Canonical bases arising from quantum symmetric pairs and Kazhdan-Lusztig theory

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Abstract

A breakthrough in representation theory is the discovery of canonical bases of quantum groups by Lusztig. In type A, the canonical bases can be used to reformulate the Kazhdan-Lusztig theory for the BGG category $\mathcal{O}$ of general linear Lie algebras, which enables further generalization to Brundan's Kazhdan-Lusztig conjecture for general linear Lie superalgebras.

In this dissertation, we first show a coideal subalgebra of the quantum group of type A and the Hecke algebra of type B satisfy a double centralizer property, generalizing the Schur-Jimbo duality. The quantum group of type A and its coideal subalgebra form a quantum symmetric pair. Then we initiate a theory of canonical bases arising from quantum symmetric pairs. We show simple integrable modules of the quantum group of type A and their tensor products admit new canonical bases different from Lusztig's canonical bases. Finally we use such new canonical bases to formulate and establish the Kazhdan-Lusztig theory for the BGG category $\mathcal{O}$ of the ortho-symplectic Lie superalgebra $\mathfrak{osp}(2m+1|2n)$ for the first time. The non-super specialization of our theory amounts to a new formulation of the classical Kazhdan-Lusztig theory for the BGG category $\mathcal{O}$ of the Lie algebras of type B/C.

Contents

Introduction

Background

A milestone in representation theory was the Kazhdan-Lusztig (KL) theory initiated in [KL] (and completed in [BB, BK]), which offered a powerful solution to the difficult problem of determining the irreducible characters in the BGG category $\mathcal{O}$ of a semisimple Lie algebra $\mathfrak{g}$. The Hecke algebra $\mathcal{H}_W$ associated to the Weyl group W of $\mathfrak{g}$ plays a central role in the KL formulation, which can be paraphrased as follows: the simple modules of the principal block in $\mathcal{O}$ correspond to the Kazhdan-Lusztig basis of $\mathcal{H}_W$ while the Verma modules correspond to the standard basis of $\mathcal{H}_W$. The characters of the simple modules in singular blocks are then determined from those in the principal block via translation functors [So1], and the characters of tilting modules were subsequently determined in [So2, So3].

Though the classification of finite-dimensional simple Lie superalgebras over $\mathbb{C}$ was achieved in 1970's by [Kac], the study of representation theory such as the BGG category $\mathcal{O}$ for a Lie superalgebra turns out to be very challenging and the progress has been made only in recent years. One fundamental reason is that the Weyl group (of the even part) of a Lie superalgebra alone is not sufficient to control the linkage principle in $\mathcal{O}$, and hence the corresponding Hecke algebra can not play a crucial role as in the classical Kazhdan-Lusztig theory. Among all basic Lie superalgebras, the infinite series $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ are arguably the most fundamental ones. Since these Lie superalgebras specialize to Lie algebras when one of the parameters m or n is zero, any possible (conjectural) approach on the irreducible character problem in the BGG category of such a Lie superalgebra has to first provide a new formulation for a classical Lie algebra in which the Hecke algebra does not feature directly.

Brundan [Br1] in 2003 formulated a conjecture on the irreducible and tilting characters for the BGG category $\mathcal{O}$ for the general linear Lie superalgebra $\mathfrak{gl}(m|n)$, using Lusztig's canonical basis. In this case, fortunately Schur-Jimbo duality [Jim] between a Drinfeld-Jimbo quantum group $\mathbf{U}$ and a Hecke algebra of type A enables one to reformulate the KL theory of type A in terms of Lusztig's canonical basis on some Fock space $\mathbb{V}^{\otimes m}$, where $\mathbb{V}$ is the natural representation of $\mathbf{U}$. Brundan's formulation for $\mathfrak{gl}(m|n)$ makes a crucial use of the Fock space $\mathbb{V}^{\otimes m} \otimes \mathbb{W}^{\otimes n}$, where $\mathbb{W}$ denotes the restricted dual to $\mathbb{V}$. The longstanding conjecture of Brundan was settled in [CLW2], where a super duality approach developed earlier [CW1, CL] (cf. [CW2, Chapter 6]) plays a key role. A second and different proof of Brundan's conjecture has appeared in Brundan, Losev, and Webster [BLW].

To date, there has been no (conjectural) formulation for a solution of the irreducible character problem in the BGG category $\mathcal{O}$ of the ortho-symplectic Lie superalgebras in general. The reason should have become clear as we explain above: no alternative approach to KL theory of type BCD existed without using Hecke algebras directly.

A super duality approach was developed in [CLW1] which solves the irreducible character problem for some *distinguished* parabolic BGG categories of the $\mathfrak{osp}$ Lie superalgebras, but it was not sufficient to attack the problem in the full BGG category. In these cases, a Brundan-type Fock space formulation was not available. One of the implications of the super duality which is important to us though is that the linkage for the distinguished parabolic categories of $\mathfrak{osp}(2m+1|2n)$-modules is controlled by Hecke algebra of type B_∞, and so one hopes that it remains to be so for the full BGG category of $\mathfrak{osp}(2m+1|2n)$-modules.

The goal

The goal of this dissertation is to give a complete solution to the irreducible character problem in the BGG category $\mathcal{O}$ of modules of integer and half-integer weights for the ortho-symplectic Lie superalgebras $\mathfrak{osp}(2m+1|2n)$ of type $B(m,n)$. In particular, the non-super specialization of our work amounts to a new approach to Kazhdan-Lusztig theory of Lie algebras of classical type.

To achieve the goal, we are led to develop in Part 1 a new theory of canonical bases (called *$\imath$-canonical basis*) arising from *quantum symmetric pairs* $(\mathbf{U}, \mathbf{U}^\imath)$. A new formulation of the KL theory for Lie algebras of type B is then made possible by our new duality that the coideal subalgebra $\mathbf{U}^\imath$ of $\mathbf{U}$ and the Hecke algebra of type B_m form double centralizers on $\mathbb{V}^{\otimes m}$, generalizing the Schur-Jimbo duality. Part 1 (which consists of Chapters 1-6) has nothing to do with Lie superalgebras and should be of independent interest, even though there was no particular motivation to do so without being desperately demanded from the super representation theory – the powerful Kazhdan-Lusztig theory in its original form has worked well after all.

We develop in Part 2 an infinite-rank version of the constructions in Part 1, and then relate the $\imath$-canonical basis to the BGG category $\mathcal{O}_{\mathbf{b}}$ of $\mathfrak{osp}(2m+1|2n)$-modules of (half-)integer weights relative to a Borel subalgebra whose type is specified by a 0^m1^n-sequence $\mathbf{b}$. In this approach, the role of Kazhdan-Lusztig basis is played by the (dual) $\imath$-canonical basis for a suitable completion of the $\mathbf{U}^\imath$-module $\mathbb{T}^{\mathbf{b}}$ associated to $\mathbf{b}$; Here $\mathbb{T}^{\mathbf{b}}$ is a tensor space which is a variant of $\mathbb{V}^{\otimes m} \otimes \mathbb{W}^{\otimes n}$. This dissertation is largely based on the preprint [BW13].

An overview of Part 1

Our starting point is actually natural and simple. The generalization of Schur duality beyond type A in the literature is not suitable to our goal, since it replaces the Lie algebra/group of type A by its classical counterpart and modifies the symmetric group to become a Brauer algebra (or a quantum version of such). For our purpose, as we look for a substitute for KL theory where the Hecke algebras have played a key role, we ask for some quantum group like object with a coproduct (not Schur type algebra) which centralizes the Hecke algebra of type B_n when acting on $\mathbb{V}^{\otimes n}$. We found the answer and recognized it as a coideal subalgebra of the quantum group $\mathbf{U}$, a quantum version of the enveloping algebra of the subalgebra of $\mathfrak{sl}(\mathbb{V})$ fixed by some involution, which forms a quantum symmetric pair with $\mathbf{U}$.

Note that the formulation of Part 1 is in the setting that $\mathbb{V}$ is finite-dimensional, while it is most natural to set $\mathbb{V}$ to be infinite-dimensional when making connection with category $\mathcal{O}$ in Part 2.

The structure theory of quantum symmetric pairs was systematically developed by Letzter and then Kolb (see [Le], [Ko] and the references therein). Though our coideal subalgebra can be identified with some particular examples in literature by an explicit (anti-)isomorphism, the particular form of our presentation and its embedding into $\mathbf{U}$ are different and new. The coideal subalgebra in our presentation manifestly admits a bar involution, and the specialization at $q = 1$ of our presentation has a natural interpretation in terms of translation functors in category $\mathcal{O}$. Depending on whether the dimension of $\mathbb{V}$ is even or odd, we denote the (right) coideal subalgebra by $\mathbf{U}^{\imath}$ or $\mathbf{U}^{\jmath}$, respectively. The two cases are similar but also have quite some differences, and the case of $\mathbf{U}^{\imath}$ is more challenging as it contains an unconventional generator which we denote by t (besides the Chevalley-like generators e_{α_i} and f_{α_i}). We mainly restrict our discussion to $\mathbf{U}^{\imath}$ (and so $\dim \mathbb{V}$ is even) below. The bar involutions on the coideal subalgebra $\mathbf{U}^{\jmath}$ and a variant of the coideal subalgebra $\mathbf{U}^{\imath}$ have been observed independently in [ES], where the generators of these algebras have been interpreted as translation functors of certain parabolic category $\mathcal{O}$ of type D.

Recall that the coproduct $\Delta : \mathbf{U} \rightarrow \mathbf{U} \otimes \mathbf{U}$ is not compatible with the bar involution ψ on $\mathbf{U}$ and $\psi \otimes \psi$ on $\mathbf{U} \otimes \mathbf{U}$, and Lusztig's quasi-$\mathcal{R}$-matrix Θ is designed to intertwine Δ and $\overline{\Delta}$, where $\overline{\Delta}(u) := (\psi \otimes \psi)\Delta(\psi(u))$, for $u \in \mathbf{U}$. Lusztig's construction of Θ is a variant of Drinfeld's construction of universal $\mathcal{R}$-matrix [Dr], and it takes great advantage of the triangular decomposition and a natural bilinear form of $\mathbf{U}$. The bar involution on $\mathbb{V}^{\otimes m}$ was then constructed by means of the quasi-$\mathcal{R}$-matrix Θ. Inspired by the type A reformulation of KL theory (cf., e.g., [VV1, Br1, CLW2]), as an alternative of the Kazhdan-Lusztig theory without using Hecke algebras we ask for a canonical basis theory arising from the quantum symmetric pair.

The embedding $\imath : \mathbf{U}^\imath \rightarrow \mathbf{U}$ which makes $\mathbf{U}^\imath$ a coideal subalgebra of $\mathbf{U}$ does not commute with the bar involution $\psi_\imath$ on $\mathbf{U}^\imath$ and ψ on $\mathbf{U}$. We have a coproduct of the coideal form $\Delta : \mathbf{U}^\imath \rightarrow \mathbf{U}^\imath \otimes \mathbf{U}$. Define $\overline{\Delta} : \mathbf{U}^\imath \rightarrow \mathbf{U}^\imath \otimes \mathbf{U}$ by $\overline{\Delta}(u) = (\psi_\imath \otimes \psi)\Delta(\psi_\imath(u))$, for all $u \in \mathbf{U}^\imath$. Note that the $\overline{\Delta}$ here is not a restriction of Lusztig's $\overline{\Delta}$. Toward our goal, in place of Lusztig's quasi-$\mathcal{R}$-matrix for $\mathbf{U}$ one would need a quasi-$\mathcal{R}$-matrix $\Theta^\imath$ which intertwines Δ and $\overline{\Delta}$ for $\mathbf{U}^\imath$. The problem here is that $\mathbf{U}^\imath$ does not have any obvious triangular decomposition or bilinear form as for $\mathbf{U}$.

Our key strategy is to ask first for some suitable intertwiner Υ which intertwines $\imath$ and $\bar{\imath} : \mathbf{U}^\imath \rightarrow \mathbf{U}$, where $\bar{\imath}(u) := \psi\big(\imath(\psi_\imath(u))\big)$, for $u \in \mathbf{U}^\imath$; note the remarkable analogy with a key property of Lusztig's Θ. We succeed in constructing such an intertwiner

Υ in some completion of the negative half $\mathbf{U}^-$ of $\mathbf{U}$ and show that it is unique up to a scalar multiple (see Theorem 2.3.1). Then by combining Υ with Lusztig's Θ we are able to construct the quasi-$\mathcal{R}$-matrix $\Theta^\imath$, which lies in some completion of $\mathbf{U}^\imath \otimes \mathbf{U}^-$. The crucial properties $\Upsilon\overline{\Upsilon} = 1$ and $\Theta^\imath\overline{\Theta^\imath} = 1$ hold. The intertwiner Υ can also be applied to turn an involutive $\mathbf{U}$-module into an $\imath$-involutive $\mathbf{U}^\imath$-module (see Proposition 3.4.2, Definitions 1.4.3 and 3.4.1).

It turns out to be a subtle problem to show that Υ lies in (a completion of) the integral $\mathcal{A}$-form $\mathbf{U}^-_{\mathcal{A}}$, where $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. We are led to study the simple lowest weight $\mathbf{U}$-modules ${}^\omega L(\lambda)$ for $\lambda \in \Lambda^+$ regarded as $\mathbf{U}^\imath$-modules. By a detailed study on the behavior of the generator t in $\mathbf{U}^\imath$ in the rank one case, we show that Υ preserves the $\mathcal{A}$-form ${}^\omega L_{\mathcal{A}}(\lambda)$ for all $\lambda \in \Lambda^+$, and this eventually allows us to establish the integrality of Υ (see Theorem 4.4.2). We then construct the *$\imath$-canonical basis* of ${}^\omega L(\lambda)$ which is $\psi_\imath$-invariant and admits a triangular decomposition with respect to Lusztig's canonical basis on ${}^\omega L(\lambda)$ with coefficients in $\mathbb{Z}[q]$ (see Theorem 4.5.2). Consequently, we construct an $\imath$-canonical basis for any tensor product of several finite-dimensional simple $\mathbf{U}$-modules, which differs from Lusztig's canonical basis on the same tensor product.

Generalizing the Schur-Jimbo duality in type A, we show that the action of the coideal algebra $\mathbf{U}^\imath$ and Hecke algebra $\mathcal{H}_{B_m}$ on $\mathbb{V}^{\otimes m}$ form double centralizers, where $\mathbb{V}$ is the natural representation of $\mathbf{U}$ (see Theorem 5.2.3). With Υ and $\Theta^\imath$ at hand, we are able to construct a bar involution $\psi_\imath$ on the $(\mathbf{U}^\imath, \mathcal{H}_{B_m})$-bimodule $\mathbb{V}^{\otimes m}$ which is compatible with the bar involutions on $\mathbf{U}^\imath$ and $\mathcal{H}_{B_m}$ (see Theorem 5.3.2). In particular, the $\imath$-canonical basis on the involutive $\mathbf{U}^\imath$-module $(\mathbb{V}^{\otimes m}, \psi_\imath)$ alone is sufficient to reformulate the KL theory of type B.

An overview of Part 2

Part 2 is very close to [CLW2] in spirit, where the category $\mathcal{O}$ of $\mathfrak{gl}(m|n)$-modules was addressed. In this Part, we take the $\mathbb{Q}(q)$-space $\mathbb{V}$ to be the direct limit as $r \mapsto \infty$ of the $2r$-dimensional ones considered in Part 1. Also let $\mathbf{U}$ and $\mathbf{U}^\imath$ be the corresponding infinite-rank limits of their finite-rank counterparts in Part 1.

For an $0^m 1^n$-sequence $\mathbf{b}$ (which consists of m zeros and n ones), we define a tensor space $\mathbb{T}^{\mathbf{b}}$ using m copies of $\mathbb{V}$ and n copies of $\mathbb{W}$ with the tensor order prescribed by $\mathbf{b}$ (with 0 corresponds to $\mathbb{V}$); for instance, associated to $\mathbf{b}^{\rm st} = (0, \ldots, 0, 1, \ldots, 1)$, we have $\mathbb{T}^{\mathbf{b}^{\rm st}} = \mathbb{V}^{\otimes m} \otimes \mathbb{W}^{\otimes n}$. Such a tensor space (called Fock space) was an essential ingredient in the formulation of Kazhdan-Lusztig-type conjecture for $\mathfrak{gl}(m|n)$ and its generalizations [Br1, Ku, CLW2]. In this approach, $\mathbb{T}^{\mathbf{b}}$ at $q = 1$ (denoted by $\mathbb{T}^{\mathbf{b}}_{\mathbb{Z}}$) is identified with the Grothendieck group of the BGG category of $\mathfrak{gl}(m|n)$-modules (relative to a Borel subalgebra of type $\mathbf{b}$), and the (dual) canonical bases of the $\mathbf{U}$-module $\mathbb{T}^{\mathbf{b}}$ play the role of Kazhdan-Lusztig basis which solves the irreducible and tilting character problem in the BGG category for $\mathfrak{gl}(m|n)$.

Now with the intertwiner Υ and the quasi-$\mathcal{R}$-matrix $\Theta^\imath$ for the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ at disposal, we are able to construct the $\imath$-canonical and dual $\imath$-canonical bases for $\mathbb{T}^{\mathbf{b}}$ (or rather in its suitable completion with respective to a Bruhat ordering); see Theorem 9.2.4. In the finite-rank setting, this was already proved in Part 1. Nevertheless, the infinite-rank setting requires much care and extra work to deal with suitable completions, similar to [CLW2] (see also [Br1]). A simple but crucial fact is that the partial ordering for $\mathbb{T}^{\mathbf{b}}$ used in [CLW2] is coarser than the one used in this paper and this allows various constructions in *loc. cit.* to carry over to the current setting. We will ignore the completion issue completely in the remainder of the Introduction.

Our main theorem (Theorem 11.6.1), which will be referred to as (**b**-KL) here, states that there exists an isomorphism between the Grothendieck group of the BGG category $\mathcal{O}_{\mathbf{b}}$ of $\mathfrak{osp}(2m+1|2n)$-modules of integer weights (relative to a Borel subalgebra of type **b**) and $\mathbb{T}^{\mathbf{b}}_{\mathbb{Z}}$, which sends the Verma, simple, and tilting modules to the standard monomial, dual $\imath$-canonical, and $\imath$-canonical bases (at $q = 1$), respectively. In other words, the entries of the transition matrix between (dual) $\imath$-canonical basis and monomial basis play the role of Kazhdan-Lusztig polynomials in our category $\mathcal{O}_{\mathbf{b}}$.

Granting the existence of the (dual) $\imath$-canonical bases of $\mathbb{T}^{\mathbf{b}}$, the overall strategy of a proof of (**b**-KL) follows the one employed in [CLW2] in establishing Brundan's Kazhdan-Lusztig-type conjecture, which is done by induction on n with the base case solved by the classical Kazhdan-Lusztig theory of type B [KL, BB, BK] (as reformulated above in terms of the $\imath$-involutive $\mathbf{U}^{\imath}$-module $\mathbb{V}^{\otimes m}$). There are two main steps in the proof. First, we need (an easy generalization of) the super duality developed in [CLW1] for $\mathfrak{osp}$, which is an equivalence of parabolic categories of $\mathfrak{osp}(2m+1|2n+\infty)$-modules and $\mathfrak{osp}(2m+1|n|\infty)$-modules. We establish the corresponding combinatorial super duality which states that there is an $\mathbf{U}^\imath$-isomorphism between $\mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{V}$ and $\mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{W}$, which matches the corresponding standard monomial, $\imath$-canonical, and dual $\imath$-canonical bases.

The second step is a comparison of ($\mathbf{b}$-KL) and ($\mathbf{b}'$-KL) for adjacent sequences $\mathbf{b}$ and $\mathbf{b}'$ (here "adjacent" means differing exactly by an adjacent pair 01). Let us assume for simplicity that the first entries of $\mathbf{b}$ and $\mathbf{b}'$ are both 0 here (see Remarks 10.2.1 and 11.6.3 for the removal of this assumption), as this is sufficient in solving the irreducible and tilting character problems for $\mathfrak{osp}(2m+1|2n)$-modules. Thanks to the coideal property of $\mathbf{U}^\imath$, the iterated coproduct for $\mathbf{U}^\imath$ has images in $\mathbf{U}^\imath \otimes \mathbf{U} \otimes \ldots \otimes \mathbf{U}$. Therefore the comparison of ($\mathbf{b}$-KL) and ($\mathbf{b}'$-KL) for adjacent $\mathbf{b}$ and $\mathbf{b}'$ can be carried out exactly as in the type A setting [CLW2] since the exchange of the adjacent 0 and 1 does not affect the first tensor factor and hence will not use $\mathbf{U}^\imath$. The upshot is that the validity of the statement ($\mathbf{b}$-KL) for one 0^m1^n-sequence implies the validity for an arbitrary 0^m1^n-sequence.

The infinite-rank version of the other quantum symmetric pairs $(\mathbf{U}, \mathbf{U}^\jmath)$ and its $\jmath$-canonical basis theory is used to solve a variant of the BGG category $\mathcal{O}$ of $\mathfrak{osp}(2m+1|2n)$-modules, now of *half*-integer weights; see Chapter 12.

Some further works

One influential and persuasive philosophy in the last two decades, supported by the quiver variety construction of Nakajima and reinforced by the categorification program of Chuang, Rouquier, Khovanov and Lauda, is that "all constructions" are of "type A" locally. A general philosophical message of this dissertation is that there exists a whole range of new yet classical $\imath$-constructions, algebraic, geometric and categorical, which are of "type A with involution". This dissertation (and [BW13]) will serve as a new starting point in several (closely related) directions.

While we have developed adequately a theory for $\imath$-canonical basis for quantum symmetric pairs to solve the irreducible character problem in the category $\mathcal{O}_{\mathbf{b}}$, a full fledged theory of canonical bases for quantum symmetric pairs remains to be developed. The quantum symmetric pairs $(\mathbf{U}, \mathbf{U}^\imath)$ and $(\mathbf{U}, \mathbf{U}^\jmath)$ are just two examples of general quantum symmetric pairs in the Kac-Moody setting (see [Ko]). The existence of bar involutions on general quantum symmetric pairs was mentioned explicitly in [BW13], and a detailed proof has been given in [BK]. The most significant quantum symmetric pairs beyond $\mathbf{U}^\imath$ and $\mathbf{U}^\jmath$ in our view would be the ones associated to the quantum group of affine type A.

In their influential work [BLM], Beilinson, Lusztig and MacPherson gave a geometric realization of the modified quantum group associated to $\mathfrak{gl}_n$ using partial flag varieties of type A. A geometric realization of the Jimio-Schur duality has been provided in [GL]. It is natural to ask for a geometric interpretation of the modified coideal subalgebras $\dot{\mathbf{U}}^\imath$ and $\dot{\mathbf{U}}^\jmath$, the type B duality as well as $\imath$-canonical bases developed algebraically and categorically in this paper. This turns out to have a classical answer which is provided in [BKLW] and [LW]. A geometric Schur duality of type D has been provided in the subsequent paper [FL].

The constructions of this dissertation will be adapted to deal with the BGG category $\mathcal{O}$ for Lie superalgebras $\mathfrak{osp}(2m|2n)$ in the future following the blue print of this dissertation.

In [KLa, Ro], Khovanov, Lauda and independently Rouquier introduced the KLR algebras, whose module categories categorify halves of the quantum groups. Lusztig's canonical basis for simply laced types was matched with indecomposable projective modules in those categories in [Ro, VV2]. Khovanov and Lauda ([KLa]) categorified the modified algebra $\dot{\mathbf{U}}$, which admits a geometric 2-representation on the "flag category", in terms of partial flag varieties of type A. We expect to categorify the modified coideal subalgebra $\dot{\mathbf{U}}^\jmath$ (as well as $\dot{\mathbf{U}}^\imath$) based on the geometric framework of [BKLW].

Organization

The dissertation is divided into two parts. Part 1, which consists of Chapters 1-6, provides various foundational constructions on quantum symmetric pairs, where $\dim \mathbb{V}$ is assumed to be finite. Part 2, which consists of Chapters 7-12, extends the $\imath$-canonical basis and dual $\imath$-canonical basis to the setting where $\mathbb{V}$ is infinite-dimensional and uses this to solve the irreducible and tilting character problems of category $\mathcal{O}$ for Lie superalgebra $\mathfrak{osp}(2m+1|2n)$.

In the preliminary Chapter 1, we review various basic constructions for quantum group $\mathbf{U}$. We also introduce the involution θ on the root system and integral weight lattice of $\mathbf{U}$ and a "weight lattice" Λ_θ which will be used in quantum symmetric pairs.

In Chapter 2, we introduce the right coideal subalgebra $\mathbf{U}^\imath$ of $\mathbf{U}$ and an algebra embedding $\imath : \mathbf{U}^\imath \rightarrow \mathbf{U}$. The algebra $\mathbf{U}^\imath$ is endowed with a natural bar involution. Then we construct an intertwiner $\Upsilon = \sum_\mu \Upsilon_\mu$, which lies in a completion $\widehat{\mathbf{U}}^-$, for the two bar involutions on $\mathbf{U}^\imath$ and $\mathbf{U}$ under $\imath$, and show it is unique once we fix the normalization $\Upsilon_0 = 1$. We prove that $\Upsilon\overline{\Upsilon} = 1$. The intertwiner Υ is used to construct a $\mathbf{U}^\imath$-module isomorphism $\mathcal{T}$ on any finite-dimensional $\mathbf{U}$-module, which should be viewed as an analogue of $\mathcal{R}$-matrix on the tensor product of $\mathbf{U}$-modules.

In Chapter 3, we define a quasi-$\mathcal{R}$-matrix $\Theta^\imath$ for $\mathbf{U}^\imath$, which will play an analogous role as Lusztig's quasi-$\mathcal{R}$-matrix for $\mathbf{U}$. Our first definition of $\Theta^\imath$ is simply obtained by combining the intertwiner Υ and Θ. More detailed analysis is required to show that (a normalized version of) $\Theta^\imath$ lies in a completion of $\mathbf{U}^\imath \otimes \mathbf{U}^-$. We prove that $\Theta^\imath\overline{\Theta^\imath} = 1$. Then we use Υ to construct an $\imath$-involutive module structure on an involutive $\mathbf{U}$-module, and then use $\Theta^\imath$ to construct an involution on a tensor product of a $\mathbf{U}^\imath$-module with a $\mathbf{U}$-module.

In Chapter 4, we first study the rank one case of $\mathbf{U}$ and $\mathbf{U}^\imath$ in detail, which turns out to be nontrivial. In the rank one setting, we easily show that Υ is integral and then construct the $\imath$-canonical bases for simple $\mathbf{U}$-modules ${}^\omega L(s)$ for $s \geq 0$. We formulate a $\mathbf{U}^\imath$-homomorphism from ${}^\omega L(s+2)$ to ${}^\omega L(s)$ and use it to study the relation of $\imath$-canonical bases on ${}^\omega L(s+2)$ and ${}^\omega L(s)$, which surprisingly depends on the parity of s. This allows us to establish the $\imath$-canonical basis for $\mathbf{U}^\imath$ in two parities, which is shown to afford integrality and should be regarded as "divided powers" of the generator t.

Then we apply the rank one results to study the general higher rank case. We show that the intertwiner Υ is integral and hence the bar involution $\psi_\imath$ on the simple $\mathbf{U}$-module ${}^\omega L(\lambda)$ preserves its $\mathcal{A}$-form. Then we construct the $\imath$-canonical basis for ${}^\omega L(\lambda)$ for $\lambda \in \Lambda^+$.

In Chapter 5, we recall Schur-Jimbo duality between quantum group $\mathbf{U}$ and Hecke algebra of type A. Then we establish a commuting action of $\mathbf{U}^\imath$ and Hecke algebra $\mathcal{H}_{B_m}$ of type B on $\mathbb{V}^{\otimes m}$, and show that they form double centralizers. Just as Jimbo showed that the generators of Hecke algebra of type A are realized by $\mathcal{R}$-matrices, we show that the extra generator of Hecke algebra of type B is realized via the $\mathbf{U}^\imath$-homomorphism $\mathcal{T}$ introduced in Chapter 2. We then show the existence of a bar involution on $\mathbb{V}^{\otimes m}$ which is compatible with the bar involutions on $\mathbf{U}^\imath$ and $\mathcal{H}_{B_m}$. This allows a reformulation of Kazhdan-Lusztig theory for Lie algebras of type B/C via the involutive $\mathbf{U}^\imath$-module $\mathbb{V}^{\otimes m}$ (without referring directly to the Hecke algebra).

In Chapter 6, we consider the other quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\jmath)$ with $\mathbf{U}$ of type A_{2r}, so its natural representation $\mathbb{V}$ is odd-dimensional. We formulate the counterparts of the main results from Chapter 2 through Chapter 5 where $\mathbf{U}$ was of type A_{2r+1} and $\dim \mathbb{V}$ was even. The proofs are similar and often simpler for $\mathbf{U}^\jmath$ since it does not contain a generator t as $\mathbf{U}^\imath$ does, and hence will be omitted almost entirely.

In Part 2, which consists of Chapters 7-12, we switch to infinite-dimensional $\mathbb{V}$ and infinite-rank quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$.

In the preliminary Chapter 7, we set up variants of BGG categories of the ortho-symplectic Lie superalgebras, allowing possibly infinite-rank and/or parabolic versions.

In Chapter 8, we formulate precisely the infinite-rank limit of various constructions in Part 1, such as $\mathbb{V}$, $\mathbf{U}$, $\mathbf{U}^\imath$, Υ, $\psi_\imath$, and so on. We transport the Bruhat ordering from the BGG category $\mathcal{O}_{\mathbf{b}}$ for $\mathfrak{osp}(2m+1|2n)$ to the Fock space $\mathbb{T}^{\mathbf{b}}$ by noting a canonical bijection of the indexing sets. We formulate the q-wedge versions of the Fock spaces, which correspond to parabolic versions of the BGG categories.

In Chapter 9, we construct the $\imath$-canonical bases and dual $\imath$-canonical bases in various completed Fock spaces, where the earlier detailed work on completion of Fock spaces in [CLW2] plays a fundamental role.

In Chapter 10, we are able to compare (dual) $\imath$-canonical bases in three different settings: a tensor space versus its (partially) wedge subspace, a Fock space versus an adjacent one, and a Fock space with a tensoring factor $\wedge^\infty \mathbb{V}$ versus another with $\wedge^\infty \mathbb{W}$.

In Chapter 11, we show that the coideal subalgebra $\mathbf{U}^\imath$ at $q = 1$ is realized by translation functors in the BGG categories. This underlies the importance of the coideal subalgebra $\mathbf{U}^\imath$. Then we put all the results in earlier chapters of Part 2 together to prove the main theorem which solves the irreducible and tilting character problem for $\mathfrak{osp}(2m+1|2n)$-modules of integer weights.

The last Chapter 12 deals with a variant of the BGG category of $\mathfrak{osp}(2m+1|2n)$-modules with half-integer weights. The Kazhdan-Lusztig theory of this half-integer variant is formulated and solved by the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\jmath)$, an infinite-rank version of the ones formulated in the last chapter of Part 1.

Convention and notation. We shall denote by $\mathbb{N}$ the set of nonnegative integers, and by $\mathbb{Z}_{>0}$ the set of positive integers. In Part 1, where $\dim \mathbb{V} = 2r+2$ (except in Chapter 6 where $\dim \mathbb{V} = 2r+1$), r is fixed and so will not show up in most of the notations (such as $\mathbb{V}$, $\mathbf{U}, \mathbf{U}^\imath$, Υ, $\psi_\imath$ and so on). In Part 2 (more precisely in Chapter 8-9), subscripts and superscripts are added to the notation used in Part 1 to indicate the dependence on r (e.g., $\mathbb{V}_r$, $\mathbf{U}_{2r+1}$, $\mathbf{U}^\imath_r$, $\Upsilon^{(r)}$, $\psi_\imath^{(r)}$ and so on). In this way we shall consider $\mathbb{V}$ as a direct limit $\varinjlim \mathbb{V}_r$, and various constructions including the intertwiner Υ as well as the bar involution $\psi_\imath$ arise as limits of their counterparts in Part 1.

Part I

Canonical bases arising from quantum symmetric pairs

Chapter 1

Preliminaries on quantum groups

In this preliminary chapter, we review some basic definitions and constructions on quantum groups from Lusztig's book, including the braid group action, canonical basis and quasi-$\mathcal{R}$-matrix. We also introduce the involution θ and the lattice Λ_θ which will be used in quantum symmetric pairs.

1.1 The involution θ and the lattice Λ_θ

Let q be an indeterminate. For $r \in \mathbb{N}$, we define the following index sets:

$$\begin{aligned} \mathbb{I}_{2r+1} &= \{i \in \mathbb{Z} \mid -r \le i \le r\}, \\ \mathbb{I}_{2r} &= \Big\{i \in \mathbb{Z} + \frac{1}{2} \mid -r < i < r\Big\}. \end{aligned} \tag{1.1.1}$$

Set $k = 2r+1$ or $2r$, and we use the shorthand notation $\mathbb{I} = \mathbb{I}_k$ in the remainder of Chapter 1. Let

$$\Pi = \Big\{\alpha_i = \varepsilon_{i-\frac{1}{2}} - \varepsilon_{i+\frac{1}{2}} \mid i \in \mathbb{I}\Big\}$$

be the simple system of type A_k, and let Φ be the associated root system. Denote by

$$\Lambda = \sum_{i \in \mathbb{I}} \left(\mathbb{Z}\varepsilon_{i-\frac{1}{2}} + \mathbb{Z}\varepsilon_{i+\frac{1}{2}} \right)$$

the integral weight lattice, and denote by $(\cdot, \cdot)$ the standard bilinear pairing on Λ such that $(\varepsilon_a, \varepsilon_b) = \delta_{ab}$ for all a, b. For any $\mu = \sum_i c_i \alpha_i \in \mathbb{N}\Pi$, set $\mathrm{ht}(\mu) = \sum_i c_i$.

Let θ be the involution of the weight lattice Λ such that

$$\theta(\varepsilon_{i-\frac{1}{2}}) = -\varepsilon_{-i+\frac{1}{2}}, \qquad \text{for all } i \in \mathbb{I}.$$

We shall also write $\lambda^\theta = \theta(\lambda)$, for $\lambda \in \Lambda$. The involution θ preserves the bilinear form $(\cdot, \cdot)$ on the weight lattice Λ and induces an automorphism on the root system Φ such that $\alpha_i^\theta = \alpha_{-i}$ for all $i \in \mathbb{I}$.

Denote by $\Lambda^\theta = \{\mu \in \Lambda \mid \mu^\theta = \mu\}$ the subgroup of θ-fixed points in Λ. It is easy to see that the quotient group

$$\Lambda_\theta := \Lambda / \Lambda^\theta \tag{1.1.2}$$

is a lattice. For $\mu \in \Lambda$, denote by $\overline{\mu}$ the image of μ under the quotient map. There is a well-defined bilinear pairing $\mathbb{Z}[\alpha_i - \alpha_{-i}]_{i \in \mathbb{I}} \times \Lambda_\theta \to \mathbb{Z}$, such that $(\sum_{i>0} a_i(\alpha_i - \alpha_{-i}), \overline{\mu}) := \sum_{i>0} a_i(\alpha_i - \alpha_{-i}, \mu)$ for any $\overline{\mu} \in \Lambda_\theta$ with any preimage $\mu \in \Lambda$.

1.2 The algebras $'\mathbf{f}$, $\mathbf{f}$ and $\mathbf{U}$

Consider a free $\mathbb{Q}(q)$-algebra $'\mathbf{f}$ generated by F_{α_i} for $i \in \mathbb{I}$ associated with the Cartan datum of type $(\mathbb{I}, (\cdot, \cdot))$ [Lu2]. As a $\mathbb{Q}(q)$-vector space, $'\mathbf{f}$ has a direct sum decomposition as

$$'\mathbf{f} = \bigoplus_{\mu \in \mathbb{N}\Pi} {'\mathbf{f}_\mu},$$

where F_{α_i} has weight α_i for all $i \in \mathbb{I}$. For any $x \in {'\mathbf{f}_\mu}$, we set $|x| = \mu$.

For each $i \in \mathbb{I}$, we define $r_i, {}_ir$ to be the unique $\mathbb{Q}(q)$-linear maps on $'\mathbf{f}$ such that

$$\begin{aligned} &r_i(1) = 0, \quad r_i(F_{\alpha_j}) = \delta_{ij}, \quad r_i(xx') = xr_i(x') + q^{(\alpha_i,\mu')} r_i(x)x', \\ &{}_ir(1) = 0, \quad {}_ir(F_{\alpha_j}) = \delta_{ij}, \quad {}_ir(xx') = q^{(\alpha_i,\mu)} x\, {}_ir(x') + {}_ir(x)x', \end{aligned} \tag{1.2.1}$$

for all $x \in {'\mathbf{f}_\mu}$ and $x' \in {'\mathbf{f}_{\mu'}}$. The following lemma is well known (see [Lu2] and [Jan, Section 10.1]).

Lemma 1.2.1. *The $\mathbb{Q}(q)$-linear map r_j and ${}_ir$ commute; that is, $r_j\, {}_ir = {}_ir\, r_j$ for all $i, j \in \mathbb{I}$.*

Proposition 1.2.2. *[Lu2] There is a unique symmetric bilinear form $(\cdot, \cdot)$ on $'\mathbf{f}$ which satisfies that, for all $x, x' \in {'\mathbf{f}}$,*

1. $(F_{\alpha_i}, F_{\alpha_j}) = \delta_{ij}(1 - q^{-2})^{-1}$,
2. $(F_{\alpha_i}x, x') = (F_{\alpha_i}, F_{\alpha_i})(x, {}_ir(x'))$,
3. $(xF_{\alpha_i}, x') = (F_{\alpha_i}, F_{\alpha_i})(x, r_i(x'))$.

Remark 1.2.3. Our version of bilinear form differs by some scalars from Lusztig's bilinear form, and coincides with the one used in [Jan].

Let $\mathbf{I}$ be the radical of the bilinear form $(\cdot,\cdot)$ on $'\mathbf{f}$. It is known in [Lu2] that $\mathbf{I}$ is generated by the quantum Serre relators S_{ij}, for $i \neq j \in \mathbb{I}$, where

$$S_{ij} = \begin{cases} F_{\alpha_i}^2 F_{\alpha_j} + F_{\alpha_j} F_{\alpha_i}^2 - (q+q^{-1}) F_{\alpha_i} F_{\alpha_j} F_{\alpha_i}, & \text{if } |i-j| = 1; \\ F_{\alpha_i} F_{\alpha_j} - F_{\alpha_j} F_{\alpha_i}, & \text{if } |i-j| > 1. \end{cases} \tag{1.2.2}$$

Let $\mathbf{f} = {}'\mathbf{f}/\mathbf{I}$. By [Lu2], we have

$$r_\ell(S_{ij}) = {}_\ell r(S_{ij}) = 0, \qquad \forall \ell, i, j \in \mathbb{I}\ (i \neq j). \tag{1.2.3}$$

Hence r_ℓ and ${}_\ell r$ descend to well-defined $\mathbb{Q}(q)$-linear maps on $\mathbf{f}$.

We introduce the divided power $F_{\alpha_i}^{(a)} = F_{\alpha_i}^a/[a]!$, where $a \geq 0$, $[a] = (q^a - q^{-a})/(q - q^{-1})$ and $[a]! = [1][2]\cdots[a]$. Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Let $\mathbf{f}_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of $\mathbf{f}$ generated by $F_{\alpha_i}^{(a)}$ for various $a \geq 0$ and $i \in \mathbb{I}$.

The quantum group $\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}(k+1))$ is defined to be the associative $\mathbb{Q}(q)$-algebra generated by E_{α_i}, F_{α_i}, K_{α_i}, $K_{\alpha_i}^{-1}$, $i \in \mathbb{I}$, subject to the following relations for

$i,\, j \in \mathbb{I}$:

$$
\begin{aligned}
K_{\alpha_i} K_{\alpha_i}^{-1} &= K_{\alpha_i}^{-1} K_{\alpha_i} = 1, \\
K_{\alpha_i} K_{\alpha_j} &= K_{\alpha_j} K_{\alpha_i}, \\
K_{\alpha_i} E_{\alpha_j} K_{\alpha_i}^{-1} &= q^{(\alpha_i,\alpha_j)} E_{\alpha_j}, \\
K_{\alpha_i} F_{\alpha_j} K_{\alpha_i}^{-1} &= q^{-(\alpha_i,\alpha_j)} F_{\alpha_j}, \\
E_{\alpha_i} F_{\alpha_j} - F_{\alpha_j} E_{\alpha_i} &= \delta_{i,j} \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q - q^{-1}}, \\
E_{\alpha_i}^2 E_{\alpha_j} + E_{\alpha_j} E_{\alpha_i}^2 &= (q + q^{-1}) E_{\alpha_i} E_{\alpha_j} E_{\alpha_i}, && \text{if } |i - j| = 1, \\
E_{\alpha_i} E_{\alpha_j} &= E_{\alpha_j} E_{\alpha_i}, && \text{if } |i - j| > 1, \\
F_{\alpha_i}^2 F_{\alpha_j} + F_{\alpha_j} F_{\alpha_i}^2 &= (q + q^{-1}) F_{\alpha_i} F_{\alpha_j} F_{\alpha_i}, && \text{if } |i - j| = 1, \\
F_{\alpha_i} F_{\alpha_j} &= F_{\alpha_j} F_{\alpha_i}, && \text{if } |i - j| > 1.
\end{aligned}
$$

Let $\mathbf{U}^+$, $\mathbf{U}^0$ and $\mathbf{U}^-$ be the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}$ generated by E_{α_i}, $K_{\alpha_i}^{\pm 1}$, and F_{α_i} respectively, for $i \in \mathbb{I}$. Following [Lu2], we can identify $\mathbf{f} \cong \mathbf{U}^-$ by matching the generators in the same notation. This identification induces a bilinear form $(\cdot, \cdot)$ on $\mathbf{U}^-$ and $\mathbb{Q}(q)$-linear maps $r_i, {}_ir$ $(i \in \mathbb{I})$ on $\mathbf{U}^-$. Under this identification, we let $\mathbf{U}^-_{-\mu}$ be the image of $\mathbf{f}_\mu$, and let $\mathbf{U}^-_{\mathcal{A}}$ be the image of $\mathbf{f}_{\mathcal{A}}$. The following Serre relation holds in $\mathbf{U}^-$:

$$
S_{ij} = 0, \qquad \forall i, j \in \mathbb{I}\ (i \neq j). \tag{1.2.4}
$$

Similarly we have $\mathbf{f} \cong \mathbf{U}^+$ by identifying each generator F_{α_i} with E_{α_i}. Similarly we let $\mathbf{U}^+_{\mathcal{A}}$ denote the image of $\mathbf{f}_{\mathcal{A}}$ under this isomorphism, which is generated by all divided powers $E_{\alpha_i}^{(a)} = E_{\alpha_i}^a / [a]!$.

Proposition 1.2.4. *1. There is an involution ω on the $\mathbb{Q}(q)$-algebra $\mathbf{U}$ such that $\omega(E_{\alpha_i}) = F_{\alpha_i}$, $\omega(F_{\alpha_i}) = E_{\alpha_i}$, and $\omega(K_{\alpha_i}) = K_{\alpha_i}^{-1}$ for all $i \in \mathbb{I}$.*

2. There is an anti-linear ($q \mapsto q^{-1}$) bar involution of the $\mathbb{Q}$-algebra $\mathbf{U}$ such that $\overline{E}_{\alpha_i} = E_{\alpha_i}$, $\overline{F}_{\alpha_i} = F_{\alpha_i}$, and $\overline{K}_{\alpha_i} = K_{\alpha_i}^{-1}$ for all $i \in \mathbb{I}$.

(Sometimes we denote the bar involution on $\mathbf{U}$ by ψ.)

Recall that $\mathbf{U}$ is a Hopf algebra with a coproduct

$$\begin{aligned}\Delta :&\mathbf{U} \longrightarrow \mathbf{U} \otimes \mathbf{U}, \\ \Delta(E_{\alpha_i}) &= 1 \otimes E_{\alpha_i} + E_{\alpha_i} \otimes K_{\alpha_i}^{-1}, \\ \Delta(F_{\alpha_i}) &= F_{\alpha_i} \otimes 1 + K_{\alpha_i} \otimes F_{\alpha_i}, \\ \Delta(K_{\alpha_i}) &= K_{\alpha_i} \otimes K_{\alpha_i}.\end{aligned} \tag{1.2.5}$$

There is a unique $\mathbb{Q}(q)$-algebra homomorphism $\epsilon : \mathbf{U} \to \mathbb{Q}(q)$, called counit, such that $\epsilon(E_{\alpha_i}) = 0$, $\epsilon(F_{\alpha_i}) = 0$, and $\epsilon(K_{\alpha_i}) = 1$.

1.3 Braid group action and canonical basis

Let $W := W_{A_k} = \mathfrak{S}_{k+1}$ be the Weyl group of type A_k. Recall [Lu2] for each α_i and each finite-dimensional $\mathbf{U}$-module M, a linear operator T_{α_i} on M is defined by, for $\lambda \in \Lambda$ and $m \in M_\lambda$,

$$T_{\alpha_i}(m) = \sum_{a,b,c \geq 0; -a+b-c=(\lambda,\alpha_i)} (-1)^b q^{b-ac} E_{\alpha_i}^{(a)} F_{\alpha_i}^{(b)} E_{\alpha_i}^{(c)} m.$$

These T_{α_i}'s induce automorphisms of $\mathbf{U}$, denoted by T_{α_i} as well, such that

$$T_{\alpha_i}(um) = T_{\alpha_i}(u)T_{\alpha_i}(m), \qquad \text{for all } u \in \mathbf{U}, m \in M.$$

As automorphisms on $\mathbf{U}$ and as $\mathbb{Q}(q)$-linear isomorphisms on M, the T_{α_i}'s satisfy the braid group relation ([Lu2, Theorem 39.4.3]):

$$T_{\alpha_i}T_{\alpha_j} = T_{\alpha_j}T_{\alpha_i}, \qquad \text{if } |i-j| > 1,$$

$$T_{\alpha_i}T_{\alpha_j}T_{\alpha_i} = T_{\alpha_j}T_{\alpha_i}T_{\alpha_j}, \qquad \text{if } |i-j| = 1,$$

Hence for each $w \in W$, T_w can be defined independent of the choices of reduced expressions of w. (The T_{α_i} here is consistent with T_{α_i} in [Jan], and it is $T''_{i,+}$ in [Lu2]).

Denote by $\ell(\cdot)$ the length function of W, and let w_0 be the longest element of W.

Lemma 1.3.1. *The following identities hold:*

$$T_{w_0}(K_{\alpha_i}) = K_{\alpha_{-i}}^{-1}, \quad T_{w_0}(E_{\alpha_i}) = -F_{\alpha_{-i}}K_{\alpha_{-i}}, \quad T_{w_0}(F_{\alpha_{-i}}) = -K_{\alpha_i}^{-1}E_{\alpha_i}, \quad \text{for } i \in \mathbb{I}.$$

Proof. The identity $T_{w_0}(K_{\alpha_i}) = K_{\alpha_{-i}}^{-1}$ is clear (see [Lu2] or [Jan]).

Let us show that $T_{w_0}(E_{\alpha_i}) = -F_{\alpha_{-i}}K_{\alpha_{-i}}$, for any given $i \in \mathbb{I}$. Indeed, we can always write $w_0 = ws_i$ with $\ell(w) = \ell(w_0) - 1$. Then we have $T_{w_0} = T_wT_{s_i}$, and

$$T_{w_0}(E_{\alpha_i}) = T_w(T_{s_i}(E_{\alpha_i})) = T_w(-F_{\alpha_i}K_{\alpha_i}) = -T_w(F_{\alpha_i})K_{\alpha_{-i}} = -F_{\alpha_{-i}}K_{\alpha_{-i}},$$

where the last identity used $w(-\alpha_i) = w_0(\alpha_i) = -\alpha_{-i}$ and [Jan, Proposition 8.20].

The identity $T_{w_0}(F_{\alpha_{-i}}) = -K_{\alpha_i}^{-1}E_{\alpha_i}$ can be similarly proved. □

Let

$$\Lambda^+ = \{\lambda \in \Lambda \mid 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i) \in \mathbb{N}, \forall i \in \mathbb{I}\}$$

be the set of dominant weights. Note that $\mu \in \Lambda^+$ if and only if $\mu^\theta \in \Lambda^+$, since the bilinear pairing $(\cdot, \cdot)$ on Λ is invariant under $\theta : \Lambda \to \Lambda$.

Let $M(\lambda)$ be the Verma module of $\mathbf{U}$ with highest weight $\lambda \in \Lambda$ and with a highest weight vector denoted by η or η_λ. We define a $\mathbf{U}$-module ${}^\omega M(\lambda)$, which has the same underlying vector space as $M(\lambda)$ but with the action twisted by the involution ω given in Proposition 1.2.4. When considering η as a vector in ${}^\omega M(\lambda)$, we shall denote it by ξ or $\xi_{-\lambda}$. The Verma module $M(\lambda)$ associated to dominant $\lambda \in \Lambda^+$ has a unique finite-dimensional simple quotient $\mathbf{U}$-module, denoted by $L(\lambda)$. Similarly we define the $\mathbf{U}$-module ${}^\omega L(\lambda)$. For $\lambda \in \Lambda^+$, we let $L_\mathcal{A}(\lambda) = \mathbf{U}^-_\mathcal{A}\eta$ and ${}^\omega L_\mathcal{A}(\lambda) = \mathbf{U}^+_\mathcal{A}\xi$ be the $\mathcal{A}$-submodules of $L(\lambda)$ and ${}^\omega L(\lambda)$, respectively.

In [Lu1, Lu2] and [Ka], the canonical basis $\mathbf{B}$ of $\mathbf{f}_\mathcal{A}$ is constructed. Recall that we can identify $\mathbf{f}$ with both $\mathbf{U}^-$ and $\mathbf{U}^+$. For any element $b \in \mathbf{B}$, when considered as an element in $\mathbf{U}^-$ or $\mathbf{U}^+$, we shall denote it by b^- or b^+, respectively. In [Lu2], subsets $\mathbf{B}(\lambda)$ of $\mathbf{B}$ is also constructed for each $\lambda \in \Lambda^+$, such that $\{b^-\eta_\lambda \mid b \in \mathbf{B}(\lambda)\}$ gives the canonical basis of $L_\mathcal{A}(\lambda)$. Similarly $\{b^+\xi_{-\lambda} \mid b \in \mathbf{B}(\lambda)\}$ gives the canonical basis of ${}^\omega L(\lambda)$. By [Lu2, Proposition 21.1.2], we can identify ${}^\omega L(\lambda)$ with $L(\lambda^\theta) = L(-w_0\lambda)$ such that the set $\{b^+\xi_{-\lambda} \mid b \in \mathbf{B}(\lambda)\}$ is identified with the set $\{b^-\eta_{\lambda^\theta} \mid b \in \mathbf{B}(\lambda^\theta)\} = \{b^-\eta_{-w_0\lambda} \mid b \in \mathbf{B}(-w_0\lambda)\}$. We shall identify ${}^\omega L(\lambda)$ with $L(\lambda^\theta)$ in this way throughout this paper.

1.4 Quasi-$\mathcal{R}$-matrix Θ

Proposition 1.4.1. *[Lu2, Theorem 4.1.2] There exists a unique family of elements Θ_μ in $\mathbf{U}^+_\mu \otimes \mathbf{U}^-_{-\mu}$ with $\mu \in \mathbb{N}\Pi$, such that $\Theta_0 = 1 \otimes 1$ and the following identities hold for all μ and all i:*

$$(1 \otimes E_{\alpha_i})\Theta_\mu + (E_{\alpha_i} \otimes K_{\alpha_i}^{-1})\Theta_{\mu-\alpha_i} = \Theta_\mu(1 \otimes E_{\alpha_i}) + \Theta_{\mu-\alpha_i}(E_{\alpha_i} \otimes K_{\alpha_i}),$$

$$(F_{\alpha_i} \otimes 1)\Theta_\mu + (K_{\alpha_i} \otimes F_{\alpha_i})\Theta_{\mu-\alpha_i} = \Theta_\mu(F_{\alpha_i} \otimes 1) + \Theta_{\mu-\alpha_i}(K_{\alpha_i}^{-1} \otimes F_{\alpha_i}),$$

$$(K_{\alpha_i} \otimes K_{\alpha_i})\Theta_\mu = \Theta_\mu(K_{\alpha_i} \otimes K_{\alpha_i}).$$

Remark 1.4.2. We adopt the convention in this paper that Θ_μ lies in $\mathbf{U}^+ \otimes \mathbf{U}^-$ due to our different choice of the coproduct Δ from [Lu2]. (In contrast the Θ_μ in [Lu2] lies in $\mathbf{U}^- \otimes \mathbf{U}^+$.) The convention here is adopted in order to be more compatible with the application to category $\mathcal{O}$ in Part 2.

Lusztig's quasi-$\mathcal{R}$-matrix for $\mathbf{U}$ is defined to be

$$\Theta := \sum_{\mu \in \mathbb{N}\Pi} \Theta_\mu. \tag{1.4.1}$$

For any finite-dimensional $\mathbf{U}$-modules M and M', the action of Θ on $M \otimes M'$ is well defined. Proposition 1.4.1 implies that

$$\Delta(u)\Theta(m \otimes m') = \Theta\overline{\Delta(\overline{u})}(m \otimes m'), \quad \text{for all } m \in M, m' \in M', \text{ and } u \in \mathbf{U}. \tag{1.4.2}$$

By [Lu2, Corollary 4.1.3], we have

$$\Theta\overline{\Theta}(m \otimes m') = m \otimes m', \quad \text{for all } m \in M \text{ and } m' \in M'. \tag{1.4.3}$$

In [Lu2, 32.1.5], a $\mathbf{U}$-module isomorphism

$$\mathcal{R} = \mathcal{R}_{M,M'} : M' \otimes M \longrightarrow M \otimes M'$$

is constructed. As an operator, $\mathcal{R}$ can be written as $\mathcal{R} = \Theta \circ \widetilde{g} \circ P$ where $\widetilde{g} : M \otimes M' \to M \otimes M'$ is the map $\widetilde{g}(m \otimes m') = q^{(\lambda,\mu)} m \otimes m'$ for all $m \in M_\lambda, m' \in M'_\mu$, and $P : M' \otimes M \to M \otimes M'$ is a $\mathbb{Q}(q)$-linear isomorphism such that $P(m \otimes m') = m' \otimes m$.

Definition 1.4.3. A $\mathbf{U}$-module M equipped with an anti-linear involution ψ is called *involutive* if

$$\psi(um) = \psi(u)\psi(m), \qquad \forall u \in \mathbf{U}, m \in M.$$

Given two involutive $\mathbf{U}$-modules (M, ψ_1) and (M_2, ψ_2), following Lusztig we define a map ψ on $M_1 \otimes M_2$ by

$$\psi(m \otimes m') := \Theta(\psi_1(m) \otimes \psi_2(m')). \tag{1.4.4}$$

By Proposition 1.4.1, we have $\psi(u(m \otimes m')) = \psi(u)\psi(m \otimes m')$ for all $u \in \mathbf{U}$, and the identity (1.4.3) implies that the map ψ on $M_1 \otimes M_2$ is an anti-linear involution. This proves the following result of Lusztig (though the terminology of involutive modules is new here).

Proposition 1.4.4. *[Lu2, 27.3.1] Given two involutive* $\mathbf{U}$*-modules* (M, ψ_1) *and* (M_2, ψ_2)*,* $(M_1 \otimes M_2, \psi)$ *is an involutive* $\mathbf{U}$*-module with* ψ *given in* (1.4.4)*.*

It follows by induction that $M_1 \otimes \cdots \otimes M_s$ is naturally an involutive $\mathbf{U}$-module for given involutive $\mathbf{U}$-modules $M_1, \ldots, M_s$; see [Lu2, 27.3.6].

As in [Lu2], there is a unique anti-linear involution ψ on ${}^{\omega}L(\lambda)$ such that $\psi(u\xi) = \psi(u)\xi$ for all $u \in \mathbf{U}$. Similarly there is a unique anti-linear involution ψ on $L(\lambda)$ such that $\psi(u\eta) = \psi(u)\eta$ for all $u \in \mathbf{U}$. Therefore ${}^{\omega}L(\lambda)$ and $L(\lambda)$ are both involutive $\mathbf{U}$-modules.

Chapter 2

Intertwiner for a quantum symmetric pair

In Chapters 2-5, we will formulate and study in depth the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ for $\mathbf{U}$ of type A_k with $k = 2r+1$ being an odd integer. In these chapters, we shall use the shorthand notation

$$\mathbb{I} = \mathbb{I}_{2r+1} = \{-r, \ldots, -1, 0, 1, \ldots, r\}$$

as given in (1.1.1), and set

$$\mathbb{I}^\imath := \mathbb{Z}_{>0} \cap \mathbb{I} = \{1, \ldots, r\}. \tag{2.0.1}$$

In this chapter, we will introduce the right coideal subalgebra $\mathbf{U}^\imath$ of $\mathbf{U}$ and an algebra embedding $\imath : \mathbf{U}^\imath \to \mathbf{U}$. Then we construct an intertwiner Υ for the two bar involutions on $\mathbf{U}^\imath$ and $\mathbf{U}$ under $\imath$, and use it to construct a $\mathbf{U}^\imath$-module isomorphism

$\mathcal{T}$ on any finite-dimensional $\mathbf{U}$-module.

2.1 Definition of the algebra $\mathbf{U}^\imath$

The algebra $\mathbf{U}^\imath = \mathbf{U}^\imath_r$ is defined to be the associative algebra over $\mathbb{Q}(q)$ generated by e_{α_i}, f_{α_i}, k_{α_i}, $k_{\alpha_i}^{-1}$ $(i \in \mathbb{I}^\imath)$, and t, subject to the following relations for i, $j \in \mathbb{I}^\imath$:

$$\begin{aligned}
k_{\alpha_i} k_{\alpha_i}^{-1} &= k_{\alpha_i}^{-1} k_{\alpha_i} = 1, \\
k_{\alpha_i} k_{\alpha_j} &= k_{\alpha_j} k_{\alpha_i}, \\
k_{\alpha_i} e_{\alpha_j} k_{\alpha_i}^{-1} &= q^{(\alpha_i - \alpha_{-i}, \alpha_j)} e_{\alpha_j}, \\
k_{\alpha_i} f_{\alpha_j} k_{\alpha_i}^{-1} &= q^{-(\alpha_i - \alpha_{-i}, \alpha_j)} f_{\alpha_j}, \\
k_{\alpha_i} t k_{\alpha_i}^{-1} &= t, \\
e_{\alpha_i} f_{\alpha_j} - f_{\alpha_j} e_{\alpha_i} &= \delta_{i,j} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q - q^{-1}}, \\
e_{\alpha_i}^2 e_{\alpha_j} + e_{\alpha_j} e_{\alpha_i}^2 &= (q + q^{-1}) e_{\alpha_i} e_{\alpha_j} e_{\alpha_i}, && \text{if } |i - j| = 1, \\
e_{\alpha_i} e_{\alpha_j} &= e_{\alpha_j} e_{\alpha_i}, && \text{if } |i - j| > 1, \\
f_{\alpha_i}^2 f_{\alpha_j} + f_{\alpha_j} f_{\alpha_i}^2 &= (q + q^{-1}) f_{\alpha_i} f_{\alpha_j} f_{\alpha_i}, && \text{if } |i - j| = 1, \\
f_{\alpha_i} f_{\alpha_j} &= f_{\alpha_j} f_{\alpha_i}, && \text{if } |i - j| > 1, \\
e_{\alpha_i} t &= t e_{\alpha_i}, && \text{if } i > 1, \\
e_{\alpha_1}^2 t + t e_{\alpha_1}^2 &= (q + q^{-1}) e_{\alpha_1} t e_{\alpha_1}, \\
t^2 e_{\alpha_1} + e_{\alpha_1} t^2 &= (q + q^{-1}) t e_{\alpha_1} t + e_{\alpha_1}, \\
f_{\alpha_i} t &= t f_{\alpha_i}, && \text{if } i > 1,
\end{aligned}$$

$$f_{\alpha_1}^2 t + t f_{\alpha_1}^2 = (q+q^{-1}) f_{\alpha_1} t f_{\alpha_1},$$

$$t^2 f_{\alpha_1} + f_{\alpha_1} t^2 = (q+q^{-1}) t f_{\alpha_1} t + f_{\alpha_1}.$$

We introduce the divided powers $e_{\alpha_i}^{(a)} = e_i^a/[a]!$, $f_{\alpha_i}^{(a)} = f_i^a/[a]!$ for $a \geq 0$, $i \in \mathbb{I}^\imath$.

Lemma 2.1.1. *1. The $\mathbb{Q}(q)$-algebra $\mathbf{U}^\imath$ has an involution $\omega_\imath$ such that $\omega_\imath(k_{\alpha_i}) = k_{\alpha_i}^{-1}$, $\omega_\imath(e_{\alpha_i}) = f_{\alpha_i}$, $\omega_\imath(f_{\alpha_i}) = e_{\alpha_i}$, and $\omega_\imath(t) = t$ for all $i \in \mathbb{I}^\imath$.*

2. The $\mathbb{Q}(q)$-algebra $\mathbf{U}^\imath$ has an anti-involution $\tau_\imath$ such that $\tau_\imath(e_{\alpha_i}) = e_{\alpha_i}, \tau_\imath(f_{\alpha_i}) = f_{\alpha_i}, \tau_\imath(t) = t$, and $\tau_\imath(k_{\alpha_i}) = k_{\alpha_i}^{-1}$ for all $i \in \mathbb{I}^\imath$.

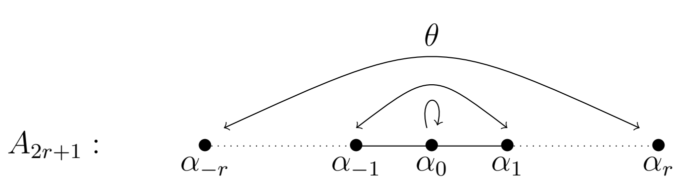
3. The $\mathbb{Q}$-algebra $\mathbf{U}^\imath$ has an anti-linear ($q \mapsto q^{-1}$) bar involution such that $\overline{k}_{\alpha_i} = k_{\alpha_i}^{-1}$, $\overline{e}_{\alpha_i} = e_{\alpha_i}$, $\overline{f}_{\alpha_i} = f_{\alpha_i}$, and $\overline{t} = t$ for all $i \in \mathbb{I}^\imath$.

(Sometimes we denote the bar involution on $\mathbf{U}^\imath$ by $\psi_\imath$.)

Proof. Follows by a direct computation from the definitions. □

2.2 Quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$

The Dynkin diagram of type A_{2r+1} together with the involution θ can be depicted as follows:



A general theory of quantum symmetric pairs via the notion of coideal subalgebras was developed systematically by Letzter [Le] (also see [KP, Ko]). As the properties in Propositions 2.2.1 and 2.2.4 below indicate, the algebra $\mathbf{U}^\imath$ is a (right) coideal subalgebra of $\mathbf{U}$ and that $(\mathbf{U}, \mathbf{U}^\imath)$ forms a quantum symmetric pair.

Proposition 2.2.1. *There is an injective $\mathbb{Q}(q)$-algebra homomorphism $\imath : \mathbf{U}^\imath \to \mathbf{U}$ which sends*

$$
\begin{aligned}
k_{\alpha_i} &\mapsto K_{\alpha_i} K_{\alpha_{-i}}^{-1}, \\
e_{\alpha_i} &\mapsto E_{\alpha_i} + K_{\alpha_i}^{-1} F_{\alpha_{-i}}, \\
f_{\alpha_i} &\mapsto F_{\alpha_i} K_{\alpha_{-i}}^{-1} + E_{\alpha_{-i}}, \\
t &\mapsto E_{\alpha_0} + q F_{\alpha_0} K_{\alpha_0}^{-1} + K_{\alpha_0}^{-1}
\end{aligned}
$$

for all $i \in \mathbb{I}^\imath$.

Proof. This proposition is a variant of a general property for quantum symmetric pairs which can be found in [Le, Theorem 7.1]. Hence we will not repeat the proof, except noting how to covert the result therein to the form used here.

It follows from a direct computation that $\imath$ is a homomorphism of $\mathbb{Q}(q)$-algebras.

We shall compare $\imath$ with the embedding in [KP, Proposition 4.1] (as modified by [KP, Remark 4.2]), which is a version of [Le, Theorem 7.1]. Set $\mathbf{U}_\mathbb{C} = \mathbb{C}(q^{\frac{1}{2}}) \otimes_{\mathbb{Q}(q)} \mathbf{U}$. Recall from [KP, §4] a $\mathbb{Q}(q)$-subalgebra $U_q'(\mathfrak{k})$ of $\mathbf{U}_\mathbb{C}$ with a generating set $\mathfrak{S}$ consisting of $F_{\alpha_0} - K_{\alpha_0}^{-1} E_{\alpha_0} + q^{-\frac{1}{2}} K_{\alpha_0}^{-1}$, $K_{\alpha_i} K_{\alpha_{-i}}^{-1}$, $F_{\alpha_{-i}} - K_{\alpha_{-i}}^{-1} E_{\alpha_i}$, $F_{\alpha_i} - E_{\alpha_{-i}} K_{\alpha_i}^{-1}$, for all $0 \neq i \in \mathbb{I}^\imath$.

Claim. The algebras $\mathbb{C}(q^{\frac{1}{2}}) \otimes_{\mathbb{Q}(q)} \imath(\mathbf{U}^\imath)$ and $\mathbb{C}(q^{\frac{1}{2}}) \otimes_{\mathbb{Q}(q)} U_q'(\mathfrak{k})$ are anti-isomorphic.

Consider the $\mathbb{C}(q^{\frac{1}{2}})$-algebra anti-automorphism $\kappa : \mathbf{U}_{\mathbb{C}} \to \mathbf{U}_{\mathbb{C}}$ such that

$$E_{\alpha_i} \mapsto \sqrt{-1}F_{\alpha_{-i}}, \quad F_{\alpha_i} \mapsto -\sqrt{-1}E_{\alpha_{-i}}, \quad K_{\alpha_i} \mapsto K_{\alpha_{-i}}, \quad \text{for all } 0 \neq i \in \mathbb{I},$$

$$E_{\alpha_0} \mapsto \sqrt{-1}q^{\frac{1}{2}}F_{\alpha_0}, \quad F_{\alpha_0} \mapsto -\sqrt{-1}q^{-\frac{1}{2}}E_{\alpha_0}, \quad K_{\alpha_0} \mapsto K_{\alpha_0}.$$

A direct computation shows that κ sends

$$K_{\alpha_i}K_{\alpha_{-i}}^{-1} \mapsto K_{\alpha_i}K_{\alpha_{-i}}^{-1},$$

$$E_{\alpha_i} + K_{\alpha_i}^{-1}F_{\alpha_{-i}} \mapsto \sqrt{-1}(F_{\alpha_{-i}} - K_{\alpha_{-i}}^{-1}E_{\alpha_i}),$$

$$F_{\alpha_i}K_{\alpha_{-i}}^{-1} + E_{\alpha_{-i}} \mapsto \sqrt{-1}(F_{\alpha_i} - E_{\alpha_{-i}}K_{\alpha_i}^{-1}),$$

$$E_{\alpha_0} + qF_{\alpha_0}K_{\alpha_0}^{-1} + K_{\alpha_0}^{-1} \mapsto \sqrt{-1}q^{\frac{1}{2}}(F_{\alpha_0} - K_{\alpha_0}^{-1}E_{\alpha_0} + q^{-\frac{1}{2}}K_{\alpha_0}^{-1}).$$

Hence, κ restricts to an anti-isomorphism between the algebras $\mathbb{C}(q^{\frac{1}{2}}) \otimes_{\mathbb{Q}(q)} \imath(\mathbf{U}^\imath)$ and $\mathbb{C}(q^{\frac{1}{2}}) \otimes_{\mathbb{Q}(q)} U_q'(\mathfrak{k})$, whence the claim.

We observe that [KP, Proposition 4.1] provides a presentation of the algebra $U_q'(\mathfrak{k})$ with the generating set $\mathfrak{G}$ and a bunch of relations, which correspond under κ exactly to (the images of) the defining relations of $\mathbf{U}^\imath$. In other words, the composition $\mathbb{C}(q^{\frac{1}{2}}) \otimes_{\mathbb{Q}(q)} \mathbf{U}^\imath \xrightarrow{\imath} \mathbb{C}(q^{\frac{1}{2}}) \otimes_{\mathbb{Q}(q)} \imath(\mathbf{U}^\imath) \xrightarrow{\kappa} \mathbb{C}(q^{\frac{1}{2}}) \otimes_{\mathbb{Q}(q)} U_q'(\mathfrak{k})$ is an anti-isomorphism. Hence $\imath : \mathbf{U}^\imath \to \mathbf{U}$ must be an embedding. □

Remark 2.2.2. Note that the coproduct for $\mathbf{U}$ used in [KP] follows Lusztig [Lu2] and hence differs from the one used in this paper; this leads to somewhat different presentations of the quantum symmetric pairs. Our choices are determined by the application we have in mind: the $(\mathbf{U}^\imath, \mathcal{H}_{B_m})$-duality in Chapter 5 and the translation functors for category $\mathcal{O}$ in Part 2. One crucial advantage of our presentation is the existence of a natural bar involution as given in Lemma 2.1.1(3).

Any $\mathbf{U}$-module M can be naturally regarded as a $\mathbf{U}^\imath$-module via the embedding $\imath$.

Remark 2.2.3. The bar involution on $\mathbf{U}^\imath$ and the bar involution on $\mathbf{U}$ are *not* compatible through $\imath$, i.e., $\overline{\imath(u)} \neq \imath(\overline{u})$ for $u \in \mathbf{U}^\imath$ in general. For example,

$$\imath(\overline{e}_{\alpha_i}) = \imath(e_{\alpha_i}) = E_{\alpha_i} + K_{\alpha_i}^{-1} F_{\alpha_{-i}},$$

$$\overline{\imath(e_{\alpha_i})} = \overline{E_{\alpha_i} + F_{\alpha_{-i}} K_{\alpha_i}^{-1}} = E_{\alpha_i} + F_{\alpha_{-i}} K_{\alpha_i}.$$

Note that $E_{\alpha_i}(K_{\alpha_i}^{-1} F_{\alpha_{-i}}) = q^2 (K_{\alpha_i}^{-1} F_{\alpha_{-i}}) E_{\alpha_i}$ for all $0 \neq i \in \mathbb{I}$. Using the quantum binomial formula [Lu2, 1.3.5], we have, for all $i \in \mathbb{I}^\imath$, $a \in \mathbb{N}$,

$$\imath(e_{\alpha_i}^{(a)}) = \sum_{j=0}^{a} q^{j(a-j)} F_{\alpha_{-i}}^{(j)} K_{\alpha_i}^{-j} E_{\alpha_i}^{(a-j)}, \tag{2.2.1}$$

$$\imath(f_{\alpha_i}^{(a)}) = \sum_{j=0}^{a} q^{j(a-j)} F_{\alpha_i}^{(j)} K_{\alpha_{-i}}^{-j} E_{\alpha_{-i}}^{(a-j)}. \tag{2.2.2}$$

Proposition 2.2.4. *The coproduct $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ restricts via the embedding $\imath$ to a $\mathbb{Q}(q)$-algebra homomorphism $\Delta : \mathbf{U}^\imath \mapsto \mathbf{U}^\imath \otimes \mathbf{U}$ such that, for all $i \in \mathbb{I}^\imath$,*

$$\Delta(k_{\alpha_i}) = k_{\alpha_i} \otimes K_{\alpha_i} K_{\alpha_{-i}}^{-1},$$

$$\Delta(e_{\alpha_i}) = 1 \otimes E_{\alpha_i} + e_{\alpha_i} \otimes K_{\alpha_i}^{-1} + k_{\alpha_i}^{-1} \otimes K_{\alpha_i}^{-1} F_{\alpha_{-i}},$$

$$\Delta(f_{\alpha_i}) = k_{\alpha_i} \otimes F_{\alpha_i} K_{\alpha_{-i}}^{-1} + f_{\alpha_i} \otimes K_{\alpha_{-i}}^{-1} + 1 \otimes E_{\alpha_{-i}},$$

$$\Delta(t) = t \otimes K_{\alpha_0}^{-1} + 1 \otimes q F_{\alpha_0} K_{\alpha_0}^{-1} + 1 \otimes E_{\alpha_0}.$$

Similarly, the counit ϵ *of* $\mathbf{U}$ *induces a* $\mathbb{Q}(q)$*-algebra homomorphism* $\epsilon : \mathbf{U}^\imath \to \mathbb{Q}(q)$ *such that* $\epsilon(e_{\alpha_i}) = \epsilon(f_{\alpha_i}) = 0$*,* $\epsilon(t) = 1$*, and* $\epsilon(k_{\alpha_i}) = 1$ *for all* $i \in \mathbb{I}^\imath$*.*

Proof. This follows from a direct computation. □

Remark 2.2.5. Propositions 2.2.1 and 2.2.4 imply that $\mathbf{U}^\imath$ (or rather $\imath(\mathbf{U}^\imath)$) is a *(right) coideal subalgebra* of $\mathbf{U}$ in the sense of [Le]. There exists a $\mathbb{Q}(q)$-algebra embedding $\imath_L : \mathbf{U}^\imath \to \mathbf{U}$ which makes $\mathbf{U}^\imath$ (or rather $\imath_L(\mathbf{U}^\imath)$) a *left coideal subalgebra* of $\mathbf{U}$; that is, the coproduct $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ restricts via $\imath_L$ to a $\mathbb{Q}(q)$-algebra homomorphism $\Delta : \mathbf{U}^\imath \to \mathbf{U} \otimes \mathbf{U}^\imath$. We will not use the left variant in this paper.

Remark 2.2.6. The pair $(\mathbf{U}, \mathbf{U}^\imath)$ forms a quantum symmetric pair in the sense of [Le]. At the limit $q \mapsto 1$, it reduces to a classical symmetric pair $(\mathfrak{sl}(2r+2), \mathfrak{sl}(2r+2)^{w_0})$; here w_0 is the involution on $\mathfrak{gl}(2r+2)$ which sends $E_{i,j}$ to $E_{-i,-j}$ and its restriction to $\mathfrak{sl}(2r+2)$ if we label the rows and columns of $\mathfrak{sl}(2r+2)$ by $\{-r-1/2, \dots, -1/2, 1/2, \dots, r+1/2\}$.

The following corollary follows immediately from the Hopf algebra structure of $\mathbf{U}$.

Corollary 2.2.7. *Let* $m : \mathbf{U} \otimes \mathbf{U} \to \mathbf{U}$ *denote the multiplication map. Then we have*

$$m(\epsilon \otimes 1)\Delta = \imath : \mathbf{U}^\imath \longrightarrow \mathbf{U}.$$

The map $\Delta : \mathbf{U}^\imath \mapsto \mathbf{U}^\imath \otimes \mathbf{U}$ is clearly coassociative, i.e., we have $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta : \mathbf{U}^\imath \longrightarrow \mathbf{U}^\imath \otimes \mathbf{U} \otimes \mathbf{U}$. This Δ will be called the *coproduct* of $\mathbf{U}^\imath$, and $\epsilon : \mathbf{U}^\imath \to \mathbb{Q}(q)$ will be called the *counit* of $\mathbf{U}^\imath$. The counit map ϵ makes $\mathbb{Q}(q)$ a $\mathbf{U}^\imath$-module. We shall call this *the trivial representation* of $\mathbf{U}^\imath$.

Remark 2.2.8. The 1-dimensional space $\mathbb{Q}(q)$ can be realized as $\mathbf{U}^\imath$-modules in different (non-isomorphic) ways. For example, we can consider the $\mathbb{Q}(q)$-algebras homomorphism $\epsilon' : \mathbf{U}^\imath \to \mathbb{Q}(q)$, such that $\epsilon'(e_{\alpha_i}) = \epsilon'(f_{\alpha_i}) = 0$, $\epsilon'(k_{\alpha_i}) = 1$ for all $i \in \mathbb{Z}_{>0}$, and $\epsilon'(t) = x$ for any $x \in \mathbb{Q}(q)$. We shall only consider the one induced by ϵ as the trivial representation of $\mathbf{U}^\imath$, which is compatible with the trivial representation of $\mathbf{U}$ via $\imath$.

2.3 The intertwiner Υ

Let $\widehat{\mathbf{U}}$ be the completion of the $\mathbb{Q}(q)$-vector space $\mathbf{U}$ with respect to the following descending sequence of subspaces $\mathbf{U}^+\mathbf{U}^0\big(\sum_{\mathrm{ht}(\mu)\geq N} \mathbf{U}^-_{-\mu}\big)$, for $N \geq 1$. Then we have the obvious embedding of $\mathbf{U}$ into $\widehat{\mathbf{U}}$. We let $\widehat{\mathbf{U}}^-$ be the closure of $\mathbf{U}^-$ in $\widehat{\mathbf{U}}$, and so $\widehat{\mathbf{U}}^- \subseteq \widehat{\mathbf{U}}$. By continuity the $\mathbb{Q}(q)$-algebra structure on $\mathbf{U}$ extends to a $\mathbb{Q}(q)$-algebra structure on $\widehat{\mathbf{U}}$. The bar involution $^-$ on $\mathbf{U}$ extends by continuity to an anti-linear involution on $\widehat{\mathbf{U}}$, also denoted by $^-$. Recall the bar involutions on $\mathbf{U}^\imath$ and $\mathbf{U}$ are not compatible via the embedding $\imath : \mathbf{U}^\imath \to \mathbf{U}$, by Remark 2.2.3.

Theorem 2.3.1. *There is a unique family of elements $\Upsilon_\mu \in \mathbf{U}^-_{-\mu}$ for $\mu \in \mathbb{N}\Pi$ such that $\Upsilon = \sum_\mu \Upsilon_\mu \in \widehat{\mathbf{U}}^-$ intertwines the bar involutions on $\mathbf{U}^\imath$ and $\mathbf{U}$ via the embedding $\imath$ and $\Upsilon_0 = 1$; that is, Υ satisfies the following identity (in $\widehat{\mathbf{U}}$):*

$$\imath(\overline{u})\Upsilon = \Upsilon\, \overline{\imath(u)}, \qquad \text{for all } u \in \mathbf{U}^\imath. \tag{2.3.1}$$

Moreover, $\Upsilon_\mu = 0$ unless $\mu^\theta = \mu$.

Remark 2.3.2. Define $\bar{\imath} : \mathbf{U}^\imath \to \mathbf{U}$, where $\bar{\imath}(u) := \psi\big(\imath(\psi_\imath(u))\big)$, for $u \in \mathbf{U}^\imath$. Then the identity (2.3.1) can be equivalently reformulated as

$$\imath(u)\Upsilon = \Upsilon \; \bar{\imath}(u), \qquad \text{for all } u \in \mathbf{U}^\imath. \tag{2.3.2}$$

This reformulation makes it more transparent to observe the remarkable analogy with Lusztig's Θ; see (1.4.2).

Sometimes it could be confusing to use $^-$ to denote the two distinct bar involutions on $\mathbf{U}$ and $\mathbf{U}^\imath$. Recall that we set in Section 1.2 that $\psi(u) = \overline{u}$ for all $u \in \mathbf{U}$, and set in Section 2.1 that $\psi_\imath(u) = \overline{u} \in \mathbf{U}^\imath$ for $u \in \mathbf{U}^\imath$. In the ψ-notation the identities (2.3.1) and (2.3.2) read

$$\imath(\psi_\imath(u))\Upsilon = \Upsilon\psi(\imath(u)), \quad \imath(u)\Upsilon = \Upsilon\psi\big(\imath(\psi_\imath(u))\big), \qquad \text{for all } u \in \mathbf{U}^\imath.$$

Definition 2.3.3. The element Υ in Theorem 2.3.1 is called *the intertwiner for the quantum symmetric pair* $(\mathbf{U}, \mathbf{U}^\imath)$.

As we shall see, the intertwiner Υ leads to the construction of what we call quasi-$\mathcal{R}$-matrix for $\mathbf{U}^\imath$, which plays an analogous role as Lusztig's quasi-$\mathcal{R}$-matrix for $\mathbf{U}$. We shall prove later on that $\Upsilon_\mu \in \mathbf{U}_{\mathcal{A}}^-$ for all μ; see Theorem 4.4.2.

The proof of Theorem 2.3.1 will be given in §2.4 below. Here we note immediately a fundamental property of Υ.

Corollary 2.3.4. *We have* $\Upsilon \cdot \overline{\Upsilon} = 1$.

Proof. Clearly Υ is invertible in $\widehat{\mathbf{U}}$. Multiplying Υ^{-1} on both sides of the identity (2.3.1) in Theorem 2.3.1, we have

$$\Upsilon^{-1}\imath(\overline{u}) = \overline{\imath(u)}\Upsilon^{-1}, \qquad \forall u \in \mathbf{U}^{\imath}.$$

Applying $^{-}$ to the above identity and replacing $\overline{u}$ by u, we have

$$\overline{\Upsilon}^{-1}\overline{\imath(u)} = \imath(\overline{u})\overline{\Upsilon}^{-1}, \qquad \forall u \in \mathbf{U}^{\imath}.$$

Hence $\overline{\Upsilon}^{-1}$ (in place of Υ) satisfies the identity (2.3.1) as well. Thanks to the uniqueness of Υ in Theorem 2.3.1, we must have $\overline{\Upsilon}^{-1} = \Upsilon$, whence the corollary. □

2.4 Constructing Υ

The goal here is to construct Υ and establish Theorem 2.3.1.

The set of all $u \in \mathbf{U}^{\imath}$ that satisfy the identity (2.3.1) is clearly a subalgebra of $\mathbf{U}^{\imath}$. Hence it suffices to consider the identity (2.3.1) when u is one of the generators e_{α_i}, f_{α_i}, k_{α_i}, and t in $\mathbf{U}^{\imath}$, that is, the following identities for all $\mu \in \mathbb{N}\Pi$ and $0 \neq i \in \mathbb{I}$:

$$K_{\alpha_i}K_{\alpha_{-i}}^{-1}\Upsilon_{\mu} = \Upsilon_{\mu}K_{\alpha_i}K_{\alpha_{-i}}^{-1},$$

$$F_{\alpha_i}K_{\alpha_{-i}}^{-1}\Upsilon_{\mu-\alpha_i-\alpha_{-i}} + E_{\alpha_{-i}}\Upsilon_{\mu} = \Upsilon_{\mu-\alpha_i-\alpha_{-i}}F_{\alpha_i}K_{\alpha_{-i}} + \Upsilon_{\mu}E_{\alpha_{-i}},$$

$$qF_{\alpha_0}K_{\alpha_0}^{-1}\Upsilon_{\mu-2\alpha_0} + K_{\alpha_0}^{-1}\Upsilon_{\mu-\alpha_0} + E_{\alpha_0}\Upsilon_{\mu} = q^{-1}\Upsilon_{\mu-2\alpha_0}F_{\alpha_0}K_{\alpha_0} + \Upsilon_{\mu-\alpha_0}K_{\alpha_0} + \Upsilon_{\mu}E_{\alpha_0}.$$

Using [Lu2, Proposition 3.1.6], we can rewrite the above identities in terms of ${}_{-i}r$ and r_{-i} as follows:

$$K_{\alpha_i}K_{\alpha_{-i}}^{-1}\Upsilon_{\mu} - \Upsilon_{\mu}K_{\alpha_i}K_{\alpha_{-i}}^{-1} = 0, \tag{2.4.1}$$

$$(q^{-1}-q)q^{(\alpha_{-i},\mu-\alpha_{-i}-\alpha_i)}\Upsilon_{\mu-\alpha_i-\alpha_{-i}}F_{\alpha_i}+\,{}_{-i}r(\Upsilon_\mu)=0, \tag{2.4.2}$$

$$(q^{-1}-q)q^{(\alpha_{-i},\mu-\alpha_{-i}-\alpha_i)}F_{\alpha_i}\Upsilon_{\mu-\alpha_i-\alpha_{-i}}+\,r_{-i}(\Upsilon_\mu)=0, \tag{2.4.3}$$

$$(q^{-1}-q)q^{(\alpha_0,\mu-\alpha_0)}(q^{-1}\Upsilon_{\mu-2\alpha_0}F_{\alpha_0}+\Upsilon_{\mu-\alpha_0})+\,{}_0r(\Upsilon_\mu)=0, \tag{2.4.4}$$

$$(q^{-1}-q)q^{(\alpha_0,\mu-\alpha_0)}(q^{-1}F_{\alpha_0}\Upsilon_{\mu-2\alpha_0}+\Upsilon_{\mu-\alpha_0})+r_0(\Upsilon_\mu)=0. \tag{2.4.5}$$

Recall the non-degenerate bilinear form $(\cdot,\cdot)$ on $\mathbf{U}^-$ in Section 1.2; see Proposition 1.2.2. The identities (2.4.2)-(2.4.5) can be shown to be equivalent to the following identities (2.4.6)-(2.4.9):

$$(\Upsilon_\mu, F_{\alpha_{-i}}z)=(1-q^{-2})^{-1}q^{(\alpha_{-i},\mu-\alpha_{-i}-\alpha_i)+1}(\Upsilon_{\mu-\alpha_i-\alpha_{-i}}, r_i(z)), \tag{2.4.6}$$

$$(\Upsilon_\mu, zF_{\alpha_{-i}})=(1-q^{-2})^{-1}q^{(\alpha_{-i},\mu-\alpha_{-i}-\alpha_i)+1}(\Upsilon_{\mu-\alpha_i-\alpha_{-i}},\,{}_ir(z)), \tag{2.4.7}$$

$$(\Upsilon_\mu, F_{\alpha_0}z)=(1-q^{-2})^{-1}q^{(\alpha_0,\mu-\alpha_0)}(\Upsilon_{\mu-2\alpha_0}, r_0(z))+q^{(\alpha_0,\mu-\alpha_0)+1}(\Upsilon_{\mu-\alpha_0}, z), \tag{2.4.8}$$

$$(\Upsilon_\mu, zF_{\alpha_0})=(1-q^{-2})^{-1}q^{(\alpha_0,\mu-\alpha_0)}(\Upsilon_{\mu-2\alpha_0},\,{}_0r(z))+q^{(\alpha_0,\mu-\alpha_0)+1}(\Upsilon_{\mu-\alpha_0}, z), \tag{2.4.9}$$

for all $z\in\mathbf{U}^-_{-\nu}$, $\nu\in\mathbb{N}\Pi$, $\mu\in\mathbb{N}\Pi$, and $0\neq i\in\mathbb{I}$. For example, the equivalence between (2.4.6) and (2.4.2) is shown as follows:

$$\begin{aligned}
(2.4.2) &\Leftrightarrow (\,{}_{-i}r(\Upsilon_\mu), z)=-(q^{-1}-q)q^{(\alpha_{-i},\mu-\alpha_{-i}-\alpha_i)}(\Upsilon_{\mu-\alpha_i-\alpha_{-i}}F_{\alpha_i}, z) \quad \forall z,\\
&\Leftrightarrow (F_{\alpha_{-i}}, F_{\alpha_{-i}})^{-1}(\Upsilon_\mu, F_{\alpha_{-i}}z)\\
&\qquad = -(q^{-1}-q)q^{(\alpha_{-i},\mu-\alpha_{-i}-\alpha_i)}(F_{\alpha_i}, F_{\alpha_i})(\Upsilon_{\mu-\alpha_i-\alpha_{-i}}, r_i(z)) \quad \forall z,\\
&\Leftrightarrow (2.4.6) \quad \forall z.
\end{aligned}$$

The remaining cases are similar.

Summarizing, we have established the following.

Lemma 2.4.1. *1. The validity of the identity* (2.3.1) *is equivalent to the validity of the identities* (2.4.1) *and* (2.4.2)-(2.4.5).

2. The validity of the identity (2.3.1) *is equivalent to the validity of the identities* (2.4.1) *and* (2.4.6)-(2.4.9).

Let $'\mathbf{f}^*$ (respectively, $(\mathbf{U}^-)^*$) be the non-restricted dual of $'\mathbf{f}$ (respectively, of $\mathbf{U}^-$). In light of Lemma 2.4.1(2), we define Υ_L^* and Υ_R^* in $'\mathbf{f}^*$, inductively on weights, by the following formulas:

$$\begin{aligned}
\Upsilon_L^*(1) &= \Upsilon_R^*(1) = 1, \\
\Upsilon_L^*(F_{\alpha_{-i}} z) &= (1-q^{-2})^{-1} q^{(\alpha_{-i}, \nu-\alpha_i)+1} \Upsilon_L^*(r_i(z)), \\
\Upsilon_L^*(F_{\alpha_0} z) &= (1-q^{-2})^{-1} q^{(\alpha_0, \nu)} \Upsilon^*(r_0(z)) + q^{(\alpha_0,\nu)+1} \Upsilon^*(z), \\
\Upsilon_R^*(z F_{\alpha_{-i}}) &= (1-q^{-2})^{-1} q^{(\alpha_{-i}, \nu-\alpha_i)+1} \Upsilon_L^*({}_i r(z)), \\
\Upsilon_R^*(z F_{\alpha_0}) &= (1-q^{-2})^{-1} q^{(\alpha_0, \nu)} \Upsilon^*({}_0 r(z)) + q^{(\alpha_0,\nu)+1} \Upsilon^*(z),
\end{aligned} \tag{2.4.10}$$

for all $i \in \mathbb{I}$ and $z \in \mathbf{f}_\nu$ with $\nu \in \mathbb{N}\Pi$. (The formulas (2.4.10) are presented here only for the sake of latter reference as they also make sense in the case of $\mathbf{U}^\jmath$.)

Note that since $(\alpha_i, \alpha_{-i}) = 0$ for all $i \neq 0$, we can simplify the definition (2.4.10) of Υ^*_L and Υ^*_R as follows:

$$
\begin{aligned}
\Upsilon^*_L(1) &= \Upsilon^*_R(1) = 1,\\
\Upsilon^*_L(F_{\alpha_{-i}} z) &= (1-q^{-2})^{-1} q^{(\alpha_{-i},\nu)+1} \Upsilon^*_L(r_i(z)),\\
\Upsilon^*_L(F_{\alpha_0} z) &= (1-q^{-2})^{-1} q^{(\alpha_0,\nu)} \Upsilon^*(r_0(z)) + q^{(\alpha_0,\nu)+1}\Upsilon^*(z),\\
\Upsilon^*_R(z F_{\alpha_{-i}}) &= (1-q^{-2})^{-1} q^{(\alpha_{-i},\nu)+1} \Upsilon^*_L({}_i r(z)),\\
\Upsilon^*_R(z F_{\alpha_0}) &= (1-q^{-2})^{-1} q^{(\alpha_0,\nu)} \Upsilon^*({}_0 r(z)) + q^{(\alpha_0,\nu)+1}\Upsilon^*(z),
\end{aligned}
\tag{2.4.11}
$$

for all $i \in \mathbb{I}$ and $z \in \mathbf{f}_\nu$ with $\nu \in \mathbb{N}\Pi$.

Lemma 2.4.2. *For all $x \in {}'\mathbf{f}_\mu$ with $\mu^\theta \neq \mu$, we have $\Upsilon^*_L(x) = \Upsilon^*_R(x) = 0$.*

Proof. We will only prove that $\Upsilon^*_L(x) = 0$ for all $x \in {}'\mathbf{f}_\mu$ with $\mu^\theta \neq \mu$, as the proof for the identity $\Upsilon^*_R(x) = 0$ is the same. By definition of Υ^*_L (2.4.11), the value of $\Upsilon^*_L(x)$ for $x \in {}'\mathbf{f}_\mu$ is equal to (up to some scalar multiple) $\Upsilon^*_L(x')$ for some $x' \in {}'\mathbf{f}_{\mu'}$, where $\mu' = \mu - \alpha_i - \alpha_{-i}$ for some i; here we recall $\theta(\alpha_i) = \alpha_{-i}$. Also by definition (2.4.11), we have $\Upsilon^*_L(F_{\alpha_i}) = 0$ for all $i \in \mathbb{I}$. Now the claim follows by an induction on weights. □

Lemma 2.4.3. *We have $\Upsilon^*_L = \Upsilon^*_R$.*

Proof. We shall prove the identity $\Upsilon^*_L(x) = \Upsilon^*_R(x)$ for all homogeneous elements $x \in {}'\mathbf{f}$, by induction on $\mathrm{ht}(|x|)$.

When $\mathrm{ht}(|x|) = 0$ or 1, this is trivial by definition. Assume the identity holds for all x with $\mathrm{ht}(|x|) \leq k$, for $k \geq 1$. Let $x' = F_{\alpha_{-i}} x'' F_{\alpha_{-j}} \in {}'\mathbf{f}_{\nu+\alpha_{-i}+\alpha_{-j}}$ with

$\mathrm{ht}(|x'|) = k+1 \geq 2$. We can further assume that $\theta(\nu+\alpha_{-i}+\alpha_{-j}) = \nu+\alpha_{-i}+\alpha_{-j}$, since otherwise $\Upsilon^*_L(x') = \Upsilon^*_R(x') = 0$ by Lemma 2.4.2. The proof is divided into four cases (1)-(4).

(1) Assume that $i, j \neq 0$. Then we have

$$\Upsilon^*_L(x') = (1-q^{-2})^{-1}q^{(\alpha_{-i},\nu+\alpha_{-j})+1}\Upsilon^*_L(r_i(x''F_{\alpha_{-j}})) = L_1 + L_2,$$

where

$$L_1 = (1-q^{-2})^{-1}q^{(\alpha_{-i},\nu+\alpha_{-j})+(\alpha_i,\alpha_{-j})+1}\Upsilon^*_L(r_i(x'')F_{\alpha_{-j}}),$$

$$L_2 = (1-q^{-2})^{-1}q^{(\alpha_{-i},\nu+\alpha_{-j})+1}\delta_{i,-j}\Upsilon^*_L(x'').$$

We also have

$$\Upsilon^*_R(x') = (1-q^{-2})^{-1}q^{(\alpha_{-j},\nu+\alpha_{-i})+1}\Upsilon^*_R({}_jr(F_{\alpha_{-i}}x'')) = R_1 + R_2,$$

where

$$R_1 = (1-q^{-2})^{-1}q^{(\alpha_{-j},\nu+\alpha_{-i})+(\alpha_j,\alpha_{-i})+1}\Upsilon^*_R(F_{\alpha_{-i}}\ {}_jr(x'')),$$

$$R_2 = (1-q^{-2})^{-1}q^{(\alpha_{-j},\nu+\alpha_{-i})+1}\delta_{i,-j}\Upsilon^*_R(x'').$$

Applying the induction hypothesis to $r_i(x'')F_{\alpha_{-j}}$ and $F_{\alpha_{-i}}\ {}_jr(x'')$ gives us

$$L_1 = (1-q^{-2})^{-2}q^{(\alpha_{-i},\nu+\alpha_{-j})+(\alpha_i,\alpha_{-j})+(\alpha_{-j},\nu-\alpha_i)+2}\Upsilon^*_L({}_jr(r_i(x'')))$$

$$= (1-q^{-2})^{-2}q^{(\alpha_{-i},\nu)+(\alpha_{-j},\nu)+(\alpha_{-i},\alpha_{-j})+2}\Upsilon^*_L({}_jr(r_i(x'')));$$

$$R_1 = (1-q^{-2})^{-2}q^{(\alpha_{-j},\nu+\alpha_{-i})+(\alpha_j,\alpha_{-i})+(\alpha_{-i},\nu-\alpha_j)+2}\Upsilon^*_R(r_i({}_jr(x'')))$$

$$= (1-q^{-2})^{-2}q^{(\alpha_{-i},\nu)+(\alpha_{-j},\nu)+(\alpha_{-j},\alpha_{-i})+2}\Upsilon^*_R(r_i({}_jr(x''))).$$

Note that ${}_jr(r_i(x'')) = r_i({}_jr(x''))$ by Lemma 1.2.1 and $\mathrm{ht}(|{}_jr(r_i(x''))|) < \mathrm{ht}(|x'|)$. By the induction hypothesis, $\Upsilon_L^*({}_jr(r_i(x''))) = \Upsilon_R^*(r_i({}_jr(x'')))$. Hence $L_1 = R_1$.

By the induction hypothesis, we also have $\Upsilon_L^*(x'') = \Upsilon_R^*(x'')$. When $i = -j$, we have $\nu^\theta = \nu$, and hence

$$\begin{aligned}
(1-q^{-2})^{-1}q^{(\alpha_{-i},\nu+\alpha_{-j})+1} &= (1-q^{-2})^{-1}q^{(\alpha_{-i},\nu+\alpha_i)+1} \\
&= (1-q^{-2})^{-1}q^{(\alpha_{-i}^\theta,\nu^\theta+\alpha_i^\theta)+1} \\
&= (1-q^{-2})^{-1}q^{(\alpha_i,\nu+\alpha_{-i})+1} \\
&= (1-q^{-2})^{-1}q^{(\alpha_{-j},\nu+\alpha_{-i})+1}.
\end{aligned}$$

Hence we have $L_2 = R_2$.

Summarizing, we have $\Upsilon_L^*(x') = L_1 + L_2 = R_1 + R_2 = \Upsilon_R^*(x')$ in this case.

(2) Assume that $i = 0$ and $j \neq 0$. Then we have

$$\begin{aligned}
&\Upsilon_L^*(x') \\
&= (1-q^{-2})^{-1}q^{(\alpha_0,\nu+\alpha_{-j})}\Upsilon_L^*(r_0(x''F_{\alpha_{-j}})) + q^{(\alpha_0,\nu+\alpha_{-j})+1}\Upsilon_L^*(x''F_{\alpha_{-j}}) \\
&= (1-q^{-2})^{-1}q^{(\alpha_0,\nu+\alpha_{-j})}\Upsilon_L^*(q^{(\alpha_0,\alpha_{-j})}r_0(x'')F_{\alpha_{-j}}) + q^{(\alpha_0,\nu+\alpha_{-j})+1}\Upsilon_L^*(x''F_{\alpha_{-j}}) \\
&= (1-q^{-2})^{-1}q^{(\alpha_0,\nu+\alpha_{-j})+(\alpha_0,\alpha_{-j})}\Upsilon_L^*(r_0(x'')F_{\alpha_{-j}}) + q^{(\alpha_0,\nu+\alpha_{-j})+1}\Upsilon_L^*(x''F_{\alpha_{-j}})
\end{aligned}$$

Applying the induction hypothesis to $r_0(x'')F_{\alpha_{-j}}$ and $x''F_{\alpha_{-j}}$, we have

$$\Upsilon_L^*(r_0(x'')F_{\alpha_{-j}}) = (1-q^{-2})^{-1}q^{(\alpha_{-j},\nu-\alpha_0)+1}\Upsilon_L^*({}_jr(r_0(x''))),$$

$$\Upsilon_L^*(x''F_{\alpha_{-j}}) = (1-q^{-2})^{-1}q^{(\alpha_{-j},\nu)+1}\Upsilon_L^*({}_jr(x'')).$$

Hence we obtain

$$\begin{aligned}\Upsilon_L^*(x') =&(1-q^{-2})^{-2}q^{(\alpha_0,\nu)+(\alpha_{-j},\nu)+(\alpha_{-j},\alpha_0)+1}\Upsilon_L^*({}_jr(r_0(x''))\\&+(1-q^{-2})^{-1}q^{(\alpha_0,\nu)+(\alpha_{-j},\nu)+(\alpha_{-j},\alpha_0)+2}\Upsilon_L^*({}_jr(x'')).\end{aligned}$$

From a similar computation we obtain

$$\begin{aligned}\Upsilon_R^*(x') =&(1-q^{-2})^{-2}q^{(\alpha_0,\nu)+(\alpha_{-j},\nu)+(\alpha_0,\alpha_{-j})+1}\Upsilon_R^*(r_0({}_jr(x''))\\&+(1-q^{-2})^{-1}q^{(\alpha_0,\nu)+(\alpha_{-j},\nu)+(\alpha_0,\alpha_{-j})+2}\Upsilon_R^*({}_jr(x'')).\end{aligned}$$

It follows by Lemma 1.2.1 that $r_0({}_jr(x'')) = {}_jr(r_0(x'')$. Then, by the induction hypothesis on $r_0({}_jr(x''))$, ${}_jr(r_0(x'')$, and ${}_jr(x'')$, we obtain $\Upsilon_L^*(x') = \Upsilon_R^*(x')$ in this case.

(3) Similar computation works for the case where $j = 0, i \neq 0$ as in Case (2).

(4) At last, consider the case where $i = j = 0$.

$$\begin{aligned}\Upsilon_L^*(x') &= (1-q^{-2})^{-1}q^{(\alpha_0,\nu+\alpha_0)}\Upsilon_L^*(r_0(x''F_{\alpha_0})) + q^{(\alpha_0,\nu+\alpha_0)+1}\Upsilon_L^*(x''F_{\alpha_0})\\&= (1-q^{-2})^{-1}q^{(\alpha_0,\nu+\alpha_0)+(\alpha_0,\alpha_0)}\Upsilon_L^*(r_0(x'')F_{\alpha_0})\\&\quad (1-q^{-2})^{-1}q^{(\alpha_0,\nu+\alpha_0)}\Upsilon_L^*(x'') + q^{(\alpha_0,\nu+\alpha_0)+1}\Upsilon_L^*(x''F_{\alpha_0}).\end{aligned}$$

Applying the induction hypothesis to $r_0(x'')F_{\alpha_0}$ and $x''F_{\alpha_0}$, we have

$$\begin{aligned}\Upsilon_L^*(r_0(x'')F_{\alpha_0}) &= (1-q^{-2})^{-1}q^{(\alpha_0,\nu-\alpha_0)}\Upsilon_L^*({}_0r(r_0(x''))) + q^{(\alpha_0,\nu-\alpha_0)+1}\Upsilon_L^*(r_0(x'')),\\ \Upsilon_L^*(x''F_{\alpha_0}) &= (1-q^{-2})^{-1}q^{(\alpha_0,\nu)}\Upsilon_L^*({}_0r(x'')) + q^{(\alpha_0,\nu)+1}\Upsilon_L^*(x'').\end{aligned}$$

Hence we have

$$
\begin{aligned}
&\Upsilon_L^*(x') \\
&= (1-q^{-2})^{-2} q^{(\alpha_0,\nu)+(\alpha_0,\nu)+(\alpha_0,\alpha_0)} \Upsilon_L^*({}_0r(r_0(x''))) \\
&\quad + (1-q^{-2})^{-1} q^{(\alpha_0,\nu)+(\alpha_0,\nu)+(\alpha_0,\alpha_0)+1} \Upsilon_L^*(r_0(x'')) + (1-q^{-2})^{-1} q^{(\alpha_0,\nu+\alpha_0)} \Upsilon_L^*(x'') \\
&\quad + (1-q^{-2})^{-1} q^{(\alpha_0,\nu)+(\alpha_0,\nu)+(\alpha_0,\alpha_0)+1} \Upsilon_L^*({}_0r(x'')) + q^{(\alpha_0,\nu)+(\alpha_0,\nu)+(\alpha_0,\alpha_0)+2} \Upsilon_L^*(x'').
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&\Upsilon_R^*(x') \\
&= (1-q^{-2})^{-2} q^{(\alpha_0,\nu)+(\alpha_0,\nu)+(\alpha_0,\alpha_0)} \Upsilon_R^*(r_0({}_0r(x''))) \\
&\quad + (1-q^{-2})^{-1} q^{(\alpha_0,\nu)+(\alpha_0,\nu)+(\alpha_0,\alpha_0)+1} \Upsilon_R^*({}_0r(x'')) + (1-q^{-2})^{-1} q^{(\alpha_0,\nu+\alpha_0)} \Upsilon_R^*(x'') \\
&\quad + (1-q^{-2})^{-1} q^{(\alpha_0,\nu)+(\alpha_0,\nu)+(\alpha_0,\alpha_0)+1} \Upsilon_R^*(r_0(x'')) + q^{(\alpha_0,\nu)+(\alpha_0,\nu)+(\alpha_0,\alpha_0)+2} \Upsilon_R^*(x'').
\end{aligned}
$$

Therefore $\Upsilon_L^*(x') = \Upsilon_R^*(x')$ in this case too by induction and by Lemma 1.2.1.

This completes the proof of Lemma 2.4.3. □

We shall simply denote $\Upsilon_L^* = \Upsilon_R^*$ by Υ^* thanks to Lemma 2.4.3. Recall $'\mathbf{f}/\mathbf{I} = \mathbf{U}^-$, where $\mathbf{I} = \langle S_{ij} \rangle$.

Lemma 2.4.4. *We have* $\Upsilon^*(\mathbf{I}) = 0$*; hence we may regard* $\Upsilon^* \in (\mathbf{U}^-)^*$.

Proof. Recall $r_k(S_{ij}) = {}_kr(S_{ij}) = 0$, for all i, j, k. Any element in $\mathbf{I}$ is a $\mathbb{Q}(q)$-linear combination of elements of the form $F_{\alpha_{m_1}} \dots F_{\alpha_{m_h}} S_{ij} F_{\alpha_{n_1}} \dots F_{\alpha_{n_l}}$. So it suffices to prove $\Upsilon^*(F_{\alpha_{m_1}} \dots F_{\alpha_{m_h}} S_{ij} F_{\alpha_{n_1}} \dots F_{\alpha_{n_l}}) = 0$, by induction on $h + l$.

Recall the Serre relator S_{ij}, for $i \neq j \in \mathbb{I}$, from (1.2.2). Let us verify that $\Upsilon^*(S_{ij}) = 0$, which is the base case of the induction. If $|i-j| = 1$, the weight of S_{ij} is $-2\alpha_i - \alpha_j$, which is not θ-invariant. If $|i-j| > 1$, the weight of S_{ij} is $-\alpha_i - \alpha_j$, which is not θ-invariant unless $i = -j$. In case of $i = -j$, a quick computation by definition (2.4.11) gives us that $\Upsilon^*(S_{ij}) = 0$. In the remaining cases, it follows by Lemma 2.4.2 that $\Upsilon^*(S_{ij}) = 0$.

If $h > 0$, by (2.4.11), (1.2.1) and (1.2.3) we have

$$\begin{aligned}
&\Upsilon^*(F_{\alpha_{m_1}} \dots F_{\alpha_{m_h}} S_{ij} F_{\alpha_{n_1}} \dots F_{\alpha_{n_l}}) \\
&\quad = \Upsilon^*(r_{-m_1}(F_{\alpha_{m_2}} \dots F_{\alpha_{m_h}} S_{ij} F_{\alpha_{n_1}} \dots F_{\alpha_{n_l}})) \\
&\quad = \Upsilon^*\Big(\sum c_{m'n'} F_{\alpha_{m'_1}} \dots F_{\alpha_{m'_{h'}}} S_{ij} F_{\alpha_{n'_1}} \dots F_{\alpha_{n'_{l'}}}\Big) \\
&\qquad + \delta_{-m_1,0} c' \Upsilon^*(F_{\alpha_{m_2}} \dots F_{\alpha_{m_h}} S_{ij} F_{\alpha_{n_1}} \dots F_{\alpha_{n_l}}),
\end{aligned}$$

for some scalars $c_{m'n'}$ and c'. Similarly if $l > 0$, we have

$$\begin{aligned}
&\Upsilon^*(F_{\alpha_{m_1}} \dots F_{\alpha_{m_h}} S_{ij} F_{\alpha_{n_1}} \dots F_{\alpha_{n_l}}) \\
&\quad = \Upsilon^*({}_{-n_l}r(F_{\alpha_{m_1}} \dots F_{\alpha_{m_h}} S_{ij} F_{\alpha_{n_1}} \dots F_{\alpha_{n_{l-1}}})) \\
&\quad = \Upsilon^*\Big(\sum c_{m''n''} F_{\alpha_{m''_1}} \dots F_{\alpha_{m''_{h''}}} S_{ij} F_{\alpha_{n''_1}} \dots F_{\alpha_{n''_{l''}}}\Big) \\
&\qquad + \delta_{-n_l,0} c'' \Upsilon^*(F_{\alpha_{m_1}} \dots F_{\alpha_{m_h}} S_{ij} F_{\alpha_{n_1}} \dots F_{\alpha_{n_{l-1}}}).
\end{aligned}$$

for some scalars $c_{m''n''}$ and c''. In either case, we have $h' + l' = h'' + l'' < h + l$. Therefore by induction on $h + l$, Lemma 2.4.4 is proved. $\square$

Now we are ready to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. We first prove the existence of Υ satisfying the identity (2.3.1). Set $\Upsilon_\mu = 0$ if $\mu \notin \mathbb{N}\Pi$. Let $B = \{b\}$ be a basis of $\mathbf{U}^-$ such that $B_\mu = B \cap \mathbf{U}^-_{-\mu}$ is a basis of $\mathbf{U}^-_{-\mu}$. Let $B^* = \{b^*\}$ be the dual basis of B with respect to the bilinear pairing $(\cdot,\cdot)$ in Section 1.2. Define Υ by

$$\Upsilon := \sum_{b \in B} \Upsilon^*(b^*) b = \sum_{\mu} \Upsilon_\mu.$$

As functions on $\mathbf{U}^-$, $(\Upsilon, \cdot) = \Upsilon^*$. Clearly Υ is in $\widehat{\mathbf{U}}^-$ and $\Upsilon_0 = 1$. Also Υ satisfies the identities in (2.4.6)-(2.4.9) by the definition of Υ^*. For any $x \in \mathbf{U}^-_\nu$, it follows by Lemma 2.4.2 that $\Upsilon^*_L(x) = \Upsilon^*_R(x) = 0$ if $\nu^\theta \neq \nu$. It follows that (2.4.1) is satisfied. Therefore, by Lemma 2.4.1(2), Υ satisfies the desired identity (2.3.1) in the theorem.

By Lemma 2.4.1(1) and the definition of Υ, the identity (2.4.2) holds for Υ, and so ${}_{-i}r(\Upsilon_\mu)$ is determined by Υ_ν with weight $\nu \prec \mu$. By [Lu2, Lemma 1.2.15], if an element $x \in \mathbf{U}^-_{-\nu}$ with $\nu \neq 0$ satisfies ${}_{-i}r(x) = 0$ for all $i \in \mathbb{I}$ then $x = 0$. Therefore, by induction on weight, the identity (2.4.2) together with $\Upsilon_0 = 1$ imply the uniqueness of Υ.

The Υ as constructed satisfies the additional property that $\Upsilon_\mu = 0$ unless $\mu^\theta = \mu$, by Lemmas 2.4.2, 2.4.3 and 2.4.4. The theorem is proved. $\square$

2.5 The isomorphism $\mathcal{T}$

Consider a function ζ on Λ such that

$$\begin{aligned} \zeta(\mu+\alpha_0) &= -q\zeta(\mu), \\ \zeta(\mu+\alpha_i) &= -q^{(\alpha_i-\alpha_{-i},\mu+\alpha_i)}\zeta(\mu), \\ \zeta(\mu+\alpha_{-i}) &= -q^{(\alpha_{-i},\mu+\alpha_{-i})-(\alpha_i,\mu)}\zeta(\mu), \quad \forall \mu\in\Lambda,\ i\in\mathbb{I}^\imath. \end{aligned} \tag{2.5.1}$$

Noting that $(\alpha_i,\alpha_{-i})=0$ for all $i\in\mathbb{I}^\imath$, we see that ζ satisfying (2.5.1) is equivalent to ζ satisfying

$$\begin{aligned} \zeta(\mu+\alpha_0) &= -q\zeta(\mu), \\ \zeta(\mu+\alpha_i) &= -q^{(\alpha_i,\mu+\alpha_i)-(\alpha_{-i},\mu)}\zeta(\mu), \quad \forall \mu\in\Lambda,\ 0\neq i\in\mathbb{I}. \end{aligned} \tag{2.5.2}$$

Such ζ clearly exists. For any weight $\mathbf{U}$-module M, define a $\mathbb{Q}(q)$-linear map on M

$$\begin{aligned} \widetilde{\zeta} &: M \longrightarrow M, \\ \widetilde{\zeta}(m) &= \zeta(\mu)m, \quad \forall m\in M_\mu. \end{aligned} \tag{2.5.3}$$

Recall that w_0 is the longest element of W and T_{w_0} is the associated braid group element from Section 1.3.

Theorem 2.5.1. *For any finite-dimensional* $\mathbf{U}$*-module* M*, the composition map*

$$\mathcal{T} := \Upsilon\circ\widetilde{\zeta}\circ T_{w_0} : M \longrightarrow M$$

is a $\mathbf{U}^\imath$*-module isomorphism.*

Proof. The map $\mathcal{T}$ is clearly a $\mathbb{Q}(q)$-linear isomorphism. So it remains to verify that $\mathcal{T}$ commutes with the action of $\mathbf{U}^\imath$; we shall check this on generators of $\mathbf{U}^\imath$ by applying repeatedly Lemma 1.3.1.

Let $m \in M_{w0(\mu)}$ and $i \in \mathbb{I}^\imath$. Then we have

$$\begin{aligned}
\mathcal{T}(k_{\alpha_i} m) &= \Upsilon \circ \widetilde{\zeta} \circ T_{w_0}(\imath(k_{\alpha_i}))T_{w_0}(m) \\
&= \Upsilon \circ \widetilde{\zeta} \circ T_{w_0}(K_{\alpha_i} K_{\alpha_{-i}}^{-1})T_{w_0}(m) \\
&= \Upsilon \circ \widetilde{\zeta} K_{\alpha_i} K_{\alpha_{-i}}^{-1} T_{w_0}(m) \\
&= (K_{\alpha_i} K_{\alpha_{-i}}^{-1}) \Upsilon \circ \widetilde{\zeta} \circ T_{w_0}(m) \\
&= k_{\alpha_i} \mathcal{T}(m).
\end{aligned}$$

We also have

$$\begin{aligned}
\mathcal{T}(e_{\alpha_i} m) &= \Upsilon \circ \widetilde{\zeta}(T_{w_0}(\imath(e_{\alpha_i}))T_{w_0}(m)) \\
&= \Upsilon \circ \widetilde{\zeta}(T_{w_0}(E_{\alpha_i} + K_{\alpha_i}^{-1} F_{\alpha_{-i}})T_{w_0}(m)) \\
&= -\Upsilon \circ \widetilde{\zeta}(K_{\alpha_i}^{-1}(K_{\alpha_i} F_{\alpha_{-i}} + E_{\alpha_i})K_{\alpha_{-i}} T_{w_0}(m)) \\
&= -\Upsilon(\zeta(\mu - \alpha_{-i}))q^{(\alpha_{-i},\mu)-(\alpha_i,\mu-\alpha_{-i})} K_{\alpha_i} F_{\alpha_{-i}} T_{w_0}(m)) \\
&\quad - \Upsilon(\zeta(\mu + \alpha_i)q^{(\alpha_{-i},\mu)-(\alpha_i,\mu+\alpha_i)} E_{\alpha_i} T_{w_0}(m)) \\
&\overset{(a)}{=} \Upsilon(E_{\alpha_i} + K_{\alpha_i} F_{\alpha_{-i}})\zeta(\mu) T_{w_0}(m) \\
&\overset{(b)}{=} (E_{\alpha_i} + K_{\alpha_i}^{-1} F_{\alpha_{-i}}) \Upsilon \circ \widetilde{\zeta} \circ T_{w_0}(m) \\
&= e_{\alpha_i} \mathcal{T}(m).
\end{aligned}$$

The identity (a) above follows from the definition of ζ and the identity (b) follows from the definition of Υ.

By a similar computation we have $\mathcal{T} f_{\alpha_i}(m) = f_{\alpha_i}\mathcal{T}(m)$.

For the generator t, we have

$$\begin{aligned}
\mathcal{T}(t\, m) &= \Upsilon \circ \widetilde{\zeta} \circ T_{w_0}(\imath(t))T_{w_0}(m) \\
&= \Upsilon \circ \widetilde{\zeta} \circ T_{w_0}(E_{\alpha_0} + qF_{\alpha_0}K_{\alpha_0}^{-1} + K_{\alpha_0}^{-1})T_{w_0}(m) \\
&= \Upsilon \circ \widetilde{\zeta}(-F_{\alpha_0}K_{\alpha_0} - q^{-1}E_{\alpha_0} + K_{\alpha_0})T_{w_0}(m) \\
&= \Upsilon(-q\zeta(\mu - \alpha_0)q^{-1}F_{\alpha_0}K_{\alpha_0} - q^{-1}\zeta(\mu + \alpha_0)E_{\alpha_0} + \zeta(\mu)K_{\alpha_0})T_{w_0}(m) \\
&\overset{(c)}{=} \Upsilon(q^{-1}F_{\alpha_0}K_{\alpha_0} + E_{\alpha_0} + K_{\alpha_0})\zeta(\mu)T_{w_0}(m) \\
&\overset{(d)}{=} (E_{\alpha_0} + qF_{\alpha_0}K_{\alpha_0}^{-1} + K_{\alpha_0}^{-1})\Upsilon \circ \widetilde{\zeta} \circ T_{w_0}(m) \\
&= t\mathcal{T}(m).
\end{aligned}$$

Here the identity (c) follows from the definition of ζ and identity (d) follows from the definition of Υ. Hence the theorem is proved. $\square$

Chapter 3

Quasi-$\mathcal{R}$-matrix for a quantum symmetric pair

In this chapter, we define a quasi-$\mathcal{R}$-matrix $\Theta^\imath$ for $\mathbf{U}^\imath$, which will play an analogous role as Lusztig's quasi-$\mathcal{R}$-matrix for $\mathbf{U}$. Our $\Theta^\imath$ is constructed from the intertwiner Υ and Θ.

3.1 Definition of $\Theta^\imath$

Recall Lusztig's quasi-$\mathcal{R}$-matrix Θ from (1.4.1). It follows by Theorem 2.3.1 that Υ is a well-defined operator on finite-dimensional $\mathbf{U}$-modules. For any finite-dimensional $\mathbf{U}$-modules M and M', the action of Υ on $M \otimes M'$ is also well defined. So we shall use the formal notation $\Upsilon^{\vartriangle}$ to denote the action of Υ on $M \otimes M'$. Hence the operator

$$\Theta^{\imath} := \Upsilon^{\vartriangle}\Theta(\Upsilon^{-1} \otimes 1) \tag{3.1.1}$$

on $M \otimes M'$ is well defined. Note that $\Theta^{\imath}$ lies in (a completion of) $\mathbf{U} \otimes \mathbf{U}$. We shall prove in Proposition 3.2.3 that it actually lies in (a completion of) $\mathbf{U}^{\imath} \otimes \mathbf{U}$.

Definition 3.1.1. The element $\Theta^{\imath}$ is called the quasi-$\mathcal{R}$-matrix for the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^{\imath})$.

Recall that we set in Section 1.2 that $\psi(u) = \overline{u}$ for all $u \in \mathbf{U}$, and in Section 2.1 that $\psi_{\imath}(x) := \overline{x} \in \mathbf{U}^{\imath}$ for $x \in \mathbf{U}^{\imath}$. We shall also set $\psi(x) := \overline{\imath(x)} \in \mathbf{U}$ for $x \in \mathbf{U}^{\imath}$.

Define $\overline{\Delta} : \mathbf{U}^{\imath} \to \mathbf{U}^{\imath} \otimes \mathbf{U}$ by $\overline{\Delta}(u) = (\psi_{\imath} \otimes \psi)\Delta(\psi_{\imath}(u))$, for all $u \in \mathbf{U}^{\imath}$. Recall that the bar involution on $\mathbf{U}^{\imath}$ is not compatible with the bar involution on $\mathbf{U}$ through $\imath$ (see Remark 2.2.3); in particular the $\overline{\Delta}$ here does not coincide with the restriction to $\mathbf{U}^{\imath}$ of the map in the same notation $\overline{\Delta} : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ in [Lu2, 4.1.1].

Proposition 3.1.2. *Let M and M' be finite-dimensional $\mathbf{U}$-modules. As linear operators on $M \otimes M'$, we have $\Delta(u)\Theta^{\imath} = \Theta^{\imath}\,\overline{\Delta}(u)$, for all $u \in \mathbf{U}^{\imath}$.*

Proof. For $u \in \mathbf{U}^{\imath}$, we set $\Delta(\overline{u}) = \sum u_{(1)} \otimes u_{(2)} \in \mathbf{U}^{\imath} \otimes \mathbf{U}$. Then, for $m \in M$, $m' \in M'$, we have

$$\begin{aligned}\Upsilon^{\triangle}\Theta(\Upsilon^{-1}\otimes 1)\overline{\Delta(\overline{u})}(m\otimes m') &= \Upsilon^{\triangle}\Theta(\sum \Upsilon^{-1}\imath(\overline{u_{(1)}})\otimes \overline{u_{(2)}})(m\otimes m') \\ &\overset{(a)}{=} \Upsilon^{\triangle}\Theta(\sum \overline{\imath(u_{(1)})}\otimes \overline{u_{(2)}})(\Upsilon^{-1}\otimes 1)(m\otimes m') \\ &\overset{(b)}{=} \Upsilon^{\triangle}\Delta\big(\overline{\imath(\overline{u})}\big)\Theta(\Upsilon^{-1}\otimes 1)(m\otimes m') \\ &\overset{(c)}{=} \Delta(u)\Upsilon^{\triangle}\Theta(\Upsilon^{-1}\otimes 1)(m\otimes m').\end{aligned}$$

The identities (a) and (c) follow from Theorem 2.3.1 and the identity (b) follows from (1.4.2). Note that the bar-notation above translates into the ψ-notation as follows:

$$\overline{u}=\psi_\imath(u),\ \overline{u_{(1)}}=\psi_\imath(u_{(1)}),\ \overline{u_{(2)}}=\psi(u_{(2)}),\ \overline{\imath(u_{(1)})}=\psi(\imath(u_{(1)})),\ \overline{\imath(\overline{u})}=\psi(\imath(\psi_\imath(u))).$$

The proposition is proved. □

3.2 Normalizing $\Theta^\imath$

Our next goal is to understand $\Theta^\imath$ in a precise sense as an element in a completion of $\mathbf{U}\otimes\mathbf{U}^-$ instead of merely as well-defined operators on $M\otimes M'$ for finite-dimensional $\mathbf{U}$-modules M, M'.

Let $B=\{b\}$ be a basis of $\mathbf{U}^-$ such that $B_\mu = B\cap \mathbf{U}^-_{-\mu}$ is a basis of $\mathbf{U}^-_{-\mu}$ for each μ. Let $B^*=\{b^*\}$ be the basis of $\mathbf{U}^-$ dual to B with respect to the bilinear form $(\cdot,\cdot)$ in Section 1.2. For each $N\in\mathbb{N}$, define the $\mathbb{Q}(q)$-linear truncation map $tr_{\leq N} : {}'\mathbf{f}\to {}'\mathbf{f}$

such that, for any $i_1, \dots, i_k \in \mathbb{I}$,

$$tr_{\leq N}(F_{\alpha_{i_1}} \dots F_{\alpha_{i_k}}) = \begin{cases} F_{\alpha_{i_1}} \dots F_{\alpha_{i_k}}, & \text{if } k \leq N, \\ \\ 0, & \text{if } k > N. \end{cases} \tag{3.2.1}$$

This induces a truncation map on $\mathbf{U}^- = {}'\mathbf{f}/\mathbf{I}$, also denoted by $tr_{\leq N}$, since $\mathbf{I}$ is homogeneous. Recalling Θ from (1.4.1), we denote

$$\Theta_{\leq N} := \sum_{\mathrm{ht}(\mu) \leq N} \Theta_\mu.$$

Then we define

$$\Theta^\imath_{\leq N} := \sum_\mu id \otimes tr_{\leq N}(\Delta(\Upsilon_\mu)\Theta_{\leq N}(\Upsilon^{-1} \otimes 1)), \tag{3.2.2}$$

which is actually a finite sum, and hence $\Theta^\imath_{\leq N} \in \mathbf{U} \otimes \mathbf{U}^-$ and $\Theta^\imath_{\leq 0} = 1 \otimes 1$. Define

$$\Theta^\imath_N := \Theta^\imath_{\leq N} - \Theta^\imath_{\leq N-1} = \sum_{b_\mu \in B_\mu, \mathrm{ht}(\mu) = N} a^\mu \otimes b_\mu \in \mathbf{U} \otimes \mathbf{U}^-, \tag{3.2.3}$$

where it is understood that $\Theta^\imath_{\leq -1} = 0$. The following lemma is clear from weight consideration.

Lemma 3.2.1. *Let M and M' be finite-dimensional $\mathbf{U}$-modules. For all $m \in M$ and $m' \in M'$, we have*

$$\Theta^\imath(m \otimes m') = \Theta^\imath_{\leq N}(m \otimes m'), \quad \textit{for } N \gg 0.$$

Note that any finite-dimensional $\mathbf{U}$-module is also a $\widehat{\mathbf{U}}$-module.

Lemma 3.2.2. *Let $u \in \widehat{\mathbf{U}}$ be an element that acts as zero on all finite-dimensional $\mathbf{U}$-modules. Then $u = 0$.*

Proof. It is well known that any element $u \in \mathbf{U}$ that acts as zero on all finite-dimensional $\mathbf{U}$-modules has to be 0 (see [Lu2, Proposition 3.5.4]). Hence the lemma follows by weight consideration. □

We have the following fundamental property of $\Theta^\imath_N$.

Proposition 3.2.3. *For any $N \in \mathbb{N}$, we have $\Theta^\imath_N \in \imath(\mathbf{U}^\imath) \otimes \mathbf{U}^-$.*

Proof. The identity in Proposition 3.1.2 for u being one of the generators k_{α_i}, e_{α_i}, f_{α_i}, and t of $\mathbf{U}^\imath$ can be rewritten as the following identities (valid for all $N \geq 0$):

$$(k_{\alpha_i} \otimes K_{\alpha_i} K^{-1}_{\alpha_{-i}}) \Theta^\imath_N (m \otimes m') = \Theta^\imath_N (k_{\alpha_i} \otimes K_{\alpha_i} K^{-1}_{\alpha_{-i}})(m \otimes m'),$$

$$\begin{aligned}&((k_{\alpha_i} \otimes F_{\alpha_i} K^{-1}_{\alpha_{-i}}) \Theta^\imath_{N-1} + (f_{\alpha_i} \otimes K^{-1}_{\alpha_{-i}}) \Theta^\imath_N + (1 \otimes E_{\alpha_{-i}}) \Theta^\imath_{N+1})(m \otimes m') \\ &\quad = (\Theta^\imath_{N-1}(k^{-1}_{\alpha_i} \otimes F_{\alpha_i} K_{\alpha_{-i}}) + \Theta^\imath_N (f_{\alpha_i} \otimes K_{\alpha_{-i}}) + \Theta^\imath_N (1 \otimes E_{\alpha_{-i}}))(m \otimes m'),\end{aligned}$$

$$\begin{aligned}&((k^{-1}_{\alpha_i} \otimes K^{-1}_{\alpha_i} F_{\alpha_{-i}}) \Theta^\imath_{N-1} + (e_{\alpha_i} \otimes K^{-1}_{\alpha_i}) \Theta^\imath_N + (1 \otimes E_{\alpha_i}) \Theta^\imath_{N+1})(m \otimes m') \\ &\quad = (\Theta^\imath_{N-1}(k_{\alpha_i} \otimes K_{\alpha_i} F_{\alpha_{-i}}) + \Theta^\imath_N (e_{\alpha_i} \otimes K_{\alpha_i}) + \Theta^\imath_{N+1}(1 \otimes E_{\alpha_i}))(m \otimes m'),\end{aligned}$$

$$\begin{aligned}&((1 \otimes q F_{\alpha_0} K^{-1}_{\alpha_0}) \Theta^\imath_{N-1} + (t \otimes K^{-1}_{\alpha_0}) \Theta^\imath_N + (1 \otimes E_{\alpha_0}) \Theta^\imath_{N+1})(m \otimes m') \\ &\quad = (\Theta^\imath_{N-1}(1 \otimes q^{-1} F_{\alpha_0} K_{\alpha_0}) + \Theta^\imath_N (t \otimes K_{\alpha_0}) + \Theta^\imath_{N+1}(1 \otimes E_{\alpha_0}))(m \otimes m'),\end{aligned}$$

for all $0 \neq i \in \mathbb{I}^\imath$, $m \in M$ and $m' \in M'$, where M, M' are finite-dimensional $\mathbf{U}$-modules. Write

$$\Theta^\imath_N = \sum_{b_\mu \in B_\mu, \mathrm{ht}(\mu) = N} a^\mu \otimes b_\mu \in \mathbf{U} \otimes \mathbf{U}^-,$$

where a_μ's are fixed once B is chosen. Thanks to Lemma 3.2.2, the above four identities for all M, M' are equivalent to the following four identities:

$$\sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N}} \imath(k_{\alpha_i})a^\mu \otimes K_{\alpha_i}K_{\alpha_{-i}}^{-1}b_\mu = \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N}} a^\mu \imath(k_{\alpha_i}) \otimes b_\mu K_{\alpha_i}K_{\alpha_{-i}}^{-1}, \tag{3.2.4}$$

$$\sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} \imath(k_{\alpha_i})a^{\mu''} \otimes F_{\alpha_i}K_{\alpha_{-i}}^{-1}b_{\mu''} + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} \imath(f_{\alpha_i})a^{\mu'} \otimes K_{\alpha_{-i}}^{-1}b_{\mu'} + \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes E_{\alpha_{-i}}b_\mu \tag{3.2.5}$$

$$= \sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} a^{\mu''}\imath(k_{\alpha_i}^{-1}) \otimes b_{\mu''}F_{\alpha_i}K_{\alpha_{-i}} + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} a^{\mu'}\imath(f_{\alpha_i}) \otimes b_{\mu'}K_{\alpha_{-i}} + \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes b_\mu E_{\alpha_{-i}},$$

$$\sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} \imath(k_{\alpha_i}^{-1})a^{\mu''} \otimes K_{\alpha_i}^{-1}F_{\alpha_{-i}}b_{\mu''} + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} \imath(e_{\alpha_i})a^{\mu'} \otimes K_{\alpha_i}^{-1}b_{\mu'} + \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes E_{\alpha_i}b_\mu \tag{3.2.6}$$

$$= \sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} a^{\mu''}\imath(k_{\alpha_i}) \otimes b_{\mu''}K_{\alpha_i}F_{\alpha_{-i}} + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} a^{\mu'}\imath(e_{\alpha_i}) \otimes b_{\mu'}K_{\alpha_i} + \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes b_\mu E_{\alpha_i},$$

$$\sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} a^{\mu''} \otimes qF_{\alpha_0}K_{\alpha_0}^{-1}b_{\mu''} + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} \imath(t)a^{\mu'} \otimes K_{\alpha_0}^{-1}b_{\mu'} + \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes E_{\alpha_0}b_\mu \tag{3.2.7}$$

$$= \sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} a^{\mu''} \otimes b_{\mu''}q^{-1}F_{\alpha_0}K_{\alpha_0} + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} a^{\mu'}\imath(t) \otimes b_{\mu'}K_{\alpha_0} + \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes b_\mu E_{\alpha_0}.$$

A straighforward rewriting of (3.2.5)-(3.2.7) involves the commutators $[E_{\alpha_k}, b_\mu]$ for various $k \in \mathbb{I}$, which can be expressed in terms of ${}_k r$ and r_k by invoking [Lu2, Proposition 3.1.6]. In this way, using the PBW theorem for $\mathbf{U}$ we rewrite the three identities (3.2.5)-(3.2.7) as the following six identities:

$$\sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} \imath(k_{\alpha_i}) a^{\mu''} \otimes F_{\alpha_i} b_{\mu''} + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} \imath(f_{\alpha_i}) a^{\mu'} \otimes b_{\mu'} + \frac{q^{(\alpha_{-i}, \mu+\alpha_{-i})}}{q^{-1}-q} \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes r_{-i}(b_\mu) = 0, \tag{3.2.8}$$

$$\sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} a^{\mu''} \imath(k_{\alpha_i}^{-1}) \otimes b_{\mu''} F_{\alpha_i} + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} a^{\mu'} \imath(f_{\alpha_i}) \otimes b_{\mu'} + \frac{q^{(\alpha_{-i}, \mu+\alpha_{-i})}}{q^{-1}-q} \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes {}_{-i}r(b_\mu) = 0,$$

$$\sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} \imath(k_{\alpha_i}^{-1}) a^{\mu''} \otimes F_{\alpha_{-i}} b_{\mu''} + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} \imath(e_{\alpha_i}) a^{\mu'} \otimes b_{\mu'} + \frac{q^{(\alpha_i, \mu+\alpha_i)}}{q^{-1}-q} \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes r_i(b_\mu) = 0, \tag{3.2.9}$$

$$\sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} a^{\mu''} \imath(k_{\alpha_i}) \otimes b_{\mu''} F_{\alpha_{-i}} + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} a^{\mu'} \imath(e_{\alpha_i}) \otimes b_{\mu'} + \frac{q^{(\alpha_i, \mu+\alpha_i)}}{q^{-1}-q} \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes {}_i r(b_\mu) = 0,$$

$$\sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} a^{\mu''} \otimes q^{-1} F_{\alpha_0} b_{\mu''} + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} \imath(t) a^{\mu'} \otimes b_{\mu'} + \frac{q^{(\alpha_0, \mu+\alpha_0)}}{q^{-1}-q} \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes r_0(b_\mu) = 0, \tag{3.2.10}$$

$$\sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} a^{\mu''} \otimes q^{-1} b_{\mu''} F_{\alpha_0} + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} a^{\mu'} \imath(t) \otimes b_{\mu'} + \frac{q^{(\alpha_0, \mu+\alpha_0)}}{q^{-1}-q} \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes_0 r(b_\mu) = 0.$$

So far we have the flexibility in choosing the dual bases B and B^* of $\mathbf{U}^-$. Now let us be more specific by fixing $B^* = \{b^*\}$ to be a monomial basis of $\mathbf{U}^-$ which consists of monomials in the Chevalley generators F_{α_i}; for example, we can take the $\mathbf{U}^-$-variant of the basis $\{E((\mathbf{c}))\}$ in [Lu1, pp.476] where Lusztig worked with $\mathbf{U}^+$. Let $B = \{b\}$ be the dual basis of B^* with respect to $(\cdot,\cdot)$, and write $B_\mu = B \cap \mathbf{U}_{-\mu} = \{b_\mu\}$ as before. Fix an arbitrary basis element $\tilde{b}_\mu \in B_\mu$ (with $\mu \neq 0$), with its dual basis element written as $\tilde{b}^*_\mu = xF_{\alpha_{-i}}$, for some $x \in \mathbf{U}^-$ and some i. We now apply $1 \otimes (x, \cdot)$ to the identities (3.2.8), (3.2.9) and (3.2.10), depending on whether i is positive, zero or negative.

We will treat in detail the case when i is positive, while the other cases are similar. Applying $1 \otimes (x, \cdot)$ to the identity (3.2.8) above, we have

$$\sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} \imath(k_{\alpha_i}) a^{\mu''} \otimes (x, F_{\alpha_i} b_{\mu''}) + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} \imath(f_{\alpha_i}) a^{\mu'} \otimes (x, b_{\mu'})$$
$$+ \frac{q^{(\alpha_{-i},\mu+\alpha_{-i})}}{q^{-1}-q} \sum_{\substack{b_\mu \\ \mathrm{ht}(\mu)=N+1}} a^\mu \otimes (x, r_{-i}(b_\mu)) = 0.$$

Since $(x, r_{-i}(b_\mu)) = (1-q^{-2})(xF_{\alpha_{-i}}, b_\mu) = (1-q^{-2})\delta_{b_\mu, \tilde{b}_\mu}$, we have

$$\sum_{\substack{b_{\mu''} \\ \mathrm{ht}(\mu'')=N-1}} \imath(k_{\alpha_i}) a^{\mu''} (x, F_{\alpha_i} b_{\mu''}) + \sum_{\substack{b_{\mu'} \\ \mathrm{ht}(\mu')=N}} \imath(f_{\alpha_i}) a^{\mu'} (x, b_{\mu'}) - q^{(\alpha_{-i},\mu+\alpha_{-i})-1} \tilde{a}^\mu = 0. \tag{3.2.11}$$

By an easy induction on height based on (3.2.11) (where the base case is $\Theta^\imath_0 = 1 \otimes 1$), we conclude that $a^\mu \in \imath(\mathbf{U}^\imath)$ for all μ; that is, $\Theta^\imath_N \in \imath(\mathbf{U}^\imath) \otimes \mathbf{U}^-$. □

By Proposition 3.2.3 we have $\imath^{-1}(\Theta^\imath_N) \in \mathbf{U}^\imath \otimes \mathbf{U}$ for each N. For any finite-dimensional $\mathbf{U}$-modules M and M', the action of $\imath^{-1}(\Theta^\imath_N)$ coincides with the action of $\Theta^\imath_N$ on $M \otimes M'$.

As we only need to use $\imath^{-1}(\Theta^\imath_N) \in \mathbf{U}^\imath \otimes \mathbf{U}$ rather than $\Theta^\imath_N$, we shall write $\Theta^\imath_N$ in place of $\imath^{-1}(\Theta^\imath_N)$ and regard $\Theta^\imath_N \in \mathbf{U}^\imath \otimes \mathbf{U}$ from now on.

3.3 Properties of $\Theta^\imath$

Let $(\mathbf{U}^\imath \otimes \mathbf{U}^-)^\wedge$ be the completion of the $\mathbb{Q}(q)$-vector space $\mathbf{U}^\imath \otimes \mathbf{U}^-$ with respect to the following descending sequence of subspaces

$$H^\imath_N := \mathbf{U}^\imath \otimes \Big(\sum_{\mathrm{ht}(\mu) \geq N} \mathbf{U}^-_{-\mu} \Big), \qquad \text{for } N \geq 1.$$

The $\mathbb{Q}(q)$-algebra structure on $\mathbf{U}^\imath \otimes \mathbf{U}^-$ extends by continuity to a $\mathbb{Q}(q)$-algebra structure on $(\mathbf{U}^\imath \otimes \mathbf{U}^-)^\wedge$, and we have an embedding $\mathbf{U}^\imath \otimes \mathbf{U}^- \hookrightarrow (\mathbf{U}^\imath \otimes \mathbf{U}^-)^\wedge$.

The actions of $\sum_{N \geq 0} \Theta^\imath_N$ (which is well defined by Lemma 3.2.1) and of $\Theta^\imath$ coincide on any tensor product of finite-dimensional $\mathbf{U}$-modules. From now on, we may and shall identify

$$\Theta^\imath = \sum_{N \geq 0} \Theta^\imath_N \in (\mathbf{U}^\imath \otimes \mathbf{U}^-)^\wedge, \tag{3.3.1}$$

(or alternatively, one may regard this as a normalized definition of $\Theta^\imath$).

The following theorem is a generalization of Proposition 3.1.2.

Theorem 3.3.1. *Let L be a finite-dimensional $\mathbf{U}^\imath$-module and M be a finite-dimensional $\mathbf{U}$-module. Then as linear operators on $L \otimes M$, we have*

$$\Delta(u)\Theta^\imath = \Theta^\imath \overline{\Delta}(u), \qquad \textit{for all } u \in \mathbf{U}^\imath.$$

Proof. By the identities (3.2.4)-(3.2.7) in the proof of Proposition 3.2.3, there exists $N_0 > 0$ (depending on L and M) such that for $N \geq N_0$ we have

$$\Delta(u)\Theta^\imath_{\leq N} - \Theta^\imath_{\leq N}\overline{\Delta}(u) = 0 \qquad \text{on } L \otimes M, \tag{3.3.2}$$

where u is one of the generators k_{α_i}, e_{α_i}, f_{α_i}, and t of $\mathbf{U}^\imath$. We then note that, for $u_1, u_2 \in \mathbf{U}^\imath$,

$$\begin{aligned} &\Delta(u_1u_2)\Theta^\imath_{\leq N} - \Theta^\imath_{\leq N}\overline{\Delta}(u_1u_2) \\ &\quad = \Delta(u_1)\big(\Delta(u_2)\Theta^\imath_{\leq N} - \Theta^\imath_{\leq N}\overline{\Delta}(u_2)\big) + \big(\Delta(u_1)\Theta^\imath_{\leq N} - \Theta^\imath_{\leq N}\overline{\Delta}(u_1)\big)\overline{\Delta}(u_2). \end{aligned} \tag{3.3.3}$$

Then by an easy induction using (3.3.3), we conclude that (3.3.2) holds for all $u \in \mathbf{U}^\imath$ and $N \geq N_0$. The theorem now follows from (3.3.1). □

Proposition 3.3.2. *We have* $\Theta^\imath\overline{\Theta^\imath} = 1$ *(an identity in* $\widehat{\mathbf{U}}^-$*).*

Proof. By construction, $\Theta^\imath = \sum_{N\geq 0}\Theta^\imath_N$ (with $\Theta^\imath_0 = 1 \otimes 1$) is clearly invertible in $(\mathbf{U}^\imath \otimes \mathbf{U}^-)^\wedge$. Write ${}'\Theta^\imath = (\Theta^\imath)^{-1}$.

Multiplying ${}'\Theta^\imath$ on both sides of the identity in Theorem 3.3.1, we have

$${}'\Theta^\imath\Delta(\overline{u}) = \overline{\Delta(u)}\,{}'\Theta^\imath, \qquad \forall u \in \mathbf{U}^\imath.$$

Applying $^-$ to the above identity and replacing $\overline{u}$ by u, we have

$$\overline{{}'\Theta^\imath}\,\overline{\Delta(u)} = \Delta(\overline{u})\,\overline{{}'\Theta^\imath}, \qquad \forall u \in \mathbf{U}^\imath.$$

Hence $\overline{{}'\Theta^\imath}$ (in place of ${}'\Theta^\imath$) satisfies the same identity in Theorem 3.3.1 as well; note that $\overline{{}'\Theta^\imath} \in (\mathbf{U}^\imath \otimes \mathbf{U}^-)^\wedge$ has constant term $1 \otimes 1$.

By reexamining the proof of Proposition 3.2.3 and especially (3.2.11), we note that the element $\Theta^\imath \in (\mathbf{U}^\imath \otimes \mathbf{U}^-)^\wedge$ (with constant term $1 \otimes 1$) satisfying the identity in Proposition 3.1.2 (and thus Theorem 2.3.1) is unique. Hence we must have $\Theta^\imath = \overline{\Theta^\imath}^{-1}$, and equivalently, $\Theta^\imath \overline{\Theta^\imath} = 1$. □

Recall that $m(\epsilon \otimes 1)\Delta = \imath$ from Corollary 2.2.7, where ϵ is the counit and m denotes the multiplication in $\mathbf{U}$.

Corollary 3.3.3. *The intertwiner Υ can be recovered from the quasi-$\mathcal{R}$-matrix $\Theta^\imath$ as $m(\epsilon \otimes 1)(\Theta^\imath) = \Upsilon$.*

Proof. Applying $m(\epsilon \otimes 1)$ to the identities (3.2.4)-(3.2.7), we obtain an identity in $\widehat{\mathbf{U}}$:

$$\imath(\overline{u})\Big(\sum_{N \geq 0} m(\epsilon \otimes 1)(\Theta^\imath_N)\Big) = \Big(\sum_{N \geq 0} m(\epsilon \otimes 1)(\Theta^\imath_N)\Big)\overline{\imath(u)}, \quad \text{for all } u \in \mathbf{U}^\imath. \tag{3.3.4}$$

The corollary now follows from (3.3.1), (3.3.4) and the uniqueness of Υ in Theorem 2.3.1, as clearly we have $m(\epsilon \otimes 1)(\Theta^\imath_0) = 1$. □

3.4 The bar map on $\mathbf{U}^\imath$-modules

In this section we shall assume all the modules are finite dimensional. Recall the bar map on $\mathbf{U}$ and on its modules is denoted by ψ, and the bar map on $\mathbf{U}^\imath$ is also denoted by $\psi_\imath$. It is also understood that $\psi(u) = \psi(\imath(u))$ for $u \in \mathbf{U}^\imath$.

Definition 3.4.1. A $\mathbf{U}^\imath$-module M equipped with an anti-linear involution $\psi_\imath$ is called *involutive* (or *ı-involutive* to avoid possible ambiguity) if

$$\psi_\imath(um) = \psi_\imath(u)\psi_\imath(m), \qquad \forall u \in \mathbf{U}^\imath, m \in M.$$

Proposition 3.4.2. *Let M be an involutive $\mathbf{U}$-module. Then M is an $\imath$-involutive $\mathbf{U}^\imath$-module with involution $\psi_\imath := \Upsilon \circ \psi$.*

Proof. By Theorem 2.3.1, we have $\imath(\psi_\imath(u))\Upsilon = \Upsilon\psi(u),$ for all $u \in \mathbf{U}^\imath$. By definition the action of $\psi_\imath(u)$ on M is the same as the action of $\imath(\psi_\imath(u))$ on M. Therefore we have

$$\psi_\imath(um) = \Upsilon\psi(um) = \Upsilon\psi(u)\psi(m) = \imath(\psi_\imath(u))\Upsilon\psi(m) = \psi_\imath(u)\psi_\imath(m),$$

for all $u \in \mathbf{U}^\imath$ and $m \in M$.

It remains to verify that $\psi_\imath$ is an involution on M. Indeed, for $m \in M$, we have

$$\psi_\imath(\psi_\imath(m)) = \Upsilon\psi(\Upsilon\psi(m)) = \Upsilon\overline{\Upsilon}\psi(\psi(m)) = \Upsilon\overline{\Upsilon}m = m,$$

where the last identity follows from Corollary 2.3.4. □

Corollary 3.4.3. *Regarded as $\mathbf{U}^\imath$-modules, $L(\lambda)$ and ${}^\omega L(\lambda)$ are $\imath$-involutive, for $\lambda \in \Lambda^+$.*

Remark 3.4.4. We can and will choose $\xi_{-\lambda} \in {}^\omega L(\lambda)$ to be ψ-invariant. It follows that $\xi_{-\lambda}$ is also $\psi_\imath$-invariant, since $\psi_\imath = \Upsilon\psi$ and Υ lies in a completion of $\mathbf{U}^-$ with constant term 1. Because of this, it is more convenient to work with a lowest weight vector instead of a highest weight vector in a finite-dimensional simple $\mathbf{U}$-module.

Recall the quasi-$\mathcal{R}$-matrix $\Theta^\imath$ from (3.1.1). Given an involutive $\mathbf{U}^\imath$-module L and an involutive $\mathbf{U}$-module M, we define $\psi_\imath : L \otimes M \to L \otimes M$ by letting

$$\psi_\imath(l \otimes m) := \Theta^\imath(\psi_\imath(l) \otimes \psi(m)), \qquad \text{for all } l \in L, m \in M. \tag{3.4.1}$$

Proposition 3.4.5. *Let L be an involutive $\mathbf{U}^\imath$-module and let M be an involutive $\mathbf{U}$-module. Then $(L \otimes M, \psi_\imath)$ is an involutive $\mathbf{U}^\imath$-module.*

Proof. For all $l \in L$, $m \in M$, $u \in \mathbf{U}^\imath$, using (3.4.1) twice we have

$$\begin{aligned} \psi_\imath(u(l \otimes m)) &= \Theta^\imath\big(\overline{\Delta(u)}(\psi_\imath(l) \otimes \psi(m))\big) \\ &= \Delta(\overline{u})\Theta^\imath(\psi_\imath(l) \otimes \psi(m)) \\ &= \psi_\imath(u)\, \psi_\imath(l \otimes m). \end{aligned}$$

The second equality in the above computation uses Theorem 3.3.1 and the first equality holds since L and M are involutive modules.

It remains to verify that $\psi_\imath$ is an involution on $L \otimes M$. It is occasionally convenient to use the bar-notation to denote the involution $\psi_\imath \otimes \psi$ on $\mathbf{U}^\imath \otimes \mathbf{U}$ below. Indeed, for $l \in L$ and $m \in M$, using (3.4.1) twice we have

$$\begin{aligned} \psi_\imath(\psi_\imath(l \otimes m)) &= \Theta^\imath(\psi_\imath \otimes \psi).\big(\Theta^\imath(\psi_\imath(l) \otimes \psi(m))\big) \\ &= \Theta^\imath\overline{\Theta^\imath}(\psi_\imath^2(l) \otimes \psi^2(m)) = l \otimes m, \end{aligned}$$

where the last equality follows from Proposition 3.3.2 and the second equality holds since L and M are involutive modules. $\square$

Remark 3.4.6. Given two involutive **U**-modules (M_1, ψ_1) and (M_2, ψ_2), the **U**-module $M_1 \otimes M_2$ is involutive with the involution given by $\Theta \circ (\psi_1 \otimes \psi_2)$, (see [Lu2, 27.3.1] or Proposition 1.4.4). Now there are two natural ways to define an anti-linear involution on the $\mathbf{U}^\imath$-module $M_1 \otimes M_2$:

(i) apply Proposition 3.4.2 to the involutive **U**-module $(M_1 \otimes M_2, \Theta \circ (\psi_1 \otimes \psi_2))$;

(ii) apply Proposition 3.4.5 by regarding M_1 as an $\imath$-involutive $\mathbf{U}^\imath$-module with involution $\Upsilon \circ \psi_1$.

One checks that the resulting involutions on the $\mathbf{U}^\imath$-module $M_1 \otimes M_2$ in two different ways coincide.

The following proposition implies that different bracketings on the tensor product of several involutive **U**-modules give rise to the same $\psi_\imath$. (Recall a similar property holds for Lusztig's bar involution on tensor products of **U**-modules [Lu2].)

Proposition 3.4.7. *Let M_1, …, M_k be involutive* **U***-modules with $k \geq 2$. We have*

$$\psi_\imath(m_1 \otimes \cdots \otimes m_k) = \Theta^\imath(\psi_\imath(m_1 \otimes \cdots \otimes m_{k'}) \otimes \psi(m_{k'+1} \otimes \cdots \otimes m_k)),$$

for any $1 \leq k' < k$.

Proof. Recall $\Theta^\imath = \Upsilon^\Delta \Theta(\Upsilon^{-1} \otimes 1)$. Unraveling the definition $\psi_\imath = \Upsilon \psi$ on $M_1 \otimes \cdots \otimes$

$M_{k'}$, we have

$$\begin{aligned}
&\Theta^{\imath}(\psi_{\imath}(m_1 \otimes \cdots \otimes m_{k'}) \otimes \psi(m_{k'+1} \otimes \cdots \otimes m_k)) \\
&\quad = \Upsilon^{\vartriangle}\Theta(\Upsilon^{-1} \otimes 1)(\Upsilon\psi(m_1 \otimes \cdots \otimes m_{k'}) \otimes \psi(m_{k'+1} \otimes \cdots \otimes m_k)) \\
&\quad = \Upsilon^{\vartriangle}\Theta(\psi(m_1 \otimes \cdots \otimes m_{k'}) \otimes \psi(m_{k'+1} \otimes \cdots \otimes m_k)) \\
&\quad = \Upsilon^{\vartriangle}\psi(m_1 \otimes \cdots \otimes m_{k'} \otimes m_{k'+1} \otimes \cdots \otimes m_k) \\
&\quad = \psi_{\imath}(m_1 \otimes \cdots \otimes m_k).
\end{aligned}$$

The proposition follows. □

Chapter 4

The integrality of Υ and the $\imath$-canonical basis of ${}^{\omega}L(\lambda)$

In this chapter, we first construct the $\imath$-canonical bases for simple $\mathbf{U}$-modules and then for the algebra $\mathbf{U}^{\imath}$ in the rank one case. Then we use the rank one results to study the general higher hank case. We show that the intertwiner Υ is integral and construct the $\imath$-canonical basis for ${}^{\omega}L(\lambda)$ for $\lambda \in \Lambda^{+}$.

4.1 The homomorphism $\pi_{\lambda,\mu}$

Though only the rank one case of the results in this section will be needed in this paper, it is natural and causes no extra work to formulate in the full generality below.

Lemma 4.1.1. *Let* $\lambda \in \Lambda^{+}$. *We have* $\mathbf{U}^{\imath}\xi_{-\lambda} = {}^{\omega}L(\lambda)$ *and* $\mathbf{U}^{\imath}\eta_{\lambda} = L(\lambda)$.

Proof. We shall only prove $\mathbf{U}^\imath \xi_{-\lambda} = {}^\omega L(\lambda)$. The proof for the second identity is similar and will be skipped.

We write $\xi = \xi_{-\lambda}$. Let $h \in {}^\omega L(\lambda)_\mu$. We shall prove $h \in \mathbf{U}^\imath \xi$ by induction on $\mathrm{ht}(\mu + \lambda)$. When $\mathrm{ht}(\mu + \lambda) = 0$, the claim is clear since h must be a scalar multiple of ξ. Thanks to $\mathbf{U}^+ \xi = {}^\omega L(\lambda)$, there exists $y \in \mathbf{U}^+$ such that $y\xi = h$. Writing y as a linear combination of PBW basis elements for $\mathbf{U}^+$ and replacing E_{α_0}, E_{α_i}, $E_{\alpha_{-i}}$ (for all $i \in \mathbb{I}^\imath$) by t, e_{α_i}, f_{α_i} in such a linear combination, respectively, we obtain an element $u = u(y) \in \mathbf{U}^\imath$. Setting $\imath(u) = y + z$ for $z \in \mathbf{U}$, we have $u\xi = h + z\xi$. By construction, $z\xi$ is a $\mathbb{Q}(q)$-linear combination of elements in ${}^\omega L(\lambda)$ of weight lower than h. Hence by the induction hypothesis, we have $z\xi \in \mathbf{U}^\imath \xi$, and so is $h = u\xi - z\xi$. $\square$

Recall from Section 1.3 that ${}^\omega L(\lambda)$ for $\lambda \in \Lambda^+$ is identified with $L(\lambda^\theta) = L(-w_0\lambda)$, ξ_λ is the lowest weight vector of ${}^\omega L(\lambda)$, and η_{λ^θ} is the highest weight vector of $L(\lambda^\theta)$.

Lemma 4.1.2. *For $\lambda \in \Lambda^+$, there is an isomorphism of $\mathbf{U}^\imath$-modules*

$$\mathcal{T} : {}^\omega L(\lambda) \longrightarrow {}^\omega L(\lambda) = L(\lambda^\theta)$$

such that $\mathcal{T}(\xi_\lambda) = \sum_{b \in \mathbf{B}(\lambda)} g_b b^- \eta_{\lambda^\theta}$ where $g_b \in \mathbb{Q}(q)$ and $g_1 = 1$. Moreover, the isomorphism $\mathcal{T}$ is uniquely determined by the image $\mathcal{T}(\xi_\lambda)$.

Proof. Recall the isomorphism $\mathcal{T} = \Upsilon \circ \widetilde{\zeta} \circ T_{w_0} : {}^\omega L(\lambda) \to {}^\omega L(\lambda)$ of $\mathbf{U}^\imath$-modules from Theorem 2.5.1. The existence of $\mathcal{T}$ satisfying the lemma follows by fixing the weight function ζ such that $\mathcal{T}(\xi_\lambda) = \eta_{\lambda^\theta} +$ terms in lower weights.

The uniqueness of such $\mathcal{T}$ follows from Lemma 4.1.1. $\square$

The following proposition can be found in [Lu2, Chapter 25].

Proposition 4.1.3. *Let* λ, $\lambda' \in \Lambda^+$.

1. *There exists a unique homomorphism of* $\mathbf{U}$*-modules*

$$\chi = \chi_{\lambda,\lambda'} : {}^{\omega}L(\lambda + \lambda') \longrightarrow {}^{\omega}L(\lambda) \otimes {}^{\omega}L(\lambda')$$

   such that $\chi(\xi_{-\lambda-\lambda'}) = \xi_{-\lambda} \otimes \xi_{-\lambda'}$.

2. *For* $b \in \mathbf{B}(\lambda + \lambda')$, *we have* $\chi(b^+\xi_{-\lambda-\lambda'}) = \sum_{b_1,b_2} f(b; b_1, b_2) b_1^+\xi_{-\lambda} \otimes b_2^+\xi_{-\lambda'}$, *summed over* $b_1 \in \mathbf{B}(\lambda)$ *and* $b_2 \in \mathbf{B}(\lambda')$, *with* $f(b; b_1, b_2) \in \mathbb{Z}[q]$. *If* $b^+\xi_{-\lambda'} \neq 0$, *then* $f(b; 1, b) = 1$ *and* $f(b; 1, b_2) = 0$ *for any* $b_2 \neq b$. *If* $b^+\xi_{-\lambda'} = 0$, *then* $f(b; 1, b_2) = 0$ *for any* b_2.

3. *There is a unique homomorphism of* $\mathbf{U}$*-modules* $\delta = \delta_\lambda : L(\lambda) \otimes {}^{\omega}L(\lambda) \to \mathbb{Q}(q)$, *where* $\mathbb{Q}(q)$ *is the trivial representation of* $\mathbf{U}$, *such that* $\delta(\eta_\lambda \otimes \xi_{-\lambda}) = 1$. *Moreover, for* $b_1, b_2 \in \mathbf{B}(\lambda)$, $\delta(b_1^-\eta_\lambda \otimes b_2^+\xi_{-\lambda})$ *is equal to* 1 *if* $b_1 = b_2 = 1$ *and is in* $q\mathbb{Z}[q]$ *otherwise. In particular,* $\delta(b_1^-\eta_\lambda \otimes b_2^+\xi_{-\lambda}) = 0$ *if* $|b_1| \neq |b_2|$.

Proposition 4.1.4. *Let* $\lambda,\mu \in \Lambda^+$. *There is a unique homomorphism of* $\mathbf{U}^\imath$*-modules*

$$\pi_{\lambda,\mu} : {}^{\omega}L(\mu^\theta + \mu + \lambda) \longrightarrow {}^{\omega}L(\lambda)$$

such that $\pi_{\lambda,\mu}(\xi_{-\mu^\theta-\mu-\lambda}) = \xi_{-\lambda}$.

Proof. The uniqueness of the map is clear, thanks to Lemma 4.1.1.

We shall prove the existence of $\pi_{\lambda,\mu}$. Recall that any homomorphism of $\mathbf{U}$-modules is naturally a homomorphism of $\mathbf{U}^\imath$-modules. Note that ${}^\omega L(\mu^\theta) = L(-w_0\mu^\theta) = L(\mu)$. Let $\pi_{\lambda,\mu}$ be the composition of the following homomorphisms of $\mathbf{U}^\imath$-modules:

$$\begin{array}{ccccc}
{}^\omega L(\mu^\theta+\mu+\lambda) & \xrightarrow{\chi} & {}^\omega L(\mu^\theta+\mu)\otimes {}^\omega L(\lambda) & \xrightarrow{\chi\otimes \mathrm{id}} & {}^\omega L(\mu^\theta)\otimes {}^\omega L(\mu)\otimes {}^\omega L(\lambda) \\
 & & & & \big\downarrow {\scriptstyle \mathcal{T}\otimes\mathrm{id}\otimes\mathrm{id}} \\
 & \searrow^{\pi_{\lambda,\mu}} & & & L(\mu)\otimes {}^\omega L(\mu)\otimes {}^\omega L(\lambda) \\
 & & & & \big\downarrow {\scriptstyle \delta\otimes\mathrm{id}} \\
 & & & & {}^\omega L(\lambda)
\end{array}$$

where $\mathcal{T}$ is the map from Lemma 4.1.2. First, we have

$$(\chi\otimes \mathrm{id})\chi(\xi_{-\mu^\theta-\mu-\lambda}) = \xi_{-\mu^\theta}\otimes\xi_{-\mu}\otimes\xi_{-\lambda}.$$

Then applying $\mathcal{T}\otimes\mathrm{id}\otimes\mathrm{id}$, by Lemma 4.1.2 we have

$$\begin{aligned}
&(\mathcal{T}\otimes \mathrm{id}\otimes\mathrm{id})(\xi_{-\mu^\theta}\otimes\xi_{-\mu}\otimes\xi_{-\lambda}) \\
&\qquad = \eta_\mu\otimes\xi_{-\mu}\otimes\xi_{-\lambda} + \sum_{1\neq b\in\mathbf{B}(\mu)} g(1;b)b^-\eta_\mu\otimes\xi_\mu\otimes\xi_{-\lambda}.
\end{aligned}$$

Applying $\delta\otimes 1$ to the above identity, we conclude that $\pi_{\lambda,\mu}(\xi_{-\mu^\theta-\mu-\lambda}) = \xi_{-\lambda}$. □

Lemma 4.1.5. *Retain the notation in Proposition 4.1.4. The homomorphism $\pi_{\lambda,\mu}$ commutes with the involution $\psi_\imath$; that is, $\pi_{\lambda,\mu}\psi_\imath = \psi_\imath\pi_{\lambda,\mu}$.*

Proof. In this proof, we write $\pi=\pi_{\lambda,\mu}$, $\xi=\xi_{-\mu^\theta-\mu-\lambda}$, and $\xi'=\xi_{-\lambda}$. Then $\pi(\xi)=\xi'$ by Proposition 4.1.4. An arbitrary element in ${}^\omega L(\mu^\theta+\mu+\lambda)$ is of the form $u\xi$ for some $u\in\mathbf{U}^\imath$, by Lemma 4.1.1. Since ξ and ξ' are both $\psi_\imath$-invariant (see Remark 3.4.4), we have

$$\pi\psi_\imath(u\xi) = \pi\psi_\imath(u)(\xi) = \psi_\imath(u)\pi(\xi) = \psi_\imath(u)\xi'.$$

On the other hand, we have

$$\psi_\imath \pi(u\xi) = \psi_\imath(u\xi') = \psi_\imath(u)\psi_\imath(\xi') = \psi_\imath(u)\xi'.$$

The lemma is proved. □

4.2 The $\imath$-canonical bases at rank one

In this section we shall consider the rank 1 case of the algebra $\mathbf{U}^\imath$, i.e., $\mathbf{U}^\imath = \mathbb{Q}(q)[t]$, the polynomial algebra in t. In order to simplify the notation, we shall write $E = E_{\alpha_0}$, $F = F_{\alpha_0}$, and $K = K_{\alpha_0}$ for the generators of $\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_2)$. By Proposition 2.2.1, we have an algebra embedding $\imath : \mathbb{Q}(q)[t] \to \mathbf{U}_q(\mathfrak{sl}_2)$ such that $\imath(t) = E + qFK^{-1} + K^{-1}$.

In the rank one case, Λ^+ can be canonically identified with $\mathbb{N}$. The finite-dimensional irreducible $\mathbf{U}$-modules are of the form ${}^\omega L(s)$ of lowest weight $-s$, with $s \in \mathbb{N}$. Recall [Lu2] the canonical basis of ${}^\omega L(s)$ consists of $\{E^{(a)}\xi_{-r} \mid 0 \leq a \leq s\}$. We denote by ${}^\omega\mathcal{L}(s)$ the $\mathbb{Z}[q]$-submodule of ${}^\omega L(s)$ generated by $\{E^{(a)}\xi_{-s} \mid 0 \leq a \leq s\}$. Also denote by ${}^\omega L_{\mathcal{A}}(s)$ the $\mathcal{A}$-submodule of ${}^\omega L(s)$ generated by $\{E^{(a)}\xi_{-s} \mid 0 \leq a \leq s\}$.

In the current rank one setting, we can write the intertwiner $\Upsilon = \sum_{k\geq 0} \Upsilon_k$, with $\Upsilon_k = \Upsilon_{k\alpha_0} = c_k F^{(k)}$ for $c_k \in \mathbb{Q}(q)$, and $c_0 = 1$.

Lemma 4.2.1. *We have $\Upsilon_k \in \mathbf{U}^-_{\mathcal{A}}$, for $k \geq 0$.*

Proof. It is equivalent to prove that $c_k \in \mathcal{A} = \mathbb{Z}[q, q^{-1}]$ for all $k \geq 0$. The equation (2.3.1) for $u = t$ implies that

$$qFK^{-1}\Upsilon_{k-2} + K^{-1}\Upsilon_{k-1} + E\Upsilon_k = q^{-1}\Upsilon_{k-2}FK + \Upsilon_{k-1}K + \Upsilon_k E,$$

for all $k \geq 0$. Solving this equation, we have the following recursive formula for c_k:

$$c_k = (-q^{k-1})(q^{-1} - q)(q^{-1}[k-1]c_{k-2} + c_{k-1}), \quad \text{for all } k \geq 1,$$

where $c_{-1} = 0$ and $c_0 = 1$. Then it follows by induction on k that $c_k \in \mathcal{A}$. □

One can show by the recursive relation in the above proof that

$$\Upsilon = \sum_{k \geq 0} q^{k(k+1)} \Big(\prod_{i=1}^{k} (q^{2i-1} - q^{1-2i}) F^{(2k)} + \prod_{i=1}^{k+1} (q^{2i-1} - q^{1-2i}) F^{(2k+1)} \Big). \tag{4.2.1}$$

Proposition 4.2.2. *Let* $s \in \mathbb{N}$.

1. *The* $\mathbf{U}^\imath$*-module* ${}^\omega L(s)$ *admits a unique* $\mathbb{Q}(q)$*-basis* $\mathbf{B}^\imath(s) = \{T^s_a \mid 0 \leq a \leq s\}$ *which satisfies* $\psi_\imath(T^s_a) = T^s_a$ *and*

$$T^s_a = E^{(a)}\xi_{-s} + \sum_{a' < a} t^s_{a;a'} E^{(a')}\xi_{-s}, \tag{4.2.2}$$

where $t^s_{a;a'} \in q\mathbb{Z}[q]$. *(We also set* $t^s_{a;a} = 1$.*)*

2. $\mathbf{B}^\imath(s)$ *forms an* $\mathcal{A}$*-basis for the* $\mathcal{A}$*-lattice* ${}^\omega L_{\mathcal{A}}(s)$.

3. $\mathbf{B}^\imath(s)$ *forms a* $\mathbb{Z}[q]$*-basis for the* $\mathbb{Z}[q]$*-lattice* ${}^\omega \mathcal{L}(s)$.

We call $\mathbf{B}^\imath(s)$ the *ı-canonical basis* of the $\mathbf{U}^\imath$-module ${}^\omega L(s)$.

Proof. Parts (2) and (3) follow immediately from (1) by noting (4.2.2).

It remains to prove (1). Since $\psi_\imath = \Upsilon\psi$ and $\psi(E^{(a)}\xi_{-s}) = E^{(a)}\xi_{-s}$, we have

$$\psi_\imath(E^{(a)}\xi_{-s}) = \Upsilon(E^{(a)}\xi_{-s}) = E^{(a)}\xi_{-s} + \sum_{a' < a} \rho^s_{a;a'} E^{(a')}\xi_{-s},$$

for some scalars $\rho^s_{a;a'} \in \mathcal{A}$. As $\psi_\imath$ is an involution, Part (1) follows by an application of [Lu2, Lemma 24.2.1] to our setting. □

Lemma 4.2.3. *Write* $x \equiv x'$ *if* $x - x' \in q\, {}^{\omega}\mathcal{L}(s)$ *with* $s \in \mathbb{N}$. *The* $\mathbf{U}^{\imath}$*-homomorphism* $\pi = \pi_{s,1} : {}^{\omega}L(s+2) \to {}^{\omega}L(s)$ *from Proposition 4.1.4 satisfies that, for* $a \geq 0$,

$$\pi(E^{(a)}\xi_{-s-2}) \equiv \begin{cases} E^{(a-1)}\xi_{-s}, & \text{if } s = a-1; \\ E^{(a)}\xi_{-s}, & \text{otherwise.} \end{cases}$$

Proof. Recall Proposition 4.1.4, Proposition 4.1.3, and $\pi = (\delta \otimes \mathrm{id})(\mathcal{T} \otimes \mathrm{id} \otimes \mathrm{id})(\chi \otimes \mathrm{id})\chi$. It is easy to compute the action of $\mathcal{T}$ on ${}^{\omega}L(1) = L(1)$ is given by

$$\mathcal{T}(\xi_{-1}) = E\xi_{-1} - (q^{-1} - q)\xi_{-1} \quad \text{and} \quad \mathcal{T}(E\xi_{-1}) = \xi_{-1}.$$

For the map $\delta \otimes \mathrm{id} : L(1) \otimes {}^{\omega}L(1) \otimes {}^{\omega}L(s) \to {}^{\omega}L(s)$, it is easy to compute that

$$\delta(E\xi_{-1} \otimes \xi_{-1}) = 1, \delta(\xi_{-1} \otimes E\xi_{-1}) = -q, \text{ and } \delta(\xi_{-1} \otimes \xi_{-1}) = \delta(E\xi_{-1} \otimes E\xi_{-1}) = 0.$$

For the map $(\chi \otimes \mathrm{id})\chi : {}^{\omega}L(s+2) \to {}^{\omega}L(1) \otimes {}^{\omega}L(1) \otimes {}^{\omega}L(s)$, we have

$$\begin{aligned}
&(\chi \otimes \mathrm{id})\chi(E^{(a)}\xi_{-s-2}) \\
=& \sum_{a_1+a_2+a_3=a} q^{-a_1a_2-a_1a_3-a_2a_3+a_1+sa_1+sa_2} E^{(a_1)}\xi_{-1} \otimes E^{(a_2)}\xi_{-1} \otimes E^{(a_3)}\xi_{-s} \\
=& \xi_{-1} \otimes \xi_{-1} \otimes E^{(a)}\xi_{-s} + q^{-a+1+s}\xi_{-1} \otimes E\xi_{-1} \otimes E^{(a-1)}\xi_{-s} \\
&+ q^{-a+2+s}E\xi_{-1} \otimes \xi_{-1} \otimes E^{(a-1)}\xi_{-s} + q^{2s-2a+4}E\xi_{-1} \otimes E\xi_{-1} \otimes E^{(a-2)}\xi_{-s}.
\end{aligned}$$

Then by applying $\mathcal{T} \otimes \mathrm{id} \otimes \mathrm{id}$, we have

$$
\begin{aligned}
&(\mathcal{T} \otimes \mathrm{id} \otimes \mathrm{id})(\chi \otimes \mathrm{id})\chi(E^{(a)}\xi_{-s-2}) \\
=&E\xi_{-1} \otimes \xi_{-1} \otimes E^{(a)}\xi_{-s} - (q^{-1} - q)\xi_{-1} \otimes \xi_{-1} \otimes E^{(a)}\xi_{-s} \\
&+ q^{-a+2+s}E\xi_{-1} \otimes E\xi_{-1} \otimes E^{(a-1)}\xi_{-s} - q^{-a+1+s}(q^{-1} - q)\xi_{-1} \otimes E\xi_{-1} \otimes E^{(a-1)}\xi_{-s} \\
&+ q^{-a+1+s}\xi_{-1} \otimes \xi_{-1} \otimes E^{(a-1)}\xi_{-s} + q^{2s-2a+4}\xi_{-1} \otimes E\xi_{-1} \otimes E^{(a-2)}\xi_{-s}.
\end{aligned}
$$

At last, by applying $\delta \otimes 1$, we have

$$
\begin{aligned}
&\pi(E^{(a)}\xi_{-s-2}) \\
=&E^{(a)}\xi_{-s} + 0 + 0 + q^{-a+2+s}(q^{-1} - q)E^{(a-1)}\xi_{-s} + 0 - q^{2s-2a+5}E^{(a-2)}\xi_{-s} \\
=&E^{(a)}\xi_{-s} + q^{-a+1+s}E^{(a-1)}\xi_{-s} - q^{-a+3+s}E^{(a-1)}\xi_{-s} - q^{2s-2a+5}E^{(a-2)}\xi_{-s}.
\end{aligned}
$$

The lemma follows. □

We adopt the convention that $T_a^s = 0$ if $s < a$.

Proposition 4.2.4. *The homomorphism* $\pi = \pi_{s,1} : {}^{\omega}L(s+2) \to {}^{\omega}L(s)$ *sends* $\imath$-*canonical basis elements to* $\imath$-*canonical basis elements or zero. More precisely, we have*

$$
\pi(T_a^{s+2}) = \begin{cases} T_{a-1}^s, & \text{if } s = a-1; \\ T_a^s, & \text{otherwise.} \end{cases}
$$

Proof. By Proposition 4.2.2 and Lemma 4.2.3, the difference of the two sides of the identity in the proposition lies in $q\,{}^{\omega}\mathcal{L}(s)$ and hence is a $q\mathbb{Z}[q]$-linear combination of

$\mathbf{B}^\imath(s)$. Lemma 4.1.5 implies that such a difference is fixed by the anti-linear involution $\psi_\imath$ and hence it must be zero. The proposition follows. □

Lemma 4.2.5. *Let $f(t) \in \mathbf{U}^\imath = \mathbb{Q}(q)[t]$ be nonzero. Then $f(t)\xi_{-s} \neq 0$ for all $s \geq \deg f$.*

Proof. We write $\xi = \xi_{-s}$. Write $a = \deg f$, and $f(t) = \sum_{i=0}^{a} c_i t^i$ with $c_a \neq 0$. Then $\imath(f(t)) = c_a E^a + x$, where x is a linear combination of elements in $\mathbf{U}$ with weights lower than that of E^a. It follows that $f(t)\xi = c_a E^a \xi + x\xi \neq 0$ for $s \geq a$, since $c_a E^a \xi \neq 0$ and it cannot be canceled out by $x\xi$ for weight reason. □

Proposition 4.2.6. *There exists a unique $\mathbb{Q}(q)$-basis $\{T_a^{\mathrm{odd}} \mid a \in \mathbb{N}\}$ of $\mathbf{U}^\imath = \mathbb{Q}(q)[t]$ with $\deg T_a^{\mathrm{odd}} = a$ such that*

$$T_a^{\mathrm{odd}} \xi_{-s} = \begin{cases} T_{a-1}^s, & \text{if } s = a-1; \\ T_a^s, & \text{otherwise,} \end{cases} \tag{4.2.3}$$

for each $s \in 2\mathbb{N}+1$. Moreover, we have $\overline{T_a^{\mathrm{odd}}} = T_a^{\mathrm{odd}}$.

Proof. By going over carefully the proof of Lemma 4.1.1 in the rank one case, we can prove the following refinement of Lemma 4.1.1:

($\heartsuit_a^s$) *Whenever $s \geq a$, there exists a* unique *element $T_a(s) \in \mathbf{U}^\imath = \mathbb{Q}(q)[t]$* of degree *$a$ such that $T_a(s)\xi_{-s} = T_a^s$.*

Let $s \geq a$ and take $l \geq 0$. Since $\pi_{s,2l}$ is a $\mathbf{U}^\imath$-homomorphism with $\pi_{s,2l}(\xi_{-(s+2l)}) =$

ξ_{-s} (see Proposition 4.1.4), we have by Proposition 4.2.4

$$T_a(s+2l)\xi_{-s} = \pi_{s,l}(T_a(s+2l)\xi_{-(s+2l)}) \overset{\heartsuit_a^{s+2l}}{=} \pi_{s,l}(T_a^{s+2l}) = T_a^s \overset{\heartsuit_a^s}{=} T_a(s)\xi_{-s}.$$

Hence $T_a(s+2l) = T_a(s)$ for all $l \geq 0$ and $s \geq a$, thanks to the uniqueness of $T_a(s)$ in ($\heartsuit_a^s$). Hence,

$$T_a^{\mathrm{odd}} := \lim_{l \mapsto \infty} T_a(1+2l) \in \mathbf{U}^\imath$$

is well defined. It follows by Proposition 4.2.4 that T_a^{odd} satisfies (4.2.3).

We now show that T_a^{odd} is unique (for a given a). Let $'T_a^{\mathrm{odd}}$ be another such element satisfying (4.2.3). Then $(T_a^{\mathrm{odd}} - {'T_a^{\mathrm{odd}}})\xi_{-s} = 0$ for all $s \in 2\mathbb{N}+1$. It follows by Lemma 4.2.5 that $T_a^{\mathrm{odd}} = {'T_a^{\mathrm{odd}}}$.

Applying $\psi_\imath$ to both sides of (4.2.3) and using Corollary 3.4.3, we conclude that $\overline{T_a^{\mathrm{odd}}}$ satisfies (4.2.3) as well. Hence by the uniqueness we have $T_a^{\mathrm{odd}} = \overline{T_a^{\mathrm{odd}}}$. □

A similar argument gives us the following proposition.

Proposition 4.2.7. *There exists a unique* $\mathbb{Q}(q)$*-basis* $\{T_a^{\mathrm{ev}} \mid a \in \mathbb{N}\}$ *of* $\mathbf{U}^\imath = \mathbb{Q}(q)[t]$ *with* $\deg T_a^{\mathrm{ev}} = a$ *such that*

$$T_a^{\mathrm{ev}}\xi_{-s} = \begin{cases} T_{a-1}^s, & \text{if } a = s+1; \\ T_a^s, & \text{otherwise,} \end{cases}$$

for each $s \in 2\mathbb{N}$. *Moreover, we have* $\overline{T_a^{\mathrm{ev}}} = T_a^{\mathrm{ev}}$.

Clearly we have $T_0^{\mathrm{odd}} = T_0^{\mathrm{ev}} = 1$. It is also easy to see that T_a^{odd} and T_a^{ev} for $a \geq 1$ are both of the form

$$\frac{t^a}{[a]!} + g(t), \quad \text{where } \deg g < a. \tag{4.2.4}$$

We have the following conjectural formula (which is not needed in this paper).

Conjecture 4.2.8. *For $a \in \mathbb{N}$, we have*

$$\begin{aligned}
T_{2a}^{\mathrm{odd}} &= \frac{t(t-[-2a+2])(t-[-2a+4])\cdots(t-[2a-4])(t-[2a-2])}{[2a]!},\\
T_{2a+1}^{\mathrm{odd}} &= \frac{(t-[-2a])(t-[-2a+2])\cdots(t-[2a-2])(t-[2a])}{[2a+1]!},\\
T_{2a}^{\mathrm{ev}} &= \frac{(t-[-2a+1])(t-[-2a+3])\cdots(t-[2a-3])(t-[2a-1])}{[2a]!},\\
T_{2a+1}^{\mathrm{ev}} &= \frac{t(t-[-2a+1])(t-[-2a+3])\cdots(t-[2a-3])(t-[2a-1])}{[2a+1]!}.
\end{aligned}$$

4.3 Integrality at rank one

Lemma 4.3.1. *Let $s, l \in \mathbb{N}$.*

1. *There exists a unique homomorphism of $\mathbf{U}^\imath$-modules*

$$\pi^- = \pi^-_{s,l} : {}^\omega L(s+2l) \longrightarrow L(l) \otimes {}^\omega L(s+l)$$

such that $\pi^-(\xi_{-s-2l}) = \eta_l \otimes \xi_{-s-l}$.

2. *π^- induces a homomorphism of $\mathcal{A}$-modules*

$$\pi^- = \pi^-_{s,l} : {}^\omega L_{\mathcal{A}}(s+2l) \longrightarrow L_{\mathcal{A}}(l) \otimes {}^\omega L_{\mathcal{A}}(s+l).$$

Proof. The uniqueness of such a homomorphism is clear, since $\mathbf{U}^\imath \xi_{-s-2l} = {}^\omega L(s+2l)$ by Lemma 4.1.1.

We let $\pi^- = \mathfrak{T}^{-1}\chi$ be the composition of the $\mathbf{U}^\imath$-homomorphisms

$$ {}^\omega L(s+2l) \xrightarrow{\chi} {}^\omega L(l) \otimes {}^\omega L(s+l) \xrightarrow{\mathfrak{T}^{-1}\otimes 1} L(l) \otimes {}^\omega L\,(s+l), $$

where χ is the $\mathbf{U}^\imath$-homomorphism from Proposition 4.1.3 and $\mathfrak{T} = \Upsilon \circ \widetilde{\zeta} \circ T_{w_0}$ is the $\mathbf{U}^\imath$-homomorphism from Theorem 2.5.1. As the automorphism T_{w_0} preserves the $\mathcal{A}$-forms, we can choose the weight function ζ in (2.5.2) with suitable value $\zeta(l) \in q^{\mathbb{Z}}$ such that $T_{w_0}^{-1}\widetilde{\zeta}^{-1}(\xi_{-l}) = \eta_l$. It follows by (2.5.2) that ζ must be $\mathcal{A}$-valued. Then $\pi^- = \mathfrak{T}^{-1}\chi$ is the map satisfying (1) since $\chi(\xi_{-s-2l}) = \xi_{-l} \otimes \xi_{-s-l}$.

By Proposition 4.1.3 χ maps ${}^\omega L_{\mathcal{A}}(s+2l)$ to $L_{\mathcal{A}}(l) \otimes {}^\omega L_{\mathcal{A}}(s+l)$. It is also well known that T_{w_0} is an automorphism of the $\mathcal{A}$-form ${}^\omega L_{\mathcal{A}}(l)$. By Lemma 4.2.1, $\Upsilon^{-1} = \overline{\Upsilon}$ preserves the $\mathcal{A}$-form ${}^\omega L_{\mathcal{A}}(l)$ as well. As a composition of all these maps, $\pi^- = (\Upsilon \circ \widetilde{\zeta} \circ T_{w_0})^{-1}\chi$ preserves the $\mathcal{A}$-forms, whence (2). □

The following lemma is a variant of Lemma 4.3.1 and can be proved in the same way.

Lemma 4.3.2. *Let $s, l \in \mathbb{N}$.*

1. *There exists a unique homomorphism of $\mathbf{U}^\imath$-modules*

$$ \pi^+ = \pi^+_{s,l} : {}^\omega L(s+2l) \longrightarrow L(s+l) \otimes {}^\omega L(l), $$

such that $\pi^+(\xi_{-s-2l}) = \eta_{s+l} \otimes \xi_{-l}$.

2. π^+ *induces a homomorphism of* $\mathcal{A}$*-modules*

$$\pi^+ : {}^\omega L_\mathcal{A}(s+2l) \longrightarrow L_\mathcal{A}(s+l) \otimes {}^\omega L_\mathcal{A}(l).$$

Recall that a modified $\mathbb{Q}(q)$-algebra $\dot{\mathbf{U}}$ as well as its $\mathcal{A}$-form $\dot{\mathbf{U}}_\mathcal{A}$ are defined in [Lu2, Chapter 23]. Any finite-dimensional unital $\dot{\mathbf{U}}$-module is naturally a weight $\mathbf{U}$-module, and vice versa (see [Lu2, 23.1.4]). In the rank one setting, $\dot{\mathbf{U}}$ (or $\dot{\mathbf{U}}_\mathcal{A}$) is generated by E, F and the idempotents $\mathbf{1}_s$ for $s \in \mathbb{Z}$. As $\dot{\mathbf{U}}$ is naturally a $\mathbf{U}$-bimodule, $\imath(T_a^{\mathrm{odd}})\mathbf{1}_s$ and $\imath(T_a^{\mathrm{ev}})\mathbf{1}_s$ make sense as elements in $\dot{\mathbf{U}}\mathbf{1}_s$, for $a \in \mathbb{N}$ and $s \in \mathbb{Z}$.

Proposition 4.3.3. *1. We have* $\imath(T_a^{\mathrm{odd}})\mathbf{1}_s \in \dot{\mathbf{U}}_\mathcal{A}$*, for all* $a \in \mathbb{N}$*,* $s \in 2\mathbb{Z}+1$*.*

2. We have $\imath(T_a^{\mathrm{ev}})\mathbf{1}_s \in \dot{\mathbf{U}}_\mathcal{A}$*, for all* $a \in \mathbb{N}$*,* $s \in 2\mathbb{Z}$*.*

Proof. (1). Let $s \in 2\mathbb{N}+1$. Fix an arbitrary $a \in \mathbb{N}$. Recall Lusztig's canonical basis $\{b\diamondsuit_{-s}b'\}$ of $\dot{\mathbf{U}}1_{-s}$ in [Lu2, Theorem 25.2.1]. We write

$$\imath(T_a^{\mathrm{odd}})\mathbf{1}_{-s} = \sum_{b,b'} c_{b,b'} b\diamondsuit_{-s}b',$$

for some scalars $c_{b,b'} \in \mathbb{Q}(q)$. Consider the map

$$\pi^- : {}^\omega L_\mathcal{A}(s+2l) \longrightarrow L_\mathcal{A}(l) \otimes {}^\omega L_\mathcal{A}(s+l)$$

in Lemma 4.3.1 for all $l \geq 0$. We have $T_a^{\mathrm{odd}}\xi_{-s-2l} \in {}^\omega L_\mathcal{A}(s+2l)$ by Propositions 4.2.2 and 4.2.6. Therefore we have

$$\imath(T_a^{\mathrm{odd}})\mathbf{1}_{-s}(\eta_l \otimes \xi_{-s-l}) = T_a^{\mathrm{odd}}(\eta_l \otimes \xi_{-s-l}) = \pi^-(T_a^{\mathrm{odd}}\xi_{-s-2l}) \in L_\mathcal{A}(l) \otimes {}^\omega L_\mathcal{A}(s+l).$$

Hence we have (in Lusztig's notation [Lu2, Theorem 25.2.1])

$$\sum_{(b,b')} c_{b,b'}(b\diamondsuit b')_{l,s+l} = \imath(T_a^{\mathrm{odd}})\mathbf{1}_{-s}(\eta_l \otimes \xi_{-s-l}) \in L_{\mathcal{A}}(l) \otimes {}^{\omega}L_{\mathcal{A}}(s+l).$$

Since this holds for all l and $(b\diamondsuit b')_{l,s+l} \neq 0$ for $l \gg 0$, all $c_{b,b'}$ must belong to $\mathcal{A}$. Hence $\imath(T_a^{\mathrm{odd}})\mathbf{1}_{-s} \in \dot{\mathbf{U}}_{\mathcal{A}}$.

By considering the map

$$\pi^+ : {}^{\omega}L_{\mathcal{A}}(s+2l) \longrightarrow L_{\mathcal{A}}(s+l) \otimes {}^{\omega}L_{\mathcal{A}}(l)$$

in Lemma 4.3.2 for all $l \geq 0$, we can show that $\imath(T_a^{\mathrm{odd}})\mathbf{1}_s \in \dot{\mathbf{U}}_{\mathcal{A}}$ for $s \in 2\mathbb{N}+1$ in a similar way. This proves (1). The proof of (2) is similar and will be skipped. □

4.4 The integrality of Υ

Back to the general higher rank case, we are now ready to prove the following crucial lemma with the help of Proposition 4.3.3.

Lemma 4.4.1. *For each $\lambda \in \Lambda^+$, we have $\Upsilon({}^{\omega}L_{\mathcal{A}}(\lambda)) \subseteq {}^{\omega}L_{\mathcal{A}}(\lambda)$.*

Proof. We write $\xi = \xi_{-\lambda}$. We shall prove that $\Upsilon x \in {}^{\omega}L_{\mathcal{A}}(\lambda)$ by induction on the height $\mathrm{ht}(\mu+\lambda)$, for an arbitrary weight vector $x \in {}^{\omega}L_{\mathcal{A}}(\lambda)_{\mu}$. It suffices to consider x of the form $x = E^{(a_1)}_{\alpha_{i_1}} E^{(a_2)}_{\alpha_{i_2}} \cdots E^{(a_s)}_{\alpha_{i_s}} \xi$ which is ψ-invariant.

The base case when $\mathrm{ht}(\mu+\lambda) = 0$ is clear, since $x = \xi$ and $\Upsilon\xi = \xi$.

Denote $x' = E^{(a_2)}_{\alpha_{i_2}} \cdots E^{(a_s)}_{\alpha_{i_s}} \xi \in {}^{\omega}L_{\mathcal{A}}(\lambda)$, and so $x = E^{(a_1)}_{\alpha_{i_1}} x'$. The induction step is divided into three cases depending on whether $i_1 > 0$, $i_1 < 0$, or $i_1 = 0$. Recall that, for any $u \in \mathbf{U}^{\imath}$, the actions of u and $\imath(u)$ on ${}^{\omega}L_{\mathcal{A}}(\lambda)$ are the same by definition.

(1) Assume that $i_1 > 0$ (i.e., $i_1 \in \mathbb{I}^\imath$). Replacing $E_{\alpha_{i_1}}^{(a_1)}$ in the expression of x by $e_{\alpha_{i_1}}^{(a_1)}$, we introduce a new element $x'' = e_{\alpha_{i_1}}^{(a_1)} x'$ which lies in ${}^\omega L_{\mathcal{A}}(\lambda)$ thanks to (2.2.1). Then $y := x'' - x \in {}^\omega L_{\mathcal{A}}(\lambda)$ is a linear combination of elements of weights lower than the weight of x.

We shall consider $\psi_\imath(x'')$ in two ways. By Corollary 3.4.3, ${}^\omega L_{\mathcal{A}}(\lambda)$ is $\imath$-involutive. Since $e_{\alpha_{i_1}}^{(a_1)}$ is $\psi_\imath$-invariant and $\psi_\imath = \Upsilon\psi$, we have

$$\psi_\imath(x'') = \psi_\imath(e_{\alpha_{i_1}}^{(a_1)} x') = e_{\alpha_{i_1}}^{(a_1)} \psi_\imath(x') = e_{\alpha_{i_1}}^{(a_1)} \Upsilon\psi(x').$$

It is well known (cf. [Lu2]) that ψ preserves ${}^\omega L_{\mathcal{A}}(\lambda)$, and so $\psi(x') \in {}^\omega L_{\mathcal{A}}(\lambda)$. Since $\psi(x')$ has weight lower than x, we have $\Upsilon\psi(x') \in {}^\omega L_{\mathcal{A}}(\lambda)$ by the induction hypothesis. Equation (2.2.1) implies that $\psi_\imath(x'') = e_{\alpha_{i_1}}^{(a_1)} \Upsilon\psi(x') \in {}^\omega L_{\mathcal{A}}(\lambda)$.

On the other hand, we have

$$\psi_\imath(x'') = \psi_\imath(x) + \psi_\imath(y) = \Upsilon\psi(x) + \Upsilon\psi(y) = \Upsilon x + \Upsilon\psi(y).$$

Since $\psi(y) \in {}^\omega L_{\mathcal{A}}(\lambda)$ has weight lower than x, we have $\Upsilon\psi(y) \in {}^\omega L_{\mathcal{A}}(\lambda)$ by the induction hypothesis. Therefore we conclude that $\Upsilon x = \psi_\imath(x'') - \Upsilon\psi(y) \in {}^\omega L_{\mathcal{A}}(\lambda)$.

(2) Assume that $i_1 < 0$. In this case, replacing $E_{\alpha_{i_1}}^{(a_1)}$ in the expression of x by $f_{\alpha_{-i_1}}^{(a_1)}$ instead, we consider a new element $x'' = f_{\alpha_{i_1}}^{(a_1)} x'$ which also lies in ${}^\omega L_{\mathcal{A}}(\lambda)$ by (2.2.2). Then an argument parallel to (1) shows that $\Upsilon x \in {}^\omega L_{\mathcal{A}}(\lambda)$.

(3) Now consider the case where $i_1 = 0$. Set $\beta = \sum_{p=2}^{s} a_i \alpha_{i_p} - \lambda$. We decide into two subcases (i)-(ii), depending on whether (α_0, β) is odd or even.

Subcase (i). Assume that (α_0, β) is an odd integer. Replacing $E_{\alpha_{i_1}}^{(a_1)}$ in the expression of x by the element $T_{a_1}^{\mathrm{odd}}$ defined in Proposition 4.2.6, we introduce a new element $x'' = T_{a_1}^{\mathrm{odd}} x'$, which belongs to ${}^\omega L_{\mathcal{A}}(\lambda)$ by Proposition 4.3.3 (as we can write $x'' = T_{a_1}^{\mathrm{odd}} \mathbf{1}_{(\alpha_0,\beta)} x'$). Thanks to (4.2.4), $y := x'' - x \in {}^\omega L_{\mathcal{A}}(\lambda)$ is a linear combination of elements of weights lower than x. Then similarly as in case (1), we have

$$\psi_\imath(x'') = \psi_\imath(T_{a_1}^{\mathrm{odd}} x') = T_{a_1}^{\mathrm{odd}} \psi_\imath(x') = T_{a_1}^{\mathrm{odd}} \Upsilon \psi(x').$$

As in (1), we have $\Upsilon\psi(x') \in {}^\omega L_{\mathcal{A}}(\lambda)$. Recall from Theorem 2.3.1 that $\Upsilon = \sum_\mu \Upsilon_\mu$, where $\Upsilon_\mu \neq 0$ only if $\mu^\theta = \mu$. Note that (α_0, μ) must be an even integer if $\mu^\theta = \mu$. Hence $(\alpha_0, \mu + \beta)$ is always odd whenever $\mu^\theta = \mu$. Therefore by Proposition 4.3.3, we have

$$\psi_\imath(x'') = T_{a_1}^{\mathrm{odd}} \Upsilon \psi(x') = \sum_{\mu : \mu^\theta = \mu} T_{a_1}^{\mathrm{odd}} \mathbf{1}_{(\alpha_0, \mu+\beta)} \Upsilon_\mu \psi(x') \in {}^\omega L_{\mathcal{A}}(\lambda).$$

Now by the induction hypothesis we have $\Upsilon\psi(y) \in {}^\omega L_{\mathcal{A}}(\lambda)$, and hence $\Upsilon x = \psi_\imath(x'') - \Upsilon\psi(y) \in {}^\omega L_{\mathcal{A}}(\lambda)$.

Subcase (ii). Assume that (α_0, β) is an even integer. In this subcase, we replace $E_{\alpha_{i_1}}^{(a_1)}$ by $T_{a_1}^{\mathrm{ev}}$. The rest of the argument is the same as Subcase (i) above.

This completes the induction and the proof of the lemma. □

Theorem 4.4.2. *We have* $\Upsilon_\mu \in \mathbf{U}_{\mathcal{A}}^-$*, for all* $\mu \in \mathbb{N}\Pi$.

Proof. Recall Lusztig's canonical basis $\mathbf{B}$ of $\mathbf{f}$ in Section 1.3 with $\mathbf{B}_\mu = \mathbf{B} \cap \mathbf{f}_\mu$. We write $\Upsilon_\mu = \sum_{b \in \mathbf{B}_\mu} c_b b^-$ for some scalars $c_b \in \mathbb{Q}(q)$. By Lemma 4.4.1, we have

$$\Upsilon_\mu \eta_\lambda = \sum_{b \in \mathbf{B}_\mu} c_b b^- \eta_\lambda \in L_{\mathcal{A}}(\lambda), \qquad \text{for all } \lambda \in \Lambda^+.$$

For an arbitrarily fixed $b \in \mathbf{B}_\mu$, $b^- \eta_\lambda \neq 0$ for λ large enough, and hence we must have $c_b \in \mathcal{A}$. Therefore $\Upsilon_\mu \in \mathbf{U}^-_{\mathcal{A}}$. □

4.5 The $\imath$-canonical basis of ${}^\omega L(\lambda)$

By Corollary 3.4.3, ${}^\omega L(\lambda)$ for $\lambda \in \Lambda^+$ is an $\imath$-involutive $\mathbf{U}^\imath$-module with involution $\psi_\imath = \Upsilon\psi$.

Lemma 4.5.1. *The bar map $\psi_\imath$ preserves the $\mathcal{A}$-form ${}^\omega L_{\mathcal{A}}(\lambda)$, for $\lambda \in \Lambda^+$.*

Proof. It is well known (cf. [Lu2]) that ψ preserves ${}^\omega L_{\mathcal{A}}(\lambda)$. As ${}^\omega L_{\mathcal{A}}(\lambda)$ is preserved by Υ by Lemma 4.4.1, it is also preserved by $\psi_\imath = \Upsilon\psi$. □

Define a partial ordering $\preceq$ on the set $\mathbf{B}(\lambda)$ of canonical basis for $\lambda \in \Lambda^+$ as follows:

$$b_1 \preceq b_2 \quad \Leftrightarrow \quad \text{the images of } |b_1|, |b_2| \text{ are the same in } \Lambda_\theta \text{ and } |b_2| - |b_1| \in \mathbb{N}\Pi. \tag{4.5.1}$$

(Recall that $|b|$ denotes the weight of b as in §1.2).

For any $b \in \mathbf{B}(\lambda)$, we have

$$\psi_\imath(b^+\xi_{-\lambda}) = \Upsilon\psi(b^+\xi_{-\lambda}) = \Upsilon(b^+\xi_{-\lambda}) = \sum_{b' \in \mathbf{B}(\lambda)} \rho_{b;b'} b'^+ \xi_{-\lambda}, \tag{4.5.2}$$

where $\rho_{b;b'} \in \mathcal{A}$ by Theorem 4.4.2. Since Υ lies in a completion of $\mathbf{U}^-$ satisfying $\Upsilon_\mu = 0$ unless $\mu^\theta = \mu$ (see Theorem 2.3.1), we have $\rho_{b;b} = 1$ and $\rho_{b;b'} = 0$ unless $b' \preceq b$. As $\psi_\imath$ is an involution, we can apply [Lu2, Lemma 24.2.1] to our setting to establish the following theorem, which is a generalization of Proposition 4.2.2 in the rank one case.

Theorem 4.5.2. *Let $\lambda \in \Lambda^+$.*

1. *The $\mathbf{U}^\imath$-module ${}^\omega L(\lambda)$ admits a unique basis*

$$\mathbf{B}^\imath(\lambda) := \{T_b^\lambda \mid b \in \mathbf{B}(\lambda)\}$$

which is $\psi_\imath$-invariant and of the form

$$T_b^\lambda = b^+\xi_{-\lambda} + \sum_{b' \prec b} t_{b;b'}^\lambda b'^+\xi_{-\lambda}, \qquad \text{for } t_{b;b'}^\lambda \in q\mathbb{Z}[q].$$

2. *$\mathbf{B}^\imath(\lambda)$ forms an $\mathcal{A}$-basis for the $\mathcal{A}$-lattice ${}^\omega L_{\mathcal{A}}(\lambda)$.*

3. *$\mathbf{B}^\imath(\lambda)$ forms a $\mathbb{Z}[q]$-basis for the $\mathbb{Z}[q]$-lattice ${}^\omega\mathcal{L}(\lambda)$.*

Definition 4.5.3. $\mathbf{B}^\imath(\lambda)$ is called the $\imath$-canonical basis of the $\mathbf{U}^\imath$-module ${}^\omega L(\lambda)$.

Remark 4.5.4. The $\imath$-canonical basis $\mathbf{B}^\imath(\lambda)$ is not homogenous in terms of the weight lattice Λ, though it is homogenous in terms of Λ_θ.

Remark 4.5.5. Lusztig's canonical basis $\mathbf{B}(\lambda)$ is computable algorithmically. As Υ is constructed recursively in §2.4, there is an algorithm to compute the structure constants $\rho_{b;b'}$ in (4.5.2) and then $t_{b;b'}^\lambda$.

Set $t_{b;b}^\lambda = 1$, and $t_{b;b'}^\lambda = 0$ if $b,\ b' \in \mathbf{B}(\lambda)$ satisfy $b' \not\preceq b$. We conjecture that $t_{b;b'}^\lambda \in \mathbb{N}[q]$, for $b, b' \in \mathbf{B}(\lambda)$.

Recall [Lu2, Chapter 27] has developed a theory of based **U**-modules (M, B) (for general quantum groups **U** of finite type). The basis B generates a $\mathbb{Z}[q]$-submodule $\mathcal{M}$ and an $\mathcal{A}$-submodule ${}_{\mathcal{A}}M$ of M. Applying the same argument for Theorem 4.5.2 above, we have established the following.

Theorem 4.5.6. *Let (M, B) be a finite-dimensional based* **U***-module.*

1. *The $\mathbf{U}^{\imath}$-module M admits a unique basis (called $\imath$-canonical basis) $B^{\imath} := \{T_b \mid b \in B\}$ which is $\psi_{\imath}$-invariant and of the form*

$$T_b = b + \sum_{b' \in B, b' \prec b} t_{b;b'} b', \qquad \textit{for } \; t_{b;b'} \in q\mathbb{Z}[q]. \tag{4.5.3}$$

2. *$B^{\imath}$ forms an $\mathcal{A}$-basis for the $\mathcal{A}$-lattice ${}_{\mathcal{A}}M$, and $B^{\imath}$ forms a $\mathbb{Z}[q]$-basis for the $\mathbb{Z}[q]$-lattice $\mathcal{M}$.*

Recall that a tensor product of finite-dimensional simple **U**-modules is a based **U**-module by [Lu2, Theorem 27.3.2]. Theorem 4.5.6 implies now the following.

Theorem 4.5.7. *Let $\lambda_1, \ldots, \lambda_r \in \Lambda^+$. The tensor product of finite-dimensional simple* **U***-modules ${}^{\omega}L(\lambda_1) \otimes \ldots \otimes {}^{\omega}L(\lambda_r)$ admits a unique $\psi_{\imath}$-invariant basis of the form* (4.5.3) *(called $\imath$-canonical basis).*

Chapter 5

The $(\mathbf{U}^\imath, \mathcal{H}_{B_m})$-duality and compatible bar involutions

In this chapter, we recall Schur-Jimbo duality between quantum group $\mathbf{U}$ and Hecke algebra of type A. Then we establish a duality between $\mathbf{U}^\imath$ and Hecke algebra $\mathcal{H}_{B_m}$ of type B acting on $\mathbb{V}^{\otimes m}$, and show the existence of a bar involution on $\mathbb{V}^{\otimes m}$ which is compatible with the bar involutions on $\mathbf{U}^\imath$ and $\mathcal{H}_{B_m}$. This allows a reformulation of Kazhdan-Lusztig theory for Lie algebras of type B/C via the involutive $\mathbf{U}^\imath$-module $\mathbb{V}^{\otimes m}$.

5.1 Schur-Jimbo duality

Recall the notation $\mathbb{I}_{2r}$ from (1.1.1), and we set

$$I = \mathbb{I}_{2r+2} = \Big\{ -r-\frac{1}{2}, \dots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots, r+\frac{1}{2} \Big\}.$$

Let the $\mathbb{Q}(q)$-vector space $\mathbb{V} := \sum_{a \in I} \mathbb{Q}(q) v_a$ be the natural representation of $\mathbf{U}$. We shall call $\mathbb{V}$ the natural representation of $\mathbf{U}^\imath$ (by restriction) as well. For $m \in \mathbb{Z}_{>0}$, the tensor space $\mathbb{V}^{\otimes m}$ is naturally a $\mathbf{U}$-module (and a $\mathbf{U}^\imath$-module) via the coproduct Δ. The $\mathbf{U}$-module $\mathbb{V}$ is involutive with ψ defined by

$$\psi(v_a) := v_a, \qquad \text{for all } a \in I.$$

Then $\mathbb{V}^{\otimes m}$ is an involutive $\mathbf{U}$-module and hence an $\imath$-involutive $\mathbf{U}^\imath$-module by Proposition 3.4.2 and Remark 3.4.6.

We view $f \in I^m$ as a function $f : \{1, \dots, m\} \to I$. For any $f \in I^m$, we define

$$M_f := v_{f(1)} \otimes \cdots \otimes v_{f(m)}.$$

Then $\{M_f \mid f \in I^m\}$ forms a basis for $\mathbb{V}^{\otimes m}$.

Let W_{B_m} be the Coxeter groups of type B_m with simple reflections $s_j, 0 \leq j \leq m$, where the subgroup generated by s_i, $1 \leq i \leq m$ is isomorphic to $W_{A_{m-1}} \cong \mathfrak{S}_m$. The group W_{B_m} and its subgroup S_m act naturally on I^m on the right as follows: for any $f \in I^m$, $1 \leq i \leq m$, we have

$$f \cdot s_j = \begin{cases} (\dots, f(j+1), f(j), \dots), & \text{if } j > 0, \\ (-f(1), f(2), \dots, f(m)), & \text{if } j = 0. \end{cases} \tag{5.1.1}$$

Let $\mathcal{H}_{B_m}$ be the Iwahori-Hecke algebra of type B_m over $\mathbb{Q}(q)$. It is generated by $H_0, H_1, H_2, \ldots, H_{m-1}$, subject to the following relations,

$$
\begin{aligned}
(H_i - q^{-1})(H_i + q) &= 0, && \text{for } i \geq 0,\\
H_i H_{i+1} H_i &= H_{i+1} H_i H_{i+1}, && \text{for } i > 0,\\
H_i H_j &= H_j H_i, && \text{for } |i-j| > 1,\\
H_0 H_1 H_0 H_1 &= H_1 H_0 H_1 H_0.
\end{aligned}
$$

Associated to $\sigma \in W_{B_m}$ with a reduced expression $\sigma = s_{i_1} \cdots s_{i_k}$, we define $H_\sigma := H_{i_1} \cdots H_{i_k}$. The bar involution on $\mathcal{H}_{B_m}$ is the unique anti-linear automorphism defined by $\overline{H_\sigma} = H^{-1}_{\sigma^{-1}}, \overline{q} = q^{-1}$, for all $\sigma \in W_{B_m}$.

There is a right action of the Hecke algebra $\mathcal{H}_{B_m}$ on the $\mathbb{Q}(q)$-vector space $\mathbb{V}^{\otimes m}$ as follows:

$$
M_f H_a = \begin{cases}
q^{-1} M_f, & \text{if } a > 0, f(a) = f(a+1);\\
M_{f \cdot s_a}, & \text{if } a > 0, f(a) < f(a+1);\\
M_{f \cdot s_a} + (q^{-1} - q) M_f, & \text{if } a > 0, f(a) > f(a+1);\\
M_{f \cdot s_0}, & \text{if } a = 0, f(1) > 0;\\
M_{f \cdot s_0} + (q^{-1} - q) M_f, & \text{if } a = 0, f(1) < 0.
\end{cases} \tag{5.1.2}
$$

Identified as the subalgebra generated by $H_1, H_2, \ldots, H_{m-1}$ of $\mathcal{H}_{B_m}$, the Hecke algebra $\mathcal{H}_{A_{m-1}}$ inherits a right action on $\mathbb{V}^{\otimes m}$. Note that the bar involution on $\mathcal{H}_{A_{m-1}}$ is just the restriction of the bar involution on $\mathcal{H}_{B_m}$.

Recall from Section 1.4 the operator $\mathcal{R}$. We define the following operator on $\mathbb{V}^{\otimes m}$ for each $1 \leq i \leq m-1$:

$$\mathcal{R}_i := id^{i-1} \otimes \mathcal{R} \otimes id^{m-i-1} : \mathbb{V}^{\otimes m} \longrightarrow \mathbb{V}^{\otimes m}.$$

The following basic result was due to Jimbo.

Proposition 5.1.1. *[Jim]*

1. *The action of $\mathcal{R}_i^{-1}$ coincides with the action of H_i on $\mathbb{V}^{\otimes m}$ for $1 \leq i \leq m-1$.*
2. *The actions of $\mathbf{U}$ and $\mathcal{H}_{A_{m-1}}$ on $\mathbb{V}^{\otimes m}$ commute with each other, and they form double centralizers.*

5.2 The $(\mathbf{U}^\imath, \mathcal{H}_{B_m})$-duality

Introduce the $\mathbb{Q}(q)$-subspaces of $\mathbb{V}$:

$$\mathbb{V}_- = \bigoplus_{0 \leq i \leq r} \mathbb{Q}(q)(v_{-i-\frac{1}{2}} - q^{-1} v_{i+\frac{1}{2}}),$$

$$\mathbb{V}_+ = \bigoplus_{0 \leq i \leq r} \mathbb{Q}(q)(v_{-i-\frac{1}{2}} + q v_{i+\frac{1}{2}}).$$

Lemma 5.2.1. *The subspace $\mathbb{V}_-$ is a $\mathbf{U}^\imath$-submodule of $\mathbb{V}$ generated by $v_{-\frac{1}{2}} - q^{-1} v_{\frac{1}{2}}$ and $\mathbb{V}_+$ is a $\mathbf{U}^\imath$-submodule of $\mathbb{V}$ generated by $v_{-\frac{1}{2}} + q v_{\frac{1}{2}}$. Moreover, we have $\mathbb{V} = \mathbb{V}_- \oplus \mathbb{V}_+$.*

Proof. Follows by a direct computation. □

Now we fix the function ζ in (2.5.2) with $\zeta(\varepsilon_{-r-\frac{1}{2}}) = 1$ so that

$$\zeta(\varepsilon_{r+\frac{1}{2}-i}) = (-q)^{i-2r-1}, \qquad \text{for } 0 \leq i \leq 2r+1.$$

Let us compute the $\mathbf{U}^\imath$-homomorphism $\mathcal{T} = \Upsilon \circ \widetilde{\zeta} \circ T_{w_0}$ (see Theorem 2.5.1) on the $\mathbf{U}$-module $\mathbb{V}$; we remind that w_0 here is associated to $\mathbf{U}$ instead of W_{B_m} or $W_{A_{m-1}}$.

Lemma 5.2.2. *The $\mathbf{U}^\imath$-isomorphism $\mathcal{T}^{-1}$ on $\mathbb{V}$ acts as a scalar $(-q)id$ on the submodule $\mathbb{V}_-$ and as $q^{-1}id$ on the submodule $\mathbb{V}_+$.*

Proof. First one computes that the action of T_{w_0} on $\mathbb{V}$ is given by

$$T_{w_0}(v_{-r-\frac{1}{2}+i}) = (-q)^{2r+1-i} v_{r+\frac{1}{2}-i}, \qquad \text{for } 0 \le i \le 2r+1.$$

Hence

$$\widetilde{\zeta} \circ T_{w_0}(v_a) = v_{a \cdot s_0}, \quad \text{for all } a \in I. \tag{5.2.1}$$

We have $\Upsilon_{\alpha_0} = -(q^{-1} - q)F_{\alpha_0}$ from the proof of Theorem 2.3.1 in §2.4. Therefore, using $\mathcal{T} = \Upsilon \circ \widetilde{\zeta} \circ T_{w_0}$ we have

$$\mathcal{T}^{-1}(v_{-\frac{1}{2}} - q^{-1} v_{\frac{1}{2}}) = -q(v_{-\frac{1}{2}} - q^{-1} v_{\frac{1}{2}}), \tag{5.2.2}$$

$$\mathcal{T}^{-1}(v_{-\frac{1}{2}} + q v_{\frac{1}{2}}) = q^{-1}(v_{-\frac{1}{2}} + q v_{\frac{1}{2}}). \tag{5.2.3}$$

The lemma now follows from Lemma 5.2.2 since $\mathcal{T}^{-1}$ is a $\mathbf{U}^\imath$-isomorphism. □

We have the following generalization of Schur-Jimbo duality in Proposition 5.1.1.

Theorem 5.2.3 ($(\mathbf{U}^\imath, \mathcal{H}_{B_m})$-duality)**.**

1. *The action of $\mathcal{T}^{-1} \otimes id^{m-1}$ coincides with the action of $H_0 \in \mathcal{H}_{B_m}$ on $\mathbb{V}^{\otimes m}$.*

2. *The actions of $\mathbf{U}^\imath$ and $\mathcal{H}_{B_m}$ on $\mathbb{V}^{\otimes m}$ commute with each other, and they form double centralizers.*

Proof. Part (1) follows from Lemma 5.2.2 and the action (5.1.2) of $H_0 \in \mathcal{H}_{B_m}$ on $\mathbb{V}^{\otimes m}$.

By Proposition 5.1.1, the actions of $\mathbf{U}^\imath$ and $\mathcal{H}_{A_{m-1}}$ on $\mathbb{V}^{\otimes m}$ commute with each other. The action of $\mathbf{U}^\imath$ on $\mathbb{V}^{\otimes m}$ comes from the iterated coproduct $\mathbf{U}^\imath \to \mathbf{U}^\imath \otimes \mathbf{U}^{\otimes m-1}$. Since $\mathcal{T}^{-1} : \mathbb{V} \to \mathbb{V}$ is a $\mathbf{U}^\imath$-homomorphism, we conclude that the actions of $\mathcal{T}^{-1} \otimes \mathrm{id}^{m-1}$ and $\mathbf{U}^\imath$ on $\mathbb{V}^{\otimes m}$ commute with each other. Hence by (1) the actions of $\mathbf{U}^\imath$ and $\mathcal{H}_{B_m}$ on $\mathbb{V}^{\otimes m}$ commute with each other.

The double centralizer property is equivalent to a multiplicity-free decomposition of $\mathbb{V}^{\otimes m}$ as an $\mathbf{U}^\imath \otimes \mathcal{H}_{B_m}$-module. The latter follows by the same multiplicity-free decomposition claim at the specialization $q \mapsto 1$, in which case $\mathbf{U}^\imath$ specializes to the enveloping algebra of $\mathfrak{sl}(r+1) \oplus \mathfrak{gl}(r+1)$ and $\mathcal{H}_{B_m}$ to the group algebra of W_{B_m}. Then $\mathbb{V} = \mathbb{V}_+ \oplus \mathbb{V}_-$ at $q = 1$ becomes the natural module of $\mathfrak{sl}(r+1) \oplus \mathfrak{gl}(r+1)$, on which $s_0 \in W_{B_m}$ acts as $(\mathrm{id}_{\mathbb{V}_+}, -\mathrm{id}_{\mathbb{V}_-})$. A multiplicity-free decomposition of $\mathbb{V}^{\otimes m}$ at $q = 1$ can be established by a standard method with the simples parameterized by ordered pairs of partitions (λ, μ) such that $\ell(\lambda) \leq r+1, \ell(\mu) \leq r+1$ and $|\lambda| + |\mu| = m$. □

Remark 5.2.4. The homomorphism $\mathcal{T}$ (or $\mathcal{T}^{-1}$) is not needed in Theorem 5.2.3(2), as one can check directly that the action of H_0 commutes with the action of $\mathbf{U}^\imath$. However, it is instructive to note that the action of H_0 arises from $\mathcal{T}$ which plays an analogous role as the $\mathcal{R}$-matrix.

Remark 5.2.5. A version of the duality in Theorem 5.2.3 was given in [Gr], where a Schur-type algebra was in place of $\mathbf{U}^\imath$ here. For the applications to BGG categories in Part 2, it is essential for us to work with the "quantum group" $\mathbf{U}^\imath$ equipped with a coproduct.

5.3 Bar involutions and duality

Definition 5.3.1. An element $f \in I^m$ is called *anti-dominant* (or *$\imath$-anti-dominant*), if $0 < f(1) \leq f(2) \leq \ldots \leq f(m)$.

Theorem 5.3.2. *There exists an anti-linear bar involution $\psi_\imath : \mathbb{V}^{\otimes m} \to \mathbb{V}^{\otimes m}$ which is compatible with both the bar involution of $\mathcal{H}_{B_m}$ and the bar involution of $\mathbf{U}^\imath$; that is, for all $v \in \mathbb{V}^{\otimes m}$, $\sigma \in W_{B_m}$, and $u \in \mathbf{U}^\imath$, we have*

$$\psi_\imath(uvH_\sigma) = \psi_\imath(u)\,\psi_\imath(v)\overline{H}_\sigma. \tag{5.3.1}$$

Such a bar involution is unique by requiring $\psi_\imath(M_f) = M_f$ for all $\imath$-anti-dominant f.

Proof. Applying the general construction in §3.4 to our setting, we have an $\imath$-involutive $\mathbf{U}^\imath$-module $(\mathbb{V}^{\otimes m}, \psi_\imath)$; in other words, we have constructed an anti-linear involution $\psi_\imath : \mathbb{V}^{\otimes m} \to \mathbb{V}^{\otimes m}$ which is compatible with the bar involution of $\mathbf{U}^\imath$.

As the $\mathcal{H}_{B_m}$-module $\mathbb{V}^{\otimes m}$ is a direct sum of permutation modules of the form $\mathcal{H}_{B_m}/\mathcal{H}_J$ for various Hecke subalgebras $\mathcal{H}_J$, there exists a unique anti-linear involution on $\mathbb{V}^{\otimes m}$, denoted by $\psi'_\imath$, such that

1. $\psi'_\imath(M_f) = M_f$, if f is $\imath$-anti-dominant;

2. $\psi'_\imath(M_g H_\sigma) = \psi'_\imath(M_g)\overline{H}_\sigma$, for all $g \in I^m$ and $\sigma \in W_{B_m}$.

To prove the compatibility of $\psi_\imath$ with the bar involution of $\mathcal{H}_{B_m}$, it suffices to prove $\psi_\imath$ satisfies the conditions (1)-(2) above; note that it suffices to consider σ in (2) to be the simple reflections.

By the construction in §3.4, the bar involution $\psi_\imath : \mathbb{V}^{\otimes m} \to \mathbb{V}^{\otimes m}$ is given by $\psi_\imath = \Upsilon\psi$, where $\psi : \mathbb{V}^{\otimes m} \to \mathbb{V}^{\otimes m}$ is a bar involution of type A. The following compatibility of the bar involutions in the type A setting is well known (see, e.g., [Br1]) (Here we note that our $\imath$-anti-dominant condition is stronger than the type A anti-dominant condition):

(1′) $\psi(M_f) = M_f$, if f is $\imath$-anti-dominant;

(2′) $\psi(M_g H_\sigma) = M_g \overline{H}_\sigma$, for any $g \in I^m$ and any $H_\sigma \in \mathcal{H}_{A_{m-1}}$.

The $\mathbf{U}$-weight of M_f is $\mathtt{wt}(f) := \sum_{a=1}^m \varepsilon_{f(a)} \in \Lambda$. Define the $\mathbf{U}^\imath$-weight of M_f $\mathtt{wt}_0(f) := \sum_{a=1}^m \overline{\varepsilon}_{f(a)} \in \Lambda_\theta$, which is the image of $\mathtt{wt}(f)$ in $\Lambda_\theta = \Lambda/\Lambda^\theta$ (here we have denoted by $\overline{\varepsilon}_k$ the image of ε_k in Λ_θ). Defined the following partial ordering $\preceq$ on I^m (which is only used in this proof):

$$g \preceq f \quad \Leftrightarrow \quad \mathtt{wt}_0(g) = \mathtt{wt}_0(f) \text{ and } \mathtt{wt}(gf) - \mathtt{wt}(g) \in \mathbb{N}\Pi.$$

Applying the intertwiner $\Upsilon = \sum_{\mu \in \mathbb{N}\Pi} \Upsilon_\mu$ from Theorem 2.3.1, we can write for any $f \in I^m$ that

$$\Upsilon(M_f) = \sum_{g \in I^m} c_g M_g, \qquad \text{for } c_g \in \mathbb{Q}(q).$$

Here the sum can be restricted to g with $\mathtt{wt}_0(g) = \mathtt{wt}_0(f)$ (since $\Upsilon_\mu = 0$ unless $\mu^\theta = \mu$ by Theorem 2.3.1); hence we have $\mathtt{wt}(gf) - \mathtt{wt}(g) \in \mathbb{N}\Pi$ (since $\Upsilon_\mu \in \mathbf{U}^-$). Therefore we have

$$\Upsilon(M_f) = M_f + \sum_{g \prec f} c_g M_g, \qquad \text{for } c_g \in \mathbb{Q}(q).$$

So if f is $\imath$-anti-dominant then we have $\Upsilon(M_f) = M_f$, and thus by Proposition 3.4.2 and (1′) above, $\psi_\imath(M_f) = \Upsilon\psi(M_f) = \Upsilon(M_f) = M_f$. Hence $\psi_\imath$ satisfies Condition (1).

To verify Condition (2) for $\psi_\imath$, let us first consider the special case when $m = 1$. Note that $\psi(v_a) = v_a$ and hence $\psi_\imath(v_a) = \Upsilon(v_a)$ for all a. By Definition 5.3.1, a is $\imath$-anti-dominant if and only if $a > 0$. Thus we have

$$\psi_\imath(v_a) = v_a = \psi'_\imath(v_a), \qquad \text{for } a > 0. \tag{5.3.2}$$

On the other hand, by (5.2.1) and Lemma 5.2.2 we have

$$\begin{aligned} \psi_\imath(v_a) &= \Upsilon(v_a) = \Upsilon \circ \widetilde{\zeta} \circ T_{w_0}(v_{a\cdot s_0}) \\ &= \mathcal{T}(v_{a\cdot s_0}) = v_{a\cdot s_0} H_0^{-1} = \psi'_\imath(v_a), \qquad \text{for } a < 0. \end{aligned} \tag{5.3.3}$$

Hence $\psi_\imath = \psi'_\imath$ and (5.3.1) holds when $m = 1$.

Now consider general $m \in \mathbb{Z}_{>0}$. For $1 \leq i \leq m-1$, by applying Proposition 3.4.2, the identity (2′) above, and Proposition 5.1.1 in a row, we have, for $g \in I^m$,

$$\psi_\imath(M_g H_i) = \Upsilon\psi(M_g H_i) = \Upsilon(\psi(M_g)\overline{H}_i) = \psi_\imath(M_g)\overline{H}_i.$$

When $i = 0$, we write $M_g = v_{g(1)} \otimes M_{g'}$, and hence

$$
\begin{aligned}
\psi_\imath(M_g H_0) &= \psi_\imath(v_{g(1)} H_0 \otimes M_{g'}) \\
&= \Theta^\imath(\psi_\imath(v_{g(1)} H_0) \otimes \psi(M_{g'})) && \text{by Proposition 3.4.7,} \\
&= \Theta^\imath(\psi_\imath(v_{g(1)}) \overline{H}_0 \otimes \psi(M_{g'})) && \text{by (5.3.1) in case } m = 1, \\
&= \Theta^\imath(\psi_\imath(v_{g(1)}) \otimes \psi(M_{g'})) \overline{H}_0 && \text{by Theorem 5.2.3,} \\
&= \psi_\imath(M_g) \overline{H}_0 && \text{by Proposition 3.4.7.}
\end{aligned}
$$

This proves $\psi_\imath = \psi'_\imath$ in general, and hence completes the proof of the compatibility of all these bar involutions.

The uniqueness of $\psi_\imath$ in the theorem follows from the uniqueness of $\psi'_\imath$ above. □

Remark 5.3.3. The anti-linear involution $\psi_\imath$ defined on $\mathbb{V}^{\otimes m}$ from the Hecke algebra side gives rise to the Kazhdan-Lusztig theory of type B. Theorem 5.3.2 implies that the (induced) Kazhdan-Luszig basis on $\mathbb{V}^{\otimes m}$ coincides with its $\imath$-canonical basis (see Theorem 4.5.7). Hence Kazhdan-Lusztig theory of type B can be reformulated from the algebra $\mathbf{U}^\imath$ side through $\psi_\imath$ without referring to the Hecke algebra; see Theorem 11.4.1.

Remark 5.3.4. It follows by (5.3.2) and (5.3.3) that $\{v_{i+\frac{1}{2}}, (v_{-i-\frac{1}{2}} - q^{-1} v_{i+\frac{1}{2}}) \mid 0 \le i \le r\}$ forms a $\psi_\imath$-invariant basis of $\mathbb{V}$. Also $\{v_{i+\frac{1}{2}}, (v_{-i-\frac{1}{2}} + q v_{i+\frac{1}{2}}) \mid 0 \le i \le r\}$ forms another $\psi_\imath$-invariant basis of $\mathbb{V}$, which must be the $\imath$-canonical basis by the characterization in Theorem 4.5.2.

Chapter 6

The quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\jmath)$

In this chapter we consider the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\jmath)$ with $\mathbf{U}$ of type A_{2r}. We formulate the counterparts of the main results from Chapter 2 through Chapter 5 where $\mathbf{U}$ was of type A_{2r+1}. The proofs are similar and often simpler for $\mathbf{U}^\jmath$ since it does not contain a generator t as $\mathbf{U}^\imath$ does, and hence will be omitted almost entirely.

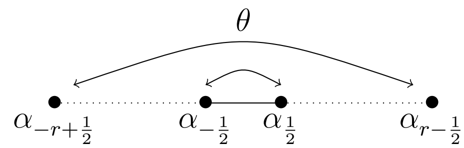
6.1 The coideal subalgebra $\mathbf{U}^\jmath$

We shall write $\mathbb{I} = \mathbb{I}_{2r}$ as given in (1.1.1) in this chapter. We define

$$\mathbb{I}^\jmath = \mathbb{I}^\jmath_r = (\frac{1}{2} + \mathbb{N}) \cap \mathbb{I} = \Big\{\frac{1}{2}, \frac{3}{2}, \dots, r - \frac{1}{2}\Big\}.$$

The Dynkin diagram of type A_{2r} together with the involution θ are depicted as follows:

$A_{2r}:$



The algebra $\mathbf{U}^\imath$ is defined to be the associative algebra over $\mathbb{Q}(q)$ generated by e_{α_i}, f_{α_i}, k_{α_i}, $k_{\alpha_i}^{-1}$, $i \in \mathbb{I}^\imath$, subject to the following relations for $i, j \in \mathbb{I}^\imath$:

$$
\begin{aligned}
k_{\alpha_i} k_{\alpha_i}^{-1} &= k_{\alpha_i}^{-1} k_{\alpha_i} = 1, \\
k_{\alpha_i} k_{\alpha_j} &= k_{\alpha_j} k_{\alpha_i}, \\
k_{\alpha_i} e_{\alpha_j} k_{\alpha_i}^{-1} &= q^{(\alpha_i - \alpha_{-i}, \alpha_j)} e_{\alpha_j}, \\
k_{\alpha_i} f_{\alpha_j} k_{\alpha_i}^{-1} &= q^{-(\alpha_i - \alpha_{-i}, \alpha_j)} f_{\alpha_j}, \\
e_{\alpha_i} f_{\alpha_j} - f_{\alpha_i} e_{\alpha_j} &= \delta_{i,j} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q - q^{-1}}, && \text{if } i, j \neq \frac{1}{2}, \\
e_{\alpha_i}^2 e_{\alpha_j} + e_{\alpha_j} e_{\alpha_i}^2 &= (q + q^{-1}) e_{\alpha_i} e_{\alpha_j} e_{\alpha_i}, && \text{if } |i - j| = 1, \\
f_{\alpha_i}^2 f_{\alpha_j} + f_{\alpha_j} f_{\alpha_i}^2 &= (q + q^{-1}) f_{\alpha_i} f_{\alpha_j} f_{\alpha_i}, && \text{if } |i - j| = 1, \\
e_{\alpha_i} e_{\alpha_j} &= e_{\alpha_j} e_{\alpha_i}, && \text{if } |i - j| > 1, \\
f_{\alpha_i} f_{\alpha_j} &= f_{\alpha_j} f_{\alpha_i}, && \text{if } |i - j| > 1, \\
f_{\alpha_{\frac{1}{2}}}^2 e_{\alpha_{\frac{1}{2}}} + e_{\alpha_{\frac{1}{2}}} f_{\alpha_{\frac{1}{2}}}^2 &= (q + q^{-1}) \Big(f_{\alpha_{\frac{1}{2}}} e_{\alpha_{\frac{1}{2}}} f_{\alpha_{\frac{1}{2}}} - q f_{\alpha_{\frac{1}{2}}} k_{\alpha_{\frac{1}{2}}}^{-1} - q^{-1} f_{\alpha_{\frac{1}{2}}} k_{\alpha_{\frac{1}{2}}} \Big), \\
e_{\alpha_{\frac{1}{2}}}^2 f_{\alpha_{\frac{1}{2}}} + f_{\alpha_{\frac{1}{2}}} e_{\alpha_{\frac{1}{2}}}^2 &= (q + q^{-1}) \Big(e_{\alpha_{\frac{1}{2}}} f_{\alpha_{\frac{1}{2}}} e_{\alpha_{\frac{1}{2}}} - q^{-1} k_{\alpha_{\frac{1}{2}}} e_{\alpha_{\frac{1}{2}}} - q k_{\alpha_{\frac{1}{2}}}^{-1} e_{\alpha_{\frac{1}{2}}} \Big).
\end{aligned}
$$

We introduce the divided powers $e_{\alpha_i}^{(a)} = e_{\alpha_i}^a / [a]!$, $f_{\alpha_i}^{(a)} = f_{\alpha_i}^a / [a]!$.

The following is a counterpart of Lemma 2.1.1.

Lemma 6.1.1. *1. The algebra* $\mathbf{U}^\imath$ *has an involution* $\omega_\imath$ *such that*

$\omega_\jmath(k_{\alpha_i}) = q^{-\delta_{i,\frac{1}{2}}} k_{\alpha_i}^{-1}$, $\omega_\jmath(e_{\alpha_i}) = f_{\alpha_i}$, *and* $\omega_\jmath(f_{\alpha_i}) = e_{\alpha_i}$, *for all* $i \in \mathbb{I}^\jmath$.

2. *The algebra* $\mathbf{U}^\jmath$ *has an anti-involution* $\tau_\jmath$ *such that*

   $\tau_\jmath(e_{\alpha_i}) = e_{\alpha_i}$, $\tau_\jmath(f_{\alpha_i}) = f_{\alpha_i}$, *and* $\tau_\jmath(k_{\alpha_i}) = q^{-\delta_{i,\frac{1}{2}}} k_{\alpha_i}^{-1}$, *for all* $i \in \mathbb{I}^\jmath$.

3. *The algebra* $\mathbf{U}^\jmath$ *has an anti-linear* ($q \mapsto q^{-1}$) *bar involution such that*

   $\overline{k}_{\alpha_i} = k_{\alpha_i}^{-1}$, $\overline{e}_{\alpha_i} = e_{\alpha_i}$, *and* $\overline{f}_{\alpha_i} = f_{\alpha_i}$, *for all* $i \in \mathbb{I}^\jmath$.

   (Sometimes we denote the bar involution on $\mathbf{U}^\jmath$ *by* $\psi_\jmath$*.)*

The following is a counterpart of Proposition 2.2.1, the proof of which relies on [KP, Proposition 4.1] and [Le, Theorem 7.1].

Proposition 6.1.2. *There is an injective* $\mathbb{Q}(q)$*-algebra homomorphism* $\jmath : \mathbf{U}^\jmath \to \mathbf{U}$ *defined by, for all* $i \in \mathbb{I}^\jmath$,

$$k_{\alpha_i} \mapsto K_{\alpha_i} K_{\alpha_{-i}}^{-1},$$

$$e_{\alpha_i} \mapsto E_{\alpha_i} + K_{\alpha_i}^{-1} F_{\alpha_{-i}},$$

$$f_{\alpha_i} \mapsto F_{\alpha_i} K_{\alpha_{-i}}^{-1} + E_{\alpha_{-i}}.$$

Note that $E_{\alpha_i}(K_{\alpha_i}^{-1} F_{\alpha_{-i}}) = q^2 (K_{\alpha_i}^{-1} F_{\alpha_{-i}}) E_{\alpha_i}$ for $i \in \mathbb{I}^\jmath$. We have for $i \in \mathbb{I}^\jmath$,

$$\jmath(e_{\alpha_i}^{(a)}) = \sum_{j=0}^{a} q^{j(a-j)} \frac{(K_{\alpha_i}^{-1} F_{\alpha_{-i}})^j}{[j]!} \frac{E_{\alpha_i}^{a-j}}{[a-j]!}, \tag{6.1.1}$$

$$\jmath(f_{\alpha_i}^{(a)}) = \sum_{j=0}^{a} q^{j(a-j)} \frac{(F_{\alpha_i} K_{\alpha_{-i}}^{-1})^j}{[j]!} \frac{E_{\alpha_{-i}}^{a-j}}{[a-j]!}. \tag{6.1.2}$$

The following is a counterpart of Proposition 2.2.4.

Proposition 6.1.3. *The coproduct* $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ *restricts under the embedding* $\jmath$ *to a* $\mathbb{Q}(q)$*-algebra homomorphism* $\Delta : \mathbf{U}^\jmath \mapsto \mathbf{U}^\jmath \otimes \mathbf{U}$ *such that for all* $i \in \mathbb{I}^\jmath$,

$$\Delta(k_{\alpha_i}) = k_{\alpha_i} \otimes K_{\alpha_i} K_{\alpha_{-i}}^{-1},$$

$$\Delta(e_{\alpha_i}) = 1 \otimes E_{\alpha_i} + e_{\alpha_i} \otimes K_{\alpha_i}^{-1} + k_{\alpha_i}^{-1} \otimes K_{\alpha_i}^{-1} F_{\alpha_{-i}},$$

$$\Delta(f_{\alpha_i}) = k_{\alpha_i} \otimes F_{\alpha_i} K_{\alpha_{-i}}^{-1} + f_{\alpha_i} \otimes K_{\alpha_{-i}}^{-1} + 1 \otimes E_{\alpha_{-i}}.$$

Similarly, the counit ϵ *of* $\mathbf{U}$ *induces a* $\mathbb{Q}(q)$*-algebra homomorphism* $\epsilon : \mathbf{U}^\jmath \to \mathbb{Q}(q)$ *such that* $\epsilon(e_{\alpha_i}) = \epsilon(f_{\alpha_i}) = 0$ *and* $\epsilon(k_{\alpha_i}) = 1$ *for all* $i \in \mathbb{I}^\jmath$.

It follows by Proposition 6.1.3 that $\mathbf{U}^\jmath$ is a (right) coideal subalgebra of $\mathbf{U}$. The map $\Delta : \mathbf{U}^\jmath \to \mathbf{U}^\jmath \otimes \mathbf{U}$ will be called the coproduct of $\mathbf{U}^\jmath$ and $\epsilon : \mathbf{U}^\jmath \to \mathbb{Q}(q)$ will be called the counit of $\mathbf{U}^\jmath$. The coproduct $\Delta : \mathbf{U}^\jmath \to \mathbf{U}^\jmath \otimes \mathbf{U}$ is coassociative, i.e., $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta : \mathbf{U}^\jmath \to \mathbf{U}^\jmath \otimes \mathbf{U} \otimes \mathbf{U}$. The counit map ϵ makes $\mathbb{Q}(q)$ a (trivial) $\mathbf{U}^\jmath$-module. Let $m : \mathbf{U} \otimes \mathbf{U} \to \mathbf{U}$ denote the multiplication map. Just as in Corollary 2.2.7, we have $m(\epsilon \otimes 1)\Delta = \jmath : \mathbf{U}^\jmath \longrightarrow \mathbf{U}$.

6.2 The intertwiner Υ and the isomorphism $\mathcal{T}$

As in §2.3, we let $\widehat{\mathbf{U}}$ be the completion of the $\mathbb{Q}(q)$-vector space $\mathbf{U}$. We have the obvious embedding of $\mathbf{U}$ into $\widehat{\mathbf{U}}$. By continuity the $\mathbb{Q}(q)$-algebra structure on $\mathbf{U}$ extends to the $\mathbb{Q}(q)$-algebra structure on $\widehat{\mathbf{U}}$. The bar involution $^-$ on $\mathbf{U}$ extends by continuity to an anti-linear involution on $\widehat{\mathbf{U}}$, which is also denoted by $^-$. The following is a counterpart of Theorem 2.3.1.

Theorem 6.2.1. *There is a unique family of elements* $\Upsilon_\mu \in \mathbf{U}^-_{-\mu}$ *for* $\mu \in \mathbb{N}\Pi$ *such that* $\Upsilon = \sum_\mu \Upsilon_\mu \in \widehat{\mathbf{U}}$ *intertwines the bar involutions on* $\mathbf{U}^\jmath$ *and* $\mathbf{U}$ *via the embedding* $\jmath$ *and* $\Upsilon_0 = 1$*; that is,* Υ *satisfies the following identity (in* $\widehat{\mathbf{U}}$*):*

$$\jmath(\overline{u})\Upsilon = \Upsilon\ \overline{\jmath(u)}, \qquad \text{for all } u \in \mathbf{U}^\jmath. \tag{6.2.1}$$

Moreover, $\Upsilon_\mu = 0$ *unless* $\mu^\theta = \mu$*.*

The following is a counterpart of Corollary 2.3.4.

Corollary 6.2.2. *We have* $\Upsilon \cdot \overline{\Upsilon} = 1$*.*

Consider a function ζ on Λ such that

$$\begin{aligned} \zeta(\mu + \alpha_i) &= -q^{(\alpha_i - \alpha_{-i}, \mu + \alpha_i)} \zeta(\mu), \\ \zeta(\mu + \alpha_{-i}) &= -q^{(\alpha_{-i}, \mu + \alpha_{-i}) - (\alpha_i, \mu)} \zeta(\mu), \end{aligned} \tag{6.2.2}$$

for all $\mu \in \Lambda$, $i \in \mathbb{I}^\jmath$. Such ζ exists. For any $\mathbf{U}$-module M, define a $\mathbb{Q}(q)$-linear map $\widetilde{\zeta} : M \to M$ by

$$\widetilde{\zeta}(m) = \zeta(\mu) m, \qquad \text{for all } m \in M_\mu.$$

Let w_0 denote the longest element of the Weyl group W of type A_{2r}. As in Section 1.3 we denote by T_{w_0} the braid group element. The following is a counterpart of Theorem 2.5.1.

Theorem 6.2.3. *Given any finite-dimensional* $\mathbf{U}$*-module* M*, the composition map*

$$\mathcal{T} := \Upsilon \circ \widetilde{\zeta} \circ T_{w_0} : M \longrightarrow M$$

is an isomorphism of $\mathbf{U}^\jmath$*-modules.*

6.3 Quasi-$\mathcal{R}$ matrix on $\mathbf{U}^\jmath$

It follows by Theorem 6.2.1 that Υ is a well-defined operator on finite-dimensional $\mathbf{U}$-modules. For any finite-dimensional $\mathbf{U}$-modules M and M', we shall use the formal notation $\Upsilon^\vartriangle$ to denote the well-defined action of Υ on $M \otimes M'$. Hence the operator

$$\Theta^\jmath := \Upsilon^\vartriangle \Theta(\Upsilon^{-1} \otimes 1) \tag{6.3.1}$$

on $M \otimes M'$ is well defined. Define

$$\overline{\Delta} : \mathbf{U}^\jmath \longrightarrow \mathbf{U}^\jmath \otimes \mathbf{U}$$

by letting $\overline{\Delta}(u) = \overline{\Delta(\overline{u})}$, for all $u \in \mathbf{U}^\jmath$.

The construction in §3.2 carries over with little modification, and we will be sketchy. For each $N \in \mathbb{N}$, we have a truncation map $tr_{\leq N}$ on $\mathbf{U}^-$ as in (3.2.1). Then the same formulas as in (3.2.2) and (3.2.3) give us $\Theta^\jmath_{\leq N}$ and $\Theta^\jmath_N$ in $\mathbf{U} \otimes \mathbf{U}^-$. The following is a counterpart of Proposition 3.2.3.

Proposition 6.3.1. *For any $N \in \mathbb{N}$, we have $\Theta^\jmath_N \in \jmath(\mathbf{U}^\jmath) \otimes \mathbf{U}^-$.*

Proposition 6.3.1 allows us to make sense of $\jmath^{-1}(\Theta^\jmath_N) \in \mathbf{U}^\jmath \otimes \mathbf{U}$ for each N. For any finite-dimensional $\mathbf{U}$-modules M and M', the action of $\jmath^{-1}(\Theta^\jmath_N)$ coincides with the action of $\Theta^\jmath_N$ on $M \otimes M'$. **As we will only need to use $\jmath^{-1}(\Theta^\jmath_N) \in \mathbf{U}^\jmath \otimes \mathbf{U}$ rather than $\Theta^\jmath_N$, we will simply write $\Theta^\jmath_N$ for $\jmath^{-1}(\Theta^\jmath_N)$ and regard $\Theta^\jmath_N \in \mathbf{U}^\jmath \otimes \mathbf{U}$ from now on.** Similarly, it is now understood that $\Theta^\jmath_{\leq N} = \sum_{r=0}^N \Theta^\jmath_r \in \mathbf{U}^\jmath \otimes \mathbf{U}$. The actions of $\sum_{N \geq 0} \Theta^\jmath_N$ and of $\Theta^\jmath$ coincide on any tensor product of finite-dimensional

U-modules. From now on, we may and shall identify

$$\Theta^\jmath = \sum_{N\geq 0} \Theta^\jmath_N$$

(or alternatively, use this as a normalized definition of $\Theta^\jmath$) as an element in a completion $(\mathbf{U}^\jmath \otimes \mathbf{U}^-)^\wedge$ of $\mathbf{U}^\jmath \otimes \mathbf{U}^-$.

The following is a counterpart of Theorem 3.3.1.

Theorem 6.3.2. *Let L be a finite-dimensional $\mathbf{U}^\jmath$-module and let M be a finite-dimensional $\mathbf{U}$-modules. Then as linear operators on $L \otimes M$, we have*

$$\Delta(u)\Theta^\jmath = \Theta^\jmath \overline{\Delta}(u), \qquad \textit{for } u \in \mathbf{U}^\jmath.$$

The following is the counterpart of Proposition 3.3.2.

Proposition 6.3.3. *We have $\Theta^\jmath \overline{\Theta^\jmath} = 1$.*

The following is the counterpart of Corollary 3.3.3.

Corollary 6.3.4. *We have $m(\epsilon \otimes 1)\Theta^\jmath = \Upsilon$.*

6.4 The $\jmath$-involutive modules

In this chapter we shall assume all modules are finite dimensional. Recall the bar map on **U** and its modules is also denoted by ψ, and the bar map on $\mathbf{U}^\jmath$ is also denoted by $\psi_\jmath$. It is also understood that $\psi(u) = \psi(\jmath(u))$ for $u \in \mathbf{U}^\jmath$.

Definition 6.4.1. A $\mathbf{U}^\jmath$-module M equipped with an anti-linear involution $\psi_\jmath$ is called *involutive* (or *$\jmath$-involutive*) if

$$\psi_\jmath(um) = \psi_\jmath(u)\psi_\jmath(m), \qquad \forall u \in \mathbf{U}^\jmath, m \in M.$$

The following is a counterpart of Proposition 3.4.2.

Proposition 6.4.2. *Let M be an involutive $\mathbf{U}$-module. Then M is an $\jmath$-involutive $\mathbf{U}^\jmath$-module with involution $\psi_\jmath := \Upsilon \circ \psi$.*

The following is a counterpart of Corollary 3.4.3.

Corollary 6.4.3. *Let $\lambda \in \Lambda^+$. Regarded as $\mathbf{U}^\jmath$-modules, $L(\lambda)$ and ${}^\omega L(\lambda)$ are $\jmath$-involutive.*

Given an involutive $\mathbf{U}^\jmath$-module L and an involutive $\mathbf{U}$-module M, we define $\psi_\jmath : L \otimes M \to L \otimes M$ by letting

$$\psi_\jmath(l \otimes m) := \Theta^\jmath(\psi_\jmath(l) \otimes \psi(m)), \qquad \text{for all } l \in L, m \in M. \tag{6.4.1}$$

The following is a counterpart of Proposition 3.4.5.

Proposition 6.4.4. *Let L be an involutive $\mathbf{U}^\jmath$-module and let M be an involutive $\mathbf{U}$-module. Then $(L \otimes M, \psi_\jmath)$ is an involutive $\mathbf{U}^\jmath$-module.*

Remark 6.4.5. Given two involutive $\mathbf{U}$-modules (M_1, ψ_1) and (M_2, ψ_2), the two different ways, via Proposition 6.4.2 or Proposition 6.4.4, of defining an $\jmath$-involutive $\mathbf{U}^\jmath$-module structure on $M_1 \otimes M_2$ coincide; compare with Remark 3.4.6.

The following proposition, which is a counterpart of Proposition 3.4.7, implies that different bracketings on the tensor product of several involutive **U**-modules give rise to the same $\psi_\jmath$.

Proposition 6.4.6. *Let $M_1, \ldots, M_k$ be involutive* **U***-modules with $k \geq 2$. We have*

$$\psi_\jmath(m_1 \otimes \cdots \otimes m_k) = \Theta^\jmath(\psi_\jmath(m_1 \otimes \cdots \otimes m_{k'}) \otimes \psi(m_{k'+1} \otimes \cdots \otimes m_k)),$$

for any $1 \leq k' < k$.

6.5 Integrality of Υ

Similar to Lemma 4.1.1 for $\mathbf{U}^\imath$, we can show that

$$\mathbf{U}^\jmath \xi_{-\lambda} = {}^\omega L(\lambda), \qquad \mathbf{U}^\jmath \eta_\lambda = L(\lambda).$$

The following is a counterpart of Lemma 4.1.2.

Lemma 6.5.1. *For any $\lambda \in \Lambda^+$, there is a unique isomorphism of* $\mathbf{U}^\jmath$*-modules*

$$\mathcal{T} : {}^\omega L(\lambda) \longrightarrow {}^\omega L(\lambda) = L(\lambda^\theta),$$

such that $\mathcal{T}(\xi_\lambda) = \sum_{b \in \mathbf{B}(\lambda)} g_b b^- \eta_{\lambda^\theta}$ for $g_b \in \mathbb{Q}(q)$ and $g_1 = 1$.

Proposition 6.5.2. *Let $\lambda, \mu \in \Lambda^+$. There is a unique homomorphism of* $\mathbf{U}^\jmath$*-modules*

$$\pi_{\lambda,\mu} : {}^\omega L(\mu^\theta + \mu + \lambda) \longrightarrow {}^\omega L(\lambda),$$

such that $\pi_{\lambda,\mu}(\xi_{-\mu^\theta-\mu-\lambda}) = \xi_{-\lambda}$.

Recall that ${}^\omega L(\mu^\theta + \mu + \lambda)$ and ${}^\omega L(\lambda)$ are both $\jmath$-involutive $\mathbf{U}^\jmath$-modules with $\psi_\jmath = \Upsilon \circ \psi$. Similar to Lemma 4.1.5, the $\mathbf{U}^\jmath$-homomorphism $\pi_{\lambda,\mu}$ commutes with the bar involution $\psi_\jmath$, i.e., $\pi_{\lambda,\mu}\psi_\jmath = \psi_\jmath\pi_{\lambda,\mu}$.

The following is a counterpart of Lemma 4.4.1, with a much easier proof. Indeed, since the identities (6.1.1) and (6.1.2) give us all the divided powers we need, we can bypass the careful study of the rank one case as in §4.2 for $\mathbf{U}^\imath$.

Lemma 6.5.3. *For each $\lambda \in \Lambda^+$, we have $\Upsilon({}^\omega L_{\mathcal{A}}(\lambda)) \subseteq {}^\omega L_{\mathcal{A}}(\lambda)$.*

The following is a counterpart of Theorem 4.4.2.

Theorem 6.5.4. *We have $\Upsilon_\mu \in \mathbf{U}^-_{\mathcal{A}}$, for all $\mu \in \mathbb{N}\Pi$.*

6.6 The $\jmath$-canonical basis of ${}^\omega L(\lambda)$

The following is a counterpart of Lemma 4.5.1, which now follows from Theorem 6.5.4 and Proposition 6.4.2. Note that we do not need the input from the rank one case here.

Lemma 6.6.1. *The bar map $\psi_\jmath$ preserves the $\mathcal{A}$-form ${}^\omega L_{\mathcal{A}}(\lambda)$, for $\lambda \in \Lambda^+$.*

Recall a partial ordering $\preceq$ on the set $\mathbf{B}(\lambda)$ of canonical basis for $\lambda \in \Lambda^+$ from (4.5.1). For any $b \in \mathbf{B}(\lambda)$, recalling $\psi_\jmath = \Upsilon\psi$, we have

$$\psi_\jmath(b^+\xi_{-\lambda}) = \Upsilon(b^+\xi_{-\lambda}) = \sum_{b' \in \mathbf{B}(\lambda)} \rho_{b;b'} b'^+\xi_{-\lambda} \tag{6.6.1}$$

where $\rho_{b;b'} \in \mathcal{A}$ by Theorem 6.5.4. Since Υ lies in a completion of $\mathbf{U}^-$ satisfying $\Upsilon_\mu = 0$ unless $\mu^\theta = \mu$ (see Theorem 6.2.1), we have $\rho_{b;b} = 1$ and $\rho_{b;b'} = 0$ unless $b' \preceq b$. Since $\psi_\jmath$ is an involution, we can apply [Lu2, Lemma 24.2.1] to our setting to establish the following counterpart of Theorem 4.5.2.

Theorem 6.6.2. *Let $\lambda \in \Lambda^+$. The $\mathbf{U}^\jmath$-module ${}^\omega L(\lambda)$ admits a unique basis*

$$\mathbf{B}^\jmath(\lambda) := \{T_b^\lambda \mid b \in \mathbf{B}(\lambda)\}$$

which is $\psi_\jmath$-invariant and of the form

$$T_b^\lambda = b^+\xi_{-\lambda} + \sum_{b' \prec b} t_{b;b'}^\lambda b'^+\xi_{-\lambda}, \qquad \textit{for } \ t_{b;b'}^\lambda \in q\mathbb{Z}[q].$$

Definition 6.6.3. $\mathbf{B}^\jmath(\lambda)$ is called the $\jmath$-canonical basis of the $\mathbf{U}^\jmath$-module ${}^\omega L(\lambda)$.

Similar to Section 4.5, we can generalize Theorem 6.6.2 to any based $\mathbf{U}$-module (M, B). Thus we establish the following counterparts of Theorem 4.5.6 and 4.5.7.

Theorem 6.6.4. *Let (M, B) be a finite-dimensional based $\mathbf{U}$-module.*

1. *The $\mathbf{U}^\jmath$-module M admits a unique basis (called $\jmath$-canonical basis) $B^\jmath := \{T_b \mid b \in B\}$ which is $\psi_\jmath$-invariant and of the form*

$$T_b = b + \sum_{b' \in B, b' \prec b} t_{b;b'} b', \qquad \textit{for } \ t_{b;b'} \in q\mathbb{Z}[q]. \tag{6.6.2}$$

2. *$B^\jmath$ forms an $\mathcal{A}$-basis for the $\mathcal{A}$-lattice ${}_\mathcal{A}M$, and $B^\jmath$ forms a $\mathbb{Z}[q]$-basis for the $\mathbb{Z}[q]$-lattice $\mathcal{M}$.*

Theorem 6.6.5. *Let $\lambda_1, \dots, \lambda_r \in \Lambda^+$. The tensor product of finite-dimensional simple* $\mathbf{U}$*-modules $^{\omega}L(\lambda_1) \otimes \dots \otimes {}^{\omega}L(\lambda_r)$ admits a unique $\psi_\jmath$-invariant basis of the form* (6.6.2) *(called $\jmath$-canonical basis).*

6.7 The $(\mathbf{U}^\jmath, \mathcal{H}_{B_m})$-duality

Again in this section $\mathbf{U}$ is of type A_{2r} with simple roots parametrized by $\mathbb{I}_{2r}$ in (1.1.1). Recall the notation $\mathbb{I}_{2r+1}$ from (1.1.1), and we set

$$I = \mathbb{I}_{2r+1} = \{-r, \dots, -1, 0, 1, \dots, r\}.$$

Let the $\mathbb{Q}(q)$-vector space $\mathbb{V} := \sum_{a \in I} \mathbb{Q}(q) v_a$ be the natural representation of $\mathbf{U}$, hence a $\mathbf{U}^\jmath$-module. We shall call $\mathbb{V}$ the natural representation of $\mathbf{U}^\jmath$ as well. For $m \in \mathbb{Z}_{>0}$, $\mathbb{V}^{\otimes m}$ becomes a natural $\mathbf{U}$-module (hence a $\mathbf{U}^\jmath$-module) via the iteration of the coproduct Δ. Note that $\mathbb{V}$ is an involutive $\mathbf{U}$-module with ψ defined as

$$\psi(v_a) := v_a, \quad \text{for all } a \in I.$$

Therefore $\mathbb{V}^{\otimes m}$ is an involutive $\mathbf{U}$-module and hence a $\jmath$-involutive $\mathbf{U}^\jmath$-module by Proposition 6.4.6.

For any $f \in I^m$, we define $M_f = v_{f(1)} \otimes \cdots \otimes v_{f(m)}$. The Weyl group W_{B_m} acts on I^m by (5.1.1) as before. Now the Hecke algebra $\mathcal{H}_{B_m}$ acts on the $\mathbb{Q}(q)$-vector space

$\mathbb{V}^{\otimes m}$ as follows:

$$M_f H_a = \begin{cases} q^{-1} v_f, & \text{if } a > 0, f(a) = f(a+1); \\ M_{f \cdot s_a}, & \text{if } a > 0, f(a) < f(a+1); \\ M_{f \cdot s_a} + (q^{-1} - q) M_f, & \text{if } a > 0, f(a) > f(a+1); \\ M_{f \cdot s_0}, & \text{if } a = 0, f(1) > 0; \\ M_{f \cdot s_0} + (q^{-1} - q) M_f, & \text{if } a = 0, f(1) < 0; \\ q^{-1} M_f, & \text{if } a = 0, f(1) = 0. \end{cases} \tag{6.7.1}$$

Identified as the subalgebra generated by $H_1, H_2, \ldots, H_{m-1}$ of $\mathcal{H}_{B_m}$, the Hecke algebra $\mathcal{H}_{A_{m-1}}$ inherits a right action on $\mathbb{V}^{\otimes m}$. The Schur-Jimbo duality as formulated in Proposition 5.1.1 remains to be valid in the current setting, i.e., the actions of $\mathbf{U}$ and $\mathcal{H}_{A_{m-1}}$ on $\mathbb{V}^{\otimes m}$ commute with each other and they form double centralizers.

Introduce the $\mathbb{Q}(q)$-subspaces of $\mathbb{V}$:

$$\mathbb{V}_- = \bigoplus_{1 \leq i \leq r} \mathbb{Q}(q)(v_{-i} - q^{-1} v_i),$$

$$\mathbb{V}_+ = \mathbb{Q}(q) v_0 \bigoplus \bigoplus_{1 \leq i \leq r} \mathbb{Q}(q)(v_{-i} + q v_i).$$

The following is a counterpart of Lemma 5.2.1.

Lemma 6.7.1. *$\mathbb{V}_-$ is a $\mathbf{U}^\jmath$-submodule of $\mathbb{V}$ generated by $v_{-1} - q^{-1} v_1$ and $\mathbb{V}_+$ is a $\mathbf{U}^\jmath$-submodule of $\mathbb{V}$ generated by v_0. Moreover, we have $\mathbb{V} = \mathbb{V}_- \oplus \mathbb{V}_+$.*

Now we fix ζ in (6.2.2) such that $\zeta(\varepsilon_{-r}) = 1$. It follows that

$$\zeta(\varepsilon_{r-i}) = \begin{cases} (-q)^{-2r+i}, & \text{if } i \neq r; \\ q \cdot (-q)^{-r}, & \text{if } i = r. \end{cases}$$

Let us compute the $\mathbf{U}^\jmath$-homomorphism $\mathcal{T} = \Upsilon \circ \widetilde{\zeta} \circ T_{w_0}$ (see Theorem 6.2.3) on the $\mathbf{U}$-module $\mathbb{V}$; we remind that w_0 here is associated to $\mathbf{U}$ instead of W_{B_m} or $W_{A_{m-1}}$.

Lemma 6.7.2. *The $\mathbf{U}^\jmath$-isomorphism $\mathcal{T}^{-1}$ on $\mathbb{V}$ acts as a scalar $(-q)id$ on the submodule $\mathbb{V}_-$ and as $q^{-1}id$ on the submodule $\mathbb{V}_+$.*

Proof. First one computes that the action of T_{w_0} on $\mathbb{V}$ is given by

$$T_{w_0}(v_{-r+i}) = (-q)^{2r-i} v_{r-i}, \qquad \text{for } 0 \leq i \leq 2r.$$

Hence

$$\widetilde{\zeta} \circ T_{w_0}(v_a) = \begin{cases} v_{a \cdot s_0}, & \text{if } a \neq 0; \\ q v_0, & \text{if } a = 0. \end{cases} \tag{6.7.2}$$

One computes that $\Upsilon_{\alpha_{-\frac{1}{2}} + \alpha_{\frac{1}{2}}} = -(q^{-1} - q) F_{\alpha_{\frac{1}{2}}} F_{\alpha_{-\frac{1}{2}}}$. Therefore using $\mathcal{T} = \Upsilon \circ \widetilde{\zeta} \circ T_{w_0}$ we have

$$\mathcal{T}^{-1} v_0 = q^{-1} v_0, \tag{6.7.3}$$

$$\mathcal{T}^{-1}(v_{-1} - q^{-1} v_1) = (-q)(v_{-1} - q^{-1} v_1), \tag{6.7.4}$$

$$\mathcal{T}^{-1}(v_{-1} + q v_1) = q^{-1}(v_{-1} + q v_1). \tag{6.7.5}$$

The lemma now follows from Lemma 6.7.2 since $\mathcal{T}^{-1}$ is a $\mathbf{U}^\jmath$-isomorphism. ∎

We have the following generalization of Schur-Jimbo duality, which is a counterpart of Theorem 5.2.3. The proof is almost identical as the one for Theorem 5.2.3, and for Part (1) we now use Lemma 6.7.2 and the action (6.7.1) of $H_0 \in \mathcal{H}_{B_m}$ on $\mathbb{V}^{\otimes m}$.

Theorem 6.7.3 ($(\mathbf{U}^\jmath, \mathcal{H}_{B_m})$-duality)**.** *1. The action of $\mathcal{T}^{-1} \otimes id^{m-1}$ coincides with the action of $H_0 \in \mathcal{H}_{B_m}$ on $\mathbb{V}^{\otimes m}$.*

2. The actions of $\mathbf{U}^\jmath$ and $\mathcal{H}_{B_m}$ on $\mathbb{V}^{\otimes m}$ commute with each other, and they form double centralizers.

Definition 6.7.4. An element $f \in I^m$ is anti-dominant (or $\jmath$-anti-dominant) if

$$0 \leq f(1) \leq f(2) \leq \cdots \leq f(m).$$

The following is the counterpart of Theorem 5.3.2.

Theorem 6.7.5. *There exists an anti-linear involution $\psi_\jmath : \mathbb{V}^{\otimes m} \to \mathbb{V}^{\otimes m}$ which is compatible with both the bar involution of $\mathcal{H}_{B_m}$ and the bar involution of $\mathbf{U}^\jmath$; that is, for all $v \in \mathbb{V}^{\otimes m}$, $H_\sigma \in \mathcal{H}_{B_m}$, and $u \in \mathbf{U}^\jmath$, we have*

$$\psi_\jmath(uvH_\sigma) = \psi_\jmath(u)\, \psi_\jmath(v)\overline{H}_\sigma.$$

Such a bar involution is unique by requiring $\psi_\jmath(M_f) = M_f$ for all $\jmath$-anti-dominant f.

Part II

Kazhdan-Lusztig theory

In this Part 2, we shall focus on the infinite-rank limit ($r \to \infty$) of the algebras and spaces formulated in Part 1. In Chapter 7 through Chapter 11 we will mainly treat in detail the counterparts of Chapter 2 through Chapter 5 where $\mathbf{U}$ was of type A_{2r+1} in Part 1. In Chapter 12 we deal with a variation of BGG category with half-integer weights which corresponds to the second quantum symmetric pair $(\mathbf{U}, \mathbf{U}^{\jmath})$ in Chapter 6 where $\mathbf{U}$ was of type A_{2r}.

As it becomes necessary to keep track of the finite ranks, we shall add subscripts and superscripts to various notation introduced in Part 1 to indicate the dependence on $r \in \mathbb{N}$. Here are the new notations in place of those in Part 1 without superscripts/subscripts (Chapter 2 through Chapter 5):

Λ_{2r+1}, Π_{2r+1}, $\mathbb{I}_{2r+1}$, $\mathbb{I}^{\imath}_r$, $\mathbf{U}_{2r+1}$, $\mathbf{U}^{\imath}_r$, $\Upsilon^{(r)}$, $\mathbb{V}_r$, $\mathbb{W}_r$, $\psi^{(r)}$, $\psi_{\imath}^{(r)}$, $\Theta^{(r)}$.

Part 2 of this paper follows closely [CLW2] with new input from Part 1. The notations here often have different meaning from the same notations used in [CLW2], as the current ones are often "of type B".

Chapter 7

BGG categories for ortho-symplectic Lie superalgebras

In this chapter, we recall the basics on the ortho-symplectic Lie superalgebras such as linkage principle and Bruhat ordering. We formulate various versions of (parabolic) BGG categories and truncation functors.

7.1 The Lie superalgebra $\mathfrak{osp}(2m+1|2n)$

We recall some basics on ortho-symplectic Lie superalgebras and set up notations to be used later on (cf. [CW2] for more on Lie superalgebras). Fix integers $m \geq 1$ and $n \geq 0$ throughout this paper.

Let $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$. Let $\mathbb{C}^{2m+1|2n}$ be a superspace of dimension $(2m+1|2n)$ with basis $\{e_i \mid 1 \leq i \leq 2m+1\} \cup \{e_{\overline{j}} \mid 1 \leq j \leq 2n\}$, where the $\mathbb{Z}_2$-grading is given by the following parity function:

$$p(e_i) = \overline{0}, \qquad p(e_{\overline{j}}) = \overline{1} \quad (\forall i, j).$$

Let B be a non-degenerate even supersymmetric bilinear form on $\mathbb{C}^{2m+1|2n}$. The general linear Lie superalgebra $\mathfrak{gl}(2m+1|2n)$ is the Lie superalgebra of linear transformations on $\mathbb{C}^{2m+1|2n}$ (in matrix form with respect to the above basis). For $s \in \mathbb{Z}_2$, we define

$$\mathfrak{osp}(2m+1|2n)_s := \{g \in \mathfrak{gl}(2m+1|2n)_s \mid B(g(x), y) = -(-1)^{s \cdot p(x)} B(x, g(y))\},$$

$$\mathfrak{osp}(2m+1|2n) := \mathfrak{osp}(2m+1|2n)_{\overline{0}} \oplus \mathfrak{osp}(2m+1|2n)_{\overline{1}}.$$

We now give a matrix realization of the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$. Take the supersymmetric bilinear form B with the following matrix form, with respect to the basis $(e_1, e_2, \ldots, e_{2m+1}, e_{\overline{1}}, e_{\overline{2}}, \ldots, e_{\overline{2n}})$:

$$\mathcal{J}_{2m+1|2n} := \begin{pmatrix} 0 & I_m & 0 & 0 & 0 \\ I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^n \\ 0 & 0 & 0 & -I^n & 0 \end{pmatrix}$$

Let $E_{i,j}$, $1 \leq i, j \leq 2m+1$, and $E_{\overline{k},\overline{h}}$, $1 \leq k, h \leq 2n$, be the (i,j)th and $(\overline{k}, \overline{h})$th elementary matrices, respectively. The Cartan subalgebra of $\mathfrak{osp}(2m+1|2n)$ of diagonal matrices is denoted by $\mathfrak{h}_{m|n}$, which is spanned by

$$H_i := E_{i,i} - E_{m+i,m+i}, \quad 1 \le i \le m,$$

$$H_{\overline{j}} := E_{\overline{j},\overline{j}} - E_{\overline{n+j},\overline{n+j}}, \quad 1 \le j \le n.$$

We denote by $\{\epsilon_i, \epsilon_{\overline{j}} \mid 1 \le i \le m, 1 \le j \le n\}$ the basis of $\mathfrak{h}^*_{m|n}$ such that

$$\epsilon_a(H_b) = \delta_{a,b}, \qquad \text{for } a, b \in \{i, \overline{j} \mid 1 \le i \le m, 1 \le j \le n\}.$$

We denote the lattice of integral weights of $\mathfrak{osp}(2m+1|2n)$ by

$$X(m|n) := \sum_{i=1}^{m} \mathbb{Z}\epsilon_i + \sum_{j=1}^{n} \mathbb{Z}\epsilon_{\overline{j}}. \tag{7.1.1}$$

The supertrace form on $\mathfrak{osp}(2m+1|2n)$ induces a non-degenerate symmetric bilinear form on $\mathfrak{h}^*_{m|n}$ determined by $(\cdot|\cdot)$, such that

$$(\epsilon_i|\epsilon_a) = \delta_{i,a}, \quad (\epsilon_{\overline{j}}|\epsilon_a) = -\delta_{\overline{j},a}, \qquad \text{for } a \in \{i, \overline{j} \mid 1 \le i \le m, 1 \le j \le n\}.$$

We have the following root system of $\mathfrak{osp}(2m+1|2n)$ with respect to $\mathfrak{h}_{m|n}$

$$\Phi = \Phi_{\overline{0}} \cup \Phi_{\overline{1}} = \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_p, \pm\epsilon_{\overline{k}} \pm \epsilon_{\overline{l}}, \pm 2\epsilon_{\overline{q}}\} \cup \{\pm\epsilon_p \pm \epsilon_{\overline{q}}, \pm\epsilon_{\overline{q}}\},$$

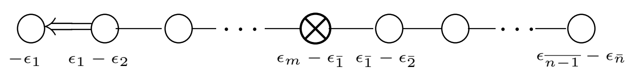
where $1 \le i < j \le n$, $1 \le p \le n$, $1 \le q \le m$, $1 \le k < l \le m$.

In this paper we shall need to deal with various Borel subalgebras, hence various simple systems of Φ. Let $\mathbf{b} = (b_1, b_2, \dots, b_{m+n})$ be a sequence of $m+n$ integers such that m of the b_i's are equal to 0 and n of them are equal to 1. We call such a sequence a 0^m1^n-*sequence*. Associated to each 0^m1^n-sequence $\mathbf{b} = (b_1, \dots, b_{m+n})$, we have the following fundamental system $\Pi_{\mathbf{b}}$, and hence a positive system $\Phi^+_{\mathbf{b}} = \Phi^+_{\mathbf{b},\bar{0}} \cup \Phi^+_{\mathbf{b},\bar{1}}$, of the root system Φ of $\mathfrak{osp}(2m+1|2n)$:

$$\Pi_{\mathbf{b}} = \{-\epsilon_1^{b_1}, \epsilon_i^{b_i} - \epsilon_{i+1}^{b_{i+1}} \mid 1 \leq i \leq m+n-1\},$$

where $\epsilon_i^0 = \epsilon_x$ for some $1 \leq x \leq m$, $\epsilon_j^1 = \epsilon_{\overline{y}}$ for some $1 \leq y \leq n$, such that $\epsilon_x - \epsilon_{x+1}$ and $\epsilon_{\overline{y}} - \epsilon_{\overline{y+1}}$ are always positive. It is clear that $\Pi_{\mathbf{b}}$ is uniquely determined by these restrictions. The Weyl vector is defined to be $\rho_{\mathbf{b}} := \frac{1}{2}\sum_{\alpha \in \Phi^+_{\mathbf{b},\bar{0}}} \alpha - \frac{1}{2}\sum_{\alpha \in \Phi^+_{\mathbf{b},\bar{1}}} \alpha$.

Corresponding to $\mathbf{b}^{\mathrm{st}} = (0, \ldots, 0, 1, \ldots, 1)$, we have the following standard Dynkin diagram associated to $\Pi_{\mathbf{b}^{\mathrm{st}}}$:

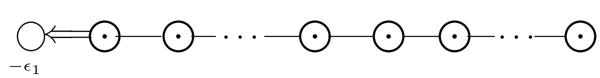


As usual, $\bigotimes$ stands for an isotropic simple odd root, $\bigcirc$ stands for an even simple root, and $\bullet$ stands for a non-isotropic odd simple root. A direct computation shows that

$$\rho_{\mathbf{b}^{\mathrm{st}}} = -\frac{1}{2}\epsilon_1 - \frac{3}{2}\epsilon_2 - \ldots - (m - \frac{1}{2})\epsilon_m + (m - \frac{1}{2})\epsilon_{\bar{1}} + \ldots + (m - n + \frac{1}{2})\epsilon_{\bar{n}}. \quad (7.1.2)$$

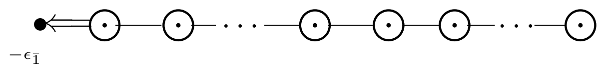
More generally, associated to a sequence $\mathbf{b}$ which starts with 0 is a Dynkin diagram which always starts on the left with a short *even* simple root:

$(\star)$



Here $\odot$ stands for either $\bigotimes$ or $\bigcirc$ depending on $\mathbf{b}$. Associated to a sequence $\mathbf{b}$ which starts with 1 is a Dynkin diagram which always starts on the left with a non-isotropic *odd* simple root:

(★★)



Remark 7.1.1. For general $\mathbf{b}$, one checks that $\rho_{\mathbf{b}}$ has a summand $(m-n+\frac{1}{2})\epsilon_{\bar{n}}$ as for $\rho_{\mathbf{b}^{\mathrm{st}}}$ in (7.1.2) if the Dynkin diagram associated to $\mathbf{b}$ has $\bigcirc$ as its rightmost node, and that $\rho_{\mathbf{b}}$ has a summand $(m-n-\frac{1}{2})\epsilon_{\bar{n}}$ if the Dynkin diagram associated to $\mathbf{b}$ has $\bigotimes$ as its rightmost node.

Now we can write the non-degenerate symmetric bilinear form on Φ as follows:

$$(\epsilon_i^{b_i}|\epsilon_j^{b_j}) = (-1)^{b_i}\delta_{ij}, \qquad 1 \leq i,j \leq m+n.$$

We define $\mathfrak{n}_{\mathbf{b}}^{\pm}$ to be the nilpotent subalgebra spanned by the positive/negative root vectors in $\mathfrak{osp}(2m+1|2n)$. Then we obtain a triangular decomposition of $\mathfrak{osp}(2m+1|2n)$:

$$\mathfrak{osp}(2m+1|2n) = \mathfrak{n}_{\mathbf{b}}^{+} \oplus \mathfrak{h}_{m|n} \oplus \mathfrak{n}_{\mathbf{b}}^{-},$$

with $\mathfrak{n}_{\mathbf{b}}^{+} \oplus \mathfrak{h}_{m|n}$ as a Borel subalgebra.

Fix a 0^m1^n-sequence $\mathbf{b}$ and hence a positve system $\Phi_{\mathbf{b}}^{+}$. We denote by $Z(\mathfrak{osp}(2m+1|2n))$ the center of the enveloping algebra $U(\mathfrak{osp}(2m+1|2n))$. There exists a standard projection $\phi : U(\mathfrak{osp}(2m+1|2n)) \to U(\mathfrak{h}_{m|n})$ which is consistent with the PBW basis associated to the above triangular decomposition ([CW2, §2.2.3]). For $\lambda \in \mathfrak{h}^*_{m|n}$, we define the central character χ_λ by letting

$$\chi_\lambda(z) := \lambda(\phi(z)), \qquad \text{for } z \in Z(\mathfrak{osp}(2m+1|2n)).$$

Denote the Weyl group of (the even subalgebra of) $\mathfrak{osp}(2m+1|2n)$ by $W_{\mathfrak{osp}}$, which is isomorphic to $(\mathbb{Z}_2 \rtimes \mathfrak{S}_m) \times (\mathbb{Z}_2 \rtimes \mathfrak{S}_n)$. Then for $\mu,\ \nu \in \mathfrak{h}^*_{m|n}$, we say $\mu,\ \nu$ are linked and denote it by $\mu \sim \nu$, if there exist mutually orthogonal isotropic odd roots $\alpha_1, \alpha_2, \ldots, \alpha_l$, complex numbers $c_1, c_2, \ldots, c_l$, and an element $w \in W_{\mathfrak{osp}}$ satisfying

$$\mu + \rho_{\mathbf{b}} = w(\nu + \rho_{\mathbf{b}} - \sum_{i=1}^{l} c_i \alpha_i), \quad (\nu + \rho_{\mathbf{b}} | \alpha_j) = 0, \quad j = 1 \ldots, l.$$

It is clear that $\sim$ is an equivalent relation on $\mathfrak{h}^*_{m|n}$. Versions of the following basic fact went back to Kac, Sergeev, and others.

Proposition 7.1.2. *[CW2, Theorem 2.30] Let λ, $\mu \in \mathfrak{h}^*_{m|n}$. Then λ is linked to μ if and only if $\chi_\lambda = \chi_\mu$.*

We define the *Bruhat ordering* $\preceq_{\mathbf{b}}$ on $\mathfrak{h}^*_{m|n}$ and hence on $X(m|n)$ as follows:

$$\lambda \preceq_{\mathbf{b}} \mu \Leftrightarrow \mu - \lambda \in \mathbb{N}\Pi_{\mathbf{b}} \text{ and } \lambda \sim \mu, \qquad \text{for } \lambda, \mu \in \mathfrak{h}^*_{m|n}. \tag{7.1.3}$$

7.2 Infinite-rank Lie superalgebras

We shall define the infinite-rank Lie superalgebras $\mathfrak{osp}(2m+1|2n+\infty)$ and $\mathfrak{osp}(2m+1|2n|\infty)$. Define the sets

$$\widetilde{\mathbb{J}} := \{1,2,\ldots,2m+1,\overline{1},\overline{2},\ldots,\overline{2n}\} \cup \{\underline{\tfrac{1}{2}},\underline{1},\underline{\tfrac{3}{2}},\ldots\} \cup \{\underline{\tfrac{1}{2}}',\underline{1}',\underline{\tfrac{3}{2}}',\ldots\},$$

$$\mathbb{J} := \{1,2,\ldots,2m+1,\overline{1},\overline{2},\ldots,\overline{2n}\} \cup \{\underline{1},\underline{2},\ldots\} \cup \{\underline{1}',\underline{2}',\ldots\},$$

$$\breve{\mathbb{J}} := \{1,2,\ldots,2m+1,\overline{1},\overline{2},\ldots,\overline{2n}\} \cup \{\underline{\tfrac{1}{2}},\underline{\tfrac{3}{2}},\ldots\} \cup \{\underline{\tfrac{1}{2}}',\underline{\tfrac{3}{2}}',\ldots\}.$$

Let $\widetilde{V}$ be the infinite-dimensional superspace over $\mathbb{C}$ with the basis $\{e_i \mid i \in \widetilde{\mathbb{J}}\}$, whose $\mathbb{Z}_2$-grading is specified as follows:

$$p(e_i) = \overline{0}\ (1 \le i \le 2m+1), \qquad p(e_{\overline{j}}) = \overline{1}\ (1 \le j \le 2n),$$

$$p(e_{\underline{s}'}) = p(e_{\underline{s}}) = \overline{0}\ (s \in \mathbb{Z}_{>0}), \qquad p(e_{\underline{t}'}) = p(e_{\underline{t}}) = \overline{1}\ (t \in \frac{1}{2}+\mathbb{N}).$$

With respect to this basis, a linear map on $\widetilde{V}$ may be identified with a complex matrix $(a_{rs})_{r,s\in\widetilde{\mathbb{J}}}$. Let $\mathfrak{gl}(\widetilde{V})$ be the Lie superalgebra consisting of $(a_{rs})_{r,s\in\widetilde{\mathbb{J}}}$ with $a_{rs}=0$ for almost all but finitely many a_{rs}'s. The standard Cartan subalgebra of $\mathfrak{gl}(\widetilde{V})$ is spanned by $\{E_{rr} \mid r \in \widetilde{\mathbb{J}}\}$, with dual basis $\{\epsilon_r \mid r \in \widetilde{\mathbb{J}}\}$. The superspaces V and $\breve{V}$ are defined to be the subspaces of $\widetilde{V}$ with basis $\{e_i\}$ indexed by $\mathbb{J}$ and $\breve{\mathbb{J}}$ respectively. Similarly we can define $\mathfrak{gl}(V)$ and $\mathfrak{gl}(\breve{V})$.

Recall the supersymmetric non-degenerate bilinear form B define in §7.1. We can easily identify $\mathbb{C}^{2m+1|2n}$ as a subspace of $\widetilde{V}$. Define a supersymmetric non-degenerate bilinear form $\tilde{B}$ on $\widetilde{V}$ by

$$\tilde{B}(e_s, e_t) = B(e_s, e_t), \qquad \tilde{B}(e_s, e_{\underline{x}}) = \tilde{B}(e_s, e_{\underline{x}'}) = 0,$$

$$\tilde{B}(e_{\underline{x}}, e_{\underline{y}}) = \tilde{B}(e_{\underline{x}'}, e_{\underline{y}'}) = 0, \qquad \tilde{B}(e_{\underline{x}}, e_{\underline{y}'}) = \delta_{x,y} = (-1)^{p(e_{\underline{x}})p(e_{\underline{y}'})}\tilde{B}(e_{\underline{y}'}, e_{\underline{x}}),$$

where $s, t \in \{i, \bar{j} \mid 1 \le i \le 2m{+}1, 1 \le j \le 2n\}$ and $x, y \in \{\underline{\frac{1}{2}}, \underline{1}, \underline{\frac{3}{2}} \ldots\}$. By restriction, we can obtain a supersymmetric non-degenerate bilinear form on V and $\breve{V}$.

Following §7.1, we define $\mathfrak{osp}(V)$ and $\mathfrak{osp}(\breve{V})$ to be the subalgebra of $\mathfrak{gl}(V)$ and $\mathfrak{gl}(\breve{V})$ preserving the bilinear forms, respectively. With respect to the standard basis of V and $\breve{V}$, we identify

$$\mathfrak{osp}(2m+1|2n|\infty) = \mathfrak{osp}(V), \qquad \mathfrak{osp}(2m+1|2n+\infty) = \mathfrak{osp}(\breve{V}).$$

The standard Cartan subalgebras of $\mathfrak{osp}(2m+1|2n|\infty)$ and $\mathfrak{osp}(2m+1|2n+\infty)$ are obtained by taking the intersection of the standard Cartan subalgebra of $\mathfrak{gl}(\widetilde{V})$ with $\mathfrak{osp}(2m{+}1|2n|\infty)$ and $\mathfrak{osp}(2m{+}1|2n{+}\infty)$, respectively, which are denoted by $\mathfrak{h}_{m|n|\infty}$ and $\mathfrak{h}_{m|n+\infty}$. For any 0^m1^n-sequence $\mathbf{b}$, we assign the following simple system to the Lie superalgebra $\mathfrak{osp}(2m+1|2n|\infty)$:

$$\Pi_{\mathbf{b},0} := \{-\epsilon_1^{b_1}, \epsilon_i^{b_i} - \epsilon_{i+1}^{b_{i+1}}, \epsilon_{m+n}^{b_{m+n}} - \epsilon_{\underline{1}}^0, \epsilon_{\underline{j}}^0 - \epsilon_{\underline{j+1}}^0 \mid 1 \le i \le m+n-1, 1 \le j\}.$$

Similarly, we assign the following simple system to $\mathfrak{osp}(2m+1|2n+\infty)$:

$$\Pi_{\mathbf{b},1} := \{-\epsilon_1^{b_1}, \epsilon_i^{b_i} - \epsilon_{i+1}^{b_{i+1}}, \epsilon_{m+n}^{b_{m+n}} - \epsilon_{\underline{1}}^1, \epsilon_{\underline{j}}^1 - \epsilon_{\underline{j+1}}^1 \mid 1 \le i \le m+n-1, 1 \le j\}.$$

The $\epsilon_i^{b_i}$'s are defined in the same way as in §7.1 and it is understood that $\epsilon_{\underline{j}}^1 := \epsilon_{\underline{j-\frac{1}{2}}}$,

$\epsilon^0_{\underline{j}} := \epsilon_{\underline{j}}$ for any $1 \le j$. We introduce the following formal symbols:

$$\epsilon^0_\infty := \sum_{j\ge1} \epsilon^0_{\underline{j}}, \qquad \epsilon^1_\infty := \sum_{j\ge1} \epsilon^1_{\underline{j}}.$$

Let $\mathcal{P}$ be the set of all partitions. We define

$$X^{\infty,+}_{\mathbf{b},0} := \{\sum_{i=1}^{m+n} \lambda_i \epsilon_i^{b_i} + \sum_{1\le j} {}^+\lambda_{\underline{j}} \epsilon^0_{\underline{j}} + d\epsilon^0_\infty \mid d, \lambda_i \in \mathbb{Z}, ({}^+\lambda_{\underline{1}}, {}^+\lambda_{\underline{2}}, \dots) \in \mathcal{P}\}, \tag{7.2.1}$$

$$X^{\infty,+}_{\mathbf{b},1} := \{\sum_{i=1}^{m+n} \lambda_i \epsilon_i^{b_i} + \sum_{1\le j} {}^+\lambda_{\underline{j}} \epsilon^1_{\underline{j}} + d\epsilon^1_\infty \mid d, \lambda_i \in \mathbb{Z}, ({}^+\lambda_{\underline{1}}, {}^+\lambda_{\underline{2}}, \dots) \in \mathcal{P}\}. \tag{7.2.2}$$

7.3 The BGG categories

We shall define various parabolic BGG categories for ortho-symplectic Lie superalgebras.

Definition 7.3.1. Let $\mathbf{b}$ be a 0^m1^n-sequence. The Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}_\mathbf{b}$ is the category of $\mathfrak{h}_{m|n}$-semisimple $\mathfrak{osp}(2m+1|2n)$-modules M such that

(i) $M = \bigoplus_{\mu\in X(m|n)} M_\mu$ and $\dim M_\mu < \infty$;

(ii) there exist finitely many weights ${}^1\lambda, {}^2\lambda, \dots, {}^k\lambda \in X(m|n)$ (depending on M) such that if μ is a weight in M, then $\mu \in {}^i\lambda - \sum_{\alpha\in\Pi_\mathbf{b}} \mathbb{N}\alpha$, for some i.

The morphisms in $\mathcal{O}_\mathbf{b}$ are all (not necessarily even) homomorphisms of $\mathfrak{osp}(2m+1|2n)$-modules.

Similar to [CLW2, Proposition 6.4], all these categories $\mathcal{O}_\mathbf{b}$ are identical for various $\mathbf{b}$, since the even subalgebras of the Borel subalgebras $\mathfrak{n}^+_\mathbf{b} \oplus \mathfrak{h}_{m|n}$ are identical and the odd parts of these Borels always act locally nilpotently.

Denote by $M_{\mathbf{b}}(\lambda)$ the $\mathbf{b}$-Verma modules with highest weight λ. Denote by $L_{\mathbf{b}}(\lambda)$ the unique simple quotient of $M_{\mathbf{b}}(\lambda)$. They are both in $\mathcal{O}_{\mathbf{b}}$.

It is well known that the Lie superalgebra $\mathfrak{gl}(2m+1|2n)$ has an automorphism τ given by the formula:

$$\tau(E_{ij}) := -(-1)^{p(i)(p(i)+p(j))}E_{ji}.$$

The restriction of τ on $\mathfrak{osp}(2m+1|2n)$ gives an automorphism of $\mathfrak{osp}(2m+1|2n)$. For an object $M = \oplus_{\mu \in X(m|n)} M_\mu \in \mathcal{O}_{\mathbf{b}}$, we let

$$M^\vee := \oplus_{\mu \in X(m|n)} M_\mu^*$$

be the restrictd dual of M. We define the action of $\mathfrak{osp}(2m+1|2n)$ on $M^\vee$ by $(g \cdot f)(x) := -f(\tau(g) \cdot x)$, for $f \in M^\vee, g \in \mathfrak{osp}(2m+1|2n)$, and $x \in M$. We denote the resulting module by M^τ.

An object $M \in \mathcal{O}_{\mathbf{b}}$ is said to have a $\mathbf{b}$-Verma flag (respectively, dual $\mathbf{b}$-Verma flag), if M has a filtration $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_t = M$, such that $M_i/M_{i-1} \cong M_{\mathbf{b}}(\gamma_i), 1 \le i \le t$ (respectively, $M_i/M_{i-1} \cong M_{\mathbf{b}}^\tau(\gamma_i)$) for some $\gamma_i \in X(m|n)$.

Definition 7.3.2. Associated to each $\lambda \in X(m|n)$, a $\mathbf{b}$-tilting module $T_{\mathbf{b}}(\lambda)$ is an indecomposable $\mathfrak{osp}(2m+1|2n)$-module in $\mathcal{O}_{\mathbf{b}}$ characterized by the following two conditions:

(i) $T_{\mathbf{b}}(\lambda)$ has a $\mathbf{b}$-Verma flag with $M_{\mathbf{b}}(\lambda)$ at the bottom;

(ii) $\mathrm{Ext}^1_{\mathcal{O}_{\mathbf{b}}}(M_{\mathbf{b}}(\mu), T_{\mathbf{b}}(\lambda)) = 0$, for all $\mu \in X(m|n)$.

Recall the definition of the infinite-rank Lie superalgebras in §7.2. For a nonempty 0^m1^n-sequence $\mathbf{b} = (b_1, b_2, \ldots, b_{m+n})$ and $k \in \mathbb{N} \cup \{\infty\}$, consider the extended sequence $(\mathbf{b}, 0^k) = (b_1, b_2, \ldots, b_{m+n}, 0, \ldots, 0)$. This sequence corresponds to the following simple system of the Lie superalgebra $\mathfrak{osp}(2m+2k+1|2n)$, which we shall denote by $\mathfrak{osp}(2m+1|2n|2k)$ throughout this paper to indicate the choice of $\Pi_{(\mathbf{b},0^k)}$:

$$\Pi_{(\mathbf{b},0^k)} = \{-\epsilon_1^{b_1}, \epsilon_i^{b_i} - \epsilon_{i+1}^{b_{i+1}} \mid 1 \le i \le m+n+k\}, \quad \text{where } b_i = 0 \text{ for } i > m+n.$$

Let $\Pi^{\underline{k}}_{\mathbf{b},0} = \{\epsilon_i^{b_i} - \epsilon_{i+1}^{b_{i+1}} \mid i > m+n\}$. Define the following Levi subalgebra and parabolic subalgebra of $\mathfrak{osp}(2m+1|2n|2k)$:

$$\mathfrak{l}^{\underline{k}}_{\mathbf{b},0} := \sum_{\alpha \in \mathbb{Z}\Pi^{\underline{k}}_{\mathbf{b},0}} \mathfrak{osp}(2m+1|2n|2k)_\alpha,$$

$$\mathfrak{p}^{\underline{k}}_{\mathbf{b},0} := \sum_{\alpha \in \Phi^+_{(\mathbf{b},0^k)} \cup \mathbb{Z}\Pi^{\underline{k}}_{\mathbf{b},0}} \mathfrak{osp}(2m+1|2n|2k)_\alpha.$$

Let $L_0(\lambda)$ be the irreducible $\mathfrak{l}^{\underline{k}}_{\mathbf{b},0}$-module with highest weight λ. It can be extended trivially to a $\mathfrak{p}^{\underline{k}}_{\mathbf{b},0}$-module. We form the parabolic Verma module

$$M^{\underline{k}}_{\mathbf{b},0}(\lambda) := \mathrm{Ind}_{\mathfrak{p}^{\underline{k}}_{\mathbf{b},0}}^{\mathfrak{osp}(2m+1|2n|2k)} L_0(\lambda).$$

For $k \in \mathbb{N}$, we define

$$X^{\underline{k},+}_{\mathbf{b},0} := \left\{ \sum_{i=1}^{m+n} \lambda_i \epsilon_i^{b_i} + \sum_{j=1}^{k} {}^+\lambda_{\underline{j}} \epsilon^0_{\underline{j}} + d \sum_{j=1}^{k} \epsilon^0_{\underline{j}} \mid d, \lambda_i \in \mathbb{Z}, ({}^+\lambda_{\underline{1}}, {}^+\lambda_{\underline{2}}, \ldots) \in \mathcal{P} \right\}, \quad (7.3.1)$$

$$X^{\underline{k},+}_{\mathbf{b},1} := \left\{ \sum_{i=1}^{m+n} \lambda_i \epsilon_i^{b_i} + \sum_{j=1}^{k} {}^+\lambda_{\underline{j}} \epsilon^1_{\underline{j}} + d \sum_{j=1}^{k} \epsilon^1_{\underline{j}} \mid d, \lambda_i \in \mathbb{Z}, ({}^+\lambda_{\underline{1}}, {}^+\lambda_{\underline{2}}, \ldots) \in \mathcal{P} \right\}. \quad (7.3.2)$$

Recall the definition of $X^{\underline{\infty},+}_{\mathbf{b},0}$ and $X^{\underline{\infty},+}_{\mathbf{b},1}$ from (7.2.1)-(7.2.2).

Definition 7.3.3. Let $\mathbf{b}$ be a 0^m1^n-sequence and $k \in \mathbb{N} \cup \{\infty\}$. Let $\mathcal{O}^{k}_{\mathbf{b},0}$ be the category of $\mathfrak{h}_{m|n|k}$-semisimple $\mathfrak{osp}(2m+1|2n|2k)$-modules M such that

(i) $M = \bigoplus_\mu M_\mu$ and $\dim M_\mu < \infty$;

(ii) M decomposes over $\mathfrak{l}^{\infty}_{\mathbf{b},0}$ into a direct sum of $L_0(\lambda)$ for $\lambda \in X^{k,+}_{\mathbf{b},0}$;

(iii) there exist finitely many weights ${}^1\lambda, {}^2\lambda, \ldots, {}^k\lambda \in X^{k,+}_{\mathbf{b},0}$ (depending on M) such that if μ is a weight in M, then $\mu \in {}^i\lambda - \sum_{\alpha\in\Pi_{(\mathbf{b},0^k)}} \mathbb{N}\alpha$, for some i.

The morphisms in $\mathcal{O}^{k}_{\mathbf{b},0}$ are all (not necessarily even) homomorphisms of $\mathfrak{osp}(2m+1|2n|2k)$-modules.

Let $\lambda \in X^{k,+}_{\mathbf{b},0}$. We shall denote by $L^{k}_{\mathbf{b},0}(\lambda)$ the simple module in $\mathcal{O}^{k}_{\mathbf{b},0}$ with highest weight λ. Following Definition 7.3.2, we can define the tilting module $T^{k}_{\mathbf{b},0}(\lambda)$ in $\mathcal{O}^{k}_{\mathbf{b},0}$.

Similar construction exists for the sequence $(\mathbf{b}, 1^k)$, where we consider the Lie superalgebras $\mathfrak{osp}(2m+1|2n+2k)$ for $k \in \mathbb{N}\cup\{\infty\}$ with the following simple systems:

$$\Pi_{(\mathbf{b},1^k)} = \{-\epsilon^{b_1}_1, \epsilon^{b_i}_i - \epsilon^{b_{i+1}}_{i+1} \mid 1 \le i \le m+n+k\}, \quad \text{where } b_i = 1 \text{ for } i > m+n.$$

Let $\Pi^{k}_{\mathbf{b},1} = \{\epsilon^{b_i}_i - \epsilon^{b_{i+1}}_{i+1} \mid i > m+n\}$. Define the following Levi subalgebra and parabolic subalgebra of $\mathfrak{osp}(2m+1|2n|2k)$:

$$\mathfrak{l}^{k}_{\mathbf{b},1} := \sum_{\alpha\in\mathbb{Z}[\Pi^{k}_{\mathbf{b},1}]} \mathfrak{osp}(2m+1|2n|2k)_\alpha,$$

$$\mathfrak{p}^{k}_{\mathbf{b},1} := \sum_{\alpha\in\Phi^+_{(\mathbf{b},1^k)}\cup\mathbb{Z}[\Pi^{k}_{\mathbf{b},1}]} \mathfrak{osp}(2m+1|2n|2k)_\alpha.$$

Let $L_1(\lambda)$ be the simple $\mathfrak{l}^{\underline{k}}_{\mathbf{b},1}$-module with highest weight λ. It can be extended trivially to a $\mathfrak{p}^{\underline{k}}_{\mathbf{b},1}$-module. Similarly we can define the parabolic Verma module

$$M^{\underline{k}}_{\mathbf{b},1}(\lambda) := \mathrm{Ind}^{\mathfrak{osp}(2m+1|2n+2k)}_{\mathfrak{p}^{\underline{k}}_{\mathbf{b},1}} L_1(\lambda).$$

Definition 7.3.4. For $k \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{O}^{\underline{k}}_{\mathbf{b},1}$ be the category of $\mathfrak{h}_{2m+1|2n+2k}$-semisimple $\mathfrak{osp}(2m+1|2n+2k)$-modules M such that

(i) $M = \bigoplus_{\mu} M_{\mu}$ and $\dim M_{\mu} < \infty$;

(ii) M decomposes over $\mathfrak{p}^{\underline{k}}_{\mathbf{b},1}$ into a direct sum of $L_1(\lambda)$ for $\lambda \in X^{\underline{k},+}_{\mathbf{b},1}$;

(iii) there exist finitely many weights ${}^1\lambda, {}^2\lambda, \ldots, {}^k\lambda \in X^{\underline{k},+}_{\mathbf{b},1}$ (depending on M) such that if μ is a weight in M, then $\mu \in {}^i\lambda - \sum_{\alpha \in \Pi_{(\mathbf{b},1^k)}} \mathbb{N}\alpha$, for some i.

The morphisms in $\mathcal{O}^{\underline{k}}_{\mathbf{b},1}$ are all (not necessarily even) homomorphisms of $\mathfrak{osp}(2m+1|2n+2k)$-modules.

For $\xi \in X^{\underline{k},+}_{\mathbf{b},1}$, we shall denote by $L^{\underline{k}}_{\mathbf{b},1}(\xi)$ the simple module in $\mathcal{O}^{\underline{k}}_{\mathbf{b},1}$ with highest weight ξ. Following Definition 7.3.2, we can define the tilting module $T^{\underline{k}}_{\mathbf{b},1}(\xi)$ in $\mathcal{O}^{\underline{k}}_{\mathbf{b},1}$.

7.4 Truncation functors

Recall the definition of $X^{\underline{k},+}_{\mathbf{b},0}$ and $X^{\underline{k},+}_{\mathbf{b},1}$ in (7.3.1) and (7.3.2). For any $\lambda = \sum_{i=1}^{m+n} \lambda_i \epsilon_i^{b_i} + \sum_{1 \leq j} {}^{+}\lambda_{\underline{j}} \epsilon^s_{\underline{j}} + d\epsilon^s_{\infty} \in X^{\underline{\infty},+}_{\mathbf{b},s}$, we define

$$\lambda^{\underline{k}} := \sum_{i=1}^{m+n} \lambda_i \epsilon_i^{b_i} + \sum_{j=1}^{k} {}^{+}\lambda_{\underline{j}} \epsilon^s_{\underline{j}} + d \sum_{j=1}^{k} \epsilon^s_{\underline{j}} \in X^{\underline{k},+}_{\mathbf{b},s}, \qquad \text{for } s \in \{0,1\}.$$

Let $M^{\infty}_{\mathbf{b},0} \in \mathcal{O}^{\infty}_{\mathbf{b},0}$ and $M^{\infty}_{\mathbf{b},1} \in \mathcal{O}^{\infty}_{\mathbf{b},1}$. Then we have the weight space decompositions

$$M^{\infty}_{\mathbf{b},0} = \bigoplus_{\mu} M^{\infty}_{\mathbf{b},0,\mu}, \qquad M^{\infty}_{\mathbf{b},1} = \bigoplus_{\mu} M^{\infty}_{\mathbf{b},1,\mu}.$$

We define an exact functor $\mathfrak{tr}_0 : \mathcal{O}^{\infty}_{\mathbf{b},0} \to \mathcal{O}^{k}_{\mathbf{b},0}$ by

$$\mathfrak{tr}_0(M^{\infty}_{\mathbf{b},0}) := \bigoplus_{\mu} M^{\infty}_{\mathbf{b},0,\mu},$$

where μ satisfies $(\mu, \epsilon^0_{\underline{j}} - \epsilon^0_{\underline{j+1}}) = 0$, $\forall j \geq k+1$ and $j \in \mathbb{N}$. Similarly, we define an exact functor $\mathfrak{tr}_1 : \mathcal{O}^{\infty}_{\mathbf{b},1} \to \mathcal{O}^{k}_{\mathbf{b},1}$ by

$$\mathfrak{tr}_1(M^{\infty}_{\mathbf{b},1}) := \bigoplus_{\mu} M^{\infty}_{\mathbf{b},1,\mu},$$

where μ satisfies $(\mu, \epsilon^1_{\underline{j}} - \epsilon^1_{\underline{j+1}}) = 0$, $\forall j \geq k+1$ and $j \in \mathbb{N}$. The following has been known [CW1, CLW1]; also see [CW2, Proposition 6.9].

Proposition 7.4.1. *For $s = 0, 1$, the functors $\mathfrak{tr}_s : \mathcal{O}^{\infty}_{\mathbf{b},s} \to \mathcal{O}^{k}_{\mathbf{b},s}$ satisfy the following: for $Y = M, L, T$, $\lambda = \sum_{i=1}^{m+n} \lambda_i \epsilon_i^{b_i} + \sum_{1 \leq j} {}^{+}\lambda_{\underline{j}} \epsilon^s_{\underline{j}} + d\epsilon^s_{\infty} \in X^{\infty,+}_{\mathbf{b},s}$, we have*

$$\mathfrak{tr}_s\left(Y^{\infty}_{\mathbf{b},s}(\lambda)\right) = \begin{cases} Y^{k}_{\mathbf{b},s}(\lambda^{k}), & \text{if } l({}^{+}\lambda) \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

Chapter 8

Fock spaces and Bruhat orderings

In this chapter, we formulate the infinite-rank variants of the basic constructions in Part 1. We set up various Fock spaces which are the q-versions of Grothendieck groups, and transport Bruhat ordering from the BGG categories to the corresponding Fock spaces.

8.1 Infinite-rank constructions

Let us first set up some notations which will be used often in Part 2. We set

$$\mathbb{I} = \cup_{r=0}^{\infty} \mathbb{I}_{2r+1} = \mathbb{Z}, \qquad \mathbb{I}^{\imath} = \cup_{r=0}^{\infty} \mathbb{I}^{\imath}_{r} = \mathbb{Z}_{>0}, \qquad I = \mathbb{Z} + \frac{1}{2}. \tag{8.1.1}$$

Recall from Chapter 2 the finite-rank quantum symmetric pair $(\mathbf{U}_{2r+1}, \mathbf{U}^{\imath}_{r})$. We have the natural inclusions of $\mathbb{Q}(q)$-algebras:

$$\cdots \subset \mathbf{U}_{2r-1} \subset \mathbf{U}_{2r+1} \subset \mathbf{U}_{2r+3} \subset \cdots,$$

$$\cdots \subset \mathbf{U}^{\imath}_{r-1} \subset \mathbf{U}^{\imath}_{r} \subset \mathbf{U}^{\imath}_{r+1} \subset \cdots.$$

Define the following $\mathbb{Q}(q)$-algebras:

$$\mathbf{U}^{\imath} := \bigcup_{r=0}^{\infty} \mathbf{U}^{\imath}_{r} \quad \text{and} \quad \mathbf{U} := \bigcup_{r=0}^{\infty} \mathbf{U}_{2r+1}.$$

It is easy to see that $\mathbf{U}^{\imath}$ is generated by $\{e_{\alpha_i}, f_{\alpha_i}, k^{\pm 1}_{\alpha_i}, t \mid i \in \mathbb{I}^{\imath}\}$, and $\mathbf{U}$ is generated by $\{E_{\alpha_i}, F_{\alpha_i}, K^{\pm 1}_{\alpha_i} \mid i \in \mathbb{I}\}$. The embeddings of $\mathbb{Q}(q)$-algebras $\imath : \mathbf{U}^{\imath}_{r} \to \mathbf{U}_{2r+1}$ induce an embedding of $\mathbb{Q}(q)$-algebras, denoted also by $\imath : \mathbf{U}^{\imath} \longrightarrow \mathbf{U}$. Again $\mathbf{U}$ is naturally a Hopf algebra with coproduct Δ, and its restriction under $\imath$, $\Delta : \mathbf{U}^{\imath} \to \mathbf{U}^{\imath} \otimes \mathbf{U}$, makes $\mathbf{U}^{\imath}$ (or more precisely $\imath(\mathbf{U}^{\imath})$) naturally a (right) coideal subalgebra of $\mathbf{U}$. The anti-linear bar involutions on $\mathbf{U}^{\imath}_{r}$ and $\mathbf{U}_{2r+1}$ induce anti-linear bar involutions on $\mathbf{U}^{\imath}$ and $\mathbf{U}$, respectively, both denoted by $\bar{\ }$ as well. As in Part 1, in order to avoid confusion, we shall sometimes set $\psi(u) := \overline{u}$ for $u \in \mathbf{U}$, and set $\psi_{\imath}(u) := \overline{u}$ for $u \in \mathbf{U}^{\imath}$.

Recall Π_{2r+1} denotes the simple system of $\mathbf{U}_{2r+1}$. Then $\Pi := \bigcup_{r=0}^{\infty} \Pi_{2r+1}$ is a simple system of $\mathbf{U}$. Recall we denote the integral weight lattice of $\mathbf{U}_{2r+1}$ by Λ_{2r+1}. Then

$$\Lambda := \oplus_{i \in \frac{1}{2}+\mathbb{Z}} \mathbb{Z}[\varepsilon_i] = \bigcup_{r=0}^{\infty} \Lambda_{2r+1}$$

is the integral weight lattice of $\mathbf{U}$. Following §1.1, we have the quotient lattice Λ_{θ} of the lattice Λ.

Recall the intertwiner of the pair $(\mathbf{U}_{2r+1}, \mathbf{U}^{\imath}_r)$ in §2.3, which we shall denote by $\Upsilon^{(r)}$. We have $\Upsilon^{(r)} = \sum_{\mu \in \mathbb{N}\Pi_{2r+1}} \Upsilon^{(r)}_\mu$ in a completion of $\mathbf{U}^-_{2r+1}$ with $\Upsilon^{(r)}_0 = 1$. Following the construction of $\Upsilon^{(r)}$ in Theorem 2.3.1, we see that

$$\Upsilon^{(r+1)}_\mu = \Upsilon^{(r)}_\mu, \qquad \text{for } \mu \in \mathbb{N}\Pi_{2r+1}.$$

Hence we can define an element $\Upsilon_\mu \in \mathbf{U}^-_\mu$, for $\mu \in \mathbb{N}\Pi$ by letting

$$\Upsilon_\mu := \lim_{r \to \infty} \Upsilon^{(r)}_\mu.$$

Define the formal sum Υ (which lies in some completion of $\mathbf{U}^-$) by

$$\Upsilon := \sum_{\mu \in \mathbb{N}\Pi} \Upsilon_\mu. \tag{8.1.2}$$

We shall view Υ as a well-defined operator on $\mathbf{U}$-modules that we are concerned.

8.2 The Fock space $\mathbb{T}^{\mathbf{b}}$

Let $\mathbb{V} := \sum_{a \in I} \mathbb{Q}(q) v_a$ be the natural representation of $\mathbf{U}$, where the action of $\mathbf{U}$ on $\mathbb{V}$ is defined as follows (for $i \in \mathbb{I}$, $a \in I$):

$$E_{\alpha_i} v_a = \delta_{i+\frac{1}{2},a} v_{a-1}, \quad F_{\alpha_i} v_a = \delta_{i-\frac{1}{2},a} v_{a+1}, \quad K_{\alpha_i} v_a = q^{(\alpha_i, \varepsilon_a)} v_a.$$

Let $\mathbb{W} := \mathbb{V}^*$ be the restricted dual module of $\mathbb{V}$ with basis $\{w_a \mid a \in I\}$ such that $\langle w_a, v_b \rangle = (-q)^{-a} \delta_{a,b}$. The action of $\mathbf{U}$ on $\mathbb{W}$ is given by the following formulas (for $i \in \mathbb{I}$, $a \in I$):

$$E_{\alpha_i} w_a = \delta_{i-\frac{1}{2},a} w_{a+1}, \quad F_{\alpha_i} w_a = \delta_{i+\frac{1}{2},a} w_{a-1}, \quad K_{\alpha_i} w_a = q^{-(\alpha_i, \varepsilon_a)} w_a.$$

By restriction through the embedding ι, $\mathbb{V}$ and $\mathbb{W}$ are naturally $\mathbf{U}^\imath$-modules.

Fix a 0^m1^n-sequence $\mathbf{b} = (b_1, b_2, \dots, b_{m+n})$. We have the following tensor space over $\mathbb{Q}(q)$, called the $\mathbf{b}$-*Fock space* or simply *Fock space*:

$$\mathbb{T}^{\mathbf{b}} := \mathbb{V}^{b_1} \otimes \mathbb{V}^{b_2} \otimes \cdots \otimes \mathbb{V}^{b_{m+n}}, \tag{8.2.1}$$

where we denote

$$\mathbb{V}^{b_i} := \begin{cases} \mathbb{V}, & \text{if } b_i = 0, \\ \mathbb{W}, & \text{if } b_i = 1. \end{cases}$$

The tensors here and in similar settings later on are understood to be over the field $\mathbb{Q}(q)$. Note that both algebras $\mathbf{U}$ and $\mathbf{U}^\imath$ act on $\mathbb{T}^{\mathbf{b}}$ via an iterated coproduct.

For $f \in I^{m+n}$, we define

$$M^{\mathbf{b}}_f := \mathsf{v}^{b_1}_{f(1)} \otimes \mathsf{v}^{b_2}_{f(2)} \otimes \cdots \otimes \mathsf{v}^{b_{m+n}}_{f(m+n)}, \tag{8.2.2}$$

where we use the notation $\mathsf{v}^{b_i} := \begin{cases} v, \text{ if } b_i = 0, \\ w, \text{ if } b_i = 1. \end{cases}$ We refer to $\{M^{\mathbf{b}}_f \mid f \in I^{m+n}\}$ as the *standard monomial basis* of $\mathbb{T}^{\mathbf{b}}$.

For $r \in \mathbb{N}$, we shall denote the natural representation of $\mathbf{U}_{2r+1}$ by $\mathbb{V}_r$ now, where $\mathbb{V}_r$ admits a natural basis $\{v_a \mid a \in \mathbb{I}_{2r+2}\}$. Let $\mathbb{W}_r$ be the dual of $\mathbb{V}_r$ with basis $\{w_a \mid a \in \mathbb{I}_{2r+2}\}$ such that $\langle w_a, v_b \rangle = (-q)^{-a}\delta_{a,b}$. We have natural inclusions of $\mathbb{Q}(q)$-spaces

$$\cdots \subset \mathbb{V}_{r-1} \subset \mathbb{V}_r \subset \mathbb{V}_{r+1} \cdots, \quad \text{and} \quad \cdots \subset \mathbb{W}_{r-1} \subset \mathbb{W}_r \subset \mathbb{W}_{r+1} \cdots.$$

Similarly we can define the space

$$\mathbb{T}_r^{\mathbf{b}} := \mathbb{V}_r^{b_1} \otimes \mathbb{V}_r^{b_2} \otimes \cdots \otimes \mathbb{V}_r^{b_{m+n}},$$

where we denote

$$\mathbb{V}_r^{b_i} := \begin{cases} \mathbb{V}_r, & \text{if } b_i = 0, \\ \\ \mathbb{W}_r, & \text{if } b_i = 1. \end{cases}$$

Then $\{M_f^{\mathbf{b}} \mid f \in \mathbb{I}_{2r+2}^{m+n}\}$ forms the *standard monomial basis* of $\mathbb{T}_r^{\mathbf{b}}$. In light of the standard monomial bases, we may view

$$\cdots \subset \mathbb{T}_r^{\mathbf{b}} \subset \mathbb{T}_{r+1}^{\mathbf{b}} \subset \cdots, \qquad \text{and} \qquad \mathbb{T}^{\mathbf{b}} = \cup_{r \in \mathbb{N}} \mathbb{T}_r^{\mathbf{b}}. \tag{8.2.3}$$

Definition 8.2.1. For $f \in \mathbb{I}_{2r+2}^{m+n}$, let $\mathtt{wt}_{\mathbf{b}}(f)$ be the $\mathbf{U}^\imath$-weight of $M_f^{\mathbf{b}}$, i.e., the image of the $\mathbf{U}$-weight in Λ_θ.

8.3 The q-wedge spaces

Recall from §5 the right action on $\mathbb{V}^{\otimes k}$ on the Hecke algebra $\mathcal{H}_{B_k}$, where $\mathbb{V}$ is now of infinite dimension. We take $\wedge^k \mathbb{V}$ as the quotient of $\mathbb{V}^{\otimes k}$ by the sum of the kernel of the operators $H_i - q^{-1}$, $1 \le i \le k-1$. The $\wedge^k \mathbb{V}$ is naturally a $\mathbf{U}$-module, hence also a $\mathbf{U}^\imath$-module. For any $v_{p_1} \otimes v_{p_2} \otimes \cdots \otimes v_{p_k}$ in $\mathbb{V}^{\otimes k}$, we denote its image in $\wedge^k \mathbb{V}$ by $v_{p_1} \wedge v_{p_2} \wedge \cdots \wedge v_{p_k}$.

For $d \in \mathbb{Z}$ and $l \geq k$, consider the $\mathbb{Q}(q)$-linear maps

$$\wedge_d^{k,l} : \wedge^k \mathbb{V} \longrightarrow \wedge^l \mathbb{V}$$

$$v_{p_1} \wedge \cdots \wedge v_{p_k} \mapsto v_{p_1} \wedge \cdots \wedge v_{p_k} \wedge v_{d+\frac{1}{2}-k-1} \wedge v_{d+\frac{1}{2}-k-2} \wedge \cdots \wedge v_{d+\frac{1}{2}-l}.$$

Let $\wedge_d^\infty \mathbb{V} := \varinjlim \wedge^k \mathbb{V}$ be the direct limit of the $\mathbb{Q}(q)$-vector spaces with respect to the maps $\wedge_d^{k,l}$, which is called the dth sector of the semi-infinite q-wedge space $\wedge^\infty \mathbb{V}$; that is,

$$\wedge^\infty \mathbb{V} = \bigoplus_{d \in \mathbb{Z}} \wedge_d^\infty \mathbb{V}.$$

Note that for any fixed $u \in \mathbf{U}$ and fixed $d \in \mathbb{Z}$, we have

$$\wedge_d^{k,l} u = u \wedge_d^{k,l} : \wedge^k \mathbb{V} \longrightarrow \wedge^l \mathbb{V}, \quad \text{for } l \geq k \gg 0.$$

Therefore $\wedge_d^\infty \mathbb{V}$ and hence $\wedge^\infty \mathbb{V}$ become both $\mathbf{U}$-modules and $\mathbf{U}^\imath$-modules.

We can think of elements in $\wedge^\infty \mathbb{V}$ as linear combinations of infinite q-wedges of the form

$$v_{p_1} \wedge v_{p_2} \wedge v_{p_3} \wedge \cdots ,$$

where $p_1 > p_2 > p_3 > \cdots$, and $p_i - p_{i+1} = 1$ for $i \gg 0$. Alternatively, the space $\wedge^\infty \mathbb{V}$ has a basis indexed by pairs of a partition and an integer given by

$$|\lambda, d\rangle := v_{\lambda_1+d-\frac{1}{2}} \wedge v_{\lambda_2+d-\frac{3}{2}} \wedge v_{\lambda_3+d-\frac{5}{2}} \wedge \cdots ,$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$ runs over the set $\mathcal{P}$ of all partitions, and d runs over $\mathbb{Z}$. Clearly we can realize $\wedge_d^\infty \mathbb{V}$ as the subspace of $\wedge^\infty \mathbb{V}$ spanned by $\{|\lambda, d\rangle \mid \lambda \in \mathcal{P}\}$, for $d \in \mathbb{Z}$.

In the rest of this paper, we shall index the q-wedge spaces by $[\underline{k}] := \{\underline{1}, \underline{2}, \dots, \underline{k}\}$ and $[\underline{\infty}] := \{\underline{1}, \underline{2} \dots\}$. More precisely, let

$$I_+^k = \{f : [\underline{k}] \to I \mid f(\underline{1}) > f(\underline{2}) > \cdots > f(\underline{k})\}, \quad \text{for } k \in \mathbb{N},$$
$$I_+^\infty = \{f : [\underline{\infty}] \to I \mid f(\underline{1}) > f(\underline{2}) > \cdots ; f(\underline{t}) - f(\underline{t+1}) = 1 \text{ for } t \gg 0\}.$$

For $f \in I_+^k$, we denote

$$\mathcal{V}_f = v_{f(\underline{1})} \wedge v_{f(\underline{2})} \wedge \cdots \wedge v_{f(\underline{k})}.$$

Then $\{\mathcal{V}_f \mid f \in I_+^k\}$ is a basis of $\wedge^k \mathbb{V}$, for $k \in \mathbb{Z}_{>0} \cup \{\infty\}$.

For $k \in \mathbb{Z}_{>0}$, we let $w_0^{(k)}$ be the longest element in $\mathfrak{S}_k$. Define

$$L_{w_0^{(k)}} := \sum_{w \in \mathfrak{S}_k} (-q)^{l(w) - l(w_0^{(k)})} H_w \in \mathcal{H}_{A_{k-1}}.$$

It is well known [KL, So2] that $\overline{L_{w_0^{(k)}}} = L_{w_0^{(k)}}$. The right action by $L_{w_0^{(k)}}$ define a $\mathbb{Q}(q)$-linear map (the q-skew-symmetrizer) $\mathrm{SkSym}_k : \mathbb{V}^{\otimes k} \to \mathbb{V}^{\otimes k}$. Then the q-wedge space $\wedge^k \mathbb{V}$ can also be regarded as a subspace $\mathrm{Im}(\mathrm{SkSym}_k)$ of $\mathbb{V}^{\otimes k}$ while identifying $\mathcal{V}_f \equiv M^{(0^k)}_{f \cdot w_0^{(k)}} L_{w_0^{(k)}}$ for $f \in I_+^k$ (cf. e.g. [CLW2, §4.1]).

Similar construction gives rise to $\wedge^\infty \mathbb{W}$. For each $d \in \mathbb{Z}$ and $l \geq k$, consider the $\mathbb{Q}(q)$-linear maps

$$\wedge_d^{k,l} : \wedge^k \mathbb{W} \longrightarrow \wedge^l \mathbb{W} \tag{8.3.1}$$
$$w_{p_1} \wedge \cdots \wedge w_{p_k} \mapsto w_{p_1} \wedge \cdots \wedge w_{p_k} \wedge w_{d-\frac{1}{2}+k+1} \wedge w_{d-\frac{1}{2}+k+2} \wedge \cdots \wedge w_{d-\frac{1}{2}+l}.$$

Let $\wedge_d^\infty \mathbb{W} := \varinjlim \wedge^k \mathbb{W}$ be the direct limit of the $\mathbb{Q}(q)$-vector spaces with respect to the maps $\wedge_d^{k,l}$. Define

$$\wedge^\infty \mathbb{W} := \bigoplus_{d\in\mathbb{Z}} \wedge_d^\infty \mathbb{W}.$$

Note that for any fixed $u \in \mathbf{U}$ and fixed $d \in \mathbb{Z}$, we have

$$\wedge_d^{k,l} u = u \wedge_d^{k,l} : \wedge^k \mathbb{W} \to \wedge^l \mathbb{W}, \quad \text{for } l \geq k \gg 0.$$

Therefore $\wedge_d^\infty \mathbb{W}$ and hence $\wedge^\infty \mathbb{W}$ become both $\mathbf{U}$-modules and $\mathbf{U}^\imath$-modules.

We can think of elements in $\wedge^\infty \mathbb{W}$ as linear combinations of infinite q-wedges of the form

$$w_{p_1} \wedge w_{p_2} \wedge w_{p_3} \wedge \cdots ,$$

where $p_1 < p_2 < p_3 < \cdots$, and $p_i - p_{i+1} = -1$, for $i \gg 0$. Alternatively, the space $\wedge^\infty \mathbb{W}$ has a basis indexed by partitions given by

$$|\lambda_*, d\rangle := w_{d+\frac{1}{2}-\lambda_1} \wedge w_{d+\frac{3}{2}-\lambda_2} \wedge w_{d+\frac{5}{2}-\lambda_3} \wedge \cdots ,$$

where $\lambda = (\lambda_1, \lambda_2, \cdots)$ runs over the set $\mathcal{P}$ of all partitions, and d runs over $\mathbb{Z}$. Clearly we can realize $\wedge_d^\infty \mathbb{W}$ as the subspace of $\wedge^\infty \mathbb{W}$ spanned by $\{|\lambda_*, d\rangle \mid \lambda \in \mathcal{P}\}$, for $d \in \mathbb{Z}$.

Let

$$I_-^k = \{f : [\underline{k}] \to I \mid f(\underline{1}) < f(\underline{2}) < \cdots < f(\underline{k})\}, \text{ for } k \in \mathbb{N},$$

$$I_-^\infty = \{f : [\underline{\infty}] \to I \mid f(\underline{1}) < f(\underline{2}) < \cdots ; f(\underline{t}) - f(\underline{t+1}) = -1 \text{ for } t \gg 0\}.$$

For $f \in I_-^k$, we denote

$$\mathcal{W}_f = w_{f(\underline{1})} \wedge w_{f(\underline{2})} \wedge \cdots \wedge w_{f(\underline{k})}.$$

Then $\{\mathcal{W}_f \mid f \in I_-^k\}$ is a basis of $\wedge^k \mathbb{W}$, for $k \in \mathbb{N} \cup \{\infty\}$.

Remark 8.3.1. The semi-infinite q-wedge spaces considered in this paper will involve all sectors, while only the 0th sector was considered and needed in [CLW2, §2.4].

8.4 Bruhat orderings

Let $\mathbf{b} = (b_1, \cdots, b_{m+n})$ be an arbitrary 0^m1^n-sequence. We first define a partial ordering on I^{m+n}, which depends on the sequence $\mathbf{b}$. There is a natural bijection $I^{m+n} \leftrightarrow X(m|n)$ (recall $X(m|n)$ from (7.1.1)), defined as

$$f \mapsto \lambda_f^{\mathbf{b}}, \text{ where } \lambda_f^{\mathbf{b}} = \sum_{i=1}^{m+n} (-1)^{b_i} f(i) \epsilon_i^{b_i} - \rho_{\mathbf{b}}, \quad \text{for } f \in I^{m+n}, \tag{8.4.1}$$

$$\lambda \mapsto f_\lambda^{\mathbf{b}}, \text{ where } f(i) = (\lambda + \rho_{\mathbf{b}} | \epsilon_i^{b_i}), \qquad \text{for } \lambda \in X(m|n). \tag{8.4.2}$$

We transport the Bruhat ordering (7.1.3) on $X(m|n)$ by the above bijection to I^{m+n}.

Definition 8.4.1. The Bruhat ordering or $\mathbf{b}$-Bruhat ordering $\preceq_{\mathbf{b}}$ on I^{m+n} is defined as follows: For f, $g \in I^{m+n}$, $f \preceq_{\mathbf{b}} g$ if $\lambda_f^{\mathbf{b}} \preceq_{\mathbf{b}} \lambda_g^{\mathbf{b}}$. We also say $f \sim g$ if $\lambda_f^{\mathbf{b}} \sim \lambda_g^{\mathbf{b}}$.

The following lemma follows immediately from the definition.

Lemma 8.4.2. *Given $f, g \in I^{m+n}$ such that $g \preceq_{\mathbf{b}} f$, then the set $\{h \in I^{m+n} \mid g \preceq_{\mathbf{b}} h \preceq_{\mathbf{b}} f\}$ is finite.*

Recalling the weight $\mathtt{wt}_{\mathbf{b}}(\cdot)$ on I^{m+n} from Definition 8.2.1, we set

$$\mathtt{wt}_{\mathbf{b}}(\lambda) := \mathtt{wt}_{\mathbf{b}}(f_\lambda^{\mathbf{b}}), \qquad \text{for } \lambda \in X(m|n). \tag{8.4.3}$$

We have the following analogue of [Br1, Lemma 4.18].

Lemma 8.4.3. *For any $f, g \in I^{m+n}$, $f \sim g$ if and only if $\mathtt{wt}_{\mathbf{b}}(f) = \mathtt{wt}_{\mathbf{b}}(g)$.*

Proof. This proof is analogous to [CW2, Theorem 2.30]. Assume $f \sim g$ at first. Recall §7.1, this means

$$\lambda_g^{\mathbf{b}} + \rho_{\mathbf{b}} = w(\lambda_f^{\mathbf{b}} + \rho_{\mathbf{b}} - \sum_{i=1}^{l} c_i\alpha_i), \quad (\lambda_f^{\mathbf{b}} + \rho_{\mathbf{b}}|\alpha_j) = 0, j = 1, \ldots, l.$$

where α_i's are mutually orthogonal isotropic odd roots. Recall the Weyl group of $\mathfrak{osp}(2m+1|2n)$ is isomorphic to $(\mathbb{Z}_2 \rtimes \mathfrak{S}_m) \times (\mathbb{Z}_2 \rtimes \mathfrak{S}_n)$. Thanks to Definition 2.2.4 and the actions the k_{α_i}'s on $\mathbb{V}$ and $\mathbb{W}$, we have $\mathtt{wt}_{\mathbf{b}}(w(\lambda_f^{\mathbf{b}} + \rho_{\mathbf{b}} - \sum_{i=1}^{l} c_i\alpha_i)) = \mathtt{wt}_{\mathbf{b}}(\lambda_f^{\mathbf{b}} + \rho_{\mathbf{b}} - \sum_{i=1}^{l} c_i\alpha_i)$. Isotropic odd roots of Φ are of the form $\pm\epsilon_x^{b_x} \pm \epsilon_y^{b_y}$, where b_x and b_y are distinct. We shall discuss one case here, as the others will be similar.

Let $\alpha = \epsilon_s^{b_s} - \epsilon_t^{b_t} = \epsilon_s^0 - \epsilon_t^1$ be an isotropic odd root such that $(\lambda_f^{\mathbf{b}} + \rho_{\mathbf{b}}|\alpha) = (\sum_{i=1}^{m+n}(-1)^{b_i} f(i)\epsilon_i^{b_i}|\alpha) = 0$. Therefore, $f(s) = f(t)$. Hence we have $\mathtt{wt}_{\mathbf{b}}(\lambda_f^{\mathbf{b}} + \rho_{\mathbf{b}} + c\alpha) = \mathtt{wt}_{\mathbf{b}}(\ldots, f(s-1), f(s)+c, f(s+1), \ldots, f(t-1), f(t)+c, f(t+1), \ldots) = \mathtt{wt}_{\mathbf{b}}(f)$, where the last equality comes from the actions of k_{α_i}'s on $\mathbb{V}$ and $\mathbb{W}$. Therefore $\mathtt{wt}_{\mathbf{b}}(f) = \mathtt{wt}_{\mathbf{b}}(g)$.

Now suppose $\mathtt{wt}_{\mathbf{b}}(f) = \mathtt{wt}_{\mathbf{b}}(g)$. We have

$$\sum_{i=1}^{m+n}(-1)^{b_i}\underline{\varepsilon_{f(i)}} = \sum_{i=1}^{m+n}(-1)^{b_i}\underline{\varepsilon_{g(i)}}. \tag{8.4.4}$$

For distinct b_{i_a}, b_{j_a} $(i_a \neq j_a)$, if $f(i_a) = \pm f(j_a)$, $(-1)^{b_{i_a}}\underline{\varepsilon_{f(i_a)}} + (-1)^{b_{j_a}}\underline{\varepsilon_{f(j_a)}} = 0$ (recall that $\underline{\varepsilon_{f(s)}} = \underline{\varepsilon_{-f(s)}}$). Similar results hold for g. After canceling all such pairs (all i_a and all j_a are distinct) on both sides of (8.4.4), the survived terms match bijectively up to signs. More precisely, for any survived $f(x)$, there exists a suvived $g(y)$, such that $g(y) = \pm f(x), b_x = b_y$. Hence the same number of pairs cancelled on both sides, say l pairs. Therefore we have $\lambda^{\mathbf{b}}_f + \rho_{\mathbf{b}} - \sum_{a=1}^{l} c_a(\epsilon^0_{i_a} - s_a\epsilon^1_{j_a}) = w(\lambda^{\mathbf{b}}_g + \rho_{\mathbf{b}})$ for some $w \in (\mathbb{Z}_2 \rtimes \mathfrak{S}_m) \times (\mathbb{Z}_2 \rtimes \mathfrak{S}_n)$, $s_a \in \{\pm\}$. The s_a's are chosen to satisfy

$$(\lambda^{\mathbf{b}}_f + \rho_{\mathbf{b}} | \epsilon^0_{i_a} - s_a\epsilon^1_{j_a}) = 0.$$

Therefore $\lambda^{\mathbf{b}}_f \sim \lambda^{\mathbf{b}}_g$ by the definition in §7.1. Hence $f \sim g$.

This completes the proof of the lemma. □

Now let us define partial orderings on the sets $I^{m+n} \times I^{\infty}_{\pm}$, which again depend on $\mathbf{b}$. Recall (7.2.1) and (7.2.2) for the definitions of $X^{\infty,+}_{\mathbf{b},0}$ and $X^{\infty,+}_{\mathbf{b},1}$. We define a map

$$X^{\infty,+}_{\mathbf{b},0} \longrightarrow I^{m+n} \times I^{\infty}_{+}, \quad \lambda \mapsto f^{\mathbf{b}0}_{\lambda}, \tag{8.4.5}$$

by sending each $\lambda = \sum_{i=1}^{m+n} \lambda_i \epsilon^{b_i}_i + \sum_{1 \leq j} {}^{+}\lambda_{\underline{j}} \epsilon^0_{\underline{j}} + d\epsilon^0_{\infty}$ to the element $f^{\mathbf{b}0}_{\lambda} = f^{(\mathbf{b},0^{\infty})}_{\lambda}$ given below (which is consistent with the ρ-shift associated to a simple system of the type ($\star$) in §7.1 by Remark 7.1.1):

$$\begin{aligned} f^{\mathbf{b}0}_{\lambda}(i) &= f^{\mathbf{b}}_{\lambda}(i), \quad \text{if } i \in [m+n] := \{1, 2, \ldots, m+n\}, \\ f^{\mathbf{b}0}_{\lambda}(\underline{j}) &= {}^{+}\lambda_{\underline{j}} + d + n - m + \frac{1}{2} - j, \quad \text{if } 1 \leq j. \end{aligned} \tag{8.4.6}$$

This map is a bijection, where the inverse sends $f \in I^{m+n} \times I^{\infty}_{+}$ to

$$\lambda^{\mathbf{b}0}_f := \sum_{i=1}^{m+n} \lambda^{\mathbf{b}}_{f,i} \epsilon^{b_i}_i + \sum_{1 \leq j} {}^{+}\lambda_{f,\underline{j}} \epsilon^0_{\underline{j}} + d_f \epsilon^0_{\infty}.$$

Similarly we define a bijection

$$X^{\infty,+}_{\mathbf{b},1} \longrightarrow I^{m+n} \times I^{\infty}_{-}, \quad \lambda \mapsto f^{\mathbf{b}1}_{\lambda}, \tag{8.4.7}$$

by sending each $\lambda = \sum_{i=1}^{m+n} \lambda_i \epsilon_i^{b_i} + \sum_{1\leq j} {}^+\lambda_{\underline{j}} \epsilon_{\underline{j}}^1 + d\epsilon_\infty^1$ to the element $f_\lambda^{\mathbf{b}1} = f_\lambda^{(\mathbf{b},1^\infty)}$ given below:

$$
\begin{aligned}
f_\lambda^{\mathbf{b}1}(i) &:= f_\lambda^{\mathbf{b}}(i), \quad \text{if } i \in [m+n], \\
f_\lambda^{\mathbf{b}1}(\underline{j}) &:= -{}^+\lambda_{\underline{j}} + d + n - m - \frac{1}{2} + j, \quad \text{if } 1 \leq j.
\end{aligned}
\tag{8.4.8}
$$

The inverse sends $f \in I^{m+n} \times I_-^\infty$ to $\lambda_f^{\mathbf{b}1} := \sum_{i=1}^{m+n} \lambda_{f,i}^{\mathbf{b}} \epsilon_i^{b_i} + \sum_{1\leq j} {}^+\lambda_{f,\underline{j}} \epsilon_{\underline{j}}^1 + d_f \epsilon_\infty^1$.

Note that for $s \in \{0,1\}$, the sum $\sum_{i=1}^{m+n} \lambda_{f,i}^{\mathbf{b}} \epsilon_i^{b_i} + \sum_{1\leq j} {}^+\lambda_{f,\underline{j}} \epsilon_{\underline{j}}^s$ lies in the root system of a finite-rank Lie superalgebra. Hence the following definitions make sense.

Definition 8.4.4. For $f, g \in I^{m+n} \times I_+^\infty$, we say $f \sim g$ if

$$
d_f = d_g \quad \text{and} \quad (\sum_{i=1}^{m+n} \lambda_{f,i}^{\mathbf{b}} \epsilon_i^{b_i} + \sum_{1\leq j} {}^+\lambda_{f,\underline{j}} \epsilon_{\underline{j}}^0) \sim (\sum_{i=1}^{m+n} \lambda_{g,i}^{\mathbf{b}} \epsilon_i^{b_i} + \sum_{1\leq j} {}^+\lambda_{g,\underline{j}} \epsilon_{\underline{j}}^0)
$$

in the sense of §7.1. We say $f \preceq_{\mathbf{b},0} g$ if

$$
f \sim g \quad \text{and} \quad \lambda_g^{\mathbf{b},0} - \lambda_f^{\mathbf{b},0} \in \mathbb{N}\Pi_{\mathbf{b},0}.
$$

We similarly define an equivalence $\sim$ and a partial ordering $\preceq_{\mathbf{b},1}$ on $I^{m+n} \times I_-^\infty$.

Definition 8.4.5. For $f, g \in I^{m+n} \times I_-^\infty$, we say $f \sim g$ if

$$
d_f = d_g \quad \text{and} \quad (\sum_{i=1}^{m+n} \lambda_{f,i}^{\mathbf{b}} \epsilon_i^{b_i} + \sum_{1\leq j} {}^+\lambda_{f,\underline{j}} \epsilon_{\underline{j}}^1) \sim (\sum_{i=1}^{m+n} \lambda_{g,i}^{\mathbf{b}} \epsilon_i^{b_i} + \sum_{1\leq j} {}^+\lambda_{g,\underline{j}} \epsilon_{\underline{j}}^1)
$$

in the sense of §7.1. We say $f \preceq_{\mathbf{b},1} g$ if

$$
f \sim g \quad \text{and } \lambda_g^{\mathbf{b},1} - \lambda_f^{\mathbf{b},1} \in \mathbb{N}\Pi_{\mathbf{b},1}.
$$

The following lemma follows from Definition 8.4.4, Definition 8.4.5, and Lemma 8.4.2.

Lemma 8.4.6. *1. Given* $f, g \in I^{m+n} \times I_{+}^{\infty}$ *such that* $g \preceq_{\mathbf{b},0} f$*, the set* $\{h \in I^{m+n} \times I_{+}^{\infty} \mid g \preceq_{\mathbf{b},0} h \preceq_{\mathbf{b},0} f\}$ *is finite.*

2. Given $f, g \in I^{m+n} \times I_{-}^{\infty}$ *such that* $g \preceq_{\mathbf{b},1} f$*, the set* $\{h \in I^{m+n} \times I_{-}^{\infty} \mid g \preceq_{\mathbf{b},1} h \preceq_{\mathbf{b},1} f\}$ *is finite.*

The following lemma is an infinite-rank analogue of Lemma 8.4.3.

Lemma 8.4.7. *For any* f*,* $g \in I^{m+n} \times I_{+}^{\infty}$ *(respectively,* $I^{m+n} \times I_{-}^{\infty}$*),* $f \sim g$ *if and only if* $\boldsymbol{wt}_{\mathbf{b},0}(f) = \boldsymbol{wt}_{\mathbf{b},0}(g)$ *(respectively,* $\boldsymbol{wt}_{\mathbf{b},1}(f) = \boldsymbol{wt}_{\mathbf{b},1}(g)$*).*

Proof. The lemma follows from Definition 8.4.4, Definition 8.4.5, and Lemma 8.4.3.

□

Chapter 9

$\imath$-Canonical bases and Kazhdan-Lusztig-type polynomials

In this chapter, suitably completed Fock spaces are constructed and shown to admit $\imath$-canonical as well as dual $\imath$-canonical bases. We introduce truncation maps to study the relations among bases for different Fock spaces, which then allow us to formulate $\imath$-canonical bases in certain semi-infinite Fock spaces.

9.1 The B-completion and Υ

Let $\mathbf{b}$ be a 0^m1^n-sequence. For $r \in \mathbb{N}$, we let

$$\pi_r : \mathbb{T}^{\mathbf{b}} \longrightarrow \mathbb{T}^{\mathbf{b}}_r \tag{9.1.1}$$

be the natural projection map with respect to the standard basis $\{M_f^{\mathbf{b}} \mid f \in I^{m+n}\}$ of $\mathbb{T}^{\mathbf{b}}$ (see (8.2.3)). We then let $\widetilde{\mathbb{T}}^{\mathbf{b}}$ be the completion of $\mathbb{T}^{\mathbf{b}}$ with respect to the descending sequence of subspaces $\{\ker \pi_r \mid r \geq 1\}$. Formally, every element in $\widetilde{\mathbb{T}}^{\mathbf{b}}$ is a possibly infinite linear combination of M_f, with $f \in I^{m+n}$. We let $\widehat{\mathbb{T}}^{\mathbf{b}}$ denote the subspace of $\widetilde{\mathbb{T}}^{\mathbf{b}}$ spanned by elements of the form

$$M_f + \sum_{g \prec_{\mathbf{b}} f} c_{gf}^{\mathbf{b}}(q) M_g, \qquad \text{for } c_{gf}^{\mathbf{b}}(q) \in \mathbb{Q}(q). \tag{9.1.2}$$

Definition 9.1.1. The $\mathbb{Q}(q)$-vector spaces $\widetilde{\mathbb{T}}^{\mathbf{b}}$ and $\widehat{\mathbb{T}}^{\mathbf{b}}$ are called the *A-completion* and *B-completion* of $\mathbb{T}^{\mathbf{b}}$, respectively.

Remark 9.1.2. The B-completion we defined here is different from the one defined in [CLW2], since they are based on *different partial orderings*. However, observing that the partial ordering used in [CLW2] is coarser than the partial ordering here, our B-completion here contains the B-completion in [CLW2, Definition 3.2] as a subspace. This fact very often allows us to cite directly the results therein.

Lemma 9.1.3. *Let $f \in I_r^{m+n}$. Then we have $M_f \in \mathbb{T}_r^{\mathbf{b}}$, and*

$$\pi_r(\Upsilon^{(l)} M_f) = \Upsilon^{(r)} M_f, \qquad \textit{for all } \ l \geq r.$$

Proof. Note that $\mathbb{N}\Pi_{2r+1} \subset \mathbb{N}\Pi_{2l+1}$, for $l \geq r$. It is clear from the construction of $\Upsilon^{(r)}$ in Theorem 2.3.1 that we have

$$\Upsilon^{(l)} = \Upsilon^{(r)} + \sum_{\mu \in \mathbb{N}\Pi_{2l+1} \backslash \mathbb{N}\Pi_{2r+1}} \Upsilon_\mu^{(l)}.$$

By $\mathbf{U}$-weight consideration, it is easy to see $\pi_r(\Upsilon^{(l)}_\mu M_f) = 0$ if $\mu \notin \mathbb{N}\Pi_{2r+1}$. Therefore

$$\pi_r(\Upsilon^{(l)} M_f) = \pi_r(\Upsilon^{(r)} M_f) = \Upsilon^{(r)} M_f.$$

The lemma follows. □

It follows from Lemma 9.1.3 that $\lim_{r\to\infty} \Upsilon^{(r)} M_f$, for any $f \in I^{m+n}$, is a well-defined element in $\widetilde{\mathbb{T}}^{\mathbf{b}}$. Therefore we have

$$\Upsilon M_f = \lim_{r\to\infty} \Upsilon^{(r)} M_f,$$

where Υ is the operator defined in (8.1.2).

Lemma 9.1.4. *For $f \in I^{m+n}$, we have*

$$\Upsilon M_f = M_f + \sum_{g \prec_{\mathbf{b}} f} r'_{gf}(q) M_g, \qquad \textit{for } r'_{gf}(q) \in \mathcal{A}. \tag{9.1.3}$$

In particular, we have $\Upsilon : \mathbb{T}^{\mathbf{b}} \to \widehat{\mathbb{T}}^{\mathbf{b}}$.

Proof. For any $u \in \mathbf{U}^-$ with $\mathbf{U}^\imath$-weight 0, $f \in I^{m+n}$, let $uM_f = \sum_g c_{gf} M_g$. Fix any g with $c_{gf} \neq 0$. Since u has $\mathbf{U}^\imath$-weight 0, we know by Lemma 8.4.3 that $g \sim f$ and so $\lambda^{\mathbf{b}}_g \sim \lambda^{\mathbf{b}}_f$. By a direct computation (by writing u in terms of Chevalley generator F's), it is easy to see that $u \in \mathbf{U}^-$ implies that $\lambda^{\mathbf{b}}_f - \lambda^{\mathbf{b}}_g \in \mathbb{N}\Pi_{\mathbf{b}}$. Hence we have $g \preceq_{\mathbf{b}} f$.

Recall that $\Upsilon_\mu \in \mathbf{U}^-$ for all μ and $\Upsilon_\mu \neq 0$ only if $\mu = \mu^\theta$, i.e., μ is of $\mathbf{U}^\imath$-weight 0. Hence we have the identity (9.1.3), where $r'_{gf}(q) \in \mathcal{A}$ follows from Theorem 4.4.2. The lemma follows. □

Lemma 9.1.5. *The map* $\Upsilon : \mathbb{T}^{\mathbf{b}} \to \widehat{\mathbb{T}}^{\mathbf{b}}$ *extends uniquely to a* $\mathbb{Q}(q)$*-linear map* $\Upsilon : \widehat{\mathbb{T}}^{\mathbf{b}} \to \widehat{\mathbb{T}}^{\mathbf{b}}$.

Proof. We adapt the proof of [CLW2, Lemma 3.7] here. To show that the map Υ extends to $\widehat{\mathbb{T}}^{\mathbf{b}}$ we need to show that if $y = M_f + \sum_{g \prec_{\mathbf{b}} f} r_g(q) M_g \in \widehat{\mathbb{T}}^{\mathbf{b}}$ for $r_g(q) \in \mathbb{Q}(q)$ then $\Upsilon y \in \widehat{\mathbb{T}}^{\mathbf{b}}$. By Lemma 9.1.4 and the definition of $\widehat{\mathbb{T}}^{\mathbf{b}}$, it remains to show that $\Upsilon y \in \widetilde{\mathbb{T}}^{\mathbf{b}}$. To that end, we note that if the coefficient of M_h in Υy is nonzero, then there exists $g \preceq_{\mathbf{b}} f$ such that $r'_{hg}(q) \neq 0$. Thus we have $h \preceq_{\mathbf{b}} g \preceq_{\mathbf{b}} f$. However, by Lemma 8.4.2 there are only finitely many such g's. Thus, only finitely many g's can contribute to the coefficient of M_h in Υy, and hence $\Upsilon y \in \widetilde{\mathbb{T}}^{\mathbf{b}}$. □

9.2 ı-Canonical bases

Anti-linear maps $\psi : \mathbb{T}^{\mathbf{b}}_r \to \mathbb{T}^{\mathbf{b}}_r$ and $\psi : \mathbb{T}^{\mathbf{b}} \to \widehat{\mathbb{T}}^{\mathbf{b}}$ were defined in [CLW2, §3.3] (recall Remark 9.1.2 that our B-completion contains the one therein as a subspace, so $\widehat{\mathbb{T}}^{\mathbf{b}}$ here can and will be understood in the sense of this paper). We define the map $\psi_\imath : \mathbb{T}^{\mathbf{b}} \to \widehat{\mathbb{T}}^{\mathbf{b}}$ by

$$\psi_\imath(M_f) := \Upsilon \psi(M_f). \tag{9.2.1}$$

Recall from §3.4 that $\mathbb{T}^{\mathbf{b}}_r$ is an ı-involutive $\mathbf{U}^\imath_r$-module with anti-linear involution $\psi_i^{(r)}$.

Lemma 9.2.1. *We have* $\pi_r(\psi_\imath(M_f)) = \psi_i^{(r)}(M_f)$, *for* $f \in I_r^{m+n}$.

Proof. Recall that $\psi_i^{(r)} = \Upsilon^{(r)} \psi^{(r)}$. By a variant of Lemma 9.1.3, we have

$$\pi_r(\psi_\imath(M_f)) = \pi_r(\Upsilon^{(r)} \psi(M_f))$$

by a **U**-weight consideration. Therefore we have

$$\pi_r(\psi_\imath(M_f)) = \Upsilon^{(r)}\pi_r(\psi(M_f)) = \Upsilon^{(r)}\psi^{(r)}(M_f),$$

where the last identity follows from [CLW2, Lemma 3.4]. The lemma follows. □

It follows immediately that we have

$$\psi_\imath(M_f) = \lim_{r\to\infty} \psi_\imath^{(r)}(M_f), \quad \text{for } f \in I^{m+n}. \tag{9.2.2}$$

Lemma 9.2.2. *Let $f \in I^{m+n}$. Then we have*

$$\psi_\imath(M_f) = M_f + \sum_{g \prec_{\mathbf{b}} f} r_{gf}(q) M_g, \quad \textit{for } r_{gf}(q) \in \mathcal{A}.$$

Hence the anti-linear map $\psi_\imath : \mathbb{T}^{\mathbf{b}} \to \widehat{\mathbb{T}}^{\mathbf{b}}$ extends to a map $\psi_\imath : \widehat{\mathbb{T}}^{\mathbf{b}} \to \widehat{\mathbb{T}}^{\mathbf{b}}$. Moreover $\psi_\imath$ is independent of the bracketing orders for the tensor product $\mathbb{T}^{\mathbf{b}}$.

Proof. Following [CLW2, Proposition 3.6] and Remark 9.1.2, we have

$$\psi(M_f) = M_f + \sum_{g \prec_{\mathbf{b}} f} r''_{gf}(q) M_g, \quad \text{for } r''_{gf}(q) \in \mathcal{A}.$$

Hence the first part of the lemma follows from Lemma 9.1.4.

We can show that the map $\psi_\imath : \mathbb{T}^{\mathbf{b}} \to \widehat{\mathbb{T}}^{\mathbf{b}}$ extends to a map $\psi_\imath : \widehat{\mathbb{T}}^{\mathbf{b}} \to \widehat{\mathbb{T}}^{\mathbf{b}}$ by applying the same argument used in the proof of Lemma 9.1.5. Since ψ is independent from the bracketing orders for the tensor product $\mathbb{T}^{\mathbf{b}}$ by [CLW2, Proposition 3.5], so is $\psi_\imath$. □

Lemma 9.2.3. *The map $\psi_\imath : \widehat{\mathbb{T}}^{\mathbf{b}} \longrightarrow \widehat{\mathbb{T}}^{\mathbf{b}}$ is an anti-linear involution.*

Proof. In order to prove the lemma, we need to prove that for fixed $f, h \in I^{m+n}$ with $h \prec_{\mathbf{b}} f$, we have

$$\sum_{h \preceq_{\mathbf{b}} g \preceq_{\mathbf{b}} f} r_{hg}(q)\overline{r_{gf}(q)} = \delta_{hf}.$$

By Lemma 8.4.2, there is only finitely many g such that $h \preceq_{\mathbf{b}} g \preceq_{\mathbf{b}} f$. Recall §3.4. We know $\psi_\imath^{(r)}$ is an involution. By (9.2.2), this is equivalent to the same identities in the finite-dimensional space $\mathbb{T}_r^{\mathbf{b}}$ with $r \gg 0$. Then the lemmas follows from Proposition 3.4.2. □

Thanks to Lemmas 9.2.2 and 9.2.3, we are in a position to apply [Lu2, Lemma 24.2.1] to the anti-linear involution $\psi_\imath : \widehat{\mathbb{T}}^{\mathbf{b}} \to \widehat{\mathbb{T}}^{\mathbf{b}}$ to establish the following.

Theorem 9.2.4. *The $\mathbb{Q}(q)$-vector space $\widehat{\mathbb{T}}^{\mathbf{b}}$ has unique $\psi_\imath$-invariant topological bases*

$$\{T_f^{\mathbf{b}} \mid f \in I^{m+n}\} \text{ and } \{L_f^{\mathbf{b}} \mid f \in I^{m+n}\}$$

such that

$$T_f^{\mathbf{b}} = M_f + \sum_{g \preceq_{\mathbf{b}} f} t_{gf}^{\mathbf{b}}(q) M_g^{\mathbf{b}}, \quad L_f^{\mathbf{b}} = M_f + \sum_{g \preceq_{\mathbf{b}} f} \ell_{gf}^{\mathbf{b}}(q) M_g^{\mathbf{b}},$$

with $t_{gf}^{\mathbf{b}}(q) \in q\mathbb{Z}[q]$, and $\ell_{gf}^{\mathbf{b}}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$, for $g \preceq_{\mathbf{b}} f$. (We shall write $t_{ff}^{\mathbf{b}}(q) = \ell_{ff}^{\mathbf{b}}(q) = 1$, $t_{gf}^{\mathbf{b}}(q) = \ell_{gf}^{\mathbf{b}}(q) = 0$ for $g \not\preceq_{\mathbf{b}} f$.)

Definition 9.2.5. $\{T_f^{\mathbf{b}} \mid f \in I^{m+n}\}$ and $\{L_f^{\mathbf{b}} \mid f \in I^{m+n}\}$ are call the *ı-canonical basis* and *dual ı-canonical basis* of $\widehat{\mathbb{T}}^{\mathbf{b}}$, respectively. The polynomials $t_{gf}^{\mathbf{b}}(q)$ and $\ell_{gf}^{\mathbf{b}}(q)$ are called *ı-Kazhdan-Lusztig (or ı-KL) polynomials.*

Conjecture 9.2.6. *1. (Positivity) We have $t_{gf}^{\mathbf{b}}(q) \in \mathbb{N}[q]$.*

2. The sum $T^{\mathbf{b}}_f = M_f + \sum_{g \preceq_{\mathbf{b}} f} t^{\mathbf{b}}_{gf}(q) M^{\mathbf{b}}_g$ *is finite, for all* $f \in I^{m+n}$.

Note that $t^{\mathbf{b}}_{gf}(1) \in \mathbb{N}$ and the finite sum claim in (2) at the $q = 1$ specialization holds by Theorem 11.6.1. Hence, the validity of the positivity conjecture (1) implies the validity of (2). We also raise the question on a possible positivity of the coefficients in the expansion of the $\imath$-canonical basis elements here relative to the (type A) canonical basis on $\widehat{\mathbb{T}}^{\mathbf{b}}$ as constructed in [CLW2].

9.3 Bar involution and q-wedges of $\mathbb{W}$

Let $k \in \mathbb{N} \cup \{\infty\}$. For $f = (f_{[m+n]}, f_{[\underline{k}]}) \in I^{m+n} \times I^k_+$, set

$$M^{\mathbf{b},0}_f := M^{\mathbf{b}}_{f_{[m+n]}} \otimes \mathcal{V}_{f_{[\underline{k}]}} \in \mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{V}.$$

Then $\{M^{\mathbf{b},0}_f \mid f \in I^{m+n} \times I^k_+\}$ forms a basis, called the *standard monomial basis*, of the $\mathbb{Q}(q)$-vector space $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{V}$. Similarly, $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W}$ admits a *standard monomial basis* given by

$$M^{\mathbf{b},1}_g := M^{\mathbf{b}}_{g_{[m+n]}} \otimes \mathcal{W}_{g_{[\underline{k}]}} \in \mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W},$$

where $g = (g_{[m+n]}, g_{[\underline{k}]}) \in I^{m+n} \times I^k_-$. Following [CLW2, §4], here we shall focus on the case $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W}$, while the case $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{V}$ is similar.

Let us consider $k \in \mathbb{N}$ first. As in [CLW2, §4], $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W}$ can be realized as a subspace of $\mathbb{T}^{\mathbf{b}} \otimes \mathbb{W}^{\otimes k} = \mathbb{T}^{(\mathbf{b},1^k)}$. Therefore we can define a B-completion of $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W}$, denoted by $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$, as the closure of the subspace $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W} \subset \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \mathbb{W}^{\otimes k} = \widehat{\mathbb{T}}^{(\mathbf{b},1^k)}$

with respect to the linear topology $\{\ker \pi_r \mid r \geq 1\}$ defined in §9.1. By construction $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ is invariant under the involution $\psi_\imath$, i.e., we have

$$\psi_\imath(M_f^{\mathbf{b},1}) = M_f^{\mathbf{b},1} + \sum_{g \prec_{(\mathbf{b},1^k)} f} r_{gf}(q) M_g^{\mathbf{b},1},$$

where $r_{gf}(q) \in \mathcal{A}$, and the sum running over $g \in I^{m+n} \times I_-^k$ is possibly infinite.

Remark 9.3.1. If $k = 0$, $M_f^{\mathbf{b},0}$ and $M_g^{\mathbf{b},1}$ are understood as $M_f^{\mathbf{b}}$ and $M_g^{\mathbf{b}}$, respectively; also, $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^0 \mathbb{W}$ and $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^0 \mathbb{V}$ are understood as $\widehat{\mathbb{T}}^{\mathbf{b}}$.

Recall the linear maps $\wedge_d^{k,l}$ defined in (8.3.1). For $l \geq k$ and each $d \in \mathbb{Z}$, define the $\mathbb{Q}(q)$-linear map

$$\mathrm{id} \otimes \wedge_d^{k,l} : \mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W} \longrightarrow \mathbb{T}^{\mathbf{b}} \otimes \wedge^l \mathbb{W}.$$

It is easy to check that the map $\mathrm{id} \otimes \wedge_d^{k,l}$ extends to the B-completions; that is, we have

$$\mathrm{id} \otimes \wedge_d^{k,l} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W} \longrightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^l \mathbb{W}.$$

Let $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge_d^\infty \mathbb{W} := \varinjlim \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ be the direct limit of the $\mathbb{Q}(q)$-vector spaces with respect to the linear maps $\mathrm{id} \otimes \wedge_d^{k,l}$. It is easy to see that $\mathbb{T}^{\mathbf{b}} \otimes \wedge_d^\infty \mathbb{W} \subset \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge_d^\infty \mathbb{W}$. Define the B-completion of $\mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{W}$ as follows:

$$\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{W} := \bigoplus_{d \in \mathbb{Z}} \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge_d^\infty \mathbb{W}. \tag{9.3.1}$$

By the same argument as in §8.3, we see that $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge_d^\infty \mathbb{W}$ and $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{W}$ are (topological) $\mathbf{U}$-modules, hence (topological) $\mathbf{U}^\imath$-modules.

Following the definitions of the partial orderings in Definition 8.4.1 and Definition 8.4.5, we see that $\mathbb{T}^{\mathbf{b}}\widehat{\otimes} \wedge^{\infty} \mathbb{W}$ is spanned by elements of the form

$$M_f^{\mathbf{b},1} + \sum_{g \prec_{\mathbf{b},1} f} c_{gf}(q) M_g^{\mathbf{b},1}, \quad \text{for } g, f \in I^{m+n} \times I_-^{\infty}.$$

Following [CLW2, §4.1], we can extend the anti-linear involution $\psi : \mathbb{T}^{\mathbf{b}}\widehat{\otimes} \wedge^k \mathbb{W} \to \mathbb{T}^{\mathbf{b}}\widehat{\otimes} \wedge^k \mathbb{W}$ to an anti-linear involution $\psi : \mathbb{T}^{\mathbf{b}}\widehat{\otimes} \wedge^{\infty} \mathbb{W} \to \mathbb{T}^{\mathbf{b}}\widehat{\otimes} \wedge^{\infty} \mathbb{W}$ such that

$$\psi(M_f^{\mathbf{b},1}) = M_f^{\mathbf{b},1} + \sum_{g \prec_{\mathbf{b},1} f} r''_{gf}(q) M_f^{\mathbf{b},1}, \quad \text{for } r''_{gf}(q) \in \mathcal{A}.$$

Here we have used the fact that our B-completion contains the B-completion in *loc. cit.* as a subspace (see Remark 9.1.2).

Following the definition of the B-completion $\mathbb{T}^{\mathbf{b}}\widehat{\otimes} \wedge^{\infty} \mathbb{W}$, we have Υ as a well-defined operator on $\mathbb{T}^{\mathbf{b}}\widehat{\otimes} \wedge^{\infty} \mathbb{W}$ such that

$$\Upsilon(M_f^{\mathbf{b},1}) = M_f^{\mathbf{b},1} + \sum_{g \prec_{\mathbf{b},1} f} r'_{gf}(q) M_f^{\mathbf{b},1}, \quad \text{for } r'_{gf}(q) \in \mathcal{A}.$$

Therefore we can define the anti-linear map

$$\psi_\imath := \Upsilon\psi : \mathbb{T}^{\mathbf{b}}\widehat{\otimes} \wedge^{\infty} \mathbb{W} \longrightarrow \mathbb{T}^{\mathbf{b}}\widehat{\otimes} \wedge^{\infty} \mathbb{W},$$

such that

$$\psi_\imath(M_f^{\mathbf{b},1}) = M_f^{\mathbf{b},1} + \sum_{g \prec_{\mathbf{b},1} f} r_{gf}(q) M_f^{\mathbf{b},1}, \quad \text{for } r_{gf}(q) \in \mathcal{A}.$$

Lemma 9.3.2. *Let $k \in \mathbb{N} \cup \{\infty\}$. The map $\psi_\imath : \mathbb{T}^{\mathbf{b}}\widehat{\otimes} \wedge^k \mathbb{W} \longrightarrow \mathbb{T}^{\mathbf{b}}\widehat{\otimes} \wedge^k \mathbb{W}$ is an involution.*

Proof. For $k \in \mathbb{N}$, the lemma was already established. For $k = \infty$, the lemma can be proved in the same way as Lemma 9.2.3 with the help of Lemma 8.4.6. □

9.4 Truncations

In this section we shall again only focus on $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W}$ for $k \in \mathbb{N} \cup \{\infty\}$. We shall use $f^{\underline{k}} \in I^{m+n} \times I^k_{\pm}$ as a short-hand notation for the restriction of $f_{[m+n]\cup[\underline{k}]}$ of a function $f \in I^{m+n} \times I^{\infty}_{\pm}$.

Now let us define the *truncation map* $\mathtt{Tr} : \mathbb{T}^{\mathbf{b}} \otimes \wedge^{\infty}\mathbb{W} \to \mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W}$, for $k \in \mathbb{N}$, as follows:

$$\mathtt{Tr}(m \otimes \mathcal{W}_h) = \begin{cases} m \otimes \mathcal{W}_{h_{[\underline{k}]}}, & \text{if } h(\underline{i}) - h(\underline{i+1}) = -1, \text{ for } i \geq k+1, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 9.4.1. *Let $k \in \mathbb{N}$. The truncation map $\mathtt{Tr} : \mathbb{T}^{\mathbf{b}} \otimes \wedge^{\infty}\mathbb{W} \to \mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W}$ is compatible with the partial orderings, and hence extends naturally to a $\mathbb{Q}(q)$-linear map $\mathtt{Tr} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{W} \to \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$.*

Proof. Let $f, g \in I^{m+n} \times I^{\infty}_{-}$ with $g \preceq_{\mathbf{b},1} f$. According to Definition 8.4.5, this means $f(\underline{i}) = g(\underline{i})$ for all $i \gg 0$. If $\mathtt{Tr}(M^{\mathbf{b},1}_f) \neq 0$ and $\mathtt{Tr}(M^{\mathbf{b},1}_g) \neq 0$, we must have $g(\underline{i}) = f(\underline{i})$, $\forall i \geq k+1$. Hence we have $\lambda^{(\mathbf{b},1^k)}_{g^{\underline{k}}} \preceq_{(\mathbf{b},1^k)} \lambda^{(\mathbf{b},1^k)}_{f^{\underline{k}}}$ by comparing Definition 8.4.5 with Definition 8.4.1. Thanks to Lemma 8.4.3 and Lemma 8.4.7, we have $g^{\underline{k}} \sim f^{\underline{k}}$ as well. Therefore we have $g^{\underline{k}} \preceq_{(\mathbf{b},1^k)} f^{\underline{k}}$.

Now suppose $\mathtt{Tr}(M^{\mathbf{b},1}_f) = 0$ and $g \preceq_{\mathbf{b},1} f$. If $f_{[\underline{\infty}]} = g_{[\underline{\infty}]}$, then $\mathtt{Tr}(M^{\mathbf{b},1}_g) = 0$. If not, choose $\underline{i}$ with i maximal such that $f(\underline{i}) \neq g(\underline{i})$. If $i \leq k$, then again we have $\mathtt{Tr}(M^{\mathbf{b},1}_g) = 0$. So suppose $i \geq k+1$. Since $g \preceq_{\mathbf{b},1} f$, we have $g(\underline{j}) = f(\underline{j})$ for $j \gg 0$ and $g(\underline{i}) < f(\underline{i})$. Hence there must be some $t \geq k+1$ such that $g(\underline{t}) - g(\underline{t+1}) \geq$

$f(\underline{t}) - f(\underline{t+1}) > -1$. Therefore $\mathtt{Tr}(M_g^{\mathbf{b},1}) = 0$. The lemma follows. □

Lemma 9.4.2. *The truncation map* $\boldsymbol{Tr} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{W} \to \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ *commutes with the anti-linear involution* $\psi_\imath$*, that is,*

$$\psi_\imath(\boldsymbol{Tr}(M_f^{\mathbf{b},1})) = \boldsymbol{Tr}(\psi_\imath(M_f^{\mathbf{b},1})), \quad \text{for } f \in I^{m+n} \times I_-^\infty.$$

Proof. Following [CLW2, Lemma 4.2], we know $\mathtt{Tr}$ commutes with ψ. As shown in the proof of [CLW2, Lemma 4.2], $\mathtt{Tr}$ is a homomorphism of $\mathbf{U}^-$-modules. By (8.1.2), we have $\Upsilon = \sum_{\mu \in \Lambda} \Upsilon_\mu$, where $\Upsilon_\mu \in \mathbf{U}^-$. The lemma follows. □

Proposition 9.4.3. *Let* $k \in \mathbb{N} \cup \{\infty\}$*. The anti-linear map* $\psi_\imath : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W} \to \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ *is an involution. Moreover, the space* $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ *has unique* $\psi_\imath$*-invariant topological bases*

$$\{T_f^{\mathbf{b},1} \mid f \in I^{m+n} \times I_-^k\} \quad \text{and} \quad \{L_f^{\mathbf{b},1} \mid f \in I^{m+n} \times I_-^k\}$$

such that

$$T_f^{\mathbf{b},1} = M_f^{\mathbf{b},1} + \sum_{g \prec_{(\mathbf{b},1^k)} f} t_{gf}^{\mathbf{b},1}(q) M_g^{\mathbf{b},1}, \quad L_f^{\mathbf{b},1} = M_f^{\mathbf{b},1} + \sum_{g \prec_{(\mathbf{b},1^k)} f} \ell_{gf}^{\mathbf{b},1}(q) M_g^{\mathbf{b},1}$$

with $t_{gf}^{\mathbf{b},1} \in q\mathbb{Z}[q]$*, and* $\ell_{gf}^{\mathbf{b},1}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$*. (We shall write* $t_{ff}^{\mathbf{b},1} = \ell_{ff}^{\mathbf{b},1}(q) = 1$*, and* $t_{gf}^{\mathbf{b},1} = \ell_{gf}^{\mathbf{b},1} = 0$*, for* $g \not\preceq_{(\mathbf{b},1^k)} f$*.)*

We call $\{T_f^{\mathbf{b},1}\}$ and $\{L_f^{\mathbf{b},1}\}$ the *ı-canonical* and *dual ı-canonical bases* of $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$. We conjecture that $t_{gf}^{\mathbf{b},1} \in \mathbb{N}[q]$.

Proposition 9.4.4. *Let $k \in \mathbb{N}$. The truncation map $\boldsymbol{Tr} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{W} \to \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^{k} \mathbb{W}$ preserves the standard, ı-canonical, and dual ı-canonical bases in the following sense: for $Y = M, L, T$ and $f \in I^{m+n} \times I_{-}^{\infty}$ we have*

$$\boldsymbol{Tr}\left(Y_f^{\mathbf{b},1}\right) = \begin{cases} Y_{f^{\underline{k}}}^{\mathbf{b},1}, & \text{if } f(\underline{i}) - f(\underline{i+1}) = -1, \text{ for } i \geq k+1, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, we have $t_{gf}^{\mathbf{b},1}(q) = t_{g^{\underline{k}} f^{\underline{k}}}^{\mathbf{b},1}(q)$ and $\ell_{gf}^{\mathbf{b},1}(q) = \ell_{g^{\underline{k}} f^{\underline{k}}}^{\mathbf{b},1}(q)$, for $g, f \in I^{m+n} \times I_{-}^{\infty}$ such that $f(\underline{i}) - f(\underline{i+1}) = g(\underline{i}) - g(\underline{i+1}) = -1$, for $i \geq k+1$.

Proof. The statement is true for $Y = M$ by definition. Lemma 9.4.1 and Lemma 9.4.2 now imply the statement for $Y = T$, L. □

9.5 Bar involution and q-wedges of $\mathbb{V}$

The constructions and statements in §9.3 and §9.4 have counterparts for $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{V}$, $k \in \mathbb{N} \cup \{\infty\}$. We shall state them without proofs. Let $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V}$ be the B-completion of $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{V}$. For $k \in \mathbb{N}$, we define the truncation map $\mathtt{Tr} : \mathbb{T}^{\mathbf{b}} \otimes \wedge^{\infty} \mathbb{V} \to \mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{V}$ by

$$\mathtt{Tr}(m \otimes \mathcal{V}_h) = \begin{cases} m \otimes \mathcal{V}_{h_{[k]}}, & \text{if } h(\underline{i}) - h(\underline{i+1}) = 1, \text{ for } i \geq k+1, \\ 0, & \text{otherwise .} \end{cases}$$

The truncation map $\mathtt{Tr}$ extends to the B-completions.

Proposition 9.5.1. *Let $k \in \mathbb{N} \cup \{\infty\}$. The bar map $\psi_\imath : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V} \to \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V}$ is an* *involution. Moreover, the space* $\mathbb{T}^{\mathbf{b}}\widehat{\otimes}\wedge^k\mathbb{V}$ *has unique* $\psi_\imath$*-invariant topological bases*

$$\{T^{\mathbf{b},0}_f \mid f \in I^{m+n}\times I^k_+\} \quad \text{and} \quad \{L^{\mathbf{b},0}_f \mid f \in I^{m+n}\times I^k_+\}$$

such that

$$T^{\mathbf{b},0}_f = M^{\mathbf{b},0}_f + \sum_{g\prec_{(\mathbf{b},0^k)} f} t^{\mathbf{b},0}_{gf}(q) M^{\mathbf{b},0}_g, \qquad L^{\mathbf{b},0}_f = M^{\mathbf{b},0}_f + \sum_{g\prec_{(\mathbf{b},0^k)} f} \ell^{\mathbf{b},0}_{gf}(q) M^{\mathbf{b},0}_g,$$

with $t^{\mathbf{b},0}_{gf}(q) \in q\mathbb{Z}[q]$, *and* $\ell^{\mathbf{b},0}_{gf}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$. *(We will write* $t^{\mathbf{b},0}_{ff}(q) = \ell^{\mathbf{b},0}_{ff}(q) = 1$, $t^{\mathbf{b},0}_{gf} = \ell^{\mathbf{b},0}_{gf} = 0$, *for* $g \not\preceq_{(\mathbf{b},0^k)} f$*.)*

We shall refer to the basis $\{T^{\mathbf{b},0}_f\}$ as the *ı-canonical basis* and refer to the basis $\{L^{\mathbf{b},0}_f\}$ the *dual ı-canonical basis* for $\mathbb{T}^{\mathbf{b}}\widehat{\otimes}\wedge^k\mathbb{V}$. Also we shall call the polynomials $t^{\mathbf{b},0}_{gf}(q)$, $t^{\mathbf{b},1}_{gf}(q)$, $\ell^{\mathbf{b},0}_{gf}(q)$ and $\ell^{\mathbf{b},1}_{gf}(q)$ the *ı-KL polynomials.*

Proposition 9.5.2. *Let* $k \in \mathbb{N}$*. The truncation map* $\boldsymbol{Tr} : \mathbb{T}^{\mathbf{b}}\widehat{\otimes}\wedge^\infty\mathbb{V} \to \mathbb{T}^{\mathbf{b}}\widehat{\otimes}\wedge^k\mathbb{V}$ *preserves the standard, ı-canonical, and dual ı-canonical bases in the following sense: for* $Y = M, L, T$ *and* $f \in I^{m+n}\times I^\infty_+$ *we have*

$$\boldsymbol{Tr}\left(Y^{\mathbf{b},0}_f\right) = \begin{cases} Y^{\mathbf{b},0}_{f^{\underline{k}}}, & \text{if } f(\underline{i}) - f(\underline{i+1}) = 1, \text{ for } i \geq k+1, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, we have $t^{\mathbf{b},0}_{gf}(q) = t^{\mathbf{b},0}_{g^{\underline{k}}f^{\underline{k}}}(q)$ *and* $\ell^{\mathbf{b},0}_{gf}(q) = \ell^{\mathbf{b},0}_{g^{\underline{k}}f^{\underline{k}}}(q)$, *for* $g, f \in I^{m+n}\times I^\infty_+$ *such that* $f(\underline{i}) - f(\underline{i+1}) = g(\underline{i}) - g(\underline{i+1}) = 1$, *for* $i \geq k+1$.

Chapter 10

Comparisons of $\imath$-canonical bases in different Fock spaces

In this chapter, we study the relations of $\imath$-canonical and dual $\imath$-canonical bases between three different pairs of Fock spaces.

10.1 Tensor versus q-wedges

As explained in §8.3, we can and will regard $\wedge^k \mathbb{V}$ as a subspace of $\mathbb{V}^{\otimes k}$, for a finite k.

Let $\mathbf{b}$ be a fixed $0^m 1^n$-sequence and $k \in \mathbb{N}$. We shall compare the $\imath$-canonical and dual $\imath$-canonical bases of $\mathbb{T}^{\mathbf{b}} \otimes \mathbb{V}^{\otimes k}$ and its subspace $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{V}$.

Let $f \in I^{m+n} \times I_+^k$. As before, we write the dual $\imath$-canonical basis element $L_f^{(\mathbf{b},0^k)}$

in $\mathbb{T}^{\mathbf{b}}\widehat{\otimes}\mathbb{V}^{\otimes k}$ and the corresponding dual $\imath$-canonical basis element $L_f^{\mathbf{b},0}$ in $\mathbb{T}^{\mathbf{b}}\widehat{\otimes}\wedge^k\mathbb{V}$ as

$$L_f^{(\mathbf{b},0^k)} = \sum_{g\in I^{m+n}\times I^k} \ell_{gf}^{(\mathbf{b},0^k)}(q) M_g^{(\mathbf{b},0^k)}, \tag{10.1.1}$$

$$L_f^{\mathbf{b},0} = \sum_{g\in I^{m+n}\times I_+^k} \ell_{gf}^{\mathbf{b},0}(q) M_g^{\mathbf{b},0}. \tag{10.1.2}$$

The following proposition states that the $\imath$-KL polynomials ℓ's in $\mathbb{T}^{\mathbf{b}}\widehat{\otimes}\wedge^k\mathbb{V}$ coincide with their counterparts in $\mathbb{T}^{\mathbf{b}}\widehat{\otimes}\mathbb{V}^{\otimes k}$.

Proposition 10.1.1. *Let $f,g\in I^{m+n}\times I_+^k$. Then $\ell_{gf}^{\mathbf{b},0}(q)=\ell_{gf}^{(\mathbf{b},0^k)}(q)$.*

Proof. The same argument in [CLW2, Proposition 4.9] applies here. □

Let $f\in I^{m+n}\times I_+^k$. Similarly as before we write the canonical basis element $T_f^{(\mathbf{b},0^k)}$ in $\mathbb{T}^{\mathbf{b}}\widehat{\otimes}\mathbb{V}^{\otimes k}$ and the canonical basis element $T_f^{\mathbf{b},0}$ in $\mathbb{T}^{\mathbf{b}}\widehat{\otimes}\wedge^k\mathbb{V}$ respectively as

$$T_f^{(\mathbf{b},0^k)} = \sum_{g\in I^{m+n}\times I^k} t_{gf}^{(\mathbf{b},0^k)}(q) M_g^{(\mathbf{b},0^k)}, \tag{10.1.3}$$

$$T_f^{\mathbf{b},0} = \sum_{g\in I^{m+n}\times I_+^k} t_{gf}^{\mathbf{b},0}(q) M_g^{\mathbf{b},0}. \tag{10.1.4}$$

Proposition 10.1.2. *For $f,\ g\in I^{m+n}\times I_+^k$, we have*

$$t_{gf}^{\mathbf{b},0}(q) = \sum_{\tau\in\mathfrak{S}_k} (-q)^{\ell(w_0^{(k)}\tau)} t_{g\cdot\tau, f\cdot w_0^{(k)}}^{(\mathbf{b},0^k)}(q).$$

Proof. Similar proof as for [CLW2, Proposition 4.10] works there.

Via identifying $\mathcal{V}_{g_{[\underline{k}]}}\equiv M^{(0^k)}_{g_{[\underline{k}]}\cdot w_0^{(k)}} L_{w_0^{(k)}}$, we have, as in [Br1, Lemma 3.8],

$$T_f^{\mathbf{b},0} = T^{(\mathbf{b},0^k)}_{f\cdot w_0^{(k)}} L_{w_0^{(k)}}.$$

A straightforward variation of [Br1, Lemma 3.4] using (10.1.3) gives us

$$
\begin{aligned}
T_f^{\mathbf{b},0} = T^{(\mathbf{b},0^k)}_{f\cdot w_0^{(k)}} L_{w_0^{(k)}} &= \sum_g t^{(\mathbf{b},0^k)}_{g,f\cdot w_0^{(k)}} M_g^{(\mathbf{b},0^k)} L_{w_0^{(k)}} \\
&= \sum_{\tau\in\mathfrak{S}_k} \sum_{g\in I^{m+n}\times I_+^k} t^{(\mathbf{b},0^k)}_{g\cdot\tau,f\cdot w_0^{(k)}} M_{g\cdot\tau}^{(\mathbf{b},0^k)} L_{w_0^{(k)}} \\
&= \sum_{\tau\in\mathfrak{S}_k} \sum_{g\in I^{m+n}\times I_+^k} t^{(\mathbf{b},0^k)}_{g\cdot\tau,f\cdot w_0^{(k)}} (-q)^{\ell(\tau^{-1}w_0^{(k)})} M_g^{\mathbf{b},0} \\
&= \sum_{g\in I^{m+n}\times I_+^k} \left(\sum_{\tau\in\mathfrak{S}_k} t^{(\mathbf{b},0^k)}_{g\cdot\tau,f\cdot w_0^{(k)}} (-q)^{\ell(w_0^{(k)}\tau)} \right) M_g^{\mathbf{b},0}.
\end{aligned}
$$

The proposition now follows by comparing with (10.1.4). □

Remark 10.1.3. The counterparts of Propositions 10.1.1 and 10.1.2 hold if we replace $\mathbb{V}$ by $\mathbb{W}$.

10.2 Adjacent $\imath$-canonical bases

Two 0^m1^n-sequences $\mathbf{b}$, $\mathbf{b}'$ of the form $\mathbf{b} = (\mathbf{b}^1, 0, 1, \mathbf{b}^2)$ and $\mathbf{b}' = (\mathbf{b}^1, 1, 0, \mathbf{b}^2)$ are called *adjacent*. Now we compare the $\imath$-canonical as well as dual $\imath$-canonical bases in Fock spaces $\widehat{\mathbb{T}}^{\mathbf{b}}$ and $\widehat{\mathbb{T}}^{\mathbf{b}'}$, for adjacent 0^m1^n-sequences $\mathbf{b}$ and $\mathbf{b}'$.

In type A setting, a strategy was developed in [CLW2, §5] for such a comparison of canonical basis in adjacent Fock spaces. We observe that the strategy applies to our current setting essentially without any change, *under the assumption that* $\mathbf{b}^1$ *is nonempty*. So we will need not copy over all the details from *loc. cit.* to this paper.

Let us review the main ideas in type A from [CLW2, §5]. We will restrict the discussion here to the case of canonical basis while the case of dual canonical basis is entirely similar. The starting point is to start with the rank two setting and compare the canonical bases in the B-completions of $\mathbb{V} \otimes \mathbb{W}$ and $\mathbb{W} \otimes \mathbb{V}$. These canonical bases can be easily computed: they are either standard monomials or a sum of two standard monomials with some q-power coefficients. The problem is that the partial orderings on $\mathbb{V}\widehat{\otimes}\mathbb{W}$ and $\mathbb{W}\widehat{\otimes}\mathbb{V}$ are not compatible. This problem is overcome by a simple observation that matching up the canonical bases directly is actually a $\mathbf{U}$-module isomorphism of their respective linear spans, which is denoted by $\mathcal{R} : \mathbb{U} \stackrel{\cong}{\to} \mathbb{U}'$. So the idea is to work with these smaller spaces $\mathbb{U}$ and $\mathbb{U}'$ instead of the B-completions directly. We use $\mathbb{U}$ and $\mathbb{U}'$ to build up smaller completions of the *adjacent* $\mathbb{T}^{\mathbf{b}}$ and $\mathbb{T}^{\mathbf{b}'}$, which are used to match the canonical bases by $T^{\mathbf{b}}_f \mapsto T^{\mathbf{b}'}_{f^{\mathbb{U}}}$. Here the index shift $f \mapsto f^{\mathbb{U}}$ is shown to correspond exactly under the bijection $I^{m+n} \leftrightarrow X(m|n)$ to the shift $\lambda \mapsto \lambda^{\mathbb{U}}$ on $X(m|n)$ in Remark 10.2.2 below (which occurs when comparing the tilting modules relative to adjacent Borel subalgebras of type $\mathbf{b}$ and $\mathbf{b}'$).

Now we restrict ourselves to two adjacent sequences $\mathbf{b}$ and $\mathbf{b}'$, where $\mathbf{b}^1$ is nonempty; this is sufficient for the main application of determining completely the irreducible and tilting characters in category $\mathcal{O}_{\mathbf{b}}$ for $\mathfrak{osp}(2m+1|2n)$-modules (see however Remark 10.2.1 below for the removal of the restriction). We will compare two Fock spaces of the form $\mathbb{T}^{\mathbf{b}^1} \otimes \mathbb{V} \otimes \mathbb{W} \otimes \mathbb{T}^{\mathbf{b}^2}$ and $\mathbb{T}^{\mathbf{b}^1} \otimes \mathbb{W} \otimes \mathbb{V} \otimes \mathbb{T}^{\mathbf{b}^2}$, where $\mathbf{b}^1$ is nonempty. The coideal property of the coproduct of the algebra $\mathbf{U}^\imath$ in Proposition 2.2.4 allows us to consider $\mathbb{V} \otimes \mathbb{W}$ and $\mathbb{W} \otimes \mathbb{V}$ as $\mathbf{U}$-modules (not as $\mathbf{U}^\imath$-modules), and so the type

A strategy of [CLW2, §5] applies verbatim to our setting.

Remark 10.2.1. Now we consider $\mathbb{V} \otimes \mathbb{W}$ and $\mathbb{W} \otimes \mathbb{V}$ as $\mathbf{U}^\imath$-modules (instead of $\mathbf{U}$-modules). The $\imath$-canonical bases on their respective B-completions can be computed explicitly, though the computation in this case (corresponding to the BGG category of $\mathfrak{osp}(3|2)$) is much more demanding; the formulas are much messier and many more cases need to be considered, in contrast to the easy type A case of $\mathfrak{gl}(1|1)$. Denote by $\mathbb{U}_\flat$ and $\mathbb{U}'_\flat$ the linear spans of these canonical bases respectively. We are able to verify by a direct computation that matching the canonical bases suitably produces a $\mathbf{U}^\imath$-module isomorphism $\mathbb{U}_\flat \to \mathbb{U}'_\flat$. (The details will take quite a few pages and hence will be skipped.) Accepting this, the strategy of [CLW2, §5] is adapted to work equally well for comparing the (dual) $\imath$-canonical bases between arbitrary adjacent Fock spaces $\widehat{\mathbb{T}}^{\mathbf{b}}$ and $\widehat{\mathbb{T}}^{\mathbf{b}'}$.

Remark 10.2.2. Let $\mathbf{b} = (\mathbf{b}^1, 0, 1, \mathbf{b}^2)$ and $\mathbf{b}' = (\mathbf{b}^1, 1, 0, \mathbf{b}^2)$ be adjacent 0^m1^n-sequences. Let α be the isomorphic simple root of $\mathfrak{osp}(2m+1|2n)$ corresponding to the pair $0, 1$ in $\mathbf{b}$. Following [CLW2, §6], we introduce the notation associated to $\lambda \in X(m|n)$:

$$\lambda^{\mathbb{L}} = \begin{cases} \lambda, & \text{if } (\lambda, \alpha) = 0 \\ \lambda - \alpha, & \text{if } (\lambda, \alpha) \neq 0, \end{cases} \qquad \lambda^{\mathbb{U}} = \begin{cases} \lambda - 2\alpha, & \text{if } (\lambda, \alpha) = 0 \\ \lambda - \alpha, & \text{if } (\lambda, \alpha) \neq 0. \end{cases}$$

Then we have the following identification of simple and tilting modules (see [PS] and

[CLW2, Lemma 6.2, Theorem 6.10]):

$$L_{\mathbf{b}}(\lambda) = L_{\mathbf{b}'}(\lambda^{\mathbb{L}}), \quad T_{\mathbf{b}}(\lambda) = T_{\mathbf{b}'}(\lambda^{\mathbb{U}}), \qquad \text{for } \lambda \in X(m|n).$$

10.3 Combinatorial super duality

For a partition $\mu = (\mu_1, \mu_2, \ldots)$, we denote its conjugate partition by $\mu' = (\mu'_1, \mu'_2, \ldots)$. We define a $\mathbb{Q}(q)$-linear isomorphism $\natural : \wedge^{\infty}_{d}\mathbb{V} \longrightarrow \wedge^{\infty}_{d}\mathbb{W}$ (for each $d \in \mathbb{Z}$), or equivalently define $\natural : \wedge^{\infty}\mathbb{V} \to \wedge^{\infty}\mathbb{W}$ by

$$\natural(|\lambda, d\rangle) = |\lambda'_{*}, d\rangle, \qquad \text{for } \lambda \in \mathcal{P}, d \in \mathbb{Z}.$$

The following is a straightforward generalization of [CWZ, Theorem 6.3].

Proposition 10.3.1. *The map $\natural : \wedge^{\infty}_{d}\mathbb{V} \longrightarrow \wedge^{\infty}_{d}\mathbb{W}$ (for each $d \in \mathbb{Z}$) or $\natural : \wedge^{\infty}\mathbb{V} \longrightarrow \wedge^{\infty}\mathbb{W}$ is an isomorphism of* **U***-modules.*

Proof. It is a well-known fact that $\wedge^{\infty}_{d}\mathbb{V}$ and $\wedge^{\infty}_{d}\mathbb{W}$ as **U**-modules are both isomorphic to the level one integrable module associated to the dth fundamental weight (by the same proof as for [CWZ, Proposition 6.1]; also see the references therein).

Now the proof of the proposition is the same as for [CWZ, Theorem 6.3], which is our special case with $d = 0$. □

This isomorphism of **U**-modules $\natural : \wedge^{\infty}\mathbb{V} \to \wedge^{\infty}\mathbb{W}$ induces an isomorphism of **U**-modules

$$\natural_{\mathbf{b}} := \mathrm{id} \otimes \natural : \mathbb{T}^{\mathbf{b}} \otimes \wedge^{\infty}\mathbb{V} \longrightarrow \mathbb{T}^{\mathbf{b}} \otimes \wedge^{\infty}\mathbb{W}.$$

Let $f \in I^{m+n} \times I_+^\infty$. There exists unique $\lambda \in \mathcal{P}$ and $d \in \mathbb{Z}$ such that $|\lambda, d\rangle = \mathcal{V}_{f_{[\infty]}}$. We define $f^\natural$ to be the unique element in $I^{m+n} \times I_-^\infty$ determined by $f^\natural(i) = f(i)$, for $i \in [m+n]$, and $\mathcal{W}_{f^\natural_{[\infty]}} = |\lambda'_*, d\rangle$. The assignment $f \mapsto f^\natural$ gives a bijection (cf. [CWZ])

$$\natural : I^{m+n} \times I_+^\infty \longrightarrow I^{m+n} \times I_-^\infty. \tag{10.3.1}$$

If we write $\lambda_f^{\mathbf{b},0} = \sum_{i=1}^{m+n} \lambda^{\mathbf{b}}_{f,i} \epsilon_i^{b_i} + \sum_{1 \le j} {}^+\lambda_{f,\underline{j}} \epsilon^0_{\underline{j}} + d_f \epsilon^0_\infty \in X^{\infty,+}_{\mathbf{b},0}$ under the bijection defined in (8.4.5), then we have

$$\lambda_{f^\natural}^{\mathbf{b},1} = \sum_{i=1}^{m+n} \lambda^{\mathbf{b}}_{f,i} \epsilon_i^{b_i} + \sum_{1 \le j} {}^+\lambda'_{f,\underline{j}} \epsilon^1_{\underline{j}} + d_f \epsilon^1_\infty \in X^{\infty,+}_{\mathbf{b},1}. \tag{10.3.2}$$

The following is the combinatorial counterpart of the super duality on representation theory in Theorem 11.5.1 . We refer to [CLW2, Theorem 4.8] for a type A version, on which our proof below is based.

Theorem 10.3.2. *Let* $\mathbf{b}$ *be a* 0^m1^n*-sequence.*

1. *The isomorphism* $\natural_{\mathbf{b}}$ *respects the Bruhat orderings and hence extends to an isomorphism of the* B*-completions* $\natural_{\mathbf{b}} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{V} \rightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{W}$.

2. *The map* $\natural_{\mathbf{b}}$ *commutes with the bar involutions.*

3. *The map* $\natural_{\mathbf{b}}$ *preserves the* $\imath$*-canonical and dual* $\imath$*-canonical bases. More precisely, we have* $\natural_{\mathbf{b}}(M_f^{\mathbf{b},0}) = M_{f^\natural}^{\mathbf{b},1}$, $\natural_{\mathbf{b}}(T_f^{\mathbf{b},0}) = T_{f^\natural}^{\mathbf{b},1}$, *and* $\natural_{\mathbf{b}}(L_f^{\mathbf{b},0}) = L_{f^\natural}^{\mathbf{b},1}$, *for* $f \in I^{m+n} \times I_+^\infty$.

4. *We have the following identifications of* $\imath$*-KL polynomials:* $\ell_{gf}^{\mathbf{b},0}(q) = \ell_{g^\natural f^\natural}^{\mathbf{b},1}(q)$, *and* $t_{gf}^{\mathbf{b},0}(q) = t_{g^\natural f^\natural}^{\mathbf{b},1}(q)$, *for all* $g,\ f \in I^{m+n} \times I_+^\infty$.

Proof. The statements (2)-(4) follows from (1) by the same argument as [CLW2, Theorem 4.8]. It remains to prove (1).

Recall the definition of the partial orderings in Definitions 8.4.4 and 8.4.5. To prove (1), we need to show for any f, $g \in I^{m+n} \times I_+^{\infty}$, $g \preceq_{\mathbf{b},0} f$ if and only if $g^{\natural} \preceq_{\mathbf{b},1} f^{\natural}$. This is equivalent to say that $f \sim g$ and $\lambda_g^{\mathbf{b},0} \preceq_{\mathbf{b},0} \lambda_f^{\mathbf{b},0}$ if and only if $f^{\natural} \sim g^{\natural}$ and $\lambda_{g^{\natural}}^{\mathbf{b},1} \preceq_{\mathbf{b},1} \lambda_{f^{\natural}}^{\mathbf{b},1}$ by Definitions 8.4.4 and 8.4.5.

Since $\natural_{\mathbf{b}} : \mathbb{T}^{\mathbf{b}} \otimes \wedge^{\infty}\mathbb{V} \to \mathbb{T}^{\mathbf{b}} \otimes \wedge^{\infty}\mathbb{W}$ is an isomorphism of $\mathbf{U}^{\imath}$-modules, by Lemma 8.4.7, we have $f \sim g$ if and only if $f^{\natural} \sim g^{\natural}$. We shall assume that $f \sim g$, hence $f^{\natural} \sim g^{\natural}$ for the rest of this proof.

We shall only prove that $\lambda_g^{\mathbf{b},0} \preceq_{\mathbf{b},0} \lambda_f^{\mathbf{b},0}$ implies $\lambda_{g^{\natural}}^{\mathbf{b},1} \preceq_{\mathbf{b},1} \lambda_{f^{\natural}}^{\mathbf{b},1}$ here, as the converse is entirely similar. We write

$$\lambda_f^{\mathbf{b},0} - \lambda_g^{\mathbf{b},0} = a(-\epsilon_1^{b_1}) + \sum_{i=1}^{m+n-1} a_i(\epsilon_i^{b_i} - \epsilon_{i+1}^{b_{i+1}}) + a_{m+n}(\epsilon_{m+n}^{b_{m+n}} - \epsilon_{\underline{1}}^{0}) + \sum_{i=1} a_{\underline{i}}(\epsilon_{\underline{i}}^{0} - \epsilon_{\underline{i+1}}^{0}),$$

where all coefficients are in $\mathbb{N}$ and $a_{\underline{i}} = 0$ for all but finitely many i. Set

$$\lambda_h^{\mathbf{b},0} := \lambda_f^{\mathbf{b},0} - a(-\epsilon_1^{b_1})$$

for some $h \in I^{m+n} \times I_+^{\infty}$. Apparently we have $\lambda_g^{\mathbf{b},0} \preceq_{\mathbf{b},0} \lambda_h^{\mathbf{b},0} \preceq_{\mathbf{b},0} \lambda_f^{\mathbf{b},0}$.

Note that $\lambda_h^{\mathbf{b},0}$ actually dominates $\lambda_g^{\mathbf{b},0}$ with respect to the Bruhat ordering of type A defined in [CLW2, §2.3]. Therefore following [CLW2, Theorem 4.8] and Remark 9.1.2, we have

$$\lambda_{g^{\natural}}^{\mathbf{b},1} \preceq_{\mathbf{b},1} \lambda_{h^{\natural}}^{\mathbf{b},1}. \tag{10.3.3}$$

On the other hand, by definitions of $\lambda_h^{\mathbf{b},0}$ and the isomorphism of $\natural$, we have that

$\lambda_{h^\natural}^{\mathbf{b},1} = \lambda_{f^\natural}^{\mathbf{b},1} - a(-\epsilon_1^{b_1})$, and hence $\lambda_{h^\natural}^{\mathbf{b},1} \preceq_{\mathbf{b},1} \lambda_{f^\natural}^{\mathbf{b},1}$. Combining this with (10.3.3) implies that $\lambda_{g^\natural}^{\mathbf{b},1} \preceq_{\mathbf{b},1} \lambda_{f^\natural}^{\mathbf{b},1}$. The statement (1) is proved. $\square$

Chapter 11

Kazhdan-Lusztig theory of type B and $\imath$-canonical basis

In this chapter, we formulate connections between Fock spaces and Grothendieck groups of various BGG categories. We establish relations of simple as well as tilting modules between a BGG category and its parabolic subcategory. We show that $\mathbf{U}^\imath$ at $q = 1$ are realized as translation functors in the BGG category. Finally, we establish the Kazhdan-Lusztig theory for $\mathfrak{osp}(2m+1|2n)$, which is the main goal of the paper.

11.1 Grothendieck groups and Fock spaces

Recall the Fock space $\mathbb{T}^{\mathbf{b}}$ in §8.2. Starting with an $\mathcal{A}$-lattice $\mathbb{T}^{\mathbf{b}}_{\mathcal{A}}$ spanned by the standard monomial basis of the $\mathbb{Q}(q)$-vector space $\mathbb{T}^{\mathbf{b}}$, we define $\mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} = \mathbb{Z} \otimes_{\mathcal{A}} \mathbb{T}^{\mathbf{b}}_{\mathcal{A}}$ where $\mathcal{A}$ acts on $\mathbb{Z}$ with $q = 1$. For any u in the $\mathcal{A}$-lattice $\mathbb{T}^{\mathbf{b}}_{\mathcal{A}}$, we denote by $u(1)$ its image in $\mathbb{T}^{\mathbf{b}}_{\mathbb{Z}}$.

Recall the category $\mathcal{O}_{\mathbf{b}}$ from §7.3. Let $\mathcal{O}^{\Delta}_{\mathbf{b}}$ be the full subcategory of $\mathcal{O}_{\mathbf{b}}$ consisting of all modules possessing a finite **b**-Verma flag. Let $[\mathcal{O}^{\Delta}_{\mathbf{b}}]$ be its Grothendieck group. The following lemma is immediate from the bijection $I^{m+n} \leftrightarrow X(m|n)$ $(\lambda \leftrightarrow f^{\mathbf{b}}_{\lambda})$ given by (8.4.1) and (8.4.2).

Lemma 11.1.1. *The map*

$$\Psi : [\mathcal{O}^{\Delta}_{\mathbf{b}}] \longrightarrow \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}}, \qquad [M_{\mathbf{b}}(\lambda)] \mapsto M_{f^{\mathbf{b}}_{\lambda}}^{\mathbf{b}}(1),$$

defines an isomorphism of $\mathbb{Z}$-modules.

Recall the category $\mathcal{O}^{\underline{k}}_{\mathbf{b},0}$ from §7.3. We shall denote $\mathcal{O}^{\underline{k},\Delta}_{\mathbf{b},0}$ the full subcategory of $\mathcal{O}^{\underline{k}}_{\mathbf{b},0}$ consisting of all modules possessing finite parabolic Verma flags. Recall in §8.3, we defined the q-wedge spaces $\wedge^k\mathbb{V}$ and $\wedge^k\mathbb{W}$. Recall a bijection $X^{\underline{\infty},+}_{\mathbf{b},0} \to I^{m+n} \times I^{\infty}_{+}, \lambda \mapsto f^{\mathbf{b}0}_{\lambda}$ from (8.4.5). Similarly, we have a bijection

$$X^{\underline{k},+}_{\mathbf{b},0} \longrightarrow I^{m+n} \times I^{k}_{+}, \quad \lambda \mapsto f^{\mathbf{b}0}_{\lambda}.$$

(Here $f^{\mathbf{b}0}_{\lambda}$ is understood as the natural restriction to the part $[m+n] \times \underline{k}$.) Now the following lemma is clear.

Lemma 11.1.2. *For $k \in \mathbb{N} \cup \{\infty\}$, the map*

$$\Psi : [\mathcal{O}^{\underline{k},\Delta}_{\mathbf{b},0}] \longrightarrow \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} \otimes \wedge^k \mathbb{V}_{\mathbb{Z}}, \qquad [M^{\underline{k}}_{\mathbf{b},0}(\lambda)] \mapsto M^{\mathbf{b},0}_{f^{\mathbf{b}0}_{\lambda}}(1),$$

defines an isomorphism of $\mathbb{Z}$-modules.

We have abused the notation Ψ for all the isomorphisms unless otherwise specified, since they share the same origin. For $k \in \mathbb{N}\cup\{\infty\}$, we define $[[\mathcal{O}^{\underline{k},\Delta}_{\mathbf{b},0}]]$ as the completion of $[\mathcal{O}^{\underline{k},\Delta}_{\mathbf{b},0}]$ such that the extensions of Ψ

$$\Psi : [[\mathcal{O}^{\underline{k},\Delta}_{\mathbf{b},0}]] \longrightarrow \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} \widehat{\otimes} \wedge^k \mathbb{V}_{\mathbb{Z}} \tag{11.1.1}$$

are isomorphism of $\mathbb{Z}$-modules. Recall the category $\mathcal{O}^{\underline{k}}_{\mathbf{b},1}$ from §7.3. We shall denote $\mathcal{O}^{\underline{k},\Delta}_{\mathbf{b},1}$ the full subcategory of $\mathcal{O}^{\underline{k}}_{\mathbf{b},1}$ consisting of all modules possessing parabolic Verma flags. Recall a bijection $X^{\infty,+}_{\mathbf{b},1} \longrightarrow I^{m+n} \times I^{\infty}_{-}, \lambda \mapsto f^{\mathbf{b}1}_{\lambda}$ from (8.4.7). Similarly, we have a bijection

$$X^{\underline{k},+}_{\mathbf{b},1} \longrightarrow I^{m+n} \times I^{k}_{-}, \quad \lambda \mapsto f^{\mathbf{b}1}_{\lambda}.$$

(Here $f^{\mathbf{b}1}_{\lambda}$ is understood as the natural restriction to the part $[m+n] \times \underline{k}$.) Now the following lemma is clear.

Lemma 11.1.3. *For $k \in \mathbb{N} \cup \{\infty\}$, the map*

$$\Psi : [\mathcal{O}^{\underline{k},\Delta}_{\mathbf{b},1}] \longrightarrow \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} \otimes \wedge^k \mathbb{W}_{\mathbb{Z}}, \qquad [M^{\underline{k}}_{\mathbf{b},1}(\lambda)] \mapsto M^{\mathbf{b},1}_{f^{\mathbf{b}1}_{\lambda}}(1),$$

is an isomorphism of $\mathbb{Z}$-modules.

For $k \in \mathbb{N} \cup \{\infty\}$, we define $[[\mathcal{O}^{\underline{k},\Delta}_{\mathbf{b},1}]]$ as the completion of $[\mathcal{O}^{\underline{k},\Delta}_{\mathbf{b},1}]$ such that the extensions of Ψ

$$\Psi : [[\mathcal{O}^{\underline{k},\Delta}_{\mathbf{b},1}]] \longrightarrow \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} \widehat{\otimes} \wedge^k \mathbb{W}_{\mathbb{Z}} \tag{11.1.2}$$

are isomorphism of $\mathbb{Z}$-modules.

Proposition 11.1.4. *The truncation maps defined here are compatible under the isomorphism ψ with the truncations in Propositions 9.4.4 and 9.5.2. More precisely, we have the following commutative diagrams,*

$$\begin{array}{ccc} [[\mathcal{O}^{\infty,\Delta}_{\mathbf{b},0}]] & \xrightarrow{\Psi} & \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}}\widehat{\otimes} \wedge^{\infty} \mathbb{V}_{\mathbb{Z}} \\ \downarrow \mathfrak{tr}_0 & & \downarrow Tr \\ [[\mathcal{O}^{k,\Delta}_{\mathbf{b},0}]] & \xrightarrow{\Psi} & \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}}\widehat{\otimes} \wedge^{k} \mathbb{V}_{\mathbb{Z}} \end{array} \qquad \begin{array}{ccc} [[\mathcal{O}^{\infty,\Delta}_{\mathbf{b},1}]] & \xrightarrow{\Psi} & \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}}\widehat{\otimes} \wedge^{\infty} \mathbb{W}_{\mathbb{Z}} \\ \downarrow \mathfrak{tr}_1 & & \downarrow Tr \\ [[\mathcal{O}^{k,\Delta}_{\mathbf{b},1}]] & \xrightarrow{\Psi} & \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}}\widehat{\otimes} \wedge^{k} \mathbb{W}_{\mathbb{Z}} \end{array}$$

Proof. The proposition follows by a direct computation using the respective standard bases $\{[M^{\infty}_{\mathbf{b},0}(\lambda)]\}$ and $\{[M^{\infty}_{\mathbf{b},1}(\lambda)]\}$, and applying Propositions 9.4.4, 9.5.2, and 7.4.1.

□

11.2 Comparison of characters

Let $\mathbf{b}$ be a fix 0^m1^n-sequence. For $k \in \mathbb{N}$, consider the extended sequences $(\mathbf{b}, 0^k)$ and $(\mathbf{b}, 1^k)$. Associated to the extended sequences, we introduced in Chapter 7 the categories $\mathcal{O}^{m+k|n}_{(\mathbf{b},0^k)}$ and $\mathcal{O}^{m|n+k}_{(\mathbf{b},1^k)}$, as well as the parabolic categories $\mathcal{O}^{k}_{\mathbf{b},0}$ and $\mathcal{O}^{k}_{\mathbf{b},1}$, respectively.

For $\lambda \in X^{k,+}_{\mathbf{b},0}$, we can express the simple module $[L_{(\mathbf{b},0^k)}(\lambda)]$ in terms of Verma modules as follows:

$$[L_{(\mathbf{b},0^k)}(\lambda)] = \sum_{\mu \in X(m+k|n)} a_{\mu\lambda}[M_{(\mathbf{b},0^k)}(\mu)], \qquad \text{for } a_{\mu\lambda} \in \mathbb{Z}.$$

Since the simple modules $\{L_{(\mathbf{b},0^k)}(\lambda) = L^{k}_{\mathbf{b},0}(\lambda) \mid \lambda \in X^{k,+}_{\mathbf{b},0}\}$ also lie in the parabolic category $\mathcal{O}^{\underline{k}}_{\mathbf{b},0}$, we can express them in terms of parabolic Verma modules as follows:

$$[L_{(\mathbf{b},0^k)}(\lambda)] = \sum_{\nu \in X^{\underline{k},+}_{\mathbf{b},0}} b_{\nu\lambda}[M^{\underline{k}}_{\mathbf{b},0}(\nu)], \quad \text{for } b_{\nu\lambda} \in \mathbb{Z}.$$

Recall that $M^{\underline{k}}_{\mathbf{b},0}(\lambda) = \mathrm{Ind}^{\mathfrak{osp}(2m+1|2n|2k)}_{\mathfrak{p}^{\underline{k}}_{\mathbf{b},0}} L_0(\lambda)$. By the Weyl character formula applied to $L_0(\lambda)$, we obtain that $a_{\nu\lambda} = b_{\nu\lambda}$, for $\nu, \lambda \in X^{\underline{k},+}_{\mathbf{b},0}$. This proves the following.

Proposition 11.2.1. *Let $\lambda \in X^{\underline{k},+}_{\mathbf{b},0}$ and let $\xi \in X^{\underline{k},+}_{\mathbf{b},1}$. Then we have*

$$[L_{(\mathbf{b},0^k)}(\lambda)] = \sum_{\mu \in X(m+k|n)} a_{\mu\lambda}[M_{(\mathbf{b},0^k)}(\mu)] = \sum_{\nu \in X^{\underline{k},+}_{\mathbf{b},0}} a_{\nu\lambda}[M^{\underline{k}}_{\mathbf{b},0}(\nu)].$$

$$[L_{(\mathbf{b},1^k)}(\xi)] = \sum_{\mu \in X(m|n+k)} a'_{\mu\xi}[M_{(\mathbf{b},1^k)}(\mu)] = \sum_{\eta \in X^{\underline{k},+}_{\mathbf{b},1}} a'_{\eta\xi}[M^{\underline{k}}_{\mathbf{b},1}(\eta)].$$

Now we proceed with the tilting modules. Let $\lambda \in X^{\underline{k},+}_{\mathbf{b},0}$ and $\xi \in X^{\underline{k},+}_{\mathbf{b},1}$. We can express the tilting modules $T_{(\mathbf{b},0^k)}(\lambda)$ and $T_{(\mathbf{b},0^k)}(\xi)$ in terms of Verma modules as follows:

$$[T_{(\mathbf{b},0^k)}(\lambda)] = \sum_{\mu \in X(m+k|n)} c_{\mu\lambda}[M_{(\mathbf{b},0^k)}(\mu)], \quad \text{for } c_{\mu\lambda} \in \mathbb{Z},$$

$$[T_{(\mathbf{b},1^k)}(\xi)] = \sum_{\eta \in X(m|n+k)} c'_{\eta\xi}[M_{(\mathbf{b},1^k)}(\eta)], \quad \text{for } c'_{\eta\xi} \in \mathbb{Z}.$$

Recall the tilting modules $T^{\underline{k}}_{\mathbf{b},0}(\lambda)$ and $T^{\underline{k}}_{\mathbf{b},1}(\xi)$ in the parabolic categories $\mathcal{O}^{\underline{k}}_{\mathbf{b},0}$ and $\mathcal{O}^{\underline{k}}_{\mathbf{b},1}$. The following proposition is a counterpart of [CLW2, Proposition 8.7] with the same proof, which is based on [So3, Br2]. Recall $w_0^{(k)}$ denotes the longest element in $\mathfrak{S}_k$.

Proposition 11.2.2. *1. Let $\lambda \in X^{\underline{k},+}_{\mathbf{b},0}$, and write $T^{\underline{k}}_{\mathbf{b},0}(\lambda) = \sum_{\nu \in X^{\underline{k},+}_{\mathbf{b},0}} d_{\nu\lambda} M^{\underline{k}}_{\mathbf{b},0}(\nu)$. Then we have $d_{\nu\lambda} = \sum_{\tau \in \mathfrak{S}_k} (-1)^{\ell(\tau w_0^{(k)})} c_{\tau\cdot\nu, w_0^{(k)}\cdot\lambda}$.*

2. *Let* $\xi \in X^{\underline{k},+}_{\mathbf{b},1}$, *and write* $T^{\underline{k}}_{\mathbf{b},1}(\xi) = \sum_{\eta \in X^{\underline{k},+}_{\mathbf{b},1}} d'_{\eta\xi} M^{\underline{k}}_{\mathbf{b},1}(\eta)$. *Then we have*

$d'_{\eta\xi} = \sum_{\tau \in \mathfrak{S}_k} (-1)^{\ell(\tau w_0^{(k)})} c'_{\tau\cdot\eta, w_0^{(k)}\cdot\lambda}$.

11.3 Translation functors

In [Br1], Brundan established a $\mathbf{U}$-module isomorphism between the Grothendieck group of the category $\mathcal{O}$ of $\mathfrak{gl}(m|n)$ and a Fock space (at $q = 1$), where some properly defined translation functors act as Chevalley generators of $\mathbf{U}$ at $q = 1$. Here we shall develop a type B analogue in the setting of $\mathfrak{osp}(2m+1|2n)$.

Let V be the natural $\mathfrak{osp}(2m+1|2n)$-module. Notice that V is self-dual. Recalling §7.1, we have the following decomposition of $\mathcal{O}_{\mathbf{b}}$:

$$\mathcal{O}_{\mathbf{b}} = \bigoplus_{\chi_\lambda} \mathcal{O}_{\mathbf{b},\chi_\lambda},$$

where χ_λ runs over all the integral central characters. Thanks to Lemma 8.4.3, we can set $\mathcal{O}_{\mathbf{b},\gamma} := \mathcal{O}_{\mathbf{b},\chi_\lambda}$, if $\mathrm{wt}_{\mathbf{b}}(\lambda) = \gamma$ (recall $\mathrm{wt}_{\mathbf{b}}$ from (8.4.3)). For $r \geq 0$, let $S^r V$ be the rth supersymmetric power of V. For $i \in \mathbb{I}^\imath$, $M \in \mathcal{O}_{\mathbf{b},\gamma}$, we define the following translation functors in $\mathcal{O}_{\mathbf{b}}$:

$$f^{(r)}_{\alpha_i} M := \mathrm{pr}_{\gamma - r(\varepsilon_{i-\frac{1}{2}} - \varepsilon_{i+\frac{1}{2}})}(M \otimes S^r V), \tag{11.3.1}$$

$$e^{(r)}_{\alpha_i} M := \mathrm{pr}_{\gamma + r(\varepsilon_{i-\frac{1}{2}} - \varepsilon_{i+\frac{1}{2}})}(M \otimes S^r V), \tag{11.3.2}$$

$$tM := \mathrm{pr}_{\gamma}(M \otimes V), \tag{11.3.3}$$

where pr_μ is the natural projection from $\mathcal{O}_{\mathbf{b}}$ to $\mathcal{O}_{\mathbf{b},\mu}$.

Note that the translation functors naturally induce operators on the Grothendieck group $[\mathcal{O}_{\mathbf{b}}^{\Delta}]$, denoted by $f_{\alpha_i}^{(r)}$, $e_{\alpha_i}^{(r)}$, and t as well. The following two lemmas are analogues of [Br1, Lemmas 4.23 and 4.24]. Since they are standard, we shall skip the proofs.

Lemma 11.3.1. *On the category $\mathcal{O}_{\mathbf{b}}$, the translation functors $f_{\alpha_i}^{(r)}$, $e_{\alpha_i}^{(r)}$, and t are all exact. They commute with the τ-duality.*

Lemma 11.3.2. *Let $\nu_1, \dots, \nu_N$ be the set of weights of S^rV ordered so that $v_i > v_j$ if and only if $i < j$. Let $\lambda \in X(m|n)$. Then $M_{\mathbf{b}}(\lambda) \otimes S^rV$ has a multiplicity-free Verma flag with subquotients isomorphic to $M_{\mathbf{b}}(\lambda+\nu_1), \dots, M_{\mathbf{b}}(\lambda+\nu_N)$ in the order from bottom to top.*

Denote by $\mathbf{U}_{\mathbb{Z}} = \mathbb{Z} \otimes_{\mathcal{A}} \mathbf{U}_{\mathcal{A}}$ the specialization of the $\mathcal{A}$-algebra $\mathbf{U}_{\mathcal{A}}$ at $q = 1$. Hence we can view $\mathbb{T}_{\mathbb{Z}}^{\mathbf{b}}$ as a $\mathbf{U}_{\mathbb{Z}}$-module. Thanks to (2.2.1) and (2.2.2), we know $\iota(f_{\alpha_i}^{(r)})$ and $\iota(e_{\alpha_i}^{(r)})$ lie in $\mathbf{U}_{\mathcal{A}}$, hence their specializations at $q = 1$ in $\mathbf{U}_{\mathbb{Z}}$ act on $\mathbb{T}_{\mathbb{Z}}^{\mathbf{b}}$.

Proposition 11.3.3. *Under the identification $[\mathcal{O}_{\mathbf{b}}^{\Delta}]$ and $\mathbb{T}_{\mathbb{Z}}^{\mathbf{b}}$ via the isomorphism Ψ, the translation functors $f_{\alpha_i}^{(r)}$, $e_{\alpha_i}^{(r)}$, and t act in the same way as the specialization of $f_{\alpha_i}^{(r)}$, $e_{\alpha_i}^{(r)}$, and t in $\mathbf{U}^{\imath}$.*

Proof. Let us show in detail that the actions match for $r = 1$ (i.e. ignoring the higher divided powers). Set

$$\lambda + \rho_{\mathbf{b}} = \sum_{j=1}^{m+n} a_j \epsilon_j^{b_j} \in X(m|n) \text{ and } \gamma = \mathtt{wt}_{\mathbf{b}}(\lambda).$$

Then we have $M_{\mathbf{b}}(\lambda) \in \mathcal{O}_{\mathbf{b},\gamma}$. By Lemma 11.3.2, $M_{\mathbf{b}}(\lambda) \otimes V$ has a multiplicity-free Verma flag with subquotients isomorphic to $M_{\mathbf{b}}(\lambda + \epsilon_1)$, $\dots$, $M_{\mathbf{b}}(\lambda + \epsilon_{m+n})$, $M_{\mathbf{b}}(\lambda)$, $M_{\mathbf{b}}(\lambda - \epsilon_{m+n})$, $\dots$, $M_{\mathbf{b}}(\lambda - \epsilon_1)$. Applying the projection $\mathrm{pr}_{\gamma-(\varepsilon_{i-\frac{1}{2}}-\varepsilon_{i+\frac{1}{2}})}$ to the filtration, we obtain that $f_{\alpha_i} M_{\mathbf{b}}(\lambda)$ has a multiplicity-free Verma flag with subquotients isomorphic to $M_{\mathbf{b}}(\lambda \pm \epsilon_j)$ such that $a_j = \pm(i - \frac{1}{2})$ respectively.

On the other hand, we have $\Psi(M_{\mathbf{b}}(\lambda)) = M^{\mathbf{b}}_{f^{\mathbf{b}}_{\lambda}}(1)$. Recall the formulas for the embedding $\imath$ from Proposition 2.2.1. Suppose $\iota(f_{\alpha_i}) M^{\mathbf{b}}_{f^{\mathbf{b}}_{\lambda}}(1) = \sum_g M^{\mathbf{b}}_g(1)$, for $i \in \mathbb{I}^{\imath}$. It is easy to see that for $M^{\mathbf{b}}_g$ to appear in the summands, we must have $\lambda^{\mathbf{b}}_g + \rho_{\mathbf{b}} = \lambda + \rho_{\mathbf{b}} \pm \epsilon_j$ such that $a_j = \pm(i - \frac{1}{2})$ respectively. Hence the action of $\iota(f_{\alpha_i})$ on $\mathbb{T}^{\mathbf{b}}_{\mathbb{Z}}$ matchs with the translation functor f_{α_i} on $\left[\mathcal{O}^{\Delta}_{\mathbf{b}}\right]$ under Ψ.

Similar argument works for the translation functor e_{α_i}.

Applying the projection pr_{γ} to the Verma flag filtration of $M_{\mathbf{b}}(\lambda) \otimes V$, we obtain that $tM_{\mathbf{b}}(\lambda)$ from (11.3.2) has a multiplicity-free Verma flag with subquotients isomorphic to $M_{\mathbf{b}}(\lambda)$ and $M_{\mathbf{b}}(\lambda \pm \epsilon_j)$ such that $a_j = \mp\frac{1}{2}$ respectively. Then one checks that the action of $\iota(t)$ on $\mathbb{T}^{\mathbf{b}}_{\mathbb{Z}}$ matchs with the translation functor t on $\left[\mathcal{O}^{\Delta}_{\mathbf{b}}\right]$ under Ψ.

For the general divided powers, the proposition follows from a direct computation using Lemma 11.3.2 , [Br1, Corollary 4.25], and the expressions of $\iota(f^{(r)}_{\alpha_i})$ and $\iota(e^{(r)}_{\alpha_i})$ in (2.2.1) and (2.2.2). We leave the details to the reader. $\square$

11.4 Classical KL theory reformulated

The following is a reformulation of the Kazhdan-Lusztig theory for Lie algebra of type B, which was established by [BB, BK, So1, So3]; also see [Vo]. Recall for $\mathbf{b} = (0^m)$ we have $\mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} = \mathbb{V}^{\otimes m}_{\mathbb{Z}}$.

Theorem 11.4.1. *Let* $\mathbf{b} = (0^m)$ *and let* $k \in \mathbb{N} \cup \{\infty\}$. *Then the isomorphism* $\Psi : [[\mathcal{O}^{k,\Delta}_{\mathbf{b},0}]] \rightarrow \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} \widehat{\otimes} \wedge^k \mathbb{V}_{\mathbb{Z}}$ *in* (11.1.1) *satisfies*

$$\Psi([L^{k}_{\mathbf{b},0}(\lambda)]) = L^{\mathbf{b},0}_{f^{\mathbf{b}0}_{\lambda}}(1), \qquad \Psi([T^{k}_{\mathbf{b},0}(\lambda)]) = T^{\mathbf{b},0}_{f^{\mathbf{b}0}_{\lambda}}(1), \qquad \textit{for } \lambda \in X^{k,+}_{\mathbf{b},0}.$$

Proof. For $k \in \mathbb{N}$, the theorem follows easily from Remark 5.3.3 that the parabolic Kazhdan-Lusztig basis is matched with the $\imath$-canonical basis. The case with $k = \infty$ follows from Proposition 9.5.2 and Proposition 7.4.1. □

11.5 Super duality and Fock spaces

Theorem 11.5.1. *[CLW2, Theorem 7.2] There is an equivalence of categories (called super duality)* $\mathsf{SD} : \mathcal{O}^{\infty,\Delta}_{\mathbf{b},0} \rightarrow \mathcal{O}^{\infty,\Delta}_{\mathbf{b},1}$ *such that the induced map* $\mathsf{SD} : [[\mathcal{O}^{\infty,\Delta}_{\mathbf{b},0}]] \rightarrow [[\mathcal{O}^{\infty,\Delta}_{\mathbf{b},1}]]$ *satisfies, for any* $Y = M$, L, *or* T,

$$\mathsf{SD}[Y^{\infty}_{\mathbf{b},0}(\lambda)] = [Y^{\infty}_{\mathbf{b},1}(\lambda^{\natural})], \qquad \textit{for } \lambda \in X^{\infty,+}_{\mathbf{b},0}.$$

Proposition 11.5.2. *Let* $\mathbf{b}$ *be any* 0^m1^n*-sequence. Assume that the isomorphism* $\Psi : [[\mathcal{O}^{\infty,\Delta}_{\mathbf{b},0}]] \rightarrow \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} \widehat{\otimes} \wedge^{\infty} \mathbb{V}_{\mathbb{Z}}$ *in* (11.1.1) *satisfies*

$$\Psi([L^{\infty}_{\mathbf{b},0}(\lambda)]) = L^{\mathbf{b},0}_{f^{\mathbf{b}0}_{\lambda}}(1), \qquad \Psi([T^{\infty}_{\mathbf{b},0}(\lambda)]) = T^{\mathbf{b},0}_{f^{\mathbf{b}0}_{\lambda}}(1), \qquad \textit{for } \lambda \in X^{\infty,+}_{\mathbf{b},0}.$$

Then the isomorphism $\Psi : [[\mathcal{O}^{\infty,\Delta}_{\mathbf{b},1}]] \to \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} \widehat{\otimes} \wedge^\infty \mathbb{W}_{\mathbb{Z}}$ *satisfies*

$$\Psi([L^{\infty}_{\mathbf{b},1}(\lambda)]) = L^{\mathbf{b},1}_{f^{\mathbf{b}1}_{\lambda}}(1), \qquad \Psi([T^{\infty}_{\mathbf{b},1}(\lambda)]) = T^{\mathbf{b},1}_{f^{\mathbf{b}1}_{\lambda}}(1), \qquad \textit{for } \lambda \in X^{\infty,+}_{\mathbf{b},1}.$$

Proof. By the combinatorial super duality in Theorem 10.3.2, we have the following isomorphism

$$\natural_{\mathbf{b}} : \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} \widehat{\otimes} \wedge^\infty \mathbb{V}_{\mathbb{Z}} \longrightarrow \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} \widehat{\otimes} \wedge^\infty \mathbb{W}_{\mathbb{Z}},$$

which preserves the $\imath$-canonical and dual $\imath$-canonical bases. Combining this with the super duality, we have the following diagram:

$$\begin{array}{ccc} [[\mathcal{O}^{\infty,\Delta}_{\mathbf{b},0}]] & \xrightarrow{\Psi} & \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} \widehat{\otimes} \wedge^\infty \mathbb{V}_{\mathbb{Z}} \\ \downarrow{\scriptstyle \mathsf{SD}} & & \downarrow{\scriptstyle \natural_{\mathbf{b}}} \\ [[\mathcal{O}^{\infty,\Delta}_{\mathbf{b},1}]] & \xrightarrow{\Psi} & \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} \widehat{\otimes} \wedge^\infty \mathbb{W}_{\mathbb{Z}} \end{array} \tag{11.5.1}$$

where SD is the super duality from Theorem 11.5.1.

With the help of the basis $\{[M^{\infty}_{\mathbf{b},0}(\lambda)]\}$, it is easy to check that the diagram (11.5.1) commutes. Hence we have the following two commutative diagrams:

$$\begin{array}{ccc} [L^{\infty}_{\mathbf{b},0}(\lambda)] & \longmapsto & L^{\mathbf{b},0}_{f^{\mathbf{b}0}_{\lambda}}(1) \\ \downarrow & & \downarrow \\ [L^{\infty}_{\mathbf{b},1}(\lambda^\natural)] & \longmapsto & L^{\mathbf{b},1}_{f^{\mathbf{b}1}_{\lambda^\natural}}(1) \end{array} \qquad \begin{array}{ccc} [T^{\infty}_{\mathbf{b},0}(\lambda)] & \longmapsto & T^{\mathbf{b},0}_{f^{\mathbf{b}0}_{\lambda}}(1) \\ \downarrow & & \downarrow \\ [T^{\infty}_{\mathbf{b},1}(\lambda^\natural)] & \longmapsto & T^{\mathbf{b},1}_{f^{\mathbf{b}1}_{\lambda^\natural}}(1) \end{array}$$

The two horizontal arrows on the bottom give us the proposition. □

11.6 $\imath$-KL theory for $\mathfrak{osp}$

We can now formulate and prove the main result of Part 2, which is a generalization of [CLW2, Theorem 8.11] (Brundan's conjecture [Br1]) to the ortho-symplectic Lie superalgebra $\mathfrak{osp}(2m+1|2n)$.

Theorem 11.6.1. *For any 0^m1^n-sequence $\mathbf{b}$ starting with* 0*, the isomorphism* Ψ : $[[\mathcal{O}^{\Delta}_{\mathbf{b}}]] \to \widehat{\mathbb{T}}^{\mathbf{b}}_{\mathbb{Z}}$ *in* (11.1.1) *(with $k=0$) satisfies*

$$\Psi([L_{\mathbf{b}}(\lambda)]) = L^{\mathbf{b}}_{f^{\mathbf{b}}_{\lambda}}(1), \qquad \Psi([T_{\mathbf{b}}(\lambda)]) = T^{\mathbf{b}}_{f^{\mathbf{b}}_{\lambda}}(1), \qquad \textit{for } \lambda \in X(m|n).$$

The following proposition is a counterpart of [CLW2, Theorem 8.8]. It can now be proved in the same way as in *loc. cit.* as we have done all the suitable preparations in §10.2 (as in [CLW2, §6]). We will skip the details.

Proposition 11.6.2. *Let $\mathbf{b} = (\mathbf{b}^1, 0, 1, \mathbf{b}^2)$ and $\mathbf{b}' = (\mathbf{b}^1, 1, 0, \mathbf{b}^2)$ be adjacent 0^m1^n-sequences with nonempty $\mathbf{b}^1$ starting with* 0*. Then Theorem 11.6.1 holds for $\mathbf{b}$ if and only if it holds for $\mathbf{b}'$.*

Remark 11.6.3. The assumption "nonempty $\mathbf{b}^1$ starting with 0" in Proposition 11.6.2 is removable, if we apply the observation in Remark 10.2.1. Subsequently, we can also remove a similar assumption on $\mathbf{b}$ from Proposition 11.5.2 and Theorem 11.6.1. Theorem 11.6.1 in its current form already solves completely the irreducible and tilting character problem on $\mathcal{O}_{\mathbf{b}}$ for an arbitrary $\mathbf{b}$, since $\mathcal{O}_{\mathbf{b}}$ is independent of $\mathbf{b}$ and the relations between the simple/tilting characters in $\mathcal{O}_{\mathbf{b}}$ for different $\mathbf{b}$ are understood (see Remark 10.2.2).

Proof of Theorem 11.6.1. The overall strategy of the proof is by induction on n, following the proof of Brundan's KL-type conjecture in [CLW2]. The inductive procedure, denoted by $\imath\mathtt{KL}(m|n)\,\forall m \geq 1 \Longrightarrow \imath\mathtt{KL}(m|n+1)$, is divided into the following steps:

$$
\begin{aligned}
\imath\mathtt{KL}(m+k|n)\ \ \forall k &\Longrightarrow \imath\mathtt{KL}(m|n|k)\ \ \forall k,\ \text{by changing Borels} &(11.6.1)\\
&\Longrightarrow \imath\mathtt{KL}(m|n|\underline{k})\ \ \forall k,\ \text{by passing to parabolic} &(11.6.2)\\
&\Longrightarrow \imath\mathtt{KL}(m|n|\underline{\infty}),\ \text{by taking } k\mapsto\infty &(11.6.3)\\
&\Longrightarrow \imath\mathtt{KL}(m|n+\underline{\infty}),\ \text{by super duality} &(11.6.4)\\
&\Longrightarrow \imath\mathtt{KL}(m|n+1)\ \ \forall m,\ \text{by truncation.} &(11.6.5)
\end{aligned}
$$

It is instructive to write down the Fock spaces corresponding to the steps above:

$$
\begin{aligned}
\mathbb{V}^{\otimes(m+k)}\otimes\mathbb{W}^{\otimes n}\ \ \forall k &\Longrightarrow \mathbb{V}^{\otimes m}\otimes\mathbb{W}^{\otimes n}\otimes\mathbb{V}^{\otimes k}\ \ \forall k\\
&\Longrightarrow \mathbb{V}^{\otimes m}\otimes\mathbb{W}^{\otimes n}\otimes\wedge^{k}\mathbb{V}\ \ \forall k\\
&\Longrightarrow \mathbb{V}^{\otimes m}\otimes\mathbb{W}^{\otimes n}\otimes\wedge^{\infty}\mathbb{V}\\
&\Longrightarrow \mathbb{V}^{\otimes m}\otimes\mathbb{W}^{\otimes n}\otimes\wedge^{\infty}\mathbb{W}\\
&\Longrightarrow \mathbb{V}^{\otimes m}\otimes\mathbb{W}^{\otimes(n+1)}\quad \forall m\geq 1.
\end{aligned}
$$

A complete proof would be simply a copy from the proof of [CLW2, Theorem 8.10], as we are in a position to take care of each step of (11.6.1)–(11.6.5). Here we will be contented with specifying how each step follows and refer the reader to the proof of [CLW2, Theorem 8.10] for details.

Thanks to Theorem 5.3.2, the base case for the induction, $\imath\mathtt{KL}(m|0)$, is equivalent to the original Kazhdan-Lusztig conjecture [KL] for $\mathfrak{so}(2m+1)$, which is a theorem of [BB] and [BK] (and extended to all singular weights by [So1]); The tilting module characters were due to [So2, So3].

Step (11.6.1) is a special case of Proposition 11.6.2.

Step (11.6.2) is based on §10.1 (Propositions 10.1.1 and 10.1.2) and §11.2 (Propositions 11.2.1 and 11.2.2.

Step (11.6.3) is based on Proposition 11.1.4.

Step (11.6.4) is based on Proposition 11.5.2.

Step (11.6.5) is based on Propositions 7.4.1, 11.1.4, and 9.4.4 (with $k = 1$ therein). The theorem is proved. $\square$

Remark 11.6.4. There is a similar Fock space formulation for various parabolic subcategories of $\mathfrak{osp}(2m+1|2n)$-modules, which can be derived as a corollary to Theorem 11.6.1 and Remark 11.6.3. Theorem 11.6.1 also raises the natural question on Koszul graded lift for $\mathcal{O}_{\mathfrak{b}}$; cf. [BGS].

Chapter 12

BGG category of $\mathfrak{osp}(2m+1|2n)$-modules of half-integer weights

In this chapter we shall deal with a version of BGG category for $\mathfrak{osp}(2m+1|2n)$ associated with a *half-integer* weight set $'X(m|n)$. The relevant quantum symmetric pair turns out to be the $r \mapsto \infty$ limit of $(\mathbf{U}_{2r}, \mathbf{U}^{\jmath}_{r})$ established in Chapter 6. This chapter is a variant of Chapters 7-11, in which we will formulate the main theorems while skipping the identical proofs.

12.1 Setups for half-integer weights

Let us first set up some notations. Switching the sets of integers and half-integers in (8.1.1), we set

$$\mathbb{I} = \cup_{r=0}^{\infty}\mathbb{I}_{2r} = \mathbb{Z} + \frac{1}{2}, \qquad \mathbb{I}^{\jmath} = \cup_{r=0}^{\infty}\mathbb{I}_r^{\jmath} = \mathbb{N} + \frac{1}{2}, \qquad I = \mathbb{Z}. \tag{12.1.1}$$

Recall from Chapter 6 the finite-rank quantum symmetric pairs $(\mathbf{U}_{2r}, \mathbf{U}_r^{\jmath})$ with embedding $\jmath : \mathbf{U}_r^{\jmath} \to \mathbf{U}_{2r}$. Let

$$\mathbf{U}^{\jmath} := \bigcup_{r=0}^{\infty} \mathbf{U}_r^{\jmath}, \qquad \mathbf{U} := \bigcup_{r=0}^{\infty} \mathbf{U}_{2r}.$$

The pair $(\mathbf{U}, \mathbf{U}^{\jmath})$ forms a quantum symmetric pair as well, with the obvious induced embedding $\jmath : \mathbf{U}^{\jmath} \to \mathbf{U}$. Let $\Pi := \bigcup_{r=0}^{\infty} \Pi_{2r}$ be the simple system of $\mathbf{U}$. Recall the intertwiner $\Upsilon^{(r)}$ of the pair $(\mathbf{U}_{2r}, \mathbf{U}_r^{\jmath})$. Note that $\Upsilon_{\mu}^{(r+1)} = \Upsilon_{\mu}^{(r)}$, for $\mu \in \mathbb{N}\Pi_{2r}$, and this allows us to define

$$\Upsilon_{\mu} = \lim_{r \to \infty} \Upsilon_{\mu}^{(r)}, \qquad \text{for } \mu \in \mathbb{N}\Pi.$$

We then define the formal sum (which lies in some completion of $\mathbf{U}^{-}$)

$$\Upsilon := \sum_{\mu \in \mathbb{N}\Pi} \Upsilon_{\mu}, \tag{12.1.2}$$

which shall be viewed as a well-defined operator on $\mathbf{U}$-modules that we are concerned.

Introduce the following set of half-integer weights

$${}'X(m|n) := \sum_{i=1}^{m} (\mathbb{Z} + \frac{1}{2})\epsilon_i + \sum_{j=1}^{n} (\mathbb{Z} + \frac{1}{2})\epsilon_{\overline{j}}. \tag{12.1.3}$$

Let $\mathbf{b} = (b_1, \ldots, b_{m+n})$ be an arbitrary $0^n 1^m$-sequence. We first define a partial ordering on I^{m+n}, which depends on the sequence $\mathbf{b}$. There is a natural bijection $I^{m+n} \leftrightarrow {}'X(m|n)$, $f \mapsto \lambda_f^{\mathbf{b}}$ and $\lambda \mapsto f_\lambda^{\mathbf{b}}$, defined formally by the same formulas (8.4.1)-(8.4.2) for the bijection $I^{m+n} \leftrightarrow X(m|n)$ therein, though I here has a different meaning.

Recall the Bruhat ordering $\preceq_{\mathbf{b}}$ given by (7.1.3) on $\mathfrak{h}_{m|n}$ and hence on ${}'X(m|n)$. We now transport the ordering on ${}'X(m|n)$ by the above bijection to I^{m+n}.

Definition 12.1.1. The Bruhat ordering or $\mathbf{b}$-Bruhat ordering $\preceq_{\mathbf{b}}$ on I^{m+n} is defined as follows: For $f, g \in I^{m+n}$, $f \preceq_{\mathbf{b}} g \Leftrightarrow \lambda_f^{\mathbf{b}} \preceq_{\mathbf{b}} \lambda_g^{\mathbf{b}}$. We also say $f \sim g$ if $\lambda_f^{\mathbf{b}} \sim \lambda_g^{\mathbf{b}}$.

A BGG category ${}'\mathcal{O}_{\mathbf{b}}$ of $\mathfrak{osp}(2m+1|2n)$-modules with weight set ${}'X(m|n)$ is defined in the same way as in Definition 7.3.1, where the weight set was taken to be $X(m|n)$. Again, the category ${}'\mathcal{O}_{\mathbf{b}}$ contains several distinguished modules: the $\mathbf{b}$-Verma modules $M_{\mathbf{b}}(\lambda)$, simple modules $L_{\mathbf{b}}(\lambda)$, and tilting modules $T_{\mathbf{b}}(\lambda)$, for $\lambda \in' X(m|n)$.

12.2 Fock spaces and $\jmath$-canonical bases

Let $\mathbb{V} := \sum_{a \in I} \mathbb{Q}(q) v_a$ be the natural representation of $\mathbf{U}$. Let $\mathbb{W} := \mathbb{V}^*$ be the restricted dual module of $\mathbb{V}$ with the basis $\{w_a \mid a \in I\}$ such that $\langle w_a, v_b \rangle = (-q)^{-a} \delta_{a,b}$. By restriction through the embedding $\jmath$, $\mathbb{V}$ and $\mathbb{W}$ are naturally $\mathbf{U}^\jmath$-modules. For a given $0^m 1^n$-sequence $\mathbf{b} = (b_1, b_2, \ldots, b_{m+n})$, we again define the Fock space $\mathbb{T}^{\mathbf{b}}$ by the formula (8.2.1) and the standard monomial basis M_f, for $f \in I^{m+n}$, by the formula

(8.2.2). Following §9.1, we define the B-completion of the Fock space $\mathbb{T}^{\mathbf{b}}$ with respect to the Bruhat ordering defined in Definition 12.1.1.

Following §9.1 and §9.2, we define an anti-linear involution

$$\psi_\jmath := \Upsilon\psi : \widehat{\mathbb{T}}^{\mathbf{b}} \longrightarrow \widehat{\mathbb{T}}^{\mathbf{b}},$$

where Υ is the operator defined in (12.1.2), such that

$$\psi_\jmath(M_f) = M_f + \sum_{g \prec_{\mathbf{b}} f} r_{gf}(q) M_g, \quad \text{for } r_{gf}(q) \in \mathcal{A}.$$

Therefore we have the following counterpart of Theorem 9.2.4 (here we emphasize that the index set I here is different from the same notation used therein and $\mathbf{U}^\jmath$ is a different algebra than $\mathbf{U}^\imath$).

Theorem 12.2.1. *The $\mathbb{Q}(q)$-vector space $\widehat{\mathbb{T}}^{\mathbf{b}}$ has unique $\psi_\jmath$-invariant topological bases*

$$\{T^{\mathbf{b}}_f \mid f \in I^{m+n}\} \quad \text{and} \quad \{L^{\mathbf{b}}_f \mid f \in I^{m+n}\}$$

such that

$$T^{\mathbf{b}}_f = M_f + \sum_{g \preceq_{\mathbf{b}} f} t^{\mathbf{b}}_{gf}(q) M^{\mathbf{b}}_g, \quad L^{\mathbf{b}}_f = M_f + \sum_{g \preceq_{\mathbf{b}} f} \ell^{\mathbf{b}}_{gf}(q) M^{\mathbf{b}}_g,$$

with $t^{\mathbf{b}}_{gf}(q) \in q\mathbb{Z}[q]$, and $\ell^{\mathbf{b}}_{gf}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$, for $g \preceq_{\mathbf{b}} f$. (We shall write $t^{\mathbf{b}}_{ff}(q) = \ell^{\mathbf{b}}_{ff}(q) = 1$, $t^{\mathbf{b}}_{gf}(q) = \ell^{\mathbf{b}}_{gf}(q) = 0$ for $g \npreceq_{\mathbf{b}} f$.)

$\{T^{\mathbf{b}}_f \mid f \in I^{m+n}\}$ and $\{L^{\mathbf{b}}_f \mid f \in I^{m+n}\}$ are call the *$\jmath$-canonical basis* and *dual $\jmath$-canonical basis* of $\widehat{\mathbb{T}}^{\mathbf{b}}$, respectively. The polynomials $t^{\mathbf{b}}_{gf}(q)$ and $\ell^{\mathbf{b}}_{gf}(q)$ are called *$\jmath$-Kazhdan-Lusztig (or $\jmath$-KL) polynomials.*

12.3 KL theory and $\jmath$-canonical basis

Starting with an $\mathcal{A}$-lattice $\mathbb{T}^{\mathbf{b}}_{\mathcal{A}}$ spanned by the standard monomial basis of the $\mathbb{Q}(q)$-vector space $\mathbb{T}^{\mathbf{b}}$, we define $\mathbb{T}^{\mathbf{b}}_{\mathbb{Z}} = \mathbb{Z} \otimes_{\mathcal{A}} \mathbb{T}^{\mathbf{b}}_{\mathcal{A}}$ where $\mathcal{A}$ acts on $\mathbb{Z}$ with $q = 1$. For any u in the $\mathcal{A}$-lattice $\mathbb{T}^{\mathbf{b}}_{\mathcal{A}}$, we denote by $u(1)$ its image in $\mathbb{T}^{\mathbf{b}}_{\mathbb{Z}}$.

Let $'\mathcal{O}^{\Delta}_{\mathbf{b}}$ be the full subcategory of $'\mathcal{O}_{\mathbf{b}}$ consisting of all modules possessing a finite $\mathbf{b}$-Verma flag. Let $\left[{}'\mathcal{O}^{\Delta}_{\mathbf{b}}\right]$ be its Grothendieck group. The following lemma is immediate from the bijection $I \leftrightarrow {}'X(m|n)$.

Lemma 12.3.1. *The map*

$$\Psi : \left[{}'\mathcal{O}^{\Delta}_{\mathbf{b}}\right] \longrightarrow \mathbb{T}^{\mathbf{b}}_{\mathbb{Z}}, \qquad [M_{\mathbf{b}}(\lambda)] \mapsto M^{\mathbf{b}}_{f^{\mathbf{b}}_{\lambda}}(1),$$

defines an isomorphism of $\mathbb{Z}$-modules.

Denote by $\mathbf{U}^{\jmath}_{\mathcal{A}}$ the $\mathcal{A}$-form of $\mathbf{U}^{\jmath}$ generated by the divided powers, and set $\mathbf{U}^{\jmath}_{\mathbb{Z}} = \mathbb{Z} \otimes_{\mathcal{A}} \mathbf{U}^{\jmath}_{\mathcal{A}}$.

Remark 12.3.2. The map Ψ is actually a $\mathbf{U}^{\jmath}_{\mathbb{Z}}$-module isomorphism, where $\mathbf{U}^{\jmath}_{\mathbb{Z}}$ acts on $\left[{}'\mathcal{O}^{\Delta}_{\mathbf{b}}\right]$ via translation functors analogous to Proposition 11.3.3.

We define $\left[\left[{}'\mathcal{O}^{\Delta}_{\mathbf{b}}\right]\right]$ as the completion of $\left[{}'\mathcal{O}^{\Delta}_{\mathbf{b}}\right]$ such that the extension of Ψ

$$\Psi : \left[\left[{}'\mathcal{O}^{\Delta}_{\mathbf{b}}\right]\right] \longrightarrow \widehat{\mathbb{T}}^{\mathbf{b}}$$

is an isomorphism of $\mathbb{Z}$-modules. We have the following counterpart of Theorem 12.3.3 with the same proof.

Theorem 12.3.3. *For any* 0^m1^n*-sequence* $\mathbf{b}$ *starting with* 0*, the isomorphism* $\Psi : \big[[{}'\mathcal{O}^{\Delta}_{\mathbf{b}}]\big] \to \widehat{\mathbb{T}}^{\mathbf{b}}_{\mathbb{Z}}$ *satisfies*

$$\Psi([L_{\mathbf{b}}(\lambda)]) = L^{\mathbf{b}}_{f^{\mathbf{b}}_{\lambda}}(1), \qquad \Psi([T_{\mathbf{b}}(\lambda)]) = T^{\mathbf{b}}_{f^{\mathbf{b}}_{\lambda}}(1), \qquad \textit{for } \lambda \in {}'X(m|n).$$

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