On Rank Gradient and p-Gradient of Finitely Generated Groups

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Abstract

Rank gradient and p-gradient are group invariants that assign some real number greater than or equal to -1 to a finitely generated group. Though the invariants originated in the study of topology (3-manifold groups), there is growing interest among group theorists. For most classes of groups for which rank gradient and p-gradient have been computed, the value is zero. The research presented consists of two main parts. First, for any prime number p and any positive real number α, we construct a finitely generated group Γ with p-gradient equal to α. This construction is used to show that there exist uncountably many pairwise non-commensurable groups that are finitely generated, infinite, torsion, non-amenable, and residually-p. Second, rank gradient and p-gradient are calculated for free products, free products with amalgamation over an amenable subgroup, and HNN extensions with an amenable associated subgroup using various methods. The notion of cost of a group is used to obtain lower bounds for the rank gradient of amalgamated free products and HNN extensions. For p-gradient, the Kurosh subgroup theorems for amalgamated free products and HNN extensions are used.

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Chapter 1

Introduction

The research covered consists of two independent sets of results. The first is showing that any non-negative real number is the p-gradient of some finitely generated group. The second is showing how to compute rank gradient and p-gradient of free products, free products with amalgamation over an amenable subgroup, and HNN extensions with an amenable associated subgroup. We begin by giving some background on rank gradient and p-gradient followed by a summary of the new results.

1.1 Definitions and Overview

Let Γ be a finitely generated group and let $d(\Gamma)$ denote the minimal number of generators of Γ. In combinatorial group theory one often wants to know, or at least bound, $d(\Gamma)$. For most groups this is a hard question to answer and few tools are presently known to help answer this question. However, for a finite index subgroup H of Γ an upper bound for $d(H)$ is known. The Schreier index formula states that if H is a finite index subgroup of a finitely generated group Γ, then $d(H)-1 \leq (d(\Gamma)-1)[\Gamma : H]$ and if Γ is free of finite rank, then H is free and $d(H)-1 = (d(\Gamma)-1)[\Gamma : H]$. The Schreier index formula can be proved using the Reidemeister-Schreier method: Given the presentation $\Gamma = \langle X \mid R \rangle$ one uses a special set of coset representatives of H in Γ called a *Schreier transversal* to construct a presentation for H. Let T be a Schreier transversal for H in Γ, then H is generated by $Y = \{tx(\overline{tx})^{-1} \neq 1 \mid t \in T, x \in X\}$, where $\overline{tx}$ is the element in T representing the coset containing tx. The cardinality of Y is $|Y| = [\Gamma : H](|X| - 1|) + 1$.

The absolute rank gradient of a finitely generated group is, in a sense, a measure of how far the Schreier index formula is from being an equality rather than an inequality. If the absolute rank gradient of Γ is significantly smaller than $d(\Gamma)-1$ then there must exist some finite index subgroup H of Γ that has many fewer generators than provided by the Reidemeister-Schreier method. Said differently, absolute rank gradient can help answer the question: How optimal is the presentation given by the Reidemeister-Schreier method?

The *absolute rank gradient* of Γ is defined by

$$RG(\Gamma) = \inf_{H} \frac{d(H) - 1}{[\Gamma : H]}$$

where the infimum is taken over all finite index subgroups H of Γ.

By the definition of rank gradient it is clear that $-1 \leq RG(\Gamma) \leq d(\Gamma) - 1$. Rank gradient can easily be calculated for the class of free groups. If F_n is a non-abelian free group on n generators, then $RG(F_n) = n - 1$. This follows by the Schreier index formula for free groups discussed above. There is a variation of rank gradient called p-gradient, where p is a prime number. The definition of p-gradient is given below, but first some history on rank gradient and p-gradient is given.

In 2004, Mark Lackenby first introduced rank gradient [20] and p-gradient [21] of finitely generated groups as means to study 3-manifold groups. Lackenby was attempting to form a program to study the Virtually Haken Conjecture, which was a major open problem in 3-manifold theory. The Virtually Haken Conjecture asserts that a compact orientable irreducible 3-manifold with infinite fundamental group is virtually Haken. There is a stronger conjecture than the Virtually Haken Conjecture, called the Largeness Conjecture, that is essentially group theoretic. The Largeness Conjecture asserts that the fundamental group of a compact orientable hyperbolic 3-manifold is *large*. A group Γ is said to be *large* if it contains a subgroup of finite index that maps onto a non-abelian free group. There is a related notion of a group being *p-large* for a prime number p if it contains a normal subgroup of p-power index that maps onto a non-abelian free group. It is clear that if a group is p-large then it is large. Lackenby [22] proved that if Γ is finitely presented and has positive p-gradient for some prime p, then Γ is p-large. In this way, one can see how p-gradient and rank gradient were used by Lackenby to try and solve the Virtually Haken Conjecture through the Largeness Conjecture.

Lackenby was ultimately unsuccessful in solving the Virtually Haken Conjecture (Ian Agol [3] recently solved the conjecture in 2012 using geometric arguments); however, Lackenby used rank gradient and p-gradient to get some nice group theoretic results, some of which have yet to be improved upon using other methods. For example, he related rank gradient and property (τ). Lackenby's work has led to rank gradient and p-gradient gaining interest among group theorists. Miklos Abert, Andrei Jaikin-Zapirain, and Nikolay Nikolov [1] were among the first to study rank gradient in the context of group theory. One of main results of [1] is that finitely generated infinite discrete amenable groups have rank gradient equal to zero with respect to any normal chain with trivial intersection. Lackenby [20] first proved the result for finitely presented groups. As a simple corollary, there is a corresponding result concerning p-gradient. Namely, if Γ has positive p-gradient for some prime p, then Γ is not amenable. The fact that rank gradient and p-gradient are zero for amenable groups will play an important role in some of the new results presented in this dissertation.

Denis Osin [32] and Jan-Christoph Schlage-Puchta [36] constructed residually finite torsion groups with positive rank gradient. An immediate consequence is that these groups are infinite and non-amenable. Both constructions are among the simplest discovered for groups that are infinite, residually finite, non-amenable, and torsion, which shows the effectiveness of rank gradient as a tool in group theory. Many of the arguments used to prove arbitrary p-gradient values (to be discussed later) are similar to those used by Osin and Schlage-Puchta.

Abert and Nikolov also showed that rank gradient has connections with an invariant called *cost*, which is used in the area of analysis called *orbit equivalence theory*. Damien Gaboriau [11] proved a connection between *cost* and another invariant, called L^2-*Betti numbers*, which primarily arise in topology. In particular, if Γ is a finitely generated residually finite group, then

$$RG(\Gamma) \geq \operatorname{cost}(\Gamma) - 1 \geq \beta_1^{(2)}(\Gamma) - \frac{1}{|\Gamma|}.$$

Abert and Nikolov [2] proved the first part of the inequality and the second part was proved by Gaboriau [11]. The relationship between rank gradient and cost is not limited to the above inequality. Abert and Nikolov [2] related two open problems about cost to rank gradient as well: the fixed price problem and the multiplicativity of cost-1 problem. This inequality will be discussed in more detail in Section 3.1.

Rank gradient is often difficult to work with and to calculate. It is often more convenient to compute the rank gradient of the pro-p completion, $\Gamma_{\widehat{p}}$, of the group Γ for some fixed prime p. Profinite and pro-p groups are defined and discussed in Section 3.3. When dealing with profinite groups the notion of topologically finitely generated is used instead of (abstractly) finitely generated. The p-gradient of the group Γ, denoted $RG_p(\Gamma)$, can be defined as the rank gradient of $\Gamma_{\widehat{p}}$. The notion of p-gradient of a group for a prime number p is also referred to in the literature as *mod-p rank gradient* or *mod-p homology gradient*. The reader should be careful as some authors define p-gradient differently [23]. The fact that $RG_p(\Gamma) = RG(\Gamma_{\widehat{p}})$ is proved in Section 3.3. A more explicit definition of p-gradient is provided below:

Let p be a prime. The *absolute p-gradient* of Γ is defined by

$$RG_p(\Gamma) = \inf_{\substack{H \trianglelefteq \Gamma \\ [\Gamma:H]=p^k}} \frac{d_p(H) - 1}{[\Gamma : H]}$$

where $d_p(H) = d\left(H/[H,H]H^p\right)$ and the infimum is taken over all normal subgroups of p-power index in Γ.

One can also define rank gradient and p-gradient relative to a lattice of subgroups. A set of subgroups $\{H_n\}$ of Γ is called a *lattice* if it is closed under finite intersections. In particular any descending chain of subgroups is a lattice. *Rank gradient* (resp. *p-gradient*) *relative to a lattice* $\{H_n\}$ of finite index (resp. p-power index) subgroups is denoted $RG(\Gamma, \{H_n\})$ (resp. $RG_p(\Gamma, \{H_n\})$). Usually it is assumed that the lattice is a strictly descending chain of finite index normal subgroups with trivial intersection.

To prove results about rank gradient (analogously p-gradient) with respect to a lattice $\{H_n\}$ of normal subgroups of finite index in Γ, it is enough to prove the result for a descending chain of subgroups from the lattice. The argument for this is shown in Lemma 3.1.3. Specifically, one can use the chain: $H_1 \geq H_1 \cap H_2 \geq H_1 \cap H_2 \cap H_3 \geq \ldots$ One of the fundamental open questions in the theory of rank gradient and p-gradient is whether rank gradient or p-gradient depends on the chain if the chain is a descending chain of normal subgroups with trivial intersection. If rank gradient and p-gradient do not depend on the choice of the chain, then the theory will be greatly simplified as one will not need to distinguish between absolute rank gradient (resp. absolute p-gradient) and rank gradient (resp. p-gradient) with respect to a given chain.

Since rank gradient and p-gradient are difficult to compute in general, there are not many classes of groups for which these invariants have been computed. For the majority of classes of groups where rank gradient has been calculated the rank gradient is zero. This research adds to the few computations that exist for rank gradient and p-gradient by showing that any non-negative real number is the p-gradient of some finitely generated group and by giving formulas for rank gradient and p-gradient of free products with amalgamation over an amenable subgroup and HNN extensions with an amenable associated subgroup.

1.2 Summary of New Results

Since Lackenby first defined rank gradient of a finitely generated group [20], the following conjecture has remained open:

Conjecture. *For every real number $\alpha > 0$ there exists a finitely generated group Γ such that $RG(\Gamma) = \alpha$.*

Although this question is still open, we were able to answer the analogous question for p-gradient. A group is called *residually-p* if the intersection of all normal subgroups of p-power index is trivial.

Theorem 1.2.1 (Main Result). *For every real number $\alpha > 0$ and any prime p, there exists a finitely generated residually-p group Γ (which can be made torsion) such that $RG_p(\Gamma) = \alpha$.*

Section 4.3 contains the complete proof, but the following is an outline: Given a prime p and an $\alpha > 0 \in \mathbb{R}$, consider a free group F of finite rank greater than $\alpha + 1$. Let Λ be the set of all residually-p groups that are homomorphic images of F that have p-gradient greater than or equal to α. Partially order this set by $\Gamma_1 \succcurlyeq \Gamma_2$ if Γ_1 surjects onto Γ_2. To prove that every chain has a minimal element, we use direct limits of groups. The following lemma was inspired by Pichot's similar result for L^2-Betti numbers [33].

Lemma 1.2.2. *Let $\Gamma_\infty = \varinjlim \Gamma_i$ be a direct limit of finitely generated groups and let p be a prime. Then, $\limsup RG_p(\Gamma_i) \leq RG_p(\Gamma_\infty)$.*

By a Zorn's Lemma argument the set Λ has a minimal element, call it Γ. We show $RG_p(\Gamma) = \alpha$ by contradiction by constructing an element which is less than Γ with respect to the partial order. To construct this smaller element the following theorem is used, which was proved using slightly different language and a different method by Barnea and Schlage-Puchta [5], but was formulated and proved independently by the author as well.

Theorem 1.2.3. *Let Γ be a finitely generated group, p some fixed prime, and $x \in \Gamma$. Then $RG_p(\Gamma/\langle\langle x^{p^k}\rangle\rangle) \geq RG_p(\Gamma) - \frac{1}{p^k}$.*

The notation $\langle\langle X\rangle\rangle$ means the normal subgroup generated by the set X in the group Γ. The methods used to prove this result are similar to those used by Schlage-Puchta in his work on p-deficiency and p-gradient [36] and Osin in his work on rank gradient [32].

Using the same argument outlined above but starting with a torsion group with positive p-gradient instead of a free group allows us to make Γ torsion. One of the primary goals of Osin's [32] and Schlage-Puchta's [36] papers was to provide a simple construction of non-amenable, torsion, residually finite groups. Theorem 1.2.1 shows that there exist such groups with arbitrary p-gradient.

The construction given in Theorem 1.2.1 has a few immediate applications. First, when Γ is torsion, Theorem 1.2.1 provides another way to construct a counter example to the General Burnside Problem. The second concerns commensurable groups. Two groups are called *commensurable* if they have isomorphic subgroups of finite index. A simple consequence of Theorem 1.2.1 is the following.

Theorem 1.2.4. *There exist uncountably many pairwise non-commensurable groups that are finitely generated, infinite, torsion, non-amenable, and residually-p.*

Showing two groups are non-commensurable is usually harder than showing two groups are non-isomorphic. However, in this case rank gradient and p-gradient can distinguish non-commensurable groups and non-isomorphic groups with the same amount of work. This shows another way in which rank gradient and p-gradient are useful invariants.

The second half of the research presented here concerns computing rank gradient and p-gradient of free products, free products with amalgamation over an amenable subgroup, and HNN extensions with an amenable associated subgroup. Abert, Jaikin-Zapirain, and Nikolov [1] computed rank gradient of a free product of finitely generated residually finite groups relative to a descending chain of finite index normal subgroups:

$$RG(\Gamma_1 * \Gamma_2, \{H_n\}) = RG(\Gamma_1, \{\Gamma_1 \cap H_n\}) + RG(\Gamma_2, \{\Gamma_2 \cap H_n\}) + 1.$$

By using a similar method, we compute the analogous result for absolute rank gradient ($RG(\Gamma_1 * \Gamma_2) = RG(\Gamma_1) + RG(\Gamma_2) + 1$) and absolute p-gradient (analogous) without requiring the groups be residually finite.

The difficulty with extending this result to free products with amalgamation or HNN extensions is getting a lower bound on the minimal number of generators of a finite index subgroup. Abert and Nikolov [2] proved a connection between rank gradient and cost that will be used to help get around this issue. Their actual result is more general than the one given below, but the following is all that was needed. Let Γ be a finitely generated group and $\{H_n\}$ a lattice of normal subgroups of finite index in Γ such that $\bigcap H_n = 1$. Let $\widehat{\Gamma}_{(H_n)}$ be the profinite completion of Γ with respect to $\{H_n\}$. Then

$$RG(\Gamma, \{H_n\}) = Cost(\Gamma, \widehat{\Gamma}_{(H_n)}) - 1.$$

Using this relationship between rank gradient and cost and the work of Gaboriau [10], we establish a lower bound for the rank gradient of amalgamated free products and HNN extensions over amenable subgroups. To prove a lower bound for rank gradient, we prove and use the following lower bound for cost:

Proposition 1.2.5. *Let Γ be a finitely generated group and L be a subgroup of Γ. Let $\{H_n\}$ be a set of finite index normal subgroups of Γ such that $\bigcap H_n = 1$. Let $\widehat{\Gamma}_{(H_n)}$ be the profinite completion of Γ with respect to $\{H_n\}$ and define $\widehat{L}_{(L\cap H_n)}$ similarly. Then $Cost(L, \widehat{\Gamma}_{(H_n)}) \geq Cost(L, \widehat{L}_{(L\cap H_n)})$.*

An upper bound for the rank gradient of amalgamated free products had already been proved in [1]. Combining the upper bound and lower bound for the rank gradient of amalgamated free products over an amenable subgroup leads to the following result:

Theorem 1.2.6. *Let $\Gamma = \Gamma_1 *_A \Gamma_2$ be finitely generated and residually finite with A amenable. Let $\{H_n\}$ be a lattice of normal subgroups of finite index in Γ such that $\bigcap H_n = 1$. Then*

$$RG(\Gamma, \{H_n\}) = RG(\Gamma_1, \{\Gamma_1 \cap H_n\}) + RG(\Gamma_2, \{\Gamma_2 \cap H_n\}) + \frac{1}{|A|}.$$

In particular, $RG(\Gamma) \geq RG(\Gamma_1) + RG(\Gamma_2) + \frac{1}{|A|}$.

Let K be a finitely generated group with isomorphic subgroups $A \simeq \varphi(A)$. We denote the corresponding HNN extension of K by $K*_A = \langle K, t \mid t^{-1}At = \varphi(A)\rangle$. To compute the rank gradient for HNN extensions with amenable associated subgroup a similar method was used. We show a lower bound using cost and an upper bound using the Kurosh subgroup theorem for HNN extensions.

Theorem 1.2.7. *Let* $\Gamma = K*_A = \langle K, t \mid t^{-1}At = B\rangle$ *be a finitely generated and residually finite HNN extension with* A *amenable. Let* $\{H_n\}$ *be a lattice of finite index normal subgroups with* $\bigcap H_n = 1$. *Then*

$$RG(\Gamma, \{H_n\}) = RG(K, \{K \cap H_n\}) + \frac{1}{|A|}.$$

In particular, $RG(\Gamma) \geq RG(K) + \frac{1}{|A|}$.

The analogous results for p-gradient of amalgamated free products and HNN extensions are so similar to the rank gradient results that we omit the statements of the results at this time. The theorems are stated and proved in Chapter 6.

Since there is no corresponding relationship between p-gradient and cost, the analogous results for the p-gradient of amalgamated free products and HNN extensions are proved differently. In fact, p-gradient is much easier to compute since $d_p(\Gamma) = d(\Gamma/[\Gamma, \Gamma]\Gamma^p)$ is easier to bound than $d(\Gamma)$. To compute p-gradient for amalgamated free products and HNN extensions we use the Kurosh subgroup theorems for amalgamated free products and HNN extensions [8]. If Γ is an amalgamated free product or HNN extension, the Kurosh subgroup theorem states that a subgroup H of Γ is an HNN group with base subgroup a "tree product" (iterated amalgamated free product).

Gaboriau [10] proved a lower bound for the cost of amalgamated free products and HNN extensions of groups over amenable subgroups. The results given here are similar to the analogous results for cost. Lück [27] proved the corresponding equality of Theorem 1.2.6 for the first L^2-Betti number of amalgamated free products and his result only requires that the first L^2-Betti number of the amalgamated subgroup is zero.

Chapter 2

Group Theory Background

2.1 Finitely Generated Groups

Combinatorial group theory is the study of groups using presentations by generators and relations. Geometric group theory is more broad and connects algebraic properties of groups and geometric properties of spaces. This arises naturally in two ways: studying the Cayley graph of the group, or letting the group act on a certain space (topological space, probability space, geometric objects by symmetries, etc.). Combinatorial group theory is older than geometric group theory and today geometric group theory is getting more attention than combinatorial group theory. Both areas are important, not entirely disjoint, and in certain ways very complementary. Both combinatorial and geometric group theory are used in this research in critical ways.

In combinatorial and geometric group theory a common restriction on the group is that it is finitely generated. In combinatorial group theory this allows the use of the fact that the group is the quotient of a free group of finite rank. In geometric group theory, this assumption is even more natural as finitely generated groups act naturally on graphs and other spaces.

We introduce finitely generated groups by giving the definition, some examples, and fundamental properties. In particular we present the Schreier index formula and a method of proof called the Reidemeister-Schreier method, as this will be used often in our study of rank gradient and p-gradient.

Definition. A group Γ is called *finitely generated* if there exists a finite set S such that $\Gamma = \langle S \rangle$. That is, every element of Γ can be written as a word in the elements of $S \cup S^{-1}$.

Example 2.1.1.

- A free group of finite rank and any quotient is finitely generated
- $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d, \in \mathbb{Z},\ ad - bc = 1 \right\}$ is generated by two elements: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

- $\mathbb{Q}$ is not finitely generated as a group under addition.
- A free group on two generators $F_2 = \langle x, y\rangle$ has the subgroup generated by $\langle y^n x y^{-n} \mid n \in \mathbb{N}\rangle$, which is a free group on infinitely many generators.

The above example shows that in general a subgroup of a finitely generated group need not be finitely generated. However, every finite index subgroup of a finitely generated group is finitely generated. The following theorem is fundamental in studying rank gradient and p-gradient since it gives a bound on the minimal number of generators for any finite index subgroup. For any finitely generated group Γ, let $d(\Gamma) =$ minimal number of generators of Γ.

Theorem 2.1.2 (Schreier Index Formula)**.** *Let Γ be a finitely generated group and let H be a finite index subgroup of Γ. Then*

$$d(H) - 1 \leq (d(\Gamma) - 1)[\Gamma : H].$$

In particular, if Γ is free of finite rank, then H is free and

$$d(H) - 1 = (d(\Gamma) - 1)[\Gamma : H].$$

Proof. Because of its importance in the study of rank gradient, we outline two proofs of this result. The first proof is topological in nature and uses fundamental groups. The second proof uses the Reidemeister-Schreier method for obtaining a presentation of H from the presentation for Γ. The reader is referred to [28, Proposition 12.1] or [35, Theorem 12.25] for the complete topological proof and [28, Proposition 4.1] for the Reidemeister-Schreier method proof. We will often reference the Reidemeister-Schreier method while proving other results later in the dissertation.

- **Fundamental Groups:** Let S be a finite generating set of Γ, with $|S| = n$. Let λ be a graph with vertex set the set of cosets Hg for $g \in \Gamma$ with edges given by (Hg, Hgs) for each $s \in S \cup S^{-1}$. Let w be a word in S. From the vertex H there is a unique path p in the edges of λ such that p ends at Hw. Note that p is a loop if and only if $w \in H$. Since H is finite index, there are only $[\Gamma : H] < \infty$ many vertices and so there is a finite set of loops $\{p_1, \ldots p_m\}$ such that any other loop at H can be generated by this set. The fundamental group $\pi(\Gamma, H)$ is thus a free group on m generators (free group on elements given by the paths $p_1, \ldots p_m$). It follows that H is generated by the image of these elements in Γ (if h cannot be written in these elements, then h would define an additional loop in λ). Therefore H is generated by no more than m elements.

  Recall that for a finite graph X, the Euler characteristic is defined to be $\chi(X) = |\text{vertices}| - |\text{undirected edges}|$. It is well known that if X is connected, then $1 - \chi(X)$ gives the rank of $\pi(X)$. Since λ is finite and connected and $\pi(\Gamma, H)$ is free on m generators, then $m = |\text{undirected edges}| - |\text{vertices}| + 1$, but there are at most n undirected edges per vertex and $[\Gamma : H]$ vertices. Therefore $m \leq [\Gamma : H]n - [\Gamma : H] + 1$. It follows that $d(H) - 1 \leq [\Gamma : H](d(\Gamma) - 1)$.

- **Reidemeister-Schreier Method:** Let Γ be finitely generated and H a subgroup of finite index. Given the presentation $\Gamma = \langle X \mid R \rangle$ one uses a special set of coset representatives of H in Γ called a Schreier transversal to construct a presentation for H. A *Schreier transversal* for H in Γ is a set T of coset representatives of H in Γ such that every initial segment of an element of T is in T and $1 \in T$. That is, if $t = z_1 \cdots z_n \in T$ then $z_1 \cdots z_i \in T$ for any $1 \leq i \leq n$. Let T be a Schreier transversal for H in Γ, then H has a presentation $H = \langle Y \mid trt^{-1} \text{ for every } t \in T,\ r \in R \rangle$ with $Y = \{tx(\overline{tx})^{-1} \neq 1 \mid t \in T, x \in X\}$, where $\overline{tx}$ is the element in T representing the coset containing tx. By a slight (but standard) abuse of notation, we use Y in two different ways. First, Y is a subset of H and second, Y is a generating set for the free group that surjects onto H to give this presentation.

  Since T is a Schreier transversal, every nontrivial $t \in T$ can be uniquely written as $t = t'x'$ for some $t' \in T$ and $x' \in X \cup X^{-1}$ and thus $t'x'(\overline{t'x'})^{-1} = 1$. Therefore the cardinality of Y is $|Y| = |T||X| - (|T| - 1) = [\Gamma : H](|X| - 1|) + 1$. We also note that H has $[\Gamma : H]|R|$ many relations.

□

Finitely generated groups are countable but not all countable groups are finitely generated as seen in the example of a non-finitely generated subgroup of the free group on two generators. G. Higman, B.H. Neumann, and H. Neumann [14] proved that every countable group can be embedded in a 2-generated group. Moreover, B.H. Neumann proved that there are uncountably many 2-generated groups (this is actually true for any $n \geq 2$). Thus, the class of finitely generated groups is large and contains groups of differing complexity.

2.2 Profinite and Pro-p Groups

Let Γ be a finitely generated group. The pro-p completion of Γ for some prime p will be denoted by $\Gamma_{\widehat{p}}$. Let $d(\Gamma)$ denote the minimal number of abstract generators of a group Γ if the group is not profinite and the minimal number of topological generators if the group is profinite. If a group is profinite, the term "finitely generated" will be used to mean "topologically finitely generated". The reader is referred to any standard text on profinite groups for more details about the results in this section [9, 39].

Definition. The following are all equivalent definitions of profinite groups:

1. A profinite group is a compact Hausdorff totally disconnected topological group.
2. A profinite group G is (topologically) isomorphic to $\varprojlim G/H$, where the inverse limit is taken over all open normal subgroups of G. Moreover, the inverse limit of every inverse system of finite groups is profinite.

3. A compact Hausdorff topological group is profinite if the neighborhoods of the identity are normal subgroups and form a basis for the topology.

4. For any group Γ one can define the profinite topology on Γ by taking all normal subgroups of finite index as a basis for the neighborhoods of the identity.

Example 2.2.1. There are a few easily stated examples of profinite groups:

1. Finite groups with the discrete topology.

2. Galois groups of finite or infinite field extensions.

3. The profinite completion of Γ, that is, the (standard topological) completion in the profinite topology defined above.

An assumption we will often make about the groups throughout this dissertation is that they are residually finite or residually-p.

Definition. A group Γ is called *residually finite* if the intersection of all (normal) subgroups of finite index is trivial. A group Γ is called *residually-p* if the intersection of all normal subgroups of p-power index is trivial.

Residually finite and residually-p groups are natural classes of groups to study because they are the groups that embed into their profinite and pro-p completions respectively.

Definition. For any group Γ one can define the *profinite completion*, $\widehat{\Gamma} \simeq \varprojlim \Gamma/H$, where the inverse limit is taken over all finite index normal subgroups of Γ. The group $\widehat{\Gamma}$ is profinite (the inverse limit of compact Hausdorff totally disconnected spaces is again compact Hausdorff and totally disconnected).

There is a natural map $\varphi : \Gamma \to \widehat{\Gamma}$ and $\varphi(\Gamma)$ is dense in $\widehat{\Gamma}$. The kernel of φ is the intersection of all finite index normal subgroups of Γ. Hence, φ is injective if and only if Γ is residually finite.

Profinite completions satisfy the following universal property: Given a profinite group G and any homomorphism $\psi : \Gamma \to G$ there exists a continuous homomorphism $\widehat{\psi} : \widehat{\Gamma} \to G$ such that $\widehat{\psi} \circ \varphi = \psi$.

The following is given as a proposition in [9]. This proposition contains many of the basic properties of profinite groups.

Proposition 2.2.2. *Let G be a profinite group.*

1. *Every open subgroup of G is closed, has finite index in G, and contains an open normal subgroup of G. The intersection of all open subgroup of G is trivial.*

2. *A closed subgroup of G is open if and only if it has finite index.*

3. *A subset of G is open if and only if it is the union of cosets of open normal subgroups.*

4. *Let H be a closed subgroup of G. Then H, given the subspace topology, is a profinite group. Every open subgroup of H is of the form $H \cap K$ with K an open subgroup of G.*

5. *Let N be a closed normal subgroup of G. Then G/N, given the quotient topology, is a profinite group. The natural homomorphism $\varphi : G \to G/N$ is an open and closed continuous homomorphism.*

Remark 2.2.3. One of the most important and fundamental results concerning profinite group was proved recently by Nikolay Nikolov and Dan Segal [30]. The result, called Serre's Conjecture, states that in a finitely generated profinite group all finite index subgroups are open. None of the work given in this dissertation depends on this fact since we are defining rank gradient of a profinite group to be over open normal subgroups, but it should be noted that this result does allow the standard definition of rank gradient (over finite index subgroups) to carry over to profinite groups without any alteration. The reason we alter the definition of rank gradient when moving to the profinite case is because open normal subgroups in a profinite group play the same role that finite index subgroups do in a discrete group.

Definition. Let G be a profinite group. The Frattini subgroup of G is

$$\Phi(G) = \bigcap \{M \mid M \text{ is a maximal proper open subgroup of } G\}.$$

The following proposition is taken from [9].

Proposition 2.2.4. *Let G be a profinite group. For a subset X of G, we say that X generates G topologically if $G = \overline{\langle X \rangle}$, where $\overline{\langle X \rangle}$ means the topological closure of the subgroup generated by X in G. The following are equivalent:*

1. *X generates G topologically.*

2. *$X \cup \Phi(G)$ generates G topologically.*

3. *$X\Phi(G)/\Phi(G)$ generates $G/\Phi(G)$ topologically.*

We now turn our attention to a specific class of profinite groups called pro-p groups. Pro-p groups will be our focus for the results using topological groups in this dissertation.

Definition. A pro-p group is a profinite group in which every open normal subgroup has p-power index.

Definition. Let Γ be a group. The pro-p completion of Γ can be defined as $\Gamma_{\widehat{p}} = \varprojlim \Gamma/H$ where the inverse limit is taken over normal subgroups of p-power index. The topological group $\Gamma_{\widehat{p}}$ is a pro-p group.

There is a natural map $\varphi : \Gamma \to \Gamma_{\widehat{p}}$ and $\varphi(\Gamma)$ is dense in $\Gamma_{\widehat{p}}$. The kernel of φ is the intersection of all normal subgroups of p-power index in Γ. Hence, φ is injective if and only if Γ is residually-p.

Pro-p completions satisfy the following universal property: Given a pro-p group G and any homomorphism $\psi : \Gamma \to G$ there exists a continuous homomorphism $\widehat{\psi} : \Gamma_{\widehat{p}} \to G$ such that $\widehat{\psi} \circ \varphi = \psi$.

Example 2.2.5. A classic example of a pro-p group (also a pro-p completion) is the group of p-adic integers. The p-adic integers are defined by $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^k\mathbb{Z}$ where the inverse limit runs over all natural numbers k.

A subgroup H of Γ is called *subnormal of length* n if there exists a chain of subgroups $H = H_0 \leq \cdots \leq H_n = \Gamma$ such that H_i is normal in H_{i+1} for each i. It is a well-known fact that subnormal subgroups of p-power index in a group Γ form a base for the pro-p topology, but the author is unaware of any reference so we provide a proof here.

Lemma 2.2.6. *Let Γ be a group and p a prime. Let H be a subnormal subgroup of p-power index in Γ. There exists a normal subgroup N of p-power index in Γ such that $N \leq H$.*

Proof. We will prove this by induction on the subnormal length of H. Assume that H is 2-subnormal. Then $H \leq K \leq \Gamma$ with each group normal and of p-power index in the next group. Let T be a transversal for H in Γ such that $1 \in T$. Consider $N = \bigcap_{t \in T} tHt^{-1}$.

Let $g \in \Gamma$. Then, $gt = sk_t$ with $s \in T$ and $k_t \in K$ and as t runs over T so does s. Thus, $gNg^{-1} \subseteq \bigcap_{t \in T} gtH(gt)^{-1} = \bigcap_{t \in T}(sk_t)H(sk_t)^{-1}$. Since H is normal in K, we have that $\bigcap_{t \in T}(sk_t)H(sk_t)^{-1} \subseteq \bigcap_{t \in T} sHs^{-1} = N$. Therefore N is normal in Γ.

Since $H \leq K$ and K is normal in Γ, it implies that $tHt^{-1} \subseteq tKt^{-1} \subseteq K$. Thus for every $t \in T$ we have that $tHt^{-1} \subseteq K$. Let $k \in K$, then since H is normal in K and K is normal in Γ we have that $ktHt^{-1}k^{-1} = t(t^{-1}kt)H(t^{-1}k^{-1}t)t^{-1} = tk_0Hk_0^{-1}t^{-1} \subseteq tHt^{-1}$. Thus for each $t \in T$, we know tHt^{-1} is normal in K. Therefore, there is an injection $K/N \to \prod_{t \in T} K/tHt^{-1}$, which implies that $|K/N|$ divides $|\prod_{t \in T} K/tHt^{-1}|$. Thus $[K : N]$ divides $[\Gamma : K][K : H] = [\Gamma : H]$. Therefore, $[K : N]$ is a p-power, which implies that $[\Gamma : N]$ is a p-power.

Now assume that H is subnormal of length n. Then there exist subgroups $H = H_0, H_1, H_2, \ldots, H_n = \Gamma$ such that $H_i \leq H_{i+1}$ is normal and $[H_{i+1} : H_i]$ is a p-power. In particular $H \leq H_1 \leq H_2$ and therefore there exists an $M \leq H$ which is normal in H_2 and has p-power index in H_2. Now, $M \leq H_2 \leq \cdots \leq H_{n-1} \leq \Gamma$ with each group normal in the next group and so M has subnormal length $< n$. Thus by induction, there exists an $N \leq M$, which is normal in Γ and has p-power index in Γ. Since $N \leq M$, then in particular $N \leq H$. □

Proposition 2.2.7. *Let Γ be a group and p a prime number. The set of subnormal subgroups of p-power index form a base of neighborhoods of the identity for the pro-p topology on Γ.*

Proof. By definition of pro-p completion, we know that the collection of normal subgroups of p-power index in Γ is a base for neighborhoods of the identity in the pro-p topology on Γ. Call this set $\mathcal{K}$ and the corresponding topology τ_p. Consider now the collection of all subnormal subgroups of p-power index in Γ. Call this set $\mathcal{H}$. Then $\mathcal{H}$ forms a base for neighborhoods of the identity for some other topology τ_p^* on Γ. Since multiplication is a homeomorphism in profinite groups, to compare topologies we only need to compare the bases for neighborhoods of the identity.

First, we note that $\mathcal{K} \subseteq \mathcal{H}$, which implies that $\tau_p \subseteq \tau_p^*$. Let $H \in \mathcal{H}$. By Lemma 2.2.6 there exists an $K \in \mathcal{K}$ such that $K \subseteq H$. This shows us that $\tau_p^* \subseteq \tau_p$. □

2.3 Free Products, Amalgams, and HNN Extensions

As will be evident later, we will need to know the structure of subgroups of free products, amalgamated free products, and HNN extensions. In this section we discuss these types of groups as well as the structure of their subgroups of finite index. The structure theorems for these groups are usually referred to as Kurosh subgroup theorems. We will begin with free products.

Definition. Let Γ_1 and Γ_2 be two groups. Consider the set of words $x_1 \cdots x_n$ in the elements of Γ_1 and Γ_2. A word is *reduced* if:

- Any instance of the identity element of Γ_1 or Γ_2 is removed from the word.
- If x_i and x_{i+1} are in Γ_j, then replace it with its product from Γ_j.

The *free product* of Γ_1 and Γ_2 is the group of all reduced words in Γ_1 and Γ_2 with the group operation being word concatenation. Said differently, if $\Gamma_1 = \langle X_1 \mid R_1 \rangle$ and $\Gamma_2 = \langle X_2 \mid R_2 \rangle$ then the free product of Γ_1 and Γ_2 is the group with presentation $\Gamma_1 * \Gamma_2 = \langle X_1, X_2 \mid R_1, R_2 \rangle$.

The following theorem is well known. The reader is referred to any standard text in group theory for a proof [7, 28, 35, 37].

Theorem 2.3.1 (Kurosh Subgroup Theorem For Free Products). *Let $\Gamma = \Gamma_1 * \Gamma_2$ be the free product of Γ_1 and Γ_2. Let $H \leq \Gamma$ be a subgroup. Then there exists a set X_i of double coset representatives of $H\backslash\Gamma/\Gamma_i$ such that*

$$H = F * \left(*_{x\in X_1} H \cap x\Gamma_1 x^{-1}\right) * \left(*_{y\in X_2} H \cap y\Gamma_2 y^{-1}\right)$$

where F is a free group. Moreover, if H is finite index in Γ then $|X_i| = |H\backslash\Gamma/\Gamma_i|$ for $i = 1, 2$ and F is a free group of rank $[\Gamma : H] - |H\backslash\Gamma/\Gamma_1| - |H\backslash\Gamma/\Gamma_2| + 1$.

Amalgamated free products and HNN extensions are certain quotients of free products and are closely related concepts. Amalgamated free products, sometimes referred to as amalgams, and HNN extensions are important constructions in combinatorial group theory. It should be noted that amalgams and HNN extensions also arise in other areas of mathematics. For example, in topology both constructions arise naturally as fundamental groups of certain spaces. As will be seen in the Kurosh subgroup theorem for amalgamated free products, every subgroup of an amalgamated free product has the structure of an HNN extension. Because of this we will define both amalgamated free products and HNN extensions now.

Definition. Let Γ_1 and Γ_2 be two groups and $\varphi : A \to B$ be an isomorphism between the subgroups $A \leq \Gamma_1$ and $B \leq \Gamma_2$. The *amalgamated free product* of Γ_1 and Γ_2 over $A \simeq B$ is the group $\Gamma_1 *_A \Gamma_2 \simeq (\Gamma_1 * \Gamma_2)/\langle\langle a\varphi(a)^{-1} \text{ for every } a \in A\rangle\rangle$. The subgroup $A \simeq B$ is usually referred to as the *amalgamated subgroup* of Γ.

HNN extensions are named after G. Higman, B.H. Neumann, and H. Neumann who constructed the groups in 1949. The construction of HNN extensions answered some important embedding questions for groups. The original paper [14] proved that any countable group G can be embedded in a countable group Γ in which all elements of the same order in G are conjugate in Γ. The construction of HNN extensions can also be used to show that every countable group can be embedded in a 2-generated group.

Definition. Let K be a group with isomorphic subgroups $A \simeq \varphi(A)$. The *HNN extension* of K over A is the group with presentation $K*_A = (K*\langle t\rangle)/\langle\langle tat^{-1}\varphi(a)^{-1}\rangle\rangle$. The group K is referred to as the *base group*, the element t is referred to as the *stable letter*, and the subgroup $A \simeq B$ is referred to as the *associated subgroup.* This group is typically written as $K*_A = \langle K, t \mid t^{-1}At = \varphi(A)\rangle$.

The subgroup structure theorems for amalgamated free products and HNN extensions were first proved by Karrass and Solitar [18, 19]. Karrass, Pietrowski, and Solitar improved the result for HNN groups using the Reidemeister-Schreier method [17]. D.E. Cohen [8] proved the same results for amalgamated free products and HNN groups independently from Karrass and Solitar using Bass-Serre theory.

The double coset representatives given in the following theorems are constructed in a specific way. The reader is referred to [8, 17–19] for the constructions. These double coset representatives are called a *cress* (compatible regular extended Schreier system) in [18], and a *semi-cress* in [8]. For our purposes it will not matter what form the representatives take.

In the theorems below the term "tree product" is used. The following description of a tree product follows that of Karrass and Solitar [18]. Let $\{\Gamma_i\}$ be a collection of groups and suppose that for certain pairs of indices $i \neq j$ there exist isomorphic subgroups A_{ij} and A_{ji} of Γ_i and Γ_j respectively. Then the *partial generalized free product* is the group Γ which has as a presentation the union of the presentations of the amalgamated free products $\langle \Gamma_i, \Gamma_j \mid A_{ij} = A_{ji}\rangle$. It is known that Γ is independent of the presentation used for the Γ_i. We associate to Γ a graph which has as vertices the groups Γ_i and an edge joins Γ_i and Γ_j if there exist isomorphic subgroups A_{ij} and A_{ji} (given above) of Γ_i and Γ_j respectively. The group Γ is called a *tree product* of the Γ_i with the subgroups A_{ij} and A_{ji} amalgamated if this graph is a tree. Tree products are usually denoted $\prod^*(\Gamma_i \mid A_{ij} = A_{ji})$.

Theorem 2.3.2 (**Kurosh Subgroup Theorem For Amalgamated Free Products** - Cohen, Karrass and Solitar)**.** *Let $\Gamma = \Gamma_1 *_A \Gamma_2$. Let H be a subgroup of Γ. One can choose the following:*

1. *$\{d_\alpha\}$, a double coset representative system for $H\backslash\Gamma/\Gamma_1$,*
2. *$\{e_u\}$, a double coset representative system for $(d_\alpha\Gamma_1 d_\alpha^{-1} \cap H)\backslash\Gamma_1/A$ for each d_α,*
3. *$\{d_\beta\}$, a double coset representative system for $H\backslash\Gamma/\Gamma_2$,*
4. *$\{e_v\}$, a double coset representative system for $(d_\beta\Gamma_2 d_\beta^{-1} \cap H)\backslash\Gamma_2/A$ for every d_β,*

such that $\{d_\beta e_v\}$ and $\{d_\alpha e_u\}$ are double coset representative systems for $H\backslash\Gamma/A$. Given d_β and e_v there exists a unique d_α, corresponding e_u, and element $x \in A$ such that $d_\beta e_v \in H d_\alpha e_u x$. Let $t_{\beta v} = d_\beta e_v (d_\alpha e_u x)^{-1} \in H$.

Then H is the HNN group

$$H = \langle L, t_{\beta v} \mid t_{\beta v}(d_\alpha e_u A (d_\alpha e_u)^{-1} \cap H) t_{\beta v}^{-1} = d_\beta e_v A (d_\beta e_v)^{-1} \cap H\rangle.$$

In this expression we take all non trivial $t_{\beta v}$ with corresponding d_α and e_u.

The group L is the tree product of the groups $d_\alpha\Gamma_1 d_\alpha^{-1} \cap H$ and $d_\beta\Gamma_2 d_\beta^{-1} \cap H$ with two such groups being adjacent if $d_\alpha = d_\beta = 1$ or $d_\alpha = d_\beta b$ or $d_\beta = d_\alpha a$ with $a \in \Gamma_1$ and $b \in \Gamma_2$. The subgroup amalgamated between these two adjacent groups is $A \cap H$ (in $\Gamma_1 \cap H$) or $d_\alpha A d_\alpha^{-1} \cap H$ (in $d_\alpha\Gamma_1 d_\alpha^{-1} \cap H$) or $d_\beta A d_\beta^{-1} \cap H$ (in $d_\beta\Gamma_2 d_\beta^{-1} \cap H$) respectively.

Moreover, if H has finite index in Γ, then the number of nontrivial $t_{\beta v}$ is equal to $|H\backslash\Gamma/A| - |H\backslash\Gamma/\Gamma_1| - |H\backslash\Gamma/\Gamma_2| + 1$.

Remark 2.3.3. For our purposes we are only interested in applying this theorem to normal subgroups of finite index. In this case we can restate the theorem as follows: Every normal subgroup H of finite index in the amalgamated free product $\Gamma = \Gamma_1 *_A \Gamma_2$ is an HNN group with base subgroup L and $n = |H\backslash\Gamma/A| - |H\backslash\Gamma/\Gamma_1| - |H\backslash\Gamma/\Gamma_2| + 1$ free generators with each associated subgroup being isomorphic to $A \cap H$. Specifically,

$$H = \langle L, t_1, \ldots, t_n \mid t_i(A \cap H)t_i^{-1} = \varphi_i(A) \cap H\rangle$$

where the φ_i are appropriate embeddings from A to L.

Further, L is an amalgamated free product of $|H\backslash\Gamma/\Gamma_1|$ groups that are isomorphic to $\Gamma_1 \cap H$ and $|H\backslash\Gamma/\Gamma_2|$ groups that are isomorphic to $\Gamma_2 \cap H$ with at most $|H\backslash\Gamma/\Gamma_1| + |H\backslash\Gamma/\Gamma_2| - 1$ amalgamations each of which is isomorphic to $A \cap H$.

Theorem 2.3.4 (**Kurosh Subgroup Theorem For HNN Groups** - Cohen, Karrass, Pietrowski, and Solitar)**.** *Let $\Gamma = \langle K, t \mid t^{-1}At = B\rangle$. Let H be a subgroup of Γ. One can choose the following:*

1. *$\{d_\kappa\}$, a double coset representative system for $H\backslash\Gamma/K$,*
2. *$\{e_\alpha\}$, a double coset representative system for $(d_\kappa K d_\kappa^{-1} \cap H)\backslash K/A$ for each d_κ,*
3. *$\{e_\beta\}$, another double coset representative system for $(d_\kappa K d_\kappa^{-1} \cap H)\backslash K/B$ for every d_κ,*

such that $\{d_\kappa e_\alpha\}$ and $\{d_\kappa e_\beta\}$ are double coset representative systems for $H\backslash\Gamma/A$. Given d_κ and e_α there exists a unique $d_\gamma \in \{d_\kappa\}$, corresponding e_β, and element $x \in A$ such that $d_\kappa e_\alpha t \in H d_\gamma e_\beta x$. Let $t_{\kappa\alpha} = d_\kappa e_\alpha t (d_\gamma e_\beta x)^{-1} \in H$.

Then H is the HNN group

$$H = \langle L, t_{\kappa\alpha} \mid t_{\kappa\alpha}(d_\gamma e_\beta A (d_\gamma e_\beta)^{-1} \cap H)t_{\kappa\alpha}^{-1} = d_\kappa e_\alpha B (d_\kappa e_\alpha)^{-1} \cap H\rangle.$$

In this expression we take all non trivial $t_{\kappa\alpha}$ with corresponding d_γ and e_β.

The group L is the tree product of the groups $d_\kappa K d_\kappa^{-1} \cap H$, where $d_\kappa K d_\kappa^{-1} \cap H$ and $d_\gamma K d_\gamma^{-1} \cap H$ (d_γ shorter than d_κ) are adjacent if $d_\kappa = d_\gamma e_\alpha t$ or $d_\kappa = d_\gamma e_\beta t^{-1}$. The amalgamated subgroup between these two adjacent subgroups is $d_\kappa A d_\kappa^{-1} \cap H$ or $d_\kappa B d_\kappa^{-1} \cap H$ respectively.

Moreover, if H has finite index in Γ, then the number of nontrivial $t_{\kappa\alpha}$ is equal to $|H\backslash\Gamma/A| - |H\backslash\Gamma/K| + 1$.

Remark 2.3.5. As in the case of amalgamated free products, for our purposes we are only interested in applying this theorem to normal subgroups of finite index. In this case we can restate the theorem as follows: Every normal subgroup H of finite index in the HNN extension $\Gamma = \langle K, t \mid tAt^{-1} = \varphi(A)\rangle$ is an HNN group with base subgroup L and $n = |H\backslash\Gamma/A| - |H\backslash\Gamma/K| + 1$ free generators with each associated subgroup being isomorphic to $A \cap H$. Specifically,

$$H = \langle L, t_1, \dots, t_n \mid t_i(A \cap H)t_i^{-1} = \varphi_i(A) \cap H \rangle$$

where the φ_i are appropriate embeddings from A to L.

Further, L is an amalgamated free product of $|H\backslash\Gamma/K|$ groups that are isomorphic to $K \cap H$ with at most $|H\backslash\Gamma/K| - 1$ amalgamations each of which is isomorphic to $A \cap H$.

2.4 Amenable Groups

There are numerous definitions of amenable groups depending on what context one is considering and what property of amenable groups is needed. In this section, two definitions of amenable groups will be given as well as some examples.

Definition (invariant measure definition)**.** A discrete group Γ is *amenable* if there exists a finitely-additive left-invariant probability measure on Γ. That is, there exists a measure μ on Γ such that:

1. Finitely-additive: Let $A_1, \dots, A_n$ be disjoint subsets of Γ. Then $\mu(A_1 \cup \dots \cup A_n) = \mu(A_1) + \dots + \mu(A_n)$.
2. Left-invariant: $\mu(gA) = \mu(A)$ for any $g \in \Gamma$ and any subset A.
3. Probability measure: $\mu(\Gamma) = 1$.

Example 2.4.1. Using the invariant measure definition it is easy to see that any finite group Γ is amenable. Let A be any subset of Γ and let $\mu(A) = \frac{|A|}{|\Gamma|}$ (normalized counting measure).

If the discrete group Γ is finitely generated we can give a different definition of amenability. The following definition is slightly different from what is normally considered as a Følner sequence, but it is equivalent. This particular version is found in [1]. First some notation. Let Γ be generated by a finite set S and let A be a finite subset of Γ. The *boundary of A with respect to S* is defined as

$$\partial_S(A) = \{(a, sa) \mid a \in A, s \in S, sa \notin A\}.$$

The set A is called *ε-invariant* with respect to S if $|\partial_S(A)| \leq \varepsilon|S||A|$.

Definition (Følner sequence definition)**.** A finitely generated discrete group Γ is *amenable* if there exists a sequence of finite subsets $\{A_n\}$ of Γ and a sequence of real numbers a_n such that A_n is a_n-invariant for each n and $\lim_{n\to\infty} a_n = 0$.

Example 2.4.2. Using the Følner sequence definition one can easily show that the group of integers is amenable. Let $S = \{-1, 1\}$ and let $A_n = \{-n, \dots, n\}$ for every $n \in \mathbb{N}$. Then $\partial_S(A_n) = \{-n, n\}$ and thus $|\partial_S(A_n)| = 2$. Since $|S||A| = 4n$ it follows that A_n is $\frac{1}{2n}$-invariant for every n.

A summary of some of the important facts about amenable groups is given in the following theorem.

Theorem 2.4.3. *The following groups are amenable:*

1. *Finite groups,*
2. *Solvable (hence nilpotent, abelian) groups,*
3. *Subgroups of amenable groups,*
4. *Quotients of amenable groups,*
5. *Direct products of amenable groups,*
6. *Direct limits of amenable groups,*
7. *Virtually amenable groups, that is, a group containing a finite index subgroup that is amenable.*

Amenable groups should be thought of as "small" in some sense as non-amenable groups are *paradoxical*.

Definition. Let Γ be a discrete group. Γ is said to have a *paradoxical decomposition* if there exist disjoint subsets $A_1, \dots, A_n, B_1, \dots B_m$ and elements $g_1, \dots, g_n, h_1, \dots h_m \in \Gamma$ such that

$$\Gamma = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j.$$

Proposition 2.4.4. *A discrete amenable group is not paradoxical.*

Proof. Let Γ be a discrete amenable group and let μ be a finitely-additive left-invariant probability measure on Γ. Assume that Γ has paradoxical decomposition $\Gamma = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j$. Since μ is finitely-additive and left-invariant,

$$1 = \mu(\Gamma) = \mu\left(\bigcup_{i=1}^{n} g_i A_i\right) = \sum_{i=1}^{n} \mu(g_i A_i) = \sum_{i=1}^{n} \mu(A_i) = \mu\left(\bigcup_{i=1}^{n} A_i\right)$$

and

$$1 = \mu(\Gamma) = \mu\left(\bigcup_{j=1}^{m} h_j B_j\right) = \sum_{j=1}^{m} \mu(h_j B_j) = \sum_{j=1}^{m} \mu(B_j) = \mu\left(\bigcup_{j=1}^{m} B_j\right).$$

Therefore,

$$\mu\left(\bigcup_{i=1}^{n} A_i \cup \bigcup_{j=1}^{m} B_j\right) = \mu\left(\bigcup_{i=1}^{n} A_i\right) + \mu\left(\bigcup_{j=1}^{m} B_j\right) = 1 + 1 = 2$$

which is clearly a contradiction since μ is a probability measure. □

Example 2.4.5. Any discrete group containing the free group on two generators F_2 is not amenable. Since amenability is closed under subgroups, we only need to show that F_2 is not amenable. Let $F_2 = \langle a, b \rangle$. Denote by $W(a)$ the set of all words in F_2 that are reduced and start with the letter a and define $W(a^{-1}), W(b)$, and $W(b^{-1})$ similarly. These subsets are all disjoint. The set $a\,W(a^{-1})$ contains all words that do not start with a since each word in $W(a^{-1})$ must be reduced. Similarly, $b\,W(b^{-1})$ contains all words that do not start with b. It follows that

$$F_2 = a\,W(a^{-1}) \cup W(a) = b\,W(b^{-1}) \cup W(b).$$

Thus F_2 has a paradoxical decomposition and therefore is not amenable.

Chapter 3

Rank Gradient and p-Gradient

3.1 Relationship Between Rank Gradient and p-Gradient With Other Group Invariants

Rank gradient is closely related to two other group invariants: L^2-*Betti numbers* and *cost*. If Γ is a finitely generated residually finite group, it is known that

$$RG(\Gamma) \geq \text{cost}(\Gamma) - 1 \geq \beta_1^{(2)}(\Gamma) - \frac{1}{|\Gamma|} \tag{3.1.1}$$

where we use the standard convention that $\frac{1}{|\Gamma|} = 0$ if Γ is infinite. The notation $\beta_1^{(2)}(\Gamma)$ stands for the first L^2-Betti number of Γ. Abert and Nikolov [2] proved the first part of the inequality and the second part was proved by Gaboriau [11]. It is not known whether or not the inequalities can be strict. Rank gradient, cost, and first L^2-Betti number are all equal in every case in which they have been computed. The relationship between rank gradient and cost is not limited to the above inequality. Abert and Nikolov related two open problems about cost to rank gradient as well: the fixed price problem and the multiplicativity of cost-1 problem [2]. This inequality relates the three main branches of mathematics: algebra (asymptotic group theory), analysis (orbit equivalence theory), and topology (homology) and emphasizes the importance of rank gradient to other areas of mathematics.

Lück proved [25] that one can compute the first L^2-Betti number of a finitely presented residually finite group Γ as follows:

Theorem 3.1.1 (Lück)**.** *Let Γ be a finitely presented residually finite group. Let $\{N_i\}$ be a descending chain of finite index normal subgroups of Γ such that $\bigcap N_i = 1$. Then the first L^2-Betti number of Γ is*

$$\beta_1^{(2)}(\Gamma) = \lim_{i\to\infty} \frac{rk(N_i^{ab})}{[\Gamma : N_i]},$$

where $rk(N_i^{ab})$ is the torsion free rank of the abelianization of N_i.

The value $rk(N_i^{ab})$ is called the *ordinary first Betti number* of N_i and is typically denoted $b_1(N_i)$. It is important to note that by definition, L^2-Betti numbers are independent of the choice of chain. However, Lück's calculation of $\beta_1^{(2)}(\Gamma)$ given above shows that for finitely presented residually finite groups, the quantity $\lim_{i\to\infty} \frac{rk(N_i^{ab})}{[\Gamma:N_i]}$ is also independent of the choice of chain. This fact is very non-trivial and interesting in its own right.

Remark 3.1.2. There are two notions of L^2-Betti numbers for a group Γ: L^2-Betti numbers of Γ and L^2-Betti numbers of matrices over the rational group ring of Γ (L^2-Betti numbers arising from Γ). The L^2-Betti numbers referred to in this dissertation are of the first type. In general, Betti numbers are topological invariants and more information on Betti numbers can be found in [26].

As will be covered later, cost is used to help compute rank gradient of free products amalgamated over an amenable subgroup and HNN extensions with an amenable associated subgroup. Let Γ be a group acting on a Borel probability space X, and consider the equivalence relation defined on X by $x \sim y$ if x and y are in the same orbit. The cost of the action of Γ on X, denoted $Cost(\Gamma, X)$, is a number that represents the amount of information needed to build this equivalence relation. Cost is often studied in the context of orbit equivalence theory and as far as the author is aware the use of cost given in this dissertation is one of the first applications of cost to prove a result in group theory. The notion of cost was first introduced by Levitt [24] and more information on cost can be found in [10, 12].

We can easily prove that $RG(\Gamma, \{H_i\}) \geq RG_p(\Gamma) \geq \beta_1^{(2)}(\Gamma)$ if $\{H_i\}$ is a normal chain of p-power index subgroups, which is a special case of Inequality 3.1.1. Before proving the inequality we define explicitly rank gradient and p-gradient relative to a lattice of subgroups.

Definition. 1. The *rank gradient relative to a lattice* $\{H_i\}$ of finite index subgroups is defined as

$$RG(\Gamma, \{H_i\}) = \inf_i \frac{d(H_i) - 1}{[\Gamma : H_i]}.$$

2. The *p-gradient relative to a lattice* $\{H_i\}$ of normal subgroups of p-power index is defined as

$$RG_p(\Gamma, \{H_i\}) = \inf_i \frac{d_p(H_i) - 1}{[\Gamma : H_i]}.$$

Often, the lattice is a descending chain of subgroups. In this case, we have the following useful lemma.

Lemma 3.1.3. *Let Γ be a finitely generated group. If $\{H_i\}_1^\infty$ is a descending chain of finite index subgroups and $\{K_i\}_1^\infty$ is a descending chain of normal subgroups of* *p-power index, then*

$$\left\{\frac{d(H_i)-1}{[\Gamma:H_i]}\right\}_1^{\infty} \quad and \quad \left\{\frac{d_p(K_i)-1}{[\Gamma:K_i]}\right\}_1^{\infty}$$

are non-increasing sequences and

$$RG(\Gamma,\{H_i\}) = \lim_{i\to\infty}\frac{d(H_i)-1}{[\Gamma:H_i]} \quad and \quad RG_p(\Gamma,\{K_i\}) = \lim_{i\to\infty}\frac{d_p(K_i)-1}{[\Gamma:K_i]}.$$

Proof. Since Γ is finitely generated and H_i is of finite index, then by the Schreier index formula H_i is finitely generated and $d(H_{i+1})-1 \leq (d(H_i)-1)[H_i:H_{i+1}]$ for each i. This implies that for each i,

$$\frac{d(H_{i+1})-1}{[\Gamma:H_{i+1}]} \leq \frac{(d(H_i)-1)[H_i:H_{i+1}]}{[\Gamma:H_{i+1}]} = \frac{d(H_i-1)}{[\Gamma:H_i]}.$$

Therefore,

$$RG(\Gamma,\{H_i\}) = \inf_i \frac{d(H_i)-1}{[\Gamma:H_i]} = \lim_{i\to\infty}\frac{d(H_i)-1}{[\Gamma:H_i]}.$$

The corresponding result for p-gradient is proved similarly using the fact that for a finitely generated group Γ and a normal subgroup of p-power index K, the inequality $d_p(K)-1 \leq (d_p(\Gamma)-1)[\Gamma:K]$ holds. This Schreier index formula for d_p is proved in Lemma 6.1.3 using the Schreier index formula for finitely generated pro-p groups. □

Remark 3.1.4. To prove results about rank gradient (analogously p-gradient) with respect to a lattice $\{H_n\}$ of normal subgroups of finite index in Γ, it is enough to prove the result for a descending chain of subgroups from the lattice. The argument for this is shown in Lemma 3.1.3. Specifically, one can use the chain: $H_1 \geq H_1 \cap H_2 \geq H_1 \cap H_2 \cap H_3 \geq \dots$

Proposition 3.1.5. *Let Γ be an infinite finitely presented residually-p group. Let $\{H_i\}$ be an infinite lattice of normal subgroups of p-power index. Then $RG(\Gamma,\{H_i\}) \geq RG_p(\Gamma) \geq \beta_1^{(2)}(\Gamma)$.*

Proof. Since Γ is residually-p, it follows that for every infinite lattice of normal subgroups of p-power index, $\{H_i\}$, there exists a descending chain $\{H_i'\}$ with trivial intersection such that H_i' is contained in H_i for all i. By the proof of Lemma 3.1.3 we have that $\frac{d(H_i)-1}{[\Gamma:H_i]} \geq \frac{d(H_i')-1}{[\Gamma:H_i']}$ for each i. Therefore, $RG(\Gamma,\{H_i\}) \geq RG(\Gamma,\{H_i'\})$.

Thus, it suffices to prove the result in the case when $\{H_i\}$ is a descending chain with trivial intersection. For every i we have $d(H_i) \geq d_p(H_i) \geq rk(H_i^{ab})$, which implies

$$\frac{d(H_i)}{[\Gamma:H_i]} \geq \frac{d_p(H_i)}{[\Gamma:H_i]} \geq \frac{rk((H_i^{ab})}{[\Gamma:H_i]}.$$

By Theorem 3.1.1, taking the limit of the above inequality yields $RG(\Gamma,\{H_i\}) \geq RG_p(\Gamma,\{H_i\}) \geq \beta_1^{(2)}(\Gamma)$. By definition $RG_p(\Gamma,\{H_i\}) \geq RG_p(\Gamma)$ and thus $RG(\Gamma,\{H_i\}) \geq RG_p(\Gamma) \geq \beta_1^{(2)}(\Gamma)$. □

3.2 Some Properties of Rank Gradient and p-Gradient

In this section useful results concerning rank gradient and p-gradient are collected.

Theorem 3.2.1. *Let Γ be a finitely generated group and let H be a finite index subgroup. Then $RG(\Gamma) = \frac{RG(H)}{[\Gamma:H]}$. If Γ is finite, then $RG(\Gamma) = -\frac{1}{|\Gamma|}$.*

Proof. Let $K \leq H \leq \Gamma$. Then $[\Gamma : K]$ is finite if and only if $[H : K]$ is finite. Since $\frac{d(K)-1}{[\Gamma:K]} = \frac{1}{[\Gamma:H]}\frac{d(K)-1}{[H:K]}$ it follows that

$$\inf_{[\Gamma:K]<\infty} \frac{d(K)-1}{[\Gamma:K]} \leq \inf_{\substack{[\Gamma:K]<\infty \\ K\leq H}} \frac{d(K)-1}{[\Gamma:K]} = \frac{1}{[\Gamma:H]} \inf_{[H:K]<\infty} \frac{d(K)-1}{[H:K]}.$$

Therefore, $RG(\Gamma) \leq \frac{RG(H)}{[\Gamma:H]}$.

It is clear that $\{K \leq H \mid [H : K] < \infty\} = \{H \cap K \mid [\Gamma : K] < \infty\}$. Note that $[K : H \cap K]$ is finite and so $[\Gamma : H \cap K] = [\Gamma : K][K : H \cap K]$ and by the Schreier index formula $d(H \cap K) - 1 \leq (d(K) - 1)[K : H \cap K]$. Therefore,

$$\frac{d(H\cap K)-1}{[\Gamma:H\cap K]} \leq \frac{(d(K)-1)[K:H\cap K]}{[\Gamma:H\cap K]} = \frac{d(K)-1}{[\Gamma:K]}.$$

It follows that

$$\begin{aligned} \inf_{[\Gamma:K]<\infty} \frac{d(K)-1}{[\Gamma:K]} &\geq \inf_{[\Gamma:H\cap K]<\infty} \frac{d(H\cap K)-1}{[\Gamma:H\cap K]} \\ &= \inf_{[H:H\cap K]<\infty} \frac{d(H\cap K)-1}{[\Gamma:H\cap K]} = \frac{1}{[\Gamma:H]} \inf_{[H:H\cap K]<\infty} \frac{d(H\cap K)-1}{[H:H\cap K]}. \end{aligned}$$

Therefore, $RG(\Gamma) \geq \frac{RG(H)}{[\Gamma:H]}$.

If Γ is finite, then using $H = \{1\}$, it follows that $RG(\Gamma) = \frac{-1}{|\Gamma|}$. □

Computing rank gradient for a free group of finite rank is easy by the Schreier index formula.

Lemma 3.2.2. *Let F be a non-abelian free group of finite rank and let p be a prime number. Then $RG(F) = RG_p(F) = rank(F) - 1$.*

Proof. For any free group F and any prime p, we know $d_p(F) = d(F)$. Let H be a finite index (resp. p-power index and normal) subgroup of F. Since H is free, $d_p(H) = d(H)$ and by the Schreier index formula, $d(H) - 1 = (d(F) - 1)[F : H]$, which implies in this case $d_p(H) - 1 = (d_p(F) - 1)[F : H]$. Therefore,

$$RG(F) = \inf_{[F:H]<\infty} \frac{d(H)-1}{[F:H]} = \inf_{[F:H]<\infty} (d(F)-1) = \text{rank}(F) - 1,$$

$$RG_p(F) = \inf_{\substack{H \text{ normal,} \\ p\text{-power}}} \frac{d_p(H)-1}{[F:H]} = \inf_{\substack{H \text{ normal,} \\ p\text{-power}}} (d_p(F)-1) = \text{rank}(F) - 1.$$

□

The difficulty of computing the rank gradient in general is due to the fact that $d(\Gamma)$ is hard to estimate from below. However, there is one very natural lower bound for $RG(\Gamma)$ if the group is finitely presented.

Proposition 3.2.3. *Suppose* $\Gamma = \langle X \mid R \rangle$ *is finitely presented. If* $m = |X| - |R| > 0$, *then* $RG(\Gamma) \geq m - 1$.

Proof. Let $n = |X|$ and $r = |R|$. If F_n is a free group of rank n, then $\Gamma = F_n / \langle\langle R \rangle\rangle$. Now, $d(\Gamma) \geq d(\Gamma^{ab})$ and $\Gamma^{ab} = \mathbb{Z}^n / \langle\langle R \rangle\rangle$. By the Fundamental Theorem of Finitely Generated Abelian Groups, it follows that the free rank of Γ^{ab} is $\geq n - r$, which implies that $d(\Gamma) \geq n - r$.

Let H be a finite index subgroup of Γ. By the Reidemeister-Schreier method (see the proof of Theorem 2.1.2), H has a presentation with $[\Gamma : H](n-1)+1$ generators and $[\Gamma : H]r$ relations. Therefore,

$$d(H) \geq [\Gamma : H](n-1) + 1 - [\Gamma : H]r = [\Gamma : H](n - r - 1) + 1.$$

Thus, for every finite index subgroup H of Γ, we have $\frac{d(H)-1}{[\Gamma:H]} \geq n - r - 1$, which implies $\displaystyle\inf_{[\Gamma:H]<\infty} \frac{d(H)-1}{[\Gamma:H]} \geq n - r - 1$. Thus, $RG(\Gamma) \geq n - r - 1 = m - 1$. □

As the following proposition shows, it is not difficult to produce groups with rational rank gradient. Whether an irrational number can be the rank gradient of some finitely generated group remains an open question. We will show later that for every prime p, every positive real number is the p-gradient for some finitely generated group.

Proposition 3.2.4. *Let* $\frac{m}{n} > 0 \in \mathbb{Q}$. *There exists a finitely presented group* Γ *such that* $RG(\Gamma) = \frac{m}{n}$.

Proof. Let F_{m+1} be a non-abelian free group of rank $m+1$ and let A be any group of order n. Consider $\Gamma = F_{m+1} \times A$. Since F_{m+1} has index n in Γ, then $RG(\Gamma) = \frac{RG(H)}{[\Gamma:H]} = \frac{m}{n}$ by Theorem 3.2.1. □

Using the theory of groups acting on trees, Abert and Nikolov [2] proved the following proposition, which can be used to show that absolute rank gradient and rank gradient relative to a lattice are not always equal.

Proposition 3.2.5 (Abert and Nikolov)**.** *There exists a virtually free group* Γ *and an interval* $[x, y) \subset \mathbb{R}$ *such that for every* $\alpha \in [x, y)$, *there exists a subnormal chain of subgroups* $\Gamma = H_0 > H_1 > H_2 > \cdots$ *with trivial intersection, such that* $RG(\Gamma, \{H_i\}) = \alpha$.

Since Γ is virtually free it contains a free subgroup of finite rank that is finite index in Γ, call it F. By Theorem 3.2.1 we know $RG(\Gamma) = \frac{RG(F)}{[\Gamma:F]}$ which is clearly rational. What is important here is that the groups in the chain are not normal. The fact the the chain must intersect in the identity is also vital to determining whether rank gradient depends on the chain.

Lemma 3.2.6. *Let $A \times B$ be the direct product of two finitely generated groups. Then $d(A) \leq d(A \times B) \leq d(A) + d(B)$.*

Proof. Since $A \times B = (A * B)/N$, then $d(A \times B) \leq d(A * B) \leq d(A) + d(B)$. Since, $A = (A \times B)/(\{1\} \times B)$ it implies that $d(A) \leq d(A \times B)$. □

Example 3.2.7. Consider the group $\Gamma = F_m \times F_n$ with $m \neq n$. Let $\{A_i\}$ be an infinite descending chain of normal subgroups of finite index in F_m. Then $\{A_i \times F_n\}$ is an infinite descending chain of normal subgroups of finite index in Γ. By Lemma 3.2.6,

$$\inf_i \frac{d(A_i) - 1}{[F_m : A_i]} \leq \inf_i \frac{d(A_i \times F_n) - 1}{[\Gamma : A_i \times F_n]} \leq \inf_i \frac{d(A_i) + d(F_n) - 1}{[F_m : A_i]} = \inf_i \frac{d(A_i) - 1}{[F_m : A_i]}$$

Therefore, using the Schreier index formula for free groups

$$RG(\Gamma, \{A_i \times F_n\}) = \inf_i \frac{d(A_i \times F_n) - 1}{[\Gamma : A_i \times F_n]} = \inf_i \frac{d(A_i) - 1}{[F_m : A_i]} = d(F_m) - 1.$$

Similarly, let $\{B_i\}$ be an infinite descending chain of normal subgroups of finite index in F_n. Then

$$RG(\Gamma, \{F_m \times B_i\}) = \inf_i \frac{d(F_m \times B_i) - 1}{[\Gamma : F_m \times B_i]} = \inf_i \frac{d(B_i) - 1}{[F_n : B_i]} = d(F_n) - 1.$$

Since $m \neq n$, then $d(F_m) - 1 \neq d(F_n) - 1$. Thus in this case the rank gradient of Γ does depend on the chain.

Open Question. *Let Γ be a finitely generated group. Does the rank gradient of Γ depend on the chain of subgroups if the chain consists of finite index normal subgroups with trivial intersection?*

3.3 Rank Gradient and p-Gradient of Profinite Groups

We will prove that a group and its pro-p completion have the same p-gradient and that the p-gradient of a group equals the rank gradient of its pro-p completion. When dealing with pro-p completions of a group, it is often convenient to assume that the group is residually-p since in this case the group will embed in its pro-p completion. To show why this type of assumption will not influence any result about the p-gradient, the following lemma is given.

Definition. Let Γ be a group and p a prime. Let $\mathcal{N}$, the p-residual of Γ, be the intersection of all normal subgroups of p-power index in Γ. The *p-residualization* of Γ is the quotient $\Gamma/\mathcal{N}$. Note that the p-residualization of Γ is isomorphic to the image of Γ in its pro-p completion, $\Gamma_{\widehat{p}}$, and is residually-p.

Lemma 3.3.1. *Let Γ be a group and p a prime number. Let $\widetilde{\Gamma}$ be the p-residualization of Γ. Then*

1. $RG_p(\Gamma) = RG_p(\widetilde{\Gamma})$.
2. $\Gamma_{\widehat{p}} \simeq \widetilde{\Gamma}_{\widehat{p}}$.

Proof. 1. Note that every normal subgroup of p-power index in Γ contains $\mathcal{N}$. Therefore, there is a bijective correspondence between normal subgroups of p-power index in $\widetilde{\Gamma}$ and normal subgroups of p-power index in Γ. Let the correspondence be $\widetilde{H} \leftrightarrow H$ with $\widetilde{H} \leq \widetilde{\Gamma}$ and $H \leq \Gamma$. Then $[\widetilde{\Gamma} : \widetilde{H}] = [\Gamma : H]$ and $\widetilde{H} \simeq H/\mathcal{N}$. Therefore, $\widetilde{H}/[\widetilde{H}, \widetilde{H}]\widetilde{H}^p \simeq H/([H,H]H^p\mathcal{N}) \simeq H/[H,H]H^p$ since $[H,H]H^p$ is a p-power index normal subgroup of Γ and thus contains $\mathcal{N}$. Therefore, $d_p(\widetilde{H}) = d_p(H)$.

Since there is a bijection $\widetilde{H} \leftrightarrow H$ between all normal subgroups of p-power index in $\widetilde{\Gamma}$ and Γ with $[\widetilde{\Gamma} : \widetilde{H}] = [\Gamma : H]$ and $d_p(\widetilde{H}) = d_p(H)$, then $RG_p(\widetilde{\Gamma}) = RG_p(\Gamma)$.

2. By the proof of (1) above there is a bijective correspondence, $\widetilde{H} \leftrightarrow H$, between normal subgroups of p-power index in $\widetilde{\Gamma}$ and normal subgroups of p-power index in Γ. By the inverse limit definition of pro-p completions, $\Gamma_{\widehat{p}} \simeq \varprojlim \Gamma/H$ and $\widetilde{\Gamma}_{\widehat{p}} \simeq \varprojlim \widetilde{\Gamma}/\widetilde{H}$, are inverse limits over the same indexing set $\{H \leq \Gamma \mid H \text{ normal}, [\Gamma, H] = p\text{-power}\}$. However, for every such H in Γ it follows that $\widetilde{\Gamma}/\widetilde{H} \simeq (\Gamma/\mathcal{N})/(H/\mathcal{N}) \simeq \Gamma/H$ and therefore $\Gamma_{\widehat{p}} \simeq \widetilde{\Gamma}_{\widehat{p}}$.

□

With the following proposition, we will be able to prove that a group and its pro-p completion have the same p-gradient. The notation $\overline{X}$ will mean the closure of the set X in the given topological space.

Proposition 3.3.2. *Let Γ be a finitely generated group and p a prime. Let $\varphi : \Gamma \to \Gamma_{\widehat{p}}$ be the natural map from Γ to its pro-p completion. Let H be a normal subgroup of p-power index of Γ. The following hold:*

1. $\varphi(H) = \varphi(\Gamma) \cap \overline{\varphi(H)}$.
2. $\overline{\varphi} : \Gamma/H \to \Gamma_{\widehat{p}}/\overline{\varphi(H)}$ *given by* $\overline{\varphi}(xH) = \varphi(x)\overline{\varphi(H)}$ *is an isomorphism.*
3. *There exists an index preserving bijection between normal subgroups of p-power index in Γ and open normal subgroups of $\Gamma_{\widehat{p}}$.*

4. *$\overline{\varphi(H)} \simeq H_{\widehat{p}}$ as pro-p groups.*

5. $RG(\Gamma_{\widehat{p}}) = \dfrac{RG(H_{\widehat{p}})}{[\Gamma : H]}$.

Proof. Parts (1)-(3) are proved in [34, Proposition 3.2.2], but we provide proofs here as well. For notational simplicity, assume Γ is residually-p and thus φ is injective. The case of Γ not residually-p is proved similarly.

1. Clearly $H \subseteq \Gamma \cap \overline{H}$. Now, $\Gamma_{\widehat{p}} \simeq \varprojlim \Gamma/K$, where the inverse limit is taken over all normal subgroups of p-power index of Γ. Thus $\Gamma_{\widehat{p}} \simeq \{\prod \Gamma/K \mid gL = \pi_{LK}(gK),\ K \subseteq L\}$ where $\pi_{LK} : \Gamma/K \to \Gamma/L$. Let $x \in \Gamma - H$. Consider $\mathcal{U} = \prod U$ with $U = \{xK\}$ when $H = K$ and $U = \Gamma/K$ otherwise. Then $\mathcal{U}$ is open in the product topology, which implies that $V = \mathcal{U} \cap \Gamma_{\widehat{p}}$ is open in $\Gamma_{\widehat{p}}$. Clearly $x = (xK) \in V$. If $h \in H$, then $h \notin V$ since $hK = K$, when $K = H$. Therefore $x \notin \overline{H}$. Therefore, $\Gamma - H \subseteq \Gamma - (\Gamma \cap \overline{H})$. Thus $\Gamma \cap \overline{H} \subseteq H$.

2. Since Γ is dense in $\Gamma_{\widehat{p}}$ it follows that $\overline{\varphi}(\Gamma/H)$ is dense in $\Gamma_{\widehat{p}}/\overline{H}$, but $\Gamma_{\widehat{p}}/\overline{H}$ is finite, which implies the map is surjective. Now, $\overline{\varphi}(xH) = \overline{\varphi}(yH)$ implies $x\overline{H} = y\overline{H}$ and thus $y^{-1}x \in \overline{H}$. But $\Gamma \cap \overline{H} = H$ by (1) and therefore $y^{-1}x \in H$. Thus $xH = yH$ and the map is injective.

3. The bijection is as follows: If H is a normal subgroup of p-power index in Γ. then send $H \to \overline{H}$. This is index preserving by (2).

   Injective: If $\overline{H} = \overline{K}$, then $H \subseteq \overline{K}$ and $K \subseteq \overline{H}$. Thus, $\Gamma \cap H \subseteq \Gamma \cap \overline{K}$ and $\Gamma \cap K \subseteq \Gamma \cap \overline{H}$, which implies $H \subseteq K$ and $K \subseteq H$. Therefore, $H = K$.

   Surjective: Let L be an open normal subgroup of $\Gamma_{\widehat{p}}$. Since $\Gamma_{\widehat{p}}$ is a finitely generated pro-p group, all open normal subgroups have p-power index. Consider $H = \Gamma \cap L$. Since $H \subseteq L$ then $\overline{H} \subseteq \overline{L}$, which implies $\overline{H} \subseteq L$ since L is open and thus closed. Let $\ell \in L$. Since Γ is dense in $\Gamma_{\widehat{p}}$, it follows that $\ell \in \overline{\Gamma}$. Thus for every open neighborhood U of ℓ the intersection $U \cap \Gamma$ is nonempty. Since L is open, $U \cap L$ is an open neighborhood of ℓ and thus $U \cap L \cap \Gamma \neq \emptyset$, which implies $U \cap H \neq \emptyset$. Therefore $\ell \in \overline{H}$ and thus $L \subseteq \overline{H}$. Therefore, $\overline{H} = L$. It remains to show $H = \Gamma \cap L$ is normal of p-power index. Clearly $H \leq \Gamma$ is normal since $L \leq \Gamma_{\widehat{p}}$ is normal. By (2) we know $[\Gamma : H] = [\Gamma_{\widehat{p}} : \overline{H}] = [\Gamma_{\widehat{p}} : L] = p$-power.

4. We only need to show that the pro-p topology on Γ induces the pro-p topology on the subspace H of Γ. By Proposition 2.2.7, subnormal subgroups of p-power index in Γ form a base for the pro-p topology. If K is subnormal of p-power index in H it implies that K is subnormal of p-power index in Γ. Thus the subspace topology on H and the pro-p topology are the same. Therefore, $\overline{\varphi(H)} \simeq H_{\widehat{p}}$ as pro-p groups.

5. By (2) and (4) it follows that $\Gamma/H \simeq \Gamma_{\widehat{p}}/H_{\widehat{p}}$ and therefore $[\Gamma : H] = [\Gamma_{\widehat{p}} : H_{\widehat{p}}]$. Thus, by Theorem 3.2.1 we have $RG(\Gamma_{\widehat{p}}) = \dfrac{RG(H_{\widehat{p}})}{[\Gamma_{\widehat{p}} : H_{\widehat{p}}]} = \dfrac{RG(H_{\widehat{p}})}{[\Gamma : H]}$.

□

Theorem 3.3.3. *If G is a (topologically) finitely generated pro-p group, then $RG_p(G) = RG(G)$.*

Proof. In a finitely generated pro-p group all finite index normal subgroups are open normal subgroups and have index a power of p [9]. Moreover, if H is a finite index subgroup of G, then H is also a finitely generated pro-p group. The Frattini subgroup of a finitely generated pro-p group H is $\Phi(H) = [H, H]H^p$ and by Theorem 2.2.4, $d_p(H) = d(H/\Phi(H)) = d(H)$. Therefore,

$$RG_p(G) = \inf_{\substack{H \trianglelefteq G \\ [G:H]=p^k}} \frac{d_p(H) - 1}{[G : H]} = \inf_{\substack{H \trianglelefteq G \\ [G:H]<\infty}} \frac{d(H) - 1}{[G : H]} = RG(G).$$

□

It is now possible to prove the relationship between the p-gradient of a group and its pro-p completion.

Theorem 3.3.4. *Let Γ be a finitely generated group and p a fixed prime. Let $\Gamma_{\widehat{p}}$ be the pro-p completion of Γ. Then $RG_p(\Gamma) = RG_p(\Gamma_{\widehat{p}}) = RG(\Gamma_{\widehat{p}})$.*

Proof. We start by assuming the case that Γ is residually-p. Then there is an injective map $\varphi : \Gamma \to \Gamma_{\widehat{p}}$ such that $\varphi(\Gamma) = \Gamma$ is dense in $\Gamma_{\widehat{p}}$. Therefore, if Γ is finitely generated then $\Gamma_{\widehat{p}}$ is finitely generated as a pro-p group. In a finitely generated pro-p group all finite index subgroups are open normal subgroups and have index a power of p [9]. Throughout this proof, the notation $\overline{X}$ will mean the closure of X in $\Gamma_{\widehat{p}}$. By Proposition 3.3.2.3 we know that $H \to \overline{H}$ is an index preserving bijection between the normal subgroups of p-power index in Γ and the normal subgroups of p-power index in $\Gamma_{\widehat{p}}$.

Since $RG_p(\Gamma) = RG_p(\Gamma_{\widehat{p}})$ if $d_p(H) = d_p(\overline{H})$ for all p-power index normal subgroups $H \leq \Gamma$, it suffices to show

$$H/[H, H]H^p \simeq \overline{H}/\left[\overline{H}, \overline{H}\right] \overline{H}^p.$$

By Proposition 3.3.2.4, $\overline{H} \simeq H_{\widehat{p}}$ as pro-p groups. Also, H is residually-p and thus the natural map $\psi : H \to H_{\widehat{p}}$ is injective. Therefore, by Proposition 3.3.2.2 we have $H/[H, H]H^p \simeq H_{\widehat{p}}/\operatorname{closure}_{H_{\widehat{p}}}([H, H]H^p)$. Since $[H, H]H^p \subseteq H$ it implies $\overline{[H, H]H^p} \subseteq \overline{H}$. Therefore,

$$H_{\widehat{p}}/\operatorname{closure}_{H_{\widehat{p}}}([H, H]H^p) \simeq \overline{H}/\left(\overline{H} \cap \overline{[H, H]H^p}\right) \simeq \overline{H}/\overline{[H, H]H^p}.$$

Thus, $H/[H,H]H^p \simeq \overline{H}/\,\overline{[H,H]H^p}$ and so its remains to show

$$[\overline{H},\overline{H}]\overline{H}^p = \overline{[H,H]H^p}.$$

"$\supseteq$" Clearly, $\Phi(\overline{H}) = [\overline{H},\overline{H}]\overline{H}^p \supseteq [H,H]H^p$ with $\Phi(\overline{H})$ the Frattini subgroup of $\overline{H}$. We note that $\Phi(\overline{H})$ is open and thus closed. Thus, $[\overline{H},\overline{H}]\overline{H}^p \supseteq \overline{[H,H]H^p}$.

"$\subseteq$" For ease of notation let $B = [H,H]H^p$. We know $\overline{H}/\overline{B} \simeq H/B$. Thus,

$$\frac{[\overline{H},\overline{H}]\overline{H}^p}{\overline{B}} \simeq \left[\frac{\overline{H}}{\overline{B}},\frac{\overline{H}}{\overline{B}}\right]\left(\frac{\overline{H}}{\overline{B}}\right)^p \simeq \left[\frac{H}{B},\frac{H}{B}\right]\left(\frac{H}{B}\right)^p \simeq \frac{[H,H]H^p}{B} = 1.$$

Therefore, we have $[\overline{H},\overline{H}]\overline{H}^p \subseteq \overline{[H,H]H^p}$.

For a residually-p group $RG_p(\Gamma) = RG_p(\Gamma_{\widehat{p}})$. However, if Γ is not residually-p, let $\widetilde{\Gamma}$ be the p-residualization of Γ. Then by Lemma 3.3.1 we know $RG_p(\Gamma) = RG_p(\widetilde{\Gamma})$ and $\widetilde{\Gamma}_{\widehat{p}} \simeq \Gamma_{\widehat{p}}$. Therefore, $RG_p(\Gamma) = RG_p(\widetilde{\Gamma}) = RG_p(\widetilde{\Gamma}_{\widehat{p}}) = RG_p(\Gamma_{\widehat{p}})$.

The fact that $RG_p(\Gamma) = RG(\Gamma_{\widehat{p}})$, where $\Gamma_{\widehat{p}}$ is the pro-p completion of Γ, follows by the above remarks and Theorem 3.3.3. □

The above two theorems provide some useful corollaries.

Corollary 3.3.5. *If Γ is a finite group, then $RG_p(\Gamma) = -\dfrac{1}{|\Gamma_{\widehat{p}}|}$.*

Proof. If Γ is finite, then so is $\Gamma_{\widehat{p}}$ and thus $RG_p(\Gamma) = RG(\Gamma_{\widehat{p}}) = -\frac{1}{|\Gamma_{\widehat{p}}|}$ by Theorem 3.2.1. □

Theorem 3.3.6. *Fix a prime p and let Γ be a finitely generated group. Assume $H \leq \Gamma$ is a p-power index subnormal subgroup. Then $RG_p(\Gamma) = \dfrac{RG_p(H)}{[\Gamma:H]}$.*

Proof. Since H is subnormal of p-power index, then there exist subgroups $H = H_0, H_1, H_2, \ldots, H_n = \Gamma$ such that $H_i \leq H_{i+1}$ is normal and $[H_{i+1} : H_i]$ is a p-power. We will induct on the subnormal length of H. Assume H is 1-subnormal and thus H is normal in Γ. By Proposition 3.3.2.5 and Corollary 3.3.3 it follows that

$$RG_p(\Gamma) = RG(\Gamma_{\widehat{p}}) = \frac{RG(H_{\widehat{p}})}{[\Gamma:H]} = \frac{RG_p(H)}{[\Gamma:H]}.$$

Now, assume H is n-subnormal. Then H_{n-1} is normal in Γ and therefore, $RG_p(\Gamma) = \dfrac{RG_p(H_{n-1})}{[\Gamma:H_{n-1}]}$. Also, H is (n-1)-subnormal in H_{n-1} and thus by induction $RG_p(H_{n-1}) = \dfrac{RG_p(H)}{[H_{n-1}:H]}$. Therefore, $RG_p(\Gamma) = \dfrac{1}{[\Gamma:H_{n-1}]}\dfrac{RG_p(H)}{[H_{n-1}:H]} = \dfrac{RG_p(H)}{[\Gamma:H]}$. □

3.4 Groups With Zero Rank Gradient

There are large classes of groups that have zero rank gradient: infinite discrete amenable groups [1,2], ascending HNN extensions [2], direct products of infinite residually finite groups, mapping class groups of genus bigger than 1 [16], $Aut(F_n)$ for all n [16], $Out(F_n)$ for $n \geq 3$ [16], and any Artin group whose underlying graph is connected [16].

It is easy to show why the direct product of infinite residually finite groups has zero rank gradient. Let $\Gamma = G \times H$ where G and H are infinite residually finite groups. Let $\{A_i\}$ and $\{B_i\}$ be infinite descending chains of normal subgroups of finite index in G and H respectively with trivial intersection. Then $\{A_i \times B_i\}$ is an infinite descending chain of normal subgroups of finite index in Γ with trivial intersection. Now,

$$\begin{aligned}
&RG(\Gamma, \{A_i \times B_i\}) = \lim_{i\to\infty} \frac{d(A_i \times B_i) - 1}{[\Gamma : A_i \times B_i]} \leq \lim_{i\to\infty} \frac{d(A_i) + d(B_i) - 1}{[G : A_i][H : B_i]} \\
&= \lim_{i\to\infty} \left(\frac{1}{[H : B_i]} \frac{d(A_i) - 1}{[G : A_i]} + \frac{1}{[G : A_i]} \frac{d(B_i) - 1}{[H : B_i]} + \frac{1}{[G : A_i][H : B_i]} \right) \\
&= \lim_{i\to\infty} \frac{1}{[H : B_i]} \lim_{i\to\infty} \frac{d(A_i) - 1}{[G : A_i]} + \lim_{i\to\infty} \frac{1}{[G : A_i]} \lim_{i\to\infty} \frac{d(B_i) - 1}{[H : B_i]} + \lim_{i\to\infty} \frac{1}{[G : A_i][H : B_i]} \\
&= 0 \cdot RG(G, \{A_i\}) + 0 \cdot RG(H, \{B_j\}) + 0.
\end{aligned}$$

Therefore $RG(\Gamma) \leq 0$, which implies $RG(\Gamma) = 0$.

Another class of groups with zero rank gradient is the class of polycyclic groups. A group is called *polycyclic* if it contains a finite subnormal series with cyclic quotients. By definition, it is easy to see that polycyclic groups are finitely generated. Let Γ be polycyclic, then there exists $k \in \mathbb{N}$ such that $d(H) \leq k$ for every finite index subgroup $H \leq \Gamma$. K. Hirsch [15, Theorem 3.25] proved that polycyclic groups are residually finite and therefore Γ is residually finite. Thus Γ contains a lattice of subgroups of arbitrarily large index. Therefore,

$$0 \leq RG(\Gamma) \leq RG(\Gamma, \{H_n\}) = \lim_{n\to\infty} \frac{d(H_n) - 1}{[\Gamma : H_n]} \leq \lim_{n\to\infty} \frac{k - 1}{[\Gamma : H_n]} = 0.$$

An infinite finitely generated nilpotent group is polycyclic [13, Theorem 10.2.4]. It should be noted that finitely generated is essential for an infinite finitely generated nilpotent group to be polycyclic: since every quotient in the lower central series is finitely generated abelian, one can "fill in" the lower central series with additional subgroups so that every quotient is cyclic.

This result can be generalized to the class of amenable groups. However, the proof is more complex than the proof for polycyclic groups. Abert, Jaikin-Zapirain, and Nikolov [1] proved that discrete infinite amenable groups have rank gradient zero with respect to any normal chain of finite index subgroups with trivial intersection.

Lackenby first proved the result for finitely presented groups [20]. The proof of this result uses the following theorem of B. Weiss [38].

Theorem 3.4.1 (Weiss)**.** *Let* Γ *be an amenable group generated by a finite set* S *and let* $\{H_n\}$ *be a normal chain in* Γ *with trivial intersection. Then for every* $\varepsilon > 0$ *there exists* $k \in \mathbb{N}$ *and a transversal* T *of* H_k *in* Γ *such that* $|TS \setminus T| < \varepsilon|T|$.

Proof. For a full proof the reader is referred to [1]. The idea of the proof is as follows: Let $\delta = \frac{0.1}{1.1e}$. Since Γ is amenable, then by the Følner sequence definition of amenability there exists a δ-invariant subset A with respect to the generating set S. Since the chain $\{H_n\}$ has trivial intersection, there exists an H_j such that the image of every element of A in $\overline{\Gamma} = \Gamma/H_j$ is unique. One can show that there exists a set X of Γ such that $|\overline{AX}| \geq (1 - \frac{1}{e})|\overline{\Gamma}|$. Let B be a subset of AX such that $\overline{B} = \overline{AX}$. Then $|\partial_S(B)| \leq \frac{1.4}{e-1}|S||B|$. By adding elements to B one can obtain a transversal T_1 for H_j in Γ with $|\partial_S(T_1)| \leq \frac{2.4}{e}|S||T_1|$. Letting $c = \frac{2.4}{e}$ we have that T_1 is c-invariant.

Applying the above argument to H_j yields a subgroup H_ℓ and a transversal T_2 of H_ℓ in H_j such that T_2 is c-invariant with respect to the generating set $S_1 = \{\overline{st}^{-1}st \mid (t, st) \in \partial_S(T_1)\}$ (where $\overline{st} \in T_2$). The set S_1 is generating by the Reidemeister-Schreier theorem. One then shows that $T' = T_1T_2$ is a transversal of H_ℓ in Γ with

$$|\partial_S(T')| = |\partial_{S_1}(T_2)| \leq c|S_1||T_2| = c|\partial_S(T_1)||T_2| \leq c^2|S||T|.$$

Iterating this process r times will yield a subgroup H_k and a transversal T that is c^r-invariant. The result follows. □

Theorem 3.4.2 (Abert, Jaikin-Zapirain, Nikolov)**.** *Finitely generated discrete infinite amenable groups have rank gradient zero with respect to any normal chain with trivial intersection.*

Proof. We will use the notation from the Følner definition of amenability and Theorem 3.4.1. Let Γ be a finitely generated infinite amenable group and let $\{H_n\}$ be a normal chain of finite index subgroups of Γ with trivial intersection. By Theorem 3.4.1, for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ and a transversal T of H_k in Γ such that $|TS \setminus T| < \varepsilon|T|$. The Reidemeister-Schreier theorem shows that $\{\overline{st}^{-1}st \mid (t, st) \in \partial_S(T)\}$ is a generating set of H_k. The size of this set is $|\partial_S(T)| = |TS \setminus T| < \varepsilon|T|$ and therefore, $d(H_k) - 1 \leq \varepsilon[\Gamma : H_k]$. It follows that

$$RG(\Gamma, \{H_n\}) = \lim_{n\to\infty} \frac{d(H_n) - 1}{[\Gamma : H_n]} \leq \frac{d(H_k) - 1}{[\Gamma : H_k]} < \varepsilon.$$

Since $RG(\Gamma, \{H_n\}) < \varepsilon$ for every $\varepsilon > 0$, then $RG(\Gamma, \{H_n\}) = 0$. □

As a simple corollary, we provide a corresponding, albeit weaker, result concerning p-gradient.

Corollary 3.4.3. *If $RG_p(\Gamma) > 0$ for some prime p, then Γ is not amenable.*

Proof. Let Γ be a finitely generated group with $RG_p(\Gamma) > 0$. Let $\widetilde{\Gamma}$ be the p-residualization of Γ. Then $0 < RG_p(\Gamma) = RG_p(\widetilde{\Gamma})$. Let $\{H_i\}$ be a descending chain of normal subgroups of p-power index in $\widetilde{\Gamma}$ with trivial intersection. Then,

$$0 < RG_p(\widetilde{\Gamma}) \leq \inf_i \frac{d_p(H_i) - 1}{[\widetilde{\Gamma} : H_i]} \leq \inf_i \frac{d(H_i) - 1}{[\widetilde{\Gamma} : H_i]} = RG(\widetilde{\Gamma}, \{H_i\}).$$

Therefore, $\widetilde{\Gamma}$ is not amenable by Theorem 3.4.2. This implies that Γ is not amenable since a quotient of an amenable group is amenable. □

We end this section with some simple consequences of the Restricted Burnside Problem, which states that a finitely generated residually finite group with finite exponent is finite. A group has *finite exponent* if every element has finite order and the maximum of the orders is also finite. The Restricted Burnside Problem was proved by Efim Zelmanov for which he received a Fields Medal in 1994.

Theorem 3.4.4. *If Γ is finitely generated and has finite exponent, then $RG_p(\Gamma) < 0$ for any prime p.*

Proof. Fix a prime p. Let $\widetilde{\Gamma}$ be the p-residualization of Γ. Then $\widetilde{\Gamma}$ is finitely generated, residually finite, and has finite exponent. By the positive solution to the Restricted Burnside Problem, $\widetilde{\Gamma}$ is finite. By Lemma 3.3.1, $RG_p(\Gamma) = RG_p(\widetilde{\Gamma}) = \frac{-1}{|\widetilde{\Gamma}_{\widehat{p}}|} < 0$. □

A simple consequence is that if Γ is finitely generated and there exists a prime p such that $RG_p(\Gamma) \geq 0$, then Γ does not have finite exponent.

Theorem 3.4.5. *If Γ is finitely generated, residually finite, and of finite exponent, then $RG(\Gamma) < 0$.*

Proof. This is equivalent to the Restricted Burnside Problem. □

Chapter 4

Arbitrary p-Gradient Values

In this chapter we will prove the main result that there exists a finitely generated group Γ with $RG_p(\Gamma) = \alpha$ for each $\alpha > 0 \in \mathbb{R}$. To prove this, we need some technical results.

4.1 A Lower Bound for p-Gradient

The following lemma is similar to a lemma of Osin [32, Lemma 2.3] concerning deficiency of a finitely presented group. In the next lemma, the notation $\langle X \rangle^G$ means the normal subgroup generated by the set X in the group G.

Lemma 4.1.1. *Let Γ be a finitely generated group and fix a prime p. Let x be some non-trivial element of Γ. Let H be a finite index normal subgroup of Γ such that $x^m \in H$, but no smaller power of x is in H. Let $\pi : \Gamma \to \Gamma / \langle x^m \rangle^\Gamma$ be the standard projection homomorphism.*

1. *If T is a right transversal for $\langle x \rangle H$ in Γ, then $\langle x^m \rangle^\Gamma =$ $\langle t x^m t^{-1} \mid t \in T \rangle^H$.*
2. *If $H = \langle Y \mid R \rangle$, then $\pi(H) = \langle Y \mid R \cup \{t x^m t^{-1} \mid t \in T\} \rangle$.*
3. *$|T| = \dfrac{[\Gamma : H]}{m}$.*
4. *If $\mathfrak{q}(H) = \dfrac{d_p(H)}{[\Gamma : H]}$, then $\mathfrak{q}(\pi(H)) \geq \mathfrak{q}(H) - \frac{1}{m}$.*

Proof. Since x^m is in H, then $[\pi(\Gamma) : \pi(H)] = [\Gamma : H]$.

1. The inclusion $\supseteq$ is clear. Let $g x^m g^{-1} \in \langle x^m \rangle^\Gamma$. Then $g = t x^j h$ for some $t \in T$

and $h \in H$ and therefore

$$\begin{aligned} gx^m g^{-1} &= (tx^j h)x^m (tx^j h)^{-1} \\ &= t(x^j h x^{-j})(x^j x^m x^{-j})(x^j h^{-1} x^{-j})t^{-1} \\ &= th_0 x^m h_0^{-1} t^{-1} (\text{since H is normal}) \\ &= (th_0 t^{-1}) tx^m t^{-1} (th_0^{-1} t^{-1}) \\ &= \tilde{h}_0 tx^m t^{-1} \tilde{h}_0^{-1} (\text{again, since H is normal}) \end{aligned}$$

This shows that the inclusion $\subseteq$ holds.

2. This holds by (1) and the fact that $\pi(H) = H/(H \cap \langle x^m \rangle^\Gamma) = H/\langle x^m \rangle^\Gamma$, since $x^m \in H$ and H is normal in Γ.

3. Since $H \subseteq \langle x \rangle H \subseteq \Gamma$, then $[\Gamma : H] = [\Gamma : \langle x \rangle H][\langle x \rangle H : H]$. Therefore, $|T| = [\Gamma : \langle x \rangle H] = \frac{[\Gamma:H]}{[\langle x \rangle H:H]}$. Since $x^m \in H$ but no smaller power of x is in H, then $V = \{1, x, x^2, \ldots, x^{m-1}\}$ is a transversal for H in $\langle x \rangle H$ and thus $[\langle x \rangle H : H] = m$. Therefore, $|T| = \frac{[\Gamma:H]}{m}$.

4. First, note that (2) and (3) imply that a presentation for $\pi(H)$ is obtained from a presentation for H by adding in $\frac{[\Gamma:H]}{m}$ relations. Now, $\mathfrak{q}(\pi(H)) \geq \mathfrak{q}(H) - \frac{1}{m}$ if and only if $d_p(\pi(H)) \geq d_p(H) - \frac{[\Gamma:H]}{m}$. If H has presentation $H = \langle Y \mid R \rangle$ then $\pi(H)$ has presentation $\pi(H) = \langle Y \mid R \cup \{tx^m t^{-1} \text{ for all } t \in T\} \rangle$. For notational simplicity let $C = \{[y_1, y_2] \mid y_1, y_2 \in Y\}$. Then,

$$H/([H,H]H^p) = \langle Y \mid R,\ C,\ w^p \text{ for all } w \in F(Y) \rangle$$

where $F(Y)$ is the free group on Y and

$$\begin{aligned} \pi(H)/([\pi(H), \pi(H)]\pi(H)^p) = \langle Y \mid R,\ C,\ w^p &\text{ for all } w \in F(Y), \\ &tx^m t^{-1} \text{for all } t \in T \rangle. \end{aligned}$$

Therefore, a presentation for $\pi(H)/([\pi(H), \pi(H)]\pi(H)^p)$ is obtained from a presentation for $H/([H,H]H^p)$ by adding in $\frac{[\Gamma:H]}{m}$ relations.

Note: $\Gamma/[\Gamma,\Gamma]\Gamma^p$ can be considered as a vector space over $\mathbb{F}_p$ and $d_p(\Gamma)$ is the dimension of this vector space. Therefore, $\pi(H)/([\pi(H), \pi(H)]\pi(H)^p)$ is a vector space satisfying $\frac{[\Gamma:H]}{m}$ more equations than the vector space $H/([H,H]H^p)$. Thus $d_p(\pi(H)) \geq d_p(H) - \frac{[\Gamma:H]}{m}$.

□

Lemma 4.1.2. *Let Γ be a finitely generated group. Let $x \in \Gamma$ and suppose there exists a normal subgroup of finite index H_0 such that the order of x in Γ/H_0 is m. Then for every normal subgroup K of finite index such that $x^m \in K$, there exists a normal subgroup L of finite index in Γ such that $x^m \in L$, $L \subseteq K$, and the order of x in Γ/L is m.*

Proof. Consider $L = K \cap H_0$. Since x^m is in K and H_0 then x^m is in L. Since x^m is in L, then the order of x in Γ/L must divide m, say the order is r. Then $x^r \in L \subseteq H_0$ and thus x^r is 1 in Γ/H_0 which is a contradiction to the order of x in Γ/H_0. □

A lower bound for the p-gradient when taking the quotient by the normal subgroup generated by an element raised to a p-power follows by the above lemmas.

Theorem 4.1.3. *Let Γ be a finitely generated group, p some fixed prime, and $x \in \Gamma$. Then $RG_p(\Gamma/\langle\langle x^{p^k}\rangle\rangle) \geq RG_p(\Gamma) - \frac{1}{p^k}$.*

Proof. Case 1: There exists a normal subgroup H_0 of p-power index such that the order of x in Γ/H_0 is at least p^k.

In any p-group K, we can construct an ascending chain of normal subgroups $1 = K_0 \leq K_1 \leq \cdots \leq K_n = K$ such that each subgroup has index p in the next subgroup. Since H_0 is a normal subgroup of p-power index, then Γ/H_0 is a p-group and applying this to Γ/H_0 and then taking full preimages will result in a chain of normal subgroups of p-power index in Γ, $H_0 \leq H_1 \leq \cdots \leq H_n = \Gamma$ such that each subgroup has index p in the next subgroup. Thus there is a subgroup H_i in this chain such that x has order precisely p^k in Γ/H_i.

Since H_0 is a normal subgroup of p-power index, then by the above remark we may assume without loss of generality that the order of x in Γ/H_0 is exactly p^k. Let $\overline{H}$ be a normal subgroup of p-power index in $\overline{\Gamma} = \Gamma/\langle\langle x^{p^k}\rangle\rangle$. Let $H \leq \Gamma$ be the full preimage of $\overline{H}$. Then H is a p-power index normal subgroup in Γ which contains $\langle\langle x^{p^k}\rangle\rangle$. Let $L_H = H \cap H_0$. Then L_H is a normal subgroup in Γ such that $x^{p^k} \in L_H$, $L_H \subseteq H$, and the order of x in Γ/L_H is p^k. Note that L_H is normal and of p-power index in Γ since both H and H_0 are normal and of p-power index. Thus by Lemma 4.1.1, $\mathfrak{q}(\overline{H}) \geq \mathfrak{q}(\overline{L_H}) \geq \mathfrak{q}(L_H) - \frac{1}{p^k}$, which by definition is greater than or equal to $RG_p(\Gamma) - \frac{1}{p^k}$. Therefore, $\mathfrak{q}(\overline{H}) \geq RG_p(\Gamma) - \frac{1}{p^k}$. Thus $RG_p(\Gamma/\langle\langle x^{p^k}\rangle\rangle) \geq RG_p(\Gamma) - \frac{1}{p^k}$.

Case 2: For every normal subgroup H of p-power index, the order of x in Γ/H is less than p^k.

It will be shown that $RG_p(\Gamma/\langle\langle x^{p^k}\rangle\rangle) = RG_p(\Gamma)$ in this case. There exists an $\ell < k$ such that $x^{p^\ell} \in H$ for every normal subgroup H of p-power index in Γ. Then x^{p^ℓ} is in the kernel of the natural map from Γ to its pro-p completion $\varphi : \Gamma \to \Gamma_{\widehat{p}}$. Therefore, $x^{p^k} = (x^{p^\ell})^{p^{k-\ell}} \in \ker\varphi$. Let $M = \langle\langle x^{p^k}\rangle\rangle$. Then $M \subseteq \ker\varphi$. This implies that there is a bijective correspondence between all normal subgroups of p-power index in Γ and Γ/M given by $N \to N/M$. Since $\Gamma/N \simeq (\Gamma/M)/(N/M)$ for all such N, then by the inverse limit definition of pro-p completions, $\Gamma_{\widehat{p}} \simeq (\Gamma/M)_{\widehat{p}}$ as pro-p groups. Therefore, $RG_p(\Gamma/\langle\langle x^{p^k}\rangle\rangle) = RG_p(\Gamma/M) = RG_p((\Gamma/M)_{\widehat{p}}) = RG_p(\Gamma_{\widehat{p}}) = RG_p(\Gamma)$. □

Remark 4.1.4. The above theorem was independently stated and proved using different language and a different method by Barnea and Schlage-Puchta [5, Theorem 3]).

Corollary 4.1.5. *Let Γ be a finitely generated group, p a fixed prime, and let $x \in \Gamma$. Then $RG_p(\Gamma/\langle\langle x\rangle\rangle) \geq RG_p(\Gamma) - 1$.*

4.2 p-Gradient and Direct Limits

Let $(I, \leq)$ be a totally ordered set with smallest element 0 and let $\{\Gamma_i \mid \pi_{ij}\}$ be a direct system of finitely generated groups with surjective homomorphisms $\pi_{ij} : \Gamma_i \to \Gamma_j$ for every $i \leq j \in I$.

Let $\Gamma_\infty = \varinjlim \Gamma_i$ be the direct limit of this direct system. Let $\pi_i : \Gamma_i \to \Gamma_\infty$ be the map obtained from the direct limit. Because all the maps in the direct system are surjective, then so are the π_i. Let $\Gamma = \Gamma_0$.

Another direct system $\{M_i \mid \mu_{ij}\}$ can be defined over the same indexing set I, where $M_i = \Gamma$ for each i and μ_{ij} is the identity map. The direct limit of this set is clearly $\Gamma = \varinjlim M_i$ and the map obtained from the direct limit $\mu_i : M_i \to \Gamma$ is the identity map.

A homomorphism $\Phi : \{M_i \mid \mu_{ij}\} \to \{\Gamma_i \mid \pi_{ij}\}$ is by definition a family of group homomorphisms $\varphi_i : M_i \to \Gamma_i$ such that $\varphi_j \circ \mu_{ij} = \pi_{ij} \circ \varphi_i$ whenever $i \leq j$. Then Φ defines a unique homomorphism $\varphi = \varinjlim \varphi_i : \varinjlim M_i \to \varinjlim \Gamma_i$ such that $\varphi \circ \mu_i = \pi_i \circ \varphi_i$ for all $i \in I$ [4].

The surjection $\varphi_i : \Gamma \to \Gamma_i$ is the map π_{0i} in this case. It is clear that $\varphi = \varinjlim \varphi_i$. Since each φ_i is surjective, it implies that $\ker \varphi_i \subseteq \ker \varphi_j$ for every $i \leq j$. In this situation,

$$\ker \varphi = \varinjlim \ker \varphi_i = \bigcup_{i \in I} \ker \varphi_i.$$

Let $H \leq \Gamma$ be a subgroup. For every i, let $H_i = \varphi_i(H)$.

Lemma 4.2.1. *Keep the notation defined above. Fix a prime p. For each $K \leq \Gamma_\infty$ normal of p-power index, there exists an $H' \leq \Gamma$ normal of p-power index such that:*

1. $K = \varinjlim H'_i$.
2. $[\Gamma_\infty : K] = \lim_{i \in I} [\Gamma_i : H'_i]$.
3. $d_p(K) = \lim_{i \in I} d_p(H'_i)$.

Proof. Let $K \leq \Gamma_\infty$ be a p-power index normal subgroup. Since $\varphi : \Gamma \to \Gamma_\infty$ is surjective then $\Gamma_\infty \simeq \Gamma/\ker \varphi$. Let $H' = \varphi^{-1}(K)$. Then H' is normal in Γ and since $K \simeq H'/\ker \varphi$ then $[\Gamma_\infty : K] = [\Gamma : H']$ and so H' is of p-power index.

1. $K = \varphi(H') = \varinjlim \varphi_i(H') = \varinjlim H'_i$.

2. Since each $\varphi_i : \Gamma \to \Gamma_i$ is surjective, then $\Gamma_i \simeq \Gamma / \ker \varphi_i$ and since H' contains $\ker \varphi$, then H' contains $\ker \varphi_i$ for each i. Thus, $H'_i \simeq H' / \ker \varphi_i$. Therefore for every i,
$$\Gamma_i / H'_i \simeq \Gamma / H' \simeq \Gamma_\infty / K.$$
Thus, $[\Gamma_\infty : K] = [\Gamma_i : H'_i]$ for every i.

3. For any group A, let $Q(A) = A/[A, A]A^p$. It is known that $K \simeq H' / \ker \varphi$ and $H'_i \simeq H' / \ker \varphi_i$ and therefore,
$$Q(K) \simeq H'/[H', H'](H')^p \ker \varphi \simeq Q(H')/M$$
where $M = [H', H'](H')^p \ker \varphi / [H', H'](H')^p$, and
$$Q(H'_i) \simeq H'/[H', H'](H')^p \ker \varphi_i \simeq Q(H')/M_i$$
where $M_i = [H', H'](H')^p \ker \varphi_i / [H', H'](H')^p$. Since $\ker \varphi_i \subseteq \ker \varphi_j$ for each $i \leq j$ then $M_i \subseteq M_j$ for each $i \leq j$. Now, $Q(H')$ is finitely generated abelian and torsion and therefore is finite. Thus $Q(H')$ can only have finitely many non-isomorphic subgroups. Since $\{M_i\}$ is an ascending set of subgroups, there must exist an $n \in I$ such that $M_i = M_n$ for every $i \geq n$. Since $\ker \varphi_i \subseteq \ker \varphi_j$ for each $i \leq j$ and $\bigcup \ker \varphi_i = \ker \varphi$, it follows that $M_i \subseteq M_j$ for every $i \leq j$ and $\bigcup M_i = M$. Therefore, $M = \bigcup M_i = M_n$. Thus for each $i \geq n$, $M = M_i$.

   Therefore, $Q(K) \simeq Q(H'_i)$ for each $i \geq n$ which implies $d_p(K) = d_p(H'_i)$ for each $i \geq n$. Thus, $d_p(K) = \lim_{i \in I} d_p(H'_i)$.

□

The following lemma is similar to Pichot's related result for L^2-Betti numbers where convergence is in the space of marked groups [33].

Lemma 4.2.2. *For each prime p, $\limsup RG_p(\Gamma_i) \leq RG_p(\Gamma_\infty)$.*

Proof. Fix a prime p. Let $K \leq \Gamma_\infty$ be a normal subgroup of p-power index. By Lemma 4.2.1 we obtain the subgroups H' and H'_i for each i. Now,
$$\limsup RG_p(\Gamma_i) = \limsup \inf_{\substack{N \trianglelefteq \Gamma_i \\ p\text{-power}}} \frac{d_p(N) - 1}{[\Gamma_i : N]} \leq \limsup \frac{d_p(H'_i) - 1}{[\Gamma_i : H'_i]}$$
and by Lemma 4.2.1
$$\limsup \frac{d_p(H'_i) - 1}{[\Gamma_i : H'_i]} = \lim_{i \in I} \frac{d_p(H'_i) - 1}{[\Gamma_i : H'_i]} = \frac{d_p(K) - 1}{[\Gamma_\infty : K]}.$$
Therefore, for each $K \leq \Gamma_\infty$ normal of p-power index, $\limsup RG_p(\Gamma_i) \leq \frac{d_p(K)-1}{[\Gamma_\infty : K]}$. This implies $\limsup RG_p(\Gamma_i) \leq RG_p(\Gamma_\infty)$. □

4.3 The Main Result and Applications

It is now possible to prove the main result that every nonnegative real number is realized as the p-gradient of some finitely generated group.

Theorem 4.3.1 (**Main Result**). *For every real number $\alpha > 0$ and any prime p, there exists a finitely generated group Γ such that $RG_p(\Gamma) = \alpha$.*

Proof. Fix a prime p and a real number $\alpha > 0$. Let F be the free group on $\lceil \alpha \rceil + 1$ generators. Let

$$\Lambda = \{G \mid F \text{ surjects onto } G,\, G \text{ is residually-}p\text{, and } RG_p(G) \geq \alpha\}.$$

Since for any free group $d(F) = d_p(F)$ it is clear that $RG_p(F) = \text{rank}(F) - 1$ and therefore, Λ is not empty since $F \in \Lambda$. Partially order Λ by $G_1 \succcurlyeq G_2$ if there is an epimorphism from G_1 to G_2, denoted $G_1 \twoheadrightarrow G_2$. This order is antisymmetric since each group in this set is Hopfian.

Let $\mathcal{C} = \{G_i\}$ be a chain in Λ. Each chain forms a direct system of groups over a totally ordered indexing set. Any chain can be extended so that it starts with the element $F = G_0$. Let $G_\infty = \varinjlim G_i$.

By Lemma 4.2.2, $RG_p(G_\infty) \geq \limsup RG_p(G_i) \geq \alpha$. Let $\widetilde{G}_\infty$ be the p-residualization of G_∞. By Lemma 3.3.1, $RG_p(\widetilde{G}_\infty) = RG_p(G_\infty)$. Therefore, $RG_p(\widetilde{G}_\infty) \geq \alpha$ and $\widetilde{G}_\infty$ is residually-p. Moreover, for each i, $G_i \twoheadrightarrow G_\infty$ and in particular $F \twoheadrightarrow G_\infty \twoheadrightarrow \widetilde{G}_\infty$. Thus $\widetilde{G}_\infty \in \Lambda$ and $G_i \succcurlyeq \widetilde{G}_\infty$ for each i. Thus, each chain $\mathcal{C}$ in Λ has a lower bound in Λ and therefore by Zorn's Lemma, Λ has a minimal element, call it Γ.

Since Γ and its p-residualization $\widetilde{\Gamma}$ have the same p-gradient and Γ surjects onto $\widetilde{\Gamma}$, it implies that $\widetilde{\Gamma} \in \Lambda$ and $\Gamma \succcurlyeq \widetilde{\Gamma}$. Thus Γ must be residually-p, otherwise $\widetilde{\Gamma}$ contradicts the minimality of Γ.

<u>Note:</u> Γ does not have finite exponent.

If Γ had finite exponent then since Γ is finitely generated and residually finite it must be finite by the positive solution to the Restricted Burnside Problem [40]. This would imply $RG_p(\Gamma) < 0$ by Corollary 3.3.5. This contradicts that Γ is in Λ.

Therefore, Γ is a finitely generated residually-p group with infinite exponent such that $RG_p(\Gamma) \geq \alpha$.

<u>Claim:</u> $RG_p(\Gamma) = \alpha$.

Assume not. Then there exists a $k \in \mathbb{N}$ such that $RG_p(\Gamma) - \frac{1}{p^k} \geq \alpha$. Since Γ is residually-p, the order of every element is a power of p and since Γ has infinite exponent, there exists an $x \in \Gamma$ whose order is greater than p^k.

Consider $\Gamma' = \Gamma / \langle\langle x^{p^k} \rangle\rangle$. Since $x^{p^k} \neq 1$ it implies that $\Gamma' \not\cong \Gamma$. By Theorem 4.1.3, $RG_p(\Gamma') \geq RG_p(\Gamma) - \frac{1}{p^k} \geq \alpha$. If Γ' is not residually-p, replace it with its p-residualization, which will have the same p-gradient. Then $\Gamma' \in \Lambda$ and $\Gamma \succcurlyeq \Gamma'$, which contradicts the minimality of Γ. $\square$

The result of Theorem 4.3.1 can be strengthened without much effort.

Theorem 4.3.2. *Fix a prime p. For every real number $\alpha > 0$ there exists a finitely generated residually-p torsion group Γ such that $RG_p(\Gamma) = \alpha$.*

Proof. Barnea and Schlage-Puchta showed [5, Corollary 4] that for any $\alpha > 0$ there exists a torsion group $\mathcal{G}$ with $RG_p(\mathcal{G}) \geq \alpha$. Applying the construction in Theorem 4.3.1, replacing the free group F with the p-residualization of $\mathcal{G}$, will result in a group Γ that is torsion, residually-p, and $RG_p(\Gamma) = \alpha$.

For completeness we provide a full proof that there exists a finitely generated torsion group with p-gradient greater than α, which was proved independently from Barnea and Schlage-Puchta.

Let $\beta = \alpha + \frac{1}{p-1}$. By Theorem 4.3.1 there exists a finitely generated group $\mathcal{G}$ such that $RG_p(\mathcal{G}) = \beta$. Since $\mathcal{G}$ is countable, let $x_1, x_2, \dots$ be the non-torsion elements of $\mathcal{G}$. Let $k_1 \in \mathbb{N}$ be such that $\beta - \frac{1}{p^{k_1}} > \alpha$. By Case 1 of the proof of Theorem 4.3.1, we can construct $\mathcal{G}_1 = \mathcal{G}/\langle\langle x_1^{p^{k_1}} \rangle\rangle$ and $RG_p(\mathcal{G}_1) \geq \beta - \frac{1}{p^{k_1}}$. Let $k_2 \in \mathbb{N}$ be such that $RG_p(\mathcal{G}_1) - \frac{1}{p^{k_2}} > \alpha$. By abuse of notation, let $x_2, x_3, \dots$ represent the image of the x_i in $\mathcal{G}_1$. Applying the same process gives $\mathcal{G}_2$ such that $\mathcal{G}_2 = \mathcal{G}_1/\langle\langle x_2^{p^{k_2}} \rangle\rangle$ and $RG_p(\mathcal{G}_2) \geq RG_p(\mathcal{G}_1) - \frac{1}{p^{k_2}} \geq \beta - \frac{1}{p^{k_1}} - \frac{1}{p^{k_2}}$. Continuing this way, we have $k_i \in \mathbb{N}$ such that $RG_p(\mathcal{G}_i) \geq RG_p(\mathcal{G}_{i-1}) - \frac{1}{p^{k_i}} > \alpha$ with $\mathcal{G}_i = \mathcal{G}_{i-1}/\langle\langle x_i^{p^{k_i}} \rangle\rangle$. Moreover, $RG_p(\mathcal{G}_i) \geq \beta - \displaystyle\sum_{j=1}^{i} \frac{1}{p^{k_j}}$. Thus we have the following

$$\mathcal{G} \twoheadrightarrow \mathcal{G}_1 \twoheadrightarrow \mathcal{G}_2 \twoheadrightarrow \cdots .$$

Let $\mathcal{G}_\infty = \varinjlim \mathcal{G}_i$. If $\mathcal{G} = \langle X \mid R \rangle$ is a presentation for $\mathcal{G}$, then by construction $\mathcal{G}_\infty = \langle X \mid R \cup \{x_1^{p^{k_1}}, x_2^{p^{k_2}}, \dots\} \rangle$ is a presentation for $\mathcal{G}_\infty$. Thus $\mathcal{G}_\infty$ is torsion.

By Lemma 4.2.2 we know $RG_p(\mathcal{G}_\infty) \geq \limsup_{i\to\infty} RG_p(\mathcal{G}_i)$ and therefore,

$$\begin{aligned} RG_p(\mathcal{G}_\infty) &\geq \limsup_{i\to\infty} \left(\beta - \sum_{j=1}^{i} \frac{1}{p^{k_j}} \right) \geq \beta - \sum_{j=1}^{\infty} \frac{1}{p^{k_j}} \\ &\geq \beta - \sum_{j=1}^{\infty} \left(\frac{1}{p}\right)^j = \beta - \frac{1}{p-1} = \alpha. \end{aligned}$$

Let $\widetilde{\mathcal{G}}_\infty$ be the p-residualization of $\mathcal{G}_\infty$. Then $\widetilde{\mathcal{G}}_\infty$ is residually-p, torsion, and $RG_p(\widetilde{\mathcal{G}}_\infty) = RG_p(\mathcal{G}_\infty) \geq \alpha$ by Lemma 3.3.1.

If we now apply the construction given in Theorem 4.3.1 replacing the free group F with the p-residualization of $\mathcal{G}_\infty$, the resulting group Γ will be torsion, residually-p and $RG_p(\Gamma) = \alpha$. $\square$

Y. Barnea and J.C. Schlage-Puchta [5] proved a result similar to Theorem 4.3.2 (inequality instead of equality) albeit in a slightly different way.

The construction given in Theorem 4.3.1 has a few immediate applications. First, it is noted that Theorem 4.3.2 gives a known counterexample to the General Burnside Problem, which asks if every finitely generated torsion group is finite. The second application is more general and shows that there exist uncountably many pairwise non-commensurable groups that are finitely generated, infinite, torsion, non-amenable, and residually-p.

The application of the construction used in Theorem 4.3.1 concerning commensurable groups is given below.

Definition. Two groups are called *commensurable* if they have isomorphic subgroups of finite index.

Lemma 4.3.3. *Fix a prime p. Let Γ be a p-torsion group (every element has order a power of p). Then every finite index subgroup $H \leq \Gamma$ is subnormal of p-power index.*

Proof. Let $H \leq \Gamma$ be a finite index subgroup. Let $N \leq \Gamma$ be normal of finite index such that $N \leq H$. Consider Γ/N. Since Γ is p-torsion it implies Γ/N is also p-torsion and by assumption $|\Gamma/N|$ is finite. Let q be a prime that divides $|\Gamma/N|$ and let Q be a Sylow-q subgroup of Γ/N. For any $y \in Q$, the order of y is both a power of p and a power of q, which implies p=q. Thus $[\Gamma : N] = |\Gamma/N| = p$-power and therefore $[\Gamma : H] = p$-power.

Let $\overline{H}$ be the image of H in Γ/N. Since Γ/N is a finite p-group then all subgroups are subnormal, thus $\overline{H}$ is subnormal in Γ/N. Let $\overline{H} = \overline{H_0} \trianglelefteq \overline{H_1} \trianglelefteq \cdots \trianglelefteq \overline{H_k} = \Gamma/N$ be the subnormal chain. Lift these subgroups to Γ to get the chain $H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_k = \Gamma$. Therefore, H is subnormal. □

Theorem 4.3.4. *There exist uncountably many pairwise non-commensurable groups that are finitely generated, infinite, torsion, non-amenable, and residually-p.*

Proof. Let p be a fixed prime number. By Theorem 4.3.2 it is known that for every real number $\alpha > 0$ there exists a finitely generated, residually-p, infinite, torsion group Γ such that $RG_p(\Gamma) = \alpha$. By Corollary 3.4.3 these groups are all non-amenable. Since each of these groups is residually-p and torsion, they are all p-torsion. Thus by Lemma 4.3.3, every subgroup of finite index in these groups is subnormal of p-power index.

By Theorem 3.3.6 if any two of these groups are commensurable, then the p-gradient of each group is a rational multiple of the other. Since there are uncountably many positive real numbers that are not rational multiples of each other, the result can be concluded. □

Chapter 5

Rank Gradient of Free Products, Amalgams, and HNN Extensions

We calculate rank gradient and p-gradient of free products, free products with amalgamation over an amenable group, and HNN extensions with an amenable associated subgroup. For rank gradient, the notion of cost is used to obtain lower bounds for the rank gradient of amalgamated free products and HNN extensions. We will discuss the notion of cost in Section 5.4 as it will be central to our calculation of the rank gradient of amalgamated free products and HNN extensions.

5.1 Rank Gradient of Free Products

We begin this section by computing the rank gradient of the free product of a finite number of finite groups and a free group. We compute the rank gradient in this case using an Euler characteristic and it shows a different approach than what is used in the more general case.

Definition. Let Ω be the smallest class of groups such that

i. Ω contains the trivial group $\{1\}$ and $\mathbb{Z}$.

ii. Ω is closed under finite direct products.

iii. Ω is closed under finite free products.

iv. Ω is closed under taking finite index subgroups and finite index supergroups.

For each group $\Gamma \in \Omega$ one can uniquely define the *rational Euler characteristic*, $\chi(\Gamma) \in \mathbb{Q}$, such that the following properties hold:

1. $\chi(\{1\}) = 1$ and $\chi(\mathbb{Z}) = 0$.

2. $\chi(\Gamma * G) = \chi(\Gamma) + \chi(G) - 1$ for any $\Gamma, G \in \Omega$.

3. $\chi(\Gamma \times G) = \chi(\Gamma)\chi(G)$ for any $\Gamma, G \in \Omega$.

4. If $\Gamma \in \Omega$ and H is a subgroup of index m in Γ, then $\chi(H) = m\chi(\Gamma)$.

This definition of rational Euler characteristic is related to the notion of Euler characteristic in topology. That is, if Γ is a group with a finite CW-complex X as its classifying space, then it should be that $\chi(\Gamma) = \chi(X)$ and we extend this to a larger class of groups.

Lemma 5.1.1. *Let $\Gamma = F_r * A_1 * \cdots * A_n$ be the free product of finitely many finite groups and a free group of rank r. Let $G = A_1 \times \cdots \times A_n$ be the direct product of the A_i. Let $\varphi : \Gamma \to G$ be the natural epimorphism sending F_r to $\{1\}$. The kernel of φ is called the Cartesian subgroup of Γ.*

1. *$C = \ker \varphi$ is free.*

2. *$rank(C) - 1 = \left(\prod_{i=1}^{n} |A_i|\right)\left(r + n - 1 - \sum_{i=1}^{n} \frac{1}{|A_i|}\right)$.*

Proof. 1. By the Kurosh Subgroup Theorem for free products (Theorem 2.3.1), $C = F_r * B_1 * \cdots * B_n$ with each B_i conjugate in Γ to F_r or one of the A_i. If B_i is conjugate to A_i then $B_i = wA_iw^{-1}$ for some $w \in \Gamma$. Then for every $b \in B_i$ there exists an $a \in A_i$ such that $b = waw^{-1}$, which implies $0 = \varphi(b) = \varphi(w)\varphi(a)\varphi(w)^{-1} = \varphi(a)$ and thus $a \in C$. This implies $A_i \subseteq C$, which is not true. Thus each B_i is trivial or conjugate to F_r and thus is free. Therefore, C is free.

2. Since $\Gamma/C \simeq A_1 \times \cdots \times A_n$, then $[\Gamma : C] = \prod_1^n |A_i|$. Therefore we know that $\chi(C) = \chi(\Gamma)\prod_1^n |A_i|$. Since C is free, it is the free product of rank(C) copies of $\mathbb{Z}$ and therefore $\chi(C) = -(\text{rank}(C) - 1)$. We calculate that $\chi(A_i) = \frac{1}{|A_i|}$ and thus, $\chi(\Gamma) = \chi(F_r) + \chi(A_1 * \cdots * A_n) - 1 = -r + 1 + \sum_1^n \frac{1}{|A_i|} - (n-1) - 1 = -(r + n - 1 - \sum_1^n \frac{1}{|A_i|})$. Therefore, $-(\text{rank}(C) - 1) = -\prod_{i=1}^n |A_i|(r + n - 1 - \sum_1^n \frac{1}{|A_i|})$ and the result follows.

□

Proposition 5.1.2. *Let $\Gamma = F_r * A_1 * \cdots * A_n$ be the free product of finitely many finite groups and a free group of rank r. Then*

$$RG(\Gamma) = r + n - 1 - \sum_{i=1}^{n} \frac{1}{|A_i|}.$$

Proof. Let C be the Cartesian subgroup of Γ as defined above. By Theorem 3.2.1 it follows that $RG(\Gamma) = \frac{RG(C)}{[\Gamma:C]}$. By Lemma 5.1.1, the group C is free and therefore

$$RG(C) = \text{rank}(C) - 1 = \left(\prod_{i=1}^{n} |A_i|\right)\left(r + n - 1 - \sum_{i=1}^{n} \frac{1}{|A_i|}\right).$$

Thus,

$$RG(\Gamma) = \frac{\left(\prod_{i=1}^{n} |A_i|\right)\left(r+n-1-\sum_{i=1}^{n}\frac{1}{|A_i|}\right)}{[\Gamma : C]} = r+n-1-\sum_{i=1}^{n}\frac{1}{|A_i|}.$$

□

Abert, Jaikin-Zapirain, and Nikolov [1] computed the rank gradient of a free product of residually finite groups relative to a descending chain of normal subgroups using Bass-Serre theory.

Theorem 5.1.3 (Abert, Jaikin-Zapirain, and Nikolov)**.** *Let Γ_1 and Γ_2 be finitely generated and residually finite. Let $\{H_n\}$ be a normal chain of finite index subgroups in $\Gamma = \Gamma_1 * \Gamma_2$. Then*

$$RG(\Gamma, \{H_n\}) = RG(\Gamma_1, \{\Gamma_1 \cap H_n\}) + RG(\Gamma_2, \{\Gamma_2 \cap H_n\}) + 1.$$

We can prove a similar result to Theorem 5.1.3 by following a similar method of proof. Namely we prove the result for the absolute rank gradient and p-gradient of arbitrary finitely generated groups. By following the proof that Abert, Jaikin-Zapirain, and Nikolov used in [1], one can actually stop after the first paragraph of the proof below. However, we provide the complete proof since the argument also gives the result for p-gradient without much alteration.

Theorem 5.1.4. *Let Γ_1 and Γ_2 be finitely generated groups. Let $\Gamma = \Gamma_1 * \Gamma_2$. Then $RG(\Gamma) = RG(\Gamma_1) + RG(\Gamma_2) + 1$.*

Proof. Let $H_i \leq \Gamma_i$ be a finite index subgroup. Let $\varphi : \Gamma \to \Gamma_1 \times \Gamma_2$ be the natural map. Let $C = \ker \varphi$ be the cartesian subgroup of Γ. First, note that $H_1 \times H_2 \leq \Gamma_1 \times \Gamma_2$ is finite index. Let $A = \varphi^{-1}(H_1 \times H_2)$. Since A is the inverse image of a finite index subgroup, then A is finite index in Γ. Moreover, $C \cap \Gamma_i = 1$. Let $a \in A \cap \Gamma_1$. Then $a \in \Gamma_1$ which implies $\varphi(a) = (a, 1) \in \Gamma_1 \times \{1\}$, but $\varphi(a) \in H_1 \times H_2$. Therefore $(a, 1) \in H_1 \times H_2$ which implies $a \in H_1$. Clearly $H_1 \subseteq A \cap \Gamma_1$ and thus $A \cap \Gamma_1 = H_1$. Similarly $A \cap \Gamma_2 = H_2$. Therefore, for every finite index subgroup $H_i \leq \Gamma_i$ there exists a finite index subgroup $A \leq \Gamma$ such that $A \cap \Gamma_i = H_i$.

Let $H \leq \Gamma$ be a finite index subgroup and let $H_i = H \cap \Gamma_i$, which are finite index. Let $A = \varphi^{-1}(H_1 \times H_2)$. Again, $A \leq \Gamma$ is finite index and $A \cap \Gamma_i = H_i$. Let $A_H = H \cap A$. Then A_H is finite index in Γ, A_H is contained in H, and $A_H \cap \Gamma_i = H_i$. Moreover, by the Schreier index formula,

$$\frac{d(A_H) - 1}{[\Gamma : A_H]} \leq \frac{d(H) - 1}{[\Gamma : H]}.$$

Note that in the case where we start with $H_i \leq \Gamma_i$ and get $A' = \varphi^{-1}(H_1 \times H_2)$, then the procedure described gives $A = A_{A'}$. Moreover, every subgroup of finite index in Γ_i can be obtained from such A_H subgroups of Γ.

Therefore, the rank gradient of Γ can be computed by only looking at the A_H subgroups. Using the Kurosh subgroup theorem for free products (Theorem 2.3.1), Bass-Serre theory, and the Grushko-Neumann theorem ($d(A * B) = d(A) + d(B)$), it follows that if $[\Gamma : A_H] = n$ and $H_i = A_H \cap \Gamma_i$ with $[\Gamma_i : H_i] = k_i$, then

$$d(A_H) = \frac{n}{k_1} d(H_1) + \frac{n}{k_2} d(H_2) + n - \frac{n}{k_1} - \frac{n}{k_2} + 1.$$

This implies

$$\frac{d(A_H) - 1}{[\Gamma : A_H]} = \frac{d(H_1) - 1}{[\Gamma_1 : H_1]} + \frac{d(H_2) - 1}{[\Gamma_2 : H_2]} + 1.$$

Therefore,

$$\begin{aligned} RG(\Gamma) = \inf_{A_H} \frac{d(A_H) - 1}{[\Gamma : A_H]} &= \inf_{A_H} \frac{d(H_1) - 1}{[\Gamma_1 : H_1]} + \inf_{A_H} \frac{d(H_2) - 1}{[\Gamma_2 : H_2]} + 1 \\ &= \inf_{H_1} \frac{d(H_1) - 1}{[\Gamma_1 : H_1]} + \inf_{H_2} \frac{d(H_2) - 1}{[\Gamma_2 : H_2]} + 1 \\ &= RG(\Gamma_1) + RG(\Gamma_2) + 1. \end{aligned}$$

□

Corollary 5.1.5. *Let $\Gamma = \Gamma_1 * \Gamma_2$ be the free product of finitely generated groups and let C be the Cartesian subgroup of Γ. Then*

$$RG(\Gamma) = \inf_{\substack{C \le H \le \Gamma \\ [\Gamma:H] < \infty}} \frac{d(H) - 1}{[\Gamma : H]}.$$

Proof. If $\varphi : \Gamma \to \Gamma_1 \times \Gamma_2$, then $C = \ker \varphi$. Let H be a subgroup of finite index in Γ containing C. Then $H = \varphi^{-1}(H_1 \times H_2)$ where $H_i \le \Gamma_i$ is of finite index. Then by the proof of Theorem 5.1.4,

$$\begin{aligned} \inf_{\substack{C \le H \le \Gamma \\ [\Gamma:H] < \infty}} \frac{d(H) - 1}{[\Gamma : H]} &= \inf_{\substack{C \le H \le \Gamma \\ [\Gamma:H] < \infty}} \frac{d(H_1) - 1}{[\Gamma_1 : H_1]} + \inf_{\substack{C \le H \le \Gamma \\ [\Gamma:H] < \infty}} \frac{d(H_2) - 1}{[\Gamma_2 : H_2]} + 1 \\ &= \inf_{H_1} \frac{d(H_1) - 1}{[\Gamma_1 : H_1]} + \inf_{H_2} \frac{d(H_2) - 1}{[\Gamma_2 : H_2]} + 1 \\ &= RG(\Gamma_1) + RG(\Gamma_2) + 1 = RG(\Gamma). \end{aligned}$$

□

Corollary 5.1.6. *Let $\Gamma = \Gamma_1 * \cdots * \Gamma_k$ be the free product of finitely many finitely generated groups. Then $RG(\Gamma) = k - 1 + \sum_{i=1}^{k} RG(\Gamma_i)$.*

5.2 Rank Gradient of Amalgams Over Finite Subgroups

In this section we compute the rank gradient of amalgamated free products of finite groups using an argument not used in any other case. The argument given in this case is elementary and can be done strictly by computation. We then move to the case of two infinite groups amalgamated over a finite subgroup where a different argument is used.

Lemma 5.2.1. *Let Γ_1 and Γ_2 be finite groups. Let $\Gamma = \Gamma_1 *_A \Gamma_2$ be the amalgamated free product of Γ_1 and Γ_2 over the subgroup A. Then Γ contains a free subgroup H such that*

1. $[\Gamma : H] = \dfrac{|\Gamma_1||\Gamma_2|}{|A|}$.

2. *rank*$(H) = \dfrac{|\Gamma_1||\Gamma_2|}{|A||A|} - \dfrac{|\Gamma_2|}{|A|} - \dfrac{|\Gamma_1|}{|A|} + 1$.

Proof. Fix coset representatives for Γ_1/A and Γ_2/A. Let $\mathcal{S} = \Gamma_1/A \times \Gamma_2/A \times A$ and note that $|\mathcal{S}|$ is finite. Let $a \in \Gamma_1, b \in \Gamma_2$, and $c \in A$. Define an action of Γ_1 and Γ_2 on $\mathcal{S}$ by

$$\begin{aligned} a \cdot (a_i, b_j, c_k) &= (a_s, b_j, c_t) \quad \text{where} \quad a_i c_k a = a_s c_t \\ b \cdot (a_i, b_j, c_k) &= (a_i, b_r, c_q) \quad \text{where} \quad b_j c_k b = b_r c_q \\ c \cdot (a_i, b_j, c_k) &= (a_i, b_j, c_k c). \end{aligned}$$

This action is well-defined and unambiguous on A. This action permutes $\mathcal{S}$ and thus $\varphi : \Gamma \to \mathrm{Symm}(\mathcal{S})$, given by $w \cdot \mathcal{S}$, is a homomorphism. If $\varphi(a) = 1_{\mathrm{Symm}(\mathcal{S})}$ then $a \cdot (a_i, b_j, c_k) = (a_i, b_j, c_k)$, which implies $a_i c_k a = a_i c_k$. It follows that $a = 1_{\Gamma_1}$. Therefore, φ restricted to Γ_1 is injective. It follows similarly that φ restricted to Γ_2 is injective.

Let $H = \ker \varphi$. For every $g \in \Gamma$ consider $H \cap g\Gamma_1 g^{-1}$. If $h \in H \cap g\Gamma_1 g^{-1}$ then $h = gag^{-1}$ with $a \in \Gamma_1$ and $1_{\mathrm{Symm}(\mathcal{S})} = \varphi(h)$. Since φ restricted to Γ_1 is injective, $a = 1_{\Gamma_1}$. Therefore, $h = 1_\Gamma$ and so $H \cap g\Gamma_1 g^{-1} = 1_\Gamma$ for every $g \in \Gamma$. Similarly $H \cap g\Gamma_2 g^{-1} = 1_\Gamma$ and thus $H \cap g\Gamma g^{-1} = 1_\Gamma$. By the Kurosh Subgroup Theorem for amalgamated free products (Theorem 2.3.2), it follows that H is a free group.

1. The group $\varphi(\Gamma)$ acts simply transitively on $\mathcal{S}$ by the following argument: Let (a_i, b_j, c_k) and (a_r, b_s, c_t) be in $\mathcal{S}$. Consider

$$\begin{aligned} a \cdot (a_i, b_j, c_k) = (a_r, b_j, c_k) &\implies a_i c_k a = a_r c_k \implies a = c_k^{-1} a_i^{-1} a_r c_k \\ b \cdot (a_i, b_j, c_k) = (a_i, b_s, c_k) &\implies b_j c_k b = b_s c_k \implies b = c_k^{-1} b_j^{-1} b_s c_k \\ c \cdot (a_i, b_j, c_k) = (a_i, b_j, c_t) &\implies c_k c = c_t \implies c = c_k^{-1} c_t. \end{aligned}$$

Let $w = cab$ with a, b, c as above. Then, $w \cdot (a_i, b_j, c_k) = (a_r, b_s, c_t)$, which shows that $\varphi(\Gamma)$ acts simply transitively on $\mathcal{S}$.

Since φ acts simply transitively on $\mathcal{S}$, it implies $|\varphi(\Gamma)| = |\mathcal{S}| = \frac{|\Gamma_1||\Gamma_2|}{|A|}$ and therefore $[\Gamma : H] = \frac{|\Gamma_1||\Gamma_2|}{|A|}$.

2. To find the rank of H, we use Bass-Serre theory [7,37].

   The group H acts on a tree with vertices $X^0 = (\Gamma/A)\coprod(\Gamma/\Gamma_1)\coprod(\Gamma/\Gamma_2)$ and oriented edges $X^1_+ = (\Gamma/A \times \{1\})\coprod(\Gamma/A \times \{2\})$. The initial and terminal vertices of an edge $(\gamma A, i)$ are γ and $\gamma\Gamma_i$ respectively. H acts on this tree by left multiplication.

   Let $Y = H\backslash X$ be the factor graph and T a maximal subtree of Y. Let $(\widetilde{Y}, \widetilde{T})$ be a lift of (Y, T) in X. Then $|Y^1 - T^1| = |\widetilde{Y}^1 - \widetilde{T}^1|$ and $|T^1| = |T^0| - 1$ since T is a tree. Well,

$$\begin{aligned}|T^0| = |Y^0| &= |H\backslash\Gamma/A| + |H\backslash\Gamma/\Gamma_1| + |H\backslash\Gamma/\Gamma_2| \\ &= \frac{[\Gamma : H]}{|A|} + \frac{[\Gamma : H]}{|\Gamma_1|} + \frac{[\Gamma : H]}{|\Gamma_2|} \\ &= \frac{|\Gamma_1||\Gamma_2|}{|A||A|} + \frac{|\Gamma_2|}{|A|} + \frac{|\Gamma_1|}{|A|}\end{aligned}$$

   and therefore,

$$|T^1| = \frac{|\Gamma_1||\Gamma_2|}{|A||A|} + \frac{|\Gamma_2|}{|A|} + \frac{|\Gamma_1|}{|A|} - 1.$$

   Now,

$$|Y^1| = |H\backslash\Gamma/A| + |H\backslash\Gamma/A| = 2\frac{|\Gamma_1||\Gamma_2|}{|A||A|}.$$

   By the proof of the Kurosh Subgroup Theorem in Bogopolski [7], it follows that $|Y^1 - T^1|$ is the rank of free group H (elements of $\widetilde{Y}^1 - \widetilde{T}^1$ are in one-to-one correspondence with the free generators of H). Therefore,

$$rank(H) = |Y^1 - T^1| = \frac{|\Gamma_1||\Gamma_2|}{|A||A|} - \frac{|\Gamma_2|}{|A|} - \frac{|\Gamma_1|}{|A|} + 1.$$

□

Theorem 5.2.2. *Let* Γ_1 *and* Γ_2 *be finite groups. Let* $\Gamma = \Gamma_1 *_A \Gamma_2$ *be the amalgamated free product of* Γ_1 *and* Γ_2 *over the subgroup* A. *Then* $RG(\Gamma) = RG(\Gamma_1) + RG(\Gamma_2) + \frac{1}{|A|}$.

Proof. Let H be as in Lemma 5.2.1. Then by Theorem 3.2.1,

$$\begin{aligned}RG(\Gamma) = \frac{RG(H)}{[\Gamma : H]} &= \frac{\frac{|\Gamma_1||\Gamma_2|}{|A||A|} - \frac{|\Gamma_2|}{|A|} - \frac{|\Gamma_1|}{|A|}}{\frac{|\Gamma_1||\Gamma_2|}{|A|}} = -\frac{1}{|\Gamma_1|} + -\frac{1}{|\Gamma_2|} + \frac{1}{|A|} \\ &= RG(\Gamma_1) + RG(\Gamma_2) + \frac{1}{|A|}.\end{aligned}$$

□

Computing the rank gradient of the free product of two infinite groups with amalgamation over a finite group was proved by the author and simultaneously and independently by the team of Kar and Nikolov [16, Proposition 2.2]. We provide our proof here.

Theorem 5.2.3. *Let* $\Gamma = \Gamma_1 *_A \Gamma_2$ *be the amalgamated free product of* Γ_1 *and* Γ_2 *over the finite subgroup* A*. If* Γ *is residually finite, then* $RG(\Gamma) = RG(\Gamma_1) + RG(\Gamma_2) + \frac{1}{|A|}$.

Proof. Claim: There exists a normal subgroup N of Γ with finite index such that $N \cap A = 1$.

Since Γ is residually finite, then for every nontrivial $a \in A$ there exists an N_a normal of finite index in Γ such that $a \notin N_a$. Let $N = \bigcap_{1 \neq a \in A} N_a$. Since A is finite, N is the intersection of finitely many normal subgroups of finite index and thus is normal and of finite index. Clearly, $N \cap A = 1$.

Since N is normal of finite index in Γ with $N \cap xAx^{-1} = N \cap A = 1$ for any $x \in \Gamma$, then by Kurosh Subgroup Theorem for amalgamated free products (Theorem 2.3.2), N is isomorphic to a free product of a free group of finite rank along with intersections of conjugates of the Γ_i's. Namely,

$$N \simeq F * (*_{x \in N\backslash\Gamma/\Gamma_1} N \cap x\Gamma_1 x^{-1}) * (*_{y \in N\backslash\Gamma/\Gamma_2} N \cap y\Gamma_2 y^{-1}),$$

where $N\backslash\Gamma/\Gamma_i$ denotes a set of double coset representatives. By Corollary 5.1.6, it follows that

$$\begin{aligned} RG(N) &= RG(F) + \sum_{x \in N\backslash\Gamma/\Gamma_1} RG(N \cap x\Gamma_1 x^{-1}) + \sum_{y \in N\backslash\Gamma/\Gamma_2} RG(N \cap y\Gamma_2 y^{-1}) \\ &\quad + (1 + |N\backslash\Gamma/\Gamma_1| + |N\backslash\Gamma/\Gamma_2|) - 1 \\ &= RG(F) + |N\backslash\Gamma/\Gamma_1|\, RG(N \cap \Gamma_1) + |N\backslash\Gamma/\Gamma_2|\, RG(N \cap \Gamma_2) \\ &\quad + |N\backslash\Gamma/\Gamma_1| + |N\backslash\Gamma/\Gamma_2|. \end{aligned}$$

Using Bass-Serre theory [7, 37], F is a free group of rank $|N\backslash\Gamma/A| - |N\backslash\Gamma/\Gamma_1| - |N\backslash\Gamma/\Gamma_2| + 1$ and therefore $RG(F) = |N\backslash\Gamma/A| - |N\backslash\Gamma/\Gamma_1| - |N\backslash\Gamma/\Gamma_2|$. We thus have

$$\begin{aligned} RG(N) &= |N\backslash\Gamma/A| + |N\backslash\Gamma/\Gamma_1|\, RG(N \cap \Gamma_1) + |N\backslash\Gamma/\Gamma_2|\, RG(N \cap \Gamma_2) \\ &= \frac{[\Gamma : N]}{[A : N \cap A]} + \frac{[\Gamma : N]}{[\Gamma_1 : N \cap \Gamma_1]} RG(N \cap \Gamma_1) + \frac{[\Gamma : N]}{[\Gamma_2 : N \cap \Gamma_2]} RG(N \cap \Gamma_2) \\ &= [\Gamma : N] \left(\frac{1}{[A : N \cap A]} + \frac{RG(N \cap \Gamma_1)}{[\Gamma_1 : N \cap \Gamma_1]} + \frac{RG(N \cap \Gamma_2)}{[\Gamma_2 : N \cap \Gamma_2]} \right) \\ &= [\Gamma : N] \left(\frac{1}{|A|} + RG(\Gamma_1) + RG(\Gamma_2) \right), \end{aligned}$$

since $N \cap A = 1$ and $N \cap \Gamma_i$ is finite index in Γ_i for $i = 1, 2$. Therefore, since N is finite index in Γ, we have $RG(\Gamma) = \frac{RG(N)}{[\Gamma:N]}$. The result follows. □

5.3 Invariant Measures on Homogenous Spaces

We will need the following theory of invariant measures on homogenous spaces when we begin to discuss cost of an equivalence relation. Most of this material can be found in [6] or [29].

Definition. Let Γ be a locally compact topological group and let $\mathcal{B}(\Gamma)$ be the Borel σ-algebra of Γ generated by all compact subsets of Γ. A left (resp. right) *Haar measure* on Γ is a nontrivial regular Borel measure μ on Γ, which is left (resp. right) invariant. That is, $\mu(gB) = \mu(B)$ for every $B \in \mathcal{B}(\Gamma)$ (resp. $\mu(Bg) = \mu(B)$).

Theorem 5.3.1. *For every locally compact group Γ, there exists a left (and a right) invariant Haar measure on Γ, which is unique up to multiplication by a positive constant.*

Not every left Haar measure is right Haar measure. By the uniqueness in Theorem 5.3.1 we can define a function which determines the right invariance of a left Haar measure.

Definition. Let μ be a left Haar measure on Γ. For every $g \in \Gamma$ there exists a $\Delta_\Gamma(g) \in \mathbb{R}_{>0}$ such that $\mu(Bg) = \mu(B)\Delta_\Gamma(g)$. The function $\Delta_\Gamma : \Gamma \to \mathbb{R}_{>0}$ is called the *modular function*. Since any two left Haar measures differ by a constant, we see that the modular function does not depend on the initial choice of μ.

A function is called *unimodular* if $\Delta_\Gamma(g) = 1$ for every $g \in \Gamma$.

Note. Γ is unimodular if and only if every left Haar measure is also a right Haar measure.

Proposition 5.3.2. *For any locally compact group Γ, the modular function $\Delta_\Gamma : \Gamma \to \mathbb{R}_{>0}$ is a continuous group homomorphism.*

Proof. Using the defining equation $\mu(Bg) = \mu(B)\Delta_\Gamma(g)$ one can easily show that Δ_Γ is a group homomorphism. For continuity, see [6]. □

Proposition 5.3.3. *Compact groups are unimodular.*

Proof. By Proposition 5.3.2, $\Delta_\Gamma(\Gamma)$ is a compact subgroup of $\mathbb{R}_{>0}$ and therefore $\Delta_\Gamma(\Gamma) = \{1\}$. □

The following theorem about invariant measures on homogenous spaces will be needed to prove how the cost of the relation changes when restricting to a subspace. The reader is referred to [6] or [29] for more.

Theorem 5.3.4. *Let Γ be a locally compact group and H a closed subgroup of Γ. Let $H\backslash\Gamma$ be the space of right cosets of H. A nontrivial regular right invariant measure on $H\backslash\Gamma$ exists if and only if $\Delta_\Gamma|_H = \Delta_H$, where Δ denotes the modular function on* *the respective group. If the latter condition holds, such a measure $\mu_{H\backslash\Gamma}$ on $H\backslash\Gamma$ is unique up to multiplication by a positive constant. For a suitable choice of such a constant, the measure $\mu_{H\backslash\Gamma}$ satisfies the following condition:*

$$\int_\Gamma f(x)\, d\mu_\Gamma(x) = \int_{H\backslash\Gamma}\int_H f(hx)\, d\mu_H(h)\, d\mu_{H\backslash\Gamma}(Hx)$$

where μ_Γ and μ_H are Haar measures on Γ and H respectively.

Remark 5.3.5. The formula in Theorem 5.3.4 makes sense only when the function $\varphi : \Gamma \to \mathbb{C}$ given by $\varphi(x) = \int_H f(hx)\, d\mu_H(h)$ is constant on right cosets of H. The fact that φ is constant on right cosets of H follows from the fact that μ_H is right invariant.

5.4 Cost of Restricted Actions

To get a lower bound for the rank gradient of amalgamated free products and HNN extensions over amenable subgroups we use the notion of cost. The notion of cost was first introduced by Levitt [24] and for more information the reader is referred to [2, 10, 24]. The following explanation of cost closely follows [2].

Let Γ be a countable group that acts on a standard Borel probability space (X, μ) by measure preserving Borel-automorphisms. Define the equivalence relation E on X by

$$xEy \text{ if there exists } \gamma \in \Gamma \text{ with } y = \gamma x.$$

The relation E is a Borel equivalence relation and every equivalence class is countable. Since E is a subset of $X \times X$, we can consider E as a graph on X.

Definition. A *Borel subgraph* of E is a directed graph on X such that the edge set is a Borel subset of E.

Definition. A subgraph S of E is said to *span* E if for any $(x, y) \in E$ with $x \neq y$ there exists a path from x to y in S, where a *path* from x to y in S is defined as a sequence $x_0, x_1, \dots, x_k \in X$ such that: $x_0 = x$, $x_k = y$; and $(x_i, x_{i+1}) \in S$ or $(x_{i+1}, x_i) \in S$ $(0 \leq i \leq k-1)$.

Definition. S is called a *graphing* of E if it is a Borel subgraph of E that spans E.

The edge-measure of a Borel subgraph S of E is defined as

$$e(S) = \int_{x\in X} deg_S(x)\, d\mu$$

where $deg_S(x)$ is the number of edges in S with initial vertex x:

$$deg_S(x) = |\{y \in X \mid (x, y) \in S\}|.$$

Note that $e(S)$ may be infinite.

Definition. Let Γ be a countable group acting on a standard Borel probability space X by measure preserving Borel-automorphism. Let E denote the equivalence relation of this action. The *cost* of E is defined as

$$Cost(E) = Cost(\Gamma, X) = \inf e(S)$$

where the infimum is taken over all graphings S of E.

Recall that Theorem 3.4.2 (Abert, Jaikin-Zapirain, and Nikolov) states that finitely generated infinite amenable groups have rank gradient equal to zero with respect to any normal chain with trivial intersection. A similar result holds for the cost of amenable groups. An equivalence relation E on a space X is called *hyperfinite* if it can be written (up to measure zero) as an ascending union of finite relations E_n: For every $x \in X$ and for every $n \in \mathbb{N}$, each equivalence class $E_n[x]$ is finite, $E_n[x] \subset E_{n+1}[x]$, and $E[x] = \bigcup_n E_n[x]$. Ornstein and Weiss [31] proved that an amenable group acting essentially freely on a standard Borel probability space X by measure preserving Borel-automorphism is a hyperfinite action. Moreover an infinite hyperfinite action is orbit equivalent to a free action of $\mathbb{Z}$. Since orbit equivalent actions have the same cost, then an infinite hyperfinite action has cost equal to 1 [10]. We summarize this information in a theorem as we will use it multiple times when computing rank gradient and p-gradient of amalgamated free products and HNN extensions over amenable subgroups.

Theorem 5.4.1 (Gaboriau - Ornstein and Weiss)**.** *Let A be an amenable group acting essentially freely on a Borel probability space X. Then $Cost(A, X) = 1 - \frac{1}{|A|}$.*

Gaboriau [10] describes how one can decompose relations coming from amalgamated free products and HNN extensions. We start with amalgamated products and then give the corresponding definitions and results for HNN extensions.

Amalgamated Products:

Let E be a relation on a Borel space X. Let E_1 and E_2 be sub-relations of E that generate E, that is to say that E is the smallest relation containing E_1 and E_2. Let E_3 be a sub-relation of both E_1 and E_2.

Definition. A sequence $(x_1, x_2, \ldots, x_n)$ of points in X is said to be *reduced* if

- (x_i, x_{i+1}) belongs to one of the factors E_1 or E_2,
- (x_{i-1}, x_i) and (x_i, x_{i+1}) belong to different factors,
- $n > 2$, then any (x_i, x_{i+1}) does not belong to E_3,
- $n = 2$, then $x_1 \neq x_2$.

Definition. We say that E is the *amalgamated product* of E_1 and E_2 over the sub-relation E_3, denoted $E = E_1 *_{E_3} E_2$ if for each reduced sequence $(x_1, \ldots, x_n)$ (up to a set of measure zero) we have $x_1 \neq x_n$.

Using the definition of a reduced element in an amalgamated free product of groups, it follows immediately by definition that if E is a relation arising as a free action of an amalgamated free product of groups $\Gamma = \Gamma_1 *_{\Gamma_3} \Gamma_2$, then $E = E|_{\Gamma_1} *_{E|_{\Gamma_3}} E|_{\Gamma_2}$ [10, Example 4.8].

Theorem 5.4.2 (Gaboriau, Theorem 4.15 [10])**.** *Let $E = E_1 *_{E_3} E_2$ be an amalgamated product relation on a space X and assume E_3 is hyperfinite. Then $Cost(E) = Cost(E_1) + Cost(E_2) - Cost(E_3)$.*

HNN Extensions:

Let E be a relation on the space X with a sub-relation E_1. Let $f : A \to B$ be a measure preserving isomorphism between Borel sets of X. Let E_3 be a sub-relation of E_1 that is trivial outside of B, that is, $E_3[x] = [x]$ for any $x \in X - B$. Consider the relation, which we denote by $f^{-1}(E_3)$, that is trivial outside of A and is defined on A by $(x, y) \in f^{-1}(E_3)$ if and only if $(f(x), f(y)) \in E_3$. Assume that $f^{-1}(E_3)$ is a sub-relation of E_1 and that E is generated by E_1 and f, that is, E is the smallest relation containing E_1 and the graph of f.

Let $\varepsilon_i = \pm 1$. We denote by $x_{2i} \xrightarrow{f^{\varepsilon_i}} x_{2i+1}$ the following:

$$x_{2i} \in A \text{ and } f(x_{2i}) = x_{2i+1} \text{ if } \varepsilon_i = 1$$
$$x_{2i} \in B \text{ and } f^{-1}(x_{2i}) = x_{2i+1} \text{ if } \varepsilon_i = -1.$$

Definition. A sequence $(x_1, x_2, \ldots, x_{2n-1}, x_{2n})$ of points of X that satisfies

$$x_1 \overset{E_1}{\sim} x_2 \xrightarrow{f^{\varepsilon_1}} x_3 \overset{E_1}{\sim} \ldots \xrightarrow{f^{\varepsilon_{i-1}}} x_{2i-1} \overset{E_1}{\sim} x_{2i} \xrightarrow{f^{\varepsilon_i}} x_{2i+1} \overset{E_1}{\sim} \ldots$$
$$x_{2n-2} \xrightarrow{f^{\varepsilon_{n-1}}} x_{2n-1} \overset{E_1}{\sim} x_{2n}$$

is said to be *reduced* if

- $n \geq 2$, and there does not exist a sub-sequence

$$x_{2i-2} \xrightarrow{f} x_{2i-1} \overset{E_1}{\sim} x_{2i} \xrightarrow{f^{-1}} x_{2i+1} \text{ with } x_{2i-1} \overset{E_1}{\sim} x_{2i}, \text{ or}$$
$$x_{2i-2} \xrightarrow{f^{-1}} x_{2i-1} \overset{E_1}{\sim} x_{2i} \xrightarrow{f} x_{2i+1} \text{ with } x_{2i-1} \overset{E_1}{\sim} x_{2i}.$$

- $n = 1$, and $x_1 \neq x_2$.

Definition. We say that E is an *HNN extension* of E_1 over the sub-relation E_3 via the map f, denoted $E = E_1 *_{f,E_3}$ if for each reduced sequence $(x_1, x_2, \ldots, x_{2n})$ (up to a set of measure zero) we have $x_1 \neq x_n$.

Let $\Gamma = \Gamma_1 *_{\Gamma_3} = \langle \Gamma_1, t \mid t\Gamma_3 t^{-1} = \varphi(\Gamma_3) \rangle$ be the HNN extension of Γ_1 with associated subgroup Γ_3. Using the definition of a reduced element in an HNN extension of groups, it follows by definition that if E is a relation arising as a free action of Γ, then $E = E|_{\Gamma_1} *_{f,E|_{\Gamma_3}}$ via the isomorphism f defined on X by the action of the element $t \in \Gamma$ [10, Example 4.21].

Theorem 5.4.3 (Gaboriau, Corollary 4.25 [10])**.** *Let* $E = E_1 *_{f,E_3}$ *be an HNN extension of relations on a space* (X, μ) *via* $f : A \to B$ *and assume* E_3 *is hyperfinite. Then* $Cost(E) = Cost(E_1) + \mu(A) - Cost(E_3)$*.*

Abert and Nikolov proved the following connection between rank gradient and cost. Their actual result [2, Theorem 1] is more general than the special case given below, but the following is all that will be needed here.

Theorem 5.4.4 (Abert and Nikolov)**.** *Let* Γ *be a finitely generated residually finite group and* $\{H_n\}$ *be a lattice of normal subgroups of finite index such that* $\bigcap H_n = 1$*. Then*

$$RG(\Gamma, \{H_n\}) = Cost(E) - 1 = Cost(\Gamma, \widehat{\Gamma}_{(H_n)}) - 1,$$

where E *is the equivalence relation coming from the action of* Γ *on* $\widehat{\Gamma}_{(H_n)}$ *(profinite completion of* Γ *with respect to the lattice of subgroups* $\{H_n\}$*) by left multiplication and* $\widehat{\Gamma}_{(H_n)}$ *has its natural Haar measure.*

As the above theorem indicates, we will be interested in a group acting on its profinite completion by left multiplication. This action is essentially free so we will be able to use all of the results about cost given above. Since a profinite group is a compact topological group then it is unimodular. That is, it has a unique Haar measure that is both left and right invariant.

The following lemma concerning profinite completions is well known. The proof follows from residual finiteness and [34, Corollary 1.1.8].

Lemma 5.4.5. *Let* Γ *be a finitely generated group and let* $\{H_n\}$ *be a lattice of normal subgroups of finite index in* Γ*. Let* L *be a subgroup of* Γ*. Then* $\widehat{L}_{(L\cap H_n)}$ *is isomorphic to a closed subgroup of* $\widehat{\Gamma}_{(H_n)}$*.*

The following lemma is used in order to determine the cost of a restricted action.

Lemma 5.4.6. *Let* Γ *be a finitely generated residually finite group and* $\{H_n\}$ *be a lattice of normal subgroups of finite index in* Γ *such that* $\bigcap H_n = 1$*. Let* L *be a subgroup of* Γ *acting on* $\widehat{\Gamma}_{(H_n)}$ *by left multiplication and denote the equivalence relation by* $E_L^{\widehat{\Gamma}_{(H_n)}}$*. Let* S *be a graphing of* $E_L^{\widehat{\Gamma}_{(H_n)}}$*. Let* $\{\bar{g}\}$ *denote a set of right coset representatives for* $\widehat{L}_{(L\cap H_n)}$ *in* $\widehat{\Gamma}_{(H_n)}$*. For any* $\bar{g}$*, let*

$$S_{\bar{g}} = \{(x, y) \in \widehat{L}_{(L\cap H_n)} \times \widehat{L}_{(L\cap H_n)} \mid (x\bar{g}, y\bar{g}) \in S\}.$$

Then $S_{\bar{g}}$ *is a graphing for* $E_L^{\widehat{L}_{(L\cap H_n)}}$*.*

Proof. Note that $(x, y) \in S_{\bar{g}}$ if and only if $(x\bar{g}, y\bar{g}) \in S$. Also, by Lemma 5.4.5 it follows that $\widehat{L}_{(L\cap H_n)}$ is a closed subgroup of $\widehat{\Gamma}_{(H_n)}$.

Spanning: We need to show that $S_{\bar{g}}$ spans $E_L^{\widehat{L}_{(L\cap H_n)}}$. Let $(x,y)\in E_L^{\widehat{L}_{(L\cap H_n)}}$. Then there exists $\alpha\in L$ such that $\alpha x=y$ which implies $\alpha x\bar{g}=y\bar{g}$ and therefore $(x\bar{g},y\bar{g})\in E_L^{\widehat{\Gamma}_{(H_n)}}$. Since S is a graphing of $E_L^{\widehat{\Gamma}_{(H_n)}}$, then S spans $E_L^{\widehat{\Gamma}_{(H_n)}}$. Therefore, there exists a path from $x\bar{g}$ to $y\bar{g}$ in S, call it

$$z_0, z_1, \ldots z_k.$$

By definition $z_0=x\bar{g}$, $z_k=y\bar{g}$, and (z_i,z_{i+1}) or $(z_{i+1},z_i)\in S$ for $0\le i\le k-1$. Let $z_i'=z_i\bar{g}^{-1}$. Then the path in S from $x\bar{g}$ to $y\bar{g}$ is now

$$z_0'\bar{g}, z_1'\bar{g}, \ldots, z_k'\bar{g}.$$

It follows that $z_0'\bar{g}=x\bar{g}$, $z_k'\bar{g}=y\bar{g}$, and $(z_i'\bar{g},z_{i+1}'\bar{g})$ or $(z_{i+1}'\bar{g},z_i'\bar{g})\in S$ for $0\le i\le k-1$. Thus, (z_i',z_{i+1}') or $(z_{i+1}',z_i')\in S_{\bar{g}}$ for $0\le i\le k-1$. Therefore there is a path in $S_{\bar{g}}$ from x to y and thus $S_{\bar{g}}$ spans $E_L^{\widehat{L}_{(L\cap H_n)}}$.

Borel Subgraph: We need to show that the edge set of $S_{\bar{g}}$ is a Borel subset of $E_L^{\widehat{L}_{(L\cap H_n)}}$. Let $\pi_{\bar{g}}:\widehat{L}_{(L\cap H_n)}\times\widehat{L}_{(L\cap H_n)}\to\widehat{\Gamma}_{(H_n)}\times\widehat{\Gamma}_{(H_n)}$ be given by $\pi_{\bar{g}}(x,y)=(x\bar{g},y\bar{g})$. Note that $\pi_{\bar{g}}$ is injective since $\widehat{L}_{(L\cap H_n)}\le\widehat{\Gamma}_{(H_n)}$. Since these spaces are topological groups, multiplication is a continuous map and so $\pi_{\bar{g}}$ is continuous. By definition, $S_{\bar{g}}=\pi_{\bar{g}}^{-1}(S)$. By continuity of $\pi_{\bar{g}}$ and the fact that S is a Borel subgraph of $E_L^{\widehat{\Gamma}_{(H_n)}}$, it follows that $S_{\bar{g}}$ is a Borel subgraph of $E_L^{\widehat{L}_{(L\cap H_n)}}$. □

Using the above theorem and lemma we can now prove the following result about the cost of a restricted action.

Proposition 5.4.7. *Let Γ be a finitely generated group and L be a subgroup. Let $\{H_n\}$ be a set of finite index normal subgroups of Γ such that $\bigcap H_n=1$. Let $\widehat{\Gamma}_{(H_n)}$ be the profinite completion of Γ with respect to $\{H_n\}$ and define $\widehat{L}_{(L\cap H_n)}$ similarly. Then $Cost(L,\widehat{\Gamma}_{(H_n)})\ge Cost(L,\widehat{L}_{(L\cap H_n)})$.*

Proof. Let

$$deg_R^X(x)=|\{y\in X\mid (x,y)\in R\}|$$

for any graphing R on E_G^X, where G is a group acting on the space X. Let $\widehat{\Gamma}=\widehat{\Gamma}_{(H_n)}$ and let $\widehat{L}=\widehat{L}_{(L\cap H_n)}$. By Lemma 5.4.5, $\widehat{L}$ is a closed subgroup of $\widehat{\Gamma}$.

We know that if S is a graphing of $E_L^{\widehat{\Gamma}}$, then $S_{\bar{g}}$ is a graphing of $E_L^{\widehat{L}}$ where we recall that $\{\bar{g}\}$ is a fixed set of right coset representatives of $\widehat{L}$ in $\widehat{\Gamma}$. For $g\in\widehat{\Gamma}$, there is a map $g\to(\ell_g,\widehat{L}\bar{g})\in\widehat{L}\times\widehat{L}\backslash\widehat{\Gamma}$ where $\ell_g\bar{g}=g$.

For $(\ell,\widehat{L}\bar{g})\in\widehat{L}\times\widehat{L}\backslash\widehat{\Gamma}$ set

$$deg_S^{\widehat{\Gamma}}(\ell,\widehat{L}\bar{g})=deg_S^{\widehat{\Gamma}}(\ell\bar{g})=|\{x\in\widehat{\Gamma}\mid(\ell\bar{g},x)\in S\}|.$$

Fix $\widehat{L}\bar{g} \in \widehat{L}\backslash\widehat{\Gamma}$. Then

$$\begin{aligned} deg_S^{\widehat{\Gamma}}(\ell, \widehat{L}\bar{g}) &= |\{x \in \widehat{\Gamma} \mid (\ell\bar{g}, x) \in S\}| = |\{z \in \widehat{\Gamma} \mid (\ell\bar{g}, z\bar{g}) \in S\}| \\ (*) \qquad &= |\{y \in \widehat{L} \mid (\ell\bar{g}, y\bar{g}) \in S\}| = |\{y \in \widehat{L} \mid (\ell, y) \in S_{\bar{g}}\}| \\ &= deg_{S_{\bar{g}}}^{\widehat{L}}(\ell). \end{aligned}$$

The equality $(*)$ is given by the following: Since $\widehat{L} \leq \widehat{\Gamma}$ it is clear that

$$\{y \in \widehat{L} \mid (\ell\bar{g}, y\bar{g}) \in S\} \subseteq \{z \in \widehat{\Gamma} \mid (\ell\bar{g}, z\bar{g}) \in S\}$$

and therefore we have the inequality $\geq$. Let $z \in \widehat{\Gamma}$ with $(\ell\bar{g}, z\bar{g}) \in S$. Then $\ell\bar{g}, z\bar{g} \in E_L^{\widehat{\Gamma}}$ and thus there is an $\alpha \in L$ such that $z\bar{g} = \alpha\ell\bar{g}$. Thus $z = \alpha\ell \in \widehat{L}$ since $L \subset \widehat{L}$ by assumption. The inequality $\leq$ follows. Thus, for all $\widehat{L}\bar{g} \in \widehat{L}\backslash\widehat{\Gamma}$ we have $deg_S^{\widehat{\Gamma}}(\ell, \widehat{L}\bar{g}) = deg_{S_{\bar{g}}}^{\widehat{L}}(\ell)$.

Let $\mu_{\widehat{\Gamma}}$ and $\mu_{\widehat{L}}$ be the unique normalized Haar measures on $\widehat{\Gamma}$ and $\widehat{L}$ respectively. By Lemma 5.4.5 it follows that $\widehat{L}$ is a closed subgroup of $\widehat{\Gamma}$ and therefore,

$$\begin{aligned} Cost(L, \widehat{\Gamma}) &= \inf_{\substack{S \text{ graphing} \\ \text{of } E_L^{\widehat{\Gamma}}}} \int_{\widehat{\Gamma}} deg_S^{\widehat{\Gamma}}(g)\, d\mu_{\widehat{\Gamma}}(g) \\ \text{by Theorem 5.3.4} \qquad &= \inf_S \int_{\widehat{L}\backslash\widehat{\Gamma}} \int_{\widehat{L}} deg_S^{\widehat{\Gamma}}(\ell, \widehat{L}\bar{g})\, d\mu_{\widehat{L}}(\ell)\, d\mu_{\widehat{L}\backslash\widehat{\Gamma}}(\widehat{L}\bar{g}) \\ \text{by above} \qquad &= \inf_S \int_{\widehat{L}\backslash\widehat{\Gamma}} \int_{\widehat{L}} deg_{S_{\bar{g}}}^{\widehat{L}}(\ell)\, d\mu_{\widehat{L}}(\ell)\, d\mu_{\widehat{L}\backslash\widehat{\Gamma}}(\widehat{L}\bar{g}) \\ (**) \qquad &\geq \int_{\widehat{L}\backslash\widehat{\Gamma}} Cost(L, \widehat{L})\, d\mu_{\widehat{L}\backslash\widehat{\Gamma}}(\widehat{L}\bar{g}) \\ &= Cost(L, \widehat{L})\, \mu_{\widehat{L}\backslash\widehat{\Gamma}}(\widehat{L}\backslash\widehat{\Gamma}) \\ &= Cost(L, \widehat{L}). \end{aligned}$$

The inequality $(**)$ follows from the definition of cost:

$$Cost(L, \widehat{L}) = \inf_{\substack{T \text{ graphing} \\ \text{of } E_L^{\widehat{L}}}} \int_{\widehat{L}} deg_T^{\widehat{L}}(\ell)\, d\mu_{\widehat{L}}(\ell). \qquad \square$$

5.5 Rank Gradient of Amalgams

Let $\Gamma = \Gamma_1 *_A \Gamma_2$ be residually finite and assume A is amenable. Let $\{H_n\}$ be a lattice of normal subgroups of finite index in Γ such that $\bigcap H_n = 1$. The action of Γ on the boundary of the coset tree $\partial T(\Gamma, \{H_n\})$ is the action of Γ by left multiplication on its profinite completion with respect to the lattice $\{H_n\}$ with normalized Haar measure. For notational simplicity denote this completion and measure by $\widehat{\Gamma} = \widehat{\Gamma}_{(H_n)}$ and μ respectively. Since for $i = 1, 2$, $\{\Gamma_i \cap H_n\}$ is a lattice of finite index normal subgroups of Γ_i with trivial intersection, then we have the completions $\widehat{\Gamma_i} = \widehat{\Gamma}_{i(\Gamma_i \cap H_n)}$ with measures μ_i. Similarly define $\widehat{A} = \widehat{A}_{(A \cap H_n)}$ and μ_A. Note that these completions are all profinite groups and thus are compact Hausdorff topological groups. By Lemma 5.4.5 it follows that $\widehat{\Gamma}_i \leq \widehat{\Gamma}$.

The following proposition of Abert, Jaikin-Zapirain, and Nikolov [1] established an upper bound for the rank gradient of an amalgamated free product.

Proposition 5.5.1 (Abert, Jaikin-Zapirain, and Nikolov)**.** *Let Γ be a residually finite group generated by two finitely generated subgroups Γ_1 and Γ_2 such that their intersection is infinite. Then*

$$RG(\Gamma, \{H_n\}) \leq RG(\Gamma_1, \{\Gamma_1 \cap H_n\}) + RG(\Gamma_2, \{\Gamma_2 \cap H_n\})$$

for any normal chain $\{H_n\}$ in Γ.

Our result strengthens that of Abert, Jaikin-Zapirain, and Nikolov in the case of the amalgamated subgroup being amenable.

Theorem 5.5.2. *Let $\Gamma = \Gamma_1 *_A \Gamma_2$ be finitely generated and residually finite with A amenable. Let $\{H_n\}$ be a lattice of normal subgroups of finite index in Γ such that $\bigcap H_n = 1$. Then*

$$RG(\Gamma, \{H_n\}) = RG(\Gamma_1, \{\Gamma_1 \cap H_n\}) + RG(\Gamma_2, \{\Gamma_2 \cap H_n\}) + \frac{1}{|A|}.$$

In particular, $RG(\Gamma) \geq RG(\Gamma_1) + RG(\Gamma_2) + \frac{1}{|A|}$.

Note. This theorem was independently proved by Kar and Nikolov [16, Proposition 2.2] in the case of amalgamation over a finite subgroup using Bass-Serre theory. We will thus only show the case where A is infinite amenable.

Proof. Since A is infinite we only need to show that

$$RG(\Gamma, \{H_n\}) = RG(\Gamma_1, \{\Gamma_1 \cap H_n\}) + RG(\Gamma_2, \{\Gamma_2 \cap H_n\}).$$

To simplify notation let $\widehat{\Gamma} = \widehat{\Gamma}_{(H_n)}$ and $\widehat{\Gamma}_i = \widehat{\Gamma}_{i(\Gamma_i \cap H_n)}$ for $i = 1, 2$ and let $E = E_{\Gamma}^{\widehat{\Gamma}}$, $E|_{\Gamma_i} = E_{\Gamma_i}^{\widehat{\Gamma}}$, and $E|_A = E_A^{\widehat{\Gamma}}$. Recall that by Theorem 5.4.1 an essentially free action of an infinite amenable group on a Borel probability space has cost equal to 1. Therefore, $Cost(E|_A) = 1$.

Theorem 5.4.4 states

$$RG(\Gamma, \{H_n\}) = Cost(\Gamma, \widehat{\Gamma}) - 1 = Cost(E) - 1$$

and by definition of amalgamated product of relations, it follows that $E = E|_{\Gamma_1} *_{E|_A} E|_{\Gamma_2}$. Since $E|_A$ is hyperfinite, then by Theorem 5.4.2

$$Cost(E|_{\Gamma_1} *_{E|_A} E|_{\Gamma_2}) - 1 = Cost(E|_{\Gamma_1}) + Cost(E|_{\Gamma_2}) - Cost(E|_A) - 1.$$

Thus,

$$\begin{aligned} RG(\Gamma, \{H_n\}) &= Cost(E|_{\Gamma_1} *_{E|_A} E|_{\Gamma_2}) - 1 \\ &= Cost(E|_{\Gamma_1}) + Cost(E|_{\Gamma_2}) - Cost(E|_A) - 1 \\ &= (Cost(E|_{\Gamma_1}) - 1) + (Cost(E|_{\Gamma_2}) - 1) \\ &= \Big(Cost(\Gamma_1, \widehat{\Gamma}) - 1\Big) + \Big(Cost(\Gamma_2, \widehat{\Gamma}) - 1\Big) \\ \text{by Prop 5.4.7} \quad &\geq \Big(Cost(\Gamma_1, \widehat{\Gamma_1}) - 1\Big) + \Big(Cost(\Gamma_2, \widehat{\Gamma_2}) - 1\Big) \\ \text{by Theorem 5.4.4} \quad &= RG(\Gamma_1, \{\Gamma_1 \cap H_n\}) + RG(\Gamma_2, \{\Gamma_2 \cap H_n\}). \end{aligned}$$

Therefore, $RG(\Gamma, \{H_n\}) \geq RG(\Gamma_1, \{\Gamma_1 \cap H_n\}) + RG(\Gamma_2, \{\Gamma_2 \cap H_n\})$.

The upper bound, $RG(\Gamma, \{H_n\}) \leq RG(\Gamma_1, \{\Gamma_1 \cap H_n\}) + RG(\Gamma_2, \{\Gamma_2 \cap H_n\})$ is given by Proposition 5.5.1.

The fact that $RG(\Gamma) \geq RG(\Gamma_1) + RG(\Gamma_2) + \frac{1}{|A|}$ follows by using the lattice of all subgroups of finite index in Γ and the definition of rank gradient. □

Theorem 5.5.2 is not limited to amalgamated free products of two groups and can be given for more general amalgamated free products. We denote by $\Gamma = \Gamma_1 *_{A_2} \Gamma_2 *_{A_3} \cdots *_{A_n} \Gamma_n$ the left justified iterated amalgamated free product. For example $\Gamma_1 *_{A_2} \Gamma_2 *_{A_3} \Gamma_3 = (\Gamma_1 *_{A_2} \Gamma_2) *_{A_3} \Gamma_3$.

Theorem 5.5.3. *Let $\Gamma = \Gamma_1 *_{A_2} \Gamma_2 *_{A_3} \cdots *_{A_n} \Gamma_n$ be finitely generated and residually finite with each A_i amenable. Let $\{H_r\}$ be a lattice of finite index normal subgroups of Γ such that $\bigcap H_r = 1$. Then*

$$RG(\Gamma, \{H_r\}) = \sum_{i=1}^{n} RG(\Gamma_i, \{\Gamma_i \cap H_r\}) + \sum_{i=2}^{n} \frac{1}{|A_i|}.$$

In particular, $RG(\Gamma) \geq \sum_{i=1}^{n} RG(\Gamma_i) + \sum_{i=2}^{n} \frac{1}{|A_i|}$.

Proof. Let $G_m = \Gamma_1 *_{A_2} \Gamma_2 *_{A_3} \cdots *_{A_n} \Gamma_m$ for $2 \leq m \leq n$. Then $G_n = \Gamma$ and $G_m = G_{m-1} *_{A_m} \Gamma_m$ for each m. It is clear that $G_{m-1} \leq G_m$ and $A_m \leq G_m$ for each m. Thus by Theorem 5.5.2, it follows that $RG(G_m, \{G_m \cap H_r\}) = RG(G_{m-1}, \{G_{m-1} \cap H_r\}) + RG(\Gamma_m, \{\Gamma_m \cap H_r\}) + \frac{1}{|A_m|}$. The result follows by induction on n. □

A group Γ is said to have *fixed price* if every free action of Γ on a probability space X has the same cost. It is an important open question about cost to determine if all groups have fixed price. This is related to the question of whether rank gradient depends on the chain of subgroups if the chain is a normal descending chain with trivial intersection.

Computing the absolute rank gradient of a free product with amalgamation over an amenable subgroup is easy if we assume the groups have fixed price. This is equivalent to assuming the rank gradient does not depend on the chain.

Proposition 5.5.4. *Let $\Gamma = \Gamma_1 *_A \Gamma_2$ with Γ_1, Γ_2, and A residually finite with fixed price and A amenable. Then $RG(\Gamma) = RG(\Gamma_1) + RG(\Gamma_2) + \frac{1}{|A|}$.*

Proof. If Γ has fixed price and is residually finite, then $RG(\Gamma, \{\Gamma_n\}) = cost(E) - 1 = cost(\Gamma) - 1$ using any chain of normal subgroups with trivial intersection. It follows that $RG(\Gamma, \{\Gamma_n\}) = RG(\Gamma)$.

Thus

$$\begin{aligned} RG(\Gamma) &= RG(\Gamma, \{\Gamma_n\}) = cost(E) - 1 \\ &= cost(E|_{\Gamma_1} *_{E|_A} E|_{\Gamma_2}) - 1 \\ &= cost(E|_{\Gamma_1}) + cost(E|_{\Gamma_2}) - cost(E|_A) - 1 \\ &= cost(\Gamma_1) - 1 + cost(\Gamma_2) - 1 + \frac{1}{|A|} \\ &= RG(\Gamma_1) + RG(\Gamma_2) + \frac{1}{|A|}. \end{aligned}$$

□

5.6 Rank Gradient of HNN Extensions

Let K be a finitely generated group with isomorphic subgroups $A \simeq \varphi(A)$. We denote the associated HNN extension of K by $K*_A = \langle K, t \mid t^{-1}At = \varphi(A)\rangle$. Let $\{H_n\}$ be a lattice of finite index normal subgroups in $\Gamma = K*_A$ with $\bigcap H_n = 1$. Let $\widehat{\Gamma}_{(H_n)}$ be the profinite completion of Γ with respect to $\{H_n\}$ and let μ denote the unique normalized Haar measure on $\widehat{\Gamma}_{(H_n)}$. Define $\widehat{K}_{(K\cap H_n)}$ and $\widehat{A}_{(A\cap H_n)}$ similarly.

The theorem about rank gradient of an HNN extension will be proved by first establishing the lower bound and then the upper bound.

Proposition 5.6.1 (Lower Bound)**.** *Let $\Gamma = \langle K, t \mid t^{-1}At = B\rangle$ be finitely generated and residually finite with A amenable. Let $\{H_n\}$ be a lattice of finite index normal subgroups with $\bigcap H_n = 1$. Then*

$$RG(\Gamma, \{H_n\}) \geq RG(K, \{K \cap H_n\}) + \frac{1}{|A|}.$$

Proof. For notational simplicity let $\widehat{\Gamma} = \widehat{\Gamma}_{(H_n)}$, $\widehat{K} = \widehat{K}_{(K\cap H_n)}$, and $\widehat{A} = \widehat{A}_{(A\cap H_n)}$. By Lemma 5.4.5 it follows that $\widehat{K} \leq \widehat{\Gamma}$. Theorem 5.4.4 states

$$RG(\Gamma, \{H_n\}) = Cost(\Gamma, \widehat{\Gamma}) - 1 = Cost\left(E_{\Gamma}^{\widehat{\Gamma}}\right) - 1.$$

Since Γ is an HNN extension, it follows by definition that

$$E_{\Gamma}^{\widehat{\Gamma}} = E_{\Gamma}^{\widehat{\Gamma}}|_K *_{f, E_{\Gamma}^{\widehat{\Gamma}}|_A} = E_K^{\widehat{\Gamma}} *_{E_A^{\widehat{\Gamma}}}$$

where $f : \widehat{\Gamma} \to \widehat{\Gamma}$ is multiplication by the element $t \in \Gamma$. Since $\mu(\widehat{\Gamma}) = 1$ by assumption and $E|_A^{\widehat{\Gamma}}$ is hyperfinite, then by Theorem 5.4.3

$$Cost\left(E_K^{\widehat{\Gamma}} *_{E_A^{\widehat{\Gamma}}}\right) = Cost\left(E_K^{\widehat{\Gamma}}\right) + 1 - Cost\left(E_A^{\widehat{\Gamma}}\right).$$

Recall that since A is amenable, then $Cost\left(E_A^{\widehat{\Gamma}}\right) = 1 - \frac{1}{|A|}$ by Theorem 5.4.1. Therefore,

$$\begin{aligned} RG(\Gamma, \{H_n\}) &= Cost\left(E_K^{\widehat{\Gamma}} *_{E_A^{\widehat{\Gamma}}}\right) - 1 \\ &= Cost\left(E_K^{\widehat{\Gamma}}\right) - 1 + \frac{1}{|A|} \\ \text{by Lemma 5.4.7} \quad &\geq Cost\left(E_K^{\widehat{K}}\right) - 1 + \frac{1}{|A|} \\ &= RG(K, \{K \cap H_n\}) + \frac{1}{|A|}. \end{aligned}$$

□

To prove an upper bound for the rank gradient of an HNN extension we need to be able to bound from above the minimal number of generators of any normal subgroup of finite index. The structure theorem for subgroups of an HNN group was proved by Karrass, Pietrowski, and Solitar and independently by Cohen [8, 17, 19]. A more detailed discussion of the structure of subgroups of amalgamated free products and HNN extensions was given in Section 2.3. We provide a weaker version of their result (Theorem 2.3.4) below as this is all that is needed right now.

Theorem 5.6.2 (Cohen - Karrass, Pietrowski, and Solitar)**.** *Let $\Gamma = \langle K, t \mid t^{-1}At = B\rangle$. Let H be a finite index subgroup of Γ and let $\Omega = \{d_i\}$ be a certain system of double coset representatives for $H\backslash\Gamma/K$ (as constructed in [8, 17, 19]). Then H is generated by $|H\backslash\Gamma/A| - |H\backslash\Gamma/K| + 1$ free generators and the groups $d_iKd_i^{-1} \cap H$ where $d_i \in \Omega$.*

Lemma 5.6.3. *Let H be a finite index normal subgroup of $\Gamma = \langle K, t \mid t^{-1}At = B\rangle$. Then*

$$d(H) \leq \frac{[\Gamma : H]}{[A : A\cap H]} - \frac{[\Gamma : H]}{[K : K\cap H]} + 1 + \frac{[\Gamma : H]}{[K : K\cap H]} d(K\cap H).$$

Proof. Since H is normal and finite index in Γ it follows that $|H\backslash\Gamma/K| = \frac{[\Gamma:H]}{[K:K\cap H]}$ and $|H\backslash\Gamma/A| = \frac{[\Gamma:H]}{[A:A\cap H]}$. Since H is normal in Γ then for any $g \in \Gamma$ it follows that $gKg^{-1}\cap H \simeq K\cap g^{-1}Hg \simeq K\cap H$. Theorem 5.6.2 states that H is therefore generated by $|H\backslash\Gamma/A| - |H\backslash\Gamma/K| + 1$ free generators and $|H\backslash\Gamma/K|$ copies of $K\cap H$. The result now follows. □

We now use the above result to get an upper bound for the rank gradient of an HNN extension.

Proposition 5.6.4 (Upper Bound)**.** *Let $\Gamma = K*_A = \langle K, t \mid t^{-1}At = B\rangle$ be finitely generated and residually finite. Let $\{H_n\}$ be a lattice of finite index normal subgroups with $\bigcap H_n = 1$. Then*

$$RG(\Gamma, \{H_n\}) \leq RG(K, \{K\cap H_n\}) + \frac{1}{|A|}.$$

Proof. By Remark 3.1.4, it is enough to prove the result assuming that $\{H_n\}$ is a descending chain. By Lemma 5.6.3 it follows that for every $H \in \{H_n\}$,

$$\frac{d(H)-1}{[\Gamma:H]} \leq \frac{d(K\cap H)-1}{[K:K\cap H]} + \frac{1}{[A:A\cap H]}.$$

Thus,

$$\begin{aligned} RG(\Gamma, \{H_n\}) &= \lim_{n\to\infty} \frac{d(H_n)-1}{[\Gamma:H_n]} \leq \lim_{n\to\infty} \frac{d(K\cap H_n)-1}{[K:K\cap H_n]} + \lim_{n\to\infty} \frac{1}{[A:A\cap H_n]} \\ &= RG(K, \{K\cap H_n\}) + \frac{1}{|A|}. \end{aligned}$$

□

Combining Proposition 5.6.1 and Proposition 5.6.4 yields the result:

Theorem 5.6.5. *Let $\Gamma = K*_A = \langle K, t \mid t^{-1}At = B\rangle$ be finitely generated and residually finite with A amenable. Let $\{H_n\}$ be a lattice of finite index normal subgroups with $\bigcap H_n = 1$. Then*

$$RG(\Gamma, \{H_n\}) = RG(K, \{K\cap H_n\}) + \frac{1}{|A|}.$$

In particular, $RG(\Gamma) \geq RG(K) + \frac{1}{|A|}$.

Proof. The fact that $RG(\Gamma) \geq RG(K) + \frac{1}{|A|}$ follows by using the lattice of all finite index subgroups of Γ and the definition of rank gradient. □

Recall that since $RG(A,\{A\cap H_n\}) = \frac{-1}{|A|}$ for amenable groups, Theorem 5.5.2 and Theorem 5.6.5 can be written as

$$RG(\Gamma_1 *_A \Gamma_2, \{H_n\}) = RG(\Gamma_1, \{\Gamma_1 \cap H_n\}) + RG(\Gamma_2, \{\Gamma_2 \cap H_n\}) - RG(A, \{A \cap H_n\})$$

and

$$RG(\Gamma, \{H_n\}) = RG(K, \{K \cap H_n\}) - RG(A, \{A \cap H_n\})$$

respectively.

The following example shows that the equation for amalgamated free products (Theorem 5.5.2) does not hold in general.

Example 5.6.6. Let $\Gamma_1 = F_r \times \mathbb{Z}/2\mathbb{Z}, \Gamma_2 = F_r \times \mathbb{Z}/3\mathbb{Z}$, and let $A = F_r$. Then A is finite index in both Γ_1 and Γ_2 which implies

$$\begin{aligned} RG(\Gamma_1) + RG(\Gamma_2) - RG(A) &= \frac{RG(A)}{[\Gamma_1 : A]} + \frac{RG(A)}{[\Gamma_2 : A]} - RG(A) \\ &= \frac{r-1}{2} + \frac{r-1}{3} - (r-1) = -\frac{1}{6}(r-1). \end{aligned}$$

If we let $r = 6k+1$, then $RG(\Gamma_1) + RG(\Gamma_2) - RG(A) = -k$ for any $k \in \mathbb{N}$. However, for any finitely generated group Γ, we know $RG(\Gamma) \geq -1$. Therefore, $RG(\Gamma) \neq RG(\Gamma_1) + RG(\Gamma_2) - RG(A)$ in this case. □

The condition that the amalgamated subgroup is amenable is sufficient but not necessary for the equation for amalgamated free products to hold. Using Theorem 5.6.5 we give an example illustrating this fact by writing the free product of a cyclic group and an HNN extension as an amalgamated free product.

Example 5.6.7. Let $G = \langle K, t \mid tAt^{-1} = \varphi(A) \rangle$ be a finitely generated HNN extension that is residually finite with A amenable. Let $\langle x \rangle$ and $\langle y \rangle$ be two cyclic groups. Let $\Gamma = \langle x \rangle * G$. Then it is not hard to see that Γ is the amalgamated free product

$$\Gamma \simeq \langle K * \langle x \rangle, K * \langle y \rangle \mid K * xAx^{-1} = K * y\varphi(A)y^{-1} \rangle.$$

To see this, simply use the isomorphism given by $x \to x$, $t \to y^{-1}x$, and map K identically to itself [18]. Let $\{H_n\}$ be a lattice of normal subgroups of finite index in Γ such that $\bigcap H_n = 1$. Since Γ can be thought of as a free product, [1, Proposition 8] shows that

$$RG(\Gamma, \{H_n\}) = RG(G, \{G \cap H_n\}) + 1.$$

Since G is a finitely generated and residually finite HNN extension with an amenable associated subgroup, then by Theorem 5.6.5 it follows that

$$RG(G, \{G \cap H_n\}) = RG(K, \{K \cap H_n\}) - RG(A, \{A \cap H_n\}).$$

Thus,

$$RG(\Gamma, \{H_n\}) = RG(K, \{K \cap H_n\}) - RG(A, \{A \cap H_n\}) + 1.$$

However, using [1, Proposition 8] again it follows that

$$\begin{aligned} RG(K,&\{K \cap H_n\}) - RG(A, \{A \cap H_n\}) + 1 \\ &= RG(K * \langle x\rangle, \{K * \langle x\rangle \cap H_n\}) + RG(K * \langle y\rangle, \{K * \langle y\rangle \cap H_n\}) \\ &\quad - RG(K * xAx^{-1}, \{K * xAx^{-1} \cap H_n\}). \end{aligned}$$

Therefore,

$$\begin{aligned} RG(\Gamma, \{H_n\}) = RG(K * \langle x\rangle, \{K * \langle x\rangle \cap H_n\}) + RG(K * \langle y\rangle, \{K * \langle y\rangle \cap H_n\}) \\ - RG(K * xAx^{-1}, \{K * xAx^{-1} \cap H_n\}). \end{aligned}$$

Thus, the equation from Theorem 5.5.2 holds for Γ when considered as an amalgamated free product. □

Theorem 5.6.5 is not limited to a simple HNN extension and can be given for more general HNN groups.

Theorem 5.6.8. *Let* $\Gamma = \langle K, t_1, \ldots, t_n \mid t_i A_i t_i^{-1} = \varphi_i(A_i)\rangle$ *be a finitely generated and residually finite HNN group where each associated subgroup* A_i *is amenable. Let* $\{H_r\}$ *be a lattice of normal subgroups of finite index in* Γ *with* $\bigcap H_r = 1$*. Then*

$$RG(\Gamma, \{H_r\}) = RG(K, \{K \cap H_r\}) + \sum_{i=1}^{n} \frac{1}{|A_i|}.$$

In particular, $RG(\Gamma) \geq RG(K) + \sum_{i=1}^{n} \frac{1}{|A_i|}$.

Proof. Let $\Gamma_m = \langle K, t_1, \ldots, t_m \mid t_i A_i t_i^{-1} = \varphi_i(A_i)\rangle$ for $1 \leq m \leq n$. Then $\Gamma_n = \Gamma$ and $\Gamma_1 = \langle K, t_1 \mid t_1 A_1 t_1^{-1} = \varphi_1(A_1)\rangle$. It is clear that $\Gamma_m = \langle \Gamma_{m-1}, t_m \mid t_m A_m t_m^{-1} = \varphi_m(A_m)\rangle$ is an HNN extension of Γ_{m-1} with associated subgroups A_m and $\varphi_m(A_m)$. Note that $\Gamma_{m-1} \leq \Gamma_m$ and $A_{m-1} \leq K \leq \Gamma_{m-1}$ for each m. Thus by Theorem 5.6.5,

$$RG(\Gamma_m, \{\Gamma_m \cap H_r\}) = RG(\Gamma_{m-1}, \{\Gamma_{m-1} \cap H_r\}) + \frac{1}{|A_m|}.$$

The result follows by induction on n. □

Chapter 6

p-Gradient of Free Products, Amalgams, and HNN Extensions

Throughout this chapter, p will always denote a prime number. We begin this chapter with a section covering some basic results concerning $d_p(\Gamma) = d(\Gamma/[\Gamma,\Gamma]\Gamma^p)$ followed by sections showing the calculation of p-gradient of amalgams and HNN extensions over amenable groups.

6.1 Some Bounds For $d_p(\Gamma)$

Before we give results about $d_p(\Gamma)$, we first give an important lemma.

Lemma 6.1.1. *Let Γ be a residually finite p-torsion group. Then Γ is residually-p.*

Proof. Since Γ is residually finite, then for every $1 \neq g \in \Gamma$ there exists a finite index normal subgroup H of Γ that does not contain g. Let $1 \neq \overline{g}$ denote the image of g in Γ/H. Since Γ is p-torsion, then $g^{p^k} = 1$ for some k and thus the order of $\overline{g}$ in Γ/H is a prime power. This implies that $|\Gamma/H| = p^t m$ for some m relatively prime to p and $t \geq 1$. If q is a prime factor of m, then by the Sylow subgroup theorem Γ/H contains a nontrivial element $\overline{y}$ which is of order q^a for some a. Let $y \in \Gamma$ be any element which has image $\overline{y}$ in Γ/H. Note that y is not in H. By assumption $y^{p^b} = 1$ in Γ for some b and thus $\overline{y}^{p^b} = 1$ in Γ/H. Since p and q are relatively prime this implies that $\overline{y} = 1$ and thus y is in H, which is a contradiction.

Therefore we have that every normal subgroup of finite index is actually a subgroup of p-power index. Thus Γ must be residually-p since it is residually finite. □

Since $\Gamma/[\Gamma,\Gamma]\Gamma^p$ can be thought of as a vector space over $\mathbb{F}_p$ it follows that $d_p(\Gamma)$ is the dimension of $\Gamma/[\Gamma,\Gamma]\Gamma^p$ over $\mathbb{F}_p$.

Lemma 6.1.2. *Let $\Gamma = G/N$. Then $d_p(\Gamma) \leq d_p(G)$. In particular, if $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ then $d_p(\Gamma) \leq d_p(\Gamma_1) + d_p(\Gamma_2)$.*

Proof. Since

$$\Gamma/[\Gamma,\Gamma]\Gamma^p \simeq \frac{\frac{G}{N}}{\left[\frac{G}{N},\frac{G}{N}\right]\left(\frac{G}{N}\right)^p} \simeq \frac{\frac{G}{N}}{\frac{[G,G]G^pN}{N}} \simeq G/[G,G]G^pN \simeq \left(\frac{G}{[G,G]G^p}\right)/M$$

then as a vector space over $\mathbb{F}_p$ the dimension of $\Gamma/[\Gamma,\Gamma]\Gamma^p$ is the dimension of $G/[G,G]G^p$ minus the dimension of M. Therefore $d_p(\Gamma) \leq d_p(G)$.

If $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$, then $\Gamma = (\Gamma_1 * \Gamma_2)/N$ and the result follows from above and Lemma 6.1.3. □

The following lemma is well known and will be needed when computing the p-gradient of free products.

Lemma 6.1.3. *Let p be a prime number.*

1. *Let Γ be a finitely generated group and H a p-power index normal subgroup. Then $d_p(H) - 1 \leq (d_p(\Gamma) - 1)[\Gamma : H]$.*
2. *Let $A * B$ be the free product of two finitely generated groups. Then $d_p(A * B) = d_p(A) + d_p(B)$.*

Proof.

1. By the proof of Lemma 3.3.1 we can assume without loss of generality that Γ is residually-p. Let $\Gamma_{\widehat{p}}$ be the pro-p completion of Γ. Let H be a normal subgroup of p-power index in Γ. Since Γ is residually-p, then $H_{\widehat{p}} \simeq \overline{H} \leq \Gamma_{\widehat{p}}$ is an open normal subgroup with index $[\Gamma_{\widehat{p}} : H_{\widehat{p}}] = [\Gamma : H]$. By the Schreier index formula for finitely generated pro-p groups it follows that
$$d(H_{\widehat{p}}) - 1 \leq [\Gamma_{\widehat{p}} : H_{\widehat{p}}](d(\Gamma_{\widehat{p}}) - 1).$$
However, by the proof of Proposition 3.3.4 we know $d_p(\Gamma) = d(\Gamma_{\widehat{p}})$ for any finitely generated group Γ. Therefore,
$$d_p(H) - 1 \leq [\Gamma : H](d_p(\Gamma) - 1).$$
2. Let $\Gamma = A * B$. Then $\Gamma/[\Gamma,\Gamma]\Gamma^p \simeq A/[A,A]A^p \times B/[[B,B]B^p$. Considering these groups as finite dimensional vector spaces over $\mathbb{F}_p$, it follows that $d(\Gamma/[\Gamma,\Gamma]\Gamma^p) = d(A/[A,A]A^p) + d(B/[B,B]B^p)$. Therefore, $d_p(\Gamma) = d_p(A) + d_p(B)$.

□

The following proposition will be needed when computing the p-gradient of amalgamated free products and HNN extensions.

Proposition 6.1.4. *If $\Gamma_1 *_A \Gamma_2 = \langle \Gamma_1, \Gamma_2 \mid A = \varphi(A) \rangle$ is an amalgamated free product, then*

$$d_p(\Gamma_1) + d_p(\Gamma_2) - d_p(A) \leq d_p(\Gamma_1 *_A \Gamma_2) \leq d_p(\Gamma_1) + d_p(\Gamma_2).$$

If $K *_A = \langle K, t \mid tAt^{-1} = \varphi(A) \rangle$ is an HNN extension, then

$$d_p(K) - d_p(A) + 1 \leq d_p(K *_A) \leq d_p(K) + 1.$$

Proof. In both cases assume that S is a generating set for A. Let $\pi : A \to A/[A,A]A^p$ be the natural homomorphism. Clearly $\pi(S)$ generates $A/[A,A]A^p$ as a group. Let $n = d_p(A)$. Then there exist $s_1, \dots, s_n \in S$ such that $\{\pi(s_1), \dots, \pi(s_n)\}$ is a basis of $A/[A,A]A^p$ as a vector space over $\mathbb{F}_p$. Let $T = \{s_1, \dots, s_n\}$. For every $s \in S - T$ there exist integers α_i such that $\pi(s) = \sum_{i=1}^{n} \alpha_i \pi(s_i)$. For every $s \in S - T$, let $s' = s \prod_{i=1}^{n} s_i^{-\alpha_i}$. Then $\pi(s') = 0$ and therefore $s' \in [A,A]A^p$. Let $S' = T \cup \{s' \mid s \in S - T\}$. Clearly $A = \langle S' \rangle$.

Thus for any generating set S of A, there exists a generating set S' of A such that S and S' have the same cardinality and only $d_p(A)$ elements of S' do not lie in $[A,A]A^p$.

Let Γ be either the amalgamated free product or HNN extension given above and let S be a generating set of A such that only $d_p(A)$ elements of S do not lie in $[A,A]A^p$. Then

$$\Gamma_1 *_A \Gamma_2 \simeq (\Gamma_1 * \Gamma_2)/\langle\langle s\varphi(s)^{-1} \mid s \in S \rangle\rangle$$

and

$$K*_A \simeq (K * \langle t \rangle)/\langle\langle tst^{-1}\varphi(s)^{-1} \mid s \in S \rangle\rangle.$$

A presentation for Γ is given by taking $\Gamma_1 * \Gamma_2$ or $K * \langle t \rangle$ and adding in $|S|$ relations. Since $d_p(\Gamma)$ is the dimension of $\Gamma/[\Gamma,\Gamma]\Gamma^p$ as a vector space over $\mathbb{F}_p$, adding any single relation to the group Γ adds at most one relation to the group $\Gamma/[\Gamma,\Gamma]\Gamma^p$ and therefore the dimension of the vector space drops by no more than 1. However, only $d_p(A)$ elements of S do not lie in $[A,A]A^p \leq [\Gamma,\Gamma]\Gamma^p$ and therefore at most $d_p(A)$ relations possibly get added to the group $\Gamma/[\Gamma,\Gamma]\Gamma^p$. Thus,

$$d_p(\Gamma_1 *_A \Gamma_2) \geq d_p(\Gamma_1 * \Gamma_2) - d_p(A) = d_p(\Gamma_1) + d_p(\Gamma_2) - d_p(A)$$

and

$$d_p(K*_A) \geq d_p(K * \langle t \rangle) - d_p(A) = d_p(K) - d_p(A) + 1.$$

The upper bounds follow from Lemma 6.1.2. □

6.2 p-Gradient of Free Products

Using the results from the previous section, one can now compute the p-gradient for free products.

Theorem 6.2.1. *Let Γ_1 and Γ_2 be finitely generated groups and p a prime number. Let $\Gamma = \Gamma_1 * \Gamma_2$. Then $RG_p(\Gamma) = RG_p(\Gamma_1) + RG_p(\Gamma_2) + 1$.*

Proof. Using Lemma 6.1.3 the proof is identical to the proof of Theorem 5.1.4 replacing "subgroups" with "normal subgroups" and "finite index" with "p-power index." However, for completeness we provide the full proof.

Let $H_i \leq \Gamma_i$ be a p-power index normal subgroup. Let $\varphi : \Gamma \to \Gamma_1 \times \Gamma_2$ be the natural map. Let $C = \ker \varphi$ be the cartesian subgroup of Γ. First, $H_1 \times H_2 \leq \Gamma_1 \times \Gamma_2$ is a p-power index normal subgroup. Let $A = \varphi^{-1}(H_1 \times H_2)$. Since A is the full pre-image of a p-power index normal subgroup, then A is a p-power index normal subgroup in Γ. Moreover, $C \cap \Gamma_i = \{1\}$. Let $a \in A \cap \Gamma_1$. Since $a \in \Gamma_1$, it follows that $\varphi(a) = (a, 1) \in \Gamma_1 \times \{1\}$, but since $a \in A$, then by assumption $\varphi(a) \in H_1 \times H_2$. Therefore $(a, 1) \in H_1 \times H_2$, which implies $a \in H_1$. Clearly $H_1 \subseteq A \cap \Gamma_1$ and thus $A \cap \Gamma_1 = H_1$. Similarly $A \cap \Gamma_2 = H_2$. Therefore, for every p-power index normal subgroup $H_i \leq \Gamma_i$ there exists a p-power index normal subgroup $A \leq \Gamma$ such that $A \cap \Gamma_i = H_i$.

Let $H \leq \Gamma$ be a p-power index normal subgroup and let $H_i = H \cap \Gamma_i$, which are p-power index normal subgroups. Let $A = \varphi^{-1}(H_1 \times H_2)$. Again, $A \leq \Gamma$ is s p-power index normal subgroup and $A \cap \Gamma_i = H_i$. Let $A_H = H \cap A$. Then A_H is a p-power index normal subgroup in Γ, A_H is contained in H, and $A_H \cap \Gamma_i = H_i$. Moreover by Lemma 6.1.3.1,

$$\frac{d_p(A_H) - 1}{[\Gamma : A_H]} \leq \frac{d_p(H) - 1}{[\Gamma : H]}.$$

Note that in the case where we start with $H_i \leq \Gamma_i$ and get $A' = \varphi^{-1}(H_1 \times H_2)$, using the procedure described gives $A = A_{A'}$. Moreover, every p-power index normal subgroup in Γ_i can be obtained from such A_H subgroups of Γ.

Therefore, we can compute the p-gradient of Γ by only looking at the A_H subgroups. Using the Kurosh subgroup theorem for free products (Theorem 2.3.1), Bass-Serre theory, and Lemma 6.1.3.2, if $[\Gamma : A_H] = n$ and $H_i = A_H \cap \Gamma_i$ with $[\Gamma_i : H_i] = k_i$, then

$$d_p(A_H) = \frac{n}{k_1} d_p(H_1) + \frac{n}{k_2} d_p(H_2) + n - \frac{n}{k_1} - \frac{n}{k_2} + 1.$$

This implies

$$d_p(A_H) - 1 = \frac{n}{k_1}(d_p(H_1) - 1) + \frac{n}{k_2}(d_p(H_2) - 1) + n$$

and therefore

$$\frac{d_p(A_H) - 1}{[\Gamma : A_H]} = \frac{d_p(H_1) - 1}{[\Gamma_1 : H_1]} + \frac{d_p(H_2) - 1}{[\Gamma_2 : H_2]} + 1.$$

Thus,

$$\begin{aligned} RG_p(\Gamma) &= \inf_{A_H} \frac{d_p(A_H) - 1}{[\Gamma : A_H]} = \inf_{A_H} \frac{d_p(H_1) - 1}{[\Gamma_1 : H_1]} + \inf_{A_H} \frac{d_p(H_2) - 1}{[\Gamma_2 : H_2]} + 1 \\ &= \inf_{H_1} \frac{d_p(H_1) - 1}{[\Gamma_1 : H_1]} + \inf_{H_2} \frac{d_p(H_2) - 1}{[\Gamma_2 : H_2]} + 1 \\ &= RG_p(\Gamma_1) + RG_p(\Gamma_2) + 1. \end{aligned}$$

□

Corollary 6.2.2. *Let $\Gamma = \Gamma_1 * \Gamma_2$ be the free product of finitely generated groups and p a prime number. Let C be the Cartesian subgroup of Γ. Then*

$$RG_p(\Gamma) = \inf_{\substack{C \leq H \trianglelefteq \Gamma \\ [\Gamma:H]=p^k}} \frac{d_p(H) - 1}{[\Gamma : H]}.$$

Proof. This is proved analogously to Corollary 5.1.5. □

Corollary 6.2.3. *Let $\Gamma = \Gamma_1 * \cdots * \Gamma_k$ be the free product of finitely many finitely generated groups and p a prime number. Then $RG_p(\Gamma) = k - 1 + \sum_{i=1}^{k} RG_p(\Gamma_i)$.*

6.3 p-Gradient of Amalgams

To compute the p-gradient for amalgamated free products we need the Kurosh subgroup theorem for amalgamated free products. We repeat the remark following Theorem 2.3.2 for convenience.

Remark 6.3.1. For our purposes we are only interested in applying Theorem 2.3.2 to normal subgroups of finite index. In this case we can restate the theorem as follows: Every normal subgroup H of finite index in the amalgamated free product $\Gamma = \Gamma_1 *_A \Gamma_2$ is an HNN group with base subgroup L and $n = |H\backslash\Gamma/A| - |H\backslash\Gamma/\Gamma_1| - |H\backslash\Gamma/\Gamma_2| + 1$ free generators with each associated subgroup being isomorphic to $A \cap H$. Specifically,

$$H = \langle L, t_1, \ldots, t_n \mid t_i(A \cap H)t_i^{-1} = \varphi_i(A) \cap H \rangle$$

where the φ_i are appropriate embeddings from A to L.

Further, L is an amalgamated free product of $|H\backslash\Gamma/\Gamma_1|$ groups that are isomorphic to $\Gamma_1 \cap H$ and $|H\backslash\Gamma/\Gamma_2|$ groups that are isomorphic to $\Gamma_2 \cap H$ with at most $|H\backslash\Gamma/\Gamma_1| + |H\backslash\Gamma/\Gamma_2| - 1$ amalgamations each of which is isomorphic to $A \cap H$.

We are now ready to compute the p-gradient for amalgamated free products and HNN extensions. We first make two trivial remarks about residually-p groups.

Lemma 6.3.2. *Let Γ be a residually-p group and let K be a subgroup of Γ. Then K is residually-p and if K is a finite subgroup, then $|K|$ is a p-power. In particular, if K is a finite and residually-p group, then $RG_p(K) = \frac{-1}{|K|}$.*

Proof. Let H be a normal subgroup of p-power index in Γ. Since

$$[\Gamma : H] = [\Gamma : HK][|HK : H] = [\Gamma : HK][K : K \cap H],$$

then $[K : K \cap H]$ divides $[\Gamma : H]$ and thus $[K : K \cap H]$ is a p-power. By definition since Γ is residually-p the intersection of all normal subgroups of p-power index in Γ

is trivial, that is, $\bigcap H = 1$. By above $K \cap H$ is normal of p-power index in K and clearly $\bigcap(K \cap H) = 1$. Thus K is residually-p.

Assume now that K is a finite subgroup of Γ. Then K is a finite residually-p group. Since K is finite then there are only finitely many normal subgroups of p-power index in K and thus $K_{\widehat{p}}$ is the inverse limit of finitely many p-groups and thus is itself a p-group. Since K is residually-p then there is an embedding of K into its pro-p completion $K_{\widehat{p}}$. Since K is dense in $K_{\widehat{p}}$, which is discrete since it is finite, it is clear that $|K| = |K_{\widehat{p}}|$.

For any finite group $RG_p(A) = \frac{-1}{|A_{\widehat{p}}|}$. The result follows since $|A_{\widehat{p}}| = |A|$. □

We are now ready to compute the p-gradient for amalgamated free products.

Theorem 6.3.3. *Let $\Gamma = \Gamma_1 *_A \Gamma_2$ be finitely generated and residually-p with A amenable. Let $\{H_n\}$ be a lattice of normal subgroups of p-power index in Γ such that $\bigcap H_n = 1$. Then*

$$RG_p(\Gamma, \{H_n\}) = RG_p(\Gamma_1, \{\Gamma_1 \cap H_n\}) + RG_p(\Gamma_2, \{\Gamma_2 \cap H_n\}) + \frac{1}{|A|}.$$

In particular, $RG_p(\Gamma) \geq RG_p(\Gamma_1) + RG_p(\Gamma_2) + \frac{1}{|A|}$.

Proof. By Remark 3.1.4, it is enough to prove the result assuming that $\{H_n\}$ is a descending chain. Let H be a normal subgroup of p-power index in Γ.

Upper Bound: By Remark 6.3.1, H is generated by $|H\backslash\Gamma/A| - |H\backslash\Gamma/\Gamma_1| - |H\backslash\Gamma/\Gamma_2| + 1$ free generators, $|H\backslash\Gamma/\Gamma_1|$ groups that are isomorphic to $\Gamma_1 \cap H$, and $|H\backslash\Gamma/\Gamma_2|$ groups that are isomorphic to $\Gamma_2 \cap H$. Therefore by Lemma 6.1.2,

$$\begin{aligned} d_p(H) \leq & \frac{[\Gamma : H]}{[A : A \cap H]} - \frac{[\Gamma : H]}{[\Gamma_1 : \Gamma_1 \cap H]} - \frac{[\Gamma : H]}{[\Gamma_2 : \Gamma_2 \cap H]} + 1 \\ & + \frac{[\Gamma : H]}{[\Gamma_1 : \Gamma_1 \cap H]} d_p(\Gamma_1 \cap H) + \frac{[\Gamma : H]}{[\Gamma_2 : \Gamma_2 \cap H]} d_p(\Gamma_2 \cap H), \end{aligned}$$

which implies

$$\frac{d_p(H) - 1}{[\Gamma : H]} \leq \frac{d_p(\Gamma_1 \cap H) - 1}{[\Gamma_1 : \Gamma_1 \cap H]} + \frac{d_p(\Gamma_2 \cap H) - 1}{[\Gamma_2 : \Gamma_2 \cap H]} + \frac{1}{[A : A \cap H]}.$$

The above inequality holds for all normal subgroups of p-power index in Γ and thus holds for all subgroups H_n in the descending chain. Therefore, taking the limit of both sides of the expression yields the upper bound in our result.

Lower Bound: First, note that $|H\backslash\Gamma/A| \geq |H\backslash\Gamma/\Gamma_1| + |H\backslash\Gamma/\Gamma_2| - 1$. Keeping the notation from Remark 6.3.1, by the HNN part of Proposition 6.1.4, it follows that

$$\begin{aligned} d_p(H) \geq\ & d_p(L) + |H\backslash\Gamma/A| - |H\backslash\Gamma/\Gamma_1| - |H\backslash\Gamma/\Gamma_2| + 1 \\ & - (|H\backslash\Gamma/A| - |H\backslash\Gamma/\Gamma_1| - |H\backslash\Gamma/\Gamma_2| + 1) d_p(A \cap H) \\ \geq\ & d_p(L) + |H\backslash\Gamma/A| - |H\backslash\Gamma/\Gamma_1| - |H\backslash\Gamma/\Gamma_2| + 1 - |H\backslash\Gamma/A| d_p(A \cap H). \end{aligned}$$

By the amalgamated free product part of Proposition 6.1.4, we can find a lower bound for $d_p(L)$:

$$d_p(L) \geq |H\backslash\Gamma/\Gamma_1| d_p(\Gamma_1 \cap H) + |H\backslash\Gamma/\Gamma_2| d_p(\Gamma_2 \cap H) - |H\backslash\Gamma/A| d_p(A \cap H).$$

Therefore,

$$\begin{aligned} & d_p(H) \geq \\ & \frac{[\Gamma : H]}{[\Gamma_1 : \Gamma_1 \cap H]} d_p(\Gamma_1 \cap H) + \frac{[\Gamma : H]}{[\Gamma_2 : \Gamma_2 \cap H]} d_p(\Gamma_2 \cap H) - \frac{[\Gamma : H]}{[A : A \cap H]} d_p(A \cap H) \\ & + \frac{[\Gamma : H]}{[A : A \cap H]} - \frac{[\Gamma : H]}{[\Gamma_1 : \Gamma_1 \cap H]} - \frac{[\Gamma : H]}{[\Gamma_2 : \Gamma_2 \cap H]} + 1 - \frac{[\Gamma : H]}{[A : A \cap H]} d_p(A \cap H), \end{aligned}$$

which implies

$$\frac{d_p(H) - 1}{[\Gamma : H]} \geq \frac{d_p(\Gamma_1 \cap H) - 1}{[\Gamma_1 : \Gamma_1 \cap H]} + \frac{d_p(\Gamma_2 \cap H) - 1}{[\Gamma_2 : \Gamma_2 \cap H]} + \frac{1}{[A : A \cap H]} - 2\frac{d_p(A \cap H)}{[A : A \cap H]}.$$

Replace H with H_n in the above inequality. If A is finite then there must exist an N such that $A \cap H_n = 1$ for every $n \geq N$ since we assume that the chain is strictly descending with trivial intersection. Therefore $d_p(A \cap H_n) = 0$ for every $n \geq N$. If A is infinite amenable then $\lim\limits_{n \to \infty} \dfrac{d_p(A \cap H_n)}{[A : A \cap H_n]} = RG_p(A, \{A \cap H_n\}) = 0$. Therefore taking the limit as $n \to \infty$ we get

$$RG_p(\Gamma, \{H_n\}) \geq RG_p(\Gamma_1, \{\Gamma_1 \cap H_n\}) + RG_p(\Gamma_2, \{\Gamma_2 \cap H_n\}) + \frac{1}{|A|}.$$

The fact that $RG_p(\Gamma) \geq RG_p(\Gamma_1) + RG_p(\Gamma_2) + \frac{1}{|A|}$ follows by using the lattice of all normal subgroups of p-power index in Γ and the definition of p-gradient. □

6.4 p-Gradient of HNN Extensions

To compute the p-gradient for HNN extensions we need the Kurosh subgroup theorem for HNN extensions. We repeat the remark following Theorem 2.3.4 for convenience.

Remark 6.4.1. As in the case of amalgamated free products, for our purposes we are only interested in applying Theorem 2.3.4 to normal subgroups of finite index. In this case we can restate the theorem as follows: Every normal subgroup H of finite index in the HNN extension $\Gamma = \langle K, t \mid tAt^{-1} = \varphi(A) \rangle$ is an HNN group with base subgroup L and $n = |H\backslash\Gamma/A| - |H\backslash\Gamma/K| + 1$ free generators with each associated subgroup being isomorphic to $A \cap H$. Specifically,

$$H = \langle L, t_1, \ldots, t_n \mid t_i(A \cap H)t_i^{-1} = \varphi_i(A) \cap H \rangle$$

where the φ_i are appropriate embeddings from A to L.

Further, L is an amalgamated free product of $|H\backslash\Gamma/K|$ groups that are isomorphic to $K \cap H$ with at most $|H\backslash\Gamma/K| - 1$ amalgamations each of which is isomorphic to $A \cap H$.

We now give the result for HNN extensions. The proof is analogous to that of Theorem 6.3.3.

Theorem 6.4.2. *Let $\Gamma = K*_A = \langle K, t \mid tAt^{-1} = B\rangle$ be finitely generated and residually-p with A amenable. Let $\{H_n\}$ be a lattice of normal subgroups of p-power index in Γ such that $\bigcap H_n = 1$. Then*

$$RG_p(\Gamma, \{H_n\}) = RG_p(K, \{K \cap H_n\}) + \frac{1}{|A|}.$$

In particular, $RG_p(\Gamma) \geq RG_p(K) + \frac{1}{|A|}$.

Proof. By Remark 3.1.4, it is enough to prove the result assuming that $\{H_n\}$ is a descending chain. Let H be any normal subgroup of p-power index in Γ.

<u>Upper Bound:</u> By Remark 6.4.1, H is generated by $|H\backslash\Gamma/A| - |H\backslash\Gamma/K| + 1$ free generators and $|H\backslash\Gamma/K|$ groups that are isomorphic to $K \cap H$. Therefore by Lemma 6.1.2,

$$d_p(H) \leq \frac{[\Gamma : H]}{[A : A \cap H]} - \frac{[\Gamma : H]}{[K : K \cap H]} + 1 + \frac{[\Gamma : H]}{[K : K \cap H]} d_p(K \cap H),$$

which implies

$$\frac{d_p(H) - 1}{[\Gamma : H]} \leq \frac{d_p(K \cap H) - 1}{[K : K \cap H]} + \frac{1}{[A : A \cap H]}.$$

The above inequality holds for all normal subgroups of p-power index in Γ and thus holds for all subgroups H_n in the descending chain. Therefore, taking the limit of both sides of the expression yields the upper bound of our result.

<u>Lower Bound:</u> First, note that $|H\backslash\Gamma/A| \geq |H\backslash\Gamma/K| - 1$. Keeping the notation from Remark 6.4.1, by the HNN result of Proposition 6.1.4 it follows that

$$\begin{aligned} d_p(H) &\geq d_p(L) + |H\backslash\Gamma/A| - |H\backslash\Gamma/K| + 1 - (|H\backslash\Gamma/A| - |H\backslash\Gamma/K| + 1)d_p(A \cap H) \\ &\geq d_p(L) + |H\backslash\Gamma/A| - |H\backslash\Gamma/K| + 1 - |H\backslash\Gamma/A| d_p(A \cap H). \end{aligned}$$

By the amalgamated free product part of Proposition 6.1.4, we can find a lower bound for $d_p(L)$:

$$d_p(L) \geq |H\backslash\Gamma/K| d_p(K \cap H) - |H\backslash\Gamma/A| d_p(A \cap H).$$

Therefore,

$$\begin{aligned} d_p(H) \geq\ & \frac{[\Gamma : H]}{[K : K \cap H]} d_p(K \cap H) - \frac{[\Gamma : H]}{[A : A \cap H]} d_p(A \cap H) \\ &+ \frac{[\Gamma : H]}{[A : A \cap H]} - \frac{[\Gamma : H]}{[K : K \cap H]} + 1 - \frac{[\Gamma : H]}{[A : A \cap H]} d_p(A \cap H), \end{aligned}$$

which implies

$$\frac{d_p(H)-1}{[\Gamma:H]} \geq \frac{d_p(K\cap H)-1}{[K:K\cap H]} + \frac{1}{[A:A\cap H]} - 2\frac{d_p(A\cap H)}{[A:A\cap H]}.$$

The result now follows for the same reason as in Theorem 6.3.3.

The fact that $RG_p(\Gamma) \geq RG_p(K) + \frac{1}{|A|}$ follows by using the lattice of all normal subgroups of p-power index in Γ and the definition of p-gradient. □

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