Characters for Projective Modules in the BGG Category $\mathcal{O}$ for general linear Lie superalgebras

Arun Shrinath Kannan

A Thesis Presented to the Graduate Faculty
of the University of Virginia
for the Distinguished Majors Program

Department of Mathematics

University of Virginia
April 2018

CHARACTERS FOR PROJECTIVE MODULES IN THE BGG CATEGORY $\mathcal{O}$ FOR GENERAL LINEAR LIE SUPERALGEBRAS

ARUN S. KANNAN

ABSTRACT. In this thesis, we determine the characters of certain projective modules in the BGG Category $\mathcal{O}$ for the general linear Lie superalgebras $\mathfrak{gl}(2|1)$, $\mathfrak{gl}(2|2)$, and $\mathfrak{gl}(3|1)$. In particular, we compute standard filtrations for projective modules of atypical weights by the method of translation functors.

CONTENTS

1. INTRODUCTION

A Lie superalgebra is a generalization of a Lie algebra wherein we consider a $\mathbb{Z}_2$-graded vector space endowed with a Lie superbracket, which is subject to relations analagous to those a usual Lie bracket satisfies. The Lie superalgebra as a mathematical construct is particularly suited for theoretical discussions in particle physics regarding a phenomenon known as supersymmetry. Because a Lie superalgebra now has notions of an even and odd part, the super case introduces many interesting complexities to the representation theory of semisimple Lie algebras. For our inquiries, we will look at certain representations of the general linear Lie superalgebra $\mathfrak{gl}(m|n)$, which informally can be thought of as all $(m+n)$-by-$(m+n)$ matrices.

The BGG Category $\mathcal{O}$ is a category of modules of a semisimple Lie algebra that has been well studied for its rich and deep theory (cf. [Hum08]). This category can be analogously defined for $\mathfrak{gl}(m|n)$ (cf. [CW12, Mus12]), and many results from the semisimple case extend to the super case.

Atypicality of weights is a phenomenon present in the Lie superalgebra case that has no analogue in the semisimple Lie algebra case. It allows for an integral block in $\mathcal{O}$ whose degree of atypicality is greater than 0 to have infinitely many simple modules. The principal block in $\mathcal{O}$ for $\mathfrak{gl}(m|n)$, which contains the trivial module, always has nonzero degree of atypicality when $m \geq 1, n \geq 1$.

Atypicality arises due to the presence of certain isotropic odd roots (i.e. roots of length zero) in the root system, which expand the notion of linkage beyond the orbit of the Weyl group. For $\mathfrak{gl}(m|n)$, the degree of atypicality is an integer in the range 0 to $\min(m,n)$, inclusive. The degree 0 block can be reduced to the semisimple Lie algebra case via an equivalence of categories (cf. [Gor02]). Therefore, the new cases arise primarily when the degree of atypicality is nonzero.

For $\mathfrak{gl}(2|1)$ and $\mathfrak{gl}(3|1)$, we look at blocks of degree of atypicality 1; for $\mathfrak{gl}(2|1)$, all atypical integral blocks are equivalent, whereas in $\mathfrak{gl}(3|1)$, there are infinitely many unequivalent atypical blocks. For $\mathfrak{gl}(2|2)$, we look at the most interesting case, where the degree of atypicality is 2. In this case, there is only one such integral block.

Verma flag formulae for all tilting modules (and consequently, projective modules via BGG reciprocity and Soergel duality) in the Category $\mathcal{O}$ of $\mathfrak{gl}(m|n)$ are provided in [CLW15], proving the conjecture in [Br03]. However, these formulae are given in terms of Brundan-Kazhdan-Lusztig polymomials, which do not offer concrete multiplicities explicitly. Using these polynomials, [CW08] was able to produce explicit Verma flag formulae for projectives in an atypical block of $\mathfrak{gl}(2|1)$.

In complementation of this work, we use the tool of translation functors to determine the characters of projective modules in the BGG Category $\mathcal{O}$ for the general linear Lie superalgebras $\mathfrak{gl}(2|1)$, $\mathfrak{gl}(3|1)$, and $\mathfrak{gl}(2|2)$. Specifically, we explicitly determine the standard filtrations of projective modules in atypical blocks in $\mathcal{O}$. In particular, we provide an alternative, simpler way of determining these standard filtrations.

Our general approach of using translation functors is as follows. Given some projective cover P_λ for which we wish to deduce a Verma flag, we find some P_μ with known Verma flag multiplicities and some finite-dimensional representation N such that the Verma module M_λ appears in a standard filtration of $P_\mu \otimes N$. If λ is the lowest weight appearing among all the weights linked to λ appearing in the Verma flag, then P_λ is a direct summand for the projection of $P_\mu \otimes N$ on to the block correpsonding to λ. In most cases, it is the only direct summand. See Section 3.7 for explicit details and justification.

For our explicit formulas, these standard filtrations always have Verma modules with multiplicity 1 or 2. By BGG reciprocity, these formulae determine the irreducible composition factors for Verma modules in $\mathcal{O}$.

In Section 2, we recall basic structure theorems for Lie superalgebras and explicitly detail the structure of the general linear Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$, such as a Cartan subalgebra, a root system, a fundamental system, and linkage. In Section 3, we formally define the BGG Category $\mathcal{O}$, review relevant results in the super case, and offer conditions when Verma modules appear in the standard filtration of projective modules.

The remaining sections (4 to 6) contain our original results. We find standard filtration formulae for projective modules of weights of degree of atypicality 1 when $\mathfrak{g} = \mathfrak{gl}(2|1)$ or $\mathfrak{g} = \mathfrak{gl}(3|1)$ and of degree of atypicality 2 when $\mathfrak{g} = \mathfrak{gl}(2|2)$. These arguments are justified using the results in Section 3 and by the strategy of translation functors.

2. Basic structure of Lie superalgebras and $\mathfrak{gl}(m|n)$

This exposition follows the presentation found in [CW18, Chap. 1, 2]. In this thesis, we shall presume all vector spaces are defined over the field of complex numbers $\mathbb{C}$.

2.1. **Basic Definitions.** A **vector superspace** V is a vector space endowed with a $\mathbb{Z}_2$ gradation $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $\mathbb{Z}_2$ is the two element group $\{\bar{0}, \bar{1}\}$. For $i \in \mathbb{Z}_2$, we say $v \in V_i$ is a homogeneous element, with **parity** $|v|$ (even or odd) given by the parity of i. If applicable, we naturally extend all results on homogeneous elements to general elements by linearity.

Of particular relevance will be the space $\mathrm{End}(V)$ of all endomorphisms of V. $\mathrm{End}(V)$ is a vector superspace when V is a vector superspace.

Definition 2.1. A **Lie superalgebra** is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ equipped with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the **supercommutator**) satisfying:

(1) $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ ($\mathbb{Z}_2$-grading),
(2) $[x, y] = -(-1)^{|x||y|}[y, x]$ (graded skew-symmetry),
(3) $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$ (graded Jacobi identity).

for all homogeneous $x, y, z \in \mathfrak{g}$. Related algebraic constructs like modules, subalgebras, and ideals follow naturally in the $\mathbb{Z}_2$-gradation sense. Inhereting the supercommutator on $\mathfrak{g}$, the even subspace $\mathfrak{g}_{\bar{0}}$ is a Lie algebra in the traditional sense.

The primary Lie superalgebra of focus will be the general linear Lie superalgebra.

2.2. **The General Linear Lie Superalgebra.** Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace so that $\mathrm{End}(V)$ is a vector superspace with an associative bilinear multiplication. Now, define the supercommutator on $\mathrm{End}(V)$ to be

$$[x, y] = xy - (-1)^{|x||y|}yx \tag{2.1}$$

for $x, y \in \mathrm{End}(V)$. One can check that this definition makes $\mathrm{End}(V)$ a Lie superalgebra. We call this Lie superalgebra the **general linear Lie superalgebra**, denoted $\mathfrak{gl}(V)$. When $V = \mathbb{C}^{m|n} := \mathbb{C}^m \oplus \mathbb{C}^n$, we also denote $\mathfrak{gl}(V)$ as $\mathfrak{gl}(m|n)$.

Suppose $V = \mathbb{C}^{m|n}$. Let $\{\bar{1}, \bar{2} \dots, \bar{m}\}$ and $\{1, 2, \dots, n\}$ parametrize the standard bases for the even and odd subspaces of V, $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Denote

$$I(m, n) = \{\bar{1}, \bar{2}, \dots, \bar{m}; 1, 2, \dots, n\} \tag{2.2}$$

where we impose the total order

$$\bar{1} < \cdots < \bar{m} < 0 < 1 < \cdots < n. \tag{2.3}$$

Here, 0 is introduced for convenience. It follows that a basis for $\mathfrak{gl}(m|n)$ is the set of elementary matrices $\{E_{ij} | i, j \in I(m, n)\}$. If $i, j < 0$ or $i, j > 0$, then E_{ij} is even, and if $i < 0 < j$ or $j < 0 < i$, then E_{ij} is odd. The even E_{ij} are a basis for the even subalgebra $\mathfrak{gl}(m|n)_{\bar{0}}$ and the odd E_{ij} are a basis for the odd subspace $\mathfrak{gl}(m|n)_{\bar{1}}$. We see that an element $g \in \mathfrak{gl}(m|n)$ is of the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with blocks a, b, c and d, whose dimensions are $m \times m$, $m \times n$, $n \times m$, and $n \times n$, respectively. Observe that the diagonal blocks a and d correspond to the even subalgebra, and the off-diagonal blocks b and c correspond to the odd subspace.

We define the **supertrace** on $g \in \mathfrak{gl}(m|n)$ to be:

$$\mathrm{str}(g) := \mathrm{tr}(a) - \mathrm{tr}(d). \tag{2.4}$$

In the Lie superalgebra setting, the supertrace plays a similar role to that of the trace in the Lie algebra setting.

2.3. **Structure Theorem.** Let $\mathfrak{h}$ be a Cartan subalgera of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}^*$, the dual space of $\mathfrak{h}$, let

$$\mathfrak{g}_\alpha \coloneqq \{g \in \mathfrak{g} \mid [h, g] = \alpha(h)g, \forall h \in \mathfrak{h}\}. \tag{2.5}$$

The root system Φ for $\mathfrak{g}$ is defined to be

$$\Phi \coloneqq \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0, \alpha \neq 0\}. \tag{2.6}$$

The sets of even roots $\Phi_{\bar{0}}$ and odd roots $\Phi_{\bar{1}}$ are defined to be

$$\Phi_{\bar{0}} \coloneqq \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{0}} \neq 0\}, \quad \Phi_{\bar{1}} \coloneqq \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{1}} \neq 0\}. \tag{2.7}$$

Lastly, we define the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ to be the Weyl group of the even subalgebra $\mathfrak{g}_{\bar{0}}$. A structure theorem similiar to that of semisimple Lie algebras exists for $\mathfrak{gl}(m|n)$.

Theorem 2.2. *Let $\mathfrak{g}$ be $\mathfrak{gl}(m|n)$ with a Cartan subalgebra $\mathfrak{h}$.*

(1) A root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is given by:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \qquad \textit{and } \mathfrak{g}_0 = \mathfrak{h}$$

.

(2) $\dim \mathfrak{g}_\alpha = 1, \alpha \in \Phi$.

(3) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta},\ \alpha, \beta \in \Phi$.

(4) $\Phi, \Phi_{\bar{0}}$, and $\Phi_{\bar{1}}$ are invariant under the natural action of the Weyl group $\mathcal{W}$ on $\mathfrak{h}^*$.

(5) There exists a non-degenerate even supersymmetric bilinear form $(\cdot, \cdot) :\ \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ such that $([x, y], z) = (x, [y, z])\ \forall x, y, z \in \mathfrak{g}$.

(6) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\alpha = -\beta \in \Phi$.

(7) The restriction of the bilinear form $(\cdot, \cdot)$ to $\mathfrak{h} \times \mathfrak{h}$ is non-degenerate and $\mathcal{W}$-invariant.

A root $\alpha \in \Phi$ is said to be **isotropic** if $(\alpha, \alpha) = 0$. All isotropic roots are necessarily odd; in the $\mathfrak{gl}(m|n)$ case, oddness is sufficient to be isotropic. There is no equivalent notion in the semisimple Lie algebra case. Denote the set of isotropic roots as

$$\bar{\Phi}_{\bar{1}} = \{\alpha \in \Phi_{\bar{1}} \mid (\alpha, \alpha) = 0\}. \tag{2.8}$$

In the two subsections immediately below, we will explicitly describe a bilinear form, root system, and Weyl group for $\mathfrak{gl}(m|n)$.

2.4. **Invariant Bilinear Form on $\mathfrak{gl}(m|n)$.** The supertrace as in (2.4) yields a non-degenerate supersymmetric invariant bilinear form on $\mathfrak{g} = \mathfrak{gl}(m|n)$:

$$(\cdot, \cdot) :\ \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad (x, y) = \operatorname{str}(xy), \tag{2.9}$$

where xy denotes matrix multiplication. One can restrict the form to the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices in $\mathfrak{g}$:

$$(E_{ii}, E_{jj}) = \begin{cases} 1 & \bar{1} \leq i = j \leq \overline{m} \\ -1 & 1 \leq i = j \leq n \\ 0 & i \neq j. \end{cases} \tag{2.10}$$

where $i, j \in I(m,n)$. Now, denote $\{\delta_i, \epsilon_j\}_{i,j}$ to be the basis of $\mathfrak{h}^*$ dual to $\{E_{\overline{i}\overline{i}}, E_{jj}\}_{i,j}$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. The functional δ_i can be identified with $(E_{\overline{i}\overline{i}}, \cdot)$ and ϵ_j can be identified with $-(E_{jj}, \cdot)$. Whenever convenient for $1 \leq i \leq m$, write

$$\epsilon_{\overline{i}} := \delta_i.$$

The restriction of $(\cdot,\cdot)$ to $\mathfrak{h}$ naturally induces a non-degenerate bilinear form on $\mathfrak{h}^*$, which will also be denoted by $(\cdot,\cdot)$. For $i, j \in I(m,n)$, we have:

$$(\epsilon_i, \epsilon_j) = \begin{cases} 1 & \overline{1} \leq i = j \leq \overline{m} \\ -1 & 1 \leq i = j \leq n \\ 0 & i \neq j. \end{cases} \tag{2.11}$$

Note that

$$(\delta_i - \epsilon_j, \delta_i - \epsilon_j) = 0$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. We can define the corresponding weight lattice X in $\mathfrak{h}^*$:

$$X := \bigoplus_{i \in I(m,n)} \mathbb{Z}\epsilon_i. \tag{2.12}$$

With this bilinear form on $\mathfrak{h}^*$, we can define for any $\alpha \in \Phi_{\overline{0}}$ the corresponding coroot $\alpha^\vee \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$ such that

$$\langle \lambda, \alpha^\vee \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \quad \forall \lambda \in \mathfrak{h}^*. \tag{2.13}$$

The simple reflection s_α acts on $\mathfrak{h}^*$ as expected: $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$.

2.5. **Root System for $\mathfrak{gl}(m|n)$.** Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathfrak{h}$ be the Cartan subalgebra given by the diagonal matrices in $\mathfrak{g}$. The root system $\Phi = \Phi_{\overline{0}} \cup \Phi_{\overline{1}}$ is given by:

$$\begin{aligned} \Phi_{\overline{0}} &= \{\epsilon_i - \epsilon_j \mid i \neq j \in I(m,n),\ i,j > 0 \text{ or } i,j < 0\}, \\ \Phi_{\overline{1}} &= \{\pm(\epsilon_i - \epsilon_j) \mid i,j \in I(m,n),\ i < 0 < j\}. \end{aligned} \tag{2.14}$$

The root vector corresponding to $\epsilon_i - \epsilon_j$ is E_{ij} for $i, j \in I(m,n)$. This root system induces a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^-$ consists of the lower-triangular matrices and $\mathfrak{n}^+$ consists of the upper-triangular matrices. The standard Borel subalgebra $\mathfrak{b}$ is given by $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$.

The Weyl group $\mathcal{W}$ of $\mathfrak{g}$ is by definition the Weyl group of the even subalgebra $\mathfrak{g}_{\overline{0}} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$; therefore, $\mathcal{W} \cong S_m \times S_n$, where S_k denotes the symmetric group on k letters. It is readily seen that $\mathcal{W}$ has generators corresponding to simple transpositions $(\sigma_{i,i+1}, 1)$ and $(1, \sigma_{j,j+1})$ where $1 \leq i < m-1,\ 1 \leq j < n-1$. The first component acts naturally on the $\{\delta_i\}_{i=1}^m$, and the second component acts naturally on the $\{\epsilon_j\}_{j=1}^n$.

Notions of positive systems and fundamental systems follow similarly to those of semisimple Lie algebras, though there are complexities that arise when dealing with odd simple roots. The standard positive system Φ^+ of Φ is

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid i < j \in I(m,n)\}. \tag{2.15}$$

The standard fundamental system Π of Φ is given by

$$\Pi = \{\epsilon_i - \epsilon_{i+1} \mid i \in I(m-1, n-1)\} \cup \{\epsilon_{\overline{m}} - \epsilon_1\}. \tag{2.16}$$

The transpositions of the Weyl group described above correspond to the simple reflections by simple even roots.

For later use, we define the set of positive isotropic roots

$$\bar{\Phi}_{\bar{1}}^{+} \coloneqq \bar{\Phi}_{\bar{1}} \cap \Phi^{+}. \tag{2.17}$$

Let $\Phi_{\bar{0}}^{+}$ and $\Phi_{\bar{1}}^{+}$ be the even and odd roots in Φ^{+}, respectively. The **Weyl vector** $\overline{\rho}$ is defined as

$$\overline{\rho} \coloneqq \rho_{\bar{0}} - \rho_{\bar{1}}, \tag{2.18}$$

where $\rho_{\bar{0}} = \frac{1}{2}\sum_{\alpha\in\Phi_{\bar{0}}^{+}} \alpha$ and $\rho_{\bar{1}} = \frac{1}{2}\sum_{\beta\in\Phi_{\bar{1}}^{+}} \beta$. In $\mathfrak{gl}(m|n)$, the Weyl vector is explicitly given as

$$\overline{\rho} \coloneqq \sum_{i=1}^{m}(m-i+1)\delta_i - \sum_{j=1}^{n} j\epsilon_j - \frac{m+n+1}{2}\left(\sum_{i=1}^{m}\delta_i - \sum_{j=1}^{n}\epsilon_j\right).$$

Due to a subtelty regarding half-integers, it is more convenient for us to use a "normalized" Weyl vector ρ for our purposes:

$$\rho = \sum_{i=1}^{m}(m-i+1)\delta_i - \sum_{j=1}^{n} j\epsilon_j. \tag{2.19}$$

Observe that we have only kept the first two sums; the removal of the last component will have no bearing on our considerations because the following inner product holds for all $\alpha \in \Phi_{\bar{1}}$:

$$(\sum_{i=1}^{m}\delta_i - \sum_{j=1}^{n}\epsilon_j, \alpha) = 0. \tag{2.20}$$

2.6. **Dominance.** A weight $\lambda \in \mathfrak{h}^*$ is said to be **antidominant** if $\langle \lambda+\rho, \alpha^\vee\rangle \notin \mathbb{Z}_{>0}$ and **dominant** if $\langle \lambda+\rho, \alpha^\vee\rangle \notin \mathbb{Z}_{<0}$ for all $\alpha \in \Phi_{\bar{0}}^{+}$.

2.7. **Atypicality and Linkage.** The notion of linkage in the super case is similar to that of semisimple Lie algebras. However, the key distinction is that while blocks of modules in the semisimple Lie algebra case are finite, isotropic roots allow for blocks in the super case to be infinite. This arises because of a notion called atypicality.

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of the diagonal matrices with standard basis for $\mathfrak{h}^*$ and standard choices for the root system as above.

Definition 2.3. The **degree of atypicality** of $\lambda \in \mathfrak{h}^*$, denoted $\#\lambda$, is the *maximum number of mutually orthogonal isotropic roots* $\alpha \in \bar{\Phi}_{\bar{1}}^{+}$ such that $(\lambda+\rho, \alpha) = 0$. An element $\lambda \in \mathfrak{h}^*$ is said to be **typical** (relative to Φ^+) if $\#\lambda = 0$ and is **atypical** otherwise.

A relation $\sim$ on $\mathfrak{h}^*$ can be defined as following. We say $\lambda \sim \mu$ $\lambda, \mu \in \mathfrak{h}^*$ if there exist mutually orthogonal isotropic odd roots $\alpha_1, \alpha_2, \ldots, \alpha_l$, complex numbers $c_1, c_2, \ldots, c_l$, and an element $w \in \mathcal{W}$ satisfying:

$$\mu + \rho = w\left(\lambda + \rho - \sum_{i=1}^{l} c_i\alpha_i\right), \quad (\lambda+\rho, \alpha_i) = 0, i = 1\ldots, l. \tag{2.21}$$

The weights λ and μ are said to be **linked** if $\lambda \sim \mu$. It can be shown that linkage is an equivalence relation.

Given a fundmanetal root system Π, we can establish the **Bruhat order** on $\mathfrak{h}^*$ as follows. Let $\lambda, \mu \in \mathfrak{h}^*$. We say $\lambda \geq \mu$ if $\lambda \sim \mu$ and $\lambda - \mu \in \mathbb{Z}_{\geq 0}\Pi$ (i.e the nonnegative sum of simple roots).

Atypicality may not seem clear at first, so we introduce notation to elucidate the phenomenon. There exists a natural bijection between the integral weight lattice X and $\mathbb{Z}^{m+n}$ where $\lambda \in X$ maps to $(q_1, q_2, \ldots, q_m \mid r_1, r_2, \ldots r_n) \in \mathbb{Z}^{m+n}$ if

$$\lambda = \sum_{i=1}^{m} q_i \delta_i - \sum_{j=1}^{n} r_j \epsilon_j.$$

Denote this identification with the congruence symbol $\cong$. By abuse of notation, we shall also use ρ to denote the image $(m, m-1, \ldots, 1 \mid 1, 2, \ldots, n)$ of the normalized Weyl vector under this identification; the context will make it clear to which we refer. Furthermore, the action of the Weyl group $\mathcal{W} \cong S_m \times S_n$ is clear. We can permute everything to the left of the bar and to the right of the bar, but no coefficient may cross the bar.

This bijection highlights atypicality very nicely. The degree of atypicality of the weight $(q_1, q_2, \ldots, q_m \mid r_1, r_2, \ldots r_n) - \rho$ is read by counting the number of pairs (q_i, r_j) such that $q_i = r_j$, with the important stipulation no q_i or r_j be reused. The corresponding set of mutually orthogonal roots are $\delta_i - \epsilon_j$ for each pair (i, j). The degree of the atypicality is also given by the size of the multiset $\{q_i\}_{i=1}^m \cap \{r_j\}_{j=1}^n$. In particular, if none of the q_i coincide with the r_j, the weight is typical.

2.8. **Examples.** In this subsection, we take a concrete look at roots and the weight lattice.

2.8.1. $\mathfrak{gl}(2|1)$. With the Cartan subalgebra $\mathfrak{h}$ given by the diagonal matrices, the bilinear form given by the supertrace induces a basis for $\mathfrak{h}^*$ given by $\{\delta_1, \delta_2, \epsilon\}$, where we write ϵ to abbreviate ϵ_1. By the standard convention above, the positive even root of $\mathfrak{gl}(2|1)$ is $\delta_1 - \delta_2$, and the positive odd roots are $\delta_2 - \epsilon$ and $\delta_1 - \epsilon$. The odd roots are also isotropic. The normalized Weyl vector is given by $\rho = 2\delta_1 + \delta_2 - \epsilon \cong (2, 1 \mid 1)$.

If $c \neq a, b$ are integers, observe $(a, b \mid c) - \rho$ is typical, as there are no odd roots to which $(a, b \mid c) - \rho$ is orthogonal. On the other hand, $(a, b \mid b) - \rho$ and $(a, b \mid a) - \rho$ are atypical (of degree 1). After a ρ-shift, the first case is orthogonal to the odd root $\delta_2 - \epsilon$, and the second is orthogonal to $\delta_1 - \epsilon$. There are no other types of atypicality because the two odd roots themselves are not mutually orthogonal.

The Weyl group is $\mathcal{W} \cong S_2 \times S_1$. We see that the integral atypical linkage classes are indexed by $a \in \mathbb{Z}$, with weights given by $(a, b \mid b) - \rho$ and $(b, a \mid b) - \rho$, with $b \in \mathbb{Z}$ allowed to vary.

Suppose $\lambda \cong (2, 1 \mid 1)$ and $\mu \cong (3, 2 \mid 3)$. The weights $\lambda - \rho$ and $\mu - \rho$ are linked because adding the odd root $\delta_2 - \epsilon$ (which is orthogonal to λ) to λ twice and then applying a Weyl group element yields μ.

2.8.2. $\mathfrak{gl}(3|1)$. With the Cartan subalgebra $\mathfrak{h}$ given by the diagonal matrices, the bilinear form given by the supertrace induces a basis for $\mathfrak{h}^*$ given by $\{\delta_1, \delta_2, \delta_3, \epsilon\}$, where we write ϵ to abbreviate ϵ_1. By the standard convention above, the positive even roots are $\delta_1 - \delta_2$, $\delta_2 - \delta_3$, and $\delta_1 - \delta_3$, and the positive odd roots are $\delta_3 - \epsilon$, $\delta_2 - \epsilon$ and $\delta_1 - \epsilon$. The odd roots are also isotropic. The normalized Weyl vector is given by $\rho = 3\delta_1 + 2\delta_2 + \delta_3 - \epsilon \cong (3, 2, 1 \mid 1)$.

If $d \neq a, b, c$ are integers, then $(a, b, c \mid d) - \rho$ is typical, as there are no odd roots to which this weight is orthogonal. The weights $(a, b, c \mid c) - \rho$, $(a, b, c \mid b) - \rho$, and $(a, b, c \mid a) - \rho$ are atypical of degree 1. After a ρ-shift, the first case is orthogonal to the odd root $\delta_3 - \epsilon$, the second is orthogonal to $\delta_2 - \epsilon$, and the last is orthogonal to $\delta_3 - \epsilon$. There are no other types of atypicality because none of the odd roots are pairwise orthogonal.

The Weyl group is $\mathcal{W} \cong S_3 \times S_1$. We see that the integral atypical linkage classes are indexed by $a, b \in \mathbb{Z},\ a \geq b$, with weights of the form $(a, b, c \mid c) - \rho$, $(b, a, c \mid c) - \rho$, $(a, c, b \mid c) - \rho$, $(b, c, a \mid c) - \rho$, $(c, a, b \mid c) - \rho$, and $(c, b, a \mid c) - \rho$ with $c \in \mathbb{Z}$ allowed to vary.

Suppose $\lambda \cong (2, 1, 3 \mid 1)$ and $\mu \cong (4, 3, 2 \mid 4)$. The weights $\lambda - \rho$ and $\mu - \rho$ are linked because adding the odd root $\delta_2 - \epsilon$ (which is orthogonal to λ) to λ thrice and then applying a Weyl group element yields μ.

2.8.3. $\mathfrak{gl}(2|2)$. With the Cartan subalgebra $\mathfrak{h}$ given by the diagonal matrices, the bilinear form given by the supertrace induces a basis for $\mathfrak{h}^*$ given by $\{\delta_1, \delta_2, \epsilon_1, \epsilon_2\}$. By the standard convention above, the positive even roots are $\delta_1 - \delta_2$ and $\epsilon_1 - \epsilon_2$. The positive odd roots (also isotropic) are $\delta_1 - \epsilon_1$, $\delta_1 - \epsilon_2$, $\delta_2 - \epsilon_1$, and $\delta_2 - \epsilon_2$. Observe now that we can choose two isotropic roots such that they are mutually orthogonal; one choice is $(\delta_1 - \epsilon_2, \delta_2 - \epsilon_1) = 0$ and the other is $(\delta_1 - \epsilon_1, \delta_2 - \epsilon_2) = 0$. This introduces weights of atypicality of degree 2. The Weyl vector is given by $\rho = 2\delta_1 + \delta_1 - \epsilon_1 - 2\epsilon_2 \cong (2, 1 \mid 1, 2)$.

If $a, b, c, d \in \mathbb{Z}$ and $\{a, b\} \cap \{c, d\} = \varnothing$, then $(a, b \mid c, d) - \rho$ is typical, as there are no odd roots to which this weight is is orthogonal after a ρ-shift. The difference from the previous two cases is that atypicality of degree 2 is now possible. Because there are two pairs of two mutually orthogonal roots, weights of the form $(a, b \mid b, a) - \rho$ and $(a, b \mid a, b) - \rho$ are atypical of degree 2.

The Weyl group is $\mathcal{W} \cong S_2 \times S_2$. We see that there is one integral atypical linkage class of degree 2, with weights of the form $(a, b \mid b, a) - \rho$, $(a, b \mid a, b) - \rho$, $(b, a \mid b, a) - \rho$, and $(b, a \mid a, b) - \rho$, where $a \geq b \in \mathbb{Z}$ are free to vary. For example, the weight $(2, 1 \mid 1, 2) - \rho$ is linked to $(3, 5 \mid 5, 3) - \rho$, but not to $(3, 3 \mid 5, 5) - \rho$.

Suppose $\lambda \cong (2, 1 \mid 1, 2)$ and $\mu \cong (5, 8 \mid 5\ 8)$. The weights $\lambda - \rho$ and $\mu - \rho$ are linked because adding the odd root $\delta_2 - \epsilon_1$ four times and $\delta_1 - \epsilon_2$ six times to λ and then applying a Weyl group element yields μ. Observe that these odd roots are both orthogonal to λ and are orthogonal to each other.

2.9. **Universal Enveloping Algebra and PBW Theorem.** The notion of an enveloping algebra with the universal property extends similarly from the Lie algebra case to the super case; if $\mathfrak{g}$ is a Lie superalgebra, let $U(\mathfrak{g})$ denote its universal enveloping algebra.

The Poincaré-Birkhoff-Witt (PBW) theorem for Lie superalgebras is similar to that for Lie algebras, with a slight adjustment for super phenomenon present.

Theorem 2.4 (PBW Theorem). *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional Lie superalgebra, and suppose $\{x_1, x_2, \ldots, x_p\}$ is a basis for $\mathfrak{g}_{\bar{0}}$ and $\{y_1, y_2, \ldots, y_q\}$ is a basis for $\mathfrak{g}_{\bar{1}}$. Then, the set*

$$\{x_1^{r_1} x_2^{r_2} \ldots x_p^{r_p} y_1^{s_1} y_2^{s_2} \ldots y_q^{s_q} \mid r_1, \ldots r_p \in \mathbb{Z}_{\geq 0},\ s_1, \ldots s_q \in \{0, 1\}\}$$

is a basis for $U(\mathfrak{g})$.

It is readily seen that representations for $\mathfrak{g}$ are representations for $U(\mathfrak{g})$, and vice versa.

3. The BGG category $\mathcal{O}$ of $\mathfrak{gl}(m|n)$-modules

In this section, let $\mathfrak{g} = \mathfrak{gl}(m|n) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with the standard associated bilinear form, root system, and triangular decomposition above: $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$.

3.1. **The BGG Category $\mathcal{O}$.** The primary modules of interest is a collection of modules of a Lie superalgebra $\mathfrak{g}$ known as the *BGG Category $\mathcal{O}$*.

Definition 3.1. The **BGG category $\mathcal{O}$** is defined as the full subcategory of $U(\mathfrak{g})$-modules M subject to the following three conditions:

(1) M is finitely generated.
(2) M is $\mathfrak{h}$-semisimple: $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$, where $M^\lambda = \{v \in M \mid h \cdot v = \lambda(h)v \quad \forall h \in \mathfrak{h}\}$ is a nonzero weight space.
(3) M is locally $\mathfrak{n}^+$-finite: $U(\mathfrak{n}^+) \cdot v$ is finite dimensional for $\forall v \in M$.

3.2. **Highest Weight Modules.** Let M be $U(\mathfrak{g})$ module and define nonzero $v^+ \in M$ to be a **highest weight vector** of weight $\lambda \in \mathfrak{h}^*$ if $v^+ \in M^\lambda$ and $\mathfrak{n}^+ \cdot v^+ = 0$. We say that M is a **highest weight module** of weight λ if there exists a highest weight vector $v^+ \in M^\lambda$ such that $M = U(\mathfrak{g}) \cdot v^+$. Of particular interest are a class of modules where $U(\mathfrak{n}^-)$ acts freely on v^+.

Definition 3.2. Observe that the abelian quotient algebra $\mathfrak{b}/\mathfrak{n}^+ \cong \mathfrak{h}$. Thus, any $\lambda \in \mathfrak{h}^*$ naturally defines a one-dimensional $\mathfrak{b}$-module with trivial $\mathfrak{n}^+$-action, which we denote as $\mathbb{C}_\lambda$. Specifically, if $v \in \mathbb{C}_\lambda$, then $h \cdot v = \lambda(h)v$, $\forall h \in \mathfrak{h}$. Now, define $M_\lambda := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$, where ρ is the normalized Weyl vector. This is naturally a left $\mathfrak{g}$-module. This is called a **Verma module** of highest weight $\lambda - \rho$ and is a free $U(\mathfrak{n}^-)$-module by the PBW Theorem. Clearly, a Verma module is a highest weight module.

We let L_λ denote the unique simple quotient of M_λ of highest weight $\lambda - \rho$, and use the notation $[M_\mu : L_\lambda]$ to denote the multiplicity of L_λ in a composition series of M_μ. Such a series exists for all M in $\mathcal{O}$ as $\mathcal{O}$ is both noetherian and artinian.

In the notation introduced in 2.7, if $\lambda \cong (q_1 \dots q_m \mid r_1 \dots r_m)$, let $M_{q_1 \dots q_m | r_1 \dots r_n}$ denote the Verma module M_λ. Similarly, let $L_{q_1 \dots q_m | r_1 \dots r_n}$ denote the irreducible module L_λ.

3.3. **Blocks in $\mathcal{O}$.** A block in $\mathcal{O}$ is the full subcategory of $\mathfrak{g}$-modules such that the composition series of each module consists entirely of simple modules L_λ, where the $\lambda \in \mathfrak{h}^*$ are all in the same linkage class. Therefore, blocks in $\mathcal{O}$ are indexed by linkage classes. It can be shown that any module $M \in \mathcal{O}$ can be decomposed as the direct sum of indecomposable submodules, each which belongs to a single block.

Integral blocks are particularly interesting because of interactions within blocks due to linkage. The integral blocks in $\mathcal{O}$ can be divided into typical and atypical blocks. By definition, any simple module in a typical block has typical highest weight. By Gorelik [Gor02], any integral typical block in $\mathcal{O}$ is equivalent to a block in the BGG Category of $\mathfrak{g}_{\bar{0}}$-modules. Therefore, we are interested in the atypical blocks, as determining the character of projective modules in the atypical blocks is not as straightforward.

Now, recall the examples in Section 2.8. In $\mathfrak{gl}(2|1)$, the linked weights are $(a, b \mid b) - \rho$ and $(b, a \mid b) - \rho$, with $a, b \in \mathbb{Z}$ and b allowed to vary. We can let $\mathcal{B}_a$ denote the corresponding block, which contains the simple modules $L_{a,b|b}$. In $\mathfrak{gl}(3|1)$, the linked weights are $(a, b, c \mid c) - \rho$, $(b, a, c \mid c) - \rho$, $(a, c, b \mid c) - \rho$, $(b, c, a \mid c) - \rho$, $(c, a, b \mid c) - \rho$, and $(c, b, a \mid c) - \rho$ with

$a, b, c \in \mathbb{Z}$ and c allowed to vary. We will let $\mathcal{B}_{a,b}$ denote the corresponding block. Lastly, in $\mathfrak{gl}(2|2)$, there is only one block of atypicality degree 2; we will denote it as $\mathcal{B}_0$.

3.4. **Key Results in $\mathcal{O}$.** The primary means by which the goals of this paper are achieved are by using translation functors. We restate the necessary results to justify our steps. This collection of results is justified in [Hum08, Chap. 1-3] for the BGG Category $\mathcal{O}$ for semisimple Lie algebras; similar arguments extend them to the BGG Category $\mathcal{O}$ of $\mathfrak{gl}(m|n)$-modules.

Theorem 3.3. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$. Let N be a finite dimensional $U(\mathfrak{g})$-module. For any $\lambda \in \mathfrak{h}^*$, the tensor module $T := M_\lambda \otimes N$ has a finite filtration with quotients isomorphic to Verma modules of the form $M_{\lambda+\mu}$, where μ ranges over the weights of N, each occurring* $\dim N^\mu$ *times in the filtration.*

Definition 3.4. A module $N \in \mathcal{O}$ has a **standard filtration** or a **Verma flag** if there is a sequence of submodules $0 = N_0 \subset N_1 \subset N_2 \subset \cdots N_k = N$ such that each N_i/N_{i-1} $1 \leq i \leq k$ is isomorphic to a Verma module. The number of times the Verma module M_λ appears in a standard filtration of N is denoted by $(N : M_\lambda)$.

It can be shown that the length and the Verma multiplicities in a standard filtration are independent of choice of a standard filtration. Therefore, the following informal notation to indicate a standard filtration of a module is useful. If $M_{\lambda_i},\ \lambda_i \in \mathfrak{h}^*,\ 1 \leq i \leq k$ are the Verma modules appearing in the standard filtration of a module M each appearing with multiciplity $c_i \in \mathbb{Z}_{>0}$, we write:

$$N = c_1 M_{\lambda_1} + c_2 M_{\lambda_2} + \cdots + c_k M_{\lambda_k} \tag{3.1}$$

Recall that a a module $P \in \mathcal{O}$ is said to be **projective** if given any modules $M, N \in \mathcal{O}$, any surjective homomorphism $\pi \in \mathrm{Hom}(M, N)$, and any homomorphism $\phi \in \mathrm{Hom}(P, N)$, there exists a lifting map $\psi \in \mathrm{Hom}(P, M)$ such that $\pi\psi = \phi$.

We let P_λ denote the projective cover for L_λ for all $\lambda \in \mathfrak{h}^*$, that is the indecomposable projective such that $P_\lambda \twoheadrightarrow L_\lambda \to 0$. The existence of such a projective cover is guaranteed by the proposition below, and it can also be shown to be unique up to isomorphism. Furthermore, it can be shown there exist homomorphisms such that $P_\lambda \twoheadrightarrow M_\lambda \twoheadrightarrow L_\lambda$. Lastly, all projective covers have a standard filtration. The following proposition is key in our understanding of projective modules in $\mathcal{O}$.

Proposition 3.5. *The following results hold for projective modules in $\mathcal{O}$:*

1. *All projectives have a standard filtration.*
2. *The Category $\mathcal{O}$ has enough projectives.*
3. *Each indecomposable projective in $\mathcal{O}$ is isomorphic to some P_λ.*
4. *The Verma modules M_μ which appear in a standard filtration of P_λ satisfy $\mu \geq \lambda$ in the Bruhat ordering, and M_λ appears with multiplicity* 1.

The following proposition, which follows from Theorem 3.3, is a critical part of our translation functor arguments.

Proposition 3.6. *If a projective P has a standard filtration given by $P_\lambda = \sum_\nu M_\nu$, the ν not necessarily distinct, then for any finite-dimensional representation N with weights μ, the standard filtration for $P \otimes N$ is given by $\sum_\nu \sum_\mu M_{\nu+\mu}$, where μ appears in the sum with multiplicity given by* $\dim N^\mu$.

The following lemma will also be useful in our arguments. It is predicated on the fact that the Weyl groups for $\mathfrak{gl}(2|1), \mathfrak{gl}(3|1)$, and $\mathfrak{gl}(2|2)$ are products of dihedral groups (cf. [CW18]).

Lemma 3.7. *If $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $m, n \leq 3$ and $\lambda \in X$ is typical, then the Verma modules that appear in a standard filtration of P_λ are of the form $M_{w\lambda}$, where $w \in \mathcal{W}$ such that $w\lambda \geq \lambda$, and each Verma module appears with multiplicity 1. Therefore, when λ is also dominant, we have that $P_\lambda = M_\lambda$.*

Lastly, we recall the notion of BGG Reciprocity.

Theorem 3.8 (BGG Reciprocity)**.** *Let $\lambda, \mu \in \mathfrak{h}^*$. Then, $(P_\lambda : M_\mu) = [M_\mu : L_\lambda]$.*

BGG Reciprocity us to deduce information about composition series from the standard filtrations of projectives.

3.5. **Useful Representations of $\mathfrak{gl}(m|n)$.** The strategy of using translation functors involves choosing appropriate representations to tensor with projective modules to produce new modules.

Let $V = \mathbb{C}^{m|n}$ be the **natural representation** V of $\mathfrak{g} = \mathfrak{gl}(m|n)$. The natural representation lies in $\mathcal{O}$, as it is finite-dimensional. With respect to $\mathfrak{h}$, the weights are simply $\{\epsilon_i \mid i \in I(m,n)\}$.

The **dual representation** V^* of the natural representation is given by $(g \cdot f)(v) := -(-1)^{|g||f|} f(g \cdot v)$ for any homogeneous linear functional in the dual space of V and homogeneous $v \in V$ (cf. [Mus12, Appendix A.2.3]). The weights are given by the negative of the weights of V. Specifically, these are $\{-\epsilon_i \mid i \in I(m,n)\}$.

We also make use of the **exterior algebra** of a finite-dimensional vector superspace. Let $W = W_{\bar{0}} \oplus W_{\bar{1}}$ be a vector superspace. Then, we can define the k-th exterior power of W as follows:

$$\bigwedge^k(W) := \bigoplus_{i+j=k} \left(\Lambda^i(W_{\bar{0}}) \otimes \mathrm{S}^j(W_{\bar{1}})\right) \tag{3.2}$$

where Λ^i and S^j acting on vector spaces are the i-th exterior power and j-th symmetric power in the traditional sense, respectively. When W is a representation of $\mathfrak{g}$, the exterior algebra of W and its graded components are also representations in the most natural sense. We will particularly be interested in the k-th exterior power when $k = 2$ or $k = 3$ and $W = V$ or $W = V^*$, which we refer to as wedge-squared or wedge-cubed of the natural or of the dual, respectively.

3.6. **Conditions for nonzero Verma flag multiplicities in projective modules.** We have the following proposition, which uses BGG reciprocity to reformulate the conditions for tilting modules in [CW18, Proposition 2.2] as conditions for projective modules.

Proposition 3.9. *Suppose that $\lambda \in X, \alpha_i \in \Phi^+_{\bar{0}}, 1 \leq i \leq k$, and $\beta, \gamma \in \Phi^+_{\bar{1}}$. Let $w = \prod_{i=1}^k s_{\alpha_i} \in \mathcal{W}$.*

(1) Suppose that $\langle \lambda, \alpha_1^\vee \rangle < 0$. Then $(P_\lambda : M_{s_{\alpha_1}\lambda}) > 0$.
(2) Suppose that $\langle s_{\alpha_{i-1}} \cdots s_{\alpha_1}\lambda, \alpha_i^\vee \rangle < 0 \ \forall i \in 1, 2, \ldots, k$. then $(P_\lambda : M_{w\lambda}) > 0$.
(3) Suppose that $(\lambda, \beta) = 0$. Then $(P_\lambda : M_{\lambda+\beta}) > 0$.
(4) Suppose that $(\lambda, \beta) = 0$ and $\langle s_{\alpha_{i-1}} \cdots s_{\alpha_1}(\lambda + \beta), \alpha_i^\vee \rangle < 0 \ \forall i \in 1, 2, \ldots, k$. Then $(P_\lambda : M_{w(\lambda+\beta)}) > 0$.

(5) *Suppose that* $(\lambda, \beta) = (\lambda + \beta, \gamma) = 0$ *and* $\mathrm{ht}(\beta) < \mathrm{ht}(\gamma)$. *Then* $(P_\lambda : M_{\lambda+\beta+\gamma}) > 0$.
(6) *Suppose that* $(\lambda, \beta) = (\lambda+\beta, \gamma) = 0$, $\mathrm{ht}(\beta) < \mathrm{ht}(\gamma)$, *and* $\langle s_{\alpha_{i-1}} \cdots s_{\alpha_1}(\lambda+\beta+\gamma), \alpha_i^\vee \rangle < 0\ \forall i \in 1, 2, \ldots, k$. *Then* $(P_\lambda : M_{w(\lambda+\beta+\gamma)}) > 0$.

Proof. The proposition is originally derived using the Super Jantzen sum formula (cf. [Gor02, Mus12]), giving conditions for composition factors. Theorem 3.8 (BGG Reciprocity) immediately translates the conditions from those on tilting modules to those on projective modules. □

Corollary 3.10. *Suppose* $\lambda - \rho \in \mathfrak{h}^*$ *is atypical. Then* P_λ *must have a Verma flag of length greater than* 1.

Proof. By condition 4 of Proposition 3.5, we have that M_λ appears in the standard filtration. Furthermore, because λ is atypical, there exists β such that $\beta \in \Phi^+_{\bar{1}}$ and $(\lambda, \beta) = 0$. Therefore, apply condition 4 of Proposition 3.9 to see that $M_{\lambda+\beta}$ also appears in the standard filtration. □

3.7. **Strategy.** Given an atypical $\lambda - \rho \in \mathfrak{h}^*$, we seek to deduce the standard filtration formula of P_λ. To do so, we choose a $\mu \in \mathfrak{h}^*$ such that we know a standard filtration for P_μ. This is often accomplished by letting $\mu := \lambda - \nu$, where ν is the lowest weight in some finite-dimensional representation W such that $\mu - \rho$ is typical; Lemma 3.7 tells us the structure of P_μ. Proposition 3.6 can be used to deduce the Verma modules which appear in a standard filtration of the projective $P_\mu \otimes W$, which must include M_λ. Our next step is to project $P_\mu \otimes W$ onto the block corresponding to the linkage class of $\lambda - \rho$. We denote the resulting projection as $\mathrm{Pr}_\lambda(P_\mu \otimes W)$. If M_λ has the lowest weight of all the Verma modules in the standard filtration of the projection, Proposition 3.5(3, 4) tells us that P_λ must appear in that projection as a direct summand, as the direct summands of a projective are projective. The projection itself is done by collecting all Verma modules in the standard filtration whose weights are linked to $\lambda - \rho$.

In this projection, we apply Proposition 3.9 to see which Verma modules appear in the standard filtration of P_λ. These necessarily appear in the projection because P_λ is a direct summand. Then, we generally try to argue that there is no other direct summand (i.e. P_λ is the projection). This is often done with the help of Corollary 3.10.

As a remark, it is not always necessary to take $\mu := \lambda - \nu$, where ν is the lowest weight in the representation W. This is often a good initial choice, but the key requirement is that λ be the lowest weight appearing in the standard filtration after the projection on to the block corresponding to the linkage class of $\lambda - \rho$.

4. Character Formulae for $\mathfrak{gl}(2|1)$

In this section, we determine standard filtration formulae for projective covers of simple modules of $\mathfrak{gl}(2|1)$ with integral, atypical weight of degree 1.

4.1. **Results.** Let $\mathfrak{g} = \mathfrak{gl}(2|1)$ have the standard choices of Cartan subalgebra, bilinear form, root system, positive, and fundamental system as described in Section 2. Recall the notation described in Section 2.7 to describe a weight in $\mathfrak{h}^*$. Lastly, recall Example 2.8.1 and the corresponding blocks $\mathcal{B}_a$, $a \in \mathbb{Z}$ (see Section 3.3). We have the following Theorem 4.1 that describes standard filtrations of projectives in these block. These formulae corroborate the results first proved in [CW08] by more complicated methods.

Theorem 4.1. *Let $a, b \in \mathbb{Z}$. The projective objects $P_{a,b|a}$ and $P_{b,a|a}$ in $\mathcal{B}_a$ have the following Verma flag formulae:*

(1) Case: $P_{a,b|a}$

(1.1) If $b < a$, then

$$P_{a,b|a} = M_{a,b|a} + M_{a+1,b|a+1}.$$

(1.2) If $b = a$, then

$$P_{a,a|a} = M_{a,a|a} + M_{a+1,a|a+1} + M_{a,a+1|a+1}.$$

(1.3) Suppose $b > a$.

(1.3.1) If $b = a+1$, then

$$P_{a,a+1|a} = M_{a,a+1|a} + M_{a+1,a|a} + M_{a+1,a+1|a+1}.$$

(1.3.2) If $b > a+1$, then

$$P_{a,b|a} = M_{a,b|a} + M_{a+1,b|a+1} + M_{b,a|a} + M_{b,a+1|a+1}.$$

(2) Case: $P_{b,a|a}$

(2.1) If $b < a$, then

$$P_{b,a|a} = M_{b,a|a} + M_{b,a+1|a+1} + M_{a,b|a} + M_{a+1,b|a+1}.$$

(2.2) Suppose $b > a$.

(2.2.1) If $b = a+1$, then

$$P_{a+1,a|a} = M_{a+1,a|a} + M_{a,a+1|a} + M_{a+1,a+1|a+1}.$$

(2.2.2) If $b > a+1$, then

$$P_{b,a|a} = M_{b,a|a} + M_{b,a+1|a+1}.$$

4.2. **Proof.** In this subsection, we prove Theorem 4.1. We use the statements in Section 3 and the method of translation functors. We use various representations for translation functors. Here are the weights of these representations:

Natural V:

$$\{\delta_1,\ \delta_2,\ \epsilon\}$$

Dual V^*:

$$\{-\delta_1,\ -\delta_2,\ -\epsilon\}$$

Wedge-squared of the natural $\bigwedge^2 V$:

$$\{\delta_1+\delta_2,\ \delta_1+\epsilon,\ \delta_2+\epsilon,\ 2\epsilon\}.$$

We now offer justification for the formulae above, separated into cases that have different formulae, based on the strategy in Section 3.7. Our proof be more explicit in the earlier cases and edge cases; those which lack much explanation follow the strategy almost directly.

4.2.1. *Case:* $P_\lambda, \lambda \cong (a, b \mid a)$.

(1) $b < a$.

Let $\mu \coloneqq \lambda - \epsilon$, so that $\mu \cong (a, b \mid a+1)$. We choose to subtract the weight ϵ because it is the lowest weight of the natural representation. Observe that $\mu - \rho$ is typical (and dominant), so by Lemma 3.7, we deduce that $P_u = M_{a,b|a+1}$. Now, when we consider the tensor product $T \coloneqq P_\mu \otimes V$. Proposition 3.6 tells us that Verma flag for T is given by $T = M_{a,b|a+1} + M_{a+1,b|a+1} + M_{a,b+1|a+1} + M_{a,b|a}$. Now, we project T on to $\mathcal{B}_\lambda$ to yield the projection P, another projective object. This entails keeping all the weights linked to $\lambda - \rho$, which is the highest weight of M_λ. We have:

$$P_\mu = M_{a,b|a+1},$$
$$P \coloneqq \mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{a,b|a} + M_{a+1,b|a+1}.$$

Now, because P is projective, its direct summands are also projective. The presence of M_λ in the standard filtration, with λ being the lowest weight appearing, tells us that P_λ is a direct summand (by parts 3 and 4 of Proposition 3.5. We now apply Corollary 3.10 to deduce that the other term in the filtration of P must also appear in the standard filtration for P_λ; if it didn't, P would be the direct sum of two projective objects, each with atypical weight but standard filtration length of only 1. We conclude that there is only one direct summand (i.e. $P = P_\lambda$).

(2) $b = a$.

Let $\mu \coloneqq \lambda - \epsilon$. Observe that $\mu - \rho$ is typical and dominant. We have:

$$P_\mu = M_{a,a|a+1},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{a,a|a} + M_{a,a+1|a+1} + M_{a+1,a|a+1}.$$

The presence of M_λ having the lowest weight in the Verma flag indicates that P_λ is a direct summand. We now apply condition 3 of Proposition 3.9 to deduce that $M_{a,a+1|a+1}$ and $M_{a+1,a|a+1}$ appear in a standard filtration for P_λ. We deduce that there is only one direct summand.

(3) $b > a$.

(3.1) $b = a + 1$.

Let $\mu \coloneqq \lambda - (-\delta_1)$. Observe that $-\delta_1$ is the lowest weight of the dual representation. We have:

$$P_\mu = M_{a+1,a+1|a},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes V^*) = M_{a,a+1|a} + M_{a+1,a|a} + M_{a+1,a+1|a+1}$$

As per usual, P_λ appears in the projective as a direct summand. By condition 1 of Proposition 3.9, $M_{a+1,a|a}$ appears in the standard filtration of P_λ. Corollary 3.10 tells us that $M_{a+1,a+1|a+1}$ also appears. We deduce that there is only one direct summand.

(3.2) $b > a + 1$.

Let $\mu \coloneqq \lambda - \epsilon$. Now, we have:

$$P_\mu = M_{a,b|a+1} + M_{b,a|a+1},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{a,b|a} + M_{a+1,b|a+1} + M_{b,a|a} + M_{b,a+1|a+1}$$

As per usual, P_λ appears in the projective as a direct summand. To deduce the remaining modules in the standard filtration of P_λ, Proposition 3.9 can be used.

By condition 2, $M_{a+1,b|a+1}$ appears, and by condition 1, $M_{b,a|a}$ appears. Now, apply Corollary 3.10 to see that $M_{b,a+1|a+1}$ also appears in filtration. We deduce that there is only one direct summand.

4.2.2. *Case:* $P_\lambda, \lambda \cong (b, a \mid a)$.

(1) $b < a$.
Let $\mu \coloneqq \lambda - \epsilon$. We have:

$$P_\mu = M_{b,a|a+1} + M_{a,b|a+1},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{b,a|a} + M_{b,a+1|a+1} + M_{a+1,b|a+1} + M_{a,b,|a}.$$

As per usual, P_λ appears in the projective as a direct summand. To deduce the remaining modules in the standard filtration of P_λ, Proposition 3.9 can be used. By condition 1, $M_{a,b|a+1}$ appears, and by condition 3, $M_{b,a+1|a+1}$ appears. Now, apply Corollary 3.10 to see that $M_{a+1,b|a+1}$ also appears in filtration. We deduce that there is only one direct summand.

(2) $b = a$. This is a repeat of an earlier case.

(3) $b > a$.

(3.1) $b = a + 1$.
Let $\mu \coloneqq \lambda - 2\epsilon$. Observe that 2ϵ is the lowest weight of the wedge-squared of the natural representation. We have:

$$P_\mu = M_{a+1,a|a+2},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) = M_{a+1,a|a} + M_{a+1,a+1|a+1} + M_{a+2,a+1|a+2}.$$

As per usual, P_λ appears in the projective as a direct summand, . By part 1 of Proposition 3.9, we see that $M_{a+1,a+1|a+1}$ appears in a Verma flag for P_λ. Now, apply Corollary 3.10 to see that $M_{a+2,a+1|a+2}$ must also appear in such a filtration. We deduce that there is only one direct summand.

(3.2) $b > a + 1$.
Let $\mu \coloneqq \lambda - \epsilon$. dual representation. We have:

$$P_\mu = M_{b,a|a+1}$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{b,a|a} + M_{b,a+1|a+1}$$

As per usual, P_λ appears in the projective as a direct summand. Apply Corollary 3.10 to see that there is only one direct summand.

5. Character Formulae for $\mathfrak{gl}(3|1)$

In this section, we determine standard filtration formulae for projective covers of simple modules of $\mathfrak{gl}(3|1)$ with integral, atypical weight of degree 1.

5.1. **Results.** Let $\mathfrak{g} = \mathfrak{gl}(3|1)$ have the standard choices of Cartan subalgebra, bilinear form, root system, positive, and fundamental system as described in Section 2. Recall the notation described in Section 2.7 to describe a weight in $\mathfrak{h}^*$. Lastly, recall Example 2.8.2 and the corresponding blocks $\mathcal{B}_{a,b}$, $a, b \in \mathbb{Z}$ (see Section 3.3). We have the following Theorems 5.1 to 5.6 that describe standard filtrations of projectives in these blocks.

Theorem 5.1. *Let $a, b, c \in \mathbb{Z}$ with $a \geq b$. The projective objects $P_{a,b,c|c}$ in $\mathcal{B}_{a,b}$ have the following Verma flag formulae.*

(1) Suppose $b > c$.

(1.1) If $b > c+1$, then

$$P_{a,b,c|c} = M_{a,b,c|c} + M_{a,b,c+1|c+1}.$$

(1.2) Suppose $b = c+1$.

(1.2.1) If $a > c+2$, then

$$P_{a,c+1,c|c} = M_{a,c+1,c|c} + M_{a,c+1,c+1|c+1} + M_{a,c+2,c+1|c+2}.$$

(1.2.2) If $a = c+2$, then

$$\begin{aligned} P_{c+2,c+1,c|c} = M_{c+2,c+1,c|c} &+ M_{c+2,c+2,c+1|c+2} \\ +M_{c+2,c+1,c+1|c+1} &+ M_{c+3,c+2,c+1|c+3}. \end{aligned}$$

(1.2.3) If $a = c+1$, then

$$\begin{aligned} P_{c+1,c+1,c|c} = M_{c+1,c+1,c|c} &+ M_{c+1,c+1,c+1|c+1} \\ +M_{c+1,c+2,c+1|c+2} &+ M_{c+2,c+1,c+1|c+2}. \end{aligned}$$

(2) Suppose $b = c$.

(2.1) If $a > c+1$, then

$$P_{a,c,c|c} = M_{a,c,c|c} + M_{a,c,c+1|c+1} + M_{a,c+1,c|c+1}.$$

(2.2) If $a = c+1$, then

$$\begin{aligned} P_{c+1,c,c|c} = M_{c+1,c,c|c} + M_{c+2,c+1,c|c+2} &+ M_{c+1,c,c+1|c+1} \\ +M_{c+1,c+1,c|c+1} &+ M_{c+2,c,c+1|c+2}. \end{aligned}$$

(2.3) If $a = c$, then

$$\begin{aligned} P_{c,c,c|c} = M_{c,c,c|c} &+ M_{c,c,c+1|c+1} \\ +M_{c+1,c,c|c+1} &+ M_{c+1,c,c|c+1}. \end{aligned}$$

(3) Suppose $b < c$.

(3.1) Suppose $a > c$.

(3.1.1) If $a > c+1$, then

$$\begin{aligned} P_{a,b,c|c} = M_{a,b,c|c} &+ M_{a,b,c+1|c+1} \\ +M_{a,c,b|c} &+ M_{a,c+1,b|c+1}. \end{aligned}$$

(3.1.2) If $a = c+1$, then

$$\begin{aligned} P_{c+1,b,c|c} = M_{c+1,b,c|c} + M_{c+1,b,c+1|c+1} &+ M_{c+2,b,c+1|c+2} \\ +M_{c+1,c,b|c} + M_{c+1,c+1,b|c+1} &+ M_{c+2,c+1,b|c+2}. \end{aligned}$$

(3.2) If $a = c$, then

$$\begin{aligned} P_{c,b,c|c} = M_{c,b,c|c} + M_{c,b,c+1|c+1} &+ M_{c+1,b,c|c+1} \\ +M_{c,c,b|c} + M_{c,c+1,b|c+1} &+ M_{c+1,c,b|c+1}. \end{aligned}$$

(3.3) Suppose $a < c$.

(3.3.1) If $a > b$, then

$$\begin{aligned} P_{a,b,c|c} = M_{a,b,c|c} + M_{a,b,c+1|c+1} + M_{a,c,b|c} \\ +M_{c,a,b|c} + M_{a,c+1,b|c+1} + M_{c+1,a,b|c+1} \\ +M_{c,b,a|c} + M_{c+1,b,a|c+1}. \end{aligned}$$

(3.3.2) If $a = b$, then

$$\begin{aligned} P_{b,b,c|c} = M_{b,b,c|c} + M_{b,b,c+1|c+1} + M_{b,c,b|c} \\ +M_{b,c+1,b|c+1} + M_{c,b,b|c} + M_{c+1,b,b|c+1}. \end{aligned}$$

Theorem 5.2. *Let $a, b, c \in \mathbb{Z}$ with $a \geq b$. The projective objects $P_{b,a,c|c}$ in $\mathcal{B}_{a,b}$ have the following Verma flag formulae.*

(1) Suppose $b > c$.

(1.1) If $b > c+1$ and $a > b$, then

$$P_{b,a,c|c} = M_{b,a,c|c} + M_{b,a,c+1|c+1} + M_{a,b,c|c} + M_{a,b,c+1|c+1}.$$

(1.2) Suppose $b = c+1$.

(1.2.1) If $a > c+2$, then

$$\begin{aligned} P_{c+1,a,c|c} = M_{c+1,a,c|c} + M_{c+2,a,c+1|c+2} + M_{c+1,a,c+1|c+1} \\ +M_{a,c+1,c|c} + M_{a,c+2,c+1|c+2} + M_{a,c+1,c+1|c+1}. \end{aligned}$$

(1.2.2) If $a = c+2$, then

$$\begin{aligned} P_{c+1,c+2,c|c} = M_{c+1,c+2,c|c} + M_{c+2,c+1,c|c} + M_{c+1,c+2,c+1|c+1} \\ +M_{c+2,c+1,c+1|c+1} + M_{c+2,c+2,c+1|c+2}. \end{aligned}$$

(2) Suppose $b = c$.

(2.1) If $a > c+1$, then

$$\begin{aligned} P_{c,a,c|c} = M_{c,a,c|c} + M_{c,a,c+1|c+1} + M_{c+1,a,c|c+1} \\ +M_{a,c,c|c} + M_{a,c,c+1|c+1} + M_{a,c+1,c|c+1}. \end{aligned}$$

(2.2) If $a = c+1$, then

$$\begin{aligned} P_{c,c+1,c|c} = M_{c,c+1,c|c} + M_{c,c+2,c+1|c+2} + M_{c+1,c+2,c|c+2} + M_{c,c+1,c+1|c+1} \\ +2M_{c+1,c+1,c|c+1} + M_{c+2,c,c+1|c+2} + M_{c+2,c+1,c|c+2} \\ +M_{c+1,c,c+1|c+1} + M_{c+1,c,c|c}. \end{aligned}$$

(3) Suppose $b < c$.

(3.1) Suppose $a > c$.

(3.1.1) If $a > c+1$, then

$$\begin{aligned} P_{b,a,c|c} = M_{b,a,c|c} + M_{a,b,c|c} + M_{a,c,b|c} + M_{c,a,b|c} + M_{b,a,c+1|c+1} \\ +M_{a,b,c+1|c+1} + M_{a,c+1,b|c+1} + M_{c+1,a,b|c+1}. \end{aligned}$$

(3.1.2) If $a = c + 1$, then

$$\begin{aligned} P_{b,c+1,c|c} &= M_{b,c+1,c|c} + M_{b,c+2,c+1|c+2} + M_{b,c+1,c+1|c+1} \\ &+ M_{c+1,b,c|c} + M_{c+2,b,c+1|c+2} + M_{c+1,b,c+1|c+1} \\ &+ M_{c,c+1,b|c} + M_{c+1,c+2,b|c+2} + M_{c+1,c+1,b|c+1} \\ &+ M_{c+1,c,b|c} + M_{c+2,c+1,b|c+2} + M_{c+1,c+1,b|c+1}. \end{aligned}$$

(3.2) If $a = c$, then

$$\begin{aligned} P_{b,c,c|c} &= M_{b,c,c|c} + M_{b,c,c+1|c+1} + M_{b,c+1,c|c+1} \\ &+ M_{c,b,c|c} + M_{c,b,c+1|c+1} + M_{c+1,b,c|c+1} \\ &+ M_{c,c,b|c} + M_{c,c+1,b|c+1} + M_{c+1,c,b|c+1}. \end{aligned}$$

(3.3) Suppose $a < c$ and $a > b$.

$$\begin{aligned} P_{b,a,c|c} &= M_{b,a,c|c} + M_{b,a,c+1|c+1} + M_{b,c,a|c} \\ &+ M_{b,c+1,a|c+1} + M_{a,b,c|c} + M_{a,b,c+1|c+1} \\ &+ M_{c,b,a|c} + M_{c+1,b,a|c+1} + M_{a,c,b|c} \\ &+ M_{a,c+1,b|c+1} + M_{c,a,b|c} + M_{c+1,a,b|c+1}. \end{aligned}$$

Theorem 5.3. *Let $a, b, c \in \mathbb{Z}$ with $a \geq b$. The projective objects $P_{a,c,b|c}$ in $\mathcal{B}_{a,b}$ have the following Verma flag formulae.*

(1) Suppose $b > c$.

(1.1) If $b > c + 1$, then

$$P_{a,c,b|c} = M_{a,c,b|c} + M_{a,c+1,b|c+1} + M_{a,b,c|c} + M_{a,b,c+1|c+1}.$$

(1.2) Suppose $b = c + 1$.

(1.2.1) If $a > c + 1$, then

$$P_{a,c,c+1|c} = M_{a,c,c+1|c} + M_{a,c+1,c+1|c+1} + M_{a,c+1,c|c}.$$

(1.2.2) If $a = c + 1$, then

$$\begin{aligned} P_{c+1,c,c+1|c} &= M_{c+1,c,c+1|c} + M_{c+1,c+1,c+1|c+1} + \\ &M_{c+2,c+1,c+1|c+2} + M_{c+1,c+1,c|c}. \end{aligned}$$

(2) Suppose $b < c$.

(2.1) Suppose $a > c$.

(2.1.1) If $a > c + 1$, then

$$P_{a,c,b|c} = M_{a,c,b|c} + M_{a,c+1,b|c+1}.$$

(2.1.2) If $a = c + 1$, then

$$P_{c+1,c,b|c} = M_{c+1,c,b|c} + M_{c+1,c+1,b|c+1} + M_{c+2,c+1,b|c+2}.$$

(2.2) If $a = c$, then

$$P_{c,c,b|c} = M_{c,c,b|c} + M_{c,c+1,b|c+1} + M_{c+1,c,b|c+1}.$$

(2.3) If $a < c$, then

$$P_{a,c,b|c} = M_{a,c,b|c} + M_{a,c+1,b|c+1} + M_{c,a,b|c} + M_{c+1,a,b|c+1}.$$

Theorem 5.4. *Let $a, b, c \in \mathbb{Z}$ with $a \geq b$. The projective objects $P_{b,c,a|c}$ in $\mathcal{B}_{a,b}$ have the following Verma flag formulae.*

(1) Suppose $b > c$.

(1.1) If $b > c+1$ and $a > b$, then

$$P_{b,c,a|c} = M_{b,c,a|c} + M_{b,a,c|c} + M_{a,b,c|c} + M_{a,c,b|c} + M_{b,c+1,a|c+1} \\ + M_{b,a,c+1|c+1} + M_{a,b,c+1|c+1} + M_{a,c+1,b|c+1}.$$

(1.2) Suppose $b = c+1$.

(1.2.1) If $a > c+2$, then

$$P_{c+1,c,a|c} = M_{c+1,c,a|c} + M_{c+2,c+1,a|c+2} + M_{c+1,c+1,a|c+1} \\ + M_{a,c,c+1|c} + M_{a,c+1,c+2|c+2} + M_{a,c+1,c+1|c+1} \\ + M_{c+1,a,c|c} + M_{c+1,a,c+2|c+2} + M_{c+1,a,c+1|c+1} \\ + M_{a,c+1,c|c} + M_{a,c+2,c+1|c+2} + M_{a,c+1,c+1|c+1}.$$

(1.2.2) If $a = c+2$, then

$$P_{c+1,c,c+2|c} = M_{c+1,c,c+2|c} + M_{c+2,c,c+1|c} + M_{c+1,c+1,c+2|c+1} \\ + M_{c+2,c+1,c+1|c+1} + M_{c+2,c+1,c+2|c+2} + M_{c+2,c+1,c|c} \\ + M_{c+1,c+2,c|c} + M_{c+1,c+2,c+1|c+1} + M_{c+2,c+1,c+1|c+1} \\ + M_{c+2,c+2,c+1|c+2}.$$

(2) Suppose $b = c$.

(2.1) If $a > c+1$, then

$$P_{c,c,a|c} = M_{c,c,a|c} + M_{c+1,c,a|c+1} + M_{c,c+1,a|c+1} \\ + M_{c,a,c|c} + M_{c+1,a,c|c+1} + M_{c,a,c+1|c+1} \\ + M_{a,c,c|c} + M_{a,c+1,c|c+1} + M_{a,c,c+1|c+1}.$$

(2.2) If $a = c+1$, then

$$P_{c,c,c+1|c} = M_{c,c,c+1|c} + M_{c+1,c,c|c} + M_{c,c+1,c|c} \\ + M_{c,c+1,c+1|c+1} + M_{c+1,c,c+1|c+1} + M_{c+1,c+1,c|c+1}.$$

(3) Suppose $b < c$.

(3.1) If $a > c+1$, then

$$P_{b,c,a|c} = M_{b,c,a|c} + M_{b,c+1,a|c+1} + M_{c,b,a|c} + M_{c+1,b,a|c+1} \\ + M_{a,b,c|c} + M_{a,b,c+1|c+1} + M_{a,c,b|c} + M_{a,c+1,b|c+1} \\ + M_{b,a,c|c} + M_{b,a,c+1|c+1} + M_{c,a,b|c} + M_{c+1,a,b|c+1}.$$

(3.2) Suppose $a = c+1$.

(3.2.1) If $b = c-1$, then

$$P_{c-1,c,c+1|c} = M_{c-1,c,c+1|c} + M_{c-1,c+1,c|c} + M_{c-1,c+1,c+1|c+1} \\ + M_{c,c-1,c+1|c} + M_{c+1,c-1,c|c} + M_{c+1,c-1,c+1|c+1} \\ + M_{c,c+1,c-1|c} + M_{c+1,c,c-1|c} + M_{c+1,c+1,c-1|c+1}.$$

(3.2.2) If $b < c-1$, then

$$P_{b,c,c+1|c} = M_{b,c,c+1|c} + M_{b,c+1,c|c} + M_{c,b,c+1|c} \\ +M_{c,c+1,b|c} + M_{c+1,b,c|c} + M_{c+1,b,c|c}.$$

(3.3) Suppose $a < c$ and $a > b$.

$$P_{b,c,a|c} = M_{b,c,a|c} + M_{b,c+1,a|c+1} + M_{c,b,a|c} + M_{c+1,b,a|c+1} \\ +M_{a,c,b|c} + M_{a,c+1,b|c+1} + M_{c,a,b|c} + M_{c+1,a,b|c+1}.$$

Theorem 5.5. *Let $a, b, c \in \mathbb{Z}$ with $a \geq b$. The projective objects $P_{c,a,b|c}$ in $\mathcal{B}_{a,b}$ have the following Verma flag formulae.*

(1) Suppose $b > c$.
(1.1) Suppose $b > c+1$.
(1.1.1) If $a > b$, then

$$P_{c,a,b|c} = M_{c,a,b|c} + M_{c+1,a,b|c+1} + M_{a,c,b|c} + M_{a,c+1,b|c+1} \\ M_{b,a,c|c} + M_{b,a,c+1|c+1} + M_{a,b,c|c} + M_{a,b,c+1|c+1}.$$

(1.1.2) If $a = b$, then

$$P_{c,b,b|c} = M_{c,b,b|c} + M_{c+1,b,b|c+1} + M_{b,c,b|c} \\ +M_{b,c+1,b|c+1} + M_{b,b,c|c} + M_{b,b,c+1|c+1}.$$

(1.2) Suppose $b = c+1$.
(1.2.1) If $a > c+1$, then

$$P_{c,a,c+1|c} = M_{c,a,c+1|c} + M_{c+1,a,c|c} + M_{c+1,a,c+1|c+1} \\ +M_{a,c,c+1|c} + M_{a,c+1,c|c} + M_{a,c+1,c+1|c+1}.$$

(1.2.2) If $a = c+1$, then

$$P_{c,c+1,c+1|c} = M_{c,c+1,c+1|c} + M_{c+1,c,c+1|c} \\ +M_{c+1,c+1,c|c} + M_{c+1,c+1,c+1|c+1}.$$

(2) Suppose $b < c$.
(2.1) Suppose $a > c$.
(2.1.1) If $a > c+1$, then

$$P_{c,a,b|c} = M_{c,a,b|c} + M_{c+1,a,b|c+1} + M_{a,c,b|c} + M_{a,c+1,b|c+1}.$$

(2.1.2) If $a = c+1$, then

$$P_{c,c+1,b|c} = M_{c,c+1,b|c} + M_{c+1,c+1,b|c+1}.$$

(2.2) If $a < c$, then

$$P_{c,a,b|c} = M_{c,a,b|c} + M_{c+1,a,b|c+1}.$$

Theorem 5.6. *Let $a, b, c \in \mathbb{Z}$ with $a \geq b$. The projective objects $P_{c,b,a|c}$ in $\mathcal{B}_{a,b}$ have the following Verma flag formulae.*

(1) Suppose $b > c$.

(1.1) If $b > c+1$ and $a > b$, then

$$\begin{aligned} P_{c,b,a|c} &= M_{c,b,a|c} + M_{c+1,b,a|c+1} + M_{b,c,a|c} M_{b,c+1,a|c+1} \\ &+ M_{c,a,b|c} + M_{c+1,a,b|c+1} M_{b,a,c|c} + M_{b,a,c+1|c+1} \\ &+ M_{a,c,b|c} M_{a,c+1,b|c+1} + M_{a,b,c|c} + M_{a,b,c+1|c+1}. \end{aligned}$$

(1.2) If $b = c+1$ and $a > c+1$,

$$\begin{aligned} P_{c,c+1,a|c} &= M_{c,c+1,a|c} + M_{c+1,c,a|c} + M_{c+1,c+1,a|c+1} \\ & M_{c,a,c+1|c} + M_{c+1,a,c|c} + M_{c+1,a,c+1|c+1} \\ & M_{a,c,c+1|c} + M_{a,c+1,c|c} + M_{a,c+1,c+1|c+1}. \end{aligned}$$

(2) Suppose $b < c$.

(2.1) Suppose $a > c$.

(2.1.1) If $a > c+1$, then

$$\begin{aligned} P_{c,b,a|c} &= M_{c,b,a|c} + M_{c+1,b,a|c+1} + M_{c,a,b|c} + M_{c+1,a,b|c+1} \\ & M_{a,b,c|c} + M_{a,b,c+1|c+1} + M_{a,c,b|c} + M_{a,c+1,b|c+1}. \end{aligned}$$

(2.1.2) If $a = c+1$, then

$$\begin{aligned} P_{c,b,c+1|c} &= M_{c,b,c+1|c} + M_{c,b,c|c} + M_{c+1,b,c+1|c+1} \\ & M_{c,c+1,b|c} + M_{c,b,c|c} + M_{c+1,c+1,b|c+1}. \end{aligned}$$

(2.2) If $a < c$ and $a > b$, then

$$P_{c,b,a|c} = M_{c,b,a|c} + M_{c+1,b,a|c+1} + M_{c,a,b|c} + M_{c+1,a,b,c+1}.$$

5.2. **Proof.** In this subsection, we prove Theorems 5.1 through 5.6. We use the statements in Section 3 and the method of translation functors. We use various representations for translation functors. Here are the weights of these representations:

Natural V:

$$\{\delta_1,\ \delta_2,\ \delta_3,\ \epsilon\}$$

Dual V^*:

$$\{-\delta_1,\ -\delta_2,\ -\delta_3,\ -\epsilon\}$$

Wedge-squared of the natural $\bigwedge^2 V$:

$$\begin{gathered} \{\delta_1 + \delta_2,\ \delta_2 + \delta_3,\ \delta_1 + \delta_3, \\ \delta_1 + \epsilon,\ \delta_2 + \epsilon,\ \delta_3 + \epsilon\ , 2\epsilon\} \end{gathered}$$

Wedge-cubed of the natural $\bigwedge^3 V$:

$$\begin{gathered} \{\delta_1 + \delta_2 + \delta_3,\ \delta_1 + \delta_2 + \epsilon,\ \delta_2 + \delta_3 + \epsilon, \\ \delta_1 + \delta_3 + \epsilon,\ \delta_1 + 2\epsilon,\ \delta_2 + 2\epsilon\ , \delta_3 + 2\epsilon,\ 3\epsilon\} \end{gathered}$$

Wedge-squared of the dual $\bigwedge^2 V^*$:

$$\begin{gathered} \{-\delta_1 - \delta_2,\ -\delta_2 - \delta_3,\ -\delta_1 - \delta_3, \\ -\delta_1 - \epsilon,\ -\delta_2 - \epsilon,\ -\delta_3 - \epsilon\ , -2\epsilon\} \end{gathered}$$

We now offer justification for the formulae above, separated into cases that have different formulae, based on the strategy in Section 3.7. Our proof will be more explicit in the earlier cases and cases which require more sophisticated techniques; those which lack much explanation follow the strategy almost directly.

5.2.1. *Case:* $P_\lambda,\ \lambda \cong (a, b, c \mid c)$.

(1) $b > c$.

(1.1) $b > c + 1$.

Let $\mu := \lambda - \epsilon$. By Proposition 3.7 for the Verma flag of P_μ and Proposition 3.6 for the Verma flag of the projection, we have the following:

$$P_\mu = M_{a,b,c|c+1}, \mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{a,b,c|c} + M_{a,b,c+1|c+1}.$$

By Proposition 3.5 (4), we have that P_λ is a direct summand. Apply Corollary 3.10 to deduce that it is the only summand.

(1.2) $b = c + 1$.

(1.2.1) $a > c + 2$.

Use P_μ as indicated below to produce the following standard filtration for the projection.

$$\begin{aligned} P_\mu &= M_{a,c+1,c|c+2}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) &= M_{a,c+1,c|c} + M_{a,c+1,c+1|c+1} \\ &\quad + M_{a,c+2,c+1|c+2}. \end{aligned}$$

Now, P_λ is a direct summand of the projection by Proposition 3.5 (4). We now apply Proposition 3.9. By condition 3, the module $M_{a,c+1,c+1|c+1}$ appears in the Verma flag. By condition 4, $M_{a,c+2,c+1|c+2}$ appears. We dedeuce that P_λ is the only direct summand.

(1.2.2) $a = c + 2$.

Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c+2,c+1,c|c+3}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^3 V) &= M_{c+2,c+1,c|c} + M_{c+2,c+2,c+1|c+2} \\ &\quad + M_{c+2,c+1,c+1|c+1} + M_{c+3,c+2,c+1|c+3}. \end{aligned}$$

(1.2.3) $a = c + 1$.

Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c+1,c+1,c|c+2}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) &= M_{c+1,c+1,c|c} + M_{c+1,c+1,c+1|c+1} \\ &\quad + M_{c+1,c+2,c+1|c+2} + M_{c+2,c+1,c+1|c+2}. \end{aligned}$$

(2) $b = c$.

(2.1) $a > c + 1$.

Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see

P_λ is the only direct summand in the projection.

$$P_\mu = M_{a,c,c|c+1},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{a,c,c|c} + M_{a,c,c+1|c+1} + M_{a,c+1,c|c+1}.$$

(2.2) $a = c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$P_\mu = M_{c+1,c,c|c+2},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) = M_{c+1,c,c|c} + M_{c+2,c+1,c|c+2}$$
$$+M_{c+1,c,c+1|c+1} + M_{c+1,c+1,c|c+1} + M_{c+2,c,c+1|c+2}.$$

(2.3) $a = c$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$P_\mu = M_{c,c,c|c+1},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{c,c,c|c} + M_{c,c,c+1|c+1}$$
$$+M_{c,c+1,c|c+1} + M_{c+1,c,c|c+1}.$$

(3) $b < c$

(3.1) $a > c$.

(3.1.1) $a > c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$P_\mu = M_{a,b,c|c+1} + M_{a,c,b|c+1},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{a,b,c|c} + M_{a,b,c+1|c+1}$$
$$+M_{a,c,b|c} + M_{a,c+1,b|c+1}.$$

(3.1.2) $a = c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$P_\mu = M_{c+1,b,c|c+2} + M_{c+1,c,b|c+2},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) = M_{c+1,b,c|c} + M_{c+1,b,c+1|c+1}$$
$$+M_{c+2,b,c+1|c+2} + M_{c+1,c,b|c}$$
$$+M_{c+1,c+1,b|c+1} + M_{c+2,c+1,b|c+2}.$$

(3.2) $a = c$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$P_\mu = M_{c,b,c|c+1} + M_{c,c,b|c+1},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{c,b,c|c} + M_{c,b,c+1|c+1} + M_{c+1,b,c|c+1}$$
$$+M_{c,c,b|c} + M_{c,c+1,b|c+1} + M_{c+1,c,b|c+1}.$$

(3.3) $a < c$.

(3.3.1) $a > b$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu = M_{a,b,c|c+1} + M_{a,c,b|c+1} + M_{c,a,b|c+1} \\ + M_{c,b,a|c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{a,b,c|c} + M_{a,b,c+1|c+1} + M_{a,c,b|c} \\ + M_{c,a,b|c} + M_{a,c+1,b|c+1} + M_{c+1,a,b|c+1} \\ + M_{c,b,a|c} + M_{c+1,b,a|c+1}. \end{aligned}$$

(3.3.2) $a = b$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu = M_{b,b,c|c+1} + M_{b,c,b|c+1} + M_{c,b,b|c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{b,b,c|c} + M_{b,b,c+1|c+1} + M_{b,c,b|c} \\ + M_{b,c+1,b|c+1} + M_{c,b,b|c} + M_{c+1,b,b|c+1}. \end{aligned}$$

5.2.2. *Case:* P_λ, $\lambda \cong (b, a, c \mid c)$.

(1) $b > c$.
(1.1) $b > c + 1$.
(1.1.1) $a > b$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu = M_{b,a,c|c+1} + M_{a,b,c|c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{b,a,c|c} + M_{b,a,c+1|c+1} + M_{a,b,c|c} \\ + M_{a,b,c+1|c+1}. \end{aligned}$$

(1.1.2) $a = b$.
This is a repeat of an earlier case.
(1.2) $b = c + 1$.
(1.2.1) $a > c + 2$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu = M_{c+1,a,c|c+2} + M_{c+1,a,c|c+2}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) = M_{c+1,a,c|c} + M_{c+2,a,c+1|c+2} \\ + M_{c+1,a,c+1|c+1} + M_{a,c+1,c|c} \\ + M_{a,c+2,c+1|c+2} + M_{a,c+1,c+1|c+1}. \end{aligned}$$

(1.2.2) $a = c + 2$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to

deduce the following:

$$P_\mu = M_{c+1,c+2,c|c+3} + M_{c+2,c+1,c|c+3},$$

$$P_1 := \mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^3 V) = M_{c+1,c+2,c|c} + 2M_{c+2,c+2,c+1|c+2} + M_{c+1,c+2,c+1|c+1} + M_{c+2,c+3,c+1|c+3} + M_{c+2,c+1,c|c} + M_{c+2,c+1,c+1|c+1} + M_{c+3,c+2,c+1|c+3}.$$

If we proceed with our strategy, we realize that $M_{c+1,c+2,c|c}$, $M_{c+2,c+1,c|c}$, $M_{c+1,c+2,c+1|c+1}$, $M_{c+2,c+1,c+1|c+1}$, and one copy of $M_{c+2,c+2,c+1|c+2}$ are the only modules we deduce that can appear in the standard filtration of P_λ. This leaves three terms remaining, of which the one with lowest weight is the second copy of $M_{c+2,c+2,c+1|c+2}$. This means that if there were another summand, it would be $P_{c+2,c+2,c+1|c+2}$. Observe that this is of the form $P_{\hat{c},\hat{c},\hat{b}|\hat{c}}$, where $\hat{b}, \hat{c} \in \mathbb{Z}$ and $\hat{b} < \hat{c}$ (Theorem 5.3, Item 2.2). This has a Verma flag of length 3, which would correspond to the remaining Verma modules. We realize that there may be another summand in the projection. Therefore, we try another approach.

We now let $\theta := \lambda - (\delta_2 + \epsilon)$. Observe that $\theta \cong (c+1, c+1, c \mid c+1)$ is atypical, so by looking at the same case (Theorem 5.3, Item 2.2) 2), we have the following:

$$P_\theta = M_{c+1,c+1,c|c+1} + M_{c+1,c+2,c|c+2} + M_{c+2,c+1,c|c+2},$$

$$P_2 := \mathrm{Pr}_\lambda(P_\theta \otimes \bigwedge^2 V) = 2M_{c+1,c+2,c|c} + 2M_{c+2,c+1,c|c} + 2M_{c+1,c+2,c+1|c+1} + 2M_{c+2,c+1,c+1|c+1} + 2M_{c+2,c+2,c+1|c+2}.$$

We have that P_λ also appears as a direct summand for both P_1 and P_2. However, in a Verma flag for P_2, we have the same five terms each appear twice. We deduce that P_λ cannot have more than 5 terms and that both P_1 and P_2 split up as the direct sum of two projectives. We conclude that these same five terms form the Verma flag for P_λ.

(1.2.3) $a = c+1$.
This is a repeat of an earlier case.

(2) $b = c$.

(2.1) $a > c$.

(2.1.1) $a > c+1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$P_\mu = M_{c,a,c|c+1} + M_{a,c,c|c+1},$$

$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{c,a,c|c} + M_{c,a,c+1|c+1} + M_{c+1,a,c|c+1} + M_{a,c,c|c} + M_{a,c,c+1|c+1} + M_{a,c+1,c|c+1}.$$

(2.1.2) $a = c+1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to

see P_λ is the only direct summand in the projection.

$$\begin{aligned}P_\mu &= M_{c,c+1,c|c+2} + M_{c+1,c,c|c+2},\\ \mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^3 V^*) &= M_{c,c+1,c|c} + M_{c,c+2,c+1|c+2} + M_{c+1,c+2,c|c+2}\\ &+M_{c,c+1,c+1|c+1} + 2M_{c+1,c+1,c|c+1} + M_{c+2,c,c+1|c+2}\\ &+M_{c+2,c+1,c|c+2} + M_{c+1,c,c+1|c+1} + M_{c+1,c,c|c}.\end{aligned}$$

(2.2) $a = c$.
This is a repeat of an earlier case.

(3) $b < c$.

(3.1) $a > c$.

(3.1.1) $a > c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned}P_\mu &= M_{b,a,c|c+1} + M_{a,b,c|c+1} + M_{a,c,b|c+1}\\ &+M_{c,a,b|c+1},\\ \mathrm{Pr}_\lambda(P_\mu \otimes V) &= M_{b,a,c|c} + M_{a,b,c|c} + M_{a,c,b|c} + M_{c,a,b|c}\\ &+M_{b,a,c+1|c+1} + M_{a,b,c+1|c+1} + M_{a,c+1,b|c+1}\\ &+M_{c+1,a,b|c+1}.\end{aligned}$$

(3.1.2) $a = c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned}P_\mu &= M_{b,c+1,c|c+2} + M_{c+1,b,c|c+2} + M_{c+1,c,b|c+2}\\ &+M_{c+1,a,b|c+2},\\ \mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) &= M_{b,c+1,c|c} + M_{b,c+2,c+1|c+2} + M_{b,c+1,c+1|c+1}\\ &+M_{c+1,b,c|c} + M_{c+2,b,c+1|c+2} + M_{c+1,b,c+1|c+1}\\ &+M_{c,c+1,b|c} + M_{c+1,c+2,b|c+2} + M_{c+1,c+1,b|c+1}\\ &+M_{c+1,c,b|c} + M_{c+2,c+1,b|c+2} + M_{c+1,c+1,b|c+1}.\end{aligned}$$

(3.2) $a = c$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned}P_\mu &= M_{b,c,c|c+1} + Mc,b,c|c+1 + M_{c,c,b|c+1},\\ \mathrm{Pr}_\lambda(P_\mu \otimes V) &= M_{b,c,c|c} + M_{b,c,c+1|c+1} + M_{b,c+1,c|c+1}\\ &+M_{c,b,c|c} + M_{c,b,c+1|c+1} + M_{c+1,b,c|c+1}\\ &+M_{c,c,b|c} + M_{c,c+1,b|c+1} + M_{c+1,c,b|c+1}.\end{aligned}$$

(3.3) $a < c$.

(3.3.1) $a > b$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to

see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{b,a,c|c+1} + M_{b,c,a|c+1} + M_{a,b,c|c+1} + M_{c,b,a|c+1} \\ &\quad + M_{a,c,b|c+1} + M_{c,a,b|c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) &= M_{b,a,c|c} + M_{b,a,c+1|c+1} + M_{b,c,a|c} \\ &\quad + M_{b,c+1,a|c+1} + M_{a,b,c|c} + M_{a,b,c+1|c+1} \\ &\quad + M_{c,b,a|c} + M_{c+1,b,a|c+1} + M_{a,c,b|c} \\ &\quad + M_{a,c+1,b|c+1} + M_{c,a,b|c} + M_{c+1,a,b|c+1}. \end{aligned}$$

(3.3.2) $a = b$.
This is a repeat of an earlier case.

5.2.3. *Case:* $P_\lambda,\ \lambda \cong (a, c, b \mid c)$.

(1) $b > c$.

(1.1) $b > c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{a,c,b|c+1} + M_{a,b,c|c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) &= M_{a,c,b|c} + M_{a,c+1,b|c+1} + M_{a,b,c|c} + M_{a,b,c+1|c+1}. \end{aligned}$$

(1.2) $b = c + 1$

(1.2.1) $a > c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{a+1,c+1,c+1|c}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes \textstyle\bigwedge^2 V^*) &= M_{a,c,c+1|c} + M_{a,c+1,c+1|c+1} + M_{a,c+1,c|c}. \end{aligned}$$

(1.2.2) $a = c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c+2,c+1,c+1|c}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes \textstyle\bigwedge^2 V^*) &= M_{c+1,c,c+1|c} \\ + M_{c+1,c+1,c+1|c+1} &+ M_{c+2,c+1,c+1|c+2} + M_{c+1,c+1,c|c}. \end{aligned}$$

(2) $b = c$. This is a repeat of an earlier case.

(3) $b < c$.

(3.1) $a > c$.

(3.1.1) $a > c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{a,c,b|c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) &= M_{a,c,b|c} + M_{a,c+1,b|c+1}. \end{aligned}$$

(3.1.2) $a = c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$P_\mu = M_{c+1,c,b|c+2},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) = M_{c+1,c,b|c} + M_{c+1,c+1,b|c+1} + M_{c+2,c+1,b|c+2}.$$

(3.2) $a = c$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$P_\mu = M_{c,c,b|c+1},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{c,c,b|c} + M_{c,c+1,b|c+1} + M_{c+1,c,b|c+1}.$$

(3.3) $a < c$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$P_\mu = M_{a,c,b|c+1} + M_{c,a,b|c+1},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{a,c,b|c} + M_{a,c+1,b|c+1} + M_{c,a,b|c} + M_{c+1,a,b|c+1}.$$

5.2.4. *Case:* $P_\lambda,\ \lambda \cong (b, c, a \mid c)$.

(1) $b > c$.

(1.1) $b > c + 1$.

(1.1.1) $a > b$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$P_\mu = M_{b,c,a|c+1} + M_{b,a,c|c+1} + M_{a,b,c|c+1} + M_{a,c,b|c+1},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{b,c,a|c} + M_{b,a,c|c} + M_{a,b,c|c} + M_{a,c,b|c}$$
$$+M_{b,c+1,a|c+1} + M_{b,a,c+1|c+1} + M_{a,b,c+1|c+1} + M_{a,c+1,b|c+1}.$$

(1.1.2) $a = b$.
This is a repeat of an earlier case.

(1.2) $b = c + 1$.

(1.2.1) $a > c + 2$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$P_\mu = M_{c+1,c,a|c+2} + M_{a,c,c+1|c+2} + M_{c+1,a,c|c+2} + M_{a,c+1,c|c+2},$$
$$\mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) = M_{c+1,c,a|c} + M_{c+2,c+1,a|c+2} + M_{c+1,c+1,a|c+1}$$
$$+M_{a,c,c+1|c} + M_{a,c+1,c+2|c+2} + M_{a,c+1,c+1|c+1}$$
$$+M_{c+1,a,c|c} + M_{c+1,a,c+2|c+2} + M_{c+1,a,c+1|c+1}$$
$$+M_{a,c+1,c|c} + M_{a,c+2,c+1|c+2} + M_{a,c+1,c+1|c+1}.$$

(1.2.2) $a = c + 2$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to

see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c+2,c+1,c+2|c} + M_{c+2,c+2,c+1|c}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V^*) &= M_{c+1,c,c+2|c} + M_{c+2,c,c+1|c} + M_{c+1,c+1,c+2|c+1} \\ &\quad + M_{c+2,c+1,c+1|c+1} + M_{c+2,c+1,c+2|c+2} + M_{c+2,c+1,c|c} \\ &\quad + M_{c+1,c+2,c|c} + M_{c+1,c+2,c+1|c+1} + M_{c+2,c+1,c+1|c+1} \\ &\quad + M_{c+2,c+2,c+1|c+2}. \end{aligned}$$

(1.2.3) $a = c+1$.
This is a repeat of an earlier case.

(2) $b = c$.

(2.1) $a > c+1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c,c,a|c+1} + M_{c,a,c|c+1} + M_{a,c,c|c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) &= M_{c,c,a|c} + M_{c+1,c,a|c+1} + M_{c,c+1,a|c+1} \\ &\quad + M_{c,a,c|c} + M_{c+1,a,c|c+1} + M_{c,a,c+1|c+1} \\ &\quad + M_{a,c,c|c} + M_{a,c+1,c|c+1} + M_{a,c,c+1|c+1}. \end{aligned}$$

(2.2) $a = c+1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} \mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V^*) &= P_\mu = M_{c+1,c+1,c+1|c}, \\ &M_{c,c,c+1|c} + M_{c+1,c,c|c} + M_{c,c+1,c|c} \\ + M_{c,c+1,c+1|c+1} &+ M_{c+1,c,c+1|c+1} + M_{c+1,c+1,c|c+1}. \end{aligned}$$

(2.3) $a = c$.
This is a repeat of an earlier case.

(3) $b < c$.

(3.1) $a > c+1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{b,c,a|c+1} + M_{c,b,a|c+1} + M_{a,b,c|c+1} \\ &\quad + M_{a,c,b|c+1} + M_{b,a,c|c+1} + M_{c,a,b|c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) &= M_{b,c,a|c} + M_{b,c+1,a|c+1} + M_{c,b,a|c} + M_{c+1,b,a|c+1} \\ &\quad + M_{a,b,c|c} + M_{a,b,c+1|c+1} + M_{a,c,b|c} + M_{a,c+1,b|c+1} \\ &\quad + M_{b,a,c|c} + M_{b,a,c+1|c+1} + M_{c,a,b|c} + M_{c+1,a,b|c+1}. \end{aligned}$$

(3.2) $a = c+1$.

(3.2.1) $b = c-1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to

see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c-1,c+1,c+1|c} + M_{c+1,c-1,c+1|c} + M_{c+1,c+1,c-1|c}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V^*) &= M_{c-1,c,c+1|c} + M_{c-1,c+1,c|c} + M_{c-1,c+1,c+1|c+1} \\ &+M_{c,c-1,c+1|c} + M_{c+1,c-1,c|c} + M_{c+1,c-1,c+1|c+1} \\ &+M_{c,c+1,c-1|c} + M_{c+1,c,c-1|c} + M_{c+1,c+1,c-1|c+1}. \end{aligned}$$

(3.2.2) $b < c-1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{b-1,c,c+1|c} + M_{b-1,c+1,c|c} + M_{c,b-1,c+1|c} \\ &+M_{c+1,b-1,c|c} + M_{c,c+1,b-1|c} + M_{c+1,c,b-1|c}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) &= M_{b,c,c+1|c} + M_{b,c+1,c|c} + M_{c,b,c+1|c} \\ &+M_{c,c+1,b|c} + M_{c+1,b,c|c} + M_{c+1,b,c|c}. \end{aligned}$$

(3.3) $a = c$.
This is a repeat of an earlier case.

(3.4) $a < c$.

(3.4.1) $a > b$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{b,c,a|c+1} + M_{c,b,a|c+1} + M_{a,c,b|c+1} + M_{c,a,b|c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) &= M_{b,c,a|c} + M_{b,c+1,a|c+1} + M_{c,b,a|c} + M_{c+1,b,a|c+1} \\ &+M_{a,c,b|c} + M_{a,c+1,b|c+1} + M_{c,a,b|c} + M_{c+1,a,b|c+1}. \end{aligned}$$

(3.4.2) $a = b$.
This is a repeat of an earlier case.

5.2.5. *Case:* P_λ, $\lambda \cong (c, a, b \mid c)$.

(1) $b > c$.

(1.1) $b > c+1$.

(1.1.1) $a > b$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c+1,a,b|c} + M_{a,c+1,b|c} + M_{b,a,c+1|c} + M_{a,b,c+1|c}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V^*) &= M_{c,a,b|c} + M_{c+1,a,b|c+1} + M_{a,c,b|c} + M_{a,c+1,b|c+1} \\ &M_{b,a,c|c} + M_{b,a,c+1|c+1} + M_{a,b,c|c} + M_{a,b,c+1|c+1}. \end{aligned}$$

(1.1.2) $a = b$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c+1,b,b|c} + M_{b,c+1,b|c} + M_{b,b,c+1|c}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V^*) &= M_{c,b,b|c} + M_{c+1,b,b|c+1} + M_{b,c,b|c} \\ &+M_{b,c+1,b|c+1} + M_{b,b,c|c} + M_{b,b,c+1|c+1}. \end{aligned}$$

(1.2) $b = c + 1$.

(1.2.1) $a > c + 1$.

Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c+1,a,c+1|c} + M_{a,c+1,c+1|c}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V^*) &= M_{c,a,c+1|c} + M_{c+1,a,c|c} + M_{c+1,a,c+1|c+1} \\ &\quad + M_{a,c,c+1|c} + M_{a,c+1,c|c} + M_{a,c+1,c+1|c+1}. \end{aligned}$$

(1.2.2) $a = c + 1$.

Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c+1,c+1,c+1|c}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V^*) &= M_{c,c+1,c+1|c} + M_{c+1,c,c+1|c} \\ &\quad + M_{c+1,c+1,c|c} + M_{c+1,c+1,c+1|c+1}. \end{aligned}$$

(2) $b = c$. This is a repeat of an earlier case.

(3) $b < c$.

(3.1) $a > c$.

(3.1.1) $a > c + 1$

Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c+1,a,b|c} + M_{a,c+1,b|c}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V^*) &= M_{c,a,b|c} + M_{c+1,a,b|c+1} + M_{a,c,b|c} + M_{a,c+1,b|c+1}. \end{aligned}$$

(3.1.2) $a = c + 1$

Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c+1,c+1,b|c}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V^*) &= M_{c,c+1,b|c} + M_{c+1,c+1,b|c+1}. \end{aligned}$$

(3.2) $a = c$.

This is a repeat of an earlier case.

(3.3) $a < c$.

Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c,a,b|c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) &= M_{c,a,b|c} + M_{c+1,a,b|c+1}. \end{aligned}$$

5.2.6. *Case:* $P_\lambda,\ \lambda \cong (c, b, a \mid c)$.

(1) $b > c$.

(1.1) $b > c + 1$.

(1.1.1) $a > b$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c,b,a|c+1} + M_{b,c,a|c+1} + M_{c,a,b|c+1} + M_{b,a,c|c+1} \\ &\quad + M_{a,c,b|c+1} + M_{a,b,c|c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) &= M_{c,b,a|c} + M_{c+1,b,a|c+1} + M_{b,c,a|c} \\ &\quad M_{b,c+1,a|c+1} + M_{c,a,b|c} + M_{c+1,a,b|c+1} \\ &\quad M_{b,a,c|c} + M_{b,a,c+1|c+1} + M_{a,c,b|c} \\ &\quad M_{a,c+1,b|c+1} + M_{a,b,c|c} + M_{a,b,c+1|c+1}. \end{aligned}$$

(1.1.2) $a = b$.
This is a repeat of an earlier case.

(1.2) $b = c + 1$.

(1.2.1) $a > c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c+1,c+1,a|c} + M_{c+1,a,c+1|c} + M_{a,c+1,c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V^*) &= M_{c,c+1,a|c} + M_{c+1,c,a|c} + M_{c+1,c+1,a|c+1} \\ &\quad M_{c,a,c+1|c} + M_{c+1,a,c|c} + M_{c+1,a,c+1|c+1} \\ &\quad M_{a,c,c+1|c} + M_{a,c+1,c|c} + M_{a,c+1,c+1|c+1}. \end{aligned}$$

(1.2.2) $a = c + 1$.
This is the repeat of an earlier case.

(2) $b = c$.
This is a repeat of an earlier case.

(3) $b < c$.

(3.1) $a > c$.

(3.1.1) $a > c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c,b,a|c+1} + M_{c,a,b|c+1} + M_{a,b,c|c+1} + M_{a,b,c|c+1} + M_{a,c,b|c+1}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V) &= M_{c,b,a|c} + M_{c+1,b,a|c+1} + M_{c,a,b|c} + M_{c+1,a,b|c+1} \\ &\quad M_{a,b,c|c} + M_{a,b,c+1|c+1} + M_{a,c,b|c} + M_{a,c+1,b|c+1}. \end{aligned}$$

(3.1.2) $a = c + 1$.
Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$\begin{aligned} P_\mu &= M_{c+1,b,c+1|c} + M_{c+1,c+1,b|c}, \\ \mathrm{Pr}_\lambda(P_\mu \otimes V^*) &= M_{c,b,c+1|c} + M_{c,b,c|c} + M_{c+1,b,c+1|c+1} \\ &\quad M_{c,c+1,b|c} + M_{c,b,c|c} + M_{c+1,c+1,b|c+1}. \end{aligned}$$

(3.2) $a = c$.
This is a repeat of an earlier case.

(3.3) $a < c$

(3.3.1) $a > b$.

Use P_μ as indicated below and proceed with the strategy in Section 3.7 to see P_λ is the only direct summand in the projection.

$$P_\mu = M_{c,b,a|c+1} + M_{c,a,b|c+1},$$

$$\mathrm{Pr}_\lambda(P_\mu \otimes V) = M_{c,b,a|c} + M_{c+1,b,a|c+1} + M_{c,a,b|c} + M_{c+1,a,b,c+1}.$$

(3.3.2) $a = b$.

This is a repeat of an earlier case.

6. Character Formulae for $\mathfrak{gl}(2|2)$

In this section, we determine standard filtration formulae for projective covers of simple modules of $\mathfrak{gl}(2|2)$ with integral, atypical weight of degree 2.

6.1. **Results.** Let $\mathfrak{g} = \mathfrak{gl}(2|2)$ have the standard choices of Cartan subalgebra, bilinear form, root system, positive, and fundamental system as described in Section 2. Recall the notation described in Section 2.7 to describe a weight in $\mathfrak{h}^*$. Lastly, recall Example 2.8.3 and the corresponding block $\mathcal{B}_0$ (see Section 3.3). We have the following Theorems 6.1 to **??** that describe standard filtrations of projectives in this block.

Theorem 6.1. *Let $a, b \in \mathbb{Z}$ with $a \geq b$. We have the following Verma flag formulae for the projective objects $P_{a,b|b,a}$ and $P_{a,b|a,b}$ in $\mathcal{B}_0$:*

(1) Case: $P_{a,b|b,a}$

(1.1) If $b < a-1$, then

$$P_{a,b|b,a} = M_{a,b|b,a} + M_{a+1,b|b,a+1} + M_{a,b+1|b+1,a} + M_{a+1,b+1|b+1,a+1}.$$

(1.2) If $b = a-1$, then

$$\begin{aligned} P_{a,a-1|a-1,a} = M_{a,a-1|a-1,a} + M_{a+1,a-1|a-1,a+1} + M_{a,a|a,a} \\ + M_{a,a+1|a,a+1} + 2M_{a+1,a|a,a+1} + M_{a+1,a|a+1,a} \\ + M_{a+1,a+1|a+1,a+1} + M_{a+2,a|a,a+2} + M_{a+2,a+1|a+1,a+2}. \end{aligned}$$

(1.3) If $b = a$, then

$$\begin{aligned} P_{a,a|a,a} = M_{a,a|a,a} + M_{a+1,a|a,a+1} + M_{a,a+1|a+1,a} \\ + M_{a+1,a+1|a+1,a+1} + M_{a+1,a|a+1,a} + M_{a,a+1|a,a+1}. \end{aligned}$$

(2) Case: $P_{a,b|a,b}$

(2.1) If $b < a-1$, then

$$\begin{aligned} P_{a,b|a,b} = M_{a,b|a,b} + M_{a+1,b|a+1,b} + M_{a,b+1|a,b+1} + M_{a+1,b+1|a+1,b+1} \\ + M_{a,b|b,a} + M_{a+1,b|b,a+1} + M_{a,b+1|b+1,a} + M_{a+1,b+1|b+1,a+1}. \end{aligned}$$

(2.2) If $b = a-1$, then

$$\begin{aligned} P_{a,a-1|a,a-1} = M_{a,a-1|a,a-1} + M_{a+1,a|a,a+1} + M_{a+1,a|a+1,a} + M_{a+1,a-1|a-1,a+1} \\ + M_{a+1,a-1|a+1,a-1} + M_{a,a|a,a} + M_{a,a-1|a-1,a}. \end{aligned}$$

Theorem 6.2. *Let $a, b \in \mathbb{Z}$ with $a \geq b$. We have the following Verma flag formulae for the projective objects $P_{b,a|b,a}$ and $P_{b,a|a,b}$ in $\mathcal{B}_0$.*

(1) Case: $P_{b,a|b,a}$

(1.1) If $b < a-1$, then

$$P_{b,a|b,a} = M_{b,a|b,a} + M_{b,a+1|b,a+1} + M_{b+1,a|b+1,a} + M_{b+1,a+1|b+1,a+1} \\ +M_{a,b|b,a} + M_{a+1,b|b,a+1} + M_{a,b+1|b+1,a} + M_{a+1,b+1|b+1,a+1}$$

(1.2) If $b = a-1$, then

$$P_{a-1,a|a-1,a} = M_{a-1,a|a-1,a} + M_{a,a-1|a-1,a} + M_{a,a|a,a} \\ +M_{a-1,a+1|a-1,a+1} + M_{a,a+1|a,a+1} + M_{a+1,a-1|a-1,a+1} \\ +M_{a+1,a|a,a+1}.$$

(2) Case: $P_{b,a|a,b}$

(2.1) If $b < a-1$, then

$$P_{b,a|a,b} = M_{b,a|a,b} + M_{b,a+1|a+1,b} + M_{b+1,a|a,b+1} + M_{b+1,a+1|a+1,b+1} \\ +M_{b,a|b,a} + M_{b,a+1|b,a+1} + M_{b+1,a|b+1,a} + M_{b+1,a+1|b+1,a+1} \\ +M_{a,b|a,b} + M_{a+1,b|a+1,b} + M_{a,b+1|a,b+1} + M_{a+1,b+1|a+1,b+1} \\ +M_{a,b|b,a} + M_{a+1,b|b,a+1} + M_{a,b+1|b+1,a} + M_{a+1,b+1|b+1,a+1}.$$

(2.2) If $b = a-1$, then

$$P_{a-1,a|a,a-1} = M_{a-1,a|a,a-1} + M_{a,a+1|a,a+1} + M_{a,a+1|a+1,a} + M_{a-1,a+1|a+1,a-1} \\ +M_{a-1,a+1|a-1,a+1} + M_{a-1,a|a-1,a} + M_{a,a-1|a,a-1} + M_{a+1,a|a,a+1} \\ +M_{a+1,a|a+1,a} + M_{a+1,a-1|a-1,a+1} + M_{a+1,a-1|a+1,a-1} + 2M_{a,a|a,a} \\ +M_{a,a-1|a-1,a}.$$

6.2. **Proof.** In this subsection, we prove Theorem 6.1 and Theorem 6.2. We use the statements in Section 3 and the method of translation functors. We use various representations for translation functors. Here are the weights of these representations.

Dual V^*:

$$\{-\delta_1,\ -\delta_2,\ -\epsilon_1,\ -\epsilon_2\}$$

Wedge-squared of the natural $\bigwedge^2 V$:

$$\{\delta_1+\delta_2,\ \delta_1+\epsilon_1,\ \delta_1+\epsilon_2,\ \delta_2+\epsilon_1, \\ \delta_2+\epsilon_2,\ 2\epsilon_1,\ \epsilon_1+\epsilon_2,\ 2\epsilon_2\}$$

Wedge-squared of the dual $\bigwedge^2 V^*$:

$$\{-\delta_1-\delta_2,\ -\delta_1-\epsilon_1,\ -\delta_1-\epsilon_2,\ -\delta_2-\epsilon_1, \\ -\delta_2-\epsilon_2,\ -2\epsilon_1,\ -\epsilon_1-\epsilon_2,\ -2\epsilon_2\}$$

Wedge-cubed of the natural $\bigwedge^3 V$:

$$\{\delta_1+\delta_2+\epsilon_1,\ \delta_1+\delta_2+\epsilon_2,\ \delta_1+2\epsilon_1, \\ \delta_1+\epsilon_1+\epsilon_2,\ \delta_1+2\epsilon_2,\ \delta_2+2\epsilon_1,\ \delta_2+\epsilon_1+\epsilon_2, \\ \delta_2+2\epsilon_2,\ 3\epsilon_1,\ 2\epsilon_1+\epsilon_2,\ \epsilon_1+2\epsilon_2,\ 3\epsilon_2\}$$

We now offer justification for the formulae above, separated into cases that have different formulae, based on the strategy in Section 3.7. Our proof be more explicit in the earlier cases and edge cases; those which lack much explanation follow the strategy almost directly.

6.2.1. *Case:* P_λ, $\lambda \cong (a, b \mid b, a)$.

(1) $b < a - 1$.
Let $\mu := \lambda - (\epsilon_1 + \epsilon_2)$, so that $\mu \cong (a, b \mid b+1, a+1)$. Note that $\epsilon_1 + \epsilon_2$ is not the lowest weight of $\bigwedge^2 V$ as prescribed by the strategy; this shall be justified below. First we claim:

$$P_\mu = M_{a,b|b+1,a+1},$$

$$P := \mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) = M_{a,b|b,a} + M_{a+1,b|b,a+1} + M_{a,b+1|b+1,a} + M_{a+1,b+1|b+1,a+1}.$$

Observe that μ is dominant and typical, so by Lemma 3.7, $P_\mu = M_{a,b|b+1,a+1}$. Ttensoring P_μ with the wedge-squared of the natural representation produces another projective object T in $\mathcal{O}$. By choice of μ and representation, $M_{a,b|b,a}$ appears in the standard filtration of T by Proposition 3.6.

Because each indecomposable object lives entirely in a single block in $\mathcal{O}$, we can project T onto $\mathcal{B}_\lambda$ in $\mathcal{O}$ to produce another projective P. Now, because λ is the lowest weight appearing in the Verma flag of P, by part 4 of Proposition 3.5, P_λ is a direct summand of P.

We now deduce which other Verma modules must appear in the filtration by the conditions in Proposition 3.9. The module $M_{a+1,b|b,a+1}$ and $M_{a,b+1|b+1,a}$ appear by condition 3. If the remaining module $M_{a+1,b+1|b+1,a+1}$ were to split off as a separate projective summand, Corollary 3.10 shows us such a projective would have a filtration length of at least two. Since there are no other modules remaining, we deduce that this last module must also be present in the filtration and that there is only one summand present in the projection.

(2) $b = a - 1$.
There appears no obvious choice of typical μ such that tensoring one of the representations above is effective. As a result, we allow μ to be atypical. This requires us to use a suitable translation functor to first determine P_μ.

Let $\mu := \lambda - 2\epsilon_2$, so that $\mu \cong (a, a-1 \mid a-1, a+2)$ and $\mu - \rho$ is atypical of degree 1. Let $\theta := \mu - 2\epsilon_1$ so that $\theta \cong (a, a-1 \mid a+1, a+2)$ and $\theta - \rho$ is typical. We have

$$P_\theta = M_{a,a-1|a+1,a+2},$$

$$\mathrm{Pr}_\mu(P_\theta \otimes \bigwedge^2 V) = M_{a,a-1|a-1,a+2} + M_{a,a|a,a+2} + M_{a+1,a|a+1,a+2}.$$

To determine P_μ, we first note that because M_μ present above has weight lower than that of the remaining Verma modules, P_μ is a direct summand of P. Now, we effect Proposition 3.9 to see which of the remaining Verma modules appear in a Verma flag of $P_{\mu+\rho}$. The module $M_{a,a|a,a+2}$ appears by condition 3, and $M_{a+1,a|a+1,a+2}$ appears by condition 5. We deduce that there is only one summand in the projection.

$$P_\mu = M_{a,a-1|a-1,a+2} + M_{a,a|a,a+2} + M_{a+1,a|a+1,a+2}.$$

Now, we argue a formula for P_λ by proceeding as usual. We have

$$\begin{aligned}\mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) &= M_{a,a-1|a-1,a} + M_{a+1,a-1|a-1,a+1} + M_{a,a|a,a} \\ &+M_{a,a+1|a,a+1} + 2M_{a+1,a|a,a+1} + M_{a+1,a|a+1,a} \\ &+M_{a+1,a+1|a+1,a+1} + M_{a+2,a|a,a+2} + M_{a+2,a+1|a+1,a+2}.\end{aligned}$$

The presence of $M_\lambda = M_{a,a-1|a-1}$ in the projection and because it is of lowest weight tells us that P_λ is a direct summand (Proposition 3.5, part 4). Now, we determine which of the remaining Verma modules must appear in a filtration for P_λ by using Proposition 3.9. $M_{a+1,a-1|a-1,a+1}$ and $M_{a,a|a,a}$ appear by condition 3. $M_{a,a+1|a,a+1}$, one copy of $M_{a+1,a|a,a+1}$, and $M_{a+1,a|a+1,a}$ appear by condition 5. Because we have enumerated at least 6 terms here, P_λ has a filtration length of at least 6.

Of the remaining four terms, observe that the module of lowest weight appearing is the second copy of $M_{a+1,a|a,a+1}$. This is actually of the same class $(a, a-1 \mid a-1, a)$, and so if it yielded another projective as a direct summand, there would be at least $12 = 6 + 6$ terms appearing. Since there are only ten terms, the second copy of $M_{a+1,a|a,a+1}$ also appears.

There are three terms remaining: $M_{a+1,a+1|a+1,a+1}$, $M_{a+2,a|a,a+2}$, and $M_{a+2,a+1|a+1,a+2}$. The weights of the first two are incomparable, and the third is higher than both. Therefore, these three modules cannot form a projective. But three modules cannot also be a direct sum of two or more projectives of atypical weight by a simple application of Corollary 3.10. We deduce that all three of these modules are also present in the filtration and that there is only one summand in the projection.

(3) $b = a$.
Let $\mu \coloneqq \lambda - (\epsilon_1 + \epsilon_2)$. We have

$$P_\mu = M_{a,a|a+1,a+1},$$

$$\begin{aligned}\mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) &= M_{a,a|a,a} + M_{a+1,a|a,a+1} + M_{a,a+1|a+1,a} \\ &+M_{a+1,a+1|a+1,a+1} + M_{a+1,a|a+1,a} + M_{a,a+1|a,a+1}.\end{aligned}$$

The presence of $M_{a,a|a,a}$ as the module with lowest weight implies that P_λ is a direct summand of the projection. To determine which other modules appear in a Verma flag for P_λ, we use Proposition 3.9. The modules $M_{a+1,a|a,a+1}$, $M_{a,a+1|a+1,a}$, $M_{a+1,a|a+1,a}$, and $M_{a,a+1|a,a+1}$ all appear by condition 3 of the proposition. This forces the remaining module $M_{a+1,a+1|a+1,a+1}$ to also appear by Corollary 3.10. We deduce that there is only one direct summand.

6.2.2. *Case* P_λ, $\lambda(a, b \mid a, b)$.

(1) $b < a - 1$.
Let $\mu \coloneqq \lambda - (\epsilon_1 + \epsilon_2)$. We have

$$P_\mu = M_{a,b|b+1,a+1} + M_{a,b|a+1,b+1},$$

$$\begin{aligned}\mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) &= M_{a,b|a,b} + M_{a+1,b|a+1,b} + M_{a,b+1|a,b+1} + M_{a+1,b+1|a+1,b+1} \\ &+M_{a,b|b,a} + M_{a+1,b|b,a+1} + M_{a,b+1|b+1,a} + M_{a+1,b+1|b+1,a+1}.\end{aligned}$$

The presence of $M_{a,b|a,b}$, which has lowest weight in the projection, implies P_λ is a direct summand. As before, we now apply Proposition 3.9 to see which modules appear in the standard filtration. Modules $M_{a+1,b|a+1,b}$ and $M_{a,b+1|a,b+1}$ appear by condition 3. The module $M_{a,b|b,a}$ appears by condition 1. The modules $M_{a+1,b|b,a+1}$ and $M_{a,b+1|b+1,a}$ appear by condition 4. We have 6 terms so far. The modules $M_{a+1,b+1|a+1,b+1}$ and $M_{a+1,b+1|b+1,a+1}$, are the two remaining terms, with the former of lower weight. But observe that its weight is in the same class $(a, b|a, b)$ with $b < a-1$. So if we were to have another projective, we would have at least $12 = 6+6$ terms. There are only 8 terms, so the module $M_{a+1,b+1|a+1,b+1}$ must also appear. It follows $M_{a+1,b+1|b+1,a+1}$ must appear by Corollary 3.10. We deduce that there is only one direct summand.

(2) $b = a-1$.
Let $\mu \coloneqq \lambda - (\epsilon_1 + 2\epsilon_2)$. We have

$$P_\mu = M_{a,a-1|a+1,a+1},$$

$$\mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^3 V) = M_{a,a-1|a,a-1} + M_{a+1,a|a,a+1} + M_{a+1,a|a+1,a} + M_{a+1,a-1|a-1,a+1} + M_{a+1,a-1|a+1,a-1} + M_{a,a|a,a} + M_{a,a-1|a-1,a}.$$

The presence of $M_{a,a-1|a,a-1}$, which has lowest weight in the projection, implies P_λ is a direct summand. As before, we apply Proposition 3.9 to see which modules appear in the standard filtration. The module $M_{a+1,a|a,a+1}$ must appear by condition 5. The module $M_{a+1,a-1|a-1,a+1}$ appears by condition 4. The modules $M_{a+1,a-1|a+1,a-1}$ and $M_{a,a|a,a}$ appear by condition 3. $M_{a,a-1|a-1,a}$ appears by condition 1. Lastly, $M_{a+1,a|a+1,a}$ must also appear in the standard filtration by Corollary 3.10. We deduce that there is only one direct summand.

(3) $b = a$. This is a repeat of an earlier case.

6.2.3. *Case* $P_\lambda,\ \lambda \cong (b, a \mid b, a)$.

(1) $b < a-1$. Let $\mu \coloneqq \lambda - (\epsilon_1 + \epsilon_2)$. We have

$$P_\mu = M_{b,a|b+1,a+1} + M_{a,b|b+1,a+1},$$

$$\mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) = M_{b,a|b,a} + M_{b,a+1|b,a+1} + M_{b+1,a|b+1,a} + M_{b+1,a+1|b+1,a+1} + M_{a,b|b,a} + M_{a+1,b|b,a+1} + M_{a,b+1|b+1,a} + M_{a+1,b+1|b+1,a+1}.$$

The presence of $M_{b,a|b,a}$, which has lowest weight in the projection, implies P_λ is a direct summand. We now apply Proposition 3.9 to deduce which of the other modules appear in the filtration. The modules $M_{b,a+1|b,a+1}$ and $M_{b+1,a|b+1,a}$ appear by condition 3. The module $M_{a,b|b,a}$ appears by condition 2. The modules $M_{a+1,b|b,a+1}$ and $M_{a,b+1|b+1,a}$ appear by condition 4. We have 6 terms so far. The modules $M_{b+1,a+1|b+1,a+1}$ and $M_{a+1,b+1|b+1,a+1}$, are the two remaining terms, with the former of lower weight. Observe that this weight is of the same class $(b, a|b, a)$ with $b < a-1$. So if this were to yield another projective, we would have at least $12 = 6+6$ terms. So $M_{b+1,a+1|b+1,a+1}$ must appear. It follows $M_{a+1,b+1|b+1,a+1}$ must appear by Corollary 3.10. We deduce that there is only one direct summand.

(2) $b = a-1$.
There appears no obvious choice of typical μ such that tensoring one of the representations above is effective. As a result, we allow μ to be atypical. This requires us

to use a suitable translation functor to first determine P_μ.

Let $\mu \coloneqq \lambda - (-\delta_1)$, so that $\mu \cong (a, a|a-1, a)$ and $\mu - \rho$ is atypical of degree 1. Let $\theta \coloneqq \mu - (-\delta_1 - \delta_2)$, so $\theta \cong (a+1, a+1|a-1, a)$ and $\theta - \rho$ is typical. We have the following:

$$P_\theta = M_{a+1,a+1|a-1,a},$$
$$\mathrm{Pr}_\mu(P_\theta \otimes \bigwedge^2 V^*) = M_{a,a|a-1,a} + M_{a+1,a|a-1,a+1} + M_{a,a+1|a-1,a+1}.$$

The presence of $M_{a,a|a-1,a}$, which has lowest weight in the projection, indicates that P_μ is a direct summand. Applying Proposition 3.9 to see which modules are present in the filtration for P_μ, we see that. Modules $M_{a+1,a|a-1,a+1}$ and $M_{a,a+1|a-1,a+1}$ appear by condition 3. We deduce that there is only one summand and that

$$P_\mu = M_{a,a|a-1,a} + M_{a+1,a|a-1,a+1} + M_{a,a+1|a-1,a+1}.$$

Now, we argue a formula for P_λ. We have

$$\begin{aligned}\mathrm{Pr}_\lambda(P_\mu \otimes V^*) = M_{a-1,a|a-1,a} + M_{a,a-1|a-1,a} + M_{a,a|a,a} \\ +M_{a-1,a+1|a-1,a+1} + M_{a,a+1|a,a+1} + M_{a+1,a-1|a-1,a+1} \\ +M_{a+1,a|a,a+1}.\end{aligned}$$

The presence of $M_{a-1,a|a-1,a}$, which has lowest weight in the projection, indicates that P_λ is a direct summand. We proceed as before with Proposition 3.9 to see which modules appear in the Verma flag. Module $M_{a,a-1|a-1,a}$ appears by condition 1. Modules $M_{a,a|a,a}$ and $M_{a-1,a+1|a-1,a+1}$ appear by condition 3. $M_{a+1,a-1|a-1,a+1}$ appears by condition 4, and $M_{a+1,a|a,a+1}$ appears by condition 5. It follows $M_{a,a+1|a,a+1}$ must appear by Corollary 3.10. We deduce that there is only one summand.

(3) $b = a$. This is a repeat of an earlier case.

6.2.4. *Case* $P_\lambda,\ \lambda \cong (b, a \mid a, b)$.

(1) $b < a - 1$.
Let $\mu \coloneqq \lambda - (\delta_1 + \delta_2)$. We have

$$P_\mu = M_{b,a|a+1,b+1} + M_{b,a|b+1,a+1} + M_{a,b|a+1,b+1} + M_{a,b|b+1,a+1},$$
$$\begin{aligned}\mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^2 V) = M_{b,a|a,b} + M_{b,a+1|a+1,b} + M_{b+1,a|a,b+1} + M_{b+1,a+1|a+1,b+1} \\ +M_{b,a|b,a} + M_{b,a+1|b,a+1} + M_{b+1,a|b+1,a} + M_{b+1,a+1|b+1,a+1} \\ +M_{a,b|a,b} + M_{a+1,b|a+1,b} + M_{a,b+1|a,b+1} + M_{a+1,b+1|a+1,b+1} \\ +M_{a,b|b,a} + M_{a+1,b|b,a+1} + M_{a,b+1|b+1,a} + M_{a+1,b+1|b+1,a+1}.\end{aligned}$$

The presence of $M_{b,a|a,b}$, which has lowest weight in the projection, indicates that P_λ is a direct summand. We proceed as before with Proposition 3.9 to see which modules appear in the Verma flag. Modules $M_{b,a|b,a}$ and $M_{a,b|a,b}$ appear by condition 1. $M_{a,b|b,a}$ appears by condition 2. The module $M_{b,a+1|a+1,b}$ appears by condition 3, and modules $M_{b,a+1|b,a+1}$, $M_{a+1,b|a+1,b}$, and $M_{a+1,b|b,a+1}$ appear by condition 4. The module $M_{b+1,a|a,b+1}$ appears by condition 3, and $M_{b+1,a|b+1,a}$, $M_{a,b+1|a,b+1}$, and $M_{a,b+1|b+1,a}$ appear by condition 4. $M_{b+1,a+1|a+1,b+1}$ appears by condition 5, and $M_{b+1,a+1|b+1,a+1}$,

$M_{a+1,b+1|a+1,b+1}$, and $M_{a+1,b+1|b+1,a+1}$ appear by condition 6. We deduce that there is only one summand.

(2) $b = a - 1$. Let $\mu \coloneqq \lambda - (\epsilon_1 + 2\epsilon_2)$. We have

$$P_\mu = M_{a-1,a|a+1,a+1} + M_{a,a-1|a+1,a+1},$$

$$\begin{aligned}\mathrm{Pr}_\lambda(P_\mu \otimes \bigwedge^3 V) = &M_{a-1,a|a,a-1} + M_{a,a+1|a,a+1} + M_{a,a+1|a+1,a} + M_{a-1,a+1|a+1,a-1}\\ &+M_{a-1,a+1|a-1,a+1} + M_{a-1,a|a-1,a} + M_{a,a-1|a,a-1} + M_{a+1,a|a,a+1}\\ &+M_{a+1,a|a+1,a} + M_{a+1,a-1|a-1,a+1} + M_{a+1,a-1|a+1,a-1} + 2M_{a,a|a,a}\\ &+M_{a,a-1|a-1,a}.\end{aligned}$$

The presence of $M_{a-1,a|a,a-1}$, which has lowest weight in the projection, indicates that P_λ is a direct summand. We proceed as before with Proposition 3.9 to see which modules appear in the Verma flag. The module $M_{a-1,a+1|a+1,a-1}$ and at least one copy of $M_{a,a|a,a}$ appear by condition 3. The module $M_{a,a+1|a+1,a}$ appears by condition 5. The modules $M_{a-1,a|a-1,a}$ and $M_{a,a-1|a,a-1}$ appear by condition 1. The module $M_{a,a-1|a-1,a}$ appears by condition 2. The modules $M_{a-1,a+1|a-1,a+1}$, $M_{a+1,a-1|a-1,a+1}$, and $M_{a+1,a-1|a+1,a-1}$ appear by condition 4. The modules $M_{a,a+1|a,a+1}$, $M_{a+1,a|a,a+1}$, and $M_{a+1,a|a+1,a}$ appear by condition 6. This forces the second copy of $M_{a,a|a,a}$ to appear by Corollary 3.10. We deduce that there is only one summand.

(3) $b = a$. This is a repeat of an earlier case.

References

[Br03] J. Brundan, *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$*, J. Amer. Math. Soc. **16** (2003), 185-231.

[CW08] S.-J. Cheng and W. Wang, *Brundan-Kazhdan-Lusztig and super duality conjectures*, Publ. RIMS **44** (2008), 1219-1272.

[CW12] S.-J. Cheng and W. Wang, *Dualities and Representations of Lie Superalgebras*, Graduate Studies in Mathematics **144**, Amer. Math. Soc., 2012.

[CLW15] S.-J. Cheng, N. Lam, and W. Wang, *Brundan-Kazhdan-Lusztig conjecture for general linear Lie superalgebras*, Duke Math. J. **164** (2015), 617-695.

[CW18] S.-J. Cheng and W. Wang, *Character Formulae in Category $\mathcal{O}$ for Exceptional Lie Superalgebra $G(3)$* (preprint, 2018).

[Gor02] M. Gorelik, *Strongly typical representations of the basic classical Lie superalgebras*, J. Amer. Math. Soc. **15** (2002), 167-184.

[Hum08] J. Humphreys, *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*, Graduate Studies in Mathematics **94**, Amer. Math. Soc., 2008.

[Mus12] I. Musson, *Lie Superalgebras and Enveloping Algebras*, Graduate Studies in Mathematics **131**, Amer. Math. Soc., 2012.

Department of Mathematics, University of Virginia, Charlottesville, VA 22904
E-mail address: ask9ge@virginia.edu