ON SOME PROPERTIES OF A PLANE VECTOR IN CONNECTION WITH THE STATIONARY FLOW OF ELECTRICITY IN AN INFINITE PLANE.

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ON THE PROPERTIES OF A PLANE VECTOR IN CONNECTION WITH THE STATIONARY FLOW OF ELECTRICITY IN AN INFINITE PLANE.

The development of the theory of imaginaries during the first half of the century, and in particular the geometrical interpretation of the same has found application in some of the most interesting branches of Mathematics and Physics. In fact we may almost say that this branch of the pure Mathematics was approached and investigated from the standpoint of a physicist rather than that of the mathematician. This remark is eminently true of the modern theory of Algebraic Functions as put forward during the last forty years chiefly by the labors of Riemann[1].

Lagrange[2] seems to have been one of the first who recognized the relation between the function of a complex argument and the problem of representing any portion of a spherical surface upon a plane. After Lagrange the geometrical theory was advanced at the hands of several writers, among whom Argand[3] deserves to be specially mentioned. The general problem of the representation of any part of one surface upon another surface so that corresponding figures upon the two should be ultimately similar, was proposed as a prize question by the Royal Society of Copenhagen and solved by Gauss[4]. A general solution was given, though the actual integration of the differential equations was effected only for surfaces of revolution. Kirchhoff[5] in a paper published in 1845 deduced a theory of electric flow for two dimensions founded upon Ohm's Law which formed the starting point for numerous subsequent investigations.

The appearance of Riemann's work in 1859 gave a new impetus to the subject, and while this is primarily a work of pure mathematics, it was written no doubt from a physical standpoint, and with reference to problems connected with the stationary flow of heat and electricity.[6] We should notice also a work of Haton de la Goupiliere[7] which is founded upon the theory of the complex variable.

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[1] "Grundlagen für eine Allgemeine Theorie der Functionen einer Veränderlichen Complexen Grösse." 1859. Also see Gesammelte mathematische Abhandlungen.

[2] "Sur la Construction des Cartes Géographiques."

[3] "Essai Sur une Manière de Représenter les Quantités Imaginaires dans les Constructions Géométriques." Paris, 1806.

[4] "Astronomische Abhandlungen—1825."

[5] "Ueber den Durchgang eines Elektrischen Stromes." etc. Poggendorfs Annalen. Bd. 64.

[6] See in this connection, Klein, "Ueber Riemann's Theorie der Algebraischen Functionen."

[7] "Mémoire sur une Nouvelle Théorie Générale des Lignes Isothermes et du Potentiel Cylindrique." Journal de l'École Polytechnique Cah. XXXVIII.

Of late years the subject of conform representation has received its fullest treatment at the hands of Schwarz[8] and of Holzmüller,[9] the latter having published a collection of results in text-book form.

Let us take a function of a vector (or of a complex argument) and examine briefly some of its properties.

Let

$$w = Re^{i\theta} = f(re^{i\beta}) = f(z).$$

$$\frac{dw}{dr} = \frac{dw}{dz} \cdot \frac{dz}{dr} = e^{i\beta}\frac{dw}{dz}$$

$$\frac{dw}{d\beta} = \frac{dw}{dz} \cdot \frac{dz}{d\beta} = rie^{i\beta}\frac{dw}{dz}$$

$$\therefore \frac{dw}{d\beta} = ri\frac{dw}{dr} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

which is the equation of condition that w shall be a function of the vector $re^{i\beta}$.

Differentiating both sides of the last equation we have

$$\frac{dR}{d\beta}e^{i\theta} + Rie^{i\theta}\frac{d\theta}{d\beta} = rie^{i\theta}\frac{dR}{dr} - Rre^{i\theta}\frac{d\theta}{dr}$$

Equating real and imaginary parts

$$\frac{dR}{dr} = \frac{R}{r}\frac{d\theta}{d\beta} \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\frac{dR}{d\beta} = -Rr\frac{d\theta}{dr} \quad \ldots\ldots\ldots\ldots \quad (3)$$

which may be regarded as equivalent to (1). The same results may be expressed in rectangular form as follows:

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[8] Articles in various journals, all of which appear in his "Gesammelte Math. Abhandlungen."

[9] "Theorie der Isogonalen Verwandtschaften."

$$w = u + iv = f(z) = f(x + iy)$$

$$\frac{dw}{dx} = \frac{dw}{dz}\frac{dz}{dx} = \frac{dw}{dz}$$

$$\frac{dw}{dy} = \frac{dw}{dz}\frac{dz}{dy} = i\frac{dw}{dz}$$

$$\therefore \frac{dw}{dy} = i\frac{dw}{dx} \qquad \text{(1a)}$$

Differentiating both sides

$$\frac{du}{dy} + i\frac{dv}{dy} = i\frac{du}{dx} - \frac{dv}{dx}$$

which gives

$$\frac{du}{dx} = \frac{dv}{dy} \qquad \text{(2a)}$$

$$\frac{du}{dy} = -\frac{dv}{dx} \qquad \text{(3a)}$$

From (2a) and (3a)

$$\frac{du}{dx}\frac{dv}{dx} + \frac{du}{dy}\frac{dv}{dy} = 0.$$

Hence it follows that the systems of curves defined by the equations

$$u(xy) = k \qquad\qquad v(xy) = k'$$

cut orthogonally.

Differentiating (2a) and (3a) according to x and y and adding, there follows:

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0.$$

Performing the same operations with regard to y and x

$$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}=o.$$

That is to say if $w=u+iv$ is a function of a vector, then both the real and imaginary parts satisfy the equation of Laplace. But in all cases of the stationary flow of heat and electricity in two dimensions, the potential function satisfies the same equation. Hence we see that if

$$u=f(x,y)=k$$

is the equation of the lines of equi-potential, that of the stream lines is

$$v=\varphi(x,y)=k'.$$

It is evident also that the latter may represent the lines of equipotential and the former the corresponding stream lines.

An important property of the differential co-efficient of a function of a vector appears from the following consideration. Put

$$w=Re^{i\theta}=f(re^{i\beta})=f(z)$$

$$\frac{dw}{dz}=\frac{e^{i\theta}}{e^{i\beta}}\frac{\frac{dR}{d\beta}+\frac{dR}{dr}\frac{dr}{d\beta}+Ri\left(\frac{d\theta}{d\beta}+\frac{d\theta}{dr}\frac{dr}{d\beta}\right)}{ri+\frac{dr}{d\beta}}$$

Substituting from the equations (2) and (3)

$$\frac{dw}{dz}=\frac{e^{i\theta}}{e^{i\beta}}\cdot\frac{\frac{dR}{d\beta}+\frac{R}{r}\frac{d\theta}{d\beta}\frac{dr}{d\beta}+Ri\frac{d\theta}{d\beta}-\frac{i}{r}\frac{dR}{dr}\frac{dr}{d\beta}}{ri+\frac{dr}{d\beta}}$$

$$=\frac{e^{i\theta}}{e^{i\beta}}\frac{\left(ri+\frac{dr}{d\beta}\right)\left(\frac{R}{r}\frac{d\theta}{d\beta}-\frac{i}{r}\frac{dR}{d\beta}\right)}{ri+\frac{dr}{d\beta}}$$

$$=\frac{e^{i\theta}}{e^{i\beta}}\left(\frac{R}{r}\frac{d\theta}{d\beta}-\frac{i}{r}\frac{dr}{d\beta}\right)$$

an expression which is independent of dr. Hence it follows that the *value of the differential co-efficient is independent of the direction in which the variation takes place.* Such functions were called by Cauchy *monogène* and are the only ones capable of representing the stationary flow of heat and electricity in a plane.

Suppose now that z_0 and w_0 refer to corresponding points which for convenience we take in separate planes. To z communicate small arbitrary displacements which may be represented by dz_1 and dz_2.

Let dw_1 and dw_2 be the displacements of w corresponding to the relation

$$w = f(z)$$

Then by virtue of the preceding proposition we have the equation

$$\frac{dw_1}{dz_1} = \frac{dw_2}{dz_2}$$

from which it appears that the angle between dz_1 and dz_2 is equal to that between dw_1 and dw_2 and the moduli of the vectors in the two planes bear constant ratio to one another. Hence it follows that if any system upon one surface be represented on another by means of such a function that the two systems will be ultimately similar. It is seen here that we have an extension of ordinary geometric inversion—this being simply a case of transformation by means of the function

$$w = \frac{1}{z}$$

and furthermore all the theorems relating to properties of figures in the first system can be translated into theorems expressing relations between the corresponding figures of the second system.

The *conform* character of figures connected by a function of a vector lends to the theory its importance in connection with the class of problems under consideration. For since any elementary rectangle bounded by equipotential lines and lines of flow is transformed into a similar rectangle it follows that the *representation* of any two dimension system will itself represent a possible case of electric flow. Analytically this result appears as follows:

Let $$w = u + iv = u(x, y) + iv(x, y) = f(z)$$

Then $$\nabla^2 u = o \qquad \nabla^2 v = o$$

Suppose x and y are no longer independent variables, but defined by the relation

$$z = x + iy = F(\zeta) = F(\xi + i\eta) = \varphi(\xi, \eta) + i\psi(\xi, \eta)$$

$$\nabla^2\varphi = o \qquad \nabla^2\psi = o$$

$$w = u\{\varphi(\xi,\eta), i\psi(\xi,\eta)\} + iv\{\varphi(\xi,\eta), i\psi(\xi,\eta)\}$$
$$= u_1(\xi,\eta) + i\delta v_1(\xi, n)$$

hence we have

$$\nabla^2 u_1 = o \qquad \nabla^2 v_1 = o$$

that is to say, if in any integral $u(x, y)$ of the equation

$$\nabla^2 u = o$$

we substitute for x, y any function of a vector, then will the new function be an integral of the same equation. The integral of this equation can evidently be written in the form

$$\frac{\varphi(u + iv) + \varphi_1(u - iv)}{2} = a$$

where a is the isothermal parameter of the system. If b be the parameter of the orthogonal system the condition of perpendicularity gives

$$\frac{\varphi(x + iy) - \varphi_1(x - iy)}{2i} = b$$

It is evident then that any case of stationary flow can be deduced from a known simpler case provided the proper transforming function is known. This must fulfill the ordinary conditions of a potential function. It must be continuous, single-valued and finite at all points except the electrodes. An infinite value in one plane must correspond to one or more infinite values in the other plane. As a simple illustration we may take the case of a rectangular system in which the equations of the equipotential lines and lines of flow are

$$u = a \quad \text{and} \quad v = b$$

and subject it to the transformation

$$w = \log(z)$$

Hence
$$u + iv = \log re^{i\beta}$$
$$= \log r + i\beta$$
$$u = \log r \qquad v = \beta$$
$$r = e^{u}$$

And the new system is a series of concentric circles and their orthogonals trajectories. The equipotential lines

$$x = a \qquad (a = a_1 + d, a_1 + 2d, \ldots a_1 + nd,)$$

go over into the series of circles whose radii are

$$r = e^{a}$$

and the stream lines

$$y = b \qquad (b = b_1 + f, b_1 + 2f, \ldots \ldots b_1 + nf)$$

appear as

$$\theta = b$$

Thus it will be noticed that a strip of the first plane bounded by the lines

$$x = \pm \infty, \qquad y = o, \qquad y = 2\pi$$

is represented on the entire area of the second plane. It is evident that by this transformation the Ptolemaic earth chart is deduced from that of Mercator. For $y > 2\pi$ the value of θ becomes not single but multiple: to avoid this an artifice of Riemann was to suppose the z plane composed of thin overlying sheets, on each of which the function has only a single value. In this case it is apparent that the successive figures on the overlying streets are congruent.

Let it be required to determine the flow when the positive electrodes are situated at regular intervals on the circumference of a circle, the infinite extent of the plane forming a negative electrode. We transfer the system last found by means of the function

$$w = \sqrt[n]{z}$$

If the centre of the system be taken at any point $z_0 = r_0 e^{i\beta_0}$ and if γ be the declivity of any right line passing through this point and ρ the length of any portion of it, then we may write the vector equation

$$re^{i\beta} - r_0 e^{i\beta_0} = \rho e^{i\gamma},$$

and for the conjugate figure

$$re^{-i\beta} - r_0 e^{-i\beta_0} = \rho e^{-i\gamma}.$$

Dividing and taking logarithms

$$\gamma = \frac{1}{2i} \log \frac{re^{i\beta} - r_0 e^{i\beta_0}}{re^{-i\beta} - r_0 e^{-i\beta_0}} \quad \ldots\ldots\ldots\ldots (4)$$

Performing the transformation indicated this becomes

$$\gamma = \frac{1}{2i} \log \frac{(re^{i\beta})^n - r_0 e^{i\beta_0}}{(re^{-i\beta})^n - r_0 e^{-i\beta_0}}.$$

But the expression

$$(re^{i\beta})^n - r_0 e^{i\beta_0}$$

can be regarded as the product of n factors of the form

$$re^{i\beta} - ae^{i\lambda}$$

when the last term denotes one of the n^{th} roots of the fixed centre. Hence the expression becomes

$$\gamma = \frac{1}{2i} \log \frac{\prod_{\nu=1}^{\nu=n} (re^{i\beta} - a_\nu e^{i\lambda_\nu})}{\prod_{\nu=1}^{\nu=n} (re^{-i\beta} - a_\nu e^{-i\lambda_\nu})}$$

$$= \sum_{\nu=1}^{\nu=n} \frac{1}{2i} \log \frac{re^{i\beta} - (ae^{i\lambda})_\nu}{re^{-i\beta} - (ae^{-i\lambda})_\nu}$$

But by equation (4) each term of this expression denotes the declivity of a line drawn from $re^{i\beta}$ to the corresponding root point, $ae^{i\lambda}$. Hence if $\varphi_1, \varphi_2 \ldots \varphi_n$ be the declivities of the n vectors drawn from any point of the locus to the root points,

$$\gamma = \varphi_1 + \varphi_2 + \ldots \ldots \varphi_n .$$

Similarly for the equipotential lines, the equation of one of which can be placed in the form

$$(re^{i\beta} - r_0 e^{i\beta_0})(re^{-i\beta} - r_0 e^{-i\beta_0}) = c^2$$

where $c = e^u$, u being the potential.

By transforming this becomes

$$[(re^{i\beta})^n - r_0 e^{i\beta_0})] [(re^{-i\beta})^n - r_0 e^{-i\beta_0}] = e^{2u} = c^2$$

$$= \prod_{\nu=1}^{\nu=n} [re^{i\beta} - (ae^{i\lambda})_\nu] [re^{-i\beta} - (ae^{-i\lambda})_\nu] .$$

Each factor of the expression represents the square of the distance from a point of the locus to the corresponding root point. Denoting the distances by

$l_1 \, l_2 \ldots l_n$ we have

$$l_1 \, l_2 \ldots \ldots l_n = e^u = c.$$

These two series of curves were called by Holzmüller "Regular Lemniscates and Hyperbolas of the n^{th} order." The n foci of the lemniscates are situated upon the circumference whose radius is

$$(\text{mod. } z_0)^{\frac{1}{n}} = r_0^{\frac{1}{n}}$$

For the equipotential lines of the primitive system not enclosing the origin, that is, for

$$l_1 \;\; l_2 \ldots \ldots l_n < r_0$$

we have lemniscates of n branches separated from each other and enclosing the foci.

To the circle through the origin corresponds the lemniscate of one branch

$$l_1 \ l_2 \ldots \ldots l_n = r_0$$

having a multiple point of the n^{th} order at the cèntre, while to any circle enclosing the origin corresponds the lemniscate of one branch

$$l_1 \ l_2 \ldots \ldots l_n > r_0$$

without multiple point having a circle of infinite radius as limiting form.

Any straight line passing through the centre midway between two adjacent electrodes cuts all lemniscates orthogonally. Hence it follows that the hyperbolas fall into n distinct and equal compartments of a plane. The equations of the hyperbolas being

$$\varphi_1 + \varphi_2 + \ldots \ldots \varphi_n = \gamma$$

that of thc asymptotes is

$$X = \frac{1}{n}(\gamma + \mu\pi) \qquad (\mu = o, 1, \ldots n - 1)$$

For $n = 2$ the system degenerates into the ordinary curves of Cassini and rectangular hyperbolas so familiar in optics. In the above it will be noticed that the current strength is the same for the different electrodes.

We proceed to consider a more general case. Electricity flows into an infinite plane at n given points,

$$p_1, p_2, \ldots p_n$$

and with current strengths,

$$h_1 \ h_2 \ldots \ldots h_n$$

and is conducted away at infinity. The required transformation is effected if we take w an entire rational function of z with sets of equal roots, the numbers in each set being equal to the respective current strengths. The infinite extents of the two planes then correspond. Each vanishing of the transformation function corresponds to a unit current at a root point of the z plane. The other conditions are evidently fulfiled. The transformation function then becomes

$$w = \prod_{\nu=1}^{\nu=n} (z - p_\nu)^{h_\nu}$$

The conjugate of which is

$$\bar{w} = \prod_{\nu=1}^{\nu=n} (\bar{z} - \bar{p}_\nu)^{h_\nu}.$$

Hence,

$$[\text{mod. } w]^2 = e^{2u} = \prod_{\nu=1}^{\nu=n} (z - p_\nu)^{h_\nu} (\bar{z} - \bar{p}_\nu)^{h_\nu}$$

or if as before l_ν denotes the distance from any point to the electrode of same index

$$e^{2u} = u^2 + v^2 = \prod_{\nu=1}^{\nu=n} l_\nu^{2h_\nu} .$$

Hence the expression for the potential becomes

$$u = \log (l_1^{h_1} . l_2^{h_2} \cdot \quad . \quad . \quad . \; l_n^{h_n})$$

which differs from the former in that the exponent of the radius from any electrode is equal to the strength of current entering at that point. Similarly for the stream lines

$$w = u + iv = \prod_{\nu=1}^{\nu=n} (z - p_\nu)^{h_\nu}$$

$$\bar{w} = u - iv = \prod_{\nu=1}^{\nu=n} (\bar{z} - \bar{p}_\nu)^{h_\nu} .$$

Hence as in a previous case

$$e^{2i\gamma} = \frac{\prod_{\nu=1}^{\nu=n} (z - p_\nu)^{h_\nu}}{\prod_{\nu=1}^{\nu=n} (\bar{z} - \bar{p}_\nu)^{h_\nu}}$$

$$= \prod_{\nu=1}^{\nu=n} e^{2ih_\nu \varphi_\nu} ,$$

which gives

$$\gamma = h_1 \varphi_1 + h_2 \varphi_2 + \; . \; . \; . \; . \; . \; h_n \varphi_n$$

It will be noticed that the preceding case is a special form of this.

The figure shows equipotential lines and lines of flow for the case where

$$p_1 = +1 \qquad p_2 = -1$$
$$h_1 = 2 \qquad h_2 = 1$$

the transforming function then being

$$w = (z - 1)^2 (z + 1)$$

By elimination this would give an algebraic curve of the sixth degree. Adopting the ordinary notion we have

$$l_1^2 l_2 = e^u$$

where l_1 and l_2 proceed from the root points. The line of zero potential is transformed into the lemniscate of two branches

$$l_1^2 l_2 = 1$$

The double point of the first lemniscate of one branch is easily given by the condition

$$(l_1 \pm \delta)^2 (l_2 \pm \delta) = e^u$$

Discarding the higher powers of δ this gives the equation

$$l_1^2 l_2 + 2 l_1 l_2 \delta - l_1^2 \delta = l_1^2 l_2 - 2 l_1 l_2 \delta + l_1^2 \delta$$
$$\therefore \quad l_1 = 2 l_2$$

that is, the segments of the axis formed by the double point are proportional to the current strengths at the respective electrodes.

For the trajectories we have

$$2\varphi_1 + \varphi_2 = \gamma$$

which is a hyperbola of the third order. The figure shows the form of the same for

$$\gamma = \frac{\pi}{3} \text{ and } \gamma = \frac{3\pi}{5}$$

The intersection of the asymptotes divides the axis into two segments which are inversely as the current strengths at the electrodes.

The equation of any curve in the first plane of the form

$$f(R, \theta) = o$$

appears in the second plane as

$$f(p_1^{h_1} p_2^{h_2} \ldots \ldots p_n^{h_n}, h_1 \varphi_1 + h_2 \varphi_2 + \ldots h_n \varphi_n) = o$$

For example the equation of an isoclinic of the first system is

$$R = a^{\theta}$$

and the equation of the corresponding isoclinic of the second system is

$$l_1^{h_1} l_2^{h_2} \ldots \ldots l_n^{h_n} = a^{(h_1 \varphi_1 + h_2 \varphi_2 + \ldots h_n \varphi_n}$$

If we suppose the current strengths equal, the lines of equipotential become a series of curves named by Darboux,[10] *Cassinoids.* The transforming function in this case becomes simply

$$w = \prod_{\nu=1}^{\nu=n} (z - p_\nu)$$

Since the infinite extents of the two planes correspond, it follows that through every electrode of the new system passes a curve with two infinite branches. The Hyperbola falls in n compartments formed by its n rectilinear asymptotes, any two of which enclose an angle given by $\frac{\pi}{n}$.

The asymptotes pass through a common point—the center of gravity of the electrodes.[11]

This property may also be shown by considering the mode of description of the curves. We may suppose the general curve to be the locus of the point of intersection of radii revolving about the electrodes as centres so as to fulfil the condition

$$h_1 \varphi_1 + h_2 \varphi_2 + \ldots \ldots h_n \varphi_n = \gamma$$

Let the subscript a denote the asymptotic position of any two radii so that we have

$$h_l \varphi_a = h_m \varphi_m$$

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[10] "Sur une classe remarquable de courbes et de surfaces algébraiques et sur la théorie des imaginaires." *Mem. de Bordeaux VIII, IX.* The paper was not accessible at the time its existence was brought to the writer's notice.

[11] See Lucas, "Gèometrie des polynomes," *Journal de l'École polytechnique,* Cah. XLVI.

Let each side of the equation receive a small increment, which gives

$$h_l\left(\varphi_a + \frac{\theta}{h_l}\right) = h_m\left(\varphi_a + \frac{\theta}{h_m}\right)$$

This shows that the angles moved over by the two radii are as the ratio $h_m : h_l$, that is, inversely as the current strengths of the electrodes from which they proceed. Let x_l and x_m be segments of a line at infinity which are formed by the radii in their asymptotic positions and the first point of intersection after displacement. In view of what he have just shown we then have

$$x_l : x_m = \frac{\theta}{h_l} : \frac{\theta}{h_m} = h_m : h_l$$

$$\therefore \qquad x_l h_l = x_m h_m$$

and the asymptote through the point of ultimate intersection of the radii cuts the axis of the electrodes at their centre of gravity. Continuing the process till all the electrodes are included the truth of the theorem becomes evident.

We consider next the transformation effected by the function of the form

$$w = \frac{\varphi(z)}{\psi(z)}$$

where numerator and denominator are entire rational functions. This transformation gives the most general case of electric flow in an infinite plane where both positive and negative electrodes have any given position. Each root of the equation

$$\psi(z) = o$$

corresponds to an infinite value of w and hence to a a negative electrode in the z plane. So likewise we have the corresponding values

$$w = o \qquad \qquad \varphi(z) = o$$

the roots of which latter equation are positive electrodes. Inversely stated the problem becomes the following: electricity flows into a plane at the points $p_1, p_2 \ldots\ldots p_n$ and is conducted away at points $q_1, q_2 \ldots\ldots q_m$.

If $h_1, h_2 \ldots\ldots h_n$ be the strengths of the inflowing, $k_1, k_2 \ldots\ldots k_m$ those of the outflowing currents the transformation function becomes

$$w = \frac{\prod_{\nu=1}^{\nu=n} (z - p_\nu)^{h_\nu}}{\prod_{\nu=1}^{\nu=m} (z - q_\nu)^{k_\nu}}$$

and for the conjugate system

$$\bar{w} = \frac{\prod_{\nu=1}^{\nu=n} (\bar{z} - \bar{p}_\nu)^{h_\nu}}{\prod_{\nu=1}^{\nu=m} (\bar{z} - \bar{q}_\nu)^{k_\nu}} \cdot$$

Hence by multiplication we have

$$e^{2u} \doteq (\text{mod}\, w)^2 = \frac{\prod_{\nu=1}^{\nu=n} (z - p_\nu)^{h_\nu} (\bar{z} - \bar{p}_\nu)^{h_\nu}}{\prod_{\nu=1}^{\nu=m} (z - q_\nu)^{k_\nu} (\bar{z} - \bar{q}_\nu)^{k_\nu}}$$

Each set of factors of numerator and denominator represents the $2h_\nu$ power of the distance from any point to the points p_ν and q_ν respectively. Hence we may write,

$$u = \text{pot.} = \log \frac{l_1^{h_1}\, l_2^{h_2} \ldots\ldots l_n^{h_n}}{s_1^{k_1}\, s_2^{k_2} \ldots\ldots s_m^{k_m}}$$

and the circle whose radius is e^u becomes the curve

$$\frac{l_1^{h_1}\, l_2^{h_2} \ldots\ldots l_n^{h_n}}{s_1^{k_1}\, s_2^{k_2} \ldots\ldots s_m^{k_m}} = e^u$$

Similarly for the stream lines

$$w = \frac{\prod_{\nu=1}^{\nu=n} (z - p_\nu)^{h_\nu}}{\prod_{\nu=1}^{\nu=m} (z - q_\nu)^{k_\nu}}$$

$$\bar{w} = \frac{\prod_{\nu=1}^{\nu=n} (\bar{z} - \bar{p}_\nu)^{h_\nu}}{\prod_{\nu=1}^{\nu=m} (\bar{z} - \bar{q}_\nu)^{k_\nu}}.$$

Hence for the argument of any stream line

$$e^{2i\gamma} = \frac{\prod_{\nu=1}^{\nu=n} (z - p_\nu)^{h_\nu}}{\prod_{\nu=1}^{\nu=n} (\bar{z} - \bar{p}_\nu)^{h_\nu}} \cdot \frac{\prod_{\nu=1}^{\nu=m} (\bar{z} - \bar{q}_\nu)^{k_\nu}}{\prod_{\nu=1}^{\nu=m} (z - q_\nu)^{k_\nu}}$$

which can be written in accordance with what precedes

$$e^{2i\gamma} = \prod_{\nu=1}^{\nu=m} e^{2ih_\nu \varphi_\nu} \cdot \prod_{\nu=1}^{\nu=m} e^{-2ik_\nu \psi_\nu}$$

or

$$\gamma = h_1 \varphi_1 + h_2 \varphi_2 + \ldots h_n \varphi_n - (k_1 \psi_1 + k_2 \psi_2 + \ldots k_m \psi_m)$$

In the two planes we have generally the corresponding sets of curves

$$f(R\theta) = o$$

$$f\left(\frac{l_1^{h_1}\, l_2^{h_2} \ldots l_n^{h_n}}{s_1^{k_1}\, s_2^{k_2} \ldots s_m^{k_m}},\, h_1 \varphi_1 + \ldots h_n \varphi_n - k_1 \psi_1 - \ldots k_m \psi_m\right) = o$$

It is evident that for stationary flow we must have

$$\sum_{\nu=1}^{\nu=n} h_\nu - \sum_{\nu=1}^{\nu=m} q_\nu = o$$

where the summation is extended over the entire system of electrodes all of which are supposed to lie at a finite distance.

The case may also rise where the points of outflow are partly at a finite and partly at an infinite distance. Suppose we have (considering only the finite points) the equation

$$\sum h - \sum k = a$$

Then if λ and μ be the degrees of the numerator and denominator of the transforming function we must have

$$\lambda - u = a$$

The condition of stationary flow evidently makes it necessary that a current of strength a must be conducted away at infinity. This may be assumed to occur at a electrodes each conveying unit current. The function

$$w = \frac{\varphi(z)}{\psi(z)}$$

becomes infinite for each root of the equation

$$\psi(z) = o$$

which gives immediately the finite lying points of outflow. It also becomes infinite for

$$z = \infty.$$

The function may be written in the form

$$w = \varphi_1(z) + \frac{x(z)}{\psi(z)}$$

where $\varphi_1(z)$ is of the degree $\lambda - \mu$.

For $z = \infty$ we have therefore a value of w of the form

$$\infty^{\lambda - \mu}.$$

The physical fact shows that this may be interpreted as equivalent to $\lambda - \mu$ infinite roots.