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BELLAVITIS'

METHOD OF EQUIPOLLENCES.

THESIS

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THE METHOD OF EQUIPOLLENCES, AND ITS RELATION TO THE CALCULUS OF QUATERNIONS.

The satisfactory solution of the important question, How to interpret imaginary quantities? belongs to the nineteenth century; and the honor of being the author of the most complete and general solution of this question belongs undoubtedly to Sir William R. Hamilton, the inventor of the Calculus of Quaternions.

These so-called imaginary or complex quantities seemed for a long time to form an irresistible barrier to further progress.

Wallis, toward the close of the seventeenth century, suggested that the square root of a negative quantity might be laid off by going *out of the line*, on which the real quantity would have been measured. This step was especially useful, in that it led to a very important one, the consideration of the *length* and *direction* of a line independently of one another.

Finally, at the beginning of the present century, not to mention less prominent writers on the subject, Mourey gave a geometrical interpretation to imaginary quantities. Argand had formed the same conceptions some years before; but Mourey, as well as Warren, seems to have made his invention independently. But Giusto Bellavitis first established this new system of Analytical Geometry under a really *methodical* form.

This invention, called by Bellavitis the Method of Equipollences, and founded on the theory of complex quantities, though a most remarkable and elegant method for solving plane problems, has no application to tridimensional space.

Sir William R. Hamilton was the first to discover the use of $\sqrt{-1}$ as a *geometric reality*, tied down to no particular direction in space, and this use was the foundation of the Calculus of Quaternions. (Tait's Quaternions, Chap. I). Bellavitis wrote several works on his new method, one of which, his "*Sposizione del Metodo delle Equipollenze*," 1854, was translated into French by Captain Laisant for the *Nouvelles Annales de Mathématiques*, 1873–'74 This translation was afterwards published in book form. Houel has briefly treated the subject in the second volume of his Calculus.

Let us now see what are the chief characteristics of the Method of Equipollences.

According to the definition given by Bellavitis, two lines are *equipollent*, when they are equal in magnitude, parallel, and drawn in the same direction.

Adopting the notation of Gauss, i is used for $\sqrt{-1}$; and, as in Houel's Calculus, $\bar{c}$ represents the conjugate of a complex quantity c. A straight line having the same length as a given straight line and the same *inclination*, but of contrary sign, is called the *conjugate* of the first. A line conjugate to a given line is obtained by simply changing the sign of i wherever it occurs in the given line.

Thus, the conjugate of the expression

$$c = a + ib = \lambda e^{i\theta}$$

would be

$$\bar{c} = a - ib = \lambda e^{-i\theta}.$$

Hence the modulus of the complex quantity is

$$\lambda = \sqrt{c \cdot \bar{c}}.$$

Frequent use is made of conjugate lines in eliminations and reductions. We discard, as not necessary, the special sign (≏) employed by Bellavitis to denote that two quantities are equipollent. Instead, we shall use the sign $=$, with the meaning "equipollent to," that is, equal in magnitude and drawn in the same direction, just as in Quaternions the sign $=$ has an extended meaning—is equivalent to. With this understanding, the equipollence $z = \varphi(t)$ indicates both the curve itself and its law of description. This shows us at once the great advantage this method has over the ordinary methods, giving, as it does, at one time, both the magnitude and position of diffent parts of the figure. Other advantages by which the method of equipollences is principally distinguished are

First, The number of theorems that flow at once from the general principle of the method—"Every property of points of a straight line gives a theorem relative to points of a plane by simply changing the equations into equipollences."

Second, In forming the equations to curves by this method, we are independent of any special system of co-ordinates.

Third, It enables us to apply algebraical operations and transformations to geometrical complex quantities.

In this method the equations to curves are most readily and naturally derived by considering their mode of generation. If we consider a geometric locus as generated by the motion of a point z corresponding to the time t, a term of the form at would express a motion of translation along the direction Ox, or parallel to this direction. Here O is supposed to be the origin, and Ox the *initial* line. A term of the form ae^{it} would express a motion of rotation of the straight line a around its initial extremity. Thus

$z = ae^{it}$ represents a circle,
$z = ne^{it} - e^{int}$ represents a epicycloid,
$z = ate^{it}$ represents a spiral of Archimedes.

For other examples, see Houel's Calculus, Vol. II.

Let us now briefly consider some of the fundamental processes of this method, following in the main Houel. For future use, we must state this rule (*Méthode des Equipollences*, Art 10.)

RULE: A, B, C being any three points in a plane, we have always the equipollence

$$AB + BC = AC$$

which takes indifferently the forms

$$BC = AC - AB,$$
$$AB = AC - BC,$$
$$AB + BC + CA = 0.$$

If a is a real constant, the equation

$$z = at$$

represents the axis Ox, the equation

$$z = ae^{i\mu} t = At \qquad (1)$$

represents a line passing through the origin and making an angle μ with Ox.

Making $\mu = \frac{\pi}{2}$ in (1), it becomes

$$z = iat \qquad (2)$$

a line through the origin perpendicular to the line $z = at$. In adding to each value of z a straight line constant in length and in direction B, we obtain a parallel to the line (1), of which the equation is (see *Rule*)

$$z = At + B \qquad (3)$$

Having given two lines

$$c = ae^{i\mu}, \quad c' = be^{i\pi}$$

of which the conjugates are

$$\bar{c} = ae^{-i\mu}, \quad \bar{c}' = be^{-i\pi}$$

the angle $\mu - \pi$ between the two lines c, c' is given by the equation

$$\frac{c}{\bar{c}} : \frac{c'}{\bar{c}'} = e^{2i(\mu - \pi)} \qquad (4)$$

In order that the two lines

$$x = At + B, \quad z = A't + B'$$

be parallel, the angular coefficients A A' must have the same argument, and hence a real ratio. The condition of parallelism should therefore be

$$\frac{A}{A'} = \text{a real quantity.}$$

We can now find the equation to the tangent.

This equation is of the form $\zeta = z + A\tau$ (see *Rule* above), where z is the z of the curve, that is, it is the distance of a point on the curve from the origin, and $A\tau$ is a small element measured in the direction of the curve and varying with t. Now A ought to be equal to dz multiplied by a real quantity. For dz represents in magnitude and direction the element of the arc of the curve $z = f(t)$. As τ is an arbitrary small quantity, we may suppose this real multiplier to be $\frac{1}{dt}$ (a large quantity). Therefore, putting

$$\frac{dz}{dt} = z'$$

we have for the equation to tangent at point z

$$\zeta = z + z'\tau \qquad (5)$$

In like manner the equation to the normal is

$$\zeta = z + iz'\tau \qquad (6)$$

Centre of curvature, Evolute.

In passing from one normal to the next, t and τ evidently vary. At the point of intersection of two normals, ζ ought to be the same, and consequently $d\zeta = 0$. Therefore, to obtain the centre of curvature, the intersection of two normals infinitely near, we differentiate the equation (6) of the normal with respect to t (of which z and τ are functions). Thus, we have

$$0 = z'\left(1 + i\frac{d\tau}{dt}\right) + iz''\tau \qquad (7)$$

Equating real and imaginary parts of this equation, we deduce the value of τ and $\frac{d\tau}{dt}$ for the point of intersection. Substituting the value of τ thus found in (6) the equation to the normal, and in the resulting equation considering t as a variable, we get the equation to the evolute, the locus of the centres of curvature.

The length of the radius of curvature would be given by the formula

$$\rho = \sqrt{(\zeta - z)(\bar{\zeta} - \bar{z})} = \tau\sqrt{z'\bar{z}'}$$

Next, to obtain the envelops and trajectories, let

$$z = f(t, \lambda) \tag{8}$$

be the equation to a curve containing an arbitrary parameter λ. Let us suppose now that the trajectory sought cuts the lines (8) under a constant angle ϑ. We should obtain this condition in placing the angle of two tangents to the line (8), obtained, the one in supposing λ constant, the other in supposing λ variable, equal to ϑ. Differentiating the equation on the two hypotheses, we find respectively for the angular coefficients of the two tangents

$$\frac{dz}{dt} = \left(\frac{df}{dt}\right), \quad \frac{dz}{dt} = \left(\frac{df}{dt}\right) + \left(\frac{df}{d\lambda}\right)\frac{d\lambda}{dt};$$

or, in supposing for brevity, $\left(\frac{df}{dt}\right) = z', \left(\frac{df}{d\lambda}\right) = z_1$,

$$\frac{dz}{dt} = z', \quad \frac{dz}{dt} = z' + z_1\frac{d\lambda}{dt}$$

We would have, therefore, to determine the angle of these two tangents the formula (equation 4)

$$\frac{z' + z_1\frac{d\lambda}{dt}}{\bar{z}' + \bar{z}_1\frac{d\lambda}{dt}} : \frac{z'}{\bar{z}_1} = e^{2i\vartheta}$$

or, in making $e^{2i\vartheta} = m$,

$$(mz'\bar{z}_1 - \bar{z}'z_1)\frac{d\lambda}{dt} + (m - 1)\, z'\bar{z}' = 0. \tag{9}$$

If $\vartheta = \frac{\pi}{2}$ or $m = -1$, we have, for the equation of orthogonal trajectories

$$(z'\bar{z}_1 + \bar{z}'z_1)\frac{d\lambda}{dt} + 2z'\bar{z}' = 0 \tag{10}$$

Making $\vartheta = 0$, whence $m = 1$, we have, to determine the envelop, the equation

$$z'\bar{z}_1 - \bar{z}'z_1 = 0 \tag{11}$$

Deducing from this the value of λ, and substituting it in the equation (8), we would have the equation to the envelop.

The application of these principles to some examples, will show us the remarkable simplicity and neatness of the method. Let us then derive the equations and some of the chief properties of the epicycloid and hypocycloid, as an im-perfect continuation of the articles on the cycloid in the *Méthode des Equipollences*. Here, as elsewhere, we refer to Laisant's French translation of the original work of Bellavitis. The equations can be derived at once from our knowledge of the law of description of these curves, which are formed by the composition of two uniform circular motions. In the one case, the generating circle rolls on the convex, and in the other, on the concave, side of the fixed circle. Therefore, the equation to the eyicycloid would be

$$z = ne^{i\theta} - e^{in\theta}$$

where $ne^{i\theta} =$ line from centre of fixed circle (the origin) to centre of generating circle, and $e^{in\theta} =$ line from the centre of the latter circle to the *point* on its circumference. See *Rule*. And the equation to the hypocycloid

$$z = me^{i\theta} - e^{-im\theta}$$

These equations may also be derived from the corresponding Cartesian co-ordinates, by using the formula

$$z = x + iy.$$

In Cartesian co-ordinates, the equations to the epitrochoid are

$$x = (a + b)\cos\vartheta - h\cos\frac{a+b}{b}\vartheta,$$

$$y = (a + b)\sin\vartheta - h\sin\frac{a+b}{b}\vartheta.$$

The equations to the hypotrochoid are

$$x = (a - b)\cos\vartheta + h\cos\frac{a-b}{b}\vartheta,$$

$$y = (a - b)\sin\vartheta - h\sin\frac{a-b}{b}\vartheta,$$

Transforming to equipollences

$$z = (a + b)e^{i\theta} - he^{i\frac{a+b}{b}\theta}$$

is equation to epitrochoid;

$$z = (a - b)e^{i\theta} + he^{-i\frac{a-b}{b}\theta}$$

is equation to hypotrochoid, and, therefore,

$$z = (a + b)\, e^{i\theta} - b e^{i\frac{a+b}{b}\theta}$$

is equation to epicycloid.

$$z = (a - b) e^{i\theta} + b e^{-i\frac{a-b}{b}\theta}$$

is equation to hypocycloid.

To simplify the equation to the epicycloid, let $a + b = nb$, then, dividing through by b and putting z for $\frac{z}{b}$, the equation becomes

$$z = n e^{i\theta} - e^{in\theta} \qquad (12)$$

Equation to tangent is

$$\zeta = n e^{i\theta} - e^{in\theta} + in\tau\,(e^{i\theta} - e^{in\theta}) \qquad (13)$$

and equation to normal

$$\zeta = n e^{i\theta} - e^{in\theta} - n\tau\,(e^{i\theta} - e^{in\theta}) \qquad (14)$$

Differentiating this last equation

$$0 = ni\,(e^{i\theta} - e^{in\theta}) - in\tau\,(e^{i\theta} - e^{in\theta}) - n\,(e^{i\theta} - e^{in\theta})\,\frac{d\tau}{d\vartheta} \qquad (15)$$

Equating real and imaginary parts

$$-\sin\vartheta - \sin n\vartheta + \tau\sin\vartheta + n\tau\sin n\vartheta - \cos\vartheta\,\frac{d\tau}{d\vartheta} + \cos n\vartheta\,\frac{d\tau}{d\vartheta} = 0$$

$$\cos\vartheta - \cos n\vartheta - \tau\cos\vartheta + n\tau\cos n\vartheta - \sin\vartheta\,\frac{d\tau}{d\vartheta} - \sin n\vartheta\,\frac{d\tau}{d\vartheta} = 0$$

Eliminating $\frac{d\tau}{d\vartheta}$, we get $\tau = \frac{2}{1 + n}$. Substituting in (14) the equation to the normal and reducing, we have the equation to the evolute

$$\zeta = \frac{n\,(n - 1)}{n + 1}\, e^{i\theta} + \frac{n - 1}{n + 1}\, e^{in\theta} \qquad (16)$$

an epicycloid in which the radii of the base and generating circles are as $n - 1$ to 1, or as a to b.

For radius of curvature, we have

$$\rho = \tau\sqrt{z'\bar{z}'} = \frac{2n}{n + 1}\sqrt{(ie^{i\theta} - ie^{in\theta})\,(ie^{-in\theta} - ie^{-i\theta})}$$

$$= -\frac{4n}{1 + n}\sin\frac{n - 1}{2}\vartheta = -\frac{4b\,(a + b)}{a + 2b}\cdot\sin\frac{a\vartheta}{2b}$$

Treating the hypocycloid in a similar way, we would find its evolute to be another hypocycloid.

If the radius of the generating circle be equal to that of the fixed circle, that is, if $a = b$, in the equation to the epicycloid, we have

$$z = 2ae^{i\theta} - ae^{2i\theta}$$

the Cardioid, a curve of the 4th order. The equation to the normal is

$$\zeta = 2ae^{i\theta} - ae^{2i\theta} - 2a\tau\,(e^{i\theta} - e^{2i\theta})$$

From which, we find the evolute to be

$$\zeta = \frac{2}{3}\,ae^{i\theta} + \frac{1}{3}\,ae^{2i\theta}$$

an inverted Cardioid. Also, for radius of curvature

$$\rho = \tau\sqrt{z'\bar{z}'} = -\frac{8}{3}\,a \sin\frac{\vartheta}{2}$$

If $b = \frac{a}{2}$ the equation of the epicycloid with two cusps, is

$$z = \frac{3}{2}\,ae^{i\theta} - \frac{1}{2}\,ae^{3i\theta}$$

the equation to one of the caustics of a circle, when the incident rays are parallel to a diameter. The equation to the hypotrochoid

$$z = (a - b)\,e^{i\theta} + he^{-i\frac{a-b}{b}\theta}$$

becomes for $b = \frac{a}{2}$

$$z = \frac{a}{2}\,e^{i\theta} + he^{i\theta}$$

the equation to an ellipse whose semi-axes are $\frac{a}{2} + h$ and $\frac{a}{2} - h$. An elegant application of this property of the ellipse to generate it by continuous motion, was made by Prof. Wallace, for explanation of which see Gregory's Examples, page 140.

The equation for the hypocycloid in this case becomes

$$z = \frac{a}{2}\,(e^{i\theta} + e^{-i\theta}) = a\cos\vartheta$$

a diameter of the base circle.

The equation for the hypocycloid, when $b = \frac{a}{4}$ becomes

$$z = \frac{3}{4} ae^{i\theta} + \frac{1}{4} ae^{-3i\theta}$$

corresponding to the Cartesian equation

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

The equation of the tangent to this curve is

$$\zeta = \frac{3}{4} ae^{i\theta} + \frac{1}{4} ae^{-3i\theta} + \frac{3}{4} ai\,(e^{i\theta} - e^{-3i\theta})\,\tau$$

or

$$z = \frac{3}{4}\, a\,(\cos\vartheta + i\sin\vartheta) + \frac{a}{4}(\cos 3\vartheta - i\sin 3\vartheta)$$

$$+ \frac{3}{4}\, ai\,(\cos\vartheta + i\sin\vartheta - \cos 3\vartheta + i\sin 3\vartheta)\,\tau$$

For intercept on axis of x, we must equate the imaginary part of this equation to zero, and substitute the value of τ thus found in the original equation. And, for the intercept on axis of y, we must equate the real part to zero, and substitute the value of τ thus found in the original equation. *Houel's Calculus*, Vol. II, Art. 613. Therefore for intercepts on the axes, we have

$$\zeta_a = \frac{1}{4}\, a\,(3\cos\vartheta + \cos 3\vartheta) - \frac{1}{4}\, a\,(\sin\vartheta + \sin 3\vartheta)\,\frac{\sin 3\vartheta - 3\sin\vartheta}{\cos\vartheta - \cos 3\vartheta}$$

$$= \frac{1}{2}\, a\,\frac{\sin 2\vartheta}{\sin\vartheta}$$

$$\zeta_b = \frac{1}{4}\, a\,(3\sin\vartheta - \sin 3\vartheta) + \frac{1}{4}\, a\,(\cos\vartheta - \cos 3\vartheta)\,\frac{3\cos\vartheta + \cos 3\vartheta}{\sin\vartheta + \sin 3\vartheta}$$

$$= \frac{1}{2}\, a\,\frac{\sin 2\vartheta}{\cos\vartheta}$$

Hence, h being the portion of the tangent intercepted by the axes,

$$h^2 = \zeta^2_a + \zeta^2_b = \frac{a^2}{4}\left(\frac{\sin^2 2\vartheta}{\sin^2\vartheta} + \frac{\sin^2 2\vartheta}{\cos^2\vartheta}\right) = a^2 \;\therefore\; h = a$$

and the intercepted portion is *constant*. Hence, this hypocycloid may be regarded as the *locus* of the *ultimate intersections* of a right line of *constant* length sliding between two straight lines at right angles.

Again, this hypocycloid is the *envelop* of the ellipses described with co-incident centre and axes, and having the sum of the semi-axes equal to a constant.

a and b being the semi-axes of the ellipses, $a + b = c =$ constant. Hence, a being a variable parameter, the equation to the ellipses is

$$z = a \cos \vartheta + i\,(c - a) \sin \vartheta \tag{17}$$

Now for an envelop we must have, as we have seen above,

$$z'\bar{z}_1 - \bar{z}'z_1 = 0 \tag{11}$$

From (17) we have

$$z' = -a \sin \vartheta + i\,(c - a) \cos \vartheta\,, \quad \therefore\ \bar{z}' = -a \sin \vartheta - i\,(c - a) \cos \vartheta$$
$$z_1 = \cos \vartheta - i \sin \vartheta, \qquad \therefore\ \bar{z}_1 = \cos \vartheta + i \sin \vartheta$$

Substituting in (11) we get

$$[-a \sin \vartheta + i\,(c - a) \cos \vartheta]\,(\cos \vartheta + i \sin \vartheta)$$
$$+ [a \sin \vartheta + i\,(c - a) \cos \vartheta]\,(\cos \vartheta - i \sin \vartheta) = 0$$

From this

$$a = c \cos^2 \vartheta$$

which value substituted in (17) gives for the envelop

$$z = c \cos^3 \vartheta + ic \sin^3 \vartheta$$

or

$$z = \frac{3}{4} c e^{i\theta} + \frac{1}{4} c e^{-3i\theta}.$$

the hypocycloid. Q. E. D.

These examples seem sufficient to illustrate the general method of deriving the equations and properties of plane curves by the Calculus of Equipollences.

KINEMATICS.

This method may also be readily applied to the problems of Kinematics. Thus

$$z = \varphi\,(t)$$

being the equipollent equation to an orbit

$$z' = \varphi'\,(t)$$

is the equation to hodograph, giving the law of its description as well as its form. And this shows that the vector-acceleration of a point's motion, z'' or $\frac{d^2z}{dt^2}$, is the vector-velocity in the hodograph. If the motion be in a plane curve, the equation may be written

$$z = re^{i\theta} \qquad (1)$$

$$z = r'e^{i\theta} + ire^{i\theta}\frac{d\vartheta}{dt} \qquad (2)$$

$$z'' = r''e^{i\theta} + 2ir'e^{i\theta}\frac{d\vartheta}{dt} - re^{i\theta}\left(\frac{d\vartheta}{dt}\right)^2 + ire^{i\theta}\frac{d^2\vartheta}{dt^2} \qquad (3)$$

Hence, at once from (2) and (3).

The radial velocity $= \frac{dr}{dt}$

The radial acceleration $= \frac{d^2r}{dt^2} - r\left(\frac{d\vartheta}{dt}\right)^2$

The transversal velocity $= r\frac{d\vartheta}{dt}$

The transversal acceleration $= \frac{1}{r}\cdot\frac{d}{dt}\left(r^2\frac{d\vartheta}{dt}\right)$

Compare Art. 311, Vol. III of Price's Calculus.

For uniform acceleration in a constant direction, the equation of motion is

$$z'' = a$$

$$\therefore z' = at + b$$

hence, hodograph is a straight line

$$z = \frac{at^2}{2} + bt$$

and orbit is a parabola.

If acceleration vary as distance from the origin, we have

$$z'' = \pm m^2 z$$

We integrate this by means of the Calculus of Operation, thus

$$\left(\frac{d^2}{dt^2} \mp m^2\right) z = 0$$

Therefore

$$\left(\frac{d}{dt} - m\right)\left(\frac{d}{dt} + m\right) z = 0$$

or

$$\left(\frac{d}{dt} - im\right)\left(\frac{d}{dt} + im\right) z = 0$$

These give, obvious substitutions being made

$z = c.\ Ch\ mt + d.\ Sh\ mt$, a hyperbola;
$z = c. \cos mt + d. \sin mt$, an ellipse;

where c and d are, in both cases, semi-conjugate diameters. Differentiating, the hodograph is a hyperbola or an ellipse

$$z' = cm\ Sh\ mt + dm\ Ch\ mt$$

or

$$z' = - cm \sin mt + dm \cos mt$$

All the plane theorems of Kinematics and Dynamics, can be solved by this method; but much more general solutions are made by employing Quaternions, as the motion is then not confined to plane curves. This brings us to the second part of our thesis:

THE RELATION OF EQUIPOLLENCES TO QUATERNIONS.

The Calculus of Quaternions is a more complicated system; but, once mastered, it possesses all the advantages that distinguish the Calculus of Equipollences, and is much more powerful, particularly in that it is not limited to plane problems.

Quaternions may rightly be considered as an extension to space of the Method of Equipollences; but having some special rules to which Equipollences are not subjected. A "vector" is a line represented, or supposed to be represented, by a complex quantity, and therefore two lines which are *equipollent* in the method of Bellavitis, would be represented by the same vector symbol in Hamilton's method.

To show the similarity, or rather the sameness of the methods, we take a well known example from Plane Geometry:

"If G be the point of intersection of the medians of a triangle, $A\ B\ C$, and O be any point whatever in the plane of the triangle, then

$$OA^2 + OB^2 + OC^2 = AG^2 + BG^2 + CG^2 + 3OG^2$$

First, by Equipollences we have

$$OB = OG - BG$$
$$OA = OG - AG$$
$$OC = OG - CG$$

$\therefore$ squaring and adding, we have

$$OA^2 + OB^2 + OC^2 = 3OG^2 - 2OG(AG + BG + CG) + AG^2 + BG^2 + CG^2$$

Now, recollecting that here the sign $=$ means "equipollent to,"

$$AG + BG + CG = AB + BC + CA = 0$$

therefore

$$OA^2 + OB^2 + OC^2 = 3OG^2 + AG^2 + BG^2 + CG^2 \quad \text{Q. E. D.}$$

Secondly, by Quaternions,

Let, $OA = \alpha$, $OB = \beta$, $OC = \gamma$, then, as is easily proved,

$$OG = \frac{1}{3}(\alpha + \beta + \gamma) \qquad (1)$$

Therefore

$$\alpha^2 + \beta^2 + \gamma^2 + 2S(\alpha\beta + \beta\gamma + \gamma\alpha) = 9OG^2 \qquad (2)$$

Now

$$AG = \frac{1}{3}(-2\alpha + \beta + \gamma)$$

$$BG = \frac{1}{3}(\alpha - 2\beta + \gamma) \qquad (3)$$

$$CO = \frac{1}{3}(\alpha + \beta - 2\gamma)$$

Hence, squaring and adding

$$AG^2 + BG^2 + CG^2 = \frac{2}{3}(\alpha^2 + \beta^2 + \gamma^2) - \frac{2}{3}S(\alpha\beta + \beta\gamma + \gamma\alpha). \qquad (4)$$

$$= \frac{2}{3}(\alpha^2 + \beta^2 + \gamma^2) - 3OG^2 + \frac{1}{3}(\alpha^2 + \beta^2 + \gamma^2) \text{ by (2).}$$

Therefore,

$$OA^2 + OB^2 + OC^2 = AG^2 + BG^2 + CG^2 + 3OG^2 \quad \text{Q. E. D.}$$

The only difference in the methods as here employed, is that the scalar part (resulting from the fact that multiplication is *not* commutative in Quaternions) occurs in the latter; but this, as we see, makes no essential difference. The proof by Quaternions, however, shows that the proposition is true for the point O anywhere in space.

Let us now examine a little more closely the fundamental principles involved in the two methods. Perfect symmetry is attained by Hamilton's method by considering all lines as equally imaginary, or rather equally real. The same symmetry, as far as lines entirely *in one plane* are concerned, seems to be attained by the method of Bellavitis. For, although there is always a *real* initial line expressed or implied, this fixed line may be taken anywhere in the plane, according to the requirements of the problem.

In Equipollences, the symbol $\sqrt{-1}$ which we have represented by the letter i is equal to the quotient $\frac{OI}{OH}$, of two equal lines at right angles, and it has the effect of turning a line through 90°, in the plane determined by OI and OH. This symbol, therefore, in a plane problem, performs the same function as the i (j k) of Quaternions; but in the latter system, i as an *operator* must be distinguished from i as an *operand*. ij is not the same as ji. This follows at once from the unique and fundamental principle of Quaternions, multiplication is *not* commutative. Although Hamilton's i, j, k are each separately equal to $\sqrt{-1}$, their combination by multiplication gives some important and remarkable results. They take us, as it were, into a higher field of algebraic analysis.

Since $\qquad i = \sqrt{-1}, \quad i^2 = -1 \ . \ . \ .$
and $\qquad j = \sqrt{-1}, \quad j^2 = -1, \ . \ . \ .$

we might expect to find $ij = -1$, but this, as we know, is not the case, for $ij = k$, that is, i operating on j turns it through 90° in a plane perpendicular to i, and it becomes k. Hence we see that in Quaternions, the $\sqrt{-1}$ is to be regarded as a symbol, rather than as an ordinary algebraic quantity. In Equipollences also, i is a symbol, but it is calculated exactly as the algebraic imaginary $\sqrt{-1}$.

In Hamilton's method, the product or quotient of two parallel lines is a (real) number; the product or quotient of two lines at right angles to each other, is a third line perpendicular to both. In Equipollences, in general we can affirm only one of these statements—the quotient of two parallel lines is a number. A *quaternion* is the product (or quotient) of two vectors α, β *not* at right angles, and consists of two parts, a numerical quantity and a vector perpendicular to plane of α, β thus

$$\alpha\beta = T\alpha \, T\beta \, (-\cos\vartheta + \varepsilon \sin\vartheta)$$

The product of two lines in equipollences has a similar form. Let

$$OA = me^{i\mu},\ OB = pe^{i\pi}$$

be any two lines not at right angles, then

$$OA \times OB = mpe^{i(\mu + \pi)} = mp\left[\cos(\mu + \pi) + i\sin(\mu + \pi)\right]$$

Here as before the product consists of two distinct parts—a real and an imaginary part, corresponding to the scalar and vector. m and p correspond to the tensors $T\alpha$, $T\beta$.

We also observe that either of these lines, or in fact any line not parallel or perpendicular to the initial line (such as $z = a(\cos\vartheta + i\sin\vartheta)$), has a form similar to that of the product $\alpha\beta$ given above.

Considering, therefore, Quaternions as applied only to plane problems, the essential difference between the two methods is that, in the one the commutative law of multiplication holds, while in the other this old law is void.

We might add innumerable examples to show the similarity of the two methods, as well as to illustrate the great advantage these Methods have over the Cartesian system. Instead of multiplying example, we refer to the various solutions given in works which treat of these methods, and especially to the examples in those works which we have been able to obtain—viz: *Tait's Quaternions*, "*Introduction to Quaternions*," by Kellard and Tait, *Méthode des Equipollences* by Bellavitis, Articles by Houel in the *Nouvelles Annales de Mathématiques*, July and August, 1869, and *Houel's Calculus*, *Vol. II.* For an account of the steps which led to the invention of Quaternions, we would refer to a sketch of Hamilton's life in the *North British Review* for September, 1866.

We think on the whole that plane problems can perhaps be solved more expeditiously by the Calculus of Equipollences, than by Quaternions. For such problems, the Method of Equipollences has certainly the advantage of being more easily understood and more quickly mastered. But the elegance, the logical simplicity and the extent of its applications, makes the Calculus of Quaternions immeasurably superior.

In conclusion, it has of course been impossible in these few pages, to give more than an outline of the Calculus of Equipollences, and its relation to Quaternions; and we can only hope that some of the elegant methods of the former system, and a few of the more striking points connecting the two systems, have been accurately presented.

University of Va., June, 1885.

Author born in Virginia.

ERRATA.

Page 5, bottom line, put z for x.

Page 7, 14th line, put $\frac{z'}{z'}$ for $\frac{z'}{z_1}$.

Page 8, 12th line, put $+$ for $-$ before e^{-imt}.

Page 11, 7th line, put ζ for z.

Page 15, in equation (3) put CG for CO.

Page 17, put Kelland for Kellard.