Root group data (RGD) systems of affine type for significant subgroups of isotropic reductive groups over $\mathbf{k[t, t^{-1}]}$

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Contents

Table of Contents 3

Introduction 5

I Background 9

1 General RGD system 9

1.1 Definition of general RGD system . . . 9

1.2 General RGD and Moufang twin buildings . . . 14

2 Basics on affine roots 15

3 Basics on Chevalley groups 18

3.1 Chevalley basis . . . 18

3.2 Constructing Chevalley groups . . . 21

3.3 Further comment on Chevalley groups . . . 24

3.4 Relations in Chevalley group . . . 26

4 Basics on isotropic reductive groups 29

4.1 Background in Algebraic geometry . . . 29

4.2 Algebraic groups . . . 30

4.2.1 Basics on Algebraic groups . . . 30

4.2.2 Reductive groups . . . 35

4.2.3 Tools for non-split case . . . 39

II Constructing RGD systems 43

5 RGD system in case of Chevalley group and split reductive case 43

5.1 Chevalley group case (Split semisimple case) . . . 43

5.1.1 Bring into context: Chevalley group over Laurent polynomial rings . . . 43

5.1.2 Constructing RGD system for elementary subgroup of Chevalley group over Laurent polynomial ring . . . 48

5.2 Split reductive case (Similar to Chevalley group case) 52
5.2.1 Bring into context: split reductive groups over Laurent polynomial rings . . . 52
5.2.2 Constructing RGD system for elementary subgroup of split reductive groups 54

6 RGD system in case of non-split reductive groups 58
6.1 Specific non-split case: (Special) Unitary groups of type BC 58
6.1.1 Basics of Unitary group 58
6.1.2 Construction of RGD system for Unitary groups 68
6.2 General non-split case 82
6.2.1 Bring into context: non split reductive groups over Laurent polynomial rings 82
6.2.2 Constructing RGD system for elementary subgroup of non split reductive group 94

III Appendix 107

7 Buildings 107
7.1 Basics on buildings 107
7.2 BN and building 109
7.3 A word on Saturated BN pair 111
7.4 Simplicial buildings and Weyl distance 111

8 Twin buildings 122
8.1 Definition(s) of twin building 123
8.1.1 Some more info on coconvexity and (co)projection 127
8.2 Twin BN and twin building 128
8.2.1 A word on saturated twin BN pair 133

IV References 135

Introduction

History and motivation

BN pairs are first introduced by Jacques Tits in the study of certain matrix groups (in particular, the reductive and semisimple groups). Through introducing buildings (first in [25]), Tits generalized projective geometric space; and he utilized BN pairs to reach a unified simplicity proof when studying Chevalley groups over fields. It is then that the link between BN pairs and buildings are established, introducing discrete geometric ideas into abstract group theory.

BN pairs arise from root groups, for instance in the Chevalley group over field k, the 1-parameter subgroups U_α (isomorphic to additive group of k as groups) where α is a root in the corresponding root system of the Chevalley group. To make connection between root groups and BN pairs, Tits formulated Root group data system (RGD system) that occur in algebraic groups, which system always give rise to a BN pair.

To every BN pair, there is an associated Weyl group (and Coxeter system). In the case of Chevalley groups (and more generally the isotropic reductive groups) over fields, the Weyl groups are finite, and give rise to spherical buildings (the author has recorded this at 7.2.5). It is then introduced by Iwahori and Matsumoto in [28], through Chevalley groups over local field, BN pairs (hence buildings) with infinite Weyl groups ($W_{aff}(\Psi)$ of affine type, where Ψ is the root system corresponding to the group). In [11], Tits and Bruhat then generalized the work, and provided affine BN pairs for isotropic reductive groups over local fields, which again arose from RGD systems. As an example, $k(t)$ has discrete valuations v^+ with prime element t and v^- with prime element t^{-1}, this in fact gives us two different affine BN pairs on $\mathcal{G}(k(t))$ for Chevalley group $\mathcal{G}$. Since (in the topology induced by the two discrete valuations) $k[t,t^{-1}]$ is dense in $k(t)$, the two affine BN pairs on $\mathcal{G}(k(t))$ further induces two affine BN pairs on $\mathcal{G}(k[t,t^{-1}])$. These two affine BN pairs on $\mathcal{G}(k[t,t^{-1}])$ will give rises to a twin building on the group (the author has recorded this without the context of $\mathcal{G}$ at 8.2.6).

Given a Chevally group $\mathcal{G}$, $G = \mathcal{G}(k[t,t^{-1}])$ can be seen as the $k[t,t^{-1}]$-points of a linear algebraic group, and as a Kac-Moody group over k of affine type (in Tits' sense) as well. In [27], Tits provided an RGD system of not necessarily spherical type for Kac-Moody groups of "split" type over fields (in Tits' sense, which covers the case of G we just described) which give rise to twin BN pairs and hence twin buildings. This has many applications, chief among which the study of action on twin buildings.

However, for isotropic reductive k-group $\mathcal{G}$, $G = \mathcal{G}(k[t, t^{-1}])$ is no longer a Kac-Moody group over k in Tits' sense. But Tits mentioned without concrete proof that (see [26, sec: 3.2] and [26, sec: 3.3]) his RGD axioms will also apply to this more general situation when $\mathcal{G}$ is almost simple and simply connected. Neither the proof nor the construction justifying this claim is present in the literature as of now. This brings us to the **main goal** of this thesis: Provide a concrete construction of RGD system in this general reductive case for some significant subgroups (the elementary subgroup $\mathcal{G}(k[t, t^{-1}])^+$, and a larger subgroup $\mathcal{G}(k[t, t^{-1}])^+ C_{\mathcal{G}}(S)(k))$ of G. Note, in case that $\mathcal{G}$ is simply connected, and any normal semisimple subgroup of $\mathcal{G}$ has k-rank at least 2, we in fact can see $\mathcal{G}(k[t, t^{-1}]) = \mathcal{G}(k)\mathcal{G}(k[t, t^{-1}])^+ = \mathcal{G}(k[t, t^{-1}])^+ C_{\mathcal{G}}(S)(k)$ (see [21, cor:6.2]).

Explanation and summary of content

In the Part I, we provide background information on the tools we need for the goal. All knowledge within Part I is somewhat well established, and so only needed concepts, result, and extended results will be recorded. Common references for information in Part I are: For affine roots, see [14]. For Chevalley groups, see [24]. And for isotropic reductive groups, see [8]. As knowledge about building and twin buildings is not extensively utilized, we will only include results on buildings and twin building within the Appendix (Part III). In short, twin BN pair is a "weaker" version of RGD system, however application of twin buildings that arises with twin BN pairs are plentiful. For reference on twin buildings (RGD system, and buildings as well), see [6].

In Part II, we proceed to the construction of RGD systems by utilizing the backgrounds we provided earlier in Part I: Constructing RGD system of affine type is essentially a study on, and construction of the affine root groups.

It is in [2], a concrete construction of RGD system for the case of Chevalley groups (precisely the split semisimple case) is provided. We will first (in section 5.1) detail a method seen in [2] to construct RGD system for the Chevalley group case (see 5.1.5). Then with the essentially same procedure, (in section 5.2) we construct RGD system for the split reductive groups case (see 5.2.2), this is essentially the same as the case of Chevalley groups due to 4.2.17. In both these cases, the affine root groups are constructed in a "expected" way according to "absolute pinning isomorphisms" (the isomorphisms $x_a : Add \to U_a$ where a is an absolute root, and U_a root group of a. The word "pinning" is direct translation from "épinglage" from [12]. See 4.2.13).

Then, in chapter 6, we will move on to non-split cases, which are the **main works** of this thesis:

We use the case of (Special) Unitary groups of type BC as a specific example for the non-split case: We first, in section 6.1.1, record elements and their relations constructed in [11]. Then in section 6.1.2, we use the said construction to construct affine root groups for Unitary groups (see 6.1.7). And with the constructed affine root groups, and the relations provided, we go through the axioms of RGD system in a similar method as did in the case of Chevalley groups (see 6.1.8). As a further comment, the reason we chose type BC is that it is the only irreducible non-reduced root system.

Then we will start facing on the case of non-split isotropic reductive groups: We first record, and further develop construction of "relative pinning maps" for relative roots (see 6.2.2) and its relations provided by Stavrova in [18]; In section 6.2.1, with the "relative pinning maps", we will construct affine root groups for the general case (see 6.2.1) and make observation about the affine root groups we have constructed by utilizing the relations and properties of the "relative pinning maps" (see 6.2.4, and 6.2.5). It can be seen (at 6.2.6) that axiom (RGD2) proves to be the most challenging part of the work, which pushed the author to make extra assumption of condition (**) when checking through the RGD system axioms in 6.2.8 of section 6.2.2. However, in 6.2.9, it can be seen that the condition (**) always hold and hence is not needed to be considered as "extra" assumption. Resulting in the "relaxation" on the requirement of the condition (**) in 6.2.8, and hence in the obtainment of the goal of this thesis at 6.2.10.

Noticing the constructions of RGD systems for the (Special) Unitary group case and the general non-split case do not automatically agree, we seek to see if the construction in the Unitary group case can be understood as a specific example of applying the construction in the general case. Hence, at the very end (in 6.2.11) we attempt to investigate this question, unfortunately, though with promising indications, the author for now could not fully prove the two constructions agree.

Part I

Background

Chapter 1

General RGD system

In short, as detailed in [6, ch:8], it can be seen that results for spherical RGD system "extends" to the general case where (W,S) is an arbitrary Coxeter system (As opposed to only spherical).

When we use ϵ in this chapter, we mean that $\epsilon \in \{\pm\}$ unless otherwise stated.

1.1 Definition of general RGD system

First, some preparations:

Definition 1.1.1 (Prenilpotent pair of roots). *[6, 8.41] Consider the Coxeter system (W,S), Σ as the simplicial Coxeter complex of type (W,S) and Φ the set of roots of Σ.*

*Given $\alpha, \beta \in \Phi$, the pair $\{\alpha, \beta\}$ is called **prenilpotent** if $\alpha \cap \beta$ and $(-\alpha) \cap (-\beta)$ each contain at least one chamber. In which case we set:*

$$[\alpha, \beta] := \{\gamma \in \Phi | \gamma \supset \alpha \cap \beta \; and \; -\gamma \supset (-\alpha) \cap (-\beta)\}$$

The left and(or) right "[]" replaced by "()" is constructed as the apparent analog of excluding α and(or) β from the set $[\alpha, \beta]$.

***Note**: This construction immediately implies that if $\gamma \in [\alpha, \beta]$, then $\partial\gamma \supset \partial\alpha \cap \partial\beta$.*

Some results regarding prenilpotent pairs of roots can be seen at [6, ch:8.5.3]; Specifically, [6, 8.44] helps to understand affine roots, and [6, 8.42] proves to be extremely helpful.

For ease of reference, we will clarify axioms for general RGD system:

Definition 1.1.2 (General RGD system). *Let (W, S) be an arbitrary Coxeter system, and let $\Sigma := \Sigma(W, S)$ be its Coxeter complex. We identify $\mathcal{C}(\Sigma)$ with W and take 1 to be the fundamental chamber. Let Φ denote the set of roots (or equivalently, of half apartments) of Σ (and Φ_+ (resp. Φ_-) as positive (resp. negative) set of roots defined by containing (resp. not containing) 1).*

Recall α_s for $s \in S$ denotes roots characterized by

$$\mathcal{C}(\alpha_s) = \{w \in W | l(sw) > l(w)\}$$

And we will use U_s to denote U_{α_s} and U_{-s} as $U_{-\alpha_s}$.

*An **RGD system of type** (W, S) is a triple $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ where T and all U_α's are subgroups of G satisfying the following:*

(RGD0) For all $\alpha \in \Phi$, $U_\alpha \neq 1$.

(RGD1) For all $\alpha \neq \beta$ in Φ such that $\{\alpha, \beta\}$ is prenilpotent, we have

$$[U_\alpha, U_\beta] \leq U_{(\alpha,\beta)}$$

(RGD2) For all $s \in S$ there is a function $m : U_s^* \to G$ such that for all $u, v \in U_s^*, \alpha \in \Phi$:

$$m(u) \in U_{-s}uU_{-s},\ m(u)U_\alpha m(u)^{-1} = U_{s\alpha},\ \text{and } m(u)^{-1}m(v) \in T.$$

(RGD3) For all $s \in S$, $U_{-s} \not\leq U_+ := \langle U_\alpha | \alpha \in \Phi_+\rangle$. (And U_- denotes $\langle U_\alpha | \alpha \in \Phi_-\rangle$).

(RGD4) $G = T\langle U_\alpha | \alpha \in \Phi\rangle$.

(RGD5) T normalizes each U_α; i.e.

$$T \subset \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$$

***Note**: [2, lem:5] also has axioms as above, though (RGD5) is not explicitly mentioned and was hidden within sentences before the axioms. [26, ch:3.3] has a different version of set of axioms as above, but can be shown to be equivalent ([6, 7.124], although the citation was for spherical RGD system, but the required statement of [6, 7.123(b)] has its analog holds in general RGD as shown at [6, 8.79]).*

***Note**: We can replace (RGD3) with either of following:*

(RGD3") For all $s \in S$: $U_{-s} \not\subset U_+$ & $U_s \not\subset U_-$

(RGD3) $TU_+ \cap U_- = \{1\}$*

and obtain an equivalent set of axioms for general RGD system ([6, 8.77] for (RGD3") and "hint 2" on [2, page:404] for (RGD3))*

One most important usage of the RGD system to us is the construction of the twin BN pair:

Theorem 1.1.3 (General RGD system to saturated twin BN pair)**.** *[6, 8.80] Following the notation and context of 1.1.2, consider with such given general RGD system, we construct:*

$$B_\epsilon := TU_\epsilon$$

$$N := \langle T,\ \{m(u)|u \in U_s^*,\ s \in S\}\rangle$$

It can be seen that (G, B_+, B_-, N, S) is a saturated twin BN pair with Weyl group $N/T \cong W$.

This leads to a type-preserving strongly transitive action of G on twin building $\Delta(G, B_+, B_-)$ by left translation.

We may extend the general RGD system to a larger group under some assumptions:

Lemma 1.1.4 (Extending general RGD system)**.** *Consider group G', a subgroup $T' \leq G'$. Let there be $G \leq G'$ such that there is a general RGD system $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ where $T \leq T'$ such that T' normalizes T and each U_α's; It can be seen that $(GT', (U_\alpha)_{\alpha\in\Phi}, T')$ is a general RGD system, the corresponding twin buildings associated to the two RGD systems are canonically identical, and the actions on the twin buildings are also canonically identical.*

Sketch. Checking (RGD 0) to (RGD 5) for $(GT', (U_\alpha)_{\alpha\in\Phi}, T')$ is standard and it pretty much automatically inherit from the assumed RGD system $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ and our further assumptions.

Moving along the construction of 1.1.3, the rest of the statements follow. □

Recall in the spherical case, spherical RGD systems correspond to Moufang buildings; We will extend this result here, and establish the correspondence between general RGD systems and Moufang twin buildings:

Definition 1.1.5 (Twin roots)**.** *[6, 5.190] Let $\alpha = (\alpha_+, \alpha_-)$ be a pair of roots in the twin apartment Σ. We call α a* ***twin root*** *of Σ if $op_\Sigma(\alpha) = -\alpha$. Where*

$$-\alpha := (-\alpha_+, -\alpha_-) \;\&\; op_\Sigma(\alpha) := (op_\Sigma(\alpha_-), op_\Sigma(\alpha_+))$$

$op_\Sigma(\alpha) = -\alpha$, of course, is equivalent to saying:

$$op_\Sigma(\alpha_-) = -\alpha_+ \;\&\; op_\Sigma(\alpha_+) = -\alpha_-$$

[6, 8.11] Which makes it apparent that in a fixed Σ, if a twin root α is in fact uniquely determined by any of $\alpha_\pm$. One usage of this observation is that we could consider a twin root α by looking at only α_+, and so consider the set of twin roots of twin apartment Σ identified with set of roots of Σ_+:

$$\alpha = (\alpha_+, -op_\Sigma(\alpha_+)) \longleftrightarrow \alpha_+$$

[6, 5.195] A ***Twin root of a twin building*** *is a pair $\alpha = (\alpha_+, \alpha_-)$ such that α is a twin root in some twin apartment of the twin building*

Consider for thin building Σ and its set of twin roots Φ, using the codistance context, let $M = (M_+, M_-) \subset \Sigma$, define set of twin roots:

$$\Psi(M) := \{\alpha \in \Phi | \alpha \supset M\}$$

[6, 8.11] Or equivalently:

$$\alpha_+ \supset M_+ \;\& \;-\alpha_+ \supset op_\Sigma(M_-)$$

We call a ***set of twin roots*** *$\Psi \subset \Phi$* ***Convex*** *if it has the form $\Psi = \Psi(M)$ for some M such that* *both $M_\pm$ are non-empty (i.e. contain at least one chamber). The latter of the equivalent definition of convex set of twin root makes it so that a set of twin root is convex if and only if its set of first component is convex in the sense of [6, 7.18]; Which gives us the observation that a convex set of twin roots is always finite ([6, 7.18]). This is helpful in the following way:*

[6, 8.12] There is an order-reversing one to one correspondence between convex pairs $\mathcal{M}$ and convex subsets of Φ. It is given by $\mathcal{M} \mapsto \Psi(\mathcal{M})$ and its inverse by $\Psi \mapsto \cap_{\alpha\in\Psi}\alpha$, one might want to compare this to 8.1.9

Lemma 1.1.6 (Correspondence to twin apartment containing twin root)**.** *[6, 5.198] Let $\alpha = (\alpha_+, \alpha_-)$ be a twin root, let P be a panel in building (in the codistance context) $\mathcal{C}_\epsilon$ that contains precisely one chamber $C \in \alpha_\epsilon$. Then there is a bijection $P \setminus \{C\} \to \mathcal{A}(\alpha)$ ($\mathcal{A}(\alpha)$ as **Set of all twin apartments containing** α) that assigns each $D \in P \setminus \{C\}$ the convex hull of $\{D\} \cup \alpha$. This map can be described by $D \mapsto \Sigma\{D, D'\}$ for D' is the unique chamber contained in $P' \cap \alpha_-$ (P' as the opposite panel to P in some fixed twin apartment having α as a twin root)*

***Note**: [6, 5.199] It can be seen that any twin apartment that contains a twin root α indeed has α as a twin root of it.*

Definition 1.1.7 (panel intersect twin root)**.** *[6, page:450] We are using the codistance context. Given a twin root α and a panel P of $\mathcal{C}_+$ or $\mathcal{C}_-$ that meets α. If $P \cap \alpha$ contains exactly one chamber, then we call P a **boundary panel** of α. And we call P an **interior panel** of α otherwise.*

In case that P is a boundary panel of α, we write $\mathcal{C}(P, \alpha) := P \setminus \{C\}$ where C is the chamber in $P \cap \alpha$

Definition 1.1.8 (Root group for twin context)**.** *[6, 8.16] Consider thick twin building $\mathcal{C}$ of type (W,S). For any twin root α of $\mathcal{C}$, the **root group** U_α is defined to be the set of automorphisms g of $\mathcal{C}$ such that:*

1. *g fixes g fixes α point-wise*
2. *g fixes every interior panel of α point-wise*

***Note**: $U_\alpha \leq Aut_0(\mathcal{C})$. If rank is at least 2, (1) follows from (2).*

Lemma 1.1.9 (On root group in twin context)**.** *[6, 8.17] Following the context of 1.1.8, the following is an analog to that of buildings instead of twin buildings.*

1. *For any twin root α and $g \in Aut_0(\mathcal{C})$, $gU_\alpha g^{-1} = U_{g\alpha}$*

2. *Let α be a twin root and let P be a boundary panel of α. Then the root group U_α acts on the sets $\mathcal{A}(\alpha)$ and $\mathcal{C}(P,\alpha)$ and these two actions are equivalent*

3. *If the Coxeter diagram of (W,S) has no isolated nodes then the actions in (2) are free*

1.2 General RGD and Moufang twin buildings

Definition 1.2.1 (Moufang in twin context). *[6, 8.18] Following the context of 1.1.8. If actions in (2) of 1.1.9 are transitive for every twin root of twin building $\mathcal{C}$, we say $\mathcal{C}$ is **Moufang twin building**. If in addition, these actions are simply transitive, then we say that $\mathcal{C}$ is **strictly Moufang twin building**.*

Observation 1.2.2 (Strictly Moufang twin building to general RGD system). *[6, 8.47] Let $\mathcal{C}$ be a strictly Moufang twin building of type (W,S) with the fundamental apartment Σ and the fundamental chamber $C_+ \in \Sigma_+$. We identify Σ_+ with W and the set of twin roots of Σ with the set of roots Φ of $\Sigma_+ = \Sigma(W,S)$ (See 1.1.5). For each $\alpha \in \Phi$ let $U_\alpha \leq Aut(\mathcal{C})$ be the corresponding root group. Let*

$$G := \langle U_\alpha | \alpha \in \Phi \rangle \leq Aut(\mathcal{C})$$

and

$$T = Fix_G(\Sigma)$$

then $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is a general RGD system.

Theorem 1.2.3 (General RGD system to Moufang twin building). *[6, 8.81]*

Let $\Delta = \Delta(G, B_+, B_-)$ be the twin building associated to the saturated twin BN pair (G, B_+, B_-, N, S) of type (W,S) given birth from a general RGD system $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ as in 1.1.3.

We take fundamental opposite chambers $B_\pm = C_\pm \in \mathcal{C}_\pm$ and denote the fundamental twin apartment $(\Sigma_+, \Sigma_-) = \hat{\Sigma} := \Sigma\{C_+, C_-\}$.

Recall that Φ in this context can be seen as the set of roots for apartment $\Sigma(W,S)$ and for root $\alpha \in \Phi$, as we can canonically identify apartment Σ_ϵ with $\Sigma(W,S)$, we have root α_ϵ in apartment Σ_ϵ associated to root $\alpha \in \Phi$ constructed as

$$\alpha_\epsilon := \{wB_\epsilon | w \in \alpha\}$$

and therefore the twin root $\hat{\alpha} := (\alpha_+, \alpha_-)$ (checking this is twin root is standard procedure) in the fundamental twin apartment $\hat{\Sigma}$ associated to root $\alpha \in \Phi$.

Denoted $U_{\hat{\alpha}}$ be the root group associated to the twin root $\hat{\alpha}$ as in 1.1.8.

In above set up, we have for any $\alpha \in \Phi$, U_α acts simply transitively on $\mathcal{A}(\hat{\alpha})$ and is a subgroup of $U_{\hat{\alpha}}$. Δ is a Moufang twin building. If, in addition, the Coxeter diagram of (W,S) has no isolated nodes, then $U_\alpha = U_{\hat{\alpha}}$ and Δ is strictly Moufang

***Note**: proof of this involve 2-coverings, which is not mentioned here.*

Chapter 2

Basics on affine roots

We gather some basics on affine roots we will need, a good reference is [14]. More than just recording the basics, we will bring the basics in [14] and results in [6] together to obtain some tools we need.
Note 2.0.1 (Affine roots and affine Weyl group)**.** *Consider Ψ be a root system in Euclidean space $V := \mathbb{R}^n$ with $\Pi = \{a_1, \cdots, a_n\}$ as a base of Ψ, we will use $(\cdot,\cdot)$ to denote the positive definite symmetric bilinear form (the inner product associated to the Euclidean space). We will denote $\Psi_\pm$ as the sets of $\pm$ roots respectively, and a_0 a root of maximal height in Ψ_+ (having a unique highest root requires **irreducible** condition of Ψ). In this set up, we will construct the **set of Affine roots associated to** Ψ as*

$$\Phi = \{\alpha_{a,l} | a \in \Psi,\ l \in \mathbb{Z}\} \text{ where } \alpha_{a,l} := \{v \in V | (a, v) \geq -l\}$$

An observation is that $\alpha_{a,l} = -\alpha_{-a,-l}$ by this construction.

*It can be seen that there are **affine hyperplanes** (associate to affine roots) $\partial\alpha_{a,l} = \{v \in V | (a, v) = -l\}$. We will define reflection $s_{a,l}$ to be the **affine reflection** with respect to affine hyperplane $\partial\alpha_{a,l}$.*

We will also denote $H_{a,r} := \{v \in V | (a, v) = -r\}$ for $a \in \Psi$ and $r \in \mathbb{R}$, note that $H_{a,r} = \partial\alpha_{a,r}$ if $r \in \mathbb{Z}$.

We denote $W_{aff}(\Psi) := \langle s_{a,l} | a \in \Psi, l \in \mathbb{Z}\rangle$ to be the ***Affine Weyl group*** *of Ψ.*

We will define n+1 ***simple affine roots*** *$\alpha_0 := \alpha_{-a_0,1}$ and $\alpha_i := \alpha_{a_i,0}$ for all $1 \leq i \leq n$; And further, s_i as affine reflection with respect to α_i for $0 \leq i \leq n$ respectively.*

Recall we define $< b, a > := \dfrac{2(a,b)}{(a,a)}$ for $a, b \in \Psi$ with ***dual root*** *of $a \in \Psi$ being $a^\vee := \dfrac{2a}{(a,a)}$.*

As per requirement of reduced root system that a root a's only two scalar multiples are a and $-a$, determining $e_a := \frac{a}{\sqrt{(a,a)}}$ is equivalent to determining $a \in \Psi$, and so we have a one to one correspondence (whenever $H_{a,r}$ exists) of:

$$(a, r) \leftrightarrow H_{a,r}$$

For a root $a \in \Psi$ and $r \in \mathbb{R}$

Note 2.0.2 (A Summary of results about affine roots)**.** *We give a* ***summary of results regarding affine roots and affine Weyl groups****:*

1. *[14, sec:4.1]: $s_{a,l}$ fixes $\partial\alpha_{a,l}$ point-wise and sends 0 to $-la^\vee$, this implies we may write $s_{a,l} = \tau(-la^\vee) \circ s_{a,0}$ where $\tau(\lambda)$ is* ***translation*** *$\tau(\lambda) : v \mapsto v + \lambda$ for $v, \lambda \in \mathbb{R}^n$.*

2. *It can be seen with routine calculations that:*

$$s_{a,l}(\alpha_{b,m}) = \alpha_{s_{a,0}(b), m-l<b,a>}$$

3. *(**Require the irreducible condition**) Taking S to be the set of affine reflections with respect to to simple affine roots, i.e. $S = \{\alpha_i | 0 \leq i \leq n\}$, it can be seen that $(W_{aff}(\Psi), S)$ is a Coxeter system ([14, sec:4.6]). We can understand the set of half apartments of Coxeter complex $\Sigma(W_{aff}(\Psi), S)$ as the set of affine roots Φ associated to Ψ, and the fundamental chamber being the unique one contained in $\cap_{i=0}^n \alpha_i$ (which we consider as $1 \in W_{aff}(\Psi)$ as per convention, in which case simple root α_i is the unique half apartment that contains 1 but not s_i). Consider the positive affine roots to be the ones that containing the fundamental chamber, then have $\Phi_+ = \{\alpha_{a,l} \in \Phi | (a \in \Psi_+ \text{ and } l \geq 0) \text{ or } (a \in \Psi_- \text{ and } l \geq 1)\}$ (This is not hard to see intuitively).*

4. ***(Require the reduced condition)*** *The pair* $\{\alpha_{a,l}, \alpha_{b,m}\} \subset \Phi$ *is prenilpotent if and only if* $e_a \neq -e_b$ *(Using [6, 8.42](3) with the concept of* ***Nested roots:*** $\{\alpha, \beta\}$ *is prenilpotent if and only if* $\{\alpha, -\beta\}$ *is nested (see definition in [6, 8.42](2), [6, ch:3.6.8] also provide some insight), and as stated in [6, 8.44]. It can be seen that* $\alpha_{a,l}, -\alpha_{b,m}$ *are nested if and only if* $e_a = -e_b$*). And accroding to the reduced requirement of root system, it can be seen equivalently, the pair* $\{\alpha_{a,l}, \alpha_{b,m}\} \subset \Phi$ *is prenilpotent if and only if* $a \neq -b$.

4'. ***(Relex the reduced condition)*** *When the reduced condition on* Ψ *is relexed, we instead have that* $e_a \neq -e_b$ *iff* $ka \neq -nb$ *for any* $k, n \in \mathbb{Z}_{>0}$ *(Note, by construction of root system, the only possible sets of proportional roots are among* $\{\pm a\}$, $\{\pm a, \pm\frac{1}{2}a\}$, *and* $\{\pm a, \pm 2a\}$*). And so* $\alpha_{a,l}$ *and* $\alpha_{b,m}$ *are prenilpotent if and only if* $ka \neq -nb$ *for any* $k, n \in \mathbb{Z}_{>0}$.

5. ***(Require the reduced condition)*** *In the case of* $\{\alpha_{a,l}, \alpha_{b,m}\} \subset \Phi$ *is prenilpotent and* $a \neq b$ *(Or equivalently to the satisfaction of the two conditions at the same time,* $a \neq \pm b$*), then* $[\alpha_{a,l}, \alpha_{b,m}] \supset \{\alpha_{pa+qb,pl+qm} \in \Phi | p, q \in \mathbb{Z}_{\geq 0}\}$. *Further, it can be seen that* $[\alpha_{a,l}, \alpha_{b,m}] = \{\alpha_{pa+qb,pl+qm} \in \Phi | p, q \in \mathbb{Q}_{\geq 0}\}$ *(If further require the root system* Ψ *to be* ***crystallographic****). Here is a sketch for this:*

   To see $\{\alpha_{pa+qb,pl+qm} \in \Phi | p, q \in \mathbb{R}_{\geq 0}\} \subset [\alpha_{a,l}, \alpha_{b,m}]$, *note for* $v \in V$ *and* $p, q \in \mathbb{R}_{\geq 0}$*:*

$$((a, v) \geq -l \;\&\; (b, v) \geq -m) \implies p(a, v) + q(b, v) \geq -pl - qm$$

$$((a, v) \leq -l \;\&\; (b, v) \leq -m) \implies p(a, v) + q(b, v) \leq -pl - qm$$

   Above imply that for $\alpha_{pa+qb,pl+qm} \in \{\alpha_{pa+qb,pl+qm} \in \Phi | p, q \in \mathbb{R}_{\geq 0}\}$, *we have* $\alpha_{a,l} \cap \alpha_{b,m} \subset \alpha_{pa+qb,pl+qm}$ *and* $-\alpha_{a,l} \cap -\alpha_{b,m} \subset -\alpha_{pa+qb,pl+qm}$. *This concludes right-hand side is contained in the left-hand side for the second statement and the first statement.*

   Above is in fact all we need for our sake, but as a further remark: To see the converse inclusion, we need [6, 8.45], and we develop into the two cases according to [6, 8.45]'s two different cases to do so.

5'. ***(Relex the reduced condition)*** *We will only need* $\{\alpha_{pa+qb,pl+qm} \in \Phi | p, q \in \mathbb{R}_{\geq 0}\} \subset [\alpha_{a,l}, \alpha_{b,m}]$, *for which, the proof in above still holds true in the case that* $\{\alpha_{a,l}, \alpha_{b,m}\}$ *is prenilpotent without the reduced condition (note that* $a \neq b$ *is not required here, it is only used in the second half in above item 5, which we do not really need).*

Chapter 3

Basics on Chevalley groups

All results here in this chapter are well-known, and we largely rely on [24]. One can use [24] as the reference for results below.

3.1 Chevalley basis

The main reference of results related to Chevalley basis is [24, Ch:1].

Definition 3.1.1 (F-Lie algebra). *Given a field F, an* ***F-Lie algebra*** *is an F-vector space L together with an* ***Alternating***(*i.e.* $[x, x] = 0 \forall x \in L$) ***bilinear form***(*i.e. linear in both coordinates*) *called* ***Lie bracket*** $[*, *] : L \times L \to L$ *that satisfies the* ***Jacobi identity*** $[x, [y, z]] + [y, [z, x]] + [z, [y, x]] = 0$.

Definition 3.1.2 (representation, adjoint representation, and killing form of Lie algebra, (faithful) module of Lie algebra). *A* ***representation*** *of an F-Lie-algebra L is a* ***Lie algebra homomorphism*** *(i.e. Respects the Lie bracket and is linear over F)* $\phi : L \to \mathfrak{gl}(V)$ *where V is an F-vector space, in this case, V is called an* ***L-module***. ϕ *is called* ***faithful*** *if* $\mathrm{Ker}(\phi) = 0$, *in which case V is a* ***faithful module***.

The ***adjoint representation*** *of L is defined as* $\mathrm{ad} : L \to \mathfrak{gl}(L)$ *such that* $\mathrm{ad}\, x : L \to L$ *by* $y \mapsto [x, y]$ *for* $x \in L$. *Or equivalently denoted with group action according to adjoint representation notation:* $x \cdot y = (\mathrm{ad}\, x)(y) = [x, y]$.

The ***killing form*** *of L, given L is finite dimensional over F is defined to be the symmetric F-bilinear form* $\kappa : L \times L \to F$ *by* $(x, y) \mapsto tr(\mathrm{ad}\, x \, \mathrm{ad}\, y)$

Representation ϕ *gives rise to action of L on V by* $x \cdot v = \phi(x)(v)$ *for* $x \in L, v \in V$. *We define* ϕ *to be* ***irreducible representation*** *(and V* ***irreducible L-module****) if the only* ***invariant subspaces*** *of V (i.e. subspace W of V so that for all* $w \in W$, $L \cdot w \in W$*) are V itself and* $\{0\}$.

From this point on within this chapter, we write $\mathbb{C}$ in place of the field F, but all statements holds when replacing $\mathbb{C}$ with an algebraically close field F with characteristic zero.

Definition 3.1.3 (cartan subalgebra (CSA)). *Given Lie algebra L, a Lie subalgebra H of L is called **cartan subalgebra (CSA)** if and only if $\mathrm{ad}(h)$ is **semisimple** (i.e. $\mathrm{ad}(h) \in \mathrm{End}_{\mathbb{C}}(L)$ is diagonalizable) for all $h \in H$ and H is maximal with such property*

*We **identify CSA** H **and** H^* by a vector space isomorphism via the killing form κ as the following*

$$\iota : H \to H^*$$

$$h \mapsto \kappa(h, *) : H \to \mathbb{C}$$

This will give rise to notion of $t_\alpha := \iota^{-1}(\alpha)$ (i.e. $\kappa(t_\alpha, h) = \alpha(h)$ for all $h \in H$) for $\alpha \in H^*$.

Note 3.1.4 (Cartan decomposition). *Given f.d. s.s. $\mathbb{C}$-Lie algebra L and H CSA of L. Consider $\alpha \in H^*$, we define*

$$L_\alpha := \{x \in L | [h, x] = \alpha(h)x \forall h \in H\}$$

(this is in fact a subspace of L). As an example, it should be observed that

$$L_0 = \{x \in L | [H, x] = \{0\}\} = C_L(H) = H$$

*where the second equal sign is a not trivial result. Define the **Root system of L with respect to H** by*

$$\Phi = \Phi(L, H) := \{\alpha \in H^* \setminus \{0\} | L_\alpha \neq \{0\}\}$$

*We now give the **Cartan decomposition** of L*

$$L = H \oplus (\bigoplus_{\alpha \in \Phi} L_\alpha)$$

This decomposition always exists under the above situation.

Theorem 3.1.5 (About Φ in 3.1.4). *Continuing situation in 3.1.4, let $E := \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}} = \langle \Phi \rangle_{\mathbb{R}}$ where $E_{\mathbb{Q}} := \langle \Phi \rangle_{\mathbb{Q}}$. One will see that there is a non-degenerate bilinear form on H^*, $(*, *)_{H^*} : (\alpha, \beta) \mapsto \kappa(t_\alpha, t_\beta)$ for $\alpha, \beta \in H^*$ with properties:*

(a) $(\alpha, \beta)_{H^*} \in \mathbb{Q}\ \forall \alpha, \beta \in E_{\mathbb{Q}} :=< \Phi >_{\mathbb{Q}}$

(b) $(\alpha, \alpha)_{H^*} > 0$ *for all* $\alpha \in E_{\mathbb{Q}} \setminus \{0\}$

The consequence of these properties of $(*, *)_{H^*}$ is that its $\mathbb{R}$ bilinear extension is an inner product $(*, *) : E \times E \to \mathbb{R}$ on E which makes E into a Euclidean space. From this we obtain:

(a) Φ *is a reduced root system (with the crystallographic condition) in* E *of rank* $l = \dim_{\mathbb{C}}(H)$*.*

(b) Φ *is irreducible if and only if* L *is simple. And more specifically, we have*

$$\Phi = \perp_{i=1}^{m} \Phi \leftrightarrow L = \bigoplus_{i=1}^{m} L_i$$

Where Φ_i *are irreducible components of* Φ *corresponding to* L_i *irreducible ideals of* L*.*

We will denote (per convention) $< \alpha, \beta > := (\alpha, \frac{2\beta}{(\beta,\beta)}) \in \mathbb{Z}$*. And hence the reflection in above* Φ *becomes:*

$$s_\alpha(\beta) = \beta - < \beta, \alpha > \alpha$$

Definition 3.1.6 (Chevalley basis)**.** *[24, thm:1] For f.d. s.s.* $\mathbb{C}$*-Lie algebra* L*, take CSA* H *of* L*. We have root system of* L *with respect to* H *denoted as* $\Phi(L, H)$ *with the base* $\Delta = \{\alpha_1, \cdots, \alpha_l\}$*, recall construction*

$$h_\alpha := \frac{2t_\alpha}{(\alpha, \alpha)} \text{ and } h_i := h_{\alpha_i} \ \forall 1 \leq i \leq l$$

Consider Cartan decomposition

$$L = H \oplus (\bigoplus_{\alpha \in \Phi} L_\alpha)$$

where $\dim_{\mathbb{C}}(L_\alpha) = 1 \ \forall \alpha \in \Phi$*. We define the* ***Chevalley Basis*** *of* L *with respect to* H *and* Δ *as a* $\mathbb{C}$ *basis* $B = \{h_i | 1 \leq i \leq l\} \cup \{x_\alpha | \alpha \in \Phi\}$ *of* L *so that*

(a) $[h_i, h_j] = 0 \ \forall i, j$

(b) $[h_i, x_\alpha] = < \alpha, \alpha_i > x_\alpha \ \forall i, \alpha \in \Phi$

(c) $[x_\alpha, x_{-\alpha}] = h_\alpha \in \bigoplus_{i=1}^{l} \mathbb{Z} h_i \ \forall \alpha$

(d) $[x_\alpha, x_\beta] = c_{\alpha\beta} x_{\alpha+\beta} \ \forall \alpha, \beta \in \Phi$ *so that* $\beta \neq -\alpha$*, with*

$$\mathbb{Z} \ni c_{\alpha\beta} = \begin{cases} 0 & \alpha + \beta \notin \Phi \\ \pm(r+1) & \alpha + \beta \in \Phi \text{ and } r = \max(i \in \mathbb{N}_0 | \beta - i\alpha \in \Phi) \end{cases}$$

Theorem 3.1.7 (Existence of Chevalley basis)**.** *[24, thm:1] Under condition given by 3.1.6, the Chevalley basis* B *in the definition always exists*

3.2 Constructing Chevalley groups

The main reference of results related to construction of Chevalley group is [24, Ch:1,2,3, and 5].

Definition 3.2.1 (Exp map on arbitrarry ring). *Given (not necessarily commutative) ring with 1 R so that $\mathbb{Z} \to R$ by $m \mapsto m \cdot 1$ is injective and $m \in R^* \ \forall m \in \mathbb{Z} \setminus \{0\}$. Then for any nilpotent $a \in R$ (i.e. $a^n = 0$ for some $n \in \mathbb{N}$), we define*

$$exp(a) := \sum_{k=0}^{\infty} \frac{1}{k!} a^k (= \sum_{k=0}^{n-1} \frac{1}{k!} a^k) \in R$$

To clarify, we, by convention, take $\frac{1}{0!} = 1$ and $a^0 := 1$ (hence $exp(0) = 1$).

Remark 3.2.2. *One should note that the requirements of the map on $\mathbb{Z}$ to R in 3.2.1 is given so that $\frac{1}{k!}$ can be defined in R. And naturally nilpotency is needed for well-definedness.*

Note 3.2.3 (Fact on exp). *(a) $ab = ba \implies exp(a + b) = exp(a)exp(b)$*

(b) $exp(a) \in R^*$ and $exp(a)^{-1} = exp(-a)$

In below, consider the conditions given in 3.1.6 (L, H, Φ, Δ) and the Chevalley basis B. Consider $\rho : L \to \mathfrak{gl}(V)$ a (faithful) f.d. over $\mathbb{C}$. We will denote $x \cdot v := \rho(x)(v)$ for $x \in L$ and $v \in V$. We will introduce the set-up first, then the definition of Chevalley group.

Definition 3.2.4 (Weight). *Given $\lambda \in H^*$, define the space*

$$V_\lambda := \{v \in V | h \cdot v = \lambda(h)v \ \forall h \in H\}$$

*If $V_\lambda \neq \{0\}$, the λ is a **weight** of representation ρ, in which case V_λ is the **weight space**. We will also establish the convention that*

$$\Gamma = \Gamma(\rho) := \{\lambda \in H^* | \ \lambda \ is \ a \ weight \ of \ \rho\}$$

Following should shine light on weight for adjoint representation

Exemple 3.2.5 (Of Weight). *Weight space is a generalization of L_α defined above in 3.1.4, which to be precise, we should include the CSA H as well. To make sense of this, review definition of L_α. To be explicit,* ad(x) *for $x \in H \cup \bigcup_{\alpha \in \Phi} L_\alpha$ is a weight, and the L_α's and $L_0 = H$ are weight spaces.*

Lemma 3.2.6 (Properties of weights of representation). *(a) $L_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha} \ \forall \lambda \in H^*, \alpha \in \Phi$ (see [24, Ch:2, Lemma 11])* *(b)* $\lambda_1, \cdots, \lambda_r \in H^*$ *distinct* $\implies \sum_{i=1}^{r} V_{\lambda_i} = \bigoplus_{i=1}^{r} V_{\lambda_i}$

Corollary 3.2.7. *(a)* $V = \bigoplus_{\lambda \in \Gamma} V_\lambda$

(b) $\rho(x_\alpha) \in \mathrm{End}_{\mathbb{C}}(V)$ *is nilpotent for all* $\alpha \in \Phi$

Definition 3.2.8 ((admissible) lattice)**.** *(a) A* $\mathbb{Z}$*-* ***lattice*** M *in* V *is a free abelian subgroup* M *of* V *which is spanned (as a* $\mathbb{Z}$*-module) by a* $\mathbb{C}$ *basis of* V*.*

(b) A lattice M *in* V *is* ***admissible*** *if* $\frac{1}{k!} x_\alpha^k \cdot M \subseteq M \ \forall \alpha \in \Phi, k \in \mathbb{N}_0$

Theorem 3.2.9. *Continuing in the setup of 3.1.6. Any f.d. L-module V contains an admissible lattice M. Which M can be chosen so that*

$$M = \bigoplus_{\mu \in \Gamma} M_\mu$$

for $M_\mu := M \cap V_\mu$

Lemma 3.2.10. *Continuing in the setup of 3.1.6. Consider faithful L-module V, and chosen admissible lattice* $M = \bigoplus_{\mu \in \Gamma} M_\mu$*. Set*

$$L_M := \{x \in L | x \cdot M \subseteq M\} \text{ and } H_M := \{H \cap L_M\} \supseteq \bigoplus_{i=1}^{l} \mathbb{Z} h_i$$

Then have

$$L_M = H_M \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z} x_\alpha$$

and H_M *is a lattice in H. In particular,* $L_M \cap \mathbb{C} x_\alpha = \mathbb{Z} x_\alpha \ \forall \alpha \in \Phi$*.*

Definition 3.2.11 (Partial ordering on weight and admissible basis of lattice)**.** *Continuing 3.2.10*

(a) We give ***partial ordering on*** Γ *by setting* $\mu' < \mu :\iff \mu - \mu'$ *is sum of positive roots*

(b) Given admissible lattice $M = \bigoplus_{\mu \in \Gamma} M_\mu$*. Define an* ***admissible*** $\mathbb{Z}$ ***basis of M*** *(hence a* $\mathbb{C}$ *basis of V) B as a union of* $\mathbb{Z}$ *bases of* M_μ *ordered so that for* $\mu' < \mu$ *the basis of* M_μ *occurs before basis of* $M_{\mu'}$*. We will denote the basis of* M_μ *as* B_μ*, this will make* $B = \bigcup_{\mu \in \Gamma} B_\mu$*.*

Note 3.2.12. *Consider the construction in 3.2.10, we will fix an admissible basis B of admissible lattice M, this validates a matrix (in* $M_{\dim_{\mathbb{C}}(V)}(\mathbb{C})$*) description of action of* $\frac{1}{k!} x_\alpha^k$ *for* $k \in \mathbb{N}_0, \alpha \in \Phi$ *on M with respect to B by* $X_{\alpha,k} := \frac{1}{k!} X_\alpha^k$ *where* $X_\alpha := X_{\alpha,1}$ *describing the action of* x_α *(This is* *saying let* $X_\alpha := \rho(x_\alpha)$*, and let* $X_{\alpha,k} := \frac{1}{k!}\rho(x_\alpha)^k$*). We also set*

$$d_\mu := \mathrm{rk}_{\mathbb{Z}}(M_\mu) = \dim_{\mathbb{C}}(V_\mu) \ for\ \mu \in \Gamma$$

$$d := \mathrm{rk}_{\mathbb{Z}}(M) = \dim_{\mathbb{C}}(V) = \sum_{\mu\in\Gamma} d_\mu$$

Lemma 3.2.13 (Properties of $X_{\alpha,k}$)**.** *We gather some properties:*

(a) $X_{\alpha,k} \in M_d(\mathbb{Z})\ \forall \alpha \in \Phi, k \in \mathbb{N}_0$

(b) X_α *is nilpotent and is upper (lower) triangular if* α *is positive (negative) with zero's as diagonal entries (Consequence of 3.2.6 (a)).*

(c) If $X_\alpha = (x_{ij})_{1\le i,j\le d}$ *then* $\gcd(x_{ij}|1 \le i,j \le d) = 1$

Definition 3.2.14 (Construction of $x_\alpha(t)$)**.** *Given the construction of* X_α *(Which depends on the action of* L *on* $\mathfrak{gl}(V)$ *induced by the representation under our consideration), we will introduce two equivalent methods to define* $x_\alpha(t)$*. Consider for any commutative ring with 1* R*,*

Method 1 *For* $t \in R$*, there is a unique ring homomorphism* $\phi' = \phi'_{R,t} : \mathbb{Z}[T] \to R$ *by* $m \mapsto m{\cdot}1$ *and* $T \mapsto t$ *which induces a group homomorphism* $\phi = \phi_{R,t} : SL_N(\mathbb{Z}[T]) \to SL_N(R)$ *by* $(a_{ij}) \mapsto (\phi'(a_{ij}))$*. As a consequence of 3.2.13,* $TX_\alpha \in \mathbb{Z}[T]$*, recall 3.2.1 have*

$$exp(TX_\alpha) = \sum_{k=0}^{\infty} \frac{1}{k!} T^k X_\alpha^k \in SL_d(\mathbb{Z}[T])$$

We construct $x_\alpha(t) := \phi_{R,t}(exp(TX_\alpha))$*.*

Method 2 *We can construct ring homomorphism* $\phi : \mathbb{Z} \to R$*, which is fully determined by* $1 \mapsto 1_R$*. This can be canonically extended to a ring homomorphism* $\phi_R : M_d(\mathbb{Z}) \to M_d(R)$ *by applying* ϕ *on entries. We will denote* $\overline{X_{\alpha,k}} := \phi_R(X_{\alpha,k})$*. We construct*

$$x_\alpha(t) := I_d + \sum_{k=1}^{\infty} t^k \overline{X_{\alpha,k}} \in SL_d(R) \ for\ t \in R$$

Lemma 3.2.15 (Property of $x_\alpha(t)$)**.** *For* $s,t \in R$*,* $x_\alpha(t)x_\alpha(s) = x_\alpha(s+t)$ *for all* $\alpha \in \Phi$*.*

Definition 3.2.16 (Chevalley group associated to f.d. faithful representations)**.** *For any f.d. faithful representation* $\rho : L \to \mathfrak{gl}(V)$ *and any integral domain* R *with some algebraically closed field* K *containing it (Consider, for instance, the algebraic closure for the fractional field of* R*), we* *define **Chevalley group**:*

$$\mathcal{G}_V(R) := \langle x_\alpha(t) | \alpha \in \Phi, t \in K \rangle \cap SL_d(R)$$

We further set

$$U_\alpha = U_\alpha(R) := \{x_\alpha(t) | t \in R\}$$

*which is the **root group** associated to α, and we denote $U^\pm :=< U_\alpha | \alpha \in \Phi^\pm >\leq \mathcal{G}_V(R)$*

*We define the **Elementary subgroup of Chevalley group** as the subgroup of $\mathcal{G}_V(R)$ generated by the root groups, or equivalently as*

$$\mathcal{G}_V(R)^+ := \langle x_\alpha(t) | t \in R \rangle$$

As a remark, $\mathcal{G}_V(K) = \mathcal{G}_V(K)^+$ for any algebraically closed field K. If R is commutative ring but not integral domain, we may still construct elementary subgroup of Chevalley group in the same way.

Remark 3.2.17. *By 3.2.13 (b), U^+ (U^-) is group of strictly upper (lower) triangular matrices in the sense that the diagonal entries must be 1's (Also called uni-triangular). In particular, is unipotent hence nilpotent.*

Lemma 3.2.18 (Group isomophism on R to root group). *For all $\alpha \in \Phi$, the map $(R, +) \to U_\alpha$ by $t \to x_\alpha(t)$ is a group isomorphism.*

3.3 Further comment on Chevalley groups

A good reference for the following is [24, Ch:3, Ch:5].

Definition 3.3.1 (root lattice, weight lattice, fundamental group). *[24, lem:27] Let Φ be a root system in E (as in 3.1.5) with the base $\Delta = \{\alpha_1, \cdots, \alpha_l\}$:*

(a) $\Lambda_r := \sum_{\alpha\in\Phi} \mathbb{Z}\alpha = \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i$ *as **root lattice of root system** Φ.*

(b) $\Lambda_w := \{\lambda \in E | < \lambda, \alpha >\in \mathbb{Z}\ \forall \alpha \in \Phi\}$ *is **weight lattice of root system** Φ. Recall concept of dual root system and the dual roots $\alpha^\vee$. By construction, $< \lambda, \alpha >= (\lambda, \alpha^\vee)$ for $\lambda \in \Lambda_w$ and $\alpha \in \Phi$, this implies that the weight lattice $\Lambda_w(\Phi)$ is the dual of $\Lambda_r^\vee := \Lambda_r(\Phi^\vee)$.*

(c) $\lambda \in \Lambda_w$ *is **dominant** if $(\lambda, \alpha) \geq 0\ \forall \alpha \in \Phi^+$. **Fundamental dominant weights** $\lambda_i, i \in [1, l]$*

are defined by $<\lambda_i, \alpha_j>=(\lambda_i, \alpha_j^\vee)=\delta_{ij}$ *for all* $1 \le i, j \le l$ *and this implies that*

$$\Lambda_w = \bigoplus_{i=1}^{l} \mathbb{Z}\lambda_i$$

(d) $\Lambda_r \subseteq \Lambda_w$ *as* $<\alpha, \beta> \in \mathbb{Z}\ \forall \alpha, \beta \in \Phi$. *And the* ***fundamental group of root system*** Φ *is* $\dfrac{\Lambda_w}{\Lambda_r}$.

(e) *We define the* ***weight lattice corresponding to f.d representation (or L-module V)*** $\rho : L \to \mathfrak{gl}(V)$ *as*

$$\Lambda = \Lambda(V) := <\Gamma>_{\mathbb{Z}} = \sum_{\lambda \in \Gamma} \mathbb{Z}\lambda \subseteq H^*$$

Theorem 3.3.2. *[24, lem:27] Continuing 3.3.1*

(a) $\Lambda \subseteq \Lambda_w$

(b) *if* ρ *is faithful, then* $\Lambda_r \subseteq \Lambda$

Note 3.3.3 (On Chevalley groups)**.** *Turns out, up to isomorphism,* $\mathcal{G}_V(R)$ *does not depend on the specific choices of representation, admissible lattice M, or the admissible basis. All that matters (Given L) is the weight lattice associated to L-module V denoted* $\Lambda = \Lambda(V) = <\Gamma>_{\mathbb{Z}}$. *Recall that s.s. f.d.* $\mathbb{C}$*-Lie algebra L is classified by its root lattice* Φ *up to isomorphism, we may "replace" the present of L with its root lattice* Φ *(One may see more explanation about this at [10, p23: Cor 5] and its related results). As discussed, [24, Ch:5, Existence Theorem]:* ***Chevalley group is determined by the (crystallographic) root system of Lie algebra and the weight lattice associated to the module of Lie algebra****, we will denote* $\mathcal{G}_{\Phi,\Lambda}(R) = \mathcal{G}_V(R)$ *given faithful representation* $\rho : L \to \mathfrak{gl}(V)$. *Recall that given faithful representation, we have* $\Lambda_r \subseteq \Lambda \subseteq \Lambda_w$. *If we have condition that* $\Lambda = \Lambda_r = \Lambda(L)$ *(e.g. adjoint representation),* $\mathcal{G}_{\Phi,\Lambda_r}(R)$ *is the* ***adjoint Chevalley group****. If* $\Lambda = \Lambda_w$, $\mathcal{G}_{\Phi,\Lambda_w}(R)$ *is* ***simply connected*** *or of* ***universal type****. If* $R = \mathbb{C}$, *then* $G = \mathcal{G}_{\Phi,\Lambda}(\mathbb{C})$ *is s Lie group with*

$$\pi_1(G) = \begin{cases} \{1\} & \Lambda = \Lambda_w \\ \Lambda_w/\Lambda_r & \Lambda = \Lambda_r \end{cases}$$

For some more specific cases, $\mathcal{G}_{A_{n-1},\Lambda_r}(R) = PE_n(R)$ *and* $\mathcal{G}_{A_{n-1},\Lambda_w}(R) = E_n(R)$

Theorem 3.3.4. *Given dominant weight* $\lambda \in \Lambda_w$, *there exists irreducible f.d. L-module* $V(\lambda)$ *having* λ *as the highest weight.*

Corollary 3.3.5. *For any lattice* Λ *sits in between the root and weight lattice of root system, there* *is f.d. faithful representation $\rho : L \to \mathfrak{gl}(V)$ having Λ as its weight lattice associating to L-module V.*

Theorem 3.3.6. *3.3.4 and 3.3.5 gives us that for any root system Φ and lattice Λ with $\Lambda_r \subseteq \Lambda \subseteq \Lambda_w$, there exists the Chevalley group $\mathcal{G}_{\Phi,\Lambda}(R)$ for every integral domain R*

3.4 Relations in Chevalley group

Results below are also well-known, a good reference is [24, Ch:6] where details are included; But there are also very good summaries, for instance [23]

Note 3.4.1 (Setup fore relations in Chevalley groups)**.** ***In following, we consider this Setup:*** *Take field F, elementary subgroup of Chevalley group $G = \mathcal{G}_{\Phi,\Lambda}(F)^+$, and admissible lattice $M = \bigoplus_{\mu\in\Gamma} M_\mu$ (See 3.2.9). Set $V_F := F \otimes_{\mathbb{Z}} M = \bigoplus_{\mu\in\Gamma} V_{\mu,F}$ and $V_{\mu,F} := F \otimes_{\mathbb{Z}} M_\mu$. We will consider $G \leq GL_d(F) = GL(V_F)$. For any $\alpha \in \Phi$, we have rank 1 subgroup $G_\alpha :=< U_\alpha \cup U_{-\alpha} >\leq G$ (G_α* ***can be shown to be isomorphic to either*** $SL_2(F)$ ***or*** *$PSL_2(F)$, as stated by [23, 3.6], there is onto group homomorphism on $SL_2(R)$ to G_α by $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto x_\alpha(t)$ and $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto x_{-\alpha}(t)$)*

Definition 3.4.2 (Special elements and subgroups in Chevalley group over field)**.** *We first introduce some elements in G_α, since $G_\alpha \cong SL_2(F)$ or $PSL_2(F)$, we will also list the explicit correspondence (Consider $t \in F$ for the first one and $t \in F^*$ for the rest):*

- $x_\alpha(t) \leftrightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ & $x_{-\alpha}(t) \leftrightarrow \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$
- $w_\alpha(t) := x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t) \leftrightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}$
- $w_\alpha := w_\alpha(1) \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- $h_\alpha(t) := w_\alpha(t)w_\alpha^{-1} \leftrightarrow \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$

Or as [24, Cor:6], take $\phi_\alpha : SL_2(F) \to G_\alpha$ by sending the matrices on the right hand sides above to the elements on the left side, then $\mathrm{Ker}(\phi_\alpha) = \{\mathrm{Id}\}$ or $\{\pm\mathrm{Id}\}$ *(So that, as mentioned before, $G_\alpha \cong SL_2(F)$ or $PSL_2(F)$). It should be noted that: $w_\alpha(t)^{-1} = w_\alpha(-t)$ and $w_\alpha^{-1} = w_\alpha(-1)$. We* *now introduce some subgroups of G:*

$$T := < h_\alpha(t) | \alpha \in \Phi, t \in F^* > \leq \ N := < w_\alpha(t) | \alpha \in \Phi, t \in F^* >$$

Lemma 3.4.3 (About $w_\alpha(t)$ and $h_\alpha(t)$)**.** *Given $\alpha \in \Phi$ and $t \in F^*$ (F is some field):*

(a) For $v \in V_{\mu,F}$, there is a $v' \in V_{\sigma_\alpha(\mu),F}$ (Independent of t) so that $w_\alpha(t)v = t^{-<\mu,\alpha>}v'$.

(b) $h_\alpha(t)$ acts by the multiplication with $t^{<\mu,\alpha>}$ on $V_{\mu,F}$.

Note 3.4.4 (Of 3.4.3)**.** *Recall that $< \mu, \alpha > := \dfrac{2(\mu,\alpha)}{(\alpha,\alpha)}$ and since $\mu \in \Gamma \subset < \Gamma >_{\mathbb{Z}} = \Lambda \subset \Lambda_w$ (The last inclusion by 3.3.2), and $< \mu, \alpha > \in \mathbb{Z}$. We have $\sigma_\alpha(\mu) = \mu - < \mu, \alpha > \alpha \in \Gamma$ when $\mu \in \Gamma$.*

Lemma 3.4.5 (By 3.4.3)**.** *$\alpha, \beta \in \Phi$, $s, t \in F$, $t \neq 0$:*

(a) There exists $c(\alpha,\beta) \in \{\pm 1\}$ independent of s, t so that

$$w_\alpha(t)x_\beta(s)w_\alpha(t)^{-1} = x_{\sigma_\alpha(\beta)}(c(\alpha,\beta)t^{-<\beta,\alpha>}s)$$

Moreover, $c(\alpha,\beta) = c(\alpha,-\beta)$ and $c(\alpha,\alpha) = -1$.

(b) $h_\alpha(t)x_\beta(s)h_\alpha(t)^{-1} = x_\beta(t^{<\beta,\alpha>}s)$.

(c) $w_\alpha h_\beta(t) w_\alpha^{-1} = h_{\sigma_\alpha(\beta)}(t)$.

Corollary 3.4.6. *This is given to us by 3.4.3 and 3.4.5. For all $\alpha, \beta \in \Phi$ and $t \in F^*$ (F is some field):*

(a) $w_\alpha(t)U_\beta w_\alpha(t)^{-1} = U_{\sigma_\alpha(\beta)}$.

(b) $h_\alpha(t)U_\beta h_\alpha(t)^{-1} = U_\beta$.

(c) T is commutative (being isomorphic to a subgroup of $\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} | * \in F^*\}$) and $T \trianglelefteq N$ (explicitly, $w_\alpha(s)h_\beta(t)w_\alpha(s)^{-1} = w_\alpha h_\beta(t) w_\alpha^{-1} = h_{\sigma_\alpha(\beta)}(t)$).

One will see that above 3.4.6 is key of constructing RGD system out of Chevalley group over fields.

Theorem 3.4.7 (Collection of relations)**.** *Consider $\alpha, \beta \in \Phi$, $s, t \in F$ (F is some field):*

(R1) $x_\alpha(s + t) = x_\alpha(s)x_\alpha(t)$.

(R2) $[x_\alpha(s), x_\beta(t)] = \prod_{i,j\in\mathbb{N}; i\alpha+j\beta\in\Phi} x_{i\alpha+j\beta}(\overline{c_{\alpha\beta;ij}}s^it^j)$. *With* $c_{\alpha\beta;ij} \in \mathbb{Z}$ *if* $\beta \neq \pm\alpha$

(R3) $w_\alpha h_\beta(t) w_\alpha^{-1} = h_{\sigma_\alpha(\beta)}(t)$ *for* $t \neq 0$.

(R4) $w_\alpha x_\beta(t) w_\alpha^{-1} = x_{\sigma_\alpha(\beta)}(c(\alpha,\beta)t)$ *with* $c(\alpha,\beta) = \pm 1$ *if* $t \neq 0$.

(R5) $h_\alpha(t) x_\beta(s) h_\alpha(t)^{-1} = x_\beta(t^{<\beta,\alpha>}s)$ *if* $t \neq 0$.

(R6) $h_\alpha(s) h_\alpha(t) = h_\alpha(st)$ *if* $s, t \neq 0$ *(This can be seen by 3.4.3 (b)).*

(R4)' $w_\alpha(t) x_\beta(s) w_\alpha(t)^{-1} = x_{\sigma_\alpha(\beta)}(c(\alpha,\beta)t^{-<\beta,\alpha>}s)$ *with* $c(\alpha,\beta) = \pm 1$ *if* $t \neq 0$.

***Note**: Similar relations are also stated in [17, lem:2.3,lem:2.4], but only considered over Laurent polynomial rings.*

Note 3.4.8 (Extending the relations to integral domain)**.** *In the above 3.4.1, 3.4.7, and 3.4.2. We may replace all field F with some integral domain R and same statements will still hold, we can see this by considering $\mathcal{G}_V(R)^+$ lives inside the $\mathcal{G}_V(F)^+$ for some field F containing R, in fact we can even consider F to be algebraically closed (Consider the algebraic closure of fractional field of R).*

The intention, as spoiled by the title, was to work toward a BN pair through construction of RGD system. But it does not hurt to have a taste of their benefit before that:

Theorem 3.4.9 (Steinberg)**.** *(a) If $\Lambda = \Lambda_w$ (i.e. G is of universal or equivalently simply connected type), then* $G =< x_\alpha(t), \alpha \in \Phi, t \in F | (R1) - (R6) >$

(b) If Φ has no irreducible component of rank 1 (i.e. of type A_1), then (R3) to (R5) are consequences of (R1) to (R2). In particular, in that case $G =< x_\alpha(t), \alpha \in \Phi, t \in F | (R1), (R2), (R6) >$

Chapter 4

Basics on isotropic reductive groups

As a reminder, we will consider fields $k \subset K$ where K is algebraically closed while k is not necessarily algebraically closed in this chapter.

4.1 Background in Algebraic geometry

We will use the classical view of algebraic geometry, only listing the results and concepts we need. Good references for this purpose are [4, sec:1.1] and [15, Ch: XII].

Note 4.1.1 (Basic definitions and facts in Algebraic geometry). *Consider an algebraically closed field K and a subfield k (not necessarily algebraically closed) of K:*

1. *For a subset $S \subset K[T_1, \cdots, T_n]$, $Zero(S) := \{x \in K^n | F(x) = 0, \forall F \in S\}$ is a* ***Affine (algebraic) Variety (over K)****. These affine varieties are considered to be closed sets in* ***Zariski topology*** *over K^n.*

2. *For a subset $X \subset K^n$, $Ideal(X) := \{F \in K[T_1, \cdots, T_n] | F(X) = 0\} \trianglelefteq K[T_1, \cdots, T_n]$ is the* ***Vanishing ideal of X****.*

3. *For Affine variety X, $K[X] := \frac{K[T_1, \cdots, T_n]}{Ideal(X)}$ is the* ***Associated affine algebra****.*

4. *For affine varieties $X \subset K^n$ and $Y \subset K^m$, $\phi := (\phi_1, \cdots, \phi_m) : X \to Y$ is a* ***Morphism*** *if each $\phi_i \in K[X]$ for all $i \in [1, m]$.*

5. *Affine variety X in K^n is* ***k-closed*** $:\iff X = Zero(Ideal(X) \cap k[T_1, \cdots, T_n])$.

6. *Affine variety X in K^n is* ***Defined over k*** *or equivalently a* ***k-variety***

$$:\iff Ideal(X) = (Ideal(X) \cap k[T_1, \cdots, T_n])K[T_1, \cdots, T_n]$$

$$\iff K[X] = k[X] \otimes_k K \text{ where } k[X] := \frac{k[T_1, \cdots, T_n]}{(Ideal(X) \cap k[T_1, \cdots, T_n])}$$

7. *For k-varieties $X \subset K^n$ and $Y \subset K^m$, morphism $\phi = (\phi_1, \cdots, \phi_m) : X \to Y$ is **defined over k** or equivalently a **k-morphism** if each $\phi_i \in k[X]$ for all $i \in [1, m]$.*

8. *Consider field k' such that $k \subset k' \subset K$. For affine k-closed variety X, $X(k') := X \cap (k')^n$ are the k'**-(rational) points** of X.*

9. *Consider $X_k \subset k^n \subset K^n$ **Affine closed set in** k^n (i.e. $X_k = Zero(S)$ for some $S \subset k[T_1, \cdots, T_n]$), let $Ideal_k(X_k) := \{F \in k[T_1, \cdots, T_n] | F(X_k) = 0\} \trianglelefteq K[T_1, \cdots, T_n]$. Let $X := Zero(Ideal_k(X_k)) = \overline{X_k}$ (The Zariski closure in K^n), then have $Ideal(X) = Ideal_k(X_k)K[T_1, \cdots, T_n]$, X is a k-variety, and $X_k = X(k)$.*

10. *For affine k-closed variety X in K^n, there exists a finite purely inseparable extension of k, k' such that X is a k'-variety.*

4.2 Algebraic groups

All following results and concepts are well-known, and we will follow mostly according to [8]. Of course, there are also good alternative references, including [15], [20], and [3].

4.2.1 Basics on Algebraic groups

Definition 4.2.1 (Algebraic groups). *[8, 1.1]. We will only consider the affine case: An **Affine algebraic group** is an affine algebraic vriety G together with*

(id) *An element $e \in G$*

(mult) *A morphism of affine varieties $\mu : G \times G \to G$ denoted with $(x, y) \mapsto xy$*

(inv) *A morphism of affine varieties $i : G \to G$ denoted with $x \mapsto x^{-1}$*

With respect to which G forms a group.

*G is a **k-group** if G is a k-variety and if morphisms μ and i are both defined over k. In this case, it can be shown that $e \in G(k)$, making $G(k)$ a group as well.*

*A **morphism** of algebraic groups is a morphism of varieties which is also a homomorphism of groups. The expression "$\alpha : G \to G'$ is a **k-morphism of k-groups**" means G and G' are k-groups and α is a morphism defined over k.*

Definition 4.2.2 (Connected component of e). *[8, 1.2] The* ***connected component of e (or Identity component)*** *in an algebraic group G will be denoted G^o. It can be seen that G^o is a subgroup of finite index in G, and if G is defined over k, then so is G^o.*

Definition 4.2.3 (rational representation and embedding). *A morphism of affine algebraic groups $\phi : G \to GL(V)$ for some finite dimensional vector space over K is called a* ***rational representation of*** *G.*

Further, in fact for affine k-group G, G is k-isomorphic to a closed subgroup (defined over k) of some $\mathbf{GL}_N$ (See [8, 1.10]). That is we always have a faithful k-rational representation of G.

Definition 4.2.4 (Characters). *[8, 5.2] Let G and G' be k-groups. We shall write $Mor(G, G')$ for the algebraic group morphisms on G to G' and $Mor(G, G')_k$ for the set of those that is defined over k.*

We write $X(G) := Mor(G, \mathbf{GL_1})$ and call its elements ***characters*** *of G. Since $\mathbf{GL}_1$ is commutative, $X(G)$ becomes an abelian group where $(a_1 + a_2)(g) = a_1(g)a_2(g)$ for $a_1, a_2 \in X(G)$.*

We also write $X(G)_k := Mor(G, \mathbf{GL_1})_k$.

$\chi \in X(G)$ means that $\chi \in K[G]$, $\chi(g) \neq 0$ and $\chi(gg') = \chi(g)\chi(g')$ for all $g, g' \in G$.

And $\chi \in X(G)_k$ means further that $\chi \in k[G]$.

Definition 4.2.5 (Tori). *[4, Definition 5][8, 8.4][8, 8.11] An algebraic group that is isomorphic (as an algebraic group) to $(K^*)^l$ is called a* ***Torus of dimension l***

A k-torus T is ***k-split*** *if it is k-isomorphic to $(K^*)^l$, or equivalently for following equivalent conditions:*

1. *$X(T) = X(T)_k$*
2. *Every rational representation defined over k is k diagonalizable (i.e. the image of the k-rational representation is conjugate over k to a subgroup of $D_n := \{g \in \mathbf{GL_n} | g_{ij} = 0 \text{ for } i \neq j\}$)*

As stated in [8, 18.2] for connected k-group G, it must contain a maximal torus that is defined over k. So we may choose maximal torus of a connected k-group to be defined over k.

Definition 4.2.6 (Cocharacters). *[8, 8.6] [20, sec:3.2] For k-group G, consider*

$$X_*(G) = Mor(\mathbf{GL_1}, G)$$

Elements of $X_*(G)$ *are called* ***cocharacters****. We have map*

$$X(G) \times X_*(G) \to \mathbb{Z} = X(\mathbf{GL_1})$$

by $<\gamma, \chi> := m$ *if* $(\gamma \circ \chi)(x) = x^m$ *for* $x \neq 0$*.* $X_*(G)$*, similar to* $X(G)$*, becomes abelian groups if* G *is commutative; with* $(\chi_1 + \chi_2)(\lambda) = \chi_1(\lambda)\chi_2(\lambda)$ *for* $\chi_1, \chi_2 \in X_*(G)$*. And hence the above map becomes a bilinear map of abelian groups if G is commutative. It can be seen (see more at [8, page:115]): For a torus T, we have*

$$X(T) \times X_*(T) \to \mathbb{Z}$$

which we often denote with $< \cdot, \cdot >$*. More information also present in [20, sec: 3.2]*

Definition 4.2.7 (Left and right translation)**.** *[8, 1.9] Consider* $g \in G$*,* $f \in A := K[G]$ *and* $x \in G$ *with* $f(x)$ *being the evaluation (See elements of* $K[G]$ *as functions on G). We define* ***Left and right translation by*** g *to be* λ_g *and* ρ_g *respectively with:*

1. $(\lambda_g(f))(x) = f(g^{-1}x)$
2. $(\rho_g(f))(x) = f(xg)$

In particular, λ_g *are* ρ_g *are both linear automorphisms of* $K[G]$*.*

Definition 4.2.8 (k derivation to Lie algebra of algebraic group)**.** *[8, p. AG.15.1] When a k-algebra A is commutative, we can regard an A-module M as a bi-module such that* $ax = xa$ *for* $a \in A$ *and* $x \in M$*. Under such set up, a* ***k-derivation*** *from A to M is a k-linear map* $X : A \to M$ *such that*

$$X(ab) = (X(a))b + a(X(b)) \ (a, b \in A)$$

Since $X(ab) = aX(b)$ *for* $a \in k$*, by taking* $b = 1$ *we have that* $X(a) = 0$ *for* $a \in k$*.*

The set $Der_k(A, M)$ *of all such k-derivation is an A-module which is functorial in M.*

We sometimes denote $Der_k(A, A) = Der_k(A)$ *a k-vector space of derivations of A.*

For following, we denote $A := K[G]$ *for an affine algebraic group G. We define ([8, 3.3]):*

$$Lie(G) := \{D \in Der_K(A, A) | \lambda_x \circ D = D \circ \lambda_x, \forall x \in G\}$$

where λ_x is as in 4.2.7. Note for any $y \in G$, $f \in A$, we have:

$$((\lambda_x \circ D)(f))(y) = (\lambda_x(D(f)))(y) = (D(f))(x^{-1}y)$$

$$((D \circ \lambda_x)(f))(y) = (D(\lambda_x(f)))(y)$$

So the condition in definition of $Lie(G)$ can be interpreted as:

$$(D(f))(x^{-1}y) = (D(\lambda_x(f)))(y)$$

For all $x, y \in G$.

Consider Lie(G) has Lie algebra structure over K with operation:

$$[D, D'] = D \circ D' - D' \circ D$$

*for $D, D' \in Lie(G)$. And hence we will denote $\mathcal{L}(G) := Lie(G)$ to be the **Lie algebra of algebraic group** G.*

In the following, we construct the $\mathcal{L}(G)$ through the method of tangent space:

*Note that following will work for $A := K[G]$ for affine variety G, (We mostly concern with the setup where G is an affine algebraic group, hence the notation). [3, p. 3.1.2 (iii)]: Consider for $g \in G$, the evaluation homomorphism $e_g : A \to K$ by $f \to f(g) \in K$. This helps to define the **Point derivation vector space at g**: $Der(A)_g := Der_K(A, K)$ as through e_g, K becomes an A-module by (For $c \in K$, $f \in A$) $f \cdot c := e_g(f)c$. This procedure is similar to procedure for $K[t_1, \cdots, t_n]$ instead of $A := K[G]$, see detail [3, p. 3.1.2 (i) (ii)].*

[3, 3.1.3]'s Corollary states: with D_j's as defined in [3, p. 3.1.2 (i)], we have basis $\{D_1, \cdots, D_n\}$ for $Der_K(K[t_1, \cdots, t_n])$ as $K[t_1, \cdots, t_n]$-module. Which, through evaluation, induces a K-vector space basis $\{D_1|_x, \cdots, D_n|_x\}$ (where $D_j|_x := e_x \circ D_j$ $(x \in K^n)$) for $Der(K[t_1, \cdots, t_n])_x$. As a consequence, we have a K-vector space isomorphism between $K^n \to Der(K[t_1, \cdots, t_n])_x$ by $v := (v_1, \cdots, v_n) \mapsto d_v := \sum_{i=1}^n v_i D_i|_x$.

*We define **Tangent space of G at g** by:*

$$T(G)_g := V(g, G) := \{v \in K^n | d_v(F) = 0 \; \forall F \in Ideal(G)\}$$

As stated in [3, 3.1.4], denoting $\pi : K[t_1, \cdots, t_n] \to K[G]$ the canonical homomorphism, we have

$V(g,G) \to Der(A)_g$ by $v \mapsto \widetilde{d_v} := d_v \circ \pi^{-1}$ is an isomorphism of K-vector spaces.

Now we return to requiring G to be an affine algebraic group (keeping notation $A := K[G]$) and denote its identity element $1 \in G$. [3, 3.3.3]: Denoting $\hat{e} : Der_K(A) \to Der(A)_1 \cong T(G)_1$ by $D \to e_1 \circ D$, construct $\Theta := \hat{e}|_{\mathcal{L}(G)} : \mathcal{L}(G) \to T(G)_1$. It can be seen that Θ is an K-vector space isomorphism with mutual inverse $* : Der(A)_1 \to \mathcal{L}(G)$ by $D \mapsto *D$ where for $f \in A$, $(*D)(f) = f * D \in A$ such that $(f * D)(g) = d(\lambda_{g^{-1}} f)$ for all $g \in G$ and $f \in A$. (Recall that $(\lambda_{g^{-1}} f)(y) = f(gy)$ for all $y \in G$) This brings Lie algebra structure of $\mathcal{L}(G)$ to $T(G)_1$, explicitly for $D_1, D_2 \in T(G)_1$ and $f \in A$:

$$[D_1, D_2](f) = D_1(f * D_2) - D_2(f * D_1)$$

And now we have constructed "the same" $\mathcal{L}(G)$ as $T(G)_1$.

Definition 4.2.9 (Weights and roots with respect to tori)**.** *[8, 8.17] Consider algebraic group G and a subtorus T of G.*

Let $\pi : T \to GL(V)$ be a rational representation of T. If $\alpha \in X(T)$ we write

$$V_\alpha = \{v \in V | t \cdot v = \alpha(t)v \forall t \in T\}$$

Since T is diagonalizable, V can be shown to be direct sum of V_α's:

$$V = \bigoplus_{\alpha \in X(T)} V_\alpha$$

*Those α for which $V_\alpha \neq 0$ are called the **weights (with respect to π) of T** and in which cases V_α is called a **weight spaces** (note that this can include the zero weight), they are finite in number ([8, 5.2]).*

Consider adjoint rational representation $Ad : G \to GL(\mathfrak{g} := \mathcal{L}(G))$ (by $Ad(x)(X) = x \cdot X \cdot x^{-1}$ for $x \in G$ and $X \in \mathfrak{g}$ where the multiplication on the right-hand side are matrix multiplication when embedding G as a k-subgroup of some $\mathbf{GL_N}$ and X considered within $\mathfrak{gl}_N(K)$, see [8, 3.13]). Construct weight spaces:

$$\mathfrak{g}_0 := \{x \in \mathfrak{g} | (Ad(t))(x) = x \forall t \in T\} = \mathcal{L}(C_G(T))$$

$$\mathfrak{g}_a := \{x \in \mathfrak{g} | (Ad(t))(x) = a(t)x \forall t \in T\} \neq 0$$

for $0 \neq a \in X(T)$

In this case we write $\Phi(G,T) := \{0 \neq a \in X(T) | \mathfrak{g}_a \neq 0\}$ the ***set of roots (of G) with respect to T***

Note it can be seen that:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (\bigoplus_{a \neq 0} \mathfrak{g}_a)$$

Remark 4.2.10 (Lie algebra of Chevalley group)**.** *According to [24, Ch:2, Corollary 1], [24, Ch:2, Corollary 3], and Remark above [24, Ch:5, Corollary 1]. We see that for algebraically closed field K, Lie algebra of Chevalley group $\mathcal{G}(K)$ is $L_M \otimes_{\mathbb{Z}} K$ where L_M is as defined as in 3.2.9.*

4.2.2 Reductive groups

Note 4.2.11 (Jordan decomposition)**.** *[8, p. 4.1-4.5] Consider a finite dimensional vector space V over K, and $E := End_K(V)$. We define $a \in E$ to be* ***nilpotent*** *if $a^n = 0$ for some positive n, and to be* ***unipotent*** *if $a - 1$ is nilpotent.*

We define $a \in E$ to be ***semisimple*** *if it satisfies following equivalent condition:*

1. *V is spanned by eigenvectors of a (i.e. a is diagonalizable over K).*
2. *Algebra $K[a] \subset E$ is semisimple (i.e. is a product of copies of K).*

Then for $a \in E$ we have ***Additive Jordan decomposition of** a **in** E: There exists unique a_s and a_n in E such that a_s is semisimple, a_n is nilpotent, $a_s a_n = a_n a_s$, and $a = a_s + a_n$.*

This additive Jordan decomposition in turn implies following ***Multiplicative Jordan decomposition of** g **in** $GL(V)$: Consider $g_u := Id + g_s^{-1} g_n$. We have for $g \in GL(V)$, there exists unique decomposition $g = g_s g_u = g_u g_s$ with g_s semisimple, and g_u (as constructed) unipotent.*

We construct ***Multiplicative Jordan decomposition in k-group****: We extend above definition to affine algebraic k-group G by consider a faithful k-rational representation embedding G into $GL(V)$ for some V. Consider $g \in G \subset GL(V)$ has unique multiplicative Jordan decomposition $g = g_s g_u$, one then check that $g_s, g_u \in G$ with help of [8, 3.8]:*

$$G = \{g \in GL(V) | \rho_g Ideal(G) = Ideal(G)\}$$

For affine k-group G, we define sets:

$$G_s := \{g \in G | g = g_s\}$$

$$G_u := \{g \in G | g = g_u\}$$

The ***Semi-simple and Unipotent parts of G****. In general, it can be seen that* $G_s \cap G_u = \{1_G\}$*, and* G_u *is always a k-closed subset of G.*

It can be seen that if G is commutative, G_s *and* G_u *are both subgroups of G.*

Definition 4.2.12 (Reductive group). *[8, 11.21][16, 6.13] Consider affine algebraic group G, we define the* ***radical of G*** *to be the maximal closed connected solvable normal subgroup of G, denoted* $\mathcal{R}G$*. See detail of this at [15, 19.5].*

We say G is ***semisimple algebraic group*** *if* $\mathcal{R}G = \{1\}$*, and say G is* ***reductive algebraic group*** *if* $\mathcal{R}_u G := (\mathcal{R}G)_u = \{1\}$ *(*$\mathcal{R}_u G$ *is called the* ***Unipotent radical****).*

Some examples can be seen at [16, 6.17], in particular that $\mathbf{GL}_n(K)$ *is reductive but note semi-simple.*

Definition 4.2.13 (Split reductive group and admissible isomorphism (i.e. absolute pinning isomorphism)). *[8, p. 18.6-18.7] [20, 8.1.1] Consider G, a reductive and connected k-group, Let T be a maximal torus of G, denoting* $\Phi := \Phi(G, T)$*. For each* $\alpha \in \Phi$*, there is a unique connected subgroup* $\mathcal{U}_\alpha$ *of G that is normalized by T and has property* $\mathcal{L}(\mathcal{U}_\alpha) = \mathfrak{g}_\alpha$*. We have isomorphism* $x_\alpha : Add(K) \to \mathcal{U}_\alpha$ *such that*

$$t x_\alpha(\lambda) t^{-1} = x_\alpha(\alpha(t)\lambda)$$

for all $t \in T$*, and* $\lambda \in K$*. It can be seen* $\mathcal{U}_\alpha$ *has dimensional one (see more of this at [8, 13.18]). We call these* $x_\alpha(\cdot)$*'s* ***admissible isomorphisms*** *(As [15] does), or as* ***absolute pinning isomorphisms*** *(As [18] refers them as a part of "épinglage" originated in [12, Exp XXIII, def 1.1], while we use term "absolute" to emphasize the fact that a is an absolute root).*

Recall definition of a torus splitting over k (4.2.5), we say G ***split over k*** *if we can choose a T split over k and isomorphisms* x_α *defined over k. Further, it can be checked that reductive G split over k if it has a maximal torus that splits over k ([8, 18.7]).*

By equation in (proof of) [20, 7.3.3], it can be seen that each $\mathcal{U}_\alpha \subset [G, G]$*. Further, note that* $[G, G]$ *is semisimple and* $G = [G, G]Z(G)^\circ$ *(see [20, 8.1.6], or [8, 14.2])*

Note by the same argument made in 4.2.17, the absolute pinning isomorphisms can be taken as the same ones defined for the Chevalley groups in 3.2.14.

Note that there is a surjective group homomorphism $\mathbf{SL}_2(K) \to \langle \mathcal{U}_\alpha, \mathcal{U}_{-\alpha} \rangle$ by (assuming α is positive root) $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mapsto x_\alpha(\lambda)$ *and* $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \mapsto x_{-\alpha}(\lambda)$

Note 4.2.14 (Reduced root system in reductive group)**.** *[8, 14.8] For connected reductive group G, and maximal torus T of G, $\Phi := \Phi(G, T)$ is a reduced root system, and we call it **Absolute root system**.*

*Note, for $\alpha \in \Phi$, we take $\alpha^\vee \in X_*(T) \subset V^\vee$ (see [20, 7.3.5] for justification for the containment) to be the **unique dual** as done in [20, 7.1.8] with $<\alpha, \alpha^\vee>= 2$, and that the reflection in $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ can be written as*

$$s_\alpha(v) = v- < v, \alpha^\vee > \alpha$$

For $v \in V := \mathbb{R} \otimes_{\mathbb{Z}} X(T)$ and $\alpha \in \Phi \subset V$. Note that the pairing of $X(T)$ and $X_*(T)$, $<\cdot,\cdot>$, induces a pairing of V and $V^\vee := \mathbb{R} \otimes_{\mathbb{Z}} X_*(T)$, which we will also denote with $<\cdot,\cdot>$.

We note that there is also "dual" reflection in $V^\vee$, described by:

$$s_{\alpha^\vee}(v^*) = v^*- < \alpha, v^* > \alpha^\vee$$

*Making $s_\alpha(\beta)^\vee = s_{\alpha^\vee}(\beta^\vee)$ for $\alpha, \beta \in \Phi$ (One can use the bilinearity of $<\cdot,\cdot>$ to show this). In fact, there are stronger properties and far more interesting thing to the quadruple $(\Phi, X(T), \Phi^\vee, X_*(T))$ (it is a root datum, see [20, 7.4.3]), see about **Root datum** at [20, 7.4.1], where information about "dual" reflection is also mentioned.*

*Comparing above to statements in chapter 2 one can see what we denote here $<\beta, \alpha^\vee>$ is in fact denoted $<\beta, \alpha>$ in said chapter (Note here α and β are roots, not affine roots). **Indeed, it does not hurt for us to define in current context that $<\beta, \alpha>:=<\beta, \alpha^\vee>$ to be consistent with our traditional notations** as each α has a unique $\alpha^\vee$ (We use both notations in the following equivalently, note that $\alpha \mapsto \alpha^\vee$ is not (necessarily) linear, and so $<\cdot,\cdot>$ for both coordinates being roots is not bilinear).*

As further comment, we can consider cocharacters group $X_*(T)$ identified to a lattice $\{\lambda \in V | (\lambda, X(T)) \subset \mathbb{Z}\}$ in V with correspondence $a^\vee \leftrightarrow \frac{2a}{(a,a)}$ for root a (see [20, 15.3.6], bilinear form "$(\cdot,\cdot)$" here is constructed as in [20, 7.1.7]).

Note: *When one replaces the all the maximal torus T's here with a maximal k-split torus S, the only statement that changes is the root system might not be reduced; all other statements and formulae will "stay the same" (See 4.2.15 below, or refer to [20, 15.3.8]).*

Definition 4.2.15 (k-root of reductive group)**.** *[8, 21.1] Let S be a maximal k-split torus of connected reductive k-group G. Denote ${}_k\Phi := \Phi(G,S)$ the **Set of k-roots of G with respect to S**. It can be seen in [8, p. 21.6-21.7] that ${}_k\Phi$ is a not necessarily reduced root system, we hence call it **Relative root system** or k**-root system**. More detail about the relative root system can be found in [20, sec: 15.3]. In particular, the reflections and "dual" reflections (generalization of the formulae for them given in 4.2.14 "stay the same") are described in [20, 15.3.8], which also serves as a good reference to the analog of descriptions we have given in 4.2.14 in the non-split case.*

Note 4.2.16 (Formulas for absolute pinning isomorphisms)**.** *[20, 9.2.1] Consider for connected reductive group G, T a maximal torus of G. Fixing the absolute pinning isomorphisms $(x_\alpha)_{\alpha\in\Phi(G,T)}$, consider for $\alpha,\beta\in\Phi(G,T)$ not multiple of each other, we have for unique **Structure constants** $c_{\alpha,\beta;i,j}\in K$ such that*

$$[x_\alpha(r),x_\beta(s)] = \prod_{i,j>0\&i\alpha+j\beta\in\Phi(G,T)} x_{i\alpha+j\beta}(c_{\alpha,\beta;i,j}r^is^j)$$

For $r,s\in K$. This is the analog of the Chevalley's commutator formula in this case.

Writing $w_\alpha := x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$ (This element can be seen to be an element of T), we have there exist some $d_{\alpha,\beta}\in K^*$ such that

$$w_\alpha x_\beta(r)w_\alpha^{-1} = x_{s_\alpha(\beta)}(d_{\alpha,\beta}r)$$

For $r\in K$.

As stated in [20, 9.2.5], the absolute pinning isomorphisms can be chosen so that the structure constants are in the image of integers in K, and $d_{\alpha,\beta}$'s are among ±1 (Note $d_{\alpha,\beta}d_{\alpha,-\beta}=1$ as in [20, 9.2.2]).

Denoted with $w_\alpha(r) := x_\alpha(r)x_{-\alpha}(-r^{-1})x_\alpha(r)$ for $r\in K^*$, as shown in [20, 8.1.4], we have (By considering the surjection of $\mathbf{SL}_2$ onto $\langle\mathcal{U}_\alpha,\mathcal{U}_{-\alpha}\rangle$ as in 4.2.13 and check the statement in $\mathbf{SL}_2$)

$$w_\alpha(r) = \alpha^\vee(r)w_\alpha$$

And we can see that:

$$w_\alpha(r)x_\beta(s)w_\alpha(r)^{-1} = \alpha^\vee(r)w_\alpha x_\beta(s)w_\alpha^{-1}\alpha^\vee(r)^{-1} = \alpha^\vee(r)x_{s_\alpha(\beta)}(d_{\alpha,\beta}s)\alpha^\vee(r)^{-1}$$

$$= x_{s_\alpha(\beta)}(r^{<s_\alpha(\beta),\alpha^\vee>}d_{\alpha,\beta}s) = x_{s_\alpha(\beta)}(r^{<\beta-<\beta,\alpha>\alpha,\alpha>}d_{\alpha,\beta}s) = x_{s_\alpha(\beta)}(r^{-<\beta,\alpha>}d_{\alpha,\beta}s)$$

for $r \in K^*$ and $s \in K$.

An observation is that $w_\alpha = w_\alpha(1)$.

We also have fact $w_\alpha(r) = w_{-\alpha}(-r^{-1})$: This can be shown with the sujective group homomorphism of $\mathbf{SL}_2 \to \langle \mathcal{U}_\alpha, \mathcal{U}_{-\alpha}\rangle$ mentioned in 4.2.13 by considering the share preimage of $w_\alpha(r) := x_\alpha(r)x_{-\alpha}(-r^{-1})x_\alpha(r)$ and $w_{-\alpha}(-r^{-1}) := x_{-\alpha}(-r^{-1})x_\alpha(r)x_\alpha(-r^{-1})$ by said homomorphism.

Note 4.2.17 (Existence of needed faithful rational representation for the split case)**.** *Consider a connected reductive split k-group G with maximal torus T that is k-splt, denote $\Phi = \Phi(G,T)$. There exists a faithful rational representation that is defined over k, $\rho : G \to GL_N(K)$ so that:*

Elements of $\rho(\mathcal{U}_\alpha)$ are uni-upper-triangular (resp. uni-lower-triangular) for α positive root (resp. negative root) in Φ.

Sketch. G shares both the absolute root system and hence the absolute root groups with its subgroup $[G,G]$ (That is semisimple), with argument in the last paragraph of [8, 18.6] (for any two absolute pinning isomorphisms $x_a(\lambda)$ and $y_a(\lambda)$, we must have that $x_a(\lambda) = y_a(c\lambda)$ for some $c \in K^*$), we see that the x_α's we defined for Chevalley groups in 3.2.14 (Chevalley groups are precisely the split semisimple groups) satisfy the requirement needed for absolute pinning isomorphisms in the reductive case. Recall that the statement classifying absolute root groups as uni-upper and uni-lower-triangular forms can be seen by construction of x_α's for Chevalley groups (3.2.13), this proves statement needed. □

4.2.3 Tools for non-split case

Note 4.2.18 (Set up for non-split case)**.** *Notations and setup follow closely to [4, sec: 1.2.3] and [8, sec: 21].*

1. *Let G be a connected reductive k-group that does not split. Consider a maximal k-split torus S of G, we can always find a maximal tours T of G that is defined over k and contains S (See [4, Satz 13]).*

Consider k-roots in ${}_k\Phi := \Phi(G,S)$ *and roots in* $\Phi := \Phi(G,T)$*, the inclusion of* $S \hookrightarrow T$ *implies a group projection of* $j : X(T) \to X(S)$ *by* $\chi \mapsto \chi|_S$*.*

As shown in [8, 21.8], ${}_k\Phi \subset j(\Phi) \subset {}_k\Phi \cup \{0\}$ *and* ${}_k\Phi_+ \subset j(\Phi_+) \subset {}_k\Phi_+ \cup \{0\}$*. We will denote:*

$$\eta(a') := j^{-1}(a') \cap \Phi$$

For $a' \in {}_k\Phi$*.*

2. *Before we go on, recall following fact about root systems:*

   - *Consider arbitrary root system* Ψ*, for root* $\alpha \in \Psi$*, the only possible roots in form of* $\lambda\alpha$ *for* $|\lambda| \le 1$ *are within* $\{\pm\alpha, \pm\frac{1}{2}\alpha\}$ *(See [8, 14.7]). This implies that the set of all roots in form of* $\lambda\alpha$ *for* α *can only be among* $\{\pm\frac{1}{2}\alpha, \pm\alpha\}$*,* $\{\pm\alpha\}$*, and* $\{\pm\alpha, \pm 2\alpha\}$ *(*2α *and* $\frac{1}{2}\alpha$ *can not "coexist" for* $\frac{1}{4}\cdot 2\alpha = \frac{1}{2}\alpha$*).*

   For $a' \in {}_k\Phi$ *we denote:*

$$(a') := \{\lambda a' | \lambda \in \mathbb{Z}_{>0}\}$$

   We have that:

$$(a') = \begin{cases} \{a'\} \\ \{a', 2a'\} \end{cases}$$

3. *For* $a' \in {}_k\Phi$*, we construct:*

$$U_{a'} := U_{(a')} := \langle U_a | a \in \eta((a')) \rangle$$

   Where U_a *for* $a \in \Phi$ *are as constructed in 4.2.13.*

   One can observe that (a') *is "special" in* ${}_k\Phi$ *("Special" in the sense of [8, 14.5], see more at [8, 14.7] and [8, 21.7]), this induces that* $\eta((a'))$ *is "special" in* Φ*. By [8, 14.5], we have* $U_{a'}$ *is* ***directly spanned*** *(See definition at [8, 14.3]) in any order by* $(U_a)_{a\in\eta((a'))}$*, that is for any ordering of* $\{a_1, \cdots, a_l\} = \eta((a'))$*:*

$$U_{a_1} \times \cdots \times U_{a_l} \to U_{a'} \text{ by } (u_1, \cdots, u_l) \mapsto u_1 \cdots u_l$$

   Is an isomorphism of affine varieties

4. *[8, 21.10]: Again, consider* $a' \in {}_k\Phi$*, a key observation of above set up is that if* $(a') = \{a'\}$*,*

then $U_{a'}$ is commutative.

But if $(a') = \{a', 2a'\}$, then $U_{a'}$ is no longer commutative, however, we do have $U_{2a'} = Z(U_{a'})$ ([9, 4.10])

Part II

Constructing RGD systems

Chapter 5

RGD system in case of Chevalley group and split reductive case

5.1 Chevalley group case (Split semisimple case)

5.1.1 Bring into context: Chevalley group over Laurent polynomial rings

We will inherit concepts in above: For a down to earth construction of Chevalley group over integral domains as matrix group, refer to chapter 3. We will gather some facts and notations that will be used in our context:

Note 5.1.1 (Concepts, relations, and notations for context of Chevalley group over Laurent polynomial ring). *Consider Laurent polynomial ring* $R := k[t, t^{-1}]$*, we will denote* $F := \overline{k(t)}$ *the algebraic closure of fractional field of* R*, and hence* $R \subset F$*. Consider the Chevalley group* $G := \mathcal{G}(R) := \mathcal{G}_V(R) = \mathcal{G}_{\Psi,\Lambda}(R) := \mathcal{G}_{\Psi,\Lambda}(F) \cap SL_d(R)$ *as a subgroup of* $SL_d(R)$ *(Where* $d := dim_{\mathbb{C}}(V)$*,* V *is a Lie algebra module over complex Lie algebra* $\mathcal{L}$*) and its elementary subgroup*

$E := \mathcal{G}(R)^+ := \mathcal{G}_{\Psi,\Lambda}(R)^+ \leq G$ for a fixed pair of crystallographic reduced irreducible root system Ψ and lattice Λ (between root lattice of Ψ and weight lattice of Ψ). As the pair (Ψ, Λ) is fixed, we will denote $\mathcal{G}_{\Psi,\Lambda}(\cdot)$ and $\mathcal{G}_{\Psi,\Lambda}(\cdot)^+$ with $\mathcal{G}(\cdot)$ and $\mathcal{G}(\cdot)^+$ respectively (it is fact that $\mathcal{G}(F) = \mathcal{G}(F)^+$ for any arbitrary algebraically closed field F). As convention, we will consider Ψ to be crystallographic reduced irreducible root system of simple $\mathbb{C}$-Lie algebra $\mathcal{L}$ with respect to its CSA $\mathcal{H}$, and hence roots α are elements of $\mathcal{H}^*$, and the non-degenerate bilinear form $(\cdot,\cdot)$ (defined according to killing form of $\mathcal{L}$) on $\mathbb{R} \otimes_{\mathbb{Q}} \langle \Psi \rangle_{\mathbb{Q}}$. Consider Φ to be the set of affine roots associated to Ψ, $W := W_{aff}(\Psi)$ and S as in 2.0.2(3) so that (W, S) is a Coxeter system.

We will use notation and concepts of

$$x_a(r) \text{ where } (a \in \Psi, r \in R)$$

$$w_a(r) := x_a(r)x_{-a}(-r^{-1})x_a(r) \text{ where } (a \in \Psi, r \in R^*)$$

$$w_a := w_a(1) \text{ where } (a \in \Psi)$$

$$h_a(r) := w_a(r)w_a^{-1} \text{ where } (a \in \Psi, r \in R^*)$$

(it was worth noting that $w_a(r)^{-1} = w_a(-r)$ by construction) as defined in chapter 3 in above. Recall from the same chapter following relations (re-worded to our context):

Consider $a, b \in \Psi$, $s, r \in R$:

(R1) $x_a(s + r) = x_a(s)x_a(r)$

(R2) $[x_a(s), x_b(r)] = \prod_{i,j \in \mathbb{N}; ia+jb \in \Psi} x_{ia+jb}(\overline{c_{ab;ij}} s^i r^j)$ *With* $c_{ab;ij} \in \mathbb{Z}$ *if* $b \neq \pm a$

(R3) $w_a h_b(r) w_a^{-1} = h_{s_{a,0}(b)}(r)$ *for* $r \in R^*$

(R4) $w_a x_b(r) w_a^{-1} = x_{s_{a,0}(b)}(c(a,b)r)$ *with* $c(a,b) = \pm 1$ *for* $r \in R^*$

(R5) $h_a(r) x_b(s) h_a(r)^{-1} = x_b(r^{<b,a>} s)$ *for* $r \in R^*$

(R6) $h_a(s) h_a(r) = h_a(sr)$ *for* $s, r \in R^*$

(R4)' $w_a(r) x_b(s) w_a(r)^{-1} = x_{s_{a,0}(b)}(c(a,b) r^{-<b,a>} s)$ *with* $c(a,b) = \pm 1$ *for* $r \in R^*$

We will take:

$$T := T_k := \langle h_a(r) | a \in \Psi, r \in k^* \rangle \ \& \ N := N_R := \langle w_a(r) | a \in \Psi, r \in R^* \rangle$$

We also consider:

$$H := \mathfrak{T}(F) = T_F := \langle h_a(r) | a \in \Psi, r \in F^* \rangle \leq \mathcal{G}(F) \ \& \ \mathfrak{T}(R) := H \cap SL_d(R) \leq \mathcal{G}(F) \cap SL_d(R) = G$$

$$\& \ \mathfrak{T}(k) := H \cap SL_d(k) \leq \mathcal{G}(F) \cap SL_d(k) = \mathcal{G}(k)$$

There are some further notations in 5.1.3(2). And by [24, Ch:5, Theorem 6] we know that H is a maximal torus in $\mathcal{G}(F)$.

Note 5.1.2 (Affine root subgroup)**.** *We construct **Affine root subgroups** of E:*

$$U_{\alpha_{a,l}} := \{x_a(ct^{-l}) | c \in k\}$$

for $\alpha_{a,l} \in \Phi$, and we use (R1) to check that this is indeed a subgroup of E. We will denote $U_\pm := \langle U_\alpha | \alpha \in \Phi_\pm \rangle$.

We make following observation on elements of N acting on $U_{\alpha_{b,m}}$ by conjugation:

$$w_a(ct^{-l}) U_{\alpha_{b,m}} w_a(ct^{-l})^{-1} = U_{s_{a,l}(\alpha_{b,m})}$$

Sketch. First, note by 2.0.2(2), we have

$$U_{s_{a,l}(\alpha_{b,m})} = U_{\alpha_{s_{a,0}(b), m-l<b,a>}}$$

(R4)' tells us:

$$w_a(ct^{-l}) x_b(dt^{-m}) w_a(ct^{-l})^{-1} = x_{s_{a,0}(b)}(c(a,b)(ct^{-l})^{-<b,a>}(dt^{-m}))$$

$$= x_{s_{a,0}(b)}((c(a,b)c^{-<b,a>}d)t^{l<b,a>-m}) \in U_{s_{a,0}(b), m-l<b,a>}$$

this implies $w_a(ct^{-l}) U_{\alpha_{b,m}} w_a(ct^{-l})^{-1} \subset U_{s_{a,l}(\alpha_{b,m})}$.

On the other hand, consider:

$$x_{s_{a,0}(b)}(dt^{l<b,a>-m}) = w_a(t^{-l}) x_b(c(a,b)dt^{-m}) w_a(t^{-l})^{-1}$$

to obtain $w_a(ct^{-l}) U_{\alpha_{b,m}} w_a(ct^{-l})^{-1} \supset U_{s_{a,l}(\alpha_{b,m})}$.

$\square$

Note 5.1.3 (Fact about Chevalley group over Laurent polynomial ring). *I provide summary of facts regarding Chevalley group we will use in our context:*

1. $x_a(r) \neq 1$ *if* $r \neq 0$ *and hence* $U_\alpha \neq \{1\}$ *for each* $\alpha \in \Phi$

2. *(This item is formulized in our context, but statements hold true for any algebraically closed field F) Consider the* ***root groups*** $\mathcal{U}_a(F) := \{x_a(r)|r \in F\}$. *By choosing the faithful representation* $\rho : L \to \mathfrak{gl}(V)$, *we may consider* $\mathcal{U}_\pm(F) := \langle \mathcal{U}_a(F)|a \in \Psi_\pm \rangle$ *to be consisted of uni-upper (resp. lower) triangular matrices in the sense that the diagonal entries must be 1's. Under the same faithful representation, we will have elements of* $T_A := \langle h_b(r)|b \in \Psi, r \in A^* \rangle$ *and of* $\mathfrak{T}(A) := \mathfrak{T}(F) \cap SL_d(A)$ *being diagonal matrices (For any subring A of F). And hence* $\mathcal{B}_\pm(F) := \mathfrak{T}(F)\mathcal{U}_\pm(F)$ *consist of upper (for +) and lower (for -) triangular matrices under this representation.*

   We will denote $T_A^X := \langle h(\chi)|\chi \in Hom(\Lambda, A^*) \rangle$ *(See about weight lattice* Λ *in 3.3.1, note that any element of* T_A^X *has form of* $h(\chi)$ *for some* $\chi \in Hom(\Lambda, A^*)$*) where* $h(\chi)$ *acts on elements of* $V_{\mu,A}$ *by the multiplication of* $\chi(\mu)$, *and so also have* $T_A^X = \{h(\chi)|\chi \in Hom(\Lambda, A^*)\}$ *by* $h(\chi_1)h(\chi_2) = h(\chi_1\chi_2)$ *(*$\chi_1\chi_2 \in Hom(\Lambda, A^*)$ *as it is just multiplication). This construction gives us* ***Generalized (R5)****:*

$$h(\chi)x_a(s)h(\chi)^{-1} = x_a(\chi(a)s)$$

   for any $a \in \Psi$, $s \in F$, *and* $\chi \in Hom(\Lambda, F^*)$.

   Proof. Recall the construction 3.2.14 method 2, let $\phi_F : M_d(\mathbb{Z}) \to M_d(F)$ is induced by $\mathbb{Z} \to F$ that is fully determined by $1 \mapsto 1_F$, we consider that $x_a(s) = \sum_{k=0}^{\infty} s^k \phi_F(\frac{1}{k!}\rho(x_a)^k)$ and that for $v \in V_{\mu,F}$, then have $x_a^k := \rho(x_a)^k \cdot v \in V_{\mu+ka,F}$ which action is extended though ϕ_F. We have by these facts and the construction of $h(\chi)$ action on V_F:

$$h(\chi)(x_a^k \cdot v) = \chi(\mu + ka)x_a^k \cdot v = \chi(\mu)\chi(a)^k x_a^k \cdot v$$

   This implies that:

$$h(\chi)(x_a(s) \cdot v) = h(\chi)\sum_{k=0}^{\infty} s^k \phi_F(\frac{1}{k!}\rho(x_a)^k)(v)$$

Considering $\phi_F(\frac{1}{k!}\rho(x_a)^k)(v) \in V_{\mu+ka,F}$ (see 3.2.6), we have:

$$h(\chi)(x_a(s)\cdot v) = \chi(\mu)\sum_{k=0}^{\infty} s^k\chi(a)^k\phi_F(\frac{1}{k!}\rho(x_a)^k)(v) = \chi(\mu)x_a(\chi(a)s)(v)$$

Consider by construction:

$$h(\chi)^{-1}\cdot v := \chi(\mu)^{-1}v$$

for $v \in V_{\mu,F}$. We have:

$$h(\chi)x_a(s)h(\chi)^{-1}(v) = x_a(\chi(a)s)(v)$$

Recall requirement that the action on v is induced by faithful representation ρ of our choice, the result is proven. $\square$

3. *[1, cor:2.4] For* ***semi local ring*** *A (i.e. A has finite many maximal ideals), $\mathcal{G}(A) = \mathcal{G}(A)^+T_A^X = \mathcal{G}(A)^+\mathfrak{T}(A)$ (for the second equal sign, see 5.1.4).*

Note 5.1.4 (Observation on $\mathfrak{T}$). *First note (as in 5.1.3(2)) that we can choose faithful representation $\rho : L \to \mathfrak{gl}(V)$ so that H consists of diagonal matrices, this makes H (and hence all its subgroups) commutative. Further we will denote the set of weights of representation ρ to be Γ (see 3.2.4), we construct $\Lambda_r \subset \Lambda \subset \Lambda_w$ as 3.3.1. Recall as in 3.4.3: $h_a(r)$ acts on $V_{\mu,F}$ by multiplying $r^{<\mu,a>}$ on the left where $\mu \in \Gamma$ and $<\mu, a>\in \mathbb{Z}$ (3.4.4). We make some observations regarding $\mathfrak{T}$:*

1. *On elements of T_A^X for integral domain $A \subset F$: (This is formulated for our context, but statement holds true for any arbitrary algebraically closed field F)*

   It is already known that $T_F = T_F^X = \mathfrak{T}(F)$.

   First recall that it is known that $V_F = \oplus_{\mu\in\Gamma}V_{\mu,F}$, we will denote $d_\mu := dim(V_{\mu,F})$

   By construction, for $h(\chi) \in T_A^X$, $h(\chi) = diag((\chi(\mu)Id_{d_\mu})_{\mu\in\Gamma})$ where each $\chi(\mu) \in A^*$ for $\mu \in \Gamma$.

   Since $T_F = T_F^X = \mathfrak{T}(F) \subset SL_d(F)$, it can be seen that $h(\chi) \in SL_d(A)$ as it is in $SL_d(F)$ and is diagonal matrix with entries in A^* hence is invertible in $SL_d(A)$. By construction, we have $h(\chi) \in T_F^X = \mathfrak{T}(F)$, then have $h(\chi) \in \mathfrak{T}(F) \cap SL_d(A) = \mathfrak{T}(A)$ and hence $T_A^X \subset \mathfrak{T}(A)$.

   We now consider $y = diag(y_1, \cdots, y_d) \in \mathfrak{T}(A) = T_F \cap SL_d(A)$. Recalling that elements of T_F act on $v \in V_{\mu,F}$ by scalar multiplication, we see that $y = diag((y_\mu Id_{d_\mu})_{\mu\in\Gamma})$ for $y_\mu \in A^*$.

Note as $\mathfrak{T}(F) = T_F^X$, by choosing $\chi(\mu)$ to be y_μ, we have it extended to a $\chi \in Hom(\Lambda, F^*)$ by linearly extending to $\Lambda = \langle\Gamma\rangle_{\mathbb{Z}}$. But as we have further know that $y_\mu \in A^*$, we also have $\chi \in Hom(\Lambda, A^*)$. And have that $h(\chi)$ would be y and hence $\mathfrak{T}(A) \subset T_A^X$.

To sum up, we have $\mathfrak{T}(A) = T_A^X$.

2. *$\mathfrak{T}(k)$ normalizes T and affine root groups.*

Sketch. (a) **($\mathfrak{T}(k)$ normalizes $T := T_k$):** H is commutative.

(b) **($\mathfrak{T}(k)$ normalizes affine root groups):** Consider by generalized (R5):

$$h(\chi)x_a(ct^{-l})h(\chi)^{-1} = x_a(\chi(a)ct^{-l})$$

as $\chi \in Hom(\Lambda, k^*)$, then have $\chi(a) \in k^*$ and so $x_a(\chi(a)ct^{-l}) \in U_{\alpha_{a,l}}$.

□

5.1.2 Constructing RGD system for elementary subgroup of Chevalley group over Laurent polynomial ring

Lemma 5.1.5 (RGD system for elementary subgroup of Chevalley group)**.** *Following context of section 5.1.1, $(E, (U_\alpha)_{\alpha\in\Phi}, T)$ is a RGD system of type (W,S)*

Proof. The proof will mostly follow along that of [2, lem:5], the fact that we limit to elementary subgroup of Chevalley group helps us with releasing the assumption of simply connected type in [2, lem:5]. We check the RGD axioms (RGD1) to (RGD5), replacing (RGD3) with (RGD3)*(Stronger than (RGD3)):

(RGD0) This is already shown in 5.1.3(1)

(RGD1) We will consider a pair of prenilpotent affine roots $\{\alpha_{a,l}, \alpha_{b,m}\} \subset \Phi$. This axiom is a consequence of Chevalley's commutator formula, or equivalently (R2): There are two cases: $a = b$ and $a \neq b$ for $a, b \in \Psi$.

(a) **(In case that $a \neq b$):**

$$[x_a(ct^{-l}), x_b(dt^{-m})] = \prod_{i,j\in\mathbb{Z}_{>0}\ \&\ ia+jb\in\Psi} x_{ia+jb}(\overline{c_{a,b;i,j}}(ct^{-l})^i(dt^{-m})^j)$$

$$= \prod_{i,j\in\mathbb{Z}_{>0}\ \&\ ia+jb\in\Psi} x_{ia+jb}(\overline{c_{a,b;i,j}}c^i d^j t^{-li-mj}) \in \langle U_\gamma | \gamma \in \{\alpha_{pa+qb,pl+qm} \in \Phi | p,q \in \mathbb{Z}_{>0}\}\rangle$$

Noticing by reduced condition of Ψ:

$$\{\alpha_{pa+qb,pl+qm} \in \Phi | p,q \in \mathbb{Z}_{\geq 0}\} \setminus \{\alpha_{a,l}, \alpha_{b,m}\} = \{\alpha_{pa+qb,pl+qm} \in \Phi | p,q \in \mathbb{Z}_{>0}\}$$

We have

$$\{\alpha_{pa+qb,pl+qm} \in \Phi | p,q \in \mathbb{Z}_{>0}\} \subset (\alpha_{a,l}, \alpha_{b,m})$$

following 2.0.2(5). We have $[x_a(ct^{-l}), x_b(dt^{-m})] \in \langle U_\gamma | \gamma \in (\alpha_{a,l}, \alpha_{b,m})\rangle$.

(b) **(In case that** $a = b$**)**: $[x_a(ct^{-l}), x_b(dt^{-m})] = 1$ by (R1).

(RGD2) Consider the map $m : x_a(ct^{-l}) \mapsto w_a(ct^{-l})$. We check: For all $s_i \in S$, and $u \in U_{\alpha_i} \setminus \{1\}$, we have following:

(a) $m(u) \in U_{s_i(\alpha_i)} u U_{s_i(\alpha_i)}$: We can show this more generally for all $\alpha_{a,l} \in \Phi$. We have by 2.0.2 that $s_{a,l}(\alpha_{a,l}) = \alpha_{s_{a,0}(a), l-l<a,a>} = \alpha_{-a,-l}$, this implies:

$$w_a(ct^{-l}) = w_{-a}(-\frac{1}{c}t^l) = x_{-a}(-\frac{1}{c}t^l) x_a(ct^{-l}) x_{-a}(-\frac{1}{c}t^l) \in U_{s_{a,l}(\alpha_{a,l})} x_a(ct^{-l}) U_{s_{a,l}(\alpha_{a,l})}$$

(b) $m(u) U_\alpha m(u)^{-1} = U_{s_i(\alpha)}$ for all $\alpha \in \Phi$: This is proven in 5.1.2.

(c) $m(u)T = m(u')T$ for all $u, u' \in U_{\alpha_i} \setminus \{1\}$: We will equivalently check: $w_a(ct^{-l}) \in w_a(dt^{-l})T \iff w_a(dt^{-l})^{-1} w_a(ct^{-l}) \in T$. Consider the following where the first equal sign in the first line is by (R6): (One should recall fact: $w_a(r)^{-1} = w_a(-r)$) let $c, d \in k$

$$T \ni h_a(\frac{d}{c}) = h_a(-dt^{-l}) h_a(-\frac{1}{c}t^l) = h_a(-dt^{-l}) h_a(-ct^{-l})^{-1} =$$

$$(w_a(-dt^{-l}) w_a^{-1})(w_a(-ct^{-l}) w_a^{-1})^{-1} = (w_a(-dt^{-l}) w_a^{-1})(w_a w_a(ct^{-l})) =$$

$$w_a(-dt^{-l}) w_a(ct^{-l}) = w_a(dt^{-l})^{-1} w_a(ct^{-l})$$

(RGD3)* We first make following observations by choosing representation $\rho : L \to \mathfrak{gl}(V)$ as in 5.1.3(2) and consider $\alpha_{a,l} \in \Phi$, recall that

$$\Phi_+ = \{\alpha_{a,l} \in \Phi | (a \in \Psi_+ \& l \geq 0)\ \mathit{or}\ (a \in \Psi_- \& l \geq 1)\}$$

$$\Phi_- = \{\alpha_{a,l} \in \Phi | (a \in \Psi_+ \& l \leq -1) \; or \; (a \in \Psi_- \& l \leq 0)\}$$

We break into cases according to a and l:

(a) **(When $a \in \Psi_+$ and $l \geq 0$)**: Elements $x_a(ct^{-l})$ of U_α are uni-upper-triangular matrices and all entries above diagonal in $k[t^{-1}]$, with $U_\alpha \subset \mathcal{G}(k[t^{-1}])$.

If we further require $l \geq 1$, it can be seen that as 3.2.14 method 2 with $\phi_F : M_d(\mathbb{Z}) \to M_d(F)$ is induced by $\mathbb{Z} \to F$ this is fully determined by $1 \mapsto 1_F$:

$$x_a(ct^{-l}) = Id_d + \sum_{k=1}^{\infty} (ct^{-l})^k \phi_F(\frac{1}{k!}\rho(x_a)^k)$$

with $\frac{1}{k!}\rho(x_a)^k \in M_d(\mathbb{Z})$ (see 3.2.13) implying $\phi_F(\frac{1}{k!}\rho(x_a)^k)$ is within the image of $M_d(\mathbb{Z})$ under ϕ_F in $M_d(F)$, in particular, $\phi_F(\frac{1}{k!}\rho(x_a)^k)$ never involves t.

As we have chosen ρ to be faithful, we may assume each $\rho(x_a) \neq 0$ (And has been proven to be strictly upper-triangular, i.e. with zeros on the diagonal). This means that all non-zero entries of $x_a(ct^{-l})$ is contained in $t^{-1}k[t^{-1}]$ if $l \neq 1$, $c \neq 0$.

We have if $a \in \Psi_+$ and $l \geq 1$, $x_a(ct^{-l}) \in \mathcal{G}(k[t^{-1}])$ is uni-upper-triangular with entries strictly above diagonal within $t^{-1}k[t^{-1}]$.

With similar reasoning, we have below cases

(b) **(When $a \in \Psi_+$ and $l \leq -1$)**: Elements $x_a(ct^{-l})$ of U_α are uni-upper-triangular matrices and has all entries strictly above diagonal in $tk[t]$, with $U_\alpha \subset \mathcal{G}(k[t])$.

(c) **(When $a \in \Psi_-$ and $l \geq 1$)**: Elements $x_a(ct^{-l})$ of U_α are uni-lower-triangular matrices and has all entries strictly below diagonal in $t^{-1}k[t^{-1}]$, with $U_\alpha \subset \mathcal{G}(k[t^{-1}])$.

(d) **(When $a \in \Psi_-$ and $l \leq 0$)**: Elements $x_a(ct^{-l})$ of U_α are uni-lower-triangular matrices and all entries below diagonal in $k[t]$, with $U_\alpha \subset \mathcal{G}(k[t])$.

Consider $T \subset \mathcal{G}(k)$, $U_\epsilon \subset \mathcal{G}(k[t^{-\epsilon}])$ we have:

$$TU_+ \cap U_- \subset \mathcal{G}(k[t^{-1}]) \cap \mathcal{G}(k[t]) = \mathcal{G}(k)$$

Where the equal sign is just a set-theoretic result.

Consider group homomorphisms $p_\epsilon : \mathcal{G}(k[t^{-\epsilon}]) \to \mathcal{G}(k)$ induced by $k[t^{-\epsilon}] \to k$ (defined by $t^{-\epsilon} \mapsto 0$, where $t^{\pm} := t^{\pm 1}$) entry-wise (Checking these are group homomorphism is standard). Considering preimage of subgroup under group homomorphism is subgroup, and (a) to (d) above, we see the following: (We denote $\mathfrak{U}_d^{\pm}(k)$ the upper (for +) and lower (for -) uni-triangular matrices with entries in k)

In the case (a), $p_+(U_{\alpha_{a,0}}) \subset \mathfrak{U}_d^+(k)$ and for $l \geq 1$, $p_+(U_{\alpha_{a,l}}) = Id_d$ and have U_α in case (a) is contained in $p_+^{-1}(\mathfrak{U}_d^+(k))$.

Similar reasoning give us the image of U_α in case (b) under p_- is Id_d and hence U_α is contained in $p_-^{-1}(\mathfrak{U}_d^-(k))$, the image of U_α in case (c) under p_+ is Id_d and hence U_α is contained in $p_+^{-1}(\mathfrak{U}_d^+(k))$, and U_α in case (d) is contained in $p_-^{-1}(\mathfrak{U}_d^-(k))$.

To sum up: We have $U_\epsilon \subset p_\epsilon^{-1}(\mathfrak{U}_d^\epsilon(k))$. This implies:

$$U_\epsilon \cap \mathcal{G}(k) \subset p_\epsilon^{-1}(\mathfrak{U}_d^\epsilon(k)) \cap \mathcal{G}(k) \subset \mathfrak{U}_d^\epsilon(k)$$

Note the middle part is just describing elements of $p_\epsilon^{-1}(\mathfrak{U}_d^\epsilon(k))$ with entries in k, so they have to be in $\mathfrak{U}_d^{\pm}(k)$ by the construction of p_ϵ and fact that $\mathfrak{U}_d^\epsilon(k)$ is uni-upper-triangular if $\epsilon = +$ and uni-lower-triangular if $\epsilon = -$.

Recall by construction that $T := T_k \subset \mathcal{G}(k)$. We further have following:

$$TU_+ \cap U_- = (TU_+ \cap U_-) \cap \mathcal{G}(k) = (TU_+ \cap \mathcal{G}(k)) \cap (U_- \cap \mathcal{G}(k)) =$$

$$(T(U_+ \cap \mathcal{G}(k))) \cap (U_- \cap \mathcal{G}(k)) \subset T\mathfrak{U}_d^+(k) \cap \mathfrak{U}_d^-(k)$$

But $T\mathfrak{U}_d^+(k)$ consists of upper-triangular matrices while $\mathfrak{U}_d^-(k)$ consists of uni-lower-triangular matrices, we must have $TU_+ \cap U_- = \{1\}$.

(RGD4) This hold by definition of $E := \langle x_a(r) | r \in R \rangle = \langle U_\alpha | \alpha \in \Phi \rangle$, the second equal sign can be seen by (R1).

(RGD5) This is proven by (R5).

□

Lemma 5.1.6 (Extending the RGD system on elementary subgroup). *By 5.1.5 and 1.1.4, it can be seen that $(E\mathfrak{T}(k), (U_\alpha)_{\alpha\in\Phi}, \mathfrak{T}(k))$ is a general RGD system.*

5.2 Split reductive case (Similar to Chevalley group case)

5.2.1 Bring into context: split reductive groups over Laurent polynomial rings

Due to 4.2.17, there is no essential difference between the case of split reductive group and the case of Chevalley groups.

Note 5.2.1 (The set up for split case). *We will summarize the context we work in and the tools we utilize in the split case:*

1. *We will consider our reductive k-split k-group $\mathcal{G} := \mathcal{G}(F)$ for $F := \overline{k(t)}$. Denoted $\mathfrak{T}$ will be our choice of k-split maximal torus of reductive group. We will denote $\Psi := \Phi(\mathcal{G}, \mathfrak{T})$, which is shown to be a reduced root system, we will further require it to be irreducible (requiring $\mathcal{G}$ to be almost simple will satisfy this requirement) and we fix a base $\{a_1, \cdots a_n\}$ of it. We will denote Φ to be the set of affine roots associated to Ψ, we consider $W := W_{aff}(\Psi)$ and S the set of simple affine roots, to obtain affine Coxeter system (W, S) (see 2.0.2).*

2. *We will take x_a for $a \in \Psi$ as the absolute pinning isomorphisms introduced in 4.2.13 defined over k.*

   *Denote $U_{\alpha_{a,l}} := \{x_a(ct^{-l}) | c \in k\}$ to be the **affine root group** in current context. Each of these are a non-trivial group according to the fact that each of x_a is an isomorphism.*

   Using following (see 4.2.16) in place of (R4)' in the same way as in 5.1.2

$$w_a(r)x_b(s)w_a(r)^{-1} = x_{s_{a,0}(b)}(r^{-<b,a>}d_{a,b}s)$$

   we obtain similar result that

$$w_a(ct^{-l})x_b(dt^{-m})w_a(ct^{-l})^{-1} = x_{s_{a,0}(b)}((d_{a,b}c^{-<b,a>}d)t^{l<b,a>-m})$$

$$\in U_{s_{a,0}(b),m-l<b,a>} = U_{s_{a,l}(\alpha_{b,m})}$$

   We have

$$w_a(ct^{-l})U_{\alpha_{b,m}}w_a(ct^{-l})^{-1} \subset U_{s_{a,l}(\alpha_{b,m})}$$

   On the other hand, we have similarly

$$x_{s_{a,0}(b)}(dt^{l<b,a>-m}) = w_a(t^{-l})x_b(d_{a,b}dt^{-m})w_a(t^{-l})^{-1}$$

And hence result in

$$w_a(ct^{-l})U_{\alpha_{b,m}}w_a(ct^{-l})^{-1} = U_{s_{a,l}(\alpha_{b,m})}$$

3. *Recall as in 4.2.16, the x_a's comes with an* ***analog of Chevalley's commutator formula*** *in this context (For $a, b \in \Psi$):*

$$[x_a(r), x_b(s)] = \prod_{i,j>0\&ia+jb\in\Psi} x_{ia+jb}(c_{a,b;i,j}r^i s^j)$$

For $(c_{a,b;i,j})$ structure constants that can be considered to be within the image of integers in F.

4. *We will denote $\mathcal{G}(k[t,t^{-1}])^+ := \langle U_{\alpha_{a,l}} | \alpha_{a,l} \in \Phi\rangle$ the* ***elementary subgroup (of $\mathcal{G}(k[t,t^{-1}])$)*** *in this context.*

5. *Furthermore, we will utilize the construction of $w_a(\cdot)$ for $a \in \Psi$ as in 4.2.16 and its related properties.*

As $\mathfrak{T}$ is k-split, according to 4.2.5, and 4.2.13, we have that for $t \in \mathfrak{T}(k)$, $a \in \Psi$, and $\lambda \in F$:

$$tx_a(\lambda)t^{-1} = x_a(a(t)\lambda)$$

For $a(t) \in k$. A key observation is that $\mathfrak{T}(k)$ ***normalizes each of the affine root groups (And hence the whole elementary subgroup)****.*

If we let $T := \langle a^\vee(r) | r \in k^*, a \in \Psi\rangle$, we can see by construction that $T \subset \mathfrak{T}$ as $a^\vee \in X_*(\mathfrak{T}) := Hom(GL_1, \mathfrak{T})$ (See 4.2.6, this also makes so that $a^\vee(rs) = a^\vee(r)a^\vee(s)$ for $r, s \in F$). By split over k condition, we for have each $a \in \Psi$, x_a is defined over k, and hence so is w_a (as a map); considering $a^\vee(r) = w_a(r)w_a^{-1}$ for $r \in F$ (see 4.2.16), we have $T \subset \mathfrak{T}(k)$. Note also that, with the split over k condition, $a^\vee(r) = w_a(r)w_a^{-1}$ shows that $a^\vee(r) \in \mathcal{G}(k[t,t^{-1}])^+$ for $r \in k[t,t^{-1}]^*$, and that similarly $T \subset \mathcal{G}(k)$ (T is generated by $w_a(r)w_a^{-1}$ for $r \in k^*$ and w_a has expression as product of absolute pinning isomorphisms).

As an informal comment: Comparing this to context of the Chevalley groups reveals that $a^\vee$ take the place of h_a.

We know $\mathfrak{T}(k)$ is diagonalizable and commutative, so it normalizes T.

5.2.2 Constructing RGD system for elementary subgroup of split reductive groups

There is no essential difference between above 5.1.5 and 5.2.2 in the following. The only difference is that we will use (RGD3) instead of (RGD3)* as (RGD3) will be the better option we when start looking at non-split cases, but even the difference in this choice does not contribute to any essential difference in arguments.

Lemma 5.2.2 (RGD system for elementary subgroup of split reductive group). *Let* $T := \langle a^\vee(r) | r \in k^*, a \in \Psi \rangle$, *we see*

$$(\mathcal{G}(k[t, t^{-1}])^+, (U_\alpha)_{\alpha \in \Phi}, T)$$

is a general RGD system of type (W, S)

Proof. This proof will be very similar to that of 5.1.5 We again go through the axioms:

(RGD0) The affine root groups are not trivial groups as each of x_a is an isomorphism (4.2.13).

(RGD1) This is (again) a consequence of the analog of the Chevalley's commutator formula in the current context. Consider $a, b \in \Psi$, in the case that $a = b$, $[x_a(ct^{-l}), x_b(dt^{-m})] = 1$ as x_a is isomorphism and $(\mathcal{G}, +)$ is commutative.

We will only consider $a \neq b$:

By the reduced condition of Ψ, and 2.0.2, we have:

$$\{\alpha_{pa+qb,pl+qm} \in \Phi | p, q \in \mathbb{Z}_{\geq 0}\} \setminus \{\alpha_{a,l}, \alpha_{b,m}\} = \{\alpha_{pa+qb,pl+qm} \in \Phi | p, q \in \mathbb{Z}_{>0}\} \subset (\alpha_{a,l}, \alpha_{b,m})$$

By the analog of the Chevalley's commutator formula (For $c_{a,b;i,j}$ integers in F, and hence integers in k):

$$[x_a(ct^{-l}), x_b(dt^{-m})] = \prod_{i,j \in \mathbb{Z}_{>0} \ \& \ ia+jb \in \Psi} x_{ia+jb}(c_{a,b;i,j}(ct^{-l})^i (dt^{-m})^j)$$

$$= \prod_{i,j \in \mathbb{Z}_{>0} \ \& \ ia+jb \in \Psi} x_{ia+jb}(c_{a,b;i,j} c^i d^j t^{-li-mj})$$

$$\in \langle U_\gamma | \gamma \in \{\alpha_{pa+qb,pl+qm} \in \Phi | p, q \in \mathbb{Z}_{>0}\} \rangle \subset \langle U_\gamma | \gamma \in (\alpha_{a,l}, \alpha_{b,m}) \rangle$$

(RGD2) Consider the map $m : x_a(ct^{-l}) \mapsto w_a(ct^{-l})$. We check: For all $s_i \in S$, and $u \in U_{\alpha_i} \setminus \{1\}$, we have following:

(a) $m(u) \in U_{s_i(\alpha_i)} u U_{s_i(\alpha_i)}$: We can show this more generally for all $\alpha_{a,l} \in \Phi$. We have by 2.0.2 that $s_{a,l}(\alpha_{a,l}) = \alpha_{s_{a,0}(a), l-l<a,a>} = \alpha_{-a,-l}$, this implies:

$$w_a(ct^{-l}) = w_{-a}(-\frac{1}{c}t^l) = x_{-a}(-\frac{1}{c}t^l)x_a(ct^{-l})x_{-a}(-\frac{1}{c}t^l) \in U_{s_{a,l}(\alpha_{a,l})}x_a(ct^{-l})U_{s_{a,l}(\alpha_{a,l})}$$

For the first equality, see end of 4.2.16.

(b) $m(u)U_\alpha m(u)^{-1} = U_{s_i(\alpha)}$ for all $\alpha \in \Phi$: This is proven in 5.2.1.

(c) $m(u)T = m(u')T$ for all $u, u' \in U_{\alpha_i} \setminus \{1\}$: We will equivalently check: $w_a(ct^{-l}) \in w_a(dt^{-l})T \iff w_a(dt^{-l})^{-1}w_a(ct^{-l}) \in T$. Consider following with $c, d \in k$:

$$T \ni a^\vee(\frac{d}{c}) = a^\vee(-dt^{-l})a^\vee(-\frac{1}{c}t^l) = a^\vee(-dt^{-l})a^\vee(-ct^{-l})^{-1} =$$

$$(w_a(-dt^{-l})w_a^{-1})(w_a(-ct^{-l})w_a^{-1})^{-1} = (w_a(-dt^{-l})w_a^{-1})(w_a w_a(ct^{-l})) =$$

$$w_a(-dt^{-l})w_a(ct^{-l}) = w_a(dt^{-l})^{-1}w_a(ct^{-l})$$

(RGD3) We follow the same procedure of the same step (RGD3)* for 5.1.5. The only differences being that we consider (RGD3) instead of (RGD3)*, and that we use 4.2.17 to choose the needed faithful k-rational representation, and we need to alter the arguments involving the specifically constructed x_a for Chevalley groups to that works with our absolute pinning isomorphisms, we will again use one case and example, we will restate the rest for reference purpose, the proves follow similar logic as first case:

(a) **(When $a \in \Psi_+$ and $l \geq 0$)**: Elements $x_a(ct^{-l})$ of U_α are uni-upper-triangular matrices with all entries above diagonal in $k[t^{-1}]$, with $U_\alpha \subset \mathcal{G}(k[t^{-1}])$.

If we further require $l \geq 1$, recall by 4.2.13 we have each x_a to be defined over k (i.e. each entry of x_a is taken as a polynomial with coordinates k). This means that all non-zero entries of $x_a(ct^{-l})$ is contained in $t^{-1}k[t^{-1}]$ if $l \geq 1$, and $c \neq 0$.

We have if $a \in \Psi_+$ and $l \geq 1$, $x_a(ct^{-l}) \in \mathcal{G}(k[t^{-1}])$ is uni-upper-triangular with entries strictly above diagonal within $t^{-1}k[t^{-1}]$.

(b) **(When $a \in \Psi_+$ and $l \leq -1$)**: Elements $x_a(ct^{-l})$ of U_α are uni-upper-triangular matrices and has all entries strictly above diagonal in $tk[t]$, with $U_\alpha \subset \mathcal{G}(k[t])$.

(c) **(When $a \in \Psi_-$ and $l \geq 1$)**: Elements $x_a(ct^{-l})$ of U_α are uni-lower-triangular matrices and has all entries strictly below diagonal in $t^{-1}k[t^{-1}]$, with $U_\alpha \subset \mathcal{G}(k[t^{-1}])$.

(d) **(When $a \in \Psi_-$ and $l \leq 0$)**: Elements $x_a(ct^{-l})$ of U_α are uni-lower-triangular matrices and has all entries below diagonal in $k[t]$, with $U_\alpha \subset \mathcal{G}(k[t])$.

Where (b), (c), and (d) are obtained with similar procedure as (a).

Following is exactly the same as in 5.1.5, except for we discard to consideration for T:

Consider $U_\epsilon \subset \mathcal{G}(k[t^{-\epsilon}])$ we have:

$$U_+ \cap U_- \subset \mathcal{G}(k[t^{-1}]) \cap \mathcal{G}(k[t]) = \mathcal{G}(k)$$

Where the equal sign is just a set-theoretic result.

Consider group homomorphisms $p_\epsilon : \mathcal{G}(k[t^{-\epsilon}]) \to \mathcal{G}(k)$ induced by $k[t^{-\epsilon}] \to k$ (defined by $t^{-\epsilon} \mapsto 0$, where $t^{\pm} := t^{\pm 1}$) entry-wise. Considering preimage of subgroup under group homomorphism is subgroup, and (a) to (d) above: (We denote $\mathfrak{U}^{\pm}(k)$ the upper (for +) and lower (for -) uni-triangular matrices with entries in k)

In the case (a), $p_+(U_{\alpha_{a,0}}) \subset \mathfrak{U}^+(k)$ and for $l \geq 1$, $p_+(U_{\alpha_{a,l}}) = Id$ and have U_α in case (a) is contained in $p_+^{-1}(\mathfrak{U}^+(k))$.

Similar reasoning give us the image of U_α in case (b) under p_- is Id and hence U_α is contained in $p_-^{-1}(\mathfrak{U}^-(k))$, the image of U_α in case (c) under p_+ is d and hence U_α is contained in $p_+^{-1}(\mathfrak{U}^+(k))$, and U_α in case (d) is contained in $p_-^{-1}(\mathfrak{U}^-(k))$.

To sum up: We have $U_\epsilon \subset p_\epsilon^{-1}(\mathfrak{U}^\epsilon(k))$. This implies:

$$U_\epsilon \cap \mathcal{G}(k) \subset p_\epsilon^{-1}(\mathfrak{U}^\epsilon(k)) \cap \mathcal{G}(k) \subset \mathfrak{U}^\epsilon(k)$$

Note the middle part is just describing elements of $p_\epsilon^{-1}(\mathfrak{U}^\epsilon(k))$ with entries in k, and they have to be in $\mathfrak{U}^{\pm}(k)$ by the construction of p_ϵ and fact that $\mathfrak{U}^\epsilon(k)$ is uni-upper-triangular if $\epsilon = +$ and uni-lower-triangular if $\epsilon = -$.

We further have following:

$$U_+ \cap U_- = (U_+ \cap U_-) \cap \mathcal{G}(k) =$$

$$(U_+ \cap \mathcal{G}(k)) \cap (U_- \cap \mathcal{G}(k)) \subset \mathfrak{U}^+(k) \cap \mathfrak{U}^-(k)$$

But $\mathfrak{U}^{+}(k)$ consists of uni-upper-triangular matrices while $\mathfrak{U}^{-}(k)$ consists of uni-lower-triangular matrices, we must have $U_{+} \cap U_{-\alpha_i} \subset U_{+} \cap U_{-} = \{1\}$ for we constructed so that all simple affine roots to be positive and have $U_{-\alpha_i} \subset U_{-}$.

Then (RGD0) tells us that $U_{-\alpha_i} \neq \{1\}$ and hence for all simple affine roots α_i, we have $U_{-\alpha_i} \not\leq U_{+}$.

(RGD4) Holds by construction and definition of elementary subgroup as in 5.2.1.

(RGD5) T normalizes each of the affine root groups, shown in 5.2.1.

$\square$

Corollary 5.2.3 (extending RGD system in the split case). *Considering $\mathfrak{T}(k)$ normalizes T and each of the affine root groups (see 5.2.1), with 1.1.4, we have*

$$(\mathcal{G}(k[t, t^{-1}])^{+}\mathfrak{T}(k), (U_\alpha)_{\alpha \in \Phi}, \mathfrak{T}(k))$$

is a general RGD system of type (W, S).

Chapter 6

RGD system in case of non-split reductive groups

6.1 Specific non-split case: (Special) Unitary groups of type BC

6.1.1 Basics of Unitary group

Note 6.1.1 (k and k(t) rank of semisimple k-group). *Given an arbitrary field k and a semisimple k-group $\mathcal{G}$, one can show that $rk_k\mathcal{G} = rk_{k(t)}\mathcal{G}$. This result is communicated to the author by Prof. Andrei Rapinchuk.*

Note 6.1.2 (Unitary group context). *A good reference for this can be found at [8, 23.7] together with [8, 23.8] and [8, 23.9] or at [19, ch: 2.3.3]. All fields in consideration are of characteristic non 2. Let D be a division algebra over its center $k' := Z(D)$. Let D be equipped with an involution denoted τ, and let k be the fixed field of k' under $\tau|_{k'}$ (Note, hence, D is also a division algebra over k, and can be embedded by regular representation into $M_{[k':k]n^2}(k)$, where n in this context is the index of D over k'). Let $\epsilon = \pm 1$, consider (ϵ, τ)-**hermitian form** denoted $f : V \times V \to D$ on an m dimensional (we will consider right) free module V over D (assuming non-degenerate, note, that as we are considering characteristic non 2, these are in fact "trace-hermitian" in the sense of [13, 5.1.9], [13, 5.1.10], and [13, 5.1.11]. In particular, note that $f(x,x) \in \{r + \tau(r)\epsilon | r \in D\}$ as a consequence of [13, 5.1.11]).*

*The **Unitary group** $U(V,f)$ is the group of automorphism (of V) preserving f, and the **Special Unitary group** $SU(V,f)$ is the subgroup of $U(V,f)$ containing elements with reduced norm 1. If we denote with $F = (f(e_i,e_j))$ (The matrix corresponding to hermitian form f) for a chosen basis of V over D, we may consider:*

$$U(V,f) := \{g \in GL(V) |\ {}^*gFg = F\}$$

Where for $g = (g_{\alpha,\beta})$, ${}^*g := {}^t(\tau(g_{\alpha,\beta}))$ (i.e. the transpose after τ is applied to each entry of g).

A key observation is that the condition $^*gFg = F$ is equivalent to $f(g(x), g(y)) = f(x, y)$ for all $x, y \in V$ (One can check this by checking the condition with x, y taken to be basis of V). And hence

$$SU(V, f) := \{g \in U(V, f) | Nrd_{M_m(D)/k'}(g) = 1\}$$

$U(V, f)$ and $SU(V, f)$ above can be seen as a k-points of algebraic k-group $\mathbf{U}(V, f)$ and $\mathbf{SU}(V, f)$ respectively (see [19, 2.3.3] for detail. In short, we consider $\mathbf{U}(V, f)$ (and $\mathbf{SU}(V, f)$) to be embedded in $GL_{[k':k]mn^2}(\overline{k})$ (and $SL_{[k':k]mn^2}(\overline{k})$) where m is the dimension of V over D, and n is the index of D over k')

From this point on: *We will focus on studying the case where τ is a nontrivial involution i.e. of second type, and $m > 2l$ (So that the unitary group of consideration is of type BC, the only irreducible non-reduced type). We without loss of generality assume that $\epsilon = -1$; no generality is lost because we mostly concern the case where τ is non identity on D(see [8, 23.8] for detail, we will keep writing ϵ when it does not over-complicate the situation).*

We may "extend above situation with tensor": $D^{k(t)} := D^{k'(t)} := D \otimes_{k'} k'(t)$ is central division algebra over $k'(t)$. We require that the extension of τ into $D^{k(t)}$ (We again denote with τ) fixes the transcendental element t, then have $k(t)$ is the fixed field of $k'(t)$ under $\tau|_{k'(t)}$ (Making $D^{k'(t)}$ also a division algebra over $k(t)$, so we will also denote $D^{k(t)} := D^{k'(t)}$, in fact by embedding D in $M_{n^2}(k')$ by regular representation and further in $M_{[k':k]n^2}(k)$ by $k' \hookrightarrow M_{[k':k]}(k)$, one can see that $D^{k(t)}$ indeed is $D \otimes_k k(t)$ by seeing D as an algebra over k). The (ϵ, τ)-hermitian form f can be extended to (we will keep using f as the notation) $f : V^{k(t)} \times V^{k(t)} \to D^{k(t)}$ where $V^{k(t)} := V \otimes_D D^{k(t)}$ is (by construction) m dimensional right module over $D^{k(t)}$.

This leads to an algebraic $k(t)$-groups (also k-groups), denoting $V^{\overline{k(t)}} := V^{k(t)} \otimes_{k(t)} \overline{k(t)} = V^{k(t)} \otimes_{D^{k(t)}} (D^{k(t)} \otimes_{k(t)} \overline{k(t)})$ (and f being the extended hermitian form on $V^{\overline{k(t)}} \times V^{\overline{k(t)}}$),

$$\mathbf{U}(V^{k(t)}, f) = Is(V^{\overline{k(t)}}, f) = \{g \in GL(V^{\overline{k(t)}}) | f(g(x), g(y)) = f(x, y) \forall x, y \in V^{\overline{k(t)}}\}$$

See detail at [8, 23.9], whose context is clarified in [8, 23.8]. Where, unfortunately, the way which f is extended to $V^{\overline{k(t)}} \times V^{\overline{k(t)}}$ is not clarified. To understand this extension, one may consult [19, ch: 2.3.3], where depending on type of involution of τ, one identify $M_*(D^{k(t)}) \otimes_{k(t)} \overline{k(t)}$ with $M_{**}(\overline{k(t)})$ (type 1) or with $M_{**}(\overline{k(t)}) \times M_{**}(\overline{k(t)})$ (type 2) and condition "respecting the hermitian form" is translated to conditions involving case (1) (type 1) and case (3) (type 2) on top of [19, page: 96]. As a further comment, $*$ and $**$ here are just two different place-holders for integers, in particular we are not claiming $**$ is NOT product of two $*$'s

We set $\mathbf{SU}(V^{k(t)}, f) = \mathbf{U}(V^{k(t)}, f) \cap SL(V^{k(t)})$. *Denoted with* $\mathcal{G} := \mathbf{SU}(V^{k(t)}, f)$, *is the algebraic* k*-group of consideration, and we will work on construction of RGD system for* $\mathcal{G}(k[t,t^{-1}])^+$ *(Considered to be embedded in* $SL_{2mn^2}(\overline{k(t)})$*, where* m *is the dimension of* V *over* D*, and* n *is the index of* D *over* k'*).*

Note 6.1.3 (Construction of the relative unitary (and related) elements)**.** *We will use construction provided in [11, ch: 10], refitted to our context. We will continue in the context of 6.1.2: First recall given the* (ϵ,τ)*-hermitian form* $f : V \times V \to D$*, we have a* $D_{\tau,\epsilon} := \{r - \epsilon\tau(\lambda) | \lambda \in D\}$ *quadratic form* $q : V \to D/D_{\tau,\epsilon}$ *by* $q(x) = f(x,x) + D_{\tau,\epsilon}$ *(This is because all fields in consideration are characteristic non 2, in which case we also have that* $D_{\tau,\epsilon} = \{\epsilon\tau(\lambda) = -\lambda | \lambda \in D\}$*, especially recall* q *is now fully determined by* f*). As a summary (Recall we consider characteristic non 2, so hermitian forms are trace hermitian):*

$$q(x\lambda) = \tau(\lambda)q(x)\lambda$$

$$q(x+y) = q(x) + q(y) + (f(x,y) + D_{\tau,\epsilon})$$

We will take Witt basis of V *over* D*, denoting the anisotropic kernel by* V_0*:*

$$B := \{e_{\pm 1}, \cdots, e_{\pm l}\} \sqcup \{\text{ Some Basis of } V_0\}$$

Where for $i \in [1,l]$ *(*l *being the Witt index),* $\{e_{\pm i}\}$ *generates over* D *the hyperbolic plain* $V_i := e_{-i} \cdot D \oplus e_i \cdot D$*, and we have decomposition:*

$$V = (\coprod_{i\in[1,l]} V_i) \oplus V_0$$

We introduce notation as in [11], for $i \in I := \{\pm 1, \cdots, \pm l\}$*:*

$$\epsilon(i) := \begin{cases} 1 & i > 0 \\ \epsilon & i < 0 \end{cases}$$

As a summary, we will have the following fact when evaluating f in respect to B (Consider $i, j \in I$):

$$\begin{cases} f(e_i, e_j) = q(e_i) = 0 & i \neq j \\ f(e_i, e_{-i}) = \epsilon(i) \\ f(e_i, V_0) = \{0\} \\ 0 \neq q(V_0 \setminus \{0\}) \end{cases}$$

***Note**, in above, we may replace D with $D^{k(t)}$ and V with $V^{k(t)}$ and all results remain the same. According to our construction and 6.1.1, even the Witt index with construction of B as a Witt basis of $V^{k(t)}$ over $D^{k(t)}$ remains the same. We hence have similar decomposition:*

$$V^{k(t)} = (\coprod_{i \in [1,l]} V_i^{k(t)}) \oplus V_0^{k(t)}$$

Where again $V_0^{k(t)}$ is the anisotropic kernel in this context. Also, note that since we assumed $\epsilon = -1$, we have $t \in D_{\tau,\epsilon}^{k(t)}$.

As in [11], we will denote:

$$Z := \{(z, \lambda) | z \in V_0, \lambda \in D, q(z) = \lambda + D_{\tau,\epsilon}\}$$

Now for $i \in I$, $(z, \lambda) \in Z$, we construct following elements (as [11, 10.1.2] did):

$$u_i(z, \lambda) : \begin{cases} x \mapsto x - e_{-i} f(z, x) \epsilon(i) & x \in V_0 \\ e_i \mapsto e_i + z - e_{-i} \lambda \epsilon(i) \\ e_j \mapsto e_j & j \in I \setminus \{i\} \end{cases}$$

And following further requiring $(z, \lambda) \neq (0, 0)$:

$$m_i(z, \lambda) : \begin{cases} x \mapsto x - z\lambda^{-1} f(z, x) & x \in V_0 \\ e_i \mapsto -e_{-i} \lambda \epsilon(i) \\ e_{-i} \mapsto -e_i \tau(\lambda)^{-1} \epsilon(-i) \\ e_j \mapsto e_j & j \in I \setminus \{\pm i\} \end{cases}$$

We also introduce following (again, as [11, 10.1.2] did): Consider for $i, j \in I$ with $j \neq \pm i$, $\lambda \in D$:

$$u_{ij}(\lambda) : \begin{cases} x \mapsto x & x \in V_0 \\ e_i \mapsto e_i + e_{-j}\tau(\lambda)\epsilon(-j) & \\ e_j \mapsto e_j - e_{-i}\lambda\epsilon(i) & \\ e_h \mapsto e_h & h \in I \setminus \{i, j\} \end{cases}$$

And following further requiring $\lambda \neq 0$:

$$m_{ij}(\lambda) : \begin{cases} x \mapsto x & x \in V_0 \\ e_i \mapsto e_{-j}\tau(\lambda)\epsilon(-j) & \\ e_j \mapsto -e_{-i}\lambda\epsilon(i) & \\ e_{-i} \mapsto e_j\lambda^{-1}\epsilon(i) & \\ e_{-j} \mapsto -e_i\tau(\lambda)^{-1}\epsilon(-j) & \\ e_h \mapsto e_h & h \in I \setminus \{\pm i, \pm j\} \end{cases}$$

Again, note that for above constructions, one may replace D with $D^{k(t)}$, and V with $V^{k(t)}$. To clarify, we will denote:

$$Z^{k(t)} := \{(z, \lambda) | z \in V_0^{k(t)}, \lambda \in D^{k(t)}, q(z) = r + D_{\tau,\epsilon}^{k(t)}\}$$

When working with $D^{k(t)}$ and $V^{k(t)}$. Recall that $D^{k(t)}$ can be considered as matrices with coefficients in $k(t)$, when we write $D^{k(t)}(k[t, t^{-1}])$, we mean consider $D^{k(t)}$ as matrices with coefficients in $k(t)$ and take only the ones with coefficients in $k[t, t^{-1}]$. We will use similar notation for other cases as well (When similar context applies), for instance, $D_{\tau,\epsilon}^{k(t)}(k[t, t^{-1}])$.

Note 6.1.4 (A centralizer of maximal k-split torus). *Denoting $D^{\overline{k(t)}} := D^{k(t)} \otimes_{k(t)} \overline{k(t)}$, consider:*

$$\{g \in \mathcal{G}(\overline{k(t)}) | g(\langle e_i \rangle_{D^{\overline{k(t)}}}) = \langle e_i \rangle_{D^{\overline{k(t)}}} \; \forall \pm i \in I \cup \{0\}\}$$

This set can be considered as a centralizer of a maximal k-split torus (Say, some S) of $\mathcal{G}$. And so we will denote this set with $C_{\mathcal{G}}(S)$.

Note 6.1.5 (A gathering of relations for unitary groups). *This is a collection of relations one obtain with constructions in 6.1.3 and 6.1.4. For majorities of the following, one can find (variants of) them within [11, 10.1.11] and [5, page: 108-110]. We will try our best to keep our numbering* *coherent to these references, but the author does not promise a perfect matching of numbering. To obtain these relations, one chases along the movement of elements in the Witt basis B: (It is understood that below holds for situations where $(z,c),(y,b) \in Z^{k(t)}$ (resp. $\in Z$) and $\mu,\lambda \in D^{k(t)}$ (resp. $\in D$))*

1.
$$u_i(z,\lambda)u_i(z',\lambda') = u_i(z+z',\lambda+\lambda'+f(z,z'))$$

2.
$$u_i(z,\lambda)^{-1} = u_i(-z,\epsilon\tau(\lambda))$$
$$m_i(z,c)^{-1} = m_i(-z,\epsilon\tau(c))$$

3.
$$u_{-i}(z\lambda^{-1}\epsilon(i),\tau(\lambda)^{-1})u_i(z,\lambda)u_{-i}(z\tau(\lambda)^{-1}\epsilon(-i),\tau(\lambda)^{-1}) = m_i(z,\lambda)$$

4. *For $i \neq \pm j$:*
$$u_{ij}(\lambda)u_{ij}(\lambda') = u_{ij}(\lambda+\lambda')$$

5. *For $i \neq \pm j$:*
$$u_{ij}(\lambda)^{-1} = u_{ij}(-\lambda)$$
$$m_{ij}(\lambda)^{-1} = m_{ij}(-\lambda)$$

6. *For $i \neq \pm j$:*
$$u_{-i,-j}(\tau(\lambda)^{-1}\epsilon(i)\epsilon(j))u_{ij}(\lambda)u_{-i,-j}(\tau(\lambda)^{-1}\epsilon(i)\epsilon(j)) = m_{ij}(\lambda)$$

7. *For $i \neq \pm j$:*
$$u_{ji}(\lambda) = u_{ij}(-\epsilon\tau(\lambda))$$
$$m_{ji}(\lambda) = m_{ij}(-\epsilon\tau(\lambda))$$

8.
$$[u_i(z,\lambda)u_i(z',\lambda')] = u_i(0,f(z,z')-f(z'-z))$$

9. *For* $i' \neq \pm i$*:*

$$[u_i(z,\lambda), u_{i'}(z',\lambda')] = u_{ii'}(f(z,z'))$$

10. *For* $i \neq \pm j$*:*

$$[u_{-i}(z,\lambda), u_{ij}(\lambda')] = u_j(-z\lambda'\epsilon(i), \epsilon\tau(\lambda')\tau(\lambda)\lambda')u_{-i,j}(-\lambda\lambda'\epsilon(i))$$

11. *For* $i \notin \{-i', -j'\}$*,* $i' \neq \pm j'$*:*

$$[u_i(z,\lambda), u_{i',j'}(\lambda')] = 1$$

12. *For* $i \neq \pm j$*,* $i, j \neq \pm h$*:*

$$[u_{ih}(\lambda), u_{-h,j}(\lambda')] = u_{ij}(-\lambda\lambda'\epsilon(-h))$$

13. *For* $i \neq \pm h$*:*

$$[u_{ih}(\lambda), u_{-h,i}(\lambda')] = u_i(0, (\tau(\lambda')\tau(\lambda) - \lambda\lambda'\epsilon)\epsilon(h))$$

14. *For* $j, j' \neq \pm i$*,* $j \neq -j'$*:*

$$[u_{ij}(\lambda), u_{ij'}(\lambda')] = 1$$

15. *For* $i \neq \pm j$*,* $i' \neq \pm j'$*,* $i', j' \notin \{\pm i, \pm j\}$*:*

$$[u_{ij}(\lambda), u_{i',j'}(\lambda')] = 1$$

16. *For* $s \neq \pm i$*:*

$$m_i(z,c)u_{-is}(\mu)m_i(z,c)^{-1} = u_{is}(\epsilon(-i)c\mu)$$

And equivalently

$$m_i(z,c)u_{is}(\mu)m_i(z,c)^{-1} = u_{-is}(\epsilon(-i)\tau(c)^{-1}\mu)$$

17.

$$m_i(z,c)u_i(y,b)m_i(z,c)^{-1} = u_{-i}(\epsilon(i)m_i(z,c)(y)c^{-1}, \tau(c)^{-1}bc^{-1})$$

And equivalently

$$m_i(z,c)u_{-i}(y,b)m_i(z,c)^{-1} = u_i(\epsilon(-i)m_i(z,c)(y)\tau(c), cb\tau(c))$$

18. For $j, s \notin \{\pm i\}$, $j \neq \pm s$:

$$m_i(z,c)u_{js}(\mu)m_i(z,c)^{-1} = u_{js}(\mu)$$

19. For $s \neq \pm i$:

$$m_i(z,c)u_s(y,b)m_i(z,c)^{-1} = u_s(m_i(z,c)(y),b)$$

20. For $s \notin \{\pm i, \pm j\}$, $i \neq \pm j$:

$$m_{ij}(\lambda)u_{is}(\mu)m_{ij}(\lambda)^{-1} = u_{-js}(\epsilon(j)\lambda^{-1}\mu)$$

And equivalently

$$m_{ij}(\lambda)u_{-js}(\mu)m_{ij}(\lambda)^{-1} = u_{is}(-\epsilon(j)\lambda\mu)$$

And equivalently

$$m_{ij}(\lambda)u_{-is}(\mu)m_{ij}(\lambda)^{-1} = u_{js}(\epsilon(j)\tau(\lambda)\mu)$$

And equivalently

$$m_{ij}(\lambda)u_{js}(\mu)m_{ij}(\lambda)^{-1} = u_{-is}(-\epsilon(-i)\tau(\lambda)^{-1}\mu)$$

21. For $i \neq \pm j$:

$$m_{ij}(\lambda)u_{i,-j}(\mu)m_{ij}(\lambda)^{-1} = u_{i,-j}(\lambda\tau(\mu)\tau(\lambda)^{-1})$$

And equivalently

$$m_{ij}(\lambda)u_{-i,j}(\mu)m_{ij}(\lambda)^{-1} = u_{-i,j}(\tau(\lambda)^{-1}\tau(\mu)\lambda)$$

22. For $i \neq \pm j$:

$$m_{ij}(\lambda)u_{-i,-j}(\mu)m_{ij}(\lambda)^{-1} = u_{ij}(\epsilon(-i)\lambda\tau(\mu)\lambda)$$

And equivalently

$$m_{ij}(\lambda)u_{ij}(\mu)m_{ij}(\lambda)^{-1} = u_{-i,-j}(\epsilon(-i)\tau(\lambda^{-1}\mu\lambda^{-1}))$$

23. For $i \neq \pm j$:

$$m_{ij}(\lambda)u_i(z,c)m_{ij}(\lambda)^{-1} = u_{-j}(\epsilon(-j)z\tau(\lambda)^{-1}, \lambda^{-1}c\tau(\lambda)^{-1})$$

And equivalently

$$m_{ij}(\lambda)u_{-j}(z,c)m_{ij}(\lambda)^{-1} = u_i(\epsilon(-j)z\tau(-\lambda), \lambda c\tau(\lambda))$$

And equivalently

$$m_{ij}(\lambda)u_{-i}(z,c)m_{ij}(\lambda)^{-1} = u_j(\epsilon(i)z\lambda, \tau(\lambda)c\lambda)$$

And equivalently

$$m_{ij}(\lambda)u_j(z,c)m_{ij}(\lambda)^{-1} = u_{-i}(-\epsilon(i)z\lambda, \tau(\lambda)^{-1}c\lambda^{-1})$$

24. *For $h \neq \pm s$, $h, s \notin \{\pm i, \pm j\}$:*

$$m_{ij}(\lambda)u_{hs}(\mu)m_{ij}(\lambda) = u_{hs}(\mu)$$

25. *For $s \notin \{\pm i, \pm j\}$:*

$$m_{ij}(\lambda)u_s(z,c)m_{ij}(\lambda) = u_s(z,c)$$

From this point on, it is beneficial to keep in mind that elements defined in 6.1.3 can be embedded in $SL_m(D^{k(t)})$ (where m is the dimension of $V^{k(t)}$ over $D^{k(t)}$ in this context). The same can be said with elements $d \in C_{\mathcal{G}}(S)(k)$, it can be seen that $d = diag(d_{\pm 1}, \cdots, d_{\pm l}, d_0)$, where $d_{\pm 1}, \cdots d_{\pm l}$ are elements in D and $d_0 \in Is(V_0^{k(t)})(k)$ (Note that elements of $Is(V_0^{k(t)})$ can be seen as a matrix with entries in $D^{k(t)}$ of row size the same as dimension of $V_0^{k(t)}$ over $D^{k(t)}$. While elements of $Is(V_0^{k(t)})(k)$ will be the elements we described as matrices, with the further requirement that the entries are in D). In short, as expected, d_i describes the action of d on $\langle e_i \rangle_{D^{k(t)}}$. These are adapted from [5, page: 109] (To check these, one again rely on the constructions of the elements in 6.1.3 by chasing along theirs actions), and we will adapt the same notation: We denote, when $i \neq \pm j$ with $d = d_{ij}(\lambda, \mu)$ if $d_i = \lambda$, $d_j = \mu$, and $d_s = 1$ (d_0 being the identity matrix if $s = 0$) for all $s \notin \{\pm i, \pm j\}$ (by fact that d is an element in the isometry group, we can see that $d_{-i} = \tau(\lambda)^{-1}$ when $i \neq 0$). Also, we will denote $m_i^0(z,c) := m_i(z,c)|_{V_0} \in Is(V_0^{k(t)})$. In following, we will consider $d \in C_{\mathcal{G}}(S)(k)$.

26. *For $i \neq \pm j$:*

$$du_{ij}(\lambda)d^{-1} = u_{ij}(d_{-i}\lambda d_j^{-1})$$

27.

$$du_i(z,c)d^{-1} = u_i(d_0(z)d_i^{-1}, d_{-i}cd_i^{-1})$$

28. For $i \neq \pm j$:

$$m_{ij}(\lambda)m_{ij}(\mu) = d_{ij}(-\tau(\mu\lambda^{-1}), -\lambda^{-1}\mu)$$

29.

$$m_i(z,c)m_i(y,b) = d_{0i}(m_i^0(z,c)m_i^0(y,b), \epsilon\tau(c)^{-1}b)$$

Note 6.1.6 (Context when considering type BC)**.** *As mentioned before, we will construct RGD system for $\mathcal{G}(k[t,t^{-1}])$ with k-root system (Denoted ${}_k\Psi$) of type BC_l:*

$${}_k\Psi := \{\pm(\epsilon_i' \pm \epsilon_j')|1 \leq i \neq j \leq l\} \cup \{\pm\epsilon_i'|1 \leq i \leq l\} \cup \{\pm 2\epsilon_i'|1 \leq i \leq l\}$$

As in [11, sec: 10.1], we will denote $\epsilon_i' := -\epsilon_{-i}'$. Here $\{\epsilon_1', \cdots, \epsilon_l'\}$ is considered to be the (standard) orthogonal basis of euclidean space $\mathbb{R}^l$ where our relative root system "lives".

We will take the base the same as that of the (root system of) type B_l:

$$\{a_1' := \epsilon_1' - \epsilon_2', \cdots, a_{l-1}' := \epsilon_{l-1}' - \epsilon_l', a_l' := \epsilon_l'\}$$

We have the corresponding set of positive relative roots ${}_k\Psi_+$:

$${}_k\Psi_+ := \{\epsilon_i', 2\epsilon_i', \epsilon_i' - \epsilon_j', \epsilon_i' + \epsilon_j'|i < j \in [1,l]\}$$

In which case, the highest element in the set of positive k-roots is taken to be $a_0' := 2\epsilon_1'$.

We recall the simple affine roots in this context being:

$$\{\alpha_0 := \alpha_{-a_0',1}, \alpha_1 := \alpha_{a_1',0}, \cdots, \alpha_l := \alpha_{a_l',0}\}$$

And we will denote in current context:

$$S := \{s_{\alpha_0}, s_{\alpha_1}, \cdots, s_{\alpha_l}\}$$

Following is a list of reflections that is most helpful to us (Recall that reflections are linear maps, *so below fully describe the said reflections):*

$$s_{a_0}:\begin{cases}\epsilon'_1 \mapsto \epsilon'_{-1}\\ \epsilon'_{-1} \mapsto \epsilon'_1\\ \epsilon'_i \mapsto \epsilon'_i & i \neq \pm 1\end{cases}$$

$$s_{a_l}:\begin{cases}\epsilon'_l \mapsto \epsilon'_{-l}\\ \epsilon'_{-l} \mapsto \epsilon'_l\\ \epsilon'_i \mapsto \epsilon'_{-i} & i \neq \pm l\end{cases}$$

And for any fixed $i \in [1, l-1]$:

$$s_{a_i}:\begin{cases}\epsilon'_{i+1} \mapsto \epsilon'_i\\ \epsilon'_i \mapsto \epsilon'_{i+1}\\ \epsilon'_j \mapsto \epsilon'_j & j \neq \pm i, \pm(i+1)\end{cases}$$

To obtain these above formulae, one proceed with the standard calculations. We will denote Φ to be the affine root system associated with ${}_k\Psi$.

6.1.2 Construction of RGD system for Unitary groups

Definition 6.1.7 (Affine root groups for unitary groups of type BC)**.** *Inspired by [5, Definition: 13] and [11, 10.1.2], we will construct following (for $|i| < |j| \in [1, l]$, $r \in \mathbb{Z}$):*

$$U_{\alpha_{\epsilon'_i+\epsilon'_j, r}} := \{u_{i,j}(\lambda t^{-r}) | \lambda \in D\}$$

$$U_{\alpha_{\epsilon'_i, r}} := \{u_i(zt^{-r}, ct^{-2r}) | z \in V_0, c \in q(z)\} = \{u_i(zt^{-r}, ct^{-2r}) | (z, c) \in Z\}$$

When checking $q(zt^{-r}) = ct^{-2r} + D^{k(t)}_{\tau,\epsilon}$, keep in mind that $t \in D^{k(t)}_{\tau,\epsilon}$, and that $t \in k(t)$ in the center of $D^{k(t)}$ (since we assumed $\epsilon = -1$, and all fields of consideration are of characteristic non 2). We also construct:

$$U_{\alpha_{2\epsilon'_i, r}} := \{u_i(0, ct^{-r}) | c \in D_{\tau,\epsilon}\}$$

The above construction is distinct from that of [5, Definition: 13] in the sense that we do not distinguish between if r is even or odd in the case of definition of $U_{\alpha_{2\epsilon'_i, r}}$.

To see each of above constructions are indeed groups, use equation 1, 2, 4, 5 in 6.1.5.

It is by construction that all above groups are ***non-trivial*** *($\neq Id$)*

It can be see that $\langle U_{\alpha_{\epsilon'_i+\epsilon'_j,r}}\rangle_{r\in\mathbb{Z}}$ *is* $U_{\epsilon'_i+\epsilon'_j}(k[t,t^{-1}]) := \{u_{ij}(\lambda)|\lambda \in D^{k(t)}(k[t,t^{-1}])\}$ *by 4 of 6.1.5. And similarly we have* $\langle U_{\alpha_{2\epsilon'_i,r}}\rangle_{r\in\mathbb{Z}}$ *is* $U_{2\epsilon'_i}(k[t,t^{-1}]) := \{u_i(0,\lambda)|\lambda \in D^{k(t)}_{\tau,\epsilon}(k[t,t^{-1}])\}$ *by equation 1 of 6.1.5 (Consider $z = 0$).*

Recall that since:

$$q(x+y) = q(x) + q(y) + f(x,y) + D^{k(t)}_{\tau,\epsilon}$$

We have that for any $\lambda'' \in q(x+y)$:

$$\lambda'' = \lambda + \lambda' + f(x,y) + D^{k(t)}_{\tau,\epsilon}$$

for some $\lambda \in q(x) + D^{k(t)}_{\tau,\epsilon}$ and $\lambda' \in q(y) + D^{k(t)}_{\tau,\epsilon}$. Using "induction" (break λ'' as sum of two elements, each living in $k[t]$ and $k[t^{-1}]$, then apply induction on the highest and lowest power of t respectively, finally add them back together to be λ'') and equation 1 of 6.1.5, one may see that $\langle U_{\alpha_{\epsilon'_i,r}}\rangle_{r\in\mathbb{Z}}$ *is* $U_{\epsilon'_i}(k[t,t^{-1}]) := \{u_i(z,\lambda)|(z,\lambda) \in Z^{k(t)}, z \in V_0^{k(t)}(k[t,t^{-1}]), \lambda \in D^{k(t)}(k[t^2,t^{-2}])\}$.

And hence $\langle U_{\alpha_{\epsilon'_i+\epsilon'_j,r}}\rangle_{r\in\mathbb{Z}}$, $\langle U_{\alpha_{2\epsilon'_i,r}}\rangle_{r\in\mathbb{Z}}$, *and* $\langle U_{\alpha_{\epsilon'_i,r}}\rangle_{r\in\mathbb{Z}}$ *with $i \neq \pm j$ running through $\pm 1, \cdots, \pm l$ together* ***Generate the elementary subgroup*** $\mathcal{G}(k[t,t^{-1}])^+$.

The proof of following roughly follow along that of [5, Lemma 35]:
Theorem 6.1.8 (RGD for unitary group)**.** *The triple*

$$(G := \mathcal{G}(k[t,t^{-1}])^+ C_{\mathcal{G}}(S)(k), (U_\alpha)_{\alpha\in\Phi}, C_{\mathcal{G}}(S)(k))$$

Is a general RGD system.

Proof. We will go through the axioms:

(RGD0) This is by construction, see 6.1.7

(RGD1) Recall that prenilpotent condition is equivalent to requiring for $a, b \in {}_k\Psi$, $ma \neq -kb$ for all $m, k \in \mathbb{Z}_{>0}$. We will use equation 8 to equation 15 of 6.1.5 to prove this axiom.

Recall again that, here, $\Phi := \{\alpha_{a,r}|a \in {}_k\Psi, r \in \mathbb{Z}\}$ and $(\alpha_{a,r}, \alpha_{b,r'}) \supseteq \{\alpha_{pa+qb,pr+qr'} \in \Phi|p,q \in \mathbb{Z}_{>0}\}$.

We will go through following cases:

- Equation 8:

$$[u_i(zt^{-r}, ct^{-2r})u_i(z't^{-r'}, c't^{-2r'})] = u_i(0, f(zt^{-r}, z't^{-r'}) - f(z't^{-r'} - zt^{-r}))$$

for $r, r' \in \mathbb{Z}$ with $(z, c), (z', c') \in Z$. We can see that: (Throughout, keep in mind that t is in the center of $D^{k(t)}$, and involution by definition is anti-automorphism)

$$f(zt^{-r}, z't^{-r'}) - f(z't^{-r'} - zt^{-r}) = f(z, z')t^{-r-r'} - f(z'z)t^{-r-r'}$$

Proving

$$[U_{\alpha_{\epsilon'_i,r}}, U_{\alpha_{\epsilon'_i,r'}}] \subset U_{\alpha_{2\epsilon'_i,r+r'}} \subset U_{(\alpha_{\epsilon'_i,r},\alpha_{\epsilon'_i,r'})}$$

- Equation 9: For $i' \neq \pm i$:

$$[u_i(zt^{-r}, ct^{-2r}), u_{i'}(z't^{-r'}, c't^{-2r'})] = u_{ii'}(f(zt^{-r}, z't^{-r'}))$$

for $(z, c), (z', c') \in Z$ and $r, r' \in \mathbb{Z}$. We can see that:

$$f(zt^{-r}, z't^{-r'}) = f(z, z')t^{-r-r'}$$

Proving

$$[U_{\alpha_{\epsilon'_i,r}}, U_{\alpha_{\epsilon'_{i'},r'}}] \subset U_{\alpha_{\epsilon'_i+\epsilon'_{i'},r+r'}} \subset U_{(\alpha_{\epsilon'_i,r},\alpha_{\epsilon'_{i'},r'})}$$

when $i' \neq \pm i$

- Equation 10: For $i \neq \pm j$:

$$[u_{-i}(zt^{-r}, ct^{-2r}), u_{ij}(c't^{-r'})]$$

$$= u_j(-zt^{-r}c't^{-r'}\epsilon(i), \epsilon\tau(c't^{-r'})\tau(ct^{-2r})c't^{-r'})u_{-i,j}(-ct^{-2r}c't^{-r'}\epsilon(i))$$

for $(z, c) \in Z$, $c \in D$, and $r, r' \in \mathbb{Z}$. We observe that:

$$u_j(-zt^{-r}c't^{-r'}\epsilon(i), \epsilon\tau(c't^{-r'})\tau(ct^{-2r})c't^{-r'})$$

$$= u_j(-zc't^{-r-r'}\epsilon(i), \epsilon\tau(cc')t^{-2r'-2r}) \subset U_{\alpha_{\epsilon'_j,r+r'}}$$

Note that $\epsilon'_{-i} + (\epsilon'_i + \epsilon'_j) = \epsilon'_j$. We also observe that:

$$u_{-i,j}(-ct^{-2r}c't^{-r'}\epsilon(i)) = u_{-i,j}(-cc'\epsilon(i)t^{-2r-r'}) \subset U_{\alpha_{\epsilon'_{-i}+\epsilon'_j,2r+r'}}$$

Note that $2\epsilon'_{-i} + (\epsilon'_i + \epsilon'_j) = \epsilon'_{-i} + \epsilon'_j$. And finally note that:

$$U_{\alpha_{\epsilon'_j,r+r'}} U_{\alpha_{\epsilon'_{-i}+\epsilon'_j,2r+r'}} \subset U_{(\alpha_{\epsilon'_{-i},r},\alpha_{\epsilon'_i+\epsilon'_j,r'})}$$

Proving

$$[U_{\alpha_{\epsilon'_{-i},r}}, U_{\alpha_{\epsilon'_i+\epsilon'_j,r'}}] \subset U_{(\alpha_{\epsilon'_{-i},r},\alpha_{\epsilon'_i+\epsilon'_j,r'})}$$

when $i \neq \pm j$

- Equation 11: $i \notin \{-i', -j'\}$, $i' \neq \pm j'$:

$$[u_i(z,\lambda), u_{i',j'}(\lambda')] = 1$$

This immediately shows that

$$[U_{\alpha_{\epsilon'_i,r}}, U_{\alpha_{\epsilon'_i+\epsilon'_j,r'}}] \subset U_{(\alpha_{\epsilon'_i,r},\alpha_{\epsilon'_{i'}+\epsilon'_{j'},r'})}$$

when $i \notin \{-i', -j'\}$, $i' \neq \pm j'$

- Equation 12: $i \neq \pm j$, $i, j \neq \pm h$:

$$[u_{ih}(ct^{-r}), u_{-h,j}(c't^{-r'})] = u_{ij}(-ct^{-r}c't^{-r'}\epsilon(-h)) = u_{ij}(-cc't^{-r-r'}\epsilon(-h))$$

Proves

$$[U_{\alpha_{\epsilon'_i+\epsilon'_h,r}}, U_{\alpha_{\epsilon'_{-h}+\epsilon'_j,r'}}] \subset U_{\alpha_{\epsilon'_i+\epsilon'_j,r+r'}} \subset U_{(\alpha_{\epsilon'_i+\epsilon'_h,r},\alpha_{\epsilon'_{-h}+\epsilon'_j,r'})}$$

when $i \neq \pm j$

- Equation 13: $i \neq \pm h$:

$$[u_{ih}(ct^{-r}), u_{-h,i}(c't^{-r'})] = u_i(0, (\tau(c't^{-r'})\tau(ct^{-r}) - ct^{-r}c't^{-r'}\epsilon)\epsilon(h))$$

$$= u_i(0, (\tau(cc')t^{-r-r'} - cc't^{-r-r'})\epsilon(h)) = u_i(0, (\tau(cc') - cc')t^{-r-r'}\epsilon(h))$$

Proves

$$[U_{\alpha_{\epsilon'_i+\epsilon'_h,r}}, U_{\alpha_{\epsilon'_{-h}+\epsilon'_i,r'}}] \subset U_{\alpha_{2\epsilon'_i,r+r'}} \subset U_{(\alpha_{\epsilon'_i+\epsilon'_h,r},\alpha_{\epsilon'_{-h}+\epsilon'_i,r'})}$$

when $i \neq \pm h$

- Equation 14: $j, j' \neq \pm i$, $j \neq -j'$:

$$[u_{ij}(\lambda), u_{ij'}(\lambda')] = 1$$

This proves

$$[U_{\alpha_{\epsilon'_i+\epsilon'_j,r}}, U_{\alpha_{\epsilon'_i+\epsilon'_{j'},r'}}] \subset U_{(\alpha_{\epsilon'_i+\epsilon'_j,r}, \alpha_{\epsilon'_i+\epsilon'_{j'},r'})}$$

when $j, j' \neq \pm i$, $j \neq -j'$

- Equation 15: $i \neq \pm j$, $i' \neq \pm j'$, $i', j' \notin \{\pm i, \pm j\}$:

$$[u_{ij}(\lambda), u_{i',j'}(\lambda')] = 1$$

$$[U_{\alpha_{\epsilon'_i+\epsilon'_j,r}}, U_{\alpha_{\epsilon'_{i'}+\epsilon'_{j'},r'}}] \subset U_{(\alpha_{\epsilon'_i+\epsilon'_j,r}, \alpha_{\epsilon'_{i'}+\epsilon'_{j'},r'})}$$

when $i \neq \pm j$, $i' \neq \pm j'$, $i', j' \notin \{\pm i, \pm j\}$

(RGD2) Consider for $s \in \{s_{\alpha_0}, \cdots, s_{\alpha_l}\}$, functions $m : U_s^* \to G$ as constructed below (by cases):

(a) For $c \in D_{\tau,\epsilon}$, $u_{-1}(0, ct^{-1}) \in U_{\alpha_0} = U_{\alpha_{-2\epsilon'_1,1}}$:

$$m(u_{-1}(0, ct^{-1})) := u_1(0, \tau(c^{-1}t))u_{-1}(0, ct^{-1})u_1(0, \tau(c^{-1}t)) = m_{-1}(0, ct^{-1})$$

(b) For $\lambda \in D$, $i \in [1, l-1]$, $u_{i,-(i+1)}(\lambda) \in U_{\alpha_i} = U_{\alpha_{\epsilon'_i-\epsilon'_{i+1},0}}$:

$$m(u_{i,-(i+1)}(\lambda))$$

$$:= u_{-i,i+1}(\tau(\lambda)^{-1}\epsilon(i)\epsilon(-(i+1)))u_{i,-(i+1)}(\lambda)u_{-i,i+1}(\tau(\lambda)^{-1}\epsilon(i)\epsilon(-(i+1)))$$

$$= m_{i,-(i+1)}(\lambda)$$

(c) For $(z, c) \in Z$, $u_l(z, c) \in U_{\alpha_l} = U_{\alpha_{\epsilon'_l,0}}$:

$$m(u_l(z, c)) := u_{-l}(zc^{-1}\epsilon(l), \tau(c^{-1}))u_l(z, c)u_{-l}(z\tau(c)^{-1}\epsilon(-l), \tau(c^{-1})) = m_l(z, c)$$

By these above constructions, $m(u) \in U_{-s}uU_{-s}$ is automatically satisfied in all these three cases. We will go through the above three cases and see that the construction of function m satisfies the rest of the requirements of (RGD2), being, $m(u)U_\alpha m(u)^{-1} = U_{s\alpha}$, and $m(u)^{-1}m(v) \in C_{\mathcal{G}}(S)(k)$: (Trough out, keep in mind that t commutes with elements of D, and is preserved by τ. It is beneficial to recall that $s_{a',n}(\alpha_{b',m}) = \alpha_{s_{a',0}(b'),m-n<b',a'>}$ for relative roots a', b' and arbitrary integers n, m. We will consider for the followings that $\lambda, \mu \in D$, and $(z, c), (y, b) \in Z$)

(a) We utilize number 16 to 19 and 29 of 6.1.5:

- Equation 16: For $s \neq \pm 1$:

$$m_{-1}(0, ct^{-1})u_{1,s}(\mu t^{-r})m_{-1}(0, ct^{-1})^{-1} = u_{-1,s}(c\mu t^{-1-r})$$

and equivalently

$$m_{-1}(0, ct^{-1})u_{-1,s}(\mu t^{-r})m_{-1}(0, ct^{-1})^{-1} = u_{1,s}(\tau(c)^{-1}\mu t^{1-r})$$

Recall that $< \epsilon'_{\pm 1} + \epsilon'_s, 2\epsilon'_{-1} >= \mp 1$, above shows that:

$$m(u_{-1}(0, ct^{-1}))U_{\alpha_{\epsilon'_1+\epsilon'_s,r}} m(u_{-1}(0, ct^{-1}))^{-1} = U_{\alpha_{\epsilon'_{-1}+\epsilon'_s,r+1}} = U_{s_{\alpha_0}(\alpha_{\epsilon'_1+\epsilon'_s,r})}$$

$$m(u_{-1}(0, ct^{-1}))U_{\alpha_{\epsilon'_{-1}+\epsilon'_s,r}} m(u_{-1}(0, ct^{-1}))^{-1} = U_{\alpha_{\epsilon'_1+\epsilon'_s,r-1}} = U_{s_{\alpha_0}(\alpha_{\epsilon'_{-1}+\epsilon'_s,r})}$$

- Equation 17:

$$m_{-1}(0, ct^{-1})u_{-1}(yt^{-r}, bt^{-2r})m_{-1}(0, ct^{-1})^{-1}$$

$$= u_1(-m_{-1}(0, ct^{-1})(yt^{-r})c^{-1}t, \tau(c)^{-1}tbt^{-2r}c^{-1}t)$$

$$= u_1(-m_{-1}(0, ct^{-1})(y)c^{-1}t^{1-r}, \tau(c)^{-1}bt^{2-2r})$$

and equivalently

$$m_{-1}(0, ct^{-1})u_1(yt^{-r}, bt^{-2r})m_{-1}(0, ct^{-1})^{-1}$$

$$= u_{-1}(m_{-1}(0, ct^{-1})(yt^{-r})\tau(c)t^{-1}, ct^{-1}bt^{-2r}\tau(c)t^{-1})$$

$$= u_{-1}(m_{-1}(0, ct^{-1})(y)\tau(c)t^{-1-r}, cb\tau(c)t^{-2-2r})$$

Recall that $< \epsilon'_{\pm 1}, 2\epsilon'_{-1} >= \mp 1$, above shows that:

$$m(u_{-1}(0, ct^{-1}))U_{\alpha_{\epsilon'_{-1},r}} m(u_{-1}(0, ct^{-1}))^{-1} = U_{\alpha_{\epsilon'_1,r-1}} = U_{s_{\alpha_0}(\alpha_{\epsilon'_{-1},r})}$$

$$m(u_{-1}(0,ct^{-1}))U_{\alpha_{\epsilon'_1,r}}m(u_{-1}(0,ct^{-1}))^{-1} = U_{\alpha_{\epsilon'_{-1},r+1}} = U_{s_{\alpha_0}(\alpha_{\epsilon'_1,r})}$$

By replacing yt^{-r} with 0 and bt^{-2r} with bt^{-r}, and recalling that $< 2\epsilon'_{\pm 1}, 2\epsilon'_{-1} >= \mp 2$, one can see similar equations show that:

$$m(u_{-1}(0,ct^{-1}))U_{\alpha_{2\epsilon'_{-1},r}}m(u_{-1}(0,ct^{-1}))^{-1} = U_{\alpha_{\epsilon'_1,r-2}} = U_{s_{\alpha_0}(\alpha_{2\epsilon'_{-1},r})}$$

$$m(u_{-1}(0,ct^{-1}))U_{\alpha_{2\epsilon'_1,r}}m(u_{-1}(0,ct^{-1}))^{-1} = U_{\alpha_{\epsilon'_{-1},r+2}} = U_{s_{\alpha_0}(\alpha_{2\epsilon'_1,r})}$$

- Equation 18: For $j, s \notin \{\pm 1\}$, $j \neq \pm s$:

$$m_{-1}(0,ct^{-1})u_{js}(\mu t^{-r})m_{-1}(0,ct^{-1})^{-1} = u_{js}(\mu t^{-r})$$

Recall that $< \epsilon'_j + \epsilon'_s, 2\epsilon'_{-1} >= 0$, above shows that $m(u_{-1}(0,ct^{-1}))$ stabilizes $U_{\alpha_{\epsilon'_j+\epsilon'_s,r}} = U_{s_{\alpha_0}(\alpha_{\epsilon'_j+\epsilon'_s,r})}$ by conjugation.

- Equation 19: For $s \neq \pm 1$:

$$m_{-1}(0,ct^{-1})u_s(yt^{-r},bt^{-2r})m_{-1}(0,ct^{-1})^{-1} = u_s(m_{-1}(0,ct^{-1})(yt^{-r}),bt^{-2r})$$

$$= u_s(m_{-1}(0,ct^{-1})(y)t^{-r},bt^{-2r})$$

Recall that $< \epsilon'_s, 2\epsilon'_{-1} >= 0$, above shows that: $m(u_{-1}(0,ct^{-1}))$ stabilizes $U_{\alpha_{\epsilon'_s,r}} = U_{s_{\alpha_0}(\alpha_{\epsilon'_s,r})}$ by conjugation. By replacing yt^{-r} with 0 and bt^{-2r} with bt^{-r}, on can see similar equation shows that: $m(u_{-1}(0,ct^{-1}))$ stabilizes $U_{\alpha_{2\epsilon'_s,r}} = U_{s_{\alpha_0}(\alpha_{2\epsilon'_s,r})}$ by conjugation.

- Equation 29:

$$m_{-1}(0,ct^{-1})m_{-1}(0,bt^{-1})^{-1} = m_{-1}(0,ct^{-1})m_{-1}(0,-\tau(b)t^{-1})$$

$$= d_{0,-1}(m^0_{-1}(0,ct^{-1})m^0_{-1}(0,-\tau(b)t^{-1}), \tau(c)^{-1}t\tau(b)t^{-1})$$

$$= d_{0,-1}(m^0_{-1}(0,ct^{-1})m^0_{-1}(0,-\tau(b)t^{-1}), \tau(bc^{-1}))$$

By recalling the construction of $m_i(z,c)$ (as in 6.1.3) we can see that:

$$m^0_{-1}(0,ct^{-1})m^0_{-1}(0,-\tau(b)t^{-1}) = Id \in Is(V_0^{k(t)})(k)$$

And it is by definition that $\tau(bc^{-1}) \in D$. And so we have that

$$m_{-1}(0, ct^{-1})m_{-1}(0, bt^{-1})^{-1} \in C_{\mathcal{G}}(S)(k)$$

(b) We utilize number 20 to 25 and 28 of 6.1.5:

- Equation 20: For $s \notin \{\pm i, \pm(i+1)\}$:

  $$m_{i,-(i+1)}(\lambda)u_{is}(\mu t^{-r})m_{i,-(i+1)}(\lambda)^{-1} = u_{(i+1),s}(-\lambda^{-1}\mu t^{-r})$$

  and equivalently

  $$m_{i,-(i+1)}(\lambda)u_{(i+1),s}(\mu t^{-r})m_{i,-(i+1)}(\lambda)^{-1} = u_{is}(\lambda\mu t^{-r})$$

  and equivalently

  $$m_{i,-(i+1)}(\lambda)u_{-is}(\mu t^{-r})m_{i,-(i+1)}(\lambda)^{-1} = u_{-(i+1),s}(-\tau(\lambda)\mu t^{-r})$$

  and equivalently

  $$m_{i,-(i+1)}(\lambda)u_{-(i+1),s}(\mu t^{-r})m_{i,-(i+1)}(\lambda)^{-1} = u_{-is}(\tau(\lambda)^{-1}\mu t^{-r})$$

  Above shows:

  $$m(u_{i,-(i+1)}(\lambda))U_{\alpha_{{\epsilon'}_{\pm i}+\epsilon'_s},r}m(u_{i,-(i+1)}(\lambda))^{-1} = U_{\alpha_{{\epsilon'}_{\pm(i+1)}+\epsilon'_s},r} = U_{s_{\alpha_i}(\alpha_{{\epsilon'}_{\pm i}+\epsilon'_s},r)}$$

  $$m(u_{i,-(i+1)}(\lambda))U_{\alpha_{{\epsilon'}_{\pm(i+1)}+\epsilon'_s},r}m(u_{i,-(i+1)}(\lambda))^{-1} = U_{\alpha_{{\epsilon'}_{\pm i}+\epsilon'_s},r} = U_{s_{\alpha_i}(\alpha_{{\epsilon'}_{\pm(i+1)}+\epsilon'_s},r)}$$

- Equation 21:

  $$m_{i,-(i+1)}(\lambda)u_{i,(i+1)}(\mu t^{-r})m_{i,-(i+1)}(\lambda)^{-1} = u_{i,(i+1)}(\lambda\tau(\mu)\tau(\lambda)^{-1}t^{-r})$$

  and equivalently

  $$m_{i,-(i+1)}(\lambda)u_{-i,-(i+1)}(\mu t^{-r})m_{i,-(i+1)}(\lambda)^{-1} = u_{-i,-(i+1)}(\tau(\lambda)^{-1}\tau(\mu)\lambda t^{-r})$$

Above shows:

$$m(u_{i,-(i+1)}(\lambda))U_{\alpha_{\epsilon'_{\pm i}+\epsilon'_{\pm(i+1)}},r}m(u_{i,-(i+1)}(\lambda))^{-1} = U_{\alpha_{\epsilon'_{\pm i}+\epsilon'_{\pm(i+1)}},r} = U_{s_{\alpha_i}(\alpha_{\epsilon'_{\pm i}+\epsilon'_{\pm(i+1)}},r)}$$

- Equation 22:

$$m_{i,-(i+1)}(\lambda)u_{-i,(i+1)}(\mu t^{-r})m_{i,-(i+1)}(\lambda)^{-1} = u_{i,-(i+1)}(-\lambda\tau(\mu)\lambda t^{-r})$$

and equivalently

$$m_{i,-(i+1)}(\lambda)u_{i,-(i+1)}(\mu t^{-r})m_{i,-(i+1)}(\lambda)^{-1} = u_{-i,(i+1)}(-\tau(\lambda^{-1}\mu\lambda^{-1})t^{-r})$$

Above shows:

$$m(u_{i,-(i+1)}(\lambda))U_{\alpha_{\epsilon'_{\pm i}+\epsilon'_{\mp(i+1)}},r}m(u_{i,-(i+1)}(\lambda))^{-1} = U_{\alpha_{\epsilon'_{\mp i}+\epsilon'_{\pm(i+1)}},r} = U_{s_{\alpha_i}(\alpha_{\epsilon'_{\pm i}+\epsilon'_{\mp(i+1)}},r)}$$

- Equation 23:

$$m_{i,-(i+1)}(\lambda)u_i(zt^{-r}, ct^{-2r})m_{i,-(i+1)}(\lambda)^{-1} = u_{i+1}(z\tau(\lambda)^{-1}t^{-r}, \lambda^{-1}c\tau(\lambda)^{-1}t^{-2r})$$

and equivalently

$$m_{i,-(i+1)}(\lambda)u_{i+1}(zt^{-r}, ct^{-2r})m_{i,-(i+1)}(\lambda)^{-1} = u_i(z\tau(-\lambda)t^{-r}, \lambda c\tau(\lambda)t^{-2r})$$

and equivalently

$$m_{i,-(i+1)}(\lambda)u_{-i}(zt^{-r}, ct^{-2r})m_{i,-(i+1)}(\lambda)^{-1} = u_{-(i+1)}(z\lambda t^{-r}, \tau(\lambda)c\lambda t^{-2r})$$

and equivalently

$$m_{i,-(i+1)}(\lambda)u_{-(i+1)}(zt^{-r}, ct^{-2r})m_{i,-(i+1)}(\lambda)^{-1} = u_{-i}(-z\lambda t^{-r}, \tau(\lambda)^{-1}c\lambda^{-1}t^{-2r})$$

Above shows:

$$m(u_{i,-(i+1)}(\lambda))U_{\alpha_{\epsilon'_{\pm i}},r}m(u_{i,-(i+1)}(\lambda))^{-1} = U_{\alpha_{\epsilon'_{\pm(i+1)}},r} = U_{s_{\alpha_i}(\alpha_{\epsilon'_{\pm i}},r)}$$

By replacing zt^{-r} with 0, and ct^{-2r} with ct^{-r}, one can see similar equations show:

$$m(u_{i,-(i+1)}(\lambda))U_{\alpha_{2\epsilon'_{\pm i}},r}m(u_{i,-(i+1)}(\lambda))^{-1} = U_{\alpha_{2\epsilon'_{\pm(i+1)}},r} = U_{s_{\alpha_i}(\alpha_{2\epsilon'_{\pm i}},r)}$$

- Equation 24: For $h \neq \pm s$, $h, s \notin \{\pm i, \pm(i+1)\}$:

$$m_{i,-(i+1)}(\lambda)u_{hs}(\mu t^{-r})m_{i,-(i+1)}(\lambda) = u_{hs}(\mu t^{-r})$$

Above shows:

$$m(u_{i,-(i+1)}(\lambda))U_{\alpha_{\epsilon'_h+\epsilon'_s},r}m(u_{i,-(i+1)}(\lambda))^{-1} = U_{\alpha_{\epsilon'_h+\epsilon'_s},r} = U_{s_{\alpha_i}(\alpha_{\epsilon'_h+\epsilon'_s},r)}$$

- Equation 25: For $s \notin \{\pm i, \pm(i+1)\}$:

$$m_{i,-(i+1)}(\lambda)u_s(zt^{-r}, ct^{-2r})m_{i,-(i+1)}(\lambda) = u_s(zt^{-r}, ct^{-2r})$$

Above shows:

$$m(u_{i,-(i+1)}(\lambda))U_{\alpha_{\epsilon'_s},r}m(u_{i,-(i+1)}(\lambda))^{-1} = U_{\alpha_{\epsilon'_s},r} = U_{s_{\alpha_i}(\alpha_{\epsilon'_s},r)}$$

By replacing zt^{-r} with 0, and ct^{-2r} with ct^{-r}, one can see similar a equation shows:

$$m(u_{i,-(i+1)}(\lambda))U_{\alpha_{2\epsilon'_s},r}m(u_{i,-(i+1)}(\lambda))^{-1} = U_{\alpha_{2\epsilon'_s},r} = U_{s_{\alpha_i}(\alpha_{2\epsilon'_s},r)}$$

- Equation 28:

$$m_{i,-(i+1)}(\lambda t^{-r})m_{i,-(i+1)}(\mu t^{-r})^{-1} = m_{i,-(i+1)}(\lambda t^{-r})m_{i,-(i+1)}(-\mu t^{-r})$$

$$= d_{i,-(i+1)}(\tau(\mu\lambda^{-1}), \lambda^{-1}\mu)$$

Since $\tau(\mu\lambda^{-1}), \lambda^{-1}\mu \in D$, we have:

$$m_{i,-(i+1)}(\lambda t^{-r})m_{i,-(i+1)}(\mu t^{-r})^{-1} \in C_{\mathcal{G}}(S)(k)$$

(c) We utilize number 16 to 19 and 29 of 6.1.5:

- Equation 16: For $s \neq \pm l$:

$$m_l(z,c)u_{-ls}(\mu t^{-r})m_l(z,c)^{-1} = u_{ls}(-c\mu t^{-r})$$

and equivalently

$$m_l(z,c)u_{ls}(\mu t^{-r})m_l(z,c)^{-1} = u_{-ls}(-\tau(c)^{-1}\mu t^{-r})$$

Above shows:

$$m(u_l(z,c))U_{\alpha_{\epsilon'_{\pm l}+\epsilon'_s},r}m(u_l(z,c))^{-1} = U_{\alpha_{\epsilon'_{\mp l}+\epsilon'_s},r} = U_{s_{\alpha_l}(\alpha_{\epsilon'_{\pm l}+\epsilon'_s},r)}$$

- Equation 17:

$$m_l(z,c)u_l(yt^{-r}, bt^{-2r})m_l(z,c)^{-1} = u_{-l}(m_l(z,c)(y)c^{-1}t^{-r}, \tau(c)^{-1}bc^{-1}t^{-2r})$$

and equivalently

$$m_l(z,c)u_{-l}(yt^{-r}, bt^{-2r})m_l(z,c)^{-1} = u_l(-m_l(z,c)(y)\tau(c)t^{-r}, cb\tau(c)t^{-2r})$$

Above shows:

$$m(u_l(z,c))U_{\alpha_{\epsilon'_{\pm l}},r}m(u_l(z,c))^{-1} = U_{\alpha_{\epsilon'_{\mp l}},r} = U_{s_{\alpha_l}(\alpha_{\epsilon'_{\pm l}},r)}$$

By replacing yt^{-r} with 0, and bt^{-2r} with bt^{-r}, one can see similar equations show:

$$m(u_l(z,c))U_{\alpha_{2\epsilon'_{\pm l}},r}m(u_l(z,c))^{-1} = U_{\alpha_{2\epsilon'_{\mp l}},r} = U_{s_{\alpha_l}(\alpha_{2\epsilon'_{\pm l}},r)}$$

- Equation 18: For $j, s \notin \{\pm l\}$, $j \neq \pm s$:

$$m_l(z,c)u_{js}(\mu t^{-r})m_l(z,c)^{-1} = u_{js}(\mu t^{-r})$$

Above shows:

$$m(u_l(z,c))U_{\alpha_{\epsilon'_j+\epsilon'_s},r}m(u_l(z,c))^{-1} = U_{\alpha_{\epsilon'_j+\epsilon'_s},r} = U_{s_{\alpha_l}(\alpha_{\epsilon'_j+\epsilon'_s},r)}$$

- Equation 19: For $s \neq \pm l$:

$$m_l(z,c)u_s(yt^{-r}, bt^{-2r})m_l(z,c)^{-1} = u_s(m_l(z,c)(y)t^{-r}, bt^{-2r})$$

  Above shows:

$$m(u_l(z,c))U_{\alpha_{\epsilon'_s,r}}m(u_l(z,c))^{-1} = U_{\alpha_{\epsilon'_s,r}}$$

  By replacing yt^{-r} with 0, and bt^{-2r} with bt^{-r}, one can see similar equations show:

$$m(u_l(z,c))U_{\alpha_{2\epsilon'_s,r}}m(u_l(z,c))^{-1} = U_{\alpha_{2\epsilon'_s,r}}$$

- Equation 29:

$$m_l(z,c)m_l(y,b)^{-1} = m_l(z,c)m_l(-y,-\tau(b))$$
$$= d_{0i}(m_l^0(z,c)m_l^0(-y,-\tau(b)), \tau(bc^{-1}))$$

  It can be seen that:

$$m_l^0(z,c)m_l^0(-y,-\tau(b)) : x \mapsto$$
$$x + y\tau(b)^{-1}f(y,x) - zc^{-1}f(z, x + y\tau(b)^{-1}f(y,z))$$

  Due to assumption that $x, y, z \in V_0$, and fact that
  $\tau(b)^{-1}f(y,x), c^{-1}f(z, x + y\tau(b)^{-1}f(y,x)), \tau(bc^{-1}) \in D$, we see that:

$$m_l(z,c)m_l(y,b)^{-1} \in C_{\mathcal{G}}(S)(k)$$

(RGD3) In this item, by ξ we mean $\xi = \pm$.

We note that by reordering the Witt basis (By making e_i appear before e_j whenever $0 < i < j$, and all e_i appear before the basis for V_0 if and only if $i > 0$), we can make it so that $U_{a'}$ consists of uni-upper-triangular (resp. uni-lower-triangular) matrices in $M_m(D^{k(t)})$ for $a' \in {}_k\Psi_+$ (resp. ${}_k\Psi_-$) (note, m is the dimension of V over D), the construction of elements in 6.1.3 as elements in $M_m(D^{k(t)})$ can be seen to "play nice with power of t", this allows us to utilize steps similar to the case of Chevalley groups (see 5.1.5) to prove the axiom:

Notation: Here, we will "invent" a new notation of $\mathcal{G}\{\cdot\}$, as an example, $\mathcal{G}\{D\}$ indicates consider $\mathcal{G}$ as embedded in $SL_m(D^{k(t)})$, and take the subgroup of $\mathcal{G}$ with entries in D. This notation is distinct from the intention of notation $\mathcal{G}(k)$; as $\mathcal{G}(k)$ indicates considering $\mathcal{G}$ as embedded in $SL_{2mn^2}(\overline{k(t)})$ (m is dimension of V over D, and n is index of D over k'), and take subgroup of $\mathcal{G}$ with entries in k.

In following list, $\alpha := \alpha_{a',r}$:

(a) **(When $a' \in {}_k\Psi_+$ and $r \geq 0$)**:

In case of $u_i(zt^{-r}, ct^{-2r})$, it can be seen that the element is upper-uni-triangular in the ordering of Witt basis we provided, and that it is by construction that each entry is either 1 or has t^{-2r}.

In case of $u_{ij}(\lambda t^{-r})$, it can be seen that the element is upper-uni-triangular in the ordering of Witt basis we provided, and that it is by construction that each non-zero entry is either 1 or has t^{-r}.

In general, this means that the entries strictly above diagonal for $U_{\alpha_{a',r}}$ would be contained within $t^{-r}D[t^{-1}]$.

In particular if we require $r \geq 0$, the entries are contained within $D[t^{-1}]$. That is $U_{\alpha_{a',r}} \subset \mathcal{G}\{D[t^{-1}]\}$

If we further require $r \geq 1$, the entries strictly above diagonal would be contained within $t^{-1}D[t^{-1}]$.

(b) **(When $a' \in {}_k\Psi_+$ and $r \leq -1$)**: An arbitrary element of U_α is a uni-upper-triangular matrix and has all entries strictly above diagonal in $tD[t]$, with $U_\alpha \subset \mathcal{G}\{D[t]\}$.

(c) **(When $a' \in {}_k\Psi_-$ and $r \geq 1$)**: An arbitrary element of U_α is a uni-lower-triangular matrix and has all entries strictly below diagonal in $t^{-1}D[t^{-1}]$, with $U_\alpha \subset \mathcal{G}\{D[t^{-1}]\}$.

(d) **(When $a' \in {}_k\Psi_-$ and $r \leq 0$)**: An arbitrary element of U_α is a uni-lower-triangular matrix and has all entries strictly below diagonal in $D[t]$, with $U_\alpha \subset \mathcal{G}\{D[t]\}$.

Where (b), (c), and (d) are obtained with similar procedure as (a).

Following is very similar to what we have done for Chevalley group case (5.1.5), except for we consider D in place of k: Consider $U_\xi \subset \mathcal{G}\{D[t^{-\xi}]\}$ we have:

$$U_+ \cap U_- \subset \mathcal{G}\{D[t^{-1}]\} \cap \mathcal{G}\{D[t]\} = \mathcal{G}\{D\}$$

Where the equal sign is just a set-theoretic result.

Consider group homomorphisms $p_\xi : \mathcal{G}\{D[t^{-\xi}]\} \to \mathcal{G}\{D\}$ induced by $D[t^{-\xi}] \to D$ (defined by $t^{-\xi} \mapsto 0$, where $t^{\pm} := t^{\pm 1}$) entry-wise. Considering preimage of subgroup under group homomorphism is subgroup, and (a) to (d) above, we can see the following: (We denote $\mathfrak{U}^{\pm}\{D\}$ the upper (for +) and lower (for -) uni-triangular matrices with entries in D)

In the case (a), $p_+(U_{\alpha_{a',0}}) \subset \mathfrak{U}^+\{D\}$ and for $r \geq 1$, $p_+(U_{\alpha_{a',l}}) = Id$ and have U_α in case (a) is contained in $p_+^{-1}(\mathfrak{U}^+\{D\})$.

Similar reasoning give us the image of U_α in case (b) under p_- is Id and hence U_α is contained in $p_-^{-1}(\mathfrak{U}^-\{D\})$, the image of U_α in case (c) under p_+ is Id and hence U_α is contained in $p_+^{-1}(\mathfrak{U}^+\{D\})$, and U_α in case (d) is contained in $p_-^{-1}(\mathfrak{U}^-\{D\})$.

To sum up: We have $U_\xi \subset p_\xi^{-1}(\mathfrak{U}^\xi\{D\})$. This implies:

$$U_\xi \cap \mathcal{G}\{D\} \subset p_\xi^{-1}(\mathfrak{U}^\xi\{D\}) \cap \mathcal{G}\{D\} \subset \mathfrak{U}^\xi\{D\}$$

Note the middle part is just describing elements of $p_\xi^{-1}(\mathfrak{U}^\xi\{D\})$ with entries in D, so they have to be in $\mathfrak{U}^{\pm}\{D\}$ by the construction of p_ξ and fact that $\mathfrak{U}^\xi\{D\}$ is uni-upper-triangular if $\xi = +$ and uni-lower-triangular if $\xi = -$.

We further have following:

$$U_+ \cap U_- = (U_+ \cap U_-) \cap \mathcal{G}\{D\} =$$

$$(U_+ \cap \mathcal{G}\{D\}) \cap (U_- \cap \mathcal{G}\{D\}) \subset \mathfrak{U}^+\{D\} \cap \mathfrak{U}^-\{D\}$$

But $\mathfrak{U}^+\{D\}$ consists of uni-upper-triangular matrices while $\mathfrak{U}^-\{D\}$ consists of uni-lower-triangular matrices, we must have $U_+ \cap U_{-\alpha_i} \subset U_+ \cap U_- = \{1\}$ for we constructed so that all simple affine roots to be positive and have $U_{-\alpha_i} \subset U_-$.

Then (RGD0) tells us that $U_{-\alpha_i} \neq \{1\}$ and hence for all simple affine roots α_i, we have $U_{-\alpha_i} \not\subset U_+$.

(RGD4) This is already shown in 6.1.7

(RGD5) Utilize number 26 and 27 of 6.1.5 and fact that t commutes with elements of D:

- Equation 26 provides:

$$du_{ij}(\lambda t^{-r})d^{-1} = u_{ij}(d_{-i}\lambda t^{-r}d_j^{-1}) = u_{ij}(d_{-i}\lambda d_j^{-1}t^{-r})$$

  This shows that $C_{\mathcal{G}}(S)(k)$ stabilizes $U_{\alpha_{\epsilon'_i+\epsilon'_j},r}$

- Equation 27 provides:

$$du_i(zt^{-r}, ct^{-2r})d^{-1} = u_i(d_0(zt^{-r})d_i^{-1}, d_{-i}ct^{-2r}d_i^{-1}) = u_i(d_0(z)d_i^{-1}t^{-r}, d_{-i}cd_i^{-1}t^{-2r})$$

  This shows that $C_{\mathcal{G}}(S)(k)$ stabilizes $U_{\alpha_{\epsilon'_i},r}$ and $U_{\alpha_{2\epsilon'_i},r}$ respectively.

$\square$

As a further comment, to construct the RGD system for the elementary subgroup $\mathcal{G}(k[t,t^{-1}])^+$, consider replacing $C_{\mathcal{G}}(S)(k)$'s with $C_{\mathcal{G}}(S)(k) \cap \mathcal{G}(k[t,t^{-1}])^+$'s in above 6.1.8.

If we were to generalize to the case where $\mathcal{G}$ is the unitary group instead of the special unitary group, argument in 6.1.8 still holds.

6.2 General non-split case

6.2.1 Bring into context: non split reductive groups over Laurent polynomial rings

Note 6.2.1 (Affine root group for non-split case)**.** *We now construct affine root group for non-split case, our context is: Consider S, a maximal k-split torus in isotropic reductive k-group $\mathcal{G}$. T, a maximal torus that contains S, is defined over k. By [8, 8.11], we may consider that T split over $\hat{k}$, some finite separable extension of k. We will abuse notation and write $\mathcal{G}(\overline{k(t)})$ to be the the same as $\mathcal{G}$.*

When we talk of the ***homogeneous polynomial map of degree i*** *denoted $f : V_1 \to V_2$ for (Arbitrary commutative ring R) R-module's V_1 and V_2, we mean map f has property $f(rv) = r^i f(v)$ for $v \in V_1$ and $r \in R$. Further, for any arbitrary R-algebra A, there is an induced map $A \otimes_R V_1 \to A \otimes_R V_2$ by $a \otimes v \mapsto a^i \otimes f(v)$ that we denote by the same f in an abuse of notation. This is consistent with the context of [22, sec: 6].*

We denote $R := k[t,t^{-1}]$, $\Psi := \Phi(\mathcal{G},T)$ the absolute root system, and ${}_k\Psi := \Phi(\mathcal{G},S)$ the relative *root system. We require the relative root system to be irreducible (requiring $\mathcal{G}$ to be almost simple will satisfy this requirement).*

As in [20, 15.3.6], we may understand the group of cocharacters of S to be a subgroup of group of cocharacters of T, and that we may still take the unique "coroot" for relative root in the similar fashion as in the absolute root case ([20, 7.1.8]). More details about the relative root system ${}_k\Psi$ is developed in [20, sec: 15.3].

We have following:

1.

Note 6.2.2 (Relative pinning maps). *For $a' \in {}_k\Psi$, take k-module $V_{a'}$ (k is a field so the module is free) as constructed in [18, thm: 2] (Or [22, lem: 6.1]), note $\hat{k}\otimes_k V_{a'}$ is also free over $\hat{k}$. We fix an ordered basis $M_{a'} := \{e^{a'}_{\delta}\}_{\delta\in\eta(a')}$ of $\hat{k}\otimes_k V_{a'}$ over $\hat{k}$, and let $m_{a'} := |M_{a'}| = |\eta(a')|$ be $\hat{k}\otimes_k V_{a'}$'s dimension over $\hat{k}$. Just like in [22, lem: 6.1], we take our T as the $\hat{R} := \hat{k}[t, t^{-1}]$-split torus ($T$ is contained in $C_{\mathcal{G}}(S)$, the Levi subgroup in our context (see [8, 20.4] [8, 20.6], we fix a pair of minimal opposite ("opposite" in the sense that $C_{\mathcal{G}}(S) = P^+ \cap P^-$, as in [8, 14.20]) parabolic k-subgroups $P^{\pm}$ to see $C_{\mathcal{G}}(S)$ as the Levi subgroup (as in [8, 21.11], take $P^{\pm} = C_{\mathcal{G}}(S)U_{{}_k\Psi_{\pm}}$))).*

For any $u = \sum_{\delta\in\eta(a')} c_\delta \otimes e^{a'}_{\delta} \in \hat{R}\otimes_k V_{a'}$ (we sometimes skip writing the "$\otimes$" in the simple tensor elements if the context is clear), we extend the construction of absolute pinning isomorphisms to ***Relative pinning maps*** *as following:*

$$x_{a'}(u) := (\prod_{\delta\in\eta(a')} x_\delta(c_\delta))(\prod_{\theta\in\eta(2a')} x_\theta(p^2_\theta(u)))$$

Where for $a \in \Psi$, x_a is the admissible isomorphism (i.e. absolute pinning isomorphism) as constructed in the split case (see 4.2.13); if for root $*$, $x_*(\cdot)$ does not exist, we treat the $x_*(\cdot)$ as 1. The homogeneous of degree 2 map $p^2_\theta : \hat{R}\otimes_k V_{a'} \to \hat{R}\otimes k \cong \hat{R}$ is induced by $p^2_\theta : V_{a'} \to k$.

We also consider for each relative root a', fix a generating set (the basis works) $M^k_{a'} := \{e^{a'}_1, \cdots, e^{a'}_{{}_km_{a'}}\}$ of $V_{a'}$ over k for ${}_km_{a'} \geq 1$. We may consider $V_{a'}$ embedded into $\hat{k}\otimes_k V_{a'}$ by $v \mapsto 1\otimes v$. Now, we consider $e^{a'}_i = \sum_{\delta\in\eta(a')} f_\delta e^{a'}_\gamma$ for $f_\delta \in \hat{k}$, then we have that:

$$x_{a'}(ce^{a'}_i) = x_{a'}(\sum_{\delta\in\eta(a')} c\cdot f_\delta e^{a'}_\gamma) = (\prod_{\delta\in\eta(a')} x_\delta(c\cdot f_\delta))(\prod_{\theta\in\eta(2a')} x_\theta(p^2_\theta(\sum_{\delta\in\eta(a')} c\cdot f_\delta e^{a'}_\gamma)))$$

Note above construction did not need to consider root that is more than 2 multiple as ${}_k\Psi$ is a root system; while in case of [22, lem: 6.1] and [18, thm: 2], the corresponding sets of roots are not necessarily root systems.

2.

Note 6.2.3 (Construction of Affine root groups in non-split case). *Consider $a' \in {}_k\Psi$. We define the **affine root groups in the non-split case** as:*

$$U_{\alpha_{a',l}} := \langle x_{a'}(ct^{-l} \otimes e_i^{a'}), x_{2a'}(ct^{-2l} \otimes e_j^{2a'}) | c \in k, e_i^{a'} \in M_{a'}^k, e_j^{2a'} \in M_{2a'}^k \rangle$$

In context of non-split isotropic reductive groups, we will denote Φ to be the set of affine roots associated to ${}_k\Psi$

In this construction, note that in the case of $(a') = \{a', 2a'\}$, we no longer have $U_{\alpha_{2a',l}} \leq U_{\alpha_{a',l}}$. But we still have $U_{\alpha_{a',l}} \leq U_{a'}$ for $a' \in {}_k\Psi$. See an alternative but equivalent construction that proves helpful in situations at 6.2.5.

3.

Note 6.2.4 (Important properies of relative pinning maps and affine root group). *We list some important properties of above constructions. We will apply results regarding the relative pinning maps from [22, lem: 6.2] and [18, lem: 7] in our context to obtain observations needed for our goal:*

(a) (Adaptation of [22, lem: 6.2 (i)] in our context) There exists a degree 2 homogeneous polynomial map $q_{a'}^2 : V_{a'} \oplus V_{a'} \to V_{2a'}$. For k-algebra R, any $v, w \in R \otimes_k V_{a'}$, and denoting the induced map again by $q_{a'}^2$, we have:

$$x_{a'}(v)x_{a'}(w) = x_{a'}(v + w)x_{2a'}(q_{a'}^2(v, w))$$

We apply above adaptation and make observation:

- *For k-root such that $(a') = \{a'\}$, we have for $c, d \in k$, and $e_i^{a'}, e_j^{a'} \in M_{a'}^k$:*

$$x_{a'}(ct^{-l} \otimes e_i^{a'} + dt^{-l} \otimes e_j^{a'}) = x_{a'}(ct^{-l} \otimes e_i^{a'})x_{a'}(dt^{-l} \otimes e_j^{a'}) \in U_{\alpha_{a',l}}$$

(We may in fact replace both of t^{-l} with t^{-2l} and see that a similar result still holds) For this reason, in this case, the extended $x_{a'}(\cdot)$'s commute just in like the split case.

And we also have:

$$x_{a'}(\sum_i c_i t^{-l} \otimes e_i^{a'}) = \prod_i x_{a'}(c_i t^{-l} \otimes e_i^{a'}) \in U_{\alpha_{a',l}}$$

Considering the fact that $x_{a'}(0) = 1$ *([18, thm: 2]), by above we have* $x_{a'}(v)^{-1} = x_{a'}(-v)$ *for all* $v \in R \otimes_k V_{a'}$.

- *For k-root such that* $(a') = \{a', 2a'\}$*, we have for* $c, d \in k$*, and* $e_i^{a'}, e_j^{a'} \in M_{a'}^k$*:*

$$x_{a'}(ct^{-l} \otimes e_i^{a'} + dt^{-l} \otimes e_j^{a'}) = x_{a'}(ct^{-l} \otimes e_i^{a'}) x_{a'}(dt^{-l} \otimes e_j^{a'}) x_{2a'}(q_{a'}^2(ct^{-l} \otimes e_i^{a'}, dt^{-l} \otimes e_j^{a'}))^{-1}$$

$$= x_{a'}(ct^{-l} \otimes e_i^{a'}) x_{a'}(dt^{-l} \otimes e_j^{a'}) x_{2a'}(-q_{a'}^2(ct^{-l} \otimes e_i^{a'}, dt^{-l} \otimes e_j^{a'}))$$

We also have:

$$q_{a'}^2(ct^{-l} \otimes e_i^{a'}, dt^{-l} \otimes e_j^{a'}) = (t^{-2l}) \otimes q_{a'}^2(ce_i^{a'}, de_j^{a'}) = (t^{-2l}) \otimes \sum_{h=1}^{{}_k m_{2a'}} f_h e_h^{2a'}$$

$$= \sum_{h=1}^{{}_k m_{2a'}} (f_h)(t^{-2l}) \otimes e_h^{2a'}$$

For $f_h \in k$ *(as* $e_i^{a'}, e_j^{a'}, e_h^{a'} \in M_{a'}^k$ *is considered a generating set of* $V_{a'}$ *over k, and recall that the map* $q_{a'}^2 : V_{a'} \oplus V_{a'} \to V_{2a'}$ *induces, through tensor, the map* $q_{a'}^2$ *we use in current context). Recall that the tensor product is taken over k. Together with the result in first bullet point, we see:*

$$x_{a'}(ct^{-l} \otimes e_i^{a'} + dt^{-l} \otimes e_j^{a'}) = x_{a'}(ct^{-l} \otimes e_i^{a'}) x_{a'}(dt^{-l} \otimes e_j^{a'}) x_{2a'}(\sum_{h=1}^{{}_k m_{2a'}} -(f_h)(t^{-2l}) \otimes e_h^{2a'})$$

$$\in U_{\alpha_{a',l}} \qquad (*)$$

By induction through changing one of the simple tensors into a sum of finite many tensors (And similar results as above line (*) can be reached), we will have that:

$$x_{a'}(\sum_i c_i t^{-l} \otimes e_i^{a'}) \in U_{\alpha_{a',l}}$$

- *It can be seen that in the case when* $a' \in {}_k\Psi$ *so that* $2a' \notin {}_k\Psi$ *(This is already* *mentioned in the first bullet point in above):*

$$x_{a'}(v)^{-1} = x_{a'}(-v)$$

And that in the case when $a', 2a' \in {}_k\Psi$:

$$x_{a'}(v)x_{a'}(-v) = x_{2a'}(q^2_{a'}(v, -v))$$

And so have:

$$x_{a'}(v)^{-1} = x_{a'}(-v)x_{2a'}(-q^2_{a'}(v, -v))$$

- *Note, by construction, taking $r, s \in R$, and $\delta \in \eta(a')$, we have $x_\delta(r) \in U_{a'}$. While for all $\gamma \in \eta(2a')$, we have $x_\gamma(s) \in U_{2a'} = Z(U_{a'})$. This means that:*

$$x_{a'}(v) \text{ and } x_{2a'}(w) \text{ commutes}$$

for $v \in R \otimes_k V_{a'}$ and $w \in R \otimes_k V_{2a'}$.

(b) *(Adaptation of [22, lem:6.2 (ii)] in our context) Recall again the Levi subgroup in our context is $C_{\mathcal{G}}(S)$, then have for any $g \in C_{\mathcal{G}}(S)(k)$, there exists degree i homogeneous polynomial maps $\phi^i_{g,a'} : V_{a'} \to V_{ia'}$ for $i = 1, 2$. For k-algebra R, $v \in R \otimes_k V_{a'}$, and denoting the induced maps by $\phi^i_{g,a'}$, we have:*

$$gx_{a'}(v)g^{-1} = x_{a'}(\phi^1_{g,a'}(v))x_{2a'}(\phi^2_{g,a'}(v))$$

*We will apply the above adaptation to claim that $C_{\mathcal{G}}(S)(k)$ **normalizes each affine root group in the non-split case**, we only need to consider the generators to do this: For $g \in C_{\mathcal{G}}(S)(k)$, we have:*

- *If $(a') = \{a'\}$:*

$$gx_{a'}(ct^{-l} \otimes e^{a'}_i)g^{-1} = x_{a'}(\phi^1_{g,a'}(ct^{-l} \otimes e^{a'}_i)) = x_{a'}((ct^{-l}) \otimes \phi^1_{g,a'}(e^{a'}_i))$$

Taking $\phi^1_{g,a'}(e^{a'}_i) = \sum_{h=1}^{{}_k m_{a'}} f_h e^{a'}_h$ for $f_h \in k$:

$$= x_{a'}((ct^{-l}) \otimes \sum_{h=1}^{{}_k m_{a'}} f_h e^{a'}_h) = x_{a'}(\sum_{h=1}^{{}_k m_{a'}} (cf_h)(t^{-l}) \otimes e^{a'}_h) \in U_{\alpha_{a',l}}$$

- *If* $(a') = \{a', 2a'\}$*:*

$$gx_{a'}(ct^{-l} \otimes e_i^{a'})g^{-1}$$

$$= x_{a'}(\phi^1_{g,a'}(ct^{-l} \otimes e_i^{a'}))x_{2a'}(\phi^2_{g,a'}(ct^{-l} \otimes e_i^{a'}))$$

Again, taking $\phi^1_{g,a'}(e_i^{a'}) = \sum_{h=1}^{{}_km_{a'}} f_h^1 e_h^{a'}$ *and* $\phi^2_{g,a'}(e_i^{a'}) = \sum_{h=1}^{{}_km_{2a'}} f_h^2 e_h^{2a'}$ *for* $f_h^1, f_h^2 \in k$*:*

$$= x_{a'}((ct^{-l}) \otimes \sum_{h=1}^{{}_km_{a'}} f_h^1 e_h^{a'})x_{2a'}((c^2t^{-2l}) \otimes \sum_{h=1}^{{}_km_{2a'}} f_h^2 e_h^{2a'})$$

$$= x_{a'}(\sum_{h=1}^{{}_km_{a'}} (cf_h^1)(t^{-l}) \otimes e_h^{a'})x_{2a'}(\sum_{h=1}^{{}_km_{2a'}} (c^2f_h^2)(t^{-2l}) \otimes e_h^{2a'}) \in U_{\alpha_{a',l}}$$

Note $gx_{2a'}(ct^{-2l} \otimes e_i^{2a'})g^{-1} \in U_{\alpha_{a',l}}$ *is already proven in above bullet point.*

(c) *(**Generalized Chevalley commutator formula for relative pinning maps** [22, lem:6.2 (iii)]) For* $a', b' \in {}_k\Psi$ *such that* $ma' \neq -kb'$ *for all* $m, k \geq 1$*, there exists polynomial map(s)*

$$N_{a',b';i,j} : V_{a'} \times V_{b'} \to V_{ia'+jb'}$$

for $i, j > 0$*, homogeneous of degree* i *in the first variable and of degree* j *in the second variable (implying we have* $(r_1 \otimes v, r_2 \otimes w) \mapsto r_1^i r_2^j \otimes N_{a',b';i,j}(v, w)$ *for the induced map); such that for k-algebra* R *and any* $v \in R \otimes_k V_{a'}$ *and* $w \in R \otimes_k V_{b'}$*, and denoting the induced maps with* $N_{a',b';i,j}$*, one has*

$$[x_{a'}(v), x_{b'}(w)] = \prod_{i,j>0, ia'+jb' \in {}_k\Psi} x_{ia'+jb'}(N_{a',b';i,j}(v, w))$$

This will be essential to our proof for (RGD1).

(d) *(Adaptation of [18, lem: 7] in our context) Considering* R *as an arbitrary ring extension of* k *(We will mostly utilize this result for* $R := k[t, t^{-1}]$*, but this works in general), recall construction of* $M_{a'}^k$ *forming a generating set of* $V_{a'}$ *over* k *for* $a' \in {}_k\Psi$*. For a **"unipotent closed set"*** ${}_k\Psi'$ *of* ${}_k\Psi$ *(That is a subset of* ${}_k\Psi$ *that is closed under addition when the sum is also a k-root and does not contain collinear oppositely directed k-roots. Note that "unipotent closed set" is same as "special" in the context of [8, 14.5], see [8, 14.7] for detail). Let* $U_{{}_k\Psi'} := \langle U_{a'} | a' \in_k \Psi' \rangle$*,* $U_{{}_k\Psi'}(R)$ *is generated by the collection of* $x_{a'}(r \otimes e_i^{a'})$*'s for* $r \in R$*,* $a \in {}_k\Psi'$*, and* $e_i^{a'} \in M_{a'}^k$*. We use this to obtain the following results:*

- *For $a' \in {}_k\Psi$, (a') is "unipotent closed set" and so we have $U_{(a')}(R)$ is generated by the collection of $x_{a'}(r \otimes e_i^{a'})$'s for $r \in R$, $a \in (a')$, and $e_i^{a'} \in M_{a'}^k$.*

- *$U_{{}_k\Psi_\pm}(R)$ is generated by the collection of $x_{a'}(r \otimes e_i^{a'})$'s for $r \in R$, $a \in {}_k\Psi_\pm$, and $e_i^{a'} \in M_{a'}^k$. To see this, we use ${}_k\Psi_\pm$ as the "unipotent closed set" in ${}_k\Psi$*

- *The **elementary subgroup** $\mathcal{G}(R)^+ := \langle U_{{}_k\Psi_+}(R), U_{{}_k\Psi_-}(R)\rangle$ is generated by the collection of $x_{a'}(r \otimes e_i^{a'})$'s for $r \in R$, $a \in {}_k\Psi$, and $e_i^{a'} \in M_{a'}^k$.*

2'.

Note 6.2.5 (Alternative form of affine root group in non-splt case). *This observation is largely a consequence of 6.2.4 (a). Consider for $a' \in {}_k\Psi$ such that $(a') = \{a', 2a'\}$:*

Recall from 6.2.4 (a) that:

$$x_{a'}(v) \text{ and } x_{2a'}(w) \text{ commutes}$$

For $v \in R \otimes_k V_{a'}$ and $w \in R \otimes_k V_{2a'}$

With this knowledge, we may move all terms of $x_{a'}(\cdot)$'s to the left, and all terms of $x_{2a'}(\cdot)$'s to the right. Consulting line (*) from (a) of item 3 above, we can rewrite for v_i with form $\sum_i h_i t^{-l} \otimes e_i^{a'}$ for h_i's in k and obtain that:

$$\prod_i x_{a'}(v_i) = x_{a'}(\sum_i v_i) x_{2a'}(\sum_{j=1}^{{}_k m_{2a'}} f_j t^{-2l} \otimes e_j^{2a'})$$

for f_j's in k. Furthering, with similar logic, by also involving the $x_{2a'}(\cdot)$'s, we sue (a) of 6.2.4 to obtain that any element of $U_{\alpha_{a',l}}$ has form

$$x_{a'}(\sum_{i=1}^{{}_k m_{a'}} c_i t^{-l} \otimes e_i^{a'}) x_{2a'}(\sum_{i=1}^{{}_k m_{2a'}} d_i t^{-2l} \otimes e_i^{2a'})$$

for $c_i, d_i \in k$. In other words, we may write:

$$U_{\alpha_{a',l}} \subset \{x_{a'}(\sum_{i=1}^{{}_k m_{a'}} c_i t^{-l} \otimes e_i^{a'}) x_{2a'}(\sum_{i=1}^{{}_k m_{2a'}} d_i t^{-2l} \otimes e_i^{2a'}) | c_i, d_i \in k\}$$

$$= \{x_{a'}(t^{-l} \otimes \sum_{i=1}^{{}_k m_{a'}} c_i e_i^{a'}) x_{2a'}(t^{-2l} \otimes \sum_{h=1}^{{}_k m_{2a'}} d_i e_i^{2a'}) | c_i, d_i \in k\}$$

To further replace $\subset$ with $=$, utilize the first two bullet points in (a) of 6.2.4 above. Hence, we have arrived at ***an alternative but equivalent way to construct the affine root groups in the non-split case****. Same construction holds when $(a') = \{a'\}$ as one may simply consider all $x_{2a'}(\cdot)$'s to be identity.*

4.

Note 6.2.6 (Preparation for (RGD2) in non-split case)**.** *We make following observations:*

(a) *Consider for $a, b \in \Psi := \Phi(\mathcal{G}, T)$, we have that $< b, j(a) >=< j(b), j(a) >$ as $j(b) := b|_S$ (We are viewing the pairing $< \cdot, \cdot >$ as "composition" in the sense of 4.2.6). Now consider following (calculation in $\mathcal{G}(\overline{k(t)})$):*

$$(j(a)^\vee(r^{1/2}))(x_{\pm a}(s))(j(a)^\vee(r^{1/2}))^{-1} = x_{\pm a}(r^{1/2<\pm a, j(a)>}s) = x_{\pm a}(r^{\pm 1}s)$$

Now further consider for $c' \in {}_k\Psi := \Phi(\mathcal{G}, S)$ with $j(a) = 2c'$, we also have:

$$(c'^\vee(r^{1/2}))(x_{\pm a}(s))(c'^\vee(r^{1/2}))^{-1} = x_{\pm a}(r^{1/2<\pm a, c'>}s) = x_{\pm a}(r^{1/2<2\pm c', c'>}s) = x_{\pm a}(r^{\pm 2}s)$$

(b) *In following, we generalize item (a) above a little:*

As convention for this item, $a, b, c, d, e \in \Psi$, $j(a) = a'$, $j(b) = b'$, $j(c) = 2b'$, $j(d) = s_{a'}(b')$, $j(e) = 2s_{a'}(b')$, and $\forall m, n \in \mathbb{Z}$. we consider:

$$(a'^\vee(t^{m/2}))(x_b(kt^{-n}))(a'^\vee(t^{m/2}))^{-1} = x_b(t^{(m/2)<b,a'>}kt^{-n}) = x_b(kt^{-n+(m/2)<b',a'>})$$

$$(a'^\vee(t^{m/2}))(x_c(kt^{-2n}))(a'^\vee(t^{m/2}))^{-1} = x_c(t^{(m/2)<c,a'>}kt^{-2n}) = x_c(kt^{-2n+(m/2)<2b',a'>})$$

On the other hand, we consider:

$$(a'^\vee(t^{-m/2}))(x_d(kt^{-n+(m/2)<b',a'>}))(a'^\vee(t^{-m/2}))^{-1}$$

$$= x_d(kt^{-n+(m/2)<b',a'>}t^{-(m/2)<s_{a'}(b'),a'>})$$

$$= x_d(kt^{-n+m<b'a'>})$$

$$(a'^\vee(t^{-m/2}))(x_e(kt^{-2n+(m/2)<2b',a'>}))(a'^\vee(t^{-m/2}))^{-1}$$

$$= x_e(kt^{-2n+(m/2)<2b',a'>}t^{-(m/2)<2s_{a'}(b'),a'>})$$

$$= x_e(kt^{-2n+m<2b',a'>})$$

Recall that $<\cdot,\cdot>$ is (only) linear in first coordinate (when both coordinates are k-roots) and that:

$$< s_{a'}(b'), a' > - < b', a' > = < - < b', a' > a', a' >$$

$$= - < b', a' >< a', a' > = -2 < b', a' >$$

(c) *Using item (b) and (a), we observe the following about generators of*

$$U_{\alpha_{b',n}} := \langle x_{b'}(ct^{-n} \otimes e_i^{b'}), x_{2b'}(ct^{-2n} \otimes e_j^{2b'}) | c \in k, e_i^{b'} \in M_{b'}^k, e_j^{2b'} \in M_{2b'}^k \rangle$$

When we talk about affine root groups, we consider $n \in \mathbb{Z}$; but in following, we will for now consider n to be half and full integers (So that $U_{\alpha_{b',n}} \subset \mathcal{G}(k[t^{-1/2}, t^{1/2}])$). Consider l in full integers (so that $l/2$ in half and full integers):

- *The case of $2b'$:*

$$a'^{\vee}(t^{-l/2}) x_{2b'}(ct^{-2n} \otimes e_i^{2b'}) a'^{\vee}(t^{-l/2})^{-1}$$

We write $e_i^{2b'} = \sum_{\theta \in \eta(2b')} h_\theta e_\theta^{a'}$ for $h_\theta \in \hat{k}$, and hence have that $x_{2b'}(ce_i^{2b'}) = \prod_{\theta \in \eta(2b')} x_\theta(c \cdot h_\theta)$:

$$= a'^{\vee}(t^{-l/2}) \prod_{\theta \in \eta(2b')} x_\theta(ct^{-2n} \cdot h_\theta) a'^{\vee}(t^{-l/2})^{-1}$$

$$= \prod_{\theta \in \eta(2a')} x_\theta(ct^{-l<b',a'>-2n} \cdot h_\theta) = x_{2b'}(ct^{-l<b',a'>-2n} \otimes e_i^{2b'}) \in U_{\alpha_{b',n+\frac{l<b',a'>}{2}}}$$

- *The case of b':*

$$a'^{\vee}(t^{-l/2}) x_{b'}(ct^{-n} \otimes e_i^{b'}) a'^{\vee}(t^{-l/2})^{-1}$$

We write again that $e_i^{b'} = \sum_{\delta \in \eta(b')} f_\delta e_\gamma^{b'}$ for $f_\delta \in \hat{k}$, then we have that $x_{b'}(ce_i^{b'}) = (\prod_{\delta \in \eta(b')} x_\delta(c \cdot f_\delta))(\prod_{\theta \in \eta(2b')} x_\theta(p_\theta^2(\sum_{\delta \in \eta(b')} c \cdot f_\delta e_\gamma^{b'})))$:

$$= a'^{\vee}(t^{-l/2})(\prod_{\delta \in \eta(b')} x_\delta(ct^{-n} \cdot f_\delta))(\prod_{\theta \in \eta(2b')} x_\theta(p_\theta^2(\sum_{\delta \in \eta(b')} ct^{-n} \cdot f_\delta e_\gamma^{b'}))) a'^{\vee}(t^{-l/2})^{-1}$$

Note that we can insert $a'^{\vee}(t^{-l/2}) a'^{\vee}(t^{-l/2})^{-1}$ in between each element, we observe *following:*

$$a'^{\vee}(t^{-l/2})x_{\delta}(ct^{-n}\cdot f_{\delta})a'^{\vee}(t^{-l/2})^{-1}=x_{\delta}(ct^{-\frac{l<b',a'>}{2}-n}\cdot f_{\delta})$$

Recall we have k-module $V_{b'}$*, then have* $p_{\theta}^{2}:\hat{k}[t^{1/2},t^{-1/2}]\otimes_{k}V_{b'}\rightarrow\hat{k}[t^{1/2},t^{-1/2}]\otimes_{k}k\cong\hat{k}[t^{1/2},t^{-1/2}]$ *is induced by homogeneous degree 2 map* $p_{\theta}^{2}:V_{b'}\rightarrow k$ *by* $r\otimes v\mapsto r^{2}\otimes p_{\theta}^{2}(v)$*, we may write* $p_{\theta}^{2}(\sum_{\delta\in\eta(b')}c\cdot f_{\delta}e_{\gamma}^{b'})=k_{\theta}$*, and hence* $p_{\theta}^{2}(p_{\theta}^{2}(\sum_{\delta\in\eta(b')}ct^{-m}\cdot f_{\delta}e_{\gamma}^{b'}))=p_{\theta}^{2}(t^{-m}\sum_{\delta\in\eta(b')}c\cdot f_{\delta}e_{\gamma}^{b'})=k_{\theta}t^{-2m}$ *for m in half and full integers (See detail about the homogeneous polynomial map at [22, sec: 6]), and have that:*

$$a'^{\vee}(t^{-l/2})x_{\theta}(p_{\theta}^{2}(\sum_{\delta\in\eta(b')}c\cdot f_{\delta}e_{\gamma}^{b'}))a'^{\vee}(t^{-l/2})^{-1}=a'^{\vee}(t^{-l/2})x_{\theta}(k_{\theta}t^{-2n})a'^{\vee}(t^{-l/2})^{-1}$$

$$=x_{\theta}(k_{\theta}t^{-l<b',a'>-2n})$$

Now we have that:

$$a'^{\vee}(t^{-l/2})x_{b'}(ct^{-n}\otimes e_{i}^{b'})a'^{\vee}(t^{-l/2})^{-1}$$

$$=a'^{\vee}(t^{-l/2})(\prod_{\delta\in\eta(b')}x_{\delta}(ct^{-n}\cdot f_{\delta}))(\prod_{\theta\in\eta(2b')}x_{\theta}(p_{\theta}^{2}(\sum_{\delta\in\eta(b')}ct^{-n}\cdot f_{\delta}e_{\gamma}^{b'})))a'^{\vee}(t^{-l/2})^{-1}$$

$$=(\prod_{\delta\in\eta(b')}x_{\delta}(ct^{-\frac{l<b',a'>}{2}-n}\cdot f_{\delta}))(\prod_{\theta\in\eta(2b')}x_{\theta}(k_{\theta}t^{-l<b',a'>-2n}))$$

$$=(\prod_{\delta\in\eta(b')}x_{\delta}(ct^{-\frac{l<b',a'>}{2}-n}\cdot f_{\delta}))(\prod_{\theta\in\eta(2b')}x_{\theta}(p_{\theta}^{2}(\sum_{\delta\in\eta(b')}ct^{-\frac{l<b',a'>}{2}-n}f_{\delta}e_{\delta}^{b'})))$$

$$=x_{b'}(ct^{-\frac{l<b',a'>}{2}-n}\otimes e_{i}^{b'})\in U_{\alpha_{b',\frac{l<b',a'>}{2}+n}}$$

As a summary, *we now have that* $a'^{\vee}(t^{-l/2})U_{\alpha_{b',n}}a'^{\vee}(t^{-l/2})^{-1}\subset U_{\alpha_{b',\frac{l<b',a'>}{2}+n}}$*. On the other hand, note that* $a'^{\vee}(t^{l/2})U_{\alpha_{b',\frac{l<b',a'>}{2}+n}}a'^{\vee}(t^{-l/2})\subset U_{\alpha_{b',\frac{l<b',a'>}{2}+n-\frac{l<b',a'>}{2}}}=U_{\alpha_{b',n}}$*, we hence have:*

$$a'^{\vee}(t^{-l/2})U_{\alpha_{b',n}}a'^{\vee}(t^{-l/2})^{-1}=U_{\alpha_{b',\frac{l<b',a'>}{2}+n}}\tag{!}$$

With similar procedure, it can be seen that:

$$a'^{\vee}(t^{-l/2})U_{\alpha_{-b',-n}}a'^{\vee}(t^{-l/2})^{-1}=U_{\alpha_{-b',-\frac{l<b',a'>}{2}-n}}$$

Observation 6.2.7 (Isomorphism between affine root groups by coroots). *By equation (!) above and the fact that conjugation by group element is a group isomorphism onto the image, we see that any element of $x \in U_{\alpha_{a',l}}$ (resp. more generally, $x \in U_{\alpha_{b',\frac{l<b',a'>}{2}+n}}$) can be expressed as:*

$$x = a'^{\vee}(t^{-l/2})ua'^{\vee}(t^{-l/2})^{-1}$$

for some unique $u \in U_{a'}(k)$ (resp. more generally, $u \in U_{\alpha_{b',n}}$).

(d) *Now assume that for all $a', b' \in {}_k\Psi$, and non-identity element $u \in U_{a'}(k) = U_{\alpha_{a',0}}$, there exists element $w_{a',u} \in U_{-a'}(k)uU_{-a'}(k) \subset \mathcal{G}(k) \subset \mathcal{G}(k[t,t^{-1}])$, for $\forall n \in \mathbb{Z}/2$ (half and full integers) so that:*

$$w_{a',u}U_{\alpha_{b',n}}w_{a',u}^{-1} = U_{\alpha_{s_{a'}(b'),n}} \subset \mathcal{G}(k[t^{-1/2}, t^{1/2}]) \ \&\ w_{a',u_1}w_{a',u_2}^{-1} \in C_{\mathcal{G}}(S)(k) \qquad (**)$$

*for all $u_1, u_2 \in U_{a'}(k)^*$. Requiring n to be half and full integer is indeed **necessary** for the sake of item (e) below.*

Note: *We will call the above condition "**Condition (**)**"*

Note: *if we can show $n_{a'}$ in [4, Satz: 27] (focusing on DR4 and DR5) satisfy the above equation when replacing $w_{a',u}$, then we can construct a $w_{a',u}$ needed as an element of $C_{\mathcal{G}}(S)(k)n_{a'}$ (This will involve [22, lem: 6.2] (or (b) of 6.2.4) where the conjugation by element in $C_{\mathcal{G}}(S)(k)$ is given exact form). See more on this candidate for $w_{a',u}$ in 6.2.9, where we use this candidate to help relax the requirement on condition (**) for 6.2.8.*

By using the element $w_{a',u}$ required in above, with item (b), we can define and see that for any non identity element $x = a'^{\vee}(t^{-l/2})ua'^{\vee}(t^{-l/2})^{-1} \in U_{\alpha_{a',l}}$ where $u \in U_{a'}(k)$ (see 6.2.7 above about this conjugation):

$$w_{a',x}(t^{-l}) := a'^{\vee}(t^{-l/2})w_{a',u}a'^{\vee}(t^{-l/2})^{-1} = a'^{\vee}(t^{-l/2})w_{a',u}a'^{\vee}(t^{l/2})$$

$$\in a'^{\vee}(t^{-l/2})U_{-a'}(k)uU_{-a'}(k)a'^{\vee}(t^{l/2})$$

$$= a'^{\vee}(t^{-l/2})U_{-a'}(k)a'^{\vee}(t^{l/2})(a'^{\vee}(t^{-l/2})ua'^{\vee}(t^{-l/2})^{-1})a'^{\vee}(t^{-l/2})U_{-a'}(k)a'^{\vee}(t^{l/2})$$

$$= U_{-\alpha_{a',l}}(a'^{\vee}(t^{-l/2})ua'^{\vee}(t^{-l/2})^{-1})U_{-\alpha_{a',l}} = U_{-\alpha_{a',l}}xU_{-\alpha_{a',l}}$$

The second equal sign is by $a'^{\vee}(r)^{-1} = a'^{\vee}(r^{-1})$ as $a'^{\vee}$ is a cocharacter.

To check that $a'^{\vee}(t^{-l/2})U_{a'}(k)a'^{\vee}(t^{-l/2})^{-1} = U_{\alpha_{a',l}}$, *we will consider the generators of*

$$U_{a'}(k) = U_{\alpha_{a',0}} := \langle x_{a'}(c \otimes e_i^{a'}), x_{2a'}(c \otimes e_j^{2a'}) | c \in k, e_i^{a'} \in M_{a'}^k, e_j^{2a'} \in M_{2a'}^k \rangle$$

Where the first equal sign is by [18, lem: 7]. We will assume $\{a', 2a'\} \subset {}_k\Psi$ *(Since the case of only* $\{a'\} \subset {}_k\Psi$ *can be done by ignoring the steps for* $x_{2a'}(\cdot)$*). It can be seen that this is really just a special case of item (c) above with* $n = 0$ *and* $b' = a'$ *(So that* $< b', a' >= 2$*).*

The fact $a'^{\vee}(t^{-l/2})U_{-a'}(k)a'^{\vee}(t^{-l/2})^{-1} = U_{\alpha_{-a',-l}} = U_{-\alpha_{a',l}}$ *can be checked similarly.*

(e) *With the item (b), we can see that for* $m, l \in \mathbb{Z}$, *and* $x = a'^{\vee}(t^{-l/2})ua'^{\vee}(t^{-l/2})^{-1} \in U_{\alpha_{a',l}}$ *where* $u \in U_{a'}(k)$ *(see 6.2.7):*

$$w_{a',x}(t^{-l})U_{\alpha_{b',m}}w_{a',x}(t^{-l})^{-1} \subset U_{\alpha_{s_{a'}(b'),m-l<b',a'>}} = U_{s_{a',l}(\alpha_{b',m})}$$

To verify this, we check the following:

Recall that $a'^{\vee}(r)^{-1} = a'^{\vee}(r^{-1})$ *as* $a'^{\vee}$ *is a cocharacter.*

First, we already know by item (c) above that:

$$a'^{\vee}(t^{l/2})U_{\alpha_{b',m}}a'^{\vee}(t^{l/2})^{-1} \subset U_{\alpha_{b',\frac{-l<b',a'>}{2}+m}}$$

Then (provided the existence of $w_{a',u}$*), we use the element* $w_{a',u}$ *as required in above to conjugate* $U_{\alpha_{b',\frac{-l<b',a'>}{2}+m}}$*:*

$$w_{a',u}U_{\alpha_{b',\frac{-l<b',a'>}{2}+m}}w_{a',u}^{-1} = U_{\alpha_{s_{a'}(b'),\frac{-l<b',a'>}{2}+m}}$$

Then conjugate with $a'^{\vee}(t^{-l/2})$*:*

$$a'^{\vee}(t^{-l/2})U_{\alpha_{s_{a'}(b'),\frac{-l<b',a'>}{2}+m}}a'^{\vee}(t^{-l/2})^{-1} \subset U_{\alpha_{s_{a'}(b'),\frac{l<s_{a'}(b'),a'>}{2}-\frac{l<b',a'>}{2}+m}}$$

$$= U_{\alpha_{s_{a'}(b'),m-l<b',a'>}} = U_{s_{a',l}(\alpha_{b',m})}$$

Recalling by construction that $(a'^{\vee}(r))^{-1} = a'^{\vee}(r^{-1})$, *we also have:*

$$w_{a',x}(t^{-l})^{-1}U_{s_{a,l}(\alpha_{b,m})}w_{a',x}(t^{-l}) \subset U_{\alpha_{b',m}}$$

And so have:

$$w_{a',x}(t^{-l})U_{\alpha_{b',m}}w_{a',x}(t^{-l})^{-1} = U_{s_{a,l}(\alpha_{b,m})}$$

6.2.2 Constructing RGD system for elementary subgroup of non split reductive group

Theorem 6.2.8 (Construction of RGD system for the non-split case)**.** *Let $\{a'_1, \cdots, a'_n\} \subset_k \Psi_+$ be base for the relative root system ${}_k\Psi$, let a'_0 be the highest element in ${}_k\Psi_+$. Assuming there exist elements $w_{a'_i,u}$ (For each $u \in U_{a'_i}(k)$) for all $i \in [1,n]$ and $w_{-a'_0,u}$ (For each $u \in U_{-a'_0}(k)$) as required in 6.2.6 satisfying (**). We claim that, using the construction in 6.2.1, we have:*

$$(\mathcal{G}(R)^+C_{\mathcal{G}}(S)(k), (U_{\alpha_{a',l}})_{\alpha_{a',l}\in\Phi}, C_{\mathcal{G}}(S)(k))$$

is a general RGD system

Proof. We will again go through the axioms:

(RGD0) We seek to show that $U_\alpha \neq \{1\}$ for all $\alpha \in \Phi$.

It is shown in (DR1) of [4, Satz: 27] that $U_{a'}(k) = U_{\alpha_{a',0}} \neq \{1\}$. Then we extend this fact with equation (!), and fact that conjugation by group element is group isomorphism to see that $U_{\alpha_{a',n}} \neq \{1\}$ in general.

(RGD1) Consider by generalized Chevalley commutator formula for relative pinning maps, for $a', b' \in {}_k\Psi$ such that $ma' \neq -kb'$ for all $m, k \geq 1$ (Note this is precisely equivalent to requiring $\alpha_{a',l}$ and $\alpha_{b,m}$ is prenilpotent pair), we have: For all $l, m \in \mathbb{Z}$ (keeping in mind that the tensor product is taken over k see detail 6.2.1):

$$[x_{a'}(ct^{-l}\otimes e_h^{a'}), x_{b'}(dt^{-m}\otimes e_k^{b'})] = \prod_{i,j>0, ia'+jb'\in {}_k\Psi} x_{ia'+jb'}(N_{a',b';i,j}(ct^{-l}\otimes e_h^{a'}, dt^{-m}\otimes e_k^{b'}))$$

$$= \prod_{i,j>0, ia'+jb'\in {}_k\Psi} x_{ia'+jb'}((c^id^j)(t^{-il-jm})\otimes N_{a',b';i,j}(e_h^{a'}, e_k^{b'}))$$

We may take $N_{a',b';i,j}(e_h^{a'}, e_k^{b'}) = \sum_{s=1}^{{}_km_{ia'+jb'}} f_s e_s^{ia'+jb'}$ for $f_s \in k$ (as $V_{ia'+jb'}$ is k-module) and continue the calculation above:

$$= \prod_{i,j>0, ia'+jb' \in {}_k\Psi} x_{ia'+jb'}((c^i d^j)(t^{-il-jm}) \otimes (\sum_{s=1}^{{}_k m_{ia'+jb'}} f_s e_s^{ia'+jb'}))$$

$$= \prod_{i,j>0, ia'+jb' \in {}_k\Psi} x_{ia'+jb'}(\sum_{s=1}^{{}_k m_{ia'+jb'}} f_s (c^i d^j)(t^{-il-jm}) \otimes e_s^{ia'+jb'})$$

$$\in \langle U_\beta | \beta \in \{\alpha_{pa'+qb', pl+qm} \in \Phi | p, q \in \mathbb{Z}_{>0}\}\rangle$$

For our context, more specifically, above precisely shows that, for $(a') = \{a', 2a'\}$:

$$[x_{a'}(ct^{-l} \otimes e_i^{a'}), x_{b'}(dt^{-m} \otimes e_j^{b'})] \in \langle U_\beta | \beta \in \{\alpha_{pa'+qb', pl+qm} \in \Phi | p, q \in \mathbb{Z}_{>0}\}\rangle$$

$$[x_{a'}(ct^{-l} \otimes e_i^{a'}), x_{2b'}(dt^{-2m} \otimes e_j^{2b'})] \in \langle U_\beta | \beta \in \{\alpha_{pa'+q2b', pl+q2m} \in \Phi | p, q \in \mathbb{Z}_{>0}\}\rangle$$

$$\subset \langle U_\beta | \beta \in \{\alpha_{pa'+qb', pl+qm} \in \Phi | p, q \in \mathbb{Z}_{>0}\}\rangle$$

$$[x_{2a'}(ct^{-2l} \otimes e_i^{2a'}), x_{2b'}(dt^{-2m} \otimes e_j^{2b'})] \in \langle U_\beta | \beta \in \{\alpha_{p2a'+q2b', p2l+q2m} \in \Phi | p, q \in \mathbb{Z}_{>0}\}\rangle$$

$$\subset \langle U_\beta | \beta \in \{\alpha_{pa'+qb', pl+qm} \in \Phi | p, q \in \mathbb{Z}_{>0}\}\rangle$$

Now, utilize elementary fact regarding commutator bracket:

$$[x, zy] = [x, y][x, z][[x, z], y] \ \& \ [xz, y] = [x, y][[x, y], z][z, y]$$

$$[x, y]^{-1} = [y, x]$$

Under the construction of affine root group by generators, extend above result to general elements of the affine roots groups and obtain:

$$[U_{\alpha_{a',l}}, U_{\alpha_{b',m}}] \subset \langle U_\beta | \beta \in \{\alpha_{pa'+qb', pl+qm} \in \Phi | p, q \in \mathbb{Z}_{>0}\}\rangle \subset U_{(\alpha_{a',l}, \alpha_{b',m})}$$

(RGD2) Consider the map on each $U^*_{\alpha_{a',l}}$, $m : x \in U^*_{\alpha_{a',l}} \mapsto w_{a',x}(t^{-l})$ (as constructed in 6.2.6). We check: For all $s_i := s_{\alpha_i}$ for $0 \leq i \leq n$, and $x \in U_{\alpha_i} \setminus \{1\}$ (Recall α_i being the simple affine roots, and are constructed to be among $\{\alpha_0 := \alpha_{-a'_0,1}\} \cup \{\alpha_i := \alpha_{a'_i,0} | i \in [1, n]\}$), we have following:

(a) $m(x) \in U_{s_i(\alpha_i)} x U_{s_i(\alpha_i)}$: This is shown by (d) of 6.2.6

(b) $m(x)U_\alpha m(x)^{-1} = U_{s_i(\alpha)}$ for all $\alpha \in \Phi$: This is shown by (e) of 6.2.6

(c) $m(x_1)m(x_2)^{-1} \in C_{\mathcal{G}}(S)(k)$ for all $x_1, x_2 \in U_{\alpha_i} \setminus \{1\}$: This holds by construction as (consider $x_1 = a'^\vee(t^{-l/2})u_1 a'^\vee(t^{-l/2})^{-1}, x_2 = a'^\vee(t^{-l/2})u_2 a'^\vee(t^{-l/2})^{-1} \in U_{\alpha_{a',l}} \in U_{\alpha_{a',l}}$, see 6.2.7 for detail):

$$m(x_1)m(x_2)^{-1} = w_{a',x_1}(t^{-l})w_{a',x_2}(t^{-l})$$

$$= a'^\vee(t^{-l/2})w_{a',u_1}a'^\vee(t^{-l/2})^{-1}a'^\vee(t^{-l/2})w_{a',u_2}^{-1}a'^\vee(t^{-l/2})^{-1}$$

$$= a'^\vee(t^{-l/2})w_{a',u_1}w_{a',u_2}^{-1}a'^\vee(t^{-l/2})^{-1}$$

$$\in a'^\vee(t^{-l/2})C_{\mathcal{G}}(S)(k)a'^\vee(t^{-l/2})^{-1} = C_{\mathcal{G}}(S)(k)$$

(RGD3) We no longer consider (RGD3)* as the structure of $C_{\mathcal{G}}(S)(k)$ is more complicated than that of "$T(k)$" in the Chevalley group case, however, we do again make some observations regarding the matrix structure of affine root groups just like we have for the split and Chevalley group cases. A key tool for us is following fact: (In following list, $\alpha := \alpha_{a',l}$)

([8, 21.8]) A compatible ordering always exists, in which we have:

$${}_k\Psi_+ \subset j(\Psi_+) \subset {}_k\Psi_+ \cup \{0\}$$

With this ordering, we generalize observation made in split cases: Recall that $\hat{k}$ is a finite separable extension of k over which T splits, and for each absolute root $a \in \Psi$, we may consider x_a to be defined over $\hat{k}$:

(a) **(When $a' \in {}_k\Psi_+$ and $l \geq 0$)**: For $a \in \eta(a')$, elements $x_a(ct^{-l})$'s (and $x_b(ct^{-2l})$'s if $b \in \eta(2a')$) of U_a are uni-upper-triangular matrices with all entries above diagonal in $\hat{k}[t^{-1}]$ (4.2.17). An arbitrary element of U_α, as a product of elements $x_a(ct^{-l})$'s $x_b(ct^{-2l})$'s (By construction 6.2.2 and 6.2.5, notice $c, d \in \hat{k}$, they do not necessarily stay in k) is also a uni-upper-triangular matrix with all entries above diagonal in $\hat{k}[t^{-1}]$. We have $U_\alpha \subset \mathcal{G}(\hat{k}[t^{-1}])$.

If we further require $l \geq 1$, keep in mind we have each x_a (for absolute root a) to be defined over $\hat{k}$ (i.e. each entry of x_a is taken as a polynomial with coordinates in $\hat{k}$). This means that all entries strictly above diagonal of $x_a(ct^{-l})$ is contained in $t^{-1}\hat{k}[t^{-1}]$ if $l \geq 1$, and $c \neq 0$, similarly all entries strictly above diagonal of $x_b(ct^{-2l})$ is contained in $t^{-2}\hat{k}[t^{-1}]$.

We have if $a' \in \Psi_+$ and $l \geq 1$, $x_{a'}(ct^{-l}) \in \mathcal{G}(\hat{k}[t^{-1}])$ is uni-upper-triangular with entries strictly above diagonal within $t^{-1}\hat{k}[t^{-1}]$, and $x_{2a'}(ct^{-2l})$ is uni-upper-triangular with entries strictly above diagonal within $t^{-2}\hat{k}[t^{-1}]$. That is an arbitrary element of U_α is uni-upper-triangular with entries strictly above diagonal within $t^{-1}\hat{k}[t^{-1}]$.

(b) **(When $a' \in {}_k\Psi_+$ and $l \leq -1$)**: An arbitrary element of U_α is a uni-upper-triangular matrix and has all entries strictly above diagonal in $t\hat{k}[t]$, with $U_\alpha \subset \mathcal{G}(\hat{k}[t])$.

(c) **(When $a' \in {}_k\Psi_-$ and $l \geq 1$)**: An arbitrary element of U_α is a uni-lower-triangular matrix and has all entries strictly below diagonal in $t^{-1}\hat{k}[t^{-1}]$, with $U_\alpha \subset \mathcal{G}(\hat{k}[t^{-1}])$.

(d) **(When $a' \in {}_k\Psi_-$ and $l \leq 0$)**: An arbitrary element of U_α is a uni-lower-triangular matrix and has all entries strictly below diagonal in $\hat{k}[t]$, with $U_\alpha \subset \mathcal{G}(\hat{k}[t])$.

Where (b), (c), and (d) are obtained with similar procedure as (a).

Following is very similar to what we have done for split case (and Chevalley group case), except for we consider $\hat{k}$ in place of k and discard consideration for $C_{\mathcal{G}}(S)(k)$: Consider $U_\epsilon \subset \mathcal{G}(\hat{k}[t^{-\epsilon}])$, we have:

$$U_+ \cap U_- \subset \mathcal{G}(\hat{k}[t^{-1}]) \cap \mathcal{G}(\hat{k}[t]) = \mathcal{G}(\hat{k})$$

where the equal sign is just a set-theoretic result.

Consider group homomorphisms $p_\epsilon : \mathcal{G}(\hat{k}[t^{-\epsilon}]) \to \mathcal{G}(\hat{k})$ induced by $\hat{k}[t^{-\epsilon}] \to \hat{k}$ (defined by $t^{-\epsilon} \mapsto 0$, where $t^{\pm} := t^{\pm 1}$) entry-wise (Checking these are group homomorphism is standard). Considering preimage of subgroup under group homomorphism is subgroup, and (a) to (d) above, we see the following: (We denote $\mathfrak{U}^{\pm}(\hat{k})$ the upper (for +) and lower (for -) uni-triangular matrices with entries in $\hat{k}$)

In the case (a), $p_+(U_{\alpha_{a',0}}) \subset \mathfrak{U}^+(\hat{k})$ and for $l \geq 1$, $p_+(U_{\alpha_{a',l}}) = Id$ and have U_α in case (a) is contained in $p_+^{-1}(\mathfrak{U}^+(\hat{k}))$.

Similar reasoning give us the image of U_α in case (b) under p_- is Id and hence U_α is contained in $p_-^{-1}(\mathfrak{U}^-(\hat{k}))$, the image of U_α in case (c) under p_+ is Id and hence U_α is contained in $p_+^{-1}(\mathfrak{U}^+(\hat{k}))$, and U_α in case (d) is contained in $p_-^{-1}(\mathfrak{U}^-(\hat{k}))$.

To sum up: We have $U_\epsilon \subset p_\epsilon^{-1}(\mathfrak{U}^\epsilon(\hat{k}))$. This implies:

$$U_\epsilon \cap \mathcal{G}(\hat{k}) \subset p_\epsilon^{-1}(\mathfrak{U}^\epsilon(\hat{k})) \cap \mathcal{G}(\hat{k}) \subset \mathfrak{U}^\epsilon(\hat{k})$$

Note the middle part is just describing elements of $p_\epsilon^{-1}(\mathfrak{U}^\epsilon(\hat{k}))$ with entries in $\hat{k}$, and they have to be in $\mathfrak{U}^\pm(\hat{k})$ by the construction of p_ϵ and fact that $\mathfrak{U}^\epsilon(\hat{k})$ is uni-upper-triangular if $\epsilon = +$ and uni-lower-triangular if $\epsilon = -$.

We further have following:

$$U_+ \cap U_- = (U_+ \cap U_-) \cap \mathcal{G}(\hat{k}) =$$

$$(U_+ \cap \mathcal{G}(\hat{k})) \cap (U_- \cap \mathcal{G}(\hat{k})) \subset \mathfrak{U}^+(\hat{k}) \cap \mathfrak{U}^-(\hat{k})$$

But $\mathfrak{U}^+(\hat{k})$ consists of uni-upper-triangular matrices while $\mathfrak{U}^-(\hat{k})$ consists of uni-lower-triangular matrices, we must have $U_+ \cap U_{-\alpha_i} \subset U_+ \cap U_- = \{1\}$ for we constructed so that all simple affine roots to be positive and have $U_{-\alpha_i} \subset U_-$.

Then (RGD0) tells us that $U_{-\alpha_i} \neq \{1\}$ and hence for all simple affine roots α_i, we have $U_{-\alpha_i} \nleq U_+$.

As a side note, in above we kept using $\hat{k}$ in place of k as we started our analysis with absolute pinning isomorphisms that are known to be defined over $\hat{k}$ but not necessarily k, in fact one should keep in mind the c and d in current item used above are considered within $\hat{k}$, not necessarily k. We are not claiming that the affine root groups we have defined do not land inside the $k[t, t^{-1}]$-points.

(RGD4) See 6.2.4 item (b) and (d), we see that for all elements $x_{a'}(r \otimes e_i^{a'})$'s for $r \in R$, $e_i^{a'} \in M_{a'}^k$ and all $a' \in {}_k\Psi$ are present in and generates $\langle U_\alpha | \alpha \in \Phi \rangle = \mathcal{G}(R)^+$. Consider $C_\mathcal{G}(S)(k)$ normalizes each affine root group hence the whole elementary subgroup, statement is proven.

(RGD5) See 6.2.4 item (b) where this is proven.

□

As a further comment, to construct the RGD system for the elementary subgroup $\mathcal{G}(k[t, t^{-1}])^+$, consider replacing $C_\mathcal{G}(S)(k)$'s with $C_\mathcal{G}(S)(k) \cap \mathcal{G}(k[t, t^{-1}])^+$'s in above 6.2.8.

In general, it is already shown (for instance in [4, Satz:27]) that $\mathcal{G}(k) = \mathcal{G}(k)^+ C_\mathcal{G}(S)(k)$. Now if we further require that the reductive group $\mathcal{G}$ in consideration is of "simply-connect type", in the same sense as in [21], and any normal semisimple subgroup of $\mathcal{G}$ has k-rank at least 2; then because [21, cor:6.2] provides isomorphism $\mathcal{G}(k)/\mathcal{G}(k)^+ \cong \mathcal{G}(k[t,t^{-1}])/\mathcal{G}(k[t,t^{-1}])^+$ induced by inclusion of $\mathcal{G}(k) \hookrightarrow \mathcal{G}(k[t,t^{-1}])$, we see that $\mathcal{G}(k[t,t^{-1}]) = \mathcal{G}(k[t,t^{-1}])^+\mathcal{G}(k) = \mathcal{G}(k[t,t^{-1}])^+C_{\mathcal{G}}(S)(k)$. And hence the RGD system we provided is in fact for the whole group of $\mathcal{G}(k[t,t^{-1}])$ in the case that $\mathcal{G}$ is "simply connected", and any normal semisimple subgroup of $\mathcal{G}$ has k-rank at least 2.

Note 6.2.9 (Candidate to satisfy condition (**))**.** *This candidate relies on utilizing [4, Satz: 25], and [4, satz: 27]. In particular, we will denote for $a' \in {}_k\Psi$, $n_{a'} \in N_{\mathcal{G}}(S)(k)$ as taken in c) of [4, Satz: 25] (originated in [8, 21.2], which is also recorded in [20, 15.3.5]). In particular, this element $n_{a'}$ enjoys following properties (DR4), and (DR5) in [4, Satz: 27] (We will present said properties in our context, in a way that is equivalent but most helpful to us, we may write $n_{-a'} = n_{a'}$, as according to its introduction in [8, 21.2] and [20, 15.3.5], they are the same):*

(DR4) For any $a' \in {}_k\Psi$:

$$U_{a'}(k) \setminus \{e\} \subset U_{-a'}(k)C_{\mathcal{G}}(S)(k)n_{-a'}U_{-a'}(k)$$

(DR5) For any $a', b' \in {}_k\Psi$, $w \in C_{\mathcal{G}}(S)(k)n_{-a'}$:

$$wU_b(k)w^{-1} = U_{s_{a'}(b')}(k)$$

We now make following assumption as extra condition: For any $a', b' \in {}_k\Psi$, $n_{-a'}$ as above can be chosen so that (Consider within $\mathcal{G}(\overline{k(t)})$):

$$n_{-a'}b'^{\vee}(t^{-n/2})n_{-a'}^{-1} = s_{a'}(b')^{\vee}(\lambda t^{-n/2}) \tag{***}$$

For all $n \in \mathbb{Z}/2$ and some $\lambda \in \overline{k}$ so that $\lambda^2 \in k$ (λ could change according to a', b'). Considering the equivalent statement being:

$$n_{-a'}b'^{\vee}(t^{n/2})n_{-a'}^{-1} = s_{a'}(b')^{\vee}(\lambda t^{n/2})$$

We see that $b'^{\vee}(1) = s_{a'}(b')^{\vee}(\lambda^2) = Id$, considering that $\ker(s_{a'}(b')^{\vee}) \subset \ker(s_{a'}(b') \circ s_{a'}(b')^{\vee}) = \{\pm 1\}$ as $s_{a'}(b') \circ s_{a'}(b')^{\vee} : \mu \mapsto \mu^2$ (see 4.2.6), we see that in fact $\lambda^2 = \pm 1$ (i.e. λ can only be some forth root of unity).

Note: *We will refer the above condition as **Condition (***)**. This condition is assumed in the following.*

With same logic as 6.2.6 (c), we see that for any $d' \in {}_k\Psi$, $n, l \in \mathbb{Z}/2$, and $\lambda \in \overline{k}$ so that $\lambda^2 \in k$: (Following calculation is considered within $\mathcal{G}(\overline{k(t)})$)

1. *In case of double roots:*

$$d'^{\vee}(\lambda t^{-l/2}) x_{2d'}(ct^{-2n} \otimes e_i^{2d'}) d'^{\vee}(\lambda t^{-l/2})^{-1}$$

$$= x_{2d'}(\lambda^2 ct^{-2l-2n} \otimes e_i^{2d'}) \in U_{\alpha_{2d',n+l}}$$

2. *In case of single roots:*

$$d'^{\vee}(\lambda t^{-l/2}) x_{d'}(ct^{-n} \otimes e_i^{d'}) d'^{\vee}(\lambda t^{-l/2})^{-1}$$

$$x_{d'}(\lambda^2 ct^{-l-n} \otimes e_i^{b'}) \in U_{\alpha_{d',l+n}}$$

Following the similar steps as the rest of 6.2.6 (c), we obtain that:

$$d'^{\vee}(\lambda t^{-l/2}) U_{\alpha_{d',n}} d'^{\vee}(\lambda t^{-l/2})^{-1} = U_{\alpha_{d',l+n}}$$

Now with condition (***), we can consider following: By (DR4), take any $u \in U_{a'}(k)^*$, we have that:

$$u \in U_{-a'}(k) C_{\mathcal{G}}(S)(k) n_{-a'} U_{-a'}(k)$$

We take and fix $w_{a',u} \in C_{\mathcal{G}}(S)(k) n_{-a'}$, so that we have $w_{a',u} \in U_{-a'}(k) u U_{-a'}(k)$. We will denote $w_{a',u} = h_{a',u} n_{-a'}$ for $h_{a',u} \in C_{\mathcal{G}}(S)(k)$. And by (DR5), this choice of $w_{a',u}$ enjoys property:

$$w_{a',u} U_{b'}(k) w_{a',u}^{-1} = U_{s_{a'}(b')}(k)$$

Now we extend the property brought to us by (DR5) with condition (***):

$$w_{a',u} U_{\alpha_{b',n}} w_{a',u}^{-1} = h_{a',u} n_{-a'} U_{\alpha_{b',n}} n_{-a'}^{-1} h_{a',u}^{-1} = h_{a',u} n_{-a'} b'^{\vee}(t^{-n/2}) U_{\alpha_{b',0}} b'^{\vee}(t^{-n/2})^{-1} n_{-a'}^{-1} h_{a',u}^{-1}$$

$$= h_{a',u} s_{a'}(b')^{\vee}(\lambda t^{-n/2}) n_{-a'} U_{b'}(k) n_{-a'}^{-1} s_{a'}(b')^{\vee}(\lambda t^{-n/2})^{-1} h_{a',u}^{-1}$$

Keep in mind that $U_{\alpha_{b',0}} = U_{b'}(k)$ when verifying above statement. We continue:

$$= h_{a',u} s_{a'}(b')^{\vee}(\lambda t^{-n/2}) U_{s_{a'}(b')}(k) s_{a'}(b')^{\vee}(\lambda t^{-n/2})^{-1} h_{a',u}^{-1}$$

$$= h_{a',u} U_{\alpha_{s_{a'}(b'),n}} h_{a',u}^{-1}$$

Then with the same logic in 6.2.4 (b), we can see that:

$$h_{a',u}U_{\alpha_{s_{a'}(b'),n}}h_{a',u}^{-1} \subset U_{\alpha_{s_{a'}(b'),n}}$$

That is we have shown that:

$$w_{a',u}U_{\alpha_{b',n}}w_{a',u}^{-1} \subset U_{\alpha_{s_{a'}(b'),n}}$$

With similar steps we obtain that:

$$w_{a',u}^{-1}U_{\alpha_{s_{a'}(b'),n}}w_{a',u} \subset U_{\alpha_{b',n}}$$

We in fact obtain:

$$w_{a',u}U_{\alpha_{b',n}}w_{a',u}^{-1} = U_{\alpha_{s_{a'}(b'),n}}$$

It is by construction that:

$$w_{a',u_1}w_{a',u_2}^{-1} \in C_{\mathcal{G}}(S)(k)$$

That is above shows that assuming condition (***) on the elements $n_{a'}$ taken as in [4, Satz: 25], the $w_{a',u}$ constructed accordingly above satisfies condition (**)

We then turn our sight to [8, 21.1], where it is stated that $N_{\mathcal{G}}(S)/C_{\mathcal{G}}(S)$ acts on S and induces action on the cocharacter group $X_*(S)$. Take $n \in N_{\mathcal{G}}(S)$, $\chi \in X_*(S)$, and $\lambda \in \overline{k(t)}$:

$$nC_{\mathcal{G}}(S) \cdot \chi(\lambda) := n\chi(\lambda)n^{-1}$$

should describe this action. And this action extends to an action on $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ (Since $X_*(S)$ is abelian group, it is a $\mathbb{Z}$-module).

Studying the origin of the element $n_{-a'}$ as taken in the third paragraph of proof of [8, 21.2] (its counterpart is $r_{a'}$ in [8, 21.2], or $s_{a'^\vee}$ in [20, 15.3.5-15.3.8]), one can see $n_{-a'} = n_{a'} \in N_{\mathcal{G}}(S)(k)$ is taken so that the element $n_{-a'}C_{\mathcal{G}}(S) \in N_{\mathcal{G}}(S)/C_{\mathcal{G}}(S)$ induces an orthogonal transformation on $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ while fixing $X_*(S_{a'}) \otimes_{\mathbb{Z}} \mathbb{R}$ (where $S_{a'}$ is the identity component of $Ker(a')$), and is therefore the reflection to that hyperplane (See about reflection in $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ at 4.2.14 where the formula for "dual" reflection "stay the same" when generalized to the non-split case, or with reference [20, 15.3.8]. In particular, $s_{a'}(b')^\vee = s_{a'^\vee}(b'^\vee)$ is still a consequence of bilinearity of $< \cdot, \cdot >$ when considered as "composition" of character and cocharacter, and the formula for "dual" reflection).

To sum up, this means $n_{a'}$ is taken as the reflection in respect to $X_*(S_{a'}) \otimes_{\mathbb{Z}} \mathbb{R}$, and hence satisfies *(in fact a stronger version of) the condition (***) in the sense that* $n_{-a'} b'^{\vee}(\lambda) n_{-a'}^{-1} = s_{a'}(b')^{\vee}(\lambda)$ *for all* $\lambda \in \overline{k(t)}$ *is satisfied.*

This means that the requirement of condition (**) for 6.2.8 has been "relaxed" in the sense that the condition is true in general without any further assumptions, and hence the following result 6.2.10.

Theorem 6.2.10 (6.2.8 without further requirement). *Using the construction in 6.2.1, we have*

$$(\mathcal{G}(R)^{+} C_{\mathcal{G}}(S)(k), (U_{\alpha_{a',l}})_{\alpha_{a',l} \in \Phi}, C_{\mathcal{G}}(S)(k))$$

is a general RGD system

Proof. Combine 6.2.8 and 6.2.9. □

Exemple 6.2.11 ("Semi example" of applying 6.2.8). *We will illustrate a "semi" example of applying 6.2.8 to a case that the reductive group of consideration is unitary and of type BC. This example is exactly the case we have explored in section 6.1. We call this example "semi" as some details of this procedure is not fully understood, and hence is presented as assumptions.*

As introduced in [19, sec:2.3.3] (or see section 6.1 for some sum up, in following we will use notations we have implemented in said section), in the case we are considering, the unitary group $\mathcal{G}(k[t, t^{-1}]) := \mathbf{SU}(V^{k(t)}, f)(k[t, t^{-1}])$ *can be embedded into* $SL_{2mn^2}(\overline{k(t)})$ *(n as index of D over* k'*, and m being the dimension of V over D), with relative root system* ${}_{k}\Psi$ *of* $\mathcal{G}(k[t, t^{-1}])$ *being of type* BC_l *(l being the Witt index of V over D), and absolute root system* Ψ *of* $\mathcal{G}$ *being of type* A_{mn-1} *(See case (3) of [19, prop: 2.15], it can be seen that* $\mathcal{G}$ *is* $\overline{k}$*-isomorphic to* $\mathbf{SL}_{mn}$ *in this case).*

If one can understand the projection of absolute roots of type A_{mn-1} *to relative roots of type* BC_l*, and construct absolute pinning isomorphisms* $x_a : Add(\overline{k(t)}) \to U_a$ *for* $a \in \Psi$ *so that the relative pinning maps constructed according to 6.2.2 as product of absolute pinning isomorphisms coincide with the construction by Bruhat and Tit that we cited in 6.1.3 in the sense that:*

$$x_{\epsilon_i'}(z) x_{2\epsilon'}(\mu_{z,\lambda}) = u_i(z, \lambda)$$

for $(z, \lambda) \in Z^{k(t)}$*. Where* $\mu_{z,\lambda} \in D_{\tau,\epsilon}^{k(t)}$ *is some value depended on z and* λ*,*

$$x_{2\epsilon_i'}(\lambda) = u_i(0, \lambda)$$

for $\lambda \in D^{k(t)}_{\tau,\epsilon}$, *and*

$$x_{\epsilon'_i+\epsilon'_j}(\lambda) = u_{ij}(\lambda)$$

for $\lambda \in D^{k(t)}$.

We construct the affine root groups as done in 6.2.3 and note that the construction coincide with that of 6.1.7 (One may find 6.2.5 helpful in this, since it provides a way to express the affine root groups in the general case as a set of elements with explicit forms).

Then we may consider making for a'_i *for* $i \in \{0, \cdots, l\}$ *as introduced in 6.1.6, the elements* $w_{a'_i,u}$*'s (in condition (**), in 6.2.6) to be constructed as:*

($i = 0$*) For* $\lambda \in D_{\tau,\epsilon}$*:*

$$w_{a'_0,u_1(0,\lambda)} := m_1(0,\lambda)$$

($i = 1, \cdots, l-1$*) For* $\lambda \in D$*:*

$$w_{a'_i,u_{i,-(i+1)}(\lambda)} := m_{i,-(i+1)}(\lambda)$$

($i = l$*) For* $(z, c) \in Z$*:*

$$w_{a'_l,u_l(z,c)} := m_l(z,c)$$

One may check, with step very similar to proof of (RGD2) in 6.1.8, that above construction of $w_{a'_i,u}$*'s fully satisfies the condition (**). This is very long, and is pretty much the same calculations, we will skip this here.*

Above provides another way to construct the same RGD system as provided by 6.1.8 using 6.2.8 under the assumption that the projection of absolute root to relative root, and a version of the absolute pinning isomorphisms are fully understood.

As an example in a more specific case, if the division algebra D *is identical to* $k' = k(\gamma)$ *(For instance, in the context of [5, part: III]) then we know that the special unitary group has the absolute root system of type (In our specific case)* A_{m-1}*. And we know that the relative root system is of type* BC_l*. As a reminder, we still use* $\hat{k}$ *as some finite separable extension of* k *that the maximal torus splits over, and we use* k_s *to denote separable closure over* k*. One can replace all following* $\hat{k}$*'s with* k_s*'s without issue. We start our construction of projection from the absolute* *roots to the relative roots on the base of the root systems. For A_{m-1}, we choose:*

$$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \cdots, \epsilon_{m-2} - \epsilon_{m-1}, \epsilon_{m-1} - \epsilon_m\}$$

For BC_l, we keep using:

$$\{\epsilon'_1 - \epsilon'_2, \epsilon'_2 - \epsilon'_3, \cdots, \epsilon'_{l-2} - \epsilon'_{l-1}, \epsilon'_{l-1} - \epsilon'_l, \epsilon'_l\}$$

We define the projection in a "symmetric to the middle" way by:

$$\begin{cases} \epsilon_i - \epsilon_{i+1}, \epsilon_{m-i} - \epsilon_{m-i+1} \mapsto \epsilon'_i - \epsilon'_{i+1} & i \in [1, l-1] \\ \epsilon_l - \epsilon_{l+1}, \epsilon_{m-l} - \epsilon_{m-l+1} \mapsto \epsilon'_l & \\ \epsilon_i - \epsilon_{i+1} \mapsto 0 & i \in [l+1, m-l-1] \end{cases}$$

This construction generalize to (We are only presenting the positive roots in below, adding negative sign in front all the absolute and relative roots in the following formulae will provide the projections for negative roots):

$$\begin{cases} \epsilon_i - \epsilon_{m-i+1} \mapsto 2\epsilon'_i & i \in [1, l] \\ \epsilon_i - \epsilon_{m-j+1}, \epsilon_j - \epsilon_{m-i+1} \mapsto \epsilon'_i + \epsilon'_j & i \neq j \in [1, l] \\ \epsilon_i - \epsilon_j, \epsilon_{m-j+1} - \epsilon_{m-i+1} \mapsto \epsilon'_i - \epsilon'_j & 1 \leq i < j \leq l \\ \epsilon_i - \epsilon_h \mapsto \epsilon'_i & i \in [1, l] \text{ and } h \in [l+1, m-l] \\ \epsilon_h - \epsilon_{m-i+1} \mapsto \epsilon'_i & i \in [1, l] \text{ and } h \in [l+1, m-l] \end{cases}$$

Unfortunately, the author does not know if this is indeed the unique possible projection in this case. How to consistently extend this projection to where D is not identified with k', and so the index of D over k' is not 1, proves challenging as of now. We recall the Witt basis in this construction:

$$\{e_1, \cdots, e_l\} \sqcup \{z_{l+1}, \cdots, z_{m-l}\} \sqcup \{e_{-l}, \cdots, e_{-1}\}$$

If we were to adopt the above projections and the Witt basis, we would like to see the agreement between 6.1.8 and 6.2.8 manifest in a way is (Or similar to):

For $\mu \in k$:

$$u_i(0, \mu) = x_{2\epsilon'_i}(\mu) = x_{\epsilon_i - \epsilon_{m-i+1}}(\mu)$$

In following we consider k' to be a dimensional 2 module over k embedded in $k' \otimes_k \hat{k} \cong \hat{k} \oplus \hat{k}$ (a *dimensional 2 module over $\hat{k}$). We consider $k' \otimes_k \hat{k} \cong \hat{k} \oplus \hat{k}$ to have basis elements e^R and e^I over $\hat{k}$. For $a, b \in \hat{k}$:*

$$u_{i,-j}(ae^R + be^I) = x_{\epsilon'_i - \epsilon'_j}(ae^R + be^I) = x_{\epsilon_i - \epsilon_j}(a) x_{\epsilon_{m-j+1} - \epsilon_{m-i+1}}(b)$$

$$u_{i,j}(ae^R + be^I) = x_{\epsilon'_i + \epsilon'_j}(ae^R + be^I) = x_{\epsilon_i - \epsilon_{m-j+1}}(a) x_{\epsilon_j - \epsilon_{m-i+1}}(b)$$

In following, we recall that the anisotropic kernel is a dimensional $(m-2l)$ module over k', and hence a dimensional $2(m-2l)$ module over k. We consider the anisotropic kernel (We will denote as V_0 here) as k module embedded in $V_0 \otimes_k \hat{k} \cong \hat{k}^{2(m-2l)}$. Let $f(z,z)$ play the role of the homogeneous of degree 2 maps. We again consider a basis of the module $V_0 \otimes_k \hat{k} \cong \hat{k}^{2(m-2l)}$ over $\hat{k}$ to be $\{e_h^R, e_h^I\}_{h \in [l+1, m-l]}$ so that k' can be embedded within $\langle e_h^R, e_h^I \rangle_{\hat{k}}$ for all h (the same way k' is embedded in $k' \otimes_k \hat{k} \cong \hat{k} \oplus \hat{k}$). For $a_h, b_h \in \hat{k}$ and $z = \sum_{h=l+1}^{m-l} z_h(a_h e_h^R + b_h e_h^I) \in V_0$:

$$u_i(z, \lambda) = x_{\epsilon'_i}\Big(\sum_{h=l+1}^{m-l} z_h(a_h e_h^R + b_h e_h^I)\Big) x_{2\epsilon'_i}(\lambda - f(z,z))$$

$$= \Big(\prod_{h=l+1}^{m-l} x_{\epsilon_i - \epsilon_h}(a_h) x_{\epsilon_h - \epsilon_{m-i+1}}(b_h)\Big) x_{\epsilon_i - \epsilon_{m-i+1}}(f(z,z)) x_{2\epsilon'_i}(\lambda - f(z,z))$$

Even in this case, it appears difficult to see how one may define the absolute pinning ismorphisms to achieve above desired result, and at the same time agree with, through some correspondences, to the form of $(A,\ {}^tA^{-1})$ for $A \in \mathbf{SL}_m(\overline{k}(t))$ as described right above [19, prop: 2.35]. By last sentence of [8, 18.6], we know for any two absolute pinning ismorphisms $x_a(\lambda)$ and $y_a(\lambda)$, we must have that $x_a(\lambda) = y_a(c\lambda)$ for some $c \in \overline{k}$; knowing the well-known absolute pinning ismorphisms of $\mathbf{SL}_m$ by $x_{\epsilon_i - \epsilon_j}(\lambda) = E_{ij}(\lambda)$, we see that this absolute pinning isomorphism by elementary matrices could suggest a good start in attempting to construct the absolute pinning ismorphisms needed.

Part III

Appendix

We append the following chapter to assist in understanding by discussing some known results related to buildings and twin buildings, all results can be found in further detail at [6].

Chapter 7

Buildings

7.1 Basics on buildings

Definition 7.1.1 (Type for simplicial view). *A **Type function** on chamber complex Δ with values in set I is a function τ that assign each vertex v to an element $\tau(v)$ in I. This assignment must map the vertices of every chamber bijectively onto I.*

*This validates the concept of **Type** $\tau(A)$ for simplex A in Δ. And further concept of **Cotype** $I \setminus \tau(A)$ of simplex A in Δ. It is by construction that a panel will have cotype of $\{i\}$ for $i \in I$, in which occasion, we will address said panel as an **i-panel**. This allows us to define the concept* *of **i-adjacent** for two chambers that share an i-panel. Further with concept of gallery, we can extend this concept to **J-equivalent** (with $J \subseteq I$) for two chambers that is connected by a gallery $C_0, \cdots, C_l$ where C_{k-1} and C_k are j-adjacent for some $j \in J$ for all $k \in [1, l]$. It can be seen that J-equivalence is indeed an equivalent relationship, hence we may define the equivalent classes of chambers under J-equivalent as **J-residue**.*

The concepts introduced above can often assist us in constructing chamber complex.

Definition 7.1.2 (Coxeter system). *A review on Coxeter system can be found in [6, ch: 1,2,3].*

*Given Coxeter system (W, S), a **standard coset** in W is $w < J >$ for $w \in W$ and $J \subseteq S$. $\Sigma(W, S)$ is the poset of standard cosets in W, ordered by reverse inclusion (making the maximal simplices known as **Chambers** to be $\{w\}$ for $w \in W$), called the **Coxeter complex** associated to (W, S). $\Sigma(W, S)$ is **Spherical** if W is finite. Coxeter complex Σ is addressed to be the **Coxeter complex of type** (W, S) if Σ is equipped with type function with values in S, and there is type-preserving isomorphism $\Sigma \to \Sigma(W, S)$.*

Recall 7.1.3 (Canonical type function from simplical view). *$\Sigma(W, S)$ has **canonical type function** by $\tau(w < J >) = S \setminus J$. $\Sigma(W, S)$ is a thin chamber complex of rank $|S|$, colorable, and action of W on it by left translation is type-preserving. See Chapter 3.1 of [6] for details.*

Definition 7.1.4 (Building as simplicial complex). *We also refer building constructed as following "Building in simplical context" ;A **Building** is a simplicial complex Δ that can be expressed as the union of subcomplexes Σ's called **Apartments** satisfying following axioms*

- ***B0*** *Each apartment is a Coxeter complex*
- ***B1*** *For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them*
- ***B2*** *If for A and B simplices in Δ both contained in apartment Σ and Σ', then there is an isomorphism $\Sigma \to \Sigma'$ fixing both A and B point-wise.*

We will often use Δ as building and Σ as apartment

Recall 7.1.5. *Under assumption of **B0** and **B1**, **B2** is equivalent to following **B2'** and **B2"***

- ***B2'*** *Σ and Σ' be apartments containing both simplex A and chamber C of Σ. Then there is isomorphism $\Sigma \to \Sigma'$ fixing A and C point-wise.*
- ***B2"*** *Let Σ and Σ' be apartments containing chamber C of Σ. Then there is an isomorphism $\Sigma \to \Sigma'$ fixing every simplex in $\Sigma \cap \Sigma'$.*

*Isomorphisms in **B2**, **B2'**, and **B2"** can be taken as the same and be type-preserving. See chapter*

*4.1 of [6] for detail. Equivalence of **B2** to **B2"** is essential to showing above statement regarding the isomorphisms. In this view, with some more work, we may note that any arbitrary apartment Σ in a building Δ is isomorphic to $\Sigma(W, S)$ by a type-preserving isomorphism ψ for some Coxeter system (W, S). This makes all apartments in Δ Coxeter complex of type (W, S).*

Recall 7.1.6. *A chamber map $\Delta \to \Sigma$ that is identity on Σ is a **Retraction**, in which case we say Σ is a **Retraction** of Δ. It can be shown for building Δ, every apartment of it is a retraction of said building [6, 4.33]. The proof of which gives us **Canonical retraction** of Δ onto Σ centered at chamber $C \in \Sigma$, denoted $\rho_{\Sigma,C} : \Delta \to \Sigma$. $\rho_{\Sigma,C}$ is the unique chamber map that fixes C point-wise and maps every apartment containing C isomorphically onto Σ.*

*Consider type-preserving isomorphism $\psi : \Sigma \to \Sigma(W, S)$ at the end of 7.1.5, $\psi \circ \rho_{\Sigma,C}$ gives us a type function for Δ that agrees with those of its apartments. This justifies addressing Δ as **Building of type** (W, S).*

Recall 7.1.7. *For a building Δ, the simplices of Δ are in one to one correspondence with the residues as subsets of $\mathcal{C}(\Delta)$ ordered by reverse inclusion by (for simplex A) $A \leftrightarrow \mathcal{C}_{\geq A}$ where $\mathcal{C}_{\geq A}$ is collection of chambers having A as a face. Note that we call $\mathcal{C}_{\geq A}$ a residue as it is proven that it is a J-residue for cotype J of A in [6, A.20]. To see full proof of this result (which correspond to [6, 4.11]), one need to understand concepts and results regarding **link** as in chapter A.1 in [6] and proposition [6, 4.9].*

7.2 BN and building

Definition 7.2.1 (BN pair)**.** *[6, 6.55] We say that a pair of subgroups B and N of a group G is a **BN pair** if B and N generates G, the intersection $T := B \cap N$ is normal in N, and the quotient $W := N/T$ admits a set of generators S such that following holds:*

(BN1) For $s \in S$ and $s \in W$:

$$sBw \subset BswB \cup BwB$$

(BN2) For $s \in S$:

$$sBs^{-1} \nleq B$$

*The group W will be called the **Weyl group associated to the BN pair** and the **type of BN pair** as stated is (W, S). One also state in this situation that the quadruple (G, B, N, S) is a Tits system.*

Definition 7.2.2 (Strongly transitive). *A G action on a building is* ***Strongly transitive with respect to system of apartments*** *$\mathcal{A}$ if G acts transitively on set of pairs in form of (Σ, C) with apartment $\Sigma \in \mathcal{A}$ and chamber $C \in \Sigma$. And equivalently,*

(a) G acts transitively on $\mathcal{A}$ and the stabilizers of an apartment $\Sigma \in A$ acts transitive on $C(\Sigma)$ (set of chambers in Σ).

(b) G acts transitively on $C(\Delta)$. and given chamber C, G acts transitively on set of apartments containing C.

Strongly transitive actions are ***Chamber transitive****, namely, transitive on the set of chambers as (b) in above*

Theorem 7.2.3 (BN pair thick building iff). *A group admits BN pair of type (W,S) if and only if G acts strongly transitively on a thick building of type (W,S) (As group of type-preserving automorphisms)*

Proof. It is quite long and requires prerequisites. In short, it consists of following two parts: 7.2.4 and 7.2.5 □

Proposition 7.2.4 (thick building to BN pair). *Given group G acting strongly transitive on thick building Δ of type (W, S) (as group of type-preserving simplicial automorphisms), then G admits BN pair of type (W, S)*

Proposition 7.2.5 (BN pair to thick building). *G admits BN pair of type (W, S) gives rise to thick building of type (W, S), on which G act strongly transitively*

Proof. Checking statements are lengthy, but it is important to be familiar with following construction 7.2.7. As the construction $\Delta(G, B)$ will be the building which G acts strongly transitively on with action given by left multiplication. One thing to note here is that N might not be the full stabilizer of fundamental apartment see more about this at [6, p:318] □

Definition 7.2.6 (Standard parabolic subgroup in context of BN pair). ***Parabolic subgroup*** *P of G is defined by existence of some $g \in G$ so that $gBg^{-1} \leq P$. And a parabolic subgroup is* ***Standard parabolic subgroup*** *if $B \leq P$. $P_J := \langle B \cup N_J \rangle = BN_JB$ ($N_J := \pi^{-1}(W_J)$ for canonical group projection $\pi : N \to W$ by $n \mapsto nT$) and the P_J's cam be identified as standard parabolic subgroups as expected. In this context, $C(w) := BwB = BnB$ for $w = nT \in W$, one should compare this to 7.4.36 and note this construction agrees with the other.*

Definition 7.2.7 (Building from BN). *Let $\Delta(G, B)$ be poset of cosets of form gP_J where P_J is standard parabolic subgroups ordered by reversed inclusion. Note that this construction is independent of N. See 7.4.37 for details.*

Here, by the reversed inclusion, we can see that chambers are gB for $g \in G$. And P_J as a simplex in chamber B in this building has cotype J. See 7.4.37 for more of this correspondence.

7.3 A word on Saturated BN pair

Definition 7.3.1 (Saturated BN pair). *[6, 6.57] A BN pair is* ***Saturated*** *if it satisfies that*

$$T = \bigcap_{w \in W} wBw^{-1}$$

Or equivalently that N is the full stabilizer of the fundamental apartment of $\Delta(G, B)$

Note 7.3.2 (Saturated BN correspondence). *As above, it can be seen that saturated BN pairs are in one to one correspondence with strongly transitive group actions on thick building*

Lemma 7.3.3 (When is it a BN pair). *[6, 6.59] Let (B,N) be a saturated BN pair in G, let N' be a subgroup of N. Then (B,N') is a BN pair if and only if $N'T = N$, or equivalently N' is surjective onto $W = N/T$*

Proposition 7.3.4 ("uniqueness" of Saturated BN pair). *[6, 6.60] Let (B,N) be a saturated BN pair in G and $\Delta(G, B)$ is of spherical type. If N' is a subgroup of G such that (B,N') is also a saturated BN pair, then there exists some $b \in B$ so that $N' = bNb^{-1}$*

7.4 Simplicial buildings and Weyl distance

Definition 7.4.1. *Consider Coxeter simplex Σ of type (W, S) (And hence we may identify it with $\Sigma(W, S)$, and collection of chambers $\mathcal{C}(\Sigma)$ with W), define* ***Weyl distance function*** *for Coxeter complex as $\delta_\Sigma : \mathcal{C}(\Sigma) \times \mathcal{C}(\Sigma) \to W$ by $\delta_\Sigma(C, D) = s_1 \cdots s_d$ for arbitrary gallery $s_1, \cdots, s_d$ connecting C and D.*

Definition 7.4.2. *Consider a building Δ of type (W, S), there is a function $\delta : \mathcal{C}(\Delta) \times \mathcal{C}(\Delta) \to W$ with following properties*

(1) Given a minimal gallery $\Gamma : C_0, \cdots, C_d$ of type $s(\Gamma) = (s_1, \cdots, s_d)$, $\delta(C_0, C_d) = s_1 \cdots s_d$

(2) Let $C, D \in \mathcal{C}(\Delta)$ s.t. $\delta(C, D) = w$. The function of $\Gamma \mapsto s(\Gamma)$ gives one to one correspondence between minimal galleries from C to D and reduced decomposition of W.

This δ will be called the ***Weyl distance function*** *for building.*

Sketch. This is an extension of δ_Σ to δ on Δ where Σ is an apartment in Δ through **B2** by choosing Σ to be an apartment containing both C and D. The two properties would be inherited from δ_Σ. □

Proposition 7.4.3. *The Weyl distance function δ has the following properties*

W1 $\delta(C, D) = 1$ *iff* $C = D$.

W2 *Given $\delta(C, D) = w$ and $\delta(C', C) = s \in S$ for $C' \in \mathcal{C}$. Then $\delta(C', D) = sw$ or w. If in additional, $l(sw) = l(w) + 1$, then $\delta(C', D) = sw$.*

W3 *If $\delta(C, D) = w$, then for any $s \in S$ there is a chamber $C' \in \mathcal{C}$ such that $\delta(C', C) = s$ and $\delta(C', D) = sw$. If $l(sw) = l(w) - 1$, then the C' is unique*

W4 $\delta(C, D) = \delta(D, C)^{-1}$

Sketch. [6, 4.84] has full proof regarding this. □

To read more on Weyl distance function, refer to chapter 4.8 of [6]. It can be seen through [6, ch:5.6] that a building of type (W, S) (in simplical context) with Weyl distance function δ as above has following equivalent definition as a **W-metric space**

Definition 7.4.4 (Building by Weyl distance)**.** *We also refer building constructed as following "Building in (Weyl) distance context" ;Consider a fixed Coxeter system (W, S),* ***A building of type*** *(W, S) is a pair $(\mathcal{C}, \delta)$ for set $\mathcal{C} \neq \emptyset$ containing elements called* ***chambers****. And Weyl distance function $\delta : \mathcal{C} \times \mathcal{C} \to W$ such that for all $C, D \in \mathcal{C}$* ***WD1*** *to* ***WD3*** *holds*

WD1 $\delta(C, D) = 1$ *iff* $C = D$.

WD2 *Given $\delta(C, D) = w$ and $\delta(C', C) = s \in S$ for $C' \in \mathcal{C}$. Then $\delta(C', D) = sw$ or w. If in additional, $l(sw) = l(w) + 1$, then $\delta(C', D) = sw$.*

WD3 *If $\delta(C, D) = w$, then for any $s \in S$ there is a chamber $C' \in \mathcal{C}$ such that $\delta(C', C) = s$ and $\delta(C', D) = sw$.*

*Pair $(\mathcal{C}, \delta)$ will be called **W-metric space**.*

We further clarify what is an apartment in this context [6, 5.49,5.50,5.53]: As a remark, one will see following should agree with corresponding concept for the context of simpicial buildings (Except for apartments which I will clarify later)

*Consider nonempty subset $\mathcal{M}$ of $\mathcal{C}$. $\delta_{\mathcal{M}}$ restriction of δ to $\mathcal{M} \times \mathcal{M}$. If $(\mathcal{M}, \delta_{\mathcal{M}})$ is a building of type (W,S), then it is called a **subbuilding** of $(\mathcal{C}, \delta)$.*

*A nonempty subset $\mathcal{N}$ of $\mathcal{C}$ is called **thin** (resp. **thick**) if $\mathcal{P} \cap \mathcal{N}$ has Cardinality 2 (resp. >2) for every panel $\mathcal{P}$ (See definition of panel in this context at 7.4.39) of $\mathcal{C}$ with $\mathcal{P} \cap \mathcal{N} \neq \emptyset$. And further, **weak** if $\mathcal{P} \cap \mathcal{N}$ has cardinality ≥ 2 for every $\mathcal{P}$ as above.*

*A thin subbuilding of $\mathcal{C}$ is called an **apartment** of $\mathcal{C}$.*

***Note**: apartment in context of Weyl distance is distinct from apartments in context of simplicial building in only one way: The apartments in context of Weyl distance is a set of chambers it contains. While the apartments in context of simplicial building is a Coxeter complex, which is a poset by reverse inclusion (So it contains more than just chambers), see detail [6, ch:3.1]. To reflect which context we are using, we will use notation Δ if we are referring to the simplicial building, and $(\mathcal{C}, \delta)$ if we are referring to the building of Weyl distance.*

***Note**: [6, 5.29] **Weyl distance** δ **applied to residues** can be seen as: Given residue R of type J, and residue L of type K in $\mathcal{C}$, then have*

$$\delta(R, L) := \{\delta(X, Y) | X \in R, Y \in L\} = W_J \delta(C, D) W_K \text{ for any } C \in R, D \in L$$

Note 7.4.5. *For G action on Δ is type-preserving, δ is G invariance. This can be seen through (1) of 7.4.2 and that G action would preserve the type of gallery. Note that the condition of minimal in (1) of 7.4.2 is essential. As apartment is convex (review at 7.4.12) so it must contain minimal gallery connecting two of its chambers, but it does not have to contain all galleries.*

It is nature that one would foresee following statement regarding **WD3** according to **W3** in 7.4.3

Lemma 7.4.6. *If $l(sw) = l(w) - 1$, then the C' in **WD3** is unique*

Note 7.4.7. *A useful way to recognize **WD2** and **WD3** with schematic:*

$$\begin{tikzcd} C' \arrow[d, "s"'] \arrow[dr, "\{sw{,}w\}"] & \\ C \arrow[r, "w"'] & D \end{tikzcd}$$

Lemma 7.4.8. *Given W-metric $(\mathcal{C}, \delta)$ of building with type (W, S) have for $C, D, E \in \mathcal{C}$*

(1) $\delta(C, D) = s$ iff $\delta(D, C) = s$

(2) If $\delta(C, D) = \delta(D, E) = s$ then $\delta(C, E) = \{1, s\}$

Proof. These results follow from **WD1** to **WD3**. Trying to prove them serves as a wonderful practice of applying 7.4.7. □

Recall 7.4.9. *Fixing arbitrary Coxeter system (W, S). There is unique longest element $w_0 \in W$ (which has order 2) that have following three equivalent properties that defines it*

(1) $l(w) \leq l(w_0)$ for all $w \in W$

(2) $l(sw_0) \leq l(w_0)$ for all $s \in S$

(3) $l(w_0) = l(w) + l(w^{-1}w_0)$ for all $w \in W$

w_0 will always indicate this element unless stated otherwise

It is apparent that we can recognize gallery distance as $d(C, D) = l(\delta(C, D))$ (And $d(R, L) = min(d(C, D)|C \in R, D \in L)$ for residues R,L). Following are further concepts that are needed for our discussion.

Definition 7.4.10. *Two chambers C and C' in a spherical building Δ are **opposite**, written as C op C' if $d(C, C') = diam(\Delta)$.*

Observation 7.4.11. *By construction, we can see that C op C' if and only if $\delta(C, C') = w_0$*

Definition 7.4.12. *We say a non-empty set of chambers to be **convex** if for any two chambers C and D in said set, any minimal gallery (might not be unique) from C to D is contained in the set.*

Definition 7.4.13. *Given a chamber complex, we define the **convex hull** of chambers C and D to be the smallest chamber sub complex containing both C and D that is convex.*

Lemma 7.4.14. *Consider C op C' in a spherical building, and Σ be an apartment containing both of them. Then every chamber of Σ occurs in some minimal gallery from C to C'.*

Proof. This follow immediately from property (3) of 7.4.9 and 7.4.11. □

Note 7.4.15. *There is an alternative proof for 7.4.14. Though it will require a fully different approach, a summary of reference is below:*

One will have to understand gallery distance in terms of $d_{\mathcal{H}}$ where $\mathcal{H}$ is the collection of all hyperplanes in a Euclidean vector space. The detail of which can be seen through [6, 1.54]. With [6, *1.56], this result follow as [6, 1.58].*

Above 7.4.14 automatically yields a powerful result

Corollary 7.4.16. *let Σ be apartment containing opposite chambers C op C' in building Δ. Then Σ is the convex hull of C op C' in Δ.*

Further, 7.4.16 gives us following result

Theorem 7.4.17. *A spherical building Δ admits a unique system of apartments. The apartments are precisely the convex hulls of pairs C op C' in Δ.*

Consider building of type (W, S) with type-preserving G action. Let $(\mathcal{C}, \delta)$ be the corresponding W-metric building. By result of section 4.8 of [6], we know that $\mathcal{C}$ is the set of chambers and δ is the Weyl distance function the rise from the building of type (W, S), and group G acts as group of isometries for Weyl distance on $\mathcal{C}$ (i.e. $\delta(gC, gD) = \delta(C, D)$, and this is equivalent to G action is type-preserving).

We define a **w-sphere around chamber** C in the common sense:

$$\{D \in \mathcal{C} | \delta(C, D) = w\}$$

Definition 7.4.18. *We say that G's action on Δ is* ***Weyl transitive*** *if for each $w \in W$, the action is transitive on the pair (C, D) where D is in w-sphere around C. Equivalently, if the action is chamber transitive and for any C, the action of stabilizer of C is also transitive on w-sphere around C for all $w \in W$.*

Recall 7.4.19 (Strongly transitive). *[6, 6.1] A G action is* ***strongly transitive*** *(with respect to system of apartment $\mathcal{A}$ of building Δ) if G acts transitively on the set of pairs (Σ, C) for apartment $\Sigma \in \mathcal{A}$ and chamber $C \in \Sigma$*

There is also an equivalent characterization of Weyl transitivity that does not refer Weyl distance:

Proposition 7.4.20. *Assume G action is chamber transitive. Let C be an arbitrary chamber, let Σ be some apartment containing C. Let $B = Stab_G(C)$, then G action is Weyl transitive if and only if*

$$\Delta = \bigcup_{b \in B} b\Sigma \tag{eq7.4.20}$$

Recall 7.4.21. *G action is strongly transitive, then for Σ an arbitrary apartment in Δ and B stabilizer in G of a chamber in Σ have*

$$\Delta = \bigcup_{b \in B} b\Sigma$$

Proof. This is a Corollary to [6, 6.4] in chapter 6.1. □

Corollary 7.4.22. *By 7.4.20 and 7.4.21, with respect to some apartment system, we have strong transitivity implies Weyl transitivity.*

We will start to see the relationship between strong and Weyl transitivity. We will start with recalling a helpful statement

Recall 7.4.23. *strongly transitive is equivalent to being chamber transitive and stabilizer of C is transitive on the collection of apartments containing C*

Lemma 7.4.24. *Let G action on Δ be Weyl transitive. Let Σ be an arbitrary apartment in Δ. Then the set $G\Sigma = \{g\Sigma | g \in G\}$ is a system of apartment (that might not be the complete system of apartments for Δ).*

Proposition 7.4.25. *The following conditions are equivalent:*

(1) G action on Δ is strongly transitive with respect to some apartment system.

(2) G action on Δ is Weyl transitive, and there is an apartment Σ (in the complete system of apartments) such that the stabilizer of Σ acts transitivity on $\mathcal{C}(\Sigma)$.

Proof. This is direct results from what we observed above:

(1) $\Longrightarrow$ (2) This is automatic by 7.4.22

(2) $\Longrightarrow$ (1) Let the system of apartment be $G\Sigma$ as in 7.4.24, with further premises assumed in (2), result follows by construction.

□

Proposition 7.4.26 (Strongly transitive is equivalent to Weyl transitive in spherical case)**.** *Given type-preserving G action on spherical building, the following three statements are equivalent*

(1) Action is strongly transitive

(2) Action is Weyl transitive

(3) Action is transitive on pair (C, C') of opposite chambers

Definition 7.4.27 (Axioms of Bruhat decomposition)**.** *[6, 6.2.2] Let G be a group, B its subgroup, (W,S) a Coxeter system and map*

$$C : W \to B\backslash G/B$$

Consider following conditions:

(Bru1) *$C(w)=B$ if and only if $w=1$ for $w \in W$*

(Bru2) *$C : W \to B\backslash G/B$ is onto i.e. $G = \bigcup_{w\in W} C(w)$*

(Bru3) *For any $s \in S$, $w \in W$ we have*

$$C(sw) \subset C(s)C(w) \subset C(sw) \cup C(w)$$

(Bru3') *(Right analog of (Bru3)) For any $s \in S$, $w \in W$ we have*

$$C(ws) \subset C(w)C(s) \subset C(ws) \cup C(w)$$

It can be seen that (Bru1) to (Bru3) implies (Bru3'). And (Bru1), (Bru2), and (Bru3) serves as ***Axioms of Bruhat decomposition****.*

Some results following axioms of Bruhat decomposition can be seen at [6, 6.36]. In particular, one should note that a group G with BN pair satisfies axioms for Bruhat decomposition with map $C : w \mapsto BwB$, see more about this at discussion following [6, 6.56].

Definition 7.4.28 (Bruhat decomposition from Weyl transitive action on building)**.** *Consider G action on Δ of type (W, S) be Weyl transitive. Choose fundamental chamber C and B as its stabilizer in G. Under assumption of Weyl transitivity, we have the B orbits in set of chambers $\mathcal{C}$ are in one to one correspondence with elements in W:*

$$BD \leftrightarrow w = \delta(C, D)$$

For D arbitrary chamber in Δ. For Weyl transitivity implies chamber transitivity, there is one to one correspondence between chambers in Δ with left cosets in G/B:

$$gC \leftrightarrow gB$$

Combine above two correspondences by recognizing $D = gC$ as gB through the second correspondence, we have

$$BgB \leftrightarrow w = \delta(C, gC)$$

Which defines a bijection $B\backslash G/B \to W$ between double cosets and W. We will let $C(w) := BgB \in$

$B\backslash G/B$ (This map C is not chamber C, sorry for bad notation) for $BgB \leftrightarrow w \in W$, note that by construction we have $C(w) \leftrightarrow w$ in above constructed correspondence between double cosets and W. This gives us a set-theoretic decomposition of G:

$$G = \bigsqcup_{w \in W} C(w)$$

*Which is known as **Bruhat decomposition**. Note that $C(w)$'s have to be disjoint as δ is well-defined.*

Note 7.4.29 (A sum up on Bruhat decomposition by Weyl transitive action)**.** *To sum up, we have following bijective correspondence between $B\backslash G/B$ and W:*

$$BgB \leftrightarrow \delta(C, gC); C(w) \leftrightarrow w \qquad \text{(eq7.4.28)}$$

Which justifies the construction of Bruhat decomposition

It is clear from the correspondence of $C(w) \leftrightarrow w$ that $g \in C(w)$ may be represented schematically with diagram

$$C \xrightarrow{w} gC$$

It is helpful to read this diagram as "The Weyl distance from C to gC is w".

It is easy to misunderstand this diagram as equivalent to $wC = gC$. This is a misunderstanding as g may move C out of the apartment containing C and wC. It is however correct that we may say the diagram is equivalent to $\rho_{\Sigma,C}(gC) = wC$ for canonical retraction $\rho_{\Sigma,C}$ of Δ onto Σ centered at C, or intuitively, wC is gC "up to type". See below 7.4.30 for a formal statement corresponding to [6, 6.18]

Proposition 7.4.30. *Choose an apartment Σ of Δ containing fundamental chamber C, identify it with $\Sigma(W, S)$ (making $\mathcal{C}(\Sigma)$ to be $\{wC\}$, and further $\{\{w\}\}$ if we identify C to be $\{1\}$). Then $g \in C(w)$ if and only if $\rho_{\Sigma,C}(gC) = wC$.*

Naturally, in this section we consider G action on building Δ of type (W, S) be Weyl transitive.

Note that Bruhat decomposition as constructed was fully set-theoretic, making limited use of fact that we are working with a group acting on a building. In following, we will extend its impact to groups, especially on double cosets.

First, we can see the construction gives us a new group-theoretic view on δ

Note 7.4.31. *Under assumption of Weyl transitive action of G on building Δ of type (W, S), we may identify $\mathcal{C}$ with G/B as mentioned in 7.4.28. Then we may reconstruct δ in a different, more group-theoretic view:*

$$\bar{\delta} : G/B \times G/B \to W$$

This $\bar{\delta}$ can be constructed as a composition:

$$G/B \times G/B \to B\backslash G/B \to W$$

$$(gB, hB) \mapsto Bg^{-1}hB \mapsto \delta(C, g^{-1}hC)$$

This construction of $\bar{\delta}$ agrees with the original Weyl distance function δ according to fact that $\delta(gC, hC) = \delta(C, g^{-1}hC)$ (G action is type-preserving). Through this procedure, we may reconstruct the building Δ as W-metric space $(G/B, \bar{\delta})$ as in 7.4.4. We now have reconstructed the building from the group G.

As above, we have reconstructed our building from information on the group.

We will attempt to use **WD2** and **WD3** to deduce some interesting result.

Theorem 7.4.32. *Continuing from 7.4.28 Given $s \in S$, and $w \in W$. We have*

$$C(sw) \subseteq C(s)C(w) \subseteq C(sw) \cup C(w)$$

Further $C(s)C(w)$ is either $C(sw)$ or a union of two double cosets. And if $l(sw) = l(w) + 1$, then $C(s)C(w) = C(sw)$.

Now we see what **W4** can bring us

Proposition 7.4.33. *Continuing from 7.4.28 $C(w)^{-1} = C(w^{-1})$*

Proof. Consider $\delta(C, g^{-1}C) = \delta(gC, C) = \delta(C, gC)^{-1}$ by **W4**. Then apply correspondence of $C(w) \leftrightarrow w$ in Bruhat decomposition. □

With 7.4.33 can help us restate 7.4.32 for $C(w)C(s)$ in place of $C(s)C(w)$

Note that we already reconstructed building from group data in 7.4.31 as W-metric space, in which procedure we have identified chambers. We seek to extend the construction to identify the simplices with help of identification of their stabilizers. In this section, consider G action on building as Weyl transitive. We will start with understanding the concept of **J-residue** in terms of Weyl distance

Note 7.4.34 (Residues)**.** *Consider Coxeter system (W, S), and a building of type (W, S) with* *chamber set $\mathcal{C}$ and Weyl distance function δ. Take set $J \subseteq S$, two chambers C and D are* ***J-equivalent*** *if $\delta(C, D) \in W_J$. Recall that for $w \in W$, there is a unique set $S(w) \subseteq S$ such that for any reduced form $w = s_1 \cdots s_n$, we have $S(w) = \{s_1, \cdots, s_n\}$. This tells us C and D are J-equivalent if and only if there is some gallery of type $(s_1, \cdots, s_n)$ connecting them with $s_1, \cdots, s_n \in J$ (And hence this relationship is indeed an equivalence relationship). Further, the equivalent classes are* ***J-residues****, and we may denote the J-residue containing a chosen chamber C as*

$$R_J(C) = \{C' \in \mathcal{C} | \delta(C, C') \in W_J\}$$

It is apparent that constructions in this note is equivalent to the concepts in 7.1.1 (Under context of Coxeter system).

We now identify the stabilizer of a simplex in building Δ of type (W, S)

Proposition 7.4.35 (Working to parabolic subgroups)**.** *Consider G acting Weyl transitively on building Δ of type (W, S) and fundamental chamber C in said building, take simplex $A \in C$ of cotype $J \subseteq S$. Then the stabilizer of A in G is*

$$P_J := \bigcup_{w \in W_J} C(w)$$

Which is a subgroup in G (Map C (which is not chamber C) as in 7.4.28, sorry for bad notation).

Finally, let's tie this back to terminology that we are familiar with

Definition 7.4.36 (Standard Parabolic subgroups)**.** *The subgroups P_J as in 7.4.35 are called* ***Standard parabolic subgroups****, and we will call the left cosets gP_J* ***Standard parabolic cosets****. This is from the perspective of G acting Weyl transitively on building of type (W, S). One should compare this to 7.2.6. It can be seen that $P_J = BW_JB$.*

Corollary 7.4.37 (An important correspondence)**.** *Consider G acting Weyl transitively on Δ. The building Δ is isomorphic as a poset to the set of standard parabolic cosets ordered by reverse inclusion. The G action on simplices of Δ correspond to left multiplication on the standard parabolic cosets. This serves as a clarification for 7.2.7. One may find the correspondence explained in following proof extremely helpful.*

Proof. Consider following diagram where simplex A is of cotype J, and B is of cotype J' with

$B \subseteq A$ (hence $J \subseteq J'$), both of which are in chosen fixed chamber C

$$\{C\} \longleftrightarrow C \longleftrightarrow P_\emptyset = B$$

$$\begin{array}{ccccc} \mathcal{C}_{\geq A} = R_J(C) & \longleftrightarrow & A & \longleftrightarrow & P_J \\ \cap & & \cup & & \cap \\ \mathcal{C}_{\geq B} = R_{J'}(C) & \longleftrightarrow & B & \longleftrightarrow & P_{J'} \end{array}$$

Further, we extend this out of the chamber C by considering G action on simplices in Δ as left multiplication on the standard parabolic cosets:

$$gA \leftrightarrow gP_J$$

And apparently

$$gC \leftrightarrow gB$$

The correspondence above can also be seen at [6, A.20], where it is clarified that $A = \bigcap_{D \in \mathcal{C}(A) = R_J(C)} D$

$\square$

Note: *In this context, stabilizer of A is P_J, stabilizer of gA is gP_Jg^{-1}. Indeed, stabilizer of P_J is P_J and stabilizer of gP_J is gP_Jg^{-1}. See [6, p:320] for action by conjugation instead of left translation, in which case the simplices are parabolic subgroups instead standard parabolic cosets*

Note 7.4.38 (Translating between simplical and Weyl diatnce context)**.** *Also, as an example for clarification, the term "Simplex A of cotype J in chamber C in apartment Σ" in simplicial context is equivalent to "J-residue containing chamber C (This residue is the simplex A) that intersect non-trivially (i.e. meet non-trivially) with apartment Σ ('meet non-trivially' is the 'contained in')" in Qeyl distance context.*

It is not hard to see construction in 7.4.37 agrees with that in 7.4.31.

Note 7.4.39 (Panel in term of Weyl distance)**.** *[6, 5.2] [6, 5.21] We will clarify the definition of* ***panel*** *that rise from 7.4.4: A panel is a* ***s-panel*** *for some $s \in S$, and an s-panel is a residue $\mathcal{R}_{\{s\}}(C)$ of type $\{s\}$ for some $C \in \mathcal{C}$. This will correspond to the panels as defined in 7.1.1 through 7.4.37. And the corresponding simplex to $\mathcal{R}_{\{s\}}(C)$ in 7.4.37 will have cotype $\{s\}$.*

We explore how we can interpret thickness with Bruhat decomposition (We are assuming Weyl transitivity).

Lemma 7.4.40. *The following are equivalent (Assuming Weyl transitivity and notation from 7.4.28)*

(1) *Building Δ is thick.*

(2) *Choose chamber C. For each $s \in S$ there are at least 2 chambers C''s such that $\delta(C, C') = s$.*

(3) *Let $[C(s) : B] = |\{hB | hB \subseteq C(s)\}|$, $[C(s) : B] \geq 2$ for all $s \in S$. Note that $[C(s) : B]$ can equivalently be defined as size of collection of Bh in $C(s)$.*

(4) *$C(s) \neq hB$ for $h \in C(s)$.*

(5) *$hBh^{-1} \neq B$ for $h \in C(s)$.*

(5) in 7.4.40 appear as [6, 6.26], it is "the most natural way" to see thickness in group-theoretic context.

Following a stronger version of 7.4.32 under assumption of thickness

Proposition 7.4.41. *Continuing from 7.4.28 Assume Δ to be thick, for any $s \in S$ and $w \in W$ with $l(sw) = l(w) - 1$*

$$C(s)C(w) = C(sw) \cup C(w)$$

Chapter 8

Twin buildings

This section includes well established concepts and results, most of which are taken from [6]. In essence, twin building allows us to deal with arbitrary Coxeter system instead only spherical ones.

When we use ϵ in this chapter, we intend that $\epsilon \in \{\pm\}$ unless otherwise stated.

8.1 Definition(s) of twin building

When we use ϵ, we intend that $\epsilon \in \{\pm\}$ unless otherwise stated.

Definition 8.1.1 (Twin building (Weyl distance version))**.** *[6, 5.133][6, 5.134] We will also refer twin building in this construction as "Twin building in codistance context". A* ***Twin building*** *of type (W, S) is a triple $(\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ consisting of two buildings $(\mathcal{C}_\pm, \delta_\pm)$ both of type (W, S) together with a* ***codistance*** *function:*

$$\delta^* : (\mathcal{C}_+, \mathcal{C}_-) \cup (\mathcal{C}_-, \mathcal{C}_+) \to W$$

(We consider $\mathcal{C}_+ \cap \mathcal{C}_- = \emptyset$) satisfying the following for each $\epsilon \in \{+, -\}$, any $C \in \mathcal{C}_\epsilon$ and any $D \in \mathcal{C}_{-\epsilon}$ where $w = \delta^*(C, D)$:

(Tw1) $\delta^*(C, D) = \delta^*(D, C)^{-1}$

(Tw2) If $C' \in \mathcal{C}_\epsilon$ satisfies $\delta_\epsilon(C', C) = s$ with $s \in S$ and $l(sw) < l(w)$, then $\delta^*(C', D) = sw$

(Tw3) For any $s \in S$, there exists a chamber $C' \in \mathcal{C}_\epsilon$ with $\delta_\epsilon(C', C) = s$ and $\delta^*(C', D) = sw$

We also define the ***numerical codistance*** *between C and D by:*

$$d^*(C, D) = l(\delta^*(C, D))$$

We say C and D are ***opposite****, denoted C op D if $d^*(C, D) = 0$ or equivalently $\delta^*(C, D) = 1$. Two residues $\mathcal{R}$ in $\mathcal{C}_+$ and $\mathcal{S}$ in $\mathcal{C}_-$ are* ***opposite*** *if they have same type and contain opposite chambers.*

[6, 5.171]: A ***Twin apartment*** *of a twin building $\mathcal{C}$ is a pair $\Sigma = (\Sigma_+, \Sigma_-)$ such that Σ_ϵ is an apartment of $\mathcal{C}_\epsilon$ and every chamber in $\Sigma_+ \cup \Sigma_-$ is opposite to precisely one other chamber in $\Sigma_+ \cup \Sigma_-$*

[6, 5.172]: If $\Sigma = (\Sigma_+, \Sigma_-)$ is a twin apartment, then the ***opposite involution*** *denoted by op_Σ associated to each chamber $C \in \Sigma_+ \cup \Sigma_-$ the unique chamber $C' = op_\Sigma(C) \in \Sigma_+ \cup \Sigma_-$ such that C op C'*

Note*: there is the "s on the right" version as opposed to "s on the left version" we have above for both (Tw2) and (Tw3). See more this at [6, 5.137].*

Note*: [6, 5.148]* ***The codistance*** *δ^** ***applied to residues*** *can be seen as: Given reside R of type J in $\mathcal{C}_\epsilon$ and residue L of type K in $\mathcal{C}_{-\epsilon}$. Then have*

$$\delta^*(R, L) := \{\delta^*(X, Y) | X \in R, Y \in L\} = W_J \delta^*(C, D) W_K \quad \textit{for any } C \in R, D \in L$$

Definition 8.1.2 (Twin building (opposite relation version)). *[7, page:8] [2, page:393] We will also refer twin building in following construction as "Twin building in simplical context". Consider $\Delta_\pm$ be two buildings both of type (W,S). Denoted $\mathcal{C}_\pm$ the chamber sets of $\Delta_\pm$ respectively. Consider given a symmetrical* ***opposite relation****:*

$$op \subset \mathcal{C}_+ \times \mathcal{C}_- \cup \mathcal{C}_- \times \mathcal{C}_+$$

For $c_\pm \in \mathcal{C}_\pm$, c_+ is opposite to c_- is denoted $c_+ \; op \; c_-$. A subset $\mathcal{A}$ of

$$\{(\Sigma_+, \Sigma_-)|\Sigma_\epsilon \text{ is an apartment in } \Delta_\epsilon \text{ for } \epsilon \in \{+,-\}\}$$

We call $\mathcal{A}$ ***the set of twin apartments****. We also denote:*

$$\mathcal{A}_\pm := \{\Sigma_\pm | \exists \Sigma_\mp : (\Sigma_+, \Sigma_-) \in \mathcal{A}\}$$

We define α as ***isomorphism between two twin apartments*** *Σ and Σ' consists of two type-preserving isomorphisms $\alpha_\pm : \Sigma_\pm \to \Sigma'_\pm$ of Coxeter complexes so that it respects opposite relation: $c_+ \; op \; c_- \iff \alpha_+(c_+) \; op \; \alpha_-(c_-)$ for all chambers $c_\pm \in Ch(\Sigma_\pm)$.*

A quadruple $\Delta = (\Delta_+, \Delta_-, \mathcal{A}, op)$ is called a ***pre-twin building*** *of type (W,S) if it satisfies (TA0) to (TA3):*

(TA0) $\mathcal{A}_\epsilon$ is a system of apartments for Δ_ϵ (i.e. for all $c_\epsilon, d_\epsilon \in \mathcal{C}_\epsilon$, there is a $\Sigma_\epsilon \in \mathcal{A}_\epsilon$ such that $c_\epsilon, d_\epsilon \in \Sigma_\epsilon$)

(TA1) For every $\Sigma = (\Sigma_+, \Sigma_-) \in \mathcal{A}$, the restriction of the opposition relation induces a bijection between chamber sets of Σ_+ and Σ_- and the latter a type-preserving isomorphism $op_{\Sigma_+} : \Sigma_+ \to \Sigma_-$.

> ***Note****: Not surprisingly, this op_{Σ_+} can be extended and understood "to be the same" to op_Σ as opposite involution (8.1.1) on (In simplical context) $\mathcal{C}(\Sigma_+) \cup \mathcal{C}(\Sigma_-)$*

(TA2) For all $c_+ \in \mathcal{C}_+$, $d_- \in \mathcal{C}_-$ there exists a $\Sigma \in \mathcal{A}$ with $(c_+, d_-) \in \Sigma$ (i.e. $c_+ \in \Sigma_+$ and $d_- \in \Sigma_-$)

(TA3) For all $\Sigma, \Sigma' \in \mathcal{A}$ and all $a = (a_+, a_-) \in \Sigma \cap \Sigma'$ there exists an isomorphism $\alpha : \Sigma \to \Sigma'$ of twin apartments satisfying $\alpha(a) = a$

If Δ satisfies (TA1) to (TA4) instead, it is a ***twin building*** *of type (W,S):*

(TA4) For all $\Sigma, \Sigma' \in \mathcal{A}$ such that $\Sigma \cap \Sigma'$ contains a chamber, $\Sigma \cap \Sigma'$ is coconvex (see 8.1.4) in both Σ and Σ'

***Note**: that (TA1) to (TA4) implies (TA0) to (TA3). This means that twin buildings are indeed pre-twin buildings in this context.*

Note 8.1.3 (pre-twin and twin building from opposite relation version to Weyl distance version (part 1)). *[7, lem:4] In context of pre-twin buildings defined by opposite relation (8.1.2), (TA1) to (TA3) will imply there exists a unique $\delta^* : (\mathcal{C}_+, \mathcal{C}_-) \cup (\mathcal{C}_-, \mathcal{C}_+) \to W$ defined by*

$$\delta^*(c_+, d_-) := \delta_-(op_\Sigma(c_+), d_-) =: \delta^*(d_-, c_+)^{-1}$$

(For any $\Sigma \in \mathcal{A}$ so that $(c_+, d_-) \in \Sigma$, why is this well-defined despite the choice of Σ? This is by the isomorphism that respects opposite relation in (TA3), see proof of [7, lem:4]) That satisfies $\delta^*(c_+, c_-) = 1 \iff c_+ \; op \; c_-$ and (Tw1), (Tw2)', and (Tw3) where:

(Tw2)' $\delta^*(c_\epsilon, d_{-\epsilon}) = w$, $\delta_{-\epsilon}(d_{-\epsilon}, e_{-\epsilon}) = s$ implies that $\delta^*(c_\epsilon, e_{-\epsilon}) \in \{w, ws\}$

This δ^* is what is needed to define projection and coconvexity in 8.1.4

Definition 8.1.4 (Coconvex and projection). *[7, page:5] Consider in context of 8.1.2: Let $p_\epsilon \in \Delta_\epsilon$ a panel and $c_{-\epsilon} \in \Delta_{-\epsilon}$ a chamber, then there is a unique chamber $d_\epsilon^0 \supset p_\epsilon$ such that $l(\delta^*(c_{-\epsilon}, d_\epsilon^0)) \geq l(\delta^*(c_{-\epsilon}, d_\epsilon))$ for all chambers $d_\epsilon \supset p_\epsilon$ (This δ^* is taken from 8.1.3 under context of 8.1.2). This d_ϵ^0 is called the **projection** of $c_{-\epsilon}$ onto p_ϵ and denoted by $d_\epsilon^0 = proj_{p_\epsilon} c_{-\epsilon}$.*

*Due to fact that $c_{-\epsilon}$ and p_ϵ are in opposing buildings, projection above is sometimes referred to as **coprojection***

Let $\Theta = (\Theta_+, \Theta_-)$ where $\Theta_\pm$ are subsets consists of panels and chambers in $\Delta_\pm$ respectively. Assuming at least one of $\Theta_\pm$ contains a chamber:

1. *Θ is called **coconvex** in Δ if for any panel p_ϵ that is a face of some chamber $c_\epsilon \in \Theta_\epsilon$, we have $p_\epsilon \in \Theta_\epsilon$. And that for any panel $p_\epsilon \in \Theta_\epsilon$ and $c_{-\epsilon} \in \Theta_{-\epsilon}$, $proj_{p_\epsilon} c_{-\epsilon} \in \Theta_\epsilon$*

2. ***coconvex hull** of Θ in Δ is the intersection of all coconvex pairs of Δ that contains Θ.*

To see more about projections, look at 8.1.8

Note 8.1.5 (pre-twin and twin building from opposite relation version to Weyl distance version (part 2)). *[7, thm:1] $\Delta = (\Delta_+, \Delta_-, \mathcal{A}, op)$ is a twin building in context of opposite relation (8.1.2)* *iff for δ^* obtained in 8.1.3, $(\Delta_+, \Delta_-, \delta^*)$ is a twin building in context of Weyl distance (8.1.1).*

***Note**: given a twin building as triple in context of Weyl distance $(\Delta_+, \Delta_-, \delta^*)$, we may define opposite relation as in 8.1.1 and hence the quadruple $(\Delta_+, \Delta_-, \mathcal{A}, op)$ which will be a twin building in context of opposite relation. The quadruple in turn will give us back the same δ^* (by uniqueness through 8.1.3) we had in the original triple. This means that the two definitions of twin buildings (8.1.2 and 8.1.1) are equivalent and hence can be used interchangeably.*

***The only distinction between simplicial and codistance context** being that (again) the twin apartments consist of pairs of chambers in context of codistance version while the twin apartments in context of opposite relation are pairs of posets (Being the positive and negative apartments). Again, we will denote a twin building Δ when we are referring to the opposite relation context, while $\mathcal{C}$ when we are referring the codistance context.*

Note 8.1.6 (On twin apartment)**.** *One will find [6, ch:5.8.4] very helpful. Especially [6, 5.179] where it is proven that (In context of Δ is twin building):*

a. *For a pair of chambers (C, C') so that C op C', there is one unique twin apartment containing this pair, this twin apartment is $\Sigma\{C, C'\}$ (see below) and can be shown to be the convex hull (Also the coconvex hull, see [7, lem:3]) of the pair. (I.e. convex hull of opposite chambers are twin apartments).*

b. *For any apartment Σ_+ of Δ_+ there is at most one apartment Σ_- of Δ_- so that (Σ_+, Σ_-) is a twin apartment*

These statements are proven using [6, 5.178] that states the following: In context that Δ is a twin building, let $C \in \mathcal{C}_\epsilon$, we construct

$$C^{op} := \{C' \in \mathcal{C}_{-\epsilon} | C' \ op \ C\}$$

Now consider $C' \in C^{op}$, Σ_ϵ be an apartment of Δ_ϵ so that $(C')^{op} \cap \Sigma_\epsilon = \{C\}$, then have:

1. $\Sigma_\epsilon = \Sigma(C, C') := \{D \in \mathcal{C}_\epsilon | \delta_\epsilon(C, D) = \delta^*(C', D)\}$

2. *If C" is another chamber so that $(C'')^{op} \cap \Sigma_\epsilon = \{C\}$, then C"=C'*

(1) above appears to be a very helpful description.

We will denote:

$$\Sigma\{C,C'\} := \begin{cases} (\Sigma(C,C'),\Sigma(C',C)) & \epsilon = + \\ (\Sigma(C',C),\Sigma(C,C')) & \epsilon = - \end{cases}$$

Exemple 8.1.7 (Of pre-twin building). *Let Δ' be a building of type (W,S) and $\mathcal{A}'$ its set of apartments. We may define a pre-twin building $(\Delta_+,\Delta_-,\mathcal{A},op)$ where $\Delta_+ := \Delta' =: \Delta_-$, $\mathcal{A} := \{(\Sigma,\Sigma)|\Sigma \in \mathcal{A}'\}$ and c op $d \iff c = d$ for $c,d \in \mathcal{C}(\Delta')$*

8.1.1 Some more info on coconvexity and (co)projection

Definition 8.1.8 (More about projection). *This best to be compared to 8.1.4. For an equivalent definition (that extend to projection onto not only just panels, but faces of spherical cotype) of projection in codistance context (8.1.1), see [6, 5.149] and [6, 5.150]: (At least to the author, the codistance context seem to be more friendly when it comes to the definition of projection)*

*If R is a residue of $\mathcal{C}_\epsilon$ of spherical type and D is a chamber in $\mathcal{C}_{-\epsilon}$ then there is a unique $C_1 \in R$ such that $\delta^*(C_1,D)$ is of maximal length in $\delta^*(R,D)$. This C_1 is called the **projection** (Again, sometimes called **coprojection**) of D onto R and is denoted $proj_R D$.*

In both above and 8.1.4, we have been considering a chamber projecting onto something in the opposing building, not surprisingly, this is an "extension" to definition of projection within the same building. The only major distinction is that the projection within the same building seek to minimize the distance, while the coprojection between opposing buildings seek to maximize the distance:

*[6, 5.35] Let R be residue in $\mathcal{C}_\epsilon$. Given $D \in \mathcal{C}_\epsilon$, the unique chamber $C_1 \in R$ at minimal distance from D is called the **projection** of D onto R and denoted $proj_R D$. (Of course, nobody would call this coprojection, since D and R are in the same building)*

*One might want to compare definition of coconvexity with [6, 5.150, 5.158, 5.160, 5.176, 5.179] (These citations use the codistance context) where **Convexity** (This condition is stronger than coconvexity as it has further requirement on containment of projection of chamber on panel in the same signed building. However, as an important special case, coconvex hull of a pair of chamber is in fact convex, see 8.1.9) is stated:*

1. *[6, 5.158] A pair (M_+,M_-) of nonempty subsets $M_\pm \subset \mathcal{C}_\pm$ is called **convex** if $proj_P C \in M_+ \cup M_-$ for any $C \in M_+ \cup M_-$ and panel $P \subset \mathcal{C}_+ \cup \mathcal{C}_-$ that meets $M_+ \cup M_-$*

2. *[6, 5.160] The* ***convex hull*** *of a pair* (N_+, N_-) *for nonempty* $N_\pm \subset \mathcal{C}_\pm$ *is the intersection of all convex pairs containing it. Or equivalently, the smallest convex pair containing it (pairs are ordered component-wise).*

The concept of projection can be extended to simplex onto simplex

Note 8.1.9 (Coconvexity and half-apartment). *[7, rmk:5]* ***An important observation****: Assume* $\Sigma \in \mathcal{A}$, $c_\epsilon \in \mathcal{C}(\Sigma_\epsilon)$ *and let* $\Theta = (\Theta_+, \Theta_-)$ *be the coconvex hull of* (c_+, c_-) *in* Δ*. Then it can be seen that* Θ_ϵ *is the intersection of all half-apartments of* Σ_ϵ *containing* c_ϵ *but not* $op_\Sigma(c_{-\epsilon})$*. In this case,* Θ *is in fact convex. Further,* $\Theta = \Sigma_\epsilon$ *if and only if* c_+ *op* c_-

8.2 Twin BN and twin building

Definition 8.2.1 ((Pre-)twin BN pair). *[6, 6.78][2, defn:2] Let* $B_\pm$ *and* N *be subgroups of a group* G *such that* $B_+ \cap N = B_- \cap N = T$*. Assume* $T \trianglelefteq N$ *and set* $W := N/T$ *and* S *be set of generators for W, consider following conditions:*

(TBN0) (G, B_ϵ, N, S) *are tits systems*

(TBN1)' $B_\epsilon s B_\epsilon w B_{-\epsilon} \subset B_\epsilon\{w, sw\}B_{-\epsilon}$ *for all* $w \in W$, $s \in S$, $\epsilon \in \{\pm\}$

(TBN2)' $B_+(W \setminus \{1\}) \cap B_- = \emptyset$

(TBN1) If $l(sw) < l(w)$, *then* $B_\epsilon s B_\epsilon w B_{-\epsilon} = B_\epsilon sw B_{-\epsilon}$

(TBN2) $B_+ s \cap B_- = \emptyset$ *for all* $s \in S$

If (G, B_+, B_-, N, S) *satisfies (TBN0), (TBN1)', and (TBN2)', it is a* ***Pre-twin BN pair of type (W,S)****. If it satisfies (TBN0), (TBN1), and (TBN2) instead, it is a* ***Twin BN pair of type (W,S)****, in which case we also call it a* ***Twin tits system****.*

It can be seen in [6, 6.80] that (TBN1) implies (TBN1)'. And claimed by [2, ex:4] at the end, (TBN1) and (TBN2) together implies (TBN2)'. These tells us that twin BN pairs are indeed pre-twin BN pairs.

Exemple 8.2.2 (Of pre-twin BN pair). *If (G,B,B,N,S) is a tits systems, then (G,B,B,N,S) is a pre-twin BN pair.*

Note 8.2.3 ("Act" in context of (pre-)twin building). *[6, 6.67] [2, page:396] When we say group* ***act*** *on (pre-)twin building Δ, we will mean it acts with following conditions for all $g \in G$, $C, C' \in \mathcal{C}_+$ and $D, D' \in \mathcal{C}_-$:*

1. *G acts type-preservingly on both $\Delta_\pm$, or equivalently both of the following two:*
   (a) $\delta_+(gC, gC') = \delta_+(C, C')$
   (b) $\delta_-(gD, gD') = \delta_-(D, D')$
2. *For any $\Sigma = (\Sigma_+, \Sigma_-) \in \mathcal{A}$, $g\Sigma := (g\Sigma_+, g\Sigma_-) \in \mathcal{A}$*
3. *$\delta^*(gC, gD) = \delta^*(C, D)$, or equivalently C op $D \iff gC$ op gD (see [6, 6.68])*

Definition 8.2.4 (Strongly transitive action on twin building). *[2, defn:3] [6, 6.70,6.71] We say group G action on twin building Δ is* ***Strongly transitive*** *if the action has any of following equivalent conditions (Take fundamental pair of chamber (C_+, C_-), and $\Sigma = (\Sigma_+, \Sigma_-) = \Sigma\{C_+, C_-\}$ as the convex hull of pair (C_+, C_-), see more about this at [6, 5.150, 5.158, 5.160, 5.176, 5.179], one may find convexity and coconvexity are very alike):*

1. *For any $w \in W$, G acts transitively on*
$$\{(C, D) \in \mathcal{C}_+ \times \mathcal{C}_- | \delta^*(C, D) = w\}$$
2. *For $\epsilon = +$ or $-$. G acts transitively on $\mathcal{C}_\epsilon$, and $Stab_G(C_\epsilon)$ acts transitively on $\{D \in \mathcal{C}_{-\epsilon} | \delta^*(C_\epsilon, D) = w\}$ for each $w \in W$*
3. *G acts transitively on $\{(C, C') \in \mathcal{C}_+ \times \mathcal{C}_- | C \text{ op } C'\}$*
4. *If G acts transitively on $\mathcal{A}$ and $Stab_G(\Sigma) = Stab_G(\Sigma_+) = Stab_G(\Sigma_-)$ acts transitively on $\{(C, C') \in \Sigma_+ \times \Sigma_- | C \text{ op } C'\}$*
5. *If G acts transitively on $\mathcal{A}_\epsilon$ and $Stab_G(\Sigma)$ acts transitively on $Ch(\Sigma_\epsilon)$*
6. *If G acts transitively on $\mathcal{A}$ and $Stab_G(\Sigma')$ acts transitively on $Ch(\Sigma'_\pm)$ for every $\Sigma' = (\Sigma'_+, \Sigma'_-) \in \mathcal{A}$*

Note: *above is in context of group acting on a twin building. However, for pre-twin building, the notion of* ***Strongly transitive*** *is defined by (6) (And the conditions are no longer equivalent). In context of pre-twin building, (6) implies (1) can be seen at [2, lem:1] where the set in (1) is denoted*

$\mathcal{C}_w$, set in (3) is then denoted $\mathcal{C}_1$ in this notation, see [2, rmk:4].

Note 8.2.5 (Construction of pre-twin building by pre-twin BN pair). *[2, page:395] [6, 6.82] We will extend the definition of buildings based on BN pair For ease of reference, we state again: For given pre-twin BN pair (G, B_+, B_-, N, S), we first consider thick building associated to (G, B_ϵ, N, S) (Recall that this building is independent of N):*

$$\Delta_\epsilon := \Delta(G, B_\epsilon) = \{gP_J^\epsilon | g \in G, J \subset S\} \ (P_J^\epsilon := B_\epsilon W_J B_\epsilon)$$

$$\textit{Standard Apartment as } \Sigma_\epsilon^0 := \{nP_J^\epsilon | n \in N,\ J \subset S\} \textit{ of } \Delta_\epsilon$$

$$\mathcal{A} := \{(g\Sigma_+^0, g\Sigma_-^0) | g \in G\}$$

We will inherit Then we define the opposition relation by :

$$gB_+ \ op \ hB_-$$

$$\iff gB_+ \cap hB_- \neq \emptyset \ for \ g, h \in G$$

We will denote this pre-twin building (We will give at least citation for this claim at 8.2.6) as $\Delta(G, B_+, B_-) := (\Delta_+, \Delta_-, \mathcal{A}, op)$. One should recall indeed that this construction is independent of N, [6, 6.83] can be helpful.

***Note**: One may find that above construction of opposite relation is a little out of nowhere, however one will see this is in fact the only one that works (so that opposition is equivalent to codistance being 1 and satisfy requirements for twin building) in this situation according to the uniqueness in 8.1.3 and things related to Birkhoff decomposition in 8.2.8:*

$$gB_+ \cap hB_- \neq \emptyset \ for \ g, h \in G$$

$$\iff \exists b \in B_+, b' \in B_- : g^{-1}h = bb'$$

$$\iff g^{-1}h \in B_+B_-$$

$$\iff \delta^*(gB_+, hB_-) = 1$$

The first equivalent need a little work, but it is just symbol pushing.

Proposition 8.2.6 ((Pre-)twin BN pair to (pre-)twin building). *[2, prop:1] Consider in context of 8.2.5, denoting $\Delta = \Delta(G, B_+, B_-)$ being constructed from pre-twin BN pair (G, B_+, B_-, N, S). This Δ is a pre-twin building. Further, it will be a twin building if and only if (G, B_+, B_-, N, S) is* *a twin BN pair. In particular, it can be seen that the isomorphism required by (TA3) is a given by multiplication on the left by an element of G.*

The proof of this statement in [2, prop:1] is quite interesting as it will confirm the pre-twin building statement by checking (TA0) to (TA3). While for the twin building statement, it will check (Tw2).

***Note**: [2, rmk:3] This G acting on Δ as above can be shown to be strongly transitive, in both pre-twin and twin context.*

Proposition 8.2.7 (Strongly transitive action on (pre-) twin building to (pre-) twin BN pair)**.** *[2, prop:2] Consider G act strongly transitively on a pre-twin building Δ of type (W,S), choose a twin apartment $\Sigma \in \mathcal{A}$, a pair of opposite chambers (C_+, C_-) so that $C_\epsilon \in \Sigma_\epsilon$. Set (just like in 8.2.8):*

$$N := Stab_G(\Sigma)(= Stab_G(\Sigma_+) \cap Stab_G(\Sigma_-))$$

$$B_\epsilon := Stab_G(C_\epsilon)$$

Under above set up, assuming $\Delta_\pm$ are both thick buildings, then:

1. *(G, B_+, B_-, N, S) is a pre twin BN pair*
2. *The pre-twin building $\bar{\Delta} := \Delta(G, B_+, B_-, N, S)$ associated to this pre-twin BN pair (see 8.2.5 for construction) is isomorphic to the Δ that we were given*
3. *Further, (G, B_+, B_-, N, S) is a twin BN pair if and only if Δ is a twin building (in which case it can be seen that $N = Stab_G(\Sigma_\epsilon)$ this will require 8.1.6, see related 8.2.8)*

Proof. [2, prop:2]:

Checking (1) is standard, but it is helpful to understand $g \in B_\epsilon w B_{-\epsilon} \iff \delta^*(C_\epsilon, gC_{-\epsilon}) = w$ as in 8.2.8. (2) is checked by extending the analog isomophism map $\phi_\epsilon : \bar{\Delta}_\epsilon \to \Delta_\epsilon$ by $gB_\epsilon \mapsto gC_\epsilon$ for building theory to our situation by making sure it respects the set of twin apartments and the opposite relationship. (3) is a consequence of (2) and 8.2.6. □

Note 8.2.8 (Birkhoff decomposition for group acts strongly transitively on pre-twin building)**.** *[6, ch:6.3.2] One will see this is very similar to Bruhat decomposition in buildings. Here we assume a group G acting on a pre-twin building Δ strongly transitively, for eaze, we use following convention and notations in this context:*

We first fix a fundamental chamber pair (C_+, C_-) *so that* $C_\pm \in \mathcal{C}_\pm$ *and* C_+ *op* C_-*. We take stablizer of* C_ϵ *as* B_ϵ*:*

$$B_\epsilon := Stab_G(C_\epsilon) = \{g \in G | gC_\epsilon = C_\epsilon\}$$

Take fundamental twin apartment as $\Sigma = (\Sigma_+, \Sigma_-) = \Sigma\{C_+, C_-\}$ *the convex hull (It is unique) of pair* (C_+, C_-) *(See 8.1.4 for a citation on convexity), we construct:*

$$N := Stab_G(\Sigma) = \{g \in G | g\Sigma = \Sigma\}$$

$$H := \{g \in G | gC = C \; for \; all \; C \in \Sigma_+ \cup \Sigma_-\}$$

And W *as the group of isometries of* Σ_ϵ*, there is the isomorphism* $W \cong N/(B_+ \cap N)$ *(In fact* $B_+ \cap N = B_- \cap N$*, this follows from (TA1)), and the representation* n *in* N *of some* w *in* W *does not care about the choice of* ϵ *being* $+$ *or* $-$*, see more about this at [6, page:326]. In particular note this isomorphism is induced by canonical projection of* $n \mapsto \delta^*(C_+, nC_-)$*.*

With above set up, we mimic our good old Bruhat decomposition when we are only acting on buildings: Strongly transitive condition would imply that there is one to one correspondence between B_+ *orbit in* $\mathcal{C}_-$ *with elements of* W *by:*

$$B_+D \leftrightarrow w = \delta^*(C, D)$$

for $D \in \mathcal{C}_-$*. But the* B_+ *orbit in* $\mathcal{C}_-$ *correspond to* B_+ *orbits in* G/B_- *so we get following one to one correspondence:*

$$B_+ \backslash G / B_- \to W$$

$$B_+ g B_- \mapsto \delta^*(C_+, gC_-)$$

And the inverse from W *to* $B_+ \backslash G/B_-$ *being* $w \mapsto B_+ w B_- := B_+ n B_-$ *for* n *in* N *a representative of* w *in* W*.*

Now the statement of ***Birkhoff decomposition in pre-twin building context*** *[6, 6.75]:*

If G *acts strongly transitively on a pre-twin building, then*

$$G = \coprod_{w \in W} B_\epsilon w B_{-\epsilon}$$

for each $\epsilon = \pm$ *and given* $g \in G$, $w \in W$, *we have:*

$$g \in B_\epsilon w B_{-\epsilon} \iff \delta^*(C_\epsilon, gC_{-\epsilon}) = w$$

Alternative proof of this $\iff$ *statement can be seen in proof of [2, prop:2] at top of [2, page:398]. It is a important tool in said proof.*

Note*: If further assume* Δ *to be twin building, we have*

$$N := \{g \in G | g\Sigma = \Sigma\} = \{g \in G | g\Sigma_\epsilon = \Sigma_\epsilon\}$$

$$H := \{g \in G | gC = C \text{ for all } C \in \Sigma_+ \cup \Sigma_-\} = \{g \in G | gC_\epsilon = C_\epsilon \text{ for all } C_\epsilon \in \Sigma_\epsilon\}$$

The equal signs except for the ":=" are proved in [6, 6.69], in particular, it will need [6, 5.179] (2) where it is proved that for each apartment Σ'_+, *there is at most one twin partner* Σ'_- *so that* (Σ_+, Σ_-) *is a twin apartment. Above set up (In context of Weyl distance version of twin building) can be seen at [6, ch:6.3.1]. It is worth noting that [6, 6.69]:*

$$B_+ \cap B_- = B_+ \cap N = B_- \cap N = H$$

Note 8.2.9 (Birkhoff decomposition from pre-twin BN pair). *[2, rmk:2] If* (G, B_+, B_-, N, S) *is a pre-twin BN pair, then one can deduce that G has a Birkhoff decomposition. I.e. map* $W \to B_+ \backslash G / B_-$ *by* $w \mapsto B_+ w B_-$ *is bijective.*

A proof in context of BN pair in place of pre-twin BN pair can be seen at [6, 6.80,6.81]

8.2.1 A word on saturated twin BN pair

Definition 8.2.10 (Saturated twin BN pair). *[6, 6.84] Pre-twin BN pair* (B_+, B_-, N) *is* ***saturated*** *if it satisfies*

$$T = B_+ \cap B_-$$

or equivalently it N is the full stabilizer of the fundamental twin apartment of $\Delta(G, B_+, B_-)$

Lemma 8.2.11 (When is it a twin BN pair). *[6, 6.85]*

1. *Let* (B_+, B_-, N) *be a saturated twin BN pair in group G. Let N' be some subgroup of G, then* (B_+, B_-, N') *is a twin BN pair iff* $N'T = N$

2. *If a pair of subgroups B_+, B_- of G is a part of a twin BN pair, then there is a unique subgroup $N \leq G$ such that (B_+, B_-, N) is a saturated twin BN pair.*

Note 8.2.12 (Saturated twin BN iff). *[6, page:333] (B_+, B_-, N) is saturated if and only if one (or both) of the BN pairs (B_ϵ, N) is (are) saturated*

Part IV

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