# The normal Structures of the Non-integrable Groups of Seven Parameters.

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The Normal Structures of the Non-Integrable $G_7$.

It has been proven (Lie - Theorie der Transformationsgruppe Vol III p. 757) that any non-integrable group must contain a sub-$G_3$ of the structure

$$(\mathbf{A}) \qquad (X_1, X_2) = X_1, (X_1, X_3) = 2X_2, (X_2, X_3) = X_3.$$

We shall subdivide ~~our~~ the problem into two heads according to whether the Sub $G_3$ $\mathbf{A}$ is invariant in the $G_7$ or not.

I. The Sub $G_3$ ($\mathbf{A}$) is Invariant in the $G_7$

Since by hypothesis the Sub $G_3$ is invariant in the $G_7$ the bracket alternant operation performed on $X_1$, $X_2$ or $X_3$ with $X_4, X_5, X_6$ or $X_7$ must give a linear sum of $X_1$ $X_2$ and $X_3$; therefore $X_1, X_2, X_3; X_j$ $(j = 4, 5, 6, 7)$ must form a sub $G_4$. ~~It~~ One must be a non integrable $G_4$ since it contains the sub-$G_3$ ($\mathbf{A}$). It must then have the structure [III, p. 723]

$$(2) \qquad (X_1, X_j) = (X_2, X_j) = (X_3, X_j) = 0 \qquad j = 4, 5, 6, 7$$

together with ($\mathbf{A}$).

This leaves ~~us~~ to be examined the results of the bracket alternant operation (applied to) $X_4, X_5, X_6, X_7$.

$$(X_i, X_K) = \sum_1^7 c_{iKs} X_s \qquad i, K = 4, 5, 6, 7$$

Now form Jacobi's identities

$$(x_i (x_j x_k)) + \cdot \qquad = 0$$

for $i = 1, 2, 3$; $j, k = 4, 5, 6, 7$

and we find that

Writing Jacobi's Identity with each of $X_1 X_2 X_3$

~~Put $X_1, X_2$ + $X_3$ one at a time in Jacobis Identity~~

together with

~~with~~ each pair of $X_4, X_5, X_6, X_7$

$$(X_1,(X_i,X_K)) + (X_K,(X_1,X_i)) + (X_i,(X_K,X_1)) \equiv 0,$$

the last two terms of this identity vanish by means of (2) this leaves

$$(X_1, \sum_{1}^{7} s C_{iKs} X_s) \equiv 0$$

which shows that $C_{iK2} = C_{iK3} = 0$

In like manner if we put $X_2$ in Jacobis Identity with $X_i X_K$ we find $C_{iK1} = C_{iK3} = 0$

Therefore

$$(X_i X_K) = \sum_{4}^{7} s C_{iKs} X_s, \quad i, K = 4,5,6,7$$

so that ~~consequently~~ $X_4, X_5, X_6, X_7$ form a sub $G_4$ of the $G_7$.

Th. 68 [LLC, p.732]

We know the structures of all the $G_4$'s, so that by taking each one of these together with (A) and (2) we obtain ~~will give~~ a non integrable $G_7$ in which the sub $G_3$ (A) is invariant; and these are all of the non integrable $G_7$'s of this character

Non Integrable $G_7$'s in which the Sub $G_3$ A is invariant.

(A): $(X_1, X_2) = X_1 f$, $(X_1, X_3) = 2X_2 f$, $(X_2, X_3) = X_3 f$

(2): $(X_1, X_4) = (X_1, X_5) = (X_1, X_6) = (X_1, X_7) = 0$

$(X_2, X_4) = (X_2, X_5) = (X_2, X_6) = (X_2, X_7) = 0$

$(X_3, X_4) = (X_3, X_5) = (X_3, X_6) = (X_3, X_7) = 0$

(A) + (2) are common to all the $G_7$'s of this class.

I. $(X_4, X_5) = X_4 f$
$(X_4, X_6) = 2X_5 f$
$(X_5, X_6) = X_6 f$
$(X_4, X_7) = 0$
$(X_5, X_7) = 0$
$(X_6, X_7) = 0$

II. $(X_4, X_5) = 0$
$(X_4, X_6) = 0$
$(X_5, X_6) = X_4 f$
$(X_4, X_7) = cX_4 f$
$(X_5, X_7) = X_5 f$
$(X_6, X_7) = (c-1)X_6 f$
$c \neq 1$

III. $(X_4, X_5) = 0$
$(X_4, X_6) = 0$
$(X_5, X_6) = X_4 f$
$(X_4, X_7) = 2X_4 f$
$(X_5, X_7) = X_5 f$
$(X_6, X_7) = X_5 f + X_6 f$

IV. $(X_4, X_5) = 0$
$(X_4, X_6) = 0$
$(X_5, X_6) = X_5 f$
$(X_4, X_7) = X_4 f$
$(X_5, X_7) = 0$
$(X_6, X_7) = 0$

V. $(X_4, X_5) = 0$
$(X_4, X_6) = 0$
$(X_5, X_6) = X_4 f$
$(X_4, X_7) = X_4 f$
$(X_5, X_7) = X_5 f$
$(X_6, X_7) = 0$

VI $(X_i, X_k) = 0$
$(X_4, X_7) = X_4 f$
$(X_5, X_7) = aX_5 f$
$(X_6, X_7) = cX_6 f$
$(i, k = 4, 5, 6)$

VII. $(X_i, X_k) = 0$

$(X_4, X_7) = c X_4 f$

$(X_5, X_7) = (1+c) X_5 f$

$(X_6, X_7) = X_4 + c X_6 f$

$(i, k = 4, 5, 6)$

VIII. $(X_i, X_k) = 0$

$(X_4, X_7) = \cdot X_5 f$

$(X_5, X_7) = 0$

$(X_6, X_7) = X_4 f$

$(i, k = 4, 5, 6)$

IX. $(X_i, X_k) = 0$

$(X_4, X_7) = X_4 f + X_5 f$

$(X_5, X_7) = X_5 f$

$(X_6, X_7) = X_4 f + X_6 f$

$(i, k = 4, 5, 6)$

X. $(X_i, X_k) = 0$

$(X_4, X_7) = 0$

$(X_5, X_7) = 0$

$(X_6, X_7) = X_5 f$

$(i, k = 4, 5, 6)$

XI. $(X_i, X_k) = 0$

$(X_4, X_7) = X_4 f$

$(X_5, X_7) = X_5 f$

$(X_6, X_7) = X_5 f + X_6 f$

$(i, k = 4, 5, 6)$

XII. $(X_i, X_k) = 0$

$(X_4, X_7) = 0$

$(X_5, X_7) = 0$

$(X_6, X_7) = 0$

$(i, k = 4, 5, 6)$

The structures of the sub $G_4$'s $X_4, X_5, X_6, X_7$ are taken from Lie's Transformationsgruppen Vol III §137 (58), (62) — (65), (67) — (73).

II. The sub $G_3$ (A) is <u>not</u> invariant in the $G_7$.

Under this head there are 4 different cases:

(i) The non-invariant sub $G_3$(A) is not contained in any greater sub-group of the $G_7$.

(ii) The said sub $G_3$ is contained in a sub $G_4$ which is itself not contained in any greater sub-group of the $G_7$.

(iii) The said sub. $G_3$ is contained in a sub. $G_5$ which is not contained in a sub $G_6$ of the $G_7$.

(iv) The said sub-$G_3$ is contained in a sub $G_6$ of the $G_7$.

These cases are seen to be exhaustive.

Case (i) <u>The non invariant sub $G_3$ (A) is not contained in a greater sub group of the $G_7$.</u>

Here there are $\infty^4$ sub $G_3$'s equally privileged with $X_1$, $X_2$, and $X_3$ [III p. 733]. They are therefore transformed by the adjoined group $\Gamma$ in a transitive manner (any one can be transformed into any other one). $\Gamma$, the adjoined group is isomorphous with the $G_7$ from the theory of the adjoined space [ ]

If $\Gamma$ were meroedrically isomorphus with the $G_7$, the sub $G_3(A)$ would contain a sub group $\Gamma$, which is invariant in the $G_7$ and in the sub $G_3$ [I 483]. This is clearly not the case since the $G_3(A)$ contains no sub groups [ ] and by hypothesis is itself not invariant in the $G_7$. Therefore $\Gamma$ is holoedrically isomorphous with the $G_7$.

Moreover, since the $G_3(A)$ is not contained in any larger subgroup of the $G_7$, $\Gamma$ transforms the $\infty^4$ sub $G_3$'s equally privileged with the sub $G_3(A)$ in a primitive manner [cf. III p. 733]. Hence $\Gamma$, and therefore the $G_7$, has the structure of a primitive 7 fold group in 4 variables.

There is only one such group

$$\boxed{\begin{array}{l} p_k, \quad x_4 p_1 + 2x_1 p_2 + 3x_2 p_3, \quad 2x_2 p_1 + x_3 p_2 + 3x_1 p_4, \\ {\scriptstyle k=1,2,3,4} \qquad x_1 p_1 - x_2 p_2 + 3(x_4 p_4 - x_3 p_3) \end{array}}$$

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Case (ii) <u>The non-invariant sub $G_3(A)$ is contained in a sub $G_4$ which is not contained in any larger sub group of the $G_7$.</u>

The structure of the sub $G_4$ is then (A) together with

$$(X_1 X_4) = 0 \quad (X_2 X_4) = 0 \quad (X_3 X_4) = 0 \qquad [\qquad]$$

Since the sub $G_3(A)$ is not invariant in the $G_7$ and since the sub $G_3(A)$ is the first derived group of the sub $G_4$, the sub $G_4$ cannot be invariant in the $G_7$ [ ]

As the sub $G_4$ contains only two invariant sub groups, namely the sub $G_3(A)$ and $X_4$, either (a) the sub $G_4$ contains no sub-group which is invariant in the $G_7$, or (b) it contains one subgroup, the sub $G_1$ $X_4$, which is invariant in the $G_7$.

In each case, as the sub $G_4$ is not contained in any larger sub group, there are $\infty^3$ sub $G_4$'s in the $G_7$ equally privileged with the above $G_4$

(a) <u>The sub $G_4$ contains no sub group, which is invariant in the $G_7$.</u>

Here the $\infty^3$ $G_4$'s are transformed transitively by the adjoined group $\Gamma$. Also as the Sub $G_4$ is not contained in any larger sub group of the $G_7$, the $\infty^3$ $G_4$'s are transformed primitively by the adjoined group $\Gamma$. Again, as the Sub $G_4$ does not contain any sub group which is invariant in the $G_7$, $\Gamma$ is holoedrically isomorphous with the $G_7$ [Vol I p. 483]. Hence $\Gamma$ and consequently the $G_7$ has the structure of a 7-fold primitive group in space of 3 dimensions. Therefore we obtain but the one case [III., p. 139, Th. 9.]

IV. $\boxed{p,\ q,\ r;\ yp-xq,\ zq-yr,\ xr-zp;\ xp+yq+zr}$

This is the $G_7$ of Euclidean movements and the perspective transformation.

b. <u>The Sub $G_4$ contains a Sub $G_1$, $X_4f$, which is invariant in the $G_7$.</u>

There must exist a $G_6$ $\Gamma$ which is meroedrically isomorphous with the $G_7$ [I. p 303] [$\Gamma$ is not necessarily any part of the adjoined group] but it must contain the $G_3(A)$ as a sub group [I. p 302] and is therefore non integrable and in addition, the transformation $X_4$ of the

$G_7$ must correspond to the identical transformation $I$ in $\Gamma_6$. The sub $G_4$ of the $G_7$, $X_1 X_2 X_3 X_4$, corresponds to the $\Gamma_3$ say $Y_1, Y_2, Y_3, I$ of the $\Gamma_6$. Then if $\Gamma_3$ were contained in a sub $\Gamma_4$ of $\Gamma_6$ - say $Y_1, Y_2, Y_3, Y_4\, I$, the corresponding $G_4$ of the $G_7$ would be contained in a sub $G_5$ of the $G_7$ - which is contrary to hypothesis, Therefore $\Gamma_6$ is a non integrable $G_6$ in which the sub $G_3$ (A) is not contained in any larger subgroup. There are only two normal structures of $G_6$'s of this character. The first is the same as the structure of

(α). $\boxed{p,\ xp,\ x^2p,\ q,\ yq,\ y^2q}$

The second ~~the~~ same as the structure of

(β). $\boxed{p,\ xp+yq,\ x^2p+2xyq,\ q,\ xq,\ x^2q}$

(III p 743 Th. 67 [1]+[6]).

(α) Suppose the group $G_6$, $\Gamma_6$, has the same structure as

$$\boxed{p,\ xp,\ x^2p,\ q,\ yq,\ y^2q}$$

$p, xp, x^2p + I$ form an invariant sub $G_3$ of the $G_6$, $\Gamma_6$, therefore $X_1, X_2, X_3, X_4$ the corresponding sub $G_4$ of $G_7$ is an invariant sub $G_4$ of which the sub $G_3(A)$ is the first derived group. Hence the sub $G_3(A)$ is invariant in the $G_7$. This is contrary to hypothesis. ($\alpha$) leads to no new case as we have found all the nonintegrable $G_7$'s in which the sub $G_3(A)$ is invariant.

($\beta$). The $G_6$, $\Gamma_6$, has the same structure as

$$\boxed{p,\ xp+yq,\ x^2p+2xyq,\ q,\ xq,\ x^2q}$$

The structure of this $G_6$ is

Putting $Y_1f \equiv p$ $Y_2f \equiv xp+yq$ $Y_3f \equiv x^2p+2xyq$ $Y_4f \equiv I$
$Y_5f \equiv q$ $Y_6f \equiv xq$, $Y_7f \equiv x^2q$ $f$:

$(Y_1, Y_2) = Y_1f$; $(Y_1, Y_3) = 2Y_2f$; $(Y_2, Y_3) = Y_3f$;

$(Y_1, Y_5) = 0$; $(Y_1, Y_6) = Y_5f$; $(Y_1, Y_7) = 2Y_6$;

$(Y_2, Y_5) = -Y_5f$; $(Y_2\ Y_6) = 0$; $(Y_2, Y_7) = Y_7f$

$(Y_3, Y_5) = -2Y_6f$; $(Y_3, Y_6) = -Y_7f$; $(Y_3, Y_7) = 0$;

$(Y_5, Y_6) = (Y_5, Y_7) = (Y_6, Y_7) = 0.$

The structure of the $G_7$ is therefore [I. p ]

(A) $(X_1 X_2) = X_1f$; $(X_1 X_3) = 2X_2f$; $(X_2, X_3) = X_3f$;
$(X_1 X_4) = 0$; $(X_2 X_4) = 0$; $(X_3\ X_4) = 0$;
together with

$(X_1,X_5) = a_{15} X_4 f$ ; $(X_2,X_5) = -X_5 f + a_{25} X_4 f$ ;

$(X_1,X_6) = X_5 f + a_{16} X_4 f$ ; $(X_2,X_6) = a_{26} X_4 f$ ;

$(X_1,X_7) = 2X_6 f + a_{17} X_4 f$ ; $(X_2,X_7) = X_7 f + a_{27} X_4 f$ ;

$(X_3,X_5) = -2X_6 f + a_{35} X_4 f$ ; $(X_4\, X_5) = a_{45} X_4 f$

$(X_3,X_6) = -X_7 f + a_{36} X_4 f$ ; $(X_4\, X_6) = a_{46} X_4 f$

$(X_3,X_7) = a_{37} X_4 f$ ; $(X_4\, X_7) = a_{47} X_4 f$

$(X_5\, X_6) = a_{56} X_4 f$ ; $(X_5, X_7) = a_{57} X_4 f$ ; $(X_6, X_7) = a_{67} X_4 f$.

If we put these transformations, three at a time into Jacobi's Identity we obtain the following results

$$a_{15} = a_{26} = a_{37} = a_{ik} = 0 \qquad i,k = 4,5,6,7. \quad i \neq k.$$

$$a_{16} = -a_{25} = c_1 \text{ (say)} \qquad a_{17} = -a_{35} = c_2$$

$$a_{27} = -a_{36} = c_3$$

This leaves the structure of the group in the form

$$(X_1,X_4) = (X_2,X_4) = (X_3,X_4) = (X_1,X_5) = (X_2,X_6) = (X_3,X_7) = (X_i,X_k).$$
$$i,k = 4,5,6,7; \ i \neq k$$

$(X_1,X_6) = X_5 f + c_1 X_4 f$; $(X_1,X_7) = 2X_6 f + c_2 X_4 f$; $(X_2,X_5) = -X_5 f - c_1 X_4 f$;

$(X_2,X_7) = X_7 f + c_3 X_4 f$; $(X_3,X_5) = -2X_6 f - c_2 X_4 f$; $(X_3,X_6) = -X_7 f - c_3 X_4 f$.

If we introduce a new transformation $\overline{X}_5 f = X_5 f + c_1 X_4 f$
the alternants affected will take the form

$$(X_1, X_6) = \overline{X}_5 f \; ; \; (X_2, \overline{X}_5) = -X_5 f - c_1 X_4 f = -\overline{X}_5 f ; (X_3, \overline{X}_5) = -2X_6 f - c_2 X_4 f$$

now introduce $\overline{X}_6 f = 2X_6 f + c_2 X_4 f$
The alternants affected will take the form

$$(X_1, \overline{X}_6) = 2\overline{X}_5 f ; \; (X_1, X_7) = \overline{X}_6 f ; (X_3, \overline{X}_5) = -\overline{X}_6 f .$$

In like manner introduce $\overline{X}_7 f = X_7 + c_3 X_4$
The alternants affected take the form

$$(X_1, \overline{X}_7) = \overline{X}_6 , \; (X_2, \overline{X}_7) = X_7 + c_3 X_4 = \overline{X}_7 , (X_3, X_6) = -2\overline{X}_7$$

Hence the normal structure of the $G_7$ is

I. $(X_1, X_2) = X_1 f , \; (X_1, X_3) = 2X_2 f , \; (X_2, X_3) = X_3 f \; ;$

$$(X_1, X_4) = (X_2, X_4) = (X_3, X_4) = (X_1, X_5) = (X_2, X_6) = (X_3, X_7) = (X_i, X_k) = 0$$
$$i \neq k , \quad i, k = 4, 5, 6, 7$$

$$(X_1, X_6) = 2X_5 f ; \; (X_2, X_5) = -X_5 f \; ; \; (X_3 X_5) = -X_6 f$$
$$(X_1, X_7) = X_6 f ; \; (X_2, X_7) = X_7 f \; ; \; (X_3, X_6) = -2X_7 f .$$

Case (iii). The non-invariant sub $G_3(A)$ is contained in a sub-$G_5$ of the $G_7$ which is not contained in a sub $G_6$ of the $G_7$.

In this case there are $\infty^2$ sub $G_5$'s equally privileged with the above mentioned sub $G_5$.

We make three cases, according as the $G_5$ which contains the $G_3$ has the same structure as

(a) $\boxed{p,\ q,\ xq,\ xp-yq,\ yp}$

(b) $\boxed{p,\ xp,\ x^2p,\ q,\ yq}$

(C) $\boxed{p,\ xp,\ x^2p,\ q,\ r}$

[III p. 736 Th. 66.]

In each case, if the sub $G_5$ does not contain a subgroup, which is invariant in the sub $G_5$ and the sub $G_7$ both, the $G_7$ must have the structure of a primitive $G_7$ in two variables. But there is no primitive $G_7$ in two variables. Hence, in each case, the $G_5$ must contain a subgroup which is invariant in the $G_5$ and the $G_7$.

(a.) The $G_5$ has the structure of $\boxed{p, q, xq, xp-yq, yp}$

By introducing new transformations we can throw this group into the form

$$\boxed{xq, \tfrac{1}{2}(yq-xp), -yp, \ p, \ q}$$

Here the only invariant sub group is $p, q$. Hence there must exist a group $\Gamma_5$ meroedrically isomorphous with $G_7$. $\Gamma_5$ must contain a sub $\gamma_3(A)$, and the sub $\gamma_3$ must not be contained in a sub $\gamma_4$ of $\Gamma_5$; for, if it were the corresponding $G_3(A)$ would be contained in a $G_6$ of the $G_7$.

Hence $\Gamma_5$ can only have the structure

$$(1) \quad \boxed{xq, \tfrac{1}{2}(yq-xp), -yp, I_4, I_5, p, q}$$

$I_4, I_5$ stand for the identical transformation

This is a similar group to (III, p. 736 th. 66 [76]).

Let $Y_1, Y_2, \ldots Y_7$ stand for the transformations of this group in order

Since $Y_6 Y_7$ is invariant in the $G_5$ and the corresponding transformations $X_6, X_7$ form an invariant sub $G_2$ in the $G_7$

$$(X_i X_k) = c_{ik6} X_6 + c_{ik7} X_7$$

$$i = 1, \ldots 7 \quad k = 6, 7.$$

From the theory of isomorphous groups [I ]
the structure of the $G_7$ is (1) together with

$$(X_1 X_6) = -X_7f + a_{16} X_4f + b_{16} X_5f$$
$$(X_1 X_7) = a_{17} X_4f + b_{17} X_5f$$
$$(X_2 X_6) = \tfrac{1}{2} X_6f + a_{26} X_4f + b_{26} X_5f$$
$$(X_2 X_7) = -\tfrac{1}{2} X_7f + a_{27} X_4f + b_{27} X_5f$$
$$(X_3 X_6) = a_{36} X_4f + b_{36} X_5f$$
$$(X_3 X_7) = X_6f + a_{37} X_4f + b_{37} X_5f$$
$$(X_i X_K) = a_{iK} X_4 + b_{iK} X_5$$

$i = 4, 5, 6 \quad K = 6, 7 \quad i \neq K$

But since $\boxed{X_6, X_7}$ is invariant in the $G_7$

$$a_{iK} = b_{iK} = 0 \qquad i = 1, 2, 3, 4, 5, 6 \quad K = 6, 7 \quad i \neq K$$

Therefore the structure of the $G_7$ is

[XVI] (1) together with

$$(X_1, X_6) = -X_7f \;;\; (X_2, X_6) = \tfrac{1}{2} X_6f \;;\; (X_3, X_6) = 0$$
$$(X_1, X_7) = 0 \;;\; (X_2, X_7) = -\tfrac{1}{2} X_7f \;;\; (X_3, X_7) = X_6f$$
$$(X_i, X_K) = 0 \qquad i = 4, 5, 6 \quad K = 6, 7 \quad i \neq K$$

(b.) The $G_5$ has the structure of

$$\boxed{p,\ xp,\ x^2p,\ q,\ yq}$$

The adjoined group shows that this $G_5$ has the following invariant subgroups; - and only these:

(1.) $\boxed{p,\ xp,\ x^2p,\ q}$ (2.) $\boxed{p,\ xp,\ x^2p}$ (3.) $\boxed{q,\ yq}$ (4.) $\boxed{q}$

(1) could not be invariant in the $G_7$ since its first derived group has the structure (A) therefore the $G_3$ (A) would be invariant in the $G_7$ which is contrary to hypothesis. In like manner (2) which has the structure (A) cannot be invariant in the $G_7$. Therefore the only cases left to consider are when (3.) + (4) are invariant in the $G_7$.

($\alpha$) The sub group $\boxed{q,\ yq}$ which is invariant in the $G_5$ is also invariant in the $G_7$. There must then exist a group $\Gamma_5$ which is meroedrically isomorphous with the $G_7$ and which is now integrable. It therefore contains a sub $\Gamma_3$ (A). This sub $\Gamma_3$ cannot be contained in a sub $\Gamma_4$ of the $\Gamma_5$ for then the $\Gamma_5$ would correspond to a sub $G_5$ of $G_7$ which would be contained in a sub $G_6$. But this is contrary to hypothesis. Hence $\Gamma_5$ has the structure of

$$\boxed{xq,\ -\tfrac{1}{2}(xp-yq),\ -yp,\ \bar{I}_4,\ \bar{I}_5,\ p,\ q}.$$

Let $Y_1 \dots\dots Y_7$ stand for these transformations respectively, while $X_1 \dots\dots X_7$ stand for the corresponding transformations in the $G_7$.

From the theory of isomorphism [I ]

$X_1 \dots\dots X_7$ have the following structure.

$X_1 \dots\dots X_5$ are connected like $\boxed{p, xp, x^2p, q, yq}$

$$(1)\left\{\begin{array}{lll} (X_1X_2) = X_1 & (X_1X_4) = 0 & (X_1X_5) = 0 \\ (X_1X_3) = 2X_2 & (X_2X_4) = 0 & (X_2X_5) = 0 \\ (X_2X_3) = X_3 & (X_3X_4) = 0 & (X_3X_5) = 0 \\ & (X_4X_5) = X_4 & \end{array}\right.$$

The rest of the structure of the group is

$$(X_1, X_6) = -X_7 f + a_{16} X_4 f + b_{16} X_5 f \,; \quad (X_1, X_7) = a_{17} X_4 f + b_{17} X_5 f$$

$$(X_2, X_6) = \tfrac{1}{2} X_6 f + a_{26} X_4 + b_{26} X_5 f \,; (X_2, X_7) = -\tfrac{1}{2} X_7 f + a_{27} X_4 f + b_{27} X_5 f$$

$$(X_3, X_6) = a_{36} X_4 f + b_{36} X_5 f \,; (X_3, X_7) = X_6 f + a_{37} X_4 f + b_{37} X_5 f$$

$$(X_4, X_6) = a_{46} X_4 f + b_{46} X_5 f \,; (X_4, X_7) = a_{47} X_4 f + b_{47} X_5 f$$

$$(X_5, X_6) = a_{56} X_4 f + b_{56} X_5 f \,; (X_5, X_7) = a_{57} X_4 f + b_{57} X_5 f$$

$$(X_6, X_7) = a_{67} X_4 f + b_{67} X_5 f$$

If we put this transformations, three at a time in Jacobis Identity we obtain the following results.

$a_{16} = 0$; $a_{17} = 0$; $\qquad b_{16} = 2b_{27} = a_{47} = 2c_1$ (say); $b_{17} = 0$

$a_{26} = \frac{1}{2} a_{37} = a_{56} = c_2$; $a_{27} = 0$; $\qquad b_{26} = \frac{1}{2} b_{37} = \frac{1}{2} a_{56} = \frac{1}{2} c_2$; $b_{27} = \frac{1}{2} b_{16} = c_1$

$a_{36} = 0$; $a_{37} = 2a_{26} = 2a_{56} = 2c_2$; $\quad b_{36} = 0$; $b_{37} = 2b_{26} = 2a_{56} = -a_{46} = c_2$

$a_{46} = -2b_{26} = -c_2$; $a_{47} = b_{16} = 2c_1$; $b_{46} = 0$; $b_{47} = 0$

$a_{56} = a_{26} = c_2$; $a_{57} = 0$; $b_{56} = 0$; $b_{57} = 0$

$a_{67} = -a_{56} \cdot a_{47} = -2c_1 c_2$; $b_{67} = 0$

0

After making these substitutions the structure of the $G_7$ becomes (1) together with

$(X_1 X_6) = -X_7 f + 2c_1 X_5 f$ ; $\qquad (X_1 X_7) = 0$ ;

$(X_2 X_6) = \frac{1}{2} X_6 f + c_2 X_4 f + \frac{1}{2} c_2 X_5 f$ ; $\qquad (X_2 X_7) = -\frac{1}{2} X_7 f + c_1 X_5 f$ ;

$(X_3 X_6) = 0$ ; $\qquad (X_3 X_7) = X_6 f + 2c_2 X_4 f + c_2 X_5 f$ ;

$(X_4, X_6) = -c_2 X_4 f \qquad (X_4, X_7) = 2c_1 X_4 f$

$(X_5, X_6) = \cancel{f}\, c_2 X_4 f \qquad (X_5, X_7) = 0$

$$(X_6, X_7) = -2c_1 c_2 X_4$$

Introduce as a new transformation

$$\overline{X}_7 f = X_7 f - 2c_1 X_5 f$$

$(X_1 X_6) = -\overline{X}_7 f \qquad (X_1, \overline{X}_7) = 0$

$$(X_2, \overline{X}_7) = -\tfrac{1}{2} X_7 f + c_1 X_5 f = -\tfrac{1}{2} \overline{X}_7 f$$

$$(X_3, \overline{X}_7) = X_6 f + 2c_2 X_4 f + c_2 X_5 f$$

$$(X_4, \overline{X}_7) = 2c_1 X_4 f - 2c_1 X_4 f = 0$$

$$(X_5 \overline{X}_7) = 0$$

$$(X_6, \overline{X}_7) = -2c_1 c_2 X_4 + 2c_1 c_2 X_4 = 0$$

Introduce $\overline{X}_6 = X_6 + 2c_2 X_4 f + c_2 X_5 f$

$(X_1 \overline{X}_6) = -\overline{X}_7 f \qquad (X_2 \overline{X}_6) = \frac{1}{2} X_6 f + c_2 X_4 f + \frac{1}{2} c_2 X_5 f = \frac{1}{2} \overline{X}_6 f$

$(X_3, \overline{X}_6) = 0 \qquad (X_3, \overline{X}_7) = \overline{X}_6 f$

$(X_4, \overline{X}_6) = -c_2 X_4 f + c_2 X_4 f = 0 \qquad (X_5, \overline{X}_6) = 2c_2 X_4 f - 2c_2 X_4 f = 0$

$$(\overline{X}_6, \overline{X}_7) = 0$$

now if we drop the bars the final form of the structure of the $G_7$ is

see over

$$(\overline{XVII})\quad (X_1 X_2) = X_1 f \quad (X_1 X_4) = 0 \quad (X_1 X_5) = 0$$

$$(X_1 X_3) = 2X_2 f \quad (X_2 X_4) = 0 \quad (X_2 X_5) = 0 \quad (X_4 X_5) = X_4$$

$$(X_2 X_3) = X_3 f \quad (X_3 X_4) = 0 \quad (X_3 X_5) = 0$$

$$(X_1, X_6) = -X_7 f \qquad (X_1 X_7) = 0$$

$$(X_2, X_6) = \tfrac{1}{2} X_6 f \qquad (X_2, X_7) = -\tfrac{1}{2} X_7 f$$

$$(X_3, X_6) = 0 \qquad (X_3, X_7) = X_6 f$$

$$(X_4 X_6) = (X_5, X_6) = (X_4, X_7) = (X_5, X_7) = (X_6, X_7) = 0$$

(β) The sub group $\boxed{G_1}$ which is invariant in the sub $G_5$ is also invariant in the $G_7$.

Since the subgroup which is invariant in the $G_7$ is a $G_1$ there must ~~exist~~ a meroedrically isomorphous group $\Gamma_6$ which is now integrable and in which ~~the~~ sub $\Gamma_3(A)$ is not in a sub $\Gamma_5$. For if it were contained in a $\Gamma_5$ then the corresponding $G_5$ would be contained in a $G_6$ of the $G_7$. If $\Gamma_3$ is contained in a sub $\Gamma_4$, the $\Gamma_4$ must be isomorphous with the

$G_5$ $\boxed{p \quad xp \quad x^2p \quad q, \quad yq}$. That is $X_1$ $X_2$ $X_3$ $X_4$, $X_5$ correspond respectively to $Y_1$ $Y_2$, $Y_3$, $I_4$, $Y_5$ of the $\Gamma_6^r$ [since, of course, $G_5$ is contained in itself.]

Hence we look for non integrable $\Gamma_6^r$'s in which the $\Upsilon_3(A)$ is either in no larger sub group or in a sub $\Upsilon_4^r$ of the $\Gamma_6$.

The only case is the general linear in two Variables

$$\boxed{xq, -\tfrac{1}{2}(xp-yq), -yp, \ xp+yq, \ p, \ q}$$

We must so choose the identical transformation that $X_1 \dots X_5$ has ~~the~~ same structure as $\boxed{p, xp, x^2p, q, yq}$ so we take the $\Gamma_6$ in the form

$$\boxed{xq, -\tfrac{1}{2}(xp-yq), -yp, I_4, \ xp+yq, p, q}$$

The structure of the $G_7$ is therefore [I. p ]

$$1 \begin{cases} (X_1X_2) = X_1f; (X_1X_4) = 0; (X_1X_5) = 0; \\ (X_1X_3) = 2X_2f; (X_2X_4) = 0; (X_2X_5) = 0; (X_4X_5) = X_4f. \\ (X_2X_3) = X_3f; (X_3X_4) = 0; (X_3X_5) = 0; \end{cases}$$

$(X_1 X_6) = -X_7 f + a_{16} X_4 f$ $\qquad (X_1 X_7) = a_{17} X_4$

$(X_2 X_6) = \frac{1}{2} X_6 f + a_{26} X_4 f$ $\qquad (X_2 X_7) = -\frac{1}{2} X_7 f + a_{27} X_4$

$(X_3 X_6) = a_{36} X_4 f$ $\qquad (X_3 X_7) = X_6 f + a_{37} X_4$

$(X_4 X_6) = a_{46} X_4 f$ $\qquad (X_4 X_7) = a_{47} X_4$

$(X_5 X_6) = -X_6 f + a_{56} X_4 f$ $\qquad (X_5 X_7) = -X_7 + a_{57} X_4$

$$(X_6 X_7) = a_{67} X_4$$

If we put these transformations three at a time in Jacobi's Identity we obtain the following results

$a_{16} = 2a_{27} = 2c_1$ (say); $\quad a_{17} = 0$; $\quad a_{26} = \frac{1}{2} a_{37} = c_2$

$a_{27} = \frac{1}{2} a_{16} = c_1$; $\quad a_{36} = 0$; $\quad a_{37} = 2a_{26} = 2c_2$

$a_{46} = a_{56} = 0$; $\quad a_{47} = a_{57} = a_{67} = 0$

After making these substitutions the structure of the group $G_7$ becomes (1) together with

$(X_1 X_6) = -X_7 f + 2c_1 X_4 f$ $\qquad (X_1 X_7) = 0$

$(X_2 X_6) = \frac{1}{2} X_6 f + c_2 X_4 f$ $\qquad (X_2 X_7) = -\frac{1}{2} X_7 f + c_1 X_4 f$

$(X_3 X_6) = 0$ $\qquad (X_3 X_7) = X_6 f + 2c_2 X_4 f$

$(X_4 X_6) = 0$ $\qquad (X_4 X_7) = 0$

$(X_5 X_6) = -X_6 f$ $\qquad (X_5 X_7) = -X_7 f$ $\qquad (X_6 X_7) = 0$

Introduce as a new transformation

$$\bar{X}_7 f = X_7 f - 2c_1 X_4 f$$

$$(X_1 X_6) = -\bar{X}_7$$

$$(X_1 \bar{X}_7) = 0$$

$$(X_2 \bar{X}_7) = -\tfrac{1}{2} X_7 f + c_1 X_4 f = -\tfrac{1}{2}\bar{X}_7 f$$

$$(X_3 \bar{X}_7) = X_6 f + 2c_2 X_4 f$$

$$(X_4 \bar{X}_7) = 0$$

$$(X_5 \bar{X}_7) = -X_7 f + 2c_1 X_4 f = -\bar{X}_7 f \qquad (\bar{X}_6 \bar{X}_7) = 0$$

Put $\bar{X}_6 f = X_6 f + 2c_2 X_4 f$

$$(X_1 \bar{X}_6) = -\bar{X}_7 f \qquad (X_4 \bar{X}_6) = 0$$

$$(X_2 \bar{X}_6) = \tfrac{1}{2} X_6 f + c_2 X_4 f = \tfrac{1}{2}\bar{X}_6 f \qquad (X_5 \bar{X}_6) = -X_6 f - 2c_2 X_4 f = -\bar{X}_6 f$$

$$(X_3 \bar{X}_6) = 0$$

$$(X_3 \bar{X}_7) = \bar{X}_6 f$$

$$(\bar{X}_6 \bar{X}_7) = 0$$

Dropping the bars we obtain as the final form of the structure of the group

XVIII. $(X_1 X_2) = X_1 f \quad (X_1 X_4) = 0 \quad (X_1 X_5) = 0$

$(X_1 X_3) = 2X_2 f \quad (X_2 X_4) = 0 \quad (X_2 X_5) = 0 \quad (X_4 X_5) = X_4 f$

$(X_2 X_3) = X_3 f \quad (X_3 X_4) = 0 \quad (X_3 X_5) = 0$

$(X_1 X_6) = -\bar{X}_7 f \qquad (X_1 X_7) = 0$

$(X_2 X_6) = \frac{1}{2} \bar{X}_6 f \qquad (X_2 X_7) = -\frac{1}{2} X_7 f$

$(X_3 X_6) = 0 \qquad (X_3 X_7) = X_6 f$

$(X_4 X_6) = 0 \qquad (X_4 X_7) = 0$

$(X_5 X_6) = -X_6 f \qquad (X_5 X_7) = -X_7 f$

$$(X_6 X_7) = 0$$

(C.) Suppose the sub $G_5$ has the structure of $\boxed{p, xp, x^2p, q, r}$.

Here $q$ and $r$ are equally privileged transformations. The invariant subgroups of this $G_5$ are (1) $\boxed{p, xp, x^2p, q}$ (2) $\boxed{p, xp, x^2p}$, (3) $\boxed{q, r}$ (4) $\boxed{q}$ (5) $\boxed{p, xp, x^2p, r}$ (6) $\boxed{r}$

(1), (2) or (5) could not be invariant in the $G_7$ for then the sub $G_3(A)$ would be invariant in the $G_7$ which is contrary to hypothesis. (3) and (4) are equally privileged so we need to consider only one of them, since the other would give rise to a similar structure. We then make two cases according as

($\alpha$) $\boxed{q, r}$ is invariant in the $G_5$ and $G_7$

or ($\beta$) $\boxed{q}$ is invariant in the $G_5$ and $G_7$.

$\alpha$. Suppose $\boxed{q, r}$ is invariant in the $G_5$ and $G_7$.

There must then exist a group $\Gamma_5$ which is merohedrically isomorphous with the $G_7$ and which is non integrable. It therefore a sub $V_3(A)$. This sub $\Gamma_3$ cannot be contained in a sub $\Gamma_4$ of the $\Gamma_5$ for then the $\Gamma_5$ would correspond to a sub $G_5$ of the $G_7$ which would be contained in a sub $G_6$. But this is contrary to hypothesis. Hence $\Gamma_5$ has the structure of

$$\boxed{xq - \tfrac{1}{2}(xp - yq), -yp, \bar{I}_4, \bar{I}_5, p, q}$$

Therefore just as in 6. ($\alpha$) the structure of the $G_7$ will be

$$1. \begin{cases} (X_1\ X_2) = X_1 f & (X_1 X_4) = 0 & (X_1 X_5) = 0 & \\ (X_1\ X_3) = 2X_2 f & (X_2 X_4) = 0 & (X_2 X_5) = 0 & (X_4, X_5) = 0 \\ (X_2 X_3) = X_3 f & (X_3 X_4) = 0 & (X_3 X_5) = 0 & \\ \text{since } X_1 \ldots\ldots X_5 \text{ must have the structure} \end{cases}$$

of $\boxed{p, xp, x^2p, q, r}$

$$(X_1 X_6) = -X_7 f + a_{16} X_4 f + b_{16} X_5 f \ ; \ (X_1 X_7) = a_{17} X_4 + b_{17} X_5$$

$$(X_2 X_6) = \tfrac{1}{2} X_6 f + a_{26} X_4 f + b_{26} X_5 f \ ; \ (X_2 X_7) = -\tfrac{1}{2} X_7 f + a_{27} X_4 + b_{27} X_5$$

$$(X_3 X_6) = a_{36} X_4 f + b_{36} X_5 f \ ; \ (X_3 X_7) = X_6 f + a_{37} X_4 + b_{37} X_5$$

$$(X_4 X_6) = a_{46} X_4 f + b_{46} X_5 f \ ; \ (X_4 X_7) = a_{47} X_4 + b_{47} X_5$$

$$(X_5 X_6) = a_{56} X_4 f + b_{56} X_5 f \ ; \ (X_5 X_7) = a_{57} X_4 + b_{57} X_5$$

$$(X_6 X_7) = a_{67} X_4 + b_{67} X_5$$

If we put these transformations, three at a time into the operator, Jacobi's Identity we obtain the following results:

$a_{16} = 2a_{27} = 2c_1$ ; $a_{17} = 0$ ; $b_{16} = 2b_{27} = 2c_2$ ; $b_{17} = 0$

$a_{26} = \frac{1}{2} a_{37} = c_3$ ; $a_{27} = \frac{1}{2} a_{16} = c_1$ ; $b_{26} = \frac{1}{2} b_{37} = c_4$ ; $b_{27} = \frac{1}{2} b_{16} = c_2$

$a_{36} = 0$ ; $a_{37} = 2a_{26} = 2c_3$ ; $b_{36} = 0$ ; $b_{37} = 2b_{26} = 2c_4$

$a_{46} = a_{56} = a_{47} = a_{57} = b_{46} = b_{56} = b_{47} = b_{57} = 0$ ; $a_{67} = c_5$ ; $b_{67} = c_6$.

after making these substitutions the structure of the $G_7$ becomes (1) together with

$$(X_1 X_6) = -X_7 f + 2c_1 X_4 f + 2c_2 X_5 f \,;\, (X_1 X_7) = 0$$

$$(X_2 X_6) = \tfrac{1}{2} X_6 + c_3 X_4 + c_4 X_5 \,;\, (X_2 X_7) = -\tfrac{1}{2} X_7 + c_1 X_4 + c_2 X_5$$

$$(X_3 X_6) = 0 \,;\, (X_3 X_7) = X_6 + 2c_3 X_4 + 2c_4 X_5$$

$$(X_4 X_6) = 0 \,;\, (X_4 X_7) = 0$$

$$(X_5 X_6) = 0 \,;\, (X_5 X_7) = 0$$

$$(X_6 X_7) = c_5 X_4 + c_6 X_5$$

Introduce $\bar{X}_6 = X_6 f + 2c_3 X_4 f + 2c_4 X_5 f$

$$(X_1 \bar{X}_6) = -X_7 f + 2c_1 X_4 f + 2c_2 X_5 f$$

$$(X_2, \bar{X}_6) = \tfrac{1}{2} X_6 f + c_3 X_4 f + c_4 X_5 f = \tfrac{1}{2} \bar{X}_6 f$$

$$(X_3 \bar{X}_6) = 0, \quad (X_4, \bar{X}_6) = 0, \quad (X_5 \bar{X}_6) = 0 \quad (X_3 X_7) = \bar{X}_6 f$$

$$(\bar{X}_6 X_7) = c_5 X_4 f + c_6 X_5 f$$

put $\bar{X}_7 = X_7 f - 2c_1 X_4 f - 2c_2 X_5 f$

$$(X_1 \bar{X}_6) = -\bar{X}_7 f \qquad (X_1 \bar{X}_7) = 0$$

$$(X_2 \bar{X}_7) = -\tfrac{1}{2} X_7 f + c_1 X_4 f + c_2 X_5 f = -\tfrac{1}{2} \bar{X}_7 f$$

$$(X_3 \bar{X}_7) = \bar{X}_6 f$$

$$(X_4 \bar{X}_7) = (X_5 \bar{X}_7) = 0$$

$$(\bar{X}_6 \bar{X}_7) = c_5 X_4 f + c_6 X_5 f.$$

Now since $X_4 f$ and $X_5 f$ are commutable with all the other transformations of the group we can introduce $\overline{X}_4 f = c_5 X_4 f$ and $\overline{X}_5 f = C_6 X_5 f$ as new transformations provided $c_5 \neq 0$ and $c_6 \neq 0$. We therefore have to make the following cases

(1.) $C_5 \neq 0$ $\quad$ $c_6 \neq 0$

(2.) $C_5 \neq 0$ $\quad$ $c_6 = 0$

(3.) $C_5 = 0$ $\quad$ $c_6 \neq 0$

(4) $C_5 = 0$ $\quad$ $c_6 = 0$

The final form of the structure of the $G_7$ will then be

$\underline{X}$. $(X_1 X_2) = X_1 f$ $\quad (X_1 X_4) = 0$ $\quad (X_1 X_5) = 0$

$(X_1 X_3) = 2X_2 f$ $\quad (X_2 X_4) = 0$ $\quad (X_2 X_5) = 0$ $\quad (X_4 X_5) = 0$

$(X_2 X_3) = X_3 f$ $\quad (X_3 X_4) = 0$ $\quad (X_3 X_5) = 0$

$(X_1 X_6) = -X_7 f$ $\quad (X_1 X_7) = 0$

$(X_2 X_6) = \frac{1}{2} X_6 f$ $\quad (X_2 X_7) = -\frac{1}{2} X_7 f$

$(X_3 X_6) = 0$ $\quad (X_3 X_7) = X_6 f$

$(X_4 X_6) = 0$ $\quad (X_4 X_7) = 0$

$(X_5 X_6) = 0$ $\quad (X_5 X_7) = 0$

* Since $X_4 f$ & $X_5 f$ are equally privileged this is no new case

$(X_6 X_7) = X_4 f + X_5 f$; or (2) $(X_6, X_7) = X_4 f$, or (3) $(X_6, X_7) = X_5 f$*; or $(X_6 X_7) = 0$

($\beta$). Suppose the Sub group $\boxed{g}$ which is invariant in the sub $G_5$ is also invariant in the $G_7$.

Since the subgroup which is invariant in the $G_7$ is a $G_1$ there must exist a meroedrically isomorpheus group $\Gamma_6$ which is nonintegrable and in which the sub $\gamma_3(A)$ is not in a sub $\gamma_5$. For if it were contained in a $\gamma_5$ then the corresponding $G_5$ would be contained in a $G_6$ of the $G_7$. If $\gamma_3$ is contained in a sub $\gamma_4$, the $\gamma_4$ must be isomorphus with the $G_5$ $\boxed{p, xp, x^2p, q, r}$. That is $X_1, X_2, X_3, X_4, X_5$ correspond respectively to $Y_1, Y_2, Y_3, I_4, Y_5$ of the $\Gamma_6$ [since, of course, $G_5$ is contained in itself.

Hence we look for nonintegrable $\Gamma_6$'s in which the $\gamma_3(A)$ is either in no larger sub group or in a sub $\gamma_4$ of the $\Gamma_6$.

The only case is the general linear in two variables

$$\boxed{xq, -\tfrac{1}{2}(xp - yq), -yp, xp + yq, p, q}$$

We must so choose the identical

transformation that $X_1 \ldots X_5$ has the same structure as

$$\boxed{p,\ xp,\ x^2p,\ q,\ r}$$

so we take $\Gamma_6$ in the form

$$\boxed{xq,\ \tfrac{1}{2}(xp-yq),\ -yp,\ I_4,\ xp+yq,\ p,\ q}$$

The structure of the $G_7$ is therefore

$$\begin{cases} (X_1X_2)=X_1f, & (X_1X_4)=0, \quad (X_1X_5)=0 \\ (X_1X_3)=2X_2f, & (X_2X_4)=0, \quad (X_2X_5)=0 \quad (X_4,X_5)=0 \\ (X_2X_3)=X_3f, & (X_3X_4)=0, \quad (X_3X_5)=0 \end{cases}$$

$$(X_1X_6)=-X_7f+a_{16}X_4f \qquad (X_1X_7)=a_{17}X_4f$$

$$(X_2X_6)=\tfrac{1}{2}X_6f+a_{26}X_4f \qquad (X_2X_7)=-\tfrac{1}{2}X_7f+a_{27}X_4$$

$$(X_3X_6)=a_{36}X_4f \qquad (X_3X_7)=X_6f+a_{37}X_4$$

$$(X_4X_6)=a_{46}X_4f \qquad (X_4X_7)=a_{47}X_4$$

$$(X_5X_6)=-X_6f+a_{56}X_4f \qquad (X_5X_7)=-X_7+a_{57}X_4$$

$$(X_6X_7)=a_{67}X_4$$

If we put these transformations three at a time in the operator, Jacobi's Identity, we obtain the following results:

$a_{16} = 2a_{27} = a_{57} = 2c_1$ $\qquad a_{17} = 0$

$a_{26} = \frac{1}{2} a_{37} = -\frac{1}{2} a_{56} = c_2$ $\qquad a_{27} = \frac{1}{2} a_{16} = \frac{1}{2} a_{57} = c_1$

$a_{36} = 0$ $\qquad a_{37} = 2a_{26} = a_{56} = 2c_2$

$a_{46} = 0$ $\qquad a_{47} = 0$

$a_{56} = -2a_{26} = -2c_2$ $\qquad a_{57} = 2a_{27} = 2c_1$

$$a_{67} = 0$$

This structure of the group is therefore (1) together with

$$(X_1 X_6) = -X_7 f + 2c_1 X_4 f; \quad (X_1 X_7) = 0$$

$$(X_2 X_6) = \frac{1}{2} X_6 f + c_2 X_4 f \qquad (X_2 X_7) = -\frac{1}{2} X_7 f + c_1 X_4 f$$

$$(X_3 X_6) = (X_4 X_6) = 0 \qquad (X_3 X_7) = X_6 f + 2c_2 X_4 f$$

$$(X_5 X_6) = -X_6 f\ 2c_2 X_4 f \qquad (X_4 X_7) = 0$$

$$(X_6 X_7) = 0 \qquad (X_5 X_7) = -X_7 f + 2c_1 X_4 f$$

Introduce $\bar{X}_6 f = X_6 f + 2c_2 X_4 f$ as a new transformation.

$$(X_1 \bar{X}_6) = -X_7 f + 2c_1 X_4 f$$

$$(X_2 \bar{X}_6) = \frac{1}{2} X_6 f + c_2 X_4 f = \frac{1}{2} \bar{X}_6$$

$$(X_3 \bar{X}_6) = (X_4 \bar{X}_6) = 0 \qquad (X_3 X_7) = \bar{X}_6 f$$

$$(X_5 \bar{X}_6) = -\bar{X}_6 f$$

$$(\bar{X}_6 X_7) = 0$$

now introduce $\bar{X}_7 f = X_7 f - 2c X_4$ as new transformation

$$(X_1 \bar{X}_6) = -\bar{X}_7 f \qquad (X_1 \bar{X}_7) = 0$$

$$(X_2 \bar{X}_7) = -\tfrac{1}{2} X_7 f + c X_4 f = -\tfrac{1}{2} \bar{X}_7 f$$

$$(X_3 \bar{X}_7) = \bar{X}_6 f$$

$$(X_4 \bar{X}_7) = 0$$

$$(X_5 \bar{X}_7) = -\bar{X}_7 f$$

$$(\bar{X}_6 \bar{X}_7) = 0$$

now drop the bars, and the final form of the structure of the $G_7$ is

$\boxed{IX}$.

$$(X_1 X_2) = X_1 f \qquad (X_1 X_4) = 0 \qquad (X_1 X_5) = 0$$

$$(X_1 X_3) = 2X_2 f \qquad (X_2 X_4) = 0 \qquad (X_2 X_5) = 0 \qquad (X_4 X_5) = 0$$

$$(X_2 X_3) = X_3 f \qquad (X_3 X_4) = 0 \qquad (X_3 X_5) = 0$$

$$(X_1 X_6) = -X_7 f \qquad (X_1 X_7) = 0$$

$$(X_2 X_6) = \tfrac{1}{2} X_6 f \qquad (X_2 X_7) = -\tfrac{1}{2} X_7 f$$

$$(X_3 X_6) = 0 \qquad (X_3 X_7) = X_6 f$$

$$(X_4 X_6) = 0 \qquad (X_4 X_7) = 0$$

$$(X_5 X_6) = -X_6 f \qquad (X_5 X_7) = -X_7 f$$

$$(X_6 X_7) = 0$$

Case IV. <u>The sub $G_3$ (A) is contained in a sub $G_6$ of the $G_7$.</u>

We subdivide this case into two parts

(a) The sub $G_6$ is not invariant in the $G_7$

(b) The sub $G_6$ is invariant in the $G_7$

a.) Suppose the sub$_6$ is not invariant in the $G_7$. Then the $G_6$ is transformed into $\infty'$ positions in the adjoined space. These $\infty'$ positions form manifoldness of one dimension, and can be depicted as the points on a line. The elements of this manifoldness are transformed by a gr. ($r = 1, 2, 3$) If $r = 1, 2$ at least one $G_6$ would be invariant and we could take that to be the one containing the sub $G_3$(A), otherwise $G_7$ would be integrable Therefore $r = 3$.

But if $r = 3$, the $G_7$ (III p 688) contains a sub $G_4$ which is invariant in the $G_7$ and also in the $G_6$. Hence we must look for the non integrable $G_6$'s which contain an invariant sub $G_4$ This sub $G_4$ must not contain the sub $G_3$(A) as the first derived group of the $G_4$ nor as an isolated invariant sub $G_3$ in the $G_4$, for then the $G_3$ would be invariant in the $G_7$.

Now consider the non integrable $G_6$'s III pp 743-744

[1]. $\boxed{p, xp, x^2p, q, yq, y^2q}$

contains no invariant sub $G_4$.

[2] $\boxed{p, xp, x^2p, q, r, yq + czr}$

contains 2 invariant sub $G_4$, but the first derived group is (A)

[3] $\boxed{p, xp, x^2p, q, r, yq + (y+z)r}$

contains 2 invariant sub $G_4$, " " " " " " "

[4] $\boxed{p, xp, x^2p, q, r, yr}$ " " "

contains 2 invariant sub $G_4$ " " " "

[5] $\boxed{p, xp, x^2p, r, yr, y^2r}$ " " "

contains 3 invariant sub $G_4$ " " " "

[6]. $\boxed{p, xp + yq, x^2p + 2xyq, q, xq, x^2q}$ ; [7]. $\boxed{xq, xp - yq, yp, xp + yq, p, q}$

[8]. $\boxed{xq, xp - yq, yp, p - yr, q + xr, r}$ ; [9]. $\boxed{xq, xp - yq, yp, p, q, r}$

contain no invariant sub $G_4$.

Hence there is no $G_7$ in which the sub $G_3$(A) is contained in a non invariant sub $G_6$.

b. The sub $G_6$ is invariant in the $G_7$.

The sub $G_6$ must not contain the sub $G_3(A)$ as one of its derived groups; nor must the sub $G_3(A)$ occur in the sub $G_6$ as an isolated invariant sub $G_3$; for in either case the $G_3(A)$ would be invariant in the $G_7$ which is contrary to hypothesis.

Consider again the types of non integrable $G_6$'s [1] - [9] above.

[1] contains $X_1 X_2 X_3$ as an isolated invariant sub $G_3$ of form (A)

[2], [3], [4], [5] contain $X_1 X_2 X_3$ as a derived group of form A.

Hence we need only consider [6], [7], [8], and [9].

α. The invariant sub $G_6$ has the same structure as $\boxed{p,\ xp+yq,\ x^2p+2xyq,\ q,\ xq,\ x^2q}$.

From this we know how $X_1 \dots X_6$ are connected:

$$\begin{cases} (X_1X_2)=X_1f;\ (X_1X_4)=0\ ;(X_2X_4)=-X_4f;\ (X_3X_4)=-2X_5f \\ (X_1,X_3)=2X_2f;\ (X_1X_5)=X_4f;\ (X_2X_5)=0\ ;\ (X_3X_5)=-X_6f \\ (X_2,X_3)=X_3f;\ (X_1X_6)=2X_5f;\ (X_2X_6)=X_6f;\ (X_3X_6)=0 \\ (X_4X_5)=(X_4X_6)=(X_5X_6)=0 \end{cases}$$

Since the sub $G_6(1)$ is invariant in the $G_7$ the alternants of $X_i$ $(i=1,2,3,4,5,6)$ with $X_7$ must give linear sums of $X_1 \dots X_6$. Hence:

$$(X_1, X_7) = \sum_1^6 {}^i\, a_{1i} X_i f \; ; \quad (X_2, X_7) = \sum_1^6 {}^i\, a_{2i} X_i f \; ;$$

$$(X_3, X_7) = \sum_1^6 {}^i\, a_{3i} X_i f \; ; \quad (X_4, X_7) = \sum_1^6 {}^i\, a_{4i} X_i f \; ;$$

$$(X_5, X_7) = \sum_1^6 {}^i\, a_{5i} X_i f \; ; \quad (X_6, X_7) = \sum_1^6 {}^i\, a_{6i} X_i f \; ;$$

If we put the transformations $X_1 \dots X_7$ 3 at a time into the operator, Jacobi's Identity, we obtain the following results

$a_{11} = a_{13} = a_{14} = a_{15} = a_{16} = 0 \qquad a_{12} = 2\, a_{23} = a_{45} = 2a_{56} = 2c_1$ say

$a_{21} = a_{22} = a_{25} = a_{26} = 0 \qquad a_{23} = \frac{1}{2} a_{12} = \text{etc} = c_1 \qquad a_{24} = \frac{1}{2} a_{35} = c_2$

$a_{31} = a_{32} = a_{33} = a_{34} = a_{36} = 0 \qquad a_{35} = 2a_{24} = 2c_2$

$a_{41} = a_{42} = a_{43} = a_{46} = 0 \qquad a_{44} = a_{55} = a_{66} = c_3, \quad a_{45} = a_{12} = 2c_1$

$a_{51} = a_{52} = a_{53} = a_{54} = 0 \qquad a_{55} = a_{44} = a_{66} = c_3, \quad a_{56} = \frac{1}{2} a_{12} = c_1$

$a_{61} = a_{62} = a_{63} = a_{64} = a_{65} = 0 \qquad a_{66} = a_{44} = a_{55} = c_3.$

This leaves the structure of the $G_7(1)$ together with

$$(X_1 X_7) = 2c_1 X_2 f \; ; (X_2 X_7) = c_1 X_3 + c_2 X_4 f, \; (X_3 X_7) = 2c_2 X_5 f$$

$$(X_4 X_7) = c_3 X_4 f + 2c_1 X_5 f \; ; (X_5 X_7) = c_3 X_5 f + c_1 X_6 f ; \; (X_6 X_7) = c_3 X_6 f$$

We see now that $X_4, X_5, X_6, X_7$ form a sub $g_4$ in which $X_4, X_5, X_6$ are in involution and

$$(X_4 X_7) = c_3 X_4 f + 2c_1 X_5 f \,;\, (X_5 X_7) = c_3 X_5 f + c_1 X_6 f \,;\, (X_6 X_7) = c_3 X_6 f$$

Comparing the structure of this $G_4$ with the structures of all $G_4$'s with 3 transformations in involution (III. p7) we find contradictions in all cases save 2

1. When $c_3 = 1$ and $c_1 = 0$

(2). When $c_3 = 0$ and $c_1 = 0$

1. Suppose $c_3 = 1$ and $c_1 = 0$

The $G_7$ then has the structure (L) together with

$$(X_1, X_7) = 0 \,;\, (X_2, X_7) = c_2 X_4 f \,;\, (X_3, X_7) = 2c_2 X_5 f$$

$$(X_4, X_7) = X_4 f \,;\, (X_5 X_7) = X_5 f \,;\, (X_6, X_7) = X_6 f$$

now introduce $\bar{X}_7 f = X_7 f + c_2 X_4 f$

$$(X_1\, \bar{X}_7) = 0 \quad (X_2, \bar{X}_7) = c_2 X_4 f - c_2 X_4 f = 0 \quad (X_3, \bar{X}_7) = 2c_2 X_5 f - 2c_2 X_5 f = 0$$

$$(X_4, X_7) = X_4 f \,;\, (X_5, X_7) = X_5 f \,;\, (X_6, X_7) = X_6 f$$

Therefore the structure of the group free of constants is:

XXI. $(X_1 X_2) = X_1 f$; $(X_1 X_4) = 0$; $(X_2 X_4) = -X_4 f$; $(X_3, X_4) = -2X_5 f$

$(X_1 X_3) = 2X_2 f$; $(X_1 X_5) = X_4 f$; $(X_2 X_5) = 0$; $(X_3, X_5) = -X_6 f$

$(X_2 X_3) = X_3 f$; $(X_1 X_6) = 2X_5 f$; $(X_2 X_6) = X_6 f$; $(X_3, X_6) = 0$

$(X_1 X_7) = (X_2 X_7) = (X_3 X_7) = (X_4 X_5) = (X_4, X_6) = (X_5, X_6) = 0$

$(X_4, X_7) = X_4 f$; $(X_5, X_7) = X_5 f$; $(X_6, X_7) = X_6 f$

(2) Suppose $c_3 = 0$ and $c_1 = 0$

The $G_7$ has the structure (1) together with $(X_1, X_7) = 0$; $(X_2, X_7) = c_2 X_4 f$; $(X_3, X_7) = 2c_2 X_5 f$

$(X_4, X_7) = (X_5, X_7) = (X_6, X_7) = 0.$

As before introduce $\overline{X}_7 f = X_7 f + c_2 X_4 f$

$(X_1 \overline{X}_7) = (X_2, \overline{X}_7) = (X_3, \overline{X}_7) = (X_4 \overline{X}_7) = (X_5 \overline{X}_7) = (X_6 \overline{X}_7) = 0$

The structure of the $G_7$ is therefore:

XXII. $(X_1, X_2) = X_1 f$; $(X_1, X_4) = 0$; $(X_2, X_4) = -X_4 f$; $(X_3, X_4) = -2X_5 f$

$(X_1, X_3) = 2X_2 f$; $(X_1, X_5) = X_4 f$; $(X_2, X_5) = 0$; $(X_3, X_5) = -X_6 f$

$(X_2, X_3) = X_3 f$; $(X_1, X_6) = 2X_5 f$; $(X_2, X_6) = X_6 f$; $(X_3, X_6) = 0$

$(X_4 X_5) = (X_4 X_6) = (X_5, X_6) = 0$

$(X_1 X_7) = (X_2 X_7) = (X_3 X_7) = (X_4 X_7) = (X_5 X_7) = (X_6 X_7) = 0$

13. The Sub $G_6$ has the same structure as

$$\boxed{xq,\ xp-yq,\ yp,\ xp+yq,\ p,\ q}$$

The structure of the $G_7$ is therefore

$$\left\{\begin{array}{l}(X_1,X_2) = -2X_1f \\ (X_1,X_3) = X_2f \\ (X_2,X_3) = -2X_3f\end{array}\right\}$$ This is equivalent to A as can be shown by putting $\bar{X}_2f = -\frac{1}{2}X_2f$ and $\bar{X}_3f = -X_3f$.

$(X_1X_4) = 0$ ; $(X_2X_4) = 0$ ; $(X_3X_4) = 0$ ; $(X_4X_5) = -X_5f$

$(X_1X_5) = -X_6f$ ; $(X_2X_5) = -X_5f$ ; $(X_3,X_5) = 0$ ; $(X_4X_6) = -X_6f$

$(X_1X_6) = 0$ ; $(X_2X_6) = X_6f$ ; $(X_3,X_6) = -X_5f$ ; $(X_5X_6) = 0$

Since the $G_6$ is invariant in the $G_7$

$(X_1,X_7) = \sum_1^6 a_{1i}X_if$; $(X_2,X_7) = \sum_1^6 a_{2i}X_if$ ; $(X_3,X_7) = \sum_1^6 a_{3i}X_if$;

$(X_4,X_7) = \sum_1^6 a_{4i}X_if$; $(X_5,X_7) = \sum_1^6 a_{5i}X_if$ ; $(X_6,X_7) = \sum_1^6 a_{6i}X_if$.

Put these transformations, three at a time into the operator Jacobis Identity and we obtain the following results

| | | |
|---|---|---|
| $a_{11} = -a_{33} = c_1$ (say) | $a_{21} = -2a_{32} = 2a_{56} = 2c_4$ | $a_{37} = 0$ |
| $a_{12} = -\frac{1}{2}a_{23} = a_{65} = \cancel{2c_2}\ c_2$ | $a_{22} = 0$ | $a_{32} = -\frac{1}{2}a_{21} = -c_4$ |
| $a_{13} = 0$ | $a_{23} = -2a_{12} = -2c_2$ | $a_{33} = -a_{11} = a_{55} - a_{66} = -c_1$ |
| $a_{14} = 0$ | $a_{24} = 0$ | $a_{34} = 0$ |
| $a_{15} = 0$ | $a_{25} = a_{16} = c_3$ | $a_{35} = -a_{26} = -c_5$ |
| $a_{16} = a_{25} = a_{45} = c_3$ | $a_{26} = -a_{46} = -a_{35} = c_5$ | $a_{36} = 0$ |

$a_{41} = a_{42} = a_{43} = a_{44} = 0$ ; $a_{51} = a_{52} = a_{53} = a_{54} = 0$ ; $a_{61} = a_{62} = a_{63} = a_{64} = 0$

$a_{45} = a_{16} = c_3$ $\qquad a_{55} = a_{66} + a_{33} = c_6 - c_1 \qquad a_{65} = a_{12} = c_2$

$a_{46} = -a_{26} = -c_5 \qquad a_{56} = \frac{1}{2} a_{21} = c_4 \qquad a_{66} = a_{55} - a_{33} = c_6$

This leaves the structure of the group (1) together with

$$(X_1 X_7) = c_1 X_1 f + c_2 X_2 f + c_3 X_6 f$$
$$(X_2 X_7) = 2c_4 X_1 f - 2c_2 X_3 f + c_3 X_5 f + c_5 X_6 f$$
$$(X_3 X_7) = -c_4 X_2 f - c_1 X_3 f - c_5 X_5 f$$
$$(X_4 X_7) = c_3 X_5 f - c_5 X_6 f$$
$$(X_5 X_7) = (c_6 - c_1) X_5 f + c_4 X_6 f$$
$$(X_6 X_7) = c_2 X_5 f + c_6 X_6 f.$$

This show that $X_1 f, X_2 f, X_3 f, X_5 f, X_6 f, X_7 f$ form a sub $G_6$ of the $G_7$. It is non integrable and $X_1 f, X_2 f, X_3 f, X_5 f, X_6 f$ are connected as they are in the original sub $G_6$, therefore $X_7 f$ must be connected with $X_1 f, X_2 f, X_3 f, X_5 f, X_6 f$ like $X_4 f$ is connected with them in the original sub $G_6$ since that is the only structure form of a $G_6$ with five transformations connected as $X_1, X_2, X_3, X_5, X_6$. For this to be the case we must have

$$c_1 = c_2 = c_3 = c_4 = c_5 = 0$$

$$c_6 = +1$$

This leaves the structure of the $G_7$

XXIII $(X_1 X_2) = -2X_1 f$ ; $(X_1 X_4) = 0$ ; $(X_2 X_4) = 0$ ; $(X_3 X_4) = 0$

$(X_1 X_3) = X_2 f$ ; $(X_1 X_5) = -X_6 f$ ; $(X_2 X_5) = -X_5 f$ ; $(X_3 X_5) = 0$

$(X_2 X_3) = -2X_3 f$ ; $(X_1 X_6) = 0$ ; $(X_2 X_6) = X_6 f$ ; $(X_3 X_6) = X_5 f$

$(X_4 X_5) = -X_5 f$ ; $(X_4 X_6) = -X_6 f$ ; $(X_5 X_6) = 0$

$(X_1 X_7) = 0$ $(X_2 X_7) = 0$ $(X_3 X_7) = 0$

$(X_4 X_7) = 0$ $(X_5 X_7) = +X_5 f$ $(X_6 X_7) = +X_6 f$

(V.) Suppose the sub $g_6$ has the same structure as $\boxed{xq,\ xp - yq,\ yp,\ p - yr,\ q + xr,\ r}$.

From this we know how $X_1 \dots X_6$ are connected

$$\text{g.1}\begin{cases} (X_1 X_2) = -2X_1 f;\ (X_1 X_4) = -X_5 f;\ (X_2 X_4) = -X_4 f;\ (X_3 X_4) = 0 \\ (X_1 X_3) = X_2 f;\ (X_1 X_5) = 0;\ (X_2 X_5) = X_5 f;\ (X_3 X_5) = -X_4 f \\ (X_2 X_3) = -2X_3 f;\ (X_1 X_6) = 0;\ (X_2 X_6) = 0;\ (X_3 X_6) = 0 \\ (X_4 X_5) = 2X_6;\ (X_4 X_6) = (X_5 X_6) = 0 \end{cases}$$

Since the $G_6$ is invariant in the $G_7$

$$(X_1 X_7) = \sum_1^6 a_{1i} X_i f \; ; \quad (X_2 X_7) = \sum_1^6 a_{2i} X_i f$$

$$(X_3 X_7) = \sum_1^6 a_{3i} X_i f \; ; \quad (X_4 X_7) = \sum_1^6 a_{4i} X_i f$$

$$(X_5 X_7) = \sum_1^6 a_{5i} X_i f \; ; \quad (X_6 X_7) = \sum_1^6 a_{6i} X_i f$$

If we put these seven transformations three at a time in the operator Jacobis Identity we obtain the following results.

$a_{11} = -a_{33} = a_{55} - a_{44} = c_1$ (say) $\qquad a_{21} = -2a_{32} = 2a_{45} = -2c_3$ ; $a_{32} = -a_{45} = -\frac{1}{2} a_{21} = c_3$

$a_{12} = a_{54} = c_2$ (say) $\qquad a_{23} = -2a_{12} = -2c_2$ ; $a_{33} = -a_{11} = -c_1$

$a_{13} = a_{14} = a_{15} = a_{16} = 0 \qquad a_{22} = a_{24} = a_{25} = a_{26} = 0$ ; $a_{31} = a_{34} = a_{35} = a_{36} = 0$

$a_{41} = a_{42} = a_{43} = a_{46} = 0 \qquad a_{51} = a_{52} = a_{53} = a_{56} = 0$ ; $a_{61} = a_{62} = a_{63} = a_{64} = a_{65} = 0$

$a_{44} = a_{55} - a_{11} = c_4 \qquad a_{54} = -\frac{1}{2} a_{23} = a_{12} = c_2$ ; $a_{66} = a_{55} + a_{44} = c_1 + 2c_4$

$a_{45} = \frac{1}{2} a_{21} = -c_3 \qquad a_{55} = a_{44} + a_{11} = c_4 + c_1$ ;

This leaves the structure of the $G_7$ (1) together with

$(X_1 X_7) = c_1 X_1 f + c_2 X_2 f$ ; $\qquad (X_2 X_7) = -2c_3 X_1 f - 2c_2 X_3 f$ ; $(X_3 X_7) = -c_3 X_2 f - c_1 X_3 f$

$(X_4 X_7) = c_4 X_4 f - c_3 X_5 f \qquad (X_5 X_7) = c_2 X_4 f + (c_1 + c_4) X_5 f$ ; $(X_6 X_7) = (c_1 + 2c_4) X_6 f$

From this we see that $X_1 f, X_2 f, X_3 f, X_7 f$.

form a sub $G_4$ and since it contains $X_1f$, $X_2f$ & $X_3f$ it must have a non integrable structure. But there is only one normal form for a non integrable structure $[\quad]$ $G_4$. Hence $(X_1 X_7) = 0, (X_2 X_7) = 0, (X_3 X_7) = 0$ for this to be true we must have

$$c_1 = c_2 = c_3 = 0.$$

This leaves

$$(X_1 X_7) = 0; (X_2 X_7) = 0; (X_3 X_7) = 0;$$

$$(X_4 X_7) = c_4 X_4 f; \; (X_5 X_7) = c_4 X_5 f; (X_6 X_7) = 2c_4 X_6 f$$

We now make two cases according as (1) $c_4 \neq 0$, (2) $c_4 = 0$

1. Suppose $c_4 \neq 0$

Introduce $\bar{X}_7 f = \frac{1}{c_4} X_7 f$

$$(X_1 \bar{X}_7) = 0; (X_2 \bar{X}_7) = 0 \quad (X_3 \bar{X}_7) = 0$$

$$(X_4 \bar{X}_7) = X_4 f; (X_5 \bar{X}_7) = X_5 f; (X_6 \bar{X}_7) = 2X_6 f$$

2. Suppose $c_4 = 0$

$$(X_1 X_7) = (X_2 X_7) = (X_3 X_7) = (X_4 X_7) = (X_5 X_7) = (X_6 X_7) = 0$$

This, then, gives of the structure of the group free of constants.

XXIV. $(X_1 X_2) = -2X_1 f$; $(X_1 X_4) = -X_5 f$; $(X_2 X_4) = -X_4 f$ $(X_3 X_4) = 0$

$(X_1 X_3) = X_2 f$ ; $(X_1 X_5) = 0$ ; $(X_2 X_5) = X_5 f$ $(X_3 X_5) = -X_4 f$

$(X_2 X_3) = -2X_3 f$; $(X_1 X_6) = 0$ ; $(X_2 X_6) = 0$ $(X_3 X_6) = 0$

$(X_4 X_5) = 2X_6 f$; $(X_4 X_6) = 0$; $(X_5 X_6) = 0$

$(X_1 X_7) = (X_2 X_7) = (X_3 X_7) = 0$

(1) $(X_4 X_7) = X_4 f$; $(X_5 X_7) = X_5 f$; $(X_6 X_7) = 2X_6 f$

(2) $(X_4 X_7) = (X_5 X_7) = (X_6 X_7) = 0$

(S.) Suppose the sub $G_6$ has the structure

$\boxed{xq,\ xp - yq,\ yp,\ p,\ q,\ r}$

From this we know how $X_1 \ldots X_6$ are connected

$$1\left\{\begin{array}{llll} (X_1 X_2) = -2X_1 f & (X_1 X_4) = -X_5; & (X_2 X_4) = -X_4; & (X_3 X_4) = 0 \\ (X_1 X_3) = X_2 f & (X_1 X_5) = 0; & (X_2 X_5) = X_5; & (X_3 X_5) = -X_4 \\ (X_2 X_3) = -2X_3 f & (X_1 X_6) = 0; & (X_2 X_6) = 0; & (X_3 X_6) = 0 \\ (X_4 X_5) = (X_4 X_6) = (X_5 X_6) = 0 & & & \end{array}\right.$$

Since the $G_6$ is invariant in the $G_7$

$$(X_j X_7) = \sum_1^6 {}^i a_{ji} X_i f \qquad j = 1,2,3,4,5,6$$

If we put these seven transformations three at a time into Jacobis Identity we obtain the following results:

$a_{11} = a_{21} + 2a_{32} = 0$ ; $a_{23} = -2a_{12} = -2a_{54} = -2c_1$ ; $a_{34} = -a_{25} = -c_3$

$a_{12} = -\frac{1}{2}a_{23} = a_{54} = c_1$ ; $a_{24} = a_{15} = c_2$ ; $a_{31} = a_{32} = a_{33} = a_{35} = a_{36} = 0$

$a_{15} = a_{24} = c_2$ ; $a_{25} = -a_{34} = c_3$ ;

$a_{13} = a_{14} = a_{16} = 0$ ; $a_{21} = a_{22} = a_{26} = 0$ ;

$a_{44} = a_{55} = c_4$ $a_{54} = -\frac{1}{2}a_{23} = a_{12} = c_1$ $a_{66} = c_5$

$a_{4i} = 0 \quad i = 1,2,3,5,6$ $a_{55} = a_{44} = c_4$ $a_{6i} = 0 \quad i = 1,2,3,4,5.$

$a_{5i} = 0 \quad i = 1,2,3,6$

This leaves the structure of the $G_7$ - (1) together with

$(X_1 X_7) = c_1 X_2 f + c_2 X_5 f$ $\qquad (X_4 X_7) = c_4 X_4 f$

$(X_2 X_7) = -2c_1 X_3 f + 2c_2 X_4 f + c_3 X_5 f$ $\qquad (X_5 X_7) = c_1 X_4 f + c_4 X_5 f$

$(X_3 X_7) = -c_3 X_4 f$ $\qquad (X_6 X_7) = c_5 X_6 f$

This shows that $X_1 f, X_2 f, X_3 f, X_4 f, X_5 f, X_7 f$ form a sub $G_6$ of non integrable structure, in which $X_1 \ldots X_5$ are connected as they are in the original sub $G_6$, therefore $X_7f$ must be connected with $X_1f \ldots X_5f$ like $X_6f$ is connected with them since that is the only structure form of a $G_6$ with five transformations connected like $X_1f \ldots X_5f$. For this to be the case we must have

$$c_1 = c_2 = c_3 = c_4 = 0$$

We now make two cases (1) $c_5 \neq 0$ (2) $c_5 = 0$

(1) Suppose $c_5 \neq 0$.

Introduce $\bar{X}_7 = \frac{1}{c_5} X_7$

$$(X_1 \bar{X}_7) = (X_2 \bar{X}_7) = (X_3 \bar{X}_7) = (X_4 \bar{X}_7) = (X_5 \bar{X}_7) = 0$$

$$(X_6 \bar{X}_7) = X_6 f$$

(2) Suppose $c_5 = 0$

$$(X_1 X_7) = (X_2 X_7) = (X_3 X_7) = (X_4 X_7) = (X_5 X_7) = (X_6 X_7) = 0$$

This, then, gives the structure of the $G_7$ free of constants:

XXV. $(X_1 X_2) = -2X_1 f$ ; $(X_1 X_4) = -X_5 f$ ; $(X_2 X_4) = -X_4 f$ ; $(X_3 X_4) = 0$

$(X_1 X_3) = X_2 f$ ; $(X_1 X_5) = 0$ ; $(X_2 X_5) = X_5 f$ ; $(X_3 X_5) = -X_4 f$

$(X_2 X_3) = -2X_3 f$ ; $(X_1 X_6) = 0$ ; $(X_2 X_6) = 0$ ; $(X_3 X_6) = 0$.

$(X_4 X_5) = (X_4 X_6) = (X_5 X_6) = 0$

$(X_1 X_7) = (X_2 X_7) = (X_3 X_7) = (X_4 X_7) = (X_5 X_7) = 0$

(1) $(X_6 X_7) = X_6 f$

(2) $(X_6 X_7) = 0$

In conclusion we see that there are twenty-five normal forms for the structure of the $G_7$. These are here given together.

I. The sub $G_4$ is invariant in the $G_7$

$$(X_1 X_2) = X_1 f, \quad (X_1, X_3) = 2X_2 f, \quad (X_2 X_3) = X_3 f$$

$$(X_1, X_4) = (X_1, X_5) = (X_1, X_6) = (X_1, X_7) = 0$$

$$(X_2, X_4) = (X_2 X_5) = (X_2, X_6) = (X_2, X_7) = 0$$

$$(X_3, X_4) = (X_3 X_5) = (X_3 X_6) = (X_3 X_7) = 0$$

Common to all the structures under this head

I. $(X_4, X_5) = X_4 f$
$(X_4, X_6) = 2X_5 f$
$(X_5, X_6) = X_6 f$
$(X_4, X_7) = 0$
$(X_5 X_7) = 0$
$(X_6 X_7) = 0$

II. $(X_4, X_5) = 0$
$(X_4, X_6) = 0$
$(X_5, X_6) = X_4 f$
$(X_4, X_7) = c X_4 f$
$(X_5, X_7) = X_5 f$
$c \neq 1$ $(X_6 X_7) = (c-1) X_6 f$

III. $(X_4, X_5) = 0$
$(X_4, X_6) = 0$
$(X_5, X_6) = X_4 f$
$(X_4, X_7) = 2X_4 f$
$(X_5, X_7) = X_5 f$
$(X_6, X_7) = X_5 f + X_6 f$

IV $(X_4 X_5) = 0$
$(X_4, X_6) = 0$
$(X_5, X_6) = X_5 f$
$(X_4 X_7) = X_4 f$
$(X_5 X_7) = 0$
$(X_6 X_7) = 0$

V $(X_4, X_5) = 0$
$(X_4, X_6) = 0$
$(X_5, X_6) = X_4 f$
$(X_4, X_7) = X_4 f$
$(X_5, X_7) = X_5 f$
$(X_6 X_7) = 0$

VI $(X_i X_k) = 0$
$(X_4 X_7) = X_4 f$
$(X_5 X_7) = a X_5 f$
$(X_6 X_7) = c X_6 f$
$(i, k = 4, 5, 6)$

VII $(X_i X_k)=0$

$(X_4 X_7)= cX_4 f$

$(X_5 X_7)=(1+c) X_5 f$

$(X_6, X_7) = X_4 f + cX_6 f$

$i, k = 4,5,6$

VIII. $(X_i X_k)=0$

$(X_4 X_7) = X_5 f$

$(X_5 X_7)=0$

$(X_6 X_7)=X_4 f$

$i, k = 4,5,6$

IX $(X_i, X_k)=0$

$(X_4, X_7)= X_4 f+X_5 f$

$(X_5, X_7)= X_5 f$

$(X_6, X_7)= X_4 f+X_6 f$

$i, k = 4, 5, 6$

X. $(X_i, X_k)=0$

$(X_4, X_7)= 0$

$(X_5, X_7) = 0$

$(X_6, X_7) = X_5 f$

$i, k = 4, 5, 6$

XI. $X_i X_k = 0$

$(X_4, X_7) = X_4 f$

$(X_5, X_7)= X_5 f$

$(X_6, X_7)= X_5 f + X_6 f$

$i, k = 4, 5, 6$

XII. $(X_i X_k)=0$

$(X_4 X_7) = 0$

$(X_5 X_7)=0$

$(X_6, X_7) = 0$

$i, k = 4 5 6$

II. The sub $G_3(A)$ is not invariant in the $G_7$ i The non-invariant sub $G_3(A)$ is not contained in any ~~greater~~ sub group of the $G_7$

XIII. The $G_7$ has the same structure as

$$\boxed{p_k,\ x_1 p_1 - x_2 p_2 + 3(x_4 p_4 - x_3 p_3),\ 2x_2 p_1 + x_3 p_2 + 3x_1 p_4,\ x_4 p_1 + 2x_1 p_2 + 3x_2 p_3 \quad k=1,2,3,4}$$

Here $x_1, x_2, x_3, x_4$ are the independent variables in $R_4$

$p_i \equiv \frac{\partial f}{\partial x_i}$

(ii) The non invariant Sub $G_3(A)$ is contained in a sub $G_4$ which is not contained in any larger sub group of the $G_7$

(a) The sub $G_4$ contains no subgroup which is invariant in the $G_7$

XIV The $G_7$ has the same structure as

$$\boxed{p;\ q;\ r;\ yp - xq;\ zq - yr;\ xr - zp;\ xp + yq + zr}$$

(b) The sub $G_4$ contains a sub $G_1$, $X_4 f$, which is invariant in the $G_7$.

XV. $(X_1\, X_2) = X_1 f$; $(X_1\, X_3) = 2X_2 f$; $(X_2 X_3) = X_3 f$

$(X_1 X_4) = (X_2 X_4) = (X_3 X_4) = 0 \quad (X_1, X_5) = (X_2\, X_6) = (X_3\, X_7) = 0$

$(X_i, X_k) = 0 \qquad i\ k = 4, 5, 6, 7$

$(X_1, X_6) = 2X_5 f$; $(X_2, X_5) = -X_5 f$; $(X_3, X_5) = -X_6 f$

$(X_1, X_7) = X_6 f$; $(X_2, X_7) = X_7 f$; $(X_3, X_6) = -2X_7 f$

(iii.) The non-invariant sub $G_3(A)$ is contained in a sub $G_5$ of the $G_7$ which is not contained in a sub $G_6$ of the $G_7$.

(a). The sub $G_5$ has the structure $\boxed{p, q, xq, xp-yq, yp}$

VI. $(X_1 X_2) = X_1 f \quad (X_1 X_4) = -X_5 f \quad (X_1 X_5) = 0$

$(X_1 X_3) = 2X_2 f \quad (X_2 X_4) = \frac{1}{2} X_4 f \quad (X_2 X_5) = -\frac{1}{2} X_5 f \quad (X_4 X_5) = 0$

$(X_2 X_3) = X_3 f \quad (X_3 X_4) = 0 \quad (X_3 X_5) = X_4 f$

$(X_1 X_6) = -X_7 f\,; \quad (X_2 X_6) = \frac{1}{2} X_6 f\,; \quad (X_3 X_6) = 0\,;$

$(X_1 X_7) = 0\,; \quad (X_2 X_7) = -\frac{1}{2} X_7 f\,; \quad (X_3 X_7) = X_6 f:$

$(X_i, X_k) = 0 \qquad i = 4, 5, 6 \quad k = 6, 7.$

(b) The $G_5$ has the structure $\boxed{p, xp, x^2p, q, yq}$

(α) The sub $G_2$ $\boxed{q, yq}$, which is invariant in the sub $G_5$ is also invariant in the $G_7$

VII. $(X_1, X_2) = X_1 f\,; (X_1, X_3) = 2X_2 f\,; (X_2, X_3) = X_3 f$

* $(X_1, X_4) = (X_2, X_4) = (X_3, X_4) = (X_1 X_5) = (X_2 X_5) = (X_3 X_5) = 0 \quad (X_4, X_5) = X_4 f.$

look up.

$(X_1, X_6) = -X_7 f\,; \quad (X_2, X_6) = \frac{1}{2} X_6 f\,; \quad (X_3, X_6) = 0$

$(X_1 X_7) = 0\,; \quad (X_2, X_7) = -\frac{1}{2} X_7 f\,; \quad (X_3, X_7) = X_6 f$

$(X_4, X_6) = (X_4, X_7) = (X_5, X_6) = (X_5 X_7) = (X_6, X_7) = 0$

(β.) The sub group $\boxed{q}$ which is invariant in the sub $G_5$ is also invariant in the $G_7$.

$\underline{XVIII}$. $(X_1 X_2) = X_1 f$, $(X_1 X_3) = 2X_2 f$, $(X_2 X_3) = X_3 f$

$(X_1 X_4) = (X_2 X_4) = (X_3 X_4) = (X_1 X_5) = (X_2 X_5) = (X_3 X_5) = 0$ $(X_4 X_5) = X_5 f$

$(X_1, X_6) = -X_7 f$; $(X_2, X_6) = \frac{1}{2} X_6 f$ $(X_3, X_6) = (X_4 X_6) = 0$; $(X_5 X_6) = -X_6 f$

$(X_1, X_7) = 0$ ; $(X_2, X_7) = -\frac{1}{2} X_7 f$; $(X_3 X_7) = (X_4, X_7) = 0$; $(X_5, X_7) = -X_7 f$

$(X_6 X_7) = 0$

C. The sub $G_5$ has the structure of $\boxed{p, xp, x^2p, q, r}$

(α). $\boxed{q, r}$ is invariant in the sub $G_5$ and in the $G_7$.

$\underline{XIX}$. $(X_1, X_2) = X_1 f$ ; $(X_1, X_3) = 2X_2 f$ ; $(X_2, X_3) = X_3 f$ ;

$(X_1, X_4) = (X_2, X_4) = (X_3, X_4) = (X_1, X_5) = (X_2, X_5) = (X_3, X_5) = (X_4, X_5) = 0$

$(X_1 X_6) = -X_7 f$ ; $(X_2, X_6) = \frac{1}{2} X_6 f$ ; $(X_3, X_6) = 0$

$(X_1, X_7) = 0$ ; $(X_2 X_7) = -\frac{1}{2} X_7 f$ ; $(X_3, X_7) = X_6 f$

$(X_4, X_6) = (X_5, X_6) = (X_4, X_7) = (X_5, X_7) = 0$

(1) $(X_6, X_7) = X_4 f + X_5 f$ (2) $(X_6 X_7) = X_4 f$ (3) $(X_6 X_7) = 0$

(($\beta$) The sub G. $\boxed{9}$ is invariant in the Sub $G_5$ and the $G_7$.

$\boxed{\text{XX}}$. $(X_1, X_2) = X_1 f$; $(X_1, X_3) = 2X_2 f$; $(X_2, X_3) = X_3 f$

$(X_i, X_k) = 0 \qquad i = 1,2,3,4,5 \qquad k = 4,5$

$(X_1, X_6) = -X_7 f \qquad (X_2, X_6) = \frac{1}{2} X_6 f$; $(X_3, X_6) = 0$; $(X_4, X_6) = 0$; $(X_5, X_6) = -X_6 f$

$(X_1, X_7) = 0 \qquad (X_2, X_7) = -\frac{1}{2} X_7 f$; $(X_3, X_7) = X_6 f$; $(X_4, X_7) = 0$; $(X_5, X_7) = -X_7 f$

$(X_6, X_7) = 0$

iv The sub $G_3(A)$ is contained in a Sub $G_6$ of the $G_7$.

(a) The sub $G_6$ is not invariant in the $G_7$

No new case

(b) The sub $G_6$ is invariant in the $G_7$

($\alpha$.) The invariant sub $G_6$ has the structure

$$\boxed{p,\ xp + yq,\ x^2 p + 2xyq,\ q,\ xq,\ x^2 q}$$

$\boxed{\text{XXI}}$. $(X_1\ X_2) = X_1 f$; $(X_1 X_3) = 2X_2 f$; $(X_2\ X_3) = X_3 f$.

$(X_1 X_4) = 0$; $(X_1, X_5) = X_4 f$; $(X_1, X_6) = 2X_5 f$; $(X_2\ X_4) = -X_4 f$; $(X_2, X_5) = 0$; $(X_2\ X_6) = X_6 f$

$(X_3\ X_4) = -2X_5 f$; $(X_3, X_5) = -X_6 f$; $(X_3\ X_6) = 0$

$(X_1, X_7) = (X_2, X_7) = (X_3, X_7) = (X_4, X_5) = (X_4, X_6) = (X_5, X_6) = 0$

$\overline{XXII}$. Same as $\overline{XXI}$ except

$$(X_4, X_7) = (X_5, X_7) = (X_6, X_7) = 0$$

(β. The sub $G_6$ has the structure of

$$\boxed{xq,\ xp-yq,\ yp,\ xp+yq,\ p,\ q}.$$

$\overline{XIII}$. $(X_1 X_2) = -2X_1 f$ ; $(X_1 X_3) = X_2 f$ ; $(X_2, X_3) = -2X_3 f$

$(X_1 X_4) = (X_2 X_4) = (X_3 X_4) = (X_1 X_7) = (X_2 X_7) = (X_3 X_7) = 0$

$(X_1 X_5) = -X_6 f$ ; $(X_2 X_5) = -X_5 f$ ; $(X_3 X_5) = 0$

$(X_1 X_6) = 0$ ; $(X_2 X_6) = X_6 f$ ; $(X_3 X_6) = -X_5 f$

$(X_4 X_5) = -X_5 f$ $(X_4, X_6) = -X_6 f$ ; $(X_5, X_6) = 0$

$(X_4 X_7) = 0$ $(X_5, X_7) = X_5 f$ ; $(X_6, X_7) = X_6 f$

(γ) The sub $G_6$ has the structure of

$$\boxed{xq,\ xp-yq,\ yp,\ p-yr,\ q+xr,\ r}$$

$\overline{XIV}$. $(X_1, X_2) = -2X_1 f$ ; $(X_1, X_3) = X_2 f$ ; $(X_2, X_3) = -2X_3 f$

$(X_1, X_4) = -X_5 f$ ; $(X_1, X_5) = 0$ ; $(X_1, X_6) = 0$ ; $(X_2, X_4) = -X_4 f$ ; $(X_2, X_5) = X_5 f$ ; $(X_2, X_6) = 0$

$(X_3, X_4) = 0$ ; $(X_3, X_5) = -X_4 f$ ; $(X_3, X_6) = 0$ ; $(X_4, X_5) = 2X_6 f$ ; $(X_4, X_6) = (X_5, X_6) = 0$

$(X_1, X_7) = (X_2, X_7) = (X_3, X_7) = 0$

(1) $(X_4, X_7) = X_4 f$ ; $(X_5, X_7) = X_5 f$ ; $(X_6, X_7) = 2X_6 f$

(2) $(X_4, X_7) = (X_5, X_7) = (X_6, X_7) = \ldots$

($\delta$) The sub $g_6$ has the structure of

$\boxed{xq,\ xp-yq,\ yp,\ p,\ q,\ r}$.

<u>XXV</u>. $(X_1,X_2)=-2X_1f$ ; $(X_1,X_3)=X_2f$; $(X_2,X_3)=-2X_3f$.

$(X_1,X_4)=-X_5f$; $(X_1,X_5)=0$; $(X_1,X_6)=0$; $(X_2,X_4)=-X_4f$; $(X_2,X_5)=X_5f$; $(X_2X_6)=0$

$(X_3,X_4)=0$; $(X_3,X_5)=-X_4f$; $(X_3,X_6)=0$

$(X_4,X_5)=(X_4,X_6)=(X_5,X_6)=0$

$(X_i,X_7)=0 \qquad i=1,2,3,4,5$

(1) $(X_6,X_7)=X_6f$

(2) $(X_6,X_7)=0$