Essays in Economics

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Abstract

Essays in Economics[1] is broadly concerned with the aggregate implications of micro-level heterogeneities, and this dissertation contains two chapters: 1.) Firm Dynamics over the Business Cycle[2] and 2.) Endogenous Formation of Dark Networks: Theory and Experiment.[3]

Firm Dynamics over the Business Cycle

Can a real business cycle model match the empirical patterns of selection in entry, exit, and employment dynamics over the cycle as well as the RBC stylized facts for key macroeconomic aggregates? The answer found in this paper is yes. Importantly, getting the facts right for key macro aggregates helps replicate the selection effects at the firm level, and getting the facts right for firm entry and exit improves several of the macro properties of the model. To show this, this paper builds a DSGE model with heterogeneous firms that endogenizes entry and exit decisions and features investment possibilities at both the intensive (through capital accumulation) and extensive (through the creation of new and heterogeneous firms) margins. Sunk costs required for entry and fixed costs required in production mean that not only does the number of firms fluctuate over the cycle, but so too does the character of their productivity distribution. The model replicates, both qualitatively and quantitatively, key features of entry and exit in the data: entry is procyclical, but exit is roughly flat over the cycle; and plants that enter during recessions are, on average, more productive and larger (in terms of employment) than plants entering during booms. Further, as a business cycle model, it retains the successes of standard RBC models and improves on several of its properties: in this model, the volatility of consumption relative to output is much closer to the data, output displays persistence in line with the data and features a "hump-shaped" response to a transitory shock, and the cross-correlations of output, net entry, and profits are all very close to those found in the data. Finally, to incorporate the rich dynamics for firm heterogeneity and changes in the productivity distribution over the cycle, this paper develops an algorithm to incorporate and track – exactly, and with a finite-dimensional object – a potentially infinite-dimensional distribution of firms and firm productivities.

JEL Codes: E32, E23, C63

Keywords: Business Cycles, Heterogeneous Firms, Firm Entry and Exit, Selection

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Endogenous Formation of Dark Networks: Theory and Experiment

We study a dark network whose members face two different threats to their survival: first, a chance of being directly detected and arrested by the authorities; and second, a possibility of being "arrested by association" if another member of their network is arrested. Our game-theoretic model predicts that the number of members in equilibrium network structures should vary with changes in the former, but not with changes in the latter. We test these predictions in a laboratory experiment and find evidence in favor of our first prediction: increasing the probability of detection reduces network size. However, contrary to our predictions, we also find that increasing the impact that any one arrest has for the indirect detection of other members also reduces network size. Further study of these behavioral anomalies has the potential to enrich the design and implementation of policies intended to disrupt the formation of dark networks.

JEL Classification: C72, C92, D85

Keywords: Networks, Experiments, Terrorism, Crime

Firm Dynamics over the Business Cycle

1 Introduction

Can a real business cycle model match important facts about firm-level dynamics over the business cycle? And if so, does it improve the ability of the model to match the business cycle properties of key macroeconomic aggregates? This paper builds a real business cycle model that not only matches empirical patterns of entry, exit, and employment dynamics over the cycle, but also retains most of the successes of a standard RBC model and improves on several of its dynamics and second moment properties.

This paper, therefore, contributes to an important line of research on firm selection effects and how patterns of selection at the micro-level may contribute to shaping aggregate dynamics at the macro-level. What are the key features of firm dynamics that would be important for a macroeconomic model of business cycle fluctuations? First, both net entry and profits are strongly procyclical.[4] See Figures 1 and 2. And second, the characteristics of entering firms change substantially over the cycle. Lee and Mukoyama (2012) have recently documented that plants entering during booms are both smaller and less productive on average than those that enter during recessions.[5] The changing qualitative character of new firms that enter across booms and busts affects the distributions of firm sizes and productivities, suggesting a potential mechanism relating to firm entry that might serve an as additional propagation mechanism for shocks over the cycle.

This paper builds a dynamic stochastic general equilibrium (DSGE) model with heterogeneous firms that endogenizes both the entry and exit decisions. Sunk costs required for firm entry and fixed costs required in production mean that not only does the number of firms fluctuate over the cycle, but so too does the character of their productivity distribution. Along the way, the paper addresses a difficult technical problem of having to track an aggregate productivity distribution that potentially consists of an infinite number of entering firm cohorts, each with different productivity distributions and subject to further changes over the cycle. By using a particular functional form for the distribution of entrant productivities, I show how the representation can be collapsed into a small number of sufficient statistics that can track the entire distribution *exactly* and without any approximations.[6] The model and methodology are simple and intuitive;

[4] As Bilbiie, Ghironi, and Melitz (2007) have suggested, firm entry in expectation of future profits may play an important role in GDP expansions.

[5] Differences among exiting plants are relatively small. The empirical work in Lee and Mukoyama (2012) suggests that selection at the entry margin may be more important than that at the exit margin. While Lee and Mukoyama (2012) find that the job destruction rate is more cyclical (higher in recessions) than the job creation rate, (as do Davis, Haltiwanger, and Schuh (1996), who focus on employment flows), they do not find the "cleansing effect" at the exit margin. Their work adds to several other recent papers that challenge the relative importance of the "cleansing effect" of recessions - i.e., that low productivity plants are scrapped during recessions, which in turn can increase aggregate efficiency. See, e.g., Campbell (1998). The importance of the entry margin, however, is not totally inconsistent with Caballero and Hammour (1994), since low productivity firms can be "insulated" from recessions since there is less firm creation during recessions.

[6] Pareto distributions are used to model the productivity distributions of entering firms, which also results in an aggregate productivity distribution (defined over a collection of different Pareto distributions) that is consistent with the empirical data.

and, given the successes of the model in this paper, may provide an additional framework to further this line of research.[7]

The model in this paper is a Melitz-style real business cycle model of monopolistic competition that features investment along both the intensive (through capital accumulation) and extensive (through the creation of new firms) margins. A key aspect of the model is that the creation of new firms requires irreversible (sunk) "intangible" investments, and – due to the presence of fixed costs in production – low productivity firms do not survive. Further, with fluctuations in aggregate technology, these fixed costs induce changes to the set of firms (and the distribution their productivities) that produce and survive over the cycle.[8] The model performs well in matching key features of firm dynamics. The model replicates both the size distribution of firms and the selection patterns of entry and exit: firms entering during recessions are, on average, larger and more productive than those entering during booms, but variations in exiting firms are relatively smaller. Further, entry and profits are both procyclical, but exit is relatively flat. The cross-correlations of output, net entry, and profits in the model also closely replicate patterns found in the data. The model also performs well with respect to the stylized facts on macroeconomic aggregates over the business cycle. Typically, RBC models have several problems: the volatility of consumption relative to output is too smooth relative to the data, there is not enough endogenous persistence, and most series are too procyclical. In this model, the relative volatility of consumption is much closer to the data, there is more endogenous persistence for most series, and model counterparts of consumption, investment and hours worked are no longer too procyclical. Further, the impulse response function of output features a "hump-shape" in response to a transitory shock (with firm selection effects contributing to this effect) which matches that in Cogley and Nason (1995).[9]

While standard RBC models ignore fluctuations in the number of firms, I show that this margin can be important for matching key facts about macroeconomic aggregates over the business cycle.[10] And while my model is not the first to consider the role of entry and exit in a business cycle model - in fact, there is a relatively large literature that has developed relating to the role of firm entry and exit over the business cycle - the model in this paper differs from the previous literature in important ways. My model is the first to embed endogenous selection in entry and exit with heterogeneous firms in a real business cycle model featuring both capital and labor in production that is able to match many of the stylized facts of both business cycles and those on the dynamics of firms and profits - both qualitatively and quantitatively - over the cycle.

Several recent papers have explored the issue of entry and exit over the cycle, and many of these papers

[7] As discussed below, most of the recent literature on firm dynamics relies on the Hopenhayn and Rogerson (1993) framework, but there are a number of advantages of the framework in this paper.

[8] To be clear, the Melitz (2003) model has neither aggregate fluctuations nor capital.

[9] Standard RBC model have no internal propagation mechanisms. See, e.g., Christiano (1988) and Cogley and Nason (1995).

[10] In neoclassical RBC models like Kydland and Prescott (1982), the number of firms is irrelevant or indeterminant and features perfect competition, so there is no role for firm profits. New Keynesian models like Blanchard and Kiyotaki (1987) have an exogenous number of firms and profits that are, counterfactually, countercyclical.

have built on the framework of Hopenhayn and Rogerson (1993) that features heterogeneous producers in a perfectly competitive environment.[11] Lee and Mukoyama (2012) build on this framework to construct a model that matches the entry and exit rates in the data for U.S. manufacturing plants. The model in Lee and Mukoyama (2012), however, features no capital in production (so it cannot also address stylized real business cycle facts) and it also comes less close to matching the more micro-level patterns of selection that they find in the data. Note that Samaniego (2008) also constructed a general equilibrium model build on the Hopenhayn and Rogerson (1993) framework that focused on the entry and exit dynamics of firms in response to an aggregate technology shock, but only characterized their response along a (deterministic) transition path.[12] Two other recent working papers by Clementi and Palazzo (2014) and by Clementi, Khan, Palazzo, and Thomas (2014) build a Hopenhayn and Rogerson (1993) style model similar to the one in Lee and Mukoyama (2012). (The model in Clementi, Khan, Palazzo, and Thomas (2014) builds a general equilibrium version of the model in Clementi and Palazzo (2014).) These models are similar in spirit to mine - they also look at micro-level firm heterogeneities and their implications for macro-level aggregates - but their models have no role for firm profits (their models feature perfect competition) and, even with much more complicated models, come much less close to matching the data at either the micro or macro level.[13]

The model in Bilbiie, Ghironi, and Melitz (2012) explores an alternative environment to that of perfect competition and incorporates endogenous entry into a business cycle model with monopolistic competition to study entry and exit and the dynamics of macro aggregates.[14] Earlier general equilibrium models with monopolistic competition also studied the effects of entry and exit over the business cycle - e.g., Chatterjee and Cooper (1993) and Devereux, Head, and Lapham (1996).[15] The model in Bilbiie, Ghironi, and Melitz (2012) improves on the frictionless entry in these models (where all profits would otherwise be zero), and can generate procyclical profits. However, in contrast with my model, all firms are alike (homogeneous) in their

[11] Veracierto (2002) also added aggregate productivity shocks to the Hopenhayn and Rogerson (1993) framework, but the model assumed completely exogenous entry and exit.

[12] Note that business cycle dynamics are not explored since there are no aggregate shocks in this model. Samaniego (2008) found that entry and exit respond very little to changes in the aggregate productivity of the economy, but Lee and Mukoyama (2012) have shown that this result is sensitive to their particular specification of entry costs. The model in Samaniego (2008) features convex marginal costs, so that the marginal cost of creating new firms is increasing in the number of new firms. Lee and Mukoyama (2012) find that, in a similar Hopenhayn and Rogerson (1993) type model, entry in fact responds strongly to an aggregate shock (assuming constant marginal costs of entry) and that this assumption allows their model to come closer to the data on entry rates for U.S. manufacturing plants.

[13] Another recent and related model is that of Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry (2012), which also features heterogeneity in shaping aggregate dynamics; but this model focuses on an alternative shock (uncertainty shocks) to drive business cycles.

[14] My model, like the model in Bilbiie, Ghironi, and Melitz (2012), has a framework close to those in variety-based endogenous growth models - e.g., Romer (1990), Aghion and Howitt (1991), and Grossman and Helpman (1991). Where neoclassical RBC models like Kydland and Prescott (1982) are DSGE versions of exogenous growth models that abstract from growth, my model and the Bilbiie, Ghironi, and Melitz (2012) model are a DSGE versions of the variety-based, endogenous growth models that similarly abstract from growth to focus on business cycle fluctuations.

[15] Other more recent models with frictionless entry (and zero profits) include the more recent models of Comin and Gertler (2006) and Jaimovich and Floetotto (2008).

model; and exit stemmed solely from an exogenous shock, so their framework is incapable of addressing the patterns of selection in productivity, size and employment among entering and exiting firms. Further, their baseline model features a production technology that uses only a single factor (labor). Using my model, I show that the ability to capture both the macro and micro level facts on firm entry and exit over the cycle, as well as matching the stylized facts on macro aggregates, requires both labor and capital in production. Without the possibility of investing at the intensive margin (in capital) in addition to the extensive margin (new firms), the model gets many of the standard business cycle facts wrong, and the patterns of selection in entry and exit at the firm level go the wrong way. Further, my model matches the data on the cross-correlations of net entry with output and profits much more closely than those in Bilbiie, Ghironi, and Melitz (2012), which lend further support to preferring the structure of the model in this paper.

This paper therefore models firm entry, exit, and selection effects over the business cycle, and subsequently studies their role in the propagation of fluctuations over the cycle. Importantly, the model features two avenues of investment - the possibility of investing in the capital stock (investment at the intensive margin) and the possibility of investing in new and heterogeneous firms (investment at the extensive margin). In doing so, the paper further disentangles the contributions of both of these margins - and how they interact with the changing character of the productivity distribution of firms - for the economy's response to changes in aggregate technology. The remainder of the paper is organized as follows: the structure of the model is presented in Section 2. Readers already familiar with the Melitz (2003) model and the Bilbiie, Ghironi, and Melitz (2012) model may want to jump to Part 2.8 that lays out how my model compares with those two. Section 3 details the calibration for the model. Section 4 presents the results: Part 4.2 presents the results for the dynamics of firm entry, exit, and employment; Part 4.3 presents impulse response functions to give intuition to the selection effects and the response of macro aggregates; and Part 4.5 compares the model's performance with regards to its second-moment properties with that of a standard RBC model and the baseline model in Bilbiie, Ghironi, and Melitz (2012). Section 5 concludes.

2 Model

2.1 Production and Pricing

At any given time, there is a continuum of firms $M_{A,t}$ that produce intermediate goods. Production technologies are common across firms, but are affected by firm productivities that reflect both common and idiosyncratic components. Aggregate technology is common across all firms and indexed by Z_t. Firms are are characterized by heterogeneous productivity differences, where these idiosyncratic relative productivity differences are indexed by z and remain constant over time. Firms with relative productivity level z are each associated with a variety $\omega(z) \in \Omega$, distributed according to some distribution $g_t(z)$. Each production entity is associated with only one variety or production line, so the terms "firm", "plant", and "product" could be used interchangeably here, as the model otherwise abstracts from product variety within firms. For ease of expression, production entities that produce varieties of intermediate goods are referred to as "firms".

Varieties of intermediate goods are composited into final goods Q_t, which are the only goods available for consumption C_t and investment I_t. Final goods are produced competitively with a CES production technology that combines intermediate goods but requires no capital or labor according to

$$Q_t = \left[\int_{z_{\min}}^{\infty} 1_{\omega(z)\in\Omega_t} \left[q_t (z)^{(\theta-1)/\theta} M_{A,t} g_t (z) \right] dz \right]^{\theta/(\theta-1)}, \tag{1}$$

where θ is the elasticity of substitution across varieties.[16] Letting $p_t (z)$ denote the price of variety $\omega (z)$, the consumption-based price index is given by[17]

$$P_t = \left[\int_{z_{\min}}^{\infty} 1_{\omega(z)\in\Omega_t} \left[p_t (z)^{1-\theta} M_{A,t} g_t (z) \right] dz \right]^{1/(1-\theta)}. \tag{2}$$

Demands for individual varieties $\omega (z)$ are then given by

$$q_t (z) = \left[p_t (z) / P_t \right]^{-\theta} Q_t. \tag{3}$$

Firms that produce intermediate goods, then, face residual demand curves and set prices that reflect the same proportional markups $\theta/(\theta-1)$ over marginal costs. These firms employ both capital and labor through a Cobb-Douglas production technology

$$q_t (z) = z Z_t \left[k_t (z) \right]^{\alpha} \left[l_{q,t} (z) \right]^{1-\alpha}, \tag{4}$$

so that relative productivity differences translate into different marginal costs $\lambda_t (z)$ according to

$$\lambda_t (z) = (z Z_t)^{-1} \alpha^{-a} (1-\alpha)^{-(1-\alpha)} (r_{K,t})^{\alpha} (w_t)^{1-\alpha}.$$

Prices are then given as a constant markup over marginal costs

$$p_t (z) = \theta/(\theta-1) \lambda_t (z), \tag{5}$$

so that firms with higher productivity levels set lower prices, face greater demands, and earn higher profits.[18] Firms must also hire labor to cover fixed costs $w_t f/Z_t$ each period, so profits are then given by

$$\pi_t (z) = p_t (z) q (z) - \lambda_t (z) q_t (z) - w_t f/Z_t. \tag{6}$$

2.2 Firm Entry and Exit

At the beginning of every period, there are a number of incumbent firms $M_{S,t}$ with average aggregate productivity $\tilde{z}_{S,t}$ and an unbounded mass of potential entrants. Households are responsible for creating new

[16] Note that, at any given time t, only a subset of varieties $\Omega_t \subset \Omega$ may be available.

[17] The consumption-based price index changes over time with changes in the composition of the consumption basket. Prices are therefore written in nominal terms as a convenient unit of account, but because prices are flexible, the model is solved for real variables. The aggregate price index P_t is then set to one without loss of generality.

[18] A common interpretation here is that this is reflective of product quality.

firms, and pay start-up or "blueprint" costs to finance the creation of N_t new firms each period.[19] Blueprint costs for the creation of N_t new firms requires the production of N_t blueprints b according to the technology $bN_t = Z_t L_{B,t}$.[20] Note that the cost of creating a blueprint is given by $w_t b / Z_t$ and depends on the aggregate productivity of the economy Z_t but not on the relative productivities of the entrants, which at the time of creating the blueprints, are as yet unknown.[21] After blueprints are created, new firms draw idiosyncratic productivity components z from a Pareto distribution $G(z)$ with support on $[z_{\min}, \infty)$, which thereafter remain fixed over the life of the firms.[22] There is a one-period time-to-build lag, so new entrants created at time t may only start to produce at time $t+1$.

The distribution of firms can vary over the business cycle due to firm entry and exit, both of which have endogenous components in my model. New entrants with low productivity draws may decide to exit immediately following entry and never produce.[23] Each period, then, there is some productivity threshold $z^*_{N,t}$ that determines this cutoff. Some incumbent firms may also exit at the beginning of the period before production occurs. Even though idiosyncratic relative productivities remain constant over time, profits each period do not. Firms, therefore, must consider current-period profits as well as the expected value of all their future profits when making choices to produce or exit. For example, a firm may decide not to exit - even if its current period profits were negative - if the stream of its expected future profits were positive. For the shock process used in this model, firms with sufficiently high idiosyncratic productivity components can always expect to have positive profits, and firms with sufficiently low productivities can expect to never have positive profits (exiting immediately upon entry). There is a range of firms, however, that can expect both positive and negative profits over the cycle.[24] Since the cutoff productivity levels that determine exit

[19] The blueprint costs are entirely sunk - i.e., they are not amortized and/or rolled up into the period-by-period fixed costs that firms pay. Unlike the Melitz (2003) model, the two representations are not equivalent in this model - household would *not* be indifferent between paying a one time investment cost and paying the amortized per-period portion of the cost every period. (The representations were equivalent in the Melitz (2003) model because, after firm productivities were revealed, there was no uncertainty and there was no time discounting apart from the exit-inducing shock.)

[20] The assumption that the creation of new firms does not require physical capital is the same as that in Bilbiie, Ghironi, and Melitz (2012), and follows Grossman and Helpman (1991).

[21] Entry occurs until the expected value of new entrants is equal to their blueprint costs, so a free-entry condition holds in every period as long as $N_t > 0$. Shocks in the model are small enough so that the number of new entrants N_t is positive in every period. Note, however, that the solution algorithm is programmed to handle $N_t = 0$. It must be able to do so, since while the model is being solved, $N_t = 0$ is sometimes the optimal choice.

[22] The productivity distribution from which potential firms draw their firm-specific productivity component is invariant over time, though the resulting distribution for those that continue to produce is not, as explained below.

[23] Note that without fixed costs, there would be no endogenous exit.

[24] Note that producing with negative profits can be an optimal decision for firms in this model, since - if the expected discounted value of all future profits were greater than the current period negative profits - the value of continuing and producing in the current period would be greater than the value of exiting completely. (Details of firm value functions are given in Section 2.6.) Note that this is different than the kinds of variations in profits in Ambler and Cardia (1998) and Cook (2001). In those models, zero profit conditions hold (only) in expectation, where, since the number of firms is pre-determined within a period, firms may experience negative profits following a current period shock. Recall that the firms in the related model of Bilbiie, Ghironi, and Melitz (2012) always experience positive profits since there is no firm heterogeneity or fixed costs.

for incumbent firms and new entrants, z_t^* and $z_{N,t}^*$ respectively, vary over the cycle, so too, then, does the distribution of firms and productivities.[25] After endogenous exit occurs at the beginning of the period, $\left[1 - G\left(z_{N,t}^*\right)\right] N_t$ new entrants remain, and the distribution of incumbent firms is subsequently described through $\boldsymbol{F}_{A,t}$, where $M_{A,t}$ firms with average aggregate productivity $\tilde{z}_{A,t}$ will be active and producing in the current period.

At the end of the period, both incumbent firms and new entrants face a constant probability $\delta_M \in (0,1)$ of an exit-inducing shock.[26] Note that there are two sources of exit. The first is endogenous and occurs at the beginning of the period. All incumbent firms with productivities below z_t^* and new entrants with productivities below $z_{N,t}^*$ exit after realization of Z_t but before production occurs. The second is exogenous and occurs at the end of the period. The exit-inducing shock is uncorrelated with individual firm productivities, so it does not affect the distribution of firms that remain at the end of the period.

Therefore, all firms continue to produce until they are either exogenously destroyed or because adverse productivity shocks render exit to be the optimal choice. Note that while $G(z)$ represents the distribution of new entrants, it does not represent the productivity distribution of continuing firms. The value of z_t^* and $z_{N,t}^*$ will fluctuate over time as the set of firms that choose to enter and exit change with changes in the state of the world. As in Melitz (2003), the range of firm productivity levels and the average aggregate productivity are both endogenously determined, but my model also allows the range of firm productivities and the average aggregate productivity level to be in flux over the business cycle. My model generates, as does the Melitz (2003) model, an empirical pattern where new entrants have, on average, lower productivities and higher probabilities of exit than those of incumbent firms. However, my model also incorporates movements over the cycle that matches certain key facts about firm and plant-level data highlighted in recent empirical studies.

Tracking the distribution of firms due to changes in the entry/exit threshold is, therefore, key to solving the model. The state variable $\boldsymbol{F}_{S,t}$ contains the entire productivity distribution for $M_{S,t}$ firms at the start of the period. The state variable $\boldsymbol{F}_{A,t}$ contains the distribution for the $M_{A,t}$ firms that remain after endogenous exit at the beginning of the period before production occurs. The variable $\boldsymbol{F}_{A,t}$ is constructed as a function of $\boldsymbol{F}_{S,t}$ and z_t^* so that $\boldsymbol{F}_{A,t} = f\left(\boldsymbol{F}_{S,t}, z_t^*\right)$, and $\boldsymbol{F}_{S,t+1}$ can be constructed either as a function of $\boldsymbol{F}_{A,t}$, N_t, and $z_{N,t}^*$ or as a function of $\boldsymbol{F}_{S,t}$, z_t^*, N_t, and $z_{N,t}^*$, so $\boldsymbol{F}_{S,t+1} = g\left(\boldsymbol{F}_{A,t}, N_t, z_{N,t}^*\right)$ or $\boldsymbol{F}_{S,t+1} = h\left(F_{S,t} z_t^*, N_t, z_{N,t}^*\right)$. Intuition for these constructions is given in the next section, and details of the algorithms are provided in the appendix.

[25] The model is parameterized in such a way that z_t^* and $z_{N,t}^*$ never fall below $z_{\min}$, so there is always some endogenous exit among new entrants each period, though there may or may not be exit due to changes in z_t^* among incumbent firms in every period. The amount of endogenous exit among incumbent firms depends not only on z_t^*, but also on the existing productivity distribution of incumbent firms. Note that z_t^* is not necessarily equal to $z_{N,t}^*$ because new entrants do not produce in period in which they were created.

[26] So, at a minimum, a fraction δ_M of new entrants will never produce.

2.3 Distribution of Firm Productivities and Construction of Aggregates

Solving the model is made simple by expressing aggregates in terms of averages. Melitz (2003) shows how average productivities for firms characterized by Pareto distributions are determined by cutoff productivities; and in turn, how economies characterized by a number of representative firms with these average productivities induce the same aggregate outcomes as those calculated over the entire distributions. In general terms, for firms characterized by a Pareto distribution with a lower bound threshold productivity z^*, Melitz's (2003) special average productivity is given by

$$\tilde{z}(z^*) = \left[\frac{1}{1-G(z^*)}\int_{z^*}^{\infty} z^{\theta-1} g(z)\, dz\right]^{\frac{1}{\theta-1}},$$

where $G(z) = 1 - \left(\frac{z_{\min}}{z}\right)^{\kappa}$ for $z \geq z_{\min}$. In this way, an average productivity $\tilde{z}$ for firms with a Pareto distribution is completely determined by the cutoff z^*. The average productivity $\tilde{z}$ can then be expressed as $\tilde{z} = \upsilon z^*$, using the scaling parameter $\upsilon = (\kappa/(\kappa-\theta+1))^{\frac{1}{\theta-1}}$.

Calculating the average productivity in my model requires a modification to the simple average constructed in Melitz (2003) because of how the distribution of firms changes over time. In my model, both the numbers of incumbent and entering firms change over time, and the threshold productivity cutoffs z_t^* and $z_{N,t}^*$ also change over time. Each period, changes to the cutoff productivities affect the distribution of new entrants and can also affect incumbent firms. In theory, and at first glance, it would seem that the model would need to keep track of an infinite past history of firm cohort sizes and productivity thresholds and/or averages. Practically, this is not necessary.[27] A sufficiently large (but finite) number of Pareto distributions will track this process exactly for the calibrated values of the shock process.[28] The exact details of the algorithm, and the structure and construction of $\boldsymbol{F}_{S,t}$ and $\boldsymbol{F}_{A,t}$ - the constructions that represents the entire distribution of firms and firm productivities - are detailed in the technical appendix.

In the same way that the Melitz (2003) model combines domestic and export average productivities to come up with an economy-wide aggregate for a two-country model, my model combines a number of R Pareto distributions in $\boldsymbol{F}_{S,t}$ and $\boldsymbol{F}_{A,t}$ to construct economy-wide average aggregate productivities according to

$$\tilde{z}_{S,t} = \left(\frac{1}{\sum_{r=1}^{R} M_{S_r,t}}\left[\sum_{r=1}^{R} M_{S_r,t}\left[\tilde{z}_{S_r,t}\right]^{\theta-1}\right]\right)^{\frac{1}{\theta-1}}$$

and

$$\tilde{z}_{A,t} = \left(\frac{1}{\sum_{r=1}^{R} M_{A_r,t}}\left[\sum_{r=1}^{R} M_{A_r,t}\left[\tilde{z}_{A_r,t}\right]^{\theta-1}\right]\right)^{\frac{1}{\theta-1}},$$

[27] This is a result of the very convenient property of Pareto distributions that, when truncated from below, the distribution remains Pareto. The basic idea is that whenever z_t^* increases and there is endogenous exit from more than one cohort, these cohorts of firms can now be combined and described by the same new lower bound, z_t^*. In this way, the firms comprising a cohort are not necessarily all of one age or "vintage".

[28] Note that if the cutoff productivity thresholds became continually lower and lower over time, the distribution would, in fact, have a boundlessly increasing number of cohorts to track. In a business cycle model, where thresholds fluctuate up and down over time, this does not happen.

where $\tilde{z}_{S_r,t}$ and $\tilde{z}_{A_r,t}$ represent averages for the groups of firms $M_{S,t} = \sum_{r=1}^{R} M_{S_r,t}$ and $M_{A,t} = \sum_{r=1}^{R} M_{A_r,t}$, respectively. As in Melitz (2003), this productivity average - based on weights proportional to the relative output shares of firms - summarizes all of the information about the productivity distribution that is relevant for constructing macroeconomic variables and for the production decisions made by individual firms. On the aggregate level, therefore, production occurs as if there were $M_{A,t}$ firms all with the same productivity $\tilde{z}_{A,t}$.

In particular, the price index in Equation (2) can be simply written as $P_t = M_{A,t}^{1/(1-\theta)} p_t(\tilde{z}_{A,t})$; and analogously, total output in Equation (1) can be written as $Q_t = M_{A,t}^{\theta/(\theta-1)} q_t(\tilde{z}_{A,t})$. Note that, as in Melitz (2003), the weighted average of firm productivity levels $\tilde{z}_{A,t}$ is constructed in such a way that $\tilde{\pi}_t = \pi_t(\tilde{z}_{A,t})$ - that is, average profits $\tilde{\pi}_t$ are equal to the profits at the "average" ($\tilde{z}_{A,t}$) firm. Aggregate profits are then given by $\Pi_t = \tilde{\pi}_t M_{A,t}$.

2.4 Households

The economy is populated by a unit mass of identical, infinitely-lived households. The representative household maximizes expected intertemporal utility over consumption and labor

$$E_t\left[\sum_{s=t}^{\infty} \beta^{s-t} U(C_s, L_s)\right],$$

with period utility defined as

$$U(C_t, L_t) = \ln(C_t) - (1 + 1/\varphi)^{-1} \chi L_t^{1+1/\varphi},$$

where $\beta \in (0,1)$ is the subjective discount factor, $\varphi \geq 0$ determines the elasticity of the labor supply, and $\chi > 0$ sets the disutility of labor.

Households supply labor L_t at wage rate w_t and rent capital K_t at rental rate $r_{K,t}$ to firms in competitive factor markets. Households make investments I_t to augment the capital stock K_t, which depreciates each period at rate δ_K. Investments in new firms N_t are also made by the households to augment the existing stock of firms $M_{S,t}$. Profits from firms are repatriated to the households through their ownership of shares x_t in a mutual fund that consists of all of the firms in the economy. Each period, households receive income from firm dividends $D_t = \tilde{d}_t M_{A,t}$ and the value of selling their share position $\tilde{v}_t M_{A,t}$ in their mutual fund holdings, where $\tilde{d}_t$ and $\tilde{v}_t$ represent average firm dividends and values, respectively, for the $M_{A,t}$ firms.

The representative household chooses consumption C_t, hours worked L_t, investment in physical capital I_t, investment in shares in a mutual fund of firms x_{t+1}, and investment in the creation of new firms N_t to maximize expected intertemporal utility subject to the period budget constraint

$$C_t + I_t + x_{t+1}\tilde{v}_t(M_{A,t} + N_t) = w_t L_t + r_{K,t} K_t + x_t\left(\tilde{v}_t + \tilde{d}_t\right) M_{A,t}.$$

2.5 Timing of Household Decisions

At the beginning of a period, households observe the new value of aggregate technology Z_t. The (actual) state of the world is characterized by $\boldsymbol{S}_t = \{Z_t, \boldsymbol{F}_{S,t}, K_t\}$, where Z_t is the value of aggregate productivity, K_t is the value of the capital stock, and $\boldsymbol{F}_{S,t}$ is the construction that represents the entire distribution of firms and firm productivities at the beginning of the period. Note that $\boldsymbol{S}_t = \{Z_t, \boldsymbol{F}_{S,t}, K_t\}$ could be used as a state variable that completely describes the economy, since $\boldsymbol{F}_{S,t}$ contains an exact description of firms and firm productivities. (The construction $\boldsymbol{F}_{S,t}$ has finite dimension, with element-by-element laws of motion as described in the appendix.) However, though finite, the matrix representation is still very large, and would therefore be impractical to use as a state variable given current computational resources. Further, households do not care about the details of the entire distribution of firms and firm productivities - only the aggregate stock of firms and their aggregate average productivity are relevant for the households. To reduce the dimensionality of the state space, the distribution contained in $\boldsymbol{F}_{S,t}$ is replaced by a smaller set of moments $\{M_{S,t}, \tilde{z}_{S,t}\}$, so the state of the world is summarized by the values $\{Z_t, M_{S,t}, \tilde{z}_{S,t}, K_t\}$. The value function for the households at the beginning of the period can therefore be defined as $V(Z_t, M_{S,t}, \tilde{z}_{S,t}, K_t)$, so that

$$V(Z_t, M_{S,t}, \tilde{z}_{S,t}, K_t) = \max_{C_t, L_t, I_t, N_t} \left\{ U(C_t, L_t) + \beta E_{Z_{t+1}|Z_t} \left[V(Z_{t+1}, M_{S,t+1}, \tilde{z}_{S,t+1}, K_{t+1}) \right] \right\} \tag{7}$$

Households know that, before production occurs, some firms may optimally choose to exit after observing Z_t. If some firms exit, the stock of firms that remains $M_{A,t}$ and their average aggregate productivity $\tilde{z}_{A,t}$ may be different from $M_{S,t}$ and $\tilde{z}_{S,t}$ at the beginning of the period.[29] To make optimal decisions, households must therefore map the values of $M_{A,t}$ and $\tilde{z}_{A,t}$ from state $\{Z_t, M_{S,t}, \tilde{z}_{S,t}, K_t\}$ that will result when incumbent firms exit after revelation of the current period's Z_t.[30] Households must also predict the value of $z^*_{N,t}$, which is necessary to calculate the proportion of financed entrants that will ultimately be added to the next period's incumbent firm stock and in turn calculate the laws of motion for $M_{S,t+1}$ and $\tilde{z}_{S,t+1}$. Households, therefore, use the state variables $\{Z_t, M_{S,t}, \tilde{z}_{S,t}, K_t\}$ to map values of $\left\{M_{A,t}, \tilde{z}_{A,t}, z^*_{N,t}\right\}$ that will occur in the period immediately following revelation of the shock Z_t. Note that, with z^*_t and $z^*_{N,t}$, values for $M_{A,t}$ and $\tilde{z}_{A,t}$ could be calculated directly from the element-by-element laws of motion for $\boldsymbol{F}_{S,t}$.[31] However, by replacing $\boldsymbol{F}_{S,t}$ with $\{M_{S,t}, \tilde{z}_{S,t}\}$, households instead make use of only these latter moments, together with values of Z_t and K_t, in mapping values of $M_{A,t}$, $\tilde{z}_{A,t}$, $z^*_{N,t}$.[32] It is important to note that household behavior would

[29] At the beginning of period t, there are $M_{S,t}$ firms with average aggregate productivity $\tilde{z}_{S,t}$. After the current value of Z_t is revealed, some firms may endogenously choose to exit. The variables $M_{A,t}$ and $\tilde{z}_{A,t}$ are used to denote the number of firms and their aggregate average productivity that remain active in period t after firms with low productivities have exited.

[30] Note that incumbent firms with productivities $z < z^*_t$ will immediately exit. However, if all incumbent firms have $z \geq z^*_t$, no firms exit and $M_{S,t} = M_{S,t}$ and $\tilde{z}_{A,t} = \tilde{z}_{S,t}$.

[31] Laws of motion for $\boldsymbol{F}_{A,t} = f(\boldsymbol{F}_{S,t})$ and the construction of $M_{A,t}$ and $\tilde{z}_{A.t}$ from $\boldsymbol{F}_{A,t}$ are described in the appendix.

[32] The solution method, therefore, uses a modification of the algorithm developed in Krusell and Smith (1998). These mapping functions are very accurate, and the accuracy of these mapping functions is summarized in Section 4.1, with further accuracy tests detailed in the appendix.

not change if they knew the entire distribution of firms and firm productivities, $\boldsymbol{F}_{A,t}$. Households only care about the number of firms $M_{A,t}$ and their average aggregate productivity $\tilde{z}_{A,t}$, since only these values are relevant when making choices for consumption, labor, and investment activities.[33] Together with the value of $z^*_{N,t}$, these variables are also sufficient to fully describe the law of motion for the number of firms and the average aggregate productivity that will carry over into the beginning of the next period.

At this point, the household's value function is denoted $\hat{V}\left(Z_t, M_{A,t}, \tilde{z}_{A,t}, K_t, z^*_{N,t}\right)$.[34] Households then choose how much to work L_t, how much they would like to consume C_t, investments they would like to make in the capital stock I_t, and the number of new entrants N_t they would like to finance to maximize

$$\hat{V}\left(Z_t, M_{A,t}, \tilde{z}_{A,t}, K_t, z^*_{N,t}\right) = \max_{C_t, L_t, I_t, N_t} \left\{U\left(C_t, L_t\right) + \beta E_{Z_{t+1}|Z_t}\left[V\left(Z_{t+1}, M_{S,t+1}, \tilde{z}_{S,t+1}, K_{t+1}\right)\right]\right\} \tag{8}$$

subject to firm production technologies, expressed in the aggregate as

$$Q_t = \tilde{z}_{A,t} Z_t \left[K_t\right]^{\alpha} \left[L_{Q,t}\right]^{1-\alpha} \left[M_{A,t}\right]^{\theta/(\theta-1)},$$

the law of motion for the capital stock

$$K_{t+1} = \left(1-\delta_K\right) K_t + I_t, \tag{9}$$

the law of motion for the firm stock

$$M_{S,t+1} = \left(1-\delta_M\right) M_{A,t} + \left(1-\delta_M\right)\left[1 - G\left(z^*_{N,t}\right)\right] N_t, \tag{10}$$

the law of motion for the average aggregate productivity

$$\tilde{z}_{S,t+1} = \begin{pmatrix} \left[\left(1-\delta_M\right) M_{A,t} + \left(1-\delta_M\right)\left[1 - G\left(z^*_{N,t}\right)\right] N_t\right]^{-1} \times \\ \left[\left(1-\delta_M\right) M_{A,t}\left[\tilde{z}_{A,t}\right]^{\theta-1} + \left(1-\delta_M\right)\left[1 - G\left(z^*_{N,t}\right)\right] N_t \left[\tilde{z}\left(z^*_{N,t}\right)\right]^{\theta-1}\right] \end{pmatrix}^{\frac{1}{\theta-1}}, \tag{11}$$

and clearing in the goods

$$Q_t = C_t + I_t$$

and labor markets[35]

$$L_t = L_{Q,t} + L_{F,t} + L_{B,t}.$$

Decisions are made and executed simultaneously with those of the firms, as explained below.[36]

[33] To reiterate, policy functions are based on values of $M_{A,t}$ and $z_{A,t}$ that actually results - i.e., on the firms that are actually producing during the period (not a forecast of those that will be producing).

[34] When making choices for the current period, households, therefore, take as given the number of firms $M_{A,t}$ and their average aggregate productivity $\tilde{z}_{A,t}$ that will result from endogenous exit at the beginning of the period along with the value of $z^*_{N,t}$ that will obtain.

[35] Labor used by all firms in the production process is denoted by $L_{Q,t}$, labor used to pay fixed costs by L_{Ft}, and labor used in the creation of new firms is given by $L_{B,t}$.

[36] Household decisions depend on the value of $z^*_{N,t}$ that will obtain during the period, determined from the firms' problem, which in turn depends on the values that the households choose. The solution method requires that both problems are solved simultaneously. The appendix details this procedure.

2.6 Timing of Firm Decisions

At the beginning of period t, the state of the world for firms is described by the new value of aggregate productivity Z_t and the values of K_{t-1}, $M_{A,t-1}$, and $\tilde{z}_{A,t-1}$ that summarized the aggregate state of the production environment in the previous period. At the beginning of period t, then, the value function for a firm with idiosyncratic productivity z is given by $J(z; Z_t, M_{A,t-1}, \tilde{z}_{A,t-1}, K_{t-1})$, where

$$\begin{aligned} J(z; Z_t, M_{A,t-1}, \tilde{z}_{A,t-1}, K_{t-1}) &= \pi_t(z; Z_t, M_{A,t}, \tilde{z}_{A,t}, K_t, Q_t) + \\ &\quad \beta(1-\delta_M) E_{Z_{t+1}|Z_t}\left[J(z; Z_{t+1}, M_{A,t}, \tilde{z}_{A,t}, K_t)\right]. \end{aligned} \tag{12}$$

Firms must then decide to produce and continue or exit.[37]

To make an optimal decision, firms must know what their profits would be in the current period if they produced, and whether the value of producing today and continuing tomorrow exceeds the value of producing nothing today and exiting permanently. To determine potential profits, firms must predict factor prices w_t and $r_{K,t}$ and the demands $q_t(z)$ and prices $p_t(z)$ of their products. Knowledge of the values that summarize the current state along with the households' choice for Q_t (the sum of household consumption and investment demands, C_t and I_t) is sufficient information to determine factor prices, output demands, and product prices to calculate profits as defined in Equation (6). That is, the information that the firms use summarizes everything about the production environment that is necessary to calculate demands, prices, and profits for individual firms.[38]

An implementation of the method in Krusell and Smith (1998), therefore, is also used in the firms' problem. Firms use their observation of Z_t and the values of K_{t-1}, $M_{A,t-1}$, and $\tilde{z}_{A,t-1}$ from production in the previous period to map the values for K_t, $M_{A,t}$, $\tilde{z}_{A,t}$, and Q_t that will obtain in the current period.[39] The value function at this point is denoted as $\hat{J}(z; Z_t, M_{A,t}, \tilde{z}_{A,t}, K_t, Q_t)$, where

$$\begin{aligned} \hat{J}(z; Z_t, M_{A,t}, \tilde{z}_{A,t}, K_t, Q_t) &= \pi_t(z; Z_t, M_{A,t}, \tilde{z}_{A,t}, K_t, Q_t) + \\ &\quad \beta(1-\delta_M) E_{Z_{t+1}|Z_t}\left[J(z; Z_{t+1}, M_{A,t}, \tilde{z}_{A,t}, K_t)\right]. \end{aligned} \tag{13}$$

Firms then compare the option of continuing and producing with exiting. With an exit value of zero, there is a cutoff productivity $z = z_t^*$ such that $\hat{J}(z_t^*; Z_t, M_{A,t}, \tilde{z}_{A,t}, K_t, Q_t) = 0$, specifically

$$z_t^* = \sup\left[z_t : \hat{J}(z_t; Z_t, M_{A,t}, \tilde{z}_{A,t}, K_t, Q_t) = 0\right], \tag{14}$$

so that all firms with idiosyncratic productivities $z \geq z_t^*$ stay and produce, while all firms with productivities

[37] A firm must produce if it does not exit. This means that some firms may occasionally have negative profits on a per-period basis.

[38] That is, firms care about output demands, but they do not care about what portion goes to consumption and what portion goes to investment (eaten and owned, respectively, by the households) when making production decisions.

[39] Note that these values depend on household choices in the previous period for additions to the firms stock N_{t-1} and additions to the capital stock I_{t-1}, along with the value of z_t^*, which is determined in part through the decisions the firms are making in the current period.

$z < z_t^*$ exit.[40] Of the remaining firms, a portion δ_M are exogenously destroyed at the end of the period. Those that do not exit and are not destroyed continue on to the next period.

Of course, during period t, new entrants are also being financed by the households. The timing and the entry/exit decisions for new entrants are the same as those for incumbent firms, except that new entrants do not produce in the period in which they are "born". Further, new entrants receiving low productivity draws exit immediately, where $z_{N,t}^*$ is such that

$$z_{N,t}^* = \sup \left[z_t : \beta (1 - \delta_M) E_{Z_{t+1}|Z_t} \left[J(z; Z_{t+1}, M_{A,t}, \tilde{z}_{A,t}, K_t) \right] = 0 \right]. \tag{15}$$

New entrants that survive $(1 - \delta_M)\left(1 - G\left(z_{N,t}^*\right)\right) N_t$ continue on to the next period and become part of the existing stock of incumbent firms.[41]

Note that households and firms use different sets of information about the world at the very beginning of the period (the value functions in Equations 7 and 12), but during the simulations, which are an important part of solving the model, policy functions for households and threshold cutoff productivity rules for firms (derived from Equations 8 and 13) are using the same information about the state of the world.[42]

2.7 RCE

Note that the obstacles to finding the equilibrium are non-trivial. First, it is not immediately obvious how to describe the distribution of firms and construct its law of motion as it evolves over time, though - because the

[40] That is, the choice can be expressed as

$$\max \left[0, \pi_t \left(z; Z_t, M_{A,t}, \tilde{z}_{A,t}, K_t, Q_t \right) + \beta (1 - \delta_M) E_{Z_{t+1}|Z_t} \left[J\left(z; Z_{t+1}, M_{A,t}, \tilde{z}_{A,t}, K_t \right) \right] \right].$$

[41] Note that, in the Melitz (2003) model, firm value functions were given by

$$v(z) = \max \left\{ 0, \Sigma_{t=0}^{\infty} (1 - \delta)^t \pi(z) \right\} = \max \left\{ 0, \frac{1}{\delta} \pi(z) \right\},$$

so determining z^* was simply given as a zero cutoff profit condition where $\pi(z^*) = 0$. Compare and contrast with my model. In my model, the threshold productivity cutoff z_t^* is given as

$$z_t^* = \sup \left[z_t : \hat{J}\left(z_t; Z_t, M_{A,t}, \tilde{z}_{At}, K_t, Q_t \right) = 0 \right].$$

Even though $\hat{J}\left(z_t^*; Z_t, M_{A,t}, \tilde{z}_{At}, K_t, Q_t \right) = 0$, current period profits $\pi_t \left(z_t^*; Z_t, M_{A,t}, \tilde{z}_{At}, K_t, Q_t \right)$ may or may not be zero. Note also that $z_{N,t}^*$ is found from a condition where

$$z_{N,t}^* = \sup \left[z_t : \beta (1 - \delta_M) E_{Z_{t+1}|Z_t} \left[J\left(z; Z_{t+1}, M_{A,t}, \tilde{z}_{A,t}, K_t \right) \right] = 0 \right].$$

In my model, the cutoff conditions must be determined from the firm value functions (not simply from a time-invariant profit equation), so both firm and household value functions must be found in order to solve the model.

[42] Results (available upon request) are robust to changes in the information structure of the value functions in Equations 7 and 12. The model was also solved where the information sets of the households and firms were the same throughout the period - i.e., firms use values for capital investment and blueprint creation to construct laws of motion for K_t, $M_{S,t}$, and $Z_{S,t}$, but this unnecessarily complicates the model (by requiring more states in Equation 12) and seems less intuitively appealing. Since the accuracy of the mapping functions in this implementation is similar to the ones used in the model, and because firms only use Z_t, K_t, $M_{A,t}$, $\tilde{z}_{A,t}$, and Q_t in calculating profits, the results do not differ markedly.

solution method involves simulations - the distribution of firms must be tracked over time to solve the model. An algorithm was developed to track this potentially infinite-dimensional object with a finite (and relatively small) number of dimensions *exactly* (i.e., no approximations were involved). Developing this algorithm was, therefore, a key piece in solving the model.[43] Second, even though the method of tracking the distribution uses an object with a finite number of dimensions, the number of dimensions is still too large for the object itself to serve as a state variable in household and firm value functions. Therefore, a smaller number of moments - the number of firms and their average aggregate productivity - are used instead. Finding the equilibrium, then, involves ensuring that households and firms can, with a high degree of accuracy, map certain variables from these summary values. It is, of course, not clear a priori that this obviates the need for knowledge of the full representation of the productivity distribution, so in finding the model's solution, variables mapped from this smaller number of state variables must be accurate.[44] And third, the choices of firms and households are interdependent. Value function iteration is used to solve both the firms' and households' problems simultaneously. However, as these problems are being solved, the mapping functions will change as value functions and policy functions change. It is therefore also necessary to update mapping functions by simulating the model in each iteration. The solution method is, therefore, a variant of the method used in Krusell and Smith (1998); and for these types of models, of course, it is unknown a priori if a solution even exists (and if it does, if it can be found through current numerical methods).[45]

For my model, a recursive competitive equilibrium is defined as follows.[46] An RCE is a collection of value functions $\left\{V, \hat{V}, J, \hat{J}\right\}$, policy functions $\left\{K^P, M_S^P\right\}$, threshold cutoff productivity rules $\{z^*, z_N^*\}$, mapping functions $\left\{M_A^H, \tilde{z}_A^H, z_N^{*H}, M_A^F, \tilde{z}_A^F, K^F, Q^F\right\}$, and laws of motion for $\{K, M_S, \tilde{z}_S\}$ and laws of motion $\{f, g, h\}$ for $\{\boldsymbol{F}_S, \boldsymbol{F}_A\}$ such that

1. The Household's Problem: Households use mapping functions M_A^H, $\tilde{z}_A^H$, and z_N^{*H} to construct value function $\hat{V}$ from V (Equations 7 and 8) so that, given factor demands, output prices, and entry costs, and laws of motion for K, M_S, and $\tilde{z}_S$, value functions V and $\hat{V}$, policy functions K^P and M_S^P, and forecasting functions M_A^H, $\tilde{z}_A^H$, and z_N^{*H} solve the households' problem.

2. The Firm's Problem: Firms use the mapping functions M_A^F, $\tilde{z}_A^F$, K^F, and Q^F to construct value function $\hat{J}$ from J (Equations 12 and 13) so that, given factor prices, output prices, fixed costs, and demand functions, value functions J and $\hat{J}$ and threshold productivity rules z^* and z_N^* (Equations 14 and 15) solve the firms' problem.

3. Firms and the Productivity Distribution: Firms and their productivity distribution represented by $\boldsymbol{F}_S$ and $\boldsymbol{F}_A$ are generated by laws of motion f, g, and h. Laws of motion for M_S and $\tilde{z}_S$ (Equations 10 and 11) are consistent with the law of motion g.

[43] See the appendix for details of the matrix used to describe the distribution and the algorithm used to track its law of motion during the simulations.

[44] Several different methods were used to check the accuracy of the mapping functions. See Section 4.

[45] The algorithm that details the step-by-step process of the solution method is in the appendix.

[46] A more detailed description of the RCE is provided in the appendix.

4. Aggregation: M_S and $\tilde{z}_S$ are generated from $\boldsymbol{F}_S$; and M_A and $\tilde{z}_A$ are generated from $\boldsymbol{F}_A$. Aggregate factor demands for K, L_Q, and L_F and output Q from firms with productivities represented in $\boldsymbol{F}_A$ are equivalent to those generated by M_A firms with average productivity $\tilde{z}_A$.

5. Rational Expectations: Household mapping functions M_A^H and $\tilde{z}_A^H$ accurately predict the outcomes from the laws of motion from $\boldsymbol{F}_S$ to $\boldsymbol{F}_A$ from f and the mapping function for z_N^* accurately predicts the productivity threshold z_N^* found from $\hat{J}$. Firm mapping functions M_A^F and $\tilde{z}_A^F$ accurately predict the outcomes from the laws of motion from $\boldsymbol{F}_{A,-1}$ to $\boldsymbol{F}_A$ from h (equivalently, $\boldsymbol{F}_{A,-1}$ to $\boldsymbol{F}_S$ from g and from $\boldsymbol{F}_S$ to $\boldsymbol{F}_A$ from f) and the mapping functions K^F and Q^F accurately predict outcomes from household policy functions K^P and M_S^P.

6. Factor markets are competitive and all markets clear.

2.8 Comparison with Melitz (2003) and BGM (2012)

For readers already familiar with the Melitz (2003) model from the international trade literature, it is useful to compare the structure of my model with the closed economy version in Melitz (2003) and the closely related model in Bilbiie, Ghironi, and Melitz (2012). The Melitz (2003) model builds on the classic Krugman (1980) and Hopenhayn (1992) models and incorporates firm heterogeneity to build a theoretical model to understand how opening to trade leads to reallocations of resources across firms. The model is, therefore, intended to be an open-economy general equilibrium model, but builds a closed-economy model "on-the-way". The closed-economy version of the Melitz (2003) model can be seen as a special case of my model in which firms use only a single factor (labor) in production and the aggregate technology is always equal to $Z = 1$ (that is, the model only considers a stationary or steady-state equilibrium).[47] The steady-state of my model, if labor were the only factor of production (i.e., $q(z) = zl(z)$), gives essentially the same solution and implies the same comparative statics predictions as the Melitz (2003) model.

Bilbiie, Ghironi, and Melitz (2012) build a closed-economy DSGE model that is similar to my model, but it does not include firm heterogeneity. They refer to their model in the paper as a "CES" model, as it - like mine - features monopolistic competition, consumer love of variety, and sunk entry costs. However, firms are identical in their model and there are no fixed costs. Their baseline model includes only labor in production, but they also briefly consider a version with capital. For my model, if fixed costs f were set to zero, the aggregate implications would be equivalent to those in Bilbiie, Ghironi, and Melitz (2012), for both of the

[47] The model assumes no time-discounting, but this is not particularly relevant. Note also that since Melitz (2003) only considers a stationary equilibrium, the ownership structure of firms is irrelevant since the blueprint costs of creating new firms that replace incumbent firms that die each period is exactly equal to the profits earned by the incumbent firms. If the mutual fund ownership structure of my model (and that of the Bilbiee, Ghironi, and Melitz (2012) model) were implemented in the Melitz (2003) model, it would simply create the "free entry" condition in the Melitz (2003) model, and the profits from the mutual fund would be used to create new firms, where the costs of these investments would be exactly equal to the present value of their future profits.

versions with and without capital in production. (Even with the heterogeneous firms in my model, if $f = 0$, all firms - regardless of individual productivities - would find it profitable to produce, and there would be no endogenous exit. The productivity distribution of firms would then remain constant, and induce aggregate implications equivalent to those of an economy where all firms shared the same productivity draw $\tilde{z}$.) Since all firms in their model are identical, only the amount of entry is endogenous (exit stems solely from an exogenous shock), and they cannot consider further issues relating to firm dynamics - that is, implications for the changing nature of the productivities, sizes, and employment patterns of entrants and exiting firms - as they evolve over the cycle.

Like the model in Bilbiie, Ghironi, and Melitz (2012), my model explicitly models the incentives for the creation of new firms from consumer love of variety and the profit incentives of investors. My baseline model, however, retains investment both at the extensive margin (through the creation of new goods/firms) and at the intensive margin (through capital accumulation). Unlike the model in Bilbiie, Ghironi, and Melitz (2012), my model explicitly incorporates firm heterogeneity in a way that matches patterns in the data and addresses more specifically the patterns of entry, exit, and the productivity of new firms over the business cycle. This feature lets me examine how changes at the extensive margin *and the qualitative character of entering firms* does or does not play a role in propagating shocks over the cycle.

In the results that follow, I show that including capital in production is necessary to match the data (and improve over a standard RBC model) on key macroeconomic variables, and that incorporating fixed costs is necessary to match the data on firm entry, exit, and employment dynamics. Further, I also show that, once firm heterogeneity is incorporated in the model, excluding capital as a factor in production implies the wrong patterns for firm dynamics. Including firm heterogeneity also improves some of the model's implications for several macro aggregates.

3 Calibration

Periods are interpreted as quarters, and β is set to 0.99. The elasticity of labor supply is set to $\varphi = 4$, consistent with that in King and Rebelo (1999), and the weight on the disutility of labor is normalized to $\chi = 1$. The value for θ under CES preferences is set to 3.8 from its equivalent in Bernard et al. (2003), which was estimated to fit US plant and macro trade data. The distribution for draws of firm productivities $G(z)$ is modelled as Pareto $G(z) = 1 - (z_{\min}/z)^{\kappa}$, which in turn implies that the steady-state size distribution of firms is also Pareto. Since Bernard et al. (2003) estimate that the standard deviation of log US plant sales is 1.67, corresponding to $1/(\kappa - \theta + 1)$ in the model, κ (the shape parameter) for the Pareto distribution is set to 3.4. The lower bound $z_{\min}$ (scale parameter) is normalized to 1.[48]

[48] Note that the parameterization of my model is similar to that in Ghironi and Melitz (2005) and Bilbiie, Ghironi, and Melitz (2012). However, $\theta = 6$ is common in the international trade literature, consistent with estimates for the elasticity of substitution among domestic varieties as in Broda and Weinstein (2006). A value of $\theta = 6$ gives standard markups of 20 percent over marginal cost as in Rotemberg and Woodford (1992). This would then imply that κ must be adjusted to 5.6 to

The size of the exogenous exit shock for firms δ_M is set to 0.025, which by itself implies a 10 percent annual job and production destruction rate, which is consistent with the empirical evidence and Bernard et al. (2010). Note that my model also features endogenous exit, so while the annual destruction rate is 10 percent on average, the exit rate will fluctuate over the cycle. Note that Lee and Mukoyama (2012) found smaller exit rates (around 5-6 percent) using data on plants in the ASM, but a value of $\delta_M = 0.025$ is used for ease of comparison with the model in Bilbiie, Ghironi, and Melitz (2012).[49] As is conventional for standard real business cycle models, the depreciation rate of capital δ_K is set to 0.025 and α is set to 1/3.[50]

The entry cost b is normalized to 1 and f is set to 0.05.[51] Note that changing f changes the unit of measure for the number of firms; but average aggregate productivity in the steady state is invariant to changes in f as long as the ratio f/b remains constant.[52]

Productivity in the steady-state is set to $Z = 1$ and follows an $AR(1)$ process as in King and Rebelo (1999) with persistence parameter $\phi = 0.979$ and innovation standard deviation $\sigma_{\varepsilon_Z} = 0.0072$.[53] Note that this process in King and Rebelo (1999) was constructed to coincide with the Solow residual for a Cobb-Douglas production function, but this interpretation is no longer appropriate for the production structure in my model. As in Bilbiie, Ghironi, and Melitz (2012), I opt for the same parameter values for the exogenous productivity process as that in King and Rebelo (1999), since it makes comparisons between models more transparent and it places the tests of the model's ability to outperform the other models on the transmission mechanism (and not on the choice of parameters for the exogenous driving force).

For ease of reference, a summary of the model's parameters is provided in Table 1.

4 Results

The business cycle properties of the model are explored along three different avenues. First, the model's implications for patterns of entry and exit, employment, and productivity are compared with the empirical data in Lee and Mukoyama (2012). Second, impulse responses are computed for a shock to aggregate technology to further explore the qualitative patterns of selection and understand the responses of key macro aggregates. And, lastly, select second moments are computed and compared with those in the data,

generate a standard deviation of log US plant sales equal to 1.67. The model was also solved for this parameterization; and results are available upon request.

[49] Using data on manufacturing plants from the ASM portion of the LRD, Lee and Mukoyama (2012) find average exit rates of 5.5 percent. With this alternative calibration target, the value of δ^M would be 0.01375, and the model was also solved with this parameterization.

[50] Note that α is not exactly the capital income share, due to the presense of profits.

[51] In general, English letters are used for variables and Greek letters are used for parameters, but exceptions are b and f. The blueprint entry costs can be set to one without loss of generality, but setting $f = 0.05$ is a parameterization. The qualitative results, however, are not sensitive to this particular choice for f.

[52] The model was solved for variations in f/b, but the model yields qualitatively similar results for variations in this ratio. Increases in the ratio f/b increase the cutoff productivities (and therefore average aggregate productivities), and vice-versa. The number of firms is inversely proportional to fixed costs.

[53] Tauchen's (1986) method is used to approximate this process.

those of a standard RBC model, and those of the related model in Bilbiie, Ghironi, and Melitz (2012). Before exploring the model's results, however, a few comments are in order regarding the accuracy of the model's mapping functions.

4.1 Accuracy of Mapping Functions

Since my model is one in a class of models for which it is a priori unclear if a recursive competitive equilibrium exists, it is prudent to first comment on the accuracy of the mapping functions (and therefore the solution). Convergence of value functions and mapping functions is easily determined - when both converge within predefined tolerance levels, the model is considered to be solved. However, assessing the accuracy of the mapping functions is more difficult (and a known issue for these types of models) since the magnitudes of the errors are difficult to interpret.

For the households, the state variables Z_t, $M_{S,t}$, $\tilde{z}_{S,t}$, and K_t at the beginning of the period are used to predict the values of $M_{A,t}$, $\tilde{z}_{A,t}$, and $z^*_{N,t}$ that will obtain in the current period. Households make their decisions based on the actual values of $M_{A,t}$, $\tilde{z}_{A,t}$, and $z^*_{N,t}$ that obtain; the accuracy of the mapping functions, then, speaks to the ability of the households to predict the impacts of firm exit decisions (and implicitly their own choices) on the state of the world immediately following the productivity shock at the beginning of the period. That is, the household value functions in Equation 7 are based on beginning-of-period values of Z_t, $M_{S,t}$, $\tilde{z}_{S,t}$, and K_t, so the accuracy of optimization by the households in Equation 8 and the optimality of their policy functions are then ultimately tied to how well households can predict $M_{A,t}$, $\tilde{z}_{A,t}$, and $z^*_{N,t}$ from the state variables Z_t, $M_{S,t}$, $\tilde{z}_{S,t}$, and K_t. Likewise, for firms, actual values of Z_t, $M_{A,t}$, $\tilde{z}_{A,t}$, K_t, and Q_t that obtain in the period are used when evaluating firm profits and value functions from Equation 13; but the ability to predict these values from the state variables Z_t, $M_{A,t-1}$, $\tilde{z}_{A,t-1}$, and K_{t-1} determines the accuracy of firm value functions in Equation 12 and the ultimate optimality of firm entry and exit decisions.

To evaluate the accuracy of household and firm mapping functions, two measures of fit - the R^2's and the standard deviations (in percent) of the regression errors, $\hat{\sigma}'$s - are evaluated over a simulated sample of $10,000$ observations. The accuracy of the mapping functions in my model are comparable with those in the model in Krusell and Smith (1995; 1998). The functions and measures of fit are given in Tables 2 and 3 for the firms and households, respectively.

For both the firms and the households, the measures of fit are quite good - with R^2's being all above 0.999 and the largest $\hat{\sigma}$ being just above 0.001 percent - so these functions, therefore, are very accurate. Note that the households actually know something more from $\boldsymbol{F}_{A,t} = f\left(\boldsymbol{F}_{S,t}, z^*_t\right)$: the value of $M_{A,t}$ can never be greater than the value of $M_{S,t}$, and likewise the value of $\tilde{z}_{A,t}$ can never be smaller than $\tilde{z}_{S,t}$. This knowledge is further incorporated into mapping functions, so that household mappings for $M_{A,t}$ are given as the minimum of the predicted value for $M_{A,t}$ or $M_{S,t}$, and mappings for $\tilde{z}_{A,t}$ are determined as the maximum of either the predicted value for $\tilde{z}_{A,t}$ or $\tilde{z}_{S,t}$.[54]

In Krusell and Smith (1998) and much of the subsequent literature that has implemented its methodology, the R^2's and $\hat{\sigma}$'s are standard checks on the relative merits of the mapping functions; and the mapping functions in my model are quite accurate according to these criteria. In addition, however, further accuracy tests are explored, since, as den Haan (2010) has argued, in some cases the R^2 and regression standard errors $\hat{\sigma}$'s may constitute rather weak accuracy tests. Therefore, deviations of actual versus predicted values are calculated as additional checks on the accuracy of the mapping functions. Further accuracy tests proposed by Krusell and Smith (1995; 1998) for the maximum x-quarter ahead forecast error and the den Haan (2010) measure are also calculated. The robustness of the mapping functions are also checked by evaluating the same accuracy measures for the model solved on denser grids, with longer simulation periods, and with multiple simultaneously simulated economies. The results from these additional accuracy tests are available in the appendix and confirm that the mapping functions are, in fact, very accurate.

4.2 Entry, Exit and Employment Dynamics

The model's implications for entry, exit, and employment dynamics are compared with the patterns in the empirical data documented in Lee and Mukoyama (2012). As a preview, my model replicates many of these dynamics in the data, both qualitatively and quantitatively: entry is procyclical, but exit is roughly flat over the cycle, and plants that enter during recessions are, on average, more productive and larger (in terms of employment) than plants entering during booms. The plant-level data in Lee and Mukoyama (2012) comes from the ASM portion of the LRD constructed by the US Census Bureau (from 1972 to 1997), and this data is only available at annual frequencies. In what follows, therefore, I report the annual implications of my (quarterly) model to facilitate comparisons with the patterns in the empirical data calculated by Lee and Mukoyama (2012). (For the results in this section, when lengthy expositions of variable constructions are warranted, they are detailed Section A.6 in the appendix.) Note that the baseline calibration in my model features a target for the exit rate in line with other RBC models that is higher than that found by Lee and Mukoyama (2012) for manufacturing plants in the ASM, but the model was also solved with a parameterization that induced a lower average exit rate. The two different values of δ^M produce quantitatively different results, but the patterns are qualitatively similar. When appropriate, results from both parameterizations are reported.

In the data, Lee and Mukoyama (2012) find a strongly procyclical entry rate, but exit rates that are relatively flat over the cycle. When analyzing the empirical data, Lee and Mukoyama (2012) focus on growth rates, calling year t a good year if the growth rate of output from year $(t-1)$ to t is above average; and a bad year if the growth rate is below average. According to this definition, entry rates are higher in booms than they are in recessions, with (annual) entry rates, measured as the number of entering establishments

[54] Therefore, the accuracy of household mapping functions is actually better than the simple linear measures of fit would indicate. The simple R^2's and $\hat{\sigma}$'s are reported for ease of comparison with other models using the Krusell-Smith algorithm.

as a percentage of the total, of 8.1 percent on average in "good times" but only 3.4 percent in "bad times". Differences in exit rates, however, are much smaller, calculated at 5.8 and 5.1 percent in good times and bad times, respectively. Lee and Mukoyama (2012) also provide an alternate calculation of entry and exit rates, where booms and busts follow the NBER business cycle dates, and calculate entry rates of 6.3 percent in good times and 4.1 percent in bad times, with exit rates of 5.5 and 5.0 percent in good and bad times, respectively. Regardless of the specific definition, however, the qualitative patterns are clear. Lee and Mukoyama (2012) stress that the cyclical movements of entry and exit rates are more related to "changes" than "levels", so I analyze my model for consistency with the data according to this interpretation.

Booms and busts - or "good times" and "bad times" - are defined in the model analogously with those in Lee and Mukoyama (2012), demarcated by annual increases or decreases in yearly output Q_t of more or less than average (of course, there is zero average annual growth in my model). Annual entry is calculated for all new entrants producing for the first time in a given year, and annual exit is calculated as the total exit over the year from both endogenous exit choices and firm deaths resulting from the exogenous shocks. Entry and exit rates are then calculated as a percentage of incumbent firms that were producing in (and survived from) the previous year. For the model with δ^M set to target an annual average exit rate of 5.5 percent, the results are very close to those in the data: the model delivers annual entry rates of 6.1 in good times and 5.1 in bad times, with exit rates that average 5.6 across good and bad times, respectively. Recall, however, that the baseline parameterization sets δ^M to target a higher average exit rate, but the results also match the qualitative patterns of entry and exit rates over the cycle.[55] Summary results for entry and exit rates are reported in Table 8. Overall, therefore, the model replicates the qualitative features that entry is procyclical and exit rates are roughly flat over the cycle.[56]

My model also matches patterns of selection in entry and exit. Lee and Mukoyama (2012) find that the average productivities of entering plants (relative to incumbents) are between $10 - 20$ percent lower during booms than in recessions. Patterns of selection in my model are consistent with the data, but these

[55] The annualized levels of entry and exit with $\delta^M = 0.025$ are larger than those in Lee and Mukoyama (2012), but this value of δ^M is chosen as the baseline parameterization in order to make the comparisons between the model in this paper and that in Bilbiie, Ghironi, and Melitz (2012) transparent. Recall that the 10 percent annual production destruction rate is measured both as a share of products and as a market share, as in Bilbiie, Ghironi, and Melitz (2012). This is also consistent with the 8.8 percent minimum destruction rate (measured as a market share) in Bernard et al. (2010).

[56] Note that the qualitative features of procyclical entry and flat exit are robust to changes in the cutoffs demarcating good from bad times. For example, defining good (bad) times as occuring when the annual growth rate of output is 1 percent above (below) the average yields a more strongly procyclical entry rate but retains the feature of relatively flat exit. In this case, entry rates are 6.75 and 4.51 percent in good and bad times, respectively, but exit rates remain relatively flat at 5.68 and 5.55 percent; and similarly with a larger cutoff, entry (exit) rates are 7.31 (5.73) and 3.95 (5.53) in good and bad times, respectively. The ability of the model, therefore, to match empirical entry and exit patterns is robust to small changes in this threshold; and strengthening good and bad times to a definition perhaps closer to one of booms and busts only serves to make the entry rate even more strongly procyclical. Note, however, that the flat (but slightly procyclical) exit rate may be puzzling to some readers familiar with data indicating countercyclical exit rates. The differences in the results here are due in part to variable definitions, explained in detail in the appendix.

magnitudes should be interpreted with some caution since it is not exactly clear how the Solow residual should be defined under the production structure for the model in this paper.[57] For example, the product zZ_t gives the best concept of technical productivity for firms with idiosyncratic productivity component z, as it combines both their idiosyncratic productivity and aggregate technology as a measure of total factor productivity, but this is not what is measured in the data.[58] Instead, revenue-based productivity measures are used that actually represent real revenue per unit input since, in the data, it is not possible to measure TFP at the plant level without a proper measure of prices. Lee and Mukoyama (2012) cite Foster, Haltiwanger, and Syverson (2008), noting the problems with interpreting revenue-based productivity measures in the data, so their numbers of course should also be interpreted with caution. Since the concept of interest is the productivity of new entrants *relative* to that of continuing firms, this paper suggests two methods of defining a revenue-based relative productivity measure for new entrants relative to incumbents in the model designed to pin down this idea. The constructions are detailed in the appendix, but regardless of the specific defintion, the pattern of selection is clear and consistent with the data.[59] In the model, new entrants are on average estimated to be between $4 - 21$ percent less productive (relative to incumbents) during booms than during recessions.[60]

Features of employment are also replicated in the model. Lee and Mukoyama (2012) find that, in general, employment is skewed towards large plants, and that feature is closely replicated in my model. They find that the largest plants (those with 250 or more employees) account for only 6.7 percent of plants but comprise almost 56 percent of employment, while the smallest plants (those with between $1 - 19$ employees) account for approximately 46 percent of plants but represent only about 5 percent of the total employment share. In my model, the use of the Pareto form for the productivity distribution of new entrants induces an aggregate employment distribution that is quite similar.[61] In the model, the largest 6.7 percent of firms account for almost 52 percent of total employment in production and the smallest 46 percent of firms account for just under 9 percent.[62] My model, therefore, matches the distribution of employment quite well. Lee and Mukoyama (2012) also find that hiring and firing are concentrated in large plants (large plants account for 46 percent of hiring and 39 percent of firing, while the smallest account for only 8 and 14 percent, respectively),

[57] Bilbiie, Ghironi, and Melitz (2012) make a similar point.

[58] Note that technical productivity differences over the cycle display the same pattern: on average, new entrants are 0.01 percent less productive (relative to incumbents) during booms than during recessions.

[59] The lower end of this range is biased downward, as the derivations in the appendix show.

[60] This measures are similar ($6 - 28$ percent) when the model is parameterized with smaller average entry and exit rates.

[61] Recall that the shape parameter for the Pareto distribution is calibrated to match the standard deviation of the log of U.S. plant sales.

[62] Since firm-level employment in my model is not discrete, but productivity is directly tied to size, for purposes of comparison, employment shares are calculated for firms over the steady-state distribution of productivity draws. Lee and Mukoyama (2012) report data for plants over five size categories: for plants with $1 - 19$, $20 - 49$, $50 - 99$, $100 - 249$, and 250+ employees. However, because the correspondence between the size distribution in the data (based on employment) and that in the model (ultimately tied to firm idiosyncratic productivities) is not one-to-one, I do not torture the interpretation of size beyond what can be reasonably compared for the largest firms and the smallest.

and similar patterns are found in my model (large plants account for 55 percent of hiring and 53 percent of firing, while the smallest account for only 18 and 17 percent, respectively), where the standard deviation of employment among the largest firms is almost five times that of the smallest over the model simulations.[63] (These results are summarized in Table 9).

Patterns for the dynamics of employment in Lee and Mukoyama (2012) are also replicated in the model - key dynamics are replicated qualitatively, and some are closer quantitatively than others. Lee and Mukoyama (2012) find that the job creation rate from startups is much higher during booms, but the job destruction rate from shutdowns is only slightly higher. In the data, they find that job creation rates from startups are about 45 percent higher in good times than in bad times, while job destruction rates are only 11 percent higher. Both the qualitative and quantitative patterns in my model replicate these features in the data. In my model (for the parameterization that matches exit rates in the ASM) job creation from startups is 40 percent higher in good times than in bad, and the job destruction rate is approximately 7 percent higher.[64] Details are in Table 10. As well, in the data, the correlation between the job creation rate from startups and the percentage change in output (0.368) is much larger than that for the correlation between the job destruction rate and output growth (-0.006). This feature is qualitatively replicated in the model, where the correlation for the former is 0.81 and for the latter is 0.31, so the model replicates the idea that the entry margin may be of particular importance for adjustments over the business cycle.

Additional empirical details reveal the entering plants are larger during recessions, but the average size of exiting plants is only slightly larger during recessions (and, relative to continuing firms, firms exiting during recessions are actually smaller.) These qualitative patterns are also replicated in the model, but the quantitative magnitudes are not as close. In the data, entering plants start with about 30 percent more workers in recessions than they do in booms; whereas in the model this number is 7 percent. The average size of exiting plants is larger in bad times (2.87 percent in the data, and 0.20 percent in the model), but in relative terms, exiting plants are actually smaller (8 percent in the data, and 6 percent in the model).

Further, Bilbiie, Ghironi, and Melitz (2012) model the cross correlations between real GDP, profits, and net entry, and find that, in the data, both net entry and profits are strongly procyclical, and net entry is

[63] Note that there is not a direct correspondence between hiring and firing in the data (which is discrete, and calculated on a size distribution for firms based on employment numbers) and hiring and firing in the model. In the model, there is no distinction between hiring (firing) an additional worker and increasing (decreasing) the hours of current workers. For purposes of comparison, however, I calculate an annual measure equal to

$$\sum_{q=1}^{4}\left[L_{Q,t_q}+L_{F,t_q}\right]-\sum_{q=1}^{4}\left[L_{Q,t-1_q}+L_{F,t-1_q}\right],$$

and call this "hiring" when this term is positive and "firing" when negative. Analogous measures are calculated for the largest and smallest firms in the model, where the largest firms are those with idiosyncratic productivities in the upper 6.7 percent of the steady-state productivity distribution, and the smallest are those with productivities in the lowest 46 percent.

[64] In my model, job creation rates from startups are calculated as the average total labor employed by startups over the four quarters as a percentage of the average total quarterly employment each year. Job destruction rates are calculated analogously.

strongly correlated with profits.[65] Their baseline model was able to reproduce the "tent-shapes" in these patterns, but my model comes closer to matching the patterns in the cross correlations between real output and net entry and those between profits and net entry. Figure 6 shows the cross correlations between real output and profits in the data, for my model, and for the baseline model in Bilbiie, Ghironi, and Melitz (2012), and both models here replicate the "tent shape" of data and are quite close. Figures 7 and 8, however, reveal that for the cross correlations between real output and net entry and between profits and net entry, respectively, my model is much closer to the patterns in the data.[66] The patterns of selection in entry and exit (not just the addition of capital in production) help the model match these correlations. Recall that if fixed costs are zero, there would be no selection in entry and no endogenous exit (since all firms, regardless of productivity draws, would find it profitable to produce). If fixed costs are zero, the model would be equivalent to a variant of the Bilbiie, Ghironi, and Melitz (2012) model that included capital. Incorporating rich dynamics for firm heterogeneity over the cycle, then, also allows the model to come closer to matching the correlations of these aggregates in the data. Comparisons of cross-correlations for the model with $f = 0$ are in Figures 13 and 14, discussed further in Section 4.4.

In summary, my model replicates several key features of entry, exit, and employment over the business cycle. It replicates the qualitative and quantitative feature that entry is strongly procyclical. It also replicates the qualitative patterns of selection in entry - the average productivity of new entrants is lower and their size is smaller in booms than in recessions, though the model does not deliver the quantitative magnitudes on size as large as those in the data. (In Section 4.4 below, I show that the model can be adjusted to better match the quantitative magnitudes, as well as the qualitative features, of selection by incorporating counter-cyclical fixed costs.) My model also matches the empirical data relating to exit patterns: exit rates are relatively flat over the cycle, and exit rates are higher among smaller firms, but large plants also exit. My model also matches many features of employment (that employment is skewed towards large plants) and employment dynamics - that during booms, job creation rates from new firms are higher, but job destructions rates are only slightly higher; and entering plants are larger in recessions than booms, but exiting plants are of similar size across booms and recessions. Many of these patterns in the model support that idea that the "cleansing" effect from exit during recessions may not be as important as changes at the entry margin for

[65] In the working paper version of Bilbiie, Ghironi, and Melitz (2012), the cross-correlations for GDP, net entry, and profits are calculated on HP-filtered data (in logs) for a sample period from 1947-1998. For ease of comparison, the results in the figures reporting cross-correlations use data from the same sample period.

[66] Note: I use essentially the same data sources as Bilbiie, Ghironi and Melitz (2012) to construct net entry: "New Business Incorporations" are from the Basic Economics Database, Global Insight, Inc. (1947-1998), and "Business Failures" are from the NBER Macrohistory Database (1947-1965), The Economic Report of the President (1965-1983), and the Basic Economics Database, Global Insight, Inc. (1984-1998). However, I note that monthly series were used to construct quarterly values in all series except for business failures from 1965-1978, where only yearly estimates were available, so I have estimated these values. Data for real GDP and profits are from the FRED (series GDPC1 and A053RC1Q027SBEA). My computations for the cross-correlation functions are essentially identifical to those in the 2007 working paper version of Bilbiie, Ghironi and Melitz (2012)

adjustments over the business cycle.[67] Further, these features also induce results where the model comes much closer to matching the cross-correlations of aggregate series involving net entry and profits.

4.3 Impulse Responses: Entry and Exit Dynamics

The preceding discussion has explored how the model matches some key quantitative and qualitative features of entry and exit in the empirical data, and the impulse responses discussed below explain the intuition. Incorporating investment in new firms introduces several differences in the model's impulse response functions for important macro aggregates in comparison with the standard RBC model of Kydland and Prescott (1982), and incorporating capital and fixed costs introduces further differences from the baseline model in Bilbiie, Ghironi, and Melitz (2012).

As well, it is important to compare the properties of the model with the empirical evidence in terms of data-consistent definitions by focusing on real variables deflated by a data-consistent price index. However, it is also important to keep in mind that the dynamics of the model come from optimization with respect to welfare-consistent concepts. In the analysis that follows, impulse responses to aggregate technology shocks are computed both for welfare-consistent concepts and their empirically relevant counterparts, and when appropriate, impulse responses according to both definitions are included.[68]

Impulse responses to a 1 percent positive and negative innovation to aggregate technology Z are computed to explore the entry and exit dynamics of the model and their implications for important macro aggregates. In what follows, it is important to keep in mind that the impulse responses are computed from the steady state (importantly, the steady state distribution of firms and productivities), so the responses to positive and negative shocks are not always symmetric.[69] These differences are explained below, when appropriate.

Consider the impulse responses for the cutoff productivity thresholds z^* and z_N^* for the implications for the qualitative patterns of selection. Figure 9 displays the impulse responses for these cutoff productivities along with those for average aggregate productivities. Following a 1 percent positive innovation to aggregate technology Z, both productivity thresholds z^* and z_N^* decrease on impact and remain below their steady state values for the first four quarters. The reverse happens with a negative innovation. Note that, in Figure 9, the impulse responses for the economy-wide average aggregate productivity are not complete mirrors under positive and negative innovations to aggregate technology. With a positive innovation to aggregate technology, the average aggregate productivity does not respond immediately to the shock, since, starting from the steady state distribution, there would be no endogenous exit in response to initial values of

[67] Note that the model does have an endogenous exit component, however, so unlike the model in Bilbiie, Ghironi, and Melitz (2012), there is still a role for the "cleansing effect".

[68] As pointed out with the related models in Ghironi and Melitz (2005) and Bilbiie, Ghironi and Melitz (2012), since the construction of CPI data does not adjust for the availability of new varieties at the frequency with which the price index adjusts in the model, the data-consistent counterpart of the price index is closer to $p_t = p_t\left(\tilde{z}_{A,t}\right)$ than P_t.

[69] And, of course, impulse responses computed from states other than the steady state will differ. Because, however, there would be an inherent regression back to the steady state from impulse responses computed in this way, these variants of analysis would be less instructive.

z_t^* below those of existing firms. In fact, the average aggregate productivity only falls marginally for a few quarters after the shock before it rises since new entrants are small in comparison with the existing stock of firms.[70] With a negative shock, however, the entry cutoff productivity initially rises so that firms with low productivities endogenously exit in response. The average aggregate productivity then also initially rises and features more of a delayed response before it falls.[71] Regardless, for a full year following a positive shock to technology, the average productivities of entering plants are lower than average; and for a full year following a negative shock, average productivities are higher.[72] This qualitatively matches the empirical evidence in Lee and Mukoyama (2012), and intuitively, the idea is that it might be easier for firms with lower productivities to enter during booms than during recessions.

Why do the threshold productivities initially react this way? To explore this further, I explore a brief digression to consider the differences between my model and a version that excludes capital from production.[73] Consider a negative shock to productivity.[74] There are several interrelated channels through which the addition of capital affects the impulse responses of the threshold productivities. In the models both with and without capital in production, the negative shock decreases current as well as expectations of future profits, since demand for output goods decreases. (In the model with capital, this comprises both consumption and capital investment goods, but in the model without capital, output goods are just consumption goods.) The demand for consumption goods drops on impact in both models, and though the demand for investment goods drops on impact in the model with capital in production, its immediate response is small - and hours worked to create investment goods actually increases on impact. Since, according to intertemporal substitution logic, it is a particularly bad time to devote resources to creating investment goods, household shift away from creating both blueprints and capital investment goods, but the drop in the creation of new firms is much more drastic and immediate. Since capital does not react on impact, and the immediate changes in investment are small, this mutes the eventual effects of a decrease in capital on firm profits in the near term, and the cutoff productivity z^* initially increases (as does z_N^*). (Ceteris paribus, a decrease in the capital stock increases firm profits, so this effect is small in the near-term.)

[70] Note that the impulse response here is for $\tilde{z}_{A,t}$, and not $\tilde{z}_{A,t}Z_t$, with the latter representing the combined effects of aggregate technology and the average of the firm idiosyncratic productivities. Further, data-consistent measures of productivity (from revenue-based TFP measures) still yield procyclical productivities, which is consistent with Rotemberg and Woodford (1999).

[71] These responses, of course, would look more symmetric if taken and averaged over different distributional constructions as they occur in the simulations, but since there is an inherent regression back to the steady-state, it is not particularly helpful.

[72] The average productivity of new entrants is, of course, $\tilde{z}\left(z_N^*\right)$.

[73] A version of my model that excludes capital from production is similar to the baseline model in Bilbiie, Ghironi, and Melitz (2012), but retains fixed costs and firm heterogeneity. By incorporating firm heterogeneity into the model, several key variables - including the number and average aggregate productivity of firms - can react to a shock to aggregate technology on impact, if the threshold productivity cutoff z^* also reacts. In the baseline model in Bilbiie, Ghironi, and Melitz (2012), both of these variables are predetermined. That is, the the Bilbiie, Ghironi, and Melitz (2012) model, the number of firms flexible in the long run, but not flexible within each period. The feature that the number of firms within a period is pre-determined (with flexibility only across periods) is shared by the models in Cook (2001) and Ambler and Cardia (1998).

[74] Contrary to standard practice, it is easier to see the effects from a negative shock than a positive one, since, with a negative shock, we can see the effects of endogenous exit when computing impulse responses from the steady state distribution.

The effects of continued and more pronounced drops in output over the near-term for the model that includes capital decreases profits further. Changes in the capital stock itself, then, drive changes in firm profits as well as cause a more delayed response for the drop in output (the difference being caused by the drop in capital investment). As well, firm profits result from both the markup over marginal costs and the ability to cover fixed costs. In the model with only labor in production, the impulse response for the wage rate is the same as that for aggregate technology, so payments to cover fixed costs remain the same; in the model with capital, the impulse response function for the wage rate drops more than proportionately against the drop in aggregate technology, and takes on a hump shape that more closely resembles that for output.[75] All together, these effects mean that, in the model with capital, in the near term following a negative shock, the decreased profits and expectations of further decreased near-term profits translate into increases in the cutoff productivity thresholds for about a year, making it initially more difficult for low productivity firms to enter/survive. What this also implies, relating to Section 4.2 above, is that it is necessary to include capital in the model for firm dynamics to move in a way that matches the data. For example, Lee and Mukoyama (2012) found that plants entering during recessions were more productive than those entering during booms, and the results from the model in Section 4.2 and the intuition from the impulse responses here match those patterns. The impulse responses for threshold productivities in the model without capital "go the wrong way", and, counter to the data, in that model new firms entering during booms are 3.25 percent more productive than those entering during recessions.

The effects of the shock "turn" after a year - while it is easier for firms with lower productivities to initially enter following a positive innovation to productivity, once these effects start working their way through the economy, it becomes progressively harder. Why does this happen? Consider the impulse responses under a positive innovation to Z in Figure 10. A positive innovation to aggregate technology increases aggregate output Q on impact and it remains above its steady state value until the effects of the shock die out, which, ceteris paribus, increases per-period profits for firms across all levels of idiosyncratic productivities (as does the effect of the technology shock itself). Also important for firm profits, however, are the other major macro aggregates - the capital stock and the number of firms; and an increase in either, ceteris paribus, serves to lower individual firm profits. The capital stock and number of firms cannot jump on impact; and they both take longer than output to initially reach their peaks. (As well, the average aggregate productivity does not respond on impact and falls just below its steady state value for only a few quarters, which works in the same direction.) Therefore, since the shock to aggregate technology makes output jump on impact while these other macro aggregates respond more slowly, it is *initially* easier for lower productivity firms to enter and the cutoff productivity level falls for the first few quarters.[76]

[75] The procyclical wage rate is, of course, consistent with Cooley and Prescott (1995).

[76] With a negative innovation to aggregate technology, the results are closely analogous. (See Figure 11.) Output drops on impact, which, ceteris paribus, decreases firm profits. Average aggregate productivity also increases on impact, which also serves to decrease firm profits. Again, the capital and firm stock are slower to respond, but the decreases in both of these macro aggregates, ceteris paribus, eventually increase firm profits. Note that the number of firms drops on impact with a negative

4.4 The Model with Countercyclical Fixed Costs

The model performs well in replicating many features of entry, exit and employment in the data; and introducing a countercyclical component to fixed costs allows the model to replicate further quantitative, as well as the qualitative, details of the patterns of selection in entry. The intuition that fixed costs of production might be higher, more difficult to finance, etc. during recessions (and vice versa) is reasonable, though the process itself is not modelled here explicitly. Experiments with countercyclical fixed costs, where fixed costs were specified as a constant component f plus a countercyclical component related to the deviation of aggregate productivity from its mean or steady-state productivity $\left(\bar{Z} - Z_t\right)$ equal to $f + f^p \left(\bar{Z} - Z_t\right)$, better matched size differentials of new entrants during good times versus bad times. E.g., for a value of $f^p = 0.20$, the model generated new entrants that were another 2 percentage points smaller on average in good times than in bad times.[77]

Why does this work? Consider a negative innovation to aggregate technology and the impulse responses in Figure 12. Now, the countercyclical nature of the fixed costs makes the threshold cutoff productivity higher and implies that profits in the near term are expected to be below average for a much longer period than before. As with the baseline model, a negative innovation to aggregate technology decreases aggregate output Q on impact and it remains below its steady state value until the effects of the shock die out, but eventually, the decreases in the capital stock and the number of firms (which take longer to work through the economy) serve to increase individual firm profits. In the baseline model, this means that the cutoff productivity threshold does not remain above its steady state value for long since decreases in the latter (which ceteris paribus increase firm profits) eventually overwhelm decreases in the former (which decrease firm profits, along with the effect of the technology shock itself).

The key difference from the baseline model here is the feedback between the average aggregate productivity and the cutoff productivity thresholds, and, with countercyclical fixed costs, both remain above their steady state values for the duration of the transition. The cutoff productivity threshold for incumbents jumps on impact and remains above its steady state value throughout the transition, as does, therefore, the average aggregate productivity of firms. (This is in contrast to the baseline model, where average aggregate productivity initially increases, but eventually decreases, falling below its steady state value for a majority of the transition period.) Since increases in the average aggregate productivity and increases to fixed costs both work in the same direction - to decrease firm profits - while the behavior of other macro aggregates remain essentially the same, threshold productivity cutoffs increase compared to the baseline model. Here, the selection effect is now larger and lasts longer - the average productivity of new entrants is higher during the entire transition period in response to a negative innovation to aggregate technology, whereas in the baseline model, it is only lower during the first four quarters immediately following the shock. Macro aggre-

shock, but the effects are relatively small and dominated by the changes in output and the average aggregate productivity. The cutoff productivity thresholds, therefore, initially increase before falling below their steady state values after a year.

[77] The larger f^p, the larger the differentials for the $S2$ measure of productivity, as detailed in the appendix.

gates, other than the average aggregate productivity, otherwise respond similarly as in the baseline model, as primarily only the character of the distribution of firm productivities is affected.

It is also interesting to note that with countercyclical fixed costs, average firm dividends are mostly above their steady state values for the duration of the transition in response to both a positive and a negative innovation (larger, of course, in response to a positive shock). However, since the number of firms decreases in response to a negative shock, aggregate dividends are still lower; so the model also continues to retain the aggregate feature of procyclical profits. Countercyclical fixed costs also improve the model's ability to match cross-correlations for real GDP, profits, and net entry - see Figures 13 and 14. In sum, countercyclical fixed costs both improve the ability of the model to match the quantitative, as well as qualitative, patterns of selection in firm entry and exit and improve the properties of the model with regards to its ability to match the cross-correlations in the data for real GDP, profits, and net entry.[78]

4.5 Second Moments and Aggregate Dynamics

The model also shows some improvements over existing models in its ability to match select moments for some key macroeconomic variables. To compare the performance of the model with the data, empirical moments for US data as reported in King and Rebelo (1999) are compared with their empirical counterparts in my model. The results are summarized in Table 11. The empirical moments for the data are listed first, followed by those of my model. For ease of comparison with existing models, the moments generated from King and Rebelo's baseline RBC model and those generated from a version of the baseline CES model in Bilbiie, Ghironi, and Melitz (2012) are also reported. (Aggregate implications from the model in Bilbiie, Ghironi, and Melitz (2012) can be constructed as a special case of my model where labor is the only factor used in production and fixed costs f are set to zero. For this section, I will refer to this as the BGM (2012) model.) Model-implied second moments are computed for HP-filtered variables for consistency with the data and RBC practice. For the models, output Q is constructed as production of final goods, given by the sum of consumption C and (capital) investments I; and these series are converted to their real counterparts by deflating by the price index $p(\tilde{z}_A)$. Hours is constructed from the labor used to create production goods, which corresponds to $L_P + L_F$ in the model, and hours used to create new firms L_B. Note that there are some departures in the constructions of macro aggregates from those in Bilbiie, Ghironi, and Melitz (2012). What is characterized as investment in the baseline Bilbiie, Ghironi, and Melitz (2012) model (restricted to investment in new firms - i.e., investment at the extensive margin) is not comparable to the data on investment or with standard RBC models (which features investment in the capital stock - i.e., investment at the intensive margin). Blueprints are a form of "intangible capital" that do not show up in the data, new firms are not "investment", and the value of new firms is not added in to GDP, since such changes in mutual fund valuations would also not be included in the data on real output.

The model performs well with respect to the data, making several improvements over the baseline RBC

[78] Empirical justification for the sources of countercyclical fixed costs, therefore, is worthy of future research.

and BGM (2012) models. Results are reported for the model under the two different parameterizations for δ^M. It is clearer to make comparisons between the performance of my model that that of the BGM (2012) model using the same parameterization for δ^M ($\delta^M = 0.025$, which implies an average annual exit rate of 10 percent). However, results are also reported for the smaller value of δ^M chosen to match the average exit rates of plants in the ASM found in Lee and Mukoyama (2012). Interestingly, the model-implied second moments under the smaller value for δ^M are actually closer to the data.

RBC models typically feature consumption that is too smooth relative to output, macro aggregates that do not have enough endogenous persistence, and variables that are too procyclical relative to the data. My model improves on several of these dimensions.[79] First, in the data, the relative standard deviation of consumption to output is 0.74, and in my model it is 0.67 (and equal to 0.74 with the small δ^M). These numbers are quite close, and much closer than that implied by the standard RBC model (0.44).[80]

My model also delivers more persistence. The persistence of output in the model is also much closer to that in the data - the first-order autocorrelation of output in the data is 0.84 and in my model it is 0.87 (and equal to 0.82 with the small δ^M). My model matches this feature much better than either the standard RBC model or the model in BGM (2012); the persistence of output is lower in these models, with autocorrelations of 0.72 and 0.74, respectively. This behavior can be understood by comparing the impulse response functions for my model against those from the standard RBC model and the BGM (2012) model. (Refer again to Figure 10.) In the data, the cyclical component of output displays "hump-shaped" dynamics in response to a transitory shock - see, e.g., Cogley and Nason (1995), and in my model this hump-shape is also present and pronounced.[81] In fact, the hump-shape response of output in my model is remarkably close to that in the data. The impulse responses function for output in the baseline RBC model, as well as the BGM (2012) model, display dynamics that are essentially inherited from the shock itself - jumping on impact, and then gradually declining back towards the steady-state.[82]

To explain this, it is instructive to look at the aggregate behavior of labor in the model. Equilibrium in the labor market requires that sectoral demands for labor - that used to produce consumption goods, (capital) investment goods, and that used to pay fixed costs of production for existing firms and the blueprint (sunk) costs of creating new ones - sum to aggregate labor supply. Aggregate accounting also requires that income received from rental income, wage income, and firm profits and capital gains be divided among consumption goods, investment in capital goods, and investment in shares of firms and the creation of new ones. In my

[79] Recall that, as discussed in the calibration section, the exogenous process for aggregate technology in the model does not coincide with that measured in the construction of the Solow residual for standard RBC models. It is, therefore, more useful to compare relative (as opposed to absolute) standard deviations.

[80] Note that, for the BGM (2012) model, consumption and output are the same series, as there is no capital investment.

[81] Explained, in part, by the gradual increase in the number of firms.

[82] Note that, in the baseline BGM (2012) model, the hump-shape response is found for the welfare-relevant measure of output Q, but not for the data consistent counterpart of real output Q/p. Including capital in production (and therefore a role for an investment component of GDP) is necessary to deliver the hump-shaped response of output. Further including firm heterogeneity (and therefore a role for selection in entry and exit) makes this hump-shape slightly more pronounced.

model, the relative prices of the investment goods are endogenous, and the allocation of labor across the sectors of production, investment, and firm creation reflects these prices. This kind of sectoral division of labor, then, is of key interest when tracing out the impulse responses. Consider a positive shock to aggregate technology. The impulse responses for hours (data-consistent), hours producing consumption goods, hours producing investment goods, and hours required for blueprints are in Figure 15. According to intertemporal substitution logic, labor is allocated away from consumption and into the two investment sectors. Further, it is re-allocated more towards the creation of new firms on impact and in the periods immediately following the shock, and more towards capital investment goods in later quarters. The immediate return to investing in the creation of new firms is high since profits increase immediately, and becomes higher faster than the return to investing in capital. Consumption does increase, of course, in response to the higher wage. Real output (consumption plus capital investment) increases and therefore displays a delayed peak.

The model also features variables that are less procyclical than those of the standard RBC model. The contemporaneous correlations of consumption and investment with output are, respectively, 0.88 and 0.80 in the data; and in my model, they are 0.70 and 0.87 (and equal to 0.84 and 0.86 with the small δ^M). In the standard RBC model, the correlations are too high, at 0.94 and 0.99, respectively.

With regards to labor, however, the model seems to feature some anomalies. The relative standard deviation of labor is too large and hours do not display enough persistence and perhaps this is a defect of the model. However, when considering more carefully how labor in the model matches up with hours worked in the data, there may be another interpretation of these results. In the data, we know that there is underreporting in hours worked. Where might this show up in the model? In the model, unreported hours might correlate with those used to make blueprints for firms that were created, but never actually produced. If it is reasonable to suppose that not all of the hours worked in creating such "false starts" are reported, hours worked in the model - that correlate more closely with what is in the data - might better be approximated by $L_{P,t} + L_{F,t} + \left(1 - G\left(z^*_{N,t}\right)\right)\left(1 - \delta^K\right) L_{B,t}$. Making this assumption leads to some interesting results: volatility of hours is reduced and persistence of hours is increased. Using a measure for hours worked of $L_{P,t}+L_{F,t}+L_{B,t}$ gives a relative standard deviations of hours equal to 2.31, but the measure of $L_{P,t} + L_{F,t} + \left(1 - G\left(z^*_{N,t}\right)\right)\left(1 - \delta^K\right) L_{B,t}$ gives 0.56. (In the data, the relative standard deviations is 0.99). The first-order autocorrelation of hours is equal to 0.88 in the data, and equal to 0.70 and 0.90, respectively, under these two measures in the model. Calculating hours worked with a measure that allows some hours to go unreported - an interpretation where hours spent coming up with blueprints and ideas for new firms may not always be reported, particularly when those ventures never come to fruition - allows the model to come much closer to the data here. What at first glance looks appears to be excessive volatility in hours, therefore, may be simply an issue of interpretation.

Additionally, labor used in the consumption and capital investment sectors of the economy are positively correlated. The standard RBC model delivers a negative correlation here, but in my model, the correlation between labor in used in the production of consumption goods and that for investment goods is 0.07, which is more consistent with evidence on sectoral comovement. The correlation between output and hours worked producing consumption and investment goods are -0.68 and 0.62, respectively; which is much better than standard RBC models (with respective correlations of -0.93 and 0.96) in matching the data (with correlations of 0.72 and 0.86). The fact that employment across different sectors of the economy moves together over the cycle has been a difficult issue to address within standard modelling frameworks - see Christiano and Fitzgerald (1998) for a discussion. Several other RBC models can match these facts, but generally have to use a two-sector model of the business cycle to do so.[83]

In summary, incorporating investment in new firms introduces several differences in the model's impulse response functions for important macro aggregates in comparison with the standard RBC model of Kydland and Prescott (1982), and incorporating capital and fixed costs introduces further differences from the related model in BGM (2012). In addition to matching the patterns of firm entry, exit, and employment, my model also better accounts for many of the stylized facts on real business cycles. All of the features in my model are necessary to do so. To explore this, variants of the model were solved turning off features one-at-a-time. Without fixed costs (and a role for patterns of selection and firm heterogeneity), the model is incapable of addressing the micro-level data on firm entry, exit, and employment dynamics. Further, without fixed costs, the cross-correlations for output, entry and profits do not match the data. A model that includes fixed costs and firm heterogeneity but excludes capital cannot address many of the stylized RBC facts, since there is no model counterpart of the data on investment and consumption and output are essentially the same series. As well, excluding capital from production also leads to model results that get many of the micro-level patterns of firm entry, exit and employment dynamics wrong - including the fact that in this model, the selection effects go "the wrong way".[84]

5 Conclusions

In summary, this paper builds a real business cycle model that not only matches empirical patterns of entry, exit, and employment dynamics over the cycle, but also retains most of the successes of a standard RBC model and improves on several of its dynamics and second moment properties. To do so, it is necessary to model both investment at the intensive margin (capital investments) and the extensive margin (new firms) and to include heterogeneous firms with endogenous entry and exit decisions. In building the DSGE model and to incorporate dynamics for firm entry, exit, and employment in line with the data, this paper also built an algorithm to incorporate and track – exactly, and with a finite-dimensional object – a potentially infinite-

[83] For example, in Boldrin, Christiano, and Fisher (2001), one sector produces consumption goods and the other sector produces investment goods, and labor and capital cannot be instantaneously reallocated across sectors. Their model comes very close to matching the data with respect to matching sectoral comovements in labor, but note that, in addition, their model also relies on habit preferences. Note that Long and Plosser (1983) is the seminal paper here, and that the literature on multisector models, in general, relies on either a costly reallocation of labor of capital across sectors to break the negative correlation found in standard RBC models.

[84] Further details of these variants are available upon request.

dimensional distribution of firms and firm productivities in a Melitz-style model. With a simple structure and an inventive algorithm, the model replicates, both qualitatively and quantitatively, key features of entry and exit in the data: entry is procyclical, but exit is roughly flat over the cycle, and plants that enter during recessions are, on average, more productive and larger (in terms of employment) than plants entering during booms. And at a more aggregate level, the model delivers consumption volatility (relative to output) that is much closer to the data, output that displays persistence in line with the data and features "hump-shaped" responses to transitory shocks, and cross-correlations of output, net entry, and profits that are all very close to those found in the data.

The solution method is complex, but intuitive; and the model itself it simple and understandable. Because of its successes in this paper in replicating both micro-level features of firm selection patterns in entry and exit as well as the stylized facts relating to the behavior of key macroeconomic aggregates over the business cycle, the framework of the model may be useful in further research. For example, Obstfeld and Rogoff (2001) have suggested that trade costs and the potentially endogenous nature of tradedness might be the key frictions to understanding several puzzles in international macroeconomics. A new wave of international real business cycle models has incorporated endogenous entry and trade patterns featuring firm heterogeneity and costly trade, but in general have not been able to resolve the anomalies of earlier models. These models have generated limited entry and exit, so they fail to capture the rich dynamics of firms that enter and exit the export markets frequently (though the data suggests that there are a sizable number of firms that enter and exit export markets at business cycle frequencies). Explicitly modeling these dynamics might be expected to affect movements in the set of products and prices that are traded across countries, which can thereby increase the role of international trade as a mechanism for the propagation and transmission of business cycles. The framework of the model in this paper (by extending concepts of selection in entry and exit to those for selection in exporting), therefore, would be capable of doing this.

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Tables

Table 1: Model Parameters

Parameter	Value	Short Description
α	1/3	Approximate Capital Income Share
β	0.99	Discount Rate (Quarterly)
δ_K	0.025	Capital Depreciation Rate
δ_M	0.025	Firm Destruction Rate
κ	3.4	Shape, Pareto Distribution
θ	3.8	CES Aggregation
φ	4.0	Elasticity of Labor Supply
χ	1.0	Weight on Disutility of Labor
b	1.0	Blueprint Costs
f	0.05	Fixed Costs
$z_{\min}$	1.0	Scale, Pareto Distribution

Refer to Section 3 for details of the calibration. An alternative parameterization used $\delta^M = 0.01375$.

Table 2: Household (H) and Firm (F) Mapping Functions

variable	$\mathbf{b}_0$	$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$	$\mathbf{b}_4$	$\mathbf{R}^2$	$\hat{\boldsymbol{\sigma}}$ (%)
$M_{A,t}$	0.3277	0.2050	0.8987	−0.0833	−0.0353	0.998812	0.00120
$\tilde{z}_{A,t}$	0.0086	−0.0015	0.0026	0.9928	−0.0003	0.999098	0.00005
K_t	−0.1633	0.0046	0.0937	0.3098	0.9534	0.999999	0.00008
Q_t	−1.4974	0.8310	0.9012	−2.7716	0.1522	0.999656	0.00124

Firm mapping functions for all variables x_t are given as $\log(x_t) = b_0 + b_1 \log Z_t + b_2 \log M_{A,t-1} + b_3 \log \tilde{z}_{A,t-1} + b_4 \log K_{t-1}$.

Table 3: Household (H) and Firm (F) Mapping Functions

variable	$\mathbf{b}_0$	$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$	$\mathbf{b}_4$	$\mathbf{R}^2$	$\hat{\boldsymbol{\sigma}}$ (%)
$M_{A,t}$	0.0682	0.0059	0.9912	-0.0683	0.0030	0.999977	0.00017
$\tilde{z}_{A,t}$	-0.0201	-0.0017	0.0026	-1.0201	-0.0009	0.999114	0.00005
$z^*_{N,t}$	0.2797	-0.0164	0.0243	-0.1357	$+0.0192$	0.999812	0.00003

Household mapping functions for all variables x_t are given as $\log(x_t) = b_0 + b_1 \log Z_t + b_2 \log M_{S,t} + b_3 \log \tilde{z}_{S,t} + b_4 \log K_t$.

Table 4: Household (H) and Firm (F) Mapping Functions

variable		**corr** $(\hat{x}, x)$	**ave** % **error**	**max** % **error**
$M_{A,t}$	(H)	0.999989	0.0074	0.1869
$\tilde{z}_{A,t}$	(H)	0.999581	0.0021	0.0517
$z^*_{N,t}$	(H)	0.999906	0.0021	0.0065
$M_{A,t}$	(F)	0.999406	0.0744	0.6623
$\tilde{z}_{A,t}$	(F)	0.999549	0.0025	0.0524
K_t	(F)	0.999999	0.0059	0.0451
Q_t	(F)	0.999828	0.0755	0.6410

Actual x and predicted $\hat{x}$ series for households (H) and firms (F) are constructed over a simulated sample of $10,000$ observations. Errors are constructed as the (absolute) deviation of predicted from actual values, as a percent of value.

Table 5: Household (H) and Firm (F) Forecasts, x-quarters Ahead

		1 **quarter ahead**		10 **quarters ahead**		100 **quarters ahead**	
variable		**corr** $(\hat{x}, x)$	**error**	**corr** $(\hat{x}, x)$	**error**	**corr** $(\hat{x}, x)$	**error**
$M_{A,t}$	(H)	0.999976	0.2137	0.999846	0.4557	0.999591	0.5918
$\tilde{z}_{A,t}$	(H)	0.999030	0.0606	0.992913	0.1360	0.978277	0.1746
$z^*_{N,t}$	(H)	0.999904	0.0077	0.999885	0.0099	0.999842	0.0100
$M_{A,t}$	(F)	0.998942	0.7736	0.997556	0.9471	0.997283	1.1630
$\tilde{z}_{A,t}$	(F)	0.998991	0.0639	0.992033	0.1520	0.976598	0.1879
K_t	(F)	0.999995	0.1235	0.999702	0.6345	0.998888	1.0557
Q_t	(F)	0.999684	0.7887	0.999125	1.1085	0.999051	1.0700

Quarter-Ahead Forecasts are constructed for households (H) and firms (F), where the forecasts $\hat{x}$ assume the sequence of future shocks Z_t is known with certainty. Errors refer to the max % error, constructed as the maximum (absolute) deviation of predicted from actual values, as a percent of its value.

Table 6: Approximating Series (Den Haan (2010)) v. Simulated Values

variable		**corr** $(\hat{x}, x)$	**ave error**	**max error**
$M_{A,t}$	(H)	0.999989	0.001029	0.003858
$\tilde{z}_{A,t}$	(H)	0.999581	0.000342	0.001220
$z^*_{N,t}$	(H)	0.999906	0.000029	0.000091
K_t	(H)	0.999999	0.000035	0.000225
Q_t	(H)	0.999999	0.000045	0.000504
$M_{A,t}$	(F)	0.997442	0.002154	0.012966
$\tilde{z}_{A,t}$	(F)	0.984215	0.000281	0.001490
$z^*_{N,t}$	(F)	0.997276	0.000098	0.000397
K_t	(F)	0.999029	0.002801	0.001019
Q_t	(F)	0.999152	0.024350	0.011258

Approximating series $\hat{x}$ that results from household (H) and firm (F) forecasts are generated according to an adaptation of the den Haan (2010) procedure (detailed in the Technical Appendix in Section A.5.3) and compared against model simulated values x as an additional accuracy check for the forecasting functions. Error terms are constructed as $\log(x_t) - \log(\hat{x}_t)$, as in den Haan (2010), and average and maximum errors refer to $|\log(x_t) - \log(\hat{x}_t)|$.

Table 7: Household (H) and Firm (F) Approximating Series (Den Haan (2010))

variable	**corr** $(\hat{x}^H, \hat{x}^F)$	**ave error**	**max error**
$M_{A,t}$	0.997571	0.002156	0.012191
$\tilde{z}_{A,t}$	0.992581	0.000299	0.001048
$z^*_{N,t}$	0.998633	0.000090	0.000380
K_t	0.999042	0.002798	0.010264
Q_t	0.999168	0.002438	0.011273

Approximating series from household and firm forecasts, $\hat{x}^H$ and $\hat{x}^F$ respectively, are generated according to an adaptation of the den Haan (2010) procedure (detailed in the Technical Appendix in Section A.5.3). Error terms are constructed as $\log\left(\hat{x}_t^H\right) - \log\left(\hat{x}_t^F\right)$, similar to den Haan (2010), and average and maximum errors refer to $\left|\log\left(\hat{x}_t^H\right) - \log\left(\hat{x}_t^F\right)\right|$.

Table 8: Entry and Exit Dynamics

	Data (Growth Rates)		**Data (NBER)**	
	Entry Rate	**Exit Rate**	**Entry Rate**	**Exit Rate**
Good Times	8.1	5.8	6.3	5.5
Bad Times	3.4	5.1	4.1	5.0
	Model $\delta^M = 0.025$		**Model** $\delta^M = 0.01375$	
	Entry Rate	**Exit Rate**	**Entry Rate**	**Exit Rate**
Good Times	11.0	10.3	6.1	5.6
Bad Times	9.6	10.2	5.1	5.6

Data are from Lee and Mukoyama (2012). Alternative calculations are provided for entry and exit rates in the data according to demarcations for good and bad times being either based on growth rates of output being above or below average or according to NBER recession dates. In the model, "Good Times" ("Bad Times") occur when annual output increases (decreases) by more than average (i.e., zero). In the model, entry refers to all new firms producing for the first time in a given year; and entry rates are calculated for these new entrants as a percent of the incumbent firms that survived from the previous year. Exit is calculated as the sum of total exit from both the exogenous shock and endogenous choices; and exit rates are constructed also as a percent of incumbent firms from the previous year. (Details of the constructions are in the appendix.)

Table 9: Size Distribution of Firms

	Data		**Model**	
	$1-19$	**250+**		
Plants (Numbers)	0.457	0.067	0.457	0.067
Employment	0.049	0.559	0.086	0.520
Hiring	0.082	0.460	0.180	0.548
Firing	0.142	0.385	0.170	0.533

Data are from Lee and Mukoyama (2012). For the model, since employment is not discrete, employment shares are calculated for the largest and smallest firms from the distribution of productivities in the steady-state. Productivity cutoffs that demarcate the smallest 45.7 percent of firms and the largest 6.7 percent of firms are then used to calculate "hiring" and "firing" shares for firms in these categories of the size distribution. (Further details are in Section 4.2).

Table 10: Job Creation and Destruction Rates

Job Creation from Startups	
Job Creation is x percent higher in Good Times than in Bad Times.	
Data	45 *percent higher*
Model $\left(\delta^M = 0.01375\right)$	40 *percent higher*
Model $\left(\delta^M = 0.025\right)$	28 *percent higher*
Job Destruction from Shutdowns	
Job Destruction is x percent higher in Good Times than in Bad Times.	
Data	11 *percent higher*
Model $\left(\delta^M = 0.01375\right)$	7 *percent higher*
Model $\left(\delta^M = 0.025\right)$	5 *percent higher*

Data are from Lee and Mukoyama (2012). In the data, job creation rates are calculated as the number of jobs created from new firms and firms that shut down as a percentage of total employment. In the model, (annual) job creation rates from startups are calculated as the average total labor employed by startups over the four quarters as a percentage of the average total quarterly employment each year. Job destruction rates are calculated similarly. (Further details on the level differences are in Section 4.2).

Table 11: Data and Model-Implied Moments

Relative Standard Deviation

variable	**Data**	**Model**	**Small** δ^M	**BGM**	**RBC**
Output	1.00	$--$	$--$	$--$	$--$
Consumption	0.74	0.67	0.74	1.00	0.44
Investment	2.93	3.15	2.57	$--$	2.95
Hours	0.99	2.31	2.56	1.52	0.48

First-Order Autocorrelation

variable	**Data**	**Model**	**Small** δ^M	**BGM**	**RBC**
Output	0.84	0.87	0.82	0.74	0.72
Consumption	0.80	0.58	0.64	0.74	0.79
Investment	0.87	0.95	0.94	$--$	0.71
Hours	0.88	0.70	0.72	0.66	0.71

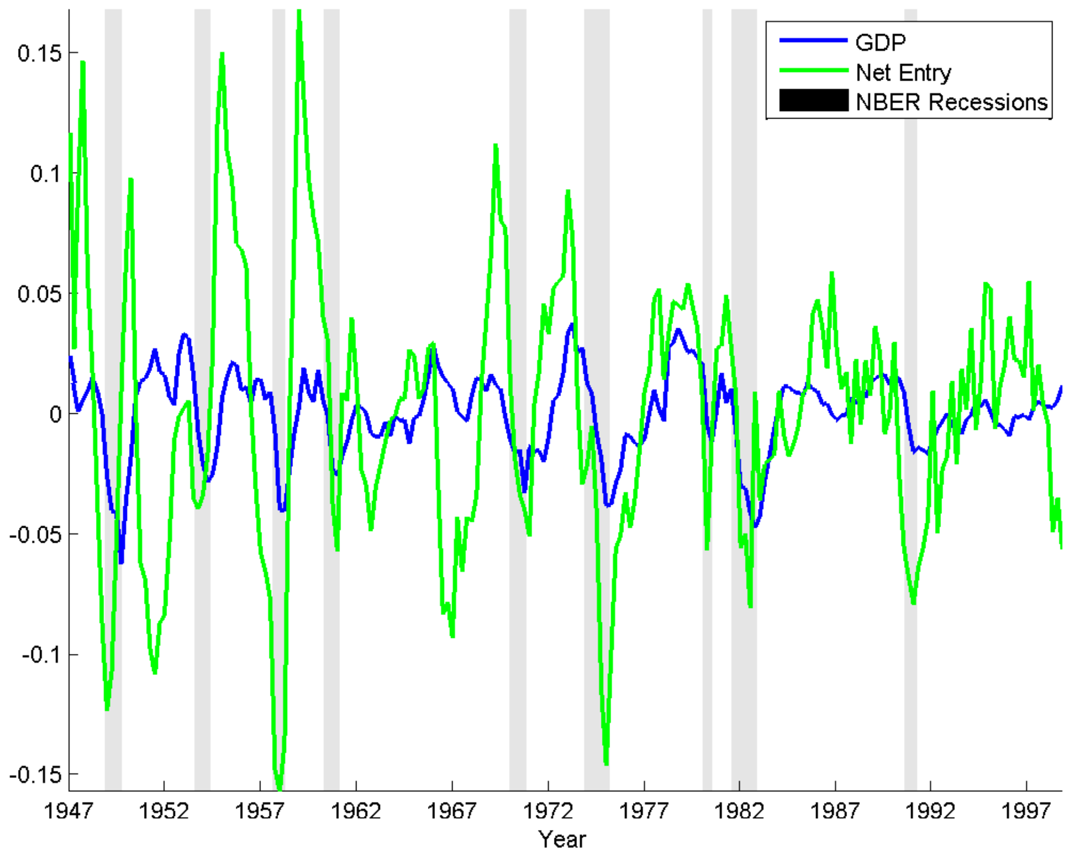
Correlation with Output

variable	**Data**	**Model**	**Small** δ^M	**BGM**	**RBC**
Output	1.00	$--$		$--$	$--$
Consumption	0.88	0.70	0.84	1.00	0.94
Investment	0.80	0.87	0.86	$--$	0.99
Hours	0.88	0.67	0.78	0.86	0.97

Data and RBC moments are from King and Rebelo (1999). Moments for the model are reported under "Model" for $\delta^M = 0.025$ and "Small δ^M" for $\delta^M = 0.01375$. BGM moments refer to a special case of my model that includes only labor in production (no capital) and no fixed costs, which has aggregate properties identical to the baseline model in Bilbiie, Ghironi, and Melitz (2012). Model-implied moments for output, consumption, and investment are constructed from their real series (in logs). All series were detrended with the HP filter.

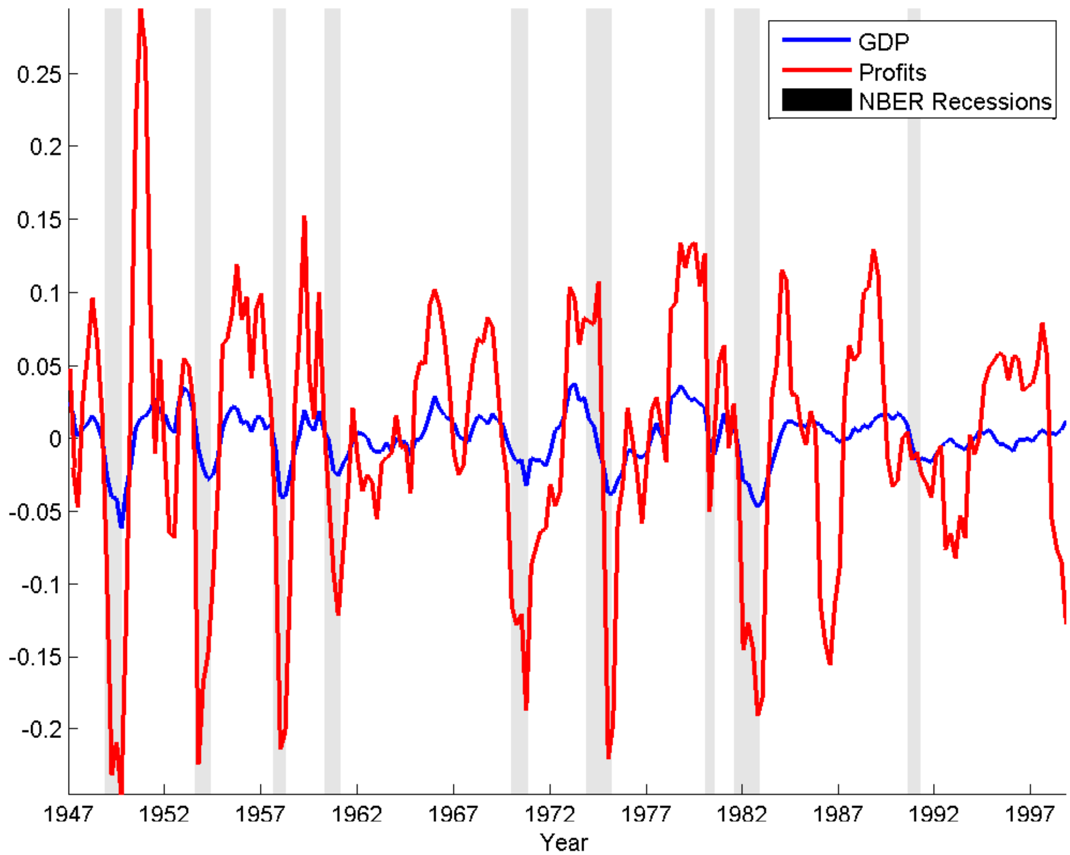
Figures

Figure 1: Net Entry is Procyclical



US quarterly HP-filtered data: 1947:1-1998:3

Figure 2: Profits are Procyclical



US quarterly HP-filtered data: 1947:1-1998:3

Figure 3: Histograms for Household Mapping Function Errors

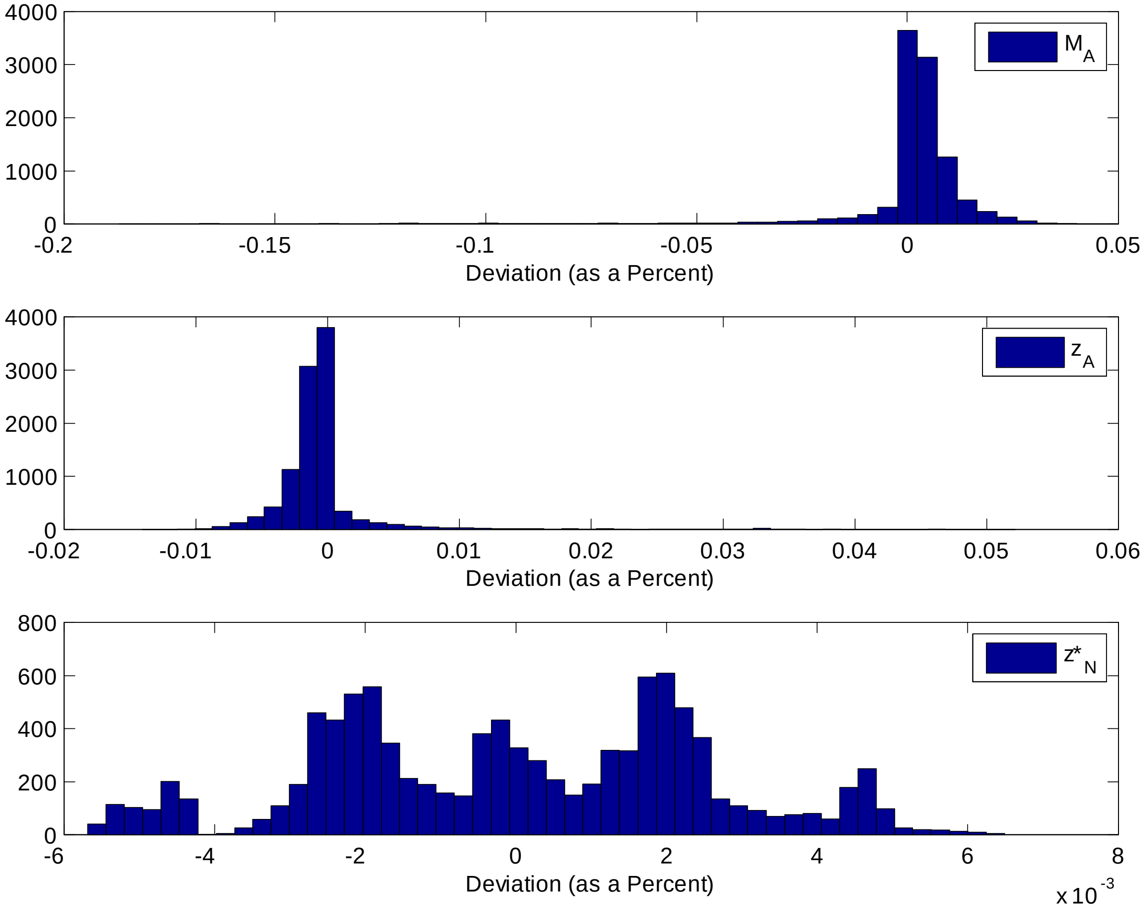


Figure 4: Histograms for Firm Mapping Function Errors

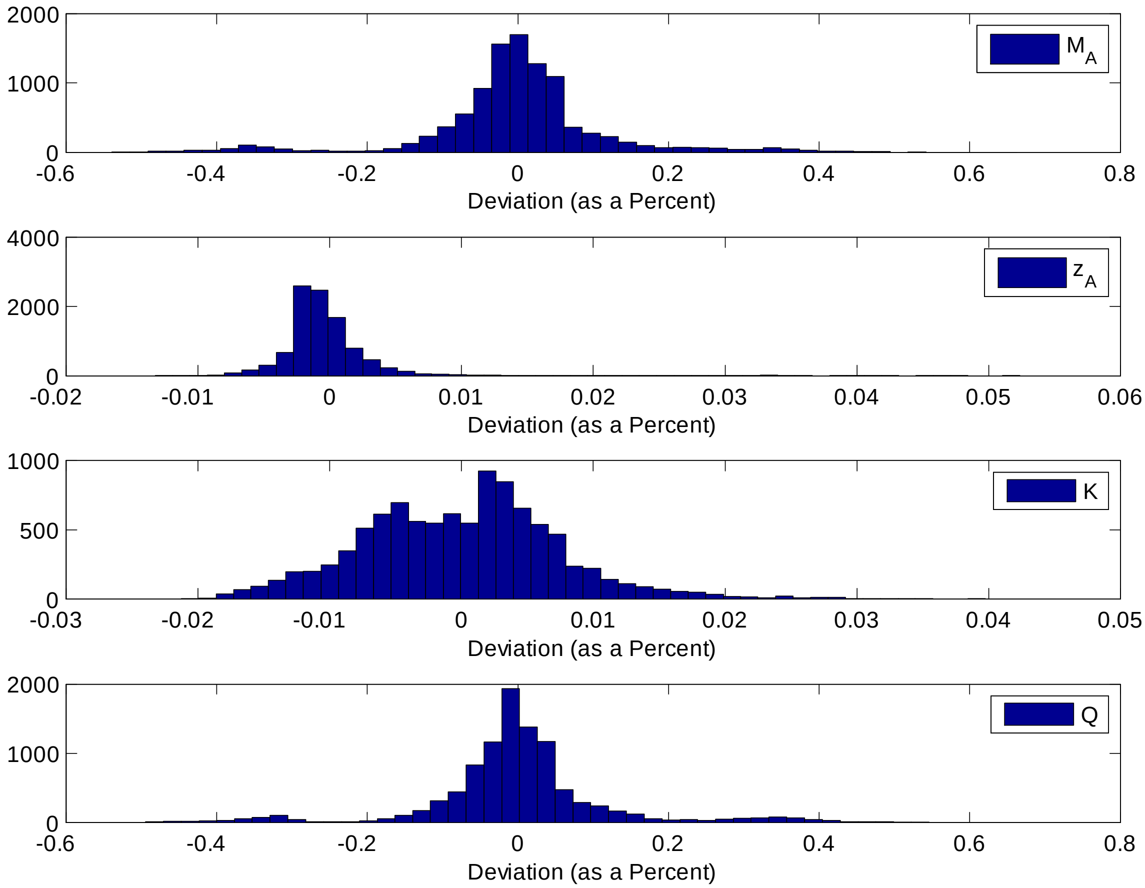


Figure 5: Data and Model Correlations: Corr(Real Output$_{t+k}$,Profits$_t$)

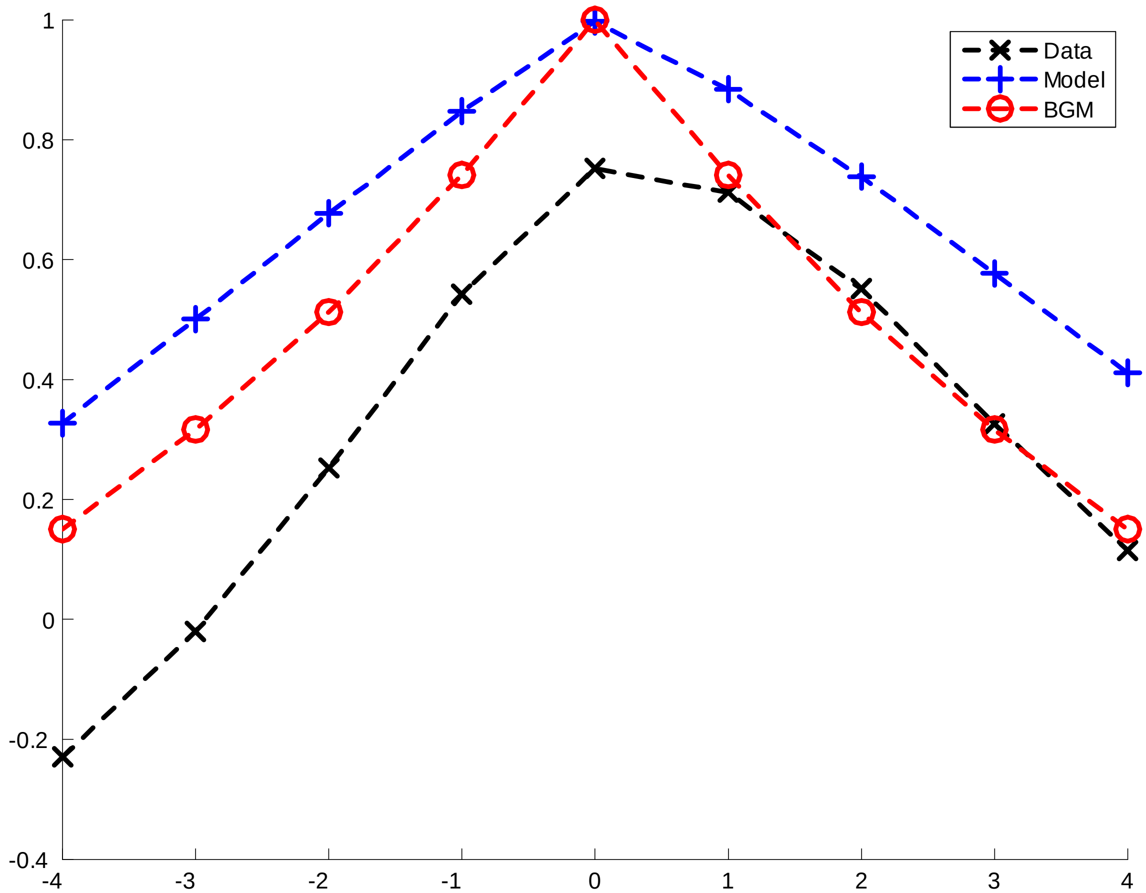


Figure 6:

Figure 7: Data and Model Correlations: Corr$\left(\text{Real Output}_{t+k}, \text{Net Entry}_t\right)$

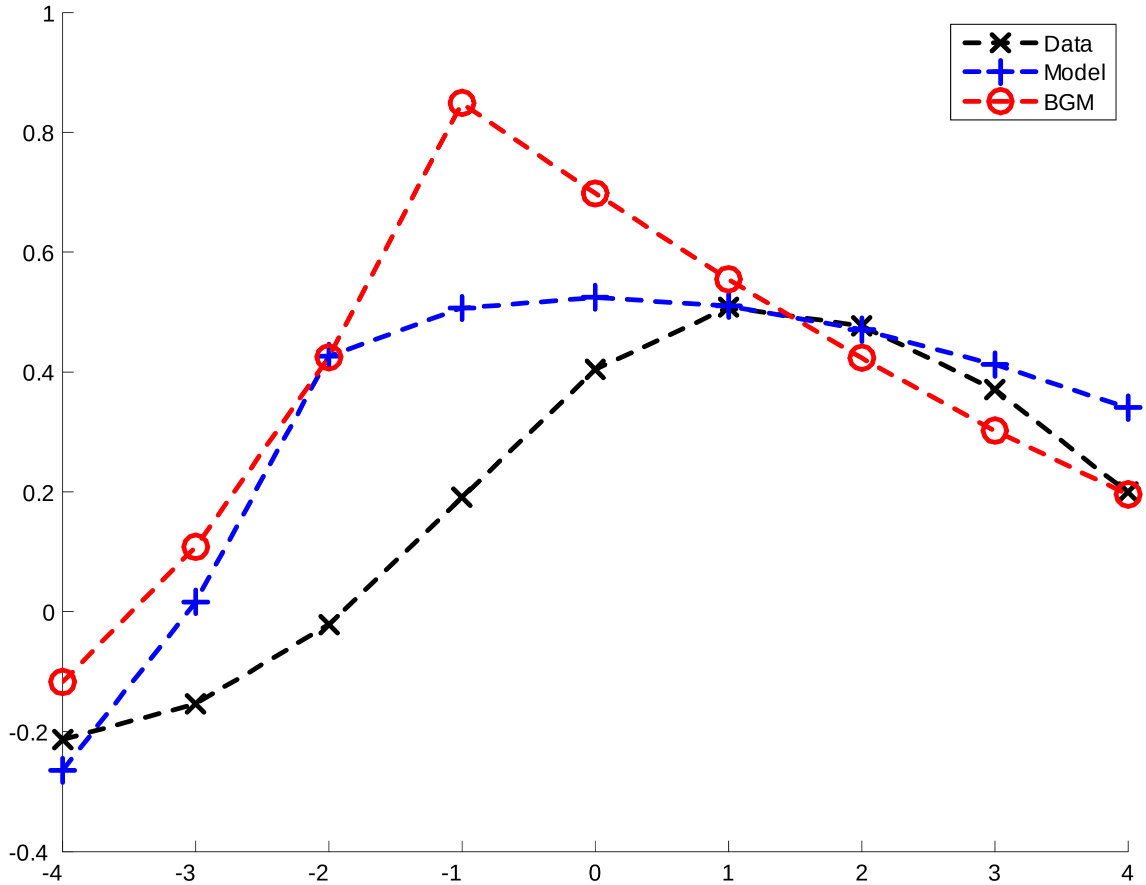


Figure 8: Data and Model Correlations: Corr(Profits$_{t+k}$,Net Entry$_t$)

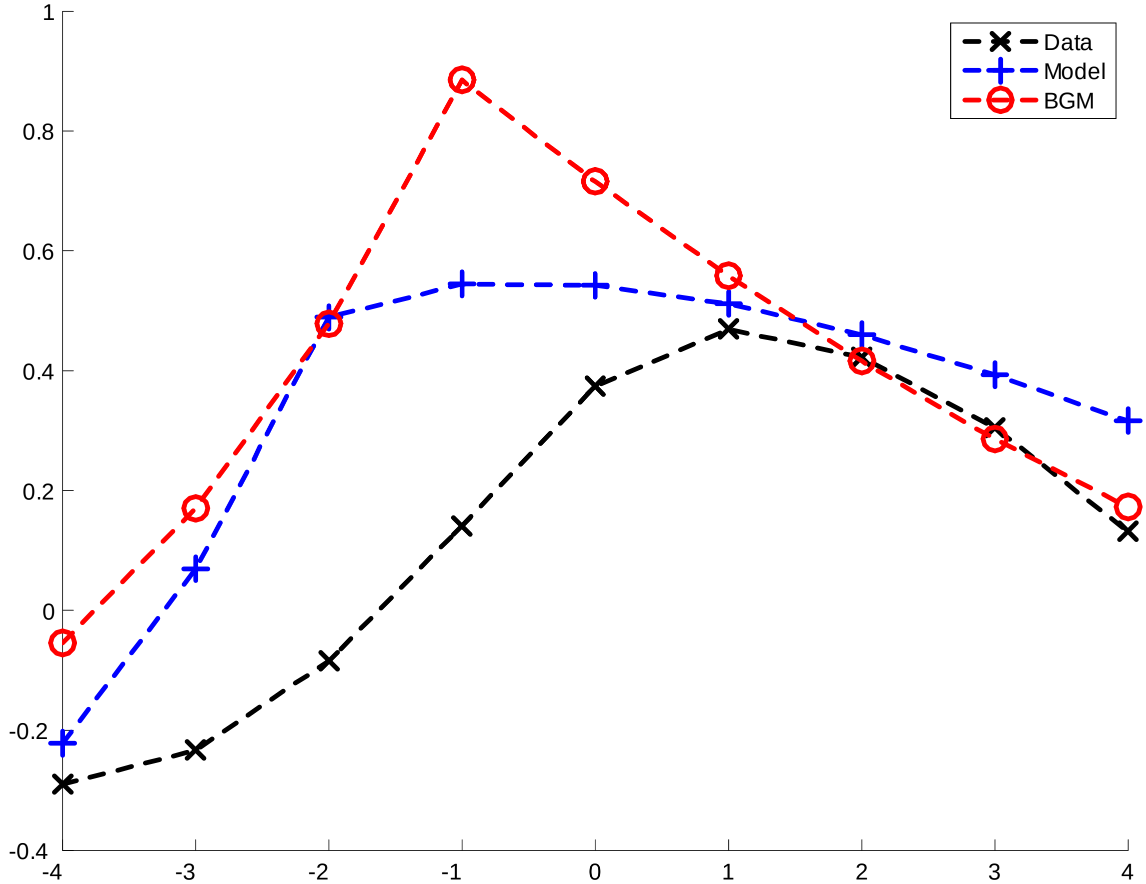


Figure 9: Impulse Responses to a Technology (Z) Shock

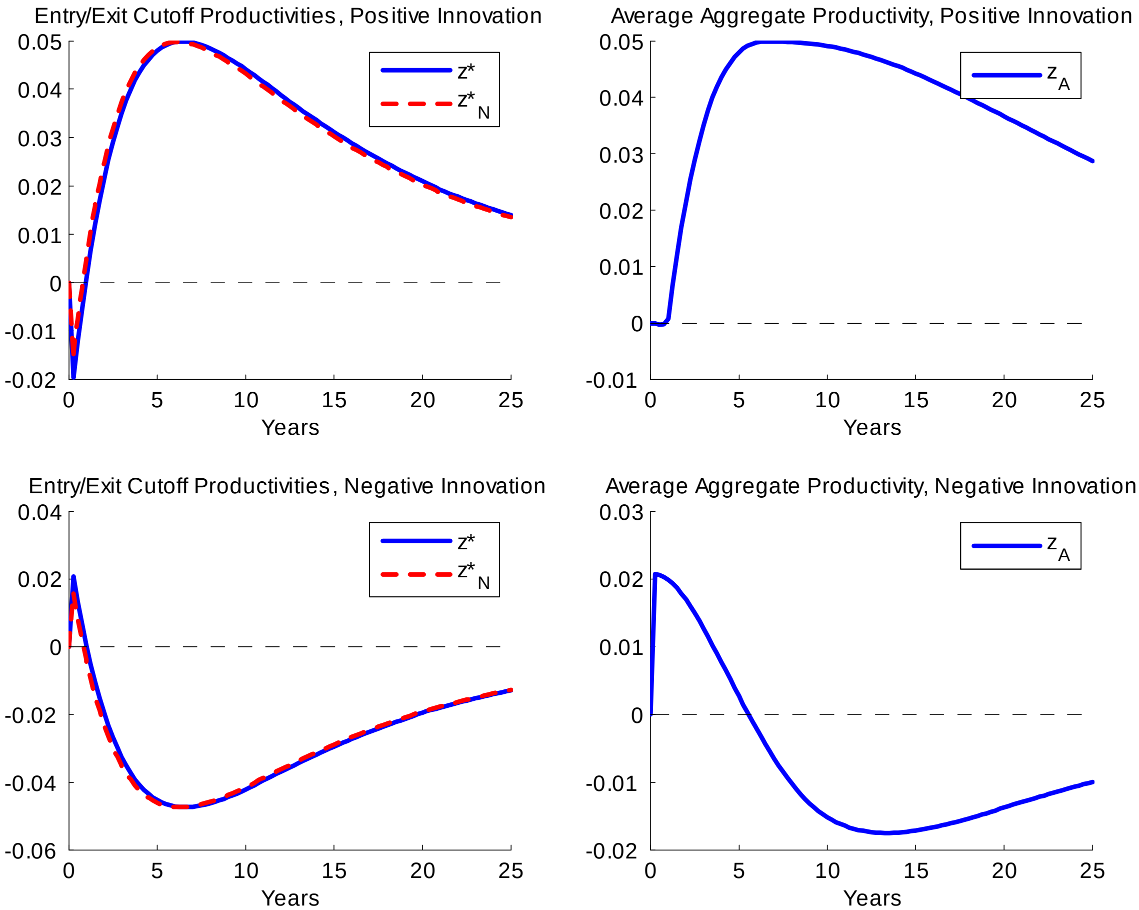


Figure 10: Impulse Responses to a Positive Technology (Z) Shock

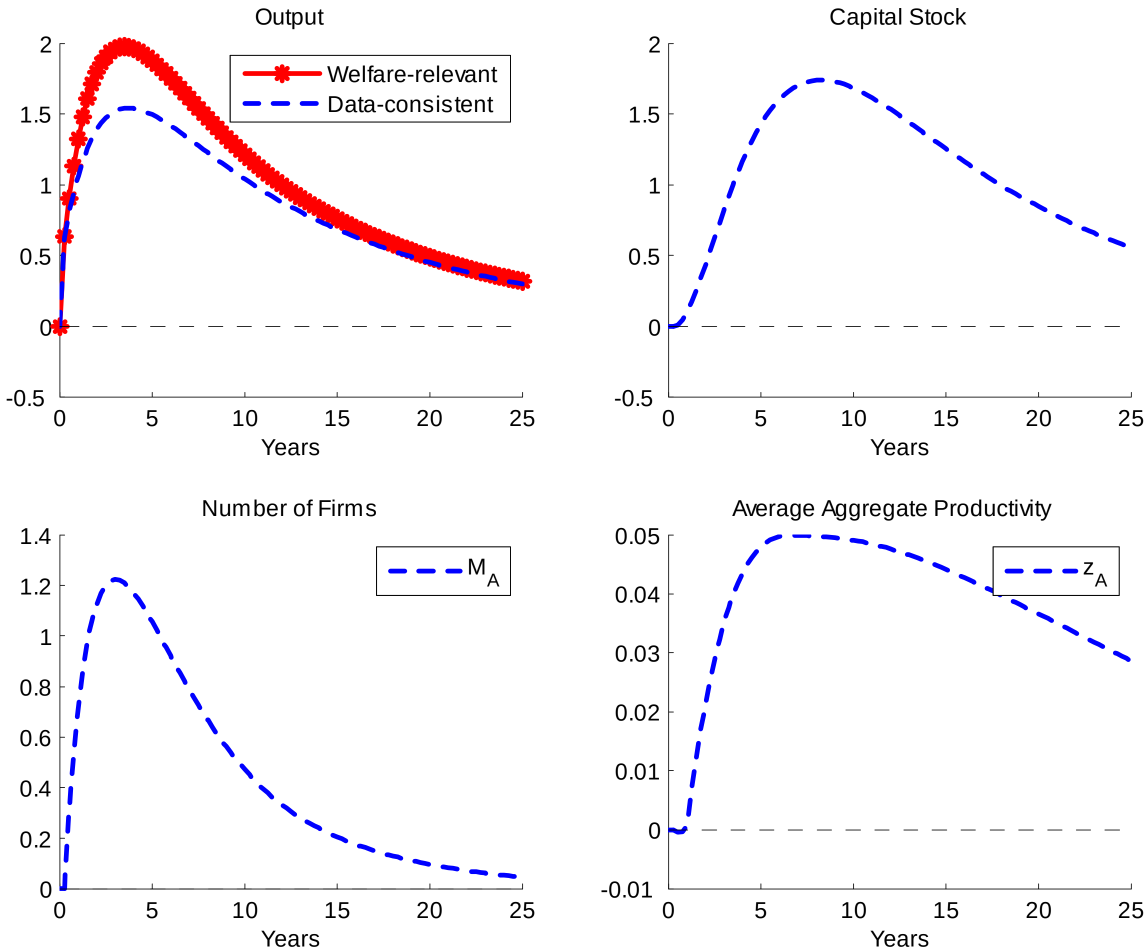


Figure 11: Impulse Responses to a Negative Technology (Z) Shock

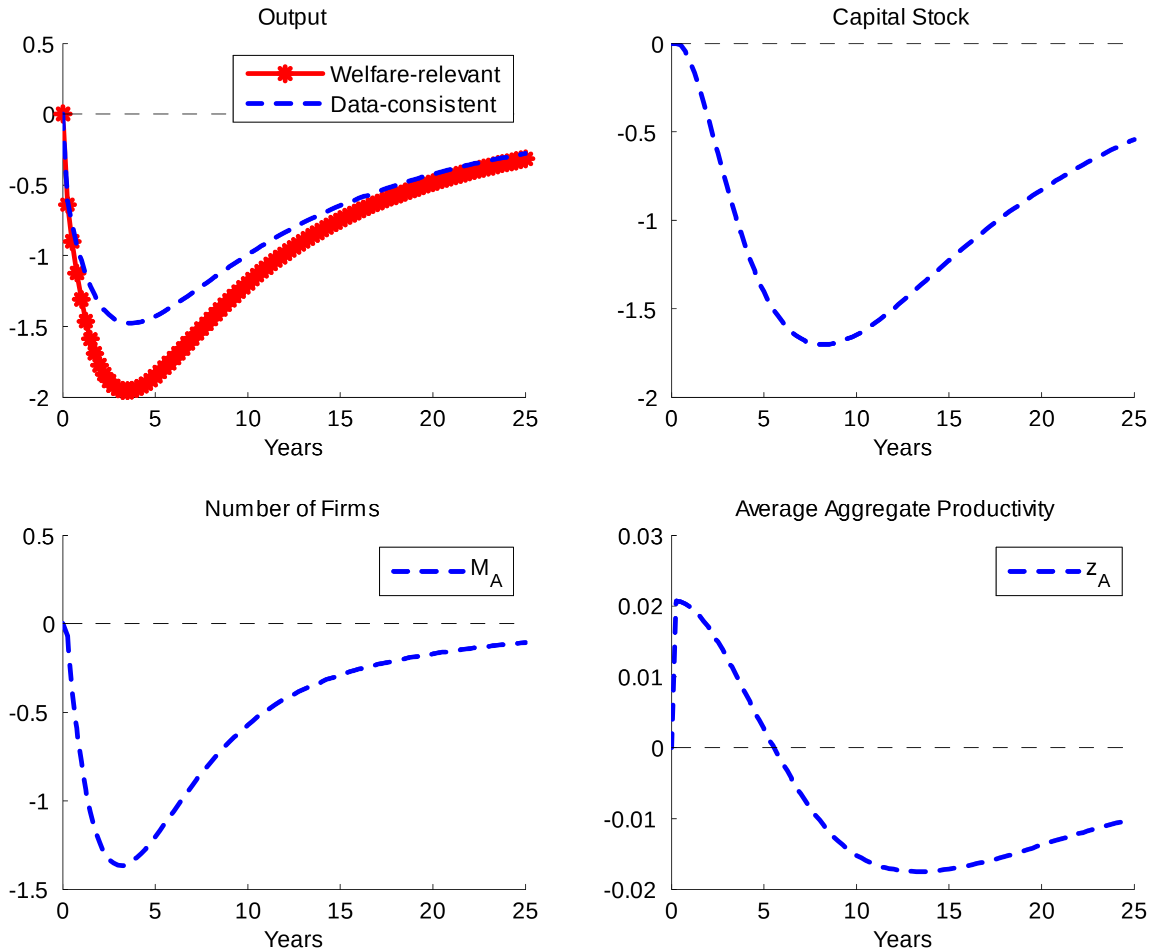
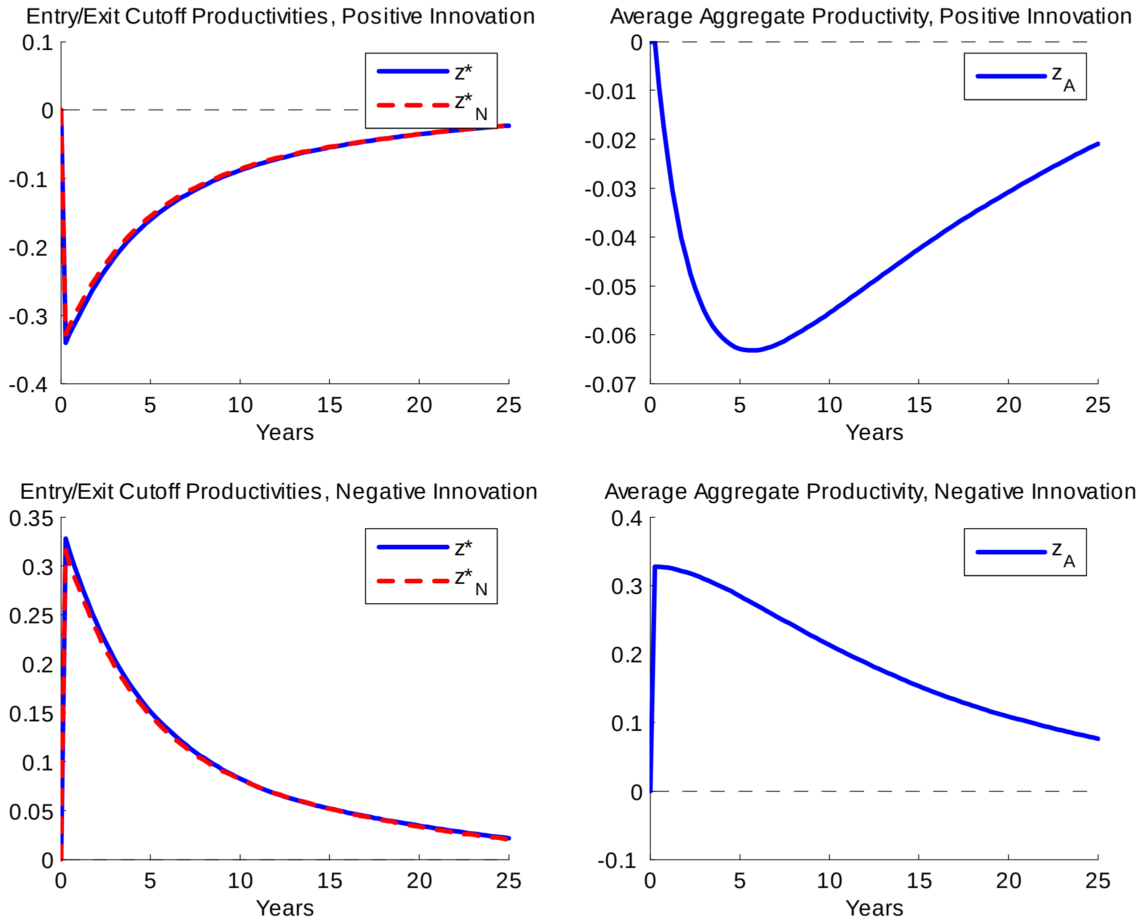
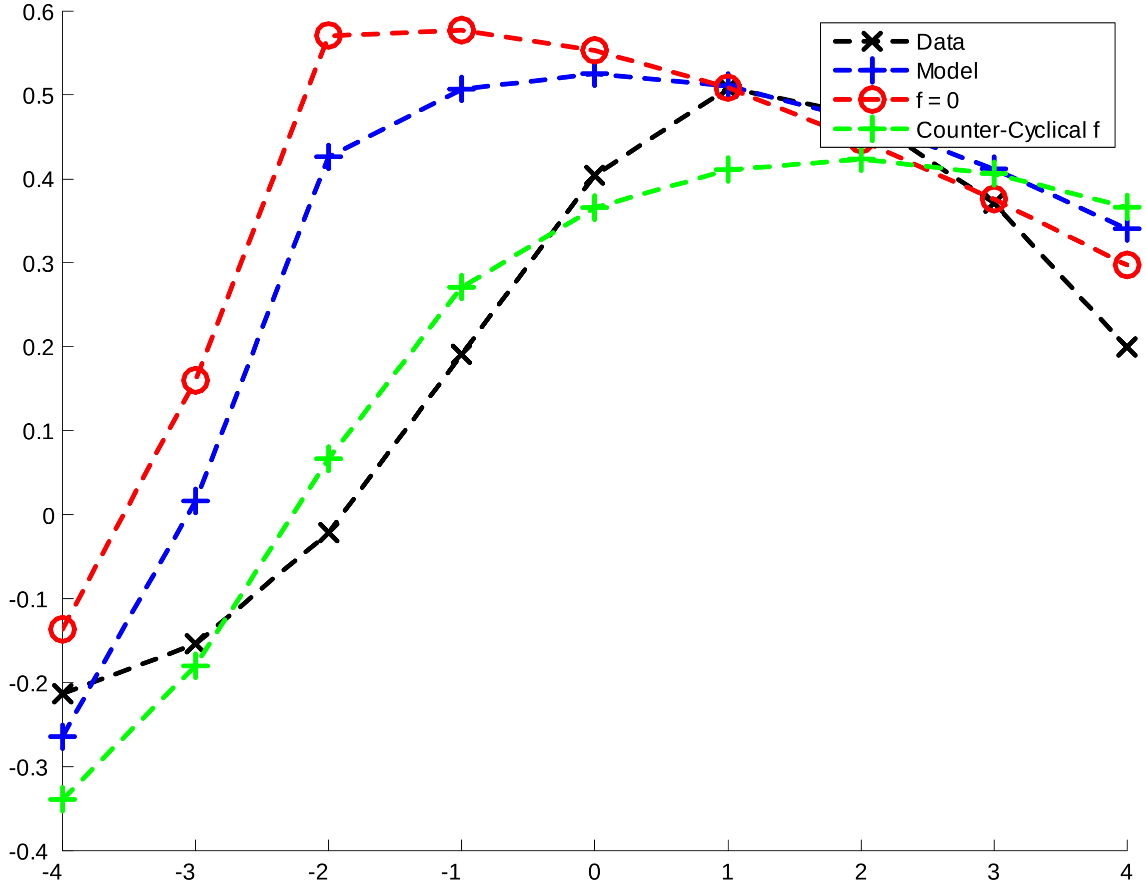


Figure 12: Impulse Responses to a Technology (Z) Shock



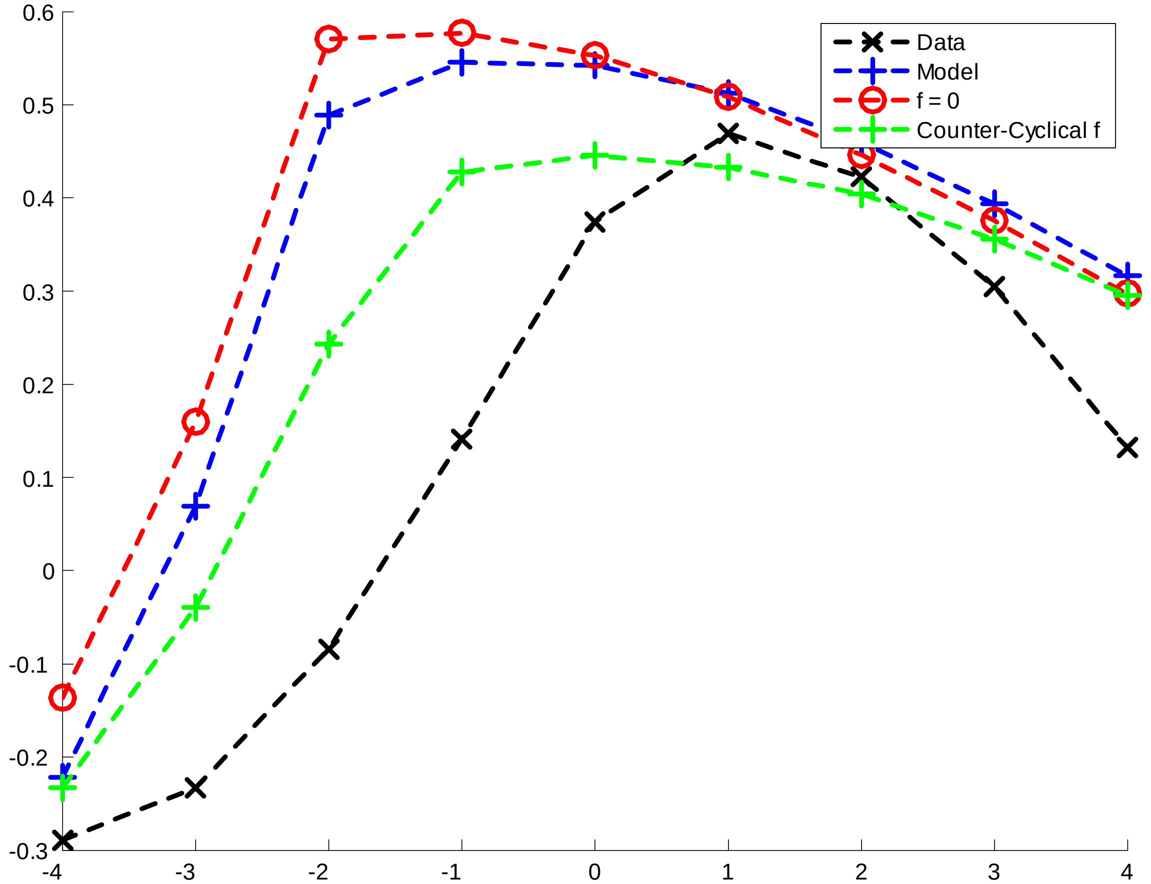
For the model with counter-cyclical fixed costs, $f^p = 0.08$.

Figure 13: Data and Model Correlations: Corr$\left(\text{Real Output}_{t+k}, \text{Net Entry}_t\right)$



For the model with counter-cyclical fixed costs, $f^p = 0.08$.

Figure 14: Data and Model Correlations: Corr(Profits$_{t+k}$,Net Entry$_t$)



For the model with counter-cyclical fixed costs, $f^p = 0.08$.

Figure 15: Impulse Responses to a Positive Technology (Z) Shock

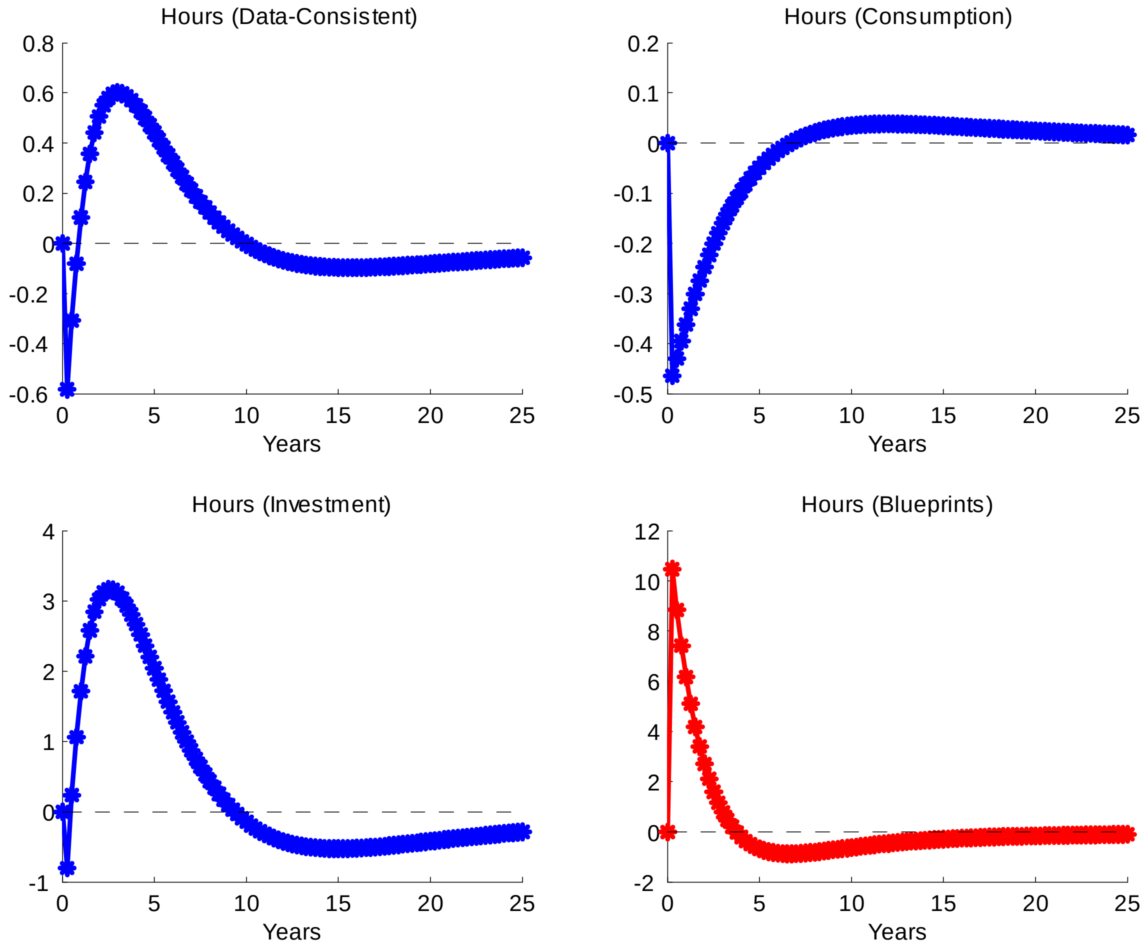


Figure 16: Impulse Responses to a Positive Productivity Shock

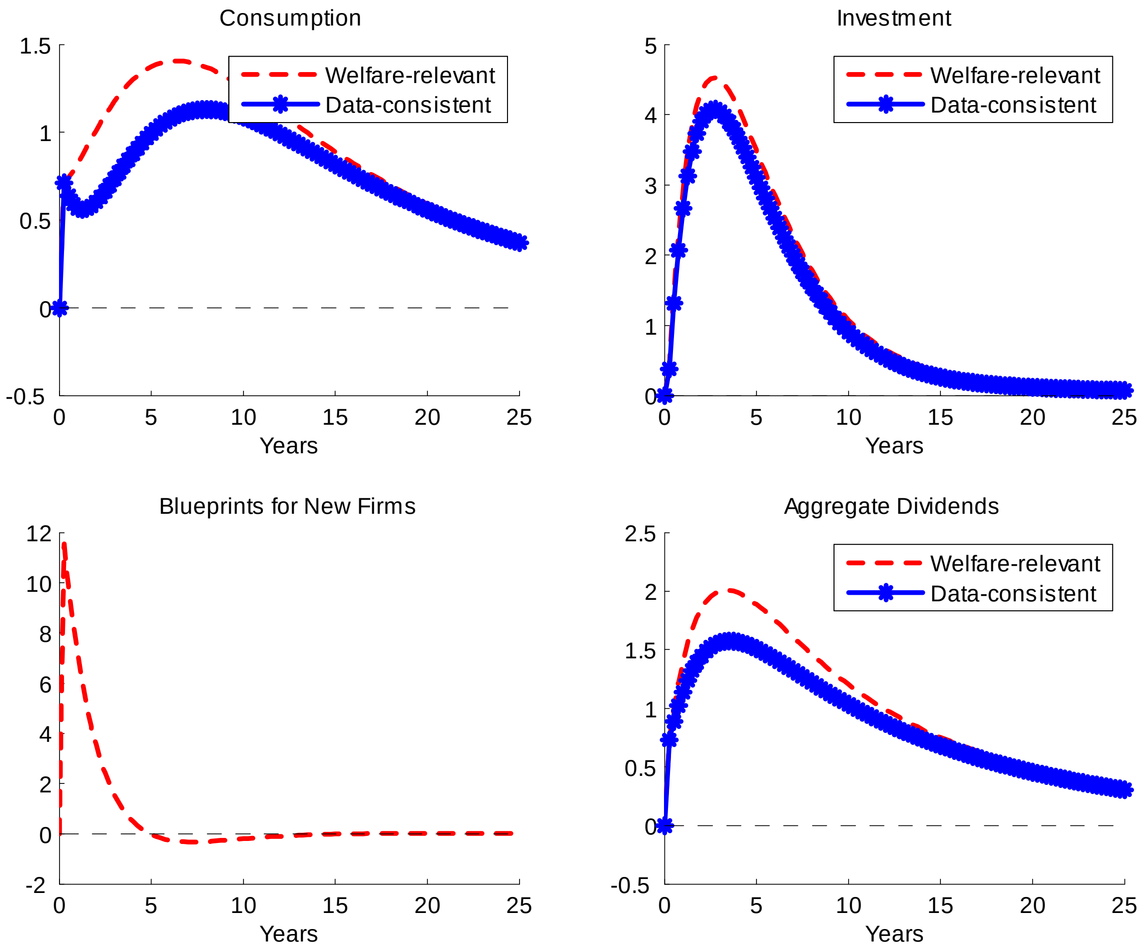


Figure 17: Histograms for Household Forecast Errors, 1-Quarter Ahead

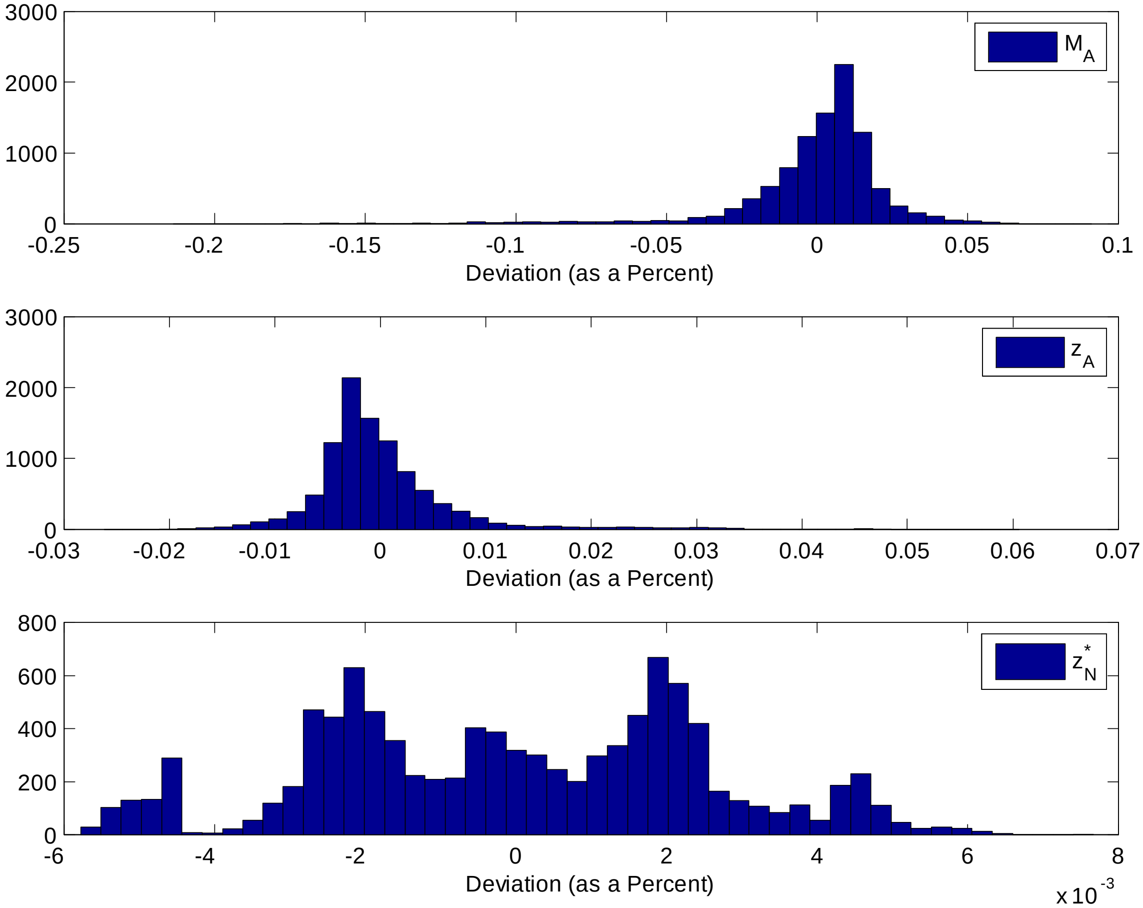


Figure 18: Histograms for Household Forecast Errors, 10-Quarters Ahead

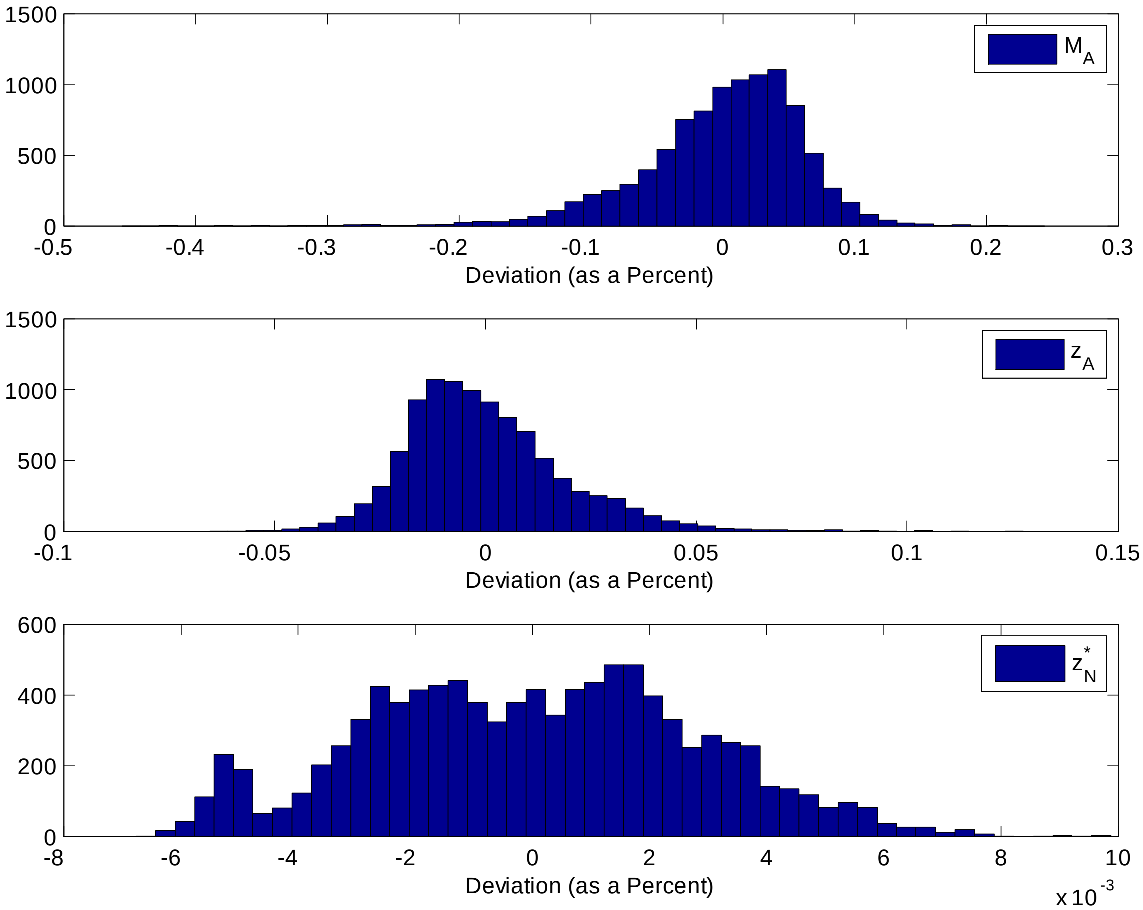


Figure 19: Histograms for Household Forecast Errors, 100-Quarters Ahead

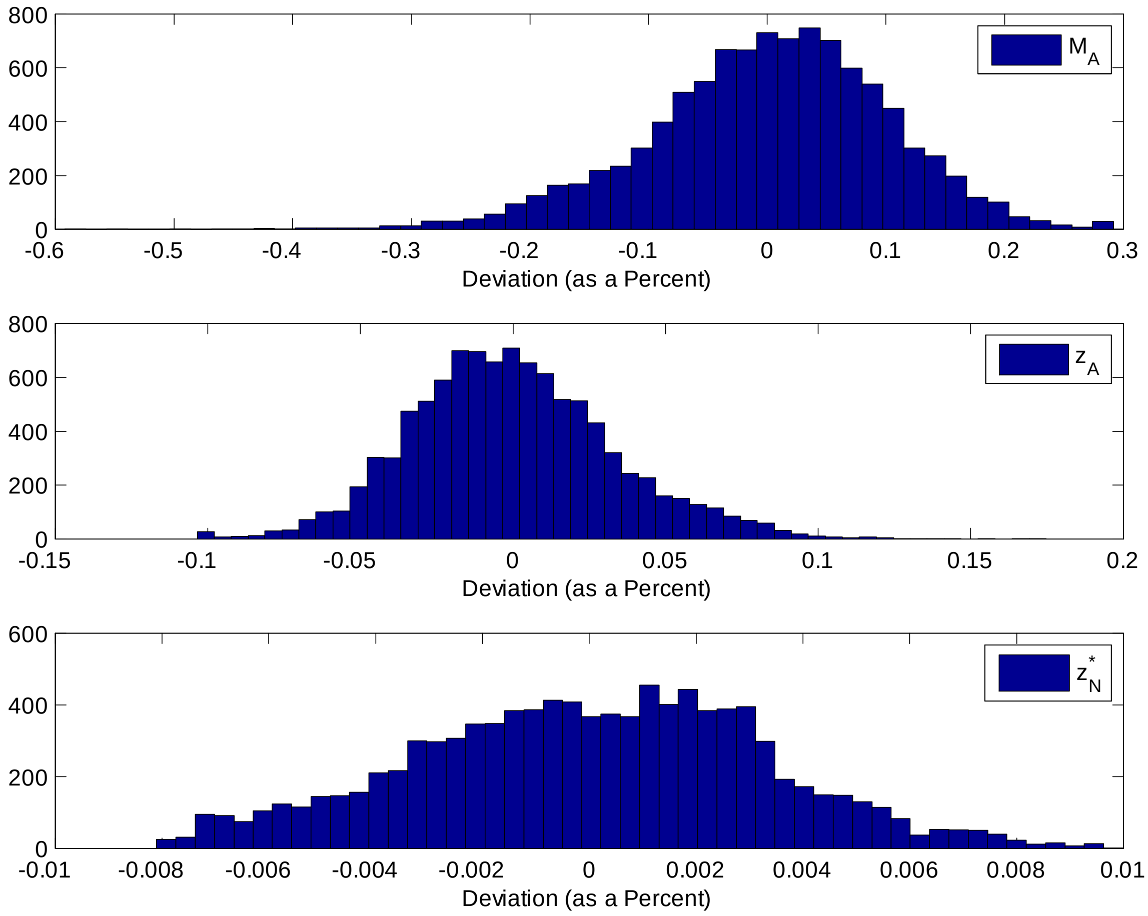


Figure 20: Histograms for Firm Forecast Errors, 1-Quarter Ahead

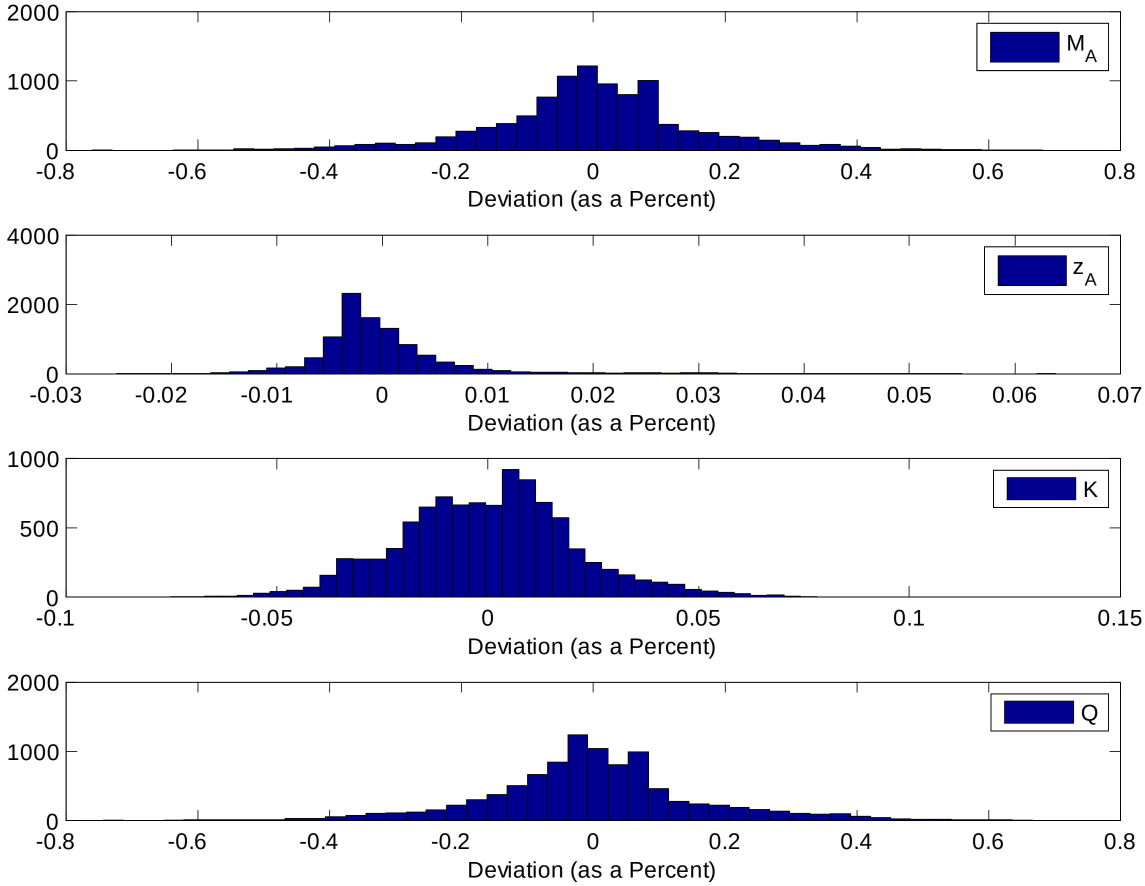


Figure 21: Histograms for Firm Forecast Errors, 10-Quarters Ahead

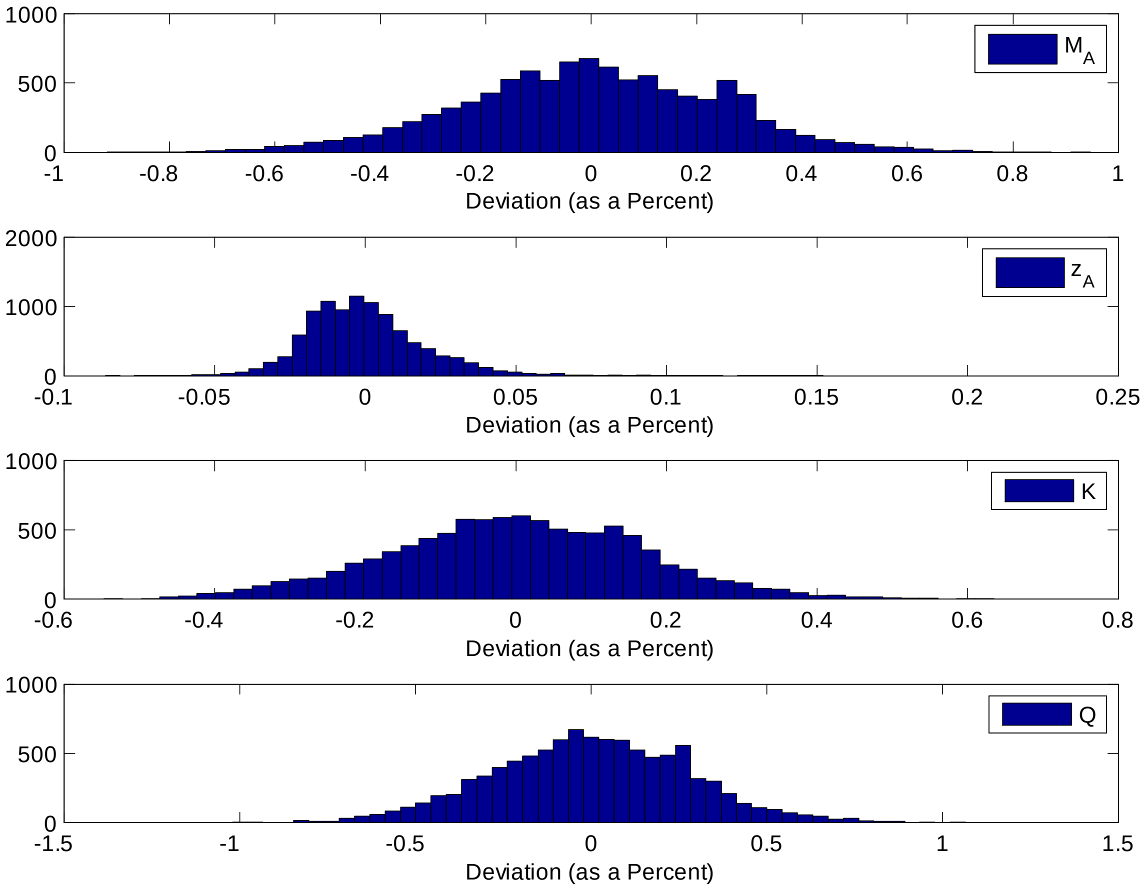


Figure 22: Histograms for Firm Forecast Errors, 100-Quarters Ahead

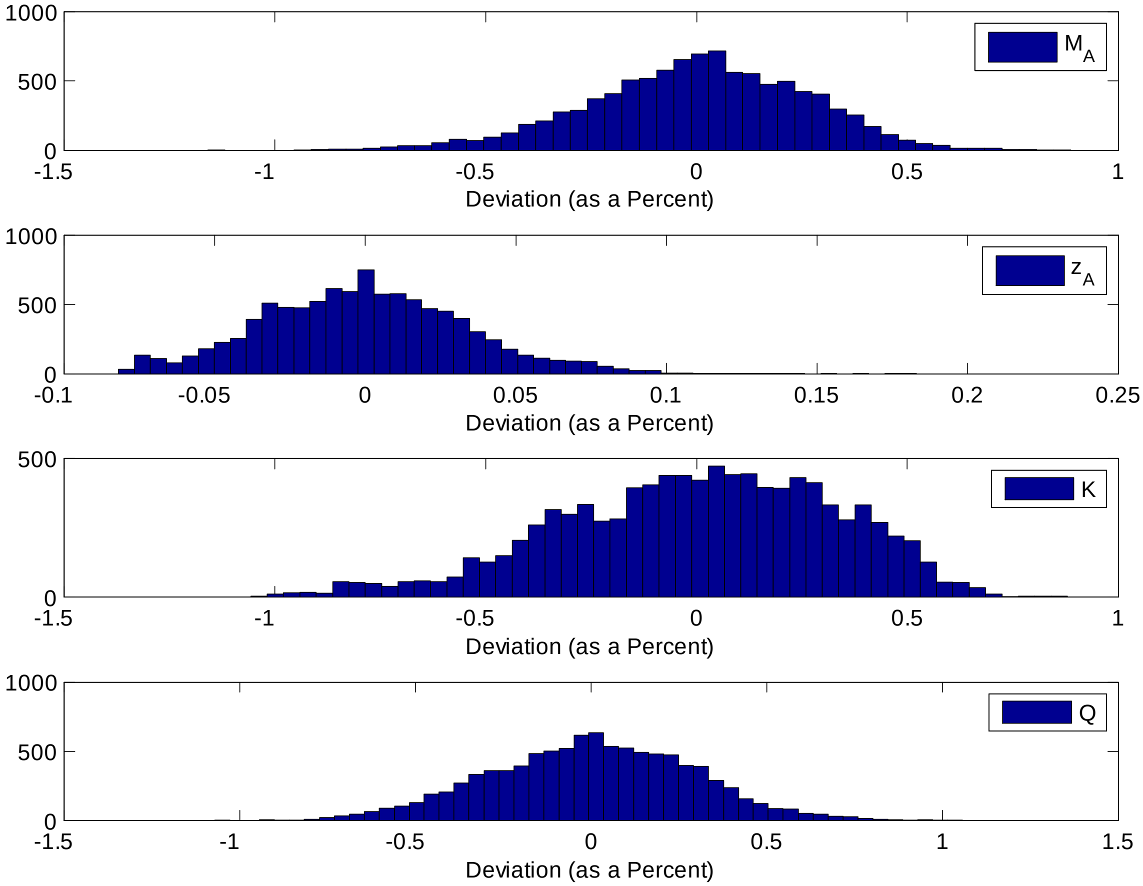
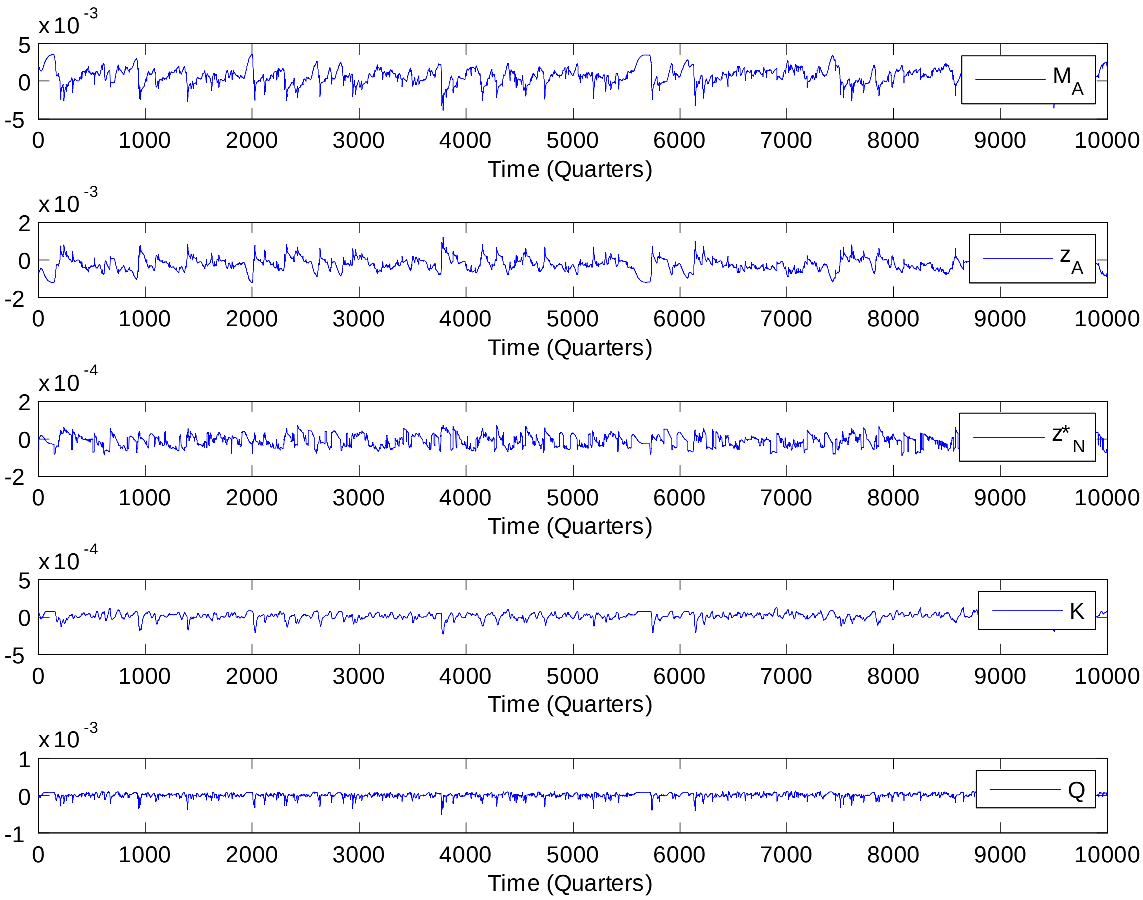
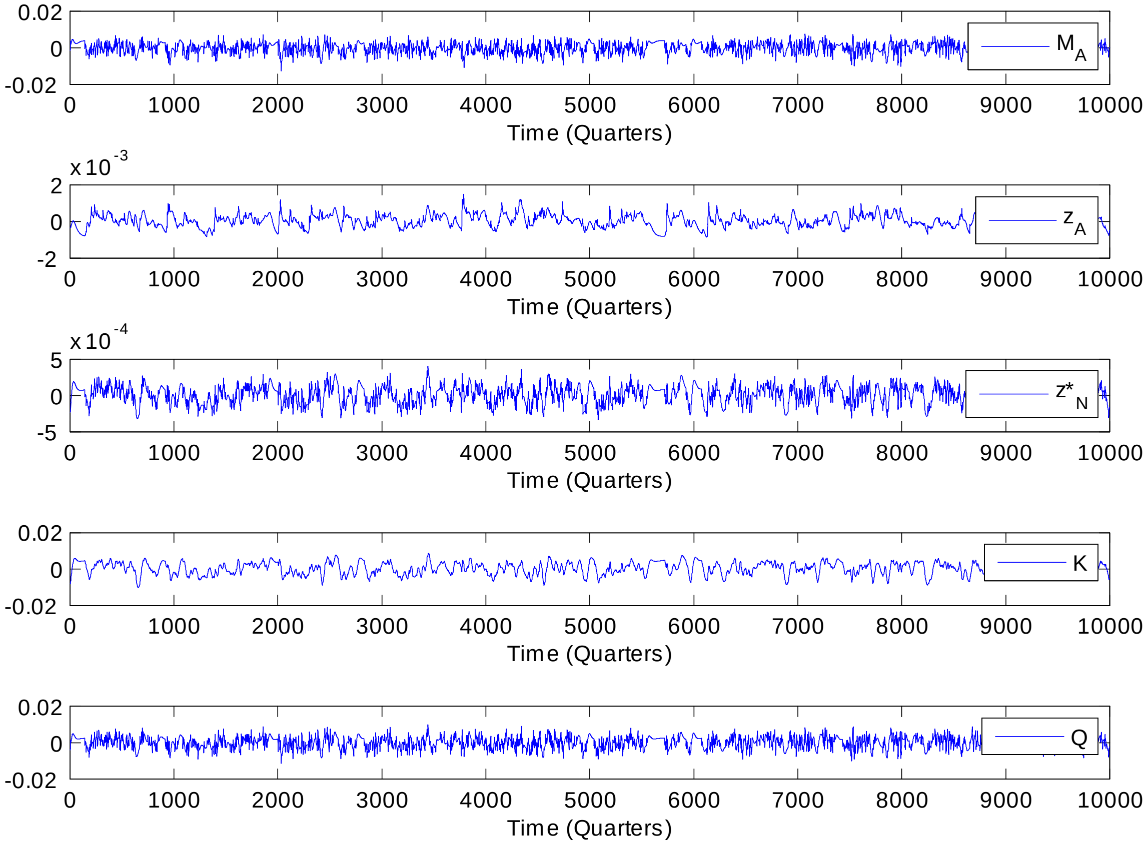


Figure 23: Deviations of Approximating Series (Households) from Simulated Values



Error terms are constructed as $\log(x_t) - \log(\hat{x}_t)$ and constitute the "essential accuracy plots", according to in den Haan (2010). Approximating series $\hat{x}^H$ are constructed according to an adaptation of the den Haan (2010) procedure (detailed in the Technical Appendix in Section A.5.3) and compared against model simulated values x.

Figure 24: Deviations of Approximating Series (Firms) from Simulated Values



Error terms are constructed as $\log(x_t) - \log(\hat{x}_t)$ and constitute the "essential accuracy plots", according to in den Haan (2010). Approximating series $\hat{x}^H$ are constructed according to an adaptation of the den Haan (2010) procedure (detailed in the Technical Appendix in Section A.5.3) and compared against model simulated values x.

A Appendix

A.1 Algorithm for Tracking the Distribution of Firms

A.1.1 Background and General Ideas

To fix ideas, consider the following:

Suppose that entering groups of new firms draw idiosyncratic relative productivities from an ex-ante continuous cumulative Pareto distribution $G(z)$ with positive support on $[z_{\min}, \infty)$. Only new entrants with $z > z^*_{N,t}$ will enter successfully. The probability of survival immediately after entry is then given by $1 - G\left(z^*_{N,t}\right)$ and the post-entry distribution of new entrants is given by $\mu_{N,t}(z)$, the conditional distribution of $g(z)$ on $\left[z^*_{N,t}, \infty\right)$, according to

$$\mu_{N,t}(z) = \begin{cases} g(z) / \left[1 - G\left(z^*_{N,t}\right)\right] & \text{if } z \geq z^*_{N,t} \\ 0 & \text{otherwise.} \end{cases}$$

The average aggregate productivity level $\tilde{z}_{N,t}$ for new entrants can then be defined as a function of the threshold productivity level $z^*_{N,t}$ according to

$$\tilde{z}_{N,t}\left(z^*_{N,t}\right) = \left[\frac{1}{1 - G\left(z^*_{N,t}\right)} \int_{z^*_{N,t}}^{\infty} z^{\theta-1} g(z)\, dz\right]^{1/(\theta-1)}.$$

Similar to Melitz (2003), the post-entry distribution for new entrants is tied to the ex-ante distribution $g(z)$, but the range of productivity levels is endogenously determined through the cutoff $z^*_{N,t}$. Melitz's (2003) special average productivity for new entrants $\tilde{z}_{N,t}$ can be completely characterized by the cutoff $z^*_{N,t}$; and $\tilde{z}_{N,t}\left(z^*_{N,t}\right)$ can be expressed simply as $\tilde{z}_{N,t}\left(z^*_{N,t}\right) = \nu z^*_{N,t}$ using the scaling parameter $\nu = \left(\kappa / \left(\kappa - \theta + 1\right)\right)^{\frac{1}{\theta-1}}$. The number and distribution of new entrants, then, can be characterized simply by reference to the summary variables $\left[1 - G\left(z^*_{N,t}\right)\right] N_t$ and $z^*_{N,t}$.

Let a group of firms distributed according to a Pareto distribution with lower bound z^* be called a "cohort", so that the c^{th} cohort has mass $M_{c,t}$ and lower bound productivity $z^*_{c,t}$. Suppose that all incumbent firms comprise a cohort, and let $M_{1,t}$ and $z^*_{1,t}$ represent the mass and lower bound productivity for these firms, respectively. New entrants comprise a second cohort, with post-entry values of $M_{2,t} = \left[1 - G\left(z^*_{N,t}\right)\right] N_t$ and $z^*_{2,t} = z^*_{N,t}$ (where $z^*_{N,t} \neq z^*_{1,t}$). Analogous to that in Melitz (2003), a weighted average function can be used to write the average aggregate productivity for the new entrants and these firms as

$$\tilde{z}_t = \left\{\frac{1}{M_{1,t} + M_{2,t}} \left[M_{1,t} \left[\tilde{z}_{1,t}\right]^{\theta-1} + M_{2,t} \left[\tilde{z}_{2,t}\right]^{\theta-1}\right]\right\}^{\frac{1}{\theta-1}},$$

where $\tilde{z}_{c,t} = \tilde{z}_{c,t}\left(z^*_{c,t}\right) = \nu z_{c,t}$ for $c = 1, 2$ are average productivities for the two groups of incumbent firms and new entrants. While the Melitz (2003) model uses a similar construction to find the weighted productivity average for all firms (domestic and foreign) competing in a single country, the same idea can be applied in my model to find the weighted productivity average for all firms in a single country, where different cohorts of firms can each be described by Pareto distributions with different cutoff productivity thresholds. Of course, incumbent firms comprise more than one cohort. Over a number of $c = 1...C$ cohorts, then, an average aggregate productivity can be defined as

$$\tilde{z}_t = \left\{ \frac{1}{\sum_{c=1}^{C} M_{c,t}} \left[\sum_{c=1}^{C} M_{c,t} \left[\tilde{z}_{c,t}\right]^{\theta-1} \right] \right\}^{\frac{1}{\theta-1}},$$

where $\tilde{z}_{c,t}$ represents the average productivity for a cohort of firms $M_{c,t}$ and the total number of firms is given by $M_t = \sum_{c=1}^{C} M_t^c$. As in Melitz (2003), this combined average productivity $\tilde{z}_t$ can completely summarize the effects of the distribution of productivity levels on the aggregate outcome.[85]

A.1.2 My Model: A Simple Example

Now, it would first appear that this representation would require keeping track of an infinite past history of cohorts, where each cohort becomes associated with each group of new entrants. This, however, is unnecessary. A finite-dimensional state variable can be used to track the distribution of firm productivities exactly due to two key features: incumbent firms and new entrants are subject to exit due to changes in z_t^* and $z_{N,t}^*$, and z_t^* and $z_{N,t}^*$ fluctuate up and down over time.

To see this, let us switch to using the notation in the paper, where S subscripts are used in describing the stock of firms and their average aggregate productivity at the beginning of a period, $M_{S,t}$ and $\tilde{z}_{S,t}$, and A subscripts are used to denote the values of those stock values that remain active for the current period after any endogenous exits, $M_{A,t}$ and $\tilde{z}_{A,t}$.

Suppose there were two existing cohorts of firms at the beginning of a period $M_{S_1,t}$ and $M_{S_2,t}$ each with lower bound productivity cutoffs $z_{S_1,t}^*$ and $z_{S_2,t}^*$, respectively, and that $z_{S_1,t}^* < z_{S_2,t}^*$. Now suppose that for the current period z_t^* is such that $z_{S_1,t}^* < z_t^* < z_{S_2,t}^*$ and there is a new group of entrants N_t that face $z_{N,t}^*$. Now, since the Pareto distribution has the convenient feature that, if the distribution is truncated from below, the new distribution remains Pareto, we can say the following about the firms that will be active in the current period and the firms that will continue on into the next period.

Of the first cohort of incumbent firms, $\left(z_t^*/z_{S_1,t}^*\right)^{\kappa} M_{S_1,t}$ remain at the beginning of the period after endogenous exit, with z_t^* now characterizing its new lower bound. Of the second incumbent cohort, $M_{S_2,t}$ remain and $z_{S_2,t}^*$ is the cutoff that still characterizes its lower bound. Firms that will be producing in the current period, then, are given by

$$M_{A,t} = \left(z_t^*/z_{S_1,t}^*\right)^{\kappa} M_{S_1,t} + M_{S_2,t}$$

and have average aggregate productivity equal to

$$\tilde{z}_{A,t} = \left\{ \frac{1}{\left(z_t^*/z_{S_1,t}^*\right)^{\kappa} M_{S_1,t} + M_{S_2,t}} \left[\left(z_t^*/z_{S_1,t}^*\right)^{\kappa} M_{S_1,t} \left[\tilde{z}_{S_1,t}\right]^{\theta-1} + M_{S_2,t} \left[\tilde{z}_{S_2,t}\right]^{\theta-1} \right] \right\}^{\frac{1}{\theta-1}},$$

[85] E.g., $P_t = M_{A,t}^{1/(1-\theta)} p_t\left(\tilde{z}_{A,t}\right)$, $Q_t = M_{A,t}^{\theta/(\theta-1)} q_t\left(\tilde{z}_{A,t}\right)$, $R_t = P_t Q_t$, etc.

where $\tilde{z}_{S_1,t} = \tilde{z}_{S_1,t}\left(z_t^*\right)$ and $\tilde{z}_{S_2,t} = \tilde{z}_{S_2,t}\left(z_{S_2,t}^*\right)$.

Recall that there is a one-period time-to-build lag for new entrants, but they are also subject to endogenous exit, so for the new entrants, $\left[1 - G\left(z_{N,t}^*\right)\right] N_t$ remain and $z_{N,t}^*$ now characterizes its lower bound.

Suppose that $z_{N,t}^* < z_t^*$. After the exogenous exit shock at the end of the period, the firms that will continue into the next period can be described through three cohorts: $M_{S_1,t+1} = (1-\delta_M)\left[1 - G\left(z_t^*\right)\right] N_t$, $M_{S_2,t+1} = (1-\delta_M)\left(z_t^*/z_{S_1,t}^*\right)^{\kappa} M_{S_1,t}$ and $M_{S_3,t+1} = (1-\delta_M) M_{S_2,t}$ with lower bounds $z_{S_1,t+1}^* = z_{N,t}^*$, $z_{S_2,t+1}^* = z_t^*$ and $z_{S_3,t+1}^* = z_{S_2,t}^*$, respectively.[86]

Importantly, note that whenever z_t^* is larger than some $z_{S_c,t}^*$, then all cohorts from 1...c for which $z_t^* > z_{S_c,t}^*$ will have the same new lower bound and can be aggregated into a single cohort. Suppose that in the next period, z_{t+1}^* was such that $z_{S_1,t+1}^* < z_{S_2,t+1}^* < z_{t+1}^* < z_{S_3,t+1}^*$. The first two cohorts of firms will now both be described by Pareto distributions with lower bounds z_{t+1}^*, so the number of previous cohorts has now been reduced by one.

Combining cohorts in this way permits the distribution of firm productivities to be represented with a finite dimensional object; and with this methodology, cohorts are not necessarily tied to vintage. Since the number of cohorts can fluctuate over time, a (sufficiently large) matrix is used to track a fixed number of C cohorts with an algorithm explained in what follows. It is easy to see that the algorithm would only fail if threshold productivities decreases monotonically each and every period from $t = 1...\infty$. For calibrated values of the shock process in a business cycle model, this does not happen. Checks are performed so that the matrix is large enough that the number of cohorts never reaches the maximum allowed by the matrix representation.

A.1.3 The Algorithm

The algorithm to track the distribution for firm productivities works as follows:

To create a state variable $\boldsymbol{F}_t$ that is sufficient to summarize the distributional information about firm productivities, set $\boldsymbol{F}_t$ as an $R \times 2$ matrix, where R is set as the maximum number of cohorts $R = C$. Let each row from $r = 1...R$ represent a cohort, so that, for the r^{th} cohort, the value in $(r, 1)$ is that cohort's threshold productivity and the value in $(r, 2)$ represents that cohort's mass. The state variable $\boldsymbol{F}_t$ is then given by

$\boldsymbol{F}_t =$

Lower Bound $z_{r,t}^*$	**Mass** $M_{r,t}$
$z_{1,t}^*$	$M_{1,t}$
$z_{2,t}^*$	$M_{2,t}$
$\vdots$	$\vdots$
$z_{R,t}^*$	$M_{R,t}$

.

Store cohorts in $\boldsymbol{F}_t$ in increasing order of their lower bounds - i.e., $z_{1,t}^* \leq z_{2,t}^* \leq ... \leq z_{R,t}^*$. If the number of cohorts C is less than R, set values for "empty cohorts" in rows $r = (C+1)...R$ with specific placeholder

[86] Cohorts are numbered in increasing order of $z_{S,c}^*$ for the algorithm as constructed in what follows.

values as follows: set $z^*_{r,t}$ equal to $z^*_{C,t}$ and set $M_{r,t}$ equal to zero for $r = (C+1)\,...R$.

For example, $\boldsymbol{F}_t$ could be initialized with the steady state distribution.[87] The state variable at time zero would then be given by

$$\boldsymbol{F}_0 = \begin{array}{|c|c|} \hline \textbf{Lower Bound } z^*_{r,0} & \textbf{Mass } M_{r,0} \\ \hline z^*_{1,0} = \bar{z} & M_{1,0} = \bar{M} \\ \hline z^*_{2,0} = \bar{z} & M_{2,0} = 0 \\ \hline \vdots & \vdots \\ \hline z^*_{R,0} = \bar{z} & M_{R,0} = 0 \\ \hline \end{array}.$$

where $\bar{z}$ and $\bar{M}$ are the steady state average aggregate productivity and number of firms, respectively.

The distribution of firms $\boldsymbol{F}_t$ changes over the period, so it is not quite sufficient to just use the notation $\boldsymbol{F}_t$. There are essentially three things that happen that affect $\boldsymbol{F}_t$ - endogenous exit at the beginning of the period, exogenous exit at the end of the period, and the addition of new entrants - and the algorithm needs to track the distribution as it changes within a period.

To do this, at the very beginning of the period, denote the distribution and number of firms by $\boldsymbol{F}_{S,t}$. The first event that happens in the period is the realization of Z_t. Following this, there may be endogenous firm exit after the realization of Z_t and determination of z^*_t: all incumbent firms with $z < z^*_t$, will exit. Let the distribution and number of firms that are left after any endogenous exit be denoted as $\boldsymbol{F}_{A,t} = f\left(\boldsymbol{F}_{S,t}, z^*_t\right)$, where laws of motion $f(\cdot)$ are defined over each element in $\boldsymbol{F}_{S,t}$. "Laws of motion" can be written as follows: for each lower bound value,[88]

$$z^*_{A_r,t} = \max\left[z^*_t, z^*_{S_r,t}\right] \text{ for } r = 1...R,$$

and for each cohort mass,[89]

$$M_{A_r,t} = \begin{cases} \left[1 - G_{r,t}\left(z^*_t\right)\right] M_{S_r,t} & \text{if } z^*_{S_r,t} < z^*_t \\ M_{S_r,t} & \text{otherwise.} \end{cases}$$

Production then takes place by the firms characterized by $\boldsymbol{F}_{A,t}$. The values for $\tilde{z}_{A,t}$ and $M_{A,t}$ are then calculated from $\boldsymbol{F}_{A,t}$ as follows:

$$\tilde{z}_{A,t} = \left\{ \frac{1}{\sum_{r=1}^{R} M_{A_r,t}} \left[\sum_{r=1}^{R} M_{A_r,t} \left[\tilde{z}_{A_r,t}\left(z^*_{A_r,t}\right)\right]^{\theta-1} \right] \right\}^{\frac{1}{\theta-1}}$$

[87] One cohort of firms that consists of the steady state number of firms $\overline{M}$ and the steady state average aggregate productivity $\overline{z}$ is assumed for convenience, but it is not necessary that the distribution is initialized at the steady state, nor that the initial distribution consists of only one cohort.

[88] Note that the algorithm is coded for $z^*_{A_r,t} = \max\left[z^{\min}, \max\left[z^*_t, z^*_{S_r,t}\right]\right]$ for $r = 1...R$, since it is possibly that, in any given period, z^*_t may fall below $z^{\min}$ - i.e., all firms over the possible range of productivity draws would find it profitably to produce. For the parameterization used in the model, however, z^*_t is always above $z^{\min}$.

[89] Note that $\left[1 - G_{r,t}\left(z^*_t\right)\right]$ is equal to $\left(z^*_t / z^*_{S_r,t}\right)^{\kappa}$, where $G_{r,t}(\cdot)$ is just the distribution as it is parameterized for the r^{th} cohort at time t.

and

$$M_{A,t} = \sum_{r=1}^{R} M_{A_r,t}.$$

Now, during the period, new entrants are being created by the households. New entrants are also subject to endogenous exit in period t, even though new entrants financed in period t may only start producing in period $t+1$. At the end of the period, all firms - new entrants and incumbents - are subject to an exogenous exit shock. The state variable at the end of the period then must reflect these additional changes, so next period's $\boldsymbol{F}_{S,t+1}$ is constructed from $\boldsymbol{F}_{A,t}$ as $\boldsymbol{F}_{S,t+1} = g\left(\boldsymbol{F}_{A,t}, N_t, z^*_{N,t}\right)$ with essentially four steps, as follows:

Step one: New entrants are subject to both endogenous and exogenous exit, so new entrants that remain at the end of the period are equal to $(1-\delta_M)\left[1 - G\left(z^*_{N,t}\right)\right] N_t$, and their distribution is Pareto with lower bound $z^*_{N,t}$.

Step two: Since all incumbent firms are subject to a probabilistic "death" shock at the end of the period, reduce $M_{A_r,t}$ to $(1-\delta_M) M_{A_r,t}$ for $r = 1...R$.

Step three: If there is more than one cohort for which $z^*_{A_r,t} < z^*_t$, then these cohorts now all share the same $z^*_{A_r,t} = z^*_t$ and can be aggregated into one cohort.[90] Aggregate these cohorts and place them into the $r = 1$ row, shifting remaining cohorts to the lowest numbered available rows.

Step four: Insert the cohort of new entrants into the $(r+1)^{th}$ row according to which $z^*_{A_r,t} < z^*_{N,t} < z^*_{A_{r+1},t}$ and move the previous $(r+1)\,...R-1$ cohorts to the next highest numbered row.[91]

It is fairly easy to see how $\boldsymbol{F}_{S,t+1}$ is constructed from $\boldsymbol{F}_{A,t}$, and the algorithm is coded for a two-stage implementation in this way since both $\boldsymbol{F}_{S,t}$ and $\boldsymbol{F}_{A,t}$ are needed to solve the model.[92]

Note that a law of motion for the state variable $\boldsymbol{F}_{S,t+1}$ can also be constructed directly from $\boldsymbol{F}_{S,t}$ as $\boldsymbol{F}_{S,t+1} = h\left(\boldsymbol{F}_{S,t}, z^*_t, N_t, z^*_{N,t}\right)$. Suppose that the state variable $\boldsymbol{F}_{S,t}$ at the beginning of the period is given by

$\boldsymbol{F}_{S,t} =$

Lower Bound $z^*_{S_r,t}$	**Mass** $M_{S_r,t}$
$z^*_{S_1,t}$	$M_{S_1,t}$
$z^*_{S_2,t}$	$M_{S_2,t}$
$\vdots$	$\vdots$
$z^*_{S_R,t}$	$M_{S_R,t}$

.

[90] Because, obviously,

$$\tilde{z}_t = \left\{ \frac{1}{\sum_{n=1}^{N} M_{n,t}} \left[\sum_{n=1}^{N} M_{n,t} \left[\tilde{z}_{n,t}\left(z^*_{n,t}\right)\right]^{\theta-1} \right] \right\}^{\frac{1}{\theta-1}} = \tilde{z}_t\left(z^*_t\right)$$

when $z^*_{n,t} = z^*_t$ for all n from $1...N$, and $M = \sum_{n=1}^{N} M_{n,t} M$.

[91] Note that, for calibrated values of the shock process, the number of cohorts C never gets close to the allowable number of rows R in the matrix F_A or F_S. Suppose, instead, that the lower bound threshold productivities did decrease monotonically over time. In this case, note that the $r = R$ cohort would have a mass only equal to $(1-\delta_M)^R$ times its initial size. For large enough values of R, then, these cohorts would be small enough to simply discard. Checks are embedded in the algorithm that alert to this possibility. Again, as long as R is sufficiently large, this never happens.

[92] The former is needed to construct the law of motion for the number of firms and their aggregate average productivity over time, and the latter is needed to construct period values of the same that determine current period production possibilities.

where $z^*_{S_1,t} \leq z^*_{S_2,t} \leq \ldots \leq z^*_{S_C,t}$ (and possibly $z^*_{S_C,t} = z^*_{S_{C+1},t} = \ldots = z^*_{S_R,t}$ and $M_{S_{C+1},t} = M_{S_{C+2},t} = \ldots = M_{S_R,t} = 0$ for some $C \in [2, 3, \ldots, R]$). Also suppose that there are N_t new entrants and the current period's threshold cutoff productivities are z^*_t and $z^*_{N,t}$. Each period there are u cohorts, with $u \in [0, 1, \ldots, R]$, that will experience endogenous exit (so $z^*_{r,t} < z^*_t$ for $r = 1...u$), and there will be some number of cohorts $v \in [0, 1, \ldots, R]$ that will have threshold productivities lower than $z^*_{N,t}$.[93] (These cohorts will therefore be at lower-numbered cohorts than the new entrant cohort.) From these facts, there are laws of motion that can be constructed for each element in $\boldsymbol{F}_{S,t}$. For example, for $r = 1$:

$$z^*_{S_1,t+1} = \begin{cases} \max\left[z^{\min}, z^*_{N,t}\right] & \text{if } v = 0 \\ \max\left[z^{\min}, z^*_t\right] & \text{if } v \neq 0 \text{ and } u > 0 \\ z^*_{S_1,t} & \text{if } v \neq 0 \text{ and } u = 0 \end{cases}$$

and

$$M_{S_1,t+1} = \begin{cases} (1-\delta_M)\left[1 - G\left(\max\left[z^{\min}, z^*_{N,t}\right]\right)\right] N_t & \text{if } v = 0 \\ (1-\delta_M)\sum_{r=1}^{u}\left[1 - G_{r,t}\left(\max\left[z^{\min}, z^*_t\right]\right)\right] M_{S_r,t} & \text{if } v \neq 0 \text{ and } u > 0 \\ (1-\delta_M) M_{S_1,t} & \text{if } v \neq 0 \text{ and } u = 0 \end{cases}.$$

The rest of the laws of motion are constructed analogously. Through a representation of the number and distribution of firms as an $R \times 2$ matrix and the constructions of these algorithms, then, both $\boldsymbol{F}_{A,t}$ and $\boldsymbol{F}_{S,t+1}$ can be found from simple "laws of motion" for $\boldsymbol{F}_{S,t}$.[94]

A.2 RCE

The following provides a more detailed description of the RCE, where functional dependencies are made explicit. For my model, a Rational Expectations Recursive Competitive Equilibrium is a collection of functions

$$\left\{V, \hat{V}, K^P, M^P_S, M^H_A, \tilde{z}^H_A, z^{*H}_N, J, \hat{J}, z^*, z^*_N, M^F_A, \tilde{z}^F_A, K^F, Q^F, f, g, h\right\}$$

such that

1. Value functions V and $\hat{V}$, policy functions K^P and M^P_S, and forecasting functions M^H_A, $\tilde{z}^H_A$, and z^{*H}_N solve the household's problem:

$$V(Z, M_S, \tilde{z}_S, K) = \max_{M'_S, K'}\left\{U(C, L) + \beta E_{Z'|Z}\left[V(Z', M'_S, \tilde{z}'_S, K')\right]\right\}$$

where

$$\hat{V}(Z, M_A, \tilde{z}_A, K, z^*_N) = \max_{M'_S, K'}\left\{U(C, L) + \beta E_{Z'|Z}\left[V(Z', M'_S, \tilde{z}'_S, K')\right]\right\}$$

[93] The value for u is equal to 0 when $z^*_t < z_{S_1,t}$, equal to R when $z^*_{S_R,t} \leq z^*_t$, or otherwise equal to $(r-1)$, where $z^*_{S_r,t} = \sup\left[z^*_{S_r,t} : z^*_t < z^*_{S_r,t}\right]$. The value of v is then equal to 0 when $z^*_{N,t} < z_{S_1,t}$, equal to R when $z^*_{S_R,t} \leq z^*_{N,t}$, or otherwise equal to r, where $z^*_{S_r,t} = \sup\left[z^*_{S_r,t} : z^*_{S_r,t} < z^*_{N,t}\right]$.

[94] The mathematical representation gets a bit tedious, but the code that implements this algorithm is available upon request.

with forecasts

$$\begin{aligned} M_A^H &= M_A^H(Z, M_S, \tilde{z}_S, K) \\ \tilde{z}_A^H &= \tilde{z}_A^H(Z, M_S, \tilde{z}_S, K) \\ z_N^{*H} &= z_N^{*H}(Z, M_S, \tilde{z}_S, K) \end{aligned}$$

subject to

$$K' = (1-\delta_K)K + I$$

$$M_S' = (1-\delta_M)M_A + (1-\delta_M)[1-G(z_N^*)]N$$

$$\tilde{z}_S' = \begin{array}{l} \Big([(1-\delta_M)M_A + (1-\delta_M)[1-G(z_N^*)]N_t]^{-1} \times \\ \Big[(1-\delta_M)M_A[\tilde{z}_A]^{\theta-1} + (1-\delta_M)[1-G(z_N^*)]N[\tilde{z}(z_N)]^{\theta-1}\Big]\Big)^{\frac{1}{\theta-1}} \end{array}$$

$$Q = \tilde{z}_A Z [K]^{\alpha}[L_Q]^{1-\alpha}[M_A]^{\theta/(\theta-1)}$$

$$Q = C + I$$

$$L = L_Q + L_F + L_B$$

so that

$$V(Z, M_S, \tilde{z}_S, K) = \hat{V}\left(Z, M_A^H, \tilde{z}_A^H, K, z_N^{*H}\right),$$

and policy functions are given as $K^P(Z, M_A, \tilde{z}_A, K, z_N)$ and $M_S^P(Z, M_A, \tilde{z}_A, K, z_N)$.[95]

2. Value functions J and $\hat{J}$ and forecasting functions M_A^F, $\tilde{z}_A^F$, K^F, and Q^F solve the firms' problem:

$$J(z; Z, M_{A,-1}, \tilde{z}_{A,-1}, K_{-1}) = \pi(z; Z, M_A, \tilde{z}_A, K, Q) + \beta(1-\delta_M)E_{Z'|Z}[J(z; Z', M_A, \tilde{z}_A, K)]$$

where

$$\hat{J}(z; Z, M_{A,}, \tilde{z}_{A,}, K, Q) = \pi(z; Z, M_A, \tilde{z}_A, K, Q) + \beta(1-\delta_M)E_{Z'|Z}[J(z; Z', M_A, \tilde{z}_A, K)]$$

with forecasts

$$\begin{aligned} M_A^F &= M_A^F(Z, M_{A,-1}, \tilde{z}_{A,-1}, K_{-1}) \\ \tilde{z}_A^F &= \tilde{z}_A^F(Z, M_{A,-1}, \tilde{z}_{A,-1}, K_{-1}) \\ K^F &= K^F(Z, M_{A,-1}, \tilde{z}_{A,-1}, K_{-1}) \\ Q^F &= Q^F(Z, M_{A,-1}, \tilde{z}_{A,-1}, K_{-1}) \end{aligned}$$

subject to

$$Q = \left[\int_{z_{\min}}^{\infty} 1_{\omega(z)\in\Omega}\left[q(z)^{(\theta-1)/\theta} M_A g(z)\right]dz\right]^{\theta/(\theta-1)}$$

[95] Policy functions could also be writen as $K^P\left(Z, M_A, \tilde{z}_A, K, z_N; M_A^H, \tilde{z}_A^H, z_N^{*H}\right)$ and $M_S^P\left(Z, M_A, \tilde{z}_A, K, z_N; M_A^H, \tilde{z}_A^H, z_N^{*H}\right)$, to make explicit their dependence on household forecasts.

$$P = \left[\int_{z_{\min}}^{\infty} 1_{\omega(z)\in\Omega} \left[p\left(z\right)^{1-\theta} M_A g\left(z\right) \right] dz \right]^{1/(1-\theta)}$$

$$q\left(z; Z, M_A, \tilde{z}_A, K, Q\right) = zZ \left[k\left(z; Z, M_A, \tilde{z}_A, K, Q\right)\right]^{\alpha} \left[l_q\left(z; Z, M_A, \tilde{z}_A, K, Q\right)\right]^{1-\alpha}$$

$$\pi\left(z; Z, M_A, \tilde{z}_A, K, Q\right) = \frac{1}{\theta - 1} \lambda\left(z; Z, M_A, \tilde{z}_A, K, Q\right) q\left(z; Z, M_A, \tilde{z}_A, K, Q\right) - wf/Z$$

so that

$$J\left(z; Z, M_{A,-1}, \tilde{z}_{A,-1}, K_{-1}\right) = \hat{J}\left(z; Z, M_{A,}^{F}, \tilde{z}_{A,}^{F}, K^F, Q^F\right)$$

and

$$z^*\left(z; Z, M_A, \tilde{z}_A, K, Q\right) = \sup\left[z : \hat{J}\left(z; Z, M_A, \tilde{z}_A, K, Q\right) = 0\right]$$

$$z_N^*\left(z; Z, M_A, \tilde{z}_A, K, Q\right) = \sup\left[z : \beta\left(1 - \delta_M\right) E_{Z'|Z}\left[J\left(z; Z', M_A, \tilde{z}_A, K\right)\right] = 0\right]$$

determine threshold productivities for entry and exit decisions.[96]

3. Firms and the Productivity Distribution: $\boldsymbol{F}_A$ is generated from $f\left(\boldsymbol{F}_S, z^*\right)$ and $\boldsymbol{F}'_S$ is generated from $g\left(\boldsymbol{F}_A, N, z_N^*\right) = g\left(f\left(\boldsymbol{F}_S, z^*\right), N, z_N^*\right) = h\left(\boldsymbol{F}_S, z^*, N, z_N^*\right)$.

4. Aggregation over $\boldsymbol{F}_S$ and $\boldsymbol{F}_A$: For the r cohorts in $\boldsymbol{F}_S$ and $\boldsymbol{F}_A$,

$$M_S = \sum_{r=1}^{R} M_{S_r}$$

$$M_A = \sum_{r=1}^{R} M_{A_r}$$

$$\tilde{z}_S = \left(\frac{1}{\sum_{r=1}^{R} M_{S_r}} \left[\sum_{r=1}^{R} M_{S_r} \left[\tilde{z}_{S_r}\right]^{\theta-1} \right] \right)^{\frac{1}{\theta-1}}$$

$$\tilde{z}_A = \left(\frac{1}{\sum_{r=1}^{R} M_{A_r}} \left[\sum_{r=1}^{R} M_{A_r} \left[\tilde{z}_{A_r}\right]^{\theta-1} \right] \right)^{\frac{1}{\theta-1}}$$

$$Q = q\left(\tilde{z}_A; Z, M_A, \tilde{z}_A, K, Q\right) M_A^{\theta/(\theta-1)}$$

$$P = p\left(\tilde{z}_A; Z, M_A, \tilde{z}_A, K, Q\right) M_A^{1/(1-\theta)}$$

$$K = k\left(\tilde{z}_A; Z, M_A, \tilde{z}_A, K, Q\right) M_A$$

$$L_Q = l_q\left(\tilde{z}_A; Z, M_A, \tilde{z}_A, K, Q\right) M_A$$

$$L_B = l_b\left(Z\right) N$$

$$L_F = l_f\left(Z\right) M_A$$

5. Rational Expectations:

[96] Again, threshold cutoff productivities cutoffs could also be written as $z^*\left(z; Z, M_A, \tilde{z}_A, K, Q; M_A^F, \tilde{z}_A^F, K^F, Q^F\right)$ and $z_N^*\left(z; Z, M_A, \tilde{z}_A, K, Q; M_A^F, \tilde{z}_A^F, K^F, Q^F\right)$ to show dependence on the forecasting functions.

(a) Households forecasts for M_A and $\tilde{z}_A$ are correct: M_A^H and $\tilde{z}_A^H$ are consistent with the law of motion f for $\boldsymbol{F}_S$ where $\boldsymbol{F}_A = f(\boldsymbol{F}_S)$; and household forecasts for z_N^* are correct: z_N^{*H} is consistent with z_N^*.

(b) Firm forecasts for M_A and $\tilde{z}_A$ are correct: M_A^F and $\tilde{z}_A^F$ are consistent with the law of motion h for $\boldsymbol{F}_S$ where $\boldsymbol{F}_S = h(\boldsymbol{F}_{S,-1}) = g(f(\boldsymbol{F}_{S,-1}))$, and firm forecasts for K and Q are correct: K^F and Q^F are consistent with K^P and M_S^P.

6. Factor markets are competitive and all markets clear.

A.3 Finding the Steady State

To solve the model, write the household's objective function as

$$\max E \sum_{j=0}^{\infty} \beta^j \left[\log(C_j) - \chi \frac{1}{1+\frac{1}{\varphi}} L_j^{\left(1+\frac{1}{\varphi}\right)} \right]$$

subject to the following constraints: the resource constraints in the final goods sector $Q_t = C_t + I_t$ and labor market $L_t = L_{B,t} + L_{F,t} + L_{Q,t}$, the laws of motion for the capital stock $K_{t+1} = (1-\delta_K) K_t + I_t$ and firms $M_{S,t+1} = (1-\delta_M) M_{A,t} + (1-\delta_M) \left[1 - G\left(z_{N,t}^*\right)\right] N_t$, and the production technologies for blueprints $bN_t = Z_t L_{B,t}$, fixed costs $fM_{A,t} = Z_t L_{F,t}$, and final goods[97] $Q_t = \tilde{z}_{A,t} Z_t \left[K_t\right]^{\alpha} \left[L_{Q,t}\right]^{1-\alpha} \left[M_{A,t}\right]^{1/(\theta-1)}$, noting of course that $\tilde{z}_{A,t}$ and $M_{A,t}$ are determined from the distribution of firms and productivities according to $f(\boldsymbol{F}_{S,t}, z_t^*)$, and that z_t^* and $z_{N,t}^*$ are determined from the firms' problem[98]. The process for technology is

[97] Note that since

$$Q_t = q_t\left(\tilde{z}_{A,t}\right)\left[M_{A,t}\right]^{\theta/(\theta-1)}$$

and

$$q_t \tilde{z}_{A,t} = \tilde{z}_{A,t} Z_t \left[k_t\left(\tilde{z}_{A,t}\right)\right]^{\alpha} \left[l_{q,t}\left(\tilde{z}_{A,t}\right)\right]^{1-\alpha},$$

aggregation results in the factor markets $K_t = k_t\left(\tilde{z}_{A,t}\right) M_{A,t}$ and $L_{Q,t} = l_t\left(\tilde{z}_{A,t}\right) M_{A,t}$ are used to write

$$Q_t = \tilde{z}_{A,t} Z_t \left[k_t\left(\tilde{z}_{A,t}\right)\right]^{\alpha} \left[l_{q,t}\left(\tilde{z}_{A,t}\right)\right]^{1-\alpha} \left[M_{A,t}\right]^{\theta/(\theta-1)}$$

simply as

$$Q_t = \tilde{z}_{A,t} Z_t \left[K_t\right]^{\alpha} \left[L_{Q,t}\right]^{1-\alpha} \left[M_{A,t}\right]^{1/(\theta-1)}.$$

[98] Recall,

$$z_t^* = \sup\left[z : \hat{J}\left(z; Z_t, M_{A,t}, \tilde{z}_{A,t}, K_t, Q_t\right) = 0\right]$$

and that

$$z_{N,t}^* = \sup\left[z : \beta\left(1-\delta_M\right) E_{Z_{t+1}|Z_t}\left[J\left(z; Z_{t+1}, M_{A,t}, \tilde{z}_{A,t}, K_t\right)\right] = 0\right].$$

$Z_{t+1} = (1-\rho)\mu + \rho Z_t + \varepsilon_t$. Simplifying and substituting constraints results in the following Lagrangian:

$$\begin{aligned}
\mathcal{L} &= \max E \sum_{t=0}^{\infty} \beta^t \left\{ \log (C_t) - \chi \frac{1}{1+\frac{1}{\varphi}} \left(L_{Q,t} + L_{B,t} + \frac{f}{Z_t} M_{A,t} \right)^{\left(1+\frac{1}{\varphi}\right)} \dots \right. \\
&\quad -\Lambda_{1,t} \left[C_t + K_{t+1} - (1-\delta_K) K_t - \tilde{z}_{A,t} Z_t [K_t]^{\alpha} [L_{Q,t}]^{1-\alpha} [M_{Q,t}]^{1/(\theta-1)} \right] \dots \\
&\quad \left. -\Lambda_{2,t} \left[\frac{M_{S,t+1}}{(1-\delta_M)\left[1-G\left(z^*_{N,t}\right)\right]} - \frac{M_{A,t}}{\left[1-G\left(z^*_{N,t}\right)\right]} - L_{B,t} \frac{Z_t}{b} \right] \right\},
\end{aligned}$$

from which the first order conditions can then be expressed as

$$\begin{aligned}
\partial/\partial C_t &: \quad \tfrac{1}{C_t} = \Lambda_{1,t} \\
\partial/\partial K_{t+1} &: \quad \Lambda_{1,t} = \beta E_t \left[\Lambda_{1,t+1} \left[(1-\delta_K) + \alpha \tilde{z}_{A,t} Z_t [K_t]^{\alpha-1} [L_{Q,t}]^{1-\alpha} [M_{A,t}]^{1/(\theta-1)} \right] \right] \\
\partial/\partial L_{Q,t} &: \quad \chi \left(L_{Q,t} + L_{B,t} + \tfrac{f}{Z_t} M_{A,t} \right)^{\frac{1}{\varphi}} = \Lambda_{1,t} (1-\alpha) \tilde{z}_{A,t} Z_t [K_t]^{\alpha} [L_{Q,t}]^{-\alpha} [M_{A,t}]^{1/(\theta-1)} \\
\partial/\partial L_{B,t} &: \quad \chi \left(L_{Q,t} + L_{B,t} + \tfrac{f}{Z_t} M_{A,t} \right)^{\frac{1}{\varphi}} = \Lambda_{2,t} \tfrac{Z_t}{b} \\
\partial/\partial M_{S,t+1} &: \quad \Lambda_{2,t} \tfrac{1}{(1-\delta_M)\left[1-G\left(z^*_{N,t}\right)\right]} = \beta E_t \left[-\chi \left(L_{Q,t} + L_{B,t} + \tfrac{f}{Z_t} M_{A,t} \right)^{\frac{1}{\varphi}} \tfrac{f}{Z_t} + \dots \right. \\
&\qquad \Lambda_{1,t+1} \tfrac{1}{\theta-1} \tilde{z}_{A,t+1} Z_{t+1} [K_{t+1}]^{\alpha} [L_{Q,t+1}]^{1-\alpha} [M_{A,t+1}]^{1/(\theta-1)-1} + \dots \\
&\qquad \left. \Lambda_{2,t+1} \tfrac{1}{\left[1-G\left(z^*_{N,t+1}\right)\right]} \right] \\
\partial/\partial \Lambda_{1,t} &: \quad \tilde{z}_{A,t} Z_t [K_t]^{\alpha} [L_{Q,t}]^{1-\alpha} [M_{A,t}]^{1/(\theta-1)} = C_t + K_{t+1} - (1-\delta_K) K_t \\
\partial/\partial \Lambda_{2,t} &: \quad \tfrac{M_{S,t+1}}{(1-\delta_M)\left[1-G\left(z^*_{N,t}\right)\right]} - \tfrac{M_{A,t}}{\left[1-G\left(z^*_{N,t}\right)\right]} = L_{B,t} \tfrac{Z_t}{b}.
\end{aligned}$$

To characterize the nonstochastic steady state, let $Z_t = Z_{t+1} = Z$ and express all steady state values of aggregate variables similarly as $X_t = X$. Note that since in the steady state, $z^*_t = z^*_{N,t} = z^*$, it is also the case that $\tilde{z}_{S,t} = \tilde{z}_{A,t} = \tilde{z}$ and $M_{S,t} = M_{A,t} = M$ - that is, there is no endogenous exit other than that for each period's entering cohort of new firms, so there is a stationary distribution of firm productivities with "average" productivity $\tilde{z}$ given by $\tilde{z} = (\kappa/(\kappa-\theta+1))^{\frac{1}{\theta-1}} z^*$ and a steady state number of firms equal to M. In the steady state, then, the first-order conditions of the households' problem yield the following five equations

$$\begin{aligned}
SS1 &: \quad 1 = \beta \left(1 - \delta_K + \alpha \tilde{z} Z [K]^{\alpha-1} [L_Q]^{1-\alpha} [M]^{1/(\theta-1)} \right) \\
SS2 &: \quad \chi \left(L_Q + L_B + \tfrac{f}{Z} M \right)^{\frac{1}{\varphi}} = \tfrac{1}{C} (1-\alpha) \tilde{z} Z [K]^{\alpha} [L_Q]^{\alpha} [M]^{1/(\theta-1)} \\
SS3 &: \quad \tfrac{b}{Z} \chi \left(L_Q + L_B + \tfrac{f}{Z} M \right)^{\frac{1}{\varphi}} \left[\tfrac{1}{(1-\delta_M)[1-G(z^*)]} - \beta \tfrac{1}{[1-G(z^*)]} \right] = \dots \\
&: \quad \quad \tfrac{1}{C} \beta \tfrac{1}{\theta-1} \tilde{z} Z [K]^{\alpha} [L_Q]^{1-\alpha} [M]^{1/(\theta-1)-1} - \beta \chi \left(L_Q + L_B + \tfrac{f}{Z} M \right)^{\frac{1}{\varphi}} \tfrac{f}{Z} \\
SS4 &: \quad C + \delta_K K = \tilde{z} Z [K]^{\alpha} [L_Q]^{1-\alpha} [M]^{1/(\theta-1)} \\
SS5 &: \quad \tfrac{\delta_M M}{(1-\delta_M)[1-G(z^*)]} = L_B \tfrac{Z}{b}
\end{aligned}$$

Together with the cutoff productivity condition from the firms' problem gives six equations in six unknowns.[99] To characterize the cutoff productivity condition from the firms' problem in the steady state, note that the

[99] Seven and seven, if you include $\tilde{z}$, but $\tilde{z}$ is just a function of z^*. Note that $(\kappa/(\kappa-\theta+1))^{\frac{1}{\theta-1}} z^*$ or $\tilde{z}(z^*)$ could be substituted for the average productivity $\tilde{z}$ in the preceding set of equations to eliminate it as another variable in the system, but a simple $\tilde{z}$ is retained for clarity of expression.

conditions

$$z_t^* = \sup\left[z : \hat{J}\left(z; Z_t, M_{A,t}, \tilde{z}_{A,t}, K_t, Q_t\right) = 0\right]$$

where

$$\hat{J}\left(\cdot\right) = \pi_t\left(z; Z_t, M_{A,t}, \tilde{z}_{A,t}, K_t, Q_t\right) + \beta\left(1 - \delta_M\right) E_{Z_{t+1}|Z_t}\left[J\left(z; Z_{t+1}, M_{A,t}, \tilde{z}_{A,t}, K_t\right)\right]$$

and

$$z_{N,t}^* = \sup\left[z : \beta\left(1 - \delta_M\right) E_{Z_{t+1}|Z_t}\left[J\left(z; Z_{t+1}, M_{A,t}, \tilde{z}_{A,t}, K_t\right)\right] = 0\right]$$

are both simply

$$z^* = \sup\left[z : \pi\left(z^*; Z, M, \tilde{z}, K, Q\right) = 0\right]$$

in the steady state. The equation $\pi\left(z^*; Z, M, \tilde{z}, K, Q\right) = 0$ can be expressed in terms of aggregate variables, after making appropriate substitutions and a bit of algebra, as

$$SS6 \quad : \quad \frac{1-\alpha}{\mu}\frac{Q}{L_Q}\frac{1}{Z}f = \left(\mu - 1\right)\mu^{-\theta}\left(\frac{1}{z^* Z}\alpha^{-\alpha}\left(1-\alpha\right)^{-(1-\alpha)}\left[\frac{\alpha}{\mu}\frac{Q}{K}\right]^{\alpha}\left[\frac{1-\alpha}{\mu}\frac{Q}{L_Q}\right]^{1-\alpha}\right)^{1-\theta} Q.$$

This last equation added to the previous five from the household's problem yields six equations that can then be solved for the six unknowns $\{z^*, C, K, L_Q, L_B, M\}$, and since all other variables can be found from these six, these equations completely characterize the steady state.[100]

A.4 Solving the DSGE Model

To solve the model:

1. Find the steady state of the model.

2. Set discrete grids and the Markov transition matrix for Z. Calibration for the process governing aggregate productivity Z_t as an $AR\left(1\right)$ process is standard in the Real Business Cycle (RBC) literature. Approximate the process for Z_t as an n-point Markov chain.[101] Simulate a sequence for Z.

3. Set discrete grids for the state variables and current-period variables that must be predicted used in household and firm value functions.[102] For the households, set discrete grids for M (used for M_S and M_A), $\tilde{z}$ (used for $\tilde{z}_S$ and $\tilde{z}_A$), K, and z_N^*. For the firms, set discrete grids for z, M_A, $\tilde{z}_A$, K, and Q.

[100] Note: The steady state was also found through a dynamic programming approach, performed as a check on the calculations. The algorithm is available upon request.

[101] The method of Tauchen and Hussey (1991) can be used for this model. Note that a small n can be used due to the trade-off between the number of grid points for Z_t and speed/memory, but robustness checks should be performed for larger values of n. A model with a large n takes much longer, on an already long runtime, to solve, but the runtime can be reduced considerably when results from a small n model can be extrapolated.

[102] Setting the range for the grid points is a bit of an art: for a given number of grid points, a smaller range will yield smaller intervals between grid points (potentially improving model accuracy), but may require extrapolation if simulated values and/or policy functions extend beyond the ends of the grids (potentially decreasing model accuracy). When the model is solved, checks are performed to ensure that simulated values remain within the range established over the grid points.

4. Initialize value functions

$$V\left(Z, M_S, \tilde{z}_S, K\right) \text{ and } \hat{V}\left(Z, M_A, \tilde{z}_A, K, z_N^*\right)$$

and policy functions

$$K^P\left(Z, M_A, \tilde{z}_A, K, z_N^*\right) \text{ and } M_S^P\left(Z, M_A, \tilde{z}_A, K, z_N^*\right)$$

for households. Initialize value functions

$$J\left(z; Z, M_A, \tilde{z}_A, K\right) \text{ and } \hat{J}\left(z; Z, M_A, \tilde{z}_A, K, Q\right)$$

for firms.[103]

5. Initialize coefficients for the forecasting functions. For household forecasts, guess M_A, $\tilde{z}_A$, and z_N^* as functions of Z, M_S, $\tilde{z}_S$, and K. For the forecasts, guess M_A, $\tilde{z}_A$, K, and Q as functions of Z, $M_{A,-1}$, $\tilde{z}_{A,-1}$, and K_{-1}.[104]

6. Note that firm profits and household utility are static in the sense that they can be calculated before any optimization loops. Therefore, to save time during these loops:

   (a) For firm profits, calculate $\pi\left(z; Z, M_A, \tilde{z}_A, K, Q\right)$ over the grids set for the firms' problem. Since Z, M_A, $\tilde{z}_A$, K, and Q are sufficient to determine wage and rental rates, marginal costs can be determined, as well as firm prices and quantities. Firm profits can then be stored in a static 6-dimensional matrix.

   (b) For household utility, note that, for given values of Z, M_A, z_A, K, and z_N^*, the optimal choices for the current period's C and L are determined by a static optimization problem, given household choices for next period's capital stock K' and firm stock M_S' . Therefore, set $u\left(\cdot\right)$ as a 7-dimensional matrix over $\{Z, M_A, z_A, K, z_N^*, M_S', K'\}$ and calculate utility over the grids defined for the household's problem.[105] Note that values for new entrants and investment must be non-negative, so values for impossible choices can be set to -9999.[106]

7. Optimization Loop.

   (a) Given an initial (or a previous iteration's) value for $J\left(z; Z, M_A, \tilde{z}_A, K\right)$, calculate

$$\hat{J}\left(z; Z, M_A, \tilde{z}_A, K, Q\right) = \pi_t\left(z; Z, M_A, \tilde{z}_A, K, Q\right) + \beta\left(1-\delta\right) E_{Z'|Z}\left[V\left(z; Z', M_A', z_A', K\right)\right].$$

[103] These can all be set to zero.

[104] A good initial guess can save a good deal of time - e.g., a better starting guess for M_A is M_S rather than the steady state value of M.

[105] Note that this speeds computational time, but the memory requirements are non-trivial here.

[106] Note that less memory is used by setting these values to zero, and then only optimizing household choices over permitted values. This also speeds computational time.

(b) Given an initial (or previous iteration's) value for $V(Z, M_S, \tilde{z}_S, K)$, calculate

$$\hat{V}(Z, M_A, \tilde{z}_A, K, z_N^*) = \max_{K', M'} U(C, L) + \beta E_{Z'|Z}\left[V(Z', M_S', \tilde{z}_S', K')\right]$$

and determine $K^P(Z, M_A, \tilde{z}_A, K, z_N^*)$ and $M_S^P(Z, M_A, \tilde{z}_A, K, z_N^*)$. This is done in several steps for computation speed:

i. First, create a gridded interpolant over V.[107]

ii. Calculate an intermediate value function

$$\hat{V}_\iota = (Z, M_A, \tilde{z}_A, K, z_N^*, M', K') = u(Z, M_A, \tilde{z}_A, K, z_N^*, M', K') + \beta E_{Z'|Z}\left[V(Z', M_S', \tilde{z}_S', K')\right].$$

iii. For all values of Z, M_A, $\tilde{z}_A$, K, and z_N^*, use spline interpolation over candidate value functions $\hat{V}_\iota(Z, M_A, \tilde{z}_A, K, z_N^*, M', K')$ (over feasible M' and K' values) to find $\hat{V}(Z, M_A, \tilde{z}_A, K, z_N^*)$, $K^P(Z, M_A, \tilde{z}_A, K, z_N^*)$, and $M_S^P(Z, M_A, \tilde{z}_A, K, z_N^*)$.[108]

(c) Simulate the model.[109]

i. Create gridded interpolants for household policy functions

$$K^P(Z, M_A, \tilde{z}_A, K, z_N^*) \text{ and } M_S^P(Z, M_A, \tilde{z}_A, K, z_N^*)$$

and firm value and profit functions

$$\hat{J}(z; Z, M_A, \tilde{z}_A, K, Q) \text{ and } \pi(z; Z, M_A, \tilde{z}_A, K, Q).$$

Initialize state variables.[110]

[107] Interpolation is needed because, even for choices of M' and K' on the grids, resulting values for $\tilde{z}'$ may not lie on a grid point. Note that values for $\tilde{z}'$ can be determined from values of M_A, $\tilde{z}_A$, z_N^*, and M'. (Again, these values of $\tilde{z}'$ can be stored in a static 4-dimensional matrix for speed.)

[108] Gridded interpolants were created over values for M' and K'. $2D$ grids consisting of 500 M' and 500 K' values were created, and the interpolants evaluated over those points. Policy functions and the value function were then simply calculated from max functions. Policy functions were also calculated using optimization routines over the gridded interpolants, but using grid computing is much faster. (Optimization makes policy functions somewhat smoother, but not notably so for interpolated grids in excess of 200 points. Calculating policy functions using optimization can be done in the final iterations, but it is not strictly necessary.) Note also that exact lower and upper bounds - not on the discrete gird points - can be appended to the submatrix before creating the gridded interpolants. (That is, this feature would allow choices of new entrants and investment to be exactly zero and would also add the possibility of choosing higher values of M' and K' beyond the upper limit of the grids.) The model was also solved with this addition, but for suitably chosen grids, the results do not differ and the simulations are unaffected. Allowing extrapolation in this way makes places in the policy functions with extreme combinations of state variables look "prettier" with this additional feature, but policies chosen during the simulations never approach these areas.

[109] The model was simulated for one run of an economy with $T = 10500$ periods, discarding the first 500 observations when calculating the forecasting functions. Note that, using parallel programming techniques, simulating more than one economy does not substantially increase computational time. This can be useful if forecasting functions seem otherwise slow to converge. This can also serve as an additional check that the model results are not an artifact of the particular sequence for Z. The model was solved in this way also, simulating 12 different economies simultaneously over 12 different realizations for $\{Z_t\}$.

[110] Values in $F_{S,0}$ can be set with $M_{S_1,0}$ and $z_{S_1,0}^*$ equal to steady state values for M and z^*, and values for $z_{S_r,0}^*$ set to z^* and $M_{S_r,0}$ set to zero for $r = 2$ to R. The value for K can be set to its steady state value.

ii. For each time period $t = 1$ to T:

A. Guess values for z_t^* and $z_{N,t}^*$.

B. Find M_A and $\tilde{z}_A$ from $f(\boldsymbol{F}_{S,t}, z_t^*)$.

C. Find M' and K' from $M_S^P(Z, M_A, \tilde{z}_A, K, z^*)$ and $K^P(Z, M_A, \tilde{z}_A, K, z^*)$.

D. Find Q.

E. Check that

$$\hat{J}(z^*; Z, M_A, \tilde{z}_A, K, Q) = 0$$

and

$$\hat{J}(z_N^*; Z, M_A, \tilde{z}_A, K, Q) - \pi(z_N^*; Z, M_A, \tilde{z}_A, K, Q) = 0.$$

If not, try other values for z_t^* and $z_{N,t}^*$ and repeat.[111]

(d) Use the simulation results to update the forecasting functions.[112]

(e) Update $J(z; Z, M_{A,-1}, \tilde{z}_{A,-1}, K_{-1})$ from values of $\hat{J}(z; Z, M_A, \tilde{z}_A, K, Q)$, using predictions for M_A, $\tilde{z}_A$, K, and Q from the updated forecasts.[113]

(f) Update $V(Z, M_S, \tilde{z}_S, K)$ from values of $\hat{V}(Z, M_A, \tilde{z}_A, K, z_N^*)$, using predictions for M_A, $\tilde{z}_A$, and z_N^* from the updated forecasts.[114]

(g) Repeat the optimization loop until value functions and forecasting functions have converged.

A.5 Mapping Functions Accuracy: Further Tests

A.5.1 Actual v. Predicted Values

In the simulations, deviations of actual from predicted values are in fact quite small. Correlations between actual and predicted series are all in excess of 0.999, errors are mean zero, and the average absolute deviations are small for all predicted series, with the maximum deviations ranging from 0.05 to 0.6 as a percent. Table 4 summarizes the accuracy of the mappings for households and firms.

[111] Checking

$$\hat{J}(z_N^*; Z, M_A, \tilde{z}_A, K, Q) - \pi(z_N^*; Z, M_A, \tilde{z}_A, K, Q) = 0$$

is equivalent to

$$\beta(1-\delta_M) E_{Z_t'|Z_t}\left[J\left(z_N^*; Z', M_A, \tilde{z}_A, K\right)\right] = 0.$$

Note that when first solving the model, the simulations will not be very accurate. Tolerances should be adjusted accordingly.

[112] Selecting an updating parameter for coefficients on the forecasting functions is a bit of an art here. E.g., since the model is not simulated very accurately in early iterations, an updating parameter that is not aggressive enough can waste time (and does not seem to work well).

[113] Predicted values for M_A, $\tilde{z}_A$, K, and Q from $Z, M_{A,-1}$, $\tilde{z}_{A,-1}$, and K_{-1} can be constructed from the forecasting functions; and the values for $J\left(z; Z, M_{A,-1}, \tilde{z}_{A,-1}, K_{-1}\right)$ must then be interpolated from $\hat{J}(z; Z, M_A, \tilde{z}_A, K, Q)$.

[114] Again, predicted values for M_A, $\tilde{z}_A$, and z_N^* from Z, M_S, $\tilde{z}_S$, and K can be constructed from the forecasting functions; and the values for $V(Z, M_S, \tilde{z}_S, K)$ must then be interpolated from $\hat{V}\left(Z, M_A, \tilde{z}_A, K, z_N^*\right)$.

Some of the maximum percentage errors in Table 4 may appear to be not all that small. However, histograms of the prediction errors reveal that most of the errors are in fact quite small and that these larger errors occur very infrequently. See Figures 3 and 4 for plots of the histograms. For example, in a simulated series of $10,000$ observations, for household predictions of $M_{A,t}$, errors in excess of 0.15 percent occur only 0.14 percent of the time and errors in excess of 0.05 percent occur in only 1.82 percent of the simulated periods. The household prediction errors for $\tilde{z}_{A,t}$ in excess of 0.05 percent occur only 0.01 percent of the time, and errors in excess of 0.01 percent occur in only 2.61 percent of the simulations. Similarly, for firms, mappings for $M_{A,t}$ and Q_t have errors larger than 0.5 percent in only 0.12 and 0.13 percent, respectively, of the simulated observations. Further, prediction errors are mean zero, the size of the errors are unrelated to the state variables, and the plots of the errors show no systematic patterns.[115]

A.5.2 Quarter-Ahead Forecasts

However, perhaps these prediction errors are in fact "large" and matter in significant ways - i.e., perhaps errors would accumulate, so that if forecasts were made on top of forecasts, deviations of actual values from predicted would be significant and predictable.[116] To further explore the accuracy of the mapping functions, x-quarter ahead forecasts are constructed for households and firms (where the forecasts assume that the sequence of future shocks Z_t is known with certainty), The results from x-quarter ahead forecasts, with x set at 1, 10, and 100 quarters are summarized in Table 5.[117] Again, deviations of actual from predicted values in the simulations are small on average - even for forecasts as long as 25 years out. Even the largest errors are still relatively small and do not occur frequently. Deviations are not systematic, and are hard to even visually detect until the 100-quarter ahead forecasts. For the 1-, 10-, and 100-quarter ahead forecasts, histograms of household forecast errors are in Figures 17 - 19, and in Figures 20 - 22 for firm forecast errors. The mapping functions are not perfect, but they are still very good, even using a metric forecasting as far as 25 years out.

[115] Note that positive and negative errors occur with roughly equal probability in all series except for household predictions of $M_{A,t}$ and $\tilde{z}_{A,t}$. This is due to the nonlinearity in the specification of the household mapping functions - that is, since households know the value of $M_{S,t}$ and $\tilde{z}_{S,t}$ at the beginning of the period, whatever value of z_t^* obtains, $M_{A,t}$ must be less than or equal to $M_{S,t}$ and $\tilde{z}_{A,t}$ must be greater than or equal to $\tilde{z}_{S,t}$. The result is that, whatever the predictions for $M_{A,t}$ and $\tilde{z}_{A,t}$, there are tighter bounds on positive errors of actual minus predicted values for $M_{A,t}$, and tighter bounds for errors for predictions of $\tilde{z}_{A,t}$ that are lower than actual values. This accounts for the skewness in the histograms for household mappings of $M_{A,t}$ and $\tilde{z}_{A,t}$.

[116] This would be the case if, as den Haan (2010) has argued, the approximating functions would push the obervations further from the truth each period. This would be missed in calculating errors from current period forecasts, as the true data generating process is used each period to put the approximating function back on track.

[117] For the same purpose, Krusell and Smith (1995, 1998) report results for 1-quarter and 25-year ahead forecasts.

A.5.3 Den Haan (2010) Measure

Den Haan (2010) has pointed to potential weaknesses in the former accuracy measures, even in the x-quarter ahead forecast comparisons of Krusell and Smith (1995). The basic criticism of using the R^2 and regression error $\hat{\sigma}^2$'s is that, each period, the forecasts are made each period from the current period's "true values" of state variables - if the approximating functions would want to push the forecasts away from the true ones each period, the error terms would understate the problem because the true data generating process is used to put the approximating function "back on track" each period. In response (mainly to this issue, but there are also several others), den Haan (2010) suggests testing the accuracy of approximating functions for laws of motion that are not "put back on track" each period, but start each period with the last period's approximate value. Then, if the approximate series followed the true one closely, the approximation would be a good one. The alternative den Haan (2010) procedure claims to be more powerful in the sense of its ability to better detect differences between the truth and approximating functions and more insightful in determining where and why the approximation fails. Therefore, the accuracy of the mapping functions in my model is further explored using an adaptation of the den Haan measure for my model.

In my model, both households and firms must create mappings for current period variables, so the accuracy of both must be checked. Not only should the mappings be compared with the "true" values, but in my model they should also be compared against each other since households and firms are each using a different set of variables from which they construct these functions. Recall that no mapping functions are relied upon in the model simulations since household policies and firm choices are based on current-period values of state variables; however, since household choices cannot be made until firm choices are known, and firm choices cannot be made until household choices are known, there is a root-finding operation that must be solved in each period of the simulation to ensure that all markets are cleared.[118] Because of this "market-clearing" aspect, I adapt the den Haan (2010) procedure by evaluating it separately over household and firm mapping functions and then comparing the two sets of approximating series against each other.

The den Haan procedure to check the accuracy of the mapping functions is adapted for my model with the following steps. To check the accuracy of the household functions:

1. Generate a times series for Z_t and choose initial values for the state variables.[119] Since the algorithm that solves the model obtains the mapping functions using simulated data, the time series used here for Z_t is a new simulation - i.e., it is not the same series as that used in the model to solve for the mapping functions.

2. Simulate the model. Recall that no mapping functions are used in the model simulations, so (following the notation in den Haan (2010)), denote these variables as X_t^{truth}.[120]

[118] See Section A.4 for details.

[119] Easy choices for the state variables K_t and the state variable describing the distribution of firms $\boldsymbol{F}_{S,t}$ are just steady state values of K_t, $\tilde{z}_t$, and M_t. Sufficiently long simulations obviate the need to simulate the model with different initial distributions.

[120] Note that, as den Haan (2010) has pointed out, X_t^{truth} may not typically be calculated without any numerical error -

3. Generate time series for variables X_t^{approx} using only the household mapping functions[121]

$$\log\left(X_t^{approx}\right) = \alpha_1 + \alpha_2 Z_t + \alpha_3 M_{S,t}^{approx} + \alpha_4 \tilde{z}_{S,t}^{approx} + \alpha_5 K_t^{approx}.$$

These series are based on the same draw for Z_t and the same initial conditions as for X_t^{truth}, but are not related to X_t^{truth} in any other way. Note that households only have to forecast $M_{A,t}$, $\tilde{z}_{A,t}$, and $z_{N,t}^*$ each period, but the approximating series that result for z_t^*, K_t, and Q_t are also of interest if their values deviate from the "true" ones.

4. Define the error term as

$$\log\left(X_t^{truth}\right) - \log\left(X_t^{approx}\right).$$

Note that where the R^2 uses $\alpha_1 + \alpha_2 Z_t + \alpha_3 M_{S,t} + \alpha_4 \tilde{z}_{S,t} + \alpha_5 K_t$ to construct forecasts of current-period variables, this accuracy test uses $\alpha_1 + \alpha_2 Z_t + \alpha_3 M_{S,t}^{approx} + \alpha_4 \tilde{z}_{S,t}^{approx} + \alpha_5 K_t^{approx}$. By using approximating values each period instead of re-setting them to the true ones, it is - as den Haan (2010) claims - much more difficult for the approximating functions to track the true values, so it is therefore a much more difficult accuracy test.

5. Report the maximum error.[122]

6. Plot the two generated series for all variables of interest, referred to in den Haan (2010) as the "essential accuracy plots".

To check the accuracy of the firm mapping functions, the procedure is similar:

1. Start with the same times series for Z_t and initial values for the state variables that were generated to check the household mappings.

2. Simulate the model. Denote these variables as X_t^{truth}. (These are the same simulated "true" values as used to check the household forecasts above, so it is really not necessary to simulate them again.)

3. Generate time series for variables X_t^{approx} using only the firm mapping functions

$$\log\left(X_t^{approx}\right) = \alpha_1 + \alpha_2 Z_t + \alpha_3 M_{A,t-1}^{approx} + \alpha_4 \tilde{z}_{A,t-1}^{approx} + \alpha_5 K_{t-1}^{approx},$$

for example, in a cases where a (finite) number of agents in a simulation is not high enough or the grids used to construct distributions are not fine enough. In my model, however, K_t, $\boldsymbol{F}_{S,t}$, and $\boldsymbol{F}_{A,t}$, (and $M_{S,t}$, $\tilde{z}_{S,t}$, $M_{A,t}$, and $\tilde{z}_{A,t}$) are tracked *exactly* with the only errors being the rounding errors typical in computational work.

[121] That is, for $t = 1$, use values of Z_1, $M_{S,1}$, $\tilde{z}_{S,1}$ and K_1 to forecast values of $M_{A,1}$, $\tilde{z}_{S,1}$, and $z_{N,1}^*$ - denote these variables as $M_{A,1}^{approx}$, $\tilde{z}_{A,1}^{approx}$, and $\left[z_{N,1}^*\right]^{approx}$. Use these values with the household policy functions to obtain values of all variables in the current period and obtain next period values for $M_{S,2}$, $\tilde{z}_{S,2}$, and $K_{S,2}$ using the laws of motion in equations 10, 11, and 9, respectively, and denote these variables as $\tilde{z}_{S,2}^{approx}$, $M_{S,2}^{approx}$, and $K_{S,2}^{approx}$. For $t = 2...T$, use values of $\tilde{z}_{S,t}^{approx}$, $M_{S,t}^{approx}$, and $K_{S,t}^{approx}$, and repeat.

[122] Note that with logged variables, no scaling is necessary.

and obtain the implied values of z_t^* and $z_{N,t}^*$. These series are based on the same draw for Z_t and the same initial conditions as for X_t^{truth}, but are not related to X_t^{truth} in any other way. Note that firms only have to predict $M_{A,t}$, $\tilde{z}_{tA,}$, K_t, and Q_t each period, but the approximating series that result for z_t^* and $z_{N,t}^*$ is also of interest.

4. Define the error term as

$$\log\left(X_t^{truth}\right) - \log\left(X_t^{approx}\right).$$

5. Report the maximum error.

6. Plot the two generated series for all variables of interest.

And finally, check the approximating series that result from household and firm mappings against each other. Of particular interest is the approximating series for z_t^* and $z_{N,t}^*$ obtained using only household mappings in comparison with those that result from using only firm mappings. (The implied series for z_t^* and $z_{N,t}^*$ that result from the firm mappings are essentially a product of all four of the firm approximating functions.)

The maximum errors for household and firm forecasting functions using this metric are also small, and den Haan's "essential accuracy plots" all look quite good. The results are summarized in Tables 6 and 7, and deviations of the approximating series from the simulated values are shown in Figures 23 and 24.[123]

A.5.4 Grid Sizes and Simulations

In the analysis above, I have tried to show that the mappings are very accurate; however, they of course are not perfect. Are they accurate enough to conclude that the model is in fact solved? Could a different specification lead to a different result? Since determining whether or not the mappings are "accurate enough" is largely a judgement call in these types of models, four additional experiments were conducted to see how grid sizes and other properties of the simulation affected their accuracy.

Value functions and policy rules for households and firms are smooth due to the incorporation of interpolation techniques into the solution algorithm, but it might be argued that households could do better - i.e., their policy functions might be more accurate - and firm value functions would be more precise if the number of grid points over which the model was solved was larger. If solving the model over denser grids changed policy functions, value functions, mapping rules, or mapping errors, it might be necessary to increase the grid sizes to find an accurate solution. The first experiment increased the number of grid points over which value functions and policy rules were defined for households and firms by 50 percent.[124] The results of this experiment demonstrate that the algorithm for solving the model does not require particularly fine grids - with a reasonable numbers of gridpoints, the interpolation embedded in the algorithm obviates the need to solve the model over a larger number of gridpoints (thus decreasing memory requirements to solve the

[123] The "essential accuracy plots" are omitted, as the deviations are difficult to see.

[124] Grid sizes were limited by memory and speed considerations.

model). Results of the simulation do not change, mapping rules are not altered, and value functions and policy rules also do not change markedly. As well, the histograms of the mapping errors do not change, so the accuracy of the mapping functions does not improve with denser grids.

Another possibility is that a n-point Markov chain may misrepresent the shock process (e.g., perhaps the process is not smooth enough) that underlies the business cycle. Perhaps forecasts would improve (and potentially even the results of the model would change) if movements in Z_t were defined over a larger n-point Markov chain. To ensure that the results of the model were not an artifact of the particular instance of the shock process used, the model was solved again using a $(2n + 1)$-point Markov chain.[125] Again, the results remain qualitatively similar and quantitatively analogous.

The final experiments considered variants on the simulations. This was an important dimension to check, since the simulation plays an important role in solving the model (and since simulations can be rather inefficient numerical procedures). One experiment considered solving the model with a longer simulation length. The basic model was solved for a simulated series of 10,500 observations, discarding the first 500 observations to mitigate the impact of the initial distribution of firms and productivities on the results. While a series of $10,000$ observations seems sufficiently long, there are some variables that are slow-moving, so perhaps the adjustments of these variables cannot be understood without a longer simulation. One potential issue to note here is that while the average aggregate productivity of firms is mean-reverting in the simulations, the composition of the actual distribution of firms and productivities that underlie those means can exhibit larger variations.[126] For concerns that some slower moving variables might not be forecasted well from a "short" simulation, the model was solved again for a simulation of $100,500$ observations (again, discarding the first 500) which is 10 times the original simulation length. Again, the results do not differ markedly.

To further explore any potential sampling uncertainty that the simulation might induce, the final experiment solved the model over several simultaneous simulations running in parallel. The model was solved again using 12 different simultaneous simulations, but again, the results do not differ markedly.[127]

In summary, the mapping functions appear to be very accurate over several different implementations of robustness checks.[128]

[125] This experiment also used a longer simulation length.

[126] Because the exact distribution of firms and productivities is kept track of during the simulations, this can be seen in the gradations of the matrices $\boldsymbol{F}_S$ and $\boldsymbol{F}_A$ used to track it. See the appendix for details.

[127] For processors with multiple cores, this process can be parallelized, so that time spent in the simulation part of the algorithm increases very little. Running one simulated economy per core is a way to take advantage of parallelization to generate more information during this part of the algorithm.

[128] Since the results from these experiments do not differ markedly, the details have been omitted. Results are available upon request.

A.6 Entry and Exit Measures

A.6.1 Entry and Exit

Annual implications of the (quarterly) model in this paper are constructed to compare the model's performance with the empirical data in Lee and Mukoyama (2012), which, again, comes from the ASM portion of the LRD constructed by the US Census Bureau (from 1972 to 1997) and is available only at annual frequences. Model counterparts of the empirical data as reported in Section 4.2 are defined as follows.

Let t_i for $i = \{1, 2, 3, 4\}$ denote the i^{th} quarter in year t. Entry in a given year t, is calculated as all new entrants producing for the first time in a given year. Note that, to match up with its counterpart in the data, only the new entrants that actually produce are relevant here - i.e., new plants that never actually produce (blueprints were created for them, but their firm-level productivity draws were below the threshold productivity cutoffs for entry) would not show up in the data. Given the 1 period time-to-build lag, entry in year t is calculated as

$$\text{Entry}_t = \left(1 - G\left(\max\left[z^*_{N,t-1_4}, z^*_{t_1}\right]\right)\right)\left(1-\delta^M\right)N_{t-1_4} + \sum_{q=2}^{4}\left(1 - G\left(\max\left[z^*_{N,t_{q-1}}, z^*_{t_q}\right]\right)\right)\left(1-\delta^M\right)N_{t_{q-1}}. \tag{16}$$

Exit in year t is calculated as the total exit during that year from both endogenous exit and the exogenous shock. The measure of exit in the model is then constructed over firms that have previously produced in at least one period (prior to the current year), calculating total exit in a given year t as

$$\begin{aligned}
\text{Exit}_t \quad = \quad & \delta^M M_{A,t_1} \\
& + \left[M_{S,t_1} - \left(1 - G\left(z^*_{N,t-1_4}\right)\right)\left(1-\delta^M\right)N_{t-1_4}\right] \\
& - \left[M_{A,t_1} - \left(1 - G\left(\max\left[z^*_{N,t-1_4}, z^*_{t,1}\right]\right)\right)\left(1-\delta^M\right)N_{t-1_4}\right] \\
& + \sum_{q=2}^{4}\Big\{\delta^M M_{A,t_q} \\
& + \left[M_{S,t_q} - \left(1 - G\left(z^*_{N,t_{q-1}}\right)\right)\left(1-\delta^M\right)N_{t_{q-1}}\right] \\
& - \left[M_{A,t_q} - \left(1 - G\left(\max\left[z^*_{N,t_{q-1}}, z^*_{t,q}\right]\right)\right)\left(1-\delta^M\right)N_{t_{q-1}}\right]\Big\},
\end{aligned}$$

This measure can be written more simply, but left in this form for now to make clear several points. First, note that, for each quarter, the first terms $\delta^M M_{A,t_q}$ capture exit from the exogenous shocks, while the remaining terms capture endogenous exit - roughly $M_{S,t_q} - M_{A,t_q}$. However, the exit measure must also adjust for the exclusions of new firms created from blueprints but never actually produce whose number and/or exit (if any) would not be included in this statistic. For example, if the cutoff productivity for new entrants in quarter t_{q-1} is $z^*_{N,t_{q-1}}$, then only $\left(1 - G\left(z^*_{N,t_{q-1}}\right)\right)\left(1-\delta^M\right)N_{t_{q-1}}$ survive into the next quarter (when they may start producing). However, if - during the following quarter - the cutoff productivity for incumbents $z^*_{t,q}$ is higher than the previous quarter's cutoff for new entrants $z^*_{N,t_{q-1}}$, then a portion of those new entrants $G\left(z^*_{t,q}\right) - G\left(z^*_{N,t_{q-1}}\right)$ would exit at the beginning of quarter t_q without ever having produced.

The measure of new entrants that survive during the quarter in which they were built but endogenously exit at the beginning of the following quarter are given as

$$\begin{aligned} & \left(1-G\left(z^*_{N,t_{q-1}}\right)\right)\left(1-\delta^M\right)N_{t_{q-1}}-\left(1-G\left(\max\left[z^*_{N,t_{q-1}},z^*_{t,q}\right]\right)\right)\left(1-\delta^M\right)N_{t_{q-1}} \\ = & \left[G\left(\max\left[z^*_{N,t_{q-1}},z^*_{t,q}\right]\right)-G\left(z^*_{N,t_{q-1}}\right)\right]\left(1-\delta^M\right)N_{t_{q-1}}. \end{aligned}$$

(The endogenous exit of new entrants in their second period of life is then not included in the measure of exit, since these firms never produced.) Exit can then be written more compactly as

$$\begin{aligned} \text{Exit}_t &= \delta^M M_{A,t_1}+M_{S,t_1}-M_{A,t_1} \\ &- \left[G\left(\max\left[z^*_{N,t-1_4},z^*_{t,1}\right]\right)-G\left(z^*_{N,t-1_4}\right)\right]\left(1-\delta^M\right)N_{t-1_4} \\ &\quad +\sum_{q=2}^{4}\left\{\delta^M M_{A,t_q}+M_{S,t_q}-M_{A,t_q}\right. \\ &\quad \left.-\left[G\left(\max\left[z^*_{N,t_{q-1}},z^*_{t,q}\right]\right)-G\left(z^*_{N,t_{q-1}}\right)\right]\left(1-\delta^M\right)N_{t_{q-1}}\right\}. \end{aligned} \tag{17}$$

A.6.2 Entry and Exit Rates

The total number of firms producing in year t is defined over the measure of firms that produced in at least one quarter. For example, an firm that produced in the first quarter t_1 but exited at the end of the period would be included, as would a firm that produced during all four quarters or a firm that produced only in the final quarter of the year. The number of firms producing in year t is then given as

$$\text{Firms}_t = M_{A,t_1}+\sum_{q=2}^{4}\left(1-G\left(\max\left[z^*_{N,t_{q-1}},z^*_{t_q}\right]\right)\right)\left(1-\delta^M\right)N_{t_{q-1}}.$$

Entry and exit rates for year t are then calculated as the number of firms that entered or exited as a percentage of the total number of firms producing in the previous year, given as

$$\text{Entry Rate}_t = \text{Entry}_t/\text{Firms}_{t-1} \tag{18}$$

and

$$\text{Exit Rate}_t = \text{Exit}_{t-1}/\text{Firms}_{t-1}. \tag{19}$$

Note, however, that the number of continuing (or incumbent) firms (that survived from time $t-1$ to time t) is given by

$$\text{Incumbents}_t = M_{A,t_1}-\left[1-G\left(\max\left[z^*_{N,t-1_4},z^*_{t,1}\right]\right)\right]\left(1-\delta^M\right)N_{t-1_4},$$

so entry and exit rates could also alternatively be expressed as a percentage of continuing (surviving) incumbent firms where

$$\text{Entry Rate}_t = \text{Entry}_t/\text{Incumbents}_t$$

and

$$\text{Exit Rate}_t = \text{Exit}_{t-1}/\text{Incumbents}_t.$$

The former measures are consistent with standard definitions and with Dunne, Roberts, and Samuelson (1998). (Note that the former measures will give lower entry and exit rates than the latter due to entry occuring within the year. The latter measures are the ones that yield entry and exit rates with a rate of, on average, δ^{M} on an annual basis.)

Lee and Mukoyama (2012), however, define entry (exit) rates as the number of entering (exiting) establishments as a percentage of the total number of establishments each period. Their definitions, then, are probably closer to

$$\text{Entry Rate}_t = \text{Entry}_t/\text{Incumbents}_t \tag{20}$$

and

$$\text{Exit Rate}_t = \text{Exit}_t/\text{Incumbents}_t. \tag{21}$$

Importantly, the cyclicality of the exit rate in the model depends on the definition used. Using either of the former definitions, e.g., that in Equation 19, where exit observable at time t occurs from entry that happened *between* the data reported for years $t-1$ and t - i.e., for firms that were producing in year $t-1$ but are not observed to be producing in year t, the exit rate is counter-cyclical. Using the latter definition in Equation 21 that defines the exit rate as exit of firms that occur *during* year t, the exit rate is procyclical.[129] The exit rate overall is relatively flat over the cycle (particularly in comparison with the entry rate), but the nature of its cyclicality is sensitive to different interpretations of the definition. This may be why Lee and Mukoyama (2012) find a slightly procyclical exit rate while other have found a countercyclical exit rate. For purposes of comparison with the data in Lee and Mukoyama (2012), calculations for entry and exit rates are constructed as in Equations 20 and 21 and are reported in Table 8.

A.6.3 Productivity Measures

First, note that it is not exactly clear how to define the model analogue of the Solow residual, since the firms of interest in this model are the intermediate-goods producers that use the production technology in Equation 4, but the aggregate output is constructed from the production of final goods from the technology in Equation 1. Using the aggregation techniques in Section 2.3, output can be written as a function of economy-wide aggregates as

$$\begin{aligned} Q_t &= q_t\left(\tilde{z}_{A,t}\right) M_{A,t}^{\theta/(\theta-1)} \\ &= \tilde{z}_{A,t} Z_t \left[k_t\left(\tilde{z}_{A,t}\right)\right]^{\alpha} \left[l_{q,t}\left(\tilde{z}_{A,t}\right)\right]^{1-\alpha} M_{A,t}^{\theta/(\theta-1)} \\ &= \tilde{z}_{A,t} Z_t \left[\frac{K_t}{M_{A,t}}\right]^{\alpha} \left[\frac{L_{Q,t}}{M_{A,t}}\right]^{1-\alpha} M_{A,t}^{\theta/(\theta-1)} \\ &= \tilde{z}_{A,t} Z_t \left[K_t\right]^{\alpha} \left[L_{Q,t}\right]^{1-\alpha} M_{A,t}^{1/(1-\theta)}, \end{aligned} \tag{22}$$

[129] The procyclical entry rate is robust to using either a measure of all firms producing in the previous year or only the ones that continue as the denominator.

but note that, even though the intermediate goods firms use a Cobb-Douglas production technology, the production of final goods is not exactly Cobb-Douglas. There are essentially two problems here: first, it is not clear how to deal with the effects of the $M_{A,t}^{1/(1-\theta)}$ term when constructing the model analogue of the Solow residual. One strategy is to ignore it, assume (as it done in the data) that production is Cobb-Douglas, re-estimate the factor shares for capital and labor based on the simulated data as $\widehat{\alpha}$ and $\widehat{\beta}$, and estimate total factor productivity as $S_{1,t}$ from the function

$$\text{Strategy 1: } P_t Q_t = S_{1,t} K_t^{\widehat{\alpha}} L_{P,t}^{\widehat{\beta}}. \tag{23}$$

(Note that the total hours used in the production $L_{P,t}$ must include not only hours from $L_{Q,t}$, but also those from $L_{F,t}$.) An alternative strategy is to assume that that the production function is as in Equation 1 (which is different than the procedure taken to the data), and estimate $S_{2,t}$ as

$$\text{Strategy 2: } P_t Q_t = S_{2,t} \left[K_t\right]^{\alpha} \left[L_{P,t}\right]^{1-\alpha} M_{A,t}^{1/(1-\theta)}. \tag{24}$$

The advantage of the first strategy is that it is closer to the procedure undertaken in the data, but it has the disadvantage of wrapping up the effects of $M_{A,t}$ into the measure of TFP (and estimating factor shares for capital and labor that are different from the actual factor shares). For the second strategy, these concerns are reversed.

The second problem, of course, relates to estimating the right-hand sides for new entrants and incumbent firms separately. While the result in equation 22 holds in the aggregate, it would not be correct to write

$$\begin{aligned} Q_t &= \tilde{z}_{A,t} Z_t \left[K_t\right]^{\alpha} \left[L_{Q,t}\right]^{1-\alpha} M_{A,t}^{1/(1-\theta)} \\ &\neq \tilde{z}_{E,t} Z_t \left[K_{N_E,t}\right]^{\alpha} \left[\left(L_Q\right)_{N_E,t}\right]^{1-\alpha} N_{E,t}^{1/(1-\theta)} + \tilde{z}_{I,t} Z_t \left[K_{M_I,t}\right]^{\alpha} \left[\left(L_Q\right)_{M_I,t}\right]^{1-\alpha} M_{I,t}^{1/(1-\theta)}, \end{aligned}$$

and for $\widehat{\alpha} + \widehat{\beta} \neq 1$,

$$S_{1,t} K_t^{\widehat{\alpha}} L_{P,t}^{\widehat{\beta}} \neq \left(S_1\right)_{N_E,t} \left[K_{N_E,t}\right]^{\widehat{\alpha}} \left[\left(L_Q\right)_{N_E,t}\right]^{\widehat{\beta}} + \left(S_1\right)_{M_I,t} \left[K_{M_I,t}\right]^{\widehat{\alpha}} \left[\left(L_Q\right)_{M_I,t}\right]^{\widehat{\beta}}.$$

It is, therefore, important to stress that these concerns place severe limitations on the productivity measures estimated from either strategy. And though each method will generate very different estimates, this paper is only concerned with estimating *relative* productivities, as explained below.

Recall that an average aggregate (idiosyncratic) productivity can be calculated both for the new entrants and continuing firms (incumbent firms that survived from the previous year) producing in a given year. However, this is not quite the right measure of productivity, since - while it indicates differences in technical efficiencies - it does not line up with the revenue-based measures of total factor productivity (TFP) calculated in the data. But since the idea is to get an estimate of how the average productivities of new entrants relative to incumbents varies over the cycle, consistent patterns for these revenue-based measures - along with consistency with the actual measures of technical efficiencies that are identifiable in the model - can pin down qualitative patterns and give some idea of the quantitative magnitudes.

To calculate the technical efficiences, the measure of new entrants and incumbent must take account of endogenous entry and exit during the year, and calculating prices, output, and factor demands for these groups separately first requires finding their average productivities. The calculations are a bit tedious, since, for example, the average productivity of new entrants producing in the first quarter of the year may change during the second quarter if there is endogenous exit among this group between the quarters. Further, each quarter a new group of entrants begins producing. Therefore, not only do the average productivities of new entrants need to be calculated each quarter, but so also do the changes in the productivities of these groups over the year. The average productivities of incumbent firms may also change over the year if there is any endogenous exit among this group.

An average productivity over the firms defined in Equation 16 is calculated as

$$\tilde{z}_{N_{E,t}} = \left[\frac{1}{\sum_{c=1}^{4} \sum_{q=c}^{4} N_{E,t_{c,q}}} \left(\sum_{c=1}^{4} \sum_{q=c}^{4} N_{E,t_{c,q}} \tilde{z}_{N_{E,t_{c,q}}}^{\theta-1} \right) \right]^{1/(\theta-1)}$$

where $N_{E,t_{c,q}}$ refers to new entrants producing in quarter q of year t whose first period of production (producing life) was in quarter c of year t (with analogous notation for $\tilde{z}_{N_{E,t_{c,q}}}^{\theta-1}$) where

$$\tilde{z}_{N_{E,t_{c,q}}} = \begin{array}{ll} \tilde{z}\left(\max\left[z_{N,t_{c-1}}^{*}, \left\{ z_{t_a}^{*} \right\}_{a=c}^{q} \right] \right) & c = \{2,3,4\} \\ \tilde{z}\left(\max\left[z_{N,t-1_4}^{*}, \left\{ z_{t_a}^{*} \right\}_{a=1}^{q} \right] \right) & c = 1 \end{array}$$

and

$$N_{E,t_{c,q}} = \begin{array}{ll} \left(1 - G\left(\max\left[z_{N,t_{c-1}}^{*}, \left\{ z_{t_a}^{*} \right\}_{a=c}^{q} \right] \right) \right) \left(1-\delta^{M}\right)^{q} N_{t_{q-1}} & c = \{2,3,4\} \\ \left(1 - G\left(\max\left[z_{N,t-1_4}^{*}, \left\{ z_{t_a}^{*} \right\}_{a=1}^{q} \right] \right) \right) \left(1-\delta^{M}\right)^{q} N_{t-1_4} & c = 1 \end{array}.$$

For incumbents, their numbers M_I and average productivities $\tilde{z}_I$ can then be found from firms actively producing in the period with adjustments for the effects of the new entrants as

$$M_{I,t_q} = M_{A,t_q} - \sum_{c=1}^{q} N_{E,t_{c,q}}$$

and

$$\tilde{z}_{I,t_q} = \left[\frac{1}{M_{I,t_q}} \left(\tilde{z}_{A,t_q}^{\theta-1} M_{A,t_q} - \sum_{c=1}^{q} \left[\tilde{z}_{N_{E,t_{c,q}}}^{\theta-1} N_{E,t_{c,q}} \right] \right) \right]^{1/(\theta-1)}.$$

Now, typically the revenue-based productivity measures in Equations 23 and 24 are estimated in logs, but consider instead estimating a relative productivity measure as ς_t, where

$$\varsigma_t = S_{N_E,t} / S_{M_I,t}.$$

Relative productivity measures can then be estimated as

$$\varsigma_{1,t_q} = \frac{p_{t_q}\left(\tilde{z}_{N_E,t_q}\right) q_{t_q}\left(\tilde{z}_{N_E,t_q}\right) N_{E,t_q}}{p_{t_q}\left(\tilde{z}_{M_I,t_q}\right) q_{t_q}\left(\tilde{z}_{M_{I,t_q}}\right) M_{I,t_q}} \times \frac{\left[K_{M_I,t_q}\right]^{\widehat{\alpha}} \left[(L_P)_{M_I,t_q}\right]^{\widehat{\beta}}}{\left[K_{N_E,t_q}\right]^{\widehat{\alpha}} \left[(L_P)_{N_E,t_q}\right]^{\widehat{\beta}}} \tag{25}$$

and

$$\varsigma_{2,t_q} = \frac{p_{t_q}\left(\tilde{z}_{N_E,t_q}\right) q_{t_q}\left(\tilde{z}_{N_E,t_q}\right) N_{E,t_q}}{p_{t_q}\left(\tilde{z}_{M_I,t_q}\right) q_{t_q}\left(\tilde{z}_{M_I,t_q}\right) M_{I,t_q}} \times \frac{\left[K_{M_I,t_q}\right]^{\alpha}\left[(L_P)_{M_I,t_q}\right]^{1-\alpha} M_{I,t_q}^{1/(1-\theta)}}{\left[K_{N_E,t_q}\right]^{\alpha}\left[(L_P)_{N_E,t_q}\right]^{1-\alpha} N_{E,t_q}^{1/(1-\theta)}} \tag{26}$$

To give some idea of the intuition behind these measures, it is instructive to provide a derivation relating these measures to the technical productivities of new entrants and incumbents. First, note that the revenue for a firm with productivity z is given by

$$r_{t_q}(z) = \left(\frac{p_{t_q}(z)}{P_{t_q}}\right)^{1-\theta} R_{t_q},$$

where R_{t_q} is total revenues, so the first terms in Equations 25 and 26 reduce to an expression that includes only the idiosyncratic productivity levels. Write ratios of real revenues for new entrants and incumbent firms as

$$\begin{aligned}
\frac{r_{t_q}\left(\tilde{z}_{N_E,t_q}\right)}{r_{t_q}\left(\tilde{z}_{M_I,t_q}\right)} &= \frac{\left(\frac{p_{t_q}\left(\tilde{z}_{N_E,t_q}\right)}{P_{t_q}}\right)^{1-\theta} R_{t_q}}{\left(\frac{p_{t_q}\left(\tilde{z}_{M_I,t_q}\right)}{P_{t_q}}\right)^{1-\theta} R_{t_q}} \\
&= \frac{\left(\mu\lambda_{t_q}\left(\tilde{z}_{N_E,t_q}\right)\right)^{1-\theta}}{\left(\mu\lambda_{t_q}\left(\tilde{z}_{M_I,t_q}\right)\right)^{1-\theta}} \\
&= \frac{\left(\mu\frac{1}{\tilde{z}_{N_E,t_q} Z_{t_q}} A r_{K,t_q}^{\alpha} w_{t_q}^{1-\alpha}\right)^{1-\theta}}{\left(\mu\frac{1}{\tilde{z}_{M_I,t_q} Z_{t_q}} A r_{K,t_q}^{\alpha} w_{t_q}^{1-\alpha}\right)^{1-\theta}} \\
&= \left[\frac{\tilde{z}_{N_E,t_q}}{\tilde{z}_{I_{t_q}}}\right]^{\theta-1},
\end{aligned}$$

so relative revenues depend on the ratio of the average productivities only. Therefore, the relative revenues of new entrants will be higher in times where new entrants are also more (relatively) technically productive.

While the second terms cannot be entirely simplified in this way, it is however instructive to consider that since

$$q_{t_q}(z) = \left(\frac{p_{t_q}(z)}{P_{t_q}}\right)^{-\theta} Q_{t_q},$$

with similar algebra,

$$\frac{q_{t_q}\left(\tilde{z}_{N_E,t_q}\right)}{q_{t_q}\left(\tilde{z}_{M_I,t_q}\right)} = \left[\frac{\tilde{z}_{N_E,t_q}}{\tilde{z}_{M_I,t_q}}\right]^{\theta};$$

so if hours worked only included L_Q (and not those used to pay fixed costs of production L_F), the relative ratios in the second terms of Equations 25 and 26 would also reduce to ratios of their average idiosyncratic productivities:

$$
\begin{aligned}
\frac{\left[k_{t_q}\left(\tilde{z}_{M_I,t_q}\right)\right]^{\alpha}}{\left[k_{t_q}\left(\tilde{z}_{N_E,t_q}\right)\right]^{\alpha}} &= \left(\frac{\frac{1}{\tilde{z}_{M_I,t_q}Z_{t_q}}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{w_{t_q}}{r_{K,t_q}}\right)^{1-\alpha} q_{t_q}\left(\tilde{z}_{M_I,t_q}\right)}{\frac{1}{\tilde{z}_{N_E,t_q}Z_{t_q}}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{w_{t_q}}{r_{K,t_q}}\right)^{1-\alpha} q_{t_q}\left(\tilde{z}_{N_E,t_q}\right)}\right)^{\alpha} \\
&= \left(\left[\frac{\tilde{z}_{N_E,t_q}}{\tilde{z}_{M_I,t_q}}\right]^{1-\theta}\right)^{\alpha}
\end{aligned}
$$

and similarly for the ratios of labor, so

$$
\frac{\left[K_{M_I,t_q}\right]^{\alpha}\left[\left(L_Q\right)_{M_I,t_q}\right]^{1-\alpha}}{\left[K_{N_E,t_q}\right]^{\alpha}\left[\left(L_Q\right)_{N_E,t_q}\right]^{1-\alpha}} = \left[\frac{\tilde{z}_{N_E,t_q}}{\tilde{z}_{M_I,t_q}}\right]^{1-\theta}\left(\frac{M_{I,t_q}}{N_{E,t_q}}\right),
$$

which also contributes to an estimated relative TFP that is higher in times when new entrants are relative more productive than incumbents.

Patterns of selection in the data show that new entrants are relatively more productive in bad times than in good, and the model also replicates this feature across both specifications in Equations 25 and 26. Note that the effects of the relative number of new entrants enters directly in Equation 26, and since entry is procyclical, the term $M_{I,t_q}^{1/(1-\theta)}/N_{E,t_q}^{1/(1-\theta)}$ is likely to bias the relative measure of productivity up in good times and down in bad times. According to this measure, then, Equation 26 is likely to be biased downward in estimating the extent to which new entrants are less productive in good times as compared with those in bad times.

Endogenous Formation of Dark Networks

1 Introduction

Covert and illegal networks (i.e., *dark networks*) operate with ever-present risks of exposure to police, militaries, and other law enforcement agencies. To survive, members of these organizations must intentionally evade investigation, and this feature makes dark networks such as drug-trafficking rings, terrorist groups, and other criminal organizations difficult to study (Borgatti et al 2006). Not only do these organizations adapt and change to mitigate detection and disruption (e.g., Everton 2012), but their hidden structure makes it difficult to even identify key players (e.g., Jordan et al 2008, Roberts and Everton 2011). Because of these difficulties in studying dark networks empirically, we instead develop a game-theoretic model of dark network formation and test our theories in a series of laboratory experiments. While we cannot attempt to explain every nuance of dark network formation, we can focus on particular aspects of individual behavioral responses and evaluate specific (dis-)incentive policies with an eye towards understanding the implications for how dark networks ultimately form and adapt.

We seek a preliminary understanding of how the structure of dark networks relates to the nature of the threats they confront. Our model considers individual responses to two different types of (dis-)incentive policies. First, we model how individuals might respond to policies that increase the likelihood of getting caught and arrested; e.g., policies that increase police presence, surveillance, and/or monitoring activities. Second, we consider how individuals might respond to policies that increase the destructive impact of any one arrest on the rest of the network. For example, policies that permit enhanced interrogation techniques or extensive electronic monitoring of phone records, bank accounts, and e-mail messages might increase the likelihood of learning the identities of other network members. Our stylized model incorporates both of these threats and constructs a basic tradeoff – there are benefits to developing larger networks, but larger networks can induce higher probabilities of being caught and arrested.

Our experiment has a 2×2 design where we vary both (i) the probability that any individual is detected and arrested and (ii) the impact that any arrest has on detecting and arresting other members in the network. Our model predicts that increased probabilities of detection should reduce the number of links that subjects form and decrease the size of equilibrium network structures, but, surprisingly, that increases in arrest impact should have no impact. Our experimental results indicate that both policy levers are effective in disrupting network formation. Increases in individual detection probabilities result in smaller networks as expected, but increases in the impacts of arrests also have a similar effect. This latter finding runs counter to our prediction, but is consistent with non-experimental empirical work (Jordan et al 2008). Interestingly, increases in both policy levers result in more subjects who simply choose not to join any network.

Game-theoretic models of network formation trace their roots to Jackson and Wolinsky (1996) and Bala and Goyal (2000), both of which have generated a large and growing literature (see Jackson 2008). Recent work has explicitly focused on networks facing threats of disruption; see, Enders and Su (2007), Baccara and Bar-Issac (2009), Enders and Jindapon (2010), Hoyer and Jaegher (2010), Goyal and Vigier (2010), and Dziubinski and Goyal (2013). Experimental analyses of network formation (e.g., Callander and Plott 2005, Carillow and Gaduh 2011, Caldara and McBride 2014, Bloxsom et al 2014) and counterterrorism policies (e.g., Arce et al. 2011) are relatively new. Three key features differentiate our work. First, our model and experiments feature decentralized network formation, i.e., networks form endogenously as the result of individual interactions and choices and not through the direction of any single planner or designer. Several other studies also consider decentralized formation (e.g., Pantz 2006, Kirchsteiger et al 2011, and Carrillo and Gaduh 2011), but none in the context of dark networks facing disruption. Second, our work is the first to consider the difference between the related, but distinct, threats of detection and arrest impact. Prior theory and experiments consider the explicit removal of identified nodes (e.g., Dziubinski and Goyal 2013, McBride and Hewitt 2013, McBride and Caldara 2013), but not how single arrests may have differing impacts. Third, our experiments feature a large number (twenty) of subjects that allow the formation of multiple network components as well as allowing individual subjects to opt out of joining any network. Previous network experiments have typically been conducted on smaller groups; for example, Callander and Plott (2005) and Carrillo and Gaduh (2011) use groups of six, and Caldara and McBride (2014) use groups of twelve. Our larger group size allows for much richer networks.

2 The Model

2.1 A Model of (Dark) Network Formation

Suppose there are n nodes (e.g., potential members in a network). Each node must decide whether and with which other node(s) to initiate a link. Let I be an $n \times n$ matrix representing link attempts, where $I_{ij} = 1(0)$ means node i did (did not) attempt to initiate a link with node j. Mutual consent is required to successfully establish a link, and we let L represent the $n \times n$ matrix of successful links, where $L_{ij} = L_{ji} = 1$ if a mutual link was attempted by both i and j, but equals zero otherwise. By construction, nodes do not form links with themselves, so $L_{ii} = 0$. Note that L is symmetric by construction ($L_{ij} = L_{ji}$ $\forall$ i and j), but I is not necessarily symmetric because attempted link initiations may be one-sided. We denote L as the *preliminary network*, so that, formally

$$L_{ij} = \begin{cases} 1, & \text{if } I_{ij} = I_{ji} = 1, \ i \neq j \\ 0, & \text{otherwise.} \end{cases} .$$

Say that there is a path between node i and node j in a network L if there is a sequence of links, e.g., $L_{ik} = 1$, $L_{kl} = 1$, $L_{lj} = 1$, connecting nodes i and j. Let $d\,(i, j|L)$ denote the length of the shortest path (or geodesic) between nodes i and j in network L, where $d\,(i, i) = 0$ and $d\,(i, j|L) = \infty$ if there is no path.

Define $S(i,d|L)$ to be the set of nodes with shortest paths to i equal to d in the preliminary network L:

$$S(i,d|L) = \{j \in L \text{ such that } d(i,j|L) = d\} M.$$

Let $\#S(i,d|L)$ be the cardinality of that set. The degree of node i in the preliminary network L, denoted $\deg_i(L_i)$ equals the number of node i's actual links:

$$\deg_i(L) = \#S(i,1|L).$$

After formation of the preliminary network L, each node is subject to random detection (by "Nature") with probability p, where $0 < p < 1$. The probability of detection of any one node is *i.i.d.* with respect to the probabilities of detection across the other nodes. Any node that is "directly detected" in this way has its links severed and is removed from the network.

Nodes that are not directly detected may also be "arrested by association" if they are linked to a node that has been directly detected. Arrests occur according to an "impact technology", where all nodes connected to a detected node i^d with shortest path less than or equal to $d = a$, given by $S(i^d, a|L)$, are also removed from the network. For example, if $a = 1$, only nodes that are immediately connected to directly detected nodes are arrested. We refer to this case as *near-neighbor impact*. If, on the other hand, $a = n-1$, all nodes that have a path of any length $d < n$ to a detected node are arrested. We refer to this case as *friend-of-a-friend impact*, since all nodes that are either directly or indirectly connected to a detected node are removed from the network. Clearly, the larger the arrest impact technology a, the larger the impact that any given detection will have on the links remaining in the network.

Detection and arrest on the preliminary network result in the *final network* $\widehat{L}$, where

$$\widehat{L}_{ij} = \left\{ \begin{array}{ll} L_{ij}, & \text{if } i \text{ and } j \text{ both avoid detection and arrest,} \\ 0, & \text{otherwise.} \end{array} \right\}$$

Given the final network $\widehat{L}$, payoffs are zero for all nodes that were detected or arrested. Nodes not detected or arrested receive a payoff $x > 0$ for surviving and additional payoffs of $y > 0$ per surviving link. Let $\deg_i\left(\widehat{L}\right)$ denote i's degree in the final network $\widehat{L}$. Then the payoff of any node i is given by u_i, where

$$u_i = \begin{cases} 0, & \text{if } i \text{ is detected or arrested,} \\ x + y \deg_i\left(\widehat{L}\right), & \text{otherwise,} \end{cases}$$

This model captures a fundamental tension in dark network formation: more links establish larger and more productive collaborations for the members of the network (captured in our model through higher payoffs), but larger networks also induce higher probabilities of detection and arrest that come with the increased exposure (captured in our model through the probability of arrest by association being an increasing function of the number of links). Though it is a very simple model, we believe that it may begin to give us better insights into how the threats of detection and arrests play out in individual behavior and the formation of networks.

2.2 Pairwise Nash Stability and Regular Networks

We use the concept of *pairwise Nash stability* in identifying stable network structures, a concept that has been widely used in examining the strategic formation of networks (see, e.g., Jackson 2010). This concept captures the idea that any collaboration requires mutual agreement, but a relationship can be unilaterally terminated by one party alone. Specifically, a network is defined to be pairwise Nash stable if two conditions are met: (PNS-i) no pair of nodes can benefit (at least one strictly) by creating a new collaboration between them, and (PNS-ii) no single node is strictly better off removing one or more existing collaborations. We make one modification in our application of pairwise Nash stability: because random detection and arrests come after the formation of the preliminary network, we modify the calculations for the costs and benefits of changing network structures to reflect changes in *expected* payoffs. Unfortunately, as is common in many models of network formation, there may exist a large number of networks that can be classified as pairwise Nash stable.

The following terminology will be useful in characterizing the classes of networks that we will consider. The *empty network* is a network with no links, i.e., each node is *isolated*. An example of an empty network with $n = 6$ is given in Figure 1(a). A *subnetwork* of a network consists of a subset of nodes and their links from the network to others in that same subset of nodes. A *component* is a non-empty subnetwork in which there is a path between every two nodes in the component, but there are no links between any nodes in the component and other nodes not in the component (see Figure 1(b)). A *complete component* (or network) has direct links between every node of the component (or network).

A special configuration is a *regular* network in which all nodes have the same degree k. Figure 1(c) depicts such a network with $k = 2$. An added feature of Figure 1(c) that will be prominent in our analysis is that each component is completely connected. This is not a feature of all regular networks (e.g., a circle network is regular but does not have a completely connected component). Figure 1(d) depicts a network with complete components that is not regular. Regular networks comprise an important class of networks. They manifest a type of symmetry that could be expected given the imposed homogeneity of the nodes in our model, and this symmetry can make the structure focal. Moreover, such networks turn out to have desirable efficiency properties in this setting.

2.3 Efficient Networks

Our first proposition identifies the degree-k regular networks as the configuration that maximizes the actor's payoffs.

Proposition 1 *(a) Let* $k^* = \max\left\{0, \frac{1}{p} - 1 - \frac{x}{y}\right\}$*. For both near-neighbor* ($a = 1$) *and friend-of-a-friend* ($a = n - 1$) *arrest impacts, the degree-*k^* *regular network with complete components yields the highest payoff to all actors among all possible networks.*

(b) The degree-k^* *regular, complete component network is pairwise Nash stable under both near-neighbor*

and friend-of-a-friend arrest impact.

(c) k^*, though identical under near-neighbor and friend-of-a-friend arrest, is decreasing in detection probability p.

All proofs are found in the appendix. According to Proposition 1, the degree-k^* regular network with complete components is the efficient network structure for both arrest impacts, and it is also pairwise stable for both arrest impacts. That this is true for the friend-of-a-friend (high) arrest impact is intuitive: the complete component is structured to remove the large negative externalities associated with the increased arrest probabilities caused by any indirect links. Forming direct links, instead of allowing indirect links to remain, does not increase the probability of arrest but does increase the payoff conditional on survival. This basic logic extends to the near-neighbor impact setting, though the logic is less obvious. As an example, consider three nodes i, j, and k, where node i is linked to node j and node j is linked to node k. The jk link does not affect node i's probability of arrest, but it does affect the expected payoff node i gets from its link with j. If node i links to node k, though node i's probability of arrest by association has increased in this case, the payoff externality disappears. With these results in place, it is straightforward to calculate the expected payoff maximizing degree k^* defined in Proposition 1(a).

These efficient networks are also pairwise Nash stable. Moreover, an important and surprising implication of Proposition 1 is that if nodes can coordinate on the efficient, regular network, then they will coordinate on structurally-identical networks under both near-neighbor and friend-of-a-friend arrest impacts. Though the efficient regular network has degree k^* that decreases as the detection probability p increases, the degree of the efficient regular network does not change as the arrest impact a changes. This result will be further examined theoretically below and will be tested in our experiment.

2.4 Degree Ranges

That the degree is decreasing in detection probability p but unchanging in arrest impact a is a special property of the efficient networks. Because of the large number of pairwise Nash stable networks, we cannot prove this to be a general result. However, we can characterize a range of actors' degrees that are possible in pairwise Nash stable networks, and this can serve as a guide to how connected dark networks can be.

Define $\overline{k}(a,p)$ and $\underline{k}(a,p)$ to be the maximum and minimum degree that an actor may have in a pairwise Nash stable network given (a,p). That is: there are at most $\underline{k}(a,p)$ actors with fewer than $\underline{k}(a,p)$ links, and no actor has more than $\overline{k}(a,p)$ links; there exists a pairwise Nash stable network in which more than $\underline{k}(a,p)$ actors have $\underline{k}(a,p)$ links; and there exists a pairwise Nash stable network in which more than $\overline{k}(a,p)$ actors have $\overline{k}(a,p)$ links. Our next result identifies the minimum and maximum number of links in pairwise Nash stable networks. We again find complete component networks to be prominent, as both $\overline{k}(a,p)$ and $\underline{k}(a,p)$ can occur in degree-k regular networks with complete components. We also again see that the arrest impact has a muted role.

Proposition 2 *Assume $k^* > 0$. Then:*

(a) $\overline{k}(1,p) = \overline{k}(n-1,p) = \lfloor \kappa_1 \rfloor$ *where κ_1 is such that*

$$(1-p)^{\kappa_1+1}(x + y\kappa_1) = (1-p)x.$$

(b) $\lceil \kappa_2 \rceil = \underline{k}(n-1,p) \leq \underline{k}(1,p) = \lceil \kappa_3 \rceil$, *where κ_2 and κ_3 are real solutions to the following equations, respectively:*

$$\begin{aligned} (1-p)^{\kappa_2+1}(x + y(\kappa_2+1)) &= (x + y\kappa_2) \\ (1-p)^{\kappa_3+1} y &= p(x + y\kappa_3). \end{aligned}$$

(c) $\underline{k}(a,p)$ *and* $\overline{k}(a,p)$ *are weakly decreasing in p.*

Part (a) says the highest degree that can be found in a pairwise Nash stable network is the same under $a = 1$ and $a = n-1$, (b) says that the lowest degree under $a = n-1$ is less than or equal to that under $a = 1$, and (c) says that both upper and lower bounds are decreasing in the detection probability. Importantly for our purposes, Proposition 2 reports that there is a clear negative relationship between the range of degrees possible in a pairwise Nash stable network and the detection probability, but that no such clear relationship exists between the degree range and the arrest impact. A change in arrest impact has no effect on the degree upper bound, though it might have an effect on the degree lower bound.

The following numerical example illustrates. Suppose there are twenty actors $n = 20$, and the payoff for avoiding detection and arrest is $x = 10$, with additional payoffs of $y = 15$ for surviving links. We can calculate the degree-k regular, complete component networks under both near-neighbor and friend-of-a-friend arrest impacts. These networks are depicted as circles in Figures 2(a) and 2(b) for detection probabilities $p = \{0, 0.1, 0.2, ..., 1\}$ - i.e., if a circle is located at point (p, k) then the network is pairwise Nash stable. The filled circles correspond to the efficient degree-k^* networks, which are the same under both arrest impacts. The circles with dots inside in Figure 2(a) correspond to those networks that are pairwise Nash stable under friend-of-a-friend, but not near-neighbor, arrest. With $n = 20$, $k = 5$ is the highest feasible k in a degree-k regular, complete component network; such a network would have four complete components. Observe that the efficient degree-k^* network has higher degree as p decreases. At $p = 0.1$, the efficient network actually would have degree larger than 5, but such a network is infeasible in this example with $n = 20$.

We interpret the above analysis as indicating that actors' degrees in a dark network will decrease as the detection probability increases but that the actors' degrees will likely remain unchanged as the arrest impact increases.

2.5 Other Results

Given the importance of degree-k regular networks with complete components in the above analysis, we here provide additional characterization (including existence) of such networks.

Proposition 3 *Assume high arrest impact ($a = n - 1$).*

(a) A Pairwise Nash Stable network (regular or irregular) is either empty or non-empty with complete components.

(b) When $\frac{p}{1-p} \geq \frac{y}{x}$, the empty network is Pairwise Nash stable.

(c) A degree-k regular, complete component network with $k \geq 1$ is pairwise Nash stable if and only if

$$1 - \left(\frac{x + ky}{x + (k+1)\,y}\right)^{\frac{1}{k+1}} \leq p \leq 1 - \left(\frac{x}{x + ky}\right)^{\frac{1}{k}}.$$

(d) Fix k, $1 \leq k < \infty$. There exists a detection probability p that supports a degree-k regular, complete component network as pairwise Nash stable.

(e) Fix p, $0 < p \leq \frac{y}{x+y}$. There exists a $k \geq 1$ for which the degree-k regular, complete component network is pairwise Nash stable.

If any two nodes are indirectly, but not directly, linked, then a new direct link between them does not affect their probability of surviving arrest by association, but it does increase their expected payoffs if they do survive. Therefore, any pairwise Nash stable network that is non-empty must have complete components. Implicit in Proposition 3 is that, for a fixed p, there can exist regular, complete component networks of different degrees that can all be pairwise Nash stable, one reason for this being the externalities in network connections. When node i considers adding a new link to node j that is not currently in her component (no indirect links), the increase in the chance of getting "arrested by association" clearly depends on the number of links that node j already has. All else equal, the more pre-existing links that node j has, the worse it is for node i: node i's likelihood of getting arrested is increasing in the number of node j's links, and the likelihood that node j avoids arrest - and thereby provides higher payoffs to node i - is also decreasing in node j's links. As an example, for a given p, a degree-3 regular, complete component network and a degree-4 regular, complete component network may both be pairwise Nash stable. A node in a degree-3 regular, complete component network may not want to link to a node in another degree-3 component, and node in a degree-4, regular, complete component network may not want to link to a node in another degree-4 component. Thus, although any pairwise Nash stable network must have complete components, differently sized complete components may exist in a pairwise Nash stable network.

We now turn to near-neighbor arrest.

Proposition 4 *Assume near-neighbor arrest impact ($a = 1$).*

(a) When $\frac{p}{1-p} > \frac{y}{x}$, the empty network is pairwise Nash stable.

(b) A degree-k regular, complete component network is pairwise Nash stable if and only if

$$(i) : \frac{p}{(1-p)^{k+1}} > \frac{y}{x + ky}$$

and

$$(ii) : \frac{y}{x} > \max\left\{\frac{1 - (1-p)^k}{k\,(1-p)^k}, \frac{p}{1-p}\right\}.$$

(c) Fix p, $0 < p < 1$. If $\frac{p}{1-p} < \frac{y}{x}$, then there exists a k for which the degree-k regular, complete component network is pairwise Nash stable; otherwise, if $\frac{p}{1-p} > \frac{y}{x}$, then the degree-$k$ regular, complete component network is not pairwise Nash stable.

Although arrests by association are dramatically different under near-neighbor (low) and friend-of-a-friend (high) impacts, there are some similarities in equilibrium network structure. In both cases, the empty network is pairwise Nash stable at high detection probabilities, and non-empty pairwise Nash stable networks exist as long as detection probabilities are sufficiently low. Complete components show up in both cases because these structures best cope with payoff externalities. For example, if node i is linked to node j, node j is linked to node k, but node i is not linked to k, then node j's link with node k imposes a cost on node i because it reduces i's expected benefits from its link with node j by increasing node j's probability of arrest. That externality disappears, however, if node i also forms a link with node k. The externality exists under both impact levels, though it is much more severe under higher impacts – so severe, in fact, that only complete components are pairwise Nash stable under friend-of-a-friend impact. With near-neighbor impact, though the externality is less severe, it is nonetheless also completely eliminated through the formation of complete components.

3 The Experiment

3.1 Design

We test our model's predictions via a laboratory experiment implementing a 2×2 between-subjects experimental design. As shown in Table 1, our treatment variables are the probability of detection and the arrest impact technology. The probability of detection p was either high (50%) or low (20%) and the level of arrest impact a was either set to arrest all possible $(n-1)$ connected nodes (friend-of-a-friend) or only directly connected nodes (near-neighbor). We ran eight sessions in total, with two sessions per treatment. Though we designed our model for its applications in studying dark network formation, we did not use any "hot" language in our experiments.

Each session consisted of 20 subjects (recruited from the student population at a large public university) that interacted over 30 rounds, and each subject could participate in at most one session.[1] Experiments lasted about one hour, and subjects averaged \$28 for participation including a show-up fee of \$7. Subjects earned "dollars per point," where exchange rates were selected to equalize expected earnings across treatments. All sessions were conducted in an experimental computer lab using zTree software (Fischbacher

[1] Subjects were drawn from the experimental laboratory's subject pool, where students receive email announcements that invite them to register online in order to enter the subject pool. Students in the subject pool then receive a sign-up email with information about a particular session (day, time, location). Subjects that wanted to attend a particular session then click on a link in the sign-up email. They get a reminder email a day before the session. Approximately 60-80% of subjects showed up that had signed up to participate.

2007). Subjects were assigned a unique numeric identifier for the experiment, but were otherwise not allowed to reveal any personal identifying information. Chats were monitored during the experiment to ensure that no such information was shared. Subjects were told the values for p and a at the start of the session, and these were fixed during the duration of the session. After reading the experiment's instructions, subjects answered practice questions to ensure their understanding of the experiment and how their earnings would be calculated.

For each of the 30 rounds, the timing of each round was as follows:

1. Subjects were allowed one minute for discussion. Each subject could send messages through a chat feature in the program software.

2. Subjects were allowed 30 seconds to privately choose other subjects with whom they desired to establish links.

3. Links were formed between subjects, and each subject was privately informed of their successful links.

4. The computer randomly detected subjects according to detection probability p and "arrested" them. Subjects that were linked to any arrested subject were also detected and "arrested by association" according to the arrest impact a.

5. Arrested subjects were privately informed of their arrest and its cause – either direct detection or arrest by association, but subjects that were arrested by association were not informed of the causal link. Subjects who survived detection were not informed of the identities of detected subjects. In this way, we avoid generating a history that could bias the formation of links in future rounds.

6. Payoffs for the round were revealed privately.

7. At the end of a round, all links were severed so that subjects would begin the next round "fresh."

We note a few features unique to our design. Previous network experiments have typically been conducted on smaller groups; for example, Callander and Plott (2005) and Carrillo and Gaduh (2011) use groups of six, and Caldara and McBride (2014) use groups of twelve. Our experiment has twenty subjects in a group, which allows us to observe the emergence of networks with multiple components. Moreover, because of these larger numbers, any social pressure to join a network that might be present in experiments with smaller groups is mitigated in our experiment, preserving a clear possibility of opting out. We also allow communication which has been shown to increase cooperation and coordination on efficient networks in other experiments (as an example, see Mobius *et al* 2005). Note that this game is essentially a large coordination game, but there is no centralized coordination mechanism, so if a network arises, it arises endogenously. The script for the experiment is provided in the appendix.

3.2 Predictions

Our benchmark predictions are that subjects will form, under both near-neighbor and friend-of-a-friend arrest impacts, degree $k = 4$ complete components when the detection probability is low ($p = 0.2$) and degree $k = 1$ complete components when the detection probability is high ($p = 0.5$). Specifically:

> **Prediction 1** For a given arrest impact a, subjects will form fewer links under the high detection probability ($p = 0.5$) than under the low detection probability ($p = 0.2$).
>
> **Prediction 2** For a given detection probability p, subjects will form the same number of links under near-neighbor ($a = 1$) impact as under friend-of-a-friend ($a = n - 1$) impact.

Efficient networks under the low probability of detection would feature four complete components of size five ($k = 4$), and efficient networks under the high probability of detection would feature ten components (complete by definition) of size 2 ($k = 1$). We realize that the former presents a more challenging coordination problem than the latter. Realizing that, we expect that the overall network structure in the former case is less likely to converge to an efficient network, though we predict subjects will form, on average, larger network components.

4 Results

We now discuss a number of results, both with respect to individual behavior and the overall network structures that emerged. We focus our attention towards the number of successful links that subjects formed, but also report results for the number of attempted link initiations. The former is the primary feature of the formed network structure, while the latter is also important for examining the extent of any coordination problems. Time-series graphs showing the average number of attempted and successful links are broken down by treatment in Figures $3 - 6$. We see several important patterns. Since we anticipated that coordination would improve over the rounds, our first results is a preliminary one on this matter.

> **Result 1** On average, subjects attempt to form more links in initial rounds, but the average number of these attempts stabilizes after about 10 rounds.

Because networks become relatively stable after about 10 rounds, our summary statistics in Table 2 are reported only for the last 20 rounds.

> **Result 2** When the probability of detection is high, across both levels of arrest impact, the average number of successful links is close to the predicted $k = 1$. However, when the probability of detection is low, the average number of successful links is lower than the predicted $k = 4$, and much lower for friend-of-a-friend (high) arrest impact than for near-neighbor (low) arrest impact.

Some of these findings align with our predictions, while others do not. When the probability of detection is high, subjects form, on average, 0.8 and 1.1 links under friend-of-a-friend and near-neighbor impacts, respectively. These results support our predictions of $k = 1$ for these treatments. When the probability of detection is low, we predict that subjects will form $k = 4$ links; but subjects form, on average, 3.0 links under low impact and only 1.4 links under high impact. Forming 4 links (as opposed to forming just 1 link) appears to be less focal and more difficult to sustain over time.

The key finding, however, is the differences across the levels of arrest impact. When both the probability of detection and the impact are low, subjects attempt to form, on average, 4.7 links. This suggests that one possible reason that subjects form only 3.0 links on average is that they may simply have trouble connecting. However, when the probability of detection is low and the impact is high (friend-of-a-friend), subjects only attempt to form 1.7 links.

Regression results (also for the last 20 rounds) reported in Table 3 also reveal that the arrest impact has a significant and negative effect for both the average number of links that subjects attempt and the ones they successfully form. Why might this be the case? Our model suggests that, holding probability of detection constant, behavior should not vary across the level of impact, but it clearly does. It appears that subjects are magnifying the threat of arrest by association. If subjects do indeed magnify the threat of arrest by association, counterterrorism policies that exploit this type of risk aversion might be particularly effective.

Result 3 Not all subjects attempt to form links. For a given level of arrest impact, more subjects choose not to attempt any links when the probability of detection is high. It also appears that, for a given probability of detection, a higher arrest impact is also associated with more subjects who choose not to attempt any links, though this effect is weaker.

Our model predicts that all subjects will attempt to form links, but, as Figure 7 shows, this is clearly not the case and the number of subjects choosing to attempt no links clearly varies with the treatment conditions. This suggests that not only might these types of policy effects be effective in reducing the size of terrorist networks, but they may also be effective in reducing the number of potential recruits.

Result 4 Network structures that emerged in our experiments roughly supported our predictions under high probabilities of detection when the equilibrium featured "easy" focal networks of size 2 components. Results were more varied under the treatments with low probabilities of detection, where the efficient network structures were less focal and more difficult to coordinate. Subjects were forming, however, larger and fewer components under these treatments, though the results for the low detection with friend-of-a-friend (high) arrest impact treatment were again somewhat anomalous.

With regards to the overall structure of the networks, we predicted that, under high probabilities of detection, the efficient network would consist of 10 components, each of size 2. Histograms for these treatments in Figures 8 and 9 reveal that components of size 2 were in fact the most frequent, and though these components numbered less than 10, the numbers of components (of all sizes) were quite high - ranging between $5-7$ for most rounds.

Under low probabilities of detection, our theory predicted that an efficient network would consist of 4 complete components, each of size 5. Given the coordination difficulties in forming complete components with 4 other subjects (note that a component of size 2 is by definition a complete component), it is not surprising that the experimental results diverged further from our predictions here. Interestingly enough, there were larger component sizes in treatments with low probabilities of detection, though they were far from complete. Under the treatment that featured both low probability of detection and near-neighbor (low) impact, there were fewer and larger components, as predicted. And though sizes of components were also slightly larger under the treatment with low probability of detection and friend-of-a-friend (high) impact, there were significantly more components than predicted.

With regards to completeness, it is not surprising that under high probabilities of detection (where the predictions were for components of size 2) networks featured 95 and 78 percent complete components for high and low impacts, respectively. Under low probability of detection and high impact, networks had 89 percent complete components; but again, this seems be due to the fact that many of these components were dyads. Under low probability of detection and low impact, only 23 percent of the components were complete, and this is mostly reflective of the dyads in the structure. Components were larger under this treatment, but they were not complete, mainly due to the larger number of connections made and the lack of size 5 components being salient. There is little evidence to suggest that subjects attempt complete components, except in the case of the dyad structures, which are complete by definition.

Because communication has been shown to facilitate coordination on efficient networks, we also looked at the chat communications between subjects. We (subjectively) categorized the chat messages and here mention a few interesting patterns. There were many instances of subjects sending out their own number (many times) in beginning rounds – and we view this as an attempt to gain visibility and distinguish themselves. Some subjects were successful at developing a leadership role early on, providing strategy and directing other subjects to link with them. Relationships seemed to hold up over the rounds and there were repeated messages confirming desires to link with the same subjects. These findings suggest that communication may have helped coordination to a degree. However, there were also many messages off-topic and unrelated to the experiment. We could see no other clear patterns with regards to the effects of communication on link formation, payoffs, or coordination on efficient outcomes (regression results not reported). Because chat messages were costless, there was a lot of noise in the data. Perhaps a modification to introduce costly communication or communication that affected detection probabilities would reduce the noise in the chats.

5 Conclusion

We have provided theoretical and experimental results that yield new insights into the formation of dark networks facing threats of detection and disruption. Our key modeling innovation identified how network structures differ when facing different strengths of detection and disruption policies; and our experimental findings revealed that each policy aspect affected the network structure, even in cases when our theory predicted it would not. Our experiments suggest that dark networks respond to changes in detection and disruption policies, which is consistent in spirit with Kenney (2003) and Enders and Su (2007) and perhaps the best known example—al Qaeda (Enders and Su 2007; see also Sageman 2004, 2008). A better understanding of this phenomenon is crucial for effective policymaking (Sawyer and Foster 2008).

While we do not argue that our model and experiment capture everything relevant in studying dark networks, our approach has certain advantages. We isolate particular aspects of detection and disruption to study the decisions people make and the aggregate implications of those decisions for the overall network. Importantly, we are able to observe the entire network at the individual level, generating data which is normally difficult to obtain in the real world.

We note an important behavioral anomaly in our study. Though the incentives in our experiment were designed so that subjects should have always attempted to form at least one link, we find that many did not. We know that some people are simply more prone to engage in these types of networks, as suggested by psychological studies of terrorists (e.g., Kruglanski et al. 2009; Merari, D. et al. 2010; and Merari, F. et al. 2010) and the neurobiological literature (Victoroff 2009). Yet, because in our experiment this number decreased as we strengthened our disruption policies, we suspect that this result may hold in the "real world" as well. A valuable next step would be to identify how this result relates to individual risk tolerance.

The type of analysis presented in this paper can never obviate the need for other kinds of studies, but the methodologies used in this paper can certainly augment them. Relatively little is known about dark networks, so using game-theoretic modelling and experimental analysis might also be particularly useful in continuing to develop theories that examine network structures for their stability and effectiveness.

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6 Tables

Table 1: 2x2 Experimental Design

Treatments		Probability (p) High	Probability (p) Low
Impact (α)	High	$\alpha = n - 1, p = 0.5$	$\alpha = n - 1, p = 0.2$
	Low	$\alpha = 1, p = 0.5$	$\alpha = 1, p = 0.2$

Table 2: Experimental Results: Summary Statistics

		High Detection		Low Detection		
		mean	(std)		mean	(std)
High Impact	Attempted	1.2	(1.57)	Attempted	1.7	(1.65)
	Actual	0.8	(0.64)	Actual	1.4	(0.72)
	Survived	0.2	(0.41)	Survived	0.7	(0.82)
		mean	(std)		mean	(std)
Low Impact	Attempted	1.9	(2.61)	Attempted	4.7	(3.78)
	Actual	1.1	(0.88)	Actual	3.0	(1.40)
	Survived	0.2	(0.42)	Survived	1.2	(1.56)

Summary statistics are reported for the last 20 rounds.

Table 3: OLS Regressions: Attempted and Successful Links

Model specification (1):

$$AL = \beta_0 + \beta_1 D_{HpH\alpha} + \beta_2 D_{HpL\alpha} + \beta_3 D_{LpH\alpha} + \varepsilon$$

Model specification (2):

$$SL = \beta_0 + \beta_1 D_{HpH\alpha} + \beta_2 D_{HpL\alpha} + \beta_3 D_{LpH\alpha} + \varepsilon$$

where

AL	=	Average Number of Attempted Links Each Subject Made
SL	=	Average Number of Successful Links Each Subject Achieved
$D_{HpH\alpha}$	=	Dummy Equal to 1 for High p, High α treatment, 0 Otherwise
$D_{HpL\alpha}$	=	Dummy Equal to 1 for High p, Low α treatment, 0 Otherwise
$D_{LpH\alpha}$	=	Dummy Equal to 1 for Low p, High α treatment, 0 Otherwise

Results:

Regression Specification	(1)	(2)
Treatment Variables		
β_0 - (Constant term; Low p, Low α)	4.70^{***}	2.97^{***}
	(0.30)	(0.11)
β_1 - High p, High α	-3.54^{***}	-2.15^{***}
	(0.43)	(0.16)
β_2 - High p, Low α	-2.78^{***}	-1.86^{***}
	(0.43)	(0.16)
β_3 - Low p, High α	-3.02^{***}	-1.62^{***}
	(0.43)	(0.16)
—		
R^2	0.35	0.59
$\overline{R}^2$	0.34	0.58
Number of Observations	160	160

Significance level: (***) denotes 1 percent. Regression results are for the last 20 rounds.

7 Figures

Figure 1: Examples of Networks

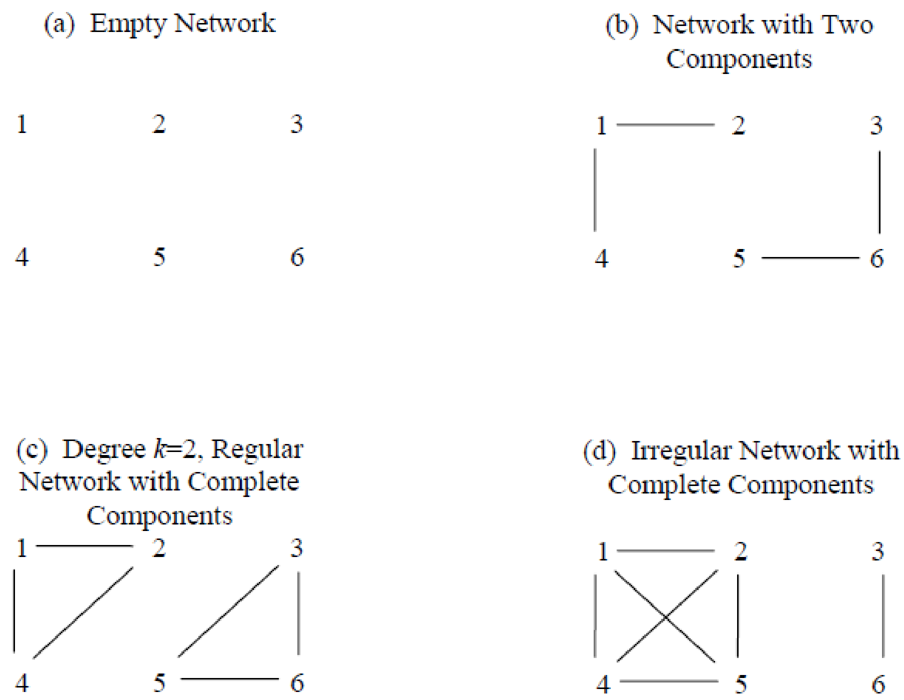
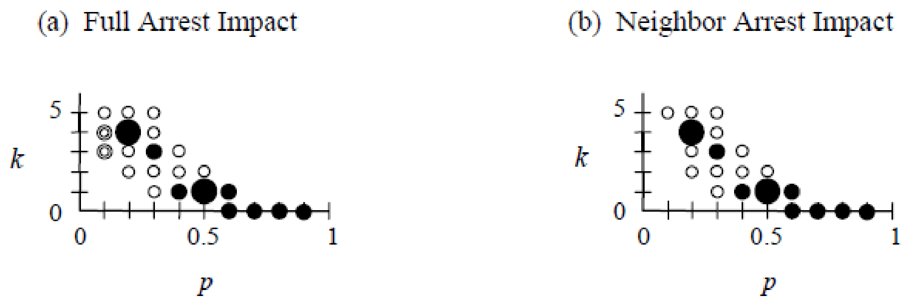
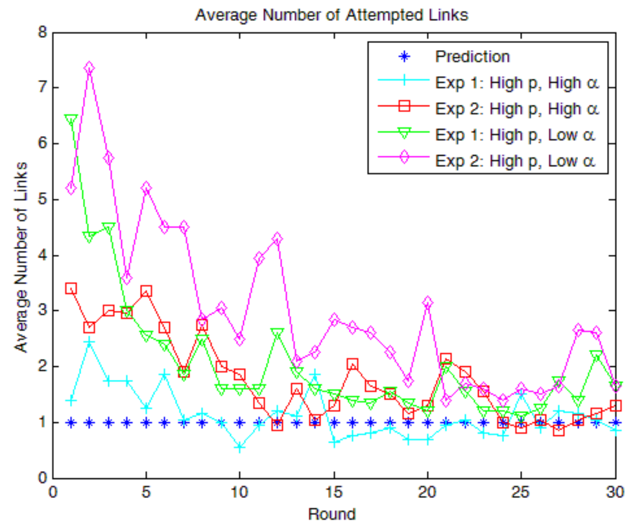
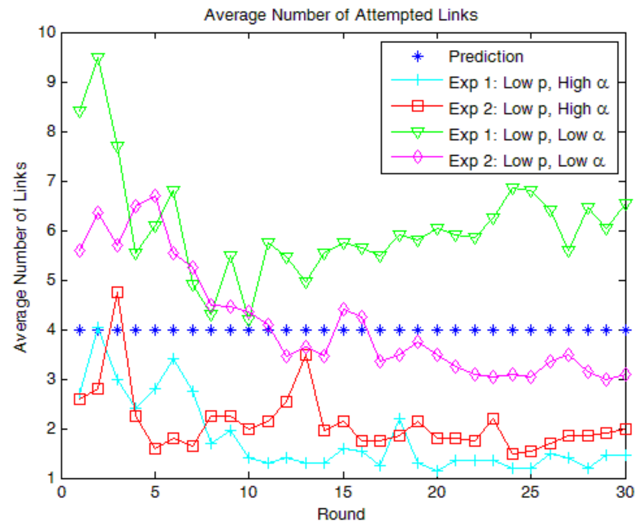


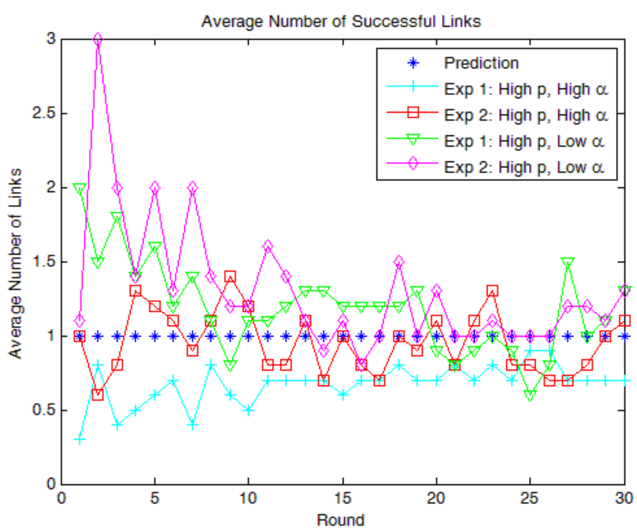
Figure 2: Degree-k Regular, Complete Component, Pairwise Stable Nash Networks with $n = 20$, $x = 10$, and $y = 15$



Figures 3-6: Experimental Results: Time-Series Graphs







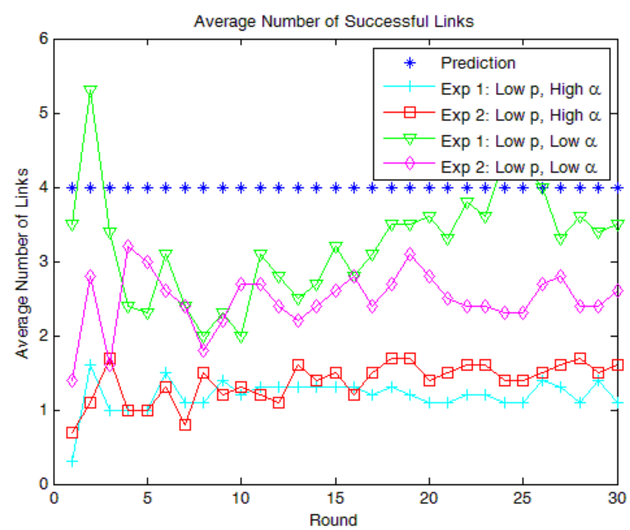
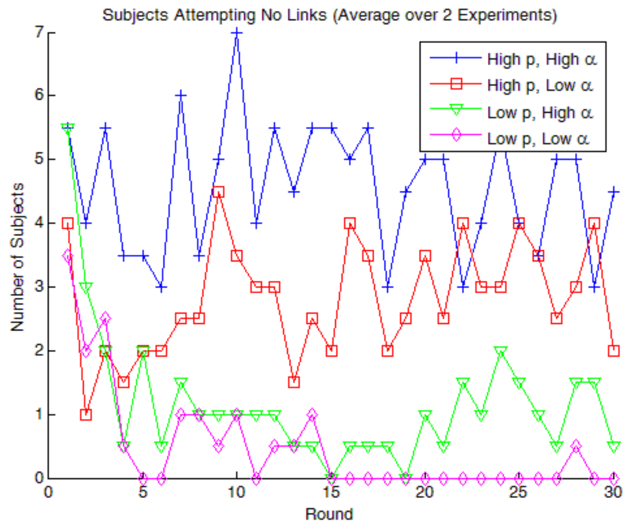


Figure 7: Number of Subjects Attempting No Links



Figures 8: Component Size Histograms

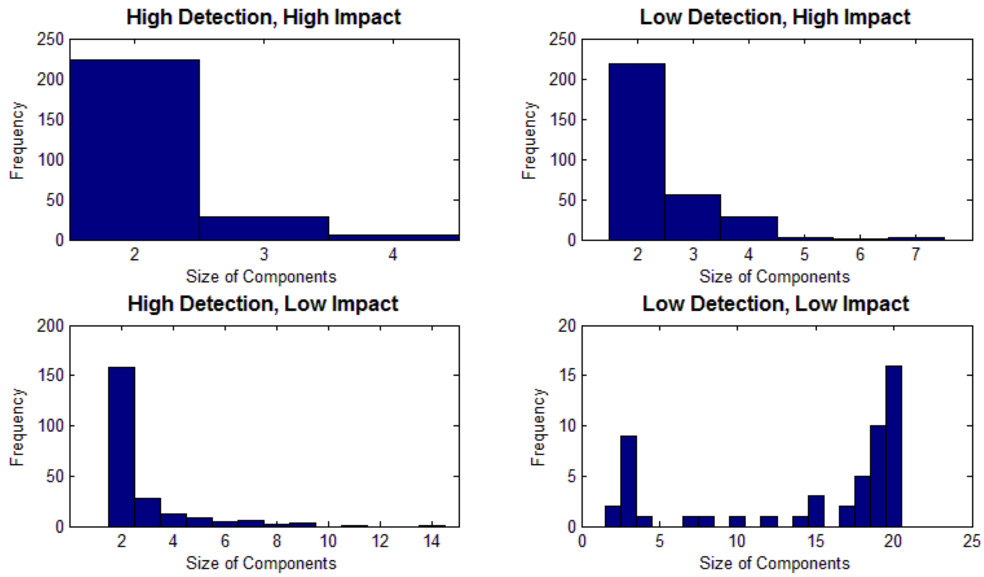
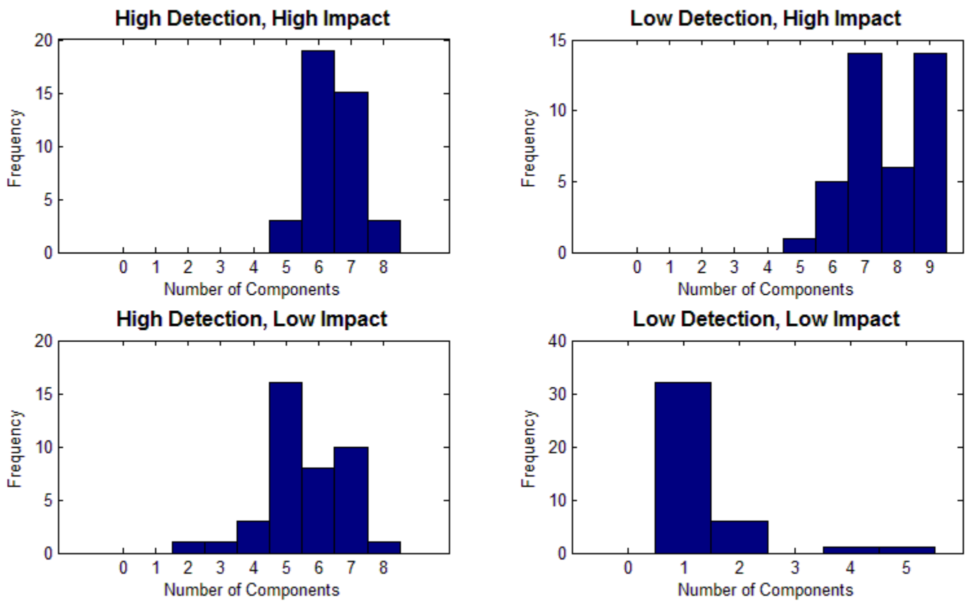


Figure 9: Component Number Histograms



A Appendix

A.1 Experiment Instructions

WELCOME

Welcome to this experiment at UC Irvine. Thank you for participating.

You are about to participate in a study of decision-making, and you will be paid for your participation in cash, privately at the end of this session. What you earn depends partly on your decisions and partly on chance.

Please turn off your cell phone.

The entire session consists of 30 rounds. You will be paid according to the outcomes of these rounds.

All rounds will take place through the computer terminals. It is important that you do not talk with any other participants during the session.

When you are ready, please click continue to go to the instructions.

INSTRUCTIONS PART 1

During each round, you will choose which of the other participants with whom to try to form a link.

If BOTH you and a particular participant try to form a link with each other, then the link is formed. We then say that you and that other participant are "partners."

If one or both of you does not try to form the link with each other, then no link forms between the two of you. You are not partners.

You may try to form a link with as many of the other participants as you prefer.

Before deciding with whom to try to form links (a.k.a., partnerships), you will have 60 seconds to chat via the computer with the other participants.

INSTRUCTIONS PART 2

After links are formed, the computer will randomly flag some of the participants.

The participants that are flagged AND the flagged participants' partners will be removed for that round. All partnerships of removed participants will be dissolved for that round. Removed participants will receive 0 points.

All remaining participants will receive 10 points for not being removed plus an additional 15 points for each remaining partnership.

For example, suppose that you successfully form X links (partnerships) in a round. Also suppose that neither you nor any of your partners are flagged in this round. However, Y of your X partners have other partners that are flagged. As a result, these Y partners are removed. Then, your payoff for this round is 10 + 15*(X-Y) points. You receive 10 points for not being removed, and 15 for each of the remaining (X-Y) partnerships.

INSTRUCTIONS PART 3

In each round, there is a [20|50]% chance that you will be flagged by the computer. This chance of being flagged does not depend on how many partnerships you form.

Each other participant also has a [20|50]% chance of being flagged.

Remember that you are removed if you or one of your partners is flagged. Note that you are not removed if a partner of one of your partners is flagged.

TEST SCREEN 1

Before proceeding, you must answer some questions. These questions test your comprehension. Remember that each participant has a [20|50]% chance of being flagged.

Please select the answer.

1. What is chance that you are removed if you have successfully formed at least one partnership?

(a) Less than [20|50]%

(b) [20|50]%

(c) More than [20|50]%

ANSWER SCREEN 1

You are [CORRECT|INCORRECT]!

Remember that each participant has a [20|50]% chance of being flagged.

1. What is chance that you are removed if you have successfully formed at least one partnership?

The correct answer is (c) More than [20|50]%.The chance that you are removed is equal to the chance that you or one of your partners is flagged. There is a [20|50]% chance that you are flagged, but when you are not flagged, there is also a chance that at least one of your partners will be flagged.

Now, please select the correct answer for the following question:

(Remember that if you are not removed, you receive 10 points plus 15 points times the number remaining partnerships.)

2. Suppose that you successfully form 5 partnerships, that you and your partners are not flagged, but that two of your partners are removed. What is your payoff?

(a) 0 points

(b) 10 points

(c) 55 points

(d) 85 points

ANSWER SCREEN 2

You are [CORRECT|INCORRECT]!

Remember that if you are not removed, you receive 10 points plus 15 points times the number remaining partnerships.

2. Suppose that you successfully form 5 partnerships, that you and your partners are not flagged, but that two of your partners are removed. What is your payoff?

The correct answer is (c) 55 points. The formula is 10 + 15(X-Y). With X=5 original partnerships, and Y=2 removed partners, the payoff is 10+15(5-2) = 55.

Click continue to proceed.

INSTRUCTIONS PART 4

You will now participate in 30 rounds.

In the first round, you will be assigned a participant ID number (different from your computer Station number). Each participant will keep the same participant ID number during the duration of the experiment.

You will receive $[0.05|0.20] for each point you earn in this session.

Remember, you are not to talk with anyone during the experiment.

Note: There are two chat rules. 1. You may not use profanity. 2. Your chats must be anonymous. That is, you must not reveal your actual name or any personal identifying information.

Chat comments that include profanity or that violate anonymity may result in you being asked to leave the experiment. In this case, you would forgo any earnings you accumulated during the experiment session.

Final note: After each chatting period, you will have 30 seconds to initiate links on the decision screen. You MUST press the "Continue" button before time runs out for your link decisions to be saved. If you do not press "Continue" in time, you will not form any links for that round.

A.2 Proofs

Proposition 1 (a) Let $k^* = \max\left\{0, \frac{1}{p} - 1 - \frac{x}{y}\right\}$. For both near-neighbor ($a = 1$) and friend-of-a-friend ($a = n-1$) arrest impacts, the degree-k^* regular network with complete components yields the highest payoff to all actors among all possible networks.

(b) The degree-k^* regular, complete component network is pairwise Nash stable under both near-neighbor and friend-of-a-friend arrest impact.

(c) k^*, though identical under near-neighbor and friend-of-a-friend arrest, is decreasing in detection probability p.

Proof (a) We first show that i having k links in a complete component yields i her highest payoff among all networks in which she has k links. Consider two networks L and L' such that actor i has k links in both L and L', and i is a member of a complete component in L but not in L'. In either network and with any value of $a \in \{1, n-1\}$, i receives a positive payoff only if neither i nor any actor with which i is connected is detected, occurring with probability $(1-p)^{k+1}$. In the network L, actor i receives a positive payoff of $(x + yk)$ if and only if neither i nor any other actor with which i is linked is detected, and otherwise receives a payoff of zero. In the network L', actor i receives a positive payoff only if neither i nor any other actor with which i is linked is detected, and receives a payoff of $(x + yk)$ only if none of the actors to which i is linked are removed. Note that since actor i is not in a complete component, there must be actors j and j' such that $d(i,j) = d(j,j') = 1$ and $d(i,j') > 1$. Thus, there is a positive probability that j' is detected, in which case actor j is removed. Therefore, actor i receives a positive payoff in network L whenever he would receive a positive payoff in L', except that there is a positive probability that actor i receives a lower payoff in L'. We conclude that actor i has a higher expected utility in L than in L'.

We next show that having $k^* = \max\left\{0, \frac{1}{p} - 1 - \frac{x}{y}\right\}$ links within a complete component yields the highest payoff to actor i among all complete-component networks for both $a = 1$ and $a = n-1$. Under both friend-of-a-friend and near-neighbor arrest impacts, the marginal increase in the expected payoff in going from a degree-k, complete component to a degree-$k+1$, complete component is:

$$\begin{aligned} & (1-p)^{k+2}(x + (k+1)y) - (1-p)^{k+1}(x + ky) \\ = \quad & (1-p)^{k+1}(-px + (1-p-pk)y). \end{aligned}$$

This marginal increase is strictly decreasing in k. It is positive when

$$\begin{aligned} (1-p)^{k+1}(-px + (1-p-pk)y) &> 0 \Rightarrow \\ \frac{1-p}{p} - \frac{x}{y} &> k. \end{aligned}$$

There is therefore a payoff-maximizing $k^* \equiv \max\left\{0, \frac{1-p}{p} - \frac{x}{y}\right\}$, and the farther k is from the optimal k^*, the lower the expected payoffs.

(b) Under both friend-of-a-friend and near-neighbor arrest impacts, the marginal increase in the expected payoff in going from a degree-k, complete component to a degree-$k+1$, complete component is:

$$\begin{aligned} & (1-p)^{k+2}(x + (k+1)y) - (1-p)^{k+1}(x + ky) \\ = \quad & (1-p)^{k+1}(-px + (1-p-pk)y). \end{aligned}$$

This marginal increase is strictly decreasing in k. It is positive when

$$\begin{aligned} (1-p)^{k+1}(-px + (1-p-pk)y) &> 0 \Rightarrow \\ \frac{1-p}{p} - \frac{x}{y} &> k. \end{aligned}$$

There is therefore a payoff-maximizing $k^* \equiv \max\left\{0, \frac{1-p}{p} - \frac{x}{y}\right\}$, and the farther k is from the optimal k^*, the lower the expected payoffs.

(c) Immediately follows from (a).

Proposition 2 Assume $k^* > 0$. Then:

(a) $\overline{k}(1,p) = \overline{k}(n-1,p) = \lfloor \kappa_1 \rfloor$ where κ_1 is such that

$$(1-p)^{\kappa_1+1}(x + y\kappa_1) = (1-p)x.$$

(b) $\lceil \kappa_2 \rceil = \underline{k}(n-1,p) \leq \underline{k}(1,p) = \lceil \kappa_3 \rceil$, where κ_2 and κ_3 are real solutions to the following equations, respectively:

$$\begin{aligned} (1-p)^{\kappa_2+1}(x+y(\kappa_2+1)) &= (x+y\kappa_2) \\ (1-p)^{\kappa_3+1}y &= p(x+y\kappa_3). \end{aligned}$$

(c) $\underline{k}(a,p)$ and $\overline{k}(a,p)$ are weakly decreasing in p.

Lemma 1 Let $a=1$. Consider an actor i in a network L. Suppose that there is a subset S of the actors to which i is linked with $|S|>1$ and an actor j to which i is not linked such that for all $j' \in S$, j is linked to j'. Then let the network L' be obtained by removing the link between j and j' for some $j' \in S$ and adding a link between j' and some other actor j'' such that $d(i,j'')>2$. Then actor i's expected utility is strictly higher in the network L' than in the network L.

Proof of Lemma 1 Let S be the set of all actors linked with both i and j and consider L and L' as in the statement of the lemma. Note that the expected utility is strictly higher in the network L' if and only if the expected number of surviving links between i and the members of S is larger under L'. That is,

$$\sum_{\kappa=0}^{|S|} \kappa q(\kappa, L') \geq \sum_{\kappa=0}^{|S|} \kappa q(\kappa, L) \tag{1}$$

where $q(\kappa, X)$ is the probability that exactly κ of the members of S are not removed given the network X. We shall prove this through induction on the size of S.

Suppose that $|S|=2$. Let k_r denote the number of links that actor r has with actors with which actor s is not linked in the network L, and let k_{rs} denote the number of actors to which both r and s are linked. Then (1) reduces to

$$\begin{aligned} & (1-p)^{k_{rs}-1}(1-p)^{k_1+1}\left(1-(1-p)^{k_2}\right)+(1-p)^{k_{rs}-1}\left(1-(1-p)^{k_1+1}\right)(1-p)^{k_2} \\ & +(1-p)^{k_{rs}-1}(1-p)^{k_1+1}(1-p)^{k_2} \\ \geq\ & (1-p)^{k_{rs}}(1-p)^{k_1}\left(1-(1-p)^{k_2}\right)+(1-p)^{k_{rs}}\left(1-(1-p)^{k_1}\right)(1-p)^{k_2} \\ & +(1-p)^{k_{rs}}(1-p)^{k_1}(1-p)^{k_2}, \end{aligned}$$

which becomes

$$\begin{aligned} & (1-p)^{k_1+1}\left(1-(1-p)^{k_2}\right)+\left(1-(1-p)^{k_1+1}\right)(1-p)^{k_2}+(1-p)^{k_1+1}(1-p)^{k_2} \\ \geq\ & (1-p)^{k_1+1}\left(1-(1-p)^{k_2}\right)+(1-p)\left(1-(1-p)^{k_1}\right)(1-p)^{k_2}+(1-p)(1-p)^{k_1+1}(1-p)^{k_2}, \end{aligned}$$

and further

$$\begin{aligned} \left(1-(1-p)^{k_1+1}\right)(1-p)^{k_2} &\geq (1-p)\left(1-(1-p)^{k_1}\right)(1-p)^{k_2} \Rightarrow \\ 1-(1-p)^{k_1+1} &\geq 1-p-(1-p)^{k_1+1} \Rightarrow \\ p &\geq 0. \end{aligned}$$

Therefore, (1) holds for $|S| = 2$.

Lemma 2 Let $a = 1$. Consider two networks L and L' with two linked actors i and j, each with a total of k links such that j does not share any neighbors (other than i) with any of i's other neighbors. Suppose that L and L' are identical except that m of i's neighbors other than j have a link with j in L but not in L'. In place of this link, those actors have another link with some other actor with which they are not linked in L. In place of his links with those actors, j has an additional m links with actors who are neither linked with i nor linked with any of i's neighbors other than j.

Proof of Lemma 2 Consider two such networks L and L'. Let m and m' denote the number of neighbors which i and j share, respectively, where $m > m'$. Note that i's expected payoff is greater in L than in L' if

$$(1-p)^{k+1}\left(x + y\sum_{\kappa=0}^{k}\kappa q(\kappa, L)\right) > (1-p)^{k+1}\left(x + y\sum_{\kappa=0}^{k}\kappa q(\kappa, L')\right)$$

where $q(\kappa, L)$ denotes the probability that exactly κ of i's links survive conditional on i's survival in the network L. This inequality holds if and only if

$$\sum_{\kappa=0}^{k}\kappa q(\kappa, L) > \sum_{\kappa=0}^{k}\kappa q(\kappa, L'),$$

that is, if the expected number of i's surviving links conditional on i's survival is higher in L than in L'. Because j shares no neighbors other than i with i's other neighbors, we may express this inequality as

$$\begin{aligned} & (1-p)^{k-m}\sum_{\kappa=0}^{k-1}(\kappa+1)\, r(\kappa, L) + \left(1-(1-p)^{k-m}\right)\sum_{\kappa=0}^{k-1}\kappa r(\kappa, L) \\ > \; & (1-p)^{k-m'}\sum_{\kappa=0}^{k-1}(\kappa+1)\, r(\kappa, L') + \left(1-(1-p)^{k-m'}\right)\sum_{\kappa=0}^{k-1}\kappa r(\kappa, L') \end{aligned}$$

where $r(\kappa, L)$ is the probability that exactly κ of i's links other than his link with j survive conditional on i's survival in the network L. This inequality is equivalent to

$$(1-p)^{k-m} + \sum_{\kappa=0}^{k-1}\kappa r(\kappa, L) > (1-p)^{k-m'} + \sum_{\kappa=0}^{k-1}\kappa r(\kappa, L').$$

Because $m > m'$, then $(1-p)^{k-m} > (1-p)^{k-m'}$, and so it is sufficient to show that $r(\kappa, L) \geq r(\kappa, L')$ for all $\kappa = 0 : k-1$. Note that i's neighbors other than j may be partitioned into a set of those whose links are identical in both L and L' and a set of those whose links are identical except that they are linked with j in L but not in L', and are linked with some other actor in L'. If one of the latter actors is such that the different actor to which they are linked is also linked to i, then the probability of their removal conditional on i's survival is identical in both networks. If, however, the different actor to which they are linked is not linked with i, then there is a greater probability of their removal conditional on i's survival in L' than in L. Thus, $r(\kappa, L) \geq r(\kappa, L')$ for all $\kappa = 0, ..., k-1$.

Lemma 3 Consider a degree-k regular network with closed components and any impact of arrest a. Then among all possible link removal deviations, the most beneficial deviation is either to remove all links or remove exactly one link. Suppose that $k^* > 0$. Then

(a) If $k \leq k^*$, no actor will prefer to remove all of his links.

(b) No actor will ever prefer to remove exactly one of his links.

Proof of Lemma 3 In the proof of Proposition 4(b), the first statement is proven. Assume that $k^* > 0$.

(a) Suppose that $k \leq k^*$. No actor prefers to remove all of his links if

$$(1-p)^{k+1}(x+yk) \geq (1-p)\,x.$$

The left hand side of this equation is quasiconcave with a peak at k^*. Evaluating the left hand side at $k = 0$ yields $(1-p)\,x$, and so it must be that at any $k \in (0, k^*)$ the inequality is satisfied.

(b) No actor prefers to remove a single link if

$$\begin{aligned}
(1-p)^{k+1}(x+yk) &\geq (1-p)^{k+1}(x+y(k-1)) + p(1-p)^k x \Rightarrow \\
(1-p)^{k+1} y &\geq p(1-p)^k x \Rightarrow \\
\frac{y}{x} &\geq \frac{p}{1-p}.
\end{aligned}$$

Note that this inequality is independent of k. Since a degree-k^* regular network with complete components is pairwise stable, then this holds for $k = k^*$, and thus for all k.

Proof of Proposition 2 The proofs of parts (a) and (b) will be conducted by finding the conditions under which a deviation is most profitable and then finding the largest/smallest number of links for which such a deviation will not occur.

(a) We will first derive the equation which defines $\overline{k}(n-1,p)$. From Proposition 3(a), we need only consider actors in complete components. The payoff to such an actor with k links is

$$(1-p)^{k+1}(x+yk).$$

Since the actor is in a complete component, removing fewer than all of his links does not change his probability of removal, rather it only reduces his payoff when he survives. Thus, the actor would only consider either adding a link or removing all of his links. Since $(1-p)^{k+1}(x+yk)$ is strictly quasiconcave with maximum at k^* (very easy to show, you just take the derivative and it is clearly positive for $k < k^*$, negative for $k > k^*$), then it is cannot be beneficial to add a link, even with actor with no other links. The actor would be indifferent between removing all his links and not if

$$(1-p)^{k+1}(x+yk) = (1-p)\,x. \tag{2}$$

Note that the left hand side of (2) is decreasing in $k > k^*$ and the right hand side is independent of k. Thus, if κ_1 satisfies (2), then $\overline{k}(n-1,p) = \lfloor \kappa_1 \rfloor$, as for any $k \in \mathbb{N}$ with $k > \lfloor \kappa_1 \rfloor$, the left hand side of (2) must be smaller than the right hand side.

Note that Lemma 3 together with the arguments above can be used to show that a degree-$\lfloor \kappa \rfloor$ regular complete components network is pairwise stable with impact of arrest $a = 1$, so $\overline{k}(1,p) \geq \overline{k}(n-1,p)$. Suppose that $\overline{k}(1,p) > \lfloor \kappa_1 \rfloor$. Then there is a pairwise stable network with an actor with $k > \lfloor \kappa_1 \rfloor$ links. From the previous proposition, the actor would be at least as well off in a complete component with k links. Because $k > \lfloor \kappa_1 \rfloor$, then

$$(1-p)^{k+1}(x+yk) < (1-p)x.$$

It follows that the actor would receive a strictly larger payoff from cutting all his links than staying in a complete component, and so he would strictly improve by cutting all his links in any network in which he has k links. This contradicts the assumption that the network was pairwise stable. Therefore, $\overline{k}(1,p) = \overline{k}(n-1,p) = \lfloor \kappa_1 \rfloor$, where κ_1 satisfies (2).

(b) We will first derive the equation which defines $\underline{k}(n-1,p)$. Based on Proposition 3(a), we need only consider networks with complete components. An actor with k links will prefer to not add a link to another actor with k' links if

$$\begin{aligned}(1-p)^{k+1}(x+yk) &> (1-p)^{k+k'+2}(x+y(k+1)) \Rightarrow \\ 0 &> (1-p)^{k'+1}(x+y(k+1)) - (x+yk). \end{aligned} \tag{3}$$

The right hand side of (3) is decreasing in both k and k'. Thus, the $\underline{k}(n-1,p)$ must be at least as large as the smallest integer k such that (3) is satisfied for $k' = k$. That is, $\underline{k}(n-1,p) \geq \lceil \kappa_2 \rceil$ where

$$(1-p)^{\kappa_2+1}(x+y(\kappa_2+1)) = (x+y\kappa_2).$$

Because $\lceil \kappa_2 \rceil \leq k^*$, then Lemma 3 implies that a degree-$\lceil \kappa_2 \rceil$ regular network with complete components is pairwise stable. Therefore $\underline{k}(n-1,p) = \lceil \kappa_2 \rceil$.

Next, we will derive the equation which defines $\underline{k}(1,p)$. With the low impact of arrest ($a = 1$), the benefit to an actor i from adding a link with another actor j is earned only if both actors survive, while the "cost" is that actor i is removed in the event that j is detected. Thus, an actor i with k links prefers to not add another link to an actor j if

$$(1-p)^{k+1}(x+yK) \geq (1-p)^{k+2+k'}(x+y(K'+1)) + \left(1-(1-p)^{k'}\right)(1-p)^{k+2}(x+yK''), \tag{4}$$

where K is the expected number of actor i's links that survive conditional on j being detected, K' is the expected number of actor i's links that survive conditional on j surviving, K'' is the expected number of actor i's links that survive conditional on j being removed but not detected, and k' is the number of actor j's neighbors that are not also neighbors of i. Note that the term $(1-p)^{k'}$ is the probability that j survives conditional on all of i's neighbors going undetected. Thus, the left hand side is actor i's expected utility conditional on no link existing between i and j, while the right hand side is actor i's expected utility conditional on a link existing between i and j.

In order to determine the smallest number of links k for which this inequality holds, we are free to pick the conditions on K, K', and K'' such that the inequality in (4) is favored most, then verify that those conditions may exist in a pairwise stable network. Thus, we will pick the network such that the left hand side of (4) is maximized and the right hand side is minimized. Based on the Lemma 1 above, the right hand side of (4) is minimized when the actor j shares no neighbors with any of i's neighbors. Based on Lemma 2 above, the right hand side of (4) is minimized when i and j share no neighbors. This implies that the minimizing value of k' is $k' = k$, in which case $K = K' = K''$. Thus, we may reduce (4) to

$$\begin{aligned}
(1-p)^{k+1}(x+yK) &\geq (1-p)^{2k+2}(x+y(K+1)) + \left(1-(1-p)^k\right)(1-p)^{k+2}(x+yK) \Rightarrow \\
(x+yK) &\geq (1-p)^{k+1}(x+y(K+1)) + \left(1-(1-p)^k\right)(1-p)(x+yK) \Rightarrow \\
(x+yK) &\geq (1-p)^{k+1}y + (1-p)(x+yK) \Rightarrow \\
p(x+yK) &\geq (1-p)^{k+1}y. \qquad (5)
\end{aligned}$$

It is clear that the left hand side of (5) is maximized at $K = k$, corresponding to i being in a complete component. Thus, the maximized left hand side is increasing in k, while the minimized right hand side is decreasing in k. Thus, the smallest value of k satisfying the following inequality (6) will be the candidate for $\underline{k}(1,p)$:

$$p(x+yk) \geq (1-p)^{k+1}y.$$

Let κ_3 satisfy (6) with equality. It follows that for any network in which at least $\lceil \kappa_3 \rceil$ actors have at most $\lceil \kappa_3 \rceil - 1$ links, at least one pair of these actors must be unlinked and benefit from forming a link together. Thus, $\underline{k}(1,p) \geq \lceil \kappa_3 \rceil$. It remains to check that there exists a pairwise stable network with $\underline{k}(1,p)$. Again from Lemma 3, a degree-$\lceil \kappa_3 \rceil$ network must be pairwise stable. Therefore, $\underline{k}(1,p) = \lceil \kappa_3 \rceil$.

It remains to be shown that $\underline{k}(n-1,p) \leq \underline{k}(1,p)$, for which it is sufficient that $\kappa_2 \leq \kappa_3$. Recall that the defining equations for κ_2 and κ_3 are

$$\begin{aligned}
(x+y\kappa_2) &= (1-p)^{\kappa_2+1}(x+y(\kappa_2+1)), \\
p(x+y\kappa_3) &= (1-p)^{\kappa_3+1}y.
\end{aligned}$$

It will be convenient to rearrange these as

$$\begin{aligned}
\frac{\left(1-(1-p)^{\kappa_2+1}\right)(x+y\kappa_2)}{(1-p)^{\kappa_2+1}} &= y, \qquad (6) \\
\frac{p(x+y\kappa_3)}{(1-p)^{\kappa_3+1}} &= y. \qquad (7)
\end{aligned}$$

Note that for $\kappa_2 = \kappa_3 = 0$, the left hand sides of the above equations are equal, evaluating to $px/(1-p)$. Since $px/(1-p) < y$ is necessary and sufficient for a nonempty network to exist, then $k^* > 0$ implies that $px/(1-p) < y$. Thus, it is sufficient to show that the derivative of the left hand side of (7) is larger than the derivative of (8) with respect to κ_i, as this will imply that the left hand side evaluates to y at a lower value of κ_i. These derivatives are as follows.

$$\begin{aligned}
\frac{d}{dk}\frac{\left(1-(1-p)^k\right)(x+yk)}{(1-p)^{k+1}} &= \frac{\left(1-(1-p)^k\right)y-\ln(1-p)(x+yk)}{(1-p)^{k+1}} \\
\frac{d}{dk}\frac{p(x+yk)}{(1-p)^{k+1}} &= \frac{py-\ln(1-p)\,p(x+yk)}{(1-p)^{k+1}}.
\end{aligned}$$

The following are equivalent.

$$\begin{aligned}
\frac{\left(1-(1-p)^k\right)y-\ln(1-p)(x+yk)}{(1-p)^{k+1}} &\geq \frac{py-\ln(1-p)\,p(x+yk)}{(1-p)^{k+1}} \\
\left(1-(1-p)^k\right)y-\ln(1-p)(x+yk) &\geq py-\ln(1-p)\,p(x+yk) \\
\left(1-(1-p)^k\right) &\geq p \\
1 &\geq p+(1-p)^k.
\end{aligned}$$

The final inequality holds since $(1-p)\leq 1$.

(c) To show that $\underline{k}(a,p)$ and $\overline{k}(a,p)$ are weakly decreasing in p, it is sufficient to show that κ_1, κ_2, and κ_3 are decreasing in p, for which we will use the implicit function theorem.

For κ_1, recall that κ_1 is defined by

$$(1-p)^{\kappa_1+1}(x+y\kappa_1)=(1-p)\,x.$$

Using the implicit function theorem, we obtain

$$\frac{\partial\kappa_1}{\partial p}=\frac{(\kappa_1+1)(1-p)^{\kappa_1}(x+yk)-x}{(1-p)^{\kappa_1+1}(\ln(1-p)(x+y\kappa_1)+y)}.$$

We will first show that the denominator is nonpositive. Recall that $\kappa_1\geq k^*$, and k^* maximizes $(1-p)^{k+1}(x+yk)$. Thus, the derivative of this expression is zero at k^*, that is,

$$\begin{aligned}
(1-p)^{k^*+1}\ln(1-p)(x+yk^*)+(1-p)^{k^*+1}y &= 0 \Rightarrow \\
\ln(1-p)(x+yk^*)+y &= 0.
\end{aligned}$$

Because $\ln(1-p)<0$, then $\ln(1-p)(x+y\kappa_1)+y$ is decreasing in k, and so it must be that the denominator is negative. The numerator is negative if and only if

$$\begin{aligned}
(\kappa_1+1)(1-p)^{\kappa_1}(x+yk) &\geq x \Rightarrow \\
(\kappa_1+1)(1-p)^{\kappa_1+1}(x+yk) &\geq (1-p)\,x.
\end{aligned}$$

Because $\kappa_1+1\geq 1$, then this is true based on the equation which defines κ_1. Thus, κ_1 is decreasing in p.

For κ_2, recall that κ_2 is defined by

$$(1-p)^{\kappa_2+1}(x+y(\kappa_2+1))=(x+y\kappa_2).$$

Again using the implicit function theorem, we obtain

$$\frac{\partial \kappa_2}{\partial p} = \frac{(\kappa_2+1)(1-p)^{\kappa_2}(x+y(\kappa_2+1))}{(1-p)^{\kappa_2+1}(\ln(1-p)(x+y(\kappa_2+1))+y)-y}.$$

The numerator is clearly positive, so we need only show that the denominator is negative. The following are equivalent.

$$\begin{aligned}
(1-p)^{\kappa_2+1}(\ln(1-p)(x+y(\kappa_2+1))+y)-y &\leq 0 \Rightarrow \\
\ln(1-p)(1-p)^{\kappa_2+1}(x+y(\kappa_2+1))+(1-p)^{\kappa_2+1}y &\leq y \Rightarrow \\
\ln(1-p)(x+y\kappa_2)+(1-p)^{\kappa_2+1}y &\leq y \Rightarrow \\
\ln(1-p)(x+y\kappa_2) &\leq \left(1-(1-p)^{\kappa_2+1}\right)y.
\end{aligned}$$

The final inequality holds trivially since the left hand side is negative and the right hand side is positive. The transition from the second to the third inequality was done by substitution using the equation which defines κ_2.

For κ_3, recall that the equation which defines κ_3 is

$$(1-p)^{\kappa_3+1}y = p(x+y\kappa_3).$$

Once more applying the implicit function theorem, we obtain

$$\frac{\partial \kappa_3}{\partial p} = \frac{(\kappa_4+1)(1-p)^{\kappa_3+1}y+(x+y\kappa_3)}{(1-p)^{\kappa_3+1}\ln(1-p)y-py}.$$

The numerator is clearly positive, while the denominator is clearly negative. Thus, κ_3 is decreasing in p.

Proposition 3 Assume high arrest impact ($a = n-1$).

(a) A Pairwise Nash Stable network (regular or irregular) is either empty or non-empty with complete components.

(b) When $\frac{p}{1-p} \geq \frac{y}{x}$, the empty network is Pairwise Nash stable.

(c) A degree-k regular, complete component network with $k \geq 1$ is pairwise Nash stable if and only if

$$1-\left(\frac{x+ky}{x+(k+1)y}\right)^{\frac{1}{k+1}} \leq p \leq 1-\left(\frac{x}{x+ky}\right)^{\frac{1}{k}}.$$

(d) Fix k, $1 \leq k < \infty$. There exists a detection probability p that supports a degree-k regular, complete component network as pairwise Nash stable.

(e) Fix p, $0 < p \leq \frac{y}{x+y}$. There exists a $k \geq 1$ for which the degree-k regular, complete component network is pairwise Nash stable.

Proof of Proposition 3 (c) Consider a non-empty network with a component that is not complete; that is, there are two nodes that are in the same component but are not directly linked. Observe that if they form a link with near-neighbor impact, then there is no decrease in the probability of survival but (with

$p < 1$) there is a strict increase in each node's expected payoff because they may now receive payments for the additional link. Hence, this network is not pairwise Nash stable, and any pairwise Nash component must be complete.

Now consider a degree-k regular, complete component network with $k \geq 1$. The payoff to node i in such a network is

$$E_p u_i = (1-p)^{k+1}(x+ky).$$

To satisfy condition (PSN-i), a node in a complete component must prefer to not form a new link to another node outside the component. Notice that any node outside the component is in another complete component of degree k. Not forming a link to a node in another component is optimal when

$$\begin{aligned}(1-p)^{k+1}(x+ky) &> (1-p)^{2k+2}(x+(k+1)y) \Rightarrow \\ p &> 1-\left(\frac{x+ky}{x+(k+1)y}\right)^{\frac{1}{k+1}}.\end{aligned}$$

To satisfy condition (PNS-ii), a node must not want to remove a link. Observe that removing some, but not all, of a node's links does not improve the node's probability of surviving, but it does strictly decrease payoffs from the links. Thus, when considering which links to remove, the only option that could potentially lead to improvement is to remove all links. Staying in the complete component is better than removing all links when

$$\begin{aligned}(1-p)^{k+1}(x+ky) &> (1-p)x \Rightarrow \\ p &< 1-\left(\frac{x}{x+ky}\right)^{\frac{1}{k}}.\end{aligned}$$

Hence, for the degree-k complete component network to be pairwise Nash stable, we must have

$$1-\left(\frac{x+ky}{x+(k+1)y}\right)^{\frac{1}{k+1}} < p < 1-\left(\frac{x}{x+ky}\right)^{\frac{1}{k}},$$

which is the condition claimed in Proposition 3(c).

(d) Holding k fixed, there always exists such a p that satisfies the condition in Proposition 3(c) because

$$\begin{aligned}1-\left(\frac{x+ky}{x+(k+1)y}\right)^{\frac{1}{k+1}} &< 1-\left(\frac{x}{x+ky}\right)^{\frac{1}{k}} \Rightarrow \\ 0 &< (k-1)xy+k^2y^2\end{aligned}$$

for all k.

(e) Fix p. The proof is obtained by consideration of the condition in Proposition 3(c). From the proof of (d), we know that the range always exists. Plugging $k=1$ into the right hand side of the condition yields

$$p < 1-\frac{x}{x+y} = \frac{y}{x+y},$$

Further note that the right hand side of the condition in (c) converges to 0 as $k \to \infty$:

$$\begin{aligned} 1-\left(\frac{x}{x+ky}\right)^{\frac{1}{k}} &= 1-e^{\ln\left(\left(\frac{x}{x+ky}\right)^{\frac{1}{k}}\right)} \\ &= 1-e^{\frac{1}{k}(\ln x+\ln(x+y))} \\ &\to 0 \text{ as } k\to\infty. \end{aligned}$$

It follows that the range in (c) always exists and converges down to $[0,0]$ as k increases from 1 to ∞. Any p such that $0 \leq p \leq \frac{x}{x+y}$ must therefore fall into this range for some $k \geq 1$.

(b) It is sufficient to show that isolation yields strictly higher payoffs in comparison with adding a link to another isolated node:

$$\begin{aligned} (1-p)\,x &> (1-p)^2\,(x+y) \Rightarrow \\ \frac{p}{1-p} &> \frac{y}{x}. \end{aligned}$$

(a) Follows from the proof of (a) and (c).

Proposition 4 Assume near-neighbor arrest impact ($a = 1$).

(a) When $\frac{p}{1-p} > \frac{y}{x}$, the empty network is pairwise Nash stable.

(b) A degree-k regular, complete component network is pairwise Nash stable if and only if

$$(i): \frac{p}{(1-p)^{k+1}} > \frac{y}{x+ky}$$

and

$$(ii): \frac{y}{x} > \max\left\{\frac{1-(1-p)^k}{k\,(1-p)^k}, \frac{p}{1-p}\right\}.$$

(c) Fix p, $0 < p < 1$. If $\frac{p}{1-p} < \frac{y}{x}$, then there exists a k for which the degree-k regular, complete component network is pairwise Nash stable; otherwise, if $\frac{p}{1-p} > \frac{y}{x}$, then the degree-$k$ regular, complete component network is not pairwise Nash stable.

Proof of Proposition 4 (b) We proceed in multiple steps.

1. We show when a node does not want to form new links. Node i's expected payoff in a degree-k regular, complete component is

$$E_p u_i = (1-p)^{k+1}\,(x+ky)\,.$$

Adding a link to node in a different degree-k regular complete component yields expected payoff

$$(1-p)^{k+2}\left(x+ky+(1-p)^k\,y\right).$$

Not forming the link is optimal when

$$\begin{aligned} (1-p)^{k+1}\,(x+ky) &> (1-p)^{k+2}\left(x+ky+(1-p)^k\,y\right) \Rightarrow \\ \frac{p}{(1-p)^{k+1}} &> \frac{y}{x+ky}, \end{aligned}$$

which is the first condition in Proposition 2(b).

2. We identify the best link removal decisions. The expected payoff from keeping a subset k' of the k links in the complete component and removing the $(k-k')$ other links is

$$(1-p)^{k'+1}\left(x+(1-p)^{k-k'}k'y\right).$$

Keeping the k links is better than dropping any selection of $(k-k')$ links when

$$\begin{aligned}
(1-p)^{k+1}(x+ky) &> \max_{k'\in\{0,\ldots,k-1\}}\left\{(1-p)^{k'+1}\left(x+(1-p)^{k-k'}k'y\right)\right\} \Rightarrow \\
(1-p)^{k+1}(x+ky) &> \max_{k'\in\{0,\ldots,k-1\}}\left\{(1-p)^{k'+1}x+(1-p)^{k+1}k'y\right\}.
\end{aligned}$$

Consider the RHS, which is the "best link removal deviation." The expected payoff to the best link removal deviation to k' links is increasing as k' goes from k' to $k'+1$ when

$$\begin{aligned}
(1-p)^{k'+2}x+(1-p)^{k+1}(k'+1)y &> (1-p)^{k'+1}x+(1-p)^{k+1}k'y \Rightarrow \\
(1-p)^{k+1}y &> (1-p)^{k'+1}(1-(1-p))x \Rightarrow \\
\frac{y}{x} &> \frac{p}{(1-p)^{k-k'}}.
\end{aligned}$$

Observe that the RHS decreases monotonically from $\frac{p}{(1-p)^k}$ to $\frac{p}{1-p}$ as k' goes from zero to $k-1$. We can thus distinguish three cases: (1) when $\frac{y}{x}>\frac{p}{(1-p)^{k-k'}}$ for all k', then $k'=n-1$ must be the best link removal deviation; (2) when $\frac{y}{x}<\frac{p}{(1-p)^{k-k'}}$ for all k', then $k'=0$ must be the best link removal deviation; and (3) when $\frac{p}{1-p}<\frac{y}{x}<\frac{p}{(1-p)^k}$, then the marginal value of the $(k'+1)$-th link is negative at low k' and positive at high k', which implies that the best link removal deviation must be either $k'=0$ or $k'=k-1$. Irrespective of these cases, the best link removal deviation is either $k'=0$ or $k'=k-1$.

3. We show when the two candidate link removal deviations are unprofitable. Keeping k links is better than deviating to $k'=0$ when

$$\begin{aligned}
(1-p)^{k+1}(x+ky) &> (1-p)x \Rightarrow \\
\frac{y}{x} &> \frac{1-(1-p)^k}{k(1-p)^k}.
\end{aligned}$$

Keeping k links is better than deviating to $k'=k-1$ when

$$\begin{aligned}
(1-p)^{k+1}(x+ky) &> (1-p)^k(x+(1-p)ky) \Rightarrow \\
\frac{y}{x} &> \frac{p}{1-p}.
\end{aligned}$$

Together, there is no profitable link removal deviation when

$$\frac{y}{x}>\max\left\{\frac{1-(1-p)^k}{k(1-p)^k},\frac{p}{1-p}\right\},$$

which is the second condition in Proposition 2(b).

(a) If $\frac{p}{1-p} > \frac{y}{x}$, then the expected payoff of isolation is strictly better than linking to another isolated node:

$$\begin{aligned}(1-p)\,x &> (1-p)^2\,(x+y) \Rightarrow \\ \frac{p}{1-p} &> \frac{y}{x}.\end{aligned}$$

(c) Fix p with $\frac{p}{1-p} < \frac{y}{x}$. We show that the conditions from Proposition 2(b) are satisfied with sufficiently large k. First consider (i):

$$\frac{p}{(1-p)^{k+1}} > \frac{y}{x+ky}.$$

The LHS and RHS converge to ∞ and zero, respectively, as $k \to \infty$, thus satisfying (i) at large k.

Now consider (ii), which becomes:

$$\frac{y}{x} > \frac{1-(1-p)^k}{k\,(1-p)^k}$$

under the assumption that $\frac{p}{1-p} < \frac{y}{x}$. In the proof of Proposition 1(e), it is shown that the RHS term converges to zero, thus satisfying the condition. All conditions for pairwise Nash stability are therefore met.

Now suppose $\frac{p}{1-p} < \frac{y}{x}$. Condition (ii) can never be satisfied, so a degree-k regular, complete component is not pairwise Nash stable.