Growth Rate Modeling Techniques for Longitudinal Data

Zhiyong Zhang
Henan, China

B.S., Renmin University of China, 2000
M.S., Renmin University of China, 2003
M.A., University of Virginia, 2006

A Dissertation Submitted to the Graduate Faculty of
the University of Virginia in Candidacy for the degree of
Doctor of Philosophy

Department of Psychology
University of Virginia
August, 2008

Abstract

Generally, at least two features are needed to characterize a growth process fully at any time point: the level of growth and the rate of growth. The level of growth represents the current status of a process at a given time point and can be viewed as a static measure of that process. The rate of growth represents how fast the level of the process is changing at that time point and can be viewed as a dynamic measure of the process. The widely used growth curve models usually focus on the analysis of the level of growth. However, techniques for analysis of rates of growth are still relatively rare. Because of the significance of rates of growth in understanding dynamic processes, a stronger and more versatile approach is proposed to model them by constructing growth rate models. The concepts of growth processes and current analytical techniques are first reviewed and both the simple rate of growth and the compound rate of growth are defined. Then, different models are developed to analyze rates of growth. Growth rate models are constructed to analyze simple rates of growth and random coefficient models are developed to analyze compound rates of growth. The proposed models are applied to analyze an empirical data set – the National Longitudinal Study of Youth (NLSY) – consisting of children's mathematics performance data and covariates of gender and behavioral problems (BPI).

Individual differences are found in both simple and compound rates of growth. BPI and gender have different relationship with simple rates of growth at different ages. BPI is also found to be negatively related to compound rates of growth. Finally, a systematic simulation study is conducted to validate the results from the analysis of the NLSY data and to investigate the performance of two main models, the quadratic growth rate model and the random coefficient latent difference score model. The simulation results support the validity of the results from the empirical data analysis. It is further found that the parameter estimates for both models are unbiased and the standard error estimates are consistent.

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Acknowledgment

I would like to gratefully and sincerely thank the members of my dissertation committee, John Nesselroade, Jack McArdle, Steven Boker, Karen Schmidt, Aki Hamagami, and Feifang Hu. Without their constructive guidance and suggestions, the completion of this dissertation would not have been possible.

My deepest gratitude is to my advisor, John Nesselroade, for his guidance, understanding, patience, and encouragement during my graduate studies at the University of Virginia. I have been amazingly fortunate to have an advisor who gave me the freedom to explore on my own, and at the same time guidance whenever I need it. I hope that one day I become as good an advisor to my students as John has been to me.

I greatly acknowledge Jack McArdle for his continuous encouragement and insightful advice at different stages of my research. I am grateful to him for holding me to a high research standard and helping me succeed.

I would like to thank Steven Boker for his insightful comments and constructive criticisms. I thank Karen Schmidt for her warm-hearted support and help. I am indebted to Aki Hamagami for numerous constructive discussions. I also thank Feifang Hu for his useful suggestions and encouragement.

I am grateful to Ryan Bowels, Sy-Miin Chow, Ryne Estabrook, Kevin Grimm, Ellen Hamaker, Nilam Ram, and many others for their friendship.

I am deeply grateful to my family for their constant love, concern, and support. I would like to thank my wife, Lijuan Wang, for all her love and support for already more than 10 years. I would like to thank my parents for their support, encouragement, and love.

1. Introduction

This dissertation is motivated by the need for general methodology to investigate growth processes more comprehensively and the need to understand better the development of mathematical ability. In this introductory chapter, I will first discuss the methodological and substantive motivations. Then, I will briefly review and introduce the background of the current study.

1.1 Motivation

1.1.1 Methodological motivation

Investigating time-related patterns of constancy and change of psychological processes is a primary reason for collecting longitudinal data. Most longitudinal repeated measures data tend to share at least three features: (1) the same entities are repeatedly observed over time; (2) the same measurements (including parallel tests) are used; and (3) the timing for each measurement is known (Baltes & Nesselroade, 1979; McArdle & Nesselroade, 2003; Zhang, Hamagami, Wang, Grimm, & Nesselroade, 2007). The need to analyze longitudinal data has stimulated the development of longitudinal data analytic techniques and models which, in turn, have further advanced the collection of longitudinal data. Growth curve models (e.g., Browne, 1993; Browne & Du Toit, 1991; Laird & Ware, 1982; McArdle & Nesselroade, 2003; Meredith & Tisak, 1990; Rao, 1958; Tucker, 1958,

1966) exemplify a widely used technique aiming to achieve the objectives of longitudinal research described by Baltes and Nesselroade (1979) – explicitly to analyze intra-individual change and inter-individual differences in intra-individual change.

Generally, at least two features are needed to fully characterize a growth process at any time point: the level of growth and the rate of growth. The level of growth represents the current status of a process at one time point and it can be viewed as a static / trajectory measure of that process. The rate of growth represents how fast the process is growing / changing at that time point, and it can be viewed as a dynamic measure of that process. Individual differences in the rate of growth are of obvious importance. Two persons of the same ability at a given time may differ markedly in ability at a future time if their rates of growth are different. If rates of growth differ, it is important to know whether there is any correspondence between degrees of intelligence and rates of growth (Freeman & Flory, 1943). Even for the same person, the growth pattern at level x can be very different from that at level y (Cattell, 1966b). Thus, to represent a process more completely, both the level of growth and the rate of growth need be considered simultaneously.

When admittedly simpler versions of growth curves were initially used to analyze individual differences of physical and intellectual growth, researchers actually emphasized the rates of growth at different development phases of children (Freeman & Flory, 1937, 1943; Scammon, 1927). However, the growth curve models developed subsequently are generally modeling the trajectories of growth processes directly instead of change and rates of growth that researchers may be more interested in. The best fitting models basically give researchers the growth functions that best represent the trajectory / curve of the longitudinal data. The modeling of rates of growth is actually the analysis of the dynamics of that process. This is one of the reasons that researchers have attempted to interpret the results from many data analyses using growth curve models in terms of rates of growth(e.g., Biesanz, Deeb-Sossa, Papadakis, Bollen, & Curran, 2004; Bollen & Curran, 2006; Kaplan, 2002; Neale & McArdle, 2000; Reynolds, Finkel, Gatz, & Pedersen, 2002; Tate, 2000).

Although rates of growth are of substantial interest to researchers, techniques for the analysis of rates of growth are, with a few exceptions, still rare. The rate of growth can only be analyzed easily and explicitly in limited growth curve models, specifically, the linear growth curve model and one variety of the quadratic growth curve model. From a hierarchical model perspective, this is mainly because in the second level of a growth curve model, the single random coefficient is actually a combination of several random coefficients forming the rate of growth. Thus, it is generally difficult or even impossible to analyze the rate of growth for more complex instances such as the exponential model in the usual growth curve modeling and estimation framework. This is a serious deficit because exponential growth is known to be very common in human development and biological growth and development. Actually, only with the linear growth curve model and the quadratic growth curve model has the rate of growth been analyzed. For the linear growth model, the latent slope parameter can be interpreted as the rate of growth directly (e.g., Kaplan, 2002; Neale & McArdle, 2000). However, the rate of growth in linear models is a constant which is not typical of human development, especially over a long period (e.g., Freeman & Flory, 1943). For the quadratic growth model, the linear random coefficients (slope parameters) at time zero can be interpreted as the instantaneous rates of growth (e.g., Bollen & Curran, 2006; Schuster & von Eye, 1998; Taylor, Graham, Cumsille, & Hansen, 2000). An obvious drawback of the instantaneous rate of growth is that time zero is usually not part of the measurement process or may not even exist.

Outside the framework of growth curve modeling, e.g., dynamic system analysis, researchers have also shown interest in developing methods and models to analyze rates of growth. For example, Sandland and McGilchrist (1979) proposed a stochastic growth curve model based on stochastic differential equations. This method focused on the analysis of a single subject. Boker and colleagues have developed a set of dynamical systems models based on differential equations and the estimation of derivatives (Boker & Nesselroade, 2002; Boker, Neale, & Rausch, 2004). These models can be used to investigate complex dynamical systems by modeling the relationship among derivatives of a system. These methods are focused on the inter-relationship among first and second derivatives. However, the emphasis here will be on how to predict the rate of growth, the first derivative, by theoretically interesting covariates.

1.1.2 Substantive motivation

This study is also motivated by substantive research questions, namely how do rates of mathematical ability develop and how are they related to interesting covariates such as gender and behavioral problems? There is an extensive history on the analysis of cognitive development through growth curves in psychological research. Thurstone and Ackerson (1929) investigated growth features of mental development based on the Binet tests. Freeman and Flory (1937) studied the individual growth curves of intellectual ability and further discussed intellectual growth in terms of rates of growth. Recently, McArdle and colleagues have applied different growth curve modeling techniques to analyze individual differences in cognitive development trajectories (e.g., McArdle & Epstein, 1987; McArdle & Anderson, 1990; McArdle, Ferrer-Caja, Hamagami, & Woodcock, 2002; McArdle & Bell, 1999).

Mathematical ability is an extensively studied cognitive ability (e.g, Douglas & Kinney, 1938; Felson & Trudeau, 1991; Hyde, Fennema, & Lamon, 1990). This may be in part because the study of mathematics can provide psychologists a well-specified domain of information to examine a wide range of cognitive processes. Longitudinal analyses regarding the development of mathematics have been also conducted (e.g., Grimm, 2005; Kowalski-Jones & Duncan, 1999; Williamson, Appelbaum, & Epanchin, 1991). These studies have found that mathematical ability develops in a curvilinear fashion with respect to time such that the rate of growth is decelerating with increasing age through childhood and into adolescence.

Relationships between mathematical ability and selected covariates have also been studied. Gender differences in mathematics performance, especially, have been investigated extensively in many studies (e.g, Douglas & Kinney, 1938; Felson & Trudeau, 1991; Hyde et al., 1990). An early review by Douglas and Kinney (1938) showed that males outperformed females significantly on mathematics tests. A recent review (Hyde et al., 1990) basically arrived at the same conclusion. However, gender differences in rates of growth (learning) of mathematics have generally been studied only in the linear growth models although mathematics development is a nonlinear process (e.g., Grimm, 2005; Kowalski-Jones & Duncan, 1999; Williamson et al., 1991). Gender differences in the nonlinear growth of mathematics still need to be studied explicitly.

Relationships between cognitive abilities and various behavior problems have also been investigated. It was found that children with higher levels of behavior problems tended to do less well in school and to have lower verbal and reading skills than their more behaviorally competent peers (e.g., Arnold, 1997; Arnold et al., 1999; McClelland, Morrison, & Holmes, 2000). In a recent longitudinal study, Bub, McCartney, and Willett (2007) found that children with higher initial levels of internalizing and externalizing behaviors at 24 months had lower cognitive ability and achievement scores in the first grade.

However, a couple of perspectives on the nature of mathematical ability development and change are still missing from the literature. First, a systematic analysis of rates of growth of mathematical ability is needed. Previous research has found that the development of mathematics is not linear over time. Thus, rates of growth of mathematics cannot be adequately analyzed using linear growth curve models. When nonlinear growth curve models are used, how to analyze the rate of growth is an area that still needs to be investigated. Second, how rates of growth of mathematics are related to covariates such as gender and behavior problems also needs much more investigation. Therefore, to contribute further to this literature, I will examine how rates of mathematical ability develop and how they relate to the covariates of gender and a general measure of behavior problems. Of course, there are many other covariates to study. By illustrating how one can tackle the analytical steps here, it is the writer's hope that future research involving a variety of covariates, the choice of which is driven by substantive concerns, will be facilitated.

Summarization of motivations

To summarize, motivated by both methodological and substantive questions, I will develop and evaluate appropriate models for the analysis of inter-individual differences in rates of mathematics growth and how the inter-individual differences in rates of growth are related to two important covariates of gender and behavioral problems discussed earlier. First, I will give a definition of growth processes in the context of current research and review some selected methods that have been used to analyze growth processes. Second, I will define two kinds of growth rates, the simple rate of growth and the compound rate of growth and review selected methods for estimating these rates of growth. Third, I will propose and examine several other more general models for analyzing rates of growth based on existing growth curve modeling methods. Fourth, I will demonstrate the application of growth rate models using a public data set consisting of children's mathematics performance scores and the covariates of gender and behaviorial problems. Finally, I will carry out a simulation study to evaluate the performance of the models used in the empirical data analysis.

1.2 Structure of the Dissertation

The dissertation is organized in the following way. In the remainder of this chapter, I will define growth process as used in the current framework and review methods that can be used to analyze growth processes. In Chapter 2, I will present two ways to define the rate of growth and then summarize and discuss different methods for estimating rates of growth. In Chapter 3, I will first review existing methods for analyzing rates of growth and then propose several other methods based on existing methods. In Chapter 4, I will apply the proposed growth rate models to analyze mathematical ability data and develop answers to the substantive questions identified above. In Chapter 5, I will discuss the simulation design and present the simulation results. In Chapter 6, I will summarize the main findings and discuss implications of this dissertation research.

1.3 Growth Processes

1.3.1 What is a growth process?

Broadly speaking, *process* is a naturally occurring or designed sequence of changes of properties or attributes of a system over time. More precisely, a process is a particular trajectory in a system's state space. A state space is a space in which all possible states of a system are represented, with each possible state of the system corresponding to one unique point in the state space. Thus, a process is a trajectory which connects the points in a state space. Clearly, the sequence in which the points are connected matters. From a behavioral perspective, Cattell (1966b, p. 394) defined a process as "a sequence of values on a variety of behavior measurements, stimulus measurements, etc. ... A process can be in one person or in several persons, constituting in the latter case a social process." Browne and Nesselroade (2005) emphasized that process involves patterns of changes that are defined across variables and organized over time (see also, Nesselroade & Molenaar, 2004).

Growth generally refers to an increase in quantity, such as size, number, value, or strength of an attribute over time. The quantity can be rather concrete (e.g., growth in height, growth in an amount of money) or abstract (e.g., a system becoming more complex, an organism becoming more mature). However, growth in this study is not limited to increases. Growth, in this context, also means change of an attribute. It also could be a negative change in an attribute, quantity, or any measurable quality. Henceforth, in this proposal the term growth can implicitly result from various dynamic states including increase, decrease, fluctuation, or even no change at all. Clearly, growth itself can be viewed as one kind of process. I use the term growth process to emphasize development-related phenomena.

The research proposed here focuses on one kind of growth process – *psychological process* – which can be defined as change of some psychological construct over time in a theoretically interesting way (Boker, 2002). Broadly speaking, these processes include repression, projection, personality development, cognitive maturation, and so on (Cattell, 1966b). As one kind of process, psychological processes are intrinsically complex issues to investigate because we need to first define these processes operationally and secondly we need to take into account issues of individual differences and dynamic processes. Boker (2002, p. 406) included examples with a wide range of time scales, "decade to decade changes in cognitive abilities (Donaldson & Horn, 1992), year to year changes in adolescent substance abuse (Boker & Graham, 1998), month to month changes in seasonal affective disorder (Sarrias, Artigas, Martínez, & Gelpí, 1989), week to week changes in self-reported mental health in recent widowhood (Bisconti, 2001), minute to minute changes in anxiety levels of children in response to perceived marital discord (Cummings & Davies, 2002), second to second changes in interpersonal coordination of gestures during conversation (Rotondo & Boker, 2002), and millisecond to millisecond changes in neurophysiological evoked response (Hari, Rif, Tiihonen, & Sams, 1992)." Other examples include day to day changes in positive and negative emotion (Lebo & Nesselroade, 1978) and diurnal changes in perceived control (Roberts & Nesselroade, 1986).

1.3.2 How does one measure a growth process?

Since a growth process is defined in terms of changes over time, it is not sufficient to characterize it using only one state of that process as in the case of cross-sectional studies. To capture more fully the characteristics of a process, multiple observation must be made at multiple time points. This operation usually generates a sequence of data points representing measurements taken at successive times, spaced at, often uniform, time intervals. Data with such a structure are usually called longitudinal data or time series data. There are no distinct differences between longitudinal data and time series data. But a single time series should provide sufficient data for statistical analysis. Time series models generally involve time series for only one experimental unit, univariately or multivariately. However, a longitudinal case may only include data from two time points but with observations from multiple subjects. For psychological research, longitudinal data collection has become a practical, commonly used routine. However, the collection of extensive time series data is still relatively rare.

1.3.3 How does one analyze a growth process?

One common method of analyzing growth process data is to model the growth process as a function of time. This is also the approach most widely used in behaviorial research. A general model can be written as,

$$y_t = f(t, \boldsymbol{b}) + e_t,$$

where y_t is the observed datum at time t, f is a function of time which can be any kind of linear or nonlinear function, $\boldsymbol{b} = (b_1, b_2, \ldots, b_p)$ is a vector of p unknown parameters with p depending on models, and e_t represents the error at time t.

For example, if the growth process is a linear one, the following linear model can be used,

$$y_t = b_0 + b_1 t + e_t,$$

with $\boldsymbol{b} = (b_0, b_1)$ representing the intercept and slope, respectively. An exponential growth process can be modeled by

$$y_t = b_0 + b_1 e^{b_2 t} + e_t,$$

where $\boldsymbol{b} = (b_0, b_1, b_2)$. b_2 is usually called the constant rate of growth and b_0 and b_1 represent the initial value and asymptote value, respectively.

Figure 1.1 plots four different growth processes representing four growth functions, namely linear growth, quadratic growth, exponential growth, and sinusoidal growth. Clearly, with different growth functions, we can fit a variety of growth processes.

As stated earlier, modeling growth processes as a function of some measure of

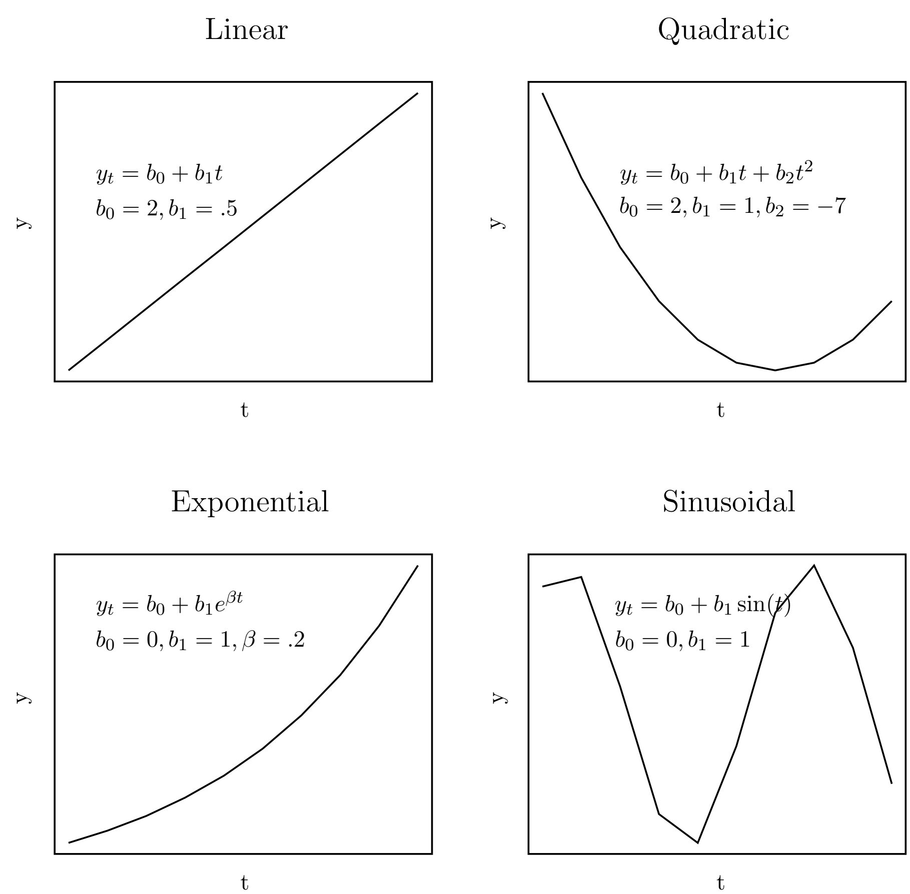

Figure 1.1. Different growth processes defined by growth functions of time.

time is the most common practice in psychological research. However, there is another way that is less familiar to psychologists and that models a growth process as a function of its previous states. This method has been widely applied in time series analysis. A general model can be written as,

$$y_t = f(y_{t-1}, \ldots, y_{t-p}, \boldsymbol{b}) + e_t,$$

where y_t is the observed score at time t, f can be any linear or nonlinear functions. The autoregressive (AR) models can be viewed as a special case of this more general model, and $\boldsymbol{b}$ and e_t are as defined above.

A simple example of the general model is the AR(1) (Hamilton, 1994) with intercept b_0,

$$y_t = b_0 + b_1 * y_{t-1} + e_t.$$

Here b_1 is usually called the autoregressive parameter. By varying the parameters b_0 and b_1, and the initial state y_0, we can produce different growth processes as illustrated in Figure 1.2.

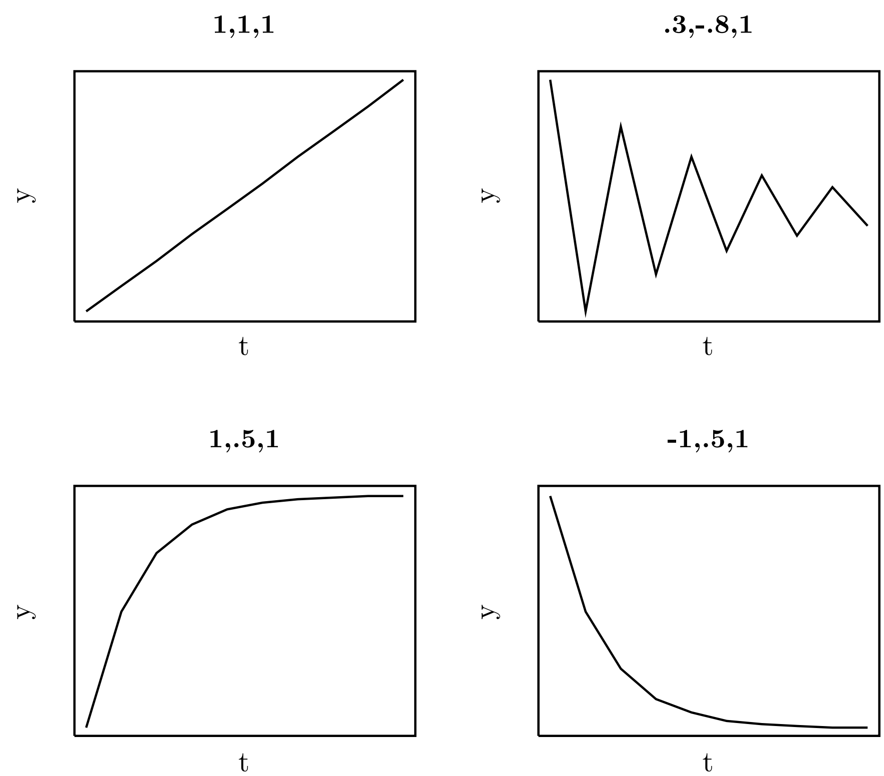

Figure 1.2. Different growth processes from which growth is defined as a function of previous state. The general form is $y_t = b_0 + b_1 * y_{t-1}$. The sequence of the parameter values on the top of each plot is b_0, b_1, y_0, respectively.

1.4 Growth Curve Models

Longitudinal research in behavioral sciences usually involves obtaining data from multiple subjects over multiple occasions. The above discussion is only focused on the analysis of a single subject. Growth curve models have emerged in psychological research as a major tool for analyzing longitudinal data representing multiple participants. Growth curve models have been widely used in the analysis of growth processes via longitudinal data in which one or more variables are measured repeatedly over multiple occasions (e.g., Browne, 1993; Browne & Du Toit, 1991; Laird & Ware, 1982; McArdle & Nesselroade, 2003; Meredith & Tisak, 1990; Rao, 1958; Tucker, 1958, 1966). From a hierarchical model standpoint, a typical growth curve model is a two level model (e.g., Bollen & Curran, 2006; Demidenko, 2004; Hox, 2002). The first level fits a growth curve with random coefficients for each individual and the second level models the relations between the random coefficients and covariates of interest.

Mathematically, the first level of a growth curve model can be written as,

$$y_{it} = f(t, \boldsymbol{b}_i) + e_{it}, t = 1, \ldots, \mathrm{T}, i = 1, \ldots, \mathrm{N}, \tag{1.1}$$

where y_{it} is the measurement for individual i at trial or occasion t, $f(t, \boldsymbol{b}_i)$ is a function of time representing the growth curve with individual parameter values $\boldsymbol{b}_i = (b_{i1}, b_{i2}, \ldots, b_{ip})$, a $p \times 1$ vector of random coefficients, and e_{it} is the error term with $E(e_{it}) = 0$ and $Cov(\boldsymbol{e}_i) = \boldsymbol{\Sigma}_{\mathrm{T}\times\mathrm{T}}$. Usually, it is simply assumed that $Cov(\boldsymbol{e}_i) = \mathbf{I}\sigma^2$.

For the second level,

$$\underset{p\times 1}{\boldsymbol{b}_i} = \underset{p\times q}{\boldsymbol{B}} \underset{q\times 1}{\boldsymbol{X}_i} + \underset{p\times 1}{\boldsymbol{u}_i}, \tag{1.2}$$

where $\boldsymbol{X}$ is the design matrix with 1s in the first column and data for the covariates in the other columns, $\boldsymbol{B}$ is the coefficient matrix, and $\boldsymbol{u}_i$ represents the errors with $E(\boldsymbol{u}_i) = 0$ and $Cov(\boldsymbol{u}_i) = \boldsymbol{D}$. As one can see, in the growth curve models, only $\boldsymbol{b}_i$, instead of a combination of $\boldsymbol{b}_i$, is used in the second level (Eq 1.2).

For estimating growth curve models, the maximum likelihood estimation (MLE) method is commonly used (e.g., Demidenko, 2004; Laird & Ware, 1982). MLE for growth curve models is embedded in commercial statistical packages, such as SAS PROC MIXED and PROC NLMIXED and Splus LME and NLME. Recently, Bayesian methods have received more and more attention as useful tools for estimating a variety of models including growth curve models, especially complex growth curve models which can be difficult or impossible to estimate in the current MLE framework using MLE based software (e.g., Lee & Chang, 2000; Lee & Liu, 2000; Wang & McArdle, 2008; Menzefricke, 1998; Pettitt, Tran, Haynes, & Hay, 2006; Seltzer, Wong, & Bryk, 1996; Zhang et al., 2007).

1.5 Multilevel Time Series Models

Multilevel time series models have not been widely used in psychological research. The main reason may be that they require intensive data collection. However, the models can be specified in the multilevel framework conveniently. Here, I only focus on the multilevel autoregressive models because they are directly related to the rate of growth to be discussed later (e.g., Goldstein, Healy, & Rasbash, 1994). This model can be expressed as

$$\begin{cases} y_{it} = f(y_{it-1}, \ldots, y_{it-p}, \boldsymbol{b}_i) + e_{it} \\ \boldsymbol{b}_i = \boldsymbol{B}\boldsymbol{X}_i + \boldsymbol{u}_i \end{cases}, \ t = 1, \ldots, \mathrm{T}, i = 1, \ldots, \mathrm{N}. \tag{1.3}$$

The symbols in this model have the same meaning as those in the growth curve models presented earlier.

A special case is the multilevel AR(1) model without covariates which can be written as

$$\begin{cases} y_{it} = b_{0i} + b_{1i} y_{it-1} + e_{it} \\ b_{0i} = \beta_0 + u_{0i} \\ b_{1i} = \beta_1 + u_{1i} \end{cases}, t = 1, \ldots, \mathrm{T}, i = 1, \ldots, \mathrm{N}, \tag{1.4}$$

where b_{0i} is the intercept and b_{1i} is the autoregressive coefficient for the ith participant. The overall means for intercept and autoregressive coefficient are β_0 and β_1. u_{0i} and u_{1i} are deviations for the individual intercept and autoregressive coefficient from their means.

2. The Rate of Growth

Estimation of the rate of growth is a primary concern of the remainder of this dissertation. Thus, in this chapter, I will discuss several methods for estimating growth rates and then propose different ways to model the growth rate in the chapters to follow. One point to reiterate here is that although I am using the term "rate of growth", the rate concept discussed here can equally work for the rate of change.

2.1 Definition of the Rate of Growth

The rate of growth is usually defined as the change in one variables with respect to one unit of change in another variable. This can be expressed as in

$$rate = r = \frac{\text{Change in y}}{\text{Change in t}} = \frac{\Delta y}{\Delta t}.$$

To distinguish it from the compound rate of growth discussed later, I refer to this rate of growth as the *simple rate of growth*[1]. If the growth process can be expressed using a continuous time function, the simple rate of growth can be obtained as the first derivative

[1]I am aware that the simple rate of growth (especially in terms of interest in finance) usually means the growth over the initial starting values (principle in terms of interest). I use the simple rate of growth in my dissertation to distinguish the rate defined here from the compound rate of growth defined later. The simple rate of growth discussed in the current framework can be a constant or vary across time.

of the growth function with respective to time as

$$r_t = y'_t = \frac{d}{dt} f(t, \boldsymbol{b}),$$

where $f(t, \boldsymbol{b})$ can be any continuous and differentiable function with population parameters $\boldsymbol{b}$. Intuitively, this is the simple rate of growth when Δt approaches 0.

Geometrically, the simple rate of growth is the slope of the tangent line of the growth function at any given time, three examples of which are shown in Figure 2.1. In this figure, the growth function represents a fluctuating growth process. At t_0, the simple rate of growth is 0. Clearly, y_{t_0} may not be 0. At t_1 and t_2, the levels of growth are the same $y_{t_1} = y_{t_2}$. However, the simple rates of growth are very different; $r_1 = .5$ and $r_2 = -1$. Thus, based only on the level of growth, one cannot distinguish the two states of growth. But, using level together with the simple rates of growth, the differences between the two states are obvious.

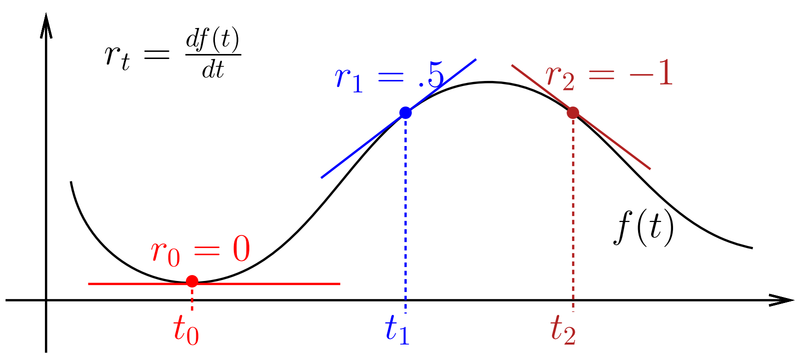

Figure 2.1. Simple rate of growth r_t is characterized by the derivative of the growth function.

One can also model a growth process as a function of its previous states as in the time series models. The rate of growth can be defined similarly in this framework. Such a rate can be defined as,

$$r_t = \frac{y_t - y_{t-1}}{y_{t-1}}. \tag{2.1}$$

This rate is the change of a process over the level of its previous state. This kind of rate is widely used in economics and finance where it is called the compound rate.

Through a simple transformation, one can relate this compound rate to the autoregressive model. Equation 2.1 can be rewritten as

$$y_t = y_{t-1} + r_t y_{t-1} = (1 + r_t) y_{t-1},$$

which has the same form as the autoregressive model in Eq 1.4 where the rate is a constant.

2.1.1 Differences between the simple rate of growth and the compound rate of growth

The major difference between the two kinds of rates lies in the consideration given to the initial states of a growth process. In the simple rate of growth, initial states are not considered so that the simple rate of growth can be the same with the same change for very different initial states. However, for the compound rate of growth, persons with large initial states will show lower compound rates of growth, given the same changes over time. Consider a simple example here. Assume $y_t^1 = 100, y_{t+1}^1 = 105$ and $y_t^2 = 10, y_{t+1}^2 = 15$, with the superscripts 1 and 2 representing two persons. If one calculates the simple rates of growth for the two persons, they will be both $r = 5$ for the two persons. However, the compound rates of growth will be $.05$ and $.5$, respectively, for the two persons.

2.2 Estimation Methods for the Rate of Growth

In this section, I will discuss three methods for estimating the simple rate of growth. They include the parametric method, the semi-parametric method, and the non-parametric method. These methods also work similarly for estimating compound rates of growth. Here, I only focus on the three methods because they represent a wide range of rate estimation methods. However, the other methods, such as the Kalman filter method (Maybeck, 1979), Savitzky-Golay filter method (Savitzky & Golay, 1964) and spline method (Schoenberg, 1946), can also be used to estimate rates of growth. These methods can be related to the three methods discussed here. For example, the filter methods can be viewed as non-parametric ways to calculate the derivatives whereas the local linear approximation methods and the spline method can be viewed as a special case of the semi-parametric method.

2.2.1 The parametric method

The first way to estimate the simple rate of growth to be discussed here can be called the parametric method. This method involves first fitting a growth function $y_t = f(t, \boldsymbol{b}) + e_t$ with some unknown parameters $\boldsymbol{b}$ to the data. Here, e_t is the error which is assumed to be independent of time. After estimating $\boldsymbol{b}$ through some methods such as the ordinary least squares (OLS) method, one can obtain the growth function $f(t, \hat{\boldsymbol{b}})$. Then based on the estimated growth function, one can calculate the simple rate of growth by using

$$r_t = \frac{d}{dt} f(t, \hat{\boldsymbol{b}}). \tag{2.2}$$

For example, if the growth process is a linear process, one can fit a linear function $y_t = b_0 + b_1 t + e_t$ to the data. The OLS method can be used to estimate b_0 and b_1. Then the simple rate of growth is

$$r_t = \frac{d}{dt} f(t, \hat{b}_0, \hat{b}_1) = y_t' \equiv \hat{b}_1. \tag{2.3}$$

In this example, the simple rate of growth is a constant over time which subsequently will be called the constant or time-invariant growth rate.

If the growth process is exponential, one can fit an exponential function $y_t = b_0 + b_1 e^{b_2 t} + e_t$. Similarly, b_0 , b_1 and b_2 can be estimated by using the OLS method. The simple rate of growth is

$$r_t = \frac{d}{dt} f(t, \hat{b}_0, \hat{b}_1) = y_t' = \hat{b_2} \hat{b}_1 e^{\hat{b_2} t}. \tag{2.4}$$

Clearly, the simple rate of growth for the exponential process is changing over time. This will be referred to as time-varying rate of growth.

One assumption underlying this method is that the growth function must be differentiable at least at the points where the derivative is to be estimated. Furthermore, it is assumed that the errors are not related to time so that their derivative will be 0. Because the growth function is assumed to represent the true growth process, one can think of the simple rate of growth based on the derivative as an error-free estimate of the true rate of growth. However, there may be specification errors in the rate estimate attributable to one's choice of the growth function.

2.2.2 The semi-parametric method

The simple rate of growth can also be calculated using semi-parametric methods. For example, one such way is to calculate the simple rate of growth in the functional data analysis framework (Ramsay & Silverman, 2005). Ramsay and colleagues suggest the use of basis functions to approximate functional data (e.g., Ramsay & Silverman, 2002, 2005). A basis function is an element of the basis for a function space which, in turn, is a set of functions of a given kind relating a set X to a set Y. Each function in the function space can be represented as a linear combination of the basis functions. In terms of the current research, any psychological process can be approximated arbitrarily well by a linear combination of a sufficiently large number of basis functions so that

$$y_t = \sum_{k=1}^{K} c_k \phi_k(t) + e_t \tag{2.5}$$

where K is the total numbers of basis functions, $\phi_k(t)$ represents a basis function, and c_k are the coefficients. If K is large enough, $e_t \equiv 0$.

There are two sets of widely used basis functions. The first is the polynomial basis function which can be used to construct a polynomial function,

$$1, t, t^2, t^3, \ldots, t^k, \ldots, \quad . \tag{2.6}$$

Another is the Fourier basis function,

$$1, \sin(\omega t), \cos(\omega t), \sin(2\omega t), \cos(2\omega t), \ldots, \sin(k\omega t), \cos(k\omega t), \ldots, \quad . \tag{2.7}$$

By choosing different K and c_k, we can approximate the growth data observed. For example, if $K = 3$, we can approximate the data using a quadratic function based on the polynomial basis functions written as,

$$\hat{y}_t = \hat{c}_1 + \hat{c}_2 t + \hat{c}_2 t^2. \tag{2.8}$$

By using the Fourier basis function, one can have

$$\hat{y}_t = \hat{c}_1 + \hat{c}_2 \sin(\omega t) + \hat{c}_2 \sin(2\omega t). \tag{2.9}$$

After deciding on values for K and c_k, the rate of growth can be estimated using the first derivative of the estimated growth function as

$$r_t = \sum_{k=1}^{K} \hat{c}_k \frac{d\phi_k(t)}{dt}. \tag{2.10}$$

By increasing K, we can approximate the observed data with increasing accuracy until it is exactly reproduced. If the estimated growth function is the true underlying growth process, there will be no error in the estimated rates of growth. Otherwise, the estimated rates involve errors. As in the parametric method, the errors are assumed to be independent of time. Ideally, the features of the basis functions should match those of the underlying growth processes. Then, with a minimum K, one can estimate the growth processes well enough to provide accurate estimates of the derivatives. However, if the basis functions do not match the underlying growth processes, serious specification errors can occur.

Both the choice of basis function forms and number of basis functions need to be given special attention. If data are periodic, the Fourier basis functions are commonly used. If data are not periodic, the polynomial basis functions are typically used. When it comes to the decision of number of basis functions, one needs to consider the power to fit the polynomial function. The choice of K amounts to a the tradeoff between data fidelity and smoothness. More discussion on this can be found in Ramsay and Silverman (2005).

2.2.3 The non-parametric method

In addition to the parametric and semi-parametric methods, non-parametric methods have been used in psychological research for estimating derivatives (Boker & Nesselroade, 2002; Boker et al., 2004). This method estimates the derivatives of the process for each occasion of measurement without fitting a growth function to data beforehand. One particular method for estimating derivatives from manifest variable processes is called Local Linear Approximations (LLA).

LLA assumes that in a short time interval, a process can be viewed as approximately linear. Thus, the derivative at a time t can be estimated using data observed within some small interval τ. To proceed, one can first calculate the slopes of the intervals $(t - \tau, t)$ and $(t, t + \tau)$. Then, the first derivative at time t is approximated by the average of these two slopes. Specifically,

$$r_t = \frac{(y_t - y_{t-\tau}) + (y_{t+\tau} - y_t)}{2\tau\Delta t} = \frac{y_{t+\tau} - y_{t-\tau}}{2\tau\Delta t}. \tag{2.11}$$

Here Δt is the time step of measurement occasions (see also, Boker & Ghisletta, 2001).

The mechanism of this method is shown in Figure 2.2. From time $t - \tau$ to time t, the slope is $s_1 = (y_t - y_{t-\tau})/\tau$ and from time t to time $t + \tau$, the slope is $s_2 = (y_{t+\tau} - y_t)/\tau$. If the growth process is a smooth one, the derivative at time t – the thick tangent line – should be somewhere between s_1 and s_2. Thus, we can approximate the derivative using the average. Notice that it is also the slope between time $t - \tau$ and $t + \tau$. Although this method seems simple, its estimation of the derivatives can do remarkably well in estimating dynamical parameters (e.g, Boker et al., 2004).

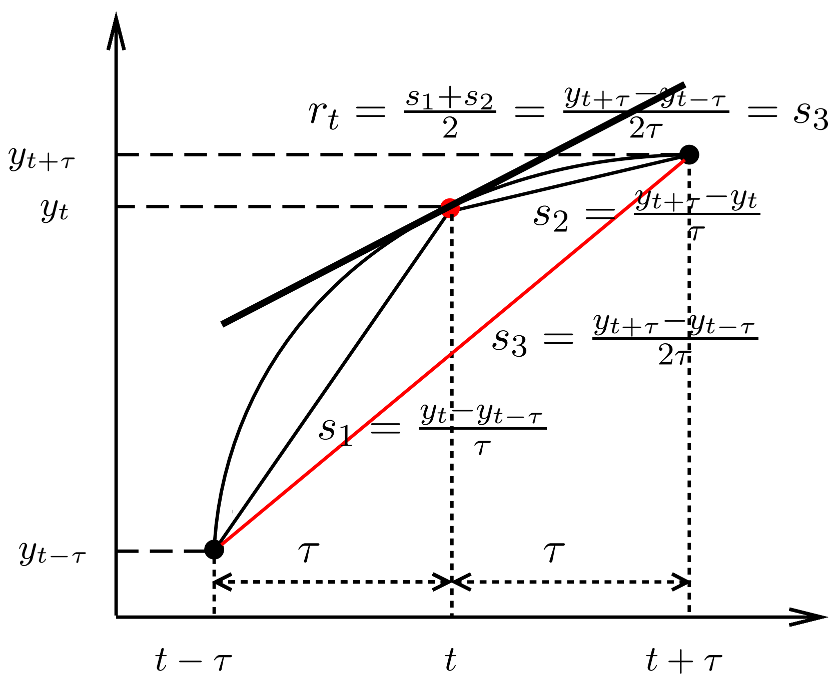

Figure 2.2. A graphical demonstration of the LLA method

The advantage of LLA is that it requires only m occasions of measurement to estimate the derivatives up to the $(m-1)$th order. Furthermore, the estimation of derivatives is model-independent. However, Boker and Nesselroade (2002) showed that it may have biased estimates of the dynamical parameters when the interval τ is not optimal. Especially when the underlying growth process between t and $t+\tau$ is not a monotonic function, the estimated rates of growth can be very misleading. Furthermore, if one is interested in the dynamics of a variable where a score cannot be directly observed, then LLA is inappropriate. In this case one may need to use the parametric or semi-parametric methods.

Boker and colleagues also developed a method – latent differential equations – to estimate the derivatives and the dynamic system models simultaneously (e.g., Boker & Nesselroade, 2002; Boker et al., 2004). The latent differential equation method (LDE) can be viewed as a way to estimate derivatives which sits between the LLA method and the semi-parametric methods. The LDE method can be viewed as a way to reduce measurement errors in the estimation of derivatives.

2.3 Additional Comments on Estimation of the Compound Rate of Growth

To estimate the compound rate of growth, one can also use the parametric and non-parametric methods. For the parametric method, we can first estimate the unknown parameters by assuming that the underlying processes are generated from models in

Equation 1.3. With the estimated model, one can calculate the compound rate of growth using Equation 2.1. This rate can be viewed as error-free under certain conditions. For the non-parametric approach, one can calculate the rate of growth directly from the observed data based on its definition in Equation 2.1. In this case, there is error in the rate estimates.

3. Methods for Analyzing the Rate of Growth

After estimating the rate of growth, the next step is to model it. Although there is lack of general models for analyzing the rate of growth, there are several models that can analyze rates of growth in a limited way. I will first review some existing models. Then I will propose and examine some more general methods.

3.1 Existing Methods for Analyzing the Rate of Growth

The linear growth curve model can be used to analyze simple rates of growth and the simplex model and the latent difference score model can be used to analyze constant compound rates of growth.

3.1.1 Linear growth curve models

The linear growth curve model is one kind of growth curve model targeted for representing linear growth processes (e.g., Bryk & Raudenbush, 1987; McArdle & Epstein, 1987; Meredith & Tisak, 1990; Rao, 1958). One form of the model can be written as,

$$\begin{cases} y_{it} = b_{0i} + b_{1i}t + e_{it} \\ b_{0i} = \beta_0 + u_{0i} \\ b_{1i} = \beta_1 + u_{1i} \end{cases}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T, \tag{3.1}$$

where $Var(e_{it}) = \sigma^2$, $Cov[(u_{0i}, u_{1i})^t] = \boldsymbol{D}$, and $\boldsymbol{\beta} = (\beta_0, \beta_1)^t$. This model is a special case of the growth curve model in Eqs 1.1 and 1.2. It is widely used because the parameters are easy to interpret and the model can be estimated using commonly available SEM software. A path diagram for the linear growth curve model is plotted in Figure 3.1 with 5 occasions of data. b_{0i} is interpreted as the initial level of the growth process and b_{1i} is interpreted as the slope of the growth process. This model can be viewed as a growth rate model because b_{1i} is equal to the simple rate of growth. Actually, this model has been frequently applied to analyze the constant simple rate of growth (e.g., Kaplan, 2002; McArdle & Nesselroade, 2003; Neale & McArdle, 2000; Taylor et al., 2000).

3.1.2 Simplex models

The compound rate of growth can be analyzed using simplex models (e.g., Loehlin, 1998; Jöreskog, Sörbom, Du Toit, & Du Toit, 2001; McArdle & Epstein, 1987). One form of this model is portrayed in Figure 3.2. In its mathematical form, the model can be written as

$$y_{it} = b_1 y_{it-1} + e_{it}, \quad i = 1, \ldots, N, t = 2, \ldots, T. \tag{3.2}$$

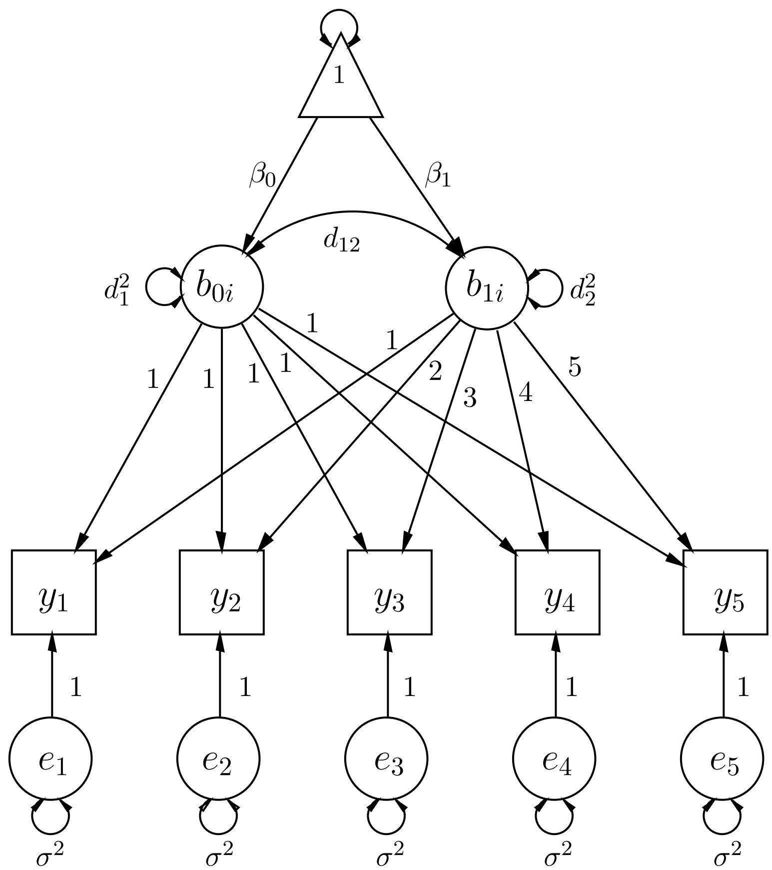

Figure 3.1. A path diagram for a linear growth curve model.

Here $Var(e_{it}) = \sigma^2$. The parameter b_1 is equal to one plus the overall compound rate of growth. This model is useful for short time series from multiple participants. However, the model assumes that there are no individual differences in the compound rate of growth.

3.1.3 Latent difference score models (LDSM)

Another model that can be used to analyze the compound rate of growth is the latent difference score model (e.g., Hamagami & McArdle, 2000; McArdle & Hamagami, 1999, 2001). The path diagram for a simple case of the model is shown in Figure 3.3. There are five occasions of observed data represented in the model. Underlying each

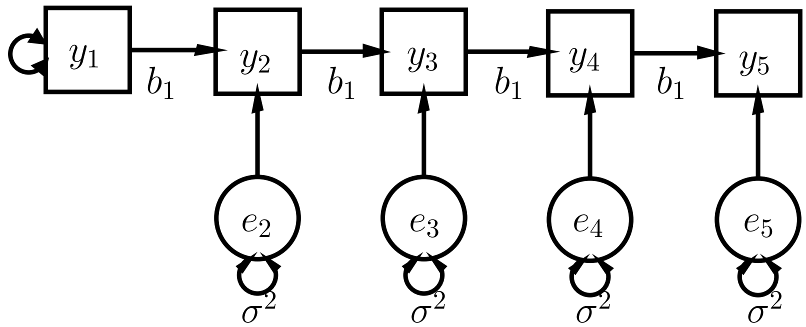

Figure 3.2. A path diagram for the simplex model.

observed variable $Y_i, i = 1, \ldots, 5$, there is an unobserved true variable $y_i, i = 1, \ldots, 5$. The difference score between any two consecutive measurements is calculated by $d_i = y_{i+1} - y_i$.

The LDSM model can be expressed as

$$\begin{cases} y_{i1} = y_{0i} \\ d_{it-1} = \beta y_{it-1} \\ y_{it} = y_{it-1} + d_{it-1} \\ Y_{it} = y_{it} + e_{it} \end{cases}, i = 1, \ldots, N, t = 2, \ldots, T. \quad (3.3)$$

In this model, d_{it-1} is the difference score between y_{it} and y_{it-1} which underlies the observed data Y_{it} and Y_{it-1}. The measurement error e_{it} has a mean 0 and variance σ^2. The initial level y_{0i} for each individual is different with mean μ_0 and variance σ_0^2. $\beta = d_{it-1}/y_{it}$ in this model can be viewed as the compound rate of growth although in the original work of McArdle and colleagues it is termed as the multiplicative change over time. As in the simplex models, β in this model is usually assumed to be the same for all participants.

Here I only present a simple case of latent difference score models to fit the current

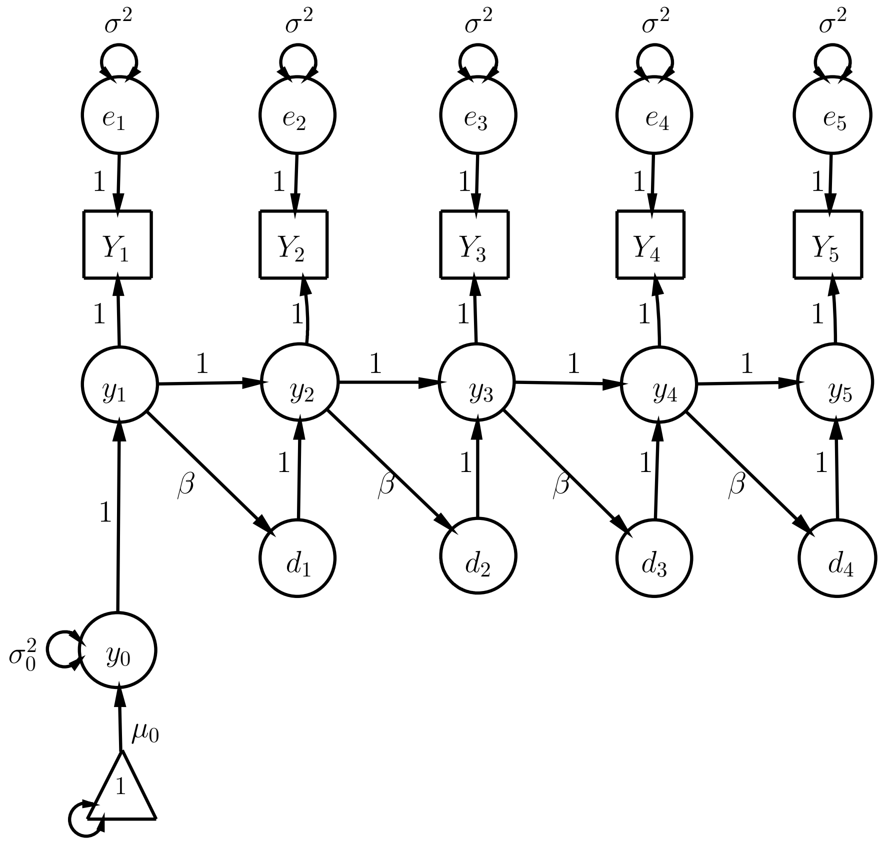

Figure 3.3. A path diagram for the univariate latent difference score model.

compound rates of growth framework. However, McArdle and Hamagami have developed much more complex models based on latent difference scores. For example, the models allow (a) the difference scores to be related to invariant slope scores and (b) the relation between initial level and slope to be tested (McArdle & Hamagami, 2001). The authors also developed multivariate latent difference score models which can be used to investigate the interrelationship across different domains (McArdle & Hamagami, 2001). More complex models have also been discussed recently (Hamagami & McArdle, 2007). These models have been used by researchers to answer many different substantive questions (e.g., Hamagami & McArdle, 2007; King et al., 2006; Malone et al., 2004; McArdle, Hamagami, Meredith, & Bradway, 2000; Schindler, Staudinger, & Nesselroade, 2006).

There are several differences between the simplex model and the latent difference score model. First, in the LDSM, β is directly the compound rate of growth. But in the simplex model, $b_1 - 1$ is equal to the compound rate of growth. Second, in the LDSM, the rate is calculated from the latent variable so that it can be viewed as error-free. Third, in the LDSM model, the intercept is considered explicitly in the model, whereas in the simplex model, it is not.

3.2 Proposed Methods for Analyzing the Rate of Growth

In this section, I will investigate several methods for analyzing the rate of growth in a more general way. Certainly, many different models can be developed to analyze rates of growth. Here, I only focus on the selected methods that can be derived directly from the existing models. The simple growth rate models can be used to analyze simple rates of growth. The multilevel simplex models and the latent difference score models can be used to analyze compound rates of growth.

3.2.1 Path analysis of rates of growth

Path analysis can be used to analyze both the simple rate of growth and the compound rate of growth in two steps. In the first step, one can estimate the rate of growth using any of the three methods discussed in the previous chapter. In the second step, a desired model can be fitted to the rates of growth for further inference.

Assume one observes longitudinal dependent data (outcome)

$y_{it},\ i = 1, \ldots, N,\ t = 1, \ldots, T$ and covariates (predictors)

$X_{it} : p \times T,\ i = 1, \ldots, N,\ t = 1, \ldots, T$. Here, the covariates can be either time-varying or time-invariant. Using the estimation methods of rates of growth, we can obtain $r_{it},\ i = 1, \ldots, N,\ t = 1, \ldots, T$. The models can be developed to investigate the relationship between rates of growth and covariates. If the rate of growth is a constant for each individual, the model will be a multiple regression model. If the rate of growth is time-varying, a path model (e.g., Loehlin, 1998), which can be viewed as a generalization of the multiple regression model, can be fitted.

The general path model for modeling the relationship between rates of growth and covariates can be expressed as

$$\underset{T\times 1}{\boldsymbol{r}_i} = \underset{T\times p}{\boldsymbol{B}}\ \underset{p\times 1}{\boldsymbol{X}_i} + \underset{T\times 1}{\boldsymbol{e}_i}, \tag{3.4}$$

where $\boldsymbol{B}$ are path coefficients and $\boldsymbol{e}$ are unexplained residuals. A path diagram of the model is portrayed in Figure 3.4. In this model, there are four occasions of rates of growth and two time-invariant covariates. This model can be estimated using a two-stage method. In the first stage, the rates of growth may be estimated by different rate estimation methods. In the second stage, the estimated rates of growth are regarded as observed variables in the path model. The path model can then be estimated conveniently using regular SEM software such as LISREL and Mplus. This method works for both simple rates of growth and compound rates of growth.

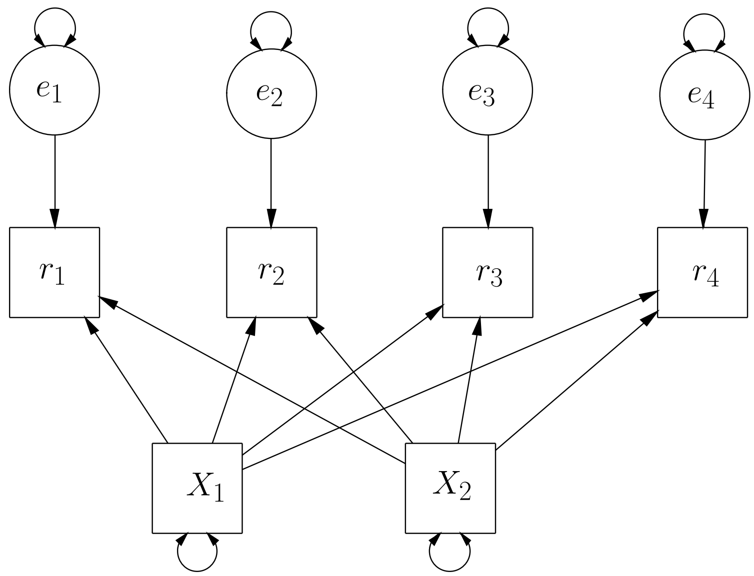

Figure 3.4. The path diagram for a simple growth rate model using path analysis

3.2.2 Simple growth rate models

Growth curve modeling can be viewed as a parametric method for estimating the rate of growth. In this case, simple growth rate models can be constructed by adding an extra equation which brings the relationship between the simple rate of growth and possible covariates into the existing growth curve models. Mathematically, a general form of such a model can be expressed as,

$$\begin{cases} y_{it} = f(t, \boldsymbol{b}_i) + e_{it} \\ \boldsymbol{b}_i = \boldsymbol{B}\boldsymbol{X}_i + \boldsymbol{u}_i \\ \frac{d}{dt} f(t, \boldsymbol{b}_i)|_t = r_{it} = \boldsymbol{c}_t \boldsymbol{b}_i = \boldsymbol{\Gamma}_t \boldsymbol{X}_i + v_{it} \end{cases}, t = 1, \ldots, \mathrm{T}, i = 1, \ldots, \mathrm{N}, \tag{3.5}$$

In this model, y_{it} is the observed datum (outcome score) for individual i at time t. f can be any continuous function. e_{it} is the residual error following a normal distribution with mean 0 and variance σ_t^2. As in the growth curve models, u_i follows a multivariate normal distribution with $\mu(\boldsymbol{u}_i) = 0$ and $Cov(\boldsymbol{u}_i) = \boldsymbol{D}$. Thus, the first part of the model is exactly the same as the growth curve model. Beyond the growth curve model, the derivative of the growth function is modeled and predicted by covariates $\boldsymbol{X}$. The derivative (rate of growth) of the growth function r_{it} can be expressed as a linear combination of the random coefficient parameters $\boldsymbol{b}_i = (b_1, \ldots, b_p)'$ and the weights $\boldsymbol{c}_t = (c_{t1}, \ldots, c_{tp})$. $\boldsymbol{\Gamma}_t$ are regression coefficients for covariates. Note that v_{it}, $E(v_{it}) = 0$ and $Var(v_{it}) = \delta_t^2$, is the residual part which is not explained by the covariates. v_{it} is assumed to be normally distributed here.

A potential problem of the model is that it requires the estimation of more parameters than can be estimated in one step. One way to overcome this problem is to transform the model. Without loss of generalization, we can assume that the jth element of $\boldsymbol{c}_t$, $c_{tj} \neq 0$, then the equation $r_{it} = \boldsymbol{c}_t \boldsymbol{b}_i$ can be written as

$$b_j = r_{it} - \boldsymbol{c}_t^j \boldsymbol{b}_i^j,$$

where $\boldsymbol{c}_t^j = (c_{t1}, \ldots, c_{tj-1}, c_{tj+1}, \ldots, c_{tp})$ and $\boldsymbol{b}_i^j = (b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_p)'$. By substituting b_j in the above formula, we have

$$\begin{cases} y_{it} = f(t, r_{it}, \boldsymbol{b}_i^j) + e_{it} \\ \boldsymbol{b}_i^j = \boldsymbol{B}^j \boldsymbol{X}_i + \boldsymbol{u}_i^j & , t = 1, \ldots, \mathrm{T}, i = 1, \ldots, \mathrm{N}, \\ r_{it} = \boldsymbol{\Gamma}_t \boldsymbol{X}_i + \boldsymbol{v}_{it} \end{cases} \tag{3.6}$$

where $\boldsymbol{B}^j = (B_1, \ldots, B_{j-1}, B_{j+1}, \ldots, B_p)$ and $\boldsymbol{u}_i^j = (u_{i1}, \ldots, u_{ij-1}, u_{ij+1}, \ldots, u_{ip})$. The unique feature of the model is that we express the growth curve as a function of the rate of growth. This model will be called the simple growth rate model henceforth in this dissertation.

To illustrate, consider the quadratic growth rate model. A quadratic growth model can be written as,

$$\begin{cases} y_{it} = b_{i1} + b_{i2}t + b_{i3}t^2 + e_{it} \\ b_{i1} = \beta_{01} + \beta_{11}x_{1i} + \beta_{21}x_{2i} + u_{i1} \\ b_{i2} = \beta_{02} + \beta_{12}x_{1i} + \beta_{22}x_{2i} + u_{i2} \\ b_{i3} = \beta_{03} + \beta_{13}x_{1i} + \beta_{23}x_{2i} + u_{i3} \end{cases}. \tag{3.7}$$

The rate of growth for the quadratic growth model is

$$r_{it} = \frac{d}{dt} f(t, \boldsymbol{b}_i)|_t = 0b_{i1} + b_{i2} + 2tb_{i3}. \tag{3.8}$$

In the case of two covariates, the rate can be modeled as,

$$r_{it} = \gamma_{t0} + \gamma_{t1}x_{1i} + \gamma_{t2}x_{2i} + v_{it}.$$

From Eq (3.8), we have $b_{i2} = r_{it} - 2tb_{i3}$. Then we have the quadratic growth rate model as

$$\begin{cases} y_{it} = b_{i1} + r_{ij}t + b_{i3}(t^2 - 2jt) + e_{it} \\ b_{i1} = \beta_{01} + \beta_{11}x_{1i} + \beta_{21}x_{2i} + u_{i1} \\ b_{i3} = \beta_{03} + \beta_{13}x_{1i} + \beta_{23}x_{2i} + u_{i3} \\ r_{ij} = \gamma_{j0} + \gamma_{j1}x_{1i} + \gamma_{j2}x_{2i} + v_{ij} \end{cases}, j = 1, \ldots, T, \tag{3.9}$$

where r_{ij} represents the rate of growth for ith individual at time j. Because the rate of growth is changing with time for the quadratic growth curve, we use the time index j to indicate it. The path diagram for the quadratic growth rate model is given in Figure 3.5.

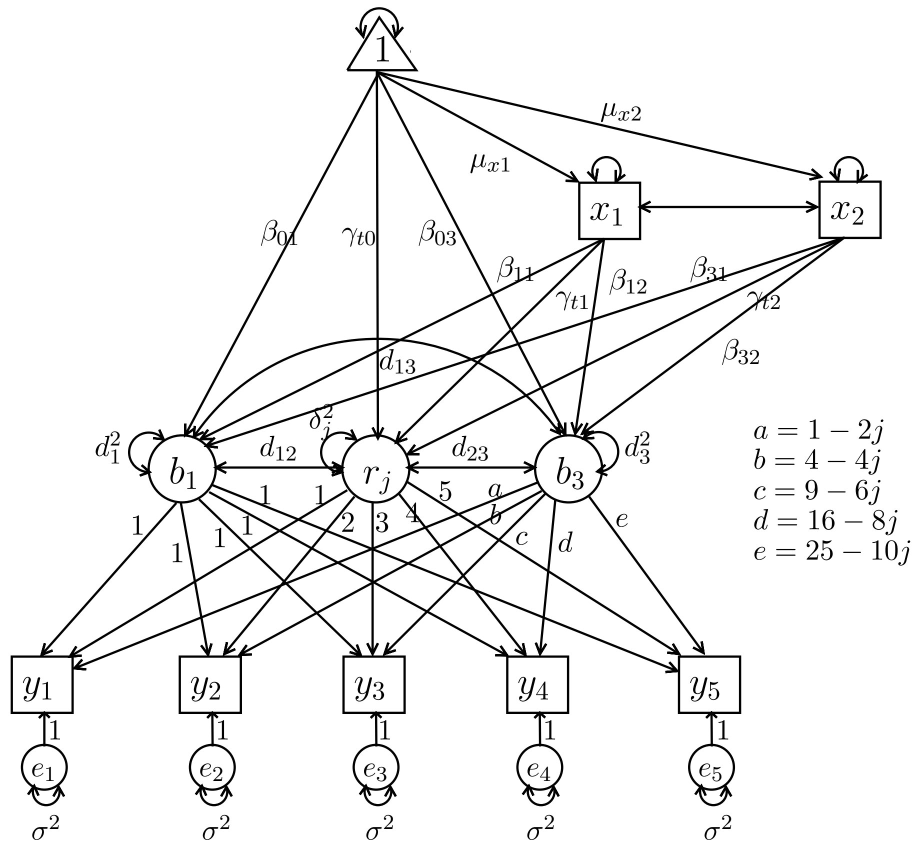

Figure 3.5. A path diagram for the quadratic growth rate model.

Rotation of growth curve

Tucker (1958, 1966) proposed applying principal components analysis to learning data (Persons $\times$ Trails) to identify meaningful dimensions that can be used to explain the individual differences in outcome scores. The widely used growth curve models can be viewed as confirmatory factor models with factors already identified. For example, for a linear growth curve model, we may hypothesize and fit a level factor and a slope factor.

For the quadratic growth curve model,

$$y_{it} = b_{i1} + b_{i2}t + b_{i3}t^2 + e_{it},$$

where b_{i1}, b_{i2}, and b_{i3} can be viewed as factor scores for ith individual for three factors, level, slope, and quadratic terms, and $(1, t, t^2)$ are factor loadings. In matrix notation, the above model can be written as,

$$\begin{pmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \\ \vdots \\ y_{iT} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ \vdots & \vdots & \vdots \\ 1 & T & T^2 \end{pmatrix} \begin{pmatrix} b_{i1} \\ b_{i2} \\ b_{i3} \end{pmatrix} + \begin{pmatrix} e_{i1} \\ e_{i2} \\ e_{i3} \\ \vdots \\ e_{iT} \end{pmatrix}$$

Sometimes, when the aim of growth curve analysis is to find the curves that best fit the data, the resulting factors may be very difficult or even impossible to interpret in a meaningful way. Researchers have suggested rotating factors to enhance their meaning. Although factor rotation methods have been well studied in the framework of factor analysis (see recent reviews by Browne, 2001; Jennrich, 2007), applying rotation to growth curves is not yet popular. Tucker (1966) suggested that rotation to simple structure may not be meaningful in the framework of learning curves. Cleary (1974, p. 938) pointed out that “a factor is considered to be nontrivial if the loadings exhibit relatively smooth changes over successive measures.” Arbuckle and Friendly (1977) further developed this factor retention idea to a rotation criterion for growth curves. The basic idea is to obtain rotated factor loadings by minimizing

$$\sum_{t=2}^{T}(l_{t-1}-l_t)^2,$$

where $l_t, t = 1, \ldots, T$ are the factor loadings. The factor loadings after rotation will have a smoother curve with respect to time.

The construction of simple growth rate models can be viewed as a method to rotate the factors of the growth curve to predefined target factors. More specifically, after rotation, one of the factors will be the rate of growth. Using the quadratic model as an example, the rotation for the factors is

$$\begin{pmatrix} b_{i1} \\ r_{ij} \\ b_{i3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2j \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{i1} \\ b_{i2} \\ b_{i3} \end{pmatrix} = \boldsymbol{Q} \begin{pmatrix} b_{i1} \\ b_{i2} \\ b_{i3} \end{pmatrix},$$

where $\boldsymbol{Q}$ here represents the rotation matrix. With this rotation matrix, the factors b_{i1} and b_{i3} remain the same but the second factor now is the rate of growth at time j. Note that the second row of elements $(= \boldsymbol{c}_j)$ is determined by the first derivative of the growth curve.

When the factors are rotated, the factor loadings need to be rotated correspondingly in order to keep the covariance structure unchanged. For the quadratic model example, one has

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ \vdots & \vdots & \vdots \\ 1 & T & T^2 \end{pmatrix} \boldsymbol{Q}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ \vdots & \vdots & \vdots \\ 1 & T & T^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2j \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1-2j \\ 1 & 2 & 4-4j \\ 1 & 3 & 9-6j \\ \vdots & \vdots & \vdots \\ 1 & T & T^2-2Tj \end{pmatrix},$$

where $\mathbf{\Lambda}$ denotes the factor loadings after rotation. Note that the elements in this matrix correspond to the loadings on the path in Figure 3.5.

It can also be shown that the rotation satisfies the criterion used by Arbuckle and Friendly (1977) as discussed earlier. Actually, for the original quadratic growth curve model, we have [1]

$$\begin{aligned} S_1 &= \sum_{t=2}^{T} \{[t^2 - (t-1)^2]\}^2 \\ &= \sum_{t=2}^{T} (2t-1)^4 \end{aligned}.$$

And for the quadratic growth rate model, we have

$$\begin{aligned} S_2 &= \sum_{t=2}^{T} \{[t^2 - 2tj - (t-1)^2 + 2(t-1)j]\}^2 \\ &= \sum_{t=2}^{T} (2t-1-2j)^4 \end{aligned}.$$

[1] Since the loadings in the first two columns are the same, one only needs to focus on the loadings in the third column.

Then,

$$S_2 - S_1 = \sum_{t=2}^{T}(2t-1-2j)^4 - \sum_{t=2}^{T}(2t-1)^4 \\ = 8jT(j-T)[(j-T)^2 + j^2 + T^2 - 1]$$

Thus, for any $j \leq T$, we have $S_2 \leq S_1$. This means that overall the loadings for the quadratic growth rate model are smoother than those of quadratic growth curve model.

Estimation of the simple growth rate models

The simple growth rate models can be estimated by SEM methods using any SEM software. All the model fit statistics and model comparison indices for SEM models, such as chi-square tests, Akaike information criterion (AIC; Akaike, 1973), Bayesian information criterion (BIC; Schwarz, 1978), and the root mean squared error of approximation (RMSEA; Browne & Cudeck, 1993), can be used to compare models and select the best fitting model.

3.2.3 Random coefficient simplex models

The simple growth rate models can be used to analyze the simple rate of growth. In this section and the section that follows, the random coefficient simplex models and the random coefficient latent difference score models will be discussed to analyze the compound rate of growth.

Although simplex models are mainly used to analyze the average compound rate of growth and ignore individual differences in the rate of growth, they can be extended easily to include random effect parameters to represent individual differences in the compound rate of growth. For example, an extension can be portrayed as shown in Figure 3.6. In this model, we estimate compound rates of growth b_i for each individual and then use a covariate X to predict the variability of the compound rate of growth.[2] Mathematically, this model can be expressed as,

$$\begin{cases} y_{it} = b_i y_{it-1} + e_{it}, \\ b_i = \beta_0 + \beta_1 X_i + v_i \end{cases} \quad i = 1, \ldots, N, t = 1, \ldots, T, \tag{3.10}$$

where y_{it} is the observed datum for ith individual at time t, β_0 and β_1 are intercept and slope, $Var(e_{it}) = \sigma^2$ is the error variance, and $Var(v_i) = d^2$ is the variability or the residual variance of compound rates of growth b_i. Both e_{it} and v_i are normally distributed.

In the path diagram in Figure 3.6, a new symbol was used (labeled $\circ$) following McArdle and Hamagami (1996). It represents a special kind of unobserved variable. This unobserved variable "has no variance and it acts only as a placeholder (augmenting the model matrices with an extra row and column)" (McArdle & Hamagami, 1996, p. 94). In this case, y_{t-1} first connects to y_t through a constant β_1 (the regression coefficient) and the covariate X. Then y_{t-1} connects to y_t through a constant d (the standard error) and the residuals v. The path diagram correctly represents the random coefficient simplex model.[3]

[2] Keep in mind that the actual compound rate of growth for this model is $b_i - 1$.

[3] I would like to thank Dr. Jack McArdle for his help in drawing the path diagram.

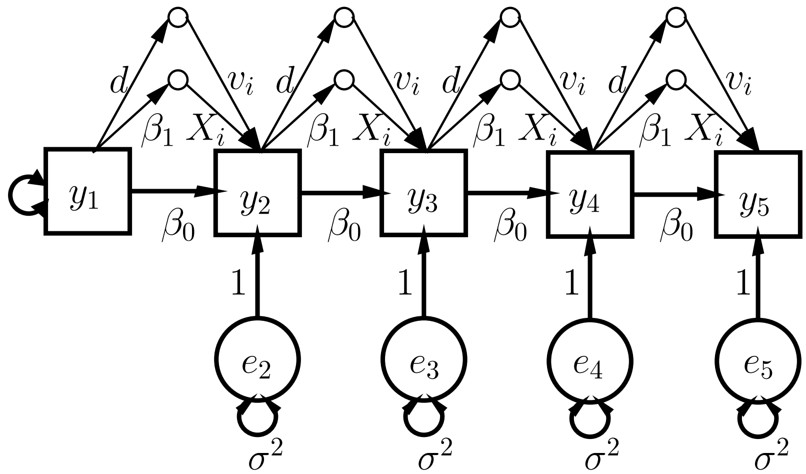

Figure 3.6. Simplex models with individual compound rates of growth. This represent the path diagram for the ith individual.

3.2.4 Random coefficient latent difference score models

Similar to the random coefficient simplex models, the LDSM can be extended to include random coefficients for β. In this case, the model can be written as,

$$\begin{cases} y_{i1} = y_{0i} \\ d_{it-1} = b_i y_{it-1} \\ y_{it} = y_{it-1} + d_{it-1} \quad , i = 1, \ldots, N, \;\; t \geq 2, \\ Y_{it} = y_{it} + e_{it} \\ b_i = \beta_0 + \beta_1 X_i + v_i \end{cases} \tag{3.11}$$

where y_{it} is the observed datum for ith individual at time t. In this model, b_i is different for each individual. The other parameters have the same meaning as in the LDSM discussed earlier. The variability of individual differences on the compound rate of growth or the unexplained residual variance of b_i can be measured by $Var(v_i) = d^2$. A path diagram of this model is given in Figure 3.7. For clearer representation of key features, some symbols on the path are omitted because they are the same as the symbols on the path in the last segment from y_4 to d_4.

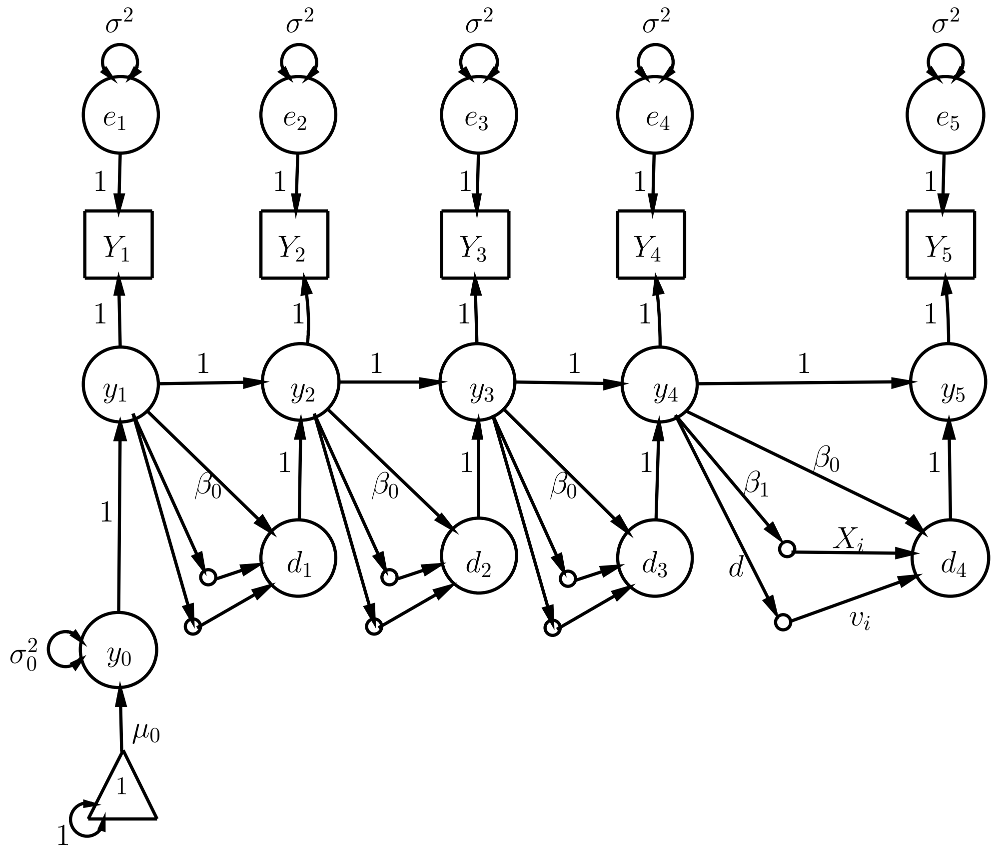

Figure 3.7. LDSM with random compound rate of growth. Represented here is the path diagram for the ith individual.

For the estimation of random coefficient simplex models and random coefficients LDSMs, Bayesian estimation methods can be used. Besides the models in Eqs 3.10 and 3.11, the other derived models can be fitted. For example, one can model the relationship between initial status and compound rate of growth. To test whether the relationship is significant, one can fit both models to check the change of the deviance information criterion (DIC, Spiegelhalter, Best, Carlin, & Linde, 2002a).

The deviance information criterion (DIC; Spiegelhalter, Best, Carlin, & Linde, 2002b) is a widely used criterion for model selection in the Bayesian framework. DIC is defined as a Bayesian measure of model fit with a penalty for model complexity p_D,

$$DIC = \overline{D(\theta)} + p_D = D(\bar{\theta}) + 2p_D,$$

where $Dbar = \overline{D(\theta)}$ is the posterior mean of -2(Loglikelihood function) and $Dhat = D(\bar{\theta})$ is the -2(Loglikelihood function) calculated at the posterior mean of θ. For more complex models, $Dbar$ and $Dhat$ become smaller but p_D becomes bigger. Overall, the smaller DIC represents the better fit of the model.

The random coefficient models discussed above are different from the multilevel factor models (e.g., Goldstein & Browne, 2002; Goldstein & McDonald, 1988; Goldstein & Browne, 2005; McDonald & Goldstein, 1989). The multilevel factor models assume that there are different factors in each level but the factor loadings within a given level are the same for all individuals. These models partition the total covariance matrix into between and within cluster covariance matrices and the respective covariance matrices are represented by separate structural equations. Therefore, model parameters are usually not random coefficients. The random coefficients models, however, assume that rates of growth, analogous to the factor loadings, are different for each individual. These models can be viewed as extensions of the random coefficient models discussed by McArdle and Hamagami (1996), where the basis coefficients (loadings) vary across groups. Because of the limitation of estimation methods and computation, McArdle and Hamagami did not estimate the basis coefficients for each individual. However, they pointed out that a further step can be taken to investigate the individual differences in the basis coefficients. This step is implemented now in the random coefficient LDSM.

4. Applications

As pointed out in the first chapter, the development of mathematical ability has been well studied but researchers have mainly focused on level, rather than rate of growth (Felson & Trudeau, 1991; Grimm, 2005; Hyde et al., 1990; Kowalski-Jones & Duncan, 1999; Williamson et al., 1991). Research on the rate of growth of mathematical ability is still rare partly because of the lack of growth rate modeling techniques. In this chapter, I will introduce an empirical set of data on mathematical ability and apply the proposed models in the previous chapter to investigate rates of growth and the nature of individual differences in the rate of growth of mathematical ability. Furthermore, many studies have investigated the relationships between mathematical ability and the covariates of gender and behavioral problems (Arnold, 1997; Arnold et al., 1999; Bub et al., 2007; Felson & Trudeau, 1991; Hyde et al., 1990). However, few studies have investigated the relationship between rate of mathematics ability growth and those covariates. Thus, the relationships between the rate of growth of mathematical ability and these two covariates will be examined in this dissertation.

4.1 Data

4.1.1 Overview of the data

All the empirical data to be used are from the National Longitudinal Survey of Youth 1979 cohort (NLSY79, Center for Human Resource Research, 2006). The NLSY79 is a multi-purpose panel survey that originally included a nationally representative sample of 12,686 men and women who were 14 to 21 years of age on December 31, 1978. Annual interviews have been completed with most of these respondents since 1979, with a shift to a biennial interview mode after 1994. Beginning in 1986, the children of NLSY79 female respondents have been interviewed and assessed every two years. The assessments measure cognitive ability (including mathematical performance), temperament, motor and social development, behavior problems, and self-competence of the children as well as the quality of their home environment.

Data from the NLSY79 child sample will be used to study the relationships between mathematical performance and gender and behaviorial problems by means of the growth rate models proposed in the previous chapter. Specifically, the models will be fitted to one cognitive variable – the Peabody Individual Achievement Test (PIAT) mathematics assessment. These measurements were collected from the year 1986 to the year 2004 with a two-year interval between testings. The Peabody Individual Achievement Test (PIAT) is a wide-ranging measure of academic achievement for children aged five and over and is widely used in research. The current study uses one of the three subtests from the full PIAT battery - the Mathematics (PIAT Math) assessments. This subtest consists of 84 multiple-choice items arranged in increasing order of difficulty. The scores of this subtest range from 0 to 84. For computation convenience, the scores were divided by 10.

The two covariates to be used in the analysis are gender and scores on the Behavior Problems Index (BPI). The two covariates are used because of their significance in the study of mathematical ability (Arnold, 1997; Arnold et al., 1999; Bub et al., 2007; Felson & Trudeau, 1991; Hyde et al., 1990). The BPI was created by Nicholas Zill and James Peterson to measure the frequency, range, and type of childhood behavior problems for children age four and over (Peterson & Zill, 1986). Many items were derived from the Achenbach Behavior Problems Checklist (Achenbach & Edelbrock, 1983) and other child behavior scales (Graham & Rutter, 1968; Kellam, Branch, Agrawal, & Ensminger., 1975; Rutter, Tizard, & Whitmore, 1970). Possible scores on the BPI range from 0 to 28. In the NLSY study, BPI was repeatedly measured. However, because of missing data, composite scores were created by averaging BPI over age and were used in the data analysis.

4.1.2 Descriptive statistics

The data are from $N = 1,233$ children ranging in age from 6 to 15 years with a 10-year span. Descriptive statistics for the measures at each age are given in Table 4.1. About 48% of the children are female. The mean scores for PIAT math are increasing with age but the rates of increase are slowing down. The longitudinal plot of PIAT math is given in Figure 4.1. From the plot, one can see that the growth trajectory of mathematics performance is not linear with time. The mean trajectories of mathematics performance for girls and boys are also plotted as shown in Figure 4.2. From the plot, it appears that there is no difference on mathematics performance for girls and boys. The data are then divided into two groups, the low BPI and high BPI groups, using the threshold of the BPI mean. The mean trajectories for low and high BPI children are given in Figure 4.3. Overall, it seems that children with low BPI outperform children with high BPI.

Table 4.1. *Descriptive Statistics*

Variable	Age	N	Mean	Std	Min	Max
Gender		1,233	0.48	0.50	0 (M)	1 (F)
BPI		1,233	7.42	5.16	0	28
Math	6	630	1.34	0.51	0	3.6
	7	498	2.00	0.77	0.5	5.2
	8	601	2.81	0.99	0.5	6.6
	9	499	3.71	1.04	0.1	6.5
	10	585	4.43	1.07	0.1	7
	11	459	4.83	1.08	0.3	7.5
	12	458	5.11	1.10	0.6	8.2
	13	302	5.35	1.11	0.1	7.6
	14	212	5.58	1.16	0.8	8.3
	15	80	5.62	1.16	2.8	8.1

Note. Std: standard deviation; Min: minimum; Max: maximum; M: male; F: female; Math: mathematical performance.

For this application, age, instead of measurement occasions, will be used as the time scale for the following reasons. First, age is a natural scale for analysis of child development. Using measurement occasions, children with different ages will be forced into one group. Second, using age as the time scale makes the interpretation of the results very easy. Thus, the time scale for the analysis will be age in years ranging from 6 to 15. Missing data appeared at each age. At age 8, about 48.9% of data were missing. At age 15 about 93.5% of data were missing. The total percentage of missingness was about 64.9%. In all following analyses, missing data will be assumed as missing at random (MAR) (e.g., Little & Rubin, 1987; McArdle & Hamagami, 1991; Rubin, 1976). The simulation study in the next chapter will show that the MAR assumption gives valid results for the current substantive data analysis.

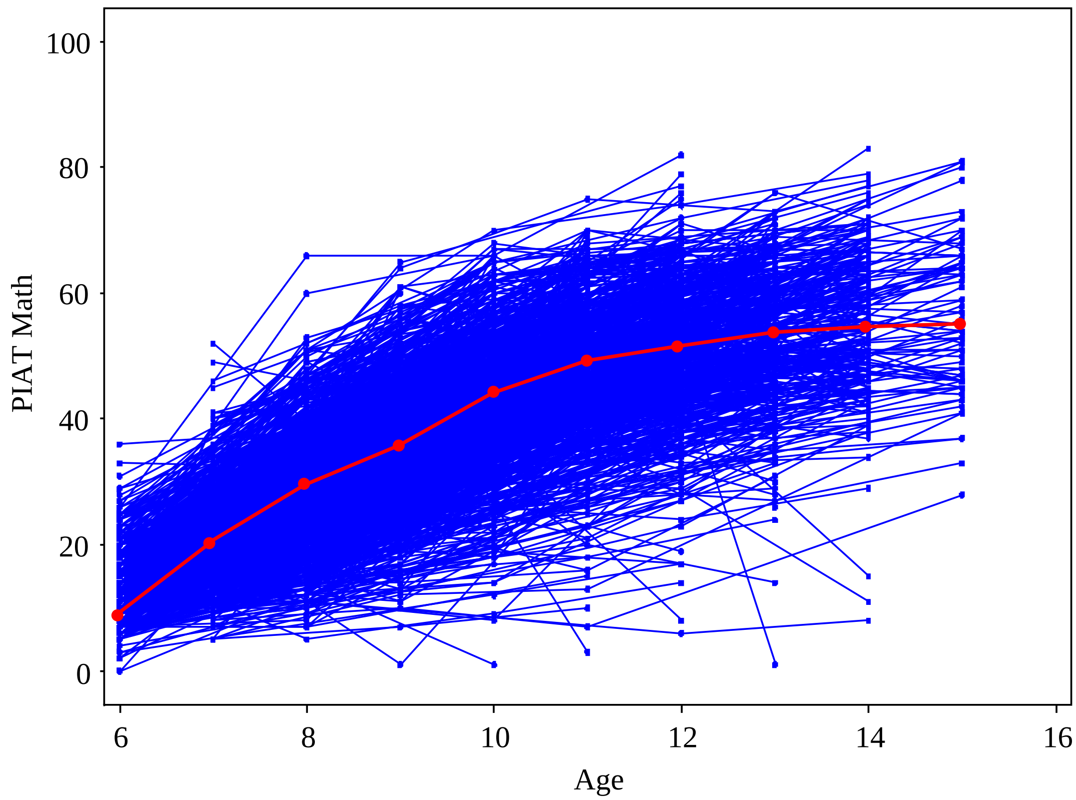

Figure 4.1. The longitudinal plot of PIAT Math. The thick line is the mean trajectory.

4.2 Goals of Analysis

Here, I reiterate the goals of the analysis of the substantive data on children's mathematical performance. For almost a century, the development of mathematical ability has been developed extensively (Bub et al., 2007; Douglas & Kinney, 1938; Grimm, 2005; Hyde et al., 1990). However, much of the research has focused on the level of development of mathematical ability. Thus, the overall goal of this research is to add to the literature by analyzing the rate of growth of mathematical ability. To achieve the goal,

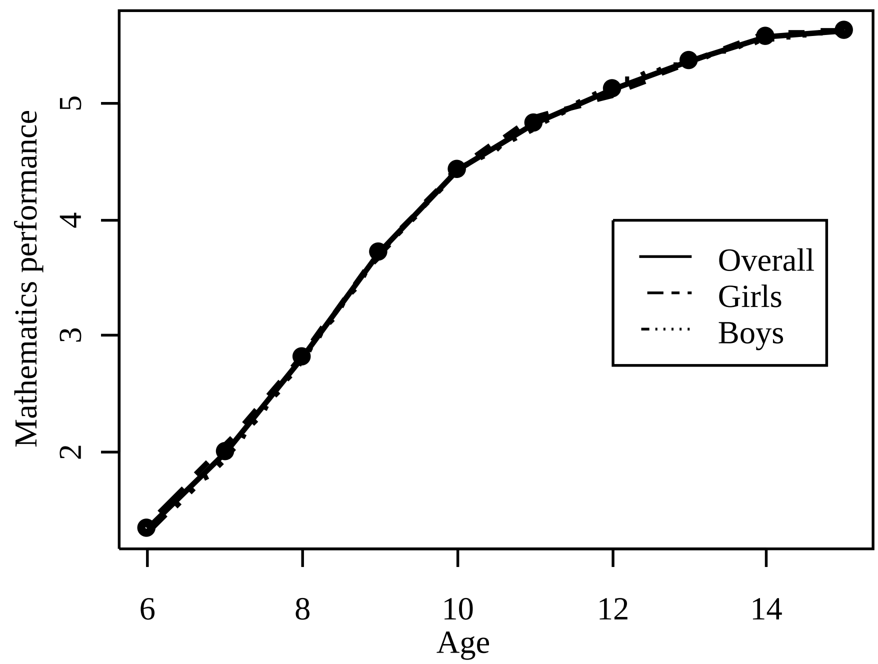

Figure 4.2. The longitudinal plot of PIAT Math for boys' and girls' data.

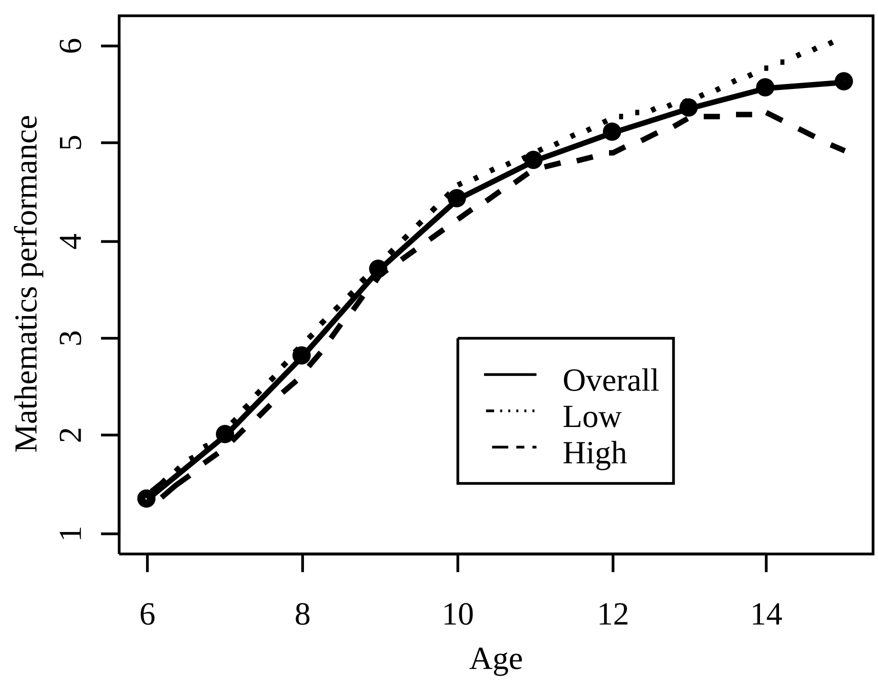

Figure 4.3. The longitudinal plot of PIAT Math for low and high BPI children.

two main questions should be answered. First, are there individual differences in the rate of growth of mathematical ability? Second, how are the individual differences related to the two important covariates of gender and BPI? In the remainder of the chapter, I will answer the two questions by analyzing both the simple rate of growth and the compound rate of growth.

4.3 Analyses of the Simple Rate of Growth of Mathematics Performance

The children's mathematics performance will be first analyzed by using simple growth rate models. To apply simple growth rate models to the data, one needs to determine generally what kind of growth curves the data follow. Here, I first fitted the linear, quadratic, and exponential growth curve models to the data to determine the best fit curve for the data based on the fitting indices. Several fit statistics including the Chi-square, AIC, BIC, and RMSEA are summarized in Table 4.2. Based on BIC and RMSEA, we can see that the quadratic growth curve model fitted the current data best. The parameter estimates from the quadratic growth curve model are presented in Table 4.3.

From the results in Table 4.3, we can draw the following conclusions. First, there were large individual differences in the level of the quadratic growth with a rough 95% confidence interval of (-.296, 1.292). Second, there were also individual differences in the slope and quadratic parameters although they were not as large as those for the level

Table 4.2. *Fit indices for the linear, quadratic, and exponential growth curve models*

	Linear	Quadratic	Exponential
Chi-square	950	**231**	294
d.f. [a]	49	45	48
AIC[b]	10728	**10017**	10074
BIC[b]	10804	**10114**	10156
RMSEA[c]	0.122	**0.058**	0.064
CI for RMSEA[c]	[0.115, 0.129]	[0.051, 0.065]	[0.058, 0.072]

[a] Degrees of freedom.
[b] Akaike information criterion (AIC) and Bayesian information criterion (BIC)
[c] Root mean squares error of approximation (RMSEA) and confidence interval (CI).

parameter. The 95% confidence intervals for the slope and quadratic terms were (.469, 1.577) and (-.084, -.008). Third, the variances of the residual were heteroscedastic with a pattern of general increases and then decreases.

Since the quadratic growth curve best represented the observed data, I can fit a quadratic growth rate model to investigate how the rates of growth change over time and how they are related to the covariates of gender and BPI. The rates of mathematics performance growth are given in Table 4.4. The average rates of growth declined linearly with age and the variability of rates of growth, however, first decreased and then increased. There were individual differences in rates of growth before age 12. To demonstrate the results intuitively, the average growth trajectory and rates of growth are plotted in Figure 4.5.

The covariates – gender and BPI variables – were then used to predict rates of growth of children's mathematics performance. Besides the main effects of gender and BPI, their interaction was also included in the models. The results are summarized in

Table 4.3. *Parameter estimates for the quadratic growth curve model* [a]

	Estimate	s.e.
Mean		
L	0.298	0.026
S	1.023	0.015
Q	-0.046	0.002
Covariance		
L-L	0.257	0.078
L-S	-0.055	0.033
L-Q	0.003	0.003
S-S	0.08	0.017
S-Q	-0.005	0.002
Q-Q	3.73E-04	1.59E-04
e-e		
1	0.042	0.033
2	0.322	0.029
3	0.44	0.031
4	0.5	0.041
5	0.367	0.034
6	0.317	0.037
7	0.297	0.037
8	0.34	0.047
9	0.33	0.068
10	0.219	0.12

[a] s.e.: standard error; L,S,Q: level, slope and quadratic terms in the quadratic model; e-e: variance of residual/measurement errors.

* Significant at alpha level .001.

Table 4.5 and Figure 4.5. The following conclusions can be drawn. First, the coefficients of the covariates for the simple rate of growth were linearly related to age. Second, the standard errors of the estimated coefficients first decreased and then increased. Third, BPI was negatively related to the rates of growth from early age until age 13. Children with a larger BPI tended to grow more slowly on mathematics performance. However, at age 14 and 15, BPI was no longer related to rates of mathematics growth. Fourth, gender

Table 4.4. *Estimated average and variability of rates of growth for mathematics performance*

Age	Mean	Variance
6	0.931 (0.012)	0.061 (0.011)
7	0.838 (0.01)	0.044 (0.007)
8	0.745 (0.007)	0.030 (0.004)
9	0.653 (0.005)	0.020 (0.002)
10	0.56 (0.005)	0.012 (0.001)
11	0.467 (0.007)	0.007 (0.002)
12	0.374 (0.009)	0.005 (0.004)
13	0.282 (0.012)	0.007 (0.008)
14	0.189 (0.015)	0.011 (0.012)
15	0.096 (0.018)	0.018 (0.018)

Note. Numbers in the parentheses are the standard errors.

differences in rates of growth appeared only at age 8 and 9. At those age, boys seemed to show quicker improvement in their performance on mathematics. Fifth, the interaction between gender and BPI evolved at age 8-11. Closer examination revealed that the positive interaction occurred. Thus, for the same increase in BPI, girls show less decrease in simple rates of growth. The MPlus codes for the model estimation are given in Appendix B.2.

Table 4.5. *Results from the growth rate analysis*

Age	Intercept	s.d.	BPI	s.d.	Gender	s.d.	B*G	s.d.
6	0.953*	0.017	-0.051*	0.017	-0.039	0.024	0.032	0.024
7	0.856*	0.013	-0.049*	0.013	-0.034	0.019	0.031	0.019
8	0.76*	0.01	-0.046*	0.009	-0.028*	0.014	0.03*	0.014
9	0.664*	0.007	-0.044*	0.007	-0.022*	0.01	0.029*	0.01
10	0.568*	0.007	-0.042*	0.007	-0.016	0.01	0.028*	0.01
11	0.472*	0.009	-0.039*	0.009	-0.01	0.012	0.027*	0.012
12	0.376*	0.012	-0.037*	0.012	-0.004	0.017	0.026	0.017
13	0.28*	0.016	-0.035*	0.016	0.002	0.022	0.025	0.022
14	0.184*	0.02	-0.032	0.019	0.008	0.028	0.024	0.028
15	0.088*	0.024	-0.03	0.023	0.014	0.034	0.023	0.034

Note. * Significant at alpha level .05.

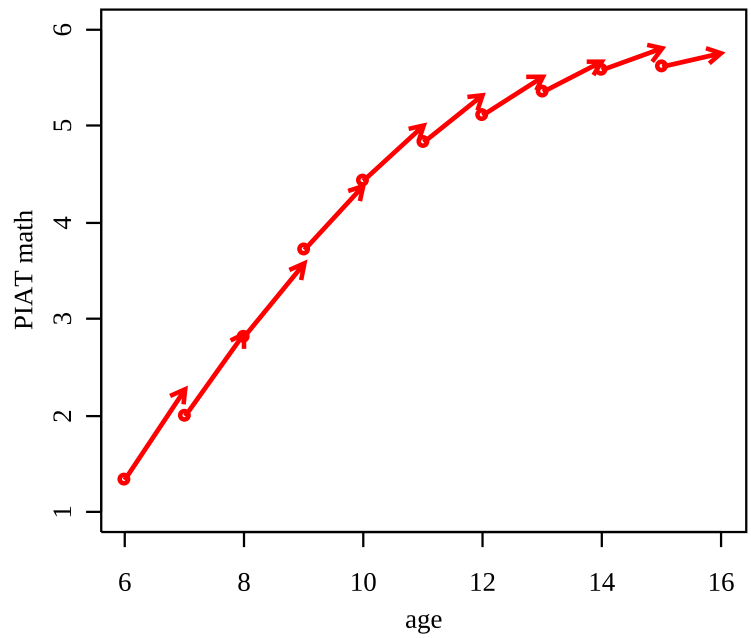

Figure 4.4. The plot of average growth trajectory and rates of growth of mathematics performance.

4.4 Analyzing the Compound Rate of Growth for Mathematics Performance

The relationship between the simple rate of growth and the covariates BPI and gender was analyzed in the previous section. In this section, how BPI and gender are related to the compound rate of growth of mathematics performance will be investigated by using the random coefficient LDSMs.

Three models were fitted to the data. First, the latent difference score model with the fixed compound rate of growth was fitted to the data. Second, the random coefficient LDSM was then employed to see whether there were individual differences in the compound rate of growth. In the third model, the covariance between the compound rate

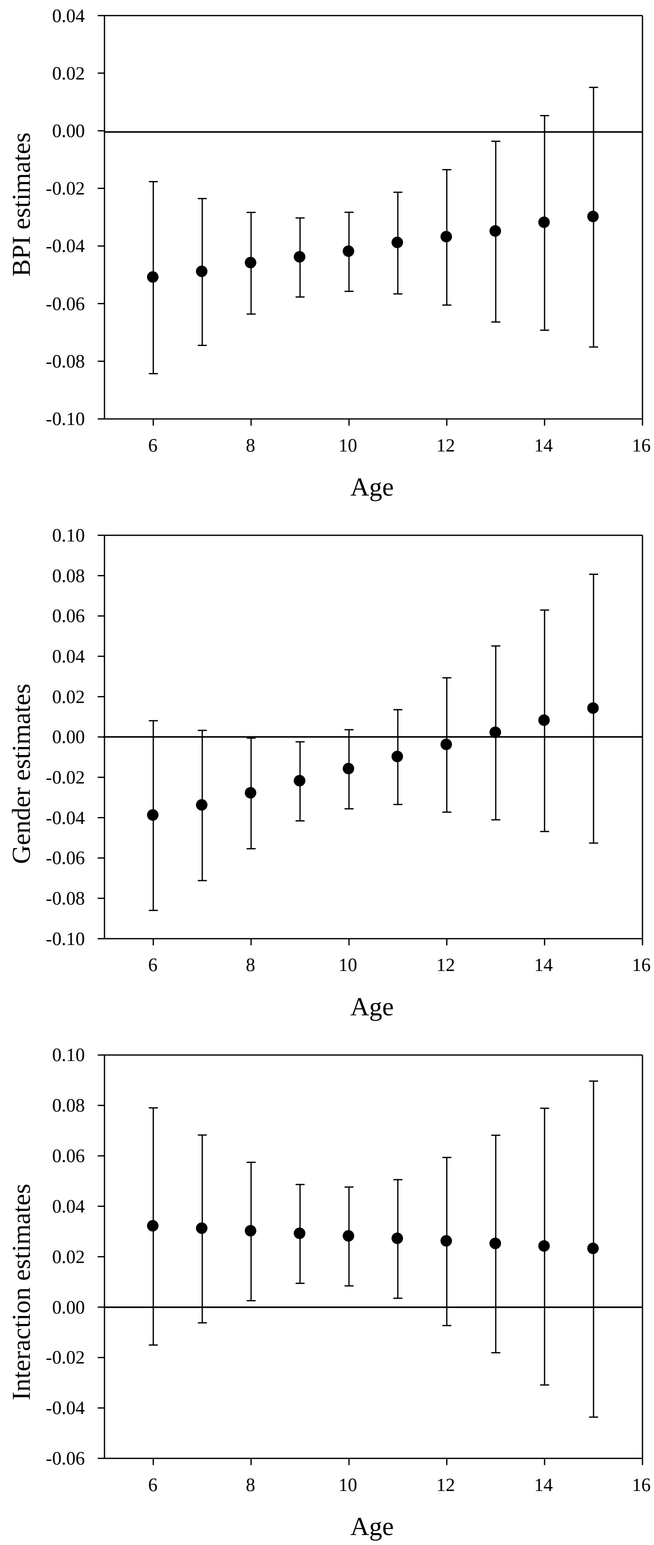

Figure 4.5. The estimated coefficients with error bars for the covariates and their interaction at each age. The error bars are calculated as the 95% confidence intervals.

of growth and the initial level was estimated. To compare the three models, the model fit index DIC was obtained and is given in Table 4.6.

Table 4.6. *DICs for the three LDSM models*

	#p	Dbar	Dhat	pD	DIC
Model 1	13	10238	9278	960	11198
Model 2	14	10160	9158	1005	11169
Model 3	15	10030	8712	1337	11386

Note. #p: number of parameters. Model 1: fixed compound rate of growth. Model 2: random compound rate of growth. Model 3: correlated compound rate of growth and initial level.

From Table 4.6, the more complex models with more parameters fitted the data better based on $Dbar$ and $Dhat$. However, the models also became more complex (see pD). Overall, the second model fitted the data best based on DIC supporting the conclusions that (a) there existed individual differences in the compound rate of growth and (b) the compound rate of growth did not covary with the initial level of children's mathematical performance. The parameter estimates for the second model are summarized in Table 4.7. The average compound rate of growth was about $.171$ and the variance of the rate of growth was about $.0001$.

To further investigate how the individual differences in the compound rate of growth were related to the covariates of BPI and gender, the model with the covariates was fitted. The results of this analysis are given in Table 4.8. The two covariates, gender and BPI, and their interaction term were used to predict the compound rate of growth. From the results, BPI was again found to be negatively related to the rates. However, gender and the interaction between gender and BPI were not found to be related to the compound rate

Table 4.7. *Parameter estimates for the random coefficient LDSM without covariates.*

		estimate	s.e.	2.50%	median	97.50%
mean initial	μ_0	1.987	0.029	1.930	1.988	2.042
variance initial	σ_0^2	0.157	0.010	0.139	0.157	0.178
compound rate	r	0.171	0.003	0.166	0.171	0.177
variance rate	d^2	0.0001	0.0000	0.0001	0.0001	0.0002
residual variance (σ_e^2)	6	0.654	0.054	0.553	0.653	0.765
	7	0.479	0.039	0.408	0.478	0.559
	8	0.523	0.036	0.456	0.522	0.600
	9	0.865	0.068	0.740	0.861	1.006
	10	0.954	0.069	0.825	0.951	1.096
	11	0.660	0.067	0.540	0.656	0.801
	12	0.153	0.043	0.069	0.151	0.238
	13	0.583	0.103	0.404	0.575	0.807
	14	2.086	0.284	1.575	2.070	2.682
	15	4.055	0.840	2.687	3.963	5.973

Note. 2.5% and 97.5%: the upper and lower limits for the confidence interval.

of growth.

4.5 Conclusions from the Empirical Analysis

To summarize, the growth rates of children's mathematical performance were analyzed through both the simple growth rate models and the compound growth rate models. The relationship between the rate of growth and two important covariates, BPI and gender, was also investigated. Now, I will discuss the results by focusing on the substantive questions of individual differences in rates of growth and their relationships to gender and BPI.

The development of mathematical performance was found to be a quadratic growth process. The simple rate of growth was a linear function of age. The individual

Table 4.8. *Results from the random coefficient LDSM with covariates*

		estimate	s.e.	2.50%	medium	97.50%
Mean initial	μ_0	1.986	0.030	1.928	1.985	2.042
Variance initial	σ_0^2	0.150	0.010	0.132	0.150	0.169
Intercept	β_0	0.172	0.003	0.166	0.172	0.179
BPI	β_1	-0.008	0.002	-0.011	-0.008	-0.004
Gender	β_2	-0.002	0.002	-0.006	-0.002	0.003
BPI*Gender	β_3	0.004	0.002	-0.001	0.004	0.008
Rate variance	d^2	0.0001	0.0000	0.0001	0.0001	0.0002
residual variance (σ_e^2)	6	0.646	0.054	0.546	0.644	0.757
	7	0.488	0.040	0.415	0.486	0.571
	8	0.528	0.037	0.460	0.526	0.603
	9	0.866	0.068	0.740	0.863	1.008
	10	0.959	0.069	0.831	0.956	1.105
	11	0.662	0.068	0.537	0.659	0.803
	12	0.165	0.044	0.084	0.163	0.256
	13	0.585	0.106	0.399	0.578	0.813
	14	2.067	0.290	1.553	2.052	2.684
	15	4.091	0.865	2.677	3.997	6.083

Note. After a burn-in iteration of 5000, the generated Markov chain converged. The results are from the WinBUGS program based on 20000 additional iterations. B*G: interaction between BPI and gender.

differences in simple rates of growth were evident based on the analysis. The simple rate of growth was related to BPI and gender in different ways at different ages. For example, BPI was negatively related to the simple rate of growth before age 13. Gender was only related to the simple rate of growth at age 8 and 9. Because the relationship between the simple rate of growth and the covariates was changing over time, it was not sufficient to investigate it at a certain time merely as relationships between covariates and the instantaneous rate of growth. Rather, the rates of growth at each time should be investigated as in the current analysis.

Individual differences in the compound rate of growth for mathematical performance were also found. Compound rate of growth was not found to be related to initial mathematical performance. This is reasonable because the calculation of the compound rate of growth has already taken initial performance into account. Compound rate of growth was negatively related to BPI but bore no relationship to gender. The disappearance of gender effect is probably because the compound rate of growth was analyzed as a time-invariant constant.

5. Simulation Study

The NLSY mathematical performance data were analyzed using both the quadratic growth rate model and the random coefficient LDSM in the previous chapter. To further validate the results from the empirical study and establish conditions where the proposed model performs appropriately, a simulation study was carried out and is reported in this chapter. The simulation study was focused directly on the quadratic growth rate model and the random coefficient latent difference score model which were used in the empirical data analysis. The general purpose was to see how well the estimation methods associated with the two models were able to recover the values of the parameters that defined the simulated data under the various design conditions.

5.1 Simulation Design

Overall, the simulation is designed to investigate whether the results from the substantive research are valid and how the proposed models and estimation methods are affected by some possibly important factors such as sample size and amount of missing data. More specific details of the design of the simulation are first presented for both the quadratic growth rate model and the random coefficient LDSM.

5.1.1 Simulation design for the quadratic growth rate model

For the quadratic growth rate model, the population parameters for the basis model in the simulation are based on the results from the empirical study but with simplification. The model used in the NYSL data can be written as

$$\begin{cases} y_{it} = b_{i1} + r_{ij}t + b_{i3}(t^2 - 2jt) + e_{it} \\ b_{i1} = \beta_{01} + \beta_{11}BPI_i + \beta_{21}GENDER_i + \beta_{31}BPI * GENDER + u_{i1} \\ b_{i3} = \beta_{03} + \beta_{13}BPI_i + \beta_{23}GENDER_i + \beta_{33}BPI * GENDER + u_{i3} \\ r_{ij} = \gamma_{j0} + \gamma_{j1}BPI_i + \gamma_{j2}GENDER_i + \gamma_{j3}BPI * GENDER + v_{ij} \end{cases}, j = 1, \ldots, T. \tag{5.1}$$

The population parameters are given by $\beta_{01} = .3$, $\beta_{11} = \beta_{21} = \beta_{31} = 0$, $\beta_{03} = -.05$, and $\beta_{13} = \beta_{23} = \beta_{33} = 0$. To determine the rate, the parameters at time $t = 1$ are used. Thus, $\gamma_{10} = 1$, $\gamma_{11} = -.05$, $\gamma_{12} = \gamma_{13} = 0$, $cov(u_1, u_3, v) = diag(.25, .0004, .06)$, and $Var(e) = .3$. As in the empirical data, two covariates, one binary variable with mean .5 and one continuous variable following the standard normal distribution, are used in the simulation. This serves as the baseline model in the simulation. The R codes for data generation are given in Appendix B.1.1.

Multiple conditioning factors for the simulation design are considered in the simulation. First, different patterns of missing data are simulated. Note that for the quadratic growth rate model, at least three occasions of data are needed for each participant. In the first case, no missing cases characterized the simulated data. The second missing data pattern included records with only three consecutive observations retained out of the total repeated measures. Note that in this missing pattern, all three occasions are not randomly selected from total occasions but the only first measurement occasion is randomly selected for each subject. The third missing data pattern included observations recorded on four consecutive occasions. The sample size and total number of study occasions were also varied in the simulation. The sample size was set at N=100, 200, 500, and 1,000. The total number of study occasions was set at 5 and 10, respectively. For each condition, 200 replications of data are generated and analyzed for a total of $3 \times 4 \times 2 = 24$ (missingness $\times$ sample size $\times$ number of occasions) conditions.

5.1.2 Simulation design for the random coefficient LDSM

For the random coefficient LDSM, the population parameters were set based on the estimates in Table 4.8. To be precise, $\mu_0 = 2$, $\beta_0 = .15$, $\beta_1 = -.01$, $\beta_2 = 0$, $\sigma_0^2 = .5$, $\sigma^2 = .5$, and $d^2 = .0001$. Three influence factors were considered in the simulation. First, the sample size was set at N=100, 200, 500, and 1,000. Second, the total number of study occasions was set at 5 and 10, respectively. Third, missing data conditions were manipulated. In the first case, all repeated measures were obtained; thus, no missingness is involved in this condition. The second missing data pattern included records with three consecutive observations. The third missing data pattern included recorded on four consecutive occasions. Thus, the simulation design is a $4 \times 2 \times 3$ (sample size $\times$ number of occasions $\times$ missingness) study with 24 cells. In each cell, 200 samples of data were simulated and analyzed. R codes for simulating the data are given in Appendix B.1.2.

5.1.3 Empirical data based simulation

To verify the results from the empirical study, the Monte Carlo study was conducted, simulating data with the same sample size of N=1233 and the same missing data pattern as the empirical data used in the previous chapter. The parameter values were set the same as in the simulation designs for the simple growth rate model and the compound growth rate model previously discussed. A total of 200 replications of data were simulated and the data were analyzed using both the simple quadratic growth rate models and the compound growth rate models (the random coefficient LDSM model).

5.2 Implementation of the simulation

In the real data analysis in the previous chapter, the simple growth rate models were estimated using the maximum likelihood estimation method and the Mplus software. The compound growth rate models were estimated using Bayesian methods and the WinBUGS software. However, because the simulation study involves the analysis of a large number of replications of data, it is neither feasible nor efficient to analyze each replication of data manually. Therefore, an automated procedure was implemented and will be described next.

5.2.1 Simulation implementation for the simple growth rate models

All the simulations for the simple growth rate models are implemented using software R (R Development Core Team, 2005) and Mplus (Muthén & Muthén, 1998-2007). The data are generated in R and Mplus is called within software R to analyze the data and obtain the parameter estimates. A simplified flowchart for this procedure is depicted in Figure 5.1 and the complete R codes are provided in Appendix B.1.1. The Mplus codes for the model estimation are provided in Appendix B.2.

The simulation procedure first sets the constant parameters to control the whole simulation. Specifically, the number of replications (R), the sample size (N), the number of occasions of observed data (T), and the number of non-missing data (M) for each individual are decided at the start of the simulation. For each replication of the simulation, the R function – `quad.gen` – is called to generate a set of data. The data are then analyzed using Mplus by calling Mplus in R using the function of `system()`. Note that although the data are generated based on the simple rate of growth at time 1, the rates of growth at the other time are also estimated in the simulation. The above procedure is repeated with a total of R times. All the results are put together in the file `res-N-T-M.txt`.

The file `res-N-T-M.txt` contains the parameter estimate and the standard error for each parameter and the model fit statistics. The R function `proc.res` is then used to process the results to calculate several summary statistics. First, the mean parameter estimates (ME) and the standard deviations (s.d.) are calculated based on the parameter estimates from all the replications. Second, the mean standard errors (m.s.e.) are calculated. Third, the mean upper and lower limits of the 95% confidence intervals are obtained. Fourth, the coverage probability, which is the percentage of the replications where the 95% confidence interval can cover the true parameter values, is also calculated for each parameter.

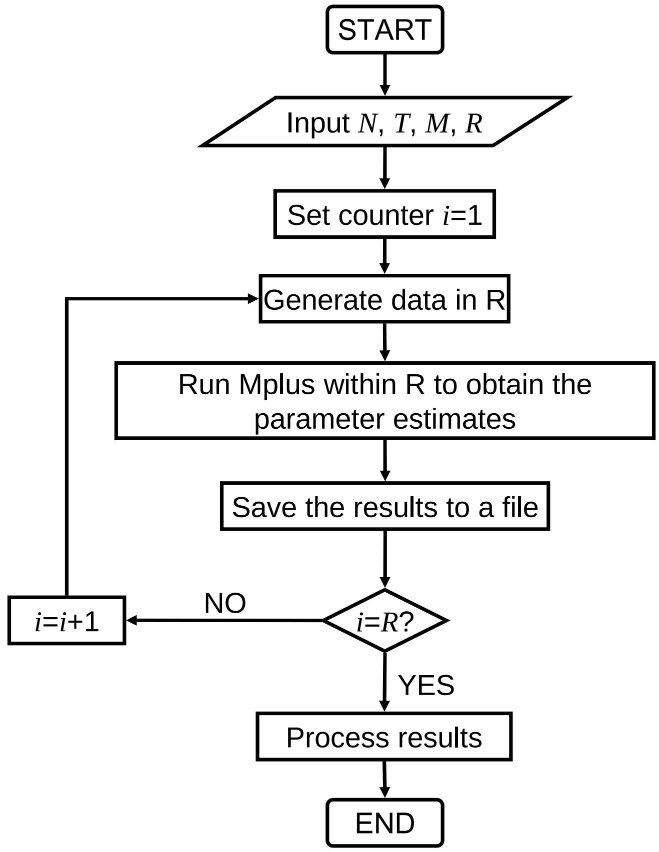

Figure 5.1. The simplified flow chart for running the simulation for the quadratic growth rate models.

5.2.2 Simulation implementation for the compound growth rate models

All the simulations for the random coefficient LDSM models are implemented using R (R Development Core Team, 2005) and JAGS (Just Another Gibbs Sampler; Plummer, 2008). JAGS is software to implement Bayesian estimation. JAGS can be viewed as a Linux version of WinBUGS (Spiegelhalter, Thomas, Best, & Lunn, 2003). JAGS is used here to take the advantage of the high performance cluster available from the Information, Technology and Communication Center at the University of Virginia.

The data are generated in R and JAGS is called to analyze the data and obtain the parameter estimates within R. A simplified flowchart for this procedure is depicted in Figure 5.2 and the complete R codes are provided in Appendix B.1.2. The JAGS or WinBUGS codes for the model estimation are provided in Appendix B.3.

The simulation procedure first sets the constant parameters to control the whole simulation. Specifically, the number of replications (R), the sample size (N), the number of occasions of observed data (T), and the number of non-missing data for each individual (M) are decided before the simulation. For each replication of the simulation, the R function – `ldsm.gen` – is called to generate a set of data. The data are then analyzed using JAGS by calling JAGS in R using the function of `system()`. All the results are put together in the file `res-N-T-M.txt`.

The file `res-N-T-M.txt` contains the parameter estimate, the standard error, the 95% confidence interval, and the convergence diagnostic statistics for each parameter. The R function `proc.res` is then used to process the results to calculate several summary statistics. First, the mean parameter estimates (ME) and the standard deviations (s.d.) are calculated based on the parameter estimates from all the replications. Second, the mean standard errors (m.s.e.) are calculated. Third, the mean upper and lower limits of the 95% confidence intervals are obtained. Fourth, the coverage probability which is the percentage of the replications where the 95% confidence interval can cover the true parameter values is also calculated for each parameter.

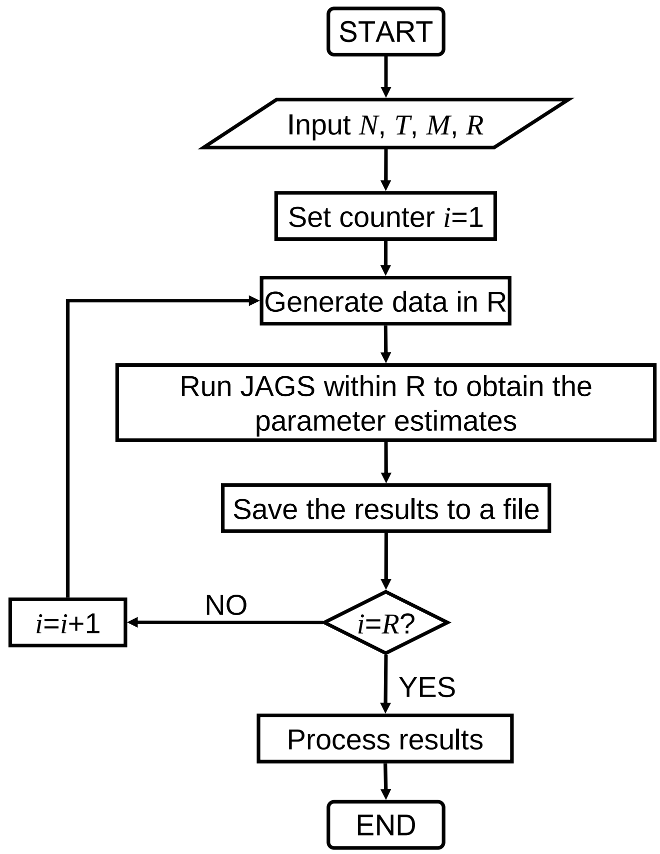

Figure 5.2. The simplified flow chart for running the simulation for the compound growth rate models.

5.3 Simulation Results

The simulation results for the quadratic growth rate models are first presented and then the results for the random coefficient LDSM are presented. Under each condition of the simulation, the average parameter estimates, the standard deviations and mean standard errors, the average upper and lower limits of the 95% confidence intervals, and the coverage probability of the confidence intervals are obtained.

Let $\boldsymbol{\theta}$ denote a vector of all parameters in the model with θ_i representing the ith parameter of a total of P parameters. The average parameter estimates ($\bar{\hat{\theta}}_i$) and standard deviation ($s.d.(\hat{\theta}_i)$) of θ_i can be calculated from the parameter estimates of the simulated data. Thus,

$$\bar{\hat{\theta}}_i = \sum_{j=1}^{R} \hat{\theta}_{ij}/R,$$

and

$$s.d.(\hat{\theta}_i) = \sqrt{\sum_{j=1}^{R} (\hat{\theta}_{ij} - \bar{\hat{\theta}}_i)^2/(R-1)},$$

with $\hat{\theta}_{ij}$ denoting the parameter estimates for the ith parameter in the jth set of simulation data and R denoting the total number of the simulation data.

Let $s.e._{ij}$ represent the standard error for the ith parameter in the jth set of simulation. The average standard error, $a.s.e.$, is calculated by

$$a.s.e_i = \sum_{j=1}^{R} s.e._{ij}/R.$$

Let u_{ij} and l_{ij} be the upper and lower limits of a confidence interval for the ith parameter in the jth set of simulation. The average upper and lower limits of the confidence intervals can be obtained as

$$m.l_i = \sum_{j=1}^{R} l_{ij}/R,$$

and

$$m.u_i = \sum_{j=1}^{R} u_{ij}/R.$$

The coverage probability of the confidence intervals for each parameter is calculated by

$$CP_i = \sum_{j=1}^{R} \text{sgn}[g(\theta_i)]/R,$$

where

$$\text{sgn}[g(\theta_i)] = \begin{cases} 1, \text{ if } u_{ij} > \theta_i \text{ and } l_{ij} < \theta_i \\ 0, \text{ otherwise} \end{cases}$$

with θ_i representing the true parameter value for the ith parameter used to generate the simulation data.

To conveniently compare the simulation results among different conditions, several more general summary statistics are employed. First, the average bias is calculated for all parameters. The bias (b_i) for the ith parameter is

$$b_i = \begin{cases} |(\bar{\hat{\theta}}_i - \theta_i)/\theta_i|, \text{ if } \theta_i \neq 0 \\ |(\bar{\hat{\theta}}_i - \theta_i)|, \text{ if } \theta_i = 0 \end{cases},$$

with θ_i representing the true parameter value for the ith parameter used to generate the simulation data. The average bias is then calculated by

$$\bar{b} = \sum_{i=1}^{P} b_i/P.$$

The smaller the bias $\bar{b}$ is, the less bias the parameter estimates are.

The model accuracy or the mean standard deviation ($m.s.d$) of parameter estimates is calculated as

$$m.s.d = \sum_{i=1}^{P} s.d._i/P.$$

The smaller the standard deviation $m.s.d$ is, the more efficient and accurate the parameter estimates are.

Finally, the average coverage probability $m.CP$ is obtained as

$$m.CP = \sum_{i=1}^{P} CP_i/P.$$

The closer the coverage probability is to the confidence level, the more accurate the confidence interval construction method.

5.3.1 Simulation results for the quadratic growth rate model

Results from the empirical data based simulation

The results for the simulation with the same sample size and the same missing data patterns as the empirical data are given in Table 5.1. The densities for all parameters are plotted in Figure 5.3. The results provide some information on the validity of the parameter estimates from the empirical data. Examining the results, one can reach the following conclusions. First, the distributions of the parameter estimates from all the replications are approximately symmetric. Second, the maximum absolute bias for all parameter estimates is about 0.009 and for most of the parameters, the absolute bias is less than $.002$. Thus, the parameter estimates can be viewed as essentially unbiased. Third, the maximum absolute difference between the standard deviation of the parameter estimates and the mean standard error of the parameter estimates is about 0.0066. Thus, the estimated standard errors for the parameter estimates can be considered to be consistent.

Fourth, for an accurate confidence interval, a 95% confidence interval should have a .95 coverage probability. Note that although the coverage probability for each parameter is not exactly equal to .95, it is very close. Overall, one can conclude that the simulation results accurately reflect the true parameter values making it reasonable to accept the results from the empirical data analysis.

Table 5.1. *Simulation results for the quadratic growth rate model with the same missing data patterns as the NLSY data (R=200)*

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.300	0.0116	0.0111	0.278	0.321	0.945
β_{01}	0.3	0.298	0.0476	0.0495	0.201	0.395	0.965
γ_{10}	1	0.999	0.0195	0.0207	0.959	1.040	0.965
β_{03}	-0.05	-0.050	0.0027	0.0027	-0.055	-0.045	0.96
β_{11}	0	-0.009	0.0430	0.0496	-0.107	0.088	0.975
β_{21}	0	0.002	0.0724	0.0701	-0.135	0.139	0.955
β_{31}	0	0.008	0.0672	0.0701	-0.129	0.145	0.945
γ_{11}	-0.05	-0.048	0.0178	0.0207	-0.089	-0.008	0.985
γ_{12}	0	-0.001	0.0289	0.0292	-0.058	0.057	0.97
γ_{13}	0	-0.001	0.0273	0.0293	-0.058	0.056	0.97
β_{13}	0	0.000	0.0025	0.0027	-0.005	0.005	0.955
β_{23}	0	0.000	0.0041	0.0038	-0.007	0.007	0.915
β_{33}	0	0.000	0.0037	0.0038	-0.008	0.007	0.97
$cov(u_1, u_1)$	0.25	0.249	0.0643	0.0609	0.129	0.368	0.945
$cov(u_1, v_1)$	0	0.000	0.0228	0.0223	-0.044	0.043	0.94
$cov(v_1, v_1)$	0.06	0.060	0.0110	0.0107	0.039	0.081	0.93
$cov(u_1, u_3)$	0	0.000	0.0028	0.0027	-0.005	0.005	0.93
$cov(v_1, u_3)$	0	0.000	0.0012	0.0012	-0.002	0.002	0.935
$cov(u_3, u_3)$	0.0004	0.0004	0.0002	0.0002	0.0001	0.0007	0.935

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Simulation results for the manipulated conditions

A total of 24 conditions with different sample sizes, measurement occasions, and missing data patterns were considered in the simulation. For each condition, $R = 200$

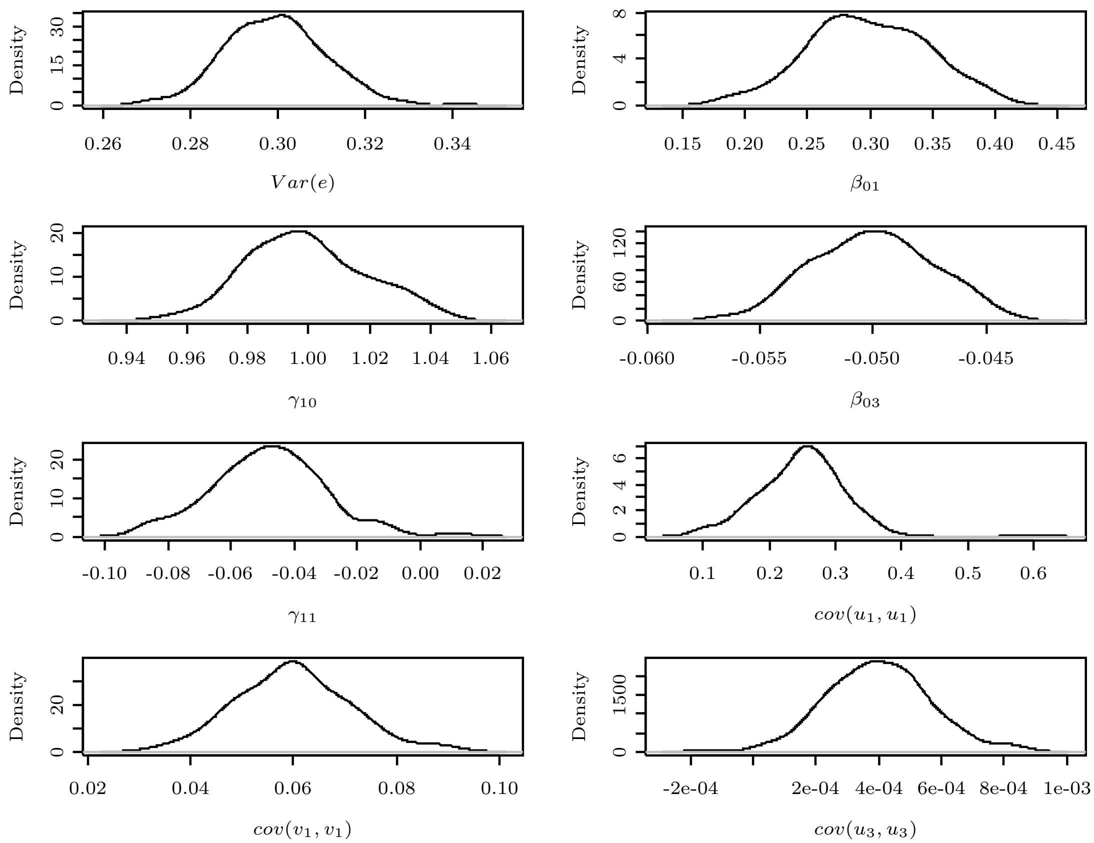

Figure 5.3. The density plot for the non-zero parameters in the quadratic growth rate model.

replications of data were generated and analyzed. The detailed results for each condition are given in the Appendix. To facilitate the discussion and simplify the demonstration, only the general summary statistics including the average bias, the average standard deviation, and the average coverage probability are provided here. These summary statistics are provided in Table 5.2. The number of times that the simulations converged out of 200 replications are also given in the table.

First, not all simulations converged, especially when the data were collected from 10 study occasions.[1] When the total number of occasions was five, most of the simulations

[1] In the simulation, the growth rates at each occasion were estimated. If at any occasion the estimation did

converged even with missing data. But when the total number of study occasions was 10, there were still 10%-15% of simulations that did not converge even without missing data. When only three data points were available among the 10 observation occasions, less than 100 replications of simulation converged, even with a sample size of $N = 500$. As few as 8 replications of the simulation converged with a sample size of $N = 100$. The bias, accuracy, and coverage probability were calculated based on the converged replications.

Table 5.2. *Simulation results for the quadratic growth rate models with manipulated conditions*

		T=5			T=10		
	Sample size	5-dp	4-dp	3-dp	10-dp	4-dp	3-dp
Convergence	100	193	200	161	198	74	8
	200	196	200	186	167	118	30
	500	198	200	200	168	194	85
	1000	198	200	200	178	199	115
Bias	100	0.214	0.231	0.461	0.023	0.076	0.358
	200	0.055	0.058	0.268	0.016	0.012	0.157
	500	0.048	0.083	0.058	0.004	0.009	0.053
	1000	0.004	0.032	0.016	0.002	0.014	0.013
Accuracy	100	0.102	0.149	0.190	0.050	0.139	0.182
	200	0.073	0.106	0.132	0.036	0.092	0.122
	500	0.046	0.067	0.083	0.023	0.058	0.073
	1000	0.033	0.047	0.060	0.016	0.040	0.051
CP	100	0.948	0.934	0.943	0.931	0.971	0.954
	200	0.940	0.926	0.933	0.938	0.960	0.972
	500	0.951	0.940	0.952	0.944	0.951	0.957
	1000	0.957	0.955	0.951	0.952	0.942	0.946

Note. 5-dp: five consecutive data points. 4-dp: four consecutive data points. 3-dp: three consecutive data points. Convergence: the number of times that the simulations had converged. Bias: the average bias. Accuracy: the average standard deviation. CP: the coverage probability.

Second, overall, with more data points and larger sample sizes, the bias of the

not converge, the whole simulation was considered as not convergent even if at all the other occasions the estimations converged.

parameter estimates was smaller. With a sample size of 1000, all biases were less than 4% even with missing data present. When the sample size was 500, the biases for most of conditions were about 5%. When the sample size was 200 and the total number of measurement occasions was 5, the biases for the complete data and missing data with 4 consecutive observations were relatively acceptable. However, with only 3 consecutive observations, the bias was as large as 26.8%. With 10 occasions of measurement, the biases were very small for the conditions of both complete and missing data with 4 consecutive observed data. With a sample size of 100, the biases were very large for the conditions with 5 consecutive observations.

Furthermore, with the same sample size, the results based on 10 occasions of complete data were much less biased than those based on 5 occasions of complete data. Even with the same amount of observed data or missing data, the results for the missing data conditions from a total of 10 study occasions were still much less biased than those from the conditions with a total of 5 occasions of measurement. Not surprisingly, these results indicate that data collected from a longer time span can provide more useful information than data from a shorter time span.

Third, with less missing data and larger sample sizes, the parameter estimates were more accurate with smaller standard deviations. It is again found that the parameter estimates were more accurate for the missing data conditions from a total of 10 measurement occasions than those from a total of 5 measurement occasions.

Fourth, for most of the conditions, the coverage probability of a 95% confidence interval was close to .95. This further supports accepting the parameter estimates and the standard error estimates at face value, at least under simulation design conditions.

Convergence problem and treatment

In the above simulation, it was found that lack of convergence was a problem especially when missing data were present. Two methods were then attempted to see whether or not the convergence rate could be improved. The first method involved increasing the number of iterations for the maximization algorithm. The second method tried was to change the starting values of the unknown parameters.

Because change in the number of iterations and the starting values are difficult to automate, only two sets of small scale simulation are conducted to examine the convergence issues. In the first simulation, complete data with $T = 10$ occasions of observations are simulated for $R = 200$ replications. In the second simulation, missing data with only 3 consecutive observations out of 10 occasions are simulated for $R = 50$ replications. After conducting the simulation using the default number of iterations (1000) and starting values, the number of iterations are first increased to 2000 to implement the simulation again. Then another two sets of starting values, the choice of which was based on the descriptive statistics of the simulated data, are used to implement the simulation.

The results for the simulation are provided in Table 5.3. With the default settings, 164 out of 200 replications of simulation converged for the complete data case and 6 out of 50 replications converged for the missing data case. Increasing the number of iterations from 1000 to 2000 did not improve the convergence. The new sets of starting values did largely improve the convergence. For example, all replications of the simulation converged for the complete data case. For the missing data case, the convergence rate increased almost threefold.

Table 5.3. *Investigation of convergence for the quadratic growth rate models*

	Default	Iterations[a]	Starting values[b]
Complete[c]	164/200	164/200	200/200
Missing[d]	6/50	6/50	23/50

Note. [a]increased iterations from 1000 to 2000, [b]change in starting values, [c]complete data case, [d]missing data case.

5.3.2 Simulation results for the random coefficient LDSM

Results for the empirical data based simulation

Just as for the quadratic growth rate model, the results of the random coefficient LDSM simulation with the same sample size and missing data pattern as in the empirical data are provided in Table 5.4. The densities for the parameters are given in Figure 5.4. First, the maximum absolute bias for all parameter estimates was about 0.002 and for most of the parameters, the absolute bias was less than $.001$. Thus, the parameter estimates can be viewed as unbiased when the non-informative priors were used. Second, the maximum absolute difference between the standard deviation of the parameter estimates and the mean standard error of the parameter estimates was about 0.0013. Thus, the estimated standard errors for the parameter estimates can be considered to be consistent. Third, the coverage probability for each parameter was mostly close to .95. One exception was the coverage probability for d^2 which was much larger than .95. Overall, the simulation results accurately reflected the true parameter values making it reasonable to accept the results from the empirical data analysis.

Table 5.4. *Simulation results for the random coefficient LDSM with the same missing data patterns as the NLSY data*

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
μ_0	2	2.0019	0.0246	0.0247	1.9538	2.0504	0.941
β_0	0.15	0.1498	0.0018	0.0021	0.1457	0.1539	0.956
β_1	-0.001	-0.0099	0.0020	0.0019	-0.0137	-0.0062	0.937
β_2	0	0.0003	0.0027	0.0027	-0.0050	0.0056	0.966
β_3	0	-0.0002	0.0027	0.0027	-0.0055	0.0050	0.941
σ_e^2	0.5	0.5007	0.0127	0.0131	0.4756	0.5270	0.951
σ_0^2	0.5	0.4982	0.0222	0.0235	0.4540	0.5460	0.951
d^2	0.0001	0.0001	0.0000	0.0000	0.0001	0.0002	0.990

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Simulation results for manipulated conditions

As was described earlier, the results for each manipulated condition of the simulation for the random coefficient LDSM were obtained through Bayesian methods. For each condition, $R = 200$ replications of data were generated and analyzed. The detailed results from all 24 conditions discussed previously in the simulation design section are given in the Appendix. Again, only the general summary statistics including the average bias, the average standard deviation, and the average coverage probability are provided here in Table 5.5. The number of times that the simulations converged out of 200 replications is also given in the table.

First, not all simulations converged, especially when there are 5 occasions of data.[2] Actually, only about 15% – 25% of the simulations converged with missing data when the

[2]The convergence was monitored on each parameter of the model by the Geweke statistics (Geweke, 1992). The Geweke statistics were calculated based on 30,000 iterations with the first 10,000 iterations discarded. If there was one or more parameters that did not converge, the overall model was viewed as not convergent. It should be expected that with more iterations the converge rate would become higher.

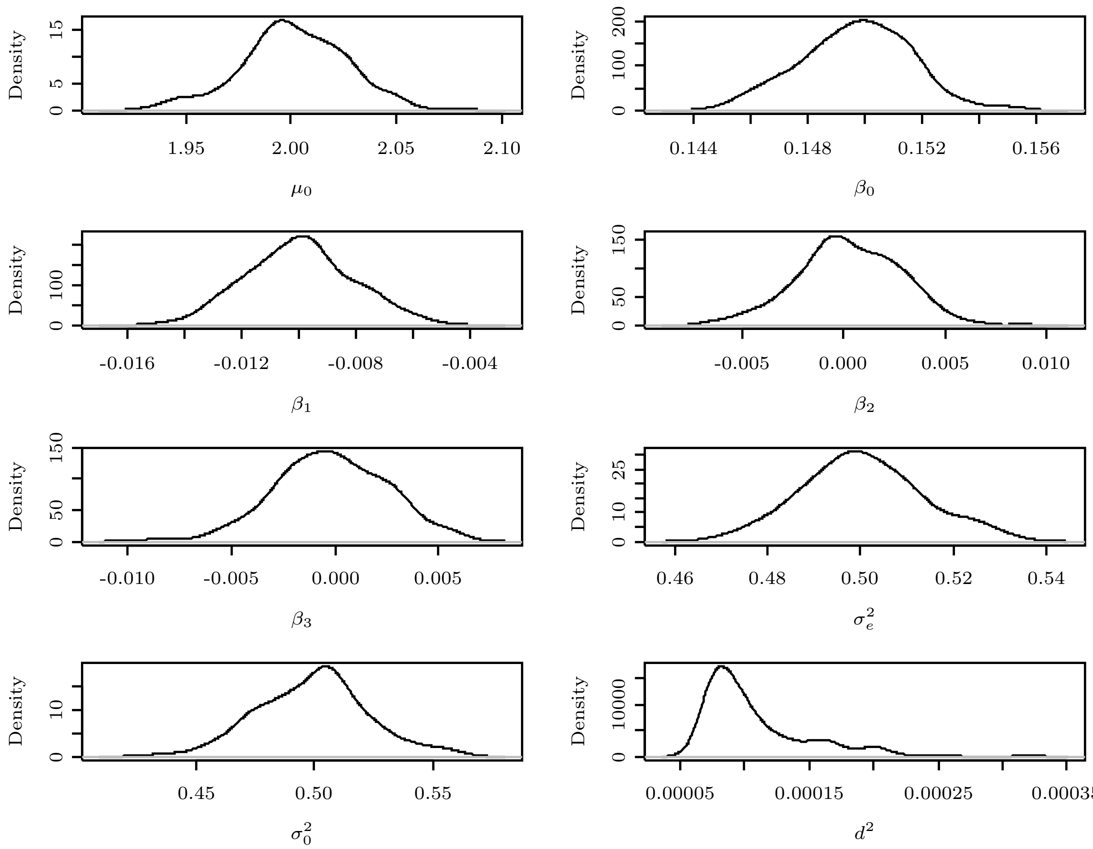

Figure 5.4. The density plot for the parameters in the random coefficient LDSM.

total number of occasions was 5. Even when the total number of occasions was ten, only about half of the simulations converged when missing data were present. The bias, accuracy, and coverage probability are calculated based on the converged replications of the simulations.

Second, overall, with more data points and larger sample sizes, the bias of the parameter estimates was smaller. For example, with the sample size of 1000, the number of occasions of 10 and without missing data, the average bias was about .6%. When the the number of occasions was 5, in only one condition with the sample size of 1000 and no missing data, the bias was less than 5%. The bias appears mainly attributable to the

Table 5.5. *Simulation results for the random coefficient LDSM with manipulated conditions*

		T=5			T=10		
	Sample size	5-dp	3-dp	2-dp	10-dp	3-dp	2-dp
Convergence	100	131	44	30	160	123	101
	200	135	59	33	171	128	105
	500	104	45	35	164	107	107
	1000	95	33	25	166	100	100
Bias	100	0.142	0.237	0.237	0.019	0.121	0.137
	200	0.103	0.163	0.303	0.004	0.081	0.092
	500	0.056	0.219	0.330	0.006	0.043	0.070
	1000	0.046	0.082	0.101	0.006	0.023	0.031
Accuracy	100	0.033	0.045	0.045	0.026	0.043	0.047
	200	0.023	0.030	0.037	0.016	0.028	0.032
	500	0.015	0.021	0.025	0.011	0.016	0.020
	1000	0.009	0.014	0.019	0.008	0.012	0.014
CP	100	0.952	0.935	0.935	0.943	0.945	0.942
	200	0.954	0.953	0.928	0.967	0.967	0.954
	500	0.946	0.942	0.911	0.957	0.956	0.968
	1000	0.960	0.939	0.945	0.953	0.960	0.951

Note. 5-dp: five consecutive data points. 3-dp: three consecutive data points. 2-dp: two consecutive data points. Convergence: the number of times that the simulations had converged. Bias: the average bias. Accuracy: the average standard deviation. CP: the coverage probability.

estimates of d^2. For data from a total of 10 study occasions, the results largely improved even with missing data. For example, with the sample size of 500 and 3 consecutive observations, the bias was only about 4.3%. Once again, it is found that even with the same amount of available data, the results from those data collected over 10 measurement occasions were better than those from the data collected over only 5 measurement occasions.

Third, with less missing data and larger sample sizes, the parameter estimates were more accurate as evidenced by smaller standard deviations. It can also be seen that the parameter estimates were more accurate for the missing data conditions with 10 measurement occasions than with 5 measurement occasions, although the same amount of observed data were available. Actually, in each missing data condition, only 2 or 3 consecutive observations were available.

Fourth, for some of the conditions with large sample sizes (≤ 500) and 10 measurement occasions, the coverage probability of a 95% confidence interval was close to .95. Depending on the conditions, the coverage probability can be overestimated or underestimated.

Overall, when estimating the random coefficient LDSM, one should pay attention to the convergence problem. To ensure better results, a larger sample size and a longer measurement span are recommended. Alternatively, one can increase the burn-in data points to ensure the convergence of the model.

Convergence problem and treatment

As was the case for the quadratic growth rate model, lack of convergence is also a problem in parameter estimation for the random coefficient LDSM. In general, convergence is still a problem deserving much investigation for Bayesian analysis (Brooks & Roberts, 1998; Cowles & Carlin, 1996). Convergence sometimes can be achieved by increasing the number of burn-in iterations (Geman & Geman, 1984; Raftery & Lewis, 1992). How the number of burn-in iterations is related to convergence is investigated briefly here.

To examine the convergence problems more fully, the following simulation is implemented. Data with $N = 200$ individuals and $T = 5$ occasions of observations are generated. Data with missing elements are also generated keeping only either 2 or 3 consecutive occasions of observation. These data are then analyzed using three different numbers of burn-in iterations, 20,000, 30,000, and 50,000. This process is then repeated for $R = 200$ times. The number of convergent replications resulting under the conditions just described is given in Table 5.6.

Table 5.6. *Number of convergent iterations for the random coefficient LDSM with different burn-ins*

	20000	30000	50000
2-dp	30 (22)	35 (28)	41 (38)
3-dp	47 (31)	51 (38)	59 (48)
Complete data	99 (38)	113 (48)	121 (67)

Note. The results are based on $R = 200$ replications. 3-dp: three consecutive data points. 2-dp: two consecutive data points. The numbers in the parentheses are the time in minutes for one replication of simulation.

Overall, with a larger number of burn-in iterations, the random coefficient LDSM is more likely to converge. For example, for the case of complete data, there are 99 out of 200 replications of simulation that converged with a burn-in of 20,000. The number of convergent replications increased to 113 with a burn-in of 30,000 and then to 122 with a burn-in of 50,000.

With an even larger number of burn-in iterations, a higher convergent rate can be expected. Certainly, this is at the cost of computational resources as demonstrated by the computation time in Table 5.6. There is no automatic way to decide how large a burn-in number is sufficient to achieve convergence. However, in real data analysis that does not require repetitions as in the simulation study, one can gradually increase the burn-in iterations to monitor and exert some control over convergence.

5.4 Summary of the Simulation Study

Simulation studies were carried out in this chapter to investigate the validity of the results from the empirical data analysis and the performance of the simple growth rate models and the compound growth rate models under a variety of conditions. The simulation with the data taking on the same structure as the empirical data showed that the results for the analysis of children's mathematics performance can be trusted. Based on the manipulated simulation conditions, one can conclude the following. (a) The convergence of both models and estimation methods should be given careful attention. The convergence of the quadratic growth rate model could be obtained by using better starting values. The convergence could be achieved simply increase the number of burn-in iterations for the random coefficient LDSM. (b) The parameter estimation can be viewed as unbiased and the standard error estimation can be viewed as consistent. (c) Data collected over a longer time span appear to provide more information than those collected over a shorter time span.

6. Summary and Discussion

6.1 Summary

In order to understand growth processes more completely, in this dissertation I have proposed investigating not only the level of growth but also the rate of growth. Although it is not the first attempt to emphasize analysis of the rate of growth (e.g. Boker & Nesselroade, 2002; Freeman & Flory, 1937; Kaplan, 2002), to my knowledge this dissertation is one of few attempts to systematically define and analyze rates of growth within the framework of longitudinal data analysis.

First, two kinds of growth rates were distinguished – the simple rate of growth and the compound rate of growth. Their main difference lies in the emphasis given to the initial status of a growth process. In the simple rate of growth, initial status is ignored whereas it is included in calculating compound rates of growth includes. Awareness of the distinction is important in selecting methods and techniques for analyzing and modeling growth processes. Three rate of growth estimation methods – parametric, semi-parametric, and nonparametric – were presented. These three methods represented a wide range of the possible methods available for estimating rates of growth.

Second, a general set of models was derived from growth curve models to represent and analyze the simple rate of growth. In simple growth rate models, the rate of growth was defined as the first derivative of the growth function. It was shown that simple growth rate models can be viewed as the results of certain rotations of growth curve models. Simple growth rate models can be estimated using SEM methods provided by software such as Mplus.

Third, several approaches for modeling compound rates of growth were discussed. Particularly, the random coefficient latent different score model was derived from the widely used latent different score models to allow for individual differences in compound rates of growth. The models can be estimated using Bayesian methods and implemented in software such as WinBUGS.

Fourth, both the simple growth rate models and the random coefficient LDSM were used to analyze the simple and compound rates of growth of children's mathematical performance data from the NLSY. The results showed individual differences in both the simple and compound rates of growth. It was also found that the covariates BPI and gender showed different relationships with the simple rates of growth at different ages. Only BPI was found to be related to the compound rate of growth in that children with a lower BPI had a larger compound rate of growth.

Fifth, simulations were used to validate the results from the analysis of the NLSY data and to investigate the performance of the proposed models, mainly the quadratic growth rate models and the random coefficient LDSM, under a variety of conditions. It was found that for both models, the results from the empirical data analysis were given additional credence by the simulation outcomes. It was further found that the parameter estimates for both models can be viewed as unbiased and the standard error estimates were consistent. For both models and their estimation methods, convergence problems are substantial and should be paid careful attention.

In summary, a set of growth rate models for both the simple rate of growth and the compound rate of growth were presented. The applications and merits of such models were demonstrated through the analysis of the NLSY data and the performances of the models were evaluated through the systematical simulation. These models can provide useful information above and beyond the analysis of the level of growth.

6.2 Implications

The findings reported in this dissertation study have several key implications from both methodological and substantive perspectives. Those implications will now be discussed in more detail.

6.2.1 Methodological implications

For those interested in understanding growth processes, the analysis of only the level of growth but not the rate of growth is not sufficient because individuals with the same levels of growth can have very different rates of growth (Cattell, 1966a; Freeman & Flory, 1937). However, the techniques for modeling rates of growth have not been much investigated compared to the widely available growth curve models for the level of growth. As the first attempt that systematically investigated models and methods for rates of growth, the current work can help lead the way to more thorough, informative changes of growth data.

It was shown that rates of growth can be constructed through different methods. The estimation of rates of growth not only is the foundation of the growth rate modeling in the current study but also provides useful methods that can be applied to broader areas such as dynamical systems analysis and functional data analysis (Boker & Nesselroade, 2002; Boker et al., 2004; Ramsay & Silverman, 2005). Thus, this study support broadening the applications and impact of growth rate modeling by both illustrating procedural aspects and supporting their validity.

Simple growth rate models can be constructed from certain rotations of the well developed growth curve models. This implies several important and useful features of simple growth rate models. (a) Simple growth rate models can be versatile. Essentially, for any growth curve model, its corresponding growth rate model can be constructed through rotation. (b) Simple growth rate models inherit the model fit statistics from their ancestral growth curve models. Thus, the model comparison methods for growth rate models are readily available. In other words, the best fitting growth rate models can be selected based on comparisons of their corresponding growth curve models. (c) Growth curve models can be viewed as special cases of growth rate models. For example, the quadratic growth curve model can be viewed as a quadratic growth rate model at time 0. For any nonlinear growth processes, the analysis of growth rates can provide information above and beyond that of growth curve models. (d) Simple growth rate models can be estimated through available methods and software. The interpretation of the models is readily meaningful in terms of rates of growth. Both of these make growth rate models practically feasible and useful.

The random coefficient LDSM is a natural extension of the LDSM. The new model not only inherits the merits of LDSM such as the direct analysis of underlying true difference scores but also allows the analysis of individual differences for compound rates of growth. The estimation of random parameters has been somewhat difficult in case of nonlinear models and modeling methods (Browne, 1993; McArdle & Hamagami, 1996; Neale & McArdle, 2000). The Bayesian estimation methods for the random coefficient LDSM in the current study can be applied to other models involving random parameters.

Finally, growth rate models were applied to children's mathematics performance data to answer substantive questions on individual differences in rates of growth and their relationships with gender and BPI. It seems worth emphasizing that the accompanying simulation study demonstrated that growth rate models can perform well under many different situations. All of these outcomes reinforce the promise and the practicality of the methodology of growth rate models for further enhancing substantive research.

6.2.2 Substantive implications

The current study also has some important substantive implications. The growth of mathematical performance was a nonlinear process, specifically, a quadratic growth process. The simple growth rate of mathematical performance was declining with increasing age. These results are consistent with previous research (Grimm, 2005; Kowalski-Jones & Duncan, 1999), but they also reinforce the additional richness of findings possible when the application of growth curve models is extended to include analysis of rates of growth enabling one to capture more dynamic change information.

Males have been shown to outperform females in school performance of mathematics (Kimball, 1989; Marsh & Yeung, 1998). However, the current study uncovered some subtleties that warrant further discussion. Gender was found to be related to the simple rate of mathematical performance only at age 8 and 9 across the age span 6 to 15 years. It appears that there is a narrow “window” during which boys separate themselves from girls because of significantly greater rates of growth in mathematical performance. However, because the rates of growth do not differ after age 9, even though they keep improving, the girls never catch up to the boys and the differences established at that early age tend to persist. This main effect of gender needs to be tempered, however, by a significant interaction between gender and BPI which will be described next.

There are positive interaction effects between gender and BPI from age 8 to age 11. For both boys and girls, higher BPI corresponds to lower simple rates of growth of mathematics performance. For the same increase in BPI, girls show less decrease in simple rates of growth. After a certain threshold of BPI, girls could show higher rates of growth in mathematics performance. However, the current observation of BPI did not exceed the threshold.

It was generally found that BPI was negatively related to rates of mathematical performance. This is consistent with the general conclusion that children with higher levels of behavior problems tended to perform less well in school (Arnold, 1997; Arnold et al., 1999; McClelland et al., 2000). Based on the current results, children performed worse seemingly in part because of the slower rates of improvement.

6.3 Limitations

Several factors limit the general conclusions that can be drawn from the present study. Substantively, it is a correlational, rather than an experimental study so conclusions regarding the direction of effects cannot be unequivocally drawn. Thus, the results from the empirical study only reflect correlational inferences instead of causal ones. In other words, although BPI was negatively related to rates of mathematical performance growth, one cannot determine whether or not higher BPI caused lower rates of growth only based on currently available data. At some point, controlled experimental designs must be used to answer such questions. It is also found that the relationships between gender and BPI and simple rates of growth were changing with age. This intriguing phenomenon needs to be explained more broadly in additional data sets.

Methodologically, it was assumed for the random coefficient LDSM that the individual compound rate of growth was a constant. However, it is possible that compound rates of growth are time-varying but the current data did not allow the analysis of time-varying compound rates of growth.

Finally, there are also limitations with regard to the simulation study. First, the high computation demand of Bayesian methods constrained the total number of iterations to be 30,000 for the random coefficient LDSM. This partly resulted in the low convergence rate of the simulations. Second, the current simulation focused on only parameter estimates. When computation power is not so limited, the simulation on the model comparisons, especially for the compound growth rate models, will be an important extension. Third, the simulation results are based on the converged replications. The results should be interpreted with cautions.

6.4 Future directions

It was proposed that analyzing both the level of growth and the rate of growth will provide a better understanding of growth processes in longitudinal data. Conceptually, this idea of the analysis of the rate of growth can be extended to any research related to change. For example, in cluster analysis, mixture models etc., one can investigate the classification of participants based on the rate of growth as well as level of growth. Here, I will simply mention several extensions of the current work that I plan to carry out in the near future.

First, the investigation of simple growth rate models has been focused on the quadratic growth rate model. The other models, such as the exponential growth rate model, will be investigated in more detail. Furthermore, the covariates used in the current study were time-invariant. The inclusion of time-varying covariates will be examined in the future. Finally, the growth rate models discussed so far are all univariate. Multivariate models that represent the relationships among rates of growth across different domains will also be examined in a later time.

Second, the random coefficient latent different score model used in the current study was an extension of a simple latent different score model. The idea of random coefficients can be applied to more complex latent different score models proposed by McArdle and colleagues (e.g., McArdle & Hamagami, 2001; ?, ?). A future study will be conducted to investigate how to apply and estimate more complex random coefficient latent difference score models.

Third, Browne (1993) has proposed a Taylor approximation method for estimating nonlinear growth curve models with random parameters. Alternatively, the Bayesian estimation method used for random coefficient latent difference score models in the current study can be readily applied to estimate such nonlinear growth curve models. In a future study, I will compare the Bayesian method and approximation method for random coefficient growth curve models to determine the relative advantages and disadvantages of the two approaches.

Fourth, based on the simulation from the two specific models, the quadratic growth rate model and the random coefficient latent difference score model, it was found that even with the same amount of data, the data collected from a longer span of time can actually provide more information than those collected from a shorter span of time. This result, although not surprising, has very important implications for the design and collection of longitudinal data. A systematical study will be carried out to investigate the mechanics of the influence of missing data in this situation in order to propose ways to gather more information with limited resources.

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A. Simulation Results

A.1 Simulation results for the quadratic growth rate models

Table A.1. $N = 100$, $T = 10$, $M = 10$, $R = 198$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.299	0.018	0.016	0.268	0.331	0.924
β_{01}	0.3	0.310	0.113	0.113	0.087	0.532	0.955
γ_{10}	1	0.995	0.050	0.047	0.904	1.087	0.899
β_{03}	-0.05	-0.050	0.004	0.004	-0.058	-0.041	0.949
β_{11}	0	0.004	0.116	0.115	-0.221	0.228	0.955
β_{21}	0	-0.017	0.157	0.160	-0.331	0.297	0.970
β_{31}	0	0.001	0.164	0.162	-0.316	0.319	0.955
γ_{11}	-0.05	-0.051	0.052	0.047	-0.144	0.041	0.914
γ_{12}	0	0.007	0.072	0.066	-0.122	0.137	0.914
γ_{13}	0	0.001	0.072	0.067	-0.130	0.131	0.939
β_{13}	0	0.001	0.004	0.004	-0.008	0.009	0.960
β_{23}	0	-0.001	0.006	0.006	-0.013	0.012	0.949
β_{33}	0	-0.001	0.006	0.006	-0.013	0.011	0.934
$cov(u_1, u_1)$	0.25	0.211	0.084	0.091	0.032	0.390	0.909
$cov(u_1, v_1)$	0	0.007	0.031	0.030	-0.051	0.065	0.914
$cov(v_1, v_1)$	0.06	0.057	0.016	0.015	0.027	0.087	0.899
$cov(u_1, u_3)$	0	-0.001	0.003	0.003	-0.006	0.004	0.919
$cov(v_1, u_3)$	0	0.000	0.001	0.001	-0.002	0.002	0.929
$cov(u_1, u_3)$	0.0004	0.0004	0.0001	0.0001	0.0001	0.0006	0.899

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.2. $N = 100$, $T = 10$, $M = 4$, $R = 74$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.301	0.027	0.030	0.242	0.360	0.959
β_{01}	0.3	0.304	0.230	0.282	-0.249	0.858	0.986
γ_{10}	1	0.990	0.076	0.094	0.806	1.174	1.000
β_{03}	-0.05	-0.049	0.009	0.011	-0.070	-0.028	0.973
β_{11}	0	-0.072	0.341	0.299	-0.657	0.514	0.932
β_{21}	0	-0.041	0.410	0.403	-0.830	0.749	0.986
β_{31}	0	0.024	0.460	0.435	-0.828	0.876	0.932
γ_{11}	-0.05	-0.027	0.104	0.098	-0.219	0.166	0.932
γ_{12}	0	0.014	0.124	0.133	-0.247	0.276	0.973
γ_{13}	0	-0.007	0.150	0.142	-0.285	0.271	0.919
β_{13}	0	-0.002	0.011	0.011	-0.024	0.020	0.959
β_{23}	0	-0.002	0.016	0.015	-0.032	0.028	0.959
β_{33}	0	-0.001	0.015	0.016	-0.032	0.031	0.946
$cov(u_1, u_1)$	0.25	0.295	0.365	0.426	-0.539	1.129	0.986
$cov(u_1, v_1)$	0	-0.029	0.121	0.155	-0.332	0.274	1.000
$cov(v_1, v_1)$	0.06	0.070	0.047	0.059	-0.046	0.187	1.000
$cov(u_1, u_3)$	0	0.006	0.014	0.025	-0.043	0.054	1.000
$cov(v_1, u_3)$	0	-0.002	0.005	0.007	-0.016	0.013	1.000
$cov(u_1, u_3)$	0.0004	0.0005	0.0006	0.0009	-0.0012	0.0023	1.000

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.3. $N = 100$, $T = 10$, $M = 3$, $R = 8$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.297	0.036	0.042	0.215	0.379	1.000
β_{01}	0.3	0.160	0.442	0.356	-0.539	0.859	0.875
γ_{10}	1	1.029	0.122	0.122	0.790	1.267	0.875
β_{03}	-0.05	-0.050	0.013	0.015	-0.079	-0.021	1.000
β_{11}	0	0.151	0.293	0.354	-0.543	0.845	0.875
β_{21}	0	-0.066	0.458	0.503	-1.052	0.921	1.000
β_{31}	0	-0.169	0.487	0.507	-1.162	0.824	0.875
γ_{11}	-0.05	-0.110	0.131	0.120	-0.345	0.126	0.875
γ_{12}	0	0.048	0.113	0.172	-0.290	0.385	1.000
γ_{13}	0	0.086	0.165	0.175	-0.256	0.428	0.875
β_{13}	0	0.005	0.011	0.015	-0.024	0.034	1.000
β_{23}	0	-0.005	0.014	0.021	-0.045	0.036	1.000
β_{33}	0	-0.014	0.015	0.022	-0.056	0.029	0.875
$cov(u_1, u_1)$	0.25	0.446	0.613	0.625	-0.779	1.671	1.000
$cov(u_1, v_1)$	0	-0.105	0.245	0.246	-0.587	0.376	1.000
$cov(v_1, v_1)$	0.06	0.115	0.089	0.098	-0.077	0.307	1.000
$cov(u_1, u_3)$	0	0.017	0.028	0.045	-0.071	0.105	1.000
$cov(v_1, u_3)$	0	-0.008	0.008	0.013	-0.034	0.018	1.000
$cov(u_1, u_3)$	0.0004	0.0015	0.0008	0.0016	-0.0017	0.0047	1.000

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.4. $N = 100$, $T = 5$, $M = 5$, $R = 193$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.303	0.029	0.030	0.243	0.362	0.959
β_{01}	0.3	0.298	0.184	0.178	-0.051	0.646	0.938
γ_{10}	1	1.000	0.092	0.092	0.820	1.180	0.969
β_{03}	-0.05	-0.050	0.021	0.021	-0.090	-0.010	0.948
β_{11}	0	-0.015	0.175	0.178	-0.363	0.333	0.964
β_{21}	0	-0.008	0.243	0.253	-0.504	0.488	0.974
β_{31}	0	0.023	0.254	0.254	-0.476	0.522	0.948
γ_{11}	-0.05	-0.044	0.089	0.092	-0.224	0.135	0.943
γ_{12}	0	0.008	0.127	0.131	-0.248	0.264	0.959
γ_{13}	0	-0.017	0.127	0.131	-0.275	0.240	0.938
β_{13}	0	-0.001	0.020	0.021	-0.042	0.039	0.959
β_{23}	0	-0.002	0.030	0.029	-0.060	0.055	0.938
β_{33}	0	0.004	0.029	0.029	-0.054	0.062	0.948
$cov(u_1, u_1)$	0.25	0.160	0.265	0.261	-0.351	0.671	0.912
$cov(u_1, v_1)$	0	0.041	0.122	0.125	-0.204	0.286	0.927
$cov(v_1, v_1)$	0.06	0.037	0.067	0.070	-0.099	0.174	0.943
$cov(u_1, u_3)$	0	-0.009	0.027	0.028	-0.063	0.045	0.933
$cov(v_1, u_3)$	0	0.005	0.014	0.015	-0.025	0.034	0.959
$cov(u_1, u_3)$	0.0004	-0.0008	0.0033	0.0037	-0.0080	0.0063	0.964

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.5. $N = 100$, $T = 5$, $M = 4$, $R = 200$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.303	0.032	0.033	0.237	0.368	0.960
β_{01}	0.3	0.295	0.268	0.244	-0.183	0.773	0.920
γ_{10}	1	1.005	0.137	0.120	0.770	1.241	0.915
β_{03}	-0.05	-0.051	0.032	0.028	-0.107	0.004	0.910
β_{11}	0	-0.006	0.262	0.252	-0.499	0.487	0.940
β_{21}	0	-0.001	0.377	0.346	-0.679	0.676	0.915
β_{31}	0	-0.023	0.385	0.357	-0.723	0.678	0.935
γ_{11}	-0.05	-0.046	0.130	0.124	-0.289	0.196	0.940
γ_{12}	0	-0.002	0.189	0.171	-0.336	0.332	0.925
γ_{13}	0	0.011	0.181	0.176	-0.333	0.356	0.960
β_{13}	0	-0.001	0.030	0.029	-0.059	0.056	0.945
β_{23}	0	0.000	0.044	0.040	-0.079	0.078	0.930
β_{33}	0	-0.001	0.043	0.041	-0.082	0.081	0.955
$cov(u_1, u_1)$	0.25	0.134	0.454	0.447	-0.742	1.009	0.935
$cov(u_1, v_1)$	0	0.048	0.218	0.220	-0.383	0.480	0.930
$cov(v_1, v_1)$	0.06	0.035	0.114	0.117	-0.196	0.265	0.930
$cov(u_1, u_3)$	0	-0.010	0.057	0.056	-0.119	0.100	0.935
$cov(v_1, u_3)$	0	0.005	0.028	0.029	-0.051	0.061	0.930
$cov(u_1, u_3)$	0.0004	-0.0009	0.0071	0.0070	-0.0147	0.0129	0.945

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.6. $N = 100$, $T = 10$, $M = 3$, $R = 161$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.296	0.044	0.042	0.214	0.379	0.925
β_{01}	0.3	0.287	0.316	0.297	-0.295	0.869	0.925
γ_{10}	1	1.007	0.150	0.142	0.728	1.286	0.938
β_{03}	-0.05	-0.052	0.037	0.035	-0.120	0.016	0.919
β_{11}	0	0.018	0.335	0.309	-0.588	0.625	0.932
β_{21}	0	0.027	0.437	0.424	-0.803	0.858	0.925
β_{31}	0	-0.007	0.485	0.437	-0.864	0.849	0.938
γ_{11}	-0.05	-0.050	0.153	0.148	-0.340	0.239	0.950
γ_{12}	0	-0.013	0.213	0.203	-0.410	0.384	0.938
γ_{13}	0	0.002	0.230	0.208	-0.406	0.410	0.963
β_{13}	0	-0.002	0.034	0.036	-0.072	0.069	0.975
β_{23}	0	0.004	0.051	0.049	-0.092	0.100	0.932
β_{33}	0	0.003	0.051	0.050	-0.095	0.102	0.950
$cov(u_1, u_1)$	0.25	0.045	0.589	0.619	-1.168	1.259	0.932
$cov(u_1, v_1)$	0	0.090	0.295	0.303	-0.504	0.685	0.932
$cov(v_1, v_1)$	0.06	0.017	0.154	0.156	-0.289	0.323	0.957
$cov(u_1, u_3)$	0	-0.025	0.081	0.092	-0.206	0.156	0.975
$cov(v_1, u_3)$	0	0.011	0.040	0.043	-0.073	0.095	0.969
$cov(u_1, u_3)$	0.0004	-0.0024	0.0102	0.0108	-0.0235	0.0188	0.950

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.7. $N = 200$, $T = 10$, $M = 10$, $R = 167$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.300	0.012	0.011	0.278	0.322	0.916
β_{01}	0.3	0.298	0.082	0.081	0.139	0.458	0.952
γ_{10}	1	1.000	0.031	0.033	0.936	1.064	0.952
β_{03}	-0.05	-0.050	0.003	0.003	-0.056	-0.044	0.964
β_{11}	0	-0.004	0.076	0.082	-0.164	0.157	0.958
β_{21}	0	0.018	0.114	0.115	-0.208	0.244	0.952
β_{31}	0	0.011	0.114	0.116	-0.216	0.239	0.958
γ_{11}	-0.05	-0.046	0.034	0.033	-0.111	0.019	0.940
γ_{12}	0	-0.002	0.047	0.046	-0.093	0.089	0.934
γ_{13}	0	-0.007	0.052	0.047	-0.099	0.085	0.946
β_{13}	0	-0.001	0.003	0.003	-0.007	0.006	0.940
β_{23}	0	0.000	0.004	0.004	-0.009	0.008	0.946
β_{33}	0	0.001	0.005	0.004	-0.007	0.010	0.934
$cov(u_1, u_1)$	0.25	0.238	0.065	0.067	0.107	0.370	0.928
$cov(u_1, v_1)$	0	0.004	0.023	0.021	-0.038	0.046	0.916
$cov(v_1, v_1)$	0.06	0.056	0.011	0.011	0.035	0.078	0.904
$cov(u_1, u_3)$	0	0.000	0.002	0.002	-0.004	0.004	0.946
$cov(v_1, u_3)$	0	0.000	0.001	0.001	-0.001	0.002	0.922
$cov(u_1, u_3)$	0.0004	0.0004	0.0001	0.0001	0.0002	0.0006	0.910

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.8. $N = 200$, $T = 10$, $M = 4$, $R = 118$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.301	0.021	0.021	0.259	0.343	0.958
β_{01}	0.3	0.308	0.205	0.192	-0.068	0.684	0.941
γ_{10}	1	0.997	0.063	0.064	0.871	1.122	0.924
β_{03}	-0.05	-0.050	0.007	0.007	-0.064	-0.035	0.941
β_{11}	0	0.000	0.181	0.192	-0.377	0.377	0.975
β_{21}	0	-0.029	0.294	0.274	-0.566	0.508	0.932
β_{31}	0	0.017	0.281	0.280	-0.532	0.566	0.932
γ_{11}	-0.05	-0.052	0.063	0.065	-0.179	0.074	0.975
γ_{12}	0	0.007	0.094	0.091	-0.172	0.186	0.924
γ_{13}	0	0.001	0.088	0.093	-0.181	0.183	0.958
β_{13}	0	0.000	0.007	0.008	-0.014	0.015	0.975
β_{23}	0	0.000	0.011	0.011	-0.021	0.021	0.941
β_{33}	0	0.000	0.011	0.011	-0.021	0.021	0.949
$cov(u_1, u_1)$	0.25	0.247	0.263	0.284	-0.309	0.802	0.992
$cov(u_1, v_1)$	0	-0.004	0.092	0.103	-0.206	0.199	0.992
$cov(v_1, v_1)$	0.06	0.059	0.037	0.040	-0.019	0.136	0.958
$cov(u_1, u_3)$	0	0.002	0.013	0.017	-0.031	0.034	0.992
$cov(v_1, u_3)$	0	0.000	0.004	0.005	-0.010	0.010	0.992
$cov(u_1, u_3)$	0.0004	0.0004	0.0005	0.0006	-0.0008	0.0016	0.992

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.9. $N = 200$, $T = 10$, $M = 3$, $R = 30$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.291	0.020	0.029	0.235	0.348	0.967
β_{01}	0.3	0.313	0.199	0.232	-0.142	0.768	0.967
γ_{10}	1	1.006	0.073	0.080	0.848	1.163	0.967
β_{03}	-0.05	-0.051	0.010	0.010	-0.070	-0.031	0.967
β_{11}	0	0.053	0.181	0.243	-0.424	0.530	1.000
β_{21}	0	0.037	0.330	0.338	-0.626	0.699	1.000
β_{31}	0	-0.044	0.287	0.354	-0.738	0.650	1.000
γ_{11}	-0.05	-0.070	0.087	0.082	-0.232	0.092	0.933
γ_{12}	0	-0.034	0.111	0.115	-0.260	0.193	0.900
γ_{13}	0	0.015	0.121	0.119	-0.220	0.249	1.000
β_{13}	0	0.002	0.009	0.010	-0.017	0.021	1.000
β_{23}	0	0.003	0.014	0.014	-0.024	0.030	0.933
β_{33}	0	0.000	0.014	0.014	-0.028	0.028	0.933
$cov(u_1, u_1)$	0.25	0.405	0.317	0.415	-0.409	1.219	1.000
$cov(u_1, v_1)$	0	-0.071	0.118	0.158	-0.381	0.239	0.967
$cov(v_1, v_1)$	0.06	0.091	0.050	0.062	-0.031	0.213	0.967
$cov(u_1, u_3)$	0	0.009	0.015	0.030	-0.050	0.068	1.000
$cov(v_1, u_3)$	0	-0.004	0.005	0.008	-0.020	0.013	0.967
$cov(u_1, u_3)$	0.0004	0.0008	0.0007	0.0010	-0.0012	0.0029	1.000

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.10. $N = 200$, $T = 5$, $M = 5$, $R = 196$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.300	0.022	0.021	0.259	0.342	0.944
β_{01}	0.3	0.327	0.133	0.128	0.076	0.577	0.934
γ_{10}	1	0.991	0.072	0.066	0.862	1.119	0.913
β_{03}	-0.05	-0.048	0.016	0.015	-0.077	-0.019	0.944
β_{11}	0	0.011	0.135	0.128	-0.239	0.262	0.929
β_{21}	0	-0.026	0.190	0.181	-0.381	0.330	0.918
β_{31}	0	0.006	0.172	0.182	-0.350	0.363	0.964
γ_{11}	-0.05	-0.054	0.065	0.066	-0.183	0.075	0.974
γ_{12}	0	0.009	0.100	0.093	-0.174	0.191	0.913
γ_{13}	0	0.002	0.089	0.093	-0.181	0.185	0.969
β_{13}	0	0.001	0.015	0.015	-0.028	0.030	0.969
β_{23}	0	-0.002	0.022	0.021	-0.043	0.039	0.949
β_{33}	0	0.000	0.020	0.021	-0.041	0.041	0.959
$cov(u_1, u_1)$	0.25	0.243	0.209	0.190	-0.129	0.616	0.944
$cov(u_1, v_1)$	0	0.003	0.099	0.091	-0.175	0.181	0.934
$cov(v_1, v_1)$	0.06	0.056	0.056	0.050	-0.043	0.154	0.908
$cov(u_1, u_3)$	0	-0.001	0.022	0.020	-0.040	0.038	0.929
$cov(v_1, u_3)$	0	0.001	0.012	0.011	-0.020	0.022	0.929
$cov(u_1, u_3)$	0.0004	0.0001	0.0029	0.0026	-0.0050	0.0053	0.929

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.11. $N = 200$, $T = 5$, $M = 4$, $R = 200$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.299	0.024	0.023	0.253	0.345	0.965
β_{01}	0.3	0.295	0.193	0.174	-0.047	0.636	0.925
γ_{10}	1	1.005	0.094	0.086	0.837	1.172	0.920
β_{03}	-0.05	-0.052	0.022	0.020	-0.091	-0.012	0.905
β_{11}	0	0.007	0.181	0.176	-0.338	0.352	0.945
β_{21}	0	0.017	0.260	0.246	-0.466	0.500	0.930
β_{31}	0	-0.009	0.262	0.248	-0.495	0.478	0.940
γ_{11}	-0.05	-0.054	0.088	0.087	-0.223	0.116	0.940
γ_{12}	0	-0.010	0.127	0.121	-0.247	0.228	0.910
γ_{13}	0	0.007	0.125	0.122	-0.232	0.246	0.925
β_{13}	0	0.000	0.020	0.020	-0.040	0.040	0.960
β_{23}	0	0.003	0.030	0.029	-0.053	0.059	0.940
β_{33}	0	-0.001	0.029	0.029	-0.057	0.056	0.930
$cov(u_1, u_1)$	0.25	0.221	0.394	0.320	-0.407	0.850	0.905
$cov(u_1, v_1)$	0	0.012	0.190	0.157	-0.297	0.320	0.910
$cov(v_1, v_1)$	0.06	0.053	0.100	0.084	-0.111	0.217	0.895
$cov(u_1, u_3)$	0	-0.002	0.047	0.040	-0.080	0.076	0.910
$cov(v_1, u_3)$	0	0.001	0.024	0.020	-0.039	0.041	0.920
$cov(u_1, u_3)$	0.0004	0.0001	0.0059	0.0050	-0.0097	0.0100	0.925

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.12. $N = 200$, $T = 5$, $M = 3$, $R = 186$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.305	0.032	0.031	0.245	0.365	0.941
β_{01}	0.3	0.304	0.235	0.208	-0.105	0.712	0.892
γ_{10}	1	0.999	0.107	0.100	0.803	1.194	0.909
β_{03}	-0.05	-0.050	0.026	0.024	-0.097	-0.003	0.925
β_{11}	0	-0.003	0.213	0.212	-0.418	0.412	0.935
β_{21}	0	-0.003	0.326	0.297	-0.585	0.579	0.914
β_{31}	0	0.016	0.319	0.302	-0.576	0.609	0.968
γ_{11}	-0.05	-0.051	0.102	0.101	-0.249	0.146	0.946
γ_{12}	0	0.001	0.155	0.142	-0.277	0.280	0.957
γ_{13}	0	-0.002	0.153	0.144	-0.284	0.281	0.946
β_{13}	0	0.000	0.025	0.024	-0.048	0.048	0.957
β_{23}	0	0.000	0.039	0.034	-0.067	0.067	0.909
β_{33}	0	0.001	0.037	0.035	-0.067	0.069	0.941
$cov(u_1, u_1)$	0.25	0.118	0.459	0.435	-0.734	0.970	0.909
$cov(u_1, v_1)$	0	0.057	0.225	0.213	-0.361	0.476	0.930
$cov(v_1, v_1)$	0.06	0.033	0.118	0.110	-0.183	0.249	0.935
$cov(u_1, u_3)$	0	-0.012	0.066	0.064	-0.139	0.114	0.941
$cov(v_1, u_3)$	0	0.006	0.031	0.030	-0.053	0.065	0.941
$cov(u_1, u_3)$	0.0004	-0.0012	0.0078	0.0075	-0.0159	0.0135	0.935

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.13. $N = 500$, $T = 10$, $M = 10$, $R = 168$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.299	0.007	0.007	0.285	0.313	0.964
β_{01}	0.3	0.302	0.048	0.052	0.201	0.404	0.982
γ_{10}	1	1.001	0.022	0.021	0.960	1.042	0.940
β_{03}	-0.05	-0.050	0.002	0.002	-0.054	-0.046	0.952
β_{11}	0	0.002	0.058	0.052	-0.099	0.104	0.940
β_{21}	0	-0.007	0.073	0.073	-0.150	0.136	0.935
β_{31}	0	-0.001	0.081	0.073	-0.144	0.143	0.911
γ_{11}	-0.05	-0.051	0.023	0.021	-0.093	-0.010	0.923
γ_{12}	0	0.002	0.030	0.030	-0.056	0.060	0.952
γ_{13}	0	0.001	0.032	0.030	-0.057	0.060	0.946
β_{13}	0	0.000	0.002	0.002	-0.004	0.004	0.929
β_{23}	0	0.000	0.003	0.003	-0.006	0.005	0.929
β_{33}	0	0.000	0.003	0.003	-0.006	0.005	0.917
$cov(u_1, u_1)$	0.25	0.250	0.041	0.043	0.165	0.334	0.958
$cov(u_1, v_1)$	0	-0.001	0.013	0.014	-0.028	0.026	0.970
$cov(v_1, v_1)$	0.06	0.060	0.007	0.007	0.046	0.073	0.958
$cov(u_1, u_3)$	0	0.000	0.001	0.001	-0.003	0.003	0.935
$cov(v_1, u_3)$	0	0.000	0.001	0.001	-0.001	0.001	0.952
$cov(u_1, u_3)$	0.0004	0.0004	0.0001	0.0001	0.0003	0.0005	0.935

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.14. $N = 500$, $T = 10$, $M = 4$, $R = 194$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.298	0.013	0.013	0.272	0.324	0.948
β_{01}	0.3	0.305	0.127	0.121	0.066	0.543	0.954
γ_{10}	1	0.998	0.040	0.040	0.919	1.077	0.943
β_{03}	-0.05	-0.049	0.004	0.005	-0.059	-0.040	0.964
β_{11}	0	0.008	0.125	0.122	-0.232	0.248	0.928
β_{21}	0	-0.015	0.172	0.172	-0.352	0.322	0.959
β_{31}	0	-0.007	0.193	0.173	-0.347	0.332	0.928
γ_{11}	-0.05	-0.052	0.042	0.041	-0.132	0.028	0.933
γ_{12}	0	0.006	0.056	0.057	-0.106	0.118	0.954
γ_{13}	0	0.003	0.063	0.058	-0.110	0.116	0.918
β_{13}	0	0.000	0.005	0.005	-0.009	0.009	0.959
β_{23}	0	-0.001	0.007	0.007	-0.014	0.012	0.948
β_{33}	0	0.000	0.007	0.007	-0.013	0.013	0.954
$cov(u_1, u_1)$	0.25	0.253	0.176	0.179	-0.097	0.604	0.959
$cov(u_1, v_1)$	0	-0.004	0.063	0.065	-0.130	0.123	0.959
$cov(v_1, v_1)$	0.06	0.061	0.024	0.025	0.012	0.109	0.954
$cov(u_1, u_3)$	0	0.001	0.010	0.011	-0.020	0.021	0.964
$cov(v_1, u_3)$	0	0.000	0.003	0.003	-0.006	0.006	0.974
$cov(u_1, u_3)$	0.0004	0.0004	0.0004	0.0004	-0.0003	0.0012	0.964

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.15. $N = 500$, $T = 10$, $M = 3$, $R = 85$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.301	0.017	0.019	0.264	0.338	0.976
β_{01}	0.3	0.331	0.141	0.146	0.044	0.617	0.941
γ_{10}	1	0.992	0.052	0.050	0.893	1.090	0.965
β_{03}	-0.05	-0.050	0.006	0.006	-0.061	-0.038	0.941
β_{11}	0	0.003	0.177	0.149	-0.289	0.296	0.941
β_{21}	0	0.004	0.207	0.207	-0.401	0.409	0.929
β_{31}	0	0.000	0.223	0.210	-0.411	0.411	0.929
γ_{11}	-0.05	-0.049	0.055	0.051	-0.148	0.050	0.953
γ_{12}	0	-0.007	0.073	0.071	-0.146	0.131	0.918
γ_{13}	0	-0.002	0.075	0.071	-0.142	0.138	0.953
β_{13}	0	-0.001	0.007	0.006	-0.012	0.011	0.953
β_{23}	0	0.001	0.009	0.008	-0.015	0.018	0.929
β_{33}	0	0.001	0.009	0.008	-0.015	0.018	0.953
$cov(u_1, u_1)$	0.25	0.297	0.227	0.240	-0.172	0.767	0.976
$cov(u_1, v_1)$	0	-0.022	0.084	0.093	-0.204	0.160	0.965
$cov(v_1, v_1)$	0.06	0.069	0.035	0.037	-0.004	0.142	0.965
$cov(u_1, u_3)$	0	0.005	0.014	0.018	-0.031	0.040	1.000
$cov(v_1, u_3)$	0	-0.002	0.004	0.005	-0.012	0.008	1.000
$cov(u_1, u_3)$	0.0004	0.0006	0.0005	0.0006	-0.0006	0.0018	0.988

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.16. $N = 500$, $T = 5$, $M = 5$, $R = 198$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.299	0.011	0.013	0.273	0.325	0.980
β_{01}	0.3	0.302	0.080	0.081	0.143	0.460	0.955
γ_{10}	1	1.001	0.040	0.042	0.919	1.082	0.939
β_{03}	-0.05	-0.049	0.009	0.009	-0.068	-0.031	0.949
β_{11}	0	0.012	0.080	0.081	-0.146	0.171	0.949
β_{21}	0	0.007	0.111	0.114	-0.217	0.230	0.949
β_{31}	0	-0.020	0.122	0.114	-0.243	0.204	0.934
γ_{11}	-0.05	-0.057	0.042	0.042	-0.138	0.025	0.949
γ_{12}	0	-0.003	0.058	0.059	-0.118	0.112	0.965
γ_{13}	0	0.010	0.062	0.059	-0.106	0.125	0.944
β_{13}	0	0.001	0.010	0.009	-0.017	0.020	0.919
β_{23}	0	0.000	0.013	0.013	-0.026	0.026	0.960
β_{33}	0	-0.002	0.014	0.013	-0.028	0.024	0.944
$cov(u_1, u_1)$	0.25	0.240	0.118	0.119	0.006	0.474	0.949
$cov(u_1, v_1)$	0	0.003	0.055	0.057	-0.109	0.115	0.944
$cov(v_1, v_1)$	0.06	0.058	0.031	0.032	-0.004	0.120	0.955
$cov(u_1, u_3)$	0	-0.001	0.012	0.013	-0.026	0.024	0.944
$cov(v_1, u_3)$	0	0.001	0.007	0.007	-0.013	0.014	0.965
$cov(u_1, u_3)$	0.0004	0.0001	0.0016	0.0017	-0.0031	0.0034	0.965

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.17. $N = 500$, $T = 5$, $M = 4$, $R = 200$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.301	0.016	0.015	0.272	0.330	0.915
β_{01}	0.3	0.292	0.113	0.110	0.076	0.507	0.940
γ_{10}	1	1.002	0.056	0.054	0.896	1.108	0.940
β_{03}	-0.05	-0.050	0.014	0.013	-0.075	-0.025	0.945
β_{11}	0	0.007	0.110	0.111	-0.210	0.224	0.965
β_{21}	0	0.008	0.162	0.156	-0.298	0.313	0.940
β_{31}	0	-0.008	0.153	0.157	-0.315	0.299	0.960
γ_{11}	-0.05	-0.055	0.055	0.054	-0.162	0.052	0.935
γ_{12}	0	0.001	0.077	0.077	-0.149	0.151	0.940
γ_{13}	0	0.005	0.078	0.077	-0.146	0.156	0.945
β_{13}	0	0.001	0.012	0.013	-0.024	0.026	0.945
β_{23}	0	-0.001	0.018	0.018	-0.036	0.034	0.950
β_{33}	0	-0.002	0.019	0.018	-0.037	0.034	0.945
$cov(u_1, u_1)$	0.25	0.217	0.214	0.202	-0.179	0.613	0.925
$cov(u_1, v_1)$	0	0.013	0.105	0.100	-0.182	0.208	0.925
$cov(v_1, v_1)$	0.06	0.053	0.055	0.053	-0.051	0.157	0.940
$cov(u_1, u_3)$	0	-0.003	0.027	0.025	-0.053	0.046	0.920
$cov(v_1, u_3)$	0	0.002	0.013	0.013	-0.024	0.027	0.945
$cov(u_1, u_3)$	0.0004	-0.0001	0.0032	0.0032	-0.0063	0.0062	0.940

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.18. $N = 500$, $T = 5$, $M = 3$, $R = 200$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.301	0.019	0.019	0.264	0.339	0.955
β_{01}	0.3	0.294	0.126	0.133	0.034	0.553	0.960
γ_{10}	1	1.000	0.059	0.063	0.876	1.124	0.970
β_{03}	-0.05	-0.050	0.014	0.015	-0.080	-0.020	0.955
β_{11}	0	0.006	0.132	0.133	-0.255	0.267	0.955
β_{21}	0	0.017	0.192	0.188	-0.352	0.385	0.930
β_{31}	0	0.012	0.188	0.189	-0.358	0.382	0.945
γ_{11}	-0.05	-0.052	0.064	0.064	-0.177	0.072	0.955
γ_{12}	0	-0.006	0.091	0.090	-0.182	0.171	0.940
γ_{13}	0	-0.001	0.089	0.090	-0.178	0.176	0.935
β_{13}	0	0.000	0.016	0.015	-0.030	0.030	0.945
β_{23}	0	0.002	0.023	0.022	-0.041	0.044	0.960
β_{33}	0	0.001	0.023	0.022	-0.042	0.043	0.940
$cov(u_1, u_1)$	0.25	0.195	0.267	0.274	-0.343	0.733	0.950
$cov(u_1, v_1)$	0	0.020	0.131	0.135	-0.244	0.284	0.960
$cov(v_1, v_1)$	0.06	0.052	0.068	0.070	-0.085	0.188	0.960
$cov(u_1, u_3)$	0	-0.003	0.038	0.041	-0.082	0.076	0.960
$cov(v_1, u_3)$	0	0.001	0.018	0.019	-0.036	0.038	0.960
$cov(u_1, u_3)$	0.0004	0.0002	0.0046	0.0047	-0.0091	0.0095	0.960

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.19. $N = 1000$, $T = 10$, $M = 10$, $R = 178$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.300	0.005	0.005	0.290	0.310	0.955
β_{01}	0.3	0.302	0.038	0.037	0.230	0.374	0.955
γ_{10}	1	1.000	0.015	0.015	0.971	1.029	0.949
β_{03}	-0.05	-0.050	0.001	0.001	-0.053	-0.047	0.944
β_{11}	0	-0.003	0.036	0.036	-0.074	0.068	0.944
β_{21}	0	0.000	0.053	0.052	-0.101	0.101	0.944
β_{31}	0	0.000	0.051	0.052	-0.101	0.101	0.938
γ_{11}	-0.05	-0.050	0.016	0.015	-0.079	-0.021	0.921
γ_{12}	0	0.001	0.022	0.021	-0.040	0.042	0.955
γ_{13}	0	0.001	0.023	0.021	-0.040	0.042	0.916
β_{13}	0	0.000	0.001	0.001	-0.003	0.003	0.938
β_{23}	0	0.000	0.002	0.002	-0.004	0.004	0.972
β_{33}	0	0.000	0.002	0.002	-0.004	0.004	0.972
$cov(u_1, u_1)$	0.25	0.249	0.030	0.031	0.189	0.309	0.961
$cov(u_1, v_1)$	0	0.000	0.009	0.010	-0.019	0.019	0.978
$cov(v_1, v_1)$	0.06	0.060	0.004	0.005	0.050	0.069	0.972
$cov(u_1, u_3)$	0	0.000	0.001	0.001	-0.002	0.002	0.955
$cov(v_1, u_3)$	0	0.000	0.000	0.000	-0.001	0.001	0.972
$cov(u_1, u_3)$	0.0004	0.0004	0.0000	0.0000	0.0003	0.0005	0.949

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.20. $N = 1000$, $T = 10$, $M = 4$, $R = 199$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.299	0.010	0.009	0.281	0.318	0.940
β_{01}	0.3	0.306	0.085	0.085	0.139	0.472	0.935
γ_{10}	1	0.996	0.028	0.028	0.940	1.051	0.940
β_{03}	-0.05	-0.050	0.003	0.003	-0.056	-0.043	0.935
β_{11}	0	-0.004	0.094	0.085	-0.171	0.163	0.925
β_{21}	0	-0.006	0.118	0.120	-0.242	0.230	0.945
β_{31}	0	0.012	0.126	0.121	-0.226	0.250	0.955
γ_{11}	-0.05	-0.047	0.030	0.028	-0.103	0.009	0.915
γ_{12}	0	0.003	0.037	0.040	-0.076	0.081	0.960
γ_{13}	0	-0.007	0.039	0.040	-0.086	0.072	0.970
β_{13}	0	0.000	0.003	0.003	-0.007	0.006	0.940
β_{23}	0	0.000	0.005	0.005	-0.009	0.009	0.930
β_{33}	0	0.001	0.005	0.005	-0.008	0.010	0.975
$cov(u_1, u_1)$	0.25	0.232	0.125	0.123	-0.009	0.473	0.930
$cov(u_1, v_1)$	0	0.006	0.046	0.045	-0.082	0.093	0.925
$cov(v_1, v_1)$	0.06	0.057	0.018	0.017	0.023	0.090	0.925
$cov(u_1, u_3)$	0	0.000	0.007	0.007	-0.014	0.015	0.955
$cov(v_1, u_3)$	0	0.000	0.002	0.002	-0.004	0.004	0.935
$cov(u_1, u_3)$	0.0004	0.0004	0.0003	0.0003	-0.0001	0.0009	0.965

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

A.2 Simulation results for the random coefficient LDSM

Table A.21. $N = 1000$, $T = 10$, $M = 3$, $R = 115$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.298	0.013	0.013	0.272	0.324	0.957
β_{01}	0.3	0.305	0.107	0.102	0.105	0.504	0.948
γ_{10}	1	0.998	0.038	0.035	0.930	1.066	0.939
β_{03}	-0.05	-0.050	0.005	0.004	-0.058	-0.042	0.913
β_{11}	0	0.000	0.115	0.103	-0.201	0.202	0.913
β_{21}	0	0.001	0.137	0.144	-0.281	0.284	0.939
β_{31}	0	-0.005	0.136	0.145	-0.290	0.279	0.965
γ_{11}	-0.05	-0.052	0.039	0.035	-0.120	0.017	0.904
γ_{12}	0	-0.002	0.049	0.049	-0.098	0.095	0.939
γ_{13}	0	0.005	0.048	0.050	-0.092	0.102	0.939
β_{13}	0	0.000	0.005	0.004	-0.008	0.008	0.904
β_{23}	0	0.001	0.006	0.006	-0.011	0.012	0.957
β_{33}	0	-0.001	0.007	0.006	-0.012	0.011	0.930
$cov(u_1, u_1)$	0.25	0.246	0.153	0.165	-0.079	0.570	0.957
$cov(u_1, v_1)$	0	0.002	0.058	0.064	-0.124	0.127	0.983
$cov(v_1, v_1)$	0.06	0.061	0.025	0.026	0.010	0.112	0.974
$cov(u_1, u_3)$	0	0.000	0.011	0.013	-0.025	0.025	0.974
$cov(v_1, u_3)$	0	0.000	0.003	0.004	-0.007	0.007	0.974
$cov(u_1, u_3)$	0.0004	0.0005	0.0004	0.0004	-0.0004	0.0013	0.974

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.22. $N = 1000$, $T = 5$, $M = 5$, $R = 198$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.299	0.009	0.009	0.281	0.318	0.960
β_{01}	0.3	0.299	0.053	0.057	0.187	0.411	0.960
γ_{10}	1	1.000	0.028	0.029	0.942	1.058	0.955
β_{03}	-0.05	-0.050	0.006	0.007	-0.063	-0.037	0.955
β_{11}	0	-0.005	0.056	0.057	-0.117	0.106	0.949
β_{21}	0	0.009	0.077	0.081	-0.149	0.167	0.980
β_{31}	0	0.008	0.084	0.081	-0.150	0.167	0.944
γ_{11}	-0.05	-0.049	0.029	0.029	-0.107	0.009	0.980
γ_{12}	0	-0.001	0.040	0.042	-0.083	0.080	0.970
γ_{13}	0	-0.005	0.041	0.042	-0.087	0.077	0.955
β_{13}	0	0.000	0.006	0.007	-0.013	0.013	0.949
β_{23}	0	0.000	0.009	0.009	-0.018	0.019	0.975
β_{33}	0	0.001	0.009	0.009	-0.017	0.019	0.949
$cov(u_1, u_1)$	0.25	0.246	0.087	0.085	0.080	0.412	0.944
$cov(u_1, v_1)$	0	0.000	0.040	0.041	-0.080	0.079	0.960
$cov(v_1, v_1)$	0.06	0.060	0.022	0.023	0.015	0.104	0.949
$cov(u_1, u_3)$	0	0.000	0.009	0.009	-0.018	0.017	0.960
$cov(v_1, u_3)$	0	0.000	0.005	0.005	-0.010	0.010	0.939
$cov(u_1, u_3)$	0.0004	0.0004	0.0011	0.0012	-0.0019	0.0027	0.949

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.23. $N = 1000$, $T = 5$, $M = 4$, $R = 200$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.300	0.012	0.010	0.280	0.321	0.945
β_{01}	0.3	0.301	0.072	0.078	0.148	0.453	0.970
γ_{10}	1	1.001	0.036	0.038	0.926	1.076	0.960
β_{03}	-0.05	-0.050	0.008	0.009	-0.068	-0.032	0.970
β_{11}	0	0.011	0.082	0.078	-0.142	0.164	0.945
β_{21}	0	0.003	0.098	0.110	-0.213	0.219	0.980
β_{31}	0	-0.011	0.123	0.110	-0.228	0.205	0.930
γ_{11}	-0.05	-0.054	0.039	0.038	-0.129	0.021	0.950
γ_{12}	0	-0.002	0.049	0.054	-0.108	0.104	0.975
γ_{13}	0	0.006	0.057	0.054	-0.100	0.113	0.945
β_{13}	0	0.001	0.009	0.009	-0.017	0.019	0.955
β_{23}	0	0.000	0.011	0.013	-0.025	0.025	0.975
β_{33}	0	-0.001	0.013	0.013	-0.026	0.024	0.940
$cov(u_1, u_1)$	0.25	0.230	0.143	0.144	-0.052	0.511	0.955
$cov(u_1, v_1)$	0	0.007	0.073	0.071	-0.132	0.145	0.945
$cov(v_1, v_1)$	0.06	0.058	0.038	0.038	-0.016	0.132	0.950
$cov(u_1, u_3)$	0	-0.002	0.017	0.018	-0.037	0.033	0.960
$cov(v_1, u_3)$	0	0.000	0.009	0.009	-0.017	0.018	0.950
$cov(u_1, u_3)$	0.0004	0.0003	0.0022	0.0023	-0.0042	0.0047	0.945

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.24. $N = 1000$, $T = 5$, $M = 3$, $R = 198$

Parameter	TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
$Var(e)$	0.3	0.300	0.013	0.013	0.274	0.326	0.945
β_{01}	0.3	0.298	0.090	0.095	0.112	0.484	0.975
γ_{10}	1	1.000	0.043	0.045	0.911	1.089	0.980
β_{03}	-0.05	-0.049	0.010	0.011	-0.071	-0.028	0.970
β_{11}	0	-0.004	0.091	0.095	-0.191	0.183	0.960
β_{21}	0	0.000	0.125	0.134	-0.263	0.262	0.965
β_{31}	0	0.013	0.140	0.135	-0.251	0.276	0.930
γ_{11}	-0.05	-0.046	0.046	0.045	-0.135	0.043	0.945
γ_{12}	0	-0.002	0.062	0.064	-0.127	0.123	0.955
γ_{13}	0	-0.008	0.070	0.064	-0.133	0.118	0.925
β_{13}	0	-0.001	0.011	0.011	-0.022	0.021	0.950
β_{23}	0	0.000	0.014	0.015	-0.030	0.030	0.955
β_{33}	0	0.002	0.016	0.015	-0.028	0.032	0.920
$cov(u_1, u_1)$	0.25	0.239	0.214	0.197	-0.147	0.624	0.930
$cov(u_1, v_1)$	0	0.005	0.102	0.096	-0.183	0.194	0.945
$cov(v_1, v_1)$	0.06	0.057	0.051	0.050	-0.040	0.154	0.950
$cov(u_1, u_3)$	0	0.000	0.031	0.029	-0.057	0.056	0.955
$cov(v_1, u_3)$	0	0.000	0.014	0.013	-0.026	0.027	0.960
$cov(u_1, u_3)$	0.0004	0.0004	0.0035	0.0034	-0.0062	0.0070	0.960

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.25. *Simulation results with N=100 and T=10*

		TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
10-10	μ_0	2	1.99708	0.07595	0.07738	1.84598	2.14966	0.956
	β_0	0.15	0.14993	0.00354	0.00370	0.14267	0.15718	0.956
	β_1	-0.001	-0.00995	0.00360	0.00364	-0.01709	-0.00281	0.963
	β_2	0	0.00021	0.00539	0.00503	-0.00966	0.01011	0.938
	β_3	0	0.00004	0.00585	0.00517	-0.01014	0.01019	0.900
	σ_e^2	0.5	0.50105	0.02485	0.02423	0.45579	0.55067	0.938
	σ_0^2	0.5	0.51993	0.08724	0.07963	0.38571	0.69666	0.900
	d^2	0.0001	0.00011	0.00004	0.00005	0.00005	0.00025	0.994
10-3	μ_0	2	2.02027	0.12772	0.11745	1.79806	2.25864	0.919
	β_0	0.15	0.14777	0.01375	0.01292	0.12272	0.17339	0.927
	β_1	-0.001	-0.00857	0.01056	0.00978	-0.02781	0.01070	0.951
	β_2	0	0.00141	0.01411	0.01360	-0.02517	0.02814	0.943
	β_3	0	-0.00170	0.01560	0.01393	-0.02912	0.02558	0.935
	σ_e^2	0.5	0.50362	0.05719	0.05145	0.41258	0.61392	0.927
	σ_0^2	0.5	0.51581	0.10207	0.09878	0.35121	0.73674	0.959
	d^2	0.0001	0.00018	0.00007	0.00020	0.00005	0.00074	1.000
10-2	μ_0	2	2.01767	0.14182	0.13740	1.76066	2.29922	0.933
	β_0	0.15	0.15047	0.01478	0.01611	0.11927	0.18240	0.942
	β_1	-0.001	-0.00982	0.01283	0.01120	-0.03174	0.01222	0.913
	β_2	0	-0.00049	0.01378	0.01533	-0.03055	0.02970	0.962
	β_3	0	0.00044	0.01837	0.01612	-0.03113	0.03220	0.904
	σ_e^2	0.5	0.49752	0.06869	0.07230	0.37545	0.65790	0.933
	σ_0^2	0.5	0.51385	0.10268	0.11136	0.33008	0.76557	0.952
	d^2	0.0001	0.00020	0.00014	0.00026	0.00005	0.00096	1.000

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.26. *Simulation results with N=100 and T=5*

		TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
5-5	μ_0	2	2.01152	0.08119	0.08537	1.84603	2.18098	0.939
	β_0	0.15	0.14895	0.01166	0.01257	0.12449	0.17383	0.962
	β_1	-0.001	-0.01163	0.01268	0.01177	-0.03455	0.01142	0.939
	β_2	0	0.00170	0.01672	0.01642	-0.03056	0.03382	0.947
	β_3	0	0.00136	0.01864	0.01679	-0.03150	0.03434	0.931
	σ_e^2	0.5	0.50624	0.03527	0.03646	0.43969	0.58245	0.962
	σ_0^2	0.5	0.50427	0.08632	0.08469	0.36144	0.69233	0.939
	d^2	0.0001	0.00019	0.00009	0.00025	0.00005	0.00092	1.000
5-3	μ_0	2	2.01966	0.09953	0.10416	1.82064	2.22839	0.977
	β_0	0.15	0.14988	0.02544	0.02183	0.10759	0.19202	0.886
	β_1	-0.001	-0.01272	0.02086	0.01992	-0.05198	0.02550	0.886
	β_2	0	0.00332	0.02809	0.02588	-0.04748	0.05341	0.932
	β_3	0	0.00777	0.03224	0.02788	-0.04732	0.06245	0.909
	σ_e^2	0.5	0.49966	0.05417	0.05124	0.40914	0.60971	0.955
	σ_0^2	0.5	0.49803	0.09721	0.09512	0.33833	0.70945	0.932
	d^2	0.0001	0.00026	0.00013	0.00049	0.00005	0.00163	1.000
5-2	μ_0	2	2.01966	0.09953	0.10416	1.82064	2.22839	0.977
	β_0	0.15	0.14988	0.02544	0.02183	0.10759	0.19202	0.886
	β_1	-0.001	-0.01272	0.02086	0.01992	-0.05198	0.02550	0.886
	β_2	0	0.00332	0.02809	0.02588	-0.04748	0.05341	0.932
	β_3	0	0.00777	0.03224	0.02788	-0.04732	0.06245	0.909
	σ_e^2	0.5	0.49966	0.05417	0.05124	0.40914	0.60971	0.955
	σ_0^2	0.5	0.49803	0.09721	0.09512	0.33833	0.70945	0.932
	d^2	0.0001	0.00026	0.00013	0.00049	0.00005	0.00163	1.000

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.27. *Simulation results with N=200 and T=10*

		TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
10-10	μ_0	2	2.00164	0.05151	0.05408	1.89599	2.10804	0.965
	β_0	0.15	0.15005	0.00246	0.00255	0.14503	0.15505	0.936
	β_1	-0.001	-0.00995	0.00231	0.00250	-0.01484	-0.00503	0.971
	β_2	0	0.00009	0.00351	0.00348	-0.00672	0.00693	0.965
	β_3	0	0.00018	0.00348	0.00353	-0.00675	0.00710	0.971
	σ_e^2	0.5	0.50236	0.01802	0.01718	0.46978	0.53709	0.959
	σ_0^2	0.5	0.50413	0.04917	0.05389	0.40890	0.61978	0.971
	d^2	0.0001	0.00010	0.00002	0.00004	0.00005	0.00019	1.000
10-3	μ_0	2	2.00332	0.08483	0.08149	1.84729	2.16657	0.930
	β_0	0.15	0.14941	0.00937	0.00895	0.13187	0.16720	0.945
	β_1	-0.001	-0.01041	0.00654	0.00673	-0.02360	0.00270	0.969
	β_2	0	-0.00016	0.00855	0.00960	-0.01897	0.01877	0.992
	β_3	0	0.00094	0.00881	0.00965	-0.01800	0.01983	0.977
	σ_e^2	0.5	0.49809	0.03851	0.03571	0.43294	0.57283	0.945
	σ_0^2	0.5	0.50337	0.06526	0.06656	0.38658	0.64689	0.977
	d^2	0.0001	0.00016	0.00007	0.00015	0.00005	0.00059	1.000
10-2	μ_0	2	2.01177	0.09804	0.09582	1.83047	2.20539	0.943
	β_0	0.15	0.14895	0.01165	0.01093	0.12772	0.17044	0.952
	β_1	-0.001	-0.00995	0.00843	0.00761	-0.02493	0.00499	0.905
	β_2	0	0.00075	0.01088	0.01055	-0.01982	0.02145	0.943
	β_3	0	0.00003	0.01069	0.01071	-0.02100	0.02104	0.971
	σ_e^2	0.5	0.50771	0.04836	0.05139	0.41687	0.61792	0.952
	σ_0^2	0.5	0.50345	0.06954	0.07547	0.37193	0.66751	0.962
	d^2	0.0001	0.00017	0.00008	0.00017	0.00005	0.00066	1.000

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.28. *Simulation results with N=200 and T=5*

		TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
5-5	μ_0	2	1.99993	0.06334	0.05981	1.88342	2.11801	0.933
	β_0	0.15	0.14911	0.00854	0.00887	0.13183	0.16666	0.956
	β_1	-0.001	-0.00901	0.00834	0.00819	-0.02498	0.00706	0.941
	β_2	0	0.00106	0.01003	0.01148	-0.02146	0.02364	0.970
	β_3	0	-0.00035	0.01153	0.01164	-0.02337	0.02243	0.933
	σ_e^2	0.5	0.50101	0.02675	0.02539	0.45358	0.55307	0.948
	σ_0^2	0.5	0.49173	0.05555	0.05754	0.39014	0.61522	0.948
	d^2	0.0001	0.00017	0.00007	0.00018	0.00005	0.00069	1.000
5-3	μ_0	2	2.00743	0.06895	0.07494	1.86256	2.15572	0.949
	β_0	0.15	0.15163	0.01484	0.01593	0.12089	0.18306	0.949
	β_1	-0.001	-0.00912	0.01493	0.01304	-0.03454	0.01672	0.881
	β_2	0	-0.00261	0.01668	0.01830	-0.03807	0.03335	0.966
	β_3	0	0.00031	0.01815	0.01884	-0.03684	0.03725	0.932
	σ_e^2	0.5	0.49707	0.03422	0.03569	0.43185	0.57166	1.000
	σ_0^2	0.5	0.50139	0.06834	0.06688	0.38349	0.64517	0.949
	d^2	0.0001	0.00022	0.00016	0.00033	0.00005	0.00115	1.000
5-2	μ_0	2	2.03651	0.09669	0.09150	1.86299	2.21890	0.848
	β_0	0.15	0.14547	0.02300	0.02192	0.10332	0.18714	0.879
	β_1	-0.001	-0.01619	0.01609	0.01612	-0.04776	0.01427	0.909
	β_2	0	-0.00457	0.02983	0.02311	-0.04978	0.04073	0.879
	β_3	0	0.00576	0.02359	0.02358	-0.03973	0.05113	0.970
	σ_e^2	0.5	0.50490	0.04590	0.05126	0.41427	0.61481	0.939
	σ_0^2	0.5	0.50971	0.06259	0.07907	0.37082	0.68022	1.000
	d^2	0.0001	0.00027	0.00015	0.00046	0.00005	0.00162	1.000

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.29. *Simulation results with N=500 and T=10*

		TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
10-10	μ_0	2	1.99946	0.03401	0.03411	1.93275	2.06644	0.951
	β_0	0.15	0.14985	0.00154	0.00160	0.14670	0.15297	0.957
	β_1	-0.001	-0.01003	0.00154	0.00155	-0.01306	-0.00700	0.939
	β_2	0	-0.00007	0.00207	0.00218	-0.00434	0.00420	0.982
	β_3	0	-0.00014	0.00219	0.00219	-0.00444	0.00416	0.945
	σ_e^2	0.5	0.50119	0.01069	0.01087	0.48029	0.52288	0.945
	σ_0^2	0.5	0.49995	0.03235	0.03351	0.43838	0.56961	0.957
	d^2	0.0001	0.00010	0.00002	0.00003	0.00006	0.00016	0.982
10-3	μ_0	2	1.99232	0.04893	0.05124	1.89377	2.09468	0.953
	β_0	0.15	0.15002	0.00552	0.00559	0.13907	0.16098	0.953
	β_1	-0.001	-0.01019	0.00426	0.00421	-0.01843	-0.00196	0.925
	β_2	0	-0.00021	0.00579	0.00587	-0.01172	0.01129	0.981
	β_3	0	0.00003	0.00644	0.00596	-0.01161	0.01168	0.935
	σ_e^2	0.5	0.50547	0.02105	0.02274	0.46283	0.55192	0.953
	σ_0^2	0.5	0.49617	0.03949	0.04110	0.42100	0.58194	0.953
	d^2	0.0001	0.00013	0.00008	0.00009	0.00005	0.00037	0.991
10-2	μ_0	2	2.00768	0.06201	0.05993	1.89290	2.12731	0.940
	β_0	0.15	0.14983	0.00730	0.00683	0.13649	0.16322	0.950
	β_1	-0.001	-0.01001	0.00537	0.00460	-0.01899	-0.00095	0.930
	β_2	0	-0.00079	0.00611	0.00653	-0.01362	0.01201	0.970
	β_3	0	-0.00021	0.00669	0.00653	-0.01306	0.01260	0.980
	σ_e^2	0.5	0.50342	0.02734	0.03196	0.44461	0.56980	0.990
	σ_0^2	0.5	0.49973	0.04582	0.04670	0.41433	0.59726	0.980
	d^2	0.0001	0.00015	0.00009	0.00012	0.00005	0.00048	1.000

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.30. *Simulation results with N=500 and T=5*

		TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
5-5	μ_0	2	2.00177	0.03673	0.03803	1.92767	2.07666	0.971
	β_0	0.15	0.15030	0.00601	0.00548	0.13965	0.16125	0.942
	β_1	-0.001	-0.01056	0.00532	0.00509	-0.02049	-0.00051	0.942
	β_2	0	-0.00145	0.00785	0.00723	-0.01569	0.01261	0.933
	β_3	0	0.00061	0.00799	0.00728	-0.01370	0.01484	0.923
	σ_e^2	0.5	0.49970	0.01754	0.01598	0.46932	0.53196	0.894
	σ_0^2	0.5	0.50383	0.03853	0.03681	0.43606	0.58023	0.962
	d^2	0.0001	0.00014	0.00004	0.00012	0.00005	0.00047	1.000
5-3	μ_0	2	2.01247	0.05186	0.04680	1.92188	2.10506	0.933
	β_0	0.15	0.14992	0.01045	0.01008	0.13051	0.16977	0.933
	β_1	-0.001	-0.01047	0.00799	0.00827	-0.02702	0.00572	0.978
	β_2	0	-0.00198	0.01196	0.01172	-0.02440	0.02137	0.956
	β_3	0	-0.00007	0.01315	0.01172	-0.02336	0.02286	0.911
	σ_e^2	0.5	0.49967	0.02436	0.02258	0.45733	0.54592	0.933
	σ_0^2	0.5	0.50177	0.04626	0.04171	0.42521	0.58856	0.933
	d^2	0.0001	0.00027	0.00046	0.00026	0.00007	0.00099	0.956
5-2	μ_0	2	2.02090	0.06289	0.05771	1.91042	2.13534	0.857
	β_0	0.15	0.14428	0.01541	0.01339	0.11857	0.17036	0.886
	β_1	-0.001	-0.00742	0.01249	0.01025	-0.02729	0.01309	0.886
	β_2	0	0.00472	0.01184	0.01376	-0.02239	0.03201	0.971
	β_3	0	-0.00400	0.01793	0.01453	-0.03198	0.02453	0.829
	σ_e^2	0.5	0.50219	0.03694	0.03203	0.44328	0.56885	0.914
	σ_0^2	0.5	0.50678	0.04403	0.04963	0.41615	0.61041	0.971
	d^2	0.0001	0.00033	0.00055	0.00032	0.00006	0.00126	0.971

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.31. *Simulation results with N=1000 and T=10*

		TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
10-10	μ_0	2	1.99884	0.02463	0.02416	1.95156	2.04631	0.958
	β_0	0.15	0.15002	0.00109	0.00113	0.14779	0.15222	0.958
	β_1	-0.001	-0.01002	0.00121	0.00109	-0.01216	-0.00788	0.934
	β_2	0	-0.00014	0.00152	0.00153	-0.00314	0.00287	0.952
	β_3	0	-0.00022	0.00155	0.00154	-0.00325	0.00280	0.940
	σ_e^2	0.5	0.50087	0.00788	0.00771	0.48596	0.51619	0.958
	σ_0^2	0.5	0.50184	0.02383	0.02374	0.45733	0.55035	0.970
	d^2	0.0001	0.00010	0.00002	0.00002	0.00006	0.00014	0.952
10-3	μ_0	2	2.00033	0.03669	0.03645	1.92981	2.07253	0.950
	β_0	0.15	0.15000	0.00414	0.00398	0.14219	0.15786	0.960
	β_1	-0.001	-0.01015	0.00269	0.00295	-0.01599	-0.00437	0.980
	β_2	0	-0.00030	0.00433	0.00414	-0.00839	0.00779	0.910
	β_3	0	-0.00017	0.00396	0.00418	-0.00836	0.00796	0.980
	σ_e^2	0.5	0.50020	0.01679	0.01590	0.47004	0.53232	0.950
	σ_0^2	0.5	0.50662	0.02686	0.02950	0.45142	0.56706	0.950
	d^2	0.0001	0.00012	0.00004	0.00007	0.00005	0.00029	1.000
10-2	μ_0	2	1.99790	0.03932	0.04251	1.91575	2.08208	0.947
	β_0	0.15	0.15050	0.00481	0.00480	0.14122	0.15993	0.958
	β_1	-0.001	-0.00991	0.00360	0.00326	-0.01627	-0.00354	0.905
	β_2	0	-0.00017	0.00438	0.00454	-0.00903	0.00877	0.958
	β_3	0	0.00027	0.00487	0.00458	-0.00870	0.00926	0.937
	σ_e^2	0.5	0.50264	0.02163	0.02249	0.46047	0.54855	0.958
	σ_0^2	0.5	0.49844	0.03345	0.03284	0.43732	0.56585	0.947
	d^2	0.0001	0.00012	0.00005	0.00007	0.00005	0.00033	1.000

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

Table A.32. *Simulation results with N=1000 and T=5*

		TRUE	Estimate	s.d.	a.s.e.	95% CI		CP
5-5	μ_0	2	2.00163	0.02529	0.02684	1.94911	2.05437	0.978
	β_0	0.15	0.15043	0.00389	0.00386	0.14293	0.15807	0.912
	β_1	-0.001	-0.01007	0.00387	0.00357	-0.01708	-0.00305	0.956
	β_2	0	-0.00011	0.00472	0.00505	-0.01000	0.00975	0.956
	β_3	0	0.00060	0.00502	0.00506	-0.00937	0.01046	0.934
	σ_e^2	0.5	0.50116	0.01032	0.01134	0.47938	0.52380	0.967
	σ_0^2	0.5	0.49894	0.02165	0.02573	0.45064	0.55147	0.978
	d^2	0.0001	0.00014	0.00005	0.00009	0.00005	0.00040	1.000
5-3	μ_0	2	2.00031	0.03307	0.03292	1.93646	2.06549	0.909
	β_0	0.15	0.14792	0.00842	0.00689	0.13416	0.16123	0.879
	β_1	-0.001	-0.00885	0.00516	0.00561	-0.01988	0.00217	0.939
	β_2	0	0.00203	0.00846	0.00812	-0.01402	0.01770	0.939
	β_3	0	-0.00139	0.00763	0.00824	-0.01744	0.01470	0.970
	σ_e^2	0.5	0.49772	0.01334	0.01580	0.46776	0.52963	1.000
	σ_0^2	0.5	0.50570	0.03479	0.02946	0.45059	0.56593	0.879
	d^2	0.0001	0.00015	0.00006	0.00014	0.00005	0.00055	1.000
5-2	μ_0	2	1.99589	0.04328	0.03872	1.92036	2.07147	0.880
	β_0	0.15	0.15259	0.01015	0.00902	0.13578	0.17094	0.920
	β_1	-0.001	-0.00880	0.00686	0.00690	-0.02241	0.00467	0.960
	β_2	0	-0.00064	0.00810	0.00971	-0.01966	0.01808	1.000
	β_3	0	-0.00251	0.01064	0.01014	-0.02229	0.01763	0.920
	σ_e^2	0.5	0.49863	0.02771	0.02236	0.45677	0.54434	0.920
	σ_0^2	0.5	0.49362	0.04402	0.03309	0.43178	0.56135	0.960
	d^2	0.0001	0.00017	0.00006	0.00016	0.00005	0.00064	1.000

Note. s.d.: empirical standard deviation for the parameter estimates based on the replications of the simulation. a.s.e.: the average standard error for the parameter estimates. 95% CI: 95% confidence interval. CP: the 95% coverage probability.

B. Scripts and Codes

B.1 R codes for simulation

B.1.1 R codes for simulating the quadratic growth rate model

```
## Set working directory
setwd("D:/zzy/research/Dissertation/Simulation/simplerate")

library(mvtnorm)
mub<-c(.3, -.05, 0)
covb<-array(c(.25, 0, 0, 0, .0004, 0, 0, 0, .06), dim=c(3,3))
gamma<-c(1, -.05, 0)
sige<-sqrt(.3)
quad.gen<-function(N, T, M){
y<-array(,dim=c(N,T))
x<-array(,dim=c(N,2))
MY<-array(9999, dim=c(N,T))
for (i in 1:N){
 ## Generate b_i
 x[i,2]<-rbinom(1,1,.5)
 x[i,1]<-rnorm(1)
 lsq<-rmvnorm(1, mub, covb)
 for (t1 in 1:T){
   ## generate data
   r<-gamma[1]+gamma[2]*x[i,1]+gamma[3]*x[i,2]+lsq[3]
   y[i,t1]<-lsq[1]+r*t1+lsq[2]*(t1*t1-2*t1)+rnorm(1,0,sige)
 }
  start<-sample(1:(T-M+1),1)
  MY[i, start:(start+M-1)]<-y[i, start:(start+M-1)]
}
Y<-cbind(MY, x)
Y
}

ptm <- proc.time()
N<-100
T<-10
M<-2
R<-200
```

```
resfile<-paste("res-",N,"-",T,"-",M,".txt", sep="")
for (i in 1:R){
   write.table(quad.gen(N,T,M), "quad_sim.txt",
               row.names=F, col.names=F)
   for (j in 1:10){
       cmd.quad<-paste("c:/programs/mplus/mplus ",
               "rate", j, ".inp out1.out", sep="")
       system(cmd.quad,invisible = TRUE)
       if (file.exists("quadest.txt")){
          results<-scan("quadest.txt", quiet=TRUE)
          cat(c(i,j))
          cat("\n")
          cat(c(i,j,results), file=resfile, append=T)
          unlink("quadest.txt")
        }else{
          cat(c(i,j))
          cat("\n")
          cat(c(i,j,9999), file=resfile, append=T)
        }
       cat("\n",  file=resfile, append=T)
   }
}
proc.time()-ptm

## Function to process the simulation results
proc.res<-function(resfile){
par.true<-c(.3, .3, 1, -.05, 0, 0, 0, -.05, 0, 0, 0,
            0, 0, .25, 0, .06, 0, 0, .0004)
results<-read.table(resfile)
ndim<-dim(results)
stat<-t(array(rep(c(N,T,M, ndim[1]/10),19), dim=c(4,19)))
stat<-cbind(par.true,stat)
for (j in 1:10){
  temp<-results[results[,2]==j,]
  temp.stat<-NULL
  for (i in 3:21){
    est<-temp[,i]
    sdest<-temp[,(19+i)]
    low.ci<-est-1.96*sdest
    up.ci<-est+1.96*sdest
    coverage<-(up.ci>par.true[i-2]) & (low.ci<par.true[i-2])
    cvg.low<-(low.ci>par.true[i-2])
    cvg.up<-(up.ci<par.true[i-2])
    temp2<-cbind(temp[,2],est,sdest,low.ci,up.ci,
```

```
                    coverage,cvg.low,cvg.up)
    temp.stat<-c(temp.stat,apply(temp2, 2, mean))
    }
    stat<-cbind(stat, t(array(temp.stat, dim=c(8,19))))
}
write.table(stat, "stat.txt",row.names=F, col.names=F, sep=',')
}

proc.res(resfile)
```

B.1.2 R codes for simulating the random coefficient LDSM model

```
library(mvtnorm)
library(coda)

## Function to generate data
ldsm.gen<-function(N, T, M){
## N: sample size
## T: No. of occasions
# Mean of the initial level
mu0<-c(2,0)
# Covariance matrix of the initial level and rate of growth
sig0<-array(c(.5,0,0,.0001),dim=c(2,2))
# standard deviaiton of the residual errors
sige<-sqrt(.5)
beta<-c(.15, -.01, 0)
y<-array(,dim=c(N,T))
Y<-array(,dim=c(N,T))
x<-array(,dim=c(N,2))
MY<-array(NA, dim=c(N,T))
b<-rep(0,N)
for (i in 1:N){
 ## Generate b_i
 x[i,2]<-rbinom(1,1,.5)
 x[i,1]<-rnorm(1)
 yb<-rmvnorm(1,mu0,sig0)
 y[i,1]<-yb[1]
 b[i]<-beta[1]+beta[2]*x[i,1]+beta[3]*x[i,2]+yb[2]
 for (t1 in 2:T){
   ## generate data
   y[i,t1]<-(1+b[i])*y[i,t1-1]
 }
}
```

```
for (i in 1:N){
  for (t1 in 1:T){
    Y[i,t1]<-y[i,t1]+rnorm(1,0,sige)
  }
  start<-sample(1:(T-M+1),1)
  MY[i, start:(start+M-1)]=Y[i, start:(start+M-1)]
}
X<-cbind(MY, x)
X
}

## Set the sample size and number of occasions
N<-100
T<-10
M<-10

## Data file name
datafile<-paste("data-",N,"-",T,"-",M,".txt", sep="")
## results file name
resfile<-paste("res-",N,"-",T,"-",M,".txt", sep="")
## batch file name
batch<-paste("batch-",N,"-",T,"-",M,".cmd", sep="")
## command to run JAGS
cmd.jags<-paste("/share/apps/contrib/JAGS-1.0.1/bin/jags
                 <", batch, sep="")

## Write the batch file to run JAGS
coda<-paste( "coda *, stem(\"", N,"-",T,"-",M,"\")\n", sep="" )
cat("model in ldsm.bug\n", file=batch)
cat("data in ", datafile, "\n", file=batch, append=T)
cat("compile\n", file=batch, append=T)
cat("inits in ini.R\n", file=batch, append=T)
cat("initialize\n", file=batch, append=T)
cat("update 10000\n", file=batch, append=T)
cat("monitor set par\n", file=batch, append=T)
cat("update 10000\n", file=batch, append=T)
cat(coda, file=batch, append=T)

for (b in 1:201){
X<-ldsm.gen(N, T, M)
dump("N", file=datafile)
dump("T", file=datafile, append=T)
dump("X", file=datafile, append=T)
system(cmd.jags)
```

```
codaindex<-paste( N, "-", T, "-", M,"index.txt", sep="")
codadata<-paste( N, "-", T, "-", M,"chain1.txt", sep="")

data<-read.coda(codadata, codaindex)
res<-summary(data)
cat(c(b,t(cbind(res$statistics, res$quantiles))),
    file=resfile, append=T)
cat("\n",  file=resfile, append=T)

## Calculate the coverage probability

cov.fun<-function(para,lower,upper){
ndim<-dim(lower)
cov.prob<-NULL
for (i in 1:ndim[2]){
cov.prob<-c(cov.prob, sum(lower[,i]<para[i] &
            para[i]<upper[,i])/ndim[1])
}
cov.prob
}


## original results
proc.res<-function(){
  res<-read.table(resfile)
  coverage<-cov.fun(c(2,.15,-.01,0,0, 0.5,.5, .0001),
                    res[,c(6,15,24,33,42,51,60,69)],
                    res[,c(10,19,28,37,46,55,64,73)])
  cbind(mean(res[,c(2,11,20,29,38,47,56,65)]),
        sd(res[,c(2,11,20,29,38,47,56,65)]),
        mean(res[,c(3,12,21,30,39,48,57,66)]),
        mean(res[,c(6,15,24,33,42,51,60,69)]),
        mean(res[,c(10,19,28,37,46,55,64,73)]), coverage)
}
}
```

B.2 An example script for quadratic growth rate models

```
TITLE: Growth Rate Model for Age 15
DATA:
    FILE IS "quad_sim.txt";
VARIABLE:
   NAMES ARE math1-math10 bpi gender;
   USEVARIABLES ARE
```

```
   math1-math10 bpi gender bsex;
   missing=all(9999);
DEFINE:
   bsex=bpi*gender;
ANALYSIS: TYPE = mean; !MISS;
          ESTIMATOR = ML;
          ITERATIONS = 2000;
   COVERAGE = .00;
MODEL:
        level BY math1@1;
        level BY math2@1;
        level BY math3@1;
        level BY math4@1;
        level BY math5@1;
        level BY math6@1;
        level BY math7@1;
        level BY math8@1;
        level BY math9@1;
        level BY math10@1;

        slope BY math1@1;
        slope BY math2@2;
        slope BY math3@3;
        slope BY math4@4;
        slope BY math5@5;
        slope BY math6@6;
        slope BY math7@7;
        slope BY math8@8;
        slope BY math9@9;
        slope BY math10@10;

        qslope BY math1@-1;
        qslope BY math2@0;
        qslope BY math3@3;
        qslope BY math4@8;
        qslope BY math5@15;
        qslope BY math6@24;
        qslope BY math7@35;
        qslope BY math8@48;
        qslope BY math9@63;
        qslope BY math10@80;
!Means
        [math1-math10@0];
        [level slope qslope*];
```

```
!Variances
        math1-math10 (1);
        level qslope slope;
        level on bpi gender bsex;
        qslope on bpi gender bsex;
        slope on bpi gender bsex;

SAVEDATA:
 RESULTS=quadest.txt;
 FORMAT is F10.5;
```

B.3 WinBUGS codes for the random coefficient LDSM

```
model{
   for (i in 1:N){
       X[i,1]~dnorm(y[i,1],theta)
       y[i,1]~dnorm(mu,psigs)
        for (t in 2:T){
             X[i,t]~dnorm(y[i,t], theta)
             y[i,t]<-(beta[i]+1)*y[i,t-1]
        }
        beta[i]~dnorm(mub[i], invsigb)
        mub[i]<-b[1]+b[2]*X[i,T+1]+b[3]*X[i,T+2]
                +b[4]*X[i,T+1]*X[i,T+2]
   }
        theta~dgamma(.001,.001)
        psigs~dgamma(.001,.001)
        invsigb~dgamma(1.0E-8,1.0E-8)
        mu~dnorm(0,.00001)
for (i in 1:4){
  b[i]~dnorm(0,1.0E-6)
}

par[1]<-mu
par[2]<-b[1]
par[3]<-b[2]
par[4]<-b[3]
par[5]<-b[4]
par[6]<-1/theta
par[7]<-1/psigs
par[8]<-1/invsigb
}
```