Commutator Formulas in Groups of Kac-Moody Type

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B.A., Wesleyan University (2013)

A Dissertation presented to the Graduate Faculty of the University of Virginia
in Candidacy for the Degree of Doctor of Philosophy

Department of Mathematics

University of Virginia

August 2022

Abstract

In this thesis, I prove that root groups associated to nested roots in certain rank 3 hyperbolic buildings must commute. Also, I give necessary and sufficient conditions for existence of nontrivial commutators between root groups associated to roots generating infinite root systems in Kac-Moody groups. Lastly, a gap in the literature regarding "reduction to rank 2" in Kac-Moody root systems is filled.

Acknowledgements

I thank my advisor Peter for his continued support.

I'd like to thank my thesis committee - Profs. Mikhail Ershov, Andrei Rapinchuk, and Kent Paschke - for their willingness to be a part of this over the summer.

Lastly, I'd like to thank my wife Kim for her endless patience and boundless love.

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Chapter 1

Introduction

Buildings - chamber complexes laced together from Coxeter complexes, which themselves are posets associated to reflection groups - were originally constructed by Jacques Tits to study the behavior of certain algebraic groups (including Chevalley groups). In this original scenario, the reflection groups (Coxeter groups) and Coxeter complexes are finite and known as "spherical" (as are the buildings). These algebraic groups generally can be realized as automorphism groups of buildings by using a "BN-pair" in the group: here the building is recovered as a coset space constructed from special subgroups B and N inside the group, and the Coxeter group is determined by the algebraic group. Upon adding a strength-of-action condition regarding a building's automorphism group (the "Moufang condition"), a class of groups with similar properties to these original algebraic groups emerge as these automorphism groups.

Allowing infinite Coxeter groups, we lose the finiteness of the Coxeter complex and thus lose the useful sense of "opposition" (which indicates two chambers at maximal distance). To deal with this, one uses twin buildings, which are pairs of buildings with a built-in opposition relation, and satisfy some of the nice properties of the spherical buildings case. One can also interpret the Moufang condition on the automorphism groups of these twin buildings, which motivates the notion of a group with an RGD system.

These groups with an RGD system were first presented in [Tit92] to study "Kac-Moody groups," which are infinite-dimensional analogues of the Chevalley groups. The Coxeter groups associated to these Kac-Moody groups are no longer generally finite (they are "nonspherical"). These Kac-Moody groups admit an RGD system, which yield a Moufang twin building by way of "twin BN-pairs" (a pair of BN-pairs in the group that interact well with each other). The conditions given to describe a group with an RGD system yield a class of groups broader than the Kac-Moody groups that they were created for, and so the potential difference invites comparison. In particular, groups with an RGD system are often referred to as "groups of Kac-Moody type," as not all groups with an RGD system are Kac-Moody groups.

One of the potential points of comparison is in the commutator formulas used in both. The commutator formulas used in defining an RGD system are written in terms of groups fixing half-Coxeter complexes ("root groups"), and are indexed by a geometric condition. The commutator formulas used in the construction of Kac-Moody groups are phrased algebraically - purely in terms of Lie algebra roots - and generally give a stronger statement than using the RGD system in the Kac-Moody group.

Peter Abramenko and Bernhard Müllherr investigated when certain root groups (those associated to a pair of nested and therefore "nonspherical" roots) commute with one another in automorphism groups of certain buildings, heretofore unpublished. We give a rigorous argument for this new result (Thm. 5.4.1 in Chapter 5), and then compare with commutator formulas of analogous root groups in Kac-Moody groups (Chapter 7). In particular, the main result of Chapter 7 (Thm. 7.3.4) yields a new condition describing precisely when these analogous root groups commute (depending only on the generalized Cartan matrix of the Lie algebra used to construct the Kac-Moody group and the characteristic). Lastly, we give an argument regarding "reduction to rank 2" in Kac-Moody algebras to fill in a gap in the literature (Prop. 6.8.11) the end of Chapter 6).

One particular situation in which one might find Thm. 7.3.4 useful can be found in Mark Schrecengost's 2020 UVa Ph.D Thesis, "Finite Generation of RGD Systems with Exceptional Links." In it, he shows important subgroups of certain groups with an RGD system are not finitely generated; in particular, one of the restrictions on these groups with an RGD system is that root groups associated to "nonspherical" pairs of roots necessarily commute (Ch5 of [Sch20]). As Theorem 7.3.4 gives (some) Kac-Moody groups over fields as a class of groups satisfying this condition, that yields a class of examples to work with.

1.1 Outline of Thesis

Chapter 2 is background information about Coxeter groups and complexes.

Chapter 3 describes buildings and BN-pairs (a slightly weaker version of RGD systems).

Chapter 4 gets into twin buildings (and their use for describing the situation when the Coxeter group W is not finite) and RGD systems.

Chapter 5 gives new results (Theorem 5.4.1) on why root groups associated to nested roots must commute for certain buildings whose Coxeter complexes correspond to certain tesselations of the hyperbolic plane by triangles.

Chapter 6 gives the classical theory of Kac-Moody algebras and the construction of Kac-Moody groups from "Kac-Moody root data." Afterwards, there is a lemma on the extent to which a "rank 2 subroot system" is the actual root system of a Kac-Moody Lie algebra (Proposition 6.8.11).

Chapter 7 details the specific commutator formulas used in the construction of Kac-Moody groups. Two ideas due to Jun Morita are discussed (these commutator formulas and "Morita pairs"), and then a result of Billig and Pianzola regarding these Morita pairs is stated. Afterwards, this result is used to characterize a necessary condition and sufficient condition (Theorem 7.3.4) of when these special root groups commute.

Chapter 2

Coxeter Groups and Coxeter Complexes

2.1 Coxeter Groups, Diagrams, and Matrices

Fundamental to our discussion is the subject of Coxeter groups. These groups figure in to all sections of this work, and while we won't need a tremendous amount of the rich theory surrounding them, we will certainly need some of it as background. The primary source we draw from for this section is [AB08].

Definition 2.1.1. A **Coxeter group** is any group W with the presentation

$$W = \langle r_1, \ldots, r_n \mid (r_i r_j)^{m_{ij}} = r_i^2 = 1 \rangle$$

where m_{ij} are integers ≥ 2 or ∞ symbols. An ∞ relation indicates no relations between r_i and r_j.

Remark 2.1.2. As a consequence of $r_i^2 = 1$, this definition implies $m_{ij} = m_{ji}$. Note that another consequence of $r_i^2 = 1$ (i.e. $r_i = r_i^{-1}$) is that if $m_{ij} = 2$, then r_i and r_j actually commute:

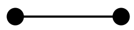
$$(r_i r_j)^2 = r_i r_j r_i r_j = 1 \iff r_i r_j = r_j r_i$$

after right multiplication by $r_j r_i$.

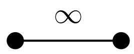
If we write $m_{ii} = 1$ for all i, the matrix $M = (m_{ij})_{i,j}$ is known as the **Coxeter matrix**. There is a definition of Coxeter groups allowing an infinite number of generators, but we will always consider our Coxeter groups as having a finite set of generators.

We also have a notion of a **Coxeter diagram**, which depends on our generators $\{r_1, \ldots, r_n\}$ and the m_{ij}: place one node for each r_i, an edge between the nodes corresponding to r_i and r_j if $m_{ij} \geq 3$, and label an edge with m_{ij} if $m_{ij} > 3$.

Example 2.1.3. $D_6 = \langle r_1, r_2 \mid (r_1 r_2)^3 = r_1^2 = r_2^2 = 1 \rangle$ is a Coxeter group with Coxeter diagram



Example 2.1.4. $D_\infty = \langle r_1, r_2 \mid r_1^2 = r_2^2 = 1 \rangle$ is a Coxeter group with Coxeter diagram



This group is known as the **infinite Dihedral group**.

The choice of generators $S = \{r_1, \ldots, r_n\}$ satisfying this definition is important since it is not, in general, uniquely determined by W (see example 2.3.8). The pair (W, S) together is called a **Coxeter system**, and $|S|$ is called the **rank** of the Coxeter system. If $|W|$ is finite, we call W and (W, S) **spherical**.

Once we have a Coxeter system, any $w \in W$ has a **length**, written $\ell(w)$, given by the minimal number of generators in S required to express w as a word in $r_1, \ldots, r_n$. In other words, $\ell(w)$ is the minimal k such that $w = r_{i_1} \cdots r_{i_k}$ for some collection of $r_{i_j} \in S$. This is useful for arguments by induction on the length of a word (and leads to a notion of "length" or "distance" in other things down the line).

If we consider a set $J \subseteq S$, we have an associated group

$$W_J = \langle J \rangle \leq W.$$

Such a group W_J is called a **standard parabolic subgroup** of W (or "standard subgroup"), while conjugates of some W_J are the more general **parabolic subgroups**. If $|W_J|$ is finite, we call it **spherical** also. One of the valuable facts about standard subgroups is that they themselves form Coxeter systems (W_J, J), thus making them one of the natural subobjects of a Coxeter system to consider. The collection of all cosets wW_J of all standard parabolic subgroups in W admits a strong geometric structure we now work towards employing. We will need a few more definitions beforehand.

2.2 Simplicial and Chamber Complexes

Definition 2.2.1. A **simplicial complex** is a poset Δ with the properties

a) For any elements $A, B \in \Delta$ there is a greatest lower bound.

b) For any $A \in \Delta$, $\Delta_{\leq A} = \{B \in \Delta \mid B \leq A\}$ is isomorphic (as a poset) to $2^{\{1,\dots,n\}}$ for some n, ordered by inclusion.

The elements of Δ are called **simplices**, and the **rank** $\mathrm{rk}(A)$ of a simplex A is the n guaranteed by b) above. If $B \leq A$, then B is called a **face** of A. The **dimension** of a simplex A is just $\dim(A) = \mathrm{rk}(A) - 1$. The phrase “simplicial complex” hints towards the fact that these posets admit a geometric realization as simplicial complexes in a topological sense. Under this realization, the dimension of a poset simplex corresponds to its counterpart's topological dimension. The elements $v \in \Delta$ of rank 1 (or dimension 0) are known as **vertices** of Δ, and in fact every element $A \in \Delta$ can be uniquely identified by its vertex set $\mathcal{V}(A)$; no two distinct simplices share the exact same vertex set. Note that by b) above, the rank of a simplex is the cardinality of its vertex set.

Remark 2.2.2. If $B \leq A$, then $\mathcal{V}(B) \subset \mathcal{V}(A)$. The combination of symbols $A \cap B$ is commonly used to refer to the element of Δ corresponding to the vertices $\mathcal{V}(B) \cap \mathcal{V}(A)$, and this is actually the greatest lower bound of A and B.

Given two simplicial complexes Δ and Δ', a **simplicial map** is a function $\phi : \Delta \to \Delta'$ induced by a mapping $\mathcal{V}(\Delta) \to \mathcal{V}(\Delta')$ that takes simplices to simplices. Note by the remark above this is necessarily a poset morphism. An isomorphism of simplicial complexes is then a simplicial map with an inverse that is also a simplicial map (simply requiring bijectivity is not sufficient). A **subcomplex** of Δ is a subset $\Delta' \subset \Delta$ that is also a simplicial complex, i.e. the injective image of a simplicial map in Δ.

While every two elements A and B are guaranteed a greatest lower bound (potentially the empty set), they are not guaranteed a least upper bound. If A and B do have a least upper bound, they are said to be **joinable**. The set $\mathrm{lk}_\Delta(A) = \mathrm{lk}(A) = \{C \in \Delta \mid C$ shares no vertices with A and is joinable to $A\}$ is known as the **link** of A, and we will include a subscript (Δ as we've written it above) when we are working with multiple simplicial complexes at once and it is not immediately clear which one we mean. A related construction, the **star** of A in Δ, is merely the collection of all simplices joinable to A.

Remark 2.2.3. There is an isomorphism of posets between $\mathrm{lk}(A)$ and $\Delta_{\geq A}$ (the simplicial complex in Δ consisting of all simplices with A as a face). One may obtain this equivalence by mapping $B \mapsto B \cup A$, where $B \in \mathrm{lk}(A)$ $B \cup A$ is the simplex corresponding to $\mathcal{V}(B) \cup \mathcal{V}(A)$.

Definition 2.2.4. A **chamber complex** is a simplicial complex in the same sense as above with the additional conditions that

a) all maximal rank simplices (called **chambers** in this context) have the same rank n, and

b) any two chambers can be connected by a "gallery."

A **gallery** is a sequence of adjacent chambers $\Gamma = (C_0, C_1, \ldots, C_k)$, where **adjacency** means that C_i and C_{i+1} intersect in a corank 1 face. In other words, two chambers are adjacent if their vertex sets differ by exactly one vertex (i.e. the possibility of two "adjacent" chambers being equal is excluded). Chamber complexes may or may not be **gallery-connected**, which is when any two simplices $A, B \in \Sigma$ can be connected by a gallery $\Gamma = (C_0, \cdots, C_k)$ such that $A \leq C_0$ and $B \leq C_k$. Not all complexes of interest have this property, so any chamber complex where this does not hold is called **gallery-disconnected**. Given two adjacent chambers, the corank 1 face they share is called a **panel**. A chamber complex is **thin** if every panel is the face of exactly 2 chambers and **thick** if every panel is the face of at least 3 chambers. We use the notation $\mathcal{C}(\Delta)$ to refer to the set of chambers of Δ.

Let Δ be a chamber complex of rank n and vertex set $\mathcal{V}(\Delta)$, and suppose I is a set of cardinality n. A **type function** τ is a function mapping $\mathcal{V}(\Delta)$ onto I in a such a way that $\tau(\mathcal{V}(C)) = I$ for any chamber C. Given a simplex $A \in \Delta$, we call the subset $\tau(A) \subset I$ the **type** of A and $I \backslash \tau(A)$ the **cotype** of A. Note that chambers have empty cotype, and panels have a cotype consisting of exactly one element of I. If the cotype of a panel P is i, then any chambers that share P as a face are said to be **i-adjacent**. The next layer of structure we add ("Coxeter complexes" below) will have galleries that can be specified by a starting chamber C_1 and a list of elements of I (each describing the cotype of a panel in the sequence of adjacencies), but not all chamber complexes have this property. The **length** of the gallery $\Gamma = (C_0, C_1, \ldots, C_k)$ from $A = C_0$ to $B = C_k$ can be thought of as the length of this list, which will always be k. If this k is minimal among all galleries from A to B, we also call k the **gallery distance** from A to B.

2.3 Coxeter Complexes

Definition 2.3.1. Given a Coxeter system (W, S), the **Coxeter complex** is the set

$$\Sigma(W, S) = \{wW_J \mid w \in W,\ J \subset S\}$$

with the natural structure of a poset ordered by reverse inclusion. By this we mean $wW_J \leq w'W_{J'} \iff wW_J \supseteq w'W_{J'}$.

Theorem 2.3.2. ([AB08] Thm. 3.5 and Defn. 3.6) $\Sigma(W,S)$ is a thin chamber complex of rank $|S|$ and admits a type function with values in S. The type function we'll use is given by

$$\tau(wW_J) = S \backslash J.$$

Remark 2.3.3. As the maximal elements of $\Sigma = \Sigma(W,S)$ correspond to cosets wW_J with $J = \varnothing$, the chambers of Σ can be thought of as elements $w \in W$ (and so W is the set of chambers of Σ). One similarly notes that the type of a chamber $w \in W$ is S. Likewise, $J = S \implies wW_S = W$. We will swap between the perspectives of Coxeter complexes as coset spaces and Coxeter complexes as simplicial complexes at will hereafter, but after justifying some of the properties of Coxeter complexes in this section through the coset space perspective, we will tend to refer to elements of Coxeter complexes as simplices A, B, C, D, etc. rather than fix cosets every time.

W acts on this set of cosets by left multiplication, so that $v(wW_J) = (vw)W_J$. This action preserves inclusion and the partial order (and thus W acts by simplicial isomorphisms): if $w'W_{J'} \subset wW_J$, then $vw'W_{J'} \subset vwW_J$ (that is, $wW_J \leq w'W_{J'} \implies vwW_J \leq vw'W_{J'}$) for all $v, w, w' \in W$. We include a useful proposition for understanding the way standard parabolic subgroups relate to each other below:

Proposition 2.3.4. ([AB08] Prop. 2.13) There exists an isomorphism of posets from the collection of sets $J \subset S$ to the collection of standard subgroups $W_J \leq W$, both ordered by (regular, unreversed) inclusion. The map is given by $J \mapsto W_J$, and the inverse is given by $W_J \mapsto W_J \cap S$.

There are some useful consequences of this proposition, one of which is definitely worth spelling out: namely, $W_J \cap W_{J'} = W_{J \cap J'}$. This is used in the proof of the Theorem at the start of this subsection.

The distance between the chamber 1 in Σ and any other chamber w is actually $\ell(w)$. This hints at the relationship between minimal galleries in the Coxeter complex and reduced decompositions of words in W : given a reduced decomposition of $w = r_{i_1} \cdots r_{i_k}$ in W, one may take a minimal gallery from 1 to w by successive adjacencies of type r_{i_j} as j increases from 1 to k (in other words, adjacencies over panels of cotype r_{i_j}). The gallery itself is then $\Gamma = (1, r_{i_1}, r_{i_1}r_{i_2}, \ldots, r_{i_1}r_{i_2} \cdots r_{i_k})$. In full generality we get that a minimal gallery from a chamber w to a chamber v corresponds to a reduced decomposition of $w^{-1}v$ (by using the W-action on Σ). It's important to note that any two chambers in

Σ can be connected by a gallery (that is: Σ is gallery-connected), so in fact any two simplices A and B can be connected by a gallery through picking any two chambers C, C' with A, B as faces and then selecting a gallery that connects C and C'.

2.3.1 Roots

Another necessary construction is that of the **root:**

Definition 2.3.5. Given a Coxeter complex $\Sigma(W, S)$, we define a **simple root** in Σ to be the set $\alpha_i = \{A \in \Sigma \mid A \leq w \in \mathcal{C}(\Sigma),\ l(r_i w) > l(w),\ r_i \in S\}$. From here we can define the set of all **roots** to be $\Phi = \{\alpha \subset \Sigma \mid \alpha = w(\alpha_i),\ w \in W,\ r_i \in S\}$ (i.e. the roots α are translates of the simple roots α_i under the $W-$action).

Proposition 2.3.6. ([AB08], Lemma 3.54) Roots are full subcomplexes of Σ. When we say α is **full**, we mean α is a chamber subcomplex containing every simplex whose vertices are all in α.

Remark 2.3.7. Roots can be thought of as "half-complexes" of Σ: in the case that Σ is spherical, they are literally half of Σ. Indeed, $|\mathcal{C}(\alpha)| = \frac{1}{2}|\mathcal{C}(\Sigma)|$ in such a situation. One can see this fairly quickly using the definition of a root, a finite chamber set, and the "reflection" simplicial isomorphisms brought up after remark 2.3.9 below.

Example 2.3.8. Suppose $W = \langle s, t \mid (st)^3 = s^2 = t^2 = 1\rangle$, the dihedral group of order 6. Then one can compute α_s in Σ fairly quickly: writing all elements of W as $\{1, s, t, st, (st)^2, sts\}$, we see $\mathcal{C}(\alpha_s) = \{1, t, stst\}$ (noting $stst = ts$, so that $l(s(stst)) = l(sts) > l(ts)$). Note that, under the action of s, α_s is translated to its complement $\{s, st, sts\}$ in Σ. This $s-$translate is also a root, as we'll see below. Lastly, we mentioned earlier that S is not uniquely determined in (W, S). As an easily accessible example of this, we may replace t in S with $r = sts$ and arrive at a parallel presentation $\langle s, r \mid (sr)^3 = s^2 = r^2 = 1\rangle$.

Remark 2.3.9. It is useful to rephrase the description of the chamber set of a root as $\mathcal{C}(\alpha_i) = \{w \in W \mid d(1, w) \leq d(r_i, w)\}$. After using the $W-$action to obtain the rest of the roots, this partially explains the notion that roots are "half-complexes," even in the case that Σ is infinite: each root is exactly the set of chambers (and their subsimplices) that are closer to some fixed C than they are to some other fixed C' adjacent to C. After forgetting the coset structure, this becomes $\alpha = \{A \in \Sigma \mid A \leq D \in \mathcal{C}(\Sigma),\ d(D, C) < d(D, C')\}$ for fixed adjacent C, C'. We could have taken this approach from the start (i.e. made no mention of cosets) by picking a **fundamental chamber** C to play the role of $1 \in W$ above; this would let us (rather arbitrarily) pick out a collection of simple roots $\{\alpha_i\}$ following the rule $\mathcal{C}(\alpha_i) = \{D \in \mathcal{C}(\Sigma) \mid d(C, D) \leq d(C', D),\ C$ and C' are $i-$adjacent$\}$. In other words, we described simple roots originally with reference to

$1 \in W$ as a fundamental chamber, but the algebraic properties of 1 are not necessary through a purely geometric description.

Each α has a "opposite" or **negative** corresponding to it called $-\alpha$, so-named because α and $-\alpha$ are images of each other under an type-preserving order 2 simplicial isomorphism. These order 2 isomorphisms will be called **reflections**, as, given a representation of these simplicial complexes in space, they can be induced on the representation of Σ by actual reflections.

We first set $-\alpha_i = r_i(\alpha_i)$ and extend this notion to all other roots in Φ by the $W-$action: if $\alpha = w(\alpha_i)$, then $-\alpha = w(-\alpha_i) = w(r_i(\alpha_i))$. For any α and $-\alpha$, their intersection $\partial\alpha = \alpha \cap -\alpha$ is called the **wall** between α and $-\alpha$. The justification for this terminology is that $\partial\alpha$, in some sense, divides Σ into α and $-\alpha$. We'll note $\partial\alpha$ contains no chambers, and thus the chambers of Σ is divided into the chambers of α and $-\alpha$. One gets this rather cheaply for simple roots by using length. We can the characterize $-\alpha_i = \{A \in \Sigma \mid A \leq w \in \mathcal{C}(\Sigma),\ l(r_i w) < l(w),\ r_i \in S\}$ by combining the definition $-\alpha_i = r_i\alpha_i$ and the original definition length-based definition of α. Since for all $w \in W$ $\ell(r_i w) > \ell(w)$ or $\ell(r_i w) < \ell(w)$, we see there cannot be any overlap in the chamber sets. After applying the $W-$action, this follows for all the other roots.

So, r_i swaps α_i and $-\alpha_i$, fixes $\partial\alpha_i$ "pointwise" (i.e. it fixes all vertices), and is of order 2 as an automorphism (since $r_i \in S$). We then write the reflection $r_\alpha \in W$ associated to $\alpha = w(\alpha_i)$ as $r_\alpha = w r_i w^{-1}$. These r_α also swap α and $-\alpha$ and fix $\partial\alpha = \alpha \cap -\alpha$ pointwise. The fact that $w(-\alpha) = -w(\alpha)$ is what gives this to us, as $r_\alpha(\alpha) = w r_i w^{-1}(w(\alpha_i)) = w r_i(\alpha_i) = w(-\alpha_i) = -w(\alpha_i) = -\alpha$. It is valuable also to note that r_α is characterized by these properties: it is the unique nontrivial automorphism of Σ that fixes $\partial\alpha$ (or stronger yet, any panel in $\partial\alpha$) pointwise (see [AB08] Lemma 3.49).

2.3.2 Products and projections

Coxeter complexes admit a semigroup structure on their sets of simplices which will be of some use in both the combinatorics and the geometry. The product in this semigroup uses the notion of a sign sequence, which is essentially a method of specifying a simplex by describing its position relative to all root walls:

Definition 2.3.10. The **sign-sequence** $\sigma(A) = (\sigma_{\partial\eta}(A))_{\eta\in\Phi}$ is defined by setting $\sigma_{\partial\eta}(A) = +, -$, or 0, depending on whether $A \in \eta, -\eta$, or $\partial\eta$.

Note that $\sigma_{\partial\eta}(A)$ is always $+, -$, or 0 for any A and η because $\eta \cup -\eta = \Sigma$; also $\sigma_{\partial\eta}(A) = + \iff \sigma_{\partial(-\eta)}(A) = -$ becomes clear by the discussion of how roots subdivide Σ above. Since $\{\mathcal{C}(\eta), \mathcal{C}(-\eta)\}$ is a partition of $\mathcal{C}(\Sigma)$, one also sees that chambers have no 0s in their sign sequences. Every simplex of rank $< n$ is contained in some wall and thus has a 0 for that part of its sign sequence. These sign sequences are essentially an alternative presentation of all poset information for elements of Σ: if we order all sign sequences by letting $\sigma(A) \leq \sigma(B) \iff \sigma_{\partial\eta}(A) \leq \sigma_{\partial\eta}(B)\ \forall \eta \in \Phi$ under the convention that $+, - > 0$, then $A \leq B \iff \sigma(A) \leq \sigma(B)$ (and $A = B \iff \sigma(A) = \sigma(B)$ (see [AB08] Proposition 3.90). So then valid, unique sign sequences determine unique simplices in Σ, and this gives us the means of defining a product of two simplices by specifying its sign sequence:

Definition 2.3.11. The **product** of A and B is defined as the simplex AB with sign-sequence

$$\sigma_{\partial\eta_+}(AB)_{\eta\in\Phi} = \begin{cases} \sigma_{\partial\eta_+}(A) & \text{if } \sigma_{\partial\eta_+}(A) \neq 0 \\ \sigma_{\partial\eta_+}(B) & \text{if } \sigma_{\partial\eta_+}(A) = 0 \end{cases}.$$

Note this definition indicates the product depends heavily on the order of simplices. Define $\mathrm{supp}(A)$, the **support** of A, as the intersection of all walls in Σ containing A. Here are some relevant properties of this product:

Proposition 2.3.12. ([AB08] Theorem 3.108, Corollaries 3.110 & 3.113, and Proposition 3.112) Every minimal gallery from A to B has AB as a face. This product is associative, and thus Σ is a semigroup. Furthermore:

1. $A \leq AB$, and $A = AB \iff \mathrm{supp}(B) \leq \mathrm{supp}(A)$
2. $\dim(AB) \geq \max\{\dim(A), \dim(B)\}$

Note the dimension condition in this proposition implies that if either of A or B is a chamber then AB is a chamber. In the case that A or B is a chamber (WLOG B), we have the familiar **gate property** for AB:

Proposition 2.3.13. ([AB08] Proposition 3.105) For any A and chambers B, C with $A \leq C$,

$$d(B, C) = d(B, AB) + d(AB, C).$$

The ability to subdivide a minimal gallery from B to C into minimal galleries to and from AB will be useful for induction arguments later. This product AB is also known as **the projection of** B **onto** A, denoted by $\mathrm{proj}_A(B)$. The word "projection" hints at the geometric situation between A, B, and AB we'll make use of in the future.

Chapter 3

Buildings

3.1 Simplicial Buildings

The rich structure of Coxeter complexes allows for a larger combinatorial structure called a **building**. By "larger," we will mean that it can be thought to contain multiple copies of a given Coxeter complex, or that it is a number of copies of a Coxeter complex $\Sigma = \Sigma(W, S)$ laced together in some way. These buildings will always be in the background of what is to come, though we will rarely ever need to refer to the whole building all at once: most of the time we will work in these "copies" of the Coxeter complex in question, which is the value of the time we spent developing the concepts of "roots" and distance in the Coxeter complex. While buildings warrant study on their own merits, they are often used to study groups that have the structure of a "BN-pair," as groups with a BN-pair structure are associated to a building in a way that is not too dissimilar to the relationship between Coxeter complexes and Coxeter groups.

Definition 3.1.1. A **building** Δ is a chamber complex that can be written as a union of chamber subcomplexes Σ (the **apartments** of the building) satisfying:

1. Each Σ is a Coxeter complex.
2. For any two simplices $A, B \in \Delta$, there is an apartment Σ that contains both of them.
3. Given two apartments Σ and Σ' that contain both A and B, there is a simplicial isomorphism $\Sigma \to \Sigma'$ that fixes the vertices of both A and B (so that A and B are fixed "pointwise").

Any collection $\mathcal{A}$ of apartments satisfying this definition is known as a **system of apartments** for Δ, and there is always a canonical system of apartments for any building.

3.2 Basic Properties and Constructions

We will say a building is **thick** if every panel is a face of at least 3 chambers and **thin** if every panel is the face of exactly 2 chambers. Note that Coxeter complexes are thin buildings (and in fact, thin buildings are Coxeter complexes). We carry over all of the terminology from our discussion of chamber complexes (e.g. "panels" as in last chapter, adjacency, galleries, links, etc.). The facts associated to these objects in the context of chamber complexes still hold, and thanks to definition 3.1.1b), we'll also often be able to use facts about Coxeter complexes to know things about buildings. Galleries in particular are very important here, as they are the primary method by which we measure distance in buildings; as before, for two chambers C and D, their **distance** $d(C, D)$ is the length of a minimal gallery between them. Δ is **gallery-connected**: recalling that Coxeter complexes Σ are gallery-connected, Definition 3.1.1b) demonstrates that every pair of chambers in Δ can be connected by a gallery. We will also shortly need the notion of **diameter** (denoted by "diam"): the maximal possible gallery length in the given chamber complex.

Proposition 3.2.1. ([AB08], Prop. 4.6) Buildings admit a type function τ, and the isomorphisms from 3.1.1c) above can be taken to be type-preserving.

Remark 3.2.2. We pick our type function on Δ to have values in some set S; it is not a coincidence that this notation mirrors the type function on Coxeter complexes from the last chapter. We do not allow S to contain any elements which are not in any image of any simplex under the type function, for convenience's sake.

Next, in any fixed apartment Σ we can recover a Coxeter matrix $M = (m(s,t))_{s,t\in S}$ by letting $m(s,t) = \text{diam}(\text{lk}_\Sigma(A))$, where A is any simplex such that $S \backslash \tau(A) = \{s, t\}$ ("the **cotype** of A is $\{s, t\}$"). Recall that, given a Coxeter matrix M and set S, there is an associated Coxeter complex $\Sigma_M = \Sigma(W_M, S)$ constructed from the associated Coxeter group as in the last chapter.

Proposition 3.2.3. ([AB08], Cor. 4.8) Given an apartment $\Sigma \subset \Delta$ with a type function induced from Δ, there is an isomorphism $\Sigma \cong \Sigma_M$ that preserves type. In particular, this isomorphism can be taken to preserve the type function on $\Sigma(W_M, S)$ described in the last chapter.

Note that, altogether, this means that every apartment has the same type (W_M, S) (which we'll now write as (W, S)) since a type-preserving isomorphism between Coxeter complexes indicates that they have the same Coxeter matrix (and thus the same Coxeter pair (W, S)). The value of this is that we've now described how Δ consists of "copies" of a Coxeter complex $\Sigma(W, S)$ as mentioned earlier; it's not clear *a priori* from the definition of Δ that every apartment is of the same type (W, S). We can actually go one step further here:

Proposition 3.2.4. ([AB08] Cor. 4.36) The Coxeter matrix M described above does not depend on our collection of apartments $\mathcal{A}$. In other words, picking a different collection of apartments $\mathcal{A}'$ satisfying the definition of a building yields the same Coxeter matrix M.

So the Coxeter matrix really is an invariant of the building itself. The Coxeter group W is also known as the **Weyl group** of Δ. Continuing in the theme of gathering results that relate properties of the building to those of an apartment, it is worthwhile to mention the following lemma:

Lemma 3.1. *([AB08], Corollary 4.34) Let C, D be chambers in Δ and Σ be an apartment containing both. If $d_\Delta(C, D)$ and $d_\Sigma(C, D)$ are the distances between C and D in Δ and Σ, respectively, then $d_\Delta(C, D) = d_\Sigma(C, D)$.*

Remark 3.2.5. Note that this means that if C and D are in Σ, then Σ contains a $\Delta-$minimal gallery from C to D. We can, as in Coxeter complexes, refer to galleries by their typing: if a gallery $\Gamma = (C_0 = C, C_1, \ldots, C_{n-1}, C_n = D)$ and each C_{i-1}, C_i is r_i-adjacent, we say the type of this gallery is $(r_1, \ldots, r_n)$.

Buildings, while interesting combinatorial objects in their own right, often arise as a certain coset space structure on groups of interest later on (much in the way that Coxeter complexes arise from Coxeter groups). We are omitting what is known as the "W-distance metric" approach to buildings, where the building is given as a set of chambers $\mathcal{C}$ with a "distance function" $\delta : \mathcal{C} \times \mathcal{C} \to W$ satisfying certain properties. While we will still use the distance function δ on Δ, we won't need this particular perspective: this definition makes no reference to apartments, and many of our arguments will need the geometric realizations of apartments themselves mentioned in Chapter 1.

3.2.1 Links

Links will be of key import later on, as they will let us consider "local," lower rank pieces of buildings - which will lead to "local" versions of relevant automorphism groups - as a tool for induction. Recall from Chapter 1 that $\mathrm{lk}_\Delta(A)$, the **link** of A in Δ, is the simplicial complex consisting of all simplices in Δ disjoint from and joinable to A. In the context of Coxeter complexes, the link $\mathrm{lk}_\Sigma(A)$ of a simplex A of cotype $J \subset S$ is again a Coxeter complex of type (W_J, J) and thus isomorphic as a chamber complex to $\Sigma(W_J, J)$ ([AB08] Prop. 3.16).

In buildings, one may consider links on the level of apartments (so that the link of a simplex in an apartment is a Coxeter complex) or on the level of the entire building itself. As one might hope, $\mathrm{lk}_\Delta(A)$ in Δ is again a building; it has a system of apartments in the Coxeter complexes $\mathrm{lk}_\Sigma(A)$ for the apartments Σ containing A ([AB08] Prop. 4.9). Note we have an alternative presentation of $\mathrm{lk}_\Sigma(A)$ as $\mathrm{lk}_\Delta(A) \cap \Sigma$ for Σ containing A. By the earlier observation about Coxeter system associated to link in a Coxeter complex, $\mathrm{lk}_\Delta(A)$ is also of type (W_J, J). **Stars** $\mathrm{st}_\Delta(A)$ and $\mathrm{st}_\Sigma(A)$, the simplices joinable to A in either Δ or Σ, are also chamber subcomplexes of Δ and Σ.

3.2.2 Pairs of roots and sphericality

It's worth mentioning what intersections of roots can look like, as they will inform the interaction between certain groups associated to roots later on. There are two possibilites, granted $\alpha \neq \pm\beta$: either both $\{\alpha, \beta\}$ and $\{\alpha, -\beta\}$ are non-nested, $\partial\alpha \cap \partial\beta$ is a chamber complex of codimension 2 in Σ, and $\mathrm{lk}_\Sigma(A)$ is spherical for all maximal simplices A in $\partial\alpha \cap \partial\beta$; or $\{\alpha, \beta\}$ or $\{\alpha, -\beta\}$ is nested, $|r_\alpha r_\beta| = \infty$, and every simplex in $\partial\alpha \cap \partial\beta$ has a nonspherical link in Σ (see [AB08] Lemma 3.164 and Prop. 3.165). We call these two cases the “spherical” and “nonspherical” cases, respectively, because these links of maximal simplices in the intersection of walls are either spherical or nonspherical according to the situation, but the intersection in the nonspherical case can also be empty. Note the nonspherical case cannot happen in a spherical Coxeter complex, as there cannot be a nonspherical (W_J, J), $J \subset S$, in a spherical (W, S).

3.2.3 Weyl-valued distance function

It was mentioned earlier that buildings admit an alternative description by considering only their chamber sets and a $W-$valued distance function δ, ostensibly “forgetting” all of the poset information. Again, we will find the simplicial definition more convenient and thus won't spend time developing the equivalence between the two, but we'll still need to talk about δ for twin buildings to make sense.

Proposition 3.2.6. ([AB08] Prop. 4.81) There exists a function $\delta : \mathcal{C}(\Delta) \times \mathcal{C}(\Delta) \to W$ such that:

1. Given a minimal gallery $\Gamma : (C_0, \dots, C_k)$ of type $(r_{i_1}, \dots, r_{i_k})$, $\delta(C_0, C_k) = r_{i_1} \cdots r_{i_k}$
2. If C and D are chambers with $\delta(C, D) = r_{i_1}, \dots, r_{i_k} = w$, then there is a bijective correspondence between minimal galleries from C to D and reduced decompositions of w.

Note this means $\ell(\delta(C, D)) = d(C, D)$. This δ is the **Weyl distance function** on Δ, and has the following properties:

Proposition 3.2.7. ([AB08] Prop. 4.84)

1. $\delta(C, D) = 1 \iff C = D$.
2. $\delta(C, D) = \delta(D, C)^{-1}$.
3. If $\delta(C', C) = s \in S$ and $\delta(C, D) = w$, then $\delta(C', D) = sw$ or w. If $\ell(sw) = \ell(w)+1$, then necessarily $\delta(C', D) = sw$.
4. If $\delta(C, D) = w$, then for any $s \in S$ there is a $C' \in \mathcal{C}(\Delta)$ such that $\delta(C', C) = s$ and $\delta(C', D) = sw$. If $\ell(sw) = \ell(w) - 1$, then there is exactly one such C'.

We won't make much use of the particulars for these last two properties, but a "$W-$valued codistance" for "twin" buildings will intentionally mirror them.

3.3 BN-pairs

Next, we explore **BN-pairs** and their connection with buildings. Many buildings of interest arise from BN-pairs, and conversely, BN-pairs arise from automorphism groups of buildings with certain properties. BN-pairs won't be the last layer of structure we impose on the automorphism groups we are interested in, but they are necessary for precise phrasing of some of the later results; they are also so closely related to RGD systems, our true goal, that it is necessary to mention them.

Definition 3.3.1. Let B, N be subgroups of G generating G such that $B \cap N \coloneqq T$ is normal in N and $W \coloneqq N/T$ admits a set of generators S such that the following hold:

1. For $s \in S$ and $w \in W$, $sBw \subseteq BswB \cup BwB$
2. For $s \in S$, $sBs^{-1} \not\leq B$.

This quotient group W is the **Weyl group** of the BN-pair, and (G, B, N, S) is called a **Tits system**. The use of "W" and "S" are, of course, not accidental: together, they'll form a Coxeter system associated to the BN-pair. We state below a result equating groups with a BN-pair and a certain class of automorphism groups of buildings, then describe how that equivalence comes about. Beforehand, we'll need the notion of a strongly transitive group action on a building and a fundamental apartment:

Definition 3.3.2. We say G acts **strongly transitively** on a building Δ if, given a $G-$invariant system of apartments $\mathcal{A}$, G acts transitively on pairs (Σ, C) of apartments Σ and chambers $C \in \Sigma$.

A **fundamental apartment** is an arbitrary choice of apartment $\Sigma \in \mathcal{A}$ containing the fundamental chamber C, which, as before, is an arbitrary choice of chamber. Just as when describing roots in a Coxeter complex without reference to the coset structure, fundamental chambers and now apartments are arbitrary (but necessary) reference points for defining other things.

Theorem 3.3.3. ([AB08] Theorem 6.56).

1. If G is a group with a BN-pair, the associated set S is uniquely determined and (W, S) is a Coxeter system. There is an associated building Δ or $\Delta(G, B)$ which admits a strongly transitive $G-$action such that B is the stabilizer of a fundamental chamber, N stabilizes a fundamental apartment, and N is transitive on the chambers of this apartment.

2. Let G be a group acting strongly transitively on a thick building Δ with fundamental chamber C and fundamental apartment Σ. Define B as being the stabilizer of C in G and let $N \leq G$ be a group that stabilizes Σ and acts transitively on chambers of Σ. Then B and N form a BN-pair in G, and Δ is canonically isomorphic to $\Delta(G, B)$.

In the case that G has a BN-pair, G has a **Bruhat decomposition**: this can be defined axiomatically as a bijection $C : W \to B\backslash G/B$ satisfying

1. $C(w) = B \iff w = 1$

2. $G = \bigcup_{w \in W} C(w)$

3. $\forall s \in S,\ \forall w \in W,\ C(sw) \subseteq C(s)C(w) \subseteq C(sw) \cup C(w)$,

which, in context, is equivalent to the usual notion of a Bruhat decomposition. Our Bruhat decomposition is given by $C(w) = B\tilde{w}B$, where $\tilde{w}$ is a representative of $w \in W$ in N (it does not depend on the choice of representative). In general, we will write "$w \in G$" when we refer to preimages of W elements in N and it does not depend on the choice of representative. Note that this second axiom actually implies $G = \coprod_{w \in W} BwB$ by virtue of working with double cosets.

There are several simplicial descriptions of $\Delta(G, B)$. The subgroups of G containing B are exactly the **standard parabolic subgroups** $P_J = BW_JB$ $(J \subset S)$; that P_J of this form are indeed subgroups is related to our Bruhat decomposition of G (see [AB08] Prop. 6.36). The simplicial building we'll use is then $\Delta(G,B) \coloneqq \{gP_J \mid g \in G, J \subset S\}$ ordered by reverse inclusion (much like Coxeter complexes) with a left $G-$action. The resulting standard fundamental apartment in this building is $\Sigma = \{nP_J \mid n \in N, J \subset S\}$, and naming the standard projection $\nu : N \twoheadrightarrow W$, and we have an isomorphism from this Σ to the Coxeter complex $\Sigma(W,S)$ via the mapping $nP_J \mapsto \nu(n)W_J$. The system of apartments $\mathcal{A}$ we take is just all $G-$translates of Σ above. In showing that this coset space is actually a building, one needs the definition of a BN pair above and properties stemming from the Bruhat decomposition.

Remark 3.3.4. In judging distance between chambers $gP_\varnothing = gB, hP_\varnothing = hB$ of such a building, we can first describe the Weyl distance between them and then take the length of that. So in this context, the Weyl distance $\delta(gB,hB) = w \iff Bg^{-1}hB = BwB$, and thus $d(gB,hB) = \ell(w)$.

Remark 3.3.5. Note that in part 2 of the definition of a BN pair we only specify N is "a" group stabilizing Σ and acting transitively on its chambers: the building $\Delta(G,B)$ actually depends only on G and B.

Example 3.3.6. $G = \mathrm{GL}_n(\mathbb{K})$ admits the structure of a BN-pair (as do $\mathrm{PGL}_n(\mathbb{K}), \mathrm{SL}_n(\mathbb{K})$, and $\mathrm{PSL}_n(\mathbb{K})$). For $\mathbb{K}^n$, take the set of all nonzero proper subspaces to be a vertex set P and say one is "incident" to another if one is contained in another. The **flag complex** Δ on $\mathbb{K}^n$ is the simplicial complex obtained by setting **flags** $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k$ of nested proper nonzero subspaces of $\mathbb{K}^n$ to be simplicies and P to be the vertex set.

G takes subspaces to subspaces (and preserves containment), so it acts on this Δ in a manner that preserves order. Set B to be the stabilizer of the standard flag

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \cdots \subset \langle e_1, \dots, e_{n-1} \rangle,$$

or the group of upper-triangular matrices (each of these subspaces is generated by "standard basis vectors" $e_i \in \mathbb{K}^n$). $N \leq G$ is then the "monomial group," or the stabilizer of the set $\{\langle e_1 \rangle, \dots \langle e_n \rangle\}$ (so it acts by permutations). One can check that the BN-pair axioms are satisfied in this group by direct matrix computation (see [AB08] Section 6.5 for more detail).

Some of the power of this structure can be seen in theorems on the simplicity of certain G with a BN pair. One such result of is

Theorem 3.3.7. ([AB08] Theorem 6.62) Suppose G has a BN pair (B,N), G is perfect, and B is solvable; also suppose that the type (W,S) of G is irreducible (so that the associated Coxeter diagram has one connected component). Every proper normal subgroup of G is contained in $Z = \bigcap_{g \in G} gBg^{-1}$, and thus G/Z is simple.

As Z is the kernel of the $G-$action on Δ, a faithful $G-$action on $\Delta(G,B)$ guarantees a simple G.

Chapter 4

Twin Buildings and RGD systems

4.1 Twin Buildings

Recalling that Coxeter groups W may be subdivided into the categories of "spherical" and "nonspherical" according to whether or not W is finite, one sees that buildings admit the same subdivision: a building is **spherical** if its apartments are spherical and **nonspherical** otherwise. As with many mathematical constructions, the finiteness brought by sphericality (in apartments, at least) affords a tractability that, on first blush, is lost when transitioning to nonspherical buildings. The way around this is to add yet another layer of structure, that of the "twin building," by introducing a "codistance" on a *pair* of buildings of type W. In what follows, Δ_ϵ are buildings of type (W, S) for $\epsilon \in \{+, -\}$, δ_ϵ are their Weyl distance functions, and $\mathcal{C}_\epsilon = \mathcal{C}(\Delta_\epsilon)$.

Definition 4.1.1. A **twin building** of type (W, S) is a triple $(\mathcal{C}_+, \mathcal{C}_-, \delta^*)$, where δ^* is a $W-$**valued codistance** function $\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$ satisfying, for $C \in \mathcal{C}_\epsilon$, $D \in \mathcal{C}_{-\epsilon}$, and $\delta^*(C, D) = w$:

1. $\delta^*(C, D) = \delta^*(D, C)^{-1}$.
2. If $C' \in \mathcal{C}_\epsilon$ satisfies $\delta_\epsilon(C', C) = s$ with $s \in S$ and $\ell(sw) < \ell(w)$, then $\delta^*(C', D) = sw$.
3. For any $s \in S$, there is a $C' \in \mathcal{C}_\epsilon$ with $\delta_\epsilon(C', C) = s$ and $\delta^*(C', D) = sw$.

Just as with the $W-$distance function, we have a **numerical codistance** $d^*(C, D) = \ell(\delta^*(C, D))$. When $\delta^*(C, D) = 1$ (i.e. $d^*(C, D) = 0$), we say C and D are **opposite**. The relationship between C and D will mimic that of two chambers at maximum distance from one another in a spherical building, which we will also call **opposite**. We denote this relationship, in both twin and spherical buildings (depending on context), by C op D

or $\text{op}(C) = D$. $\text{op}(-)$ as a function $\Sigma_\epsilon \to \Sigma_{-\epsilon}$ is an isometry of buildings: that is, it is an isomorphism that preserves Weyl distance between pairs of chambers in Σ_ϵ and codistance between pairs in $(\Sigma_\epsilon, \Sigma_{-\epsilon})$. This axiomatized opposition relation is a key purpose of twin buildings.

We collect and organize the apartments of these buildings into so-called **twin apartments** based on the opposition relation: a twin apartment of Δ is a pair (Σ_+, Σ_-) of apartments from Δ_+ and Δ_-, respectively such that every chamber in the union of Σ_+ and Σ_- is opposite exactly one other chamber in this union. Note that a twin apartment is itself a twin building. Every twin building admits a system of twin apartments satisfying:

Proposition 4.1.2. ([AB08] Prop. 5.179)

1. If C, C' are opposite chambers then there is a unique twin apartment $\Sigma(C, C')$ containing them both.
2. For each apartment $\Sigma_\epsilon \subset \Delta_\epsilon$ there is at most one apartment $\Sigma_{-\epsilon} \subset \Delta_{-\epsilon}$ forming a twin apartment.
3. Given two chambers $C, C' \in \Delta_+ \cup \Delta_-$, there is a twin apartment containing both of them.

In twin apartments we have **twin roots:** pairs of roots $\alpha = (\alpha_+, \alpha_-)$, $\alpha_\epsilon \subset \Sigma_\epsilon$, such that $\text{op}(\alpha_\epsilon) = -\alpha_{-\epsilon}$. Note here we set $-\alpha = (-\alpha_+, -\alpha_-)$, so that $\text{op}(\alpha) = -\alpha$. The set of twin roots of any twin apartment is in bijection with the set of non-twin roots in either half of the apartment, and we will thus refer to this set of twin roots also as Φ. Twin roots are negative or positive (just as with regular roots) according to whether or not they contain a choice of fundamental chamber. While the definition of a twin root depends on a choice of twin apartment, they may be considered independently: a twin root from Σ remains a twin root in any Σ' that contains it, and we may thus consider the twin roots of Δ as being just the collection of all (unique) twin roots in all apartments. Likewise, we can consider the set of all apartments $\mathcal{A}(\alpha)$ containing a given α, which will be useful later.

4.2 RGD Systems, Twin BN Pairs, and the Moufang Property

Next we describe the algebraic structure we are most interested in, that of the RGD system. We'll start with the group-theoretic definition and then build an equivalence with automorphism groups of certain twin buildings; groups with an RGD system will, in general, give us the only kind of twin buildings we are concerned with here. First, say a pair $\{\alpha, \beta\}$ of nontwin roots is **prenilpotent** if $\alpha \cap \beta$ and $(-\alpha) \cap (-\beta)$ both contain at least one chamber. As well, define $[\alpha, \beta] = \{\gamma \in \Phi \mid \alpha \cap \beta \subset \gamma, (-\alpha) \cap (-\beta) \subset -\gamma\}$ and $(\alpha, \beta) = [\alpha, \beta] \backslash \{\alpha, \beta\}$, the **closed** and **open intervals** between α and β. We will also refer to sets of **prenilpotent twin roots**, which are sets of twin roots $\{\alpha, \beta\}$ such that both components of $\alpha \cap \beta$ are nonempty. Note that if $\{\alpha, \beta\}$ are prenilpotent twin roots then $\{\alpha_+, \beta_+\}$ and $\{\alpha_-, \beta_-\}$ are prenilpotent pairs of nontwin roots.

Lemma 4.2.1. Let $\{\alpha, \beta\}$ be prenilpotent pair of roots. We have one of two cases:

1. $\{\alpha, \beta\}$ is nested, WLOG $\alpha \subset \beta$, in which case $[\alpha, \beta] = \{\gamma \in \Phi \mid \alpha \subset \gamma \subset \beta\}$.
2. Every maximal rank simplex of $\partial\alpha \cap \partial\beta$ is of codimension 2, and the links $\mathrm{lk}_\Delta(A)$ are spherical. For an A in this intersection, set $\alpha' = \alpha \cap \mathrm{lk}_\Delta(A)$ and $\beta' = \beta \cap \mathrm{lk}_\Delta(A)$. There is a bijection $[\alpha, \beta] \to [\alpha', \beta']$ defined by $\gamma \mapsto \gamma \cap \mathrm{lk}_\Delta(A)$.

This restates subsection 3.2.2, but also adds an interpretation of the closed interval in these situations. With that out of the way,

Definition 4.2.2. Let Σ be a Coxeter complex of type (W, S) with set of roots Φ and positive roots Φ_+. An **RGD system** of type (W, S) is a list $(G, (U_\alpha)_{\alpha \in \Phi}, T)$, $U_\alpha \leq G$, satisfying

(RGD0) For all $\alpha \in \Phi$, $U_\alpha \neq \{1\}$.

(RGD1) For all prenilpotent $\{\alpha, \beta\} \subset \Phi$ with $\alpha \neq \beta$, $[U_\alpha, U_\beta] \leq U_{(\alpha,\beta)} = \langle U_\gamma \mid \gamma \in (\alpha, \beta)\rangle$.

(RGD2) For every $s \in S$, there is a function $m : U_s^* \to G$ (where $U_s = U_{\alpha_s}$) such that for all $u \in U_s^*$ and $\alpha \in \Phi$, $m(u) \in U_{-s} u U_{-s}$ and $m(u) U_\alpha m(u)^{-1} = U_{s(\alpha)}$. Also, $m(u)m(v)^{-1} \in T$ for all $u, v \in U_s^*$.

(RGD3) For all $s \in S$, $U_{-s} \not\leq U_+ = U_{\Phi_+}$.

(RGD4) $G = TU_\Phi = T\langle U_\alpha \mid \alpha \in \Phi\rangle$

(RGD5) T normalizes U_α for all $\alpha \in \Phi$.

The technicality of this definition suggests the strength of it: a group with an RGD system has an associated coset space, which is not only a twin building but a twin building with a "very strong" action by its type-preserving automorphism group. We will take special interest in **(RGD1)**, the commutator formulas for root groups, as they describe the interactions within a large piece of a generating set for G: as $G = TU_\Phi$ and

T is a torus, a chunk of the complexity of these groups is encoded in the combination of these commutator formulas and the set of roots Φ. As the method by which we arrive at this building matches that of the "twin" BN pair, we will describe that first.

Definition 4.2.3. Let B_+, B_-, and N be subgroups of a group G such that $B_\epsilon \cap N = T$ for both $\epsilon \in \{+,-\}$. As with regular BN pairs, assume T is normal in N and $W \cong N/T$. The triple (B_+, B_-, N) is called a **twin BN pair** with Weyl group W if W admits a set of generators S such that for all $w \in W$, $s \in S$, and $\epsilon \in \{+,-\}$:

(TBN0) (G, B_ϵ, N, S) is a BN pair.

(TBN1) If $\ell(sw) < \ell(w)$, then $B_\epsilon s B_\epsilon w B_{-\epsilon} = B_\epsilon sw B_{-\epsilon}$.

(TBN2) $B_+ s \cap B_- = \varnothing$.

Again, $w = \tilde{w} \in N$ is a representative of $w \in W$, and we will continue to omit a tilde whenever it does not matter which representative we use. (G, B_+, B_-, N, S) is a **twin Tits system**.

As with regular BN pairs, twin buildings with a strongly-transitive actions by their automorphism groups are equivalent to twin BN pairs. By "strongly transitive" here we mean that G acts transitively on pairs of opposite chambers (there are several other equivalent definitions - see [AB08] Lemma 6.70 - but this will do for us). We have not mentioned actions on twin buildings yet, but they are just actions on both Δ_ϵ that preserve Weyl distance in each building and codistance of chambers between buildings. Groups with a twin BN pair have a **Birkhoff decomposition**: bijections $W \to B_\epsilon G B_{-\epsilon}$ satisfying $G = \coprod_{w \in W} B_\epsilon w B_{-\epsilon}$.

Definition 4.2.4. The twin building associated to a twin BN pair is the pair of buildings associated to each BN pair (G, B_ϵ, N, S), together with a Weyl codistance δ^* defined by $\delta^*(gB_\epsilon, hB_{-\epsilon}) = w \iff g^{-1}h \in B_\epsilon w B_{-\epsilon}$.

Remark 4.2.5. RGD systems are a stronger version of these twin BN pairs, and a group with an RGD system also has a twin BN pair: we set $B_\epsilon = TU_\epsilon$ and $N = \langle T, \{m(u) \mid u \in U_s^*, s \in S\}\rangle$. These three groups, together with G and S from the definition of an RGD system, give a twin BN pair and thus a twin building ([AB08] Theorem 8.80).

There is a similar statement to the above equivalence (between twin BN pairs and buildings) regarding equivalence between RGD systems and certain group actions on twin buildings, but of course the group action must also be stronger in order to give an RGD system. For a *twin* root α of a twin building Δ, we define the (geometric) **root group** U_α to be the set of automorphisms of Δ that both fix α pointwise and fix the star of every panel of $\alpha \backslash \partial\alpha$ pointwise (the first condition is redundant if $\text{rk}(\Delta) \geq 2$). As

U_α consists of automorphisms of Δ, it acts on the set of chambers that have any fixed boundary panel $\mathcal{P}$ of α as a face; omitting the one in α, we will call this set $\mathcal{C}(\mathcal{P}, \alpha)$. This action is equivalent to an action of U_α on $\mathcal{A}(\alpha)$, the set of apartments containing α.

Definition *4.2.6.* We then say Δ is **Moufang**, or "has the Moufang property" if the actions of U_α on $\mathcal{C}(\mathcal{P}, \alpha)$ and $\mathcal{A}(\alpha)$ are transitive for each twin root $\alpha \subset \Delta$ (the "Moufang condition").

If these actions are simply transitive, then we say Δ is **strictly Moufang**. A Moufang twin building with an associated Coxeter diagram that has no isolated nodes is necessarily strictly Moufang; this will inform how we phrase certain conditions later on. As well, every spherical link in a Moufang twin building is itself a spherical Moufang nontwin building (which is just the above definition of "Moufang" repeated, with the same definitions for U_α and $\mathcal{C}(\mathcal{P}, \alpha)$, in the context of non-twin buildings).

Remark *4.2.7.* The value of the Moufang property at this juncture is that it gives us the class of twin buildings to which groups with an RGD system correspond. After picking a fundamental twin apartment Σ, and a fundamental pair of chambers $C_\pm$ in it, we obtain an RGD system by taking $(U_\alpha)_{\alpha\in\Phi}$ to be the geometric root groups associated to roots in this fundamental apartment and T to be the fixer of Σ in $G = \mathrm{Aut}_0(\Delta)$ (the group of type-preserving automorphisms of Δ).

4.2.1 Algebraic and Geometric Root Groups

While groups with an RGD system yield a Moufang twin building through use of a twin BN-pair, in many cases one can say a bit more about the axiomatic root groups from the RGD system and the geometric root groups from the automorphism group of the twin building (following [AB08] Section 8.9).

Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be a group with an RGD system of type (W, S) (so that $\alpha \in \Phi$ is a root of the Coxeter complex $\Sigma(W, S)$) and let $\Delta(G, B_+, B_-)$ be the twin building associated to the twin BN-pair in G. Picking a pair of opposite fundamental chambers in Δ_ϵ determines a fundamental twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$ (see [AB08] Prop. 5.179), the halves of which are canonically isomorphic to $\Sigma(W, S)$. For this α in $\Sigma(W, S)$, we can associate roots $\alpha_\epsilon \in \Delta_\epsilon$ with chambers $\{wB_\epsilon \mid w \in \alpha\}$. Note that since we are still in the simplicial perspective, we must also throw in the faces of these chambers: cosets of the standard parabolic subgroups containing the wB_ϵ (coming from each BN pair).

The twin root associated to these α_ϵ is $\hat{\alpha} = (\alpha_+, -\alpha_-)$. These U_α act simply transitively on $\mathcal{C}(\mathcal{P}, \alpha_+)$ for each boundary panel of α_+, and thus we can realize U_α as a subgroup of $U_{\hat{\alpha}}$ in $\mathrm{Aut}(\mathcal{C})$. More specifically,

Theorem 4.2.8. ([AB08] Theorem 8.81) Let $\Delta(G, B_+, B_-)$ be the twin building associated to the twin Tits system (G, B_+, B_-, N, S) (coming from the RGD system $(G, (U_\alpha)_{\alpha\in\Phi}, T)$. Then for any $\alpha \in \Phi$, U_α acts simply transitively on $\mathcal{A}(\hat{\alpha})$ and is identifiable with a subgroup of the geometric root group $U_{\hat{\alpha}}$. In particular, Δ is a Moufang twin building. If the Coxeter diagram of (W, S) has no isolated nodes, $U_\alpha = U_{\hat{\alpha}}$, and Δ is strictly Moufang.

4.2.2 Convexity and Root Groups

Considering our notion of distance in buildings of all sorts, it is natural that some sort of "convexity" would figure into our discussion. We are most interested in what this means in twin buildings, as it has some interplay with the groups on the right side of **(RGD1)** (the group $U_{(\alpha,\beta)}$, dependent on α and β). Let $\mathcal{M} = (\mathcal{M}_+, \mathcal{M}_-) \subset (\Sigma_+, \Sigma_-)$ be a pair of sets of chambers and define $\Psi(\mathcal{M}) \coloneqq \{\alpha \in \Phi \mid \alpha \supset \mathcal{M}\}$. A set of twin roots Ξ is **convex** if $\Xi = \Psi(\mathcal{M})$ for some $\mathcal{M}$ with both $\mathcal{M}_+, \mathcal{M}_-$ nonempty. Say $\mathcal{M}$ is a **convex pair** if, given a $C \in \mathcal{M}_+ \cup \mathcal{M}_-$ and a panel $\mathcal{P}$ that meets $\mathcal{M}_+ \cup \mathcal{M}_-$, we have $\mathrm{proj}_{\mathcal{P}}(C) \in \mathcal{M}_+ \cup \mathcal{M}_-$. Here $\mathrm{proj}_{\mathcal{P}}(C)$ is taken to mean projection in either half of the building and also projection "between halves" of the building: there is a notion of projection for twin buildings that, given the right circumstances, finds the chamber having a given simplex as a face (as written above, $\mathcal{P}$) at maximal codistance from a chamber in the other half of the building (C above). Convex pairs $\mathcal{M}$ of Σ (necessarily intersections of twin roots) are in one-to-one correspondence with convex sets of twin roots in Σ, i.e. $\bigcap_{\alpha\in\Psi} \alpha \leftrightarrow \Psi$ ([AB08] 8.12). There are similar definitions as above for the spherical case (which we will use with links later). If Σ is a spherical Coxeter complex and K is a set of simplices in Σ containing at least one chamber, we say Ξ is **convex** if $\Xi = \Psi(K) = \{\alpha \in \Phi \mid \alpha \supset K\}$. As in the twin case, there is also a correspondence $\Psi(K) \leftrightarrow \bigcap_{\alpha\in\Psi(K)} \alpha$ between convex sets of roots and convex chamber subcomplexes of Σ. In both situations, twin and spherical, we have a notion of an **admissable ordering**: $\alpha_1, \ldots, \alpha_k$ is admissable if $\alpha_i, \ldots, \alpha_k$ is convex for each $1 \leq i \leq k$. Again in both situations, every convex set of roots has an admissable ordering.

The value of this ordering is the following:

Proposition 4.2.9. ([AB08] Prop. 8.33) Let Ψ be a convex set of twin roots in a fundamental twin apartment Σ, let $\alpha_1, \ldots, \alpha_m$ be an admissable ordering of Ψ, and write $U_i = U_{\alpha_i}$ for each i. Then

1. $U_\Psi = U_1 \cdots U_m$, and in fact every $u \in U_\Psi$ has a unique expression $u = u_1 \cdots u_m$ with $u_i \in U_i$.

2. If $\Psi = \Psi(\mathcal{M})$ for a specific convex pair $\mathcal{M}$ with both components nonempty, then $\mathrm{Fix}_G(\mathcal{M}) = U_\Psi T$, where $T = \mathrm{Fix}_G(\Sigma)$.

The first part of this proposition is the one we will see the most use for, as closed intervals $[\alpha, \beta]$ are actually convex sets of roots. This will allow us to write elements of the commutator $[U_\alpha, U_\beta]$ uniquely as products of elements of root groups associated to roots from the closed interval later on (thanks to **(RGD1)**).

Chapter 5

Commutator formulas in Certain Groups of Kac-Moody type

The eventual aim of this chapter is to show that, in twin buildings associated to RGD systems with certain Coxeter systems, the root groups associated to prenilpotent pairs of roots with nested positive halves commute. Aside from some combinatorial (simplicial) background we'll build up in the early parts of the chapter, the content of the main arguments is strongly geometric, and depends heavily on the associated Coxeter complexes admitting the structure of a hyperbolic plane. The blueprint/general program behind these arguments are due to Peter Abramenko and Bernhard Mullhërr.

5.1 A Lemma on Roots with Nested Positive Halves

We continue to regard a simplicial twin building $\Delta = (\Delta_+, \Delta_-)$, and restrict ourselves to Coxeter systems with no isolated nodes in the Coxeter diagram (so that a Moufang Δ is strictly Moufang). Recall that for any pair of twin roots $\{\alpha, \beta\}$, $\alpha \neq \pm\beta$, either $\partial\alpha_+ \cap \partial\beta_+$ has codimension 2 and all maximal simplices in it have spherical links, or two of the four pairs in $\{\pm\alpha_+, \pm\beta_+\}$ are nested (equivalently: one of $\{\alpha_+, \beta_+\}$ or $\{\alpha_+, -\beta_+\}$ is nested). Now assuming α_+, β_+ are nested (WLOG $\alpha_+ \subsetneq \beta_+$), recall also it follows very quickly from the definition that $\{\alpha, \beta\}$ is a prenilpotent pair. We will freely use the alternative formulations of the intervals $[\alpha, \beta] = \{\gamma \in \Phi \mid \alpha_+ \subset \gamma_+ \subset \beta_+\}$ and $(\alpha, \beta) = \{\gamma \in \Phi \mid \alpha_+ \subsetneq \gamma_+ \subsetneq \beta_+\}$ given $\alpha_+ \subsetneq \beta_+$. We have several lemmas and propositions to collect before we get to the main theorem

Lemma 5.1.1. Let Σ be a $2-$spherical rank 3 twin apartment with no isolated nodes in its Coxeter diagram. Suppose α, β satisfies $\alpha_+ \subsetneq \beta_+$ and for some $\gamma \in \Phi$ we have $\partial\gamma_+ \cap \partial\alpha_+ \neq \varnothing \neq \partial\gamma_+ \cap \partial\beta_+$. In the case that $(\alpha, \beta) \neq \varnothing$, let $\zeta \in (\alpha, \beta)$. Then *i)* $\partial\alpha_+ \cap \partial\beta_+ = \varnothing$

ii) $\gamma_+ \neq \pm\alpha_+, \pm\beta_+$ and neither $\{\gamma_+, \pm\alpha_+\}$ nor $\{\gamma_+, \pm\beta_+\}$ is nested,

iii) $\partial\zeta_+ \subset (\beta_+ \backslash \partial\beta_+) \cap (-\alpha_+ \backslash \partial(-\alpha_+))$, and

iv) $\partial\zeta_+ \cap \partial\gamma_+ \neq \varnothing$.

Proof. *i)* Suppose A is a vertex in $\partial\alpha_+ \cap \partial\beta_+$. Recalling the bijection between the walls of Σ_+ containing A and the walls of $\mathrm{lk}_{\Sigma_+}(A)$, non-equal nested α_+ and β_+ would then correspond to non-equal nested roots $\alpha'_+ = \alpha_+ \cap \mathrm{lk}_{\Sigma_+}(A)$ and $\beta'_+ = \beta_+ \cap \mathrm{lk}_{\Sigma_+}(A)$ in $\mathrm{lk}_{\Sigma_+}(A)$, contradicting $2-$sphericality.

ii) By assumption, $\partial\gamma_+$ meets both $\partial\alpha_+$ and $\partial\beta_+$. If $\gamma_+ = \pm\alpha_+$ then $\partial(\pm\alpha_+) \cap \partial\beta_+ = \partial\alpha_+ \cap \partial\beta_+ \neq \varnothing$, contradicting part *i)*. Likewise $\gamma_+ = \pm\beta_+$ would imply $\partial\alpha_+ \cap \partial(\pm\beta_+) = \partial\alpha_+ \cap \partial\beta_+ \neq \varnothing$. Note that part *i)* implies that, given Σ as in the lemma, the walls of any nested roots in Σ_+ do not meet (assuming strict containment). So again since $\partial\gamma_+$ meets $\partial\alpha_+$ and $\partial\beta_+$, neither $\{\gamma_+, \pm\alpha_+\}$ nor $\{\gamma_+, \pm\beta_+\}$ can be nested.

iii) Part *i)* and the formulation of the open interval (α, β) as $\{\zeta \in \Phi \mid \alpha_+ \subsetneq \zeta_+ \subsetneq \beta_+\}$ shows $\partial\zeta_+ \cap \partial\beta_+ = \varnothing$ and $\partial\zeta_+ \cap \partial\alpha_+ = \varnothing$. Now observe $\zeta_+ \subsetneq \beta_+ \implies \partial\zeta_+ \subsetneq \beta_+$, so $\partial\zeta_+ \subset (\beta_+ \backslash \partial\beta_+)$. To see $\partial\zeta_+ \subset (-\alpha_+ \backslash \partial(-\alpha_+))$, we note $-\zeta_+ \subsetneq -\alpha_+ \implies \partial(-\zeta_+) \subsetneq -\alpha_+$ and $\partial(-\zeta_+) \cap \partial(-\alpha_+) = \varnothing$, so that $\partial(-\zeta_+) \subset (-\alpha_+ \backslash \partial(-\alpha_+))$. So since $\partial(-\zeta_+) = \partial\zeta_+$, we have $\partial\zeta_+ \subset (-\alpha_+ \backslash \partial(-\alpha_+))$.

iv) If we suppose $\partial\zeta_+ \cap \partial\gamma_+ = \varnothing$, then either $\{\zeta_+, \gamma_+\}$ or $\{\zeta_+, -\gamma_+\}$ must be nested. This gives four cases:

Cases 1 and 2: Suppose $\zeta_+ \supsetneq \pm\gamma_+$. Then for a vertex A in $\partial\gamma_+ \cap \partial\beta_+$, the $2-$sphericality of Σ guarantees $\pm\gamma'_+ = \pm\gamma_+ \cap \mathrm{lk}_{\Sigma_+}(A)$ intersects $-\beta'_+ = \beta_+ \cap \mathrm{lk}_{\Sigma_+}(A)$ in least one chamber, and thus $\pm\gamma_+$ intersects $-\beta_+$ in at least one chamber. But $\pm\gamma_+ \subsetneq \zeta_+$ then implies ζ_+ intersects $-\beta_+$ in at least one chamber, which contradicts $\zeta_+ \subset \beta_+$ (from $\zeta \in (\alpha, \beta)$).

Cases 3 and 4: Suppose $\zeta_+ \subsetneq \pm\gamma_+$. If $\zeta_+ \subset \gamma_+$ then $-\zeta_+ \supset -\gamma_+$, and similarly $\zeta_+ \subset -\gamma_+ \implies -\zeta_+ \supset \gamma_+$. As in the previous case, examining the link of a vertex $A \in \partial\gamma_+ \cap \partial\alpha_+$ shows $\pm\gamma_+$ intersects α_+ in at least one chamber, which would imply $-\zeta_+$ intersects α_+ in at least one chamber. But again, this contradicts $\zeta \in (\alpha, \beta)$ since $-\zeta_+ \subsetneq -\alpha_+$.

So neither $\{\zeta_+, \gamma_+\}$ nor $\{\zeta_+, -\gamma_+\}$ can be nested, and thus $\partial\zeta_+ \cap \partial\gamma_+ \neq \varnothing$. □

Corollary 5.1.2. Suppose γ, α, β, and Σ are as above. If $\partial\gamma_+ \cap (-\alpha_+) \cap \beta_+$ is an edge, then $(\alpha, \beta) = \varnothing$.

Proof. Suppose $\zeta \in (\alpha, \beta)$. The edge $\partial\gamma_+ \cap (-\alpha_+) \cap \beta_+$ must have its vertices on both $\partial\alpha_+$ and $\partial\beta_+$. By part *iii*) of the lemma, $\partial\zeta_+ \subset (\beta_+ \backslash \partial\beta_+) \cap (-\alpha_+ \backslash \partial(-\alpha_+))$, so no vertex of $\partial\zeta_+$ can be on $\partial\beta_+$ or $\partial\alpha_+$. But $\partial\zeta_+$ must meet $\partial\gamma_+$ by part *iv*) of the lemma, so a vertex of $\partial\zeta_+ \cap \partial\gamma_+$ is simultaneously in $(\beta_+ \backslash \partial\beta_+) \cap (-\alpha_+ \backslash \partial(-\alpha_+))$ and $\partial\beta_+ \cup \partial\alpha_+$ (a contradiction). Thus $(\alpha, \beta) = \varnothing$.

□

5.2 A restriction isomorphism between "global" and "local" root groups

Now we need to develop a correspondence between certain collections of root groups and certain root groups associated to links in Δ (viewing a link as a lower rank building). Given a simplex A of a spherical cotype and positive codimension in Δ_+, we know its link Δ' is a spherical strictly Moufang building ([AB08] Prop. 8.21). We may therefore consider the associated RGD system $(G', (U_{\alpha'})_{\alpha' \in \Phi'}, T')$, where Φ' is taken to be set of roots of $\Sigma' := \Sigma_+ \cap \Delta'$. Recall each root α' here is an intersection of some α_+ with Σ', where α_+ is any positive half of a twin root such that $v \in \partial\alpha_+$. We actually obtain a restriction homomorphism $\rho_\alpha : U_\alpha \to U_{\alpha'}$: each $x \in U_\alpha$ has a restriction $x' := x\big|_{\Delta'}$ which satisfies all the requirements for being in the "local root group" $U_{\alpha'}$. This restriction homomorphism figures heavily in what is to come.

From this point on, $'$ after a symbol will always indicate the object is associated to a link.

Lemma 5.2.1. Let $\Delta = (\Delta_+, \Delta_-)$ be a strictly Moufang twin building. Suppose A is a simplex in Δ_+ with nonzero codimension and a spherical link Δ' that has no isolated nodes in its Coxeter diagram. Fix a twin root α with $A \in \partial\alpha_+$. Then the restriction map $\rho_\alpha : U_\alpha \to U_{\alpha'}$ is an isomorphism.

Proof. We follow along the proof of Prop 7.3.2 of [AB08]. Let $\mathcal{P}'$ be a boundary panel of α', $\mathcal{P} = \mathcal{P}' \cup \mathcal{A}$, and $\mathcal{C}(\mathcal{P}', \alpha') := \{C' \in \mathcal{C}(\Delta') \mid \mathcal{P}' \leq C',\, C' \notin \alpha'\}$. Then $\mathcal{P}$ is a boundary panel of α_+, and there is a bijection $\mathcal{C}(\mathcal{P}', \alpha') \tilde{\to} \mathcal{C}(\mathcal{P}, \alpha)$ by mapping $C' \mapsto C' \cup A$. In particular, the simply transitive action of U_α on $\mathcal{C}(\mathcal{P}, \alpha)$ (coming from Δ strictly Moufang) can be considered as a simply transitive action on $\mathcal{C}(\mathcal{P}', \alpha')$. This action coincides with the action of $\rho_\alpha(U_\alpha)$ induced via the action of $U_{\alpha'}$: more specifically, the action of U_α coming from this bijection matches the composition

$$U_\alpha \xrightarrow{\rho_\alpha} U_{\alpha'} \to \operatorname{Sym} \mathcal{C}(\mathcal{P}', \alpha')$$

(where $\mathrm{Sym}\,\mathcal{C}(\mathcal{P}',\alpha')$ is the group of permutations of $\mathcal{C}(\mathcal{P}',\alpha')$) when we look at the way the bijection $\mathcal{C}(\mathcal{P}',\alpha')\tilde{\to}\mathcal{C}(\mathcal{P},\alpha)$ defines the action. So the action of $\rho_\alpha(U_\alpha)$ is also simply transitive, and ρ_α is injective. Now since Δ' is Moufang with no isolated nodes, Δ' is strictly Moufang. $U_{\alpha'}$'s action on $\mathcal{C}(\mathcal{P}',\alpha')$ is then simply transitive. But since $\rho_\alpha(U_\alpha)$ acts simply transitively on $\mathcal{C}(\mathcal{P}',\alpha')$ and $\rho_\alpha(U_\alpha) \leq U_{\alpha'}$, $\rho_\alpha(U_\alpha) = U_{\alpha'}$. So ρ_α is surjective and thus an isomorphism. □

Proposition 5.2.2. Consider the same setup as the lemma above, but now pick out a chamber C in $\mathrm{st}_{\Sigma_+}(A)$ to serve as a fundamental chamber. The restriction isomorphisms ρ_α extend to an isomorphism of groups $\tilde{\rho} : U_+(A) \to U'_+$, where $U_+(A) \coloneqq \langle U_\alpha \mid \alpha = (\alpha_+,\alpha_-) \subset \Sigma,\ A \in \partial\alpha_+,\ C \in \alpha_+\rangle$ and $U'_+ \coloneqq \langle U_{\alpha'} \mid \alpha' \text{ a root of } \Sigma',\ C' = C \setminus A \in \alpha'\rangle$.

Proof. Surjectivity of $\tilde{\rho}$ is immediate since each ρ_{α_i} is surjective and the $U_{\alpha'_i}$ generate U'_+. Write $\Phi_+(A) \coloneqq \{\alpha \in \Phi \mid A \in \partial\alpha_+,\ C \in \alpha_+\}$, i.e. let $\Phi_+(A)$ be the set of roots defining $U_+(A)$. We will first show $\Phi_+(A)$ is a convex set of twin roots: this will let use a product decomposition for $U_+(A)$ to obtain injectivity of $\tilde{\rho}$.

Since Δ' is spherical, there is a notion of opposition in Σ'. Let $D' \coloneqq \mathrm{op}_{\Sigma'}(C')$ and $D = D' \cup A$. We claim $\Phi_+(A)$ can be expressed as a set of roots $\Phi(C,D) \coloneqq \{\alpha \in \Phi \mid C \in \alpha_+ \text{ and } D \notin \alpha_+,\ C, D \text{ chambers in } \Sigma_+\}$ (which is always convex by [AB08] prop. 8.13). Given $\alpha \in \Phi(C,D)$, we know $D \in -\alpha_+$ by $D \notin \alpha_+$ ($D \in \alpha_+ \implies D' \in \alpha'$, contradicting D' op C' in Σ'). Now $A \leq D$, $A \leq C$ implies $A \in \partial\alpha_+$, hence $\alpha \in \Phi_+(A)$. So $\Phi(C,D) \subseteq \Phi_+(A)$. If $\alpha \in \Phi_+(A)$, we need to show $D \notin \alpha$. Recalling that D' is opposite C' in Σ', we know $D' \in -\alpha'$. Through definition of $U_+(A)$ we know $A \in \partial\alpha_-$, so $D \in \alpha-$ and therefore $D \notin \alpha$. This shows $\Phi_+(A) \subseteq \Phi(C,D)$ and thus $\Phi_+(A) = \Phi(C,D)$.

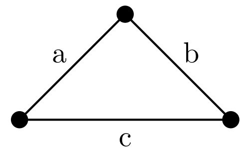
Now $\Phi_+(A)$ is convex with some admissable ordering $\alpha_1,\ldots,\alpha_k$, so each $\{\alpha_i,\ldots,\alpha_k\}$ is convex. Write $\Phi'_+ \coloneqq \{\alpha' \mid \alpha' \text{ a root of } \Sigma',\ C' \coloneqq C \setminus A \in \alpha'\}$: this is the set of roots associated to U'_+. We observe $\Phi'_+ = \{\alpha_+ \cap \Sigma' \mid \alpha = (\alpha_+,\alpha_-) \in \Phi_+(A)\}$ and claim the admissable ordering of $\Phi_+(A)$ induces an admissable ordering of Φ'_+. Each $\bigcap_{i\leq j\leq k}\alpha_j$ contains C, and therefore $\bigcap_{i\leq j\leq k}\alpha'_j$ contains C'. The set $\bigcap_{i\leq j\leq k}\alpha'_j$ is then a convex chamber subcomplex of Σ' ([AB08] props. 3.136 and 3.137), and through the correspondence mentioned above, $\{\alpha'_i,\ldots,\alpha'_k\}$ is a convex set of roots. So $\alpha'_1,\ldots,\alpha'_k$ is an admissable ordering of Φ'_+.

Thus we can write $U_+(A) = U_{\Phi_+(A)} = U_{\alpha_1}\cdots U_{\alpha_k}$ and $U'_+ = U_{\Phi'_+} = U_{\alpha'_1}\cdots U_{\alpha'_k}$; each element of $U_+(A)$ (respectively U'_+) is a uniquely expressible as product of elements of the U_{α_i} (respectively $U_{\alpha'_i}$). Given $x = x_{\alpha_1}\cdots x_{\alpha_k}$ with $x_{\alpha_i} \in U_{\alpha_i}$, we have $\tilde{\rho}(x) = \tilde{\rho}(x_{\alpha_1}\cdots x_{\alpha_k}) = \rho_{\alpha_1}(x_{\alpha_1})\cdots\rho_{\alpha_k}(x_{\alpha_k})$. Now $\rho_{\alpha_1}(x_{\alpha_1})\cdots\rho_{\alpha_k}(x_{\alpha_k})$ is a unique product decomposition (by $\rho_{\alpha_i}(x_{\alpha_i}) \in U_{\alpha'_i}$, the terms' order matches our admissable ordering) and each ρ_{α_i} is an isomorphism, so $\rho_{\alpha_1}(x_{\alpha_1})\cdots\rho_{\alpha_k}(x_{\alpha_k}) = 1 \iff \rho_{\alpha_i}(x_{\alpha_i}) = 1$ for all $i \iff x_{\alpha_i} = 1$ for all i. The map $\tilde{\rho}$ must then be injective, and $\tilde{\rho}$ is an isomorphism.

□

5.3 Triangles in a Rank 3 Hyperbolic Coxeter Complex

Let $(G, (U_\alpha)_{\alpha\in\Phi(W,S)}, T)$ be an RGD system of type (W, S) with associated Coxeter diagram



$$3 \leq a \leq b \leq c < \infty, \quad c \geq 4.$$

In this case, the Coxeter complex can be realized as a copy of hyperbolic 2-space. When we are considering a twin building of this same type later on, we will have twin apartments $\Sigma = (\Sigma_+, \Sigma_-)$ that are particular pairs of these hyperbolic spaces. We develop here a lemma about general triangles in these hyperbolic Coxeter complexes:

Lemma 5.3.1. Suppose (W, S) is a Coxeter system with a Coxeter diagram as above (triangle, at least one label is ≥ 4) and consider its geometric realization $|\Sigma(W, S)| = |\Sigma| = \mathbb{H}^2$. Suppose T is a triangle in $|\Sigma|$ such that each side of T lies on a wall of Σ. Then T is necessarily a chamber in Σ.

Proof. There are several important pieces of information to note:

1. the walls of $|\Sigma|$ are straight lines in $\mathbb{H}^2$,
2. $|\Sigma|$ is a tiling of $\mathbb{H}^2$ by triangles, where each triangle is a chamber,
3. the angles $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$ of a chamber satisfy $\frac{\pi}{a} + \frac{\pi}{b} + \frac{\pi}{c} < \pi$, and lastly
4. $\mathbb{H}^2$ admits a notion of area for polygons that is additive on polygons with disjoint interior.

This **area** of a triangle T with angles θ_1, θ_2, and θ_3 in $\mathbb{H}^2$ may be written

$$\mu(T) = \pi - \theta_1 - \theta_2 - \theta_3.$$

Since the sides of T lie on walls, T is tiled by (closed) chambers. Thus we may write $T = \bigcup_{i=1}^{n} C_i$ for chambers C_i and some n, and we would like to show $n = 1$. We first show that $n \leq 3$, then we rule out $n = 2$ and 3 individually.

To see that $n \leq 3$, we use additivity of area μ and manipulate some upper and lower bounds on $\mu(T)$. The area of T is $\mu(T) = \mu(\bigcup_{i=1}^{n} C_i) = \sum_{i=1}^{n} \mu(C_i)$ since the interiors of the C_i are pairwise disjoint. Fix a fundamental chamber C in Σ. Since each C_i is identical to C, $\mu(C_i) = \mu(C) = \pi - \frac{\pi}{a} - \frac{\pi}{b} - \frac{\pi}{c}$. Then $\mu(T) = \sum_{i=1}^{n} \mu(C_i) = n\mu(C) = n(\pi - \frac{\pi}{a} - \frac{\pi}{b} - \frac{\pi}{c})$. Again since T lies on walls of Σ, its angles must be sums of the angles $\frac{\pi}{a}, \frac{\pi}{b}$, and $\frac{\pi}{c}$. Minimizing these angles gives an upper bound for the area of T: $a \leq b \leq c$ implies $\mu(T) \leq \pi - \frac{\pi}{c} - \frac{\pi}{c} - \frac{\pi}{c} \leq \pi(\frac{c-3}{c})$. Similarly, $3 \leq a \leq b$ implies $\pi - \frac{\pi}{3} - \frac{\pi}{3} - \frac{\pi}{c} = \frac{\pi}{3} - \frac{\pi}{c} \leq \mu(C)$. Together, this yields

$$n\left(\frac{\pi}{3} - \frac{\pi}{c}\right) = n\pi\left(\frac{c-3}{3c}\right) \leq n\mu(C) = \mu(T) \leq \pi\left(\frac{c-3}{c}\right)$$

so that $n\pi(\frac{c-3}{3c}) \leq \pi(\frac{c-3}{c}) \implies n \leq 3$.

To show $n \neq 2$, we use the geometry of $\mathbb{H}^2$. If T is a hyperbolic triangle and $L_1, \ldots, L_n$ is a collection of lines that pass through the interior of T, adding the L_i one by one adds at least one more component to the interior of T at each step. So if we suppose T is a union of two chambers, there must be exactly one wall L_1 that subdivides T into these chambers. Now L_1 must meet one of the sides of T at some point p that is not a corner: if L_1 meets two corners of T, L_1 must be a side of T (and therefore L_1 would not divide T into two subtriangles). If L_1 does not meet a corner of T, it does not divide T into triangles. Hence L_1 must meet a corner and the opposite wall.

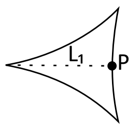


Figure 5.1: A subdivision of T into 2 triangles

The angles formed by L_1 and the wall it meets (at p) must be supplementary. As these are angles of the two chambers, this would mean that at least one of $\frac{\pi}{a}, \frac{\pi}{b}$, or $\frac{\pi}{c}$ is greater than or equal to $\frac{\pi}{2}$. This contradicts $3 \leq a \leq b \leq c$.

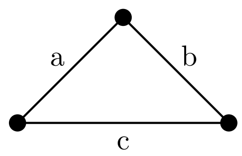
Excluding $n = 3$ is a similar process. If we suppose that T is a union of three chambers, T must have exactly two lines L_1 and L_2 passing through it. We consider the different cases for L_1 and L_2 by adding them in succession. If L_1 does not meet any corner of T, we have violated the conditions on the angles of our chambers: L_1 forms supplementary angles inside T with the sides of T, and adding L_2 cannot subdivide the angles on both ends of L_1 into allowable angles. So L_1 must pass through a corner of T and the opposite side at a point p (as in figure 1). Now L_2 must pass through this point p as well, or else we have again violated our angle conditions. But now the other end of L_2 that meets T (away from p) must violate these angle conditions: the other end of L_2 cannot meet T at a corner (else it is a side or L_1 again), so it meets T at different side and again forms supplementary angles.

The only remaining possibility is that $n = 1$, and T must be a chamber itself. $\square$

5.4 The Main Theorem

In what follows, we will have strictly Moufang twin building generated by an RGD system of type (W, S) via a twin BN pair, and we make the identification of each root α of $\Sigma(W, S)$ with a twin root α (known as $\hat{\alpha}$ in Section 4.2.1 and Theorem 4.2.8) of a fundamental twin apartment. This identification (and Δ being strictly Moufang) will allow us to refer to the geometric $U_{\hat{\alpha}}$ and RGD U_α by the same name ("U_α"), and we will do so at will. We will refer to roots of $\Sigma(W, S)$ as $\Phi(W, S)$ and the roots of the fundamental twin apartment as Φ.

Theorem 5.4.1. Let $(G, (U_\alpha)_{\alpha \in \Phi(W,S)}, T)$ be an RGD system of type (W, S) with associated Coxeter diagram



$$3 \leq a \leq b \leq c < \infty, \quad c \geq 4.$$

and suppose that each $|U_\alpha| \geq 4$. Let $\Delta = (\Delta_+, \Delta_-)$ be the associated (simplicial) strictly Moufang twin building. Then $[U_\alpha, U_\beta] = 1$ for any twin roots $\alpha = (\alpha_+, \alpha_-)$, $\beta = (\beta_+, \beta_-)$ in Φ such that $\alpha_+ \subsetneq \beta_+$.

Remark 5.4.2. Before we begin the proof of this theorem, we note that many of the arguments can be adapted to prove an analogous (but stronger) result for the affine case of $a = b = c = 3$ (which we will go through after the hyperbolic case). Just as hyperbolic geometry will be necessary for the proceeding proof, Euclidean geometry will useful for this affine version.

Remark 5.4.3. This condition on the cardinality of the root groups is stronger than what is really necessary. More specifically, it is enough that for a fundamental chamber C' chosen in the link of any vertex $\Delta' = \text{lk}_\Delta(v)$, the "opposite complex" $(\Delta')^0(C')$ is gallery-connected: the given minimum size guarantees this. We will point this out when it becomes apparent (remark 5.4.5).

Proof. Recall $\Sigma = (\Sigma_+, \Sigma_-)$ is a fundamental twin apartment containing $\beta = (\beta_+, \beta_-)$ (thus containing $\alpha = (\alpha_+, \alpha_-)$). The main argument is an induction on a notion of distance between α_+ and β_+. Let $d = d(\partial\alpha_+, \partial\beta_+) := \min\{d(x_0, y_0) \mid x_0, y_0$ chambers in Σ_+, $\exists$ panels $\mathcal{P}_x \leq x_0$, $\mathcal{P}_y \leq y_0$ with $\mathcal{P}_x \in \partial\alpha_+$, $\mathcal{P}_y \in \partial\beta_+\}$, where $d(x_0, y_0)$ is the usual notion of numerical distance between chambers. In what follows, x_0 and y_0 will always be a pair of chambers that achieves this minimum. If $d = 1$, then x_0 and y_0 must share a panel with vertices in both $\partial\alpha_+$ and $\partial\beta_+$. By corollary 5.1.2 the open interval (α, β) is then empty, and by (RGD1), $[U_\alpha, U_\beta] \subseteq U_{(\alpha,\beta)} = 1$.

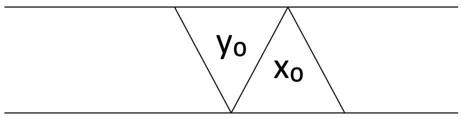


FIGURE 5.2: The $d = 1$ case

Now suppose $d > 1$, and assume the result holds for all pairs of twin roots with distance less than d. Let w_0 be the second chamber in a minimal gallery Γ from y_0 to x_0, $v = w_0 \cap y_0 \cap \partial\beta_+$, and $z_0 = \text{proj}_{x_0} v$. This z_0 has properties necessary for the main induction argument; we will view it as the product vx_0. Recall a product AB of two simplices A and B in Δ_+ may be defined via the sign-sequences $(\sigma_{\partial\eta_+}(A))_{\eta\in\Phi}$ and $(\sigma_{\partial\eta_+}(B))_{\eta\in\Phi}$, with $\sigma_{\partial\eta_+}(A) = +, -,$ or 0 depending on whether $A \in \eta_+, -\eta_+,$ or $\partial\eta_+$ (and the same for $\sigma_{\partial\eta_+}(B)$). In particular, AB is uniquely defined as the simplex with sign-sequence

$$\sigma_{\partial\eta_+}(AB)_{\eta\in\Phi} = \begin{cases} \sigma_{\partial\eta_+}(A) & \text{if } \sigma_{\partial\eta_+}(A) \neq 0 \\ \sigma_{\partial\eta_+}(B) & \text{if } \sigma_{\partial\eta_+}(A) = 0 \end{cases}.$$

It is easy to see that vx_0 is a chamber since x_0 is a chamber. Also, every minimal gallery from a chamber $C \in \text{st}_{\Sigma_+}(v)$ to x_0 must begin with vx_0 and thus the "gate property" holds: $d(x_0, C) = d(x_0, vx_0) + d(vx_0, C)$.

Because v is a vertex of z_0, two of the three walls z_0 lies on pass through v. Let γ, δ be the twin roots containing z_0 corresponding to these walls. Both γ_+ and δ_+ contain x_0: since $\sigma_{\partial\gamma_+}(v) = 0$, we know that $\sigma_{\partial\gamma_+}(z_0) = \sigma_{\partial\gamma_+}(vx_0) = \sigma_{\partial\gamma_+}(x_0) = +$ (and the same holds for δ_+). We will use induction arguments (noting the useful property of z_0 mentioned at the end of the last paragraph) in the following 3 configurations for α, γ, and δ: $(\partial\delta_+ \cap \partial\alpha_+) \neq \varnothing \neq (\partial\gamma_+ \cap \partial\alpha_+)$, $(\partial\delta_+ \cap \partial\alpha_+) = \varnothing = (\partial\gamma_+ \cap \partial\alpha_+)$, or $(\partial\delta_+ \cap \partial\alpha_+) = \varnothing \neq (\partial\gamma_+ \cap \partial\alpha_+)$.

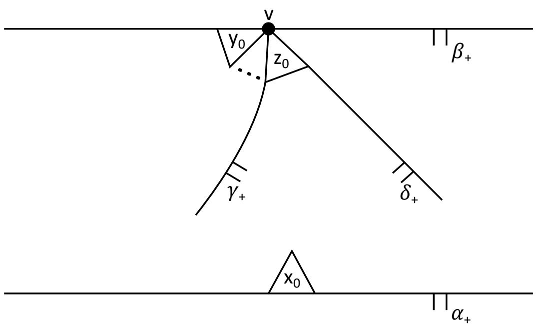


FIGURE 5.3: The general case, $d \neq 1$

Case 1: $(\partial\delta_+ \cap \partial\alpha_+) \neq \varnothing \neq (\partial\gamma_+ \cap \partial\alpha_+)$

Note that a triangle T is formed by $\partial\delta_+$, $\partial\gamma_+$, and $\partial\alpha_+$. We may then apply lemma 5.3.1 to see that T is a chamber, and since T contains z_0 it must be z_0. This means that z_0 has a panel on $\partial\alpha_+$ and a vertex on $\partial\beta_+$, and since z_0 is a chamber its intersection with $\partial\gamma_+$ must be an edge. Thus $\partial\gamma_+ \cap \beta_+ \cap (-\alpha_+)$ is an edge, and we again use corollary 5.1.2 to see $(\alpha, \beta) = \varnothing$ and $[U_\alpha, U_\beta] = 1$.

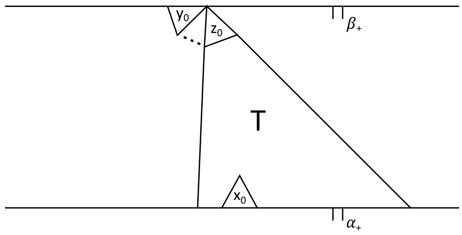


FIGURE 5.4: $(\partial\gamma_+ \cap \partial\alpha_+) \neq \varnothing \neq (\partial\delta_+ \cap \partial\alpha_+)$

Case 2: $(\partial\delta_+ \cap \partial\alpha_+) = \varnothing = (\partial\gamma_+ \cap \partial\alpha_+)$

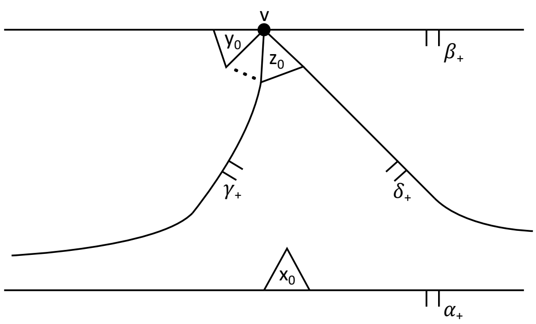


FIGURE 5.5: $(\partial\delta_+ \cap \partial\alpha_+) = \varnothing = (\partial\gamma_+ \cap \partial\alpha_+)$

Observe $\alpha_+ \subsetneq \gamma_+$ and $\alpha_+ \subsetneq \delta_+$ (otherwise z_0 does not satisfy the gate property), and recall $d(z_0, x_0) < d(y_0, x_0)$. Since z_0 has a panel on both $\partial\gamma_+$ and $\partial\delta_+$, we also know that $d(\partial\gamma_+, \partial\alpha_+) \leq d(z_0, x_0)$ and $d(\partial\delta_+, \partial\alpha_+) \leq d(z_0, x_0)$. So $d(\partial\gamma_+, \partial\alpha_+)$ and $d(\partial\delta_+, \partial\alpha_+)$ are strictly less than $d(\partial\beta_+, \partial\alpha_+)$, and we may apply the inductive step to see that $[U_\alpha, U_\gamma] = 1 = [U_\alpha, U_\delta]$. We will show $U_\beta \subset \langle U_\gamma, U_\delta \rangle$ and thus $[U_\alpha, U_\beta] = 1$.

Note we have the same setup as lemma 5.2.1 and proposition 5.2.2 with $\Delta' = \mathrm{lk}_{\Delta_+}(v)$, $A = v$, and $C = z_0$ (as a fundamental chamber). Since v is in $\partial\beta_+$, $\partial\gamma_+$, and $\partial\delta_+$, we apply lemma 5.2.1 to get $U_\beta \cong U_{\beta'}$, $U_\gamma \cong U_{\gamma'}$, and $U_\delta \cong U_{\delta'}$ (where, as before, the root groups with a prime symbol are root groups of the link). We also apply proposition 5.2.2 to see $U_+(v) \cong U'_+$. The chamber z_0 is in both γ and δ and has a panel on each of their boundaries, so γ' and δ' are the simple roots (with respect to z'_0 as a fundamental chamber) of Σ'. We also know z_0 is in β, so $\beta' \in \Phi'_+$. Then showing $\langle U_{\gamma'}, U_{\delta'}\rangle = U'_+$ would imply $U_{\beta'} \subset \langle U_{\gamma'}, U_{\delta'}\rangle$, and through the use of $U_+(v) \cong U'_+$ we would obtain $U_\beta \subset \langle U_\gamma, U_\delta\rangle$.

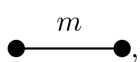
Now for any spherical building X with fundamental chamber C, define $X^0(C) := \bigcup_{D \text{ op } C} \mathrm{st}_X(D)$ (noting "opposition" here means maximal distance, as we are not regarding X as a twin building). We have a criterion for when U'_+ is generated by its root groups corresponding to simple roots:

Lemma 5.4.4 ([AM99] lemma 3). Let X be a spherical Moufang building with fundamental chamber C in some apartment Σ. Let Φ^+ be the set of positive roots in Σ and let $\alpha_1, \dots, \alpha_n$ be the simple roots in Φ^+. Write $U = \langle U_\alpha \mid \alpha \in \Phi^+\rangle$ and $U' = \langle U_{\alpha_i} \mid 1 \le i \le n\rangle$. Then the index $[U : U']$ is equal to the number of gallery-connected components of $X^0(C)$. In particular, $U = U'$ iff $X^0(C)$ is gallery-connected.

Remark 5.4.5. In our case, showing that $(\Delta')^0(z'_0)$ is gallery-connected is equivalent to showing that $U'_+ = \langle U_{\delta'}, U_{\gamma'}\rangle$. But the Moufang polygons with gallery-disconnected $X^0(C)$ have been classified: there are only a few "small" cases, which we will soon show are ruled out by the condition that $|U_\alpha| \ge 4$. Thus, the minimal necessary condition is that no links are of the types mentioned following theorem:

Theorem 5.4.6 ([AM99] Main Result). Let X be a Moufang polygon and C be a chamber of X. Then $X^0(C)$ is gallery-disconnected iff X is a building associated to one of the groups $C_2(\mathbb{F}_2) = Sp_4(\mathbb{F}_2)$, $G_2(\mathbb{F}_2)$, $G_2(\mathbb{F}_3)$, or ${}^2F_4(\mathbb{F}_2)$.

Through our choice of Δ, the cotype of v yields a Coxeter diagram



,

where $m \in \{3, 4, 6, 8\}$ (by Tits' classification of Moufang polygons). This m determines which of the above four groups we must rule out. The first three groups are Chevalley groups and thus have root groups with cardinality matching that of the base field; the Ree group ${}^2F_4(\mathbb{F}_2)$ has root groups of cardinalities 2 and 4 (see [Tit83] p.551). If $m = 3$, we have no restrictions on our $|U_\alpha|$ (since we are then in type A_2). If $m = 4$, we must have root groups of size at least 3 to guarantee Δ' is not isomorphic to the building associated to $Sp_4(\mathbb{F}_2)$. If $m = 6$, root groups of cardinality 4 are required to rule out $G_2(\mathbb{F}_3)$ (which also rules out $G_2(\mathbb{F}_2)$). Lastly, $m = 8$ requires root groups of size at least 3 to make the root groups of size 2 in ${}^2F_4(\mathbb{F}_2)$ impossible.

We can use proposition 5.2.2 to formulate the preceeding discussion into a proposition:

Proposition 5.4.7. Let Δ be a 2-spherical strictly Moufang twin building of rank at least 3. Suppose Δ' is the link of a codimension 2 simplex in Δ and there are no isolated nodes in the Coxeter diagram of Δ'. If the cardinalities of the root groups associated to Δ are at least 4, then for any chamber $C' \in \Delta'$ we must have that $(\Delta')^0(C')$ is gallery-connected.

Since by assumption our root groups have cardinality at least 4, we must have gallery-connected $(\Delta')^0(z_0')$ and thus $U_+' = \langle U_{\gamma'}, U_{\delta'} \rangle$. Hence $U_{\beta'} \subset \langle U_{\gamma'}, U_{\delta'} \rangle$, and by proposition 5.2.2 this yields $U_\beta \subset \langle U_\gamma, U_\delta \rangle$. Together with our earlier observation that $[U_\alpha, U_\gamma] = 1 = [U_\alpha, U_\delta]$, we finally arrive at $[U_\alpha, U_\beta] = 1$.

Case 3: $(\partial\delta_+ \cap \partial\alpha_+) = \varnothing \neq (\partial\gamma_+ \cap \partial\alpha_+)$

Now we have the following picture:

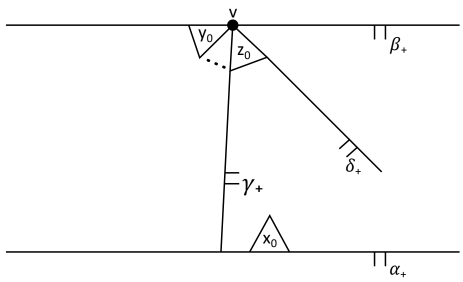


Figure 5.6

We note here that $\{\alpha, \gamma\}$ must be prenilpotent in order to make use of the commutator $[U_\alpha, U_\gamma]$. Let w be the vertex $\partial\gamma_+ \cap \partial\alpha_+$. By lemma 5.1.1 *ii*), $\{\gamma_+, \pm\alpha_+\}$ cannot be nested, so the roots corresponding to them in the link of w are also not nested. Then the intersections of γ_+ and $\pm\alpha_+$ with this link must share a chamber, and thus γ_+ and $\pm\alpha_+$ must share a chamber. Thus $\{\gamma, \alpha\}$ is prenilpotent.

Now in this new case the situation at v has not changed, so by the same arguments from case 2 we know $U_{\beta'} \subset \langle U_{\gamma'}, U_{\delta'} \rangle$ and $U_\beta \subset \langle U_\gamma, U_\delta \rangle$. The first order of business is to analyze (α, γ): while the inductive step still yields $[U_\alpha, U_\delta] = 1$, it can no longer be applied to show $[U_\alpha, U_\gamma] = 1$. If there exists $\epsilon \in (\alpha, \gamma)$ with $\partial\varepsilon_+ \cap \partial\delta_+ \neq \varnothing$, we will be able show $[U_\alpha, U_\beta] = 1$ directly. If not, we show $[[U_\alpha, U_\gamma], U_\delta] = 1$, which will then imply $[U_\alpha, U_{(\gamma,\delta]}] = 1$ (and thus $[U_\alpha, U_\beta] = 1$ since $\beta \in (\gamma, \delta) \subset (\gamma, \delta]$).

Now suppose there is an $\varepsilon \in (\alpha, \gamma)$ with $\partial\varepsilon_+ \cap \partial\delta_+ \neq \varnothing$. We observe $\partial\varepsilon_+, \partial\delta_+$, and $\partial\gamma_+$ form a triangle T and we may apply lemma 5.3.1 (see figure 7) so that $T = z_0$ is a chamber.

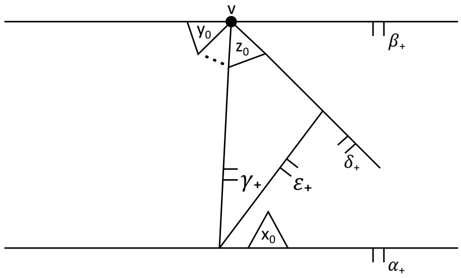


FIGURE 5.7

Since T is a chamber, $\partial\gamma_+ \cap (\beta_+ \cap (-\alpha_+))$ is an edge and we again use corollary 5.1.2 to see $(\alpha, \beta) = \varnothing$. So again $[U_\alpha, U_\beta] = 1$.

Suppose $\partial\varepsilon_+ \cap \partial\delta_+ = \varnothing$ for all ε in (α, γ). The temporary goal is to show $[[U_\alpha, U_\gamma], U_\delta] = 1$. To do so, we'll need to show for any such ε we have $\varepsilon_+ \subset \delta_+$ and $d(\partial\varepsilon_+, \partial\delta_+) \leq d(x_0, z_0)$ (in order to use the induction hypothesis and see $[U_\varepsilon, U_\delta] = 1$).

We first need that $\varepsilon_+ \subsetneq \delta_+$. Write $x_1 = r_\alpha(x_0)$ and recall α_+ is the unique root in Σ_+ that contains x_1 but not x_0. As γ_+ contains x_0, it also contains x_1. By definition of the open interval (α, γ), ε_+ must contain x_1 and thus also x_0. Now δ_+ contains x_0 by design, so since ε_+ and δ_+ share a chamber and their walls do not meet, we must have $\{\varepsilon_+, \pm\delta_+\}$ nested.

We rule out $\pm\delta_+ \subsetneq \varepsilon_+$ and $\varepsilon_+ \subsetneq -\delta_+$. By its definition, δ_+ contains z_0. Note $-\varepsilon \in (-\alpha, -\gamma)$, and $-\gamma_+$ is the unique root in Σ_+ that contains $r_\gamma(z_0) = z_1$ and not z_0 (since γ_+ contains z_0 and not z_1). As $-\alpha_+$ contains z_0 (z_0 is not in α_+), it must also contain z_1. So by definition of the open interval, $-\varepsilon_+$ must contain z_1 and thus also z_0. Hence ε_+ cannot contain z_0, and $\delta_+ \not\subset \varepsilon_+$. In the same way $-\delta_+$ is the unique root that contains $r_\delta(z_0)$ and not z_0, so since ε_+ doesn't contain z_0 it also doesn't contain $r_\delta(z_0)$. Thus $-\delta_+ \not\subset \varepsilon_+$. Finally, we know $x_0 \notin -\delta_+$ by $x_0 \in \delta_+$ so $\varepsilon_+ \not\subset -\delta_+$. Then it must be that $\varepsilon_+ \subsetneq \delta_+$.

Now ε_+ contains x_0 and not z_0, so any minimal gallery from x_0 to z_0 must cross $\partial\varepsilon_+$. But z_0 has a panel on $\partial\delta_+$ and $x_0 \in \delta_+$, so $d(\partial\varepsilon_+, \partial\delta_+) \leq d(x_0, z_0)$ and we may use the induction to show $[U_\varepsilon, U_\delta] = 1$. Finally, since $[U_\alpha, U_\gamma] \subset U_{(\alpha,\gamma)} = \langle U_\varepsilon \mid \varepsilon \in (\alpha, \gamma)\rangle$, we see $[[U_\alpha, U_\gamma], U_\delta] \subseteq [U_{(\alpha,\gamma)}, U_\delta] = 1$.

Now recall the standard fact that there is a semidirect product decomposition $U_{[\gamma,\delta]} = U_\gamma \ltimes U_{(\gamma,\delta]}$ with associated surjection $\phi : U_{[\gamma,\delta]} \twoheadrightarrow U_\gamma$ such that $\ker\phi = U_{(\gamma,\delta]}$ and $\phi|_{U_\gamma} = \mathrm{id}_{U_\gamma}$. One more characterization of $U_{[\gamma,\delta]}$ is needed in order to get a specific form for elements of $U_{(\gamma,\delta]}$. Recall we observed that $\langle U_{\gamma'}, U_{\delta'}\rangle = U'_+$ at the end of the second case. The same arguments apply here, so we know $\langle U_{\gamma'}, U_{\delta'}\rangle = U'_+ = U_{[\gamma',\delta']}$ and thus $\langle U_\gamma, U_\delta\rangle = U_+(v) = U_{[\gamma,\delta]}$. So for any $x \in U_{(\gamma,\delta]}$ we may write $x = c_1 d_1 \cdots c_r d_r$, where $c_i \in U_\gamma$ and $d_i \in U_\delta$.

We are ready to show $[U_\alpha, U_{(\gamma,\delta]}] = 1$. Let $a \in U_\alpha$, $x \in U_{(\gamma,\delta]}$ and $x = c_1 d_1 \cdots c_r d_r$ as above. By definition of ϕ

$$1 = \phi(x) = \phi(c_1 d_1 \cdots c_r d_r) = \phi(c_1)\phi(d_1)\cdots\phi(c_r)\phi(d_r) = c_1 \cdots c_r.$$

We will show $[a, x] = [a, c_1 \cdots c_r] = [a, 1] = 1$.

For elements x, y of any group let ${}^y x = yxy^{-1}$ and $x^y = y^{-1}xy$. It is straightforward to compute that ${}^w[x, yz] = {}^w[x, y]({}^{wy}[x, z])$. Specializing to the case $w = 1$, we obtain $[x, yz] = [x, y]({}^y[x, z])$.

Apply these formulas to $[a, x]$:

$$[a, x] = [a, c_1 d_1 \cdots c_r d_r] = [a, c_1]({}^{c_1}[a, d_1 c_2 d_2 \cdots c_r d_r])$$

$$= [a, c_1]({}^{c_1}[a, d_1]({}^{c_1 d_1}[a, c_2 d_2 \cdots c_r d_r])) = \cdots$$

$$= [a, c_1]({}^{c_1}[a, d_1]({}^{c_1 d_1}[a, c_2](\cdots({}^{c_1 d_1 \cdots c_r}[a, d_r])\cdots).$$

By the main inductive step (from the very beginning of this case) we know $[U_\alpha, U_\delta] = 1$ and therefore $[a, d_i] = 1$ for all i. So the above becomes

$$[a, x] = [a, c_1](({}^{c_1 d_1}[a, c_2](\cdots({}^{c_1 d_1 \cdots c_{r-1} d_{r-1}}[a, c_r])\cdots).$$

We may also remove all the d_i terms from any ${}^{c_1 d_1 \cdots d_{j-1}}[a, c_j]$. We showed $[[U_\alpha, U_\gamma], U_\delta] = 1$ earlier, so ${}^{d_{j-1}}[a, c_j] = [a, c_j]$. Recall that if G, H are subgroups of some larger group then H normalizes $[G, H]$. To see this, observe that given $g \in G$, $x, y \in H$ we may write any conjugated commutator as

$$x[g, y]x^{-1} = xgyg^{-1}y^{-1}x^{-1} = xg(x^{-1}g^{-1}gx)yg^{-1}y^{-1}x^{-1} =$$

$$(xgx^{-1}g^{-1})(gxyg^{-1}y^{-1}x^{-1}) = [x, g][g, xy] = [g, x]^{-1}[g, xy].$$

This righthand side is a product of two elements of $[G, H]$, hence the generators of $[G, H]$ are normalized by H and $[G, H]$ is normalized by H. Thus U_γ normalizes $[U_\alpha, U_\gamma]$ and

$c_{j-1}[a,c_j]c_{j-1}^{-1}$ is in $[U_\alpha, U_\gamma]$. We may then write

$$^{d_{j-2}c_{j-1}}[a,c_j] = d_{j-2}(c_{j-1}[a,c_j]c_{j-1}^{-1})d_{j-2}^{-1} = c_{j-1}[a,c_j]c_{j-1}^{-1} = {}^{c_{j-1}}[a,c_j].$$

Continuing on in this way, we have $^{c_1 d_1 \cdots c_{j-1} d_{j-1}}[a,c_j] = {}^{c_1 \cdots c_{j-1}}[a,c_j]$ and therefore

$$[a,x] = [a,c_1](^{c_1}[a,c_2](\cdots(^{c_1\cdots c_{r-1}}[a,c_r])\cdots).$$

Now collapse this righthand side: by using

$$^{c_1\cdots c_{j-2}}[a,c_{j-1}](^{c_1\cdots c_{j-2}c_{j-1}}[a,c_j\cdots c_r]) = {}^{c_1\cdots c_{j-2}}[a,c_{j-1}c_j\cdots c_r]$$

we arrive at

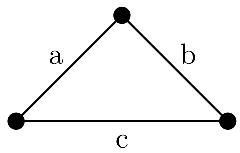
$$[a,x] = [a,c_1](^{c_1}[a,c_2](\cdots(^{c_1\cdots c_{r-1}}[a,c_r])\cdots) = [a,c_1\cdots c_r].$$

By our earlier observation that $c_1\cdots c_r = 1$, we have $[a,x]=1$. So we have finally shown $[U_\alpha, U_{(\gamma,\delta]}] = 1$, and by $U_\beta \subset U_{(\gamma,\delta]}$, we obtain $[U_\alpha, U_\beta] = 1$. □

5.5 The Affine Case

As promised, we can also develop a (much shorter) proof of the affine case, thanks to all of the relevant ideas already being introduced:

Theorem 5.5.1. Let $(G, (U_\alpha)_{\alpha\in\Phi(W,S)}, T)$ be an RGD system of type (W,S) with associated Coxeter diagram



$3 = a = b = c.$

Let $\Delta = (\Delta_+, \Delta_-)$ be the associated (simplicial) strictly Moufang twin building. Then $[U_\alpha, U_\beta] = 1$ for any twin roots $\alpha = (\alpha_+, \alpha_-)$, $\beta = (\beta_+, \beta_-)$ in Φ such that $\alpha_+ \subsetneq \beta_+$.

Remark 5.5.2. Note the absence of any condition on the size of the root groups: this was only necessary to guarantee gallery-connected opposite complexes in the link, and our links will have gallery-connected opposite complexes from the situation itself.

Proof. Similar to the hyperbolic case, we observe that the Coxeter complexes $\Sigma(W,S)$ associated to the apartments of Δ may be geometrically realized as Euclidean planes $\mathbb{E}^2$. In thinking about the affine case we lose lemma 5.3.1, but we will not need it. We repeat the same notational setup from before, as well. We have a prenilpotent pair $\{\alpha,\beta\}$ with nested positive halves, $\alpha_+ \subsetneq \beta_+$, and two distance-minimizing chambers x_0 and y_0 on a minimal gallery from $\partial\beta_+$ to $\partial\alpha_+$. We can repeat the same setup for w_0, v, and z_0: w_0 should be the first chamber along this minimal gallery (starting at y_0), v is $y_0 \cap w_0 \cap \partial\beta_+$, and $z_0 = \mathrm{proj}_{x_0}(v)$. Finally, we can select the same two roots γ and δ which contain z_0, such that z_0 has panels on $\partial\gamma_+$ and $\partial\delta_+$ and both $\partial\gamma_+, \partial\delta_+$ contain v.

Now again consider the local root groups at v (associated to $\Delta' = \mathrm{lk}_\Delta(v)$): we have an isomorphism of groups $\tilde{\rho} : U_+(v) \to U'_+$ (as in proposition 5.2.2) where $U_+(v) := \langle U_\alpha \mid \alpha = (\alpha_+, \alpha_-) \subset \Sigma,\ v \in \partial\alpha_+,\ z_0 \in \alpha_+\rangle$ and $U'_+ := \langle U_{\alpha'} \mid \alpha' \text{ a root of } \Sigma',\ z'_0 = (z_0 \setminus A) \in \alpha'\rangle$ that restricts to isomorphisms $U_\eta \to U_{\eta'}$ for $\eta \in \Phi$, and again γ' and δ' are simple roots in the link. So if we note that $(\Delta')^0(z'_0)$, the opposite complex of chamber z'_0 in Δ', is gallery-connected by way of Theorem **??** (as any vertex v is of type A_2), we then know that $U'_+ = \langle U'_\gamma, U'_\delta\rangle$. The strategy will then again be to use the fact that $U'_\beta \subset \langle U'_\gamma, U'_\delta\rangle \implies U_\beta \subset \langle U_\gamma, U_\delta\rangle$ (using the isomorphism mentioned above) and show that U_γ, U_δ both commute with U_α.

Here is where the geometry starts to come into play: as $\Sigma(W,S)$ corresponds to the tiling of the Euclidean plane by equilateral triangles, the star of any vertex is a hexagon with 6 triangles (as the angles meeting at any vertex are all $\frac{\pi}{3}$, and must add up to 2π). As well, since $v \in \partial\beta_+$ and all of the root walls correspond to lines in this geometric realization, we are in the following situation (or a left-right reflected version):

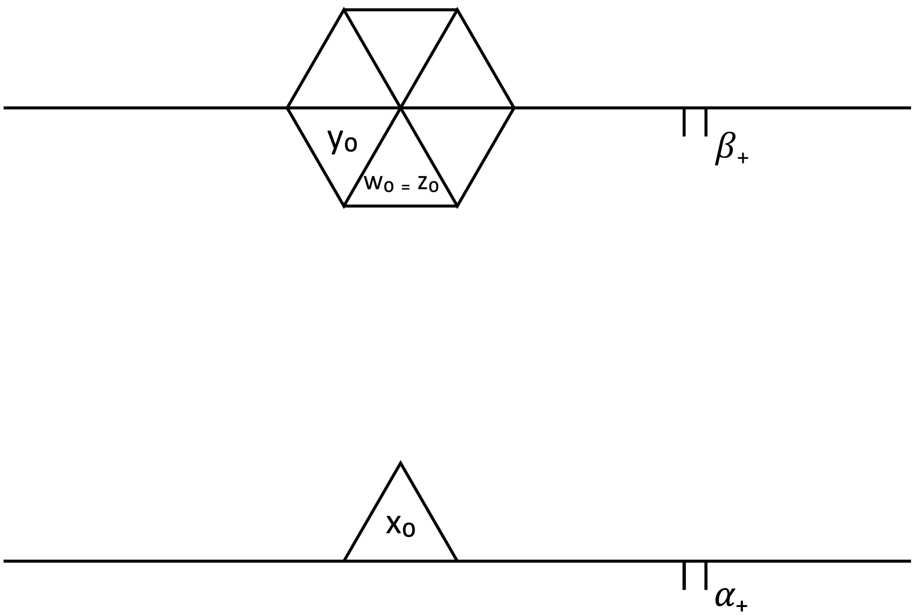


FIGURE 5.8: The general Euclidean situation

Note that $w_0 = z_0$ here, as z_0 being any other chamber in this star would imply our original minimal gallery was not minimal (and z_0 must be in this star since it is the projection of v onto x_0 and thus has v as a face). The roots α_+ and β_+ correspond to Euclidean half-planes here (all of the simplices on one side of the walls $\partial\alpha_+$ and $\partial\beta_+$), and as $\alpha_+ \subsetneq \beta_+$, α_+ and β_+ correspond to nested half-planes.

Both pairs of walls $\{\partial\beta_+, \partial\gamma_+\}$ and $\{\partial\beta_+, \partial\delta_+\}$ yield 2 pairs of angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ at their intersection (each pair on either side of $\partial\beta_+$). We can then define the angles $\theta_v(\pm\gamma_+, \pm\beta_+)$ to be the angles given by the rays emanating from v that bound each combination of $(\pm\gamma_+) \cap (\pm\beta_+)$, and we extend this definition any vertex and any pair of roots that both have walls on that vertex. In particular, our understanding of $\mathrm{st}_\Sigma(v)$ determines that both $\theta_v(\gamma_+, \beta_+) = \frac{2\pi}{3}$ and $\theta_v(\delta_+, \beta_+) = \frac{2\pi}{3}$.

As $\partial\gamma_+$ and $\partial\beta_+$ (likewise $\partial\delta_+$ and $\partial\beta_+$) are not parallel, $\partial\gamma_+$ and $\partial\delta_+$ must both cross $\partial\alpha_+$ (at p_1 and p_2, respectively). We may then infer that $\theta_{p_1}(\gamma_+, -\alpha_+) = \theta_{p_2}(\delta_+, -\alpha_+) = \frac{\pi}{3}$, as they are supplementary angles to $\theta_v(\gamma_+, \beta_+)$ and $\theta_v(\delta_+, \beta_+)$, respectively. Finally, from this we gather that $\theta_{p_1}(\gamma_+, \alpha_+) = \theta_{p_2}(\delta_+, \alpha_+) = \frac{2\pi}{3}$ (since $\theta_{p_1}(\gamma_+, \alpha_+) + \theta_{p_1}(\gamma_+, -\alpha_+) = \theta_{p_2}(\delta_+, \alpha_+) + \theta_{p_2}(\delta_+, -\alpha_+) = \pi$).

Now the star of p_1 in Σ is again a hexagon tiled by 6 copies of an equilateral triangle (each meeting at p_1), and any root η in the interval $[\alpha, \gamma]$ must have a positive half η_+ such that $(\alpha_+ \cap \gamma_+) \subset \eta_+$ and $((-\alpha_+) \cap (-\gamma_+)) \subset -\eta_+$. $\partial\eta_+$ must also contain w, as $w \in \alpha_+ \cap \gamma_+ \subset \eta_+$ and $w \in (-\alpha_+) \cap (-\gamma_+) \subset -\eta_+$. But given $\theta_{p_1}(\gamma_+, \alpha_+) = \frac{2\pi}{3}$, one can see easily from looking at such a hexagon that the only such η are α and γ themselves. Thus $(\alpha, \gamma) = \varnothing$, and a similar argument shows $(\alpha, \delta) = \varnothing$. Hence $[U_\alpha, U_\gamma] = 1 = [U_\alpha, U_\delta]$, and since we showed earlier that $U_\beta \subset \langle U_\gamma, U_\delta \rangle$, we obtain $[U_\alpha, U_\beta] \subset [U_\alpha, \langle U_\gamma, U_\delta \rangle] = 1$.

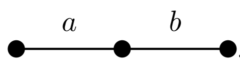
□

5.6 A Comment on the General Situation

A natural question to ask after this is:

What happens with the other 2-spherical rank 3 hyperbolic cases?

I.e., do we get commutation for root groups associated to nested roots in Moufang twin buildings of type



,

for $a, b \in \{3, 4, 6, 8\}$ with $\frac{1}{a} + \frac{1}{b} < \frac{1}{2}$ (perhaps assuming some condition bounding the size of root groups from below)? Specifically, in this case the Coxeter complex corresponds to a tiling of the hyperbolic plane by right triangles ($\frac{\pi}{a} + \frac{\pi}{b} + \frac{\pi}{2} < \pi$).

The answer is no. In Chapter 7, we find a condition on a subclass of groups with an RGD system that guarantees existence of nested roots with non-commuting associated root groups. In particular, we will be able to construct two buildings of this same type, one with and one without all root groups associated to nested roots commuting (example 7.3.9).

Chapter 6

Kac-Moody Algebras and Kac-Moody Groups

One plentiful source for groups admitting an RGD system is Kac-Moody groups over fields. We will first aggregate necessary information about Kac-Moody algebras, and then give the specific construction we use (as there are several different "versions" of Kac-Moody groups). The definition we use will coincide with (or restrict to) most constructions of so-called "minimal" Kac-Moody groups, which ignore "imaginary" roots of Kac-Moody algebras. We will make reference to the newer text [Mar18] for the convenience with which it sets up our approach to Kac-Moody groups (in expounding on [Rém02] and [Tit87]): namely, we will gather information about the classical construction of Kac-Moody algebras (which can mostly be found in [Kac90] as well), loosen up the definition a bit to a "Kac-Moody algebra of type $\mathcal{D}$" (variants of which are presented in [Rém02] and [MP95]), and then we will use this second definition to construct our groups.

6.1 Basic Definitions

We start with the notion of a generalized Cartan matrix, a loosening on the restrictions of the standard Cartan matrix from the basic structure theory of finite-dimensional Lie algebras:

Definition 6.1.1. A **generalized Cartan matrix** (GCM) is a matrix $A = [a_{ij}]_{i,j\in I}$ satisfying

(GCM1) $a_{ii} = 2$ for all $i \in I$

(GCM2) a_{ij} is a nonpositive integer $\forall i \neq j$

(GCM3) $a_{ij} = 0$ iff $a_{ji} = 0$.

We will not have cause to let this index set be anything other than finite (in general, we will set $I = \{1, \ldots, n\}$ for the rest of this chapter).

Definition 6.1.2. A **realization** of an $n \times n$ generalized Cartan matrix A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ such that

(i) $\mathfrak{h}$ is a complex vector space of dimension $2n - l$, where $l = \text{corank}(A)$

(ii) $\Pi = \{\alpha_i \mid i \in I\}$ and $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$ are linearly independent in $\mathfrak{h}^*$ (the dual space of $\mathfrak{h}$) and $\mathfrak{h}$, respectively and

(iii) $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ for all $i, j \in I$

This realization will contain the important information about roots and the Cartan subalgebra for our construction.

Definition 6.1.3. Define $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{g}}$ to be the $\mathbb{C}-$Lie algebra generated by $\mathfrak{h}$ and symbols e_i, f_i for $1 \leq i \leq n$ subject to the relations

$$[h, h'] = 0 \quad \forall h, h' \in \mathfrak{h},$$

$$[e_i, f_j] = -\delta_{ij}\alpha_i^\vee,$$

$$[h, e_i] = \alpha_i(h)e_i,$$

$$[h, f_i] = -\alpha_i(h)f_i$$

Let $\tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{n}}^-$ be the subalgebras generated by the e_i and f_i, respectively. Define $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$, the **root lattice**, and $Q^\vee = \sum_{i=1}^n \mathbb{Z}\alpha_i^\vee$, the **coroot lattice**. Set $Q_+ = \sum_{i=1}^n \mathbb{N}\alpha_i$.

Proposition 6.1.4. ([Mar18] Prop. 3.14)

1. $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}^+$

2. Mapping $e_i \mapsto f_i$, $f_i \mapsto e_i$, and $h \mapsto -h$ extends to an involution of the Lie algebra $\tilde{\mathfrak{g}}(A)$

3. $\tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{n}}^-$ are freely generated by the e_i and the f_i, respectively

4. The adjoint action of $\mathfrak{h}$ on $\tilde{\mathfrak{g}}(A)$ yields a root space decomposition

$$\tilde{\mathfrak{g}}(A) = \left(\bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \tilde{\mathfrak{g}}_\alpha \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \tilde{\mathfrak{g}}_{-\alpha} \right)$$

where each $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}} \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$. Also, $\tilde{\mathfrak{g}}_\alpha$ (resp. $\tilde{\mathfrak{g}}_{-\alpha}$) is spanned by all iterated brackets $[e_{i_1}, \ldots, e_{i_k}]$ (resp. $[f_{i_1}, \ldots, f_{i_k}]$) for $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_k} \in Q_+$ (resp. $-\alpha = -\alpha_{i_1} - \cdots - \alpha_{i_k} \in -Q_+$)

5. There is a unique maximal ideal $\mathfrak{i}$ in the collection of all ideals that intersect $\mathfrak{h}$ trivially. Also, $\mathfrak{i} = (\mathfrak{i} \cap \tilde{\mathfrak{n}}^-) \oplus (\mathfrak{i} \cap \tilde{\mathfrak{n}}^+)$

This $\tilde{\mathfrak{g}}(A)$ is an intermediate step in the usual construction, and can go in two directions: one can mod out by this maximal $\mathfrak{i}$ (this is Kac's definition from [Kac90]), or one can mod out by the Serre relations. We will go this second route.

Definition 6.1.5. The Kac-Moody algebra $\mathfrak{g}(A)$ is the quotient of $\tilde{\mathfrak{g}}(A)$ by the Serre relations

$$(ad\, e_i)^{1-a_{ij}} e_j = 0$$

$$(ad\, f_i)^{1-a_{ij}} f_j = 0$$

for $i \neq j$.

One can show that the Serre relations lie in $\mathfrak{i}$, and thus $\mathfrak{g}(A)$ admits Kac's version as a quotient. They coincide in (at least) the case that A is a symmetrizable matrix (this is a consequence of the Gabber-Kac theorem - see [MP95] Theorem 3, p.377), but it is not known if they always coincide or if there is a GCM A for which they do not. This $\mathfrak{g}(A) = \mathfrak{g}$ also admits a root space decomposition

$$\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha,$$

where these $\mathfrak{g}_\alpha$ are the same way as before in $\mathfrak{g}$. An element α of Q is a **root** if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. The set of all roots is Δ, and $\alpha \in Q_+ \cap \Delta$ is **positive** (resp. $\alpha \in -Q_+ \cap \Delta$ is **negative**). Note the previous proposition implies $\Delta = \Delta_+ \sqcup \Delta_-$. The elements α_i of Π are **simple roots**, while the $\alpha_i^\vee$ are **simple coroots**.

$\mathfrak{g}(A)$ admits a "triangular decomposition" just as $\tilde{\mathfrak{g}}(A)$ does, where, if writing $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) as the subalgebra of $\mathfrak{g}$ generated by the e_i (resp. the f_i), $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. We say n is the **rank** of $\mathfrak{g}$ (and A).

Remark 6.1.6. This root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ yields a $Q-$gradation of $\mathfrak{g}$ with $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, and defining the **height** $ht(\alpha) = ht(c_1\alpha_1 + \ldots c_n\alpha_n) = \sum_{i=1}^{n} c_i$, gives a $\mathbb{Z}-$gradation as well.

Remark 6.1.7. The $-$ sign in the relation $[e_i, f_j] = -\delta_{ij}\alpha_i^\vee$ is known as the "Tits convention," and lets us remove minus signs later on (most immediately, it will allow removal of a minus in the "integrated" form of the Weyl group.

6.2 Integrable Modules

Given a module V for $\mathfrak{g}$ (equiv. a representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$) we say an element $x \in \mathfrak{g}$ is **locally finite** if for all $v \in V$ there is a finite dimensional $W \subset V$ containing v such that $x \cdot W \subset W$, and we say $x \in \mathfrak{g}$ is **locally nilpotent** on V if for each $v \in V$ there is some m such that $x^m \cdot v = 0$. V is **$\mathfrak{h}$-diagonalizable** if it may be written as $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where $V_\lambda := \{v \in V \mid h \cdot v = \lambda(h)v \quad \forall h \in \mathfrak{h}\}$. A λ for which V_λ isn't zero is called a **weight**, and V_λ is a **weight space**. An $\mathfrak{h}$-diagonalizable module V such that all $e_i, f_i \in \mathfrak{g}$ are locally nilpotent is **integrable**. It is important to note that $\mathfrak{g}$, as a $\mathfrak{g}$-module via the adjoint representation, is integrable (see [Mar18] Lemma 4.1).

One often uses the $\mathfrak{sl}_2$ theory in describing the action of $\mathfrak{g}$ on these modules, as $\mathfrak{g}_{(i)} \cong \mathbb{C}f_i \oplus \mathbb{C}\alpha_i^\vee \oplus \mathbb{C}e_i \cong \mathfrak{sl}_2$ (i.e. it is an "$\mathfrak{sl}_2$-triple"). Thus:

Proposition 6.2.1. ([Mar18] Prop. 4.3) Let V be an integrable module for $\mathfrak{g}(A)$, and fix $1 \leq i \leq n$.

1. As a $\mathfrak{g}_{(i)}$-module, V decomposes into a direct sum of finite-dimensional irreducible $\mathfrak{h}$-invariant submodules (i.e. $(\mathfrak{g}_{(i)} + \mathfrak{h})$ modules)
2. Fix a weight $\lambda \in \mathfrak{h}^*$ of V, set $M_\lambda := \{t \in \mathbb{Z} \mid V_{\lambda + m\alpha_i} \neq 0\}$, and $m_t := mult_V(\lambda + t\alpha_i) = dim(V_{\lambda + t\alpha_i})$.
   (a) There are $p, q \in \mathbb{N} \cup \{\infty\}$ such that M_λ is the closed interval $[-p, q]$ and $p - q = \lambda(\alpha_i^\vee)$ if $p, q < \infty$. If $mult_V(\lambda) < \infty$ then $p, q < \infty$. Lastly, $\lambda(\alpha_i^\vee) \in \mathbb{Z}$ always.
   (b) $e_i : V_{\lambda + t\alpha_i} \to V_{\lambda + (t+1)\alpha_i}$ is injective for $t \in [-p, t_0)$, where $t := -\frac{1}{2}\lambda(\alpha_i^\vee)$.
   (c) The assignment $t \mapsto m_t$ is symmetric about t_0 and increasing for $t < t_0$.
   (d) If λ and $\lambda + \alpha_i$ are weights then $e \cdot V_\lambda \neq 0$.

At this point we could construct something we could call Kac-Moody groups by "integrating" the actions of these $\mathfrak{sl}_2-$triples for integrable modules: that is, we can exponentiate $\pi_{(i)} = \pi|_{\mathfrak{g}_{(i)}}$ via $\exp(\pi(x)) := \mathrm{Id}_V + \frac{1}{1!}\pi(x) + \frac{1}{2!}\pi(x)^2 + \ldots$, but there would not be much we could do with it yet. As well, integrating the action of only the $\mathfrak{sl}_2-$triples indexed only by each simple root doesn't "integrate" the *full* Cartan subalgebra but instead only the derived subalgebra $\mathfrak{g}_A := [\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}^- \oplus \sum_{i=1}^{n} \mathbb{Z}\alpha_i^\vee \oplus \mathfrak{n}^+$. We will need the Weyl group and the "generators and relations" version of Kac-Moody groups for maximum breadth.

6.3 The Weyl Group

We define the **fundamental reflections** $r_i : \mathfrak{h}^* \to \mathfrak{h}^*$ as $r_i(\lambda) := \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i = \lambda - \lambda(\alpha_i^\vee)\alpha_i$. The **Weyl group** $W = W(A)$ is then the group $\langle r_i \mid 1 \leq i \leq n \rangle$. The **dual fundamental reflections** $r_i^\vee : \mathfrak{h} \to \mathfrak{h}$ as $r_i^\vee(h) = h - \alpha_i(h)\alpha_i^\vee$ generate an isomorphic group, and we will write r_i for both. Writing $w \in W$ as both $w = r_{i_1} \cdots r_{i_k}$ and $w = r_{i_1}^\vee \cdots r_{i_k}^\vee$, the bracket is preserved: $\langle w\lambda, wh \rangle = \langle \lambda, h \rangle$ for all $\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, $w \in W$. As well, if V is an integrable $\mathfrak{g}$-module, then $mult_V(\lambda) = mult_V(w\lambda)$ (and the set of weights of V is therefore W-invariant)- see [Mar18] Lemma 4.16. Thus the set of roots Δ of $\mathfrak{g}$ is Weyl-invariant also.

Given an integrable representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, the Weyl group of $\mathfrak{g}$ also admits an "integrated form" by defining

$$r_i^\pi := \exp(\pi(f_i)) \exp(\pi(e_i)) \exp(\pi(f_i)) \in GL(V)$$

Remark 6.3.1. The Tits convention here allows the removal of a minus sign from the middle term.

Proposition 6.3.2. ([Mar18] Prop. 4.18) These r_i^π have the following properties:

1. $r_i^\pi(V_\lambda) = V_{r_i(\lambda)}$
2. $(r_i^\pi)|_{\mathfrak{h}} = r_i^\vee \in GL(\mathfrak{h})$
3. $r_i^{\text{ad}} \in Aut(\mathfrak{g}(A))$ - rather, r_i^{ad} is a Lie algebra automorphism of $\mathfrak{g}$
4. $(r_i^\pi)^2(v) = (-1)^{\langle \lambda, \alpha_i^\vee \rangle} v$ for all $v \neq 0$ in V_λ.

The Weyl group is used to define the **real roots** Δ^{re} of $\mathfrak{g}$: these are the Weyl translates of the simple roots Π, i.e. $\Delta^{re} = W \cdot \Pi$. The set $\Delta^{im} = \Delta \setminus \Delta^{re}$ is then the set of **imaginary roots** of $\mathfrak{g}$. We can also define, for each $\alpha = w(\alpha_i) \in \Delta^{re}$, its associated coroot $\alpha^\vee = w(\alpha_i^\vee)$. This gives us a definition of $r_\alpha : \mathfrak{h}^* \to *$ by $r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, and this is in W since $r_\alpha = w r_i w^{-1}$.

Proposition 6.3.3. ([Mar18] Prop. 4.22) Set $S = \{r_i \mid 1 \leq i \leq n\}$. (W, S) is a Coxeter system, and the order $m_{ij} \in \mathbb{N} \cup \{\infty\}$ of $r_i r_j$ is determined by $a_{ij} a_{ji}$ in the following way:

$a_{ij}a_{ji}$	m_{ij}
0	2
1	3
2	4
3	6
≥ 4	∞

Lastly, we will later reference the following proposition on $W^* = W^{\mathrm{ad}*} = \langle r_i^{\mathrm{ad}} \mid 1 \leq i \leq n\rangle$.

Proposition 6.3.4. ([Mar18] Prop. 4.24)

1. There is a unique surjective group homomorphism $\nu : W^* toW$ such that $\nu(r_i^{\mathrm{ad}}) = r_i$ for all $1 \leq i \leq n$.

2. For all $w^* \in W^*$, and $1 \leq i \leq n$, the pair $E_\alpha := \{w^* e_i, -w^* e_i\}$ is determined by $\alpha := \nu(w^*)\alpha_i \in \Delta$, and it does not depend on decomposition of w^* or $\alpha = \nu(v^*)\alpha_j$ $(v^* \in W)$.

6.4 More on roots

We will also need a few basic facts about roots.

Proposition 6.4.1. ([Mar18] Prop. 6.2) Given a real root α of Δ,

1. $dim(\mathfrak{g}_\alpha) = 1$

2. $k\alpha$ is a root iff $k = \pm 1$

3. If $\beta \in \Delta$, then there exists $p, q \in \mathbb{N}$ such that $p - q = \langle \beta, \alpha^\vee \rangle$, such that $\beta + k\alpha \in \Delta \cup \{0\}$ iff $-p \leq k \leq q$ for $k \in \mathbb{Z}$

4. Assuming $\alpha \notin \pm\Pi$, there is an i such that $|ht(r_i(\alpha))| < |ht(\alpha)|$.

Note this third part is just the specialization of the earlier proposition to the adjoint representation (with the additional fact that p, q are finite).

It is also important for us to note that

Proposition 6.4.2. ([MP95] Section 5.2 Prop. 8) The sign of $\langle \alpha, \beta^\vee \rangle$ matches the sign of $\langle \beta, \alpha^\vee \rangle$: i.e. $\langle \alpha, \beta^\vee \rangle = 0 \iff \langle \beta, \alpha^\vee \rangle = 0$, $\langle \alpha, \beta^\vee \rangle \in -\mathbb{N} \iff \langle \beta, \alpha^\vee \rangle \in -\mathbb{N}$, and $\langle \alpha, \beta^\vee \rangle \in \mathbb{N} \iff \langle \beta, \alpha^\vee \rangle \in \mathbb{N}$

One can also prove some useful facts about imaginary roots (like that they do not occur if A is a regular Cartan matrix from the classic theory of semisimple Lie algebras, that they can and do achieve a multiplicity higher than 1, etc.) but we will only need one:

Proposition 6.4.3. ([Mar18] Prop. 6.6) The set Δ_+^{im} is $W-$invariant.

This will matter later when discussing prenilpotent sets of roots.

6.5 A $\mathbb{Z}$-form of the Universal Enveloping Algebra

Much like the construction of Chevalley groups, we next define a $\mathbb{Z}$-form of the universal enveloping algebra of $\mathfrak{g}$ (the construction of which will carry over to the way we actually end up defining Kac-Moody groups). Here a $\mathbb{Z}$-form means a $\mathbb{Z}$-submodule $\mathcal{U}_\mathbb{Z}$ of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$ such that $\phi : \mathcal{U}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C} \to \mathcal{U}_\mathbb{C}(\mathfrak{g})$ by $\phi(u \otimes c) = cu$ is an isomorphism. We note that $\mathfrak{h}' = \bigoplus_{i=1}^{n} \mathbb{C}\alpha_i^\vee$, the part of $\mathfrak{h}$ in the derived subalgebra, admits $\mathfrak{h}'_\mathbb{Z} := \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i^\vee$ as a $\mathbb{Z}$-form.

We make the following notational choices for $u \in \mathcal{U}_\mathbb{C}(\mathfrak{g})$:

$$u^{(s)} := \frac{u^s}{s!} \quad \text{as well as} \quad \binom{u}{s} := \frac{1}{s!}u(u-1)\cdots(u-s+1)$$

Definition 6.5.1. Let $\mathcal{U}^+$, $\mathcal{U}^-$, $\mathcal{U}'$ be the $\mathbb{Z}$-subalgebras of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$ generated by $e_i^{(s)}$, $f_i^{(s)}$, and $\binom{h}{s}$ (for $1 \leq i \leq n$, $s \in \mathbb{N}$, $h \in \mathfrak{h}'_\mathbb{Z}$) respectively. Define $\mathcal{U}$ as the subalgebra of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$ generated by these 3 subalgebras together. Lastly, set $\mathcal{U}_i := \sum_{s\in\mathbb{N}} \mathbb{Z}e_i^{(s)}$, $\mathcal{U}_{-i} := \sum_{s\in\mathbb{N}} \mathbb{Z}f_i^{(s)}$, and $\mathcal{U}^0_{(i)} = \sum_{s\in\mathbb{N}} \binom{\alpha_i^\vee}{s}$.

Proposition 6.5.2. ([Mar18] Prop. 7.4)

1. $\mathcal{U}^+$, $\mathcal{U}^-$, and $\mathcal{U}^0$ are $\mathbb{Z}$-forms of $\mathcal{U}_\mathbb{C}(\mathfrak{n}^+)$, $\mathcal{U}_\mathbb{C}(\mathfrak{n}^-)$, and $\mathcal{U}_\mathbb{C}(\mathfrak{h}')$, respectively.
2. The product maps $\mathcal{U}_i \otimes \mathcal{U}^0_{(i)} \otimes \mathcal{U}_{-i} \to \mathcal{U}_{(i)}$ and $\mathcal{U}^+ \otimes \mathcal{U}^0 \otimes \mathcal{U}^- \to \mathcal{U}$ are bijective.
3. $\mathcal{U}$ is a $\mathbb{Z}$-form of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$.

Remark 6.5.3. The bijectivity of the product maps from (2) uses the Poincaré-Birkhoff-Witt theorem.

These $\mathbb{Z}$-forms are what we use to define Kac-Moody algebras over arbitrary commutative rings R. We recall that $\mathfrak{g}$ injects into $\mathcal{U}_\mathbb{C}(\mathfrak{g})$ as a tensor algebra, and we can identify its image $i(\mathfrak{g})$ with $\mathfrak{g}$ if we switch from the tensor multiplication to commutators. Then setting $\mathfrak{g}_\mathbb{Z} := \mathfrak{g} \cap \mathcal{U}$, we set $\mathfrak{g}_R = \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} R$. All previously defined substructures of $\mathfrak{g}$ and $\mathcal{U}_\mathbb{C}(\mathfrak{g})$ thereby admit R-versions: $\mathcal{U}_R = \mathcal{U} \otimes R$, $\mathfrak{n}_R^\pm = (\mathfrak{n}^\pm \cap \mathcal{U}^\pm) \otimes R$, $\mathfrak{h}'_R = \mathfrak{h}'_\mathbb{Z} \otimes R$,

$\mathfrak{g}_{\alpha R} \coloneqq (\mathfrak{g}_\alpha \cap \mathfrak{g}_\mathbb{Z}) \otimes R$, etc. As well, $\mathfrak{g}_R$ admits triangular/root-space decompositions as before:

$$\mathfrak{g}_R = \mathfrak{n}_R^- \oplus \mathfrak{h}'_R \oplus \mathfrak{n}_R^+ = \mathfrak{h}'_R \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha R} \right)$$

6.6 Kac-Moody Root Data

With all of this under our belt, we change perspective to "Kac-Moody root data", which is a mild generalization of this theory with the fixed Cartan subalgebra size. The source for this information is again the presentation in [Mar18] (Ch.7.1-7.4).

Definition 6.6.1. A **Kac-Moody root datum** is a 5-tuple

$$\mathcal{D} = (I, A, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I}),$$

where I (of cardinality n) indexes the generalized Cartan matrix $A = (a_{ij})$, Λ is a free $\mathbb{Z}$-module of finite rank with dual $\Lambda^\vee$, and the $c_i \in \Lambda$ and $h_i \in \Lambda^\vee$ satisfy $\langle c_j, h_i \rangle = a_{ij}$ for all $i, j \in I$.

Note here the minimal rank for Λ and $\Lambda^\vee$ having linearly independent c_i and h_i (i.e. the minimal rank for $\mathcal{D}$ to be **free** and **cofree**) is $2n - \text{rank}(A)$.

Definition 6.6.2. A **Kac-Moody algebra of type** $\mathcal{D}$, say $\mathfrak{g}_\mathcal{D}$, as a free product of the abelian Lie algebra $\mathfrak{h} = \mathfrak{h}_\mathcal{D} = \Lambda^\vee \otimes_\mathbb{Z} \mathbb{C}$ with the free Lie algebra on the symbols e_i, f_i subject to the relations $[e_i, f_j] = -\delta_{ij} h_i$, $[h, e_i] = \langle c_i, h \rangle e_i$, $[h, f_i] = -\langle c_i, h \rangle f_i$, $(\text{ad}(e_i))^{1-a_{ij}} e_j = 0$, and $(\text{ad}(f_i))^{1-a_{ij}} f_j = 0$.

Note in the context of the Lie algebra we identify $\Lambda^\vee = \Lambda^\vee \otimes 1 \subset \mathfrak{h}$ and $\Lambda = \Lambda \otimes 1 \subset \mathfrak{h}^*$. This may seem like a broad definition, but one can recover the earlier definition of a Kac-Moody algebra here by picking an appropriate root datum. For example, if Λ and $\Lambda^\vee$ are rank $2n - l$ (where l is the rank of A) and the c_i and h_i are linearly independent, we can recover a realization (and thus the previous definition).

Remark 6.6.3. These $\mathfrak{g}_\mathcal{D}$ algebras are just a modification of $\mathfrak{g}(A)$ by adjusting $\mathfrak{h}$: in particular, $\mathfrak{g}_\mathcal{D} = \mathfrak{n}^- \oplus \mathfrak{h}_\mathcal{D} \oplus \mathfrak{n}^+$ where $\mathfrak{n}^\pm$ are isomorphic to the $\mathfrak{n}^\pm$ in $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ (see [Mar18] remark 7.16 and section 7.3.2). There may be some concern about collapsing root spaces from linearly dependent roots into each other if the c_i, h_i are linearly dependent, but $\mathfrak{g}_\mathcal{D}$ can be equipped with an "abstract Q-gradation" by defining the "abstract root lattice" $Q = \bigoplus_{1 \le i \le n} \mathbb{Z}\alpha_i$ (where the α_i are just symbols) and setting $\deg(e_i) = \alpha_i$, $\deg(f_i) = -\alpha_i$, and $\deg(\mathfrak{h})_\mathcal{D} = 0$. That is, $\mathfrak{g}_\alpha$ is now the $\mathbb{C}$-span of brackets

$[e_{i_1}, \ldots, e_{i_k}]$ if $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_k}$ (or a similar statement for negative roots α and the f_i) instead of an actual (full) $\mathfrak{h}_{\mathcal{D}}$-eigenspace (if $\mathcal{D}$ is not free). In particular, $\mathfrak{g}(A)$ and $\mathfrak{g}_{\mathcal{D}}$ (if A is the GCM in $\mathcal{D}$) admit a common extension in degree 0 by central extensions and semi-direct extensions (i.e. only $\mathfrak{h}$ and $\mathfrak{h}_{\mathcal{D}}$ are really "extended"). One can save some mental energy here by sticking only to $\mathcal{D}$ that are free and cofree, if desired.

The $\mathbb{Z}$-forms of $\mathfrak{g}_{\mathcal{D}}$ and $\mathcal{U}_{\mathbb{C}}(\mathfrak{g}_{\mathcal{D}})$ can be defined in exactly the same way as the previous subsection, potentially using abstract gradation if $\mathcal{D}$ is not free. Thus we obtain $R-$forms of each of these things (by simply tensoring with R over $\mathbb{Z}$).

6.7 Kac-Moody Groups

6.7.1 The Torus

Given a Kac-Moody root datum $\mathcal{D}$ and commutative ring R with unit group $R^\times$, we can define a split torus scheme by $\mathfrak{T}_\Lambda(R) = \mathrm{Hom}_{gr}(\Lambda, R^\times)$ (where we have group homomorphisms here). This group functor yields groups isomorphic to an m-fold product of the additive group $(R, +)$, and we can think of its elements as maps $r^h : \Lambda \to R^\times$ defined by $r^h(\lambda) := r^{\langle \lambda, h \rangle} = r^{\lambda(h)}$. This approach gets around the problem of integrating only the $\mathfrak{h}'_{\mathcal{D}}$ from using the $\mathfrak{sl}_2$-triples by simply "integrating" (though we've not actually written any exponentials here) $\mathfrak{h}_{\mathcal{D}}$ separately.

6.7.2 The Steinberg Group Functor

Next, we build the Kac-Moody version of the Steinberg group functor (as a generalization of the Steinberg group functor from Chevalley groups) following [Mar18] 7.4.3. Here when we refer to "real roots" Δ^{re}, we are speaking of the real roots of $\mathfrak{g}(A)$. For each real root $\alpha \in \Delta^{re}$, define $\mathfrak{U}_\alpha : \mathbb{Z}\text{-alg} \to \mathrm{Grp}$ as a copy of the group functor (each separate from each other) that sends each commutative ring to its additive group. We will name elements of $\mathfrak{U}_\alpha(R)$ as $x_\alpha(r)$, as we can designate an isomorphism $x_\alpha : (R, +) \to \mathfrak{U}_\alpha(R)$ by $r \mapsto x_\alpha(r)$.

We say a set of roots $\Psi \subset \Delta$ is **prenilpotent** if there exists a $v, w \in W$ such that $w\Psi \subset \Delta_+$, $v\Psi \subset \Delta_-$ (a set of roots which is prenilpotent in this sense is necessarily finite, as a consequence of the definition).

Proposition 6.7.1. If Ψ is prenilpotent, $\Psi \subset \Delta^{re}$.

Proposition 6.7.2. If $\{\alpha, \beta\}$ is a prenilpotent pair of roots, $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta$ is prenilpotent.

Note the first proposition follows from the fact the set of imaginary positive roots are $W-$invariant (Prop. 6.4.3), and the second follows from $\{\alpha,\beta\}$ being prenilpotent (apply the $w, v \in W$ such that $w\{\alpha,\beta\} \subset \Delta_+, v\{\alpha,\beta\} \subset \Delta_-$ to $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta$).

A set of roots Ψ is **closed** if $\alpha + \beta \in \Psi$ whenever $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Delta \cup \{0\}$. Ψ is **nilpotent** if it is closed and prenilpotent. Note $[\alpha,\beta]_{\mathbb{N}_0} := (\mathbb{N}_0\alpha + \mathbb{N}_0\beta) \cap \Delta$ and $(\alpha,\beta)_{\mathbb{N}_0} := [\alpha,\beta]_{\mathbb{N}_0} \setminus \{\alpha,\beta\} = (\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta)$ are both nilpotent sets of roots. The relevant fact here is that if Ψ is nilpotent, then $\mathfrak{g}_\Psi = \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha$ is a subalgebra of $\mathfrak{g}_\mathcal{D}$, and we can further define $\mathcal{U}_\Psi := \mathcal{U}_\mathcal{D} \cap \mathcal{U}_\mathbb{C}(\mathfrak{g}_\Psi)$ (where $\mathcal{U}_\mathcal{D}$ is the $\mathbb{Z}$-form of $\mathcal{U}_\mathbb{C}(\mathfrak{g}_\mathcal{D})$).

Proposition 6.7.3. ([Mar18] Prop 7.43, or [Rém02] 9.2.2 for a proof) Let $\{\alpha,\beta\}$ be a prenilpotent pair of roots, and put an arbitrary order on $(\alpha,\beta)_\mathbb{N}$. Then there exist integers $C_{ij}^{\alpha\beta}$ depending only on α,β and the chosen order such that, in the power series ring $\mathcal{U}_{[\alpha,\beta]_{\mathbb{N}_0}}[\![t,u]\!]$,

$$[\exp(te_\alpha), \exp(ue_\beta)] = \prod_\gamma \exp(t^i u^j C_{ij}^{\alpha\beta} e^\gamma)$$

where $\gamma = i\alpha + j\beta$ runs through the chosen order of $(\alpha,\beta)_\mathbb{N}$.

Definition 6.7.4. The **Steinberg functor** associated to A is the group functor $\mathfrak{St}_A$: $\mathbb{Z}$-alg $\to$ Grp constructed with $\mathfrak{St}_A(R)$ as the free product of the $\mathfrak{U}_\gamma(R)$ for $\gamma \in \Delta^{re}$ subject to the relations

$$[x_\alpha(t), x_\beta(u)] = \prod_\gamma x_\gamma(C_{ij}^{\alpha\beta} t^i u^j)$$

6.7.3 The Weyl Action

The last bit of structure needed to define our Kac-Moody groups is the action of the Weyl group on the key players. For this section we regard the Weyl group W of $\mathfrak{g}(A)$ as an abstract Coxeter group with fundamental reflections s_i. The actions of W on Λ and $\Lambda^\vee$ by $s_i(\lambda) = \lambda - \langle\lambda, h_i\rangle c_i$ and $s_i(h) = h - \langle c_i, h\rangle h_i$ induce an action of W on $\mathfrak{T}_\Lambda(R)$ by (if $t \in \mathrm{Hom}_{gr}(\Lambda, R^\times)$) letting $s_i(t)$ be the map such that $s_i(t)(\lambda) = t(s_i(\lambda))$ (i.e. precomposing with s_i). One can also write this as $s_i(r^h) = r^{s_i(h)}$, using the characterization of elements of $\mathfrak{T}_\Lambda$ coming the subsection constructing the torus.

There is an "integrated" version of the Weyl group for these $\mathfrak{g}_\mathcal{D}$ algebras, say $W^* = \langle r_i^{\mathrm{ad}} \mid 1 \le i \le n\rangle$ (where these r_i^{ad}s have the same definition as before - $\exp(\mathrm{ad}(f_i))\exp(\mathrm{ad}(e_i))\exp(\mathrm{ad}(f_i))$ - as e_i and f_i are still locally nilpotent on $\mathfrak{g}_\mathcal{D}$). Now, one can choose, for each $\alpha \in \Delta^{re}$, a pair $E_\alpha = \{\pm e_\alpha\} \subset (\mathfrak{g}_\mathcal{D})_\alpha$ such that for any $w^* \in W^*$ we have $w^* e_\alpha = \pm e_{w\alpha}$ (this uses the $\mathfrak{g}_\mathcal{D}$ version of Prop. 6.3.4). These are **double bases**, in the same sense as [Tit87].

We use these double bases to give the action of W^* on $\mathfrak{St}_A(R)$:

$$s_i^*(x_\gamma(r)) = x_{s_i(\gamma)}(\pm r)$$

where the $\pm$ is determined by the sign given in $s_i^*(e_\gamma) = \pm e_{s_i(\gamma)}$.

Lastly, we pick out and name the following elements of $\mathfrak{St}_A(R)$:

1. for $r \in R^\times$, $\tilde{s}_i(r) \coloneqq x_i(r)x_{-i}(r^{-1})x_i(r)$ (writing x_i for x_{α_i})
2. and $\tilde{s}_i \coloneqq \tilde{s}_i(1)$.

6.7.4 The Tits Functor of Type $\mathcal{D}$

Definition 6.7.5. The **Tits functor of type** $\mathcal{D}$ is the group functor $\mathfrak{G}_\mathcal{D} : \mathbb{Z}\text{−alg}\to\text{Grp}$ defined by, for each commutative ring R, $\mathfrak{G}_\mathcal{D}(R)$ as the free product of $\mathfrak{St}_A(R) * \mathfrak{T}_\Lambda(R)$ subject to the relations (for $1 \le i \le n$, $r \in R$, $t \in \mathfrak{T}_\Lambda(R)$

$$t \cdot x_i(r) \cdot t^{-1} = x_i(t(c_i)r)$$

$$\tilde{s}_i \cdot t \cdot \tilde{s}_i^{-1} = s_i(t)$$

$$\tilde{s}_i(r^{-1}) = \tilde{s}_i \cdot r^{h_i} \quad (\text{for } r \in R^\times)$$

$$\tilde{s}_i \cdot x_\gamma(r) \cdot \tilde{s}_i^{-1} = s_i^*(x_\gamma(r)) \quad (\gamma \in \Delta^{re})$$

Some authors restrict what they call "Kac-Moody groups" to values of this functor over fields, but as our result in the next chapter will hold over commutative rings, we'll loosen this definition up to include all groups of this form.

6.7.5 Actions on Modules

While we don't particularly need them, the other "exponentiated Lie algebra action" versions of Kac-Moody groups are quotients of this $\mathfrak{G}_\mathcal{D}$. With a suitable refinement of the definition of an "integrable" module $\pi : \mathfrak{g}_\mathcal{D} \to \mathfrak{gl}(V)$ accounting for the potential for root spaces (and thus potentially weight spaces) to combine under action by $\mathfrak{g}_\mathcal{D}$, one can construct $\exp(\pi(e_i))$, etc. and all of the relations used in the above construction of $\mathfrak{G}_\mathcal{D}$ hold in these groups. In particular, one can define a representation $\hat{\pi}_R : \mathfrak{G}_\mathcal{D}(R) \to \mathrm{GL}(V_R)$ (where $V_R = V_\mathbb{Z} \otimes R$ and $V_\mathbb{Z}$ is a $\mathbb{Z}$-form stabilized by the $\mathbb{Z}$-form on $\mathcal{U}_\mathbb{C}(\mathfrak{g}_\mathcal{D})$, as in the construction of Chevalley groups) that surjects onto these exponentiated versions in $GL(V_R)$ (see Theorem 7.48 from [Mar18]).

6.7.6 Kac-Moody Groups and RGD Systems

Lastly, we make the link to Chapter 4 definition 4.2.2:

Theorem 6.7.6. ([Mar18] Thm. 7.69) $\mathfrak{G}_{\mathcal{D}}(\mathbb{K})$, for $\mathbb{K}$ a field, has an RGD system of type (W, S) via $(\mathfrak{G}_{\mathcal{D}}(\mathbb{K}), (\mathfrak{U}_\alpha(\mathbb{K}))_{\alpha\in\Delta^{re}}, \mathfrak{T}_\Lambda(\mathbb{K}))$.

Note there are some mildly differing definitions of "prenilpotent" here (between the Kac-Moody and the Coxeter definition), but upon identifying α, β with their images in $\Phi = \Delta^{re}$ (done using half-spaces in the Tits cone: see [Mar18] section 4.3 for a definition, or generally any book on Kac-Moody algebras like [Kac90]), one gets

$$w\{\alpha, \beta\} \subset \Phi_+ \iff 1 \in w\alpha, w\beta \iff w^{-1} \in \alpha \cap \beta$$

$$v\{\alpha, \beta\} \subset \Phi_- \iff 1 \notin v\alpha, v\beta \iff v^{-1} \notin \alpha, \beta \iff v^{-1} \in (-\alpha) \cap (-\beta).$$

This matches the "geometric" definition of prenilpotence given earlier. (This also uses that "positive" in the Kac-Moody sense agrees with "positive" in the Coxeter sense if we choose our positive set of geometric roots to be those containing $1 \in \Sigma(W, S)$.)

As we now know these Kac-Moody groups have RGD systems, we now have a wealth of examples of twin buildings of any type (W, S) such that the relations between generators are $m_{ij} = 2, 3, 4, 6$ and/or ∞ here (by the condition that W be the Weyl group of a Kac-Moody algebra).

6.8 Reduction to Rank 2

The last thing we do in this chapter is give an argument for when we can "reduce to rank 2," i.e. when $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Delta$ really is the root system of a rank 2 Kac-Moody algebra. Arguments using this have been in use for decades, likely relying on [MP95] Section 5.7 Thm. 1, which uses "sets of root data" - a more general notion that allows infinite bases; that there exists an actual Kac-Moody algebra of rank 2 having the desired root system has not been shown directly. We use the machinery of [MP95], as it is well-suited to the task.

Let A be a generalized Cartan matrix. We have a Kac-Moody Lie algebra $\mathfrak{g}(A)$ over $\mathbb{C}$ with roots Δ, where Δ_{re} and Δ_{im} are $\mathfrak{g}(A)$'s real and imaginary roots, respectively. $\mathfrak{g}(A)$ has a Weyl group W, where r_α will indicate the reflection about the real root α. Also, if $\alpha, \beta \in \Delta_{re}$ we write $\Delta^+(\alpha,\beta) = (\mathbb{N}_0\alpha + \mathbb{N}_0\beta) \cap \Delta_{re}$ and $\Delta' = (\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Delta_{re}$. Lastly, we state a lemma for later use:

Lemma 6.8.1. ([MP95] Ch.6.3 Lemma 3(i)) Let $\alpha, \beta \in \Delta_{re}$ such that $\langle \alpha, \beta^\vee \rangle \geq 0$. Then $\Delta^+(\alpha,\beta) \subset \{\alpha, \beta, \alpha+\beta\}$.

We start by showing

Proposition 6.8.2. Let $\alpha, \beta \in \Delta_{re}$ such that $\beta - \alpha \notin \Delta_{re}$ ($\implies \beta - \alpha \notin \Delta$) and $\langle \alpha, \beta^\vee \rangle \leq 0$. Then $\Delta' = \langle r_\alpha, r_\beta \rangle \{\alpha, \beta\}$.

Proof. Note $\langle \alpha, \beta^\vee \rangle \leq 0 \implies \langle \beta, \alpha^\vee \rangle \leq 0$ (Prop. 6.4.2). We first show $\Delta' = \Delta^+(\alpha,\beta) \cup \Delta^+(-\alpha,-\beta)$. To this end, we apply the above lemma to pairs $\{\alpha, -\beta\}$ and $\{-\alpha, \beta\}$. We see no root $\gamma = m\alpha + n\beta \in \Delta'$ can have mixed signs, i.e. $m \geq 0 \iff n \geq 0$ and $m \leq 0 \iff n \leq 0$.

Now we show $\Delta^+(\alpha,\beta) \subset \langle r_\alpha, r_\beta \rangle \{\alpha, \beta\}$. This is sufficient to prove the forward containment since $\gamma \in \langle r_\alpha, r_\beta \rangle \{\alpha, \beta\} \implies r_\gamma \in \langle r_\alpha, r_\beta \rangle \implies -\gamma \in \langle r_\alpha, r_\beta \rangle \{\alpha, \beta\}$. So if $\gamma = m\alpha + n\beta \in \Delta^+(\alpha,\beta)$ with $m, n \geq 0$, we induct on $m+n$. For our base case $m+n = 1$, we see either $m = 1$ or $n = 1 \implies \gamma \in \{\alpha, \beta\}$ and $\gamma \in \langle r_\alpha, r_\beta \rangle \{\alpha, \beta\}$.

Assume that for all real roots $\gamma' = m'\alpha + n'\beta \in \Delta^+(\alpha,\beta)$ with $m'+n' < m+n$ we have $\gamma' \in \langle r_\alpha, r_\beta \rangle \{\alpha, \beta\}$. We will show $\gamma = m\alpha + n\beta$ is Weyl-conjugate to such a root γ'. Observe $r_\alpha(\gamma) = (-m - n\langle \beta, \alpha^\vee \rangle)\alpha + n\beta$ and $r_\beta(\gamma) = m\alpha + (-n - m\langle \alpha, \beta^\vee \rangle)\beta$. Note since $m, n \geq 0$ we have $-n - m\langle \alpha, \beta^\vee \rangle, -m - n\langle \beta, \alpha^\vee \rangle \geq 0$. For the sake of contradicton, suppose $-m - n\langle \beta, \alpha^\vee \rangle \geq m$ and $-n - m\langle \alpha, \beta^\vee \rangle \geq n$ (else $r_\alpha(\gamma)$ or $r_\beta(\gamma)$ satisfies the induction hypothesis and we're done). In particular, $-n\langle \beta, \alpha^\vee \rangle \geq 2m$ and $-m\langle \alpha, \beta^\vee \rangle \geq 2n$. Using this, we see $\langle \gamma, \alpha^\vee \rangle = \langle m\alpha + n\beta, \alpha^\vee \rangle = m\langle \alpha, \alpha^\vee \rangle + n\langle \beta, \alpha^\vee \rangle = 2m + n\langle \beta, \alpha^\vee \rangle \leq -n\langle \beta, \alpha^\vee \rangle + n\langle \beta, \alpha^\vee \rangle = 0$. In a similar fashion we also observe $\langle \gamma, \beta^\vee \rangle \leq 0$.

Since γ is a real root, we know $2 = \langle \gamma, \gamma^\vee \rangle = \langle m\alpha + n\beta, \gamma^\vee \rangle = m\langle \alpha, \gamma^\vee \rangle + n\langle \beta, \gamma^\vee \rangle$. Our computations above show $\langle \gamma, \alpha^\vee \rangle, \langle \gamma, \beta^\vee \rangle \leq 0$, so $\langle \alpha, \gamma^\vee \rangle, \langle \beta, \gamma^\vee \rangle \leq 0$. This is a contradiction: we thus have at least one of $-m - n\langle \beta, \alpha^\vee \rangle < m$, $-n - m\langle \alpha, \beta^\vee \rangle < n$ and so $\Delta' \subset \langle r_\alpha, r_\beta \rangle \{\alpha, \beta\}$.

To show the other containment, we simply observe that r_α or r_β applied to a $\mathbb{Z}-$linear combination of α and β is a $\mathbb{Z}-$linear combination of α and β. Thus $\langle r_\alpha, r_\beta \rangle \{\alpha, \beta\} \subset \Delta'$. $\square$

The set $\{\alpha, \beta\}$ is actually a base for this subroot system Δ': each root is a $\mathbb{Z}-$linear combination of α, β (with strictly nonnegative or nonpositive coefficients) and also Weyl-conjugate to at least one of α, β. We next construct a rank 2 Kac-Moody algebra that has Δ' as its root system. To do this, we will need to accumulate several definitions, the first of which is a generalization of the "realization" introduced by Kac. The following definitions are from [MP95] section 4.2.

Definition 6.8.3. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ be sets and $V^\vee$ be a finite dimensional vector space over our field (with V as a dual space). We call a triple $R = (V^\vee, \Pi, \Pi^\vee)$ a **realization** of a GCM A if there are embeddings $\Pi \hookrightarrow V$, $\Pi^\vee \hookrightarrow V^\vee$ such that Π, $\Pi^\vee$ are independent in V, $V^\vee$ and $\langle \alpha_i, \alpha_j^\vee \rangle = A_{ji}$. When we say "a realization" we will mean to fix a pair of maps $\Pi \hookrightarrow V$, $\Pi^\vee \hookrightarrow V^\vee$.

The only difference between this definition and Kac's is that there is no upper bound on the dimension of V or $V^\vee$. The requirement that $\Pi, \Pi^\vee$ be independent still forces $\dim V \geq 2n - \text{rank } A$, and (as per Kac's construction) there exists a realization fitting the above definition with dimension $2n - \text{rank } A$. Note also that the Cartan subalgebra $V^\vee$ of $\mathfrak{g}(A)$ yields a realization for the GCM

$$B = \begin{bmatrix} 2 & \langle \beta, \alpha^\vee \rangle \\ \langle \alpha, \beta^\vee \rangle & 2 \end{bmatrix}$$

via the natural inclusions of $\alpha^\vee, \beta^\vee, \alpha$, and β.

Given any realization R of a GCM A we may construct several Lie algebras from generators and relations.

Definition 6.8.4. Let $R = (V, \Pi, \Pi^\vee)$ be a realization of a generalized Cartan matrix A, where $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. Let $\mathfrak{e}$ be the free Lie algebra on the set $\{e_i, f_i | 1 \leq i \leq n\}$ and give $V^\vee$ an abelian Lie algebra structure. Define the Lie algebra $\mathfrak{u}(A, R)$ as the quotient of $\mathfrak{e} * V^\vee$ by the ideal generated by all elements of the form

$$[e_i, f_j] - \delta_{i,j} \alpha_i^\vee$$

$$[h, e_i] - \langle \alpha_i, h^\vee \rangle e_i$$

$$[h, f_i] + \langle \alpha_i, h^\vee \rangle f_i$$

We obtain two more algebras of interest from this $\mathfrak{u}(A, R)$: quotienting out by the Serre relations

$$(\text{ad} e_j)^{-A_{i,j}+1} e_i = 0, \;\; i \neq j$$

$$(\text{ad} f_j)^{-A_{i,j}+1} f_i = 0, \;\; i \neq j$$

yields a new Lie algebra $\mathfrak{g}^{\dagger}(A,R)$, and quotienting by the maximal ideal of $\mathfrak{u}(A,R)$ trivially intersecting $V^{\vee}$ (the radical of $\mathfrak{u}(A,R)$) yields the algebra $\mathfrak{g}(A,R)$.

Remark 6.8.5. $\mathfrak{g}(A,R)$ is the usual Kac-Moody algebra $\mathfrak{g}(A)$ in the case that R has dimension $2n - \text{rank } A$. This is a **minimal** realization associated to A. Note also the Tits convention is not used here.

Remark 6.8.6. The elements $(\text{ad}e_j)^{-A_{i,j}+1}e_i$, $(\text{ad}f_j)^{-A_{i,j}+1}f_i$ are actually elements of $\text{rad}(\mathfrak{u}(A,R))$, so we have a series of surjective homomorphisms

$$\mathfrak{u}(A,R) \twoheadrightarrow \mathfrak{g}^{\dagger}(A,R) \twoheadrightarrow \mathfrak{g}(A,R)$$

Note also that $V^{\vee}$ is a Cartan subalgebra in each one of these algebras.

We will also need the notion of an "embedding" of realizations. This will let us define a bijective map ϕ between the root system of a rank 2 Kac-Moody algebra and Δ' such that the bracket is preserved (i.e. $\langle \alpha, \beta^{\vee} \rangle = \langle \phi(\alpha), (\phi(\beta))^{\vee} \rangle$).

Definition 6.8.7. Let $R = (V, \Pi, \Pi^{\vee}), R' = (V', \Pi, \Pi'^{\vee})$ be realizations of A and A' where A is a submatrix of A'. Index Π by a set J, Π' by a set J' (so that $J \subseteq J'$). An **embedding** of R in R' is a pair of injective homomorphisms $\eta = (\eta_L, \eta_R)$ such that

$$\eta_L : V \to V', \quad \eta_R : V^{\vee} \to V'^{\vee}$$

$$\eta_L(\Pi) \subseteq \Pi' \text{ and } \eta_L(\alpha_i) = \alpha_i' \quad \forall i \in J$$

$$\eta_R(\Pi^{\vee}) \subseteq \Pi'^{\vee} \text{ and } \eta_R(\alpha_i^{\vee}) = \alpha_i'^{\vee} \quad \forall i \in J$$

$$\forall \alpha \in V, v^{\vee} \in V^{\vee}, \ \langle \eta_L(\alpha), \eta_R(v^{\vee}) \rangle = \langle \alpha, v^{\vee} \rangle$$

Note the leftmost $\langle \cdot, \cdot \rangle$ is a pairing of V' with $V'^{\vee}$ and the rightmost pairs V with $V^{\vee}$.

Now we are in a position to construct a useful embedding of realizations. Recall we have a GCM

$$B = \begin{bmatrix} 2 & \langle \beta, \alpha^{\vee} \rangle \\ \langle \alpha, \beta^{\vee} \rangle & 2 \end{bmatrix}$$

associated to the root system $\Delta' = (\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Delta_{re} \subset V$. We first construct a minimal realization for B with dimension depending on $\text{rank}(B)$. The method of construction will make the embedding clearer.

Case 1: rank(B) = 2. In the case that B has full rank, we pick the subspace $\mathbb{C}\alpha^{\vee} \oplus \mathbb{C}\beta^{\vee}$ of $V^{\vee}$ to be the space for our realization. We define maps $\{\alpha^{\vee}, \beta^{\vee}\} \hookrightarrow \mathbb{C}\alpha^{\vee} \oplus \mathbb{C}\beta^{\vee}$ by $\alpha^{\vee} \mapsto \alpha^{\vee}, \beta^{\vee} \mapsto \beta^{\vee}$ and $\{\alpha, \beta\} \to (\mathbb{C}\alpha^{\vee} \oplus \mathbb{C}\beta^{\vee})^*$ by $\alpha \mapsto res(\alpha), \beta \mapsto res(\beta)$ (where $res(\alpha), res(\beta)$ are the restrictions of α and β to $\mathbb{C}\alpha^{\vee} \oplus \mathbb{C}\beta^{\vee}$). Since the columns of B

are linearly independent, $res(\alpha)$ and $res(\beta)$ are independent in $(\mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee)^*$. As well, $\alpha^\vee$ and $\beta^\vee$ are independent in $\mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee$ since $\alpha \neq \pm\beta$. Lastly, our pairings between $\alpha, \beta, \alpha^\vee, \beta^\vee$ are entries in B by construction, and $(\mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee, \{\alpha, \beta\}, \{\alpha^\vee, \beta^\vee\})$ is indeed a realization.

Case 2: rank(B) = 1. In the affine case, we need a slightly larger realization. The linear functionals α and β are not independent on $\mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee$, so we must pick a third vector $\gamma^\vee$ in $V^\vee$ so that α, β are linearly independent on $\mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee \oplus \mathbb{C}\gamma^\vee$. Extend $\{\alpha, \beta\}$ to a basis for V and $\{\alpha^\vee, \beta^\vee\}$ to a basis for $V^\vee$. Consider the matrix for the pairing $\langle \cdot, \cdot \rangle$ on $V \times V^\vee$ corresponding to these choices of basis. Since α and β are independent in V their columns in this matrix are independent. As such, there exists a $\gamma^\vee \in V^\vee$ such that the columns

$$\begin{bmatrix} \langle \alpha, \alpha^\vee \rangle \\ \langle \alpha, \beta^\vee \rangle \\ \langle \alpha, \gamma^\vee \rangle \end{bmatrix}, \quad \begin{bmatrix} \langle \beta, \alpha^\vee \rangle \\ \langle \beta, \beta^\vee \rangle \\ \langle \beta, \gamma^\vee \rangle \end{bmatrix}$$

are independent ($\gamma^\vee$ may be chosen as a member of the above basis for $V^\vee$). Thus there exists a $\gamma^\vee \in V^\vee$ such that the restrictions of α and β to $\mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee \oplus \mathbb{C}\gamma^\vee$ are independent. Now we may use the same sorts of maps $\{\alpha^\vee, \beta^\vee\} \hookrightarrow \mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee \oplus \mathbb{C}\gamma^\vee$ and $\{\alpha, \beta\} \hookrightarrow (\mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee \oplus \mathbb{C}\gamma^\vee)^*$ as before. $(\mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee \oplus \mathbb{C}\gamma^\vee, \{\alpha, \beta\}, \{\alpha^\vee, \beta^\vee\})$ is hence a realization for B.

Call the realization constructed above R. We have a realization $R' = (V^\vee, \{\alpha, \beta\}, \{\alpha^\vee, \beta^\vee\})$ (as noted earlier) from identifying α, β as elements of V and $\alpha^\vee, \beta^\vee$ as elements of $V^\vee$.

Lemma 6.8.8. There is an embedding $\eta = (\eta_L, \eta_R)$ of realizations $R \to R'$.

Proof. We again have two cases, depending on the rank of B. If B has full rank, we simply pick $\eta_R(\alpha^\vee) = \alpha^\vee, \eta_R(\beta^\vee) = \beta^\vee$ and $\eta_L(res(\alpha)) = \alpha$, $\eta_L(res(\beta)) = \beta$. Injectivity of both maps is clear, and since we constructed R directly from R' using restrictions of α and β, these maps preserve the pairing $\langle \cdot, \cdot \rangle$. η is then an embedding of realizations.

If B is of rank 1, we define the map η_R by the same idea as before: $\alpha^\vee \mapsto \alpha^\vee$, $\beta^\vee \mapsto \beta^\vee$, and $\gamma^\vee \mapsto \gamma^\vee$. This is injective. To define η_L, we note the map $res : V \to (\mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee \oplus \mathbb{C}\gamma^\vee)^*$ is surjective, and hence there is a $\zeta \in V$ such that $\mathbb{C}res(\alpha) \oplus \mathbb{C}res(\beta) \oplus \mathbb{C}res(\zeta) = (\mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee \oplus \mathbb{C}\gamma^\vee)^*$. We then define $\eta_L : (\mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee \oplus \mathbb{C}\gamma^\vee)^* \to V$ by $\eta_L(res(\alpha)) = \alpha, \eta_L(res(\beta)) = \beta$, and $\eta_L(res(\zeta)) = \zeta$. This is injective since $res \circ \eta_L$ is the identity map. Finally, since we've picked the images of a basis $\{res(\alpha), res(\beta), res(\zeta)\}$ under η_L to be choices of lifts in V, we have $\langle \eta_L(\delta), \eta_R(x^\vee) \rangle = \langle \delta, x^\vee \rangle$ for all $\delta \in (\mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee \oplus \mathbb{C}\gamma^\vee)^*$, $x^\vee \in \mathbb{C}\alpha^\vee \oplus \mathbb{C}\beta^\vee \oplus \mathbb{C}\gamma^\vee$. So we have an embedding in this case as well. $\square$

Now that we have an embedding $R \to R'$, we can use the earlier-introduced algebras to show the intended result. We will need to record just two more results beforehand.

Proposition 6.8.9. ([MP95] p.344) Let R be a realization of A, R' a realization of A'. Denote the α root space of $\mathfrak{u}(A, R)$ by u_α. An embedding of realizations $R \to R'$ yields an embedding of Lie algebras $\eta : \mathfrak{u}(R, A) \to \mathfrak{u}(R', A')$ such that

$$\text{i)} \quad \eta(V^\vee) = \eta_R(V^\vee) \subset V'^\vee$$

$$\text{ii)} \quad \eta(e_i) = e'_i, \eta(f_i) = f_i$$

$$\text{iii)} \quad \eta(\mathfrak{u}_\alpha) = \mathfrak{u}_{\eta_L(\alpha)}$$

Proposition 6.8.10. ([MP95] p. 345) Suppose R and R' are realizations of the generalized Cartan matrices A and A', and let $\eta = (\eta_L, \eta_R)$ be an embedding of realizations $R \to R'$. Let η be the corresponding embedding of Lie algebras $\mathfrak{u}(A, R) \to \mathfrak{u}(A', R')$. Write $\mathfrak{u}' = \mathfrak{u}(A', R')$, and $Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$. Then

$$\text{i)} \quad \eta(\text{rad}(\mathfrak{u})) = \text{rad}(\mathfrak{u}') \cap \eta(\mathfrak{u})$$

ii) *the induced homomorphism* $\mathfrak{g}(A, R) \to \mathfrak{g}(A', R')$ *is injective*

iii) $\eta_L(\Delta) = \Delta' \cap \eta_L(Q)$, *where* Δ *and* Δ' *are the root systems of* $\mathfrak{g}(A, R)$ *and* $\mathfrak{g}(A', R')$, *respectively*

iv) $\eta_L(\Delta_{re}) = \Delta'_{re} \cap \eta_L(Q)$ *and* $\eta_L(\Delta_{im}) = \Delta'_{im} \cap \eta_L(Q)$

We are finally ready to show $\Delta' = (\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Delta_{re}$ are the real roots of a rank 2 Kac-Moody algebra.

Proposition 6.8.11. Let α, β be real roots of a Kac-Moody algebra $\mathfrak{g}(A)$, $\alpha \neq \pm\beta$, such that $\beta - \alpha \notin \Delta_{re}$ and $\langle \beta, \alpha^\vee \rangle \leq 0$. Define

$$B = \begin{bmatrix} 2 & \langle \beta, \alpha^\vee \rangle \\ \langle \alpha, \beta^\vee \rangle & 2 \end{bmatrix}$$

with associated Kac-Moody algebra $\mathfrak{g}(B)$ (which has real roots $\dot{\Delta}$). Then there is a bijection $\phi : \dot{\Delta} \to (\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Delta_{re}$ such that $\langle \delta, \zeta^\vee \rangle = \langle \phi(\delta), (\phi(\zeta))^\vee \rangle$.

Proof. We have already constructed a realization R of minimal dimension (either 2 or 3) with an embedding $\eta = (\eta_L, \eta_R)$ from R to R' where R' is the realization of B derived from $\mathfrak{g}(A)$. Recall that any two choices of (minimal) realizations for a generalized Cartan matrix yield isomorphic Kac-Moody algebras, so we may regard $\mathfrak{g}(B)$ as $\mathfrak{g}(B, R)$. Next, note $\mathfrak{g}(B, R')$ has α, β as simple roots (by construction) and $V^\vee$ as a Cartan subalgebra. Since its real roots are $\langle r_\alpha, r_\beta \rangle$-translates of the roots α, β in V, we can view these real roots as $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Delta_{re}$. Next, define $\phi = \eta_L|_{\dot{\Delta}}$. Then by proposition 6.8.10 iv), we know

$$\eta_L(\dot{\Delta}) = \Delta' \cap \eta_L(Q)$$

but since $\Delta' \subset \eta_L(Q)$ (we are applying proposition 6.8.10 in the case $A = A'$) we automatically get

$$\eta_L(\dot{\Delta}) = \Delta'$$

. Since η_L is injective we then see it is a bijection between the two. Lastly, since our map ϕ is a restriction of η_L, we know ϕ preserves brackets, as desired. □

Chapter 7

Necessary and Sufficient Conditions for Commutation of Certain Root Groups in Kac-Moody Groups

7.1 RGD vs. Kac-Moody Commutators

We now turn our attention to the commutator relations in Kac-Moody groups. In what follows, we consider a Kac-Moody group $\mathcal{G}_{\mathcal{D}}(R)$ (where $\mathcal{D} = \mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ is a Kac-Moody root datum and R is a commutative ring). As a subclass of groups with an RGD system (when $R = \mathbb{K}$ is a field), the additional structure allows us to make a classification statement about when certain commutator relations occur in these groups (though the statement holds over commutative rings in general).

Recall that Kac-Moody groups have a stronger version of **(RGD1)**. For **(RGD1)** in its most general sense, the best we can do is to observe that $[\alpha, \beta] = \{\gamma \mid \alpha \subset \gamma \subset \beta\}$, i.e.

$$[U_\alpha, U_\beta] \subseteq \langle U_\gamma \mid \alpha \subsetneq \gamma \subsetneq \beta \rangle.$$

In the case of Kac-Moody groups, we obtain

$$[U_\alpha, U_\beta] \subseteq \prod_{i,j\geq 1} U_{i\alpha+j\beta},$$

where by convention $U_\gamma = 1$ if γ is not a root. We have the following explicit calculation of a commutator relation (at least one for any prenilpotent $\{\alpha, \beta\}$) used in the construction of the Steinberg group functor:

Theorem 7.1.1. [Mor87] For a prenilpotent pair $\{\alpha, \beta\}$, write $Q_{\alpha\beta} = (\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta$ (necessarily contained in Δ^{re} by Props. 6.7.2 & 6.7.1). Then one of the following 5 cases occurs:

1. if $Q_{\alpha\beta} = \varnothing$ then $[x_\alpha(t), x_\beta(u)] = 1$.

2. if $Q_{\alpha\beta} = \{\alpha + \beta\}$ then $[x_\alpha(t), x_\beta(u)] = x_{\alpha+\beta}(\pm(p+1)tu)$, where p is the integer such that $S(\alpha, \beta) = \{\beta - p\alpha, \ldots, \beta, \ldots, \beta + q\alpha\}$.

3. if $Q_{\alpha\beta} = \{\alpha + \beta, \beta + 2\alpha\}$ then $[x_\alpha(t), x_\beta(u)] = x_{\alpha+\beta}(\pm tu) x_{\beta+2\alpha}(\pm t^2 u)$.

4. if $Q_{\alpha\beta} = \{\alpha + \beta, \beta + 2\alpha, 2\beta + \alpha\}$ then $[x_\alpha(t), x_\beta(u)] =$
   $x_{\alpha+\beta}(\pm 2tu) x_{\beta+2\alpha}(\pm 3t^2 u) x_{2\beta+\alpha}(\pm 3tu^2)$.

5. if $Q_{\alpha\beta} = \{\alpha + \beta, \beta + 2\alpha, \beta + 3\alpha, 3\beta + 2\alpha\}$ then $[x_\alpha(t), x_\beta(u)] =$
   $x_{\alpha+\beta}(\pm tu) x_{\beta+2\alpha}(\pm t^2 u) x_{\beta+3\alpha}(\pm t^3 u) x_{2\beta+3\alpha}(\pm 2t^3 u^2)$.

Note there is some minor ambiguity in the sign here, but beyond that, the arguments on the righthand-side can be determined in the order given. Recall also that prenilpotence admits a meaning in terms of the root system and Weyl group in the Kac-Moody situation: we say a pair $\{\alpha, \beta\}$ ($\alpha \neq \beta$) is prenilpotent if $\exists w, v \in W$ such that $w(\alpha), w(\beta) \in \Delta^+$ and $v(\alpha), v(\beta) \in \Delta^-$. We will later see that if $\{\alpha, \beta\}$ is a set of prenilpotent roots with $|r_\alpha r_\beta| = \infty$, then $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta \subseteq \{\alpha + \beta\}$. Under these circumstances, the above statement becomes

$$[U_\alpha, U_\beta] \subset U_{\alpha+\beta}.$$

Thus we become concerned with when, exactly, $U_{\alpha+\beta} = \{1\}$.

Remark 7.1.2. It is important to note that the condition $|r_\alpha r_\beta| = \infty$, after identifying the Weyl group of the Kac-Moody algebra with the Coxeter group of the associated Coxeter complex, means exactly that the "geometric" versions of the roots α and β in $\Sigma(W, S)$ are nested (see Subsection 3.2.2). The additional fact that the geometric roots

α, β are prenilpotent rules out the possibility that $\{\alpha, -\beta\}$ is nested (as necessarily $\alpha \cap \beta$ contains a chamber).

Example 1 below illustrates how these commutator formulas are “stronger” for Kac-Moody groups.

Example 7.1.3. Suppose we have roots α, β such that α is properly contained in β. To be more concrete, we may use the Kac-Moody algebra associated to

$$A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix},$$

with $W = D_\infty$, $\alpha = -r_0 r_1(\alpha_0)$, and $\beta = \alpha_1$ (where α_0 and α_1 are both simple roots).

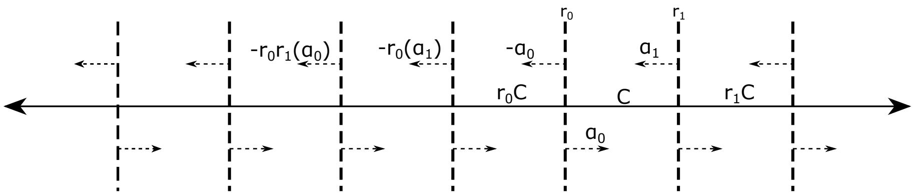


FIGURE 7.1: D_∞ Coxeter complex

Above we have a real line as the $(D_\infty, \{r_0, r_1\})$ Coxeter complex with fundamental chamber C; vertical dotted lines indicate a panel on the real line while horizontal dotted lines indicate a root (the half-line below). We compare the different commutator formulas listed above. By Figure 7.1,

$$[U_\alpha, U_\beta] \subseteq \langle U_{-r_0(\alpha_1)}, U_{-\alpha_0} \rangle = \langle U_{-\alpha_1-\alpha_0}, U_{-\alpha_0} \rangle,$$

noting our fixed GCM A lets us compute $-r_0(\alpha_1)$ explicitly. The Kac-Moody root system version nets us a bit more than this RGD system commutator formula. With it, we obtain also

$$[U_\alpha, U_\beta] \subseteq U_{\alpha+\beta} = U_{-(3\alpha_0+4\alpha_1)+\alpha_1} = U_{-3(\alpha_0+\alpha_1)}.$$

Note $\alpha_0 + \alpha_1$ is a real root (since $\langle \alpha_1, \alpha_0^\vee \rangle = -1$). Then we can already conclude $-3(\alpha_0 + \alpha_1)$ is not a root and therefore U_α and U_β commute.

7.2 Root Strings and Morita Pairs

In the end, existence of a prenilpotent pair of roots $\{\alpha, \beta\}$ such that $|r_\alpha r_\beta| = \infty$ and $[U_\alpha, U_\beta] \neq 1$ in an associated Kac-Moody group will be equivalent to existence of a special collection of simple roots satisfying nice properties, which is the aim of this work. To rephrase, we will describe an easily checkable condition for the generalized Cartan matrix that describes exactly when a pair of roots $\{\alpha, \beta\}$ as above occurs. We now cite a proposition which will be integral in the discussion to come:

Proposition 7.2.1. [BP95] For real roots α, β we can represent the α-string through β, $S(\alpha, \beta)$, as a diagram $\bullet$, $\bullet \circ \cdots \circ \bullet$, $\bullet \bullet \bullet$, or $\bullet \bullet \circ \cdots \circ \bullet \bullet$, where $\bullet$ represents a real root in the string, $\circ$ represents an imaginary root, and moving right in the diagram from dot-to-dot indicates adding α. If $S(\alpha, \beta) = \{\beta - u\alpha, \ldots, \beta + v\alpha\}$, then $\langle \beta, \alpha^\vee \rangle = u - v$. Furthermore, applying r_α to $S(\alpha, \beta)$ amounts to reflecting the string about its midpoint.

From here on we will be fixing an arbitrary prenilpotent pair $\{\alpha, \beta\}$ of real roots such that $|r_\alpha r_\beta| = \infty$.

Lemma 7.2.2. Both $S(\alpha, \beta)$ and $S(\beta, \alpha)$ can be represented by diagrams of the form $\bullet \circ \cdots \circ \bullet$ or $\bullet \bullet \circ \cdots \circ \bullet \bullet$. If a string is of the form $\bullet \bullet \circ \cdots \circ \bullet \bullet$, it contains an imaginary root (i.e., there must be at least one hollow dot). It cannot be the case that both strings are of the form $\bullet \bullet$ (with no imaginary roots).

Proof. We rule out the cases that at least one of $S(\alpha, \beta)$ or $S(\beta, \alpha)$ can be of the form $\bullet$, $\bullet \bullet \bullet$ or $\bullet \bullet \bullet \bullet$. If one string is of the form $\bullet$ then it is clear that $\langle \alpha, \beta^\vee \rangle = 0 = \langle \beta, \alpha^\vee \rangle$, which implies r_α and r_β commute and contradicts our choice of α and β. In the case that a string (WLOG $S(\alpha, \beta)$) is of the form $\bullet \bullet \bullet$ or $\bullet \bullet \bullet \bullet$, we will reduce to the rank 2 case and think of $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Delta$ as the root system of a rank 2 Kac-Moody algebra. We note $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Delta$ is the same collection of roots as $(\mathbb{Z}\alpha + \mathbb{Z}\gamma) \cap \Delta$ for $\gamma = \beta - k\alpha$, so then we may use the leftmost root γ (where k in the description of γ above is maximal) in the string $S(\alpha, \beta)$ and show $(\mathbb{Z}\alpha + \mathbb{Z}\gamma) \cap \Delta$ is spherical. This choice of γ will let us make the reduction to rank 2 (from Prop. 6.8.11).

In particular, if $S(\alpha, \beta)$ is of the form $\bullet \bullet \bullet$ or $\bullet \bullet \bullet \bullet$, then γ, α, and $\gamma + \alpha$ are real roots with $\langle \gamma, \alpha^\vee \rangle = -2$ or -3, respectively, so $\langle \alpha, \gamma^\vee \rangle = -1$. To see this, first observe $\langle \gamma, \alpha^\vee \rangle \le -2 \implies \langle \alpha, \gamma^\vee \rangle \le -1$ (see Prop. 6.4.2). Then if $\langle \alpha, \gamma^\vee \rangle \le -2$, $\gamma + 2\alpha$ is a root. Now $\langle \gamma + 2\alpha, \gamma^\vee \rangle = 2 + 2\langle \alpha, \gamma^\vee \rangle$ with $\langle \alpha, \gamma^\vee \rangle \le -2$ implies $\langle \gamma + 2\alpha, \gamma^\vee \rangle \le -2$. But then $2\gamma + 2\alpha$ is a root in $S(\gamma + 2\alpha, \gamma)$, which contradicts $\gamma + \alpha \in \Delta_{re}$. So then we must have $\langle \alpha, \gamma^\vee \rangle = -1$.

This would mean that, treating $\{\alpha, \gamma\}$ as a base for $(\mathbb{Z}\alpha + \mathbb{Z}\gamma) \cap \Delta$, $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Delta$ is a root system of type C_2 or G_2 and thus $|r_\alpha r_\beta| < \infty$, a contradiction. So the only remaining possibilities are root strings of the form $\bullet \circ \cdots \circ \bullet$ or $\bullet \bullet \circ \cdots \circ \bullet \bullet$, where there is at least one imaginary root if there are 4 real roots.

In a similar vein, we can rule out $S(\alpha,\beta)$ and $S(\beta,\alpha)$ being of the form $\bullet\ \bullet$. If β is the leftmost root, $\langle\beta,\alpha^\vee\rangle = -1$ and $\beta-\alpha\notin\Delta$, which immediately lets us realize $(\mathbb{Z}\alpha+\mathbb{Z}\beta)\cap\Delta$ as the root system of a rank 2 Kac-Moody algebra. If we suppose β is the rightmost root in $S(\alpha,\beta)$ (so that $\langle\beta,\alpha^\vee\rangle = 1$), then again $(\mathbb{Z}\alpha+\mathbb{Z}\beta)\cap\Delta = (\mathbb{Z}\alpha+\mathbb{Z}(\beta-\alpha))\cap\Delta$, and $(\mathbb{Z}\alpha+\mathbb{Z}(\beta-\alpha))\cap\Delta$ is the root system of a rank 2 Kac-Moody algebra. We can then compute $\langle\alpha,(\beta-\alpha)^\vee\rangle = \langle\alpha,r_\alpha(\beta)^\vee\rangle = \langle r_\alpha(\alpha),\beta^\vee\rangle = \langle-\alpha,\beta^\vee\rangle = -1$, and $\langle\beta-\alpha,\alpha^\vee\rangle = -1$ (by how $\beta-\alpha$ was chosen) so that $(\mathbb{Z}\alpha+\mathbb{Z}(\beta-\alpha))\cap\Delta$ is of type A_2, which is again finite type.

□

Remark 7.2.3. The cases we will care about here are those such that at least one of the root strings $S(\alpha,\beta)$ or $S(\beta,\alpha)$ is of the form $\bullet\bullet\circ\cdots\circ\bullet\bullet$. If both have only two real roots (both are of the form $\bullet\circ\cdots\circ\bullet$) then $\alpha+\beta\notin\Delta_{re}$, in which case U_α, U_β commute because $U_{\alpha+\beta}=1$. So we may assume **at least one string**, say $S(\alpha,\beta)$, is of the form $\bullet\bullet\circ\cdots\circ\bullet\bullet$ (with at least one imaginary root by Lemma 7.2.2).

Corollary 7.2.4. $(\mathbb{N}\alpha+\mathbb{N}\beta)\cap\Delta\subseteq\{\alpha+\beta\}$.

Proof. We recall that since $\{\alpha,\beta\}$ is a prenilpotent pair, $(\mathbb{N}\alpha+\mathbb{N}\beta)\cap\Delta\subset\Delta_{re}$ (Props. 6.7.1 & 6.7.2). To prove $(\mathbb{N}\alpha+\mathbb{N}\beta)\cap\Delta\subseteq\{\alpha+\beta\}$, we claim $\langle\alpha,\beta^\vee\rangle\geq 0$. First, note our only diagram options for $S(\alpha,\beta)$ and $S(\beta,\alpha)$ are of the form given in Lemma 7.2.2. Then since $(\mathbb{N}\alpha+\mathbb{N}\beta)\cap\Delta\subset\Delta_{re}$, β cannot occur to the left of imaginary roots in its strings $S(\beta,\alpha)$ and $S(\alpha,\beta)$. In particular, β must be the rightmost or second-to-rightmost root in the diagram for $S(\alpha,\beta)$: either $\bullet\bullet\circ\cdots\circ\bullet\bullet_\beta$ or $\bullet\bullet\circ\cdots\circ\bullet_\beta\bullet$. This means $S(\alpha,\beta)=\{\beta-u\alpha,\ldots,\beta,\beta+\alpha\}$ or $\{\beta-u\alpha,\ldots,\beta\}$ with $u\geq 2$, so $\langle\beta,\alpha^\vee\rangle\geq 0$ ($\implies$ $\langle\alpha,\beta^\vee\rangle\geq 0$ by Prop. 6.4.2). Finally, we have a pair of real roots $\{\alpha,\beta\}$ with $\langle\alpha,\beta^\vee\rangle\geq 0$, and this means $(\mathbb{N}\alpha+\mathbb{N}\beta)\cap\Delta\subseteq\{\alpha+\beta\}$ (Lemma 6.8.1). □

Corollary 7.2.5. If $\alpha+\beta\in S(\alpha,\beta)$ then it must be the rightmost root in the string.

Proof. This follows immediately from the earlier description of $S(\alpha,\beta)$ and the proof of the preceeding Corollary. □

We next introduce the notion of a *Morita pair*.

Definition 7.2.6. A pair of positive real roots $\{\eta,\gamma\}$ is called a **Morita pair** if

1. $\eta-\gamma\notin\Delta$
2. $\eta+\gamma\in\Delta^{re}$
3. $\forall w\in W$ *such that* $w(\eta),w(\gamma)\in\Delta_+^{re}$, $ht(\eta+\gamma)\leq ht(w(\eta+\gamma))$

Remark 7.2.7. Two important pieces of information can be gathered from the definition of a Morita pair. The first is that the pairings $\langle \eta, \gamma^\vee \rangle$ and $\langle \gamma, \eta^\vee \rangle$ must be negative, since $\eta + \gamma \in \Delta$ and $\eta - \gamma \notin \Delta$. The second is that at least one of $\langle \eta, \gamma^\vee \rangle, \langle \gamma, \eta^\vee \rangle$ is -1. This can be shown in more than one way (see [BP95], Prop. 2(i),(ii)), but we may use the fact that if $\zeta, \psi, \zeta + \psi \in \Delta_{re}$ and $\langle \psi, \zeta^\vee \rangle \leq -2$ then $\langle \zeta, \psi^\vee \rangle = -1$ (see the proof of Lemma 7.2.2). Rather, we combine this with the definition of a Morita pair and our earlier observation that the pairings are negative to our benefit.

We will need the following proposition and theorem.

Proposition 7.2.8. ([BP95] Prop. 3) Let $\gamma_1, \gamma_2 \in \Delta^{re}$ be such that $\gamma_1 + \gamma_2 \in \Delta^{re}$. Then

1. If $\gamma_1, \gamma_2 \in \Delta_+$ and $\gamma_2 - \gamma_1 \notin \Delta$, then there exists $w \in W$ such that $\{w\gamma_1, w\gamma_2\}$ is a Morita pair.

2. There exists $\sigma \in \pm W$ such that $\sigma\gamma_1 \in \Delta_+$ and $S(\sigma\gamma_1, \sigma\gamma_2) \subset \Delta_+$.

Theorem 7.2.9. ([BP95] Thm. 1) Let $\{\eta, \gamma\}$ be a Morita pair with $\langle \eta, \gamma^\vee \rangle = -a$ and $\langle \gamma, \eta^\vee \rangle = -1$. Then exactly one of the three following cases holds.

- Case F (Finite case). $a = 1, 2,$ or 3 and either
  $\eta, \gamma \in \Pi$ or
  $a = 1$ and there exist $\alpha_i, \alpha_j \in \Pi$ such that $\langle \alpha_i, \alpha_j^\vee \rangle = -3, \langle \alpha_j, \alpha_i^\vee \rangle = -1$ and $\{\eta, \gamma\} = \{\alpha_i, \alpha_i + 3\alpha_j\}$.

- Case A (Affine case). $a = 4$ and there exists a sequence of distinct simple roots $\alpha_{i_1}, \dots, \alpha_{i_l} \in \Pi$ (with $l \geq 2$) which generate an affine subroot system of type $BC_{l-1}^{(2)}$. This sequence of simple roots also satisfies
  $\langle \alpha_{i_{k+1}}, \alpha_{i_k}^\vee \rangle = -1$ for $k = 1, \dots, l-1$
  $\langle \alpha_{i_k}, \alpha_{i_m}^\vee \rangle = 0$ if $|k - m| > 1$
  $\langle \alpha_{i_1}, \alpha_{i_2}^\vee \rangle = \langle \alpha_{i_{l-1}}, \alpha_{i_l}^\vee \rangle = -2$ if $l > 2$
  $\langle \alpha_{i_1}, \alpha_{i_2}^\vee \rangle = -4$ if $l = 2$
  $\eta = \alpha_{i_1}$ and
  $\gamma = r_{i_2} \cdots r_{i_{l-1}}(\alpha_{i_l}) + n\delta$
  where δ is a nullroot of $BC_{l-1}^{(2)}$ and $n \geq 0$ (say $r_{i_1} \cdots r_{i_{l-1}} = 1$ if $l = 2$).

- Case I (Indefinite case). $a \geq 5$ and there exists a sequence of distinct simple roots $\alpha_{i_1}, \dots, \alpha_{i_l} \in \Pi$ (with $l \geq 2$) satisfying
  $\langle \alpha_{i_{k+1}}, \alpha_{i_k}^\vee \rangle = -1$ for $k = 1, \dots, l-1$
  $\langle \alpha_{i_k}, \alpha_{i_m}^\vee \rangle = 0$ if $|k - m| > 1$
  $\langle \alpha_{i_1}, \alpha_{i_2}^\vee \rangle < -1$, $\langle \alpha_{i_{l-1}}, \alpha_{i_l}^\vee \rangle < -1$

$\left|\prod_{k=1}^{l-1} \langle \alpha_{i_k}, \alpha_{i_{k+1}}^\vee \rangle\right| = a$
$\eta = \alpha_{i_1}$ and
$\gamma = r_{i_2} \cdots r_{i_{l-1}}(\alpha_{i_l})$ (again say $r_{i_2} \cdots r_{i_{l-1}} = 1$ if $l = 2$).

7.3 A Necessary and Sufficient Condition for Commutation

Remark 7.3.1. We will refer to any collection $\Psi = \{\alpha_{i_1}, \ldots, \alpha_{i_l}\}$ with the properties from case A or I of the theorem as being "of case A" (or I) for convenience.

We will need to gather another technical lemma before getting to the main result.

Lemma 7.3.2. Suppose A is a generalized Cartan matrix with simple roots Π, and that $\{\alpha_{i_1}, \ldots, \alpha_{i_l}\} \subset \Pi$ is as in case A or I from the theorem. Then the root system Δ for Π contains a Morita pair $\{\eta, \gamma\}$ with $\langle \eta, \gamma^\vee \rangle \leq -4$.

Proof. We find our Morita pair by using the techniques found in the statement and proof of Theorem 7.2.9. Define $\gamma = r_{i_2} r_{i_3} \cdots r_{i_{l-1}}(\alpha_{i_l})$ and $\eta = \alpha_{i_1}$. For later convenience, set $p_k = \langle \alpha_{i_k}, \alpha_{i_{k+1}}^\vee \rangle$. We compute

$$\langle \gamma, \eta^\vee \rangle = \langle r_{i_2} r_{i_3} \cdots r_{i_{l-1}}(\alpha_{i_l}), \alpha_{i_1}^\vee \rangle = \left\langle \sum_{k=2}^{l} \alpha_{i_k}, \alpha_{i_1} \right\rangle = \langle \alpha_{i_2}, \alpha_{i_1}^\vee \rangle = -1$$

and

$$\langle \eta, \gamma^\vee \rangle = \langle \alpha_{i_1}, r_{i_2} r_{i_3} \cdots r_{i_{l-1}}(\alpha_{i_l})^\vee \rangle = \langle r_{i_{l-1}} \cdots r_{i_2}(\alpha_{i_1}), \alpha_{i_l}^\vee \rangle$$
$$= \left\langle \sum_{m=1}^{l-1} \left(\prod_{k=1}^{m} p_k \right) \alpha_{i_m}, \alpha_{i_l} \right\rangle = \sum_{m=1}^{l-1} \left(\prod_{k=1}^{m} p_k \right) \langle \alpha_{i_m}, \alpha_{i_l}^\vee \rangle.$$

Now for each $m \neq l-1$ we have $\langle \alpha_{i_m}, \alpha_{i_l}^\vee \rangle = 0$, so in fact

$$\langle \eta, \gamma^\vee \rangle = \prod_{k=1}^{l-1} p_k = \prod_{k=1}^{l-1} \langle \alpha_{i_k}, \alpha_{i_{k+1}}^\vee \rangle \leq -4.$$

by definition of $\{\alpha_{i_1}, \ldots, \alpha_{i_l}\}$.

We claim that $\{\eta, \gamma\}$ is actually Weyl-conjugate to a Morita pair. Certainly $\eta + \gamma \in \Delta_{re}$ by $\langle \gamma, \eta^\vee \rangle = -1$, so then $w(\eta + \gamma) \in \Delta_{re}$ for any $w \in W$. Similarly $\eta - \gamma$ (and hence $w(\eta - \gamma)$) is not in Δ by $\eta - \gamma = \sum_{k=2}^{l} \alpha_{i_k} - \alpha_{i_1}$, which is a combination of signs not allowed since the α_{i_j} are in Π. So to find our Morita pair, we minimize the height of

$w(\eta+\gamma)$ among all $w \in W$ such that $w(\eta), w(\gamma) \in \Delta^+$ (see [BP95] Prop. 3i), and this yields our Morita pair $\{w(\eta), w(\gamma)\}$. □

Remark 7.3.3. Lemma 7.3.2 is a converse to cases A and I of Theorem 7.2.9. In fact, the proof of Lemma 7.3.2 indicates $\langle \eta, \gamma^\vee \rangle = -4$ exactly in case A and $\langle \eta, \gamma^\vee \rangle < -4$ in case I.

Theorem 7.3.4. Fix a Kac-Moody algebra $\mathfrak{g}_{\mathcal{D}}$ and Kac-Moody group $\mathcal{G}_{\mathcal{D}}(R)$ (for a commutative ring R).

1. (Necessary) Let $\{\alpha, \beta\}$ prenilpotent pair of real roots such that $|r_\alpha r_\beta| = \infty$. If $[U_\alpha, U_\beta] \neq 1$, then the root system has a Morita pair $\{\eta, \gamma\}$ such that $\langle \eta, \gamma^\vee \rangle \leq -4$, $\langle \gamma, \eta^\vee \rangle = -1$.

2. (Sufficient) Let $\{\eta, \gamma\}$ be a Morita pair in $\mathfrak{g}_{\mathcal{D}}$'s root system such that $\langle \eta, \gamma^\vee \rangle \leq -4$, $\langle \gamma, \eta^\vee \rangle = -1$ and $\{\alpha_{i_1}, \dots, \alpha_{i_l}\}$ is the guaranteed set of simple roots satisfying Case A or I (writing $\langle \alpha_{i_k}, \alpha_{i_{k+1}}^\vee \rangle = p_k$). Further suppose that $p_1 \cdots p_{l-1} \neq 0$ in R. Then there exists a prenilpotent pair of real roots $\{\alpha, \beta\}$ such that $|r_\alpha r_\beta| = \infty$ and $[U_\alpha, U_\beta] \neq 1$.

Proof. (1.) By remark 7.2.3, $\alpha+\beta$ is a root and we've assumed $S(\alpha, \beta)$ is of the form $\bullet\bullet\circ\cdots\circ\bullet\bullet$. Say ζ is the leftmost root in the diagram, i.e. $S(\alpha,\beta) = S(\alpha,\zeta) = \{\zeta, \zeta+\alpha, \dots, \zeta+k\alpha\}$. Then since $\alpha, \zeta, \zeta+\alpha \in \Delta_{re}$, there exists $\sigma \in \pm W$ such that $\sigma\alpha \in \Delta^+_{re}$ and $S(\sigma\alpha, \sigma\zeta) \subset \Delta^+$ (Prop. 7.2.8); we may pick σ so that $\{\sigma\alpha, \sigma\zeta\}$ is a Morita pair by minimizing height. We also see that $\langle \sigma\zeta, \sigma\alpha^\vee \rangle \leq -4$ since $\langle \zeta, \alpha^\vee \rangle \leq -4$ and $\langle \sigma\zeta, \sigma\alpha^\vee \rangle = \langle \zeta, \alpha^\vee \rangle$ (and we have at least one imaginary root in $S(\alpha, \beta)$). Also, $\langle \sigma\alpha, \sigma\zeta^\vee \rangle = -1$ since we have $\sigma\alpha, \sigma\zeta, \sigma\alpha+\sigma\zeta \in \Delta_{re}$ with $\langle \sigma\zeta, \sigma\alpha^\vee \rangle \leq -2$ (see proof of Lemma 7.2.2).

(2.) Suppose $\mathfrak{g}_{\mathcal{D}}$'s root system possesses a Morita pair $\{\gamma, \eta\}$ such that $\langle \eta, \gamma^\vee \rangle \leq -4$ and $\langle \gamma, \eta^\vee \rangle = -1$. By the definition of a Morita pair $\eta+\gamma$ is a real root. Then by the symmetry of root strings $S(\gamma, \eta)$ for real roots γ and η, we know $\eta - (\langle \eta, \gamma^\vee \rangle + 1)\gamma$ and $\eta - \langle \eta, \gamma^\vee \rangle \gamma$ are real roots. In the diagrammatic representation $S(\gamma, \eta) = \bullet\bullet\circ\cdots\circ\bullet\bullet$, η is the leftmost root, $\eta - (\langle \eta, \gamma^\vee \rangle + 1)\gamma$ is the next-to-rightmost root, and $\eta - \langle \eta, \gamma^\vee \rangle \gamma$ is the rightmost root. We claim the pair $\{\gamma, \eta - (\langle \eta, \gamma^\vee \rangle + 1)\gamma\}$ is a pair $\{\alpha, \beta\}$ as described in the proposition, and set $\alpha = \gamma$, $\beta = \eta - (\langle \eta, \gamma^\vee \rangle + 1)\gamma$ for ease of reference.

We first check prenilpotence. α and β are clearly positive real roots (since they are positive sums of the positive roots γ and η), so $1 \in W$ yields $1\{\alpha, \beta\} \subset \Delta^+$. Now we set $v = r_\gamma r_\eta r_\gamma$, and observe $v\{\alpha, \beta\} \subset \Delta^-$:

$$v(\alpha) = r_\gamma r_\eta r_\gamma(\gamma) = r_\gamma r_\eta(-\gamma) = r_\gamma(-\gamma + -\eta) = \gamma + -(\eta - \langle \eta, \gamma^\vee \rangle \gamma)$$

$$= -(\eta - (\langle \eta, \gamma^\vee \rangle + 1)\gamma) = -\beta \in \Delta^-$$

$$v(\beta) = r_\gamma r_\eta r_\gamma (\eta - (\langle \eta, \gamma^\vee \rangle + 1)\gamma) = r_\gamma r_\eta (\eta - \langle \eta, \gamma^\vee \rangle \gamma + (\langle \eta, \gamma^\vee \rangle + 1)\gamma) = r_\gamma r_\eta (\eta + \gamma)$$

$$= r_\gamma(-\eta + (\gamma + \eta)) = -\gamma = -\alpha \in \Delta^-$$

So then $\{\alpha, \beta\}$ is a prenilpotent pair.

We also check that $|r_\alpha r_\beta| = \infty$. Observe that since $\langle \eta, \gamma^\vee \rangle \leq -4$ we must have $\langle \beta, \alpha^\vee \rangle = \langle \eta - (\langle \eta, \gamma^\vee \rangle + 1)\gamma, \gamma^\vee \rangle = \langle \eta, \gamma^\vee \rangle - (\langle \eta, \gamma^\vee \rangle + 1)\langle \gamma, \gamma^\vee \rangle = \langle \eta, \gamma^\vee \rangle - 2(\langle \eta, \gamma^\vee \rangle + 1) = -\langle \eta, \gamma^\vee \rangle - 2 \geq 2$. But also $\beta - \alpha \in \Delta_{im}$ and $\beta + \alpha \in \Delta_{re}$ by reexamining the above diagrammatic description of $S(\gamma, \eta)$, so then $\alpha - \beta \in \Delta_{im}$ and $\alpha + \beta \in \Delta_{re}$ forces $S(\beta, \alpha) = \bullet\bullet\circ\cdots\circ\bullet\bullet$ where γ is the next-to-rightmost root. This means $\langle \alpha, \beta^\vee \rangle \geq 2$ so $\langle \beta, \alpha^\vee \rangle \langle \alpha, \beta^\vee \rangle \geq 4$ and hence $|r_\alpha r_\beta| = \infty$.

To obtain that $[U_\alpha, U_\beta] \neq 1$ for this pair $\{\alpha, \beta\}$, we refer to our specific construction of the Steinberg group functors. Part of our defining relations for these functors is that, after placing an order on $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta_{re}$, there are specific integers $c_{ij}^{\alpha\beta}$ (depending on this order) such that for all $x_\alpha(t) \in U_\alpha$ and $x_\beta(u) \in U_\beta$ we have

$$[x_\alpha(t), x_\beta(u)] = \prod_{\gamma = i\alpha + j\beta} x_\gamma(t^i u^j c_{ij}^{\alpha\beta}).$$

Thanks to Proposition 7.1.1, we know exactly what these coefficients are. In our case, $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta_{re} = \{\alpha + \beta\}$ implies that $[x_\alpha(t), x_\beta(u)] = x_{\alpha+\beta}(tuc_{11}^{\alpha\beta}) = x_{\alpha+\beta}(\pm(p+1)tu)$, where p is the integer such that $S(\alpha, \beta) = \{\beta - p\alpha, \ldots, \beta, \beta + \alpha\}$. By the way we found α and β, we know $p = \langle \beta, \alpha^\vee \rangle + 1 = (-\langle \eta, \gamma^\vee \rangle - 2) + 1$. By the earlier computation of brackets between Morita pairs in Lemma 7.3.2, we can write this last term as

$$\left(- \left(- \left| \prod_{k=1}^{l-1} p_i \right| + 1 \right) - 2 \right) = \left| \prod_{k=1}^{l-1} p_i \right| - 1,$$

hence

$$\pm(p+1) = \pm \left(\left(\left| \prod_{k=1}^{l-1} p_i \right| - 1 \right) + 1 \right) = \pm \left| \prod_{k=1}^{l-1} p_i \right|.$$

So, by examining

$$[x_\alpha(1), x_\beta(1)] = x_{\alpha+\beta}(\pm(p+1)) = x_{\alpha+\beta} \left(\pm \left| \prod_{k=1}^{l-1} p_i \right| \right),$$

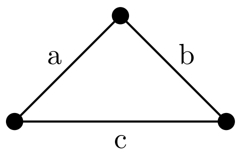
we know that $[U_\alpha, U_\beta] \neq 1$ iff $\prod_{k=1}^{l-1} p_i \neq 0$ in our ring R.

□

Remark 7.3.5. Thanks to Lemma 7.3.2, this theorem means it is only possible to get nontrivial commutator relations associated to prenilpotent $\{\alpha, \beta\}$ with $|r_\alpha r_\beta| = \infty$ in a Kac-Moody group $\mathcal{G}_\mathcal{D}(\mathbb{K})$ when there is a subset of simple roots satifying Case A or Case I from Theorem 7.2.9.

We have the following comparison to Theorem 5.4.1:

Corollary 7.3.6. Let $\mathcal{G}_\mathcal{D}(\mathbb{K})$ be a Kac-Moody group whose Weyl group has a Coxeter diagram matching



$3 \leq a \leq b \leq c < \infty.$

Then for any prenilpotent $\{\alpha, \beta\}$ with $|r_\alpha r_\beta| = \infty$, $[U_\alpha, U_\beta] = 1$.

Proof. The only question here is whether or not there are any GCMs admitting a set of simple roots of Case A or I which have a Coxeter diagram like this. We have 3 simple roots to choose from, as we have 3 simple reflections. But this Coxeter diagram necessitates $a_{ij}a_{ji} \neq 0$ for any $i \neq j$, as it is a complete graph; such a GCM has no zero entries. Since we need $\langle a_{i_k}, \alpha_{i_m}^\vee \rangle = 0$ if $|k - m| > 1$ in a set of simple roots $\{\alpha_{i_1}, \ldots, \alpha_{i_l}\}$ satisfying Case A or I, we can pick at most two simple roots here. As the size 2 sets of simple roots satisfying Case A or I $\{\alpha_{i_1}, \alpha_{i_2}\}$ require $\langle a_{i_1}, \alpha_{i_2}^\vee \rangle \langle a_{i_2}, \alpha_{i_1}^\vee \rangle \geq 4$, this would give us an ∞-label in the Coxeter diagram. Thus we cannot have a GCM admitting simple roots of Case A or I that also has this as a Coxeter diagram. □

Note the agreement with Theorem 5.4.1, with the addition of losing the condition that $|U_\alpha| \geq 4$. As well, the affine case can be lumped in with the hyperbolic case here.

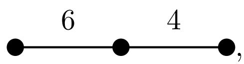
Remark 7.3.7. More generally, there are no GCMs admitting a set of simple roots satisfying Case A or I whose Coxeter diagram is a complete graph with no $\infty-$labels. Thus Kac-Moody groups whose Coxeter diagram is a complete graph with no $\infty-$labels necessarily have $[U_\alpha, U_\beta] = 1$ for prenilpotent $\{\alpha, \beta\}$ with $|r_\alpha r_\beta| = \infty$.

Example 7.3.8. We can return to the example at the beginning of the chapter (Ex. 7.1.3)to actually find a pair of roots $\{\alpha, \beta\}$ with $|r_\alpha r_\beta| = \infty$ and $[U_\alpha, U_\beta] \neq 1$. Recall we had earlier examined a GCM of type $BC_1^{(2)}$ (case A) and a pair $\{-r_0 r_1(\alpha_0), \alpha_1\}$; the resulting commutator group was trivial. Using the proof of Theorem 7.3.4, we can find a pair so that the resulting root groups do not commute. Set $\eta = \alpha_0$ and $\gamma = \alpha_1$, so that $\alpha = \gamma = \alpha_1$ and $\beta = \eta - (\langle \eta, \gamma^\vee \rangle + 1)\gamma = \alpha_0 + 3\alpha_1$. This is then the desired pair.

Example 7.3.9. One can also use Theorem 7.3.4 to explicitly find buildings whose Coxeter complexes can be realized as a tiling of the hyperbolic plane by *right* triangles with nested roots contributing nontrivial commutator relations (i.e. $\alpha \subsetneq \beta$ and $[U_\alpha, U_\beta] \neq 1$). For example, the GCM

$$\begin{bmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix}$$

yields Coxeter diagram



and its simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$ forms a collection satisfying Case I (thus leading to α, β with $|r_\alpha r_\beta| = \infty$ and $[U_\alpha, U_\beta] \neq 1$ if $\text{char}(\mathbb{K}) \neq 2, 3$). Note we say *right* triangles here as the Coxeter complex corresponds to a tesselation of $\mathbb{H}^2$ by triangles with angles $\frac{\pi}{2}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$. Not all buildings coming from Kac-Moody groups with this same Coxeter diagram have such a pair of noncommuting root groups like this: note the GCM

$$\begin{bmatrix} 2 & -1 & 0 \\ -3 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

gives the same Coxeter diagram (thus the associated building from $\mathcal{G}_\mathcal{D}(\mathbb{K})$ is of the same type), and there is no set of simple roots satisfying case A or I from Theorem 7.2.9. More specifically, the condition that all of the pairings $\langle \alpha_{i_{k+1}}, \alpha_{i_k}^\vee \rangle = -1$ cannot be satisfied, and so we really do get two different possibilities for these specific commutator relations for RGD systems with the same type (W, S).

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