Min-Max Game Theory for the Linearized Navier-Stokes Equations with Localized Internal Control and Distributed Disturbance

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Chapter 1

Introduction and Statement of Main Results

1.1 Model

We begin with the Navier-Stokes equations with non-slip Dirichlet boundary conditions, and added control u and disturbance w:

$$\begin{cases} \eta_t - \nu_0 \Delta \eta + (\eta \cdot \nabla)\eta = mu + w + f_e - \nabla p_1 & \text{in } Q = \Omega \times (0, \infty) \\ \operatorname{div} \eta = 0 & \text{in } Q \\ \eta = 0 & \text{on } \Sigma = \Gamma \times (0, \infty) \\ \eta(x, 0) = \eta_0(x) & \text{in } \Omega \end{cases} \tag{1.1.1}$$

Here, Ω is an open and bounded subset of $\mathbb{R}^d$, $d = 2, 3$ with smooth boundary Γ. The function $m = m(x)$, with $x \in \mathbb{R}^d$ is the characteristic function of ω, which is an open subset of Ω with positive measure. The functions η, u and w represent the velocity, control input and disturbance input, respectively. The initial condition $y_0 \in [L^2(\Omega)]^d$ is given. The function mu acts as an internal controller with support in $Q_\omega = \omega \times (0, \infty)$. In contrast, the disturbance, w, has support on all of $Q = \Omega \times (0, \infty)$.

We now introduce the steady state Navier Stokes equations

$$\begin{cases} -\nu\Delta y_e + (y_e\cdot\nabla)y_e = f_e + \nabla p_e & \text{in } \Omega \\ \operatorname{div}\cdot y_e = 0 & \text{in } \Omega \\ y_e = 0 & \text{on } \Gamma \end{cases} \tag{1.1.2}$$

In [3] (p 59, Theorem 7.3), it is shown that taking the bodily force, f_e in $[L^2(\Omega)]^d$ guarantees the existence of a solution pair

$$(y_e, p_e) \in ((H^2(\Omega))^d \cap V) \times H^1(\Omega) \text{ for } d = 2,\, 3. \tag{1.1.3}$$

The space V in (1.1.3) is defined by (see [3] p.9):

$$V = \left\{ y \in [H_0^1(\Omega)]^d \,:\, \nabla\cdot y = 0 \right\}, \qquad \|y\|_V = \left(\int_\Omega |\nabla y(x)|^2 \, d\Omega \right)^{1/2} \tag{1.1.4}$$

In order to linearize the equation given in (1.1.1), we translate by leting $y = \eta - y_e$ and $p = p_1 - p_e$. Using these values for the velocity and the pressure, and simplifying using the steady state equations, (1.1.2), we obtain

$$\begin{cases} y_t - \nu_0\Delta y + (y\cdot\nabla)y_e + (y_e\cdot\nabla)y + (y\cdot\nabla)y = mu + w - \nabla p & \text{in } Q \\ \operatorname{div} y = 0 & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(x,0) = \eta_0(x) - y_e & \text{in } \Omega \end{cases} \tag{1.1.5}$$

In order to eliminate the pressure from equation (1.1.5), we introduce the following orthogonal decomposition of $[L^2(\Omega)]^d$:

$$H = \left\{ f \in [L^2(\Omega)]^d : \operatorname{div} f = 0 \text{ in } \Omega, \text{ and } f\cdot\nu\big|_\Gamma = 0 \right\} \tag{1.1.6}$$

$$H^{\perp} = \left\{ f \in [L^2(\Omega)]^d : f = \nabla \phi \text{ for some } \phi \in H^1(\Omega) \right\} \tag{1.1.7}$$

where ν is the outward pointing unit normal vector on Γ, and $[L^2(\Omega)]^d = H \bigoplus H^{\perp}$. We will use P to denote the Leray-Helmholtz projector $P : [L^2(\Omega)]^d \to H$. When we apply this projector to (1.1.5), the pressure is eliminated. Since $y \in H$, we have that $Py_t = y_t$, and (1.1.5) becomes:

$$\begin{cases} y_t - \nu_0 P\Delta y + P\left[(y\cdot\nabla)y_e + (y_e \cdot \nabla)y\right] + P(y\cdot\nabla)y = P(mu) + Pw & \text{in } Q \\ y(0) = P\left(\eta_0 - y_e\right) = y_0 \in H \end{cases} \tag{1.1.8}$$

The following operators will allow us to write (1.1.8) in a more concise manner

$$Ay = -P\Delta y \qquad \mathcal{D}(A) = [H^2(\Omega)]^d \cap V \tag{1.1.9a}$$

$$A_0 y = P\left[(y\cdot\nabla)y_e + (y_e\cdot\nabla)y\right] \qquad \mathcal{D}(A_0) = V = \mathcal{D}(A^{\frac{1}{2}}) \tag{1.1.9b}$$

$$By = P\left[(y\cdot\nabla)y\right] \qquad B : V \to V', \tag{1.1.9c}$$

where V' is the dual of V with H as a pivot space. The operator A defined in (1.1.9a) is called the Stokes operator. It is positive self-adjoint with compact resolvent A^{-1} on H. Therefore, its fractional powers are well-defined and $-\nu_0 A$ generates a stable c_0 analytic semigroup on H. Using the operators from (1.1.9), we rewrite (1.1.8) as

$$\begin{cases} y_t(t) + \nu_0 Ay(t) + A_0 y(t) + By(t) = Pmu(t) + Pw(t) \quad \text{in } [\mathcal{D}(\mathcal{A}^*)]' \\ y(0) = y_0 \in H \end{cases} \tag{1.1.10}$$

where y_0 is an element of H. When we remove the nonlinear term By from (1.1.10), we obtain the linearized Navier-Stokes equations,

$$\begin{cases} y_t(t) = (-\nu_0 A - A_0)\, y(t) + Pmu(t) + Pw(t) \quad \text{in } [\mathcal{D}(\mathcal{A}^*)]' \\ \\ y(0) = y_0 \in H. \end{cases} \tag{1.1.11}$$

Finally, we define the Oseen operator $\mathcal{A}$ by

$$\mathcal{A} = -\nu_0 A - A_0 \qquad \mathcal{D}(A) = \mathcal{D}(A) = [H^2(\Omega)]^d \cap V \tag{1.1.12}$$

and insert it into (1.1.11) to obtain the final version of the linearized Navier-Stokes equations:

$$\begin{cases} y_t(t) = \mathcal{A}y(t) + Pmu(t) + Pw(t) \quad \text{in } [\mathcal{D}(\mathcal{A}^*)]' \\ \\ y(0) = y_0 \end{cases} . \tag{1.1.13}$$

Note that the Oseen operator $\mathcal{A}$, defined in (1.1.12), is a lower-order perturbation of $-\nu_0 A$. Thus, from Corollary 2.4 on page 81 of [6], we have that, like $-\nu_0 A$, $\mathcal{A}$ is the generator of a strongly continuous analytic semigroup on H. However, the Oseen operator is not positive or self-adjoint. Thus, the semigroup generated by $\mathcal{A}$ does not inherit the property of being uniformly stable from semigroup generated by $-\nu_0 A$. As a strongly continuous analytic but unstable semigroup, $e^{\mathcal{A}t}$ satisfies the following inequality for $C \geq 1$ and $\beta > 0$

$$\left\| e^{\mathcal{A}t} \right\|_{\mathcal{L}(H)} \leq Ce^{\beta t} \qquad \text{for all } t \geq 0. \tag{1.1.14}$$

1.2 Game Theory Problem

For a fixed, positive γ, we introduce the cost functional:

$$J(u, w, y_0) = \int_0^\infty \left[\|y(t)\|_H^2 + \|u(t)\|^2_{[L^2(\omega)]^d} - \gamma^2 \|w(t)\|^2_{[L^2(\Omega)]^d} \right] dt \tag{1.2.1}$$

where y is a solution to (1.1.13) for a given control $u \in L^2(0, \infty; [L^2(\omega)]^d)$ and disturbance $w \in L^2(0, \infty; [L^2(\Omega)]^d)$. Our aim is to study the following game theory problem:

$$\sup_w \inf_u J(u, w, y_0) \tag{1.2.2}$$

where the infimum is taken over all $u \in L^2(0, \infty; [L^2(\omega)]^d)$, and the supremum is taken over all $w \in L^2(0, \infty; [L^2(\Omega)]^d)$.

Remark 1.2.1. *Often, cost functionals include an observation operator Q, which may be bounded or unbounded. The image of of the state space (which is H here) under Q is in a space called the observation space. In the present case, we take Q equal to the identity, and the observation space equal to the state space, H.*

Taking the observation operator equal to the identity is one of the aspects of this problem that allows for the result (1.3.2) *and for the result* (1.3.1) *to hold for all $x, z \in H$ (see Remark 1.5.1).*

1.3 Statement of Main Results

Theorem 1.3.1. *Suppose y solves (1.1.13), with $y_0 \in H$, $u \in L^2(0,\infty;[L^2(\omega)]^d)$, and $w \in L^2(0,\infty;[L^2(\Omega)]^d)$. Then there exists a critical value, $\gamma_c \geq 0$, which can be defined explicitly in terms of the problem data such that:*

(i) If $0 < \gamma < \gamma_c$, then taking the supremum in w in the game theory problem (1.2.2) leads to positive infinity for all initial conditions $y_0 \in H$.

(ii) If $\gamma_c < \gamma$, then:

(a) For each $y_0 \in H$, there exists a unique solution $\{u^*(\,\cdot\,;y_0), w^*(\,\cdot\,;y_0), y^*(\,\cdot\,;y_0)\}$ of the game theory problem, (1.2.2).

(b) There exists a unique bounded, nonnegative, self-adjoint operator, R satisfying the following algebraic Riccati operator equation for all $x, z \in H$:

$$\begin{aligned}(R\mathcal{A}x, z)_H + (Rx, \mathcal{A}z)_H + (x,z)_H \\ = (Rx, Rz)_{[L^2(\omega)]^d} - \gamma^{-2}(Rx, Rz)_{[L^2(\Omega)]^d}\end{aligned} \tag{1.3.1}$$

Moreover, we have that

$$\mathcal{A}^* R \in \mathcal{L}(H) \tag{1.3.2}$$

(c) The following pointwise feedback relations hold:

$$u^*(t,y_0) = -mRy^*(t,y_0) \in L^2(0,\infty;[L^2(\omega)]^d) \cap C([0,\infty];[L^2(\omega)]^d) \tag{1.3.3}$$

$$\gamma^2 w^*(t, y_0) = Ry^*(t, y_0) \in L^2(0, \infty; [L^2(\Omega)]^d) \cap C([0, \infty]; [L^2(\Omega)]^d) \tag{1.3.4}$$

(d) The feedback operator $\mathcal{A}_F$ defined by:

$$\mathcal{A}_F = \mathcal{A} - PmR + \gamma^{-2}R : \mathcal{D}(A_F) \to H \tag{1.3.5}$$

$$\mathcal{D}(\mathcal{A}_F) = \mathcal{D}(\mathcal{A}) = [H^2(\Omega)]^d \cap V \tag{1.3.6}$$

generates a strongly continuous analytic semigroup, $e^{\mathcal{A}_F t}$ on H that satisfies

$$y^*(t; y_0) = e^{\mathcal{A}_F t} y_0 \in L^2(0, \infty; H) \cap C\left([0, \infty]; H\right). \tag{1.3.7}$$

Furthermore, the semigroup $e^{\mathcal{A}_F t}$ is uniformly stable on H.

(e) For any $y_0 \in H$, the cost of the game is:

$$(Ry_0, y_0)_H = \sup_w \inf_u J(u, w, y_0) \tag{1.3.8}$$

An explicit relationship expressing R (which depends on γ) as the sum of R_0 and a nonnegative, self-adjoint operator (so that, in particular, $R \geq R_0$) is given in part (v) *of Proposition 4.5.1*

(f) The operator $\mathcal{A} - PmR$ generates a strongly continuous uniformly stable analytic semigroup.

Theorem 1.3.2. *Conversely, suppose that $\hat{R} = \hat{R}^* \geq 0$ is an operator in $\mathcal{L}(H)$ such that* *(a) the operator* $A_F = \mathcal{A} - Pm\hat{R} + \gamma^{-2}\hat{R}$ *is the generator of a strongly continuous uniformly stable semigroup* $e^{A_F t}$ *on* H *for some* $\gamma > 0$*; and*

(b) $\hat{R}$ *is a solution of the corresponding ARE in* (1.3.1) *for all* $x, z \in H$ *with the property that* $\mathcal{A}^* \hat{R} \in \mathcal{L}(H)$.

Then, the min-max game problem in (1.2.2) *is finite for all* $y_0 \in H$*, and we have that* $\gamma \geq \gamma_c$.

1.4 Outline of Proof

We begin the proof of Theorem 1.3.1 by solving the following the minimization problem for a fixed disturbance $w \in L^2(0, \infty; [L^2(\Omega)]^d)$ and initial condition $y_0 \in H$

$$\inf_{u \in L^2(0,\infty;[L^2(\omega)]^d)} \int_0^\infty \left[\|y(t;y_0)\|_H^2 + \|u(t;y_0)\|_{[L^2(\omega)]^d}^2 - \gamma^2 \|w(t)\|_{[L^2(\Omega)]^d} \right] dt = J_w^0(y_0) \tag{1.4.1}$$

where $y(\,\cdot\,; y_0)$ solves (1.1.13) for the given control u, disturbance w, and initial condition y_0.

Due to the lack of uniform stability for $e^{\mathcal{A}t}$, there are combinations of control u, disturbance w, and initial condition y_0 that when inserted in (1.1.13) yield a solution y that is not an element of $L^2(0, \infty; H)$. This is a concern when solving problem (1.4.1). In Chapter 2, to deal with this concern, we solve the minimization problem

$$\inf_{u \in L^2(0,T;[L^2(\omega)]^d)} \int_0^T \left[\|y(t;y_0)\|_H^2 + \|u(t;y_0)\|_{[L^2(\omega)]^d}^2 - \gamma^2 \|w(t)\|_{[L^2(\Omega)]^d} \right] dt \tag{1.4.2}$$

where $y(\,\cdot\,;y_0) \in L^2(0,T;H)$ solves (1.1.13) on the interval $[0,T]$. Then, making use of some results from [5], which will be listed in Section 1.5, we take the limit $T \uparrow \infty$ of the minimizing control $u^0_{w,T}(\,\cdot\,;y_0)$, trajectory $y^0_{w,T}(\,\cdot\,;y_0)$, and other relevant quantities on $[0,T]$ to show the existence of a unique solution to (1.4.1). Moreover, we find expressions for the minimizing control, trajectory and other relevant quantities explicitly in terms of the fixed y_0, w, and other problem data. This includes an expression for the the minimizing trajectory $y^0_w(\,\cdot\,;y_0)$ in stable form, which is given in Proposition 2.4.1.

Alternatively, the concern arising from the behavior of $e^{\mathcal{A}t}$ as t approaches infinity could also be dealt with by using a result given in [1] (p 115, Theorem C.1.) which states that the minimization problem in (1.4.1) is equivalent to a similar minimization problem where y solves the equation

$$\begin{cases} y_t(t) = (\mathcal{A} - \lambda)y(t) + Pmu(t) + Pw(t) \quad \text{in } [\mathcal{D}(\mathcal{A}^*)]' \\ y(0) = y_0 \end{cases}, \tag{1.4.3}$$

with $\lambda > \beta$ so that the semigroup generated by $(\mathcal{A} - \lambda I)$ is uniformly stable and satisfies

$$\left\| e^{(\mathcal{A}-\lambda I)t} \right\|_{\mathcal{L}(H)} \leq Ce^{(\beta-\lambda)} \qquad \text{for all } t \geq 0. \tag{1.4.4}$$

In Chapter 3, we solve the following maximization problem for $\gamma > \gamma_c$

$$\sup_{w \in L^2(0,\infty;[L^2(\Omega)]^d)} J^0_w(y_0) = J^*(y_0), \tag{1.4.5}$$

which is equivalent to solving the min-max game theory problem in (1.2.2). We begin the chapter by finding an expression for $J^0_w(y_0 = 0)$ in terms of the problem data.

We then define the critical value γ_c and solve the maximization problem (1.4.5) for $\gamma > \gamma_c$. Note that although the critical value γ_c is defined in this chapter, part (i) of Theorem 1.3.1 is not fully proved until Section 4.9. The stable dynamics for $y_w^0(\,\cdot\,; y_0)$ from Proposition 2.4.1 give us an expression for the minimizing trajectory as a sum of terms in $L^2(0,\infty; H)$, which allows us to directly solve (1.4.5) over the infinite time interval by completing the square to characterize the optimal solution $w^*(\,\cdot\,; y_0)$ directly in terms of the problem data. We finish up the chapter by providing some regularity results for the optimal quantities.

In Chapter 4, we show that y^* satisfies the transition property

$$y^*(t+\sigma; y_0) = y^*(\sigma; y^*(t; y_0)) \underset{(\text{in } \sigma)}{\in} C\left([0,\infty]; H\right) \tag{1.4.6}$$

by using the explicit expression for the optimal disturbance w^* from Chapter 3 to demonstrate that w^* satisfies a transition property analogous to (1.4.6). Next, we define a family of operators $\Phi(t)$ by $\Phi(t)y_0 = y^*(t; y_0)$. From the transition property for y^* in (1.4.6) and with the regularity result $y^*(\,\cdot\,; y_0) \in C\left([0,\infty]; H\right)$, we deduce that $\Phi(t)$ is a strongly continuous semigroup on H.

We then define the operator $R \in \mathcal{L}(H)$ using one of the optimal quantities and show the validity of equations (1.3.3) and (1.3.4), which give expressions for $u^*(t; y_0)$ and $w^*(t; y_0)$ in terms of R and $y^*(t; y_0)$. After showing that R is self-adjoint and satisfies $(Ry_0, y_0)_H = J^*(y_0)$ for all $y_0 \in H$, we find an expression for the infinitesimal generator $\mathcal{A}_F$ of the semigroup $\Phi(t) = e^{\mathcal{A}_F t}$ by differentiating $y^*(t; y_0)$ with respect to t. Finally, we show that R satisfies (1.3.1), the Algebraic Riccati Equation, and we wrap up the proof of Theorem (1.3.1) by using the ARE_γ to demonstrate the uniform stability of the analytic semigroup generated by $\mathcal{A} - PmR$.

1.5 Results from Other Sources

We start by discussing the solution of a specific case of the infimum problem, in which we take the disturbance w equal to zero:

$$\inf_u J(u, w = 0, y_0) = \inf_u \int_0^\infty \left[\|y(t)\|_H^2 + \|u(t)\|_{[L^2(\omega)]^d}^2 \right] dt \tag{1.5.1}$$

where the infimum is taken over all u in $L^2(0, \infty; [L^2(\omega)]^d)$.

From Theorem 2.1 on page 1448 of [2], we know that for all $y_0 \in H$, there exists a control $\overline{u}_{y_0}$ depending on y_0 so that

$$y(t; \overline{u}_{y_0}; y_0) = e^{\mathcal{A}t} y_0 + \int_0^t e^{\mathcal{A}(t-\tau)} P[m\overline{u}_{y_0}(\tau)]\, d\tau \in L^2(0, \infty, H) \tag{1.5.2}$$

Thus, we have that $J(\overline{u}_{y_0}; w = 0; y_0) < \infty$, and the finite cost condition is satisfied for the problem (1.5.1). The following results follow from Theorem 2.2.1 on pages 125-126 of [5]:

Theorem 1.5.1. *Let $y_0 \in H$. We have the following results concerning problem* (1.5.1)

(i) For all $y_0 \in H$, the minimization problem without disturbance, (1.5.1)*, is uniquely solved by the pair $\{u^0_{w=0}(\,\cdot\,; y_0), y^0_{w=0}(\,\cdot\,; y_0)\}$, with*

$$u^0_{w=0}(\,\cdot\,; y_0) \in L^2(0, \infty; [L^2(\omega)]^d); \;\; \textit{and } y^0_{w=0}(\,\cdot\,; y_0) \in L^2(0, \infty; H) \tag{1.5.3}$$

(ii) There exists a nonnegative, self-adjoint operator $R_0 \in \mathcal{L}(H)$ *defined by:*

$$R_0 x = \lim_{T\uparrow\infty} R_{0,T}(t)x, \qquad x \in H,\ t \text{ fixed and arbitrary} < T \uparrow \infty \tag{1.5.4a}$$

and in fact, uniformly on compact sets $0 \le t \le T_0 < T \uparrow \infty$*. Moreover,*

$$\sup_T \sup_{0\le t\le T} \|R_{0,T}(t)\|_{\mathcal{L}(H)} \le M < \infty \tag{1.5.4b}$$

(iii) For any $y_0 \in H$*, let* $\tilde{u}^0_{w=0,T}(\,\cdot\,; y_0)$ *and* $\tilde{y}^0_{w=0,T}(\,\cdot\,; y_0)$ *denote the extension by zero of* $u^0_{w=0,T}(\,\cdot\,; y_0)$ *and* $y^0_{w=0,T}(\,\cdot\,; y_0)$*, respectively, for* $t > T$*. Then*

$$\tilde{u}^0_{w=0,T}(\,\cdot\,; y_0) \to u^0_{w=0}(\,\cdot\,; y_0) \qquad \text{in } L^2(0,\infty; [L^2(\omega)]^d) \tag{1.5.5a}$$

$$\tilde{y}^0_{w=0,T}(\,\cdot\,; y_0) \to y^0_{w=0}(\,\cdot\,; y_0) \qquad \text{in } L^2(0,\infty; H). \tag{1.5.5b}$$

Moreover,

$$R_{0,T}\tilde{y}^0_{w=0,T}(t; y_0) \to R_0 y^0_{w=0}(t; y_0) \text{ in } L^2(0,\infty; H) \tag{1.5.6}$$

(iv) The minimizing cost for (1.5.1)*, is given by:*

$$(R_0 y_0, y_0)_H = J^0_{w=0}(y_0) = \inf_{u\in L^2(0,\infty;[L^2(\omega)]^d)} J(u, w = 0, y_0) \tag{1.5.7}$$

(v) The operator R_0 *satisfies the following regularity property*

$$\mathcal{A}^* R_0 \text{ and } R_0\mathcal{A} \in \mathcal{L}(H) \tag{1.5.8}$$

(vi) Setting $\Phi_0(t)y_0 = y^0_{w=0}(t; y_0)$*, for* $y_0 \in H$*, we have that* $\Phi_0(t)$ *is a strongly continuous analytic semigroup on* H*, with infinitesimal generator:*

$$\mathcal{A}_{R_0} = \mathcal{A} - PmR_0 \tag{1.5.9}$$

Moreover, because $y^0_{w=0}(\,\cdot\,;y_0) \in L^2(0,\infty;H)$ *for all* $y_0 \in H$*, we have that*

$$\Phi_0(t)y_0 = e^{\mathcal{A}_{R_0}t}y_0 \in L^2(0,\infty;H) \qquad \textit{for all } y_0 \in H \tag{1.5.10}$$

By use of Theorem 4.1 on page 116 of [6], it follows that $\Phi_0(t) = e^{\mathcal{A}_{R_0}t}$ *is a uniformly stable semigroup on* H*. Thus,* $e^{\mathcal{A}_{R_0}t}$ *satisfies the following inequality*

$$\left\| e^{\mathcal{A}_{R_0}t} \right\|_{\mathcal{L}(H)} \leq Me^{-\alpha t} \tag{1.5.11}$$

where $M \geq 1$ *and* $\alpha > 0$

(vii) *For any* $x \in H$*, we have that*

$$y^0_{w=0}(t;x) = e^{\mathcal{A}_{R_0}t}x \in C\left([0,\infty];H\right) \tag{1.5.12a}$$

$$u^0_{w=0}(t;x) = -mR_0e^{\mathcal{A}_{R_0}t}x \in C\left([0,\infty];[L^2(\omega)]^d\right). \tag{1.5.12b}$$

(viii) *For all* $x, z \in H$*, the operator* R_0 *satisfies the following algebraic Riccati equation:*

$$\left(\mathcal{A}^*R_0x,z\right)_H + \left(R_0\mathcal{A}x,z\right)_H + (x,z)_H = (Px,z)_{[L^2(\omega)]^d} \tag{1.5.13}$$

Remark 1.5.1. *The result for part* (v) *of Theorem 1.5.1 given in [5] actually states that for any* θ *with* $0 \leq \theta < 1$*, we have*

$$\left((a-\mathcal{A})^*\right)^\theta R_0 \in \mathcal{L}(H), \tag{1.5.14}$$

where a *satisfies* $a > \beta$ *so that the fractional powers of* $(a-\mathcal{A})$ *and* $(a-\mathcal{A})^*$ *are well defined. However, because the Oseen operator* $\mathcal{A}$ *is a lower order perturbation of a* *self-adjoint operator, and our observation operator is equal to the identity on H (see Remark 1.2.1) the result in* (1.5.14) *can be extended to include* $\theta = 1$ *as well.*

1.6 Regularity of the Abstract Equation Driven by $\mathcal{A}_{R_0}$

Note: *This section hasn't been changed significantly, but it is in a different location than it was for the last draft.*

Recalling the stable generator $\mathcal{A}_{R_0}$ in (1.5.9), we define the operator $\mathcal{K}_{R_0}$ as

$$\left(\mathcal{K}_{R_0} f\right)(t) = \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)} f(\tau)\, d\tau \tag{1.6.1a}$$

$$\mathcal{K}_{R_0} : \text{ continuous } L^2(0,\infty;H) \to L^r(0,\infty;H) \cap C\left([0,\infty];H\right),$$

$$\text{for any } r \text{ with } 2 \le r \le \infty \tag{1.6.1b}$$

and its L^2-adjoint, $\mathcal{K}^*_{R_0}$ as

$$\left(\mathcal{K}^*_{R_0} v\right)(t) = \int_t^\infty e^{\mathcal{A}^*_{R_0}(\tau-t)} v(\tau)\, d\tau \tag{1.6.2a}$$

$$\mathcal{K}^*_{R_0} : \text{ continuous } L^2(0,\infty;H) \to L^r(0,\infty;H) \cap C\left([0,\infty];H\right),$$

$$\text{for any } r \text{ with } 2 \le r \le \infty. \tag{1.6.2b}$$

Additionally, we introduce the operators $\mathscr{L}_{R_0}$, $\mathcal{W}_{R_0}$, and their L^2-adjoints as:

$$\left(\mathscr{L}_{R_0} f\right)(t) = \left(\mathcal{K}_{R_0} P m f\right)(t) \qquad \left(\mathscr{L}^*_{R_0} v\right)(t) = \left(\mathcal{K}^*_{R_0} v\right)(t)\big|_\omega \tag{1.6.3a}$$

$$\left(\mathcal{W}_{R_0} f\right)(t) = \left(\mathcal{K}_{R_0} P f\right)(t) \qquad \left(\mathcal{W}_{R_0}^{*} v\right)(t) = \left(\mathcal{K}_{R_0}^{*} v\right)(t) \tag{1.6.3b}$$

with the regularity:

$$\mathscr{L}_{R_0},\, \mathcal{W}_{R_0} : \text{ continuous } L^2(0,\infty;\,\cdot\,) \to L^r(0,\infty;H) \cap C\left([0,\infty];H\right) \tag{1.6.4a}$$

$$\mathscr{L}_{R_0}^{*},\, \mathcal{W}_{R_0}^{*} : \text{ continuous } L^2(0,\infty;H) \to L^r(0,\infty;H) \cap C\left([0,\infty] :\,\cdot\,\right) \tag{1.6.4b}$$

where $\cdot$ is a place holder for the appropriate space, $[L^2(\omega)]^d$ or $[L^2(\Omega)]^d$ and, as in (1.6.1b) and (1.6.2b), we may take $2 \leq r \leq \infty$.

Chapter 2

Minimization of J_w over $u \in L^2(0,\infty;[L^2(\omega)]^d)$ for w fixed

2.1 Minimization of $J_{w,T}$ over $u \in L^2(0,T;[L^2(\omega)]^d)$ for w Fixed

We consider a cost functional on a finite time interval, $[0,T]$ with T an arbitrary positive real number:

$$J_{w,T}(u,y_0) = \int_0^T \left[\|y(t)\|_H^2 + \|u(t)\|_{[L^2(\omega)]^d}^2 - \gamma^2 \|w(t)\|_{[L^2(\Omega)]^d}^2 \right] dt \tag{2.1.1}$$

For a fixed $y_0 \in H$ and a fixed $w \in L^2(0,\infty;[L^2(\Omega)]^d)$, we study the following minimization problem:

$$\inf_{u \in L^2(0,T;[L^2(\omega)]^d)} J_{w,T}(u,y_0). \tag{2.1.2}$$

We start by defining the integral operator K_T and its $L^2(0,T;H)$-adjoint K_T^*

$$\left(K_T f\right)(t) = \int_0^t e^{\mathcal{A}(t-\tau)} f(\tau)\, d\tau \tag{2.1.3a}$$

$$K_T: \text{ continuous } L^2(0,T;H) \to L^r(0,T;H) \cap C\left([0,T];H\right) \tag{2.1.3b}$$

$$\left(K_T^* g\right)(t) = \int_t^T e^{\mathcal{A}(\tau - t)} g(\tau)\, d\tau \tag{2.1.3c}$$

$$K_T^* : \text{ continuous } L^2(0,T;H) \to L^r(0,T;H) \cap C\left([0,T];H\right). \tag{2.1.3d}$$

where we may take $1 \leq r \leq \infty$. Making use of K_T, we now define the following operators, which will be useful in solving the minimization problem on $[0,T]$ in equation (2.1.2):

$$L_T = K_T P m \qquad W_T = K_T P \qquad L_T^* = m K_T^* \qquad W_T^* = K_T^*. \tag{2.1.4}$$

Due to the regularity of K_T and K_T^* from (2.1.3b) and (2.1.3d), respectively, we have the following regularity results for L_T, W_T and their adjoints:

$$L_T,\, W_T : \text{ continuous } L^2(0,T;\,\cdot\,) \to L^r(0,T;H) \cap C\left([0,T];H\right) \tag{2.1.5a}$$

$$L_T^*,\, W_T^* : \text{ continuous } L^2(0,T;H) \to L^r(0,T;\,\cdot\,) \cap C\left([0,T];\,\cdot\,\right) \tag{2.1.5b}$$

where $\cdot$ is a place holder for the appropriate space, $[L^2(\omega)]^d$ or $[L^2(\Omega)]^d$, and as with K_T and its adjoint, we may $1 \leq r \leq \infty$. Now, we can use (2.1.4) to rewrite $y(t;y_0)$, the solution to (1.1.13) for $t \in [0,T]$, as:

$$y(t;y_0) = e^{\mathcal{A}t} y_0 + (L_T u)(t) + (W_T w)(t) \tag{2.1.6}$$

2.1.1 Existence of a Unique Optimal Pair on $[0,T]$ and Its Characterization

Theorem 2.1.1. *Consider the minimization problem,* (2.1.2)*, where y solves* (1.1.13) *for $t \in [0,T]$. For each $y_0 \in H$, and $w \in L^2(0,\infty;[L^2(\Omega)]^d)$, we have* *(i) There exists a unique minimizing pair denoted by*

$$\{u^0_{w,T}(\,\cdot\,;y_0), y^0_{w,T}(\,\cdot\,;y_0)\} \in L^2(0,T;[L^2(\omega)]^d) \times L^2(0,T;H) \tag{2.1.7}$$

that solves the minimization problem (2.1.2)*. The cost associated with this minimizing pair is denoted by* $J^0_{w,T}(y_0)$ *and given by:*

$$J^0_{w,T}(y_0) = \int_0^T \left[\|y^0_{w,T}(t;y_0)\|^2_H + \|u^0_{w,T}(t;y_0)\|^2_{[L^2(\omega)]^d} - \gamma^2 \|w(t)\|^2_{[L^2(\Omega)]^d} \right] dt \tag{2.1.8}$$

(ii) The optimal pair is related by:

$$u^0_{w,T}(\,\cdot\,;y_0) = -L^*_T y^0_{w,T}(\,\cdot\,;y_0) \qquad \text{in } L^2(0,T;[L^2(\omega)]^d) \tag{2.1.9}$$

and is characterized explicitly in terms of the problem data by the following formulae:

$$-u^0_{w,T}(\,\cdot\,;y_0) = [I + L^*_T L_T]^{-1} L^*_T \left(e^{\mathcal{A}\,\cdot}\, y_0 + W_T w\right) \in L^2(0,T;[L^2(\omega)]^d) \tag{2.1.10a}$$

$$= -u^0_{w=0,T}(\,\cdot\,;y_0) - u^0_{w,T}(\,\cdot\,;y_0=0) \tag{2.1.10b}$$

$$y^0_{w,T}(\,\cdot\,;y_0) = [I + L_T L^*_T]^{-1} \left(e^{\mathcal{A}\,\cdot}\, y_0 + W_T w\right) \in L^2(0,T;H) \tag{2.1.10c}$$

$$= y^0_{w=0,T}(\,\cdot\,;y_0) + y^0_{w,T}(\,\cdot\,;y_0=0) \tag{2.1.10d}$$

$$= e^{\mathcal{A}t} y_0 + \left(L_T u^0_{w,T}(\,\cdot\,;y_0)\right)(t) + \left(W_T w\right)(t) \tag{2.1.10e}$$

(iii) We obtain the following for the minimum cost $J^0_{w,T}(y_0)$*:*

$$J^0_{w,T}(y_0) = J^0_{w=0,T}(y_0) + J^0_{w,T}(y_0=0) + X_{w,T}(y_0) \tag{2.1.11}$$

with $J^0_{w=0,T}(y_0)$*,* $J^0_{w,T}(y_0=0)$*, and* $X_{w,T}(y_0)$ *defined by*

$$J^0_{w=0,T}(y_0) = \left(e^{\mathcal{A}\cdot}\, y_0, \left[I + L_T L_T^*\right]^{-1} e^{\mathcal{A}\cdot}\, y_0\right)_{L^2(0,T;H)} \tag{2.1.12a}$$

$$J^0_{w,T}(y_0=0) = \left(w, \left(W_T^*\left[I + L_T L_T^*\right]^{-1} W_T - \gamma^2 I\right) w\right)_{L^2(0,T;[L^2(\Omega)]^d)} \tag{2.1.12b}$$

$$X_{w,T}(y_0) = 2\left(w, W_T^*\left[I + L_T L_T^*\right]^{-1} e^{\mathcal{A}\cdot}\, y_0\right)_{L^2(0,T;[L^2(\Omega)]^d)} \tag{2.1.12c}$$

Proof. (i): The minimization problem (2.1.2) can be rewritten using squared norms in $L^2(0,T;H)$, $L^2(0,T;[L^2(\omega)]^d)$, and $L^2(0,T;[L^2(\Omega)]^d)$ as:

$$\inf_u \left(\|y(\,\cdot\,;y_0)\|^2_{L^2(0,T;H)} + \|u\|^2_{L^2(0,T;[L^2(\omega)]^d)} - \gamma^2\|w\|^2_{L^2(0,T;[L^2(\Omega)]^d)}\right) \tag{2.1.13}$$

where the infimum is taken over all $u \in L^2(0,T;[L^2(\omega)]^d)$. Using the definition of $y(t;y_0)$ given in equation (2.1.6), we can rewrite cost functional as:

$$\begin{aligned} J_{w,T}(u,y_0) &= \| y\|^2_{L^2(0,T;H)} + \|u\|^2_{L^2(0,T;[L^2(\omega)]^d)} - \gamma^2\|w\|^2_{L^2(0,T;[L^2(\Omega)]^d)} \\ &= \left\|e^{\mathcal{A}\cdot}\, y_0 + L_T u + W_T w\right\|^2_{L^2(0,T;H)} + \|u\|^2_{L^2(0,T;[L^2(\omega)]^d)} - \gamma^2\|w\|^2_{L^2(0,T;[L^2(\Omega)]^d)} \\ &= \left\|e^{\mathcal{A}\cdot}\, y_0 + W_T w\right\|^2_{L^2(0,T;H)} + \|L_T u\|^2_{L^2(0,T;H)} + 2\left(e^{\mathcal{A}\cdot}\, y_0 + W_T w,\ L_T u\right)_{L^2(0,T;H)} \\ &\qquad + \|u\|^2_{L^2(0,T;[L^2(\omega)]^d)} - \gamma^2\|w\|^2_{L^2(0,T;[L^2(\Omega)]^d)} \end{aligned} \tag{2.1.14}$$

Because w and y_0 are independent of u, we can find the infimum by finding the $u \in L^2(0,T;[L^2(\omega)]^d)$ that minimizes the following quantity

$$\begin{aligned} &J_{w,T}(u,y_0) + \gamma^2\|w\|^2_{L^2(0,T;[L^2(\Omega)]^d)} - \left\|e^{\mathcal{A}\cdot}\, y_0 + W_T w\right\|^2_{L^2(0,T;H)} \\ &= \|L_T u\|^2_{L^2(0,T;H)} + 2\left(e^{\mathcal{A}\cdot}\, y_0 + W_T w, L_T u\right)_{L^2(0,T;H)} + \|u\|^2_{L^2(0,T;[L^2(\omega)]^d)} \end{aligned} \tag{2.1.15}$$

To minimize the quantity (2.1.15) above, we will complete the square. Note that the operator $[I + L_T^* L_T]$ is self-adjoint and positive definite on $L^2(0,T;[L^2(\omega)]^d)$, so it has a well defined inverse $[I + L_T^* L_T]^{-1} \in L^2(0,T;[L^2(\omega)]^d)$. The calculations for completing the square follow

$$\|L_T u\|^2_{L^2(0,T;H)} + 2\left(e^{\mathcal{A}\cdot} y_0 + W_T w, L_T u\right)_{L^2(0,T;H)} + \|u\|^2_{L^2(0,T;[L^2(\omega)]^d)}$$

$$= ([I + L_T^* L_T]\, u, u)_{L^2(0,T;[L^2(\omega)]^d)} + 2\left(L_T^*(e^{\mathcal{A}\cdot} y_0 + W_T w), u\right)_{L^2(0,T;[L^2(\omega)]^d)} \tag{2.1.16a}$$

$$= \left([I + L_T^* L_T]\, u + L_T^*(e^{\mathcal{A}\cdot} y_0 + W_T w), u\right)_{L^2(0,T;[L^2(\omega)]^d)}$$
$$+ \left([I + L_T^* L_T]\, u, [I + L_T^* L_T]^{-1} L_T^*(e^{\mathcal{A}\cdot} y_0 + W_T w)\right)_{L^2(0,T;[L^2(\omega)]^d)} \tag{2.1.16b}$$

$$= \left([I + L_T^* L_T]\, u + L_T^*(e^{\mathcal{A}\cdot} y_0 + W_T w), u + [I + L_T^* L_T]^{-1} L_T^*(e^{\mathcal{A}\cdot} y_0 + W_T w)\right)_{L^2(0,T;[L^2(\omega)]^d)}$$
$$- \left(L_T^*(e^{\mathcal{A}\cdot} y_0 + W_T w), [I + L_T^* L_T]^{-1} L_T^*(e^{\mathcal{A}\cdot} y_0 + W_T w)\right)_{L^2(0,T;[L^2(\omega)]^d)}. \tag{2.1.16c}$$

Defining $x_u \in L^2(0,T;[L^2(\omega)]^d)$ by

$$x_u = u + [I + L_T^* L_T]^{-1} L_T^* \left(e^{\mathcal{A}\cdot} y_0 + W_T w\right) \text{ in } L^2(0,T;[L^2(\omega)]^d), \tag{2.1.17}$$

we can see that the first inner product in the last line of (2.1.16) satisfies

$$\left([I + L_T^* L_T]\, u + L_T^* \left(e^{\mathcal{A}\cdot} y_0 + W_T w\right), u + [I + L_T^* L_T]^{-1} L_T^* \left(e^{\mathcal{A}\cdot} y_0 + W_T w\right)\right)$$
$$= ([I + L_T^* L_T]\, x_u, x_u), \tag{2.1.18}$$

where both inner products above are in $L^2(0,T;[L^2(\omega)]^d)$. Since $[I + L_T^* L_T]$ is a positive definite operator, we have that the expression in (2.1.16), hence $J_{w,T}(u, y_0)$, is uniquely minimized by the $u \in L^2(0,T;[L^2(\omega)]^d)$ that satisfies $x_u = 0$. Recalling

(2.1.17), we see that the unique minimizing control $u^0_{w,T}(\,\cdot\,; y_0)$ is given explicitly by:

$$u^0_{w,T}(\,\cdot\,; y_0) = -\left[I + L_T^* L_T\right]^{-1} L_T^* \left(e^{\mathcal{A}\,\cdot}\, y_0 + W_T w\right) \tag{2.1.19a}$$

$$= u^0_{w=0,T}(\,\cdot\,; y_0) + u^0_{w,T}(\,\cdot\,; y_0 = 0) \tag{2.1.19b}$$

Moreover, because the trajectory y is uniquely determined by y_0, w and u, there is also a unique minimizing trajectory, $y = y^0_{w,T}(\,\cdot\,; y_0)$ associated with y_0, w and $u = u^0_{w,T}(\,\cdot\,; y_0)$. Thus, we have proved the existence of a unique minimizing pair $\{u^0_{w,T}(\,\cdot\,; y_0), y^0_{w,T}(\,\cdot\,; y_0)\}$ for each $y_0 \in H$ and $w \in L^2(0, \infty; [L^2(\Omega)]^d)$.

The validity of (2.1.8) follows from the definition of $J_{w,T}(u, y_0)$ in (2.1.1) with $u = u^0_{w,T}(\,\cdot\,; y_0)$.

(ii): We have already shown (2.1.10a) and (2.1.10b) in (2.1.19) above. It remains to show the relationship between $u^0_{w,T}(\,\cdot\,; y_0)$ and $y^0_{w,T}(\,\cdot\,; y_0)$ in (2.1.9), and (2.1.10c) and (2.1.10d), the two characterizing equations for $y^0_{w,T}(\,\cdot\,; y_0)$.

Step 1: To show the remaining relationships, we first introduce some results relating to the self-adjoint positive definite operators $[I + L_T^* L_T]$ and $[I + L_T L_T^*]$ on $L^2(0, T; [L^2(\omega)]^d)$ and $L^2(0, T; H)$, respectively. Note that both have well defined inverses $[I + L_T^* L_T]^{-1} \in L^2(0, T; [L^2(\omega)]^d)$ and $[I + L_T L_T^*]^{-1} \in L^2(0, T; H)$,

The first result,

$$L_T^* \left[I + L_T L_T^*\right]^{-1} = \left[I + L_T^* L_T\right]^{-1} L_T^*. \tag{2.1.20}$$

will be used to rewrite $u^0_{w,T}$ in equation (2.1.24). This result follows from rewriting

L_T^* as

$$L_T^* = [I + L_T^* L_T]^{-1} [I + L_T^* L_T] L_T^* = [I + L_T^* L_T]^{-1} [L_T^* + L_T^* L_T L_T^*]$$
$$= [I + L_T^* L_T]^{-1} L_T^* [I + L_T L_T^*], \qquad (2.1.21)$$

then applying $[I + L_T L_T^*]^{-1}$ on the right of both sides of (2.1.21) to obtain (2.1.20).

The second result,

$$I - L_T L_T^* [I + L_T L_T^*]^{-1} = [I + L_T L_T^*]^{-1}, \qquad (2.1.22)$$

will be used in showing (2.1.10c). It can be justified with the following calculations

$$I - L_T L_T^* [I + L_T L_T^*]^{-1} = [I + L_T L_T^*] [I + L_T L_T^*]^{-1} - L_T L_T^* [I + L_T L_T^*]^{-1}$$
$$= \left([I + L_T L_T^*] - L_T L_T^*\right) [I + L_T L_T^*]^{-1}$$
$$= [I + L_T L_T^*]^{-1}. \qquad (2.1.23)$$

Step 2: Now, we use these results to show the validity of the remaining equations from part (ii) of the theorem.

We start by using equation (2.1.20) to rewrite the minimizing control in (2.1.19a) as:

$$u^0_{w,T}(\,\cdot\,; y_0) = -L_T^* [I + L_T L_T^*]^{-1} (e^{\mathcal{A}\,\cdot}\, y_0 + W_T w) \qquad (2.1.24)$$

Then, using the expression for $y(\,\cdot\,; y_0)$ from equation (2.1.6), with $u = u^0_{w,T}(\,\cdot\,; y_0)$ from (2.1.24), we may express $y^0_{w,T}(\,\cdot\,; y_0)$ as

$$y^0_{w,T}(\,\cdot\,; y_0) = e^{\mathcal{A}\,\cdot}\, y_0 + L_T u^0_{w,T}(\,\cdot\,; y_0) + W_T w$$

$$= e^{\mathcal{A}\cdot} y_0 - L_T L_T^* \left[I + L_T L_T^*\right]^{-1} \left(e^{\mathcal{A}\cdot} y_0 + W_T w\right) + W_T w$$

$$= \left[I - L_T L_T^* \left[I + L_T^* L_T\right]^{-1}\right] \left(e^{\mathcal{A}\cdot} y_0 + W_T w\right)$$

$$= \left[I + L_T L_T^*\right]^{-1} \left(e^{\mathcal{A}\cdot} y_0 + W_T w\right) \tag{2.1.25a}$$

$$= y^0_{w=0,T}(\,\cdot\,; y_0) + y^0_{w,T}(\,\cdot\,; y_0 = 0), \tag{2.1.25b}$$

where we used the relation in (2.1.22) to obtain (2.1.25a). Thus, we have proved (2.1.10c) and (2.1.10d). Equation (2.1.9) is clear after comparing equations (2.1.24) and (2.1.25a).

(iii) The minimum cost on $[0, T]$, $J^0_{w,T}(y_0)$, is achieved by inserting the minimizing control and trajectory, related in equation (2.1.9) above, into the cost functional. It can be expressed in terms of $y^0_{w,T}(\,\cdot\,; y_0)$ as

$$J^0_{w,T}(y_0) = \left\|y^0_{w,T}(\,\cdot\,; y_0)\right\|^2_{L^2(0,T;H)} + \left\|-L_T^* y^0_{w,T}(\,\cdot\,; y_0)\right\|^2_{L^2(0,T;[L^2(\omega)]^d)} - \gamma^2 \|w\|^2_{L^2(0,T;[L^2(\Omega)]^d)} \tag{2.1.26a}$$

$$= \left\|y^0_{w,T}(\,\cdot\,; y_0)\right\|^2_{L^2(0,T;H)} + \left(L_T L_T^* y^0_{w,T}(\,\cdot\,; y_0), y^0_{w,T}(\,\cdot\,; y_0)\right)_{L^2(0,T;H)}$$
$$- \gamma^2 \|w\|^2_{L^2(0,T;[L^2(\Omega)]^d)} \tag{2.1.26b}$$

$$= \left(\left[I + L_T L_T^*\right] y^0_{w,T}(\,\cdot\,; y_0), y^0_{w,T}(\,\cdot\,; y_0)\right)_{L^2(0,T;H)} - \gamma^2 \|w\|^2_{L^2(0,T;[L^2(\Omega)]^d)} \tag{2.1.26c}$$

Recalling the expression for the minimizing trajectory in (2.1.25a) above, we calculate

$$J^0_{w,T}(y_0) = \left(\left(e^{\mathcal{A}\cdot} y_0 + W_T w\right), \left[I + L_T L_T^*\right]^{-1} \left(e^{\mathcal{A}\cdot} y_0 + W_T w\right)\right)_{L^2(0,T;H)} - \gamma^2 \|w\|^2_{L^2(0,T;[L^2(\Omega)]^d)}$$

$$= \left(e^{\mathcal{A}\cdot} y_0, \left[I + L_T L_T^*\right]^{-1} e^{\mathcal{A}\cdot} y_0\right)_{L^2(0,T;H)} + 2\left(W_T w, \left[I + L_T L_T^*\right]^{-1} e^{\mathcal{A}\cdot} y_0\right)_{L^2(0,T;H)}$$

$$+\left(W_T w, [I+L_T L_T^*]^{-1} W_T w\right)_{L^2(0,T;H)} - \left(\gamma^2 w, w\right)_{L^2(0,T;[L^2(\Omega)]^d)}$$

$$= \left(e^{\mathcal{A}\cdot} y_0, [I+L_T L_T^*]^{-1} e^{\mathcal{A}\cdot} y_0\right)_{L^2(0,T;H)} + 2\left(w, W_T^* [I+L_T L_T^*]^{-1} e^{\mathcal{A}\cdot} y_0\right)_{L^2(0,T;[L^2(\Omega)]^d)}$$

$$+\left(w, \left(W_T^* [I+L_T L_T^*]^{-1} W_T - \gamma^2 I\right) w\right)_{L^2(0,T;[L^2(\Omega)]^d)}. \tag{2.1.27}$$

Equations (2.1.11) and (2.1.12) follow. □

2.1.2 The Functions $p_{w,T}(\,\cdot\,; y_0)$ and $r_{w,T}(\,\cdot\,)$; the Riccati Operator $R_{0,T}(\,\cdot\,)$ when $w = 0$

For $y_0 \in H$, we define:

$$p_{w,T}(t, y_0) = \int_t^T e^{\mathcal{A}^*(\tau - t)} y^0_{w,T}(\tau; y_0)\, d\tau \in C\left([0,T]; H\right) \tag{2.1.28}$$

Let $y^0_{w,T}(\,\cdot\,; s; y_0) \in L^2(s,T;H)$ be the optimal trajectory of the optimization problem (2.1.2), except that the integral is taken over the interval $[s,T]$ rather than $[0,T]$. Then we have that $y^0_{w=0,T}(\,\cdot\,, s; y_0) \in C([s,T];H)$. We define the evolution operator

$$\Phi_{0,T}(\tau, s)x = y^0_{w=0,T}(\tau, s; x) \in C([s,T];H) \tag{2.1.29}$$

which satisfies the following equation for all $x \in H$

$$\Phi_{0,T}(\tau, s)x = \Phi_{0,T}(\tau, t)\Phi_{0,T}(t, s)x, \quad 0 \le s \le t \le \tau \le T \tag{2.1.30}$$

and corresponds to the optimization problem (2.1.2) with $w = 0$ on the interval $[s,T]$.

Additionally, we introduce the family of operators $R_{0,T}(t) \in \mathcal{L}(H)$ for each $t \in [0,T]$

$$R_{0,T}(t)x = \int_t^T e^{\mathcal{A}^*(\tau - t)} \Phi_{0,T}(\tau, t)x\, d\tau \tag{2.1.31}$$

Proposition 2.1.2. *For $R_{0,T}(\,\cdot\,)$ defined in* (2.1.31)*, we have the following results:*

(i) For each $t \in [0,T]$, $R_{0,T}(t)$ is a nonnegative self-adjoint operator.

(ii) The following regularity result regarding $R_{0,T}$ holds:

$$R_{0,T}(\,\cdot\,): \ \textit{continuous } H \to C\left([0,T];H\right) \tag{2.1.32}$$

Proof. In the proof of both parts of the proposition, we will make use of a property of $\Phi_{0,T}$, which is shown in Lemma 2.3.2.1 on page 132 of [5]

$$\Phi_{0,T}(\tau,t) = \Phi_{0,T-t}(\tau - t, 0). \tag{2.1.33}$$

(i): Using a change of variables and the relation for $\Phi_{0,T}$ from (2.1.33), we obtain the following for $R_{0,T}$

$$\begin{aligned} R_{0,T}(t)x &= \int_t^T e^{\mathcal{A}^*(\tau-t)}\Phi_{0,T}(\tau,t)x\,d\tau \\ &= \int_0^{T-t} e^{\mathcal{A}^* s}\Phi_{0,T-t}(s,0)x\,ds = R_{0,T-t}(0)x. \end{aligned} \tag{2.1.34}$$

Thus, we can prove part (i) by showing that $R_{0,T}(0)$ is self adjoint for each $T > 0$. We perform the following calculations, making use of (2.1.10e) for $y^0_{w=0,T}$ and (2.1.9) relating $y^0_{w=0,T}$ and $u^0_{w=0,T}$

$$\begin{aligned} (R_{0,T}(0)x,x)_H &= \int_0^T \left(y^0_{w=0,T}(t;x), e^{\mathcal{A}t}x\right)_H \,dt \\ \text{by (2.1.10e)} \qquad &= \int_0^T \left(y^0_{w=0,T}(t;x), y^0_{w=0,T}(t;x) - \left(L_T u^0_{w=0,T}(\,\cdot\,;x)\right)(t)\right)_H \,dt \end{aligned}$$

$$= \left\|y^0_{w=0,T}(\,\cdot\,;x)\right\|^2_{L^2(0,T;H)} - \left(L^*_T y^0_{w=0,T}(\,\cdot\,;x), u^0_{w=0,T}(\,\cdot\,;x)\right)_{L^2(0,T;H)}$$

by (2.1.9)
$$= \left\|y^0_{w=0,T}(\,\cdot\,;x)\right\|^2_{L^2(0,T;H)} + \left\|u^0_{w=0,T}(\,\cdot\,;x)\right\|_{L^2(0,T;[L^2(\omega)]^d)}. \quad (2.1.35)$$

Thus $R_{0,T}(0)$ is positive, self-adjoint for all $T > 0$.

(ii): From equation (2.1.35), we have that for $x \in H$

$$\left(R_{0,T}(0)x, x\right)_H = J^0_{w=0,T}(x). \quad (2.1.36)$$

Combining the information from (2.1.36) and (2.1.34), we have for $x \in H$ and $t \in [0, T]$

$$\left(R_{0,T}(t)x\right)_H = \left(R_{0,T-t}(0)x\right)_H = J^0_{w=0,T-t}(x) \leq J^0_{w=0,T}(x) \quad (2.1.37)$$

where the inequality in (2.1.37) follows from the fact that the pair $\left\{u^0_{w=0,T}(s;x), y^0_{w=0,T}(s;x)\right\}$ restricted to the interval $0 \leq s \leq T - t$ form a competing pair for the minimum cost on $[0, T - t]$, so that

$$\begin{aligned} J^0_{w=0,T-t}(x) &\leq \int_0^{T-t} \left[\left\|y^0_{w=0,T}(s;x)\right\|^2_H + \left\|u^0_{w=0,T}(s;x)\right\|^2_H\right] ds \\ &= \int_0^{T} \left[\left\|y^0_{w=0,T}(s;x)\right\|^2_H + \left\|u^0_{w=0,T}(s;x)\right\|^2_H\right] ds = J^0_{w=0,T}(x). \end{aligned} \quad (2.1.38)$$

Thus, we have that for each $x \in H$ and each $t \in [0, T]$

$$\left\|R_{0,T}(t)x\right\|_H \leq \left\|R_{0,T}(0)x\right\|_H \quad (2.1.39)$$

We use the definition of $R_{0,T}(t)$ in (2.1.31), the relationship $y^0_{w=0,T}(s;x) = \Phi_{0,T}(s, 0)x$ from (2.1.29), the inequality for $e^{\mathcal{A}\,\cdot}$ from (1.1.14), the Cauchy-Schwartz inequality, and the formula for $y^0_{w=0,T}$ in (2.1.10c) to calculate

$$
\begin{aligned}
\|R_{0,T}(0)x\|_H &= \left\| \int_0^T e^{\mathcal{A}s}\Phi_{0,T}(s,0)x\,ds \right\|_H \\
\text{by (2.1.29)} \qquad &\le \int_0^T \left\| e^{\mathcal{A}s} y^0_{w=0,T}(s;x) \right\|_H ds \\
\text{by (1.1.14)} \qquad &\le \int_0^T C e^{\beta s} \left\| y^0_{w=0,T}(s;x) \right\|_H ds \\
&\le C e^{\beta T}\sqrt{T} \left\| y^0_{w=0,T}(\,\cdot\,;x) \right\|_{L^2(0,T;H)} \\
\text{by (2.1.10c)} \qquad &= C e^{\beta T}\sqrt{T} \left\| [I + L_T L_T^*]^{-1} e^{\mathcal{A}\,\cdot}\, x \right\|_{L^2(0,T;H)} \\
&\le C e^{\beta T}\sqrt{T} \left\| [I + L_T L_T^*]^{-1} e^{\mathcal{A}\,\cdot} \right\|_{\mathcal{L}(L^2(0,T;H))} \|x\|_H. \qquad (2.1.40)
\end{aligned}
$$

Combining (2.1.39) and the result of (2.1.40), we see that

$$
\sup_{t\in[0,T]} \|R_{0,T}(t)x\|_H \le c_T \|x\|_H. \qquad (2.1.41)
$$

To show that $R_{0,T}(\,\cdot\,)x \in C\left([0,\infty];H\right)$, we take $h>0$ and calculate

$$
\begin{aligned}
&\|R_{0,T}(t)x - R_{0,T}(t+h)x\|_H \\
&= \left\| \int_t^T e^{\mathcal{A}^*(\tau-t)}\Phi_{0,T}(\tau,t)x\,d\tau - \int_{t+h}^T e^{\mathcal{A}^*(\tau-t-h)}\Phi_{0,T}(\tau,t+h)x\,d\tau \right\|_H \\
&\le \int_t^{t+h} \left\| e^{\mathcal{A}^*(\tau-t)}\Phi_{0,T}(\tau,t)x \right\|_H d\tau \\
&\qquad + \int_{t+h}^T \left\| e^{\mathcal{A}^*(\tau-t-h)} \left(e^{\mathcal{A}^* h}\Phi_{0,T}(\tau,t)x - \Phi_{0,T}(\tau,t+h)x \right) \right\|_H d\tau. \qquad (2.1.42)
\end{aligned}
$$

We have the following convergences for $x \in H$ fixed

$$
\lim_{h\downarrow 0} \int_t^{t+h} \left\| e^{\mathcal{A}^*(\tau-t)}\Phi_{0,T}(\tau,t)x \right\|_H d\tau = 0 \qquad (2.1.43)
$$

$$
\lim_{h\downarrow 0} \left\| e^{\mathcal{A}^* h}\Phi_{0,T}(\tau,t)x - \Phi_{0,T}(\tau,t+h)x \right\|_H = 0 \qquad \text{for } t \text{ and } \tau \text{ fixed.} \qquad (2.1.44)
$$

Thus, combining (2.1.42), (2.1.43), and (2.1.44), and using dominated convergence for the last line of (2.1.42) we see that

$$\lim_{h \downarrow 0} \|R_{0,T}(t)x - R_{0,T}(t+h)x\|_H = 0. \tag{2.1.45}$$

Similar calculations will show that $R_{0,T}(t-h)x$ converges to $R_{0,T}(t)x$ in H as $h \downarrow 0$. □

Lemma 2.1.3. *With reference to the function $p_{w,T}(t, y_0)$, and the family of operators $R_{0,T}(t)$, defined in (2.1.28), and (2.1.31), we have:*

(i) $p_{w,T}(t; y_0)$ is the unique solution of the equation:

$$\begin{cases} \dfrac{d}{dt} p_{w,T}(t; y_0) = -\mathcal{A}^* p_{w,T}(t; y_0) - y^0_{w,T}(t; y_0) & \text{in } [\mathcal{D}(\mathcal{A}^*)]' \\ \\ p_{w,T}(t; y_0) = 0 \end{cases} \tag{2.1.46}$$

(ii) The following identity holds true a.e. in t:

$$p_{w,T}(t; y_0) = R_{0,T}(t) y^0_{w,T}(t; y_0) + r_{w,T}(t) \in L^2(0, T; H) \tag{2.1.47}$$

where $r_{w,T}(t)$ is defined by:

$$r_{w,T}(t) = p_{w,T}(t; y_0 = 0) - R_{0,T}(t) y^0_{w,T}(t; y_0 = 0) \in L^2(0, T; H) \tag{2.1.48}$$

(iii) The optimizing control for the minimization problem (2.1.2) can be written as:

$$u^0_{w,T}(t; y_0) = -(p_{w,T}(t; y_0))\big|_\omega \in L^2(0, \infty; [L^2(\omega)]^d) \tag{2.1.49a}$$

$$= -\big(R_{0,T}(t) y^0_{w,T}(t; y_0) + r_{w,T}(t)\big)\big|_\omega \tag{2.1.49b}$$

Proof. (i): Differentiating (2.1.28), the equation defining $p_{w,T}$, in t, we obtain,

$$\frac{d}{dt}p_{w,T}(t,y_0) = \frac{d}{dt}\int_t^T e^{\mathcal{A}^*(\tau-t)}y^0_{w,T}(\tau;y_0)\,d\tau \tag{2.1.50a}$$

$$= -e^{\mathcal{A}^*(t-t)}y^0_{w,T}(t;y_0) + \int_t^T \frac{d}{dt}e^{\mathcal{A}^*(\tau-t)}y^0_{w,T}(\tau;y_0)\,d\tau \tag{2.1.50b}$$

$$= -y^0_{w,T}(t;y_0) + \int_t^T (-\mathcal{A}^*)\,e^{\mathcal{A}^*(\tau-t)}y^0_{w,T}(\tau;y_0)\,d\tau \tag{2.1.50c}$$

$$= -\mathcal{A}^*p_{w,T}(t;y_0) - y^0_{w,T}(t;y_0) \tag{2.1.50d}$$

where the calculations above are in $[\mathcal{D}(\mathcal{A}^*)]'$. Thus, we have shown (2.1.46).

(ii): To prove (2.1.47), we return to the definition of $p_{w,T}$ in (2.1.28), and substitute identity (2.1.10d), rewritten as $y^0_{w,T}(t;y_0) = \Phi_{0,T}(t)y_0 + y^0_{w,T}(t;y_0=0)$ using (2.1.29) Doing this, we obtain

$$p_{w,T}(t,y_0) = \int_t^T e^{\mathcal{A}^*(\tau-t)}y^0_{w,T}(\tau;y_0)\,d\tau \tag{2.1.51a}$$

$$= \int_t^T e^{\mathcal{A}^*(\tau-t)}\left[y^0_{w=0,T}(\tau;y_0) + y^0_{w,T}(\tau;y_0=0)\right]d\tau \tag{2.1.51b}$$

$$\begin{aligned} &= \int_t^T e^{\mathcal{A}^*(\tau-t)}y^0_{w=0,T}(\tau;y_0)\,d\tau \\ &\qquad + \int_t^T e^{\mathcal{A}^*(\tau-t)}y^0_{w,T}(\tau;y_0=0)\,d\tau \end{aligned} \tag{2.1.51c}$$

$$= \int_t^T e^{\mathcal{A}^*(\tau-t)}\Phi_{0,T}(\tau,0)y_0\,d\tau + p_{w,T}(t;y_0=0) \tag{2.1.51d}$$

Recalling the definitions of $\Phi_{0,T}$ in (2.1.29) and $R_{0,T}$ in (2.1.31), and using the property in (2.1.30) for $\Phi_{0,T}$, we preform the following calculations

$$p_{w,T}(t,y_0) = \int_t^T e^{\mathcal{A}^*(\tau-t)}\Phi_{0,T}(\tau,t)\Phi_{0,T}(t,0)y_0\,d\tau + p_{w,T}(t;y_0=0) \tag{2.1.52a}$$

$$= R_{0,T}(t)y^0_{w=0,T}(t;y_0)\,d\tau + p_{w,T}(t;y_0=0) \tag{2.1.52b}$$

$$= R_{0,T}(t)\left[y^0_{w,T}(t;y_0) - y^0_{w,T}(t;y_0=0)\right] + p_{w,T}(t;y_0=0) \tag{2.1.52c}$$

$$= R_{0,T}(t)y^0_{w,T}(t;y_0)$$

$$p_{w,T}(t;y_0=0) - R_{0,T}(t)y^0_{w,T}(t;y_0=0). \tag{2.1.52d}$$

from which (2.1.47) follows when we recall the definition of $r_{w,T}$ in equation (2.1.48).

(iii): To obtain (2.1.49a), we return to the relationship between $u^0_{w,T}$ and $y^0_{w,T}$ in (2.1.9) and recall the definitions of $p_{w,T}$ in (2.1.28) and L^*_T in (2.1.4)

$$u^0_{w,T}(t;y_0) = -\left(L^*_T y^0_{w,T}(\,\cdot\,;y_0)\right)(t) \tag{2.1.53a}$$

$$= -\left(\int_t^T e^{\mathcal{A}^*(\tau-t)} y^0_{w,T}(\tau;y_0)\,d\tau\right)\Big|_\omega \tag{2.1.53b}$$

$$= -(\,p_{w,T}(t;y_0))\,\Big|_\omega \tag{2.1.53c}$$

We then use (2.1.47) to obtain (2.1.49b). $\square$

Corollary 2.1.4. *With reference to* $p_{w,T}(\,\cdot\,;y_0)$ *defined in* (2.1.28) *and the operator* $\mathcal{A}_{R_0}$ *defined in* (1.5.9)*, we have*

$$\begin{cases} \dfrac{d}{dt}p_{w,T}(t;y_0) = -\mathcal{A}^*_{R_0}p_{w,T}(t;y_0) + R_0Pmu^0_{w,T}(t;y_0) - y^0_{w,T}(t;y_0) & \text{in } [\mathcal{D}(\mathcal{A}^*)]' \\ \quad p_{w,T}(t;y_0) = 0 \end{cases} \tag{2.1.54}$$

Proof. We return to the differential equation (2.1.46) for $p_{w,T}(\,\cdot\,;y_0)$ and add and subtract the quantity $R_0Pmp_{w,T}$ to obtain the following calculations in $[\mathcal{D}(\mathcal{A}^*)]'$

$$\frac{d}{dt}p_{w,T}(t;y_0) = -\mathcal{A}^*p_{w,T}(t;y_0) - y^0_{w,T}(t;y_0) \tag{2.1.55a}$$

$$= -\left(\mathcal{A}^* - R_0 Pm\right) p_{w,T}(t; y_0) - R_0 Pm p_{w,T}(t; y_0) - y^0_{w,T}(t; y_0) \quad (2.1.55b)$$

$$= -\mathcal{A}^*_{R_0} p_{w,T}(t; y_0) - R_0 Pm p_{w,T}(t; y_0) - y^0_{w,T}(t; y_0) \quad (2.1.55c)$$

$$= -\mathcal{A}^*_{R_0} p_{w,T}(t; y_0) + R_0 Pm u^0_{w,T}(t; y_0) - y^0_{w,T}(t; y_0) \quad (2.1.55d)$$

where the equality between (2.1.55c) and (2.1.55d) follows from the relationship between $u^0_{w,T}(\,\cdot\,; y_0)$ and $p_{w,T}(\,\cdot\,; y_0)$ in (2.1.49a). □

We now rewrite the definition of $p_{w,T}$ in (2.1.28) in a form that will be useful in proving Corollary 2.2.5 in the future.

Proposition 2.1.5. *We can rewrite $p_{w,T}$ defined in* (2.1.28) *as*

$$p_{w,T}(t; y_0) = \int_t^{t_0} e^{\mathcal{A}^*(\tau - t)} y^0_{w,T}(\tau; y_0)\, d\tau + e^{\mathcal{A}^*(t_0 - t)} p_{w,T}(t_0; y_0) \quad (2.1.56)$$

where t_0 is an arbitrary number in $[t, T)$.

Proof. From the definition of $p_{w,T}$ in (2.1.28), we compute

$$\begin{aligned} p_{w,T}(t; y_0) &= \int_t^{t_0} e^{\mathcal{A}^*(\tau - t)} y^0_{w,T}(\tau; y_0)\, d\tau + \int_{t_0}^{T} e^{\mathcal{A}^*(\tau - t)} y^0_{w,T}(\tau; y_0)\, d\tau \\ &= \int_t^{t_0} e^{\mathcal{A}^*(\tau - t)} y^0_{w,T}(\tau; y_0)\, d\tau + e^{\mathcal{A}^*(t_0 - t)} \int_{t_0}^{T} e^{\mathcal{A}^*(\tau - t_0)} y^0_{w,T}(\tau; y_0)\, d\tau \\ &= \int_t^{t_0} e^{\mathcal{A}^*(\tau - t)} y^0_{w,T}(\tau; y_0)\, d\tau + e^{\mathcal{A}^*(t_0 - t)} p_{w,T}(t_0; y_0) \quad (2.1.57) \end{aligned}$$

which completes the proof of the proposition. □

2.2 The Limit Process as $T \uparrow \infty$

We will now solve the infimum part of the game theory problem in (1.2.2) for $w \in L^2(0,\infty;[L^2(\Omega)]^d)$ fixed.

2.2.1 Showing That the Finite Cost Condition is Satisfied

We now consider the following minimization problem with $w \in L^2(0,\infty;[L^2(\Omega)]^d)$ and $y_0 \in H$ fixed:

$$\inf_{u \in L^2(0,\infty;[L^2(\omega)]^d)} J(u,w,y_0) \tag{2.2.1}$$

where y is the solution to (1.1.13), and $J(u,w,y_0)$ is the cost functional defined in (1.2.1). The following lemma shows that for any pair of disturbance and initial condition $\{w,y_0\} \in L^2(0,\infty;[L^2(\Omega)]^d) \times L^2(0,\infty;H)$, the finite cost condition (see section 1.5) is satisfied for the problem (2.2.1).

Lemma 2.2.1. *For each pair $\{w,y_0\}$ of disturbance and initial condition, with $w \in L^2(0,\infty;[L^2(\Omega)]^d)$ and $y_0 \in H$, there exists a control $u \in L^2(0,\infty;[L^2(\omega)]^d)$, such that the associated trajectory y solving* (1.1.13) *is in $L^2(0,\infty;H)$.*

Proof. Motivated by the dynamics (1.1.13) and recalling (1.5.9), defining the operator $\mathcal{A}_{R_0}$, a perturbation of $\mathcal{A}$ that generates a uniformly stable strongly continuous analytic semigroup on H, we define the function g by:

$$g(t) = e^{\mathcal{A}_{R_0}t}y_0 + \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)}Pw(\tau)\,d\tau, \tag{2.2.2}$$

where w and y_0 are arbitrary elements of $L^2(0,\infty;[L^2(\Omega)]^d)$ and H, respectively. From the uniform stability of $e^{\mathcal{A}_{R_0}t}$ reflected in equation (1.5.11), we deduce that $g \in L^2(0,\infty;H)$. Moreover, $g(0) = y_0$, and g is differentiable in time with

$$g'(t) = \mathcal{A}_{R_0} e^{\mathcal{A}_{R_0}t} y_0 + \mathcal{A}_{R_0} \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)} Pw(\tau)\, d\tau + Pw(t) \qquad \text{in } [\mathcal{D}(\mathcal{A}^*)]' \quad (2.2.3a)$$

$$= \mathcal{A}_{R_0} g(t) + Pw(t) \qquad \text{in } [\mathcal{D}(\mathcal{A}^*)]' \quad (2.2.3b)$$

$$= (\mathcal{A} - PmR_0)\, g(t) + Pw(t) \qquad \text{in } [\mathcal{D}(\mathcal{A}^*)]' \quad (2.2.3c)$$

where the equality from (2.2.3b) and (2.2.3c) follows from the definition of $\mathcal{A}_{R_0}$ given in (1.5.9). Defining the control $\hat{u}$ by :

$$\hat{u}(t) = -\left(R_0 e^{\mathcal{A}_{R_0}t} y_0 + R_0 \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)} Pw(\tau)\, d\tau \right)\Bigg|_{\omega} \in L^2(0,\infty;[L^2(\omega)]^d), \quad (2.2.4)$$

then recalling (2.2.2) for $g(t)$ and (2.2.3c) for $g'(t)$, we get that

$$g'(t) = \mathcal{A}g(t) + P[m\hat{u}(t)] + Pw(t) \quad (2.2.5)$$

From (2.2.5) and the fact that $g(0) = y_0$, we get that $y = g$ solves the abstract differential equation (1.1.13) with initial condition y_0, disturbance w, and control $\hat{u}$ in $L^2(0,\infty;[L^2(\omega)]^d)$ defined in terms of y_0 and w in (2.2.4). Because $g \in L^2(0,\infty;H)$, the statement of the lemma is satisfied by this control for any $w \in L^2(0,\infty;[L^2(\Omega)]^d)$ and $y_0 \in H$. $\square$

2.2.2 The Limit Process for $u^0_{w,T}(\,\cdot\,;y_0)$ and $y^0_{w,T}(\,\cdot\,;y_0)$ as $T\uparrow\infty$

Theorem 2.2.2. *With reference to the minimization problem (2.2.1), for the dynamics (1.1.13), for each $w\in L^2(0,\infty;[L^2(\Omega)]^d)$, and $y_0\in H$, we have:*

(i) There exists a unique optimal pair:

$$\left\{u^0_w(\,\cdot\,;y_0),y^0_w(\,\cdot\,;y_0)\right\}\in L^2(0,\infty;[L^2(\omega)]^d)\times L^2(0,\infty;H) \tag{2.2.6}$$

(ii) For each pair $\{w,y_0\}$, let $J^0_{w,T}(y_0)$ denote the minimum cost on the finite time interval $[0,T]$, and $J^0_w(y_0)$ denote the minimum cost on the infinite time interval $[0,\infty)$. We have that

$$\lim_{T\to\infty}J^0_{w,T}(y_0)=J^0_w(y_0) \tag{2.2.7}$$

(iii) Let $\tilde{u}^0_{w,T}(\,\cdot\,;y_0)$ and $\tilde{y}^0_{w,T}(\,\cdot\,;y_0)$ denote the extensions by zero of $u^0_{w,T}(\,\cdot\,;y_0)$ and $y^0_{w,T}(\,\cdot\,;y_0)$, respectively, for $t>T$, then:

$$\tilde{u}^0_{w,T}(\,\cdot\,;y_0)\to u^0_w(\,\cdot\,;y_0)\qquad \textit{in } L^2(0,\infty;[L^2(\omega)]^d) \tag{2.2.8a}$$

$$\tilde{y}^0_{w,T}(\,\cdot\,;y_0)\to y^0_w(\,\cdot\,;y_0)\qquad \textit{in } L^2(0,\infty;H) \tag{2.2.8b}$$

When we recall (2.1.10b) and (2.1.10d), the results in (2.2.8) above yield

$$u^0_w(\,\cdot\,;y_0)=u^0_{w=0}(\,\cdot\,;y_0)+u^0_w(\,\cdot\,;y_0=0) \tag{2.2.9a}$$

$$y^0_w(\,\cdot\,;y_0)=y^0_{w=0}(\,\cdot\,;y_0)+y^0_w(\,\cdot\,;y_0=0) \tag{2.2.9b}$$

Proof. (i): The existence of a unique minimizing pair $\{u_w^0(\,\cdot\,;y_0), y_w^0(\,\cdot\,;y_0)\}$ for the quadratic cost functional J under the finite cost condition stems from convex optimization theory. See [4] for a reference.

(ii): Let $C_{w,T}(y_0) = J_{w,T}^0(y_0) + \gamma^2\|w\|^2_{L^2(0,T;[L^2(\Omega)]^d)}$, and $C_w(y_0) = J_w^0(y_0) + \gamma^2\|w\|^2_{L^2(0,\infty;[L^2(\Omega)]^d)}$ For w, and y_0 fixed, the sequence $\{C_{w,T}(y_0)\}$ of nonnegative real numbers is nondecreasing. Additionally, because the pair $\{u_w^0(\,\cdot\,;y_0), y_w^0(\,\cdot\,;y_0)\}$ restricted to the interval $[0,T]$ is a competing pair for the minimization on $[0,T]$, we have that:

$$\begin{aligned} C_{w,T}(y_0) &\leq \int_0^T \left(\left\|y_w^0(t;y_0)\right\|_H^2 + \left\|u_w^0(t;y_0)\right\|^2_{[L^2(\omega)]^d} \right) dt \\ &\leq \int_0^\infty \left(\left\|y_w^0(t;y_0)\right\|_H^2 + \left\|u_w^0(t;y_0)\right\|^2_{[L^2(\omega)]^d} \right) dt = C_w(y_0) \end{aligned} \tag{2.2.10}$$

Thus, the sequence $\{C_{w,T}(y_0)\}$ converges because it is nondecreasing and bounded above. Moreover, by taking the limit as T approaches infinity of the inequality (2.2.10), we have that:

$$\lim_{T\to\infty} C_{w,T}(y_0) \leq C_w(y_0) \tag{2.2.11}$$

Next, we show that $C_{w,T}(y_0)$ converges to $C_w(y_0)$ as T approaches infinity by showing that $\lim\limits_{T\to\infty} C_{w,T}(y_0) \geq C_w(y_0)$. Define the functions $\tilde{u}^0_{w,T}(\,\cdot\,;y_0) \in L^2(0,\infty;[L^2(\omega)]^d)$, and $\tilde{y}^0_{w,T}(\,\cdot\,;y_0) \in L^2(0,\infty;H)$ as:

$$\tilde{u}^0_{w,T}(t;y_0) = \begin{cases} u^0_{w,T}(t;y_0) & \text{for } t \in [0,T] \\ 0 & \text{for } t > T \end{cases} \tag{2.2.12}$$

$$\tilde{y}^0_{wT}(t;y_0) = \begin{cases} y^0_{w,T}(t;y_0) & \text{for } t \in [0,T] \\ \\ 0 & \text{for } t > T \end{cases} \tag{2.2.13}$$

We now consider the sequence of functions

$$\mathscr{F} = \{\mathscr{F}_T\}_{T\geq 0} = \left\{\left(\tilde{u}^0_{w,T}(\,\cdot\,;y_0), \tilde{y}^0_{w,T}(\,\cdot\,;y_0)\right)\right\}_{T\geq 0}. \tag{2.2.14}$$

Because $C_{w,T}(y_0)$ is bounded above, we have that $\mathscr{F}$ is contained in a fixed ball depending on w and y_0 in $L^2(0,\infty;[L^2(\omega)]^d) \times L^2(0,\infty;H)$. Thus, there exists a sequence, $T_i \uparrow \infty$ as $i \to \infty$, so that the subsequence $\{\mathscr{F}_{T_i}\}_{i=1}^{\infty}$ of $\{\mathscr{F}_T\}_{T\geq 0}$ is weakly convergent in $L^2(0,\infty;[L^2(\omega)]^d) \times L^2(0,\infty;H)$:

$$\left(\tilde{u}^0_{w,T_i}(\,\cdot\,;y_0), \tilde{y}^0_{w,T_i}(\,\cdot\,;y_0)\right) \underset{\text{w}}{\longrightarrow} \left(\tilde{u}_w(\,\cdot\,;y_0), \tilde{y}_w(\,\cdot\,;y_0)\right)$$
$$\text{in } L^2(0,\infty;[L^2(\omega)]^d) \times L^2(0,\infty;H). \tag{2.2.15}$$

For $t \in [0,T]$, we have that:

$$\tilde{y}^0_{w,T}(t;y_0) = e^{\mathcal{A}t}y_0 + \left(L_T\tilde{u}^0_{w,T}(\,\cdot\,;y_0)\right)(t) + (W_T w)(t) \tag{2.2.16}$$

So for each positive T_0, we have that:

$$\tilde{y}^0_{w,T_i}(\,\cdot\,;y_0) \underset{\text{w}}{\longrightarrow} e^{\mathcal{A}\,\cdot}\,y_0 + L_{T_0}\tilde{u}_w(\,\cdot\,;y_0) + W_{T_0}w \qquad \text{in } L^2(0,T_0;H) \tag{2.2.17}$$

$$\tilde{y}^0_{w,T_i}(\,\cdot\,;y_0) \underset{\text{w}}{\longrightarrow} \tilde{y}_w(\,\cdot\,;y_0) \qquad \text{in } L^2(0,T_0;H) \tag{2.2.18}$$

Thus, by the uniqueness of the weak limit, we have $\tilde{y}_w = e^{\mathcal{A}\,\cdot}\,y_0 + L_{T_0}\tilde{u}_w + W_{T_0}w$ in $L^2(0,T_0;H)$, for all positive T_0, and $\tilde{y}_w = e^{\mathcal{A}\,\cdot}\,y_0 + L\tilde{u}_w + Ww$ in $L^2(0,\infty;H)$

The function $C(u, y) = \|y\|^2_{L^2(0,\infty;H)} + \|u\|^2_{L^2(0,\infty;[L^2(\omega)]^d)}$ is lower semicontinuous in the weak topology on $L^2(0, \infty; [L^2(\omega)]^d) \times L^2(0, \infty; H)$. Thus, we have:

$$\liminf_{T_i \to \infty} C\left(\tilde{u}^0_{w,T_i}(\,\cdot\,; y_0), \tilde{y}^0_{w,T_i}(\,\cdot\,; y_0)\right) \geq C\left(\tilde{u}_w(\,\cdot\,; y_0), \tilde{y}_w(\,\cdot\,; y_0)\right) \geq C_w(y_0) \tag{2.2.19}$$

where the last inequality is due to the fact that $\tilde{u}_w$, and $\tilde{y}_w$ are a competing pair for the minimization on the infinite time interval. This gives us that

$$\lim_{T \to \infty} C_{w,T}(y_0) \geq C_w(y_0) \tag{2.2.20}$$

When we combine the inequalities in (2.2.20) and (2.2.11), we get

$$\lim_{T \to \infty} C_{w,T}(y_0) = C_w(y_0) \tag{2.2.21}$$

Expanding equation (2.2.21) out and using the fact that $\|w\|^2_{L^2(0,T;[L^2(\Omega)]^d)}$ converges to $\|w\|^2_{L^2(0,\infty;[L^2(\Omega)]^d)}$ as $T \uparrow \infty$, we obtain:

$$\lim_{T \to \infty} J^0_{w,T}(y_0) = \lim_{T \to \infty} \int_0^T \left(\|y^0_{w,T}(t; y_0)\|^2_H + \|u^0_{w,T}(t; y_0)\|^2_{[L^2(\omega)]^d} - \gamma^2 \|w(t)\|^2_{[L^2(\Omega)]^d} \right) dt$$
$$= \int_0^\infty \left(\|y^0_w(t; y_0)\|^2_H + \|u^0_w(t; y_0)\|^2_{[L^2(\omega)]^d} - \gamma^2 \|w(t)\|^2_{[L^2(\Omega)]^d} \right) dt = J^0_w(y_0) \tag{2.2.22}$$

which concludes the proof of part (ii).

(iii): By the uniqueness of the minimizing pair, the inequalities in (2.2.19), and the equality in (2.2.21), we have that:

$$\tilde{u}_w(\,\cdot\,; y_0) = u^0_w(\,\cdot\,; y_0) \qquad \text{in } L^2(0, \infty; [L^2(\omega)]^d) \tag{2.2.23a}$$

$$\tilde{y}_w(\,\cdot\,; y_0) = y^0_w(\,\cdot\,; y_0) \qquad \text{in } L^2(0, \infty; H) \tag{2.2.23b}$$

The argument from the proof of part (ii) can be used to show that any sequence of values of T approaching infinity has a subsequence T_i so that $\tilde{u}^0_{w,T_i}(\,\cdot\,;y_0)$ and $\tilde{y}^0_{w,T_i}(\,\cdot\,;y_0)$ converge weakly to $u^0_w(\,\cdot\,;y_0)$ and $y^0_w(\,\cdot\,;y_0)$, respectively. Thus, the weak convergence in (2.2.15) can be rewritten as:

$$\tilde{u}^0_{w,T_i}(\,\cdot\,;y_0) \underset{\mathrm{w}}{\longrightarrow} u^0_w(\,\cdot\,;y_0) \qquad \text{in } L^2(0,\infty;[L^2(\omega)]^d) \tag{2.2.24a}$$

$$\tilde{y}^0_{w,T_i}(\,\cdot\,;y_0) \underset{\mathrm{w}}{\longrightarrow} y^0_w(\,\cdot\,;y_0) \qquad \text{in } L^2(0,\infty;H) \tag{2.2.24b}$$

The established convergence of the minimum cost $J^0_{w,T}(y_0) \to J^0_w(y_0)$ as $T \uparrow \infty$ provides norm-convergence:

$$\begin{aligned} &\left\|\tilde{y}^0_{w,T}(\,\cdot\,;y_0)\right\|^2_{L^2(0,\infty;H)} + \left\|\tilde{u}^0_{w,T}(\,\cdot\,;y_0)\right\|^2_{L^2(0,\infty;[L^2(\omega)]^d)} \\ &\quad \to \left\|y^0_w(\,\cdot\,;y_0)\right\|^2_{L^2(0,\infty;H)} + \left\|u^0_w(\,\cdot\,;y_0)\right\|^2_{L^2(0,\infty;[L^2(\omega)]^d)} \end{aligned} \tag{2.2.25}$$

Thus, weak convergence in (2.2.24) combined with norm-convergence in (2.2.25) provide strong convergence:

$$\tilde{u}^0_{w,T}(\,\cdot\,;y_0) \to u^0_w(\,\cdot\,;y_0) \qquad \text{in } L^2(0,\infty;[L^2(\omega)]^d) \tag{2.2.26a}$$

$$\tilde{y}^0_{w,T}(\,\cdot\,;y_0) \to y^0_w(\,\cdot\,;y_0) \qquad \text{in } L^2(0,\infty;H). \tag{2.2.26b}$$

$\square$

Corollary 2.2.3. *Recall $R_{0,T}(t)$ defined in (2.1.31) and $y^0_{w,T}(\,\cdot\,;y_0)$. Let $\tilde{R}_{0,T}(t)\tilde{y}^0_{w,T}(\,\cdot\,;y_0)$ denote the extension by zero of $R_{0,T}(t)y^0_{w,T}(\,\cdot\,;y_0)$ for $t > T$. Then we have the following convergence as $T \uparrow \infty$*

$$\tilde{R}_{0,T}(\,\cdot\,)\tilde{y}^0_{w,T}(\,\cdot\,;y_0) - R_0 y^0_w(\,\cdot\,;y_0) \to 0 \qquad \textit{in } L^2(0,\infty;H) \tag{2.2.27}$$

Proof. For $\tilde{R}_{0,T}(\,\cdot\,)\tilde{y}^0_{w,T}(\,\cdot\,;y_0) - R_0 y^0_w(\,\cdot\,;y_0)$, we have the following inequality

$$\begin{aligned}
&\left\|\tilde{R}_{0,T}(\,\cdot\,)\tilde{y}^0_{w,T}(\,\cdot\,;y_0) - R_0 y^0_w(\,\cdot\,;y_0)\right\| \\
&\leq \left\|\tilde{R}_{0,T}(\,\cdot\,)\left[\tilde{y}^0_{w,T}(\,\cdot\,;y_0) - y^0_w(\,\cdot\,;y_0)\right]\right\| + \left\|\left[\tilde{R}_{0,T}(\,\cdot\,) - R_0\right] y^0_w(\,\cdot\,;y_0)\right\|,
\end{aligned} \tag{2.2.28}$$

where the norms are in $L^2(0,\infty;H)$. Recalling the inequality in (1.5.4b) for R_0, and the convergence of $\tilde{u}^0_{w,T}(\,\cdot\,;y_0)$ to $y^0_w(\,\cdot\,;y_0)$ in $L^2(0,\infty;H)$ from (2.2.8b), we have

$$\begin{aligned}
&\left\|\tilde{R}_{0,T}(\,\cdot\,)\left[\tilde{y}^0_{w,T}(\,\cdot\,;y_0) - y^0_w(\,\cdot\,;y_0)\right]\right\| \\
&\qquad \leq M\left\|\tilde{y}^0_{w,T}(\,\cdot\,;y_0) - y^0_w(\,\cdot\,;y_0)\right\| \to 0 \text{ as } T\uparrow\infty
\end{aligned} \tag{2.2.29}$$

where the norms are still in $L^2(0,\infty;H)$. Using (1.5.4b) again, we obtain that

$$\left\|\left[\tilde{R}_{0,T}(t) - R_0\right] y^0_w(t;y_0)\right\|_H \leq \left(M + \|R_0\|_{\mathcal{L}(H)}\right) y^0_w(t;y_0) \tag{2.2.30}$$

Additionally, using (2.2.8b) again, we have that

$$\tilde{R}_{0,T}(t)y^0_w(t;y_0) \to R_0 y^0_w(t;y_0) \in H \qquad \text{a.e. in } t. \tag{2.2.31}$$

Dominated convergence applies due to the pointwise convergence a.e. in (2.2.31) and the upper bound in (2.2.30), and gives us

$$\left\|\left[\tilde{R}_{0,T}(\,\cdot\,) - R_0\right] y^0_w(\,\cdot\,;y_0)\right\|_{L^2(0,\infty;H)} \to 0 \text{ as } T\uparrow\infty. \tag{2.2.32}$$

The desired convergence follows from inequality (2.2.28) combined with the results from (2.2.32) and (2.2.29). $\square$

2.2.3 The Limit Process for $p_{w,T}(\,\cdot\,; y_0)$ and $r_{w,T}(\,\cdot\,)$ as $T \uparrow \infty$; the Equation for $p_{w,\infty}(\,\cdot\,; y_0)$

Now, we return to equation (2.1.47), relating $p_{w,T}(\,\cdot\,; y_0)$, $R_{0,T}$, and $r_{w,T}$, and take the limit as $T \uparrow \infty$. Recall that $y^0_{w,T}(\,\cdot\,; y_0) = y^0_{w=0,T}(\,\cdot\,; y_0) + y^0_{w,T}(\,\cdot\,; y_0 = 0)$, so we can invoke equation (1.5.5b) for the $y^0_{w=0;T}$ portion of the limit. For the $y^0_{w,T}(\,\cdot\,; y_0 = 0)$ portion of the limit, we need to establish a corresponding limit for $p_{w,T}(\,\cdot\,; y_0)$. To this end, we define for $y_0 \in H$:

$$
\begin{aligned}
p_{w,\infty}(t; y_0) &= \int_t^\infty e^{\mathcal{A}^*_{R_0}(\tau - t)} \left[-R_0 P m u^0_w(\tau; y_0) + y^0_w(\tau; y_0)\right] d\tau \\
&= \left(\mathcal{K}^*_{R_0}\left[-R_0 P m u^0_w(\,\cdot\,; y_0) + y^0_w(\,\cdot\,; y_0)\right]\right)(t) && (2.2.33a)\\
&\in C\left([0, \infty]; H\right) && (2.2.33b)
\end{aligned}
$$

where the regularity in (2.2.33b) follows from the smoothing property of $\mathcal{K}^*_{R_0}$ given in (1.6.2b) and the fact that the sum $-R_0 P m u^0_w(\,\cdot\,; y_0) + y^0_w(\,\cdot\,; y_0)$ is in $L^2(0, \infty; H)$, which follows from (2.2.6).

The justification for naming the quantity at the right hand side of equation (2.2.33a) $p_{w,\infty}(t; y_0)$, given the definition of $p_{w,T}(t; y_0)$ in (2.1.28) is established in the following proposition.

Proposition 2.2.4. *With reference to $p_{w,T}$ in* (2.1.28)*, and $p_{w,\infty}$ in* (2.2.33a)*, we have for $y_0 \in H$:*

$$
\left\| p_{w,\infty}(t; y_0) - p_{w,T}(t; y_0) \right\|_H \to 0 \qquad (2.2.34)
$$

as $T \uparrow \infty$, for each t fixed, and uniformly on compact t-sets. Moreover, if we let $\tilde{p}_{w,T}(\,\cdot\,; y_0)$ denote the extension by zero of $p_{w,T}(\,\cdot\,; y_0)$ for $t > T$, we have for $y_0 \in H$:

$$\|\tilde{p}_{w,T}(\,\cdot\,; y_0) - p_{w,\infty}(\,\cdot\,; y_0)\|_{C([0,\infty];H)} \to 0 \tag{2.2.35a}$$

$$\|\tilde{p}_{w,T}(\,\cdot\,; y_0) - p_{w,\infty}(\,\cdot\,; y_0)\|_{L^2(0,\infty;H)} \to 0 \tag{2.2.35b}$$

as $T \uparrow \infty$.

Proof. This proof proceeds in two steps. In the first step, we will provide a new integral equation for $p_{w,T}(\,\cdot\,; y_0)$, and in the second step we will use that equation to show the desired convergences.

Step 1: Using equation (2.1.54) of Corollary 2.1.4, we have that equation (2.1.28) defining $p_{w,T}(\,\cdot\,; y_0)$ can be rewritten in the more convenient formula

$$p_{w,T}(t; y_0) = \int_t^T e^{\mathcal{A}^*_{R_0}(\tau - t)} \left[-R_0 P\left(m u^0_{w,T}(\tau; y_0)\right) + y^0_{w,T}(\tau; y_0)\right] d\tau \tag{2.2.36a}$$

$$\in C\left([0,T]; H\right). \tag{2.2.36b}$$

where the continuity in (2.2.36b) follows from the smoothing property of the integral operator K_T given in (2.1.3b), and the fact that $u^0_{w,T}$ and $y^0_{w,T}$ are in $L^2(0,T;[L^2(\omega)]^d)$ and $L^2(0,T;H)$, respectively by (2.1.7).

Step 2: Using (2.2.36a) for $p_{w,T}$ and (2.2.33a) for $p_{w,\infty}$, with $y_0 \in H$, we obtain the following estimate for t fixed.

$$\|p_{w,\infty}(t; y_0) - p_{w,T}(t; y_0)\|_H$$

$$\leq \left\| \int_t^T e^{\mathcal{A}^*_{R_0}(\tau - t)} R_0 P \left[m u^0_w(\tau; y_0) - m u^0_{w,T}(\tau; y_0) \right] d\tau \right\|_H$$
$$+ \left\| \int_t^T e^{\mathcal{A}^*_{R_0}(\tau - t)} \left[y^0_w(\tau; y_0) - y^0_{w,T}(\tau; y_0) \right] d\tau \right\|_H$$
$$+ \left\| \int_T^\infty e^{\mathcal{A}^*_{R_0}(\tau - t)} \left[-R_0 P \left(m u^0_w(\tau; y_0) \right) + y^0_w(\tau; y_0) \right] d\tau \right\|_H \tag{2.2.37}$$

Expanded since previous draft:

We now move the norms inside the integrals and use the uniform stability of the analytic semigroup $e^{\mathcal{A}^*_{R_0} t}$ given in (1.5.11) to obtain

$$\|p_{w,\infty}(t; y_0) - p_{w,T}(t; y_0)\|_H$$
$$\leq \int_t^T M e^{-\alpha(\tau - t)} \left(c_1 \left\| u^0_w(\tau; y_0) - u^0_{w,T}(\tau; y_0) \right\|_{[L^2(\omega)]^d} + \left\| y^0_w(\tau; y_0) - y^0_{w,T}(\tau; y_0) \right\|_H \right) d\tau$$
$$+ \int_T^\infty M e^{-\alpha(\tau - t)} \left(c_1 \left\| u^0_w(\tau; y_0) \right\|_{[L^2(\omega)]^d} + \left\| y^0_w(\tau; y_0) \right\|_H \right) d\tau \tag{2.2.38}$$

Applying the Cauchy-Schwartz inequality to both integrals in (2.2.38) yields the inequality

$$\|p_{w,\infty}(t; y_0) - p_{w,T}(t; y_0)\|_H \leq$$
$$\frac{M}{\sqrt{2\alpha}} \left(c_1 \left\| u^0_w(\,\cdot\,; y_0) - u^0_{w,T}(\,\cdot\,; y_0) \right\|_{L^2(0,\infty;[L^2(\omega)]^d)} + \left\| y^0_w(\,\cdot\,; y_0) - y^0_{w,T}(\,\cdot\,; y_0) \right\|_{L^2(0,\infty;H)} \right.$$
$$\left. + c_1 \left\| u^0_w(\,\cdot\,; y_0) \right\|_{L^2(T,\infty;[L^2(\omega)]^d)} + \left\| y^0_w(\,\cdot\,; y_0) \right\|_{L^2(T,\infty;H)} \right). \tag{2.2.39}$$

Thus, recalling the convergence of $\tilde{u}^0_{w,T}(\,\cdot\,; y_0)$ to $u^0_w(\,\cdot\,; y_0)$ in $L^2(0, \infty; [L^2(\omega)]^d)$ and of $\tilde{y}^0_{w,T}(\,\cdot\,; y_0)$ to $y^0_w(\,\cdot\,; y_0)$ in $L^2(0, \infty; H)$ from equations (2.2.8a) and (2.2.8b), we obtain the convergence in (2.2.34) and (2.2.35a).

To show (2.2.35b), the convergence in $L^2(0, \infty; H)$, we define functions f, g_T, and

h_T, where f and g_T have support in $[0,\infty)$ and g_T has support on $[T,\infty)$, by

$$f(s) = Me^{-\alpha s} \qquad \text{for } s \geq 0 \tag{2.2.40a}$$

$$\begin{aligned} g_T(s) = c_1 & \left\| u_w^0(s;y_0) - u_{w,T}^0(s;y_0) \right\|_{[L^2(\omega)]^d} \\ & + \left\| y_w^0(s;y_0) - y_{w,T}^0(s;y_0) \right\|_H \qquad \text{for } s \geq 0 \end{aligned} \tag{2.2.40b}$$

$$h_T(s) = c_1 \left\| u_w^0(s;y_0) \right\|_{[L^2(\omega)]^d} + \left\| y_w^0(s;y_0) \right\|_H \qquad \text{for } s \geq T. \tag{2.2.40c}$$

Using these functions, we rewrite the inequality (2.2.38) as

$$\left\| p_{w,\infty}(t;y_0) - p_{w,T}(t;y_0) \right\|_H \leq \left(f * (g_T + f_T) \right)(t). \tag{2.2.41}$$

Taking the norm in $L^2(0,\infty;H)$ on both sides of (2.2.42), and then applying Young's Convolution Theorem (see Theorem 9.3 on page 146 of [7]), we obtain

$$\left\| p_{w,\infty}(\,\cdot\,;y_0) - p_{w,T}(\,\cdot\,;y_0) \right\|_{L^2(0,\infty;H)} \leq \left\| f * (g_T + h_T) \right\|_{L^2(0,\infty)} \tag{2.2.42}$$

$$\begin{aligned} &\leq c \left\| f \right\|_{L^1(0,\infty)} \left\| g_T + h_T \right\|_{L^2(0,\infty)} \to 0 \\ &\qquad \text{as } T \uparrow \infty \end{aligned} \tag{2.2.43}$$

where the convergence in (2.2.43) follows from the definitions of g_T and h_T in (2.2.40) and the L^2-convergence of $\tilde{u}_{w,T}^0(\,\cdot\,;y_0)$ and $\tilde{y}_{w,T}^0(\,\cdot\,;y_0)$ from equations (2.2.8a) and (2.2.8b), respectively. □

We now draw a few corollaries from the convergence of Proposition 2.2.4.

Corollary 2.2.5. *The function* $p_{w,\infty}$ *defined in* (2.2.33a) *can be rewritten as:*

$$p_{w,\infty}(t;y_0) = \int_t^{t_0} e^{\mathcal{A}^*(\tau - t)} y_w^0(\tau;y_0)\,d\tau + e^{\mathcal{A}_{R_0}^*(t_0 - t)} p_{w,\infty}(t_0;y_0) \tag{2.2.44}$$

where t_0 is an arbitrary point $t_0 \geq t$.

Proof. We return to the identity (2.1.56) in Proposition 2.1.5 and take the limit as $T \uparrow \infty$ on both sides. Invoking the L^2 convergence of $\tilde{y}^0_{w,T}(\,\cdot\,; y_0)$ to $y^0_w(\,\cdot\,; y_0)$ in (2.2.8b) and the convergence of $p_{w,T}(\,\cdot\,; y_0)$ to $p_{w,\infty}(\,\cdot\,; y_0)$ from Proposition 2.2.4, we obtain (2.2.44). $\square$

Note that Corollary 2.2.5 extends to an infinite time interval, $t \in [0, \infty)$, the idea of Proposition 2.1.5, which holds on a finite time interval $t \in [0, T]$.

Corollary 2.2.6. *With reference to $p_{w,\infty}$ defined in* (2.2.33a)*, the following identity holds true for $y_0 \in H$:*

$$p_{w,\infty}(t; y_0) = R_0 y^0_w(t; y_0) + r_{w,\infty}(t) \in L^2(0, \infty; H) \tag{2.2.45}$$

where

$$r_{w,\infty}(t) \equiv \lim_{T\uparrow\infty} r_{w,T}(t) = p_{w,\infty}(t; y_0 = 0) - R_0 y^0_w(t; y_0 = 0) \in L^2(0, \infty; H) \tag{2.2.46}$$

Proof. We return to identity (2.1.47) relating $p_{w,T}$, $y^0_{w,T}$, and $r_{w,T}$ and take the limit in $L^2(0, \infty; H)$ as $T \uparrow \infty$ after extending each function by zero for $t > T$.

$$\lim_{T\uparrow\infty} \tilde{p}_{w,T}(\,\cdot\,; y_0) = \lim_{T\uparrow\infty} \left[R_{0,T}(\,\cdot\,)\tilde{y}^0_{w,T}(\,\cdot\,; y_0) + \tilde{r}_{w,T}(\,\cdot\,)\right] \tag{2.2.47}$$

On the left, we use the convergence of $\tilde{p}_{w,T}$ to $p_{w,\infty}$ in $L^2(0, \infty; H)$, given in (2.2.35b), and on the right, we use (2.2.27), the convergence of $\tilde{R}_{0,T}(\,\cdot\,)\tilde{y}^0_{w,T}(\,\cdot\,; y_0)$ to $R_0 y^0_w(\,\cdot\,; y_0)$ in $L^2(0, \infty; H)$. Calling $r_{w,\infty}$ the limit as $T \uparrow \infty$ of $\tilde{r}_{w,T}$ in $L^2(0, \infty; H)$, we obtain

$$\lim_{T\uparrow\infty} \tilde{r}_{w,T} = r_{w,\infty} = p_{w,\infty}(\,\cdot\,; y_0) - R_0 y^0_w(\,\cdot\,; y_0) \in L^2(0, \infty; H) \tag{2.2.48}$$

We can remove y_0 from the definition of $r_{w,\infty}$ by taking the limit as $T \uparrow \infty$ of both sides of the equation defining $r_{w,T}$, (2.1.48)

$$r_{w,\infty} = \lim_{T\uparrow\infty} \tilde{r}_{w,T} = \lim_{T\uparrow\infty} \left[\tilde{p}_{w,T}(\,\cdot\,; y_0 = 0) - R_{0,T}(\,\cdot\,) y^0_{w,T}(\,\cdot\,; y_0 = 0)\right] \tag{2.2.49a}$$

$$= p_{w,\infty}(\,\cdot\,; y_0 = 0) - R_0 y^0_w(\,\cdot\,; y_0 = 0) \tag{2.2.49b}$$

This concludes the proof of the corollary. □

We next provide the differential versions of the definition of $p_{w,\infty}$ in (2.2.33a) and (2.2.44).

Corollary 2.2.7. *With reference to $p_{w,\infty}$ defined in* (2.2.33a), *we have for all $y_0 \in H$:*

$$\frac{d}{dt} p_{w,\infty}(t; y_0) = -\mathcal{A}^*_{R_0} p_{w,\infty}(t; y_0) + R_0 P m u^0_w(t; y_0) - y^0_w(t; y_0) \quad \text{in } [\mathcal{D}(\mathcal{A}^*)]' \tag{2.2.50a}$$

$$= -\mathcal{A}^* p_{w,\infty}(t; y_0) - y^0_w(t; y_0) \quad \text{in } [\mathcal{D}(\mathcal{A}^*)]' \tag{2.2.50b}$$

Proof. We take inner products of both sides of (2.2.33a) with $x \in \mathcal{D}(\mathcal{A})$ and differentiate in t to obtain:

$$\begin{aligned}
\frac{d}{dt} \left(p_{w,\infty}(t; y_0), x\right)_H &= \frac{d}{dt} \left(\int_t^\infty e^{\mathcal{A}^*_{R_0}(\tau - t)} \left[-R_0 P \left(m u^0_w(\tau; y_0)\right) + y^0_w(\tau; y_0)\right] d\tau, x \right)_H \\
&= \left(-\mathcal{A}^*_{R_0} \int_t^\infty e^{\mathcal{A}^*_{R_0}(\tau - t)} \left[-R_0 P \left(m u^0_w(\tau; y_0)\right) + y^0_w(\tau; y_0)\right] d\tau, x \right)_H \\
&\quad - \left(-R_0 P \left[m u^0_w(t; y_0)\right] + y^0_w(t; y_0), x\right)_H \\
&= -\left(\mathcal{A}^*_{R_0} p_{w,\infty}(t) - R_0 P \left[m u^0_w(t; y_0)\right] + y^0_w(t; y_0), x\right)_H
\end{aligned} \tag{2.2.51}$$

The conclusions of the corollary follow. □

Corollary 2.2.8. *With reference to* $p_{w,\infty}$ *defined in* (2.2.33a)*, and to* u_w^0 *guaranteed by* (2.2.6)*, we have:*

$$u_w^0(t;y_0) = -p_{w,\infty}(t;y_0)\big|_\omega \in L^2(0,\infty;[L^2(\omega)]^d) \tag{2.2.52}$$

Proof. From (2.2.35b), we have that

$$\|p_{w,\infty}(\,\cdot\,;y_0) - p_{w,T}(\,\cdot\,;y_0)\|_{L^2(0,\infty;[L^2(\omega)]^d)} \to 0 \quad \text{as } T \uparrow \infty \tag{2.2.53}$$

Returning to (2.1.49a), $u_{w,T}^0 = -p_{w,T}\big|_\omega$, we replace $u_{w,T}^0$ and $p_{w,T}$ by $\tilde{u}_{w,T}^0$ and $\tilde{p}_{w,T}$, their extensions by zero for $t > T$.

$$\tilde{u}_{w,T}^0(\,\cdot\,;y_0) = -\big(\tilde{p}_{w,T}(\,\cdot\,;y_0)\big)\,\big|_\omega \tag{2.2.54}$$

We then take the limit in $L^2(0,\infty;[L^2(\omega)]^d)$ as $T \uparrow \infty$ of (2.2.54) using (2.2.8a), the convergence $\tilde{u}_{w,T}^0 \to u_w^0$ as $T \uparrow \infty$ on the left hand side and (2.2.35b), the convergence $\tilde{p}_{w,T} \to p_{w,\infty}$ as $T \uparrow \infty$ to obtain (2.2.52). □

2.3 The Equation for $r_{w,\infty}(t)$

Proposition 2.3.1. *The function* $r_{w,\infty}(t)$ *defined by* (2.2.46) *satisfies the equation:*

$$\frac{d}{dt} r_{w,\infty}(t) = -\mathcal{A}_{R_0}^* r_{w,\infty}(t) - R_0 P w(t) \qquad \textit{in } [\mathcal{D}(\mathcal{A}^*)]' \tag{2.3.1}$$

and is thus given explicitly by:

$$r_{w,\infty}(t) = \int_t^\infty e^{\mathcal{A}_{R_0}^*(\tau - t)} R_0 P w(\tau)\, d\tau \in L^2(0,\infty;H) \cap C([0,\infty];H) \tag{2.3.2}$$

with terminal condition:

$$r_{w,\infty}(\infty) = 0 \tag{2.3.3}$$

Proof. We start from relation (2.2.45) defining $r_{w,\infty}$, and then differentiate in t using (2.2.50a) for $\frac{d}{dt}p_{w,\infty}(t;y_0)$, and (1.1.13) for $\frac{d}{dt}y_w^0(t;y_0)$. We take $x \in \mathcal{D}(\mathcal{A}_{R_0})$ and recall from (1.5.8) that $R_0\mathcal{A} \in \mathcal{L}(H)$. Then, using the duality pairing over H, we obtain from (2.2.45) a.e. in t,

$$\begin{aligned}
\frac{d}{dt}\left(r_{w,\infty}(t),x\right)_H &= \frac{d}{dt}\left(p_{w,\infty}(t;y_0),x\right)_H - \frac{d}{dt}\left(R_0y_w^0(t;y_0),x\right)_H \\
&= \left(-\mathcal{A}_{R_0}^*p_{w,\infty}(t;y_0) + R_0P\left[mu_w^0(t;y_0)\right] - y_w^0(t;y_0),x\right)_H \\
&\quad - \left(R_0\mathcal{A}y_w^0(t;y_0) + R_0P\left[mu_w^0(t;y_0)+w(t)\right],x\right)_H \qquad (2.3.4) \\
&= -\left(\mathcal{A}_{R_0}^*p_{w,\infty}(t;y_0) + y_w^0(t;y_0),x\right)_H \\
&\quad - \left(R_0\mathcal{A}y_w^0(t;y_0) + R_0Pw(t),x\right)_H \qquad (2.3.5)
\end{aligned}$$

Next, with $x \in H$, we invoke (1.5.13), the ARE for R_0 to obtain:

$$\begin{aligned}
\left(R_0\mathcal{A}y_w^0(t;y_0) + y_w^0(t;y_0),x\right)_H &= \left(y_w^0(t;y_0),\mathcal{A}^*R_0x + x\right)_H \\
&= \left(R_0y_w^0(t;y_0),R_0x\right)_{[L^2(\omega)]^d} - \left(y_w^0(t;y_0),R_0\mathcal{A}x\right)_H \\
&= -\left(y_w^0(t;y_0),R_0\left[\mathcal{A}-PmR_0\right]x\right)_H \qquad (2.3.6)
\end{aligned}$$

Using the definition of $\mathcal{A}_{R_0}$ provided in (1.5.9) and the fact that $R_0\mathcal{A} \in \mathcal{L}(H)$, provided in (1.5.8), we obtain,

$$\left(R_0\mathcal{A}y_w^0(t;y_0) + y_w^0(t;y_0),x\right)_H = -\left(y_w^0(t;y_0),R_0\mathcal{A}_{R_0}x\right)_H$$

$$= -\left(\mathcal{A}_{R_0}^* R_0 y_w^0(t; y_0), x\right)_H \quad x \in H \tag{2.3.7}$$

Inserting (2.3.7) into (2.3.5), we have a.e. in t:

$$\begin{aligned}
\frac{d}{dt}\left(r_{w,\infty}(t), x\right)_H &= -\left(\mathcal{A}_{R_0}^* p_{w,\infty}(t; y_0), x\right)_H - \left(R_0 P w(t), x\right)_H + \left(\mathcal{A}_{R_0}^* R_0 y_w^0(t; y_0), x\right)_H \\
&= -\left(\mathcal{A}_{R_0}^* \left[p_{w,\infty}(t; y_0) - R_0 y_w^0(t; y_0)\right], x\right)_H - \left(R_0 P w(t), x\right)_H \\
&= -\left(\mathcal{A}_{R_0}^* r_{w,\infty}(t) + R_0 P w(t), x\right)_H
\end{aligned} \tag{2.3.8}$$

Then (2.3.8) above yields the differential equation (2.3.1), which has the unique solution given by (2.3.2). Moreover, (2.3.2) implies the terminal condition, (2.3.3) at $t = \infty$ by virtue of the exponential decay, (1.5.11). □

2.4 The Stable Form of the Equation of y_w^0

In addition to the equation for the optimal dynamics,

$$\frac{d}{dt} y_w^0(t; y_0) = \mathcal{A} y_w^0(t; y_0) + P m u_w^0(t; y_0) + P w(t; y_0) \quad \text{on } [\mathcal{D}(\mathcal{A})]' \tag{2.4.1}$$

we will now present another representation, which is more useful in describing the behavior at infinity.

Proposition 2.4.1. *The minimizing solution $y_w^0(t; y_0)$ satisfies the following equation with stable generator:*

$$\frac{d}{dt} y_w^0(t; y_0) = \left(\mathcal{A} - P m R_0\right) y_w^0(t; y_0) - P m r_{w,\infty}(t) + P w(t; y_0) \quad \textit{in } [\mathcal{D}(\mathcal{A})]' \tag{2.4.2}$$

The solution of (2.4.2) *may be written in the following stable form*

$$y_w^0(t;y_0) = e^{\mathcal{A}_{R_0}t}y_0 + \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)}\left[-Pmr_{w,\infty}(\tau) + Pw(\tau)\right]d\tau \tag{2.4.3a}$$

$$= e^{\mathcal{A}_{R_0}t}y_0 - \left(\mathscr{L}_{R_0}r_{w,\infty}\right)(t) + \left(\mathcal{W}_{R_0}w\right)(t) \tag{2.4.3b}$$

with $y_w^0(\,\cdot\,;y_0) \in L^2(0,\infty;H)$.

Proof. Recall equation (2.2.45), relating y_w^0, $p_{w,\infty}$, and $r_{w,\infty}$, and equation (2.2.52), relating $p_{w,\infty}$ and u_w^0, which yield:

$$R_0y_w^0(t;y_0)\big|_\omega = -u_w^0(t;y_0) - r_{w,\infty}(t)\big|_\omega \tag{2.4.4}$$

Then adding and subtracting $PmR_0y_w^0(t;y_0) \in L^2(0,\infty;H)$ to the right hand side of (2.4.1) gives us (2.4.2). The expression for $y_w^0(\,\cdot\,;y_0)$ in (2.4.3) gives the unique solution of (2.4.2) under the condition that $y_w^0(0;y_0) = y_0$, and the regularity $y_w^0(\,\cdot\,;y_0) \in L^2(0,\infty;H)$ is a result of the regularity of $\mathscr{L}_{R_0}$ and $\mathcal{W}_{R_0}$ given in (1.6.4a). □

2.5 Collection of Explicit Formulae for $p_{w,\infty}$, $r_{w,\infty}$, and y_w^0 in Stable Form

For convenience, we collect the relevant formulae for $p_{w,\infty}$, $r_{w,\infty}$, and y_w^0 obtained in the preceding section that display a stable generator. We will make use of the operators $\mathcal{K}_{R_0}$, $\mathscr{L}_{R_0}$, $\mathcal{W}_{R_0}$, and their L^2-adjoints defined in (1.6.1) through (1.6.3).

- **Formula for $\mathbf{p}_{\mathbf{w},\infty}$** Using $\mathcal{K}^*_{R_0}$, we can rewrite formula (2.2.33a) for $p_{w,\infty}$ as:

$$p_{w,\infty}(t;y_0) = \int_t^\infty \left[-R_0 Pmu^0_w(\tau;y_0) + y^0_w(\tau;y_0)\right] d\tau \tag{2.5.1a}$$

$$= -\left(\mathcal{K}^*_{R_0} R_0 Pmu^0_w(\,\cdot\,;y_0) + \mathcal{K}^*_{R_0} y^0_w(\,\cdot\,;y_0)\right)(t) \tag{2.5.1b}$$

$$\in C\left([0,\infty];H\right) \cap L^q(0,\infty;H) \quad \forall\, q \geq 2$$

- **Formula for $\mathbf{r}_{\mathbf{w},\infty}$** Using $\mathcal{K}^*_{R_0}$, formula (2.3.2) for $r_{w,\infty}$ is rewritten as

$$r_{w,\infty}(t) = \int_t^\infty e^{\mathcal{A}^*_{R_0}(\tau-t)} R_0 Pw(\tau) d\tau \tag{2.5.2a}$$

$$= \left(\mathcal{K}^*_{R_0} R_0 Pw\right)(t) \in C\left([0,\infty];H\right) \cap L^q(0,\infty;H) \quad \forall\, q \geq 2 \tag{2.5.2b}$$

- **Formulae for $\mathbf{y}^{\mathbf{0}}_{\mathbf{w}}$** Using the operators $\mathscr{L}_{R_0}$ and $\mathcal{W}_{R_0}$, formula (2.4.3) for y^0_w is rewritten as

$$y^0(t;y_0) = e^{\mathcal{A}_{R_0}t} y_0 + \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)} \left[Pmr_{w,\infty}(\tau) + Pw(\tau)\right] d\tau \tag{2.5.3a}$$

$$= e^{\mathcal{A}_{R_0}t} y_0 - \left(\mathscr{L}_{R_0}\left[r_{w,\infty}\big|_\omega\right]\right)(t) + \left(\mathcal{W}_{R_0} w\right)(t) \tag{2.5.3b}$$

$$\in L^2(0,\infty;H)$$

Remark 2.5.1. *These stable dynamics will be used below in Section 3.2 to define the critical value γ_c, and in Section 3.3 to study the problem of maximizing $J^0_w(y_0)$ over $w \in L^2(0,\infty;[L^2(\Omega)]^d)$ directly over the infinite time interval. In this way, explicit formulae for all quantities involved will be obtained, which will involve R_0.*

Using (2.5.2) for $r_{w,\infty}$ in (2.2.45), the expression for $p_{w,\infty}$ in terms of y_w^0 and $r_{w,\infty}$, and in the expression (2.5.3b) for y_w^0, we obtain

$$p_{w,\infty}(\,\cdot\,;y_0) = R_0 y_w^0(\,\cdot\,;y_0) + \mathcal{K}_{R_0}^* R_0 P w \qquad \text{in } C\left([0,\infty];H\right) \tag{2.5.4}$$

$$y_w^0(\,\cdot\,;y_0) = e^{\mathcal{A}_{R_0}\,\cdot}\, y_0 - \mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P w + \mathcal{W}_{R_0} w \qquad \text{in } L^2(0,\infty;H) \tag{2.5.5}$$

Notice that for $w = 0$, equation (2.5.5), above gives the optimal solution $y_{w=0}^0$ *explicitly in terms of the problem data* using the (unique) Riccati operator R_0. Then, (2.5.5), inserted into (2.5.4) provides an explicit expression for $p_{w,\infty}$, which in turn provides one for $u_w^0 = -p_{w,\infty}\big|_\omega$ *directly in terms of the problem data.*

Chapter 3

Solving the Game Theory Problem (1.2.2)

3.1 Explicit Expression for the Optimal Cost $J_w^0(y_0 = 0)$ as a Quadratic Term

We introduce the bounded, self-adjoint operators, S and E_γ in $\mathcal{L}\left(L^2(0,\infty;[L^2(\Omega)]^d)\right)$ by:

$$S = R_0\mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P - \left[\mathcal{W}_{R_0}^* R_0 P + R_0 \mathcal{W}_{R_0}\right] \quad \text{in } \mathcal{L}\left(L^2(0,\infty;[L^2(\Omega)]^d)\right) \tag{3.1.1}$$

$$E_\gamma = \gamma^2 I + S \quad \text{in } \mathcal{L}\left(L^2(0,\infty;[L^2(\Omega)]^d)\right) \tag{3.1.2}$$

The boundedness of both S and E_γ follows from the regularity of the operators $\mathscr{L}_{R_0}$, $\mathcal{W}_{R_0}$, and their adjoints, given in (1.6.4). The goal of this section is to demonstrate:

Theorem 3.1.1. *With reference to E_γ defined in* (3.1.2)*, the minimum cost corresponding to a fixed $w \in L^2(0,\infty;[L^2(\Omega)]^d)$ and $y_0 = 0$, $J_w^0(y_0 = 0)$, is given by:*

$$J_w^0(y_0 = 0) = -\left(E_\gamma w, w\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.1.3}$$

Proof. The proof of Theorem 3.1.1 proceeds in two steps. In the first step, we show that:

$$-\left(Sw, w\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} = \left(Pw, p_{w,\infty}(\,\cdot\,; y_0 = 0)\right)_{L^2(0,\infty;H)} \tag{3.1.4a}$$

$$= \left(w, p_{w,\infty}(\,\cdot\,; y_0 = 0)\right)_{L^2(0,\infty;H)}. \tag{3.1.4b}$$

where the equality between (3.1.4a) and (3.1.4b) is a result of the fact that P is an orthogonal projection from $[L^2(\Omega)]^d$ onto H. In the second step, we show that:

$$J_w^0(y_0 = 0) + \gamma^2 \left\|w\right\|^2_{L^2(0,\infty;[L^2(\Omega)]^d)} = -(Sw, w)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.1.5}$$

Step 1: We use the definition of the operator S, given in (3.1.1), and the relationship between $\mathscr{L}_{R_0}$ and $\mathcal{K}_{R_0}$ from (1.6.3a) and obtain:

$$\begin{aligned}
&- \left(Sw, w\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \\
&\quad = \left(\left[\mathcal{W}^*_{R_0} R_0 P + R_0 \mathcal{W}_{R_0} - R_0 \mathscr{L}_{R_0} \mathscr{L}^*_{R_0} R_0 P\right] w, w\right)_{L^2(0,\infty;H)} \\
&\quad = \left(\left[\mathcal{K}^*_{R_0} R_0 + R_0 \mathcal{K}_{R_0} - R_0 \mathscr{L}_{R_0} \mathscr{L}^*_{R_0} R_0\right] Pw, Pw\right)_{L^2(0,\infty;H)} \\
&\quad = \left(\mathcal{K}^*_{R_0} R_0 Pw + R_0\left[-\mathcal{K}_{R_0} P\left(m \mathcal{K}^*_{R_0} R_0 Pw\right) + \mathcal{K}_{R_0} Pw\right], Pw\right)_{L^2(0,\infty;H)} \\
&\quad = \left(\mathcal{K}^*_{R_0} R_0 Pw + R_0\left[-\mathscr{L}_{R_0}\left(\left(\mathcal{K}^*_{R_0} R_0 Pw\right)\big|_\omega\right) + \mathcal{W}_{R_0} w\right], Pw\right)_{L^2(0,\infty;H)}.
\end{aligned} \tag{3.1.6}$$

Recalling equation (2.5.2), the stable form of the equation for $r_{w,\infty}$,

$$r_{w,\infty} = \mathcal{K}^*_{R_0} R_0 Pw \tag{3.1.7}$$

and (2.4.3b), the expression for the minimizing trajectory in terms of $\mathscr{L}_{R_0}$ and $\mathcal{W}_{R_0}$, we obtain the following:

$$-\left(Sw, w\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} = \left(r_{w,\infty} + R_0\left[-\mathscr{L}_{R_0}\left(r_{w,\infty}\big|_{\omega}\right) + \mathcal{W}_{R_0}w\right], Pw\right)_{L^2(0,\infty;H)} \tag{3.1.8a}$$

$$= \left(r_{w,\infty} + R_0 y^0_w(\,\cdot\,; y_0 = 0), Pw\right)_{L^2(0,\infty;H)} \tag{3.1.8b}$$

$$= \left(p_{w,\infty}(\,\cdot\,; y_0 = 0), Pw\right)_{L^2(0,\infty;H)} \tag{3.1.8c}$$

This concludes the first step of the proof.

Step 2: Our aim in this step is to prove the equality in (3.1.5) relating S and $J^0_w(y_0 = 0)$. To that end, we use

$$p_{w,\infty} + \mathcal{K}^*_{R_0} R_0 P\left(m u^0_w(\,\cdot\,; y_0 = 0)\right) = \mathcal{K}^*_{R_0} y^0_w(\,\cdot\,; y_0 = 0) \tag{3.1.9}$$

and expression (2.4.3b) for y^0_w, which can be rewritten as:

$$y^0_w(\,\cdot\,; y_0 = 0) = \mathcal{K}_{R_0} P\left[-m\mathscr{L}^*_{R_0} R_0 P w + w\right] \tag{3.1.10}$$

to compute the following, with $y^0(\,\cdot\,; y_0 = 0)$ and $u^0(\,\cdot\,; y_0 = 0)$ denoted by y^0_w and u^0_w, respectively

$$\left(y^0_w, y^0_w\right)_{L^2(0,\infty;H)}$$

$$= \left(\mathcal{K}_{R_0} P\left[-m\mathscr{L}^*_{R_0} R_0 P w + w\right], y^0_w\right)_{L^2(0,\infty;H)} \tag{3.1.11a}$$

$$= \left(P\left[-m\mathscr{L}^*_{R_0} R_0 P w + w\right], \mathcal{K}^*_{R_0} y^0_w\right)_{L^2(0,\infty;H)} \tag{3.1.11b}$$

$$= \left(P\left[-m\mathscr{L}^*_{R_0} R_0 P w + w\right], p_{w,\infty}\right)_{L^2(0,\infty;H)}$$

$$+ \left(P\left[-m\mathscr{L}^*_{R_0} R_0 P w + w\right], \mathcal{K}^*_{R_0} R_0 P\left(m u^0_w\right)\right)_{L^2(0,\infty;H)} \tag{3.1.11c}$$

$$= \left(P\left[-m\mathscr{L}_{R_0}^{*}R_0Pw + w\right], p_{w,\infty}\right)_{L^2(0,\infty;H)}$$
$$+ \left(R_0y_w^0, P\left(mu_w^0\right)\right)_{L^2(0,\infty;H)} \tag{3.1.11d}$$

Recall that $p_{w,\infty}\big|_{\omega} = -u_w^0$, and that for $f \in L^2(0,\infty;H)$, we have $\mathscr{L}_{R_0}^{*}f = \mathcal{K}_{R_0}^{*}f\big|_{\omega}$. Thus, we have that

$$\left(Pm\mathscr{L}_{R_0}^{*}R_0Pw, p_{w,\infty}\right)_{L^2(0,\infty;H)} = -\left(\mathcal{K}_{R_0}^{*}R_0Pw, u_w^0\right)_{L^2(0,\infty;[L^2(\omega)]^d)} \tag{3.1.12}$$

Using these equalities, and the stable expression for $r_{w,\infty}$, given in (3.1.7), we can rewrite the last part of equation (3.1.11) as:

$$-\left(m\mathscr{L}_{R_0}^{*}R_0Pw, p_{w,\infty}\right)_{L^2(0,\infty;H)} + \left(Pw, p_{w,\infty}\right)_{L^2(0,\infty;H)}$$
$$+ \left(R_0y_w^0, P\left(mu_w^0\right)\right)_{L^2(0,\infty;H)} \tag{3.1.13a}$$
$$= \left(\mathcal{K}_{R_0}^{*}R_0Pw, u_w^0\right)_{L^2(0,\infty;[L^2(\omega)]^d)} + \left(Pw, p_{w,\infty}\right)_{L^2(0,\infty;H)}$$
$$+ \left(R_0y_w^0, u_w^0\right)_{L^2(0,\infty;[L^2(\omega)]^d)} \tag{3.1.13b}$$
$$= \left(R_0y_w^0 + r_{w,\infty}, u_w^0\right)_{L^2(0,\infty;[L^2(\omega)]^d)} + \left(Pw, p_{w,\infty}\right)_{L^2(0,\infty;H)} \tag{3.1.13c}$$

As a result of the fact that $p_{w,\infty} = R_0y_w^0 + r_{w,\infty}$, we obtain:

$$\left(R_0y_w^0 + r_{w,\infty}, u_w^0\right)_{L^2(0,\infty;[L^2(\omega)]^d)} + \left(Pw, p_{w,\infty}\right)_{L^2(0,\infty;H)} \tag{3.1.14}$$
$$= \left(p_{w,\infty}, u_w^0\right)_{L^2(0,\infty;[L^2(\omega)]^d)} + \left(Pw, p_{w,\infty}\right)_{L^2(0,\infty;H)} \tag{3.1.15}$$
$$= -\left(u_w^0, u_w^0\right)_{L^2(0,\infty;[L^2(\omega)]^d)} + \left(Pw, p_{w,\infty}\right)_{L^2(0,\infty;H)} \tag{3.1.16}$$

Thus,

$$\left\|y_w^0\right\|^2_{L^2(0,\infty;H)} = -\left\|u_w^0\right\|^2_{L^2(0,\infty;[L^2(\omega)]^d)} + \left(Pw, p_{w,\infty}\right)_{L^2(0,\infty;H)} \tag{3.1.17}$$

so we have that

$$\begin{aligned}(Pw, p_{w,\infty})_{L^2(0,\infty;H)} &= \left\|y_w^0\right\|^2_{L^2(0,\infty;H)} + \left\|u_w^0\right\|^2_{L^2(0,\infty;[L^2(\omega)]^d)} \\ &= J_w^0(y_0 = 0) + \|w\|^2_{L^2(0,\infty;[L^2(\Omega)]^d)} \end{aligned} \tag{3.1.18}$$

Recalling (3.1.4a), the equality proved in the first step, yields

$$-\left(Sw, w\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} = (Pw, p_{w,\infty})_{L^2(0,\infty;H)} = J_w^0(y_0 = 0) + \|w\|^2_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.1.19}$$

which concludes the second step of the proof. We use the definition of E_γ given in (3.1.2) to complete the proof:

$$J_w^0(y_0 = 0) = -\left(Sw, w\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} - \|w\|^2_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.1.20a}$$

$$= -\left(E_\gamma w, w\right)^2_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.1.20b}$$

□

3.2 Definition of the Critical Value γ_c. Coercivity of E_γ for $\gamma > \gamma_c$

On the basis of Theorem 3.1.1, we now define the critical value, $\gamma_c \geq 0$, in terms of the problem data by

$$\gamma_c^2 = \sup_{\|w\|=1} (-Sw, w) = -\inf_{\|w\|=1} (Sw, w) \tag{3.2.1}$$

where both norms, $\|w\| = 1$, above are in $L^2(0,\infty;[L^2(\Omega)]^d)$. We have that $(-S) \geq 0$ is a nonnegative self-adjoint operator in $L^2(0,\infty;[L^2(\Omega)]^d)$, defined explicitly in terms of the problem data by (3.1.1). Thus, the infimum in equation (3.2.1) above gives the lowest point of the spectrum of the nonnegative, self-adjoint operator $(-S)$.

Proposition 3.2.1. *The bounded, self-adjoint operator E_γ defined in* (3.1.2) *satisfies*

$$(E_\gamma w, w)_{L^2(0,\infty;[L^2(\Omega)]^d)} \geq \left(\gamma^2 - \gamma_c^2\right) \|w\|^2_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.2.2}$$

and is strictly positive if and only if $\gamma > \gamma_c$, in which case $E_\gamma^{-1} \in \mathcal{L}(L^2(0,\infty;[L^2(\Omega)]^d))$

Proof. From the definitions of E_γ and γ_c, in (3.1.2) and (3.2.1), respectively, we have:

$$(E_\gamma w, w)_{L^2(0,\infty;[L^2(\Omega)]^d)} = \left(\left[\gamma^2 I + S\right] w, w\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.2.3}$$

$$= \gamma^2 \|w\|^2_{L^2(0,\infty;[L^2(\Omega)]^d)} + (Sw, w)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.2.4}$$

$$\geq \left(\gamma^2 - \gamma_c^2\right) \|w\|^2_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.2.5}$$

which concludes the proof of the proposition. □

3.3 Maximization of J_w^0 Over w Directly on $[0,\infty]$ for $\gamma > \gamma_c$. Explicit Expression of $w^*(\,\cdot\,; y_0)$ in Terms of the Data via E_γ^{-1}

Until the end of the chapter, we will take $\gamma > \gamma_c$ unless otherwise stated. We return to the optimal $J_w^0(y_0)$ in (2.2.1) for fixed $w \in L^2(0,\infty;[L^2(\Omega)]^d)$. In this section we consider the following optimal problem, which has two equivalent representations

$$\sup_{w\in L^2(0,\infty;[L^2(\Omega)]^d)} J^0_w(y_0) \quad \text{or} \quad \inf_{w\in L^2(0,\infty;[L^2(\Omega)]^d)} -J^0_w(y_0) \tag{3.3.1}$$

We shall first show that a unique optimal solution $w^*(\,\cdot\,;y_0)$ exists for problem (3.3.1). Next, taking advantage of the fact that y^0_w is written in stable form as in (2.4.3), we will study the maximization problem (3.3.1) directly over the infinite time interval to characterize the optimal solution w^* using the method of completing the square.

Theorem 3.3.1. *Consider the optimization problem* (3.3.1)

(i) For each $y_0 \in H$, there exists a unique optimal solution of the maximization problem (3.3.1)*, denoted by $w^*(\,\cdot\,;y_0)$.*

(ii) The maximizing disturbance, $w^*(\,\cdot\,;y_0)$ is characterized by

$$w^*(t;y_0) = \left(E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\,\cdot}\,y_0\right)(t) \in L^2(0,\infty;[L^2(\Omega)]^d) \tag{3.3.2}$$

Proof. (i): We start by noting that the optimal cost over the infinite time interval, $J^0_w(y_0)$ can be written as the sum of three terms

$$J^0_w(y_0) = J^0_w(y_0=0) + J^0_{w=0}(y_0=0) + X_w(y_0) \tag{3.3.3}$$

where the second term on the right hand side is constant in w, and the third term is linear in w. For linear term we have

$$X_w(y_0) \le C\|w\|\|y_0\| \le \varepsilon\|w\|^2 + C_\varepsilon\|y_0\|^2 \quad \text{for all } \varepsilon > 0 \tag{3.3.4}$$

where the norm on w is in $L^2(0,\infty;[L^2(\Omega)]^d)$ and the norm on y_0 is in H. By Proposition 3.2.1, the first term on the right hand side of (3.3.3) satisfies equation (3.2.2), with $(E_\gamma w, w)_{L^2(0,\infty;[L^2(\Omega)]^d)}$ a positive definite quadratic form when $\gamma > \gamma_c$. Thus, we obtain the following lower bound on $-J^0_w(y_0)$

$$-J^0_w(y_0) \geq \left[\gamma^2 - \left(\gamma_c^2 + \varepsilon\right)\right] \|w\|^2 - J^0_{w=0}(y_0) - C_\varepsilon \|y_0\|^2 \tag{3.3.5}$$

As a consequence, $-J^0_w(y_0)$ admits a unique minimum in w for $\gamma > \gamma_c$, which we will call $w^* = w^*(\,\cdot\,; y_0) \in L^2(0,\infty;[L^2(\Omega)]^d)$. This concludes the proof of part (i) of the theorem.

(ii): To characterize the optimal disturbance, we will take advantage of the stable dynamics (2.4.3) for $y^0_w(\,\cdot\,; y_0)$, so that we can study the optimization problem directly over the infinite time interval. The proof of (3.3.2) proceeds in two steps.

Step 1: To begin, we will show the following relationship for $X_w(y_0)$

$$X_w(y_0) = 2\left(e^{\mathcal{A}_{R_0}\cdot} y_0, R_0 P w\right)_{L^2(0,\infty;H)} \tag{3.3.6}$$

Recalling (3.3.3) above and the identities $y^0_w(\,\cdot\,; y_0) = y^0_w(\,\cdot\,; y_0 = 0) + y^0_{w=0}(\,\cdot\,; y_0)$, and $u^0_w(\,\cdot\,; y_0) = u^0_w(\,\cdot\,; y_0 = 0) + u^0_{w=0}(\,\cdot\,; y_0)$, given in (2.2.9), we obtain

$$\begin{aligned} X_w(y_0) = {} & 2\left(y^0_w(\,\cdot\,; y_0 = 0), y^0_{w=0}(\,\cdot\,; y_0)\right)_{L^2(0,\infty;H)} \\ & + 2\left(u^0_w(\,\cdot\,; y_0 = 0), u^0_{w=0}(\,\cdot\,; y_0)\right)_{L^2(0,\infty;[L^2(\omega)]^d)} \end{aligned} \tag{3.3.7}$$

We can simplify (3.3.7) by recalling from (1.5.12a) that $y^0_{w=0}(t; y_0) = e^{\mathcal{A}_{R_0} t} y_0$, and from (1.5.12b) that $u^0_{w=0}(t; y_0) = -R_0 e^{\mathcal{A}_{R_0} t} y_0\big|_\omega$

$$X_w(y_0) = 2\left(e^{\mathcal{A}_{R_0}\cdot} y_0, y^0_w(\,\cdot\,; y_0 = 0)\right)_{L^2(0,\infty;H)} - 2\left(R_0 e^{\mathcal{A}_{R_0}\cdot} y_0, u^0_w(\,\cdot\,; y_0 = 0)\right)_{L^2(0,\infty;[L^2(\omega)]^d)}$$

$$= 2\left(e^{\mathcal{A}_{R_0}\,\cdot}\,y_0, -R_0Pmu_w^0(\,\cdot\,;y_0=0) + y_w^0(\,\cdot\,;y_0=0)\right)_{L^2(0,\infty;H)} \tag{3.3.8}$$

Writing out the inner product in $L^2(0,\infty;H)$ and recalling the definition of $p_{w,\infty}(t;y_0)$, given in (2.2.33a), we obtain

$$\begin{aligned}
X_w(y_0) &= 2\int_0^\infty \left(e^{\mathcal{A}_{R_0}t}y_0, -R_0Pmu_w^0(t;y_0=0) + y_w^0(t;y_0=0)\right)_H dt \\
&= 2\left(y_0, \int_0^\infty e^{\mathcal{A}_{R_0}^* t}\left[-R_0Pmu_w^0(t;y_0=0) + y_w^0(t;y_0=0)\right]dt\right)_H \\
&= 2\left(y_0, p_{w,\infty}(0;y_0=0)\right)_H \qquad (3.3.9)
\end{aligned}$$

Using that $p_{w,\infty}(t;y_0) = R_0y_w^0(t;y_0) + r_{w,\infty}(t)$, and that $y_w^0(0,y_0=0)=0$, we express the cross term as

$$X_w(y_0) = 2\left(y_0, r_{w,\infty}(0)\right)_H = 2\left(y_0, \int_0^\infty e^{\mathcal{A}_{R_0}^* t}R_0Pw(t)\,dt\right)_H \tag{3.3.10}$$

$$= 2\left(e^{\mathcal{A}_{R_0}\,\cdot}\,y_0, R_0Pw\right)_{L^2(0,\infty;H)}, \tag{3.3.11}$$

which proves equation (3.3.6).

Step 2: We will now "complete the square" to find the maximizing disturbance $w^*(\,\cdot\,;y_0)$. Using the formulas in equations (1.5.7), (3.1.3), and (3.3.6) for $J_{w=0}^0(y_0)$, $J_w^0(y_0=0)$, and $X_w(y_0)$, respectively, we rewrite (3.3.3) as

$$J_w^0(y_0) = (R_0y_0,y_0)_H - (E_\gamma w, w)_{L^2(0,\infty;[L^2(\Omega)]^d)} + 2\left(e^{\mathcal{A}_{R_0}\,\cdot}\,y_0, R_0Pw\right)_{L^2(0,\infty;H)}. \tag{3.3.12}$$

Because we want to find the maximum value of $J_w^0(y_0)$ over all $w \in L^2(0,\infty;[L^2(\Omega)]^d)$ for a fixed initial condition $y_0 \in H$, we can maximize the quantity $2\left(e^{\mathcal{A}_{R_0}\,\cdot}\,y_0, R_0Pw\right)_{L^2(0,\infty;H)}$ $-(E_\gamma w, w)_{L^2(0,\infty;[L^2(\Omega)]^d)}$, which satisfies

$$2\left(e^{\mathcal{A}_{R_0}\,\cdot}\,y_0, R_0Pw\right)_{L^2(0,\infty;H)} - (E_\gamma w, w)_{L^2(0,\infty;[L^2(\Omega)]^d)}$$

$$= -\left(E_\gamma\left[w - E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0\right], w - E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0\right)_{L^2(0,\infty;[L^2(\Omega)]^d)}$$
$$+ \left(R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.3.13}$$

The following calculations show the validity of (3.3.13) by manipulating inner products and using E_γ^{-1}, which, by Proposition 3.2.1 is a well defined self-adjoint operator on $L^2(0,\infty;[L^2(\Omega)]^d)$ for $\gamma > \gamma_c$

$$2\left(e^{\mathcal{A}_{R_0}\cdot}y_0, R_0 P w\right)_{L^2(0,\infty;H)} - (E_\gamma w, w)_{L^2(0,\infty;[L^2(\Omega)]^d)}$$
$$= \left(e^{\mathcal{A}_{R_0}\cdot}y_0, R_0 P w\right)_{L^2(0,\infty;H)} - \left(E_\gamma w - R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, w\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.3.14a}$$
$$= \left(R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, w\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} - \left(w - E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, E_\gamma w\right)_{L^2(0,\infty;[L^2(\Omega)]^d)}$$
$$\pm \left(R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.3.14b}$$
$$= \left(R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, w - E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} - \left(w - E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, E_\gamma w\right)_{L^2(0,\infty;[L^2(\Omega)]^d)}$$
$$+ \left(R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.3.14c}$$
$$= -\left(E_\gamma w - R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, w - E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0\right)_{L^2(0,\infty;[L^2(\Omega)]^d)}$$
$$+ \left(R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.3.14d}$$
$$= -\left(E_\gamma\left[w - E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0\right], w - E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0\right)_{L^2(0,\infty;[L^2(\Omega)]^d)}$$
$$+ \left(R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.3.14e}$$

Because the operator E_γ is positive definite on $L^2(0,\infty;[L^2(\Omega)]^d)$ for $\gamma > \gamma_c$, the first inner product in the last equality above is always nonnegative. Thus, we maximize the quantity by taking

$$w^* = E_\gamma^{-1}R_0 e^{\mathcal{A}_{R_0}\cdot}y_0 \tag{3.3.15}$$

which gives us:

$$2\left(e^{\mathcal{A}_{R_0}\cdot}y_0, R_0 P w\right)_{L^2(0,\infty;H)} - \left(E_\gamma w, w\right)_{L^2(0,\infty;[L^2(\Omega)]^d)}$$
$$= \left(R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, E_\gamma^{-1} R_0 e^{\mathcal{A}_{R_0}\cdot}y_0\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.3.16}$$

This concludes the proof of part (ii). □

3.4 Collection of Explicit Formulae for J^*, r^*, y^*, u^*, and p^* in terms of w^* and the Problem Data

We can use this optimal disturbance, $w(\,\cdot\,;y_0) = w^*(\,\cdot\,;y_0)$, to obtain expressions for $J^*(y_0)$, $r^*(\,\cdot\,)$, $y^*(\,\cdot\,;y_0)$, $p^*(\,\cdot\,;y_0)$, and $u^*(\,\cdot\,;y_0)$ *in terms of the problem data.* For $J^*(y_0)$, we use equations (3.3.12) and (3.3.16). For $r^*(\,\cdot\,)$, $y^*(\,\cdot\,;y_0)$, $p^*(\,\cdot\,;y_0)$, and $u^*(\,\cdot\,;y_0)$, we use (2.5.1) through (2.5.5) specialized for $w = w^*$.

- **Expressions for $\mathbf{J}^*(\mathbf{y_0}) = \mathbf{J}^{\mathbf{0}}_{\mathbf{w}=\mathbf{w}^*}(\mathbf{y_0})$:** Using the definition of J from (1.2.1), we obtain (3.4.1a). We then make use of (3.3.12) and (3.3.14) to rewrite $J^*(y_0)$ strictly in terms of the problem data to obtain (3.4.1b), from which (3.4.1c) follows.

$$J^0_{w=w^*}(y_0) = \left(y^*(\,\cdot\,;y_0), y^*(\,\cdot\,;y_0)\right)_{L^2(0,\infty;H)} + \left(u^*(\,\cdot\,;y_0), u^*(\,\cdot\,;y_0)\right)_{L^2(0,\infty;[L^2(\omega)]^d)}$$
$$- \gamma^2\left(w^*(\,\cdot\,;y_0), w^*(\,\cdot\,;y_0)\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{3.4.1a}$$
$$= \left(R_0 y_0, y_0\right)_H + \left(R_0 e^{\mathcal{A}_{R_0}\cdot}y_0, P E_\gamma^{-1} R_0 e^{\mathcal{A}_{R_0}\cdot}y_0\right)_{L^2(0,\infty;H)} \tag{3.4.1b}$$

$$= \left(y_0, R_0 y_0 + \int_0^\infty e^{\mathcal{A}_{R_0}^* t} R_0 P \left(E_\gamma^{-1} R_0 e^{\mathcal{A}_{R_0} \cdot} y_0 \right)(t) \, dt \right)_H \tag{3.4.1c}$$

- **Expressions for $\mathbf{r}^*(\mathbf{t}) = \mathbf{r}_{\mathbf{w}=\mathbf{w}^*,\infty}(\mathbf{t})$:** Inserting w^* for w into (2.5.2), yields (3.4.2a). Next, we make use of the characterization of w^* in terms of the problem data in (3.3.2) to express $r^*(\,\cdot\,; y_0)$ in terms of the problem data in (3.4.2b).

$$r^*(t; y_0) = \int_t^\infty e^{\mathcal{A}_{R_0}^* (\tau - t)} R_0 P w^*(\tau; y_0) \, d\tau = \left(\mathcal{K}_{R_0}^* R_0 P w^*(\,\cdot\,; y_0) \right)(t) \tag{3.4.2a}$$

$$= \left(\mathcal{K}_{R_0}^* R_0 P E_\gamma^{-1} R_0 e^{\mathcal{A}_{R_0} \cdot} y_0 \right)(t) \tag{3.4.2b}$$

$$\in C\left([0,\infty]; H\right) \cap L^q(0,\infty; H) \quad \forall q \geq 2 \tag{3.4.2c}$$

The regularity in (3.4.2c) follows from the expression for $r^*(\,\cdot\,; y_0)$ in (3.4.2a), with $w^*(\,\cdot\,; y_0)$ in $L^2(0,\infty; [L^2(\Omega)]^d)$ and $\mathcal{K}_{R_0}^*$ a continuous operator from $L^2(0,\infty; H)$ to $C\left([0,\infty]; H\right) \cap L^q(0,\infty; H)$ (see (1.6.2b)).

- **Expressions for $\mathbf{y}^*(\mathbf{t}; \mathbf{y_0}) = \mathbf{y}^{\mathbf{0}}_{\mathbf{w}=\mathbf{w}^*}(\mathbf{t}; \mathbf{y_0})$:** Recalling the two stable forms for $y_w^0(\,\cdot\,; y_0)$ given in (2.4.3), we insert r^* for $r_{w,\infty}$ and w^* for w to obtain the formulae (3.4.3a) and (3.4.3b). Making use of (3.4.2a) allows us to express y^* in terms of w^* in (3.4.3c), which via (3.3.2) allows us to express y^* entirely in terms of the problem data in (3.4.3d) and (3.4.3e).

$$y^*(t; y_0) = e^{\mathcal{A}_{R_0} t} y_0 + \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)} \left(-Pmr^*(\tau; y_0) + Pw^*(\tau; y_0) \right) d\tau \tag{3.4.3a}$$

$$= e^{\mathcal{A}_{R_0} t} y_0 - \left\{ \mathscr{L}_{R_0} \left(r^*(\,\cdot\,; y_0)\big|_\omega \right) \right\}(t) + \left\{ \mathcal{W}_{R_0} w^*(\,\cdot\,; y_0) \right\}(t) \tag{3.4.3b}$$

$$= e^{\mathcal{A}_{R_0} t} y_0 + \left(\mathcal{W}_{R_0} w^*(\,\cdot\,; y_0) - \mathscr{L}_{R_0} \mathscr{L}_{R_0}^* R_0 P w^*(\,\cdot\,; y_0) \right)(t) \tag{3.4.3c}$$

$$= e^{\mathcal{A}_{R_0}t}y_0 + \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)}\left[-Pm\left(\mathscr{L}_{R_0}^* R_0 P E_\gamma^{-1} R_0 e^{\mathcal{A}_{R_0}\cdot} y_0\right)(\tau)\right.$$

$$\left. + P\left(E_\gamma^{-1} R_0 e^{\mathcal{A}_{R_0}\cdot} y_0\right)(\tau)\right] d\tau \tag{3.4.3d}$$

$$= e^{\mathcal{A}_{R_0}t}y_0 + \left\{\left(\mathcal{W}_{R_0} - \mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P\right)\left(E_\gamma^{-1} R_0 e^{\mathcal{A}_{R_0}\cdot} y_0\right)\right\}(t) \tag{3.4.3e}$$

- **Expressions for $\mathbf{p}^*(\mathbf{t}; \mathbf{y_0}) = \mathbf{p}_{\mathbf{w}=\mathbf{w}^*,\infty}(\mathbf{t}; \mathbf{y_0})$:** Using the expressions for $p_{w,\infty}$ in (2.2.45) with y^* and r^* inserted for y_w^0 and $r_{w,\infty}$, we obtain (3.4.4a) for p^*. Then, recalling (3.4.2a) for r^*, we obtain (3.4.4b).

$$p^*(\,\cdot\,; y_0) = R_0 y^*(\,\cdot\,; y_0) + r^*(\,\cdot\,; y_0) \tag{3.4.4a}$$

$$= R_0 y^*(\,\cdot\,; y_0) + K^* R_0 P w^*(\,\cdot\,; y_0) \tag{3.4.4b}$$

$$\in C\left([0,\infty]; H\right) \cap L^q(0,\infty; H) \quad \forall q \geq 2 \tag{3.4.4c}$$

The regularity in (3.4.4c) follows from

$$p^*(\,\cdot\,; y_0) = \mathcal{K}_{R_0}^*\left(-R_0 P m u^*(\,\cdot\,; y_0) + y^*(\,\cdot\,; y_0)\right) \tag{3.4.5}$$

with $R_0 P m u^*(\,\cdot\,; y_0)$ and $y^*(\,\cdot\,; y_0)$ in $L^2(0,\infty; H)$ and $\mathcal{K}_{R_0}^*$ a continuous operator from $L^2(0,\infty; H)$ to $C\left([0,\infty]; H\right) \cap L^q(0,\infty; H)$ (see (1.6.2b)).

- **Expressions for $\mathbf{u}^*(\mathbf{t}; \mathbf{y_0}) = \mathbf{u}^{\mathbf{0}}_{\mathbf{w}=\mathbf{w}^*}(\mathbf{t}; \mathbf{y_0})$:** Recalling (2.2.52) relating u_w^0 and $p_{w,\infty}$, and inserting w^* for w, so that u_w^0 and $p_{w,\infty}$ become u^* and p^*, respectively, we obtain (3.4.6a), from which (3.4.6b) follows immediately after using (3.4.4a)

$$u^*(t; y_0) = -p^*(t; y_0)\big|_\omega \tag{3.4.6a}$$

$$= -\big(R_0 y^*(t; y_0) + r^*(t)\big)\Big|_\omega \in L^2(0,\infty; [L^2(\omega)]^d). \tag{3.4.6b}$$

Chapter 4

The Feedback Semigroup $y^*(t; y_0) = \Phi(t)y_0$, the Riccati Operator R, and their Properties

4.1 Regularity for Optimal u^*, y^*, and w^*

Theorem 4.1.1. *Let $\gamma > \gamma_c$. Then we have the following results pertaining to $p^*(\,\cdot\,; y_0)$ from equation* (3.4.4)

$$p^*(\,\cdot\,; y_0): \quad \textit{continuous } H \to C\left([0, \infty]; H\right) \tag{4.1.1a}$$

$$\sup_{t \in [0,\infty)} \|p^*(t; y_0)\|_H = \|p^*(t; y_0)\|_{C([0,\infty]; H)} \le C \|y_0\|_H \,. \tag{4.1.1b}$$

Proof. We start by noting the following for $w^*(\,\cdot\,; y_0)$

$$\begin{aligned}
\left\|w^*(\,\cdot\,; y_0)\right\|_{L^2(0,\infty;[L^2(\Omega)]^d)} &= \left\|E_\gamma^{-1} R_0 e^{\mathcal{A}_{R_0}\,\cdot} y_0\right\|_{L^2(0,\infty;[L^2(\Omega)]^d)} \\
&\le \left\|E_\gamma^{-1}\right\| \left\|R_0 e^{\mathcal{A}_{R_0}\,\cdot}\right\| \|y_0\|_H \\
&= C_w \|y_0\|_H
\end{aligned} \tag{4.1.2}$$

where the norms on E_γ^{-1} and $R_0 e^{\mathcal{A}_{R_0}\,\cdot}$ are in $\mathcal{L}\left(L^2(0,\infty; [L^2(\Omega)]^d)\right)$ and $\mathcal{L}\left(L^2(0,\infty; H)\right)$, respectively, and the first equality follows from the characterization of w^* in (3.3.2).

Recall from Proposition 3.2.1 that for $\gamma > \gamma_c$, E_γ^{-1} is a well defined bounded linear operator on $L^2(0,\infty;[L^2(\Omega)]^d)$.

From here, we perform two calculations for r^*, defined in (3.4.2), in which we make use of (4.1.2) for w^*. The first calculation pertains to the norm in $L^2(0,\infty;H)$,

$$\begin{aligned}
\|r^*(\,\cdot\,;y_0)\|_{L^2(0,\infty;H)} &= \left\|\mathcal{K}_{R_0}^* R_0 P w^*(\,\cdot\,;y_0)\right\|_H \\
&\le \left\|\mathcal{K}_{R_0}^* R_0 P\right\| \left\|w^*(\,\cdot\,;y_0)\right\|_{L^2(0,\infty;[L^2(\Omega)]^d)} \\
&\le \left\|\mathcal{K}_{R_0}^* R_0 P\right\| C_w \left\|y_0\right\|_H \\
&= C_{1,r} \left\|y_0\right\|_H
\end{aligned} \tag{4.1.3}$$

where the norm on the operator $\mathcal{K}_{R_0}^* R_0 P$ is in $\mathcal{L}\left(L^2(0,\infty;[L^2(\Omega)]^d),\ L^2(0,\infty;H)\right)$. Note that we used the regularity of $\mathcal{K}_{R_0}^*$ from (1.6.2b) and (4.1.2) for $w^*(\,\cdot\,;y_0)$ to obtain the final inequality. The second calculation pertains to the norm of r^* in $C\left([0,\infty];H\right)$

$$\begin{aligned}
\|r^*(t;y_0)\|_H &= \left\|\left(\mathcal{K}_{R_0}^* R_0 P w^*(\,\cdot\,;y_0)\right)(t)\right\|_H \\
&\le \left\|\mathcal{K}_{R_0}^* R_0 P\right\| \left\|w^*(\,\cdot\,;y_0)\right\|_{L^2(0,\infty;[L^2(\Omega)]^d)} \\
&\le \left\|\mathcal{K}_{R_0}^* R_0 P\right\| C_w \left\|y_0\right\|_H \\
&\le C_{2,r} \left\|y_0\right\|_H
\end{aligned} \tag{4.1.4}$$

where now, the norm on $\mathcal{K}_{R_0}^* R_0 P$ is in $\mathcal{L}\left(L^2(0,\infty;[L^2(\Omega)]^d), C\left([0,\infty];H\right)\right)$ instead of on the space $\mathcal{L}\left(L^2(0,\infty;[L^2(\Omega)]^d), L^2(0,\infty;H)\right)$, as it was in the calculations in (4.1.3). As with the calculations in (4.1.3), we made use of (1.6.2b) for K^* and (4.1.2) for $w^*(\,\cdot\,; y_0)$. In particular, we used the fact that $\mathcal{K}_{R_0}^*$ is continuous as an operator from $L^2(0,\infty;H)$ to $C\left([0,\infty];H\right)$.

We will now use expression (3.4.3b) for y^* to find an inequality for y^* that is similar to the inequality in (4.1.4) for r^*. The inequalities in (4.1.2) and (4.1.3) for the L^2-norms of $w^*(\,\cdot\,; y_0)$ and $r^*(\,\cdot\,; y_0)$ combined with the smoothing properties of the operators $\mathscr{L}_{R_0}$ and $\mathcal{W}_{R_0}$ in (1.6.4a) and (1.5.11) for $e^{\mathcal{A}_{R_0}t}$ give us

$$\begin{aligned}
\left\|y^*(t;y_0)\right\|_H &= \left\|e^{\mathcal{A}_{R_0}t}y_0 - \left(\mathscr{L}_{R_0}r^*(\,\cdot\,;y_0)\right)(t) + \left(\mathcal{W}_{R_0}w^*(\,\cdot\,;y_0)\right)(t)\right\|_H \\
&\le \left\|e^{\mathcal{A}_{R_0}t}y_0\right\|_H + \left\|(\mathscr{L}_{R_0}r^*(\,\cdot\,;y_0))(t)\right\|_H + \left\|\left(\mathcal{W}_{R_0}w^*(\,\cdot\,;y_0)\right)(t)\right\|_H \\
&\le Me^{-\alpha t}\left\|y_0\right\|_H + \left\|\mathscr{L}_{R_0}\right\|\left\|r^*(\,\cdot\,;y_0)\right\|_{L^2(0,\infty;H)} + \left\|\mathcal{W}_{R_0}\right\|\left\|w^*(\,\cdot\,;y_0)\right\|_{L^2(0,\infty;[L^2(\Omega)]^d)} \\
&\le \left(Me^{-\alpha t} + C_{1,r}\left\|\mathscr{L}_{R_0}\right\| + C_w\left\|\mathcal{W}_{R_0}\right\|\right)\left\|y_0\right\|_H \\
&\le C_y\left\|y_0\right\|_H
\end{aligned} \tag{4.1.5}$$

where the norms on the operators $\mathscr{L}_{R_0}$, and $\mathcal{W}_{R_0}$, are in $\mathcal{L}\left(L^2(0,\infty;[L^2(\omega)]^d),\, C\left([0,\infty];H\right)\right)$, and $\mathcal{L}\left(L^2(0,\infty;[L^2(\Omega)]^d),\, C\left([0,\infty];H\right)\right)$, respectively.

We now show (4.1.1) by making use of the inequalities (4.1.4) and (4.1.5) for $r^*(t;y_0)$ and $y^*(t;y_0)$. Recalling the expression for p^* in (3.4.4a), we have

$$\begin{aligned}
\left\|p^*(t;y_0)\right\|_H &= \left\|R_0y^*(t;y_0) + r^*(t;y_0)\right\|_H \\
&\le \left\|R_0\right\|_{\mathcal{L}(H)}\left\|y^*(t;y_0)\right\|_H + \left\|r^*(t;y_0)\right\|_H \\
&\le \left(C_y\left\|R_0\right\|_{\mathcal{L}(H)} + C_{2,r}\right)\left\|y_0\right\|_H \\
&= C\left\|y_0\right\|_H
\end{aligned} \tag{4.1.6}$$

Because the inequality in (4.1.6) hold for all $t \in [0, \infty)$, we have completed the proof of the theorem. $\square$

Theorem 4.1.2. *For $\gamma > \gamma_c$ and $y_0 \in H$, the optimal control, trajectory and disturbance satisfy the following regularity results*

$$u^*(\,\cdot\,; y_0) \in C\left([0,\infty]; [L^2(\omega)]^d\right) \tag{4.1.7a}$$

$$y^*(\,\cdot\,; y_0) \in C\left([0,\infty]; H\right) \tag{4.1.7b}$$

$$w^*(\,\cdot\,; y_0) \in C\left([0,\infty]; [L^2(\Omega)]^d\right) \tag{4.1.7c}$$

Moreover, we have that

$$\gamma^2 w^*(\,\cdot\,; y_0) = p^*(\,\cdot\,; y_0) \in L^2(0,\infty; [L^2(\Omega)]^d) \tag{4.1.8}$$

Proof. The regularity of $u^*(\,\cdot\,; y_0)$ in (4.1.7a) is a result of the fact that $p^*(\,\cdot\,; y_0) \in C\left([0,\infty]; H\right)$, from (3.4.4c), and from the relation $u^*(\,\cdot\,; y_0) = -p^*(\,\cdot\,; y_0)\big|_\omega$ from (3.4.6).

In equation (4.1.5) in the proof of Theorem 4.1.1, we showed that there exists a constant c such that

$$\left\|y^*(t; y_0)\right\|_H \le c \left\|y_0\right\|_H . \tag{4.1.9}$$

The result in equation (4.1.7b) follows directly from (4.1.9).

In order to show (4.1.7c), we will show that (4.1.8) holds. From here, the result follows from the regularity of $p^*(\,\cdot\,; y_0)$ provided in (3.4.4c). From the characterization of w^*, in (3.3.2), and (3.1.2), the definition of E_γ, we have that

$$E_\gamma w^*(\,\cdot\,;y_0) = R_0 e^{\mathcal{A}_{R_0}\,\cdot}\,y_0 \tag{4.1.10a}$$

$$\gamma^2 w^*(\,\cdot\,;y_0) = R_0 e^{\mathcal{A}_{R_0}\,\cdot}\,y_0 - Sw^*(\,\cdot\,;y_0). \tag{4.1.10b}$$

Recalling the definition of S, provided in (3.1.1), and the relationships between $p^*(\,\cdot\,;y_0)$, $y^*(\,\cdot\,;y_0)$, and $r^*(\,\cdot\,;y_0)$, we obtain

$$\begin{aligned}\gamma^2 w^*(\,\cdot\,;y_0) &= R_0 e^{\mathcal{A}_{R_0}\,\cdot}\,y_0 - \left[R_0 \mathscr{L}_{R_0}\mathscr{L}^*_{R_0} R_0 P - \mathcal{W}^*_{R_0} R_0 P - R_0 \mathcal{W}_{R_0}\right] w^*(\,\cdot\,;y_0)\\ &= R_0\left[e^{\mathcal{A}_{R_0}\,\cdot}\,y_0 + \left(-\mathscr{L}_{R_0}\mathscr{L}^*_{R_0} R_0 P + \mathcal{W}_{R_0}\right) w^*(\,\cdot\,;y_0)\right] + \mathcal{W}^*_{R_0} R_0 P w^*(\,\cdot\,;y_0)\\ &= R_0 y^*(\,\cdot\,;y_0) + r^*(\,\cdot\,;y_0) = p^*(\,\cdot\,;y_0). \end{aligned}\tag{4.1.11}$$

Thus, we have (4.1.8), and (4.1.7c) follows. □

4.2 A Transition Property for w^* for $\gamma > \gamma_c$

We have the following important transition property for the optimizing disturbance, w^*, which instrumental in showing a similar transition property for the optimal trajectory, y^* in Section 4.4.

Theorem 4.2.1. *For $\gamma > \gamma_c$ and $y_0 \in H$, the following transition property holds for the optimal disturbance*

$$w^*(t+\sigma;y_0) = w^*(\sigma;y^*(t;y_0)) \underset{(in\ \sigma)}{\in} C\left([0,\infty];[L^2(\Omega)]^d\right) \tag{4.2.1}$$

for each t fixed, with $t \geq 0$.

Proof. The proof proceeds in three steps.

Step 1: In this step, we use $E_\gamma w^*(\,\cdot\,;y_0) = R_0 e^{\mathcal{A}_{R_0}\,\cdot}\,y_0$, provided in equation (3.3.2), and the definitions of S and E_γ from equations (3.1.1) and (3.1.2) to find a relationship between $w^*(\sigma; y^*(t;y_0))$ and $w^*(\sigma+t;y_0)$.

First, we find a relationship between $\gamma^2 w^*(\sigma; y^*(t;y_0))$ and $R_0 e^{\mathcal{A}_{R_0}\sigma} y^*(t;y_0)$ by using (3.1.1) and (3.1.2) to expand $R_0 e^{\mathcal{A}_{R_0}\sigma} y^*(t;y_0) = \left(E_\gamma w^*(\,\cdot\,; y^*(t;y_0))\right)(\sigma)$. We obtain

$$
\begin{aligned}
&R_0 e^{\mathcal{A}_{R_0}\sigma} y^*(t;y_0) \\
&= \gamma^2 w^*(\sigma; y^*(t;y_0)) + R_0 \left\{\mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P w^*(\,\cdot\,; y^*(t;y_0))\right\}(\sigma) \\
&\qquad - R_0\left\{\mathcal{W}_{R_0} w^*(\,\cdot\,; y^*(t;y_0))\right\}(\sigma) - \left\{\mathcal{W}_{R_0}^* R_0 P w^*(\,\cdot\,; y^*(t;y_0))\right\}(\sigma).
\end{aligned} \tag{4.2.2}
$$

Recalling the expression for $y^*(t;y_0)$ in terms of w^* in (3.4.3c), we obtain the following expression for $R_0 e^{\mathcal{A}_{R_0}\sigma} y^*(t;y_0)$

$$
\begin{aligned}
R_0 e^{\mathcal{A}_{R_0}\sigma} y^*(t;y_0) &= R_0 e^{\mathcal{A}_{R_0}(\sigma+t)} y_0 - R_0 e^{\mathcal{A}_{R_0}\sigma}\left\{\mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P w^*(\,\cdot\,;y_0)\right\}(t) \\
&\qquad + R_0 e^{\mathcal{A}_{R_0}\sigma}\left\{\mathcal{W}_{R_0} w^*(\,\cdot\,;y_0)\right\}(t)
\end{aligned} \tag{4.2.3}
$$

Using the fact that the left hand side of both equations (4.2.2) and (4.2.3) is $R_0 e^{\mathcal{A}_{R_0}\sigma} y^*(t;y_0)$, we set the right hand sides equal and solve for $R_0 e^{\mathcal{A}_{R_0}(\sigma+t)} y_0$, yielding

$$
\begin{aligned}
&R_0 e^{\mathcal{A}_{R_0}(\sigma+t)} y_0 \\
&= \gamma^2 w^*(\sigma; y^*(t;y_0)) + R_0\left\{\mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P w^*(\,\cdot\,; y^*(t;y_0))\right\}(\sigma) \\
&\qquad + R_0 e^{\mathcal{A}_{R_0}\sigma}\left\{\mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P w^*(\,\cdot\,;y_0)\right\}(t) - R_0 e^{\mathcal{A}_{R_0}\sigma}\left\{\mathcal{W}_{R_0} w^*(\,\cdot\,;y_0)\right\}(t)
\end{aligned}
$$

$$- R_0 \left\{\mathcal{W}_{R_0} w^*(\,\cdot\,; y^*(t;y_0))\right\}(\sigma) - \left\{\mathcal{W}_{R_0}^* R_0 P w^*(\,\cdot\,; y^*(t;y_0))\right\}(\sigma). \tag{4.2.4}$$

Now, we use (3.1.2) and (3.1.1) to expand $R_0 e^{\mathcal{A}_{R_0}(\sigma+t)} y_0 = \left(E_\gamma w^*(\,\cdot\,; y_0)\right)(\sigma+t)$ and obtain the following relationship

$$\begin{aligned}
&R_0 e^{\mathcal{A}_{R_0}(\sigma+t)} y_0 \\
&= \gamma^2 w^*(\sigma+t; y_0) + R_0 \left\{\mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P w^*(\,\cdot\,; y_0)\right\}(\sigma+t) \\
&\quad - R_0 \left\{\mathcal{W}_{R_0} w^*(\,\cdot\,; y_0)\right\}(\sigma+t) - \left\{\mathcal{W}_{R_0}^* R_0 P w^*(\,\cdot\,; y_0)\right\}(\sigma+t)
\end{aligned} \tag{4.2.5}$$

Both equations (4.2.4) and (4.2.5) above have $R_0 e^{\mathcal{A}_{R_0}(\sigma+t)} y_0$ on the left hand side. Setting their right hand sides equal gives us

$$\begin{aligned}
&\gamma^2 w^*(\sigma; y^*(t;y_0)) + R_0 \left\{\mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P w^*(\,\cdot\,; y^*(t;y_0))\right\}(\sigma) \\
&\quad + R_0 e^{\mathcal{A}_{R_0}\sigma} \left\{\mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P w^*(\,\cdot\,; y_0)\right\}(t) - R_0 e^{\mathcal{A}_{R_0}\sigma} \left\{\mathcal{W}_{R_0} w^*(\,\cdot\,; y_0)\right\}(t) \\
&\quad - R_0 \left\{\mathcal{W}_{R_0} w^*(\,\cdot\,; y^*(t;y_0))\right\}(\sigma) - \left\{\mathcal{W}_{R_0}^* R_0 P w^*(\,\cdot\,; y^*(t;y_0))\right\}(\sigma) \\
&= \gamma^2 w^*(\sigma+t; y_0) + R_0 \left\{\mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P w^*(\,\cdot\,; y_0)\right\}(\sigma+t) \\
&\quad - R_0 \left\{\mathcal{W}_{R_0} w^*(\,\cdot\,; y_0)\right\}(\sigma+t) - \left\{\mathcal{W}_{R_0}^* R_0 P w^*(\,\cdot\,; y_0)\right\}(\sigma+t)
\end{aligned} \tag{4.2.6}$$

which leads to the following relationship between $w^*(\sigma; y^*(t;y_0))$ and $w^*(\sigma+t; y_0)$:

$$\begin{aligned}
&\gamma^2 \left[w^*(\sigma; y^*(t;y_0)) - w^*(\sigma+t; y_0)\right] \\
&+ R_0 \Big[\left\{\mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P w^*(\,\cdot\,; y^*(t;y_0))\right\}(\sigma) - \left\{\mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P w^*(\,\cdot\,; y_0)\right\}(\sigma+t) \\
&\qquad + e^{\mathcal{A}_{R_0}\sigma} \left\{\mathscr{L}_{R_0}\mathscr{L}_{R_0}^* R_0 P w^*(\,\cdot\,; y_0)\right\}(t)\Big] \\
&- R_0 \Big[e^{\mathcal{A}_{R_0}\sigma} \left\{\mathcal{W}_{R_0} w^*(\,\cdot\,; y_0)\right\}(t) - \left\{\mathcal{W}_{R_0} w^*(\,\cdot\,; y_0)\right\}(\sigma+t) + \left\{\mathcal{W}_{R_0} w^*(\,\cdot\,; y^*(t;y_0))\right\}(\sigma)\Big]
\end{aligned}$$

$$+ \Big[\big\{\mathcal{W}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0)\big\}(\sigma + t) - \big\{\mathcal{W}^*_{R_0} R_0 P w^*(\,\cdot\,; y^*(t; y_0))\big\}(\sigma)\Big] = 0 \tag{4.2.7}$$

Step 2: In this step, we will show that the the following three equalities hold

$$\begin{aligned}
&R_0 \Big[\big\{\mathscr{L}_{R_0}\mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y^*(t; y_0))\big\}(\sigma) - \big\{\mathscr{L}_{R_0}\mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0)\big\}(\sigma + t) \\
&\qquad + e^{\mathcal{A}_{R_0}\sigma}\big\{\mathscr{L}_{R_0}\mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0)\big\}(t)\Big] \\
&\qquad = R_0\big\{\mathscr{L}_{R_0}\mathscr{L}^*_{R_0} R_0 P\big[w^*(\,\cdot\,; y^*(t; y_0)) - w^*(\,\cdot\,; y_0)\big]\big\}(\sigma) && (4.2.8a) \\
&R_0\Big[e^{\mathcal{A}_{R_0}\sigma}\big\{\mathcal{W}_{R_0} w^*(\,\cdot\,; y_0)\big\}(t) - \big\{\mathcal{W}_{R_0} w^*(\,\cdot\,; y_0)\big\}(\sigma + t) + \big\{\mathcal{W}_{R_0} w^*(\,\cdot\,; y^*(t; y_0))\big\}(\sigma)\Big] \\
&\qquad = \big(R_0\mathcal{W}_{R_0}\big[w^*(\,\cdot\,; y^*(t; y_0)) - w^*(t + \,\cdot\,; y_0)\big]\big)(\sigma) && (4.2.8b) \\
&\big\{\mathcal{W}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0)\big\}(\sigma + t) - \big\{\mathcal{W}^*_{R_0} R_0 P w^*(\,\cdot\,; y^*(t; y_0))\big\}(\sigma) \\
&\qquad = \big(\mathcal{W}^*_{R_0} R_0 P\big[w^*(\,\cdot\, + t; y_0) - w^*(\,\cdot\,; y^*(t; y_0))\big]\big)(\sigma) && (4.2.8c)
\end{aligned}$$

To show (4.2.8a), we use the definition of $\mathscr{L}_{R_0}$ in (1.6.3a) to write the quantity on the left hand side of (4.2.8a) as a sum of integrals, then calculate

$$\begin{aligned}
&R_0 \Big[\big\{\mathscr{L}_{R_0}\mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y^*(t; y_0))\big\}(\sigma) - \big\{\mathscr{L}_{R_0}\mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0)\big\}(\sigma + t) \\
&\qquad + e^{\mathcal{A}_{R_0}\sigma}\big\{\mathscr{L}_{R_0}\mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0)\big\}(t)\Big] \\
&= R_0\Big[\int_0^\sigma e^{\mathcal{A}_{R_0}(\sigma-\tau)} Pm\big\{\mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y^*(t; y_0))\big\}(\tau)\,d\tau \\
&\qquad - \int_0^{\sigma+t} e^{\mathcal{A}_{R_0}(t+\sigma-\tau)} Pm\big\{\mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0)\big\}(\tau)\,d\tau \\
&\qquad + \int_0^{t} e^{\mathcal{A}_{R_0}(t+\sigma-\tau)} Pm\big\{\mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0)\big\}(\tau)\,d\tau\Big] && (4.2.9a) \\
&= R_0\Big[\int_0^\sigma e^{\mathcal{A}_{R_0}(\sigma-\tau)} Pm\big\{\mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y^*(t; y_0))\big\}(\tau)\,d\tau \\
&\qquad - \int_t^{\sigma+t} e^{\mathcal{A}_{R_0}(t+\sigma-\tau)} Pm\big\{\mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0)\big\}(\tau)\,d\tau\Big] && (4.2.9b)
\end{aligned}$$

$$= R_0 \left\{ \mathscr{L}_{R_0} \mathscr{L}^*_{R_0} R_0 P \left[w^*(\,\cdot\,; y^*(t; y_0)) - w^*(\,\cdot\,; y_0) \right] \right\}(\sigma), \tag{4.2.9c}$$

where the equality between (4.2.9b) and (4.2.9c), above, comes from

$$\int_t^{\sigma+t} e^{\mathcal{A}_{R_0}(t+\sigma-\tau)} Pm \left\{ \mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0) \right\}(\tau)\, d\tau = \left(\mathscr{L}_{R_0} \mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\, + t; y_0) \right)(\sigma) \tag{4.2.10}$$

The relationship in (4.2.10) is justified using the definitions of $\mathscr{L}_{R_0}$ and $\mathscr{L}^*_{R_0}$ in (1.6.3a) and two changes of variables as follows:

$$\begin{aligned}
& && \int_t^{\sigma+t} e^{\mathcal{A}_{R_0}(t+\sigma-\tau)} Pm \left\{ \mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0) \right\}(\tau)\, d\tau \\
&(\beta = \tau - t) && = \int_0^{\sigma} e^{\mathcal{A}_{R_0}(\sigma-\beta)} Pm \left\{ \mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0) \right\}(\beta + t)\, d\beta \\
& && = \int_0^{\sigma} e^{\mathcal{A}_{R_0}(\sigma-\beta)} Pm \int_{\beta+t}^{\infty} e^{\mathcal{A}^*_{R_0}(\alpha-(\beta+t))} R_0 P w^*(\alpha; y_0)\, d\alpha\, d\beta \\
&(s = \alpha - t) && = \int_0^{\sigma} e^{\mathcal{A}_{R_0}(\sigma-\beta)} Pm \int_{\beta}^{\infty} e^{\mathcal{A}^*_{R_0}(s-\beta)} R_0 P w^*(s + t; y_0)\, ds\, d\beta \\
& && = \left(\mathscr{L}_{R_0} \mathscr{L}^*_{R_0} R_0 P w^*(\,\cdot\, + t; y_0) \right)(\sigma)
\end{aligned} \tag{4.2.11}$$

To show (4.2.8b), we rewrite its left hand side as an integral using the definition of $\mathcal{W}^*_{R_0}$ from (1.6.3b) and calculate

$$\begin{aligned}
& e^{\mathcal{A}_{R_0}\sigma} \left\{ \mathcal{W}_{R_0} w^*(\,\cdot\,; y_0) \right\}(t) - \left\{ \mathcal{W}_{R_0} w^*(\,\cdot\,; y_0) \right\}(\sigma + t) + \left\{ \mathcal{W}_{R_0} w^*(\,\cdot\,; y^*(t; y_0)) \right\}(\sigma) \\
&= e^{\mathcal{A}_{R_0}\sigma} \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)} w^*(\tau; y_0)\, d\tau - \int_0^{t+\sigma} e^{\mathcal{A}_{R_0}(t+\sigma-\tau)} w^*(\tau; y_0)\, d\tau \\
&\qquad + \int_0^{\sigma} e^{\mathcal{A}_{R_0}(\sigma-\alpha)} w^*(\alpha; y^*(t; y_0))\, d\alpha && (4.2.12a) \\
&= - \int_t^{t+\sigma} e^{\mathcal{A}_{R_0}(t+\sigma-\tau)} w^*(\tau; y_0)\, d\tau + \int_0^{\sigma} e^{\mathcal{A}_{R_0}(\sigma-\alpha)} w^*(\alpha; y^*(t; y_0))\, d\alpha && (4.2.12b)
\end{aligned}$$

$$= -\int_0^\sigma e^{\mathscr{A}_{R_0}(\sigma-\alpha)} w^*(t+\alpha; y_0)\, d\alpha + \int_0^\sigma e^{\mathscr{A}_{R_0}(\sigma-\alpha)} w^*(\alpha; y^*(t; y_0))\, d\alpha \tag{4.2.12c}$$

$$= \big(\mathcal{W}_{R_0}\left[w^*(\,\cdot\,; y^*(t; y_0)) - w^*(t+\,\cdot\,; y_0)\right]\big)(\sigma) \tag{4.2.12d}$$

where we used the change of variables $\alpha = \tau - t$ to equate (4.2.12b) and (4.2.12c) above. Multiplying by $(-R_0)$ gives (4.2.8b).

To show (4.2.8c), we rewrite its left hand side as a sum of integrals using the definition of $\mathcal{W}^*_{R_0}$ from (1.6.3b) and calculate

$$\big\{\mathcal{W}^*_{R_0} R_0 P w^*(\,\cdot\,; y_0)\big\}(\sigma+t) - \big\{\mathcal{W}^*_{R_0} R_0 P w^*(\,\cdot\,; y^*(t; y_0))\big\}(\sigma)$$

$$= \int_{\sigma+t}^\infty e^{\mathscr{A}^*_{R_0}(\beta-\sigma-t)} R_0 P w^*(\beta; y_0)\, d\beta - \int_\sigma^\infty e^{\mathscr{A}^*_{R_0}(\tau-\sigma)} R_0 P w^*(\tau; y^*(t; y_0))\, d\tau \tag{4.2.13a}$$

$$= \int_\sigma^\infty e^{\mathscr{A}^*_{R_0}(\tau-\sigma)} R_0 P \left[w^*(\tau+t; y_0) - w^*(\tau; y^*(t; y_0))\right] d\tau \tag{4.2.13b}$$

$$= \big(\mathcal{W}^*_{R_0} R_0 P \left[w^*(\,\cdot\,+t; y_0) - w^*(\,\cdot\,; y^*(t; y_0))\right]\big)(\sigma) \tag{4.2.13c}$$

where we used the change of variables $\tau = \beta - t$ to obtain equality between (4.2.13a) and (4.2.13b) above.

Step 3: Substituting the right hand side of each of the equations in (4.2.8) into (4.2.7), and recalling the definitions of S and E_γ from (3.1.1) and (3.1.2), we obtain

$$\begin{aligned} 0 = {} & \gamma^2 \left[w^*(\sigma; y^*(t; y_0)) - w^*(\sigma+t; y_0)\right] \\ & + \big(R_0 \mathscr{L}_{R_0} \mathscr{L}^*_{R_0} R_0 P \left[w^*(\,\cdot\,; y^*(t; y_0)) - w^*(\,\cdot\,; y_0)\right]\big)(\sigma) \\ & - \big(\mathcal{W}^*_{R_0} R_0 P \left[w^*(\,\cdot\,+t; y_0) - w^*(\,\cdot\,; y^*(t; y_0))\right]\big)(\sigma) \\ & - \big(R_0 \mathcal{W}_{R_0} \left[w^*(\,\cdot\,; y^*(t; y_0)) - w^*(t+\,\cdot\,; y_0)\right]\big)(\sigma) \end{aligned} \tag{4.2.14a}$$

$$= \big(E_\gamma \left[w^*(\,\cdot\,; y^*(t; y_0)) - w^*(\,\cdot\,+t; y_0)\right]\big)(\sigma) \tag{4.2.14b}$$

For $\gamma > \gamma_c$, Proposition 3.2.1 applies, and $E_\gamma^{-1} \in L^2(0,\infty;[L^2(\Omega)]^d)$. Thus, (4.2.14) above yields the desired equality (4.2.1), first in $L^2(0,\infty;[L^2(\Omega)]^d)$, and next pointwise by the C-regularity of w^* in (4.1.7c). □

4.3 A Transition Property for r^* for $\gamma > \gamma_c$

We will now use Theorem 4.2.1, which shows the transition property for w^*, (4.2.1) to show a transition for r^* when $\gamma > \gamma_c$. Like the transition property for w^*, this transition property will be useful in proving Theorem 4.4.1, the transition property for y^*.

Theorem 4.3.1. *For $\gamma > \gamma_c$ and $y_0 \in H$, the following transition property holds*

$$r^*(t+\sigma;y_0) = r^*(\sigma;y^*(t;y_0)) \underset{(in\ \sigma)}{\in} C\left([0,\infty];H\right) \tag{4.3.1}$$

for each t fixed.

Proof. We return to equation (3.4.2a), which gives r^* in terms of w^*,

$$\begin{aligned} r^*(t+\sigma;y_0) &= \int_{t+\sigma}^{\infty} e^{\mathcal{A}^*_{R_0}(\tau-(t+\sigma))} R_0 P w^*(\tau;y_0)\,d\tau \\ &= \int_{\sigma}^{\infty} e^{\mathcal{A}^*_{R_0}(\alpha-\sigma)} R_0 P w^*(\alpha+t;y_0)\,d\alpha \end{aligned} \tag{4.3.2}$$

where we used the change of variables $\alpha = \tau - t$ to get equality between the two integrals in (4.3.2), above. We can now use the transition property for w^*, (4.2.1), to rewrite this quantity as

$$r^*(t+\sigma;y_0) = \int_{t}^{\infty} e^{\mathcal{A}^*_{R_0}(\alpha-\sigma)} R_0 P w^*(\alpha;y^*(t;y_0))\,d\alpha \tag{4.3.3}$$

Finally, invoking (3.4.2a) again, we obtain

$$\int_t^\infty e^{\mathcal{A}^*_{R_0}(\alpha-\sigma)} R_0 P w^*(\alpha; y^*(t; y_0))\, d\alpha = r^*(\sigma; y^*(t; y_0)) \tag{4.3.4}$$

Thus, $r^*(t+\sigma; y_0) = r^*(\sigma; y^*(t; y_0))$ holds for all σ, t, and in $C([0,\infty]; H)$ because $r^*(\,\cdot\,; y_0) \in C([0,\infty]; H)$ for all $y_0 \in H$. $\square$

4.4 The Semigroup Property for y^*, and a Transition Property for p^* for $\gamma > \gamma_c$

We will now use Theorems 4.2.1 and 4.3.1, which show the transition properties for w^* and r^* to show a transition property for y^* when $\gamma > \gamma_c$.

Theorem 4.4.1. *For $\gamma > \gamma_c$ and $y_0 \in H$,*

(i) The following transition property holds for y^*

$$y^*(t+\sigma; y_0) = y^*(\sigma; y^*(t; y_0)) \underset{(in\ \sigma)}{\in} C([0,\infty]; H) \tag{4.4.1}$$

Thus, the operator $\Phi(t)$, which depends on γ defined by

$$\Phi(t)x = y^*(t; x), \qquad x \in H \tag{4.4.2}$$

is a strongly continuous semigroup on H.

(ii) Furthermore, $\Phi(t)$ is exponentially stable: There exist $c \geq 1$ and $k > 0$ such that

$$\|\Phi(t)\|_{\mathcal{L}(H)} \leq ce^{-kt}, \qquad t \geq 0 \tag{4.4.3}$$

Proof. (i): The proof of part (i) follows the same idea used to prove the transition property for w^* (Theorem 4.2.1). We begin by using formula (3.4.3c), which gives y^* in terms of w^*, to write $y^*(\sigma; y^*(t; y_0))$ as

$$y^*(\sigma; y^*(t; y_0)) = \\ e^{\mathcal{A}_{R_0}\sigma} y^*(t; y_0) + \int_0^\sigma e^{\mathcal{A}_{R_0}(\sigma-\tau)} \left[-Pmr^*(\tau; y^*(t; y_0)) + Pw^*(\tau; y^*(t; y_0))\right] d\tau \quad (4.4.4)$$

Then, we use formula (3.4.3c) for $y^*(t; y_0)$ to rewrite $e^{\mathcal{A}_{R_0}\sigma} y^*(t; y_0)$ as

$$e^{\mathcal{A}_{R_0}\sigma} y^*(t; y_0) = \\ e^{\mathcal{A}_{R_0}(\sigma+t)} y_0 + \int_0^t e^{\mathcal{A}_{R_0}(\sigma+t-\tau)} \left[-Pmr^*(\tau; y_0) + Pw^*(\tau; y_0)\right] d\tau \quad (4.4.5)$$

Now, $y^*(\sigma; y^*(t; y_0))$ can be written as:

$$y^*(\sigma; y^*(t; y_0)) = e^{\mathcal{A}_{R_0}(\sigma+t)} y_0 + \int_0^t e^{\mathcal{A}_{R_0}(\sigma+t-\tau)} \left[-Pmr^*(\tau; y_0) + Pw^*(\tau; y_0)\right] d\tau \\ + \int_0^\sigma e^{\mathcal{A}_{R_0}(\sigma-\tau)} \left[-Pmr^*(\tau; y^*(t; y_0)) + Pw^*(\tau; y^*(t; y_0))\right] d\tau \quad (4.4.6)$$

Using formula (3.4.3c) yet again, we write $y^*(t+\sigma; y_0)$ as

$$y^*(t+\sigma; y_0) = \\ e^{\mathcal{A}_{R_0}(t+\sigma)} y_0 + \int_0^{t+\sigma} e^{\mathcal{A}_{R_0}(t+\sigma-\tau)} \left[-Pmr^*(\tau; y_0) + Pw^*(\tau; y_0)\right] d\tau \quad (4.4.7)$$

so that, combining equations (4.4.6) and (4.4.7) yields

$$y^*(t+\sigma; y_0) - y^*(\sigma; y^*(t; y_0)) = \\ \int_t^{t+\sigma} e^{\mathcal{A}_{R_0}(t+\sigma-\tau)} \left[-Pmr^*(\tau; y_0) + Pw^*(\tau; y_0)\right] d\tau$$

$$- \int_0^\sigma e^{\mathcal{A}_{R_0}(\sigma-\tau)} \left[-Pmr^*(\tau; y^*(t; y_0)) + Pw^*(\tau; y^*(t; y_0))\right] d\tau \qquad (4.4.8)$$

Using the change of variables $\alpha = \tau - t$, and recalling equations (4.2.1) and (4.3.1), the transition properties for w^* and r^*, we obtain

$$\begin{aligned} &\int_t^{t+\sigma} e^{\mathcal{A}_{R_0}(t+\sigma-\tau)} \left[-Pmr^*(\tau; y_0) + Pw^*(\tau; y_0)\right] d\tau \\ &\qquad = \int_0^\sigma e^{\mathcal{A}_{R_0}(\sigma-\alpha)} \left[-Pmr^*(\alpha + t; y_0) + Pw^*(\alpha + t; y_0)\right] d\alpha \\ &\qquad = \int_0^\sigma e^{\mathcal{A}_{R_0}(\sigma-\alpha)} \left[-Pmr^*(\alpha; y^*(t; y_0)) + Pw^*(\alpha; y^*(t; y_0))\right] d\alpha \qquad (4.4.9) \end{aligned}$$

Thus, combining equations (4.4.8) and (4.4.9), we obtain

$$y^*(t + \sigma; y_0) - y^*(\sigma; y^*(t; y_0)) = 0 \qquad (4.4.10)$$

This implies that $\Phi(t)$ in (4.4.2) satisfies the semigroup property and is strongly continuous.

(ii): The proof of part (ii) is then a consequence of the semigroup property of (i), since $\Phi(t)x = y^*(t; x) \in L^2(0, \infty; H)$ for all $x \in H$, so that a well-known result (see [6] Theorem 4.1 on page 116) applies and yields the desired exponential stability. $\square$

Finally, as a consequence of Theorems 4.3.1 and 4.4.1, we obtain

Theorem 4.4.2. *For $\gamma > \gamma_c$ and $y_0 \in H$, we have*

$$p^*(t + \sigma; y_0) =_{(in\ \sigma)} p^*(\sigma; y^*(t; y_0)) \in C\left([0, \infty]; H\right) \qquad (4.4.11)$$

fot all t.

Proof. We return to (3.4.4a), the formula for p^* in terms of y^* and r^*, and compute

$$\begin{aligned} p^*(t+\sigma; y_0) &= R_0 y^*(t+\sigma; y_0) + r^*(t+\sigma; y_0) \\ &= R_0 y^*(\sigma; y^*(t; y_0)) + r^*(\sigma; y^*(t; y_0)) \\ &= p^*(\sigma; y^*(t; y_0)) \end{aligned} \tag{4.4.12}$$

where we have used equations (4.4.1) and (4.3.1), which give the transition properties for y^* and r^*. □

4.5 Definition of R and its Properties

With reference to $p^*(\,\cdot\,; y_0)$, we define an operator R by setting

$$Rx = p^*(0; x), \qquad x \in H \tag{4.5.1}$$

We can deuce that $R \in \mathcal{L}(H)$ from the definition of R and the inequality (4.1.1b) in Theorem 4.1.1 for $p^*(t; y_0)$. Some preliminary properties of R are collected below.

Proposition 4.5.1. *For $\gamma > \gamma_c$, and $y_0 \in H$, we have*

(i)

$$p^*(t; y_0) = Ry^*(t; y_0) = R\Phi(t)y_0 \in L^2(0, \infty; H) \cap C\left([0, \infty]; H\right) \tag{4.5.2}$$

(ii)

$$u^*(t; y_0) = -Ry^*(t; y_0)\big|_\omega \in L^2(0, \infty; [L^2(\omega)]^d) \cap C\left([0, \infty]; [L^2(\omega)]^d\right) \tag{4.5.3}$$

(iii)

$$\gamma^2 w^*(t;y_0) = Ry^*(t;y_0) \in L^2(0,\infty;[L^2(\Omega)]^d) \cap C\left([0,\infty];[L^2(\Omega)]^d\right) \tag{4.5.4}$$

(iv) For $x \in H$, the folling identity holds true:

$$Rx = \int_0^{t_0} e^{\mathcal{A}^*\tau}\Phi(\tau)x\,d\tau + e^{\mathcal{A}^* t_0}\Phi(t_0)x \tag{4.5.5}$$

where t_0 is an arbitrary point $0 < t_0 < \infty$.

(v) For $x \in H$, the following formula holds true:

$$Rx = R_0 x + \int_0^{\infty} e^{\mathcal{A}^*_{R_0}\tau} R_0 P\left(E_\gamma R_0 e^{\mathcal{A}_{R_0}\,\cdot}\, y_0\right)(\tau)\,d\tau, \tag{4.5.6}$$

which expresses R in terms of the problem data, via E_γ in (3.1.2).

(vi) For $x_1, x_2 \in H$, we have

$$(Rx_1, x_2) = (R_0 x_1, x_2) + \left(E_\gamma^{-1}\left[R_0 e^{\mathcal{A}_{R_0}\,\cdot}\, x_1\right], R_0 e^{\mathcal{A}_{R_0}\,\cdot}\, x_2\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \tag{4.5.7}$$

so that R is a positive self-adjoint operator: $R = R^* \geq R_0 \geq 0$.

Proof. (i): We use equation (4.4.11) of Theorem 4.4.2, which provides the transition property for $p^*(\,\cdot\,;y_0)$, and set $\sigma = 0$ to obtain

$$p^*(t;y_0) = p^*(0;y^*(t;y_0)) = Ry^*(t;y_0) \tag{4.5.8}$$

where in the last step above we used (4.5.1), the definition of R. The stated regularity follows from (3.4.4c).

(ii), (iii): These follow directly from equations (3.4.6) and (4.1.11), which say $u^*(\,\cdot\,;y_0) = -p^*(\,\cdot\,;y_0)\big|_\omega$ and $\gamma^2 w^*(\,\cdot\,;y_0) = p^*(\,\cdot\,;y_0)$, respectively, via the result of (i). The stated regularity follows from Theorem 4.1.2.

(iv): To obtain the relation (4.5.5), we return to a similar formula for $p^*(t;y_0)$ from Corollary 2.2.5, given in equation (2.2.44). Then we use the definition of R (equation (4.5.1)), the definition of $\Phi(t)$ (equation (4.4.2)), and (4.5.2), the result of part (i) for $t = t_0$ to rewrite (2.2.44) as (4.5.5).

(v): Returning to equation (3.4.4a) and inserting $t = 0$, we have, via (4.5.1), the definition of R,

$$\begin{aligned} Rx &= p^*(0;x) = R_0 y^*(0;x) + r^*(0;x) \\ &= R_0 x + \int_0^\infty e^{\mathcal{A}^*_{R_0}\tau} R_0 P w^*(\tau;x)\,d\tau \end{aligned} \tag{4.5.9}$$

where in the last step we have recalled equation (3.4.2a) for $r^*(0;x)$. Next, we insert (3.3.2) for w^* in equation (4.5.9) to obtain (4.5.6).

(vi): Identity (4.5.7) is an immediate consequence of (4.5.6), since R_0 is nonnegative, self-adjoint on H by (ii) of Theorem 1.5.1 and E_γ^{-1} is positive, self-adjoint on $L^2(0,\infty;[L^2(\Omega)]^d)$ by Proposition 3.2.1 for $\gamma > \gamma_c$. $\square$

Remark 4.5.1. *Part* (iii) *of Proposition 4.5.1 shows us that although the game theory problem in* (1.2.2) *involves taking a supremum over all disturbances* $w \in L^2(0,\infty;[L^2(\Omega)]^d)$*, the maximizing disturbance,* $w^*(\,\cdot\,;y_0)$ *for any* $y_0 \in H$ *is actually in* $L^2(0,\infty;H)$*, a subset of* $L^2(0,\infty;[L^2(\Omega)]^d)$*.*

Proposition 4.5.2. *Let $\gamma > \gamma_c$ and $x_1, x_2 \in H$. Then:*

(i) The following symmetric relation holds true:

$$(Rx_1, x_2) = \int_0^\infty \Big[\big(y^*(t;x_1), y^*(t;x_2)\big)_H + \big(u^*(t;x_1), u^*(t;x_2)\big)_{[L^2(\omega)]^d} \\ -\gamma^2 \left(w^*(t;x_1), w^*(t;x_2)\right)_{[L^2(\Omega)]^d} \Big] dt \tag{4.5.10}$$

(ii) Hence, for $y_0 \in H$, we have

$$\begin{aligned}(Ry_0, y_0)_H &= J^*(y_0) \\ &= \int_0^\infty \Big[\big\|y^*(t;y_0)\big\|_H^2 + \big\|u^*(t;y_0)\big\|_{[L^2(\omega)]^d}^2 - \gamma^2 \big\|w^*(t;y_0)\big\|_{[L^2(\Omega)]^d}^2 \Big] dt\end{aligned} \tag{4.5.11}$$

Proof. (i): By (4.5.1), recalling (3.4.5), the formula for p^* in terms of u^* and y^*, we obtain

$$Rx_1 = p^*(0;x_1) = \int_0^\infty e^{\mathscr{A}_{R_0}^* \tau} \left[-R_0 Pmu^*(\tau;x_1) + y^*(\tau;x_1) \right] d\tau \tag{4.5.12}$$

so that

$$\begin{aligned}(Rx_1, x_2)_H &= \int_0^\infty \Big(e^{\mathscr{A}_{R_0}^* \tau} \left[-R_0 Pmu^*(\tau;x_1) + y^*(\tau;x_1) \right], x_2 \Big)_H d\tau \\ &= \int_0^\infty \Big(-R_0 Pmu^*(\tau;x_1) + y^*(\tau;x_1), e^{\mathscr{A}_{R_0} \tau} x_2 \Big)_H d\tau\end{aligned} \tag{4.5.13}$$

Recalling (3.4.3b), we insert

$$\begin{aligned}e^{\mathscr{A}_{R_0} \cdot} x_2 &= y^*(\,\cdot\,;x_2) + \mathscr{L}_{R_0}\left(r^*(\,\cdot\,;x_2)\big|_\omega \right) - \mathcal{W}_{R_0} w^*(\,\cdot\,;x_2) \\ &= y^*(\,\cdot\,;x_2) + \mathcal{K}_{R_0}\left[Pmr^*(\,\cdot\,;x_2) - Pw^*(\,\cdot\,;x_2) \right]\end{aligned} \tag{4.5.14}$$

on the right hand side of (4.5.13), thus we have

$$
\begin{aligned}
(Rx_1, x_2)_H = &\big(-R_0 Pmu^*(\,\cdot\,; x_1) + y^*(\,\cdot\,; x_1), y^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;H)} \\
&+ \big(-R_0 Pmu^*(\,\cdot\,; x_1) + y^*(\,\cdot\,; x_1), \mathcal{K}_{R_0}\left[Pmr^*(\,\cdot\,; x_2) - Pw^*(\,\cdot\,; x_2)\right]\big)_{L^2(0,\infty;H)}
\end{aligned}
\tag{4.5.15}
$$

Recalling first equation (3.4.5) for p^*, using $\mathcal{K}^*_{R_0}$, then the relation $u^*(\,\cdot\,; x_1) = -p^*(\,\cdot\,; x_1)\big|_\omega$ from (3.4.6a), and finally the relations $p^*(\,\cdot\,; x_2) = R_0 y^*(\,\cdot\,; x_2) + r^*(\,\cdot\,; x_2)$ and $p^*(\,\cdot\,; x_1) = \gamma^2 w^*(\,\cdot\,; x_1)$ from equations (3.4.4a) and (4.1.11), respectively, we obtain

$$
\begin{aligned}
(Rx_1, x_2)_H = &\big(y^*(\,\cdot\,; x_1), y^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;H)} - \big(u^*(\,\cdot\,; x_1), R_0 y^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;[L^2(\omega)]^d)} \\
&+ \big(p^*(\,\cdot\,; x_1), Pmr^*(\,\cdot\,; x_2) - Pw^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;H)} \qquad (4.5.16a)\\
= &\big(y^*(\,\cdot\,; x_1), y^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;H)} - \big(u^*(\,\cdot\,; x_1), R_0 y^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;[L^2(\omega)]^d)} \\
&- \big(u^*(\,\cdot\,; x_1), r^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;[L^2(\omega)]^d)} - \big(p^*(\,\cdot\,; x_1), w^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;[L^2(\Omega)]^d)} \qquad (4.5.16b)\\
= &\big(y^*(\,\cdot\,; x_1), y^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;H)} - \big(u^*(\,\cdot\,; x_1), R_0 y^*(\,\cdot\,; x_2) + r^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;[L^2(\omega)]^d)} \\
&- \big(p^*(\,\cdot\,; x_1), w^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;[L^2(\Omega)]^d)} \qquad (4.5.16c)\\
= &\big(y^*(\,\cdot\,; x_1), y^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;H)} + \big(u^*(\,\cdot\,; x_1), u^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;[L^2(\omega)]^d)} \\
&- \gamma^2 \big(w^*(\,\cdot\,; x_1), w^*(\,\cdot\,; x_2)\big)_{L^2(0,\infty;[L^2(\Omega)]^d)} \qquad (4.5.16d)
\end{aligned}
$$

(ii): This result follows directly from (i) by letting $x_1 = x_2 = y_0$, and recalling (1.2.1) for J.

□

Proposition 4.5.3. *For $\gamma > \gamma_c$, we have the following regularity result pertaining to the operators $\mathcal{A}$ and R*

$$\mathcal{A}^* R,\ R\mathcal{A} \in \mathcal{L}(H) \tag{4.5.17}$$

Proof. First, we show that

$$\mathcal{A}^*_{R_0} R \in \mathcal{L}(H) \tag{4.5.18}$$

Using the formula in (4.5.9) for R, and the smoothing property of K^* from (1.6.2), for $x \in H$, we have

$$\begin{aligned} \left\|\mathcal{A}^*_{R_0} R x\right\|_H &\leq \left\|\mathcal{A}^*_{R_0} R_0 x\right\|_H + \left\|\mathcal{K}^*_{R_0} \mathcal{A}^*_{R_0} R_0 P w^*(\,\cdot\,; x)\right\|_{C([0,\infty];H)} \\ &\leq \left\|\mathcal{A}^*_{R_0} R_0 x\right\|_H + \left\|\mathcal{K}^*_{R_0}\right\| \left\|\mathcal{A}^*_{R_0} R_0 P w^*(\,\cdot\,; x)\right\|_{L^2(0,\infty;H)}, \end{aligned} \tag{4.5.19}$$

where the norm on $\mathcal{K}^*_{R_0}$ is in $\mathcal{L}\left(L^2(0,\infty;H), C\left([0,\infty];H\right)\right)$. Recalling that $\mathcal{A}^* R_0 \in \mathcal{L}(H)$ from (1.5.8), and that the L^2-norm of $\left\|w^*(\,\cdot\,; x)\right\|_{[L^2(\Omega)]^d}$ is bounded above by $\leq C_w \|x\|_H$ from (4.1.2), inequality in (4.5.19) becomes

$$\left\|\mathcal{A}^*_{R_0} R x\right\|_H \leq \left\|\mathcal{A}^*_{R_0} R_0 x\right\|_H + \left\|\mathcal{K}^*_{R_0}\right\| \left\|\mathcal{A}^*_{R_0} R_0\right\|_{\mathcal{L}(H)} C_w \|x\|_H, \tag{4.5.20}$$

where the norm on $\mathcal{K}^*_{R_0}$ is still in $\mathcal{L}\left(L^2(0,\infty;H), C\left([0,\infty];H\right)\right)$, and we dropped the projection P because $P w^*(\,\cdot\,; x) = w^*(\,\cdot\,; x)$ in $L^2(0,\infty;[L^2(\Omega)]^d)$ (see Remark 4.5.1). Thus, we have that (4.5.18) holds.

Now, we use (4.5.18) to show (4.5.17). From the definition of $\mathcal{A}_{R_0}$, we have that

$$\mathcal{A}^* R = \mathcal{A}^*_{R_0} R + R_0 P M R. \tag{4.5.21}$$

Because $\mathcal{A}^*_{R_0} R$ and $R_0 P M R$ are in $\mathcal{L}(H)$, it follows have that $\mathcal{A}^* R$ is in $\mathcal{L}(H)$, and by duality, $R\mathcal{A}$ is in $\mathcal{L}(H)$ as well. □

4.6 The Feedback Generator $\mathcal{A}_F$ and its Preliminary Properties for $\gamma > \gamma_c$

For $\gamma > \gamma_c$, we return to the strongly continuous semigroup $\Phi(t)$ that defines the optimal solution $y^*(t;x) = \Phi(t)x$ by (4.4.2), and call $\mathcal{A}_F$ its infinitesimal generator, so that

$$\Phi(t)x = e^{\mathcal{A}_F t}x, \quad x \in H; \qquad \frac{d}{dt}\Phi(t)x = \mathcal{A}_F\Phi(t)x = \Phi(t)\mathcal{A}_F x, \quad x \in \mathcal{D}(\mathcal{A}_F) \quad (4.6.1)$$

We recall from (4.4.3) that $\Phi(t)$ is uniformly stable.

We next provide information about $\mathcal{A}_F$ essentially as a consequence of (4.5.3) and (4.5.4) being inserted into equation (1.1.13) for $y^*(t;x) = \Phi(t)x$.

Theorem 4.6.1. *For $x \in H$ and $\gamma > \gamma_c$, and a.e. in $t \geq 0$*

(i) We have

$$\frac{d}{dt}\Phi(t)x = \left[\mathcal{A} - PmR + \gamma^{-2}R\right]\Phi(t)x \in [\mathcal{D}(\mathcal{A})]' \quad (4.6.2)$$

Thus, by (4.6.1)*, we have*

$$[\mathcal{A} - PmR + \gamma^{-2}R]\Phi(t)x = \mathcal{A}_F\Phi(t)x = \Phi(t)\mathcal{A}_F x \in H$$

$$x \in \mathcal{D}(\mathcal{A}_F),\ t \geq 0 \quad (4.6.3a)$$

$$[\mathcal{A} - PmR + \gamma^{-2}R]x = \mathcal{A}_F x \in H, \qquad x \in \mathcal{D}(\mathcal{A}_F) \quad (4.6.3b)$$

$$e^{(\mathcal{A} - PmR + \gamma^{-2}R)t}x = \Phi(t)x, \qquad x \in H \quad (4.6.3c)$$

(ii) As first order perturbations of the Oseen Operator, $\mathcal{A}$, the operators $\mathcal{A}_F$, $\mathcal{A} - PmR$ and $\mathcal{A} + \gamma^{-2}R$ each generate a strongly continuous analytic semigroup on H (see [6], Corollary 2.2 on page 81). (Recall from (4.4.3) *that the semigroup generated by $\mathcal{A}_F$ is uniformaly stable.)*

Proof. (i): Writing y^* as

$$y^*(t; y_0) = e^{\mathcal{A}t} y_0 + \int_0^t e^{\mathcal{A}(t-\tau)} Pmu^*(\tau; y_0)\, d\tau + \int_0^t e^{\mathcal{A}(t-\tau)} Pw^*(\tau; y_0)\, d\tau, \tag{4.6.4}$$

we take the inner product with $y \in \mathcal{D}(\mathcal{A}^*)$, and differentiate in t, with $y_0 \in \mathcal{D}(\mathcal{A}_F)$,

thus obtaining

$$\frac{d}{dt} (\Phi(t) y_0, y)_H = \frac{d}{dt} \big(y^*(t; y_0), y\big)_H \tag{4.6.5a}$$

$$= \big(\mathcal{A} y^*(t; y_0), y\big)_H + \big(Pmu^*(t; y_0), y\big)_H + \big(Pw^*(t; y_0), y\big)_H \tag{4.6.5b}$$

$$= \big(\mathcal{A} y^*(t; y_0), y\big)_H - \big(PmRy^*(t; y_0), y\big)_H + \gamma^{-2} \big(Ry^*(t; y_0), y\big)_H \tag{4.6.5c}$$

$$= \big(\big[\mathcal{A} - PmR + \gamma^{-2} R\big] \Phi(t) y_0, y\big)_H \tag{4.6.5d}$$

by using (4.5.3) for $u^*(t; y_0)$ and (4.5.4) for $w^*(t; y_0)$.

(ii): These results follow directly from a result in Pazy [6], which says that a perturbation of the generator of an analytic semigroup by a bounded linear operator is itself the generator of an analytic semigroup. □

4.7 The Operator R is a Solution of the Algebraic Riccati Equation, ARE$_\gamma$, for $\gamma > \gamma_c$

We finally obtain the ultimate goal of our analysis.

Theorem 4.7.1. *For $\gamma > \gamma_c$, the operator R defined by* (4.5.1) *satisfies the algebraic Riccati equation, ARE$_\gamma$, from* (1.3.1). *That is:*

$$\begin{aligned}(\mathcal{A}^* Rx, z)_H + (R\mathcal{A}x, z)_H + (x, z)_H \\ = (Rx, Rz)_{[L^2(\omega)]^d} - \gamma^{-2} (Rx, Rz)_{[L^2(\Omega)]^d}\end{aligned} \tag{4.7.1}$$

for all x, $z \in H$.

Proof. We will first show that (4.7.1) holds true for all $x, z \in \mathcal{D}(\mathcal{A})$. To this end, we return to (4.5.5), which we rewrite as

$$(Rx, z)_H = \left(\int_0^{t_0} e^{\mathcal{A}^*\tau} \Phi(\tau) x \, d\tau + e^{\mathcal{A}^* t_0} R \Phi(t_0) x, z \right)_H, \tag{4.7.2}$$

where we recall that t_0 is an arbitrary point $t_0 \geq 0$. We now specialize to $x, z \in \mathcal{D}(\mathcal{A})$ and differentiate the inner product (4.7.2) with respect to t_0, which yields

$$0 = \left(e^{\mathcal{A}^* t_0} \Phi(t_0) x, z\right)_H + \left(\mathcal{A}^* e^{\mathcal{A}^* t_0} R \Phi(t_0) x, z\right)_H + \left(e^{\mathcal{A}^* t_0} R \mathcal{A}_F \Phi(t_0) x, z\right)_H \tag{4.7.3}$$

for all $t_0 \geq 0$. Setting $t_0 = 0$ above yields

$$0 = (x, z)_H + (\mathcal{A}^* Rx, z)_H + (R\mathcal{A}_F x, z)_H \quad \forall \ x, z \in \mathcal{D}(\mathcal{A}) \tag{4.7.4}$$

Inserting the definition of $\mathcal{A}_F$ from (4.6.3) in equation (4.7.4), we obtain the following equation, where, due to (4.5.17), which states that $\mathcal{A}^*R$ and $R\mathcal{A}$ are both in $\mathcal{L}(H)$, all inner products are well defined

$$\begin{aligned} 0 = (x,z)_H + (\mathcal{A}^*Rx, z)_H + (R\mathcal{A}x, z)_H \\ - (Rx, Rz)_{[L^2(\omega)]^d} + \gamma^{-2}(Rx, Rz)_{[L^2(\Omega)]^d} \quad \forall\ x, z \in H. \end{aligned} \tag{4.7.5}$$

Equation (4.7.1) follows. □

4.8 The Semigroup Generated by $\mathcal{A} - PmR$ Is Uniformly Stable

We now return to the strongly continuous semigroup generated by $(\mathcal{A} - PmR)$ in part (ii) of Theorem 4.6.1. We now show that this semigroup inherits from $e^{\mathcal{A}_F t}$, the property of being uniformly stable.

Proposition 4.8.1. *Let $\gamma > \gamma_c$. The strongly continuous semigroup $e^{(\mathcal{A}-PmR)t}$ is uniformly stable on H. Thus, there exist constants $C_1 \geq 1$ and $a_1 > 0$ such that*

$$\left\| e^{(\mathcal{A}-PmR)t} \right\|_{\mathcal{L}(H)} \leq C_1 e^{-a_1 t}, \quad t \geq 0 \tag{4.8.1}$$

Proof. We write

$$z(t; z_0) = e^{(\mathcal{A}-PmR)t} z_0 \in C\left([0,T]; H\right); \quad z' = (\mathcal{A} - PmR)z, \quad z(0) = z_0 \in H \tag{4.8.2}$$

Hence, we may rewrite z' and $z(t;z_0)$ using $\mathcal{A}_F$ as

$$z' = (\mathcal{A} - PmR + \gamma^{-2}R)z - \gamma^{-2}Rz = \mathcal{A}_F z - \gamma^{-2}Rz \tag{4.8.3a}$$

$$z(t;z_0) = e^{\mathcal{A}_F t}z_0 - \gamma^{-2}\int_0^t e^{\mathcal{A}_F(t-\tau)}Rz(\tau;z_0)\,d\tau \tag{4.8.3b}$$

We now take the inner product of the z-equation in (4.8.2) with $Rz = Rz(t;z_0)$ for $t \geq 0$ and R self-adjoint, and differentiate in t

$$\frac{d}{dt}(z, Rz)_H = 2\left((\mathcal{A} - PmR)z, Rz\right)_H = 2(\mathcal{A}z, Rz)_H - 2\|Rz\|^2_{[L^2(\omega)]^d} \tag{4.8.4}$$

Invoking the ARE, (4.7.1) with $x = z = z(t;z_0)$, we obtain

$$2(\mathcal{A}z, Rz)_H + \|z\|_H^2 = \|Rz\|^2_{[L^2(\omega)]^d} - \gamma^{-2}\|Rz\|^2_{[L^2(\Omega)]^d} \tag{4.8.5}$$

Solving (4.8.5) for $2(\mathcal{A}z, Rz)_H$ and substituting in (4.8.4) results in

$$\frac{d}{dt}(z, Rz)_H = -\|Rz\|^2_{[L^2(\omega)]^d} - \gamma^{-2}\|Rz\|^2_{[L^2(\Omega)]^d} - \|z\|_H^2 \tag{4.8.6}$$

Integrating (4.8.6) over $[0,T]$ yields

$$\begin{aligned}(z_0, Rz_0)_H = \int_0^T \Big(\|Rz(t;z_0)\|^2_{[L^2(\omega)]^d} + \gamma^{-2}\|Rz(t;z_0)\|^2_{[L^2(\Omega)]^d} + \|z(t;z_0)\|_H^2\Big)dt \\ + \left(z(T;z_0), Rz(T;z_0)\right)_H \end{aligned}\tag{4.8.7}$$

Because R is positive definite by part (vi) of Proposition 4.5.1, we can drop the inner product $(z(T;z_0), Rz(T;z_0))_H$ from the right hand side of (4.8.7) to obtain the following inequality

$$\int_0^\infty \Big(\|Rz(t;z_0)\|^2_{[L^2(\omega)]^d} + \gamma^{-2}\|Rz(t;z_0)\|^2_{[L^2(\Omega)]^d} + \|z(t;z_0)\|_H^2\Big)dt \leq (z_0, Rz_0)_H \tag{4.8.8}$$

Thus, we have that

$$z(t;z_0) = e^{(\mathcal{A}-PmR)t}z_0 \in L^2(0,\infty;H), \qquad \forall\, z_0 \in H. \tag{4.8.9}$$

From here, we apply a well-known result [6] to (4.8.9) and obtain the exponential decay (4.8.1) for the semigroup $e^{(\mathcal{A}-PmR)t}$. □

4.9 The Case $0 < \gamma < \gamma_c$: $\sup J_w^0 = +\infty$

We now consider the case where $\gamma_c > 0$ and $0 < \gamma < \gamma_c$, of part (i) of Theorem 1.3.1.

Proposition 4.9.1. *Let $0 < \gamma < \gamma_c$. Then, there exists a sequence $\{w_k\}_{k=1}^{\infty}$ such that for all $y_0 \in H$, we have*

$$J^0_{w_k}(y_0) \to +\infty, \qquad \textit{as } k \to +\infty \tag{4.9.1}$$

so that for all $y_0 \in H$,

$$\sup_{w \in L^2(0,\infty;[L^2(\Omega)]^d)} J^0_w(y_0) = +\infty \tag{4.9.2}$$

Proof. Let $\gamma_c > 0$ so that $-\gamma_c^2 = \inf_{\|w\|=1}(Sw,w)$ by (3.2.1), with norm and inner product in $L^2(0,\infty;[L^2(\Omega)]^d)$. Thus, given $\varepsilon > 0$, there exists $w_\varepsilon \in L^2(0,\infty;[L^2(\Omega)]^d)$, with $\|w_\varepsilon\| = 1$ such that

$$(Sw_\varepsilon, w_\varepsilon) < -\gamma_c^2 + \varepsilon \tag{4.9.3}$$

Recalling E_γ from (3.1.2), we then obtain by (4.9.3), under the present assumption that $0 < \gamma < \gamma_c$:

$$(E_\gamma w_\varepsilon, w_\varepsilon) = \gamma^2 \|w_\varepsilon\|^2 + (Sw_\varepsilon, w_\varepsilon) < \gamma^2 - \gamma_c^2 + \varepsilon = -c_\varepsilon < 0 \tag{4.9.4}$$

after choosing ε sufficiently small. Recalling (3.3.3) and (3.1.3), we have

$$\begin{aligned} J^0_{w_\varepsilon}(y_0) &= J^0_{w_\varepsilon}(y_0=0) + J^0_{w=0}(y_0) + X_{w_\varepsilon}(y_0) \\ &= -\left(E_\gamma w_\varepsilon, w_\varepsilon\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} + \left(R_0 y_0, y_0\right)_H + \left(w_\varepsilon, a_{y_0}\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \end{aligned} \tag{4.9.5}$$

where a_{y_0} is a suitable vector in $L^2(0,\infty;[L^2(\Omega)]^d)$ depending on y_0. We now define the sequence w_k by setting $w_k = kw_\varepsilon \in L^2(0,\infty;[L^2(\Omega)]^d)$. Then (4.9.6) with w_k substituted in for w_ε becomes

$$\begin{aligned} J^0_{w_k}(y_0) &= -k^2\left(E_\gamma w_\varepsilon, w_\varepsilon\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} + \left(R_0 y_0, y_0\right)_H + k\left(w_\varepsilon, a_{y_0}\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \\ &> k^2 c_\varepsilon + \left(R_0 y_0, y_0\right)_H + k\left(w_\varepsilon, a_{y_0}\right)_{L^2(0,\infty;[L^2(\Omega)]^d)} \to \infty \text{ as } k \to \infty \end{aligned} \tag{4.9.6}$$

after using we used $c_\varepsilon = \gamma^2 - \gamma_c^2 + \varepsilon$ from the inequality in (4.9.4). $\square$

4.10 Proof of Theorem 1.3.2

We now prove Theorem 1.3.2, which is the converse of Theorem 1.3.1

Theorem 4.10.1. *Assume that $\hat{R} = \hat{R}^* \geq 0$ is an operator in $\mathcal{L}(H)$ such that*

(i) the operator $A_F = \mathcal{A} - Pm\hat{R} + \gamma^{-2}\hat{R}$ is the generator of a strongly continuous uniformly stable semigroup $e^{A_F t}$ on H for some $\gamma > 0$; and

(ii) $\hat{R}$ is a solution of the corresponding ARE in (1.3.1) for all $x, z \in H$ with the property that $\mathcal{A}^*\hat{R} \in \mathcal{L}(H)$

Then the min-max game theory problem in (1.2.2) *is finite for all* $y_0 \in H$*, so that then* $\gamma \geq \gamma_c$.

Note: To line up with $\hat{R}$, I changed $\bar{p}$ to $\hat{p}$ and so on (for w, u and y)

Proof. With such $\hat{R}$, we define the functions

$$\hat{p}(t; y_0) = \hat{R}e^{A_F t}y_0 \qquad \text{in } L^2(0,\infty; H) \cap C\left([0,\infty]; H\right) \tag{4.10.1a}$$

$$\gamma^2\hat{w}(t; y_0) = \hat{R}e^{A_F t}y_0 \qquad \text{in } L^2(0,\infty; [L^2(\Omega)]^d) \cap C\left([0,\infty]; [L^2(\Omega)]^d\right) \tag{4.10.1b}$$

$$\hat{u}(t; y_0) = -\hat{R}e^{A_F t}y_0\Big|_\omega \qquad \text{in } L^2(0,\infty; [L^2(\omega)]^d) \cap C\left([0,\infty]; [L^2(\omega)]^d\right) \tag{4.10.1c}$$

$$\hat{y}(t; y_0) = e^{A_F t}y_0 \qquad \text{in } L^2(0,\infty; H) \cap C\left([0,\infty]; H\right) \tag{4.10.1d}$$

and show that they are the are the optimal functions for the given initial condition $y_0 \in H$ and value of γ. Differentiating $\hat{y}(t; y_0)$ in t, we can see that $y = \hat{y}$ satisfies the dynamics (1.1.13) with $u = \hat{u}(\,\cdot\,; y_0)$ and $w = \hat{w}(\,\cdot\,; y_0)$

$$\frac{d}{dt}\hat{y}(t; y_0) = A_F e^{A_F t}y_0 \qquad \text{in } [\mathcal{D}(\mathcal{A}^*)]' \tag{4.10.2a}$$

$$= \mathcal{A}e^{A_F t}y_0 - Pm\hat{R}e^{A_F t}y_0 + \gamma^{-2}\hat{R}e^{A_F t}y_0 \qquad \text{in } [\mathcal{D}(\mathcal{A}^*)]' \tag{4.10.2b}$$

$$= \mathcal{A}e^{A_F t}y_0 + Pm\left(-\hat{R}e^{A_F t}y_0\Big|_\omega\right) + \gamma^{-2}\hat{R}e^{A_F t}y_0 \qquad \text{in } [\mathcal{D}(\mathcal{A}^*)]' \tag{4.10.2c}$$

$$= \mathcal{A}\hat{y}(t; y_0) + Pm\hat{u}(t; y_0) + P\hat{w}(t; y_0) \qquad \text{in } [\mathcal{D}(\mathcal{A}^*)]'. \tag{4.10.2d}$$

In the following steps, we will show that $\hat{y}(\,\cdot\,; y_0)$ satisfies the stable dynamics from (2.4.2) for $w = \hat{w}(\,\cdot\,; y_0)$, and then that $w = \hat{w}(\,\cdot\,; y_0)$ satisfies equation (3.3.2), which characterizes the maximizing $w \in L^2(0,\infty; [L^2(\Omega)]^d)$ for a given initial condition $y_0 \in H$.

Step 1: Motivated by the relationship between p^*, w^* and y^* in (3.4.4b), we will show that $\hat{p}$, $\hat{w}$ and $\hat{y}$ defined in , and (4.10.1d) satisfy the similar relation

$$\hat{p}(\,\cdot\,;y_0) - R_0\hat{y}(\,\cdot\,;y_0) = \mathcal{K}^*_{R_0}R_0P\hat{w}(\,\cdot\,;y_0) \tag{4.10.3}$$

We start by defining for each y_0 in H the function $f(\,\cdot\,;y_0)$ by

$$f(t;y_0) = \hat{p}(t;y_0) - R_0\hat{y}(t;y_0) \tag{4.10.4}$$

and the function $\hat{r}(\,\cdot\,;y_0)$ by

$$\begin{aligned} \hat{r}(t;y_0) = r_{w=\hat{w}(\,\cdot\,;y_0),\infty}(t) &= \gamma^{-2}\int_t^\infty e^{\mathcal{A}^*_{R_0}(\tau-t)}R_0\hat{R}e^{A_F\tau}y_0\,d\tau \\ &= \gamma^{-2}\left(\mathcal{K}^*_{R_0}R_0\hat{R}e^{A_F\,\cdot}\,y_0\right)(t), \end{aligned} \tag{4.10.5}$$

which is the result of replacing w by $\hat{w}(\,\cdot\,;y_0)$ in formula (2.5.2) for $r_{w,\infty}$, using (4.10.1b) for $\hat{w}$.

Note that given the definitions of $\hat{y}$ in (4.10.1d) and $\hat{p}$ in (4.10.1a), we have that

$$f(t;y_0) = \hat{R}e^{A_Ft}y_0 - R_0e^{A_Ft}y_0 = \left(\hat{R} - R_0\right)e^{A_Ft}y_0, \tag{4.10.6}$$

so that (4.10.3), which is the result we aim to show in this step, can be rephrased as:

$$\left(\hat{R} - R_0\right)e^{A_Ft}y_0 = \gamma^{-2}\left(\mathcal{K}^*_{R_0}R_0\hat{R}e^{A_F\,\cdot}\,y_0\right)(t). \tag{4.10.7}$$

Also, because $r_{w,\infty}$ satisfies equations (2.3.1) and (2.3.3), we have that $\hat{r}$ satisfies

$$\begin{aligned} \frac{d}{dt}\hat{r}(t;y_0) &= -\mathcal{A}^*_{R_0}\hat{r}(t;y_0) - R_0P\hat{w}(t;y_0) && \text{in } H \\ &= -\mathcal{A}^*_{R_0}\hat{r}(t;y_0) - \gamma^{-2}R_0\hat{R}e^{A_Ft}y_0 && \text{in } H \end{aligned} \tag{4.10.8}$$

$$\hat{r}(\infty; y_0) = 0. \tag{4.10.9}$$

Thus, the validity of (4.10.3) will follow once we show that f satisfies

$$\frac{d}{dt} f(t; y_0) = -\mathcal{A}^*_{R_0} f(t; y_0) - \gamma^{-2} R_0 \hat{R} e^{A_F t} y_0 \qquad \text{in } H \tag{4.10.10}$$

$$f(\infty; y_0) = 0. \tag{4.10.11}$$

We now differentiate in t the inner product in H of $f(t; y_0)$ and z with $z \in H$ and use the duality pairing over H and (4.10.6) for $f(t; y_0)$ to obtain

$$\begin{aligned} \frac{d}{dt} \left(f(t; y_0), z \right)_H &= \left(\left(\hat{R} - R_0 \right) A_F e^{A_F t} y_0, z \right)_H \\ &= \left(A_F e^{A_F t} y_0, \hat{R} z \right)_H - \left(A_F e^{A_F t} y_0, R_0 z \right)_H . \end{aligned} \tag{4.10.12}$$

For ease of exposition we set $x = e^{A_F t} y_0$, and preform the following calculations where we use part (i) of the theorem for A_F and the ARE_γ from (1.3.1) for $\hat{R}$ to obtain

$$\begin{aligned} \frac{d}{dt} (f(t), z)_H &= \left(\left(\mathcal{A} - Pm\hat{R} + \gamma^{-2} \hat{R} \right) x, \hat{R} z \right)_H - (A_F x, R_0 z)_H \\ &= \left(\mathcal{A} x, \hat{R} z \right)_H - \left(\hat{R} x, \hat{R} z \right)_{[L^2(\omega)]^d} + \gamma^{-2} \left(\hat{R} x, \hat{R} z \right)_H - (A_F x, R_0 z)_H \\ &= - \left(\mathcal{A}^* \hat{R} x, z \right)_H - (x, z)_H - (\mathcal{A} x, R_0 z)_H + \left(\hat{R} x, R_0 z \right)_{[L^2(\omega)]^d} \\ &\qquad - \gamma^{-2} \left(\hat{R} x, R_0 z \right)_H . \end{aligned} \tag{4.10.13}$$

Equation (1.5.13), the ARE for R_0, allows us to rewrite $-(\mathcal{A}^* \hat{R} x, z) - (x, z)$ as

$$- \left(\mathcal{A}^* \hat{R} x, z \right)_H - (x, z)_H = (\mathcal{A}^* R_0 x, z)_H - (R_0 x, R_0 z)_{[L^2(\omega)]^d} \tag{4.10.14}$$

so that (4.10.13) becomes

$$\frac{d}{dt} (f(t), z)_H = - \left(\mathcal{A}^* \hat{R} x, z \right)_H + (\mathcal{A}^* R_0 x, z)_H - (R_0 x, R_0 z)_{[L^2(\omega)]^d} + \left(\hat{R} x, R_0 z \right)_{[L^2(\omega)]^d}$$

$$
\begin{aligned}
&\quad - \gamma^{-2}\left(\hat{R}x, R_0 z\right)_H \\
&= -\left(\mathcal{A}^*\hat{R}x, z\right)_H - (R_0 P m R_0 x, z)_H + (\mathcal{A}^* R_0 x, z)_H + \left(R_0 P m \hat{R}x, z\right)_H \\
&\quad - \gamma^{-2}\left(R_0\hat{R}x, z\right)_H \\
&= -\left((\mathcal{A}^* - R_0 P m)\,\hat{R}x, z\right)_H + ((\mathcal{A}^* - R_0 P m)\,R_0 x, z)_H - \gamma^{-2}\left(R_0\hat{R}x, z\right)_H \\
&= -\left(\mathcal{A}^*_{R_0} f(t; y_0), z\right)_H - \gamma^{-2}\left(R_0\hat{R}e^{A_F t} y_0, z\right)_H. \qquad (4.10.15)
\end{aligned}
$$

This shows the validity of (4.10.10). We have thus shown (4.10.3), as the terminal condition for f in (4.10.11) follows from the uniform stability of the semigroup $e^{A_F t}$ on H.

Step 2: We now return to equation (4.10.2b) and then add and subtract the quantity $PmR_0 e^{A_F t} y_0$ to obrain

$$
\begin{aligned}
\frac{d}{dt}\hat{y}(t; y_0) &= \mathcal{A}e^{A_F t} y_0 - Pm\hat{R}e^{A_F t} y_0 + \gamma^{-2}\hat{R}e^{A_F t} y_0 && \text{in } [\mathcal{D}(\mathcal{A}^*)]' \quad (4.10.16a) \\
&= (\mathcal{A} - PmR_0)\,e^{A_F t} y_0 - Pm\left(\hat{R} - R_0\right)e^{A_F t} y_0 \\
&\quad + \gamma^{-2}\hat{R}e^{A_F t} y_0 && \text{in } [\mathcal{D}(\mathcal{A}^*)]' \quad (4.10.16b) \\
&= \mathcal{A}_{R_0}\hat{y}(t; y_0) - Pm\hat{r}(t; y_0) + P\hat{w}(t; y_0) && \text{in } [\mathcal{D}(\mathcal{A}^*)]' \quad (4.10.16c)
\end{aligned}
$$

where we have used (1.5.9) for $\mathcal{A}_{R_0}$, (4.10.1d) for $\hat{y}$, (4.10.1b) for $\hat{w}$, and both (4.10.5) and (4.10.7) for $\hat{r}$. Comparing this with (2.4.2), the stable form for the minimizing dynamics, we see that $\{\hat{u}(\,\cdot\,; y_0), \hat{y}(\,\cdot\,; y_0)\}$ is the minimizing pair for initial condition y_0 and disturbance $\hat{w}$.

Step 3: Now we make use of (4.10.3) to show that $\hat{w}(\,\cdot\,; y_0)$ is the maximizing disturbance for initial condition $y_0 \in H$. In particular, we seek to show that $\hat{w}$satisfies a relation similar to (3.3.2):

$$\left(E_\gamma \hat{w}(\,\cdot\,; y_0)\right)(t) = R_0 e^{\mathcal{A}_{R_0} t} y_0 \tag{4.10.17}$$

with E_γ the bounded operator on $L^2(0,\infty;[L^2(\Omega)]^d)$ defined in (3.1.2). Using (3.1.2) and (3.1.1) for E_γ and S, and recalling (1.6.3) defining $\mathscr{L}_{R_0}$, $\mathcal{W}_{R_0}$, and their adjoints, we calculate:

$$\begin{aligned}
E_\gamma \hat{w}(\,\cdot\,; y_0) &= \left[\gamma^2 + R_0 \mathscr{L}_{R_0} \mathscr{L}^*_{R_0} R_0 P - \mathcal{W}^*_{R_0} R_0 P - R_0 \mathcal{W}_{R_0}\right] \hat{w}(\,\cdot\,; y_0) \\
&= \left[\gamma^2 + R_0 \mathcal{K}_{R_0} P m \mathcal{K}^*_{R_0} R_0 P - \mathcal{K}^*_{R_0} R_0 P - R_0 \mathcal{K}_{R_0} P\right] \hat{w}(\,\cdot\,; y_0) \\
&= \left[\gamma^2 + R_0 \mathcal{K}_{R_0} P m \mathcal{K}^*_{R_0} R_0 P - \mathcal{K}^*_{R_0} R_0 P - R_0 \mathcal{K}_{R_0} P\right] \left[\gamma^{-2} \hat{R} e^{A_F \,\cdot} y_0\right] \\
&= \hat{R} e^{A_F \,\cdot} y_0 + \gamma^{-2} R_0 \mathcal{K}_{R_0} P m \mathcal{K}^*_{R_0} R_0 \hat{R} e^{A_F \,\cdot} y_0 - \gamma^{-2} \mathcal{K}^*_{R_0} R_0 \hat{R} e^{A_F \,\cdot} y_0 \\
&\qquad - \gamma^{-2} R_0 \mathcal{K}_{R_0} \hat{R} e^{A_F \,\cdot} y_0
\end{aligned} \tag{4.10.18}$$

where we used (4.10.1b) for $\hat{w}(\,\cdot\,; y_0)$. Recalling equation (4.10.7) allows us to insert $\left(\hat{R} - R_0\right) e^{A_F t} y_0$ into (4.10.18) to obtain

$$\begin{aligned}
E_\gamma \hat{w}(\,\cdot\,; y_0) &= \hat{R} e^{A_F \,\cdot} y_0 + \gamma^{-2} R_0 \mathcal{K}_{R_0} P m \mathcal{K}^*_{R_0} R_0 \hat{R} e^{A_F \,\cdot} y_0 - \gamma^{-2} \mathcal{K}^*_{R_0} R_0 \hat{R} e^{A_F \,\cdot} y_0 \\
&\qquad - \gamma^{-2} R_0 \mathcal{K}_{R_0} \hat{R} e^{A_F \,\cdot} y_0 \\
&= \hat{R} e^{A_F \,\cdot} y_0 + R_0 \mathcal{K}_{R_0} P m \left(\hat{R} - R_0\right) e^{A_F \,\cdot} y_0 - \left(\hat{R} - R_0\right) e^{A_F \,\cdot} y_0 \\
&\qquad - \gamma^{-2} R_0 \mathcal{K}_{R_0} \hat{R} e^{A_F \,\cdot} y_0 \\
&= R_0 \left(\mathcal{K}_{R_0} \left[P m \hat{R} - P m R_0 - \gamma^{-2} \hat{R}\right] e^{A_F \,\cdot} y_0 + e^{A_F \,\cdot} y_0\right)
\end{aligned} \tag{4.10.19}$$

Using the definitions of $\mathcal{A}_{R_0}$ in (1.5.9) and $\mathcal{K}_{R_0}$ in (1.6.1) and the definition of A_F in part (i) of the theorem, we perform the following calculations pertaining to the quantity in the last line of (4.10.19)

$$\begin{aligned}
&\left(\mathcal{K}_{R_0}\left[Pm\hat{R} - PmR_0 - \gamma^{-2}\hat{R}\right] e^{A_F\,\cdot}\,y_0 + e^{A_F\,\cdot}\,y_0\right)(t) \\
&= \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)}\left(\mathcal{A}_{R_0} - A_F\right) e^{A_F\tau} y_0\, d\tau + e^{A_F t} y_0 && \text{in } [\mathcal{D}(\mathcal{A}^*)]' \quad (4.10.20\text{a}) \\
&= \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)} \mathcal{A}_{R_0} e^{A_F\tau} y_0\, d\tau - \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)} A_F e^{A_F\tau} y_0\, d\tau \\
&\qquad + e^{A_F t} y_0 && \text{in } [\mathcal{D}(\mathcal{A}^*)]' \quad (4.10.20\text{b}) \\
&= -\, e^{\mathcal{A}_{R_0}(t-\tau)} e^{A_F\tau} y_0 \Big|_{\tau=0}^{\tau=t} + \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)} A_F e^{A_F\tau} y_0\, d\tau \\
&\qquad - \int_0^t e^{\mathcal{A}_{R_0}(t-\tau)} A_F e^{A_F\tau} y_0\, d\tau + e^{A_F t} y_0 && \text{in } [\mathcal{D}(\mathcal{A}^*)]' \quad (4.10.20\text{c}) \\
&= -e^{A_F t} y_0 + e^{\mathcal{A}_{R_0} t} y_0 + e^{A_F t} y_0 = e^{\mathcal{A}_{R_0} t} y_0 && \text{in } [\mathcal{D}(\mathcal{A}^*)]' \quad (4.10.20\text{d})
\end{aligned}$$

where we used integration by parts on the first integral in (4.10.20b) to obtain (4.10.20c). Inserting the result of (4.10.20) into the last line of (4.10.19), we see that (4.10.17) holds true.

We have shown that the quantities defined in (4.10) are the optimal quantities for $y_0 \in H$.

Since $\hat{R}$ is a solution of the ARE, with $\mathcal{A}^*\hat{R} \in \mathcal{L}(H)$, it follows that

$$\begin{aligned}
\frac{d}{dt}\left(\hat{R}e^{A_F t} y_0, e^{A_F t} y_0\right)_H &= 2\left(\hat{R}A_F e^{A_F t} y_0, e^{A_F t} y_0\right)_H \\
&= 2\left(\hat{R}\mathcal{A} e^{A_F t} y_0, e^{A_F t} y_0\right)_H - 2\left\|\hat{R}e^{A_F t} y_0\right\|^2_{[L^2(\omega)]^d} \\
&\qquad + 2\gamma^{-2}\left\|\hat{R}e^{A_F t} y_0\right\|^2_{[L^2(\Omega)]^d} \qquad (4.10.21)
\end{aligned}$$

We now apply the ARE in (1.3.1) with $x = z = e^{A_F t} y_0$ to rewrite (4.10.21) as

$$-\frac{d}{dt}\left(\hat{R}e^{A_F t}y_0, e^{A_F t}y_0\right)_H = \left\|e^{A_F t}y_0\right\|_H^2 + \left\|\hat{R}e^{A_F t}y_0\right\|_{[L^2(\omega)]^d}^2 - \gamma^{-2}\left\|\hat{R}e^{A_F t}y_0\right\|_{[L^2(\Omega)]^d}^2 \tag{4.10.22}$$

Because $e^{A_F t}$ is a uniformly stable semigroup, integration in t over $[0,\infty]$ yields

$$\left(\hat{R}y_0, y_0\right)_H = \int_0^\infty \left(\left\|e^{A_F t}y_0\right\|_H^2 + \left\|\hat{R}e^{A_F t}y_0\right\|_{[L^2(\omega)]^d}^2 - \gamma^{-2}\left\|\hat{R}e^{A_F t}y_0\right\|_{[L^2(\Omega)]^d}^2\right) dt, \quad \forall\, y_0 \in H \tag{4.10.23}$$

Since $J^*(y_0) < \infty$, Proposition 4.9.1 implies that $\gamma \geq \gamma_c$. $\square$

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