)$ and let

$$A_r = \mathrm{span}\big\{\, h_s(\underline{v}) = (h_s(v_1), \dots, h_s(v_k)) \,\big|\, s \geq r \,\big\} \leq M_{\mu_1} \times \cdots \times M_{\mu_n}, \tag{11.32}$$

a finite-dimensional vector space with

$$\cdots \subseteq A_{r+1} \subseteq A_r \subseteq \cdots \tag{11.33}$$

If $\underline{w} = (w_1, \dots, w_k) = \sum_{s \geq r} a_s h_s(\underline{v}) \in A_r$ then the map $\tilde{h}_r = \sum_{s \geq r} a_s h_s\big|_{P_r(\lambda)}$ from $P_r(\lambda)$ to M_r is a homomorphism of $\mathfrak{g}_r$-modules with $\tilde{h}_r(v_i) = w_i$. It is unique with this property. Moreover,

$$g(\tilde{h}_r(v_i)) = \sum_{s \geq r} a_s g(h_s(v_i)) = \Big(\sum_{s \geq r} a_s\Big) f(v_i) \tag{11.34}$$

for all i so that in particular

$$\sum_{s \geq r} a_s h_s(v_i) = \sum_{s \geq r} b_s h_s(v_i) \in A_r \quad \Rightarrow \quad \sum_{s \geq r} a_s = \sum_{s \geq r} b_s, \tag{11.35}$$

since $f \neq 0$ implies that there exists an i with $f(v_i) \neq 0$. We wish to find $\underline{w} \in \bigcap_{r \geq r_\lambda} A_r$ with $\sum_{s \geq r} a_s = 1$ and define $h : P(\lambda) \to M$ by $h\big|_{P_r(\lambda)} = \tilde{h}_r$.

For $r \geq r_\lambda$ let

$$B_r = \Big\{ \sum_{s \geq r} a_s h_s(\underline{v}) \,\Big|\, \sum_{s \geq r} a_s = 0 \Big\} \leq A_r. \tag{11.36}$$

Since all spaces are finite-dimensional, the short exact sequence of inverse systems

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_{r+1} & \longrightarrow & A_{r+1} & \longrightarrow & A_{r+1}/B_{r+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_r & \longrightarrow & A_r & \longrightarrow & A_r/B_r & \longrightarrow & 0 \end{array} \tag{11.37}$$

induces a short exact sequence of the inverse limits:

$$0 \longrightarrow \bigcap_{r \geq r_\lambda} B_r \longrightarrow \bigcap_{r \geq r_\lambda} A_r \longrightarrow \varprojlim A_r/B_r \longrightarrow 0. \tag{11.38}$$

Each A_r/B_r is a non-zero, finite-dimensional vector space, so $\varprojlim A_r/B_r \neq 0$ and therefore $\bigcap B_r \subsetneq \bigcap A_r$. Take $\underline{w} \in \bigcap A_r \setminus \bigcap B_r$ such that $\underline{w} = \sum a_s h_s(\underline{v})$ implies $\sum a_s = 1$. The assignment $v_i \mapsto w_i$ induces a well-defined $\mathfrak{g}_r$-module homomorphism $P_r(\lambda) \to M_r$ for any $r \geq r_\lambda$ and so induces a well-defined $\mathfrak{g}$-module homomorphism $h : P(\lambda) \to M$. Morever, $g(h(v_i)) = f(v_i)$ for all i by (11.34) and so $g \circ h = f$. Thus $P(\lambda)$ is projective.

Now we claim that $P(\lambda)$ is a projective cover of $L(\lambda)$. Indeed, from the Δ-flag of $P(\lambda)$ there is a surjection $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ and so there is an epimorphism $\pi : P(\lambda) \twoheadrightarrow L(\lambda)$. We claim that π is superfluous. Suppose $M \in \mathcal{O}^{++}$ and $f : M \to P(\lambda)$ with $\pi \circ f$ surjective. Then $\mathrm{tr}_r^\infty(\pi \circ f) = \mathrm{tr}_r^\infty(\pi) \circ f_r$ is surjective for any $r \geq r_\lambda$ and so f_r is surjective since $\mathrm{tr}_r^\infty(\pi) : P_r(\lambda) \twoheadrightarrow L_r(\lambda)$ is a projective cover. But $f = \bigcup f_r$, so f is surjective and thus π is superfluous. □

Corollary 11.2.7. *$\mathcal{O}^{++}$ is a highest weight category with weight poset $(X^{++}, \leq)$ and standard objects $\{ \Delta(\lambda) \mid \lambda \in X^{++} \}$.*

Proof. The category $\mathcal{O}^{++}$ is abelian by [CLW15, §7.2], objects have finite length by definition, and irreducibles have one-dimensional endomorphism algebras. By the theorem, $\mathcal{O}^{++}$ has enough projectives and applying duality shows it has enough injectives. Thus $\mathcal{O}^{++}$ is a Schurian category. The poset $(X^{++}, \leq)$ is interval-finite by Lemma 2.4 in *loc. cit.* and the remaining highest weight conditions follow easily from the theorem. □

11.3 Categorical action

Notation 11.3.1. If $M \in \mathcal{O}^{++}$ and $r \in \mathbb{N}$ then we will write $M_r := \mathrm{tr}_r^\infty(M)$.

For $r < \infty$ let

$$\Omega_r := \sum_{k,l=1}^{m+n+r} (-1)^{p_l} e_{kl} \otimes e_{lk}. \tag{11.39}$$

For $M_r \in \mathcal{O}_r$, let $\mathcal{F}_r M_r = M_r \otimes U_r$ and $\mathcal{E}_r M_r = M_r \otimes U_r^*$. Let $x_r \in \operatorname{End}(\mathcal{F}_r)$ be given by multiplication by Ω_r and $t_r \in \operatorname{End}(\mathcal{F}_r^2)$ be induced by the map

$$\begin{aligned} U_r \otimes U_r &\longrightarrow U_r \otimes U_r \\ u_k \otimes u_l &\longmapsto (-1)^{p_k p_l} u_l \otimes u_k. \end{aligned} \tag{11.40}$$

Theorem 11.3.2. *[BLW17, Theorem 3.10] With respect to the above actions, $\mathcal{O}_r$ is an $\mathfrak{sl}_{I_r}$-TPC of type $(\underline{r}, \underline{\epsilon})$.*

The fact that the data above defines strong $\mathfrak{sl}_2$-categorifications on $\mathcal{O}_r$ à la Chuang-Rouquier [CR08] was checked in [CW08, Proposition 5.1].

The endomorphism $x_r \in \operatorname{End}(\mathcal{F}_r)$ induces an endomorphism of $\mathcal{E}_r$, also denoted x_r, given by multiplication by $(m - n - (-1)^{\epsilon} r) - \Omega_r$. For $j \in \mathbb{Z}$, let $\mathcal{F}_{j,r}$ and $\mathcal{E}_{j,r}$ denote the generalized j-eigenspaces of x_r on $\mathcal{F}_r$ and $\mathcal{E}_r$ respectively.

For $\lambda \in X_r$, $j \in \mathbb{Z}$, and $1 \leq i \leq l+1$, let $t_j^i(\lambda) \in X_r$ be obtained from λ by applying the transposition $(j\ j+1)$ to the i^{th} row of λ, considered as a 01-matrix (λ_j^i) as described in §11.1. Then $\mathcal{F}_{j,r}\Delta_r(\lambda)$ has a Δ-flag and

$$[\mathcal{F}_{j,r}\Delta_r(\lambda)] = \sum_i \left[\Delta_r(t_j^i(\lambda))\right] \tag{11.41}$$

where the sum is taken over all $1 \leq i \leq l+1$ with $\lambda_j^i = 1$ and $\lambda_{j+1}^i = 0$. Similarly $\mathcal{E}_{j,r}\Delta_r(\lambda)$ has a Δ-flag and

$$[\mathcal{E}_{j,r}\Delta_r(\lambda)] = \sum_i \left[\Delta_r(t_j^i(\lambda))\right]. \tag{11.42}$$

where the sum is taken over all $1 \leq i \leq l+1$ with $\lambda_j^i = 0$ and $\lambda_{j+1}^i = 1$.

If $\lambda \in X_r^{++}$ and $r > |j|$ then $t_j^i(\lambda) \in X_r^{++}$ so $\mathcal{E}_{j,r}$ and $\mathcal{F}_{j,r}$ restrict to endofunctors of $\mathcal{O}_r^{++}$.

Proposition 11.3.3. *Take $M \in \mathcal{O}^{++}$ and $j \in \mathbb{Z}$. There exists $r_M > |j|$ such that the composition multiplicities of $\mathcal{E}_{j,r}M_r$ and $\mathcal{F}_{j,r}M_r$ are independent of $r \geq r_M$ in the sense of* (11.22). *We will always assume that $r_M > |j - m + n|$.*

Of course r_M depends on j as well as M. But since we rarely vary j we will not record this dependence in our notation.

Proof. Take $\lambda \in X^{++}$. It suffices to prove the claim for $M = L(\lambda)$. Take $r_M > |j|$ such that $\lambda \in X^{++}_{r_M}$ and if $\mu \in X^{++}$ with $[\Delta(t^i_j(\lambda)) : L(\mu)] \neq 0$ for some $1 \leq i \leq l+1$ then $r_M \geq r_\mu$ (c.f. Theorem 11.2.6).

Take $r \in \mathbb{N}$ and suppose $\mu \in X^{++}_r$ with $[\mathcal{F}_{r,j} L_r(\lambda) : L_r(\mu)] \neq 0$. Since $\Delta_r(\lambda)$ surjects onto $L_r(\lambda)$, $[\mathcal{F}_{r,j}\Delta_r(\lambda) : L_r(\mu)] \neq 0$ and so $[\Delta_r(t^i_j(\lambda)) : L_r(\mu)] \neq 0$ for some $1 \leq i \leq l+1$ by (11.41). This implies $[\Delta(t^i_j(\lambda)) : L(\mu)] \neq 0$ and so $\mu \in X^{++}_{r_\mu} \subseteq X^{++}_{r_M}$ by the definition of r_M. We have

$$\begin{aligned} [\mathcal{F}_{r,j} L_r(\lambda) : L_r(\mu)] &= \dim \operatorname{Hom}_{\mathfrak{g}_r}(P_r(\mu), \mathcal{F}_{j,r} L_r(\lambda)) \\ &= \dim \operatorname{Hom}_{\mathfrak{g}_r}(\mathcal{E}_{j,r} P_r(\mu), L_r(\lambda)), \end{aligned} \tag{11.43}$$

which is the multiplicity of $P_r(\lambda)$ as a direct summand of $\mathcal{E}_{j,r}P_r(\mu)$. As $r_M \geq r_\mu$, Theorem 11.2.6(i) implies that the Δ-multiplicities of $P_r(\mu)$ are independent of $r \geq r_M$. So the same is true of $\mathcal{E}_{j,r}P_r(\mu)$ by (11.42). The decomposition of $\mathcal{E}_{j,r}P_r(\mu)$ into indecomposables is uniquely determined by these multiplicities and so $[\mathcal{F}_{r,j} L_s(\lambda) : L_r(\mu)]$ is independent of $r \geq r_M$. The analogous proof works for $\mathcal{E}_{j,r}M$. □

For $r = \infty$ we define Ω, $\mathcal{F}$, $\mathcal{F}_j$, x, and t analogously. If $M \in \mathcal{O}^{++}$ and $r \in \mathbb{N}$ then $(\mathcal{F}M)_r = \mathcal{F}_r M_r$. Moreover, the action of Ω on $\mathcal{F}M$ restricts to the action of Ω_r on $\mathcal{F}_r M_r$ so if $r > |j|$ then $(\mathcal{F}_j M)_r = \mathcal{F}_{j,r} M_r$. By the proposition above and Lemma 11.1.5, $\mathcal{F}_j M \in \mathcal{O}^{++}$. By (11.41), only finitely many $\mathcal{F}_j M$ are non-zero, and so $\mathcal{F}M = \bigoplus_j \mathcal{F}_j M \in \mathcal{O}^{++}$. So $\mathcal{F}$ and $\mathcal{F}_j$ are well-defined endofunctors of $\mathcal{O}^{++}$.

It remains to define a two-sided adjoint $\mathcal{E}$ to $\mathcal{F}$. In general, $M \otimes U^* \notin \mathcal{O}^{++}$ for $M \in \mathcal{O}^{++}$

so the obvious definition will not work. This is because Proposition 11.3.3 does not hold if we replace $\mathcal{E}_{j,r}M_r$ with $\mathcal{E}_r M_r$. Instead we will define each $\mathcal{E}_j M$ as a direct limit of the $\mathcal{E}_{j,r}M_r$ and then set $\mathcal{E}M = \bigoplus_j \mathcal{E}_j M$. However, the natural inclusion $\mathcal{E}_r M_r \hookrightarrow \mathcal{E}_{r+1}M_{r+1}$ does not restrict to inclusions $\mathcal{E}_{j,r}M_r \hookrightarrow \mathcal{E}_{j,r+1}M_{r+1}$. To get around this we will define two sets of maps $\mathcal{E}_{j,r}M_r \to \mathrm{tr}_r^{r+1}(\mathcal{E}_{j,r+1}M_{r+1})$, leading to functors $\mathcal{E}_j^{\mathtt{L}}$ and $\mathcal{E}_j^{\mathtt{R}}$ on $\mathcal{O}^{++}$ that are naturally left and right adjoint $\mathcal{F}_j$ respectively. Finally we will show that $\mathcal{E}_j^{\mathtt{L}} \cong \mathcal{E}_j^{\mathtt{R}}$.

Definition 11.3.4. Take $j \in \mathbb{Z}$ and $M \in \mathcal{O}^{++}$. For $r \in \mathbb{N}$, let ψ_r be the composition of $\mathfrak{g}_r$-module homomorphisms below:

$$\mathcal{E}_{j,r}M_r \subseteq \mathcal{E}_r M_r \subseteq \mathcal{E}_{r+1}M_{r+1} \longrightarrow \mathcal{E}_{j,r+1}M_{r+1}. \tag{11.44}$$

Define $\mathcal{E}_j^{\mathtt{L}} M = \varinjlim \mathcal{E}_{j,r}M_r$, where the limit is taken over the maps ψ_r above.

Lemma 11.3.5. *Take $M \in \mathcal{O}^{++}$ and $r \geq r_M$ (see Proposition 11.3.3). Then ψ_r is an injective $\mathfrak{g}_r$-module homomorphism $\mathcal{E}_{j,r}M_r \to \mathrm{tr}_r^{r+1}(\mathcal{E}_{j,r+1}M_{r+1})$ and so $\mathcal{E}_j^{L} M \in \mathcal{O}^{++}$ by Proposition 11.3.3 and Lemma 11.1.5.*

Proof. Take $\alpha \in \mathcal{E}_r M_r$ and $d \in \mathbb{Z}$. For $t \in \mathbb{N}$, define

$$\beta_t := (d - (-1)^{\epsilon} - \Omega_{r+1})^t \alpha - (d - \Omega_r)^t \alpha. \tag{11.45}$$

We claim $\Omega_{r+1}\beta_t = 0$. Proceed by induction on t. First take $t = 1$. We have

$$\beta_1 = \Omega_r \alpha - \Omega_{r+1}\alpha - (-1)^{\epsilon}\alpha \tag{11.46}$$

Let $z := m+n+r+1$. Write $\alpha = \sum_{i=1}^{z-1} m_i \otimes u_i^*$ with $m_i \in M_r$. By direct computation,

$$\begin{aligned}\Omega_{r+1}\alpha &= -\sum_{k=1}^{z}\left(\sum_{i=1}^{z-1}(-1)^{p_i}e_{ki}m_i\right)\otimes u_k^* \\ &= \Omega_r\alpha - \left(\sum_{i=1}^{z-1}(-1)^{p_i}e_{zi}m_i\right)\otimes u_z^*,\end{aligned} \tag{11.47}$$

so

$$\Omega_{r+1}(\Omega_r\alpha - \Omega_{r+1}\alpha) = -\sum_{k=1}^{z}\sum_{i=1}^{z-1}(-1)^{p_i+\epsilon}e_{kz}e_{zi}m_i \otimes u_k^*. \tag{11.48}$$

By the definition of the supercommutator bracket,

$$\begin{aligned}e_{kz}e_{zi}m_i &= (-1)^{(p_i+\epsilon)(p_k+\epsilon)}e_{zi}e_{kz}m_i + [e_{kz}, e_{zi}]\, m_i \\ &= (-1)^{(p_i+\epsilon)(p_k+\epsilon)}e_{zi}e_{kz}m_i + e_{ki}m_i - (-1)^{p_i+\epsilon}\delta_{ik}e_{zz}m_i.\end{aligned} \tag{11.49}$$

If $k < z$ then applying e_{kz} to a weight vector in M_r yields an element of M_{r+1} whose weight has δ_z-component -1. Since $M_{r+1} \in \mathcal{O}_{r+1}^{++}$, this weight space is zero. So $e_{kz}m_i = 0$. By weight considerations, $e_{zz}m_i = 0$ also. Therefore (11.48) equals

$$-\sum_{k=1}^{z}\sum_{i=1}^{z-1}(-1)^{p_i+\epsilon}e_{ki}m_i \otimes u_k^* = (-1)^{\epsilon}\Omega_{r+1}\alpha. \tag{11.50}$$

The claim follows.

Now suppose $\Omega_{r+1}\beta_t = 0$ for some $t \in \mathbb{N}$. Let

$$\beta_t' := \left(d - (-1)^{\epsilon} - \Omega_{r+1}\right)^t (d - \Omega_r)\alpha - \left(d - \Omega_r\right)^{t+1}\alpha. \tag{11.51}$$

Note that $\Omega_{r+1}\beta'_t = 0$ by the inductive hypothesis applied to $(d - \Omega_r)\alpha \in \mathcal{E}_r M_r$. But

$$\begin{aligned}(d - (-1)^\epsilon - \Omega_{r+1})^{t+1}\alpha &= (d - (-1)^\epsilon - \Omega_{r+1})^t \left(d - (-1)^\epsilon - \Omega_{r+1}\right)\alpha \\ &= (d - (-1)^\epsilon - \Omega_{r+1})^t \left((d - \Omega_r)\alpha + \beta_1\right) \\ &= (d - \Omega_r)^{t+1}\alpha + \left(\beta'_t + (d - (-1)^\epsilon)^t\beta_1\right).\end{aligned} \tag{11.52}$$

So $\beta_{t+1} = \beta'_t + (d - (-1)^\epsilon)^t\beta_1 \in \mathrm{Ker}\ \Omega_{r+1}$.

Now take $\alpha \in \mathcal{E}_{j,r}M_r$. The module $\mathcal{E}_{j,r}M_r$ is defined to be the generalized j-eigenspace of $(m - n - (-1)^\epsilon r) - \Omega_r$, so there exists $t \in \mathbb{N}$ such that

$$(d - \Omega_r)^t\alpha = 0, \tag{11.53}$$

where $d = m - n - (-1)^\epsilon r - j$. So

$$\beta_t = (d - (-1)^\epsilon - \Omega_{r+1})^t\alpha \in \mathrm{Ker}\ \Omega_{r+1}. \tag{11.54}$$

Define β by the equation $(d - (-1)^\epsilon - \Omega_{r+1})^t\beta = \beta_t$. Then $\Omega_{r+1}\beta = 0$ and so

$$(d - (-1)^\epsilon - \Omega_{r+1})^t (\alpha - \beta) = \beta_t - (d - (-1)^\epsilon)^t\beta = 0. \tag{11.55}$$

So $\alpha = (\alpha - \beta) + \beta$ is a decomposition of $\alpha \in \mathcal{E}_r M_r \subseteq \mathcal{E}_{r+1}M_{r+1}$ into generalized $(d - (-1)^\epsilon - \Omega_{r+1})$-eigenvectors and $\alpha - \beta \in \mathcal{E}_{j,r+1}M_{r+1}$. So $\psi_r(\alpha) = \alpha - \beta$.

Suppose $\psi_r(\alpha) = 0$. Then $\alpha = \beta \in \mathrm{Ker}\ \Omega_{r+1}$ so (11.47) implies that the sum $\sum_{i=1}^{z-1}(-1)^{p_i}e_{ki}m_i$ is zero for each k and thus $\Omega_r\alpha = 0$. Generalized eigenspaces with different eigenvalues intersect trivially, so if $\alpha \neq 0$ then (11.53) implies $d = 0$, so $j = m - n - (-1)^\epsilon r$. But this contradicts the assumption $r \geq r_M > |j - m + n|$ made in Proposition 11.3.3. So ψ_r is injective. □

Now we define the right adjoint $\mathcal{E}_j^{\mathrm{R}}$ to $\mathcal{F}_j$. Take $M \in \mathcal{O}^{++}$, $r \geq r_M$, and an element

$x \in \mathrm{tr}_r^{r+1}(\mathcal{E}_{j,r+1} M_{r+1})$. We can write

$$x = \sum_{i=1}^{m+n+r+1} m_i \otimes u_i^* \tag{11.56}$$

where $m_i \in M_r$ for $1 \leq i \leq m+n+r+1$ and $m_{m+n+r+1} \in M_{r+1}$ is a sum of weight vectors whose weights have $\delta_{m+n+r+1}$-component 1. Define

$$\begin{aligned} \phi_r : \mathrm{tr}_r^{r+1}(\mathcal{E}_{j,r+1} M_{r+1}) &\longrightarrow \mathcal{E}_r M_r \\ \sum_{i=1}^{m+n+r+1} m_i \otimes u_i^* &\longmapsto \sum_{i=1}^{m+n+r} m_i \otimes u_i^*. \end{aligned} \tag{11.57}$$

This is a homomorphism of $\mathfrak{g}_r$-modules.

Lemma 11.3.6. *Take $M \in \mathcal{O}$ and $r \geq r_M$. Then*

(i) Im $\phi_r \subseteq \mathcal{E}_{j,r} M_r$;

(ii) ϕ_r is injective and so is an isomorphism by Proposition 11.3.3.

Definition 11.3.7. Let $\mathcal{E}_j^{\mathtt{R}} M = \varinjlim \mathcal{E}_{j,r} M_r$, where the direct limit is taken along the maps

$$\mathcal{E}_{j,r} M_r \xrightarrow{\phi_r^{-1}} \mathrm{tr}_r^{r+1}(\mathcal{E}_{j,r+1} M_{r+1}) \subseteq \mathcal{E}_{j,r+1} M_{r+1} \tag{11.58}$$

for $r \geq r_M$. We have $\mathcal{E}_j^{\mathtt{R}} M \in \mathcal{O}^{++}$ by Proposition 11.3.3 and Lemma 11.1.5.

Proof of Lemma 11.3.6. Take $x \in \mathrm{tr}_r^{r+1}(\mathcal{E}_{j,r+1} M_{r+1})$ and write $x = \sum_{i=1}^{z} m_i \otimes u_i^*$, where $z = m+n+r+1$. Fix $d \in \mathbb{Z}$. For $t \geq 0$, write

$$(d - (-1)^\epsilon - \Omega_{r+1})^t x = \sum_{k=1}^{z} m_{kt} \otimes u_k^* \tag{11.59}$$

so that $m_{k0} = m_k$ for $1 \leq k \leq z$. The map Ω_{r+1} preserves $\mathcal{E}_{j,r+1} M_{r+1}$ and preserves weights, so (11.59) lies in $\mathrm{tr}_r^{r+1}(\mathcal{E}_{j,r+1} M_{r+1})$.

We claim that

$$m_{k,t+1} - e_{kz}m_{z,t+1} = (d - (-1)^\epsilon)(m_{kt} - e_{kz}m_{zt}) \tag{11.60}$$

for all $1 \leq k \leq z$ and $t \geq 0$. By direct computation,

$$m_{k,t+1} = (d - (-1)^\epsilon)m_{kt} + \sum_{i=1}^{z}(-1)^{p_i}e_{ki}m_{it}. \tag{11.61}$$

So

$$\begin{aligned} m_{k,t+1} - e_{kz}m_{z,t+1} = &(d - (-1)^\epsilon)m_{kt} + \sum_{i=1}^{z}(-1)^{p_i}e_{ki}m_{it} \\ &- (d - (-1)^\epsilon)e_{kz}m_{zt} + \sum_{i=1}^{z}(-1)^{p_i}e_{kz}e_{zi}m_{it}. \end{aligned} \tag{11.62}$$

But $e_{kz}e_{zz}m_{zt} = e_{kz}m_{zt}$ by the weight of m_{zt}, and if $1 \leq i \leq z-1$ then $e_{kz}e_{zi}m_{it} = e_{ki}m_{it}$ as in (11.49). The claim follows.

Set $d = m - n - (-1)^\epsilon r - j$. Since $x \in \mathcal{E}_{j,r+1}M_{r+1}$, $(d - (-1)^\epsilon - \Omega_{r+1})^t x = 0$ for $t \gg 0$ and so $m_{kt} = 0$ for $1 \leq k \leq z$ and $t \gg 0$. As $r \geq r_M$, the assumption $r_M > |j - m + n|$ in Proposition 11.3.3 means $d - (-1)^\epsilon \neq 0$ and so (11.60) implies that $e_{kz}m_{zt} = m_{kt}$ for all k and t. Now (11.61) simplifies to

$$m_{k,t+1} = dm_{k,t} + \sum_{i=1}^{z-1} e_{ki}m_{i,t}. \tag{11.63}$$

But for $1 \leq k \leq z-1$, this is the same recursive formula as for the terms of $(d - \Omega_r)^t\phi_r(x)$. So

$$\phi_r\left((d - (-1)^\epsilon - \Omega_{r+1})^r x\right) = (d - \Omega)^t\phi_r(x) \tag{11.64}$$

In particular this implies that $(d - \Omega)^t\phi_r(x) = 0$ for $t \gg 0$, so $\phi_r(x) \in \mathcal{E}_{j,r}M_r$ and (i) holds.

Now suppose $\phi_r(x) = 0$. Then $m_i = 0$ for $1 \leq i \leq z-1$. Equation (11.63) shows that $m_{kt} = 0$ for $1 \leq k \leq z-1$ and $m_{zt} = d^t m_z$ for all $t \geq 0$. But $m_{zt} = 0$ for $t \gg 0$ and $d \neq 0$ so this implies $m_z = m_{z0} = 0$, and therefore $x = 0$. $\square$

Proposition 11.3.8. *Take $j \in \mathbb{Z}$. Then*

(i) $\mathcal{E}_j^{L}$ is left-adjoint to $\mathcal{F}_j$;

(ii) $\mathcal{E}_j^{R}$ is right-adjoint to $\mathcal{F}_j$;

(iii) $\mathcal{E}_j^{L} \cong \mathcal{E}_j^{R}$

Proof. (i) If $M, N \in \mathcal{O}^{++}$ then

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}}(M, \mathcal{F}_j N) &= \mathrm{Hom}_{\mathfrak{g}}(\varinjlim M_r, \mathcal{F}_j N) \\ &= \varprojlim \mathrm{Hom}_{\mathfrak{g}_r}(M_r, \mathcal{F}_{j,r} N_r) \\ &= \varprojlim \mathrm{Hom}_{\mathfrak{g}_r}(\mathcal{E}_{j,r} M_r, N_r) \\ &= \mathrm{Hom}_{\mathfrak{g}}(\varinjlim \mathcal{E}_{j,r} M_r, N) \end{aligned} \tag{11.65}$$

Unravelling these isomorphisms shows that the connecting maps in the final direct limit $\varinjlim \mathcal{E}_{j,r} M_r$ are just the ψ_r from Definition 11.3.4. The result follows.

(ii) Similar.

(iii) Take $M \in \mathcal{O}^{++}$ and $r \geq r_M$. Multiplication by Ω_r is a $\mathfrak{g}_{\mathfrak{r}}$-module isomorphism when restricted to $\mathcal{E}_{j,r} M_r$, so it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E}_{j,r} M_r & \xrightarrow{\quad \Omega_r \quad} & \mathcal{E}_{j,r} M_r \\ \downarrow{\scriptstyle \psi_r} & & \uparrow{\scriptstyle \phi_r} \\ \mathrm{tr}_r^{r+1}(\mathcal{E}_{j,r+1} M_{r+1}) & \xrightarrow{\Omega_{r+1}} & \mathrm{tr}_r^{r+1}(\mathcal{E}_{j,r+1} M_{r+1}) \end{array} \tag{11.66}$$

Take $\alpha \in \mathcal{E}_{j,r} M_r$. From the proof of Lemma 11.3.5, $\psi_r(\alpha) = \alpha - \beta$ for some $\beta \in \mathrm{Ker}\ \Omega_{r+1}$. So $\Omega_{r+1}\psi_r(\alpha) = \Omega_{r+1}\alpha$. Equation (11.47) implies $\phi_r(\Omega_{r+1}\alpha) = \Omega_r \alpha$, so $\phi_r(\Omega_{r+1}\psi_r(\alpha)) = \Omega_r \alpha$ as required. □

Write $\mathcal{E}_j = \mathcal{E}_j^{\mathtt{R}}$ and let $\mathcal{E} = \bigoplus_j \mathcal{E}_j$. If $M \in \mathcal{O}^{++}$ then only finitely many $\mathcal{E}_j M$ are non-zero by (11.42), so $\mathcal{E}M \in \mathcal{O}^{++}$. The biadjunctions between the $\mathcal{F}_j$ and $\mathcal{E}_j$ induce a biadjunction between $\mathcal{F}$ and $\mathcal{E}$.

Theorem 11.3.9. *With the definitions of $\mathcal{E}$, $\mathcal{F}$, x, and t above, $\mathcal{O}^{++}$ is an $\mathfrak{sl}_{\mathbb{Z}}$-TPC of type $(\underline{\infty}, \underline{\epsilon})$.*

Proof. (CA1) and (CA3) are clear. (CA2) follows from truncation and Theorem 11.3.2. (TPC1) and (TPC2) are consequences of (11.41) and (11.42) and the identification of X^{++} with $\Xi_{\underline{\infty},\underline{\epsilon}}$ described in §11.1. (CA4) is a consequence of (TPC2). □

Recall that the category $\mathcal{O}^{++}$ depends on $\epsilon \in \{0, 1\}$. We reintroduce the ϵ-dependence and write $\mathcal{O}^{++}$ as $\mathcal{O}^{++}_{\epsilon}$. A less general formulation of the following was conjectured in [CWZ08, Conjecture 6.10]. It was generalized in [CW08, Conjecture 4.18] and first proved in [CL10, Theorem 5.1].

Corollary 11.3.10. *There is a strongly equivariant equivalence*

$$\mathbb{S} : \mathcal{O}^{++}_0 \longrightarrow \mathcal{O}^{++}_1 \tag{11.67}$$

with $\mathbb{S}L(\lambda) \cong L(\lambda)$.

Proof. This follows immediately from Proposition 10.1.4, Theorem 10.2.1, and Theorem 11.3.9. The condition on irreducibles shows that this is the same functor as in [CL10]. □

Chapter 12

Graded lifts

We finish by describing how to construct graded lifts of $\mathfrak{sl}_{I_r}$-TPCs and deduce a "graded super duality". This chapter closely follows [BLW17, §5] and we refer the interested reader there for most definitions and proofs.

For $r \in \mathbb{N} \cup \{\infty\}$, let $U_q(\mathfrak{sl}_{I_r})$ be the quantum group associated to $\mathfrak{sl}_{I_r}$For $\epsilon \in \{0, 1\}$ there is a $U_q(\mathfrak{sl}_{I_r})$-module $\bigwedge_q^{\underline{r},\underline{\epsilon}} V_r$ with basis $\{ v_\lambda \mid \lambda \in \Xi_{\underline{r},\underline{\epsilon}} \}$ as before. The action of the generators on this basis is given in [BLW17, (5.3)-(5.4)].

A graded lift of an (ungraded) Schurian category $\overline{\mathcal{C}}$ is a graded abelian category $\mathcal{C}$ with a fully faithful functor $\nu : \mathcal{C}^* \to \overline{\mathcal{C}}$ such that ν is dense on projectives and $\nu \circ \langle 1 \rangle \cong \nu$, where $\langle 1 \rangle$ denotes grading shift in $\mathcal{C}$.

We define a $U_q(\mathfrak{sl}_{I_r})$-*tensor product categorification* of type $(\underline{r}, \underline{\epsilon})$ as in [BLW17, Definition 5.9]. If $\mathcal{C}$ is a $U_q(\mathfrak{sl}_{I_r})$-TPC of type $(\underline{r}, \underline{\epsilon})$ we denote the distinguished set of irreducibles in $\mathcal{C}$ by $\{ L(\lambda) \mid \lambda \in \Xi_{\underline{r},\underline{\epsilon}} \}$.

Theorem 12.0.11. *Take $r \in \mathbb{N} \cup \{\infty\}$. Suppose that $\overline{\mathcal{C}}$ is an $\mathfrak{sl}_{I_r}$-TPC of type $(\underline{r}, \underline{\epsilon})$.*

(i) There exists a graded lift $\mathcal{C}$ of $\overline{\mathcal{C}}$ such that $\mathcal{C}$ is a $U_q(\mathfrak{sl}_{I_r})$-TPC $\mathcal{C}$ of type $(\underline{r}, \underline{\epsilon})$ and the graded functors $\mathcal{E}_j$ and $\mathcal{F}_j$ and graded natural transformations x and t are all graded lifts of the corresponding data for $\overline{\mathcal{C}}$. Moreover $\mathcal{C}$ is Koszul.

(ii) If $\mathcal{C}'$ is another such graded lift of $\overline{\mathcal{C}}$ then there is a strongly equivariant graded equivalence $\mathbb{G} : \mathcal{C} \to \mathcal{C}'$ with $\nu' \circ \widehat{\mathbb{G}} \cong \nu$ and $\mathbb{G}L(\lambda) \cong L'(\lambda)$.

Proof. For $r < \infty$ this is covered by [BLW17, Theorem 5.11]. For $r = \infty$ the proof is formally identical to that outlined in [BLW17, §5.7 and §5.8] so we only provide a sketch. We adopt the notations of §10.2-10.4. Given $\overline{\mathcal{C}}$ there is an exact functor $\overline{\mathbb{U}} : \overline{\mathcal{C}} \to$ mod-H sending the indecomposable projective $\overline{P}(\lambda) \in \overline{\mathcal{C}}$ for $\lambda \in \Xi_{\underline{\infty},\underline{\epsilon}}$ to some $\overline{Y}(\lambda) \in$ mod-H. For $r \geq r_0$ the cyclotomic affine Hecke algebras H_r have compatible $\mathbb{Z}$-gradings from which we define a category of *graded stable modules* grmod-H. This is a graded lift of mod-H and carries a categorical $U_q(\mathfrak{sl}_\mathbb{Z})$-action.

For each $\lambda \in \Xi_{\underline{\infty},\underline{\epsilon}}$ we make a canonical choice of graded lift $Y(\lambda) \in$ grmod-H of $\overline{Y}(\lambda)$ as in [BLW17, Theorem 5.22]. This choice depends on the defect of λ whose definition requires a small modification in our setting. Take $r \geq r_0$ such that $\lambda \in \Xi_{\underline{r},\underline{\epsilon}}$ and define the defect by

$$\text{def}(\lambda) := \frac{1}{2}(|\kappa_r| \cdot |\kappa_r| - |\lambda|_r \cdot |\lambda|_r), \tag{12.1}$$

where $\kappa_r \in \Xi_{\underline{r},\underline{\epsilon}}$ is defined as in §10.3. The proof of [BLW17, Lemma 2.2] can be adapted to show that this is independent of the choice of r and now [BLW17, Theorem 5.22] and its proof go through unchanged.

From this information we can construct a graded lift $\mathcal{C}$ of $\overline{\mathcal{C}}$ that is a $U_q(\mathfrak{sl}_{I_r})$-TPC of type $(\underline{\infty}, \underline{\epsilon})$. The functor $\overline{\mathbb{U}}$ lifts to a graded functor $\mathbb{U} : \mathcal{C} \to$ grmod-H satisfying graded analogues of (U1)-(U3). Uniqueness follows from this as in the ungraded setting of §10.2. Koszulity follows from reduction to finite intervals where it is known, see [BLW17, Theorem 5.26]. □

Recall the $\mathfrak{sl}_\mathbb{Z}$-TPC $\mathcal{O}_\epsilon^{++}$ of type $(\underline{\infty}, \underline{\epsilon})$ from Chapter 11 and the super duality equivalence $\mathbb{S} : \mathcal{O}_0^{++} \to \mathcal{O}_1^{++}$ of Theorem 11.3.10. The following new 'graded super duality' is an immediately corollary of the theorem above.

Corollary 12.0.12. *For $\epsilon \in \{0, 1\}$, the category $\mathcal{O}_\epsilon^{++}$ has a unique Koszul graded lift ${}^{gr}\mathcal{O}_\epsilon^{++}$*

as in Theorem 12.0.11(i) and $\mathbb{S}$ *lifts to a strongly equivariant graded equivalence*

$$^{gr}\mathbb{S} : {}^{gr}\mathcal{O}_0^{++} \longrightarrow {}^{gr}\mathcal{O}_1^{++} \tag{12.2}$$

with $^{gr}\mathbb{S}L(\lambda) \cong L(\lambda)$.

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