THE HYPOELLIPTIC HEAT KERNEL OF INFINITE-DIMENSIONAL LIE GROUPS: HEISENBERG-LIKE QUASI-INVARIANCE AND THE TAYLOR ISOMORPHISM

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(ABSTRACT)

The Gaussian distribution on $\mathbb{R}^n$ translates to infinite-dimensional (separable) Banach spaces by assuming the structure of an abstract Wiener space. The equivalent of the Gaussian distribution on Lie groups is called the heat kernel measure, named for its connection to a version of the Lie group equivalent of the heat equation. In this work, we will investigate combining these ideas to define what it means for G to be a (simply connected graded nilpotent) abstract Wiener Lie group. We will impose 2 major complications. Firstly, we restrict our attention to the hypoelliptic setting, in which the diffusion is only infinitesimally generated by a subset of the possible directions, called "horizontal" directions. Secondly, we allow for the possibility that there are infinitely-many "vertical" directions. Imposing both of these restrictions simultaneously complicates the analysis, and will require specifying a generalization of the Hörmander (bracket-generating) condition. Presented here are 2 primary results. The first is a quasi-invariance result for Heisenberg-like groups, meaning that we restrict to when G is nilpotent of step-2. There, we show that, under the right conditions, the infinite-dimensional heat kernel measure is invariant under shifts of a certain group, which we call the Cameron-Martin subgroup. The second result is a Taylor isomorphism that allows for G to be of arbitrary step. It provides a classification of the "L^2 holomorphic" functions on G. While there are a number of works that illustrate similar results, this work is the first to show such results for infinite-dimensional hypoelliptic diffusions in the presence of infinitely-many vertical directions.

Contents

Chapter 1

Common Notions

1 Introduction

The $\mathbb{R}$-valued Gaussian distribution is a pivotal object of probability theory. It plays a role in the solution to the heat equation, while also approximately describing the distribution of a random walk on $\mathbb{R}$ with independent steps. But consider that the notions of heat and random motion are concepts that readily apply to other types spaces. For example, one can consider infinite-dimensional vector spaces. A vector space equipped with a nondegenerate Gaussian measure is called an *abstract Wiener space*. Such spaces can model data that is considered "high-dimensional" or consisting of many components. Additionally, this also applies to the path space of a random walk, where one tracks every position at every time-step, rather than just the endpoint distribution. Every abstract Wiener space has a Hilbert subspace called the *Cameron-Martin subspace*, whose inner product determines the Gaussian distribution, and it plays a critical role in its analysis.

On the other hand, one can consider "curved spaces," like manifolds or Lie groups. These spaces (when given a Riemannian structure) are equipped with a distribution called the *heat kernel measure*, named for its connection to (the Riemannian manifold equivalent of) the heat equation. Such spaces are capable of modelling data that has dependency. For example, a random 3-D orientation can be realized as a random element orthogonal 3×3 matrix, and while such objects are naturally 9-dimensional, they only exhibit 3 degrees of freedom, the other 6 entries being determined via (nonlinear) relationships with the first 3 (indeed, the set of possible orientations can be realized as $SO(3)$, a 3-dimensional Lie group). It stands to reason that data that both is high-dimensional and exhibits dependencies could be modelled by a distribution on a space that is both infinite-dimensional and curved.

The aim of the research presented here is to investigate the heat kernel distribution of infinite-dimensional simply connected nilpotent Lie groups. Contrary to much of the previous research, we impose 2 complications. Firstly, we restrict our attention to the hypoelliptic heat kernel, meaning that our distribution only "spreads" in a subset of the possible directions. Secondly, we allow for the possibility that there are infinitely-many other directions, called "vertical" directions. To impose both of these restrictions causes complications (even the *definition* of the heat kernel distribution itself is less obvious), for which previous methods of analysis do not suffice. To the best of the author's knowledge, this work is the first of its kind to explore this setting in the Lie group context.

It will be reiterated in Section 4 and Section 9 that there are 3 spaces that appear in our analysis:

1. G, an (simply connected graded nilpotent) abstract Wiener Lie group, with heat kernel distributed element g_t and distribution ν_t.

2. $\mathfrak{g}_{CM}$, the *Cameron-Martin subalgebra* with Lie bracket $[\cdot, \cdot]$.

3. G_{CM}, the *Cameron-Martin subgroup* with a horizontal (Carnot-Carathéodory) metric d.

In the presence of infinitely-many vertical directions, it is possible for all 3 of these spaces to be different sets from one another.

There organization of this thesis is the following. Chapter 1 will introduce some of the common background, theory, and notions that this work will use. Included are notes on Hilbert space tensor products, weakly Hilbert-Schmidt maps, abstract Wiener spaces, Lie groups, and the Hörmander condition. It is also there that we will provide a generic definition for simply connected nilpotent abstract Wiener Lie groups and provide examples. Chapter 2 will prove a quasi-invariance result, which can be interpreted as a smoothness result for the heat kernel measure ν_t. There, in order to implement generalized curvature-dimension bounds, we assume that our Lie group is "Heisenberg-like" (simply connected graded nilpotent of step 2). Chapter 3 will prove a Taylor isomorphism, which classifies the space of "L^2 holomorphic" functions. This is achieved under the assumption that our construction is simply connected, graded, nilpotent, and complex.

2 Background

2.1 Hilbert space constructions

For the entirety of this work, even if not explicitly stated, we will assume that all Hilbert spaces are separable (and thus possess a countable orthonormal basis).

2.1.1 Some functional analysis

Recall, for example from [Rud91], that the unit ball of a real Hilbert space is weakly compact. Aside from this, we will also make use of results below, which, while they are well-understood, are not always explicitly stated in literature.

Proposition 2.1. *For a bounded linear operator between Hilbert spaces, $T : H \to Z$, T^* is bounded below if and only if T is surjective.*

Proof. First recall that an operator T is invertible if and only if T^* is invertible, for $TS = ST = I$ if and only if $T^*S^* = S^*T^* = I$. Now suppose T^* is bounded below, meaning that there exists a constant $c > 0$ such that, for all $z \in Z$, $\|T^*z\|_H \geq c\|z\|_Z$. If $T^*(z_n)$ converges to some $h \in H$, then $\|z_n - z_m\|_Z \leq \frac{1}{c}\|T^*(z_n) - T^*(z_m)\|_H$, so $(z_n)_{n\in\mathbb{N}}$ must be Cauchy, and hence convergent to some $z \in Z$, for which the continuity of T^* implies $T^*(z) = h$. This shows us that $\mathrm{im}(T^*)$ is a closed subspace of H. Since T^* being bounded below implies that T^* is injective, we have that $T^* : Z \to \mathrm{im}(T^*)$ is invertible, so that $T : \mathrm{im}(T^*) \to Z$ is invertible, and hence $T : H \to Z$ must be surjective. Conversely, T being surjective implies $T : \ker(T)^\perp \to Z$ is (by the open mapping theorem) an invertible operator, which implies $T^* : Z \to \ker(T)^\perp$ is invertible, and hence bounded below. Indeed, we have

$$\|z\|_Z \;=\; \|T^{*-1}T^*z\|_Z \;\leq\; \|T^{*-1}\|\,\|T^*z\|_H$$

□

Proposition 2.2. *Suppose $v_n \in H$ is a bounded sequence such that, for a dense collection $\mathcal{J} \subseteq H^*$, for all $\alpha \in \mathcal{J}$, $\alpha(v_n) \to \alpha(v)$. Then v_n converges weakly to v.*

Proof. Say $\|v_n\|_H \leq C$ for all $n \in \mathbb{N}$. Let $\alpha \in H^*$, so there exists a sequence $(\alpha_m)_{m\in\mathbb{N}}$ in $\mathcal{J}$ such that $\alpha_m \to \alpha$ (in H^*). Then, for any $m \in \mathbb{N}$,

$$\begin{aligned}
&\lim_{n\to\infty} |\alpha(v_n) - \alpha(v)| \\
&\quad\leq \lim_{n\to\infty} \Big(\big|(\alpha - \alpha_m)(v_n)\big| + \big|\alpha_m(v_n) - \alpha_m(v)\big| + \big|(\alpha_m - \alpha)(v)\big| \Big) \\
&\quad\leq 2C\|\alpha - \alpha_m\|_{H^*} + \lim_{n\to\infty} \big|\alpha_m(v_n) - \alpha_m(v)\big| \\
&\quad= 2C\|\alpha - \alpha_m\|_{H^*}\,.
\end{aligned}$$

Then for this to be true for all $m \in \mathbb{N}$ implies that $\lim_{n\to\infty} |\alpha(v_n) - \alpha(v)| = 0$, which implies the desired weak convergence. $\square$

2.1.2 Tensor products

Given real or complex Hilbert spaces K_1, K_2, we may take the (real or complex) algebraic tensor product of K_1 and K_2 consisting of finite sums of simple tensors, or formal pairs of the form $h \otimes k$, which satisfy for all $h_1, k_1 \in K_1$, $h_2, k_2 \in K_2$, and α (in $\mathbb{R}$ or in $\mathbb{C}$),

$$\begin{aligned} h_1 \otimes h_2 + h_1 \otimes k_2 &= h_1 \otimes (h_2 + k_2) \\ h_1 \otimes h_2 + k_1 \otimes h_2 &= (h_1 + k_1) \otimes h_2 \\ (\alpha h_1) \otimes h_2 \;=\; h_1 \otimes (\alpha h_2) &= \alpha(h_1 \otimes h_2) \end{aligned}$$

We may define an inner product as the unique (real or complex) bilinear map satisfying

$$\langle h_1 \otimes h_2, k_1 \otimes k_2 \rangle_{K_1 \otimes K_2} \;=\; \langle h_1, k_1 \rangle_{K_1} \langle h_2, k_2 \rangle_{K_2} \,,$$

which determines a norm. We refer to the closure with respect to the norm as the (real or complex) *Hilbert space tensor product* of K_1 and K_2, simply denoted as $K_1 \otimes K_2$. If $\{e_{1,j}\}_{j\in\Lambda_1}$ and $\{e_{2,j}\}_{j\in\Lambda_2}$ are orthonormal bases of K_1 and K_2 respectively, then $\{e_{1,j_1} \otimes e_{2,j_2}\}_{(j_1,j_2)\in\Lambda_1\times\Lambda_2}$ is an orthonormal basis of $K_1 \otimes K_2$. Given Hilbert spaces $K_1, \ldots, K_N$, we may naturally inductively define the N-fold Hilbert space tensor product $K_1 \otimes \ldots \otimes K_N$. See [Jan97, Appendix E] for more information.

One useful example is the following: for any real (or complex) Hilbert space K, let $L^2([0,1], K)$ denote the square-integrable measurable functions from $[0,1]$ to K with typical L^2 inner product. Then $L^2([0,1], K) \cong L^2([0,1], \mathbb{R}) \otimes K$ (or $L^2([0,1], \mathbb{C}) \otimes K$). See, for example, [Jan97, Example E.12].

2.1.3 Hilbert-Schmidt and weakly Hilbert-Schmidt maps

Recall that, for $N \in \mathbb{N}$, a continuous real- or complex-linear map $M : H \to K$ is called *Hilbert-Schmidt* if, given orthonormal bases $\{e_j\}_{j\in\Lambda_H}$ and $\{f_j\}_{j\in\Lambda_K}$ of H and K respectively,

$$\sum_{j\in\Lambda_H} \left\|M(e_j)\right\|_K^2 \;=\; \sum_{j\in\Lambda_H} \sum_{\ell\in\Lambda_K} \left|\langle M(e_j), f_\ell \rangle_K\right|^2 \;<\; \infty \,.$$

We will let $HS(H,K)$ denote the set of Hilbert-Schmidt maps $H \to K$. The set of Hilbert-Schmidt maps naturally have a norm determined by the following real or complex inner product:

$$\begin{aligned}\langle M_1, M_2\rangle_{HS(H,K)} &= \sum_{j\in\Lambda_H} \big\langle M_1(e_j), M_2(e_j)\big\rangle_K \\ &= \sum_{j\in\Lambda_H}\sum_{\ell\in\Lambda_K} \langle M_1(e_j), f_\ell\rangle_K \langle M_2(e_J), f_\ell\rangle_K\,.\end{aligned}$$

For real Hilbert spaces H and K, we have a natural identification of $HS(H,K)$ with $(H\otimes K)^*$ via $H\otimes K \ni h\otimes k \mapsto \langle k, M(h)\rangle_K \in \mathbb{R}$, and thus a natural identification of $HS(H,K)$ with $H\otimes K$.

Consider the following estimate.

Proposition 2.3. *For Hilbert spaces A, B, C, if $T : A \to B$ is Hilbert-Schmidt and $L : B \to C$ is linear continuous, then $L\circ T$ is Hilbert-Schmidt, and satisfies $\|L\circ T\|_{HS(A,C)} \le \|L\|_{B,C}\|T\|_{HS(A,B)}$.*

On the other hand, if $T : B \to C$ is Hilbert-Schmidt and $L : A \to B$ is linear continuous, then $\|T\circ L\|_{HS(A,C)} \le \|T\|_{HS(B,C)}\|L\|_{A,B}$.

Proof. Let $\{a_j\}_{j\in\Lambda_A}$, $\{b_j\}_{j\in\Lambda_B}$, $\{c_j\}_{j\in\Lambda_C}$ be bases of A, B, C respectively. Then

$$\|L\circ T\|_{HS(A,C)} = \sum_{j\in\Lambda_A} \|LT(a_j)\|_C^2 \le \sum_{j\in\Lambda_A} \|L\|_{B,C}^2 \|T(a_j)\|_B^2\,,$$

which proves the first inequality. For the second,

$$\begin{aligned}\|T\circ L\|_{HS(A,C)} &= \sum_{j=1}^\infty \|TL(a_j)\|_C^2 = \sum_{j=1}^\infty\sum_{\ell=1}^\infty |\langle TL(a_k), c_\ell\rangle_C|^2 \\ &= \sum_{j=1}^\infty\sum_{\ell=1}^\infty |\langle a_k, L^*T^*(c_\ell)\rangle_A|^2 = \sum_{\ell=1}^\infty \|L^*T^*(c_\ell)\|_A^2 \\ &\le \|L^*\|_{B,A}^2 \sum_{\ell=1}^\infty \|T^*(c_\ell)\|_B^2 = \|L\|_{A,B}^2 \sum_{\ell=1}^\infty\sum_{k=1}^\infty |\langle T(b_k), c_\ell\rangle_C|^2 \\ &= \|L\|_{A,B}^2 \|T\|_{HS(B,C)}^2\,.\end{aligned}$$

□

More generally, a we say that a multilinear map $M : K_1 \times \ldots K_N \to K$ is *Hilbert-Schmidt* if, given orthonormal bases $\{e_{n,j}\}_{j\in\Lambda_n}$ of K_n, we have

$$\sum_{n=1}^N \sum_{j_n\in\Lambda_n} \big\|M(e_{1,j_1},\ldots,e_{N,j_N})\big\|_K^2 < \infty\,.$$

Hilbert-Schmidt maps will always have a continuous linear extension to the tensor product $K_1 \otimes \ldots \otimes K_N \to K$. In fact, as before, there is a natural identification of the set of such Hilbert-Schmidt maps with elements of $K_1 \otimes \ldots \otimes K_N \otimes K$.

If K is finite-dimensional, a multilinear map having an extension to $K_1 \otimes \ldots \otimes K_N \to K$ is equivalent to being Hilbert-Schmidt. We will now characterize this property of multilinear maps in a way that allows K to be infinite-dimensional. This and other information on weakly Hilbert-Schmidt maps is provided in [KR97, section 2.6].

Proposition 2.4. *Given any Hilbert spaces $K, K_1, \ldots, K_N$ with orthonormal basis $\{e_{n,j}\}_{j\in\Lambda_n} \subseteq K_n$ and any real (complex) multilinear map $M : K_1 \times \ldots \times K_N \to K$, M having a linear continuous extension $\widetilde{M} : K_1 \otimes \ldots \otimes K_N \to K$ is equivalent to the existence of a constant $C \in \mathbb{R}$ such that*

$$\sum_{n=1}^{N} \sum_{j_n\in\Lambda_n} \left|\langle v, M(e_{1,j_1}, \ldots, e_{N,j_N})\rangle_K\right|^2 \;\leq\; C\|v\|_K^2\,,$$

or equivalently,

$$C \;:=\; \sup_{\|v\|_K=1} \sum_{n=1}^{N} \sum_{j_n\in\Lambda_n} \left|\langle v, M(e_{1,j_1}, \ldots, e_{N,j_N})\rangle_K\right|^2 \;<\; \infty\,.$$

We refer to this criterion as being ***weakly Hilbert-Schmidt****. Furthermore, the extension M is surjective if and only if there exists a constant $c > 0$ such that*

$$\sum_{n=1}^{N} \sum_{j_n\in\Lambda_n} \left|\langle v, M(e_{1,j_1}, \ldots, e_{N,j_N})\rangle_K\right|^2 \;\geq\; c\|v\|_K^2\,,$$

or equivalently,

$$c \;:=\; \inf_{\|v\|_K=1} \sum_{n=1}^{N} \sum_{j_n\in\Lambda_n} \left|\langle v, M(e_{1,j_1}, \ldots, e_{N,j_N})\rangle_K\right|^2 \;>\; 0\,.$$

Proof. First assuming the existence of the constant C, we will define a map $W : K \to K_1 \otimes \ldots \otimes K_N$ as

$$Wv \;=\; \sum_{n=1}^{N} \sum_{j_n\in\Lambda_n} \langle v, M(e_{1,j_1}, \ldots, e_{N,j_N})\rangle_K \; e_{1,j_1} \otimes \ldots \otimes e_{N,j_N}\,.$$

Indeed, this will be a linear continuous map because

$$
\begin{aligned}
\|Wv\|_{K_1\otimes\ldots\otimes K_N} &= \sum_{n=1}^{N}\sum_{j_n\in\Lambda_n} \left|\langle Wv, e_{1,j_1}\otimes\ldots\otimes e_{N,j_N})\rangle_{K_1\otimes\ldots\otimes K_N}\right|^2 \\
&= \sum_{n=1}^{N}\sum_{j_n\in\Lambda_n} \left|\langle v, M(e_{1,j_1},\ldots,e_{N,j_N})\rangle_{K}\right|^2 \\
&\leq C\|v\|_K^2 .
\end{aligned}
$$

Then consider that $W^*: K \to K_1 \otimes \ldots \otimes K_N$ satisfies

$$
W^*\left(\sum_{n=1}^{N}\sum_{j_n\in\Lambda_n} a_{j_1,\ldots,j_N}\, e_{1,j_1}\otimes\ldots\otimes e_{N,j_N}\right) = \sum_{n=1}^{N}\sum_{j_n\in\Lambda_n} a_{j_1,\ldots,j_N} M(e_{1,j_1},\ldots,e_{N,j_N}) .
$$

Thus, $\widetilde{M} := W^*$ is indeed the desired linear continuous extension.

For the other direction, if $\widetilde{M}$ is continuous linear, then so is its adjoint $\widetilde{M}^*$. Then

$$
\begin{aligned}
\sum_{n=1}^{N}\sum_{j_n\in\Lambda_n} &\left|\langle v, M(e_{1,j_1},\ldots,e_{N,j_N})\rangle_{K}\right|^2 \\
&= \sum_{n=1}^{N}\sum_{j_n\in\Lambda_n} \left|\langle v, \widetilde{M}(e_{1,j_1}\otimes\ldots\otimes e_{N,j_N})\rangle_{K}\right|^2 \\
&= \sum_{n=1}^{N}\sum_{j_n\in\Lambda_n} \left|\langle \widetilde{M}^* v, e_{1,j_1}\otimes\ldots\otimes e_{N,j_N})\rangle_{K_1\otimes\ldots\otimes K_N}\right|^2 \\
&= \|\widetilde{M}^* v\|_{K_1\otimes\ldots\otimes K_N}^2 \;\leq\; \|\widetilde{M}^*\|^2\|v\|_K^2 .
\end{aligned}
$$

To discuss surjectivity, we know that, by Proposition 2.1, $\widetilde{M}$ is surjective if and only if $\widetilde{M}^*$ is bounded below, so the equality

$$
\sum_{n=1}^{N}\sum_{j_n\in\Lambda_n} \left|\langle v, M(e_{1,j_1},\ldots,e_{N,j_N})\rangle_{K}\right|^2 = \|\widetilde{M}^* v\|_{K_1\otimes\ldots\otimes K_N}^2
$$

proves the claim.

□

2.1.4 The paths of finite energy

We define the set of *paths of finite energy* and list a number of facts that will be useful throughout. For any real (complex) Hilbert space K, we will define $\mathcal{H}_0([0,1],K)$ as the set of finite-energy paths in K, that is,

$$\mathcal{H}_0([0,1],K) \;=\; \left\{ f:[0,1]\to K \;:\; f \text{ absolutely continuous}, \int_0^1 \|f'(t)\|_K^2 dt < \infty \right\},$$

which naturally has the real (complex) inner product

$$\langle f, g\rangle_{\mathcal{H}_0([0,1],K)} \;:=\; \int_0^1 \langle f'(t), g'(t)\rangle_K dt\,.$$

Then $\mathcal{H}_0([0,1],K)$ will also be a real (complex) Hilbert space.

We see that, for $t\in[0,1]$, we may interpret $\int_0^t f'(s)ds$ as a Bochner integral, for which we have $\int_0^t f'(s)ds = f(t)$. In fact, point evaluation is continuous with respect to the norm on $\mathcal{H}_0([0,1],K)$. Indeed, for $t\in[0,1]$,

$$\begin{aligned} \|f(t)\|_K^2 \;&=\; \left\| \int_0^t f'(s)ds \right\|_K^2 \;\le\; \left(\int_0^t \|f'(s)\|_K^2 ds \right)\left(\int_0^t 1 ds \right) \\ &=\; t\|f\|^2_{\mathcal{H}_0([0,t],K)}\,. \end{aligned} \tag{1.1}$$

We have a natural isomorphism with $L^2([0,1],K)$ given via the integration operation $\mathcal{I}: L^2([0,1],K)\to\mathcal{H}_0([0,1],K)$, defined as

$$\mathcal{I}f(t) \;=\; \int_0^t f(s)ds \;=\; \int_0^1 1_{[0,t]}(s)f(s)ds\,.$$

That this is an isomorphism is justified by the fundamental theorem of calculus.

2.2 Abstract Wiener space, Brownian motion, stochastic integrals

2.2.1 Definition

In [Wie23], Norbert Wiener introduced the classical Wiener space: the set of continuous functions $f:[0,1]\to\mathbb{R}$ with $f(0)=0$ equipped with the Gaussian measure induced by a standard Brownian motion $B.:\Omega\times[0,1]\to\mathbb{R}$, wherein the functions of finite energy $\mathcal{H}_0([0,1],\mathbb{R})$ played a crucial role in determining the Gaussian structure. Leonard Gross in [Gro67] generalized the notion by considering a Gaussian measure on a general Banach space W whose structure is determined by a dense Hilbert subspace H. This construction is called an *abstract Wiener space*, which we will now describe. This information is primarily derived from [Dri10; DG10; Bog14; Kuo75].

Given a real separable Banach space W, a measure μ on W is called a *Gaussian measure* if its characteristic functional satisfies, for $u \in W^*$,

$$\widehat{\mu}(u) := \int_W e^{iu(x)} d\mu(x) \;=\; e^{-\frac{1}{2}q(u,u)}$$

for some nonnegative symmetric bilinear form $q : W^* \times W^* \to \mathbb{R}$. For now, we will assume that Gaussian measures on W are nondegenerate, meaning that they are with "full support" (every open subset of W has positive measure), which corresponds precisely to q being positive definite (so an inner product) on $W^* \times W^*$.

For $w \in W$, define

$$\|w\|_H \;:=\; \sup_{u \in W^* \setminus 0} \frac{|u(w)|}{\sqrt{q(u,u)}}\,.$$

Then define $H = \{w \in W \mid \|w\|_H < \infty\}$. Then H is a subset called the *Cameron-Martin subspace* of W. By using properties of Gaussians, it can be derived that, for $u \in W^*$, we have that $u \in L^2(W)$, and in fact u is an $\mathbb{R}$-valued Gaussian random variable. Moreover, the inner product q determines the covariance across all elements of W^*, since $q(u,v) = \int_W u(x)v(x)d\mu(x) = \langle u, v\rangle_{L^2(W,\mathbb{R})}$.

We may take the completion of W^* with respect to the L^2 inner product as $\overline{W^*}^{L^2(W)}$. Define the map $J : W^* \to W$ as $J(u) = \int_W u(x)x\, d\mu(x)$ via Bochner integrals. Then J is injective, continuous with respect to $\|\cdot\|_{L^2(W)}$, and thus extends to $\overline{W^*}^{L^2(W)}$, and $J\big(\overline{W^*}^{L^2(W)}\big) = H$. Then H has a natural inner product $\langle \cdot,\cdot\rangle_H$ defined as the push-forward of $\langle\cdot,\cdot\rangle_{L^2(W)}$. Then the map J provides a natural way to view $W^* \subseteq H \subseteq W$, by identifying W^* with $J(W^*)$, under which we may conclude that the inclusions are dense. Equivalently, if we are given that H is a Hilbert space and dense subset of W, we may use the Riesz representation theorem to naturally view $W^* \subseteq H^* \cong H \subseteq W$, again concluding that both inclusions must be dense.

The triple (W, H, μ) is what is referred to as an *abstract Wiener space*. Knowing that H is a Hilbert space whose inner product determines the measure μ, we will often simply write (W, H). Having provided the definition, the next subsection is devoted to listing more properties for abstract Wiener spaces.

2.2.2 Further properties

When we view $W^* \subseteq H \subseteq W$, the elements $h \in W^* \subseteq H$ are precisely those $h \in H$ for which the map $\langle\cdot, h\rangle_H : H \to H$ has an extension to a continuous linear map

$\langle \cdot, h \rangle_H : W \to W$. Given a finite-rank projection $P : H \to H$ where $PH \subseteq W^*$, we may find a finite orthogonal set $\{h_1, \ldots, h_k\} \subset W^*$ such that

$$P = \sum_{j=1}^{k} \langle \cdot, h_j \rangle_H h_j$$

from which we see that P has a continuous linear extension $P : W \to PH$. We will denote the set of such projections (or possibly their extensions) as $\mathrm{Proj}(W)$.

By density, it is possible to choose an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ for H that lies entirely within W^*. Then we may take an increasing sequence $(n_m)_{m\in\mathbb{N}}$ in $\mathbb{N}$ and define

$$P_m = \sum_{j=1}^{n_m} \langle \cdot, e_j \rangle_H e_j \,.$$

We call $(P_m)_{m\in\mathbb{N}}$ an increasing set of finite-rank projections, and will denote the set of such sequences as $\mathrm{Proj}(W)^{\uparrow}$. It is a fact that $P_m \to I_W$, the identity on W, in the operator norm topology.

It is a fact that, for any $h \in H$, the map $\langle h, \cdot \rangle_H : H \to H$ always has a (not necessarily continuous) measurable linear extension $\langle \cdot, h \rangle_H : W \to W$ (see [Bog14, Section 3.7] for information on measurable linear maps, or see [Zha82] for the more generic "quasilinear map"). Using this, it is possible to take any linear map $A : H \to H$ and derive a measurable linear extension to $W \to W$. Related to this is the fact that if (W, H) is an abstract Wiener space and Z is a separable Hilbert space, then any Hilbert-Schmidt map $H \to Z$ has a measurable-linear extension $W \to Z$.

As remarked in [Gro67], there are 2 key examples of abstract Wiener spaces. Firstly is of course the classical Wiener space from the start of Section 2.2. Secondly, if W is a Hilbert space, then (W, H) is an abstract Wiener space if and only if the inclusion $H \hookrightarrow W$ is Hilbert-Schmidt.

The Fernique theorem tells us that, for some $\epsilon > 0$,

$$\int_W e^{\epsilon \|x\|_W^2} d\mu(x) < \infty \,,$$

which suffices to prove that $\int_W \|x\|_W^p d\mu(x) < \infty$ for all $p \in \mathbb{N}$. In particular, a Gaussian-distributed element in W is in L^p for all $p \in [1, \infty)$.

This work does not provide an introduction to Brownian motion or stochastic calculus. Instead, we will simply refer the reader to the standard reference [Øks98], which covers the finite-dimensional scenario. Following [Kuo75] or [Bog14], we have a (infinite-dimensional) notion of a Brownian motion on W, in which we may compute stochastic integrals. Though for the purposes of this work and our definitions, we will only consider stochastic integrals of the projections of this Brownian motion and limits in probability therein.

Lastly, we remark that, in everything discussed above, one may apply a complex structure. If W is a complex Banach space, and we let W^* denote the complex dual, and if we assume that we have a sesquilinear nonnegative symmetric bilinear form (or Hermitian inner product) $q : W^* \times W^* \to \mathbb{C}$, then we may replace q in the construction with the real part $\operatorname{Re} q$, for which the measure will be called a *complex* Gaussian measure. In this case, H will naturally be a complex Hilbert space.

2.3 Lie algebras, Lie groups

2.3.1 Nilpotent Lie algebras, their simply connected Lie groups, and the Baker-Campbell-Hausdorff formula

A real (complex) *Lie algebra* is a (possibly infinite-dimensional) real (complex) Banach space $\mathfrak{g}$ with a *Lie bracket* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that must satisfy the properties of being (complex) bilinear, anti-symmetric ($[x,y] = -[y,x]$) and satisfy the Jacobi identity, meaning

$$[[x,y],z] + [[z,x],y] + [[y,z],x] \;=\; 0\,.$$

When $\mathfrak{g}$ is infinite-dimensional, we will add the criterion that $[\cdot,\cdot]$ is bilinear continuous, meaning $\|[x,y]\|_{\mathfrak{g}} \leq C\|x\|_{\mathfrak{g}}\|y\|_{\mathfrak{g}}$ for some C (we will later impose an even stronger condition, namely that $[\cdot,\cdot]$ is weakly Hilbert-Schmidt, as noted in (A2.1) and (A3.1). A Lie algebra is *nilpotent* if the lower central series defined as $\mathfrak{g}_1 = \mathfrak{g}$ and $\mathfrak{g}_n = [\mathfrak{g}, \mathfrak{g}_{n-1}]$ is eventually 0 for $n \geq N$ for some N. We will refer to N as the *step* of $\mathfrak{g}$.

A *Lie group* G, on the other hand, is a smooth manifold that also has a group operation $\cdot : G \times G \to G$ such that the map $(g_1, g_2) \mapsto g_1 \cdot g_2^{-1}$ is smooth. Every *nilpotent* Lie algebra has a naturally associated *simply connected* Lie group, which is simply the vector space $\mathfrak{g}$ itself with the operator $\cdot$ defined by the Baker-Campbell-Hausdorff formula:

$$\begin{aligned} x \cdot y \;:=\; & x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] - \frac{1}{12}[y,[x,y]] + \dots \\ =\; & \sum_{n=1}^{N} \sum_{\substack{r_i+s_i>0 \\ r_i,s_i\geq 0}} \frac{(-1)^{n-1}}{n} \frac{ad_x^{r_1} \circ ad_y^{s_1} \circ \dots \circ ad_x^{r_n} ad_y^{s_n}}{\left(\sum_{j=1}^{n}(r_j+s_j)\right)\prod_{j=1}^{n} r_i! s_i!}\,, \end{aligned} \tag{1.2}$$

where we define the linear map $ad_x : \mathfrak{g} \to \mathfrak{g}$ as $ad_x(y) = [x,y]$. These are the primary types of Lie groups and Lie algebras that this thesis explores, and is worthy of several remarks.

Remark 2.5. 1. First and foremost, for the uninitiated, we remark that the exact coefficients in (1.2) are not significant. All that is important is that there exists an expression for the group operation in terms of the Lie bracket.

2. The typical framework is to begin with a (finite-dimensional) Lie group G by specifying the manifold and group operator $\cdot$. The corresponding Lie algebra, $\mathfrak{g}$, is defined as $T_e(G)$, the tangent space at $e \in G$. To define its Lie bracket, for each $h \in \mathfrak{g}$, we associate a vector field $\widetilde{h}$ on G (whose definition depends on $\cdot$), called the *left-invariant vector field* (a definition provided in Section 2.3.2), and define the Lie bracket such that $\widetilde{[h_1, h_2]} = \widetilde{h}_1\widetilde{h}_2 - \widetilde{h}_2\widetilde{h}_1$ (in likeness to taking the "Lie bracket of vector fields"). In our construction, we are able to view $\mathfrak{g}$ as being both the Lie algebra and the Lie group because we simultaneously assume that the bracket is nilpotent and that the corresponding group is simply connected.

3. In the typical setting described above, the Baker-Campbell-Hausdorff formula is still a valid consequence, but it requires interpretation. Firstly, we note that general Lie groups have an exponential map $\exp : \mathfrak{g} \to G$, and the formula (1.2) gives the expression for z when $\exp(x)\cdot\exp(y) = \exp(z)$. Secondly, for general G, this series is summed over all $n \in \mathbb{N}$, for which one must interpret this formula in the sense of formal power series by regarding x and y as formal noncommuting algebra elements with the identification $xy - yx = [x, y]$. Alternatively, it is possible to take the series somewhat literally, as it is convergent for x and y sufficiently close to 0. In our construction, there is no interpretation necessary, because we view exp as the identity map and the Baker-Campbell-Hausdorff formula is necessarily a finite sum with no question of its convergence.

4. Note that this construction makes sense even if $\mathfrak{g}$ is infinite-dimensional, in spite of Lie groups traditionally being thought of as finite-dimensional manifolds. This is simply by virtue of the fact that $\mathfrak{g}$ possessing a nilpotent Lie bracket allows one to define a group operation on $\mathfrak{g}$ itself via (1.2). When we view $\mathfrak{g}$ as a group, we will still endow it with the topology induced by $\|\cdot\|_{\mathfrak{g}}$. Assuming that $[\cdot,\cdot]$ is bilinear continuous, and knowing that $g^{-1} = -g$ for all $g \in \mathfrak{g}$, we may conclude that the map $(g_1, g_2) \mapsto g_1 \cdot g_2^{-1}$ is continuous with respect to $\|\cdot\|_{\mathfrak{g}}$, which allows us to consider $\mathfrak{g}$ to be an infinite-dimensional Lie group in a sense consistent with other literature; see, for example, [Sch10].

5. Periodically, especially when our nilpotent Lie algebra $\mathfrak{g}$ is finite-dimensional, when we wish to emphasize its group structure defined by (1.2), we will either write $\exp(\mathfrak{g})$ or G instead of $\mathfrak{g}$. Be aware that, in Chapter 2 and Chapter 3, we will define $\mathfrak{g}_{CM}$, an infinite-dimensional Lie algebra, and we will write it as $\exp(\mathfrak{g}_{CM})$ to emphasize the group structure. The symbol G_{CM} will refer to a special subgroup of $\exp(\mathfrak{g}_{CM})$, which will generally be a proper subset with

an entirely different topology. The reason for this complication is due to the hypoelliptic nature of our setup. See Section 4.2.2 and Section 9.2 for more discussion.

2.3.2 Linear derivatives, left-invariant derivatives, and the generalized Baker-Campbell-Hausdorff formula

We continue assuming the construction in the previous section, where $\mathfrak{g}$ is a (possibly infinite-dimensional) nilpotent Lie algebra, as well as its own simply connected Lie group (written as $\exp(\mathfrak{g})$). As a vector space, $\mathfrak{g}$ has a natural definition for linear derivatives. For $f : \mathfrak{g} \to \mathbb{R}$ (or $f : \mathfrak{g} \to \mathbb{C}$) , we write, for $x, h \in \mathfrak{g}$,

$$f'(x)h = \partial_h f(x) = \frac{d}{dt}\Big|_{t=0} f(x + th)$$

whenever this derivative exists. More generally, we write

$$f^{(n)}(x)(h_1, \ldots, h_n) \ := \ \partial_{h_1} \ldots \partial_{h_n} f(x) \, .$$

We will now begin to introduce an alternate notion of derivative that respects the group structure on $\exp(\mathfrak{g}) = \mathfrak{g}$. We regard $\exp(\mathfrak{g})$ as having, at every point $x \in \exp(\mathfrak{g})$, tangent spaces $T_g(\exp(\mathfrak{g}))$, each naturally isomorphic to $T_e(\exp(\mathfrak{g})) \cong \mathfrak{g}$. For $x \in \exp(\mathfrak{g})$, $v \in \mathfrak{g}$, we let $v_x \in T_x(\exp(\mathfrak{g}))$ denote the tangent vector satisfying $v_x f = f'(x)v$.

Let $L_g : \exp(\mathfrak{g}) \to \exp(\mathfrak{g})$ denote left-multiplication by g, so that $L_g h = g \cdot h$. Then for all $x \in \exp(\mathfrak{g})$, its derivative $L_{g*} : T_x(\exp(\mathfrak{g})) \to T_{gx}(\exp(\mathfrak{g}))$ satisfies

$$(L_{g*} v_x) f = (f \circ L_g)'(x) v = \frac{d}{dt}\Big|_{t=0} f(g \cdot (x + tv)) \, .$$

Equivalently, we may write $L_{g*} v_x = \frac{d}{dt}\big|_{t=0} (g \cdot (x + tv))$. It can be seen that this is independent of x and this can be worked out to be the finite sum

$$L_{g*} v_x \ = \ \frac{d}{dt}\Big|_{t=0} (g \cdot (tv)) \ = \ c_1 v + c_2 [g, v] + c_3 [g, [g, v]] + \ldots$$

where c_j denote a set of coefficients that can be derived from the Baker-Campbell-Hausdorff series, (1.2). The map $L_{g^{-1}*}$ is referred to as the Maurer-Cartan form, and we will regard it as the canonical map that identifies the tangent spaces $T_x(\exp(\mathfrak{g}))$ all as $T_e(\exp(\mathfrak{g})) \cong \mathfrak{g}$.

For $g \in \mathfrak{g}$, $h = h_e \in T_e(\exp(\mathfrak{g})) = \mathfrak{g}$, we define the vector field $\widetilde{h}$ on $\exp(\mathfrak{g})$ as $\widetilde{h}_g := L_{g*} h_e$. Then $\widetilde{h}$ is the *left-invariant vector field* on $\exp(\mathfrak{g})$ satisfying $\widetilde{h}_e = h$. It can be regarded as a first-order differential operator in which, for smooth $f : \exp(\mathfrak{g}) \to \mathbb{C}$,

$$\widetilde{h} f(g) \ := \ \widetilde{h}_g f(e) \ = \ (L_{g*} h) f(e) \ = \ \frac{d}{dt}\Big|_{t=0} f(g \cdot (th)) \, ,$$

and we call $\widetilde{h}f : \exp(\mathfrak{g}) \to \mathbb{C}$ the *left-invariant derivative* of f with respect to h. This name is given due to the property that $\widetilde{h}(f \circ L_g) = (\widetilde{h}f) \circ L_g$. More generally, we write

$$\widehat{f}(g)(h_1, \ldots, h_n) := \widetilde{h_1} \ldots \widetilde{h_n} f(g) .$$

We may use the language of left-invariant derivatives to express the following generalization to the Baker-Campbell-Hausdorff formula, which is derived in [Str87].

Theorem 2.6. *Given a path $A : [0, 1] \to \mathfrak{g}$, the solution to the differential equation*

$$\sigma'(t) = L_{\sigma(t)*} A'(t) \qquad\qquad \sigma(0) = e$$

is given by

$$\sigma(T) = \sum_{n=1}^{N} \sum_{\sigma \in \mathcal{S}_n} \left((-1)^{e(\sigma)} \Big/ n^2 \begin{bmatrix} n-1 \\ e(\sigma) \end{bmatrix} \right) \times \int_{\Delta_T^n} [\ldots [A'(t_{\sigma(1)}), A'(t_{\sigma(2)})], \ldots, A'(t_{\sigma(n)})] \quad (1.3)$$

where $\Delta_T^n = \{(t_1, \ldots, t_n) \in [0, T]^n \ : \ t_1 \leq \ldots \leq t_n\}$, $\mathcal{S}_n$ is the permutation group on n elements, $e(\sigma) = |\{k \in \{1, \ldots, n\} \ : \ \sigma(k) > \sigma(k+1)\}|$ is the number of "errors" of σ, and $\begin{bmatrix} a \\ b \end{bmatrix}$ is a choose b with repetition.

As with the Baker-Campbell-Hausdorff formula, this can be regarded as a formal infinite formal sum for general Lie groups, but will be a finite sum in our context. Also, as explained in [Str87], this can be viewed as a generalization of the Baker-Campbell-Hausdorff formula, as the coefficients in (1.2) can be recovered by taking a particular choice of A in (1.3), though it results in a different expression for the coefficients.

2.3.3 Lie group-valued Brownian motion

For any finite-dimensional Lie group G, one may consider a natural notion of a (left-invariant) Laplacian Δ_G, defined for smooth $f : G \to \mathbb{R}$ as

$$\Delta_G f = \sum_{j=1}^{n} \widetilde{x_i}^2 f$$

where $\{x_1, \ldots, x_n\} \subseteq \mathfrak{g}$ is some linearly-independent set. For any such Laplacian Δ_G, there is a naturally associated diffusion process $(g_t)_{t \geq 0}$ called G-valued Brownian motion, for which Δ_G is the infinitessimal generator. It can be realized as the solution to the Stratonovich stochastic differential equation, provided that $(B_t)_{t \geq 0}$ is a (flat)

Brownian motion in $\mathfrak{g}$ in which $\{x_1, \ldots, x_n\}$ are the principle directions of diffusion (or, more precisely, $B_t = B_t^{(1)} x_1 + \ldots + B_t^{(n)} x_n$ where each $(B_t^{(j)})_{t \geq 0}$ is an independent standard Brownian motion),

$$\delta g_t \;=\; L_{g_t *} \delta B_t \qquad\qquad\qquad g_0 = e\,.$$

When G is a simply connected nilpotent Lie group, we may use Theorem 2.6 to deduce that g_t takes on the form of the following Stratonovich stochastic integral

$$g_t \;=\; \sum_{n=1}^{N} \sum_{\sigma \in \mathcal{S}_n} \left((-1)^{e(\sigma)} \Big/ n^2 \begin{bmatrix} n-1 \\ e(\sigma) \end{bmatrix} \right) \int_{\Delta_t^n} [\ldots[\delta B_{s_{\sigma(1)}}, \delta B_{s_{\sigma(2)}}], \ldots, \delta B_{s_{\sigma(n)}}]\,.$$

The fixed-time distribution of g_t is referred to as the *heat kernel measure* on G, named for its connection to the (Lie group equivalent of the) heat equation:

$$\frac{d}{dt}\bigg|_{t=0} \mathbb{E}[f(g \cdot g_t)] = \Delta_G f(g)\,.$$

This measure acts as the Lie group equivalent of the Gaussian measure for $\mathbb{R}^n$, acting as the limiting distribution of random walks. See, for example, [Bre04].

If $\operatorname{span}\{x_1, \ldots, x_n\} = \mathfrak{g}$, then the Laplacian and associated diffusion is referred to as "nondegenerate" or "elliptic." On the other hand, if $\{x_1, \ldots, x_n\}$ fails to span all of $\mathfrak{g}$, it is often to instead assume that it satisfies the Hörmander condition, or "bracket-generating" condition, meaning

$$\operatorname{span}\big\{x_{j_1}, [x_{j_1}, x_{j_2}], \ldots, [\ldots[x_{j_1}, x_{j_2}], \ldots, x_{j_k}]\big\}_{j_1, \ldots, j_k = 1}^{n} \;=\; \mathfrak{g}\,.$$

This is what is often meant when we call a diffusion "hypoelliptic." It is a theorem of Hörmander [Hör67] that such a diffusion shares some of the nice properties of elliptic diffusions, such as smoothness and strict positivity of the density of the probability measure with respect to Haar measure.

One of the primary goals of this thesis is to make sense of hypoelliptic diffusions on Lie groups in infinite-dimensions, which will require us to combine the ideas of this section with those related to abstract Wiener space. We will begin to explore this concept in the next section, Section 3. The infinite-dimensional equivalent of the Hörmander condition, however, will be reserved for the introductions in Chapter 2 and Chapter 3.

3 Nilpotent abstract Wiener Lie groups

It should be noted that this section is not necessary to understand the main results in this work, since Chapter 2 and Chapter 3 have self-contained definitions for nilpotent abstract Wiener Lie groups (specifically in Section 7 and Section 10.3 respectively). The primary value of this section is proving that we have a general framework and examples of infinite-dimensional heat kernel measure *without* the extra assumptions of Chapter 2 and Chapter 3. However, thoroughly understanding this section requires knowledge of stochastic calculus.

In this work, we say that nilpotent abstract Wiener Lie groups consist of a pair (G, X), where

1. G, a separable Banach space on which we will define the heat kernel measure ν_t.
2. X, a vector subspace of G equipped with a nilpotent Lie bracket $[\cdot,\cdot] : X \times X \to X$ and a Hilbert subspace $(X_H, \langle \cdot, \cdot \rangle_{X_H})$.

that satisfy assumptions (A1.1) and (A1.2), defined in Section 3.1. It will be seen that the Lie bracket on X determines geometric structure on G, and in particular how random paths naturally transverse. Meanwhile, the Hilbert space X_H will determine the generating directions for the heat kernel diffusion. By allowing for the possibility that $X_H \neq X$, we set the stage for discussing hypoelliptic diffusions.

It will be seen that G has a similar role to W when (W, H) is an abstract Wiener space, because G is merely a Banach space "large enough" to contain the random variable of interest. In particular, in spite of using the letter "G," we do not assume that any kind of group structure exists on G. There are situations in which G will have some form of group structure. For example, the path space of a finite-dimensional Lie group is discussed in Example 3.5, wherein we in fact have a continuous bracket $[\cdot,\cdot]$ and group operator $\cdot$ (defined pointwise). Also, the context for Chapter 2 will suffice to discuss a measurable group action of a special subgroup on G.

First, in Section 3.1, we provide a general definition, which is independent of whether the diffusion is elliptic or otherwise. Then in Section 3.2, we will discuss criteria for determining if a space satisfies the definition. Lastly, in Section 3.3, we provide examples.

3.1 The general definition

As stated in the introduction above, we assume that G is a separable Banach space that contains a vector space X, which has a Lie bracket $[\cdot,\cdot] : X \times X \to \mathbb{R}$ and contains a (separable) Hilbert space X_H. As discussed in Section 2.3, we may view X as a simply connected nilpotent Lie group, defining the product via the Baker-Campbell-Hausdorff formula, (1.2).

We assume the following:

$$X_H \text{ determines a Gaussian measure on } G. \tag{A1.1}$$

This means that, viewing $G^* \subseteq X_H^* \cong X_H \subseteq G$, there exists a (Radon) Gaussian measure μ on G whose Fourier transform is given by, for $f \in G^*$,

$$\int_G e^{i\, f(x)} d\mu(x) \;=\; e^{-\frac{1}{2}\langle f,f\rangle_{X_H}}$$

Note that this is not the same as saying (G, X_H) is an abstract Wiener space, since we did not assert that $\langle\cdot,\cdot\rangle_{X_H}$ is positive definite on G^*, which allows for the measure to be degenerate (in fact, this will be the case for our hypoelliptic examples of interest). Instead, we may define $W = \overline{X_H}^{\|\cdot\|_G}$, and say that (W, X_H) is an abstract Wiener space.

As discussed in Section 2.2, there exists a W-valued Brownian-motion $(B_t)_{t\geq 0}$. Any finite-rank orthogonal projection $P : X_H \to X_H$ has a measurable-linear extension to $W \to PX_H$, so that we may realize PB_t as a PX_H-valued Brownian motion. By defining

$$\mathfrak{g}^P := PX_H + \operatorname{span}([PX_H, PX_H]) + \ldots + \operatorname{span}([\ldots[PX_H, PX_H], \ldots, PX_H])\,,$$

we may realize $\mathfrak{g}^P \subseteq X$ as a finite-dimensional simply connected nilpotent Lie algebra (generated by PX_H), and likewise we may realize $G^P = \mathfrak{g}^P$ as a simply connected nilpotent Lie group. Then, as a Lie group, we have G^P-valued Brownian motion, $(g_t^P)_{t\geq 0}$, which is the solution to the stochastic differential equation

$$\delta g_t^P \;=\; L_{g_t^P *}\delta PB_t \qquad\qquad g_0 \;=\; e\,.$$

As explained in Theorem 2.6 and Section 2.3.3, this must have as its solution

$$g_t^P \;=\; \sum_{n=1}^{N} \sum_{\sigma\in S_n} c_\sigma \int_{\Delta_T^n} [\ldots[\delta PB_{s_{\sigma(1)}}, \delta PB_{s_{\sigma(2)}}], \ldots, \delta PB_{s_{\sigma(n)}}] \tag{1.4}$$

for some constants c_σ, where δ corresponds to the Stratonovich stochastic integral.

Now we may state our second assumption:

For some $t > 0$, there exists a G-valued random variable g_t such that, for all $f \in G^*$, there exists an increasing sequence of finite-rank projections $(P_m)_{m\in\mathbb{N}} \in \operatorname{Proj}(W)^{\uparrow}$ such that $f(g_t^{P_m})$ converges to $f(g_t)$ in probability. (A1.2)

Or, in other words, for all $\delta > 0$, $\lim_{m\to\infty} \mathbb{P}(|f(g_t^{P_m}) - f(g_t)| \geq \delta) = 0$.

For such a Banach space G and nilpotent Lie algebra $X \subseteq G$, we say that (G, X) (or simply G, when the context is clear) is an *(infinite-dimensional simply connected) nilpotent abstract Wiener Lie Group* when (A1.1) and (A1.2) are satisfied, for which the heat kernel measure on G, sometimes denoted ν_t, is defined as the distribution of g_t. We will regard g_t as equalling, in a weak sense,

$$g_t = \sum_{n=1}^{N} \sum_{\sigma \in \mathcal{S}_n} c_\sigma \int_{\Delta_t^n} [\dots[\delta B_{s_{\sigma(1)}}, \delta B_{s_{\sigma(2)}}], \dots, \delta B_{s_{\sigma(n)}}] .$$

We remark that, in spite of the subscript "t" in g_t, (A1.2) only asks that the limit holds for a fixed value of t, though this criterion is independent of the value of t. Indeed, if $g_t^{P_m}$ converges in probability to g_t, then by replacing $(B_t)_{t\geq 0}$ with $(B_{\beta t})_{t\geq 0}$ for some $\beta > 0$, (1.4) will be equal in distribution to $g_{\beta t}^{P}$, and knowing that, as stochastic processes, $(B_{\beta t})_{t\geq 0} \sim (\sqrt{\beta} B_t)_{t\geq 0}$, we see that (1.4) would once again converge in probability to a random variable.

In the commutative case ($N = 1$), G is merely an abstract Wiener space, for which the assumption is sufficient to know that $g_t = B_t$ is the fixed-time distribution of a stochastic process $(g_t)_{t\geq 0}$, which constitutes an L^p martingale for all $p \in [1, \infty)$. It is entirely possible that this assumption implies similar properties of g_t for general N, but without the special properties and results for Gaussian distributions, such as the Fernique and Skorohod theorems, it cannot be easily deduced[1].

Parallel to how Leonard Gross first introduced abstract Wiener spaces in [Gro67], this definition is inspired by the existence of 2 key examples. The first is when G is a Hilbert space in which the inclusion $X_H \to G$ has sufficient Hilbert-Schmidt properties. The exact properties are discussed in Example 3.4, using calculations inspired by [Mel21]. The second is the path space of a finite-dimensional nilpotent Lie group. The existence of the heat kernel distributed element is worked out in [CD08], and we provide further discussion in Example 3.5.

3.2 Expressions in terms of Itô integrals

In this subsection, we will express the stochastic integrals in (1.4) in terms of Itô integrals. This will provide an alternate criterion, (A1.2′), which imposes a requirement on the Lie bracket, implies (A1.2), holds, and further implies that for any $f \in G^*$ and any choice of $(P_m)_{m\in\mathbb{N}} \in \mathrm{Proj}(W)^\uparrow$, $f(g_t^{P_m})$ converges to $f(g_t)$ in L^2. In this thesis, such L^2 convergence will be proven to be satisfied in the following cases.

[1]It may be worth noting that there will be many nice properties exhibited by the Hilbert spaces (see, for example, [DZ14]) and path space examples (see [CD08]).

1. G is a Hilbert space under a "tracial Hilbert-Schmidt" assumption on the iterated brackets $[\ldots[\cdot,\cdot],\ldots,\cdot]: X_H \times \ldots \times X_H \to G$.
2. $G = \mathcal{W}_0([0,1],A)$ is the path space of a finite-dimensional simply connected nilpotent Lie group A, for which X_H will be a subset of the finite-energy paths in A.
3. X is a step 2 Lie algebra.
4. X is complex Lie algebra and $(X_H, \langle\cdot,\cdot\rangle_{X_H})$ is a complex Hilbert space (which we model as a real Hilbert space via the real inner product $\Re\langle\cdot,\cdot\rangle_{X_H}$).

The first 2 will be discussed in Example 3.4 and Example 3.5. The third will be the context of Chapter 2, and the stochastic properties will be detailed in Section 7. The fourth is the starting point for Chapter 3, with discussion appearing in Section 10.3.

We define $\mathcal{J}_m^n = \{\alpha \in \{1,2\}^m \mid \sum_{k=1}^m \alpha_k = n\}$, and for $\alpha \in \mathcal{J}_m^n$ we let $p_\alpha := \#\{k : \alpha_k = 1\} = 2m - n$ and $q_\alpha := \#\{k : \alpha_k = 2\} = n - m$. Then let V be a Hilbert space[2]. Given a real-multilinear Hilbert-Schmidt map $M : X_H^n \to V$, and given $\alpha \in \mathcal{J}_m^n$, and a real basis $\{e_j\}_{j\in\mathbb{N}}$ of X_H, we define the α-trace $Tr_\alpha M : X_H^{p_\alpha} \to V$ as follows. If we have ordered sets $\{i_1,\ldots,i_{p_\alpha}\} = k : \alpha_k = 1\}$, $\{j_1,\ldots,j_{q_\alpha}\} = \{k : \alpha_k = 2\}$, then let $\sigma \in \mathcal{S}_m$ be such that $(\sigma(1),\ldots,\sigma(m)) = (j_1,\ldots,j_{q_\alpha},i_1,\ldots,i_{p_\alpha})$, so that $(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(m)}) = (2,\ldots 2,1,\ldots 1) \in \mathcal{J}_m^n$, and define $M_{\sigma^{-1}}(h_1,\ldots,h_n) := M(h_{\sigma^{-1}(1)},\ldots,h_{\sigma^{-1}(n)})$. Then define

$$Tr_\alpha M(h_1,\ldots,h_{p_\alpha}) \quad = \quad \sum_{\ell_1,\ldots,\ell_{q_\alpha}=1}^{\infty} M_{\sigma^{-1}}(e_{\ell_1},e_{\ell_1},\ldots,e_{\ell_{q_\alpha}},e_{\ell_{q_\alpha}},h_1,\ldots,h_{p_\alpha})\,.$$

We say $Tr_\alpha M$ is well-defined provided that this produces another Hilbert-Schmidt map, and moreover that this series converges in $HS(X_H^{p_\alpha},V)$ norm, in which case this expression is independent of basis chosen. Alternatively, we may write, for finite-rank $PX_H \to X_H$ where PX_H has basis $\{e_j\}_{1\le j\le r}$,

$$\begin{aligned} Tr_\alpha(M\circ P^{\times p_\alpha})(h_1,\ldots,h_{p_\alpha}) \;&=\; Tr_\alpha M(Ph_1,\ldots,Ph_{p_\alpha}) \\ &=\; \widetilde{M}(Ph_1^\alpha\otimes\ldots\otimes Ph_m^\alpha)\,, \end{aligned}$$

where $\widetilde{M}$ is the tensor product extension (Proposition 2.4), $Ph_{i_k}^\alpha = Ph_k$, and $Ph_{j_k}^\alpha = \sum_{\ell=1}^r e_\ell\otimes e_\ell$, and then if $(P_r)_{r\in\mathbb{N}}$ is a sequence of finite-rank projections onto $\mathrm{span}\{e_j\}_{1\le j\le r}$, define $Tr_\alpha M = \lim_{r\to\infty} Tr_\alpha(M\circ P^{p_\alpha})$, again provided that the sequence converges in $HS(X_H^n,V)$, in which case the limit will be independent of sequence $(P_r)_{r\in\mathbb{N}}$ chosen.

[2]We will mostly consider $V=\mathbb{R}$, but example Example 3.4 will use a more general V.

If $M : X_H^n \to V$ is a Hilbert-Schmidt map for which $Tr_\alpha M$ exists for all $\alpha \in \mathcal{J}_m^n$ for all $\lfloor n/2 \rfloor \leq m \leq n$, then we define the *tracial norm* as

$$\|M\|_{Tr(X_H,V)} = \max_{\lfloor n/2 \rfloor + 1 \leq m \leq n} \|Tr_\alpha M\|_{HS(X_H^{p_\alpha},V)} .$$

We now present a lemma that describes precisely how to convert the Stratonovich integrals of interest into Itô integrals.

Lemma 3.1. *Given a finite-rank projection $P : X_H \to X_H$ and $\{e_j\}_{j\in\mathbb{N}}$ a (real) basis of X_H such that $\{e_j\}_{1\leq j\leq r}$ is a basis of PX_H, let*

$$dPX_t^{\alpha,i} = \begin{cases} dPB_t & \text{if } \alpha_i = 1 \\ \sum_{\ell=1}^r e_\ell \otimes e_\ell \, dt & \text{if } \alpha_i = 2 \end{cases} .$$

Define $\Delta_t^n = \{(s_1, \ldots, s_n) \in [0,t]^n \,:\, 0 \leq s_1 \leq \ldots \leq s_n \leq 1\}$. Then we may write the following iterated Stratonovich integral into the iterated Itô integral below.

$$\int_{\Delta_t^n} \delta PB_{s_1} \otimes \ldots \otimes \delta PB_{s_n} \;=\; \sum_{m=\lfloor n/2 \rfloor}^{n} \frac{1}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_m^n} \int_{\Delta_t^n} dPX_{s_1}^{\alpha,1} \otimes \ldots \otimes dPX_{s_n}^{\alpha,n} . \quad (1.5)$$

Given a Hilbert-Schmidt multilinear map $M : X_H^n \to \mathbb{C}$ with $\|M\|_{Tr} < \infty$, we have that there exist constants $\{b_a^\alpha\}_{\alpha\in\mathcal{J}_m^n, 0\leq a\leq q_\alpha}$ and polynomials $\{f_t^\alpha : [0,t]^{p_\alpha} \to \mathbb{R}\}_{\alpha\in\mathcal{J}_m^n, 0\leq a\leq q_\alpha}$ such that

$$\int_{\Delta_t^n} M(\delta PB_{s_1}, \ldots, \delta PB_{s_n}) = \sum_{m=\lfloor n/2 \rfloor}^{n} \frac{1}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_n^m} \int_{\Delta_t^{p_\alpha}} f_t^\alpha(s) Tr_\alpha M(dPB_{s_1}, \ldots, dPB_{s_{p_\alpha}}) . \quad (1.6)$$

Furthermore, if we let Z^P denote the random variable

$$Z^P := \int_{\Delta_t^n} M(\delta PB_{s_1}, \ldots, \delta PB_{s_n}) ,$$

then, we have, for some $K > 0$

$$\left(\mathbb{E}|Z^P|^2\right)^{1/2} \leq K \|M \circ P^n\|_{Tr(X_H^n,V)} . \quad (1.7)$$

This inequality in turn tell us that, if $(P_m)_{m\in\mathbb{N}}$ is an increasing sequence of finite-rank projections, then Z^{P_m} converges in L^2 to a random variable, which we denote by

$$\int_{\Delta_t^n} M(\delta B_{s_1}, \ldots, \delta B_{s_n}) .$$

Proof. This entire calculation is inspired by (and is partially identical to) those done in [Mel21, section 4.1], for which most steps below can be checked against. By classical Stochastic calculus, for 2 independent $\mathbb{R}$-valued Brownian motions $(b_t)_{t\geq 0}$, $(b'_t)_{t\geq 0}$, the Itô formula tells us that

$$\int_0^t b_s \delta b_s \;=\; \frac{1}{2}b_t^2 \;=\; \int_0^t b_s db_s + \frac{1}{2}t\,, \qquad\qquad \int_0^t b_s \delta b'_s \;=\; \int_0^t b_s db'_s\,.$$

Then, by the Itô formula, for a bilinear map $S : PX_H \times PX_H \to \mathbb{R}$, the $PX_H \cong \mathbb{R}^r$-valued Brownian motion $(PB_s)_{t\geq 0}$, satisfies

$$\int_0^t \int_0^{s_2} S(\delta PB_{s_1}, \delta PB_{s_2}) \;=\; \int_0^t S(PB_s, \delta PB_s)$$
$$= \int_0^t S(PB_s, dPB_s) + \frac{1}{2}\sum_{j=1}^r S(e_j, e_j)\,.$$

Regarding the Stratonovich integral as an inductively-defined iteration of integrals, the equality in (1.5) will come from iteratively applying this formula.

For the second equality, first consider that

$$\int_{\Delta_t^n} M(\delta PB_{s_1}, \ldots, \delta PB_{s_n})$$
$$= \sum_{m=\lfloor n/2 \rfloor}^{n} \frac{1}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_m^n} \int_{\Delta_t^n} Tr_\alpha M(dPB_{s_{i_1^\alpha}}, \ldots, dPB_{s_{i_{p_\alpha}^\alpha}}) ds_{j_1^\alpha} \ldots ds_{j_{q_\alpha}^\alpha}\,,$$

where we again define the ordered sets

$$\{i_1^\alpha, \ldots, i_{p_\alpha}^\alpha\} \;=\; \{k \;:\; \alpha_k = 1\} \qquad\qquad \{j_1^\alpha, \ldots, j_{q_\alpha}^\alpha\} \;=\; \{k \;:\; \alpha_k = 2\}\,.$$

Then integrating over the j_k^α indices results in some polynomial $f_t^\alpha(s) = f^\alpha(s_1, \ldots, s_{p_\alpha}, t)$, arriving at equation (1.6) .

Now we address the inequality. Note that we have a constant K_1 such that

$$\mathbb{E}\Bigg| \sum_{m=n/2}^{n} \frac{1}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_n^m} \int_{\Delta_t^{p_\alpha}} f_t^\alpha(s) Tr^\alpha M(dPB_{s_1}, \ldots, dPB_{s_n}) \Bigg|^2$$
$$\leq\; K_1 \sum_{m=n/2}^{n} \frac{1}{2^{2n-2m}} \sum_{\alpha \in \mathcal{J}_n^m} \mathbb{E}\Bigg| \int_{\Delta_t^{p_\alpha}} f_t^\alpha(s) Tr^\alpha M(dPB_{s_1}, \ldots, dPB_{s_n}) \Bigg|^2.$$

And, for each α, by the Itô isometry,

$$\begin{aligned}\mathbb{E}\left|\int_{\Delta_t^{p_\alpha}} f_t^\alpha(s) Tr^\alpha M(dPB_{s_1},\dots,dPB_{s_n})\right|^2 \\ &= \int_{\Delta_t^{p_\alpha}} \|f_t^\alpha(s) Tr^\alpha(M\circ P^{\times n})\|^2_{HS(PX_H^{p_\alpha},V)} ds \\ &= \left(\int_{\Delta_t^{p_\alpha}} |f_t^\alpha(s)|^2 ds\right) \|Tr^\alpha(M\circ P^{\times n})\|^2_{HS(PX_H^{\times p_\alpha},V)} ,\end{aligned}$$

and each

$$\begin{aligned}\|Tr^\alpha(M\circ P^{\times n})\|_{HS(PX_H^{p_\alpha},V)} &= \|Tr^\alpha(M\circ P^{\times p_\alpha})\|_{HS(X_H^{p_\alpha},V)} \\ &\leq \|M\circ P^{\times p_\alpha}\|_{Tr(X_H^n,V)}\end{aligned}$$

Then (1.7) follows.

Lastly, consider that, for any increasing sequence $(P_r)_{r\in\mathbb{N}}$, we have that $\|M - M\circ P_r^{\times n}\|_{Tr(X_H^{\times n},V)} \to 0$ because, for any $\alpha \in \mathcal{J}_m^n$,

$$\begin{aligned}&\|Tr_\alpha M - Tr_\alpha(M\circ P^{\times n})\|_{HS(X_H^m,V)} \\ &\quad\leq \|Tr_\alpha M - (Tr_\alpha M)\circ P_r^{\times m}\|_{HS} + \|(Tr_\alpha M)\circ P^{\times m} - (Tr_\alpha(M\circ P^{\times n})\|_{HS} ,\end{aligned}$$

which necessarily converges to 0 by assumption. Hence, Z^{P_r} must be Cauchy, and thus convergent, in L^2. And note that the calculation above justifies that the limit is independent of the sequence $(P_r)_{r\in\mathbb{N}}$ chosen.

□

Theorem 3.2. *Suppose that, for all $f \in G^*$*

$$\max_{1\leq n\leq N}\max_{\sigma\in\mathcal{S}_n}\left\|f\left([\dots[\cdot,\cdot]_{1,1},\dots,\cdot]_{n-1,1}\circ\widetilde{\sigma}\right)\right\|_{Tr(X_H^n,\mathbb{R})} < \infty , \qquad \text{(A1.2}')$$

where we let $\widetilde{\sigma}$ denote the natural action of $\sigma \in \mathcal{S}_n$ on X_H^n. Then for all $(P_m)_{m\in\mathbb{N}} \in Proj(W)^\uparrow$ of finite-rank projections, for all $f \in G^*$, $f(g_t^{P_m})$ converges in L^2 to $f(g_t)$.

Proof. We may write

$$f(g_t^{P_m}) = \sum_{n=1}^N \sum_{\sigma\in\mathcal{S}_n} c_\sigma \int_{\Delta_t^n} f\left([\dots[\delta P_m B_{s_{\sigma(1)}}, \delta P_m B_{s_{\sigma(2)}}],\dots,\delta P_m B_{s_{\sigma(n)}}]\right)$$

and apply Lemma 3.1 to know that $f(g_t^{P_m})$ is converging in L^2 independent of the choice of $(P_m)_{m\in\mathbb{N}}$, and its limit must be the limit in probability $f(g_t)$. □

3.3 Examples

Example 3.3 (Hilbert-Schmidt bracket)**.** Let G be a Banach space and X a Hilbert space for which (G, X) is an abstract Wiener space. We also assume that we have a Lie bracket $[\cdot, \cdot] : X \times X \to X$, and impose the assumption that it is Hilbert-Schmidt. Under these assumptions, using $X_H = X$, (G, X) is a real nilpotent abstract Wiener Lie group. This object is the primary focus of [Mel21] (see Definition 4.2). In fact, [Mel21] shows in Proposition 4.1 that a version of (A1.2′) holds (essentially equivalent to (1.8) in the following example), and in Proposition 4.3 proves that $g_t^{P_m}$ converges to g_t in L^2 with respect to $\| \cdot \|_G$.

While this setup is considered generally in [Mel21], it also encapsulates previous research. Suppose further that $\operatorname{span}[X, X]$ is a finite-dimensional subspace of X. Such G have been referred to as being semi-infinite Lie groups in [Mel09], and are also the object of study in [DG08; DG10; DM13] assuming a Heisenberg-like structure (nilpotent of step 2). In this case, the assumption that the Lie bracket is Hilbert-Schmidt is critical to prove the existence of the heat kernel distribution.

It is also possible to make this example subelliptic (as suggested in [Mel21, Remark 4.6]). Still assuming that X is a Hilbert space and subset of the Banach space G with Hilbert-Schmidt Lie bracket, now suppose that we have an orthogonal decomposition $X = X_H \oplus X_V$ where (A1.1) and (A1.2′) hold (that they hold for the choice $X_H = X$ implies that they hold when X_H is instead chosen to be a closed subspace of X), and further that $[X, X] \subseteq X_V$. Then we may say that X_H consists of the "horizontal" (or generating) directions, while X_V consists of "vertical" directions (in likeness to the horizontal and vertical directions of the classical sub-Riemannian structure on the Heisenberg group). If we again impose that X_V is finite-dimensional and that X has a Heisenberg-like structure, then this describes the setup for the results in [BGM13; GM13; DEM16]. Again, the assumption that the bracket (which we may view as a map $[\cdot, \cdot] : X_H \oplus X_H \to X_V$) is Hilbert-Schmidt is a necessity when X_V is finite-dimensional.

However, the assumption that the bracket is Hilbert-Schmidt is, for the purposes of this thesis, a problematic restriction. If $[\cdot, \cdot] : X_H \times X_H \to X_V$ is Hilbert-Schmidt, then its tensor product extension (Section 2.1.3) $[\cdot] : X_H \otimes X_H \to X_V$ is a Hilbert-Schmidt operator. When X_V is infinite-dimensional, this prevents the extension from being surjective, which forces critical assumptions to fail (specifically assumption (A2.3) in Chapter 2, as explained in Section 4.2.1, and (A3.2) in Chapter 3, explained further in Section 9.4.1). Thus, this assumption is unsuitable for our hypoelliptic framework of interest.

When we develop our theorems and examples for Chapter 2 and Chapter 3, we do give all of X (called $\mathfrak{g}_{CM}$) a Hilbert space structure, but instead assume that the bracket is *weakly* Hilbert-Schmidt (see (A2.1) and (A3.1)), which does not contradict our other assumptions.

Example 3.4 (General Hilbert space)**.** Suppose that we have spaces $X_H \subseteq X \subseteq G$ where X_H is a Hilbert space, X is a vector space with a Lie bracket $[\cdot,\cdot] : X \times X \to X$, and G is a separable Hilbert space. Not only do we assume that the inclusion $X_H \hookrightarrow G$ is Hilbert-Schmidt, but we go further and assume

$$\max_{1\le n\le N} \max_{\sigma\in\mathcal{S}_n} \Big\| \underbrace{[\dots[\cdot,\cdot],\dots,\cdot]}_{n} \circ \widetilde{\sigma} \Big\|_{Tr(H^n,G)} < \infty \,, \tag{1.8}$$

where $\widetilde{\sigma}$ is the induced action of σ on H^n. Then by Lemma 3.1, we see that $g_t^{P_m}$ converges in L^2 with respect to $\|\cdot\|_G$. Thus, (G, X) is an nilpotent abstract Wiener Lie group.

Example 3.5 (Hypoelliptic path space)**.** In this example, we will consider the path space of a finite-dimensional simply connected nilpotent Lie group, which itself will be an infinite-dimensional simply connected nilpotent Lie group under pointwise multiplication and brackets. We remark that the existence of a Brownian motion on the path space of any Lie group is thoroughly answered in [CD08].

Suppose first that A is any real finite-dimensional simply connected nilpotent Lie algebra (and Lie group) with $\langle\cdot,\cdot\rangle_A$. Further suppose that we have a subspace $A_H \subseteq A$ which adopts the inner product $\langle\cdot,\cdot\rangle_{A_H} = \langle\cdot,\cdot\rangle_A|_{A_H\times A_H}$. We define the Banach space

$$\mathcal{W}_0([0,1],A) \;:=\; \{f : [0,1] \to A \mid f \text{ continuous}, f(0) = 0\}$$

with the norm $\|f\|_{\mathcal{W}_0([0,1],A)} = \sup_{s\in[0,1]} \|f(s)\|_A$, and we have a pointwise-defined bracket and group operator on $\mathcal{W}_0([0,1],A)$ defined as, for all $f, g \in \mathcal{W}_0([0,1],A)$, for $s \in [0,1]$,

$$[f,g](s) := [f(s), g(s)] \qquad\qquad (f\cdot g)(s) := f(s)\cdot g(s)\,,$$

Indeed, this Lie bracket (and thus product) of continuous functions is again continuous. Then we have that (G, X) is a nilpotent abstract Wiener Lie group, where

$$G \;=\; \mathcal{W}_0([0,1],A) \qquad X \;=\; \mathcal{H}_0([0,1],A) \qquad X_H \;=\; \mathcal{H}_0([0,1],A_H)\,,$$

where $\mathcal{H}_0$ denotes the set of finite-energy paths, as defined in Section 2.1.4. We naturally equip $X = \mathcal{H}_0([0,1],A)$ with the same pointwise-defined Lie bracket (the bracket of finite-energy paths is again of finite energy), and $X_H = \mathcal{H}_0([0,1],A_H)$ with its inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_0([0,1],A_H)}$.

We know that $(\mathcal{W}_0([0,1],A_H), \mathcal{H}_0([0,1],A_H))$ forms an abstract Wiener space (which proves (A1.1)), on which we have a Gaussian measure and may define (flat) $\mathcal{W}_0([0,1],A_H)$-valued Brownian motion $(B_t)_{t\ge0}$. We remark that point evaluations will serve as our primary elements in $G^* = \mathcal{W}_0([0,1],A)^*$, and in particular note that

$(B_t(\tau))_{t\geq 0}$ is an A_H-valued Brownian motion for every $\tau \in [0,1]$. Then the heat kernel distribution on $\mathcal{W}_0([0,1],A)$ is the distribution of g_1, which has the following pointwise definition for all $\tau \in [0,1]$.

$$g_1(\tau) \;=\; \sum_{n=1}^{N} \sum_{\sigma\in\mathcal{S}_n} c_\sigma \int_{\Delta_1^n} [\ldots[\delta B_{s_{\sigma(1)}}(\tau), \delta B_{s_{\sigma(2)}}(\tau)], \ldots, \delta B_{s_{\sigma(n)}}(\tau)]$$

Note that the differentials are taken with respect to $s \mapsto B_s(\tau)$), and with the notation defined in (1.4). Since $(B_t(\tau))_{t\geq 0}$ is finite-dimensional, this can readily be written as an Itô integral using (1.6).

The rest of this example is devoted to explicitly showing that this satisfies (A1.2′). For any partition $\mathcal{P} = \{0 = s_0 < s_1 < \ldots < s_{n+1} = 1\} \subseteq [0,1]$, let $\pi^{\mathcal{P}}$ be the map $\mathcal{H}_0([0,1],A_H) \to \mathcal{H}_0([0,1],A_H)$ defined as the piecewise linear approximation of f, that is,

$$\pi^{\mathcal{P}} f(s) \;:=\; \begin{cases} \frac{s}{s_1} f(s_1) & \text{if } 0 \leq s \leq s_1 \\ f(s_1) + \frac{s-s_1}{s_2-s_1}(f(s_2) - f(s_1) & \text{if } s_1 < s \leq s_2 \\ \vdots & \\ f(s_n) + \frac{s-s_n}{1-s_n}(f(1) - f(s_n)) & \text{if } s_n < s \leq 1 \end{cases} .$$

Then if $\mathcal{P}_m$ is a sequence of refined partition of $[0,1]$, then $(\pi^{\mathcal{P}_m} B_t)_{0\leq t\leq 1}$ converges to $(B_t)_{0\leq t\leq 1}$ almost surely and in L^2 with respect to $\mathcal{W}_0([0,1],A_H)$. Then if

$$g_1^{\mathcal{P}}(\tau) \;:=\; \sum_{n=1}^{N} \sum_{\sigma\in\mathcal{S}_n} c_\sigma \int_{\Delta_1^n} [\ldots[\delta\pi^{\mathcal{P}} B_{s_{\sigma(1)}}(\tau), \delta\pi^{\mathcal{P}} B_{s_{\sigma(2)}}(\tau)], \ldots, \delta\pi^{\mathcal{P}} B_{s_{\sigma(n)}}(\tau)]\,,$$

then for all $\tau \in P$, $g_1^{\mathcal{P}}(\tau) = g_1(\tau)$. From here, we see that for all $\tau_1, \ldots, \tau_n \in [0,1]$, there exists $\mathcal{P}_m$ such that $g_1^{\mathcal{P}_m}(\tau_n) \to g_1(\tau_n)$ for all n. But we may state something stronger.

Theorem 3.6. *For a finite set $\tau = \{\tau_1 \leq \ldots \leq \tau_K\} \subseteq [0,1]$, let $\tau^* : \mathcal{W}_0([0,1],A) \to A$ be defined as $\tau^*(f) = \frac{1}{K}\sum_{k=1}^K f(\tau_k)$. Then* $\operatorname{span}\{\langle v, \tau^*(\cdot)\rangle_A \;:\; v \in A, \tau = \{\tau_1,\ldots,\tau_k\} \subseteq [0,1]\}$ *is dense in $\mathcal{W}_0([0,1],A)^*$. Moreover,*

$$\max_{1\leq n\leq N} \max_{\sigma\in\mathcal{S}_n} \Big\| \langle v, \tau^*(\cdot)\rangle_A \circ [\ldots[\cdot,\cdot],\ldots,\cdot] \circ \widetilde{\sigma} \Big\|_{Tr(\mathcal{H}_0([0,1],A_H)^n,\mathbb{R})} \leq C \|\langle v, \tau^*(\cdot)\rangle_A\|_{\mathcal{W}_0([0,1],A)^*}\,, \quad (1.9)$$

and thus we have a continuous inclusion $\mathcal{W}_0([0,1],A)^* \hookrightarrow L^2(\mathcal{W}_0([0,1],A))$, which suffices to prove that $\big(\mathcal{W}_0([0,1],A)\,,\, \mathcal{H}_0([0,1],A)\,,\, \mathcal{H}_0([0,1],A_H)\big)$ will constitute a nilpotent abstract Wiener Lie group.

Proof. For the density, we need only consider that, for any $\phi \in \mathcal{W}_0([0,1],A)^*$, if $\{v_1,\dots,v_{\dim(A)}\}$ is a basis of A, then the following is true for all $f \in \mathcal{W}_0([0,1],A)$:

$$\begin{aligned} \mathcal{W}_0([0,1],A) \ni \qquad f(\cdot) &= \langle v_1, f(\cdot)\rangle_A v_1 + \ldots + \langle v_{\dim(A)}, f(\cdot)\rangle_A v_{\dim(A)} \\ \mathbb{R} \ni \qquad \phi(f) &= \phi(\langle v_1, f(\cdot)\rangle_A v_1) + \ldots + \phi(\langle v_{\dim(A)}, f(\cdot)\rangle_A v_{\dim(A)})\,. \end{aligned}$$

Each map of the form $\mathcal{W}_0([0,1],\mathbb{R}) \ni g \mapsto \phi(g(\cdot)v_j)$ is in $\mathcal{W}_0([0,1],\mathbb{R})^*$. And the set of piecewise-linear approximations $\pi^\tau g$ are dense in $\mathcal{W}_0([0,1],\mathbb{R})^*$. This suffices to prove the stated density.

Next, recall from Section 2.1.4 that $\mathcal{H}_0([0,1],A_H) \cong \mathcal{H}_0([0,1],\mathbb{R}) \otimes A_H$, so it has as basis $\{\mathfrak{e}_k h_i\}_{k\in\mathbb{N},1\le i\le\dim(A_H)}$, where $\{h_i\}_{1\le i\le \dim(A_H)}$ is a basis of A_H and $\{\mathfrak{e}_k\}_{k\in\mathbb{N}}$ is a basis of $\mathcal{H}_0([0,1],\mathbb{R})$. This can be taken as

$$\{\mathfrak{e}_k\}_{k\in\mathbb{N}} = \left\{\frac{1}{\sqrt{2}\pi k}(1-\cos(2\pi k t))\,,\ \frac{1}{\sqrt{2}\pi k}\sin(2\pi k t)\right\}_{k\in\mathbb{N}} \cup \Big\{t\Big\}\,,$$

which can in particular be assumed to satisfy $\sup_{t\in[0,1]}|\mathfrak{e}_k(t)| \le \frac{1}{k}$. Then take $1 \le n \le N$, $\sigma \in \mathcal{S}_n$, and without loss of generality, using the notation from Lemma 3.1, assume $\alpha \in \mathcal{J}^n_m$ takes on the form $(\alpha_1,\dots,\alpha_m) = (1,\dots,1,2,\dots,2)$, with $p = p_\alpha$ and $q = q_\alpha$ defined as before.

Then

$$\begin{aligned} &\Big\| Tr_\alpha\big(\langle v, \tau^*(\cdot)\rangle_A \circ [\ldots[\cdot,\cdot],\ldots,\cdot] \circ \widetilde{\sigma}\big)\Big\|_{HS(\mathcal{H}_0([0,1],A_H)^p,\mathbb{R})} \\ &= \sum_{i_1,\dots,i_p=1}^{\dim(A)} \sum_{k_1,\dots,k_p=1}^{\infty} \Bigg| \sum_{j_1,\dots,j_p=1}^{\dim(A)} \sum_{\ell_1,\dots,\ell_q=1}^{\infty} \Big\langle v, \tau^*(\cdot)\Big\rangle_A \Big([\ldots[\cdot,\cdot],\ldots,\cdot]\circ\widetilde{\sigma}\Big) \\ &\qquad \times \Big(\mathfrak{e}_{k_1}h_{i_1},\ldots,\mathfrak{e}_{k_p}h_{i_p},\mathfrak{e}_{\ell_1}h_{j_1},\mathfrak{e}_{\ell_1}h_{j_1},\ldots,\mathfrak{e}_{\ell_q}h_{j_q},\mathfrak{e}_{\ell_q}h_{j_q}\Big)\Bigg|^2 \\ &= \left(\sum_{i.}\left|\sum_{j.}\Big\langle v, [\ldots[\cdot,\cdot],\ldots,\cdot]\circ\widetilde{\sigma}\Big\rangle_A\Big(h_{i_1},\ldots,h_{i_p},h_{j_1},h_{j_1},\ldots,h_{j_q},h_{j_q}\Big)\right|^2\right) \\ &\qquad \times \left(\sum_{k.}\left|\sum_{\ell.}\tau^*\Big(\mathfrak{e}_{k_1}\ldots\mathfrak{e}_{k_p}\mathfrak{e}^2_{\ell_1}\ldots\mathfrak{e}^2_{\ell_q}\Big)\right|^2\right) \end{aligned}$$

The first of these 2 factors will be bounded by $C_1\|v\|_A$. This can be done either by repeatedly invoking $\dim(A) < \infty$, or by noting that its square root satisfies the properties of a norm and knowing that all norms in finite dimensions are equivalent.

As for the second factor,

$$\sum_{k_1,\dots,k_p=1}^{\infty}\left|\sum_{\ell_1,\dots,\ell_q=1}^{\infty}\frac{1}{K}\sum_{r=1}^{K}\mathfrak{e}_{k_1}(\tau_r)\dots\mathfrak{e}_{k_p}(\tau_r)\mathfrak{e}_{\ell_1}(\tau_r)^2\dots\mathfrak{e}_{\ell_q}(\tau_r)^2\right|^2$$
$$\leq \sum_{k_1,\dots,k_p=1}^{\infty}\left|\sum_{\ell_1,\dots,\ell_q=1}^{\infty}\frac{1}{k_1}\cdots\frac{1}{k_p}\frac{1}{\ell_1^2}\cdots\frac{1}{\ell_q^2}\right|^2 = \left(\frac{\pi^2}{6}\right)^{p+q}.$$

Then, knowing that $\|\langle v,\tau^*(\cdot)\rangle_A\|_{\mathcal{W}_0([0,1],A)^*} = \|v\|_A$, we have a uniform bound of the form

$$\left\|\langle v,\tau^*(\cdot)\rangle_A\circ[\dots[\cdot,\cdot],\dots,\cdot]\circ\widetilde{\sigma}\right\|_{Tr(\mathcal{H}_0([0,1],A_H)^n,\mathbb{R})}$$
$$\leq C\|v\|_A = C\|\langle v,\tau^*(\cdot)\rangle_A\|_{\mathcal{W}_0([0,1],A)^*}$$

over all of the (finitely-many) choices of n, σ, and α, (1.9) will follow. We will also have a similar bound if we replace $\langle v,\tau^*(\cdot)\rangle_A$ with an arbitrary element of $\mathcal{W}_0([0,1],A)^*$ by using the fact that any such general element can be approximated via sums of $\dim(A)$-many elements of the form $\langle v,\tau^*\rangle_A$, as described at the beginning of the proof. The conclusions of the theorems will follow by Lemma 3.1 and Theorem 3.2.

□

Chapter 2

A Quasi-invariance theorem

4 Introduction

In this chapter, the first of our 2 main results, we will extend a quasi-invariance result from [BGM13] for certain shifts of the hypoelliptic heat kernel measure on infinite-dimensional Heisenberg-like groups. Such groups, as previously defined in [DG08; BGM13; DEM16], take on the form $W \times C$, where (W, H, μ) is an abstract Wiener space containing the "horizontal" directions that generate the diffusion, and C is the center of the group, and consisting of the "vertical" directions. The longstanding assumption in these works is that, while W may be infinite-dimensional, C is restricted to being a finite-dimensional Hilbert space. Here, we provide additional structure and assumptions on these groups that allow us to discuss and prove quasi-invariance in the case when C is infinite-dimensional. As done in [BGM13], this will be achieved by proving so-called generalized curvature-dimension bounds, which first appeared in [BG17].

One often discusses the smoothness of a measure by computing its density with respect to Lebesgue measure and discussing the smoothness of the density function. However, in infinite-dimensions, there is no equivalent of Lebesgue measure, so one must take a different approach to discussing the smoothness of a measure. Instead of using a Lebesgue measure as a reference measure, we could instead compare the measure to a shifted version of itself. This is what is exhibited by the Cameron-Martin theorem for an abstract Wiener space (W, H), which states that, if μ is the Gaussian measure on W whose structure is determined by H (meaning, for $f \in W^* \subseteq H$, $\int_G e^{if(x)} d\mu(x) = e^{-\frac{1}{2}\langle f,f\rangle_H}$), then the measure μ exhibits "quasi-invariance," meaning that, if $h \in H$ and A_h denotes the map $A_h(x) = x + h$, then $\mu \circ A_h$ is absolutely continuous with respect to h, and $\frac{d\mu \circ A_h}{d\mu} = e^{-\frac{1}{2}\langle h,h\rangle_H + \langle x,h\rangle_H}$. This can be interpreted as a type of smoothness result for the measure. The quasi-invariance result that we will show is similar in nature.

4.1 Considerations

While the methods in [BGM13] are capable of handling the major complicating factor, namely hypoellipticity of the diffusion, we are presented with obstacles when trying to apply the methods when there are infinitely-many vertical directions. The quasi-invariance result requires one to discuss how the Carnot-Carotheodory distance, which determines the sub-Riemannian structure on $W \times C$, acts in infinite-dimensions. In [BGM13], it was shown that this distance is topologically equivalent to using the metric $\|\cdot\|_H + K\sqrt{\|\cdot\|_C}$ for some $K \geq 0$, which is used to show that the Carnot-Carotheodory distance is well-approximated by finite-dimensional "projected subgroups" of $W \times C$. However, it is proven in Section 6 that, if C is infinite-dimensional, then we cannot expect such a K to exist. Nevertheless, by using a different approach, an approximation result for the Carnot-Carotheodory distance is proven and used.

A second impediment comes from the group structure on $W \times C$. Let $\omega : W \times W \to C$ be a bilinear, antisymmetric map, which determines the group structure of $W \times C$. In previous works (meaning $\dim(C) < \infty$), it has been assumed that $\omega : H \times H \to C$ is Hilbert-Schmidt. Supposing for now that C is an infinite-dimensional Hilbert space, we maintain that $\omega : H \times H \to C$ Hilbert-Schmidt, as this is a necessary assumption to ensure $W \times C$-valued Brownian motion is defined. However, attempting to proceed with the methods of [BGM13] using the norm and inner product structure of C will fail. Indeed, to satisfy the desired curvature-dimension bound, we need to make use of the strictly positive constant (referred to as ρ_2 in [BGM13]):

$$\lfloor \omega \rfloor_{H \otimes H}^2 := \inf_{\|c\|_C = 1} \sum_{i,j=1}^{\infty} \langle \omega(e_i, e_j), c \rangle_C^2 > 0 .$$

The quasi-invariance result incorporates the ratio $\frac{\|\omega\|_{HS}}{\lfloor \omega \rfloor_{H \otimes H}}$ into the bound. But if ω is Hilbert-Schmidt, and if $\{c_\ell\}_{\ell \in \mathbb{N}}$ is an orthonormal basis for C, then the sequence indexed by ℓ given as

$$x_\ell := \sum_{i,j=1}^{\infty} \langle \omega(e_i, e_j), c_\ell \rangle_C^2$$

is summable, as $\sum_{\ell=1}^{\infty} x_\ell$ is equal to the Hilbert-Schmidt norm of ω, so x_ℓ converges to 0, implying that $\lfloor \omega \rfloor_{H \otimes H} = 0$. In fact, one can show that if $\lfloor \omega \rfloor_{H \otimes H} > 0$, then $\|\omega\|_{HS} = \infty$. We are forced to reconcile that, under any set of assumptions, $\frac{\|\omega\|_{HS}}{\lfloor \omega \rfloor_{H \otimes H}} = \infty$

Even so, we are able to achieve desired curvature bounds. Rather than using the structure of C alone, we also rely on the existence of a dense Hilbert subspace $Z \subsetneq C$. Under additional assumptions and with a sharper bound, we can replace $\|\omega\|_{HS}$ with an alternate constant and prove a quasi-invariance result.

4.2 Summary of assumptions and results

Let H and Z be separable Hilbert spaces. Given any skew-symmetric bilinear map $\omega : H \times H \to Z$, applying the ideas from Section 2.3, we may consider $H \times Z$ as a (potentially infinite-dimensional) graded step-2 nilpotent Lie algebra, also sometimes referred to as a "Heisenberg-like" Lie algebra, by defining the Lie-bracket as

$$[(h_1, z_1), (h_2, z_2)] \;=\; \omega(h_1, h_2)\,.$$

This, in turn, allows us to consider $H \times Z$ as a (potentially infinite-dimensional) simply connected graded step-2 nilpotent Lie group (or "Heisenberg-like" group, perhaps called $\exp(H \times Z)$ when emphasizing the group structure) by defining the product

$$(h_1, z_1) \cdot (h_2, z_2) \;=\; \left(h_1 + h_2\,,\; z_1 + z_2 + \frac{1}{2}\omega(h_1, h_2)\right).$$

We further impose technical assumptions on ω, namely that, for orthonormal bases $\{e_j\}_{j\in\mathbb{N}}$ and $\{f_\ell\}_{\ell\in\mathbb{N}}$ respectively,

$$\|\omega\|^2_{H\otimes H} := \sup_{\|z\|_Z=1} \sum_{i,j=1}^{\infty} \langle \omega(e_i, e_j), z\rangle_Z^2 \;<\; \infty \tag{A2.1}$$

$$\|\omega\|^2_{H\otimes Z} := \sup_{\|h\|_H=1} \sum_{j,\ell=1}^{\infty} \langle \omega(h, e_j), f_\ell\rangle_Z^2 \;<\; \infty \tag{A2.2}$$

$$\lfloor \omega \rfloor^2_{H\otimes H} := \inf_{\|z\|_Z=1} \sum_{i,j=1}^{\infty} \langle \omega(e_i, e_j), z\rangle_Z^2 \;>\; 0\,. \tag{A2.3}$$

We may also readily discuss the notion of group-valued Brownian motion, provided that we restrict to finite-dimensions. We assume that we have Banach spaces W and C that contain H and Z respectively as dense subsets, and that (W, H) is an abstract Wiener space, as defined in Section 2.2. This suffices to say that we have a Brownian motion $(B_t)_{t\geq 0}$ on W. Any finite-rank projection $P : H \to H$ (that is, $P \in \mathrm{Proj}(W)$) has a measurable linear extension to $P : W \to PH$, and can be used to define a finite-dimensional Brownian motion $(PB_t)_{t\geq 0}$ on PH. Then we may define the stochastic process $(g_t^P)_{t\geq 0}$, as

$$g_t^P \;=\; \left(PB_t\,,\; \int_0^t \omega(PB_s, \cdot)dPB_s\right).$$

This is essentially hypoelliptic $\exp(PH \times Z)$-valued Brownian motion, and is consistent with the definition in Section 2.3.3.

Our final technical assumption is the following.

For all $t > 0$, there exists a $W \times C$-valued random variable g_t such that, given an increasing sequence of finite-rank projections $(P_m)_{m\in\mathbb{N}} \in \mathrm{Proj}(W)^{\uparrow}$, for every $f \in (W \times C)^*$, $f(g_t^{P_m}) \to f(g_t)$ in probability. (A2.4)

We define[1] the hypoelliptic heat kernel measure on $W \times C$, denoted ν_t, to be the distribution of g_t.

The quasi-invariance result makes use of a metric subgroup of $\exp(H \times Z)$ that is intrinsically related to both the Cameron-Martin subspace of W and the subelliptic structure induced by ν_t. Let $\mathcal{AC}$ denote the set of absolutely continuous paths $\sigma : [0,1] \to H \times Z$, on which we may define the length[2] $\ell(\sigma) = \int_0^1 \|L_{\sigma(t)^{-1}*}\sigma'(t)\|_{H\times Z}dt$. We say σ is *horizontal* if $L_{\sigma(t)^{-1}*}\sigma'(t) \in H \times \{0\}$ for all $t \in [0,1]$, and we denote the set of such paths as $\mathcal{AC}_h$. Then for any $(h,z) \in H \times Z$, we define the Carnot-Carathéodory distance from the origin as

$$d(e,g) \;=\; \inf\left\{\ell(\sigma) \,\middle|\, \sigma \in \mathcal{AC}_h \,,\ \sigma(0) = e \,,\ \sigma(1) = g\right\}.$$

We will study the set of elements that are of finite horizontal distance from the origin, that is, $\{g \in H \times Z \mid d(e,g) < \infty\}$. It will be illustrated that, unlike for finite-dimensional C, we cannot expect this set to be exactly equal to $\exp(H \times Z)$. The objective of Chapter 2 is to prove Theorem 8.4, which states that, for elements g of finite horizontal distance, the shifted measure $\nu_t \circ L_{g*}$ on $W \times C$ is absolutely continuous with respect to ν_t, and we give L^p bounds on the Radon-Nikodym derivative, which be a function of $d(e,g)$.

4.2.1 The Hörmander condition

As remarked in Section 2.3.3, when discussing smoothness properties of densities of a diffusion, it is often crucial to make use of our Hörmander condition. In previous works on Heisenberg-like groups, even with infinitely-many horizontal directions and finitely-many vertical directions, this clearly corresponded to $\mathrm{span}\{\omega(e_i,e_j)\}_{i,j=1}^{\infty} = C$. It is the Hörmander condition that implies the existence of the nonzero constant

$$\inf_{\|c\|_C=1} \sum_{i,j=1}^{\infty} \langle\omega(e_i,e_j),c\rangle_C^2 \;>\; 0\,. \tag{2.1}$$

[1]Note that, in [BGM13], the heat kernel measure ν_t was essentially defined as the limiting distribution of $g_{2t}^{P_m}$. In this thesis, we will not introduce this factor of 2, in the interest of being consistent with the notation of Chapter 3, which is inline with notation for many finite-dimensional Taylor isomorphism theorems, like [DG97] and [DGS09a]. This means that formulae in this work will differ from those in [BGM13].

[2]Recall from Section 2.3.2 that $L_g : \exp(H \times Z) \to \exp(H \times Z)$ is left-multiplication by g, and that $L_{g*} : H \times Z \to H \times Z$ is its derivative.

When C is infinite-dimensional, it is not clear what the equivalent of the Hörmander condition is meant to be, but we will now provide 2 equivalent interpretations. The first is to simply declare that it is the existence of a constant resembling (2.1), which in this case is the strict positivity of the constant $\lfloor \omega \rfloor_{H \otimes H}$ (that is, assumption (A2.3)). On the other hand, in Section 5.2, we will show that (A2.1) implies that ω has an extension to the Hilbert space tensor product $\omega : H \otimes H \to Z$, and that (A2.3) implies this extension is surjective. Then, in a certain sense, $\{\omega(e_i, e_j)\}_{i,j=1}^{\infty}$ generates Z, which parallels the finite-dimesional Hörmander condition.

4.2.2 Naming convention

There are 3 major spaces that play a role in proving the quasi-invariance result. Here, we will provide their official titles and notation, as well as rationale for their titles. They are

1. $G = W \times C$, an *abstract Wiener nilpotent Lie group*, satisfying (A2.4), and thus possessing the heat kernel measure ν_t.

2. $\mathfrak{g}_{CM} = H \times Z$, the *Cameron-Martin subalgebra*, with Lie bracket determined by ω, satisfying (A2.1), (A2.2), and (A2.3).

3. $G_{CM} = \{g \in \exp(\mathfrak{g}_{CM}) : d(e, g) < \infty\}$, the *Cameron-Martin subgroup*, possessing the right-invariant metric d.

We first remark that our construction of a nilpotent abstract Wiener Lie group is consistent with the general definition provided in Section 3, in which $X = \mathfrak{g}_{CM}$ and $X_H = H$. As in the general case, the primary assumption on G is that it is "large enough" to contain the probability distribution, and its role mimics that of W when (W, H) is an abstract Wiener space. We do not assume that ω has an extension (continuous, measurable, or otherwise), but (A2.2) suffices to show that there is a measurable action of G_{CM} on G, which is necessary to even describe quasi-invariance. This will be described further in Section 7.

We can make sense of $\mathfrak{g}_{CM}$ by considering our generalized Hörmander condition. Indeed, in a sense, we may view $H \times Z$ as the Lie algebra generated by H, the Cameron-Martin subspace of W. It is not necessarily true that every element of Z is a finite linear-combination of elements in $\omega(H, H)$, but they do lie in the image of $\widetilde{\omega} : H \otimes H \to Z$, and can be written as a potentially infinite sum of elements in $\omega(H, H)$.

On the other hand, we may regard G_{CM} as the group generated by H, but this too requires some interpretation of the word "generated." Not only does G_{CM} include finite products of the form $(h_1, 0) \cdot \ldots \cdot (h_n, 0)$, but it also includes endpoints of horizontal paths $\sigma : [0,1] \to \exp(\mathfrak{g}_{CM})$, which must satisfy, for some path $A : [0,1] \to H$,

$$\sigma'(t) \;=\; L_{\sigma(t)*}A(t) \qquad\qquad \sigma(0) = e\,.$$

Such endpoints are generalized products of elements in H in the same way that the solution to this differential equation results in a generalization of the Baker-Campbell-Hausdorff formula, see Theorem 2.6 and [Str87].

The sets $\mathfrak{g}_{CM}$ and G_{CM} are only related in that they are both, in some sense, generated by H, the former by the Lie bracket and the latter by group multiplication. Their relationship with each other is not equivalent to finite-dimensional Lie groups, since we generally have $\exp(\mathfrak{g}_{CM}) \neq G_{CM}$. However, in Chapter 3, there will be some discussion in which we regard elements in $\mathfrak{g}_{CM}$ as corresponding to left-invariant vector fields on G_{CM}, see Section 11.3.

4.2.3 Further directions

In the text [Wan14, Section 5.2], the curvature-dimension bounds (as discussed on Section 5.1) are further generalized. It would be interesting if such bounds, and hence quasi-invariance, could be generalized to account for step n nilpotent Lie groups. Here is a possible setup for considering an infinite-dimensional step 3 nilpotent Lie group. We assume that we have a separable Hilbert space and Lie algebra $H = X \oplus Y \oplus Z$, where the Lie bracket $[\cdot,\cdot] : H \times H \to H$ is weakly Hilbert-Schmidt with surjectivity properties, meaning that, for bases $\{e_j\}_{j\in\Lambda_X}$, $\{f_\ell\}_{\ell\in\Lambda_Y}$, $\{g_p\}_{p\in\Lambda_Z}$ of X, Y, and Z respectively, for $y \in Y$ and $z \in Z$,

$$c_X\|y\|_Y^2 \;\le\; \sum_{i,j\in\Lambda_X} \big\langle y, [e_i, e_j]\big\rangle_Y^2 \;\le\; C_X\|y\|_Y^2$$

$$c_y\|z\|_Z^2 \;\le\; \sum_{j\in\Lambda_X,\ell\in\Lambda_Y} \big\langle z, [e_j, f_\ell]\big\rangle_Z^2 \;\le\; C_Y\|z\|_Y^2\,,$$

where C_X and C_Y play the role of $\|\omega\|_{H\otimes H}$, and c_X and c_Y play the role of $\lfloor\omega\rfloor_{H\otimes H}$. We would also assume the existence of constants K_X, K_Y that play the role of $\|\omega\|_{H\otimes Z}$, that resemble, for $x \in X$, $y \in Y$,

$$\sum_{j\in\Lambda_X,\ell\in\Lambda_Y} \big\langle f_\ell, [e_j, x]\big\rangle_Y^2 \;\le\; K_X\|x\|_X^2$$

$$\sum_{j\in\Lambda_X,p\in\Lambda_Z} \big\langle g_p, [e_j, y]\big\rangle_Z^2 \;\le\; K_Y\|y\|_Y^2\,.$$

It seems reasonable for such inequalities to give rise to further generalized curvature dimension bounds. If we define the differential operators, for sufficiently smooth

$f \in \mathcal{C}^\infty(H)$,

$$\Delta_X f \;=\; \sum_{j\in\Lambda_X} \widetilde{e_j}^2 f$$

$$\begin{aligned}
\Gamma^X(f) \;&=\; \Gamma^X(f,f) \;=\; \sum_{j\in\Lambda_X} \left(\widetilde{e_j} f\right)^2 \qquad \Gamma_2^X(f) \;=\; \Delta_X \Gamma^X(f) - 2\Gamma^X(f, \Delta_X f) \\
\Gamma^Y(f) \;&=\; \Gamma^Y(f,f) \;=\; \sum_{\ell\in\Lambda_Y} \left(\widetilde{f_\ell} f\right)^2 \qquad \Gamma_2^Y(f) \;=\; \Delta_X \Gamma^Y(f) - 2\Gamma^Y(f, \Delta_X f) \\
\Gamma^Z(f) \;&=\; \Gamma^Z(f,f) \;=\; \sum_{p\in\Lambda_Z} \left(\widetilde{g_p} f\right)^2 \qquad \Gamma_2^Z(f) \;=\; \Delta_X \Gamma^Z(f) - 2\Gamma^Z(f, \Delta_X f)\,,
\end{aligned}$$

then there may be functional inequalities that resemble (2.3). One seemingly plausible candidate is, for all $s, \nu \in \mathbb{R}$,

$$\begin{aligned}
&\Gamma_2^X(f) + s\nu \Gamma_2^Y(f) + s\nu^2 \Gamma_2^Z(f) \\
&\qquad\qquad \geq\; (A - Bs\nu)\Gamma^Y(f) + Cs\nu\Gamma^Z(f) - (D\frac{1}{s\nu} + Es\nu)\Gamma^X(f) \qquad (2.2)
\end{aligned}$$

for some positive constants $A, B, C, D, E \in \mathbb{R}$. However, it does not seem that this inequality is quite satisfied under the provided conditions. Taking inspiration from calculations in [Mel21] (see also Section 10.3), it is possible that introducing a trace-type estimate like

$$\sum_{k\in\Lambda_X} \left(\sum_{i,j\in\Lambda_X} \langle z, [[e_i, e_k], e_j] \rangle_Z \right)^2 \;\leq\; C\|z\|_Z^2$$

could give rise to another operator

$$\sum_{k\in\Lambda_X} \left(\sum_{i,j\in\Lambda_X} \widetilde{[[e_i, e_k], e_j]} f \right)^2 ,$$

whose addition would allow for an inequality resembling (2.2). Such an inequality seems to provide a reverse Poincare inequality: if $P_T f = e^{T/2\Delta_X} f = \mathbb{E}[f(\cdot g_T)]$ is the heat semi-group, then

$$\Gamma^X(P_T f) \;\leq\; \frac{C}{T}\big(P_T(f^2) - (P_T f)^2\big)\,.$$

for some constant C. The proof would require a modified argument, making use of the functional

$$\Phi(t) \;=\; a(t)P_t(\Gamma^X(P_{T-t}f)) + b(t)P_t(\Gamma^Y(P_{T-t}f)) + c(t)P_t(\Gamma^Z(P_{T-t}f))$$

with “controls” $a(t)$, $b(t)$, and $c(t)$ defined by a differential equation; see the proof of [BGM13, Proposition 2.5]. However, without transverse symmetry, as in (2.4) (or, at a minimum, some type of bound resembling (2.4)), a reverse logarithmic Sobolev inequality, and thus a quasi-invariance result, would be out of reach.

It would also be worthwhile to explore further smoothness properties of the Radon-Nikodym derivative $\frac{\nu_T \circ L_g}{\nu_T}$, like differentiability, even merely in the step-2 case, as was done in [DEM16]. This relied on examining the Radon-Nikodym derivative of ν_T with respect to $\nu_T \otimes dc$, that is, the product of a Gaussian measure on W and Lebesgue measure on the (finite-dimensional) C. When C is infinite-dimensional, such a Lebesgue measure will not exist, and a different approach would have to be taken.

5 Setup

5.1 A brief exposition to curvature-dimension bounds

The discussion below is primarily derived from [BG17] and [BGM13]. For now, let G be a finite-dimensional Lie group with operation $\cdot$ and Lie algebra $\mathfrak{g}$. For $x \in \mathfrak{g}$, we may define the corresponding left-invariant vector field $\widetilde{x}$ satisfying

$$\widetilde{x}f(g) \;=\; \frac{d}{dt}f(g\cdot(tx))\,,$$

which coincides with the notion described in Section 2.3.2

Then assume G has a left-invariant Laplacian Δ_G given as

$$\Delta_G f(g) \;=\; \sum_{i=1}^{n} \widetilde{x_i}^2 f(g)$$

for some $x_1, \ldots, x_n \in \mathfrak{g}$. Given $t \in \mathbb{R}$, we have a heat operator $e^{t/2\Delta_G} : C^\infty(G) \to C^\infty(G)$, a probability measure ν_t, a probability density ρ_t with respect to the left-invariant Haar measure dg, and a corresponding Lie group Brownian motion $(g_t)_{t\geq 0}$ all satisfying[3]

$$P_t f(h) \;=\; e^{t/2\Delta_G} f(h) \;=\; \mathbb{E}[f(h\cdot g_t)] \;=\; \int_G f(h\cdot g)d\nu_t(g) \;=\; \int_G f(h\cdot g)p_t(g)dg\,.$$

Critical to our analysis is the Carré du champ operator $\Gamma : \mathcal{C}^\infty(G)\times\mathcal{C}^\infty(G) \to \mathbb{R}$, defined as as

$$\Gamma(f_1,f_2) \;=\; \sum_{i=1}^{n}\Big(\widetilde{x}_i f_1\Big)\Big(\widetilde{x}_i f_2\Big)\,.$$

And $\Gamma_2 : \mathcal{C}^\infty(G)\times\mathcal{C}^\infty(G) \to \mathbb{R}$ as

$$\Gamma_2(f_1,f_2) \;=\; \frac{1}{2}\Big(\Delta_G\Gamma(f_1,f_2) - \Gamma(\Delta_G f_1, f_2) - \Gamma(f_1, \Delta_G f_2)\Big)\,,$$

where we abbreviate $\Gamma(f) := \Gamma(f,f)$ and $\Gamma_2(f) := \Gamma_2(f,f)$.

Suppose G is equipped with a (nondegenerate) Riemannian metric, meaning that $x_1, \ldots, x_n$ span all of $\mathfrak{g}$. Then the Ricci curvature tensor, denoted Ric, satisfies the equality below.

$$\Gamma_2(f) \;=\; \|\nabla^2 f\|^2 + 2\mathrm{Ric}(\nabla f, \nabla f)$$

[3] Note that, in [BGM13], the heat kernel measure ν_t was defined as the distribution of g_{2t}, rather than g_t. Thus, formulae in this work will differ from those in [BGM13].

By applying the Cauchy-Schwarz inequality, we see $\|\nabla^2 f\|_2^2 \geq \frac{1}{n}(\Delta f)^2$, so the inequality $\mathrm{Ric}(\nabla f, \nabla f) \geq \rho$ implies

$$\Gamma_2(f) \geq \frac{1}{n}(\Delta f)^2 + \rho\Gamma(f)\,.$$

Inequalities of this type have proven to be useful for a wide variety of applications, as demonstrated in [BÉ85; BL06; BQ99; Led00]. Dimension-independent properties can be derived with merely $\Gamma_2(f) \geq \rho\Gamma(f)$. However, having a lower bound on the Ricci curvature, meaning $\mathrm{Ric}(x,x) \geq \rho \in \mathbb{R}$ for all $x \in \mathfrak{g}$, is not immediately available in the sub-Riemannian case.

The generalized curvature-dimension bounds, first used in [BG17] and later in [BBG14; BB12], apply in sub-Riemannian contexts while still being powerful enough to emulate the classical curvature bounds. In this context, we do not assume that the "diffusion directions" $\{x_1, \ldots, x_n\}$ span all of $\mathfrak{g}$, which induces a "degenerate" geometry, but still satisfy the Hörmander condition (Section 2.3.3), meaning that

$$\mathrm{span}\Big\{x_{j_1}, [x_{j_1}, x_{j_2}], \ldots, [[x_{j_1}, x_{j_2}], \ldots, x_{j_k}]\Big\}_{j_1,\ldots,j_k=1}^{n} \;=\; \mathfrak{g}\,.$$

It is a theorem of Hörmander that such a diffusion shares some of the nice properties of elliptic diffusions, such as smoothness and strict positivity of the density.

In this case, we say that $\mathfrak{g}$ is comprised of "horizontal" x_i directions that generate the diffusion, and refer to the remaining directions as being "vertical." So suppose Δ_G is given by $\Delta_G f = \sum_{i=1}^n \widetilde{x}_i f$, where $\{x_i\}_{i=1}^n \cup \{z_i\}_{i=1}^m$ is a basis of $\mathfrak{g}$. Then we define operators

$$\Gamma^Z(f_1, f_2) \;=\; \sum_{\ell=1}^{m} \Big(\widetilde{z}_i f_1\Big)\Big(\widetilde{z}_i f_2\Big)$$

$$\Gamma_2^Z(f_1, f_2) \;=\; \frac{1}{2}\Big(\Delta_G \Gamma^Z(f_1, f_2) - \Gamma^Z(\Delta_G f_1, f_2) - \Gamma^Z(f_1, \Delta_G f_2)\Big)\,,$$

where we again abbreviate $\Gamma^Z(f) := \Gamma^Z(f,f)$ and $\Gamma_2^Z(f) := \Gamma_2^Z(f,f)$. Then the generalized curvature-dimension bounds are as follows: there exist $\alpha, \beta > 0$ such that, for all $\nu > 0$

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \;\geq\; \alpha\Gamma^Z(f) - \frac{\beta}{\nu}\Gamma(f)\,. \tag{2.3}$$

In [BG17], this was used in combination with transverse symetries

$$\Gamma(f, \Gamma^Z(f)) \;=\; \Gamma^Z(f, \Gamma(f))\,. \tag{2.4}$$

These imply a reverse logarithmic Sobolev inequality[4]

$$\Gamma(\ln P_T f) \leq \frac{2\big(1 + \frac{2\beta}{\alpha}\big)}{T}\left(\frac{P_T(f \ln f)}{P_T f} - \ln P_T f\right)$$

[4]We remark that this formula differs from those in [BGM13], due to our definition of ν_t.

A survey of different types of logarithmic Sobolev inequalities and their many uses is presented in [Led11]. It is known that a reverse logarithmic Sobolev inequality leads to Wang-type Harnack inequalities: setting $C = 2(1+\frac{2\beta}{\alpha})$, for all $f \in L^\infty(G)$,

$$(P_T f)^p(x) \;\le\; P_T f^p(y) \exp\left(\frac{Cp}{p-1}\frac{d(x,y)^2}{4T}\right).$$

Wang type Harnack inequalities are equivalent to integrated Harnack inequalities: there exists a constant c where $(P_T f)^p(x) \le c P_T f^p(y)$ if and only if

$$\left(\int_G \left(\frac{p_T(x,z)}{p_T(y,z)}\right)^{1/(p-1)} p_T(y,z)dz\right)^{p-1} \;\le\; c$$

This estimate is naturally related to quasi-invariance estimates. Indeed, if we let J_L be the Radon-Nikodyn derivative of $\nu_T \circ L_g$ with respect to ν_T, then we know that J_L corresponds to a smooth function. We may write $J_L = \frac{d\nu_T \circ L_g}{d\nu_T} = \frac{\rho_T \circ L_g}{\rho_T}$, and can write

$$\|J_L\|_{L^q(G,\nu_T} \;\le\; c$$

We compile this into a single statement.

Theorem 5.1. *Suppose that, for all $f \in \mathcal{C}^\infty(G)$, the generalized curvature dimension bounds (2.3) and transverse symmetries (2.4) hold. Then for $J^L_g(x) = \frac{d\nu_T \circ L_g}{d\nu_T}(x) = \frac{\rho_T(g\cdot x)}{\rho_T(x)}$, and assuming $\frac{1}{p}+\frac{1}{q}=1$,*

$$\|J^L_g\|_{L^p(G,\nu_T)} \le \exp\left(\left(1+\frac{2\beta}{\alpha}\right)\frac{(p+1)d(e,g)^2}{2T}\right).$$

with a similar bound holding for $J^R_g(x) = \frac{d\nu_T \circ R_g}{d\nu_T}$. In other words, using $L^q(G,\nu_T)^* \cong L^p(G,\nu_T)$, for all $f \in L^p(G,\nu_T)$,

$$\begin{aligned}\left|\int_G f(g\cdot x)d\nu_T(x)\right| &= \left|\int_G f(x)d(\nu_T \circ L_{g^{-1}})(x)\right| \\ &\le \left|\int_G f(x)d\nu_T(x)\right| \exp\left(\left(1+\frac{2\beta}{\alpha}\right)\frac{(p+1)d(e,g)^2}{2T}\right).\end{aligned}$$

So, using this approach, the key to arriving at dimension-independent quasi-invariance estimates is proving dimension-independent generalized curvature-dimension bounds.

5.2 Assumptions on the bilinear map

As discussed in Section 4.2 that we are given separable Hilbert spaces H, Z, and a bilinear map $\omega : H \times H \to Z$. For now, let $\{e_j\}_{j\in\mathbb{N}}$ and $\{f_\ell\}_{\ell\in\mathbb{N}}$ be orthonormal bases of H and Z respectively. We define (independently of basis chosen)

$$\|\omega\|^2_{H\otimes H} := \sup_{\|z\|=1} \sum_{i,j} \langle \omega(e_i, e_j), z\rangle_Z^2 < \infty . \tag{A2.1}$$

As explained in Section 2.1.3, the property that $\|\omega\|_{H\otimes H} < \infty$ is referred to as being *weakly Hilbert-Schmidt*. This property is characterized by the fact that ω extends to a linear operator $\widetilde{\omega} : H \otimes H \to Z$ with operator norm $\|\omega\|_{H\otimes H}$. Indeed,

$$\begin{aligned}\sup_{\|z\|=1} \sum_{i,j} \langle \omega(e_i, e_j), z\rangle_Z^2 &= \sup_{\|z\|=1} \sum_{i,j} \langle e_i \otimes e_j, \widetilde{\omega}^* z\rangle^2_{H\otimes H} \\ &= \|\widetilde{\omega}^*\|_{\mathcal{L}(Z, H\otimes H)} = \|\widetilde{\omega}\|_{\mathcal{L}(H\otimes H, Z)} .\end{aligned}$$

In addition to contributing to showing that $W \times C$-valued Brownian motion is well-defined, this will be used in a critical way in Section 6 when discussing finite-dimensional approximations of horizontal distance.

We also assume

$$\lfloor \omega \rfloor^2_{H\otimes H} := \inf_{\|z\|=1} \sum_{i,j} \langle \omega(e_i, e_j), z\rangle_Z^2 > 0 . \tag{A2.3}$$

The significance of $\lfloor \omega \rfloor_{H\otimes H}$ is the following: if $\|\omega\|_{H\otimes H} < \infty$, then $\lfloor \omega \rfloor_{H\otimes H}$ is the operator lower-bound of $\widetilde{\omega}^* : Z \to H \otimes H$. Indeed,

$$\begin{aligned}\|\widetilde{\omega}^*(z)\|^2_{H\otimes H} &= \sum_{i,j=1}^{\infty} \langle \widetilde{\omega}^*(z), e_i \otimes e_j\rangle^2_{H\otimes H} \\ &= \sum_{i,j=1}^{\infty} \langle z, \omega(e_i, e_j)\rangle_Z^2 \geq \lfloor \omega \rfloor^2_{H\otimes H} \|z\|_Z^2 .\end{aligned}$$

And as proven in Proposition 2.1, $\widetilde{\omega}^*$ is bounded below if and only if $\widetilde{\omega}$ is surjective. Thus, $\lfloor \omega \rfloor_{H\otimes H} > 0$, if and only if $\widetilde{\omega}$ is surjective. Importantly, this implies that $\operatorname{span}(\omega(H \times H))$ is dense in Z, but note that the converse is not necessarily true. That is, $\operatorname{span}(\omega(H \times H))$ being dense does not imply the existence of such a lower bound.

Furthermore, we assume

$$\|\omega\|^2_{H\otimes Z} := \sup_{\|h\|=1} \sum_{i,\ell} \langle \omega(h, e_i), f_\ell\rangle_Z^2 < \infty . \tag{A2.2}$$

Phrased another way, we assume that the trilinear map $\langle \omega(\cdot,\cdot),\cdot\rangle_Z : H \times H \times Z \to \mathbb{R}$ extends to a continuous map $H \times H \otimes Z \to \mathbb{R}$. This constant is not needed to define the notion of (group-valued) Brownian motion on $W \times C$, and will not play a role in Chapter 3. However, for this result, this constant tempers the group and differential structure on G, and will appear in the generalized curvature-dimension bounds.

The reader should note that ω being Hilbert-Schmidt is stronger than $\|\omega\|_{H\otimes H} < \infty$ and $\|\omega\|_{H\otimes Z} < \infty$, since $\max(\|\omega\|_{H\otimes H}, \|\omega\|_{H\otimes Z}) \le \|\omega\|_{HS(H\otimes H,Z)}$. But, as remarked in the introduction, the assumptions that $\|\omega\|_{H\otimes H} < \infty$, $\lfloor\omega\rfloor_{H\otimes H} > 0$, and $\dim(Z) = \infty$ will contradict ω being Hilbert-Schmidt.

From this, we may regard $H \times Z$ as a (infinite-dimensional) Lie algebra, where $[h_1 + z_1, h_2 + z_2] = \omega(h_1, h_2)$. By identifying $H \times Z$ with $\exp(H \times Z)$, we may also regard $H \times Z$ as a group (sometimes referred to as $\exp(H \times Z)$ when we do) with the product

$$(h_1, z_1) \cdot (h_2, z_2) \;=\; \left(h_1 + h_2, z_1 + z_2 + \frac{1}{2}\omega(h_1, h_2)\right).$$

5.3 Finite-dimensional projections and subgroups

Continuing the assumptions from the previous section, we further assume that H and Z are densely contained in Banach spaces W and C. Using the same methods as in Section 2.2, we may say that there are dense inclusions $W^* \subseteq H \subseteq W$ and $C^* \subseteq Z \subseteq C$, and that any finite rank projections $P : H \to H$ and $Q : Z \to Z$ have continuous linear extensions to $P : W \to H$ and $Q : C \to Z$. We will use $\mathrm{Proj}(W)$ and $\mathrm{Proj}(C)$ respectively to denote the sets of these projections. By density, we may also consider increasing sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_m)_{m\in\mathbb{N}}$, for which $P_n \to I_W$ and $Q_m \to I_C$ in the strong operator topology, and we denote the sets of such sequences as $\mathrm{Proj}(W)^\uparrow$ and $\mathrm{Proj}(C)^\uparrow$.

For $P \in \mathrm{Proj}(W) \cup \{I_H\}$ and $Q \in \mathrm{Proj}(C) \cup \{I_Z\}$, we define $G^{P,Q}$ as the set $PH \times QZ$. We may regard this set as a Lie algebra with bracket determined by $Q\omega : PH \times PH \to QZ$, and as a group by using the group operation $\cdot_Q$ defined as

$$(x_1, y_1) \cdot_Q (x_2, y_2) \;=\; \left(x_1 + x_2, y_1 + y_2 + \frac{1}{2}Q\omega(x_1, x_2)\right).$$

Observe that $G^{P,Q}$ is typically not a subgroup of G (it generally does not have the same group operattion!), and the continuous projection $P \times Q : G \to G^{P,Q}$ is not a homomorphism. Still, $G^{P,Q}$ will still serve as a “geometric” approximation of G. When considering projections, we will often require $\mathrm{span}(\omega(PH \times PH)) \supseteq QZ$. This is to be expected, because this corresponds to the hypoelliptic differential equation defining Brownian motion on $G^{P,Q}$ satisfying the Hörmander condition.

5.4 The group structure on $H \times Z$

In this section, we will discuss more structure on $H \times Z$ and recall some facts and notation from Section 2.3.

We regard $H \times Z$ as a Lie algebra. In fact, we will sometimes refer to it as $\mathfrak{g}_{CM}$, the Cameron-Martin subalgebra, with a Lie-bracket $[\cdot,\cdot] : \mathfrak{g}_{CM} \times \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$ defined as

$$[(h_1, z_1), (h_2, z_2)] \;=\; \big(0, \omega(h_1, h_2)\big)\,.$$

In this way, we may also regard it as a group by defining a group operation via the Baker-Campbell-Hausdorff formula, as described in Section 2.3, ultimately giving

$$(h_1, z_1) \cdot (h_2, z_2) \;=\; \left(h_1 + h_2, z_1 + z_2 + \frac{1}{2}\omega(h_1, h_2)\right).$$

When emphasizing the group structure, we will refer to $\mathfrak{g}_{CM}$ as $\exp(\mathfrak{g}_{CM})$. Note that this differs from G_{CM}, which is described in Section 6.

As noted in Section 7, we do not assume that $\omega : W \times W \to C$ is continuous. To make sense of the main result, we only need to recognize that left-multiplication by an element in $\exp(\mathfrak{g}_{CM})$ induces a measurable action on $W \times C$, see Section 7.2 for more details.

5.5 Convergence of constants

Given any $P \in \mathrm{Proj}(W) \cup \{I_H\}$, $Q \in \mathrm{Proj}(C) \cup \{I_Z\}$, if $\{e_j\}_{1 \le j \le r \le \infty}$ is a basis of PH, then we may consider $\|Q\omega\|_{PH\otimes PH}$, calculated as

$$\|Q\omega\|^2_{PH\otimes PH} \;:=\; \|Q \circ \omega \circ (P \otimes P)\|^2_{PH\otimes PH} \;:=\; \sup_{\substack{\|z\|_Z=1\\ z\in QZ}} \sum_{i,j=1}^{r} \langle \omega(e_i, e_j), z\rangle_Z^2$$

and we may similarly consider $\lfloor Q\omega \rfloor_{PH\otimes PH}$, and $\|Q\omega\|_{PH\otimes QZ}$. We will consider these "projected" constants and prove that they converge to (or are bounded by) their infinite-dimensional counterparts. The value here lies in that these constants appear in the generalized curvature-dimension bounds in Section 8.2, as well as the quasi-invariance result in Section 8.3.

For any metric space V, we let $B^V_{\le 1}$ denote the closed unit ball of V centered at 0.

Lemma 5.2. *For any weakly Hilbert-Schmidt bilinear map $\omega : H \times H \to Z$, $\lfloor \omega \rfloor_{H\otimes H} > 0$ if and only if $\widetilde{\omega}(H \otimes H) = Z$. As a consequence, if $P \in$ Proj$(W) \cup \{I_W\}$, $Q \in$ Proj(C), and if* $\mathrm{span}(Q\omega(PH \times PH)) = QZ$*, then $\lfloor Q\omega \rfloor_{PH\otimes PH} \ne 0$.*

Remark 5.3. The consequence of Lemma 5.2 above is equivalent to remark 4.1 in [BGM13].

Proof. We have that $\lfloor \omega \rfloor_{H\otimes H}$ is the lower-bound of $\widetilde{\omega}^*$. Indeed,

$$\begin{aligned} \|\widetilde{\omega}^*(z)\|^2_{H\otimes H} &= \sum_{i,j=1}^{\infty} \langle \widetilde{\omega}^*(z), e_i \otimes e_j \rangle^2_{H\otimes H} \\ &= \sum_{i,j=1}^{\infty} \langle z, \omega(e_i, e_j)\rangle^2_Z \geq \lfloor \omega \rfloor_{H\otimes H} \|z\|^2_Z \,. \end{aligned}$$

As shown in Proposition 2.1, $\widetilde{\omega}^*$ is bounded below if and only if $\widetilde{\omega}$ is surjective. From this, the general consequence follows since, if $P \in \mathrm{Proj}(W) \cup \{I_H\}$ and $Q \in \mathrm{Proj}(C)$ with $\mathrm{span}(\omega(PH \times PH)) \supseteq QZ$, then

$$Q\widetilde{\omega}(PH \otimes PH) = \widetilde{Q\omega}(PH \otimes PH) = \mathrm{span}(Q\omega(PH \times PH)) = QZ \,,$$

which implies that $\widetilde{Q\omega \circ (P \otimes P)}$ is surjective, so that $\lfloor Q\omega \rfloor_{PH\otimes PH} > 0$.

□

Lemma 5.4. *If* $(P_n)_{n\in\mathbb{N}} \in \mathit{Proj}(W)^{\uparrow}$, $Q \in \mathit{Proj}(C)^{\uparrow}$, *and* $\mathrm{span}(\omega(P_1H \times P_1H)) \supseteq QZ$, *then*

$$\lfloor Q\omega \rfloor_{P_nH\otimes P_nH} \xrightarrow[n\to\infty]{} \lfloor Q\omega \rfloor_{H\otimes H} \qquad \|Q\omega\|_{P_nH\otimes P_nH} \xrightarrow[n\to\infty]{} \|Q\omega\|_{H\otimes H} \,.$$

And if $(Q_n)_{n\in\mathbb{N}} \in \mathit{Proj}(C)^{\uparrow}$,

$$\lfloor Q_n\omega \rfloor_{H\otimes H} \xrightarrow[n\to\infty]{} \lfloor \omega \rfloor_{H\otimes H} \qquad \|Q_n\omega\|_{H\otimes H} \xrightarrow[n\to\infty]{} \|\omega\|_{H\otimes H} \,.$$

And for any $P \in \mathit{Proj}(W) \cup \{I_W\}$, $Q \in \mathit{Proj}(C) \cup \{I_C\}$, $\|Q\omega\|_{PH\otimes QZ} \leq \|\omega\|_{H\otimes Z}$.

Proof. For all n, let $\{e_j\}_{1\leq j\leq r_n}$ be an orthonormal basis of P_nH, so that $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of H. Then define the function $F_n : B^{QZ}_{\leq 1} \to \mathbb{R}$ as $F_n(z) = \sum_{i,j=1}^{r_n} \langle \omega(e_i, e_j), z\rangle^2_Z$, and $F : B^{QZ}_{\leq 1}(0) \to \mathbb{R}$ as $F(z) = \sum_{i,j=1}^{\infty} \langle \omega(e_i, e_j), z\rangle^2_Z$. Then we see that all F_n and F are continuous by the existence of $\|\omega\|_{H\otimes H} < \infty$, and that, pointwise, F_n increases and converges to F in n. Since $B^{QZ}_{\leq 1}$ is compact, by Dini's theorem (see, for example, [Rud76, Theorem 7.13]), F_n converges uniformly to F. Then

$$\sup_{z\in B^{QZ}_{\leq 1}(0)} F_n(z) \xrightarrow[n\to\infty]{} \sup_{z\in B^{QZ}_{\leq 1}(0)} F(z)$$

and we may also deduce that $-F_n$ converges uniformly to $-F$, and that

$$\inf_{z\in B^{QZ}_{\leq 1}(0)} F_n(z) = -\sup_{z\in B^{QZ}_{\leq 1}(0)} (-F_n(z)) \xrightarrow[n\to\infty]{} -\sup_{z\in B^{QZ}_{\leq 1}(0)} (-F(z)) = \inf_{z\in B^{QZ}_{\leq 1}(0)} F(z)\,.$$

This proves the first 2 claimed convergences. For the next 2, we regard F as being defined on $B^Z_{\leq 1}$ (on which it is still continuous). Consider that $B^{Q_mZ}_{\leq 1}$ is a nested, increasing sequence of sets such that $\bigcup_{m=1}^\infty B^{Q_mZ}_{\leq 1}$ is dense in $B^Z_{\leq 1}$. Thus,

$$\sup_{z\in B^{Q_mZ}_{\leq 1}} F(z) \xrightarrow[m\to\infty]{} \sup_{z\in \bigcup_m B^{Q_mZ}_{\leq 1}} F(z) = \sup_{h\in B^Z_{\leq 1}} F(z)$$

and likewise for the infimum. This proves the last 2 claimed limits.

The final inequality follows from the definitions. Indeed, if $\{e_j\}_{1\leq j\leq r}$ and $\{f_\ell\}_{1\leq \ell\leq s}$ are bases of PH and QZ respectively, then

$$\|Q\omega\|^2_{PH\otimes QZ} \;=\; \sup_{h\in B^{PH}_{\leq 1}} \sum_{j=1}^r \sum_{\ell=1}^s \langle f_\ell, \omega(h,e_j)\rangle_Z^2 \;\leq\; \|\omega\|^2_{H\otimes Z}\,.$$

□

5.6 Examples

The examples below determine the structures of H, Z, and ω. We remark that these can always be made into nilpotent abstract Wiener Lie groups as defined in Section 3 by asserting that W and C are Hilbert spaces in which the inclusions $H\to W$ and $Z\to C$ are , as discussed in Example 3.4.

Example 5.5. If both $H = \operatorname{span}\{e_1,\dots,e_n\}$ and $Z = \operatorname{span}\{f_1,\dots,f_m\}$ are finite-dimensional, then any bilinear anti-symmetric defintion of ω will suffice, provided that $\operatorname{span}\{[e_i,e_j]\}_{i,j=1}^n = Z$. If H is infinite-dimensional and Z is finite-dimensional, then this becomes the object of study in [BGM13], for which we need the additional assumption that the bracket $[\cdot,\cdot]$ is Hilbert-Schmidt. Thus, for the remaining examples, we will consider the presence of infinitely-many horizontal *and* vertical directions.

Example 5.6. Let $\{e_j\}_{j\in\mathbb{N}}$ and $\{f_\ell\}_{\ell\in\mathbb{N}}$ be orthonormal bases of H and Z respectively, and define ω as

$$\omega(e_i,e_j) \;=\; \begin{cases} f_\ell & \text{if } i=2\ell-1,\ j=2\ell \\ -f_\ell & \text{if } i=2\ell,\ j=2\ell-1 \\ 0 & \text{otherwise}\end{cases}\,.$$

Or, in other words,

$$\omega\left(\sum_{i=1}^{\infty}\alpha_i e_i\,,\ \sum_{j=1}^{\infty}\beta_j e_j\right) \;=\; \sum_{\ell=1}^{\infty}\Big(\alpha_{2\ell-1}\beta_{2\ell}-\alpha_{2\ell}\beta_{2\ell-1}\Big)f_\ell\,.$$

For any $\ell\in\mathbb{N}$, $\mathrm{span}(e_{2\ell-1},e_{2\ell},f_\ell)$ is a subgroup of $H\times Z$, and is isomorphic to the classical Heisenberg group. In this way, we may regard this example as an infinite product of Heisenberg groups.

Observe that, for any $z\in Z$,

$$\begin{aligned}\sum_{i,j=1}^{\infty}\langle z,\omega(e_i,e_j)\rangle_Z^2 \;&=\; \sum_{\ell=1}^{\infty}\sum_{j=1}^{\infty}\Big(\langle z,\omega(e_{2\ell-1},e_j)\rangle_Z^2+\langle z,\omega(e_{2\ell},e_j)\rangle_Z^2\Big)\\ &=\; \sum_{\ell=1}^{\infty}\Big(\langle z,\omega(e_{2\ell-1},e_{2\ell})\rangle_Z^2+\langle z,\omega(e_{2\ell},e_{2\ell-1})\rangle_Z^2\Big)\\ &=\; \sum_{\ell=1}^{\infty}2\langle z,f_\ell\rangle_Z^2 \;=\; 2\|z\|_Z^2\,,\end{aligned}$$

so, in this case, we have $\|\omega\|_{H\otimes H}=\lfloor\omega\rfloor_{H\otimes H}=2$. And, for $h\in H$, first consider that

$$\begin{aligned}\langle\omega(e_i,h),f_\ell\rangle_Z \;&=\; \sum_{j=1}^{\infty}\langle h,e_j\rangle_H\langle\omega(e_i,e_j),f_\ell\rangle_Z\\ &=\; \delta_{i,2\ell-1}\langle h,e_{2\ell}\rangle_H-\delta_{i,2\ell}\langle h,e_{2\ell-1}\rangle_H\,.\end{aligned}$$

and

$$\begin{aligned}\sum_{i,\ell=1}^{\infty}\langle\omega(e_i,h),f_\ell\rangle_Z^2 \;&=\; \sum_{\ell=1}^{\infty}\Big(\langle\omega(e_{2\ell-1},h),f_\ell\rangle_Z^2+\langle\omega(e_{2\ell},h),f_\ell\rangle_Z^2\Big)\\ &=\; \sum_{\ell=1}^{\infty}\Big(\langle h,e_{2\ell}\rangle_H^2+\langle h,e_{2\ell-1}\rangle_H^2\Big) \;=\; \|h\|_H^2\,,\end{aligned}$$

so that $\|\omega\|_{H\otimes Z}=1$.

Example 5.7. The previous example can be likened to taking an infinite product of the standard 3-dimensional Heisenberg group. We now generalize the idea by taking an infinite product of finite-dimensional Heisenberg-like groups, where each need not be identical to one another, nor does there need to be an upper-bound on the dimension.

Let $\{H^{(n)}\}_{n\in\mathbb{N}}$, $\{Z^{(n)}\}_{n\in\mathbb{N}}$ be sequences of finite-dimensional Hilbert spaces, and for each n, $\omega^{(n)}:H^{(n)}\times H^{(n)}\to Z^{(n)}$ an antisymmetric bilinear map. Suppose that we know

$$\inf_{n\in\mathbb{N}} \lfloor \omega^{(n)} \rfloor_{H\otimes H} > 0 \qquad \sup_{n\in\mathbb{N}} \|\omega^{(n)}\|_{H\otimes H} < \infty \qquad \sup_{n\in\mathbb{N}} \|\omega^{(n)}\|_{H\otimes Z} < \infty\,.$$

Then set $H := \bigoplus_{n=1}^{\infty} H^{(n)}$ and $Z := \bigoplus_{n=1}^{\infty} Z^{(n)}$, and define $\omega : H \times H \to Z$ as

$$\omega\Big((h^{(1)}, h^{(2)} \dots)\,,\, (k^{(1)}, k^{(2)}, \dots)\Big) \;=\; \Big(\omega^{(1)}(h^{(1)}, k^{(1)})\,,\, \omega^{(2)}(h^{(2)}, k^{(2)})\,,\, \dots\Big)\,.$$

For each $H^{(n)}$, set $\{e_j^{(n)}\}_{1\le j\le \dim(H^{(n)})}$ to be an orthonormal basis. Then we may identify each $e_j^{(n)}$ with the embedded element in H, which implies that $\{e_j^{(n)}\}_{1\le j\le \dim(H^{(n)}), n\in\mathbb{N}}$ is an orthonormal basis of H. Then we check, for $z = (z^{(1)}, z^{(2)}, \dots) \in Z$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{i,j=1}^{\dim(H^{(n)})} \Big\langle z, \omega(e_i^{(n)}, e_j^{(n)}) \Big\rangle_Z^2 &= \sum_{n=1}^{\infty} \sum_{i,j=1}^{\dim(H^{(n)})} \Big\langle z^{(n)}, \omega^{(n)}(e_i^{(n)}, e_j^{(n)}) \Big\rangle_{Z^{(n)}}^2 \\
&\le \sum_{n=1}^{\infty} \|\omega^{(n)}\|_{H\otimes H}^2 \|z^{(n)}\|_{Z^{(n)}}^2 \;\le\; \Big(\sup_{n\in\mathbb{N}} \|\omega^{(n)}\|_{H\otimes H}\Big)^2 \sum_{n=1}^{\infty} \|z^{(n)}\|_{Z^{(n)}}^2 \\
&= \Big(\sup_{n\in\mathbb{N}} \|\omega^{(n)}\|_{H\otimes H}\Big)^2 \|z\|_Z^2\,,
\end{aligned}$$

and similarly, we can arrive at the bound

$$\sum_{n=1}^{\infty} \sum_{i,j=1}^{\dim(H^{(n)})} \Big\langle (z_1, \dots), \omega(e_i^{(n)}, e_j^{(n)}) \Big\rangle_Z^2 \;\ge\; \Big(\inf_{n\in\mathbb{N}} \lfloor \omega^{(n)} \rfloor_{H\otimes H}\Big)^2 \|z\|_Z^2\,.$$

And if $\{f_\ell^{(n)}\}_{1\le i\le n}$ is a basis of $Z^{(n)}$, then if $h = (h^{(1)}, h^{(2)}, \dots) \in H$, then

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{i=1}^{\dim(H^{(n)})} \sum_{\ell=1}^{\dim(Z^{(n)})} \Big\langle f_\ell^{(n)}, \omega(h, e_i^{(n)}) \Big\rangle_Z^2 \\
&\qquad = \sum_{n=1}^{\infty} \sum_{i=1}^{\dim(H^{(n)})} \sum_{\ell=1}^{\dim(Z^{(n)})} \Big\langle f_\ell^{(n)}, \omega^{(n)}(h^{(n)}, e_i^{(n)}) \Big\rangle_{Z^{(n)}}^2 \\
&\qquad \le \sum_{n=1}^{\infty} \|\omega^{(n)}\|_{H\otimes Z}^2 \|h^{(n)}\|_{H^{(n)}}^2 \;\le\; \Big(\sup_{n\in\mathbb{N}} \|\omega^{(n)}\|_{H\otimes Z}\Big)^2 \|h\|_H^2\,.
\end{aligned}$$

Example 5.8. If H and Z are separable, infinite-dimensional Hilbert spaces with bases $\{e_j\}_{j\in\mathbb{N}}$ and $\{f_\ell\}_{\ell\in\mathbb{N}}$ respectively, then define ω by asserting

$$\langle f_\ell, \omega(e_i, e_j) \rangle_Z \;=\; \frac{1}{(|i-\ell|+1)(|j-\ell|+1)} \mathrm{sgn}(i,j)\,,$$

where we define

$$\operatorname{sgn}(i,j) \;=\; \begin{cases} 1 & \text{if } i<j \\ -1 & \text{if } i>j \\ 0 & \text{if } i=j \end{cases}\;.$$

Then this too will satisfy our assumptions. First note that, for any $\ell \in \mathbb{N}$, we have

$$\sum_{i=1}^{\infty} \frac{1}{(|i-\ell|+1)^2} \;\leq\; \frac{\pi^2}{3}\,.$$

Then

$$\begin{aligned}
\sum_{i,j=1}^{\infty} \langle z, \omega(e_i,e_j)\rangle_Z^2 \;&=\; \sum_{i,j,\ell=1}^{\infty} \langle z, f_\ell\rangle_Z^2 \langle f_\ell, \omega(e_i,e_j)\rangle_Z^2 \\
&\leq\; \sum_{i,j,\ell=1}^{\infty} \langle z, f_\ell\rangle_Z^2 \frac{1}{(|i-\ell|+1)^2(|j-\ell|+1)^2} \\
&\leq\; \sum_{\ell=1}^{\infty} \langle z, f_\ell\rangle_Z^2 \left(\sum_{i=1}^{\infty} \left(\frac{1}{|i-\ell|+1}\right)^2\right)^2 \;\leq\; \frac{\pi^4}{9}\|z\|_Z^2\,,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i,j=1}^{\infty} \langle z, \omega(e_i,e_j)\rangle_Z^2 \;&\geq\; \sum_{\ell=1}^{\infty} \langle z, f_\ell\rangle_Z^2 \frac{1}{(|(\ell+1)-\ell|+1)^2(|(\ell-1)-\ell|+1)^2} \\
&=\; \frac{1}{16}\|z\|_Z^2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i,\ell=1}^{\infty} \langle f_\ell, \omega(h,e_i)\rangle_Z^2 \;&=\; \sum_{i,j,\ell=1}^{\infty} \langle h, e_j\rangle_H^2 \langle f_\ell, \omega(e_i,e_j)\rangle_Z^2 \\
&\leq\; \sum_{i,j,\ell=1}^{\infty} \langle h, e_j\rangle_H^2 \frac{1}{(|i-\ell|+1)^2(|j-\ell|+1)^2} \\
&\leq\; \frac{\pi^2}{3} \sum_{j,\ell=1}^{\infty} \langle h, e_j\rangle_H^2 \frac{1}{(|j-\ell|+1)^2} \\
&\leq\; \frac{\pi^4}{9} \sum_{j=1}^{\infty} \langle h, e_j\rangle_H^2 \;=\; \frac{\pi^4}{9}\|h\|_H^2\,.
\end{aligned}$$

So while the previous 2 examples consist of "sparse structural constants," this example illustrates that one can have many nonzero entries, provided that they have sufficient decay.

The next 3 examples are actually nonexamples. However, they will still satisfy $\|\omega\|_{H\otimes H} < \infty$ and can be used as examples for the result in Chapter 3 (after taking the complexification as needed).

Example 5.9. For $\{e_j\}_{j\geq 0}$ a basis for H and $\{f_\ell\}_{\ell\geq 1}$ a basis for Z, we now define ω as

$$\omega(e_i, e_j) \;=\; \begin{cases} f_j & \text{if } i = 0, j \neq 0 \\ -f_i & \text{if } j = 0, i \neq 0 \\ 0 & \text{otherwise} \end{cases} .$$

Or, in other words,

$$\omega\left(\sum_{i=0}^{\infty} \alpha_i e_i\,,\, \sum_{j=0}^{\infty} \beta_j e_j\right) \;=\; \sum_{\ell=1}^{\infty} \Big(\alpha_0\beta_\ell - \alpha_\ell\beta_0\Big) f_\ell\,.$$

This can be likened to taking Example 5.6 and identifying all the odd-indexed basis vectors in H as e_0. Then

$$\begin{aligned} \sum_{i,j=0}^{\infty} \langle z, \omega(e_i, e_j)\rangle_Z^2 \;&=\; \sum_{i=1}^{\infty} \Big(\langle z, \omega(e_i, e_0)\rangle_Z^2 + \langle z, \omega(e_0, e_i)\rangle_Z^2\Big) \\ &=\; 2\sum_{i=1}^{\infty} \langle z, f_\ell\rangle_Z^2 \;=\; 2\|z\|_Z^2\,. \end{aligned}$$

But

$$\sum_{i=0}^{\infty}\sum_{\ell=1}^{\infty} \langle f_\ell, \omega(e_0, e_i)\rangle_Z^2 \;=\; \sum_{i=1}^{\infty} \langle f_i, \omega(e_0, e_i)\rangle_Z^2 \;=\; \sum_{i=1}^{\infty} \langle f_i, f_i\rangle_Z^2 \;=\; \infty\,.$$

Hence, this example satisfies $0 < \lfloor\omega\rfloor_{H\otimes H} = \|\omega\|_{H\otimes H} < \infty$, but $\|\omega\|_{H\otimes Z} = \infty$.

Example 5.10. Let $\{f_{k,\ell}\}_{k<\ell\in\mathbb{N}}$ be a (doubly-indexed) orthonormal basis of Z, and define ω such that

$$\omega(e_i, e_j) \;=\; \begin{cases} f_{i,j} & \text{if } i < j \\ -f_{i,j} & \text{if } j < i \\ 0 & \text{if } i = j \end{cases} .$$

This constitutes the free infinite-dimensional step-2 graded nilpotent Lie group. It is readily seen that

$$\sum_{i,j=1}^{\infty} \langle z, \omega(e_i, e_j)\rangle_Z^2 \;=\; 2\sum_{1\leq i<j} \langle z, f_{i,j}\rangle_Z^2 \;=\; 2\|z\|_Z^2\,.$$

But

$$\sum_{i,k,\ell=1}^{\infty} \langle f_{k,\ell}, \omega(e_1, e_i)\rangle_Z^2 \;=\; \sum_{i=2}^{\infty}\langle f_{1,i}, \omega(e_1, e_i)\rangle_Z^2 \;=\; \sum_{i=2}^{\infty}\langle f_{1,i}, f_{1,i}\rangle_Z^2 \;=\; \infty\,,$$

so again we have $\|\omega\|_{H\otimes Z} = \infty$.

Example 5.11. We now consider the path space of a finite-dimensional Heisenberg-like group. This explores paths of finite-energy, for which the notation and some theory is presented in Section 2.1.4. This is the Hilbert space structure for the graded step-2 equivalent of Example 3.5, where we use $\mathfrak{w}$ to denote the pointwise Lie-bracket. As it turns out, even though $\|\mathfrak{w}\|_{\mathcal{H}\otimes\mathcal{H}} < \infty$, it can be demonstrated that $\|\mathfrak{w}\|_{\mathcal{H}\otimes\mathcal{Z}} = \infty$.

To define this space precisely, let $H = \operatorname{span}\{e_1, \ldots, e_n\}$ and $Z = \operatorname{span}\{f_1, \ldots f_m\}$ be finite-dimensional vector spaces, with $\omega : H \times H \to Z$ continuous, bilinear. Then we may consider $H \times Z$ to be a finite-dimensional Heisenberg-like group, equal to its own Lie algebra, with Lie bracket and group operator determined by ω. We then consider $\mathcal{H} = \mathcal{H}_0([0,1], H)$ and $\mathcal{Z} = \mathcal{H}_0([0,1], Z)$, and we define $\mathfrak{w} : \mathcal{H}\times\mathcal{H} \to \mathcal{Z}$ as $\mathfrak{w}(h_1, h_2)(t) = \omega(h_1(t), h_2(t))$. Then $\mathcal{H}\times\mathcal{Z}$ is the Cameron-Martin subalgebra for an infinite-dimensional Heisenberg-like group; see Example 3.5 for more information.

Consider the bilinear map $\mathfrak{m} : \mathcal{H}_0([0,1],\mathbb{R}) \times \mathcal{H}_0([0,1],\mathbb{R}) \to \mathcal{H}_0([0,1],\mathbb{R})$ defined as $\mathfrak{m}(f,g)(t) = f(t)\cdot g(t)$. Recall that we may use $\{\sqrt{2}\sin(2\pi kt), \sqrt{2}\cos(2\pi kt)\}_{k\in\mathbb{N}} \cup \{1\}$ as an orthonormal basis of $L^2([0,1],\mathbb{R})$, so identifying $\mathcal{H}_0([0,1],\mathbb{R})$ as the set of antiderivatives on $L^2([0,1],\mathbb{R})$, $\left\{\frac{1}{\sqrt{2}\pi k}\sin(2\pi kt), \frac{1}{\sqrt{2}\pi k}\cos(2\pi kt)\right\}_{k\in\mathbb{N}} \cup \{t\}$ is an orthonormal basis for $\mathcal{H}_0([0,1],\mathbb{R})$. Since $\mathcal{H}_0([0,1],H) = \mathcal{H}_0([0,1],\mathbb{R})\otimes H$ and $\mathcal{H}_0([0,1],Z) = \mathcal{H}_0([0,1],\mathbb{R})\otimes Z$, we have $\left\{\frac{1}{\sqrt{2}\pi k}\sin(2\pi kt)e_j, \frac{1}{\sqrt{2}\pi k}(1-\cos(2\pi kt))e_j\right\}_{k\in\mathbb{N}, 1\le j\le n} \cup \{te_j\}_{1\le j\le n}$ is an orthornomal basis of $\mathcal{H}$, and we have a similar basis for $\mathcal{Z}$.

Then for a function $f \in \mathcal{H}_0([0,1],\mathbb{R})$ and an element $v \in H$, $f(\cdot)v \in \mathcal{H}$, and

$$\begin{aligned}
&\sum_{k,\ell=1}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} \left\langle \mathfrak{w}\Big(\frac{1}{\sqrt{2}\pi k}\sin(2\pi kt)e_i\,,\; f(t)v\Big)\,,\; \frac{1}{\sqrt{2}\pi k}\sin(2\pi\ell t)f_j \right\rangle_{\mathcal{H}}^2 \\
&\quad= \sum_{k,\ell=1}^{\infty} \left\langle \frac{1}{\sqrt{2}\pi k}\sin(2\pi kt)\; f(t)\,,\; \frac{1}{\sqrt{2}\pi k}\sin(2\pi\ell t) \right\rangle_{\mathcal{H}}^2 \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\langle\omega(e_i, v), f_j\rangle_Z^2\right).
\end{aligned}$$

This demonstrates the connection between $\mathfrak{w}$ and $\mathfrak{m}$. If we can show that

$$\|\mathfrak{m}\|_{\mathcal{H}_0([0,1],\mathbb{R})\otimes\mathcal{H}_0([0,1],\mathbb{R}} = \infty\,,$$

then we could conclude that $\|\mathfrak{w}\|_{\mathcal{H}\otimes\mathcal{Z}} = \infty$.

Before beginning the true calculation, we remark that, if $g \in L^2([0,1],\mathbb{R})$, then (doubly-indexed) sequences of the form $\big(\frac{1}{k}\int_0^1 g(t)\sin(2\pi kt)\sin(2\pi\ell t)\big)_{k,\ell\in\mathbb{N}}$ are square-summable. This is because

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{k^2}\Big(\int_0^1 g(t)\sin(2\pi kt)\sin(2\pi\ell t)dt\Big)^2\\
&\qquad\qquad = \sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{\ell=1}^{\infty}\big\langle g(t)\sin(2\pi kt),\sin(2\pi\ell t)\big\rangle_{L^2}^2\\
&\qquad\qquad = \sum_{k=1}^{\infty}\frac{1}{k^2}\|g(t)\sin(2\pi kt)\|_{L^2}^2 \;\le\; \sum_{k=1}^{\infty}\frac{1}{k^2}\|g(t)\|_{L^2}^2 \;=\; \frac{\pi^2}{6}\|g(t)\|_{L^2}^2\,.
\end{aligned}$$

and note that this does not change if one or both sine functions are replaced with cosine functions. Then, for any $f \in \mathcal{H}_0^1([0,1],\mathbb{R})$,

$$\begin{aligned}
&\sum_{k,\ell=1}^{\infty}\Big\langle \frac{1}{\sqrt{2}\pi k}\sin(2\pi kt), \mathfrak{m}(f(t), \frac{1}{\sqrt{2}\pi\ell}\sin(2\pi\ell t))\Big\rangle_{\mathcal{H}_0^1([0,1],\mathbb{R})}^2\\
&\quad = \sum_{k,\ell=1}^{\infty}\Big(\int_0^1 \sqrt{2}\cos(2\pi kt)\big(\sqrt{2}\cos(2\pi\ell t)f(t) + \frac{1}{\sqrt{2}\pi k}\sin(2\pi\ell t)f'(t)\big)dt\Big)^2\\
&\quad = \sum_{k,\ell=1}^{\infty}\Big(2\int_0^1 \cos(2\pi kt)\cos(2\pi\ell t)f(t)dt\\
&\qquad\qquad\qquad\qquad + \frac{1}{\pi k}\int_0^1 \cos(2\pi kt)\sin(2\pi\ell t)f'(t)dt\Big)^2.
\end{aligned}$$

Then, based on the remarks above, we know that $\big(\frac{1}{k}\int_0^1 \cos(2\pi kt)\sin(2\pi\ell t)f'(t)dt\big)_{k,\ell\in\mathbb{N}}$ is square-summable, so the summability of the expression above is determined by the square-summability of $(\int_0^1 \cos(2\pi kt)\cos(2\pi\ell t)f(t)dt)_{k,\ell\in\mathbb{N}}$.

Using integration by parts, we get

$$\begin{aligned}
&\sum_{k,\ell=1}^{\infty}\Big(\int_0^1 \cos(2\pi kt)\cos(2\pi\ell t)f(t)dt\Big)^2\\
&\qquad = \frac{1}{4}\sum_{k,\ell=1}^{\infty}\Big(\int_0^1 \big(\cos(2\pi(k+\ell)t) + \cos(2\pi(k-\ell)t)\big)f(t)dt\Big)^2\\
&\qquad \ge \frac{1}{4}\sum_{\substack{k,\ell=1\\ k\neq\ell}}^{\infty}\Big(\int_0^1 \frac{1}{2\pi(k+\ell)}\sin(2\pi(k+\ell)t)f'(t)dt\\
&\qquad\qquad\qquad + \int_0^1 \frac{1}{2\pi(k-\ell)}\sin(2\pi(k-\ell)t)f'(t)dt\Big)^2.
\end{aligned}$$

It can be seen that $\left(\frac{1}{k+\ell}\int_0^1 \sin(2\pi(k+\ell)t)f'(t)dt\right)_{k,\ell\in\mathbb{N}}$ is square summable because

$$\begin{aligned}\sum_{k,\ell=1}^{\infty} \frac{1}{(k+\ell)^2}\left(\int_0^1 \sin(2\pi(k+\ell)t)f'(t)dt\right)^2 \\ \leq \sum_{k,\ell=1}^{\infty} \frac{1}{k^2}\left(\int_0^1 \sin(2\pi kt)\cos(2\pi\ell t)f'(t)dt + \int_0^1 \sin(2\pi\ell t)\cos(2\pi kt)f'(t)dt\right)^2,\end{aligned}$$

so we need only to determine the square-summability of $\left(\frac{1}{k-\ell}\int_0^1 \sin(2\pi(k-\ell)t)f'(t)dt\right)_{k,\ell\in\mathbb{N}}$. However, if $\langle \sin(2\pi t), f(t)\rangle_{L^2} \neq 0$, then

$$\begin{aligned}\sum_{\substack{k,\ell=1\\ k\neq\ell}}^{\infty} \frac{1}{(k-\ell)^2}\left(\int_0^1 \sin(2\pi(k-\ell)t)f'(t)dt\right)^2 \\ \geq \sum_{k=1}^{\infty}\left(\int_0^1 \sin(2\pi t)f'(t)dt\right)^2 = \infty\,.\end{aligned}$$

From this, we may conclude that, in this example, $\|\mathfrak{w}\|_{\mathcal{H}\otimes\mathcal{Z}} = \infty$.

The path space of a nilpotent Lie group will be explored further in Example 10.21, where it will satisfy the criteria for Chapter 3 (upon taking the complexification if needed).

6 Horizontal distance approximations

We consider a subgroup of G called G_{CM}, the Cameron-Martin subgroup, which satisfies, for all $P \in \mathrm{Proj}(W)$, $G^P \subseteq G_{CM} \subseteq \exp(\mathfrak{g}_{CM})$. This group is intrinsically related to both the Cameron-Martin space of W and the subelliptic structure induced by Δ_G. We will first spend time defining this distance and related notions, followed by showing how this distance is well-approximated in finite dimensions.

6.1 Preliminaries

Recall that we give $\mathfrak{g}_{CM} = H \times Z$ a group structure (called $\exp(\mathfrak{g}_{CM})$ when we do so), from which we have left-invariant vector fields as described in Section 2.3.2, defined as

$$L_{g*}v \;=\; \frac{d}{dt}\Big|_{t=0}(g \cdot tv) \;=\; v + \frac{1}{2}[g, v]\,.$$

And the map $L_{g^{-1}*} : \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$, the Maurer-Cartan form, is regarded as the map that cannonically identifies the tangent spaces of $\exp(\mathfrak{g}_{CM})$ together.

With this is mind, let $\mathcal{C}^1 = \mathcal{C}^1([0,1], \exp(H \times Z))$ denote the set of continuously differentiable paths $\sigma : [0,1] \to \exp(H \times Z) = H \times Z$, on which we may define the length $\ell(\sigma) = \int_0^1 \|L_{\sigma(t)^{-1}*}\sigma'(t)\|_{H\times Z} dt$. We say σ is "horizontal" if $L_{\sigma(t)^{-1}*}\sigma'(t) \in H \times \{0\}$ for all $t \in [0,1]$, and we denote the set of such paths as $\mathcal{C}^1_h$. Then for any $(h, z) \in H \times Z$, we define the horizontal distance from the origin as

$$d(e, (h,z)) \;=\; \inf\left\{\ell(\sigma) \;\middle|\; \sigma \in \mathcal{C}^1_h\,,\ \sigma(0) = e\,,\ \sigma(1) = (h,z)\right\}.$$

The horizontal distance d is a (infinite-dimensional) example of Carnot-Carathéodory distance. This idea was first introduced (in finite dimensions) by Carathéodory in [Car09], though it wasn't known if it constituted a proper metric (specifically, that it was always finite) until being proven in Chow [Cho40] and Rashevsky [Ras38]. As a modern reference, see [ABB20, Rashevsky-Chow Theorem, Theorem 3.31], or [Mon02].

We then define the Cameron-Martin subgroup as $G_{CM} = \{g \in H \times Z \,:\, d(e, g) < \infty\}$.

Proposition 6.1.

1. *For $g = (h, z) \in G_{CM}$, $\alpha > 0$, $d(e, \delta_\alpha g) = d(e, (\alpha h, \alpha^2 z)) = \alpha d(e, (h, z))$.*
2. *For all $g_1, g_2, a \in G_{CM}$, $d(g_1, g_2) = d(g_1 \cdot a, g_2 \cdot a)$.*
3. *d is a metric.*

Proof. For a horizontal path $\sigma = (A, B) : [0,1] \to H \times Z$, $\delta_\alpha\sigma = (\alpha A, \alpha^2 B)$ satisfies

$$\begin{aligned} L_{\delta_\alpha\sigma(t)^{-1}*}(\delta_\alpha\sigma)'(t) &= \frac{d}{ds}\Big|_{s=0} (\delta_\alpha\sigma'(t)) \cdot \big(s\delta_\alpha\sigma(t)^{-1}\big) \\ &= \delta_\alpha\left(\frac{d}{ds}\Big|_{s=0} \sigma'(t) \cdot (s\sigma(t))\right) = \delta_\alpha\Big(L_{\sigma(t)^{-1}*}\sigma'(t)\Big) \\ &= \alpha L_{\sigma(t)^{-1}*}\sigma'(t)\,, \end{aligned}$$

where the last equality holds because $L_{\sigma(t)^{-1}*}\sigma'(t) \in H \times 0$. From this, we see that σ is horizontal if and only if $\delta_\alpha\sigma$ is. And

$$\begin{aligned} \ell(\delta_\alpha\sigma) &= \int_0^1 \|L_{\delta_\alpha\sigma(t)^{-1}*}\delta_\alpha\sigma'(t)\|_{H\times Z} dt \\ &= \int_0^1 \|\delta_\alpha(L_{\sigma(t)^{-1}*}\sigma'(t))\|_{H\times Z} dt \\ &= \int_0^1 \|\alpha L_{\sigma(t)^{-1}*}\sigma'(t)\|_H dt = \alpha\ell(\sigma)\,. \end{aligned}$$

Hence, there is a one-to-one correspondence between horizontal paths connecting e to g and those connecting e to $\delta_\alpha g$, and this correspondence scales the length by α. It follows that $d(e, \delta_\alpha g) = \alpha d(e, g)$.

For translation-invariance, suppose $\sigma : [0,1] \to H \times Z$ is in $\mathcal{C}^1_h$, where $\sigma(0) = g_1$, $\sigma(1) = g_2$. Then $\sigma \cdot a : [0,1] \to H \times Z$ satisfies $(\sigma \cdot a)'(t) = a \cdot \sigma'(t)$, and

$$\begin{aligned} L_{(a\cdot\sigma(t))^{-1}*}(a \cdot \sigma)'(t) &= \frac{d}{ds}\Big|_{s=0} (\sigma'(t) \cdot a) \cdot s(\sigma(t) \cdot a)^{-1}) \\ &= \frac{d}{ds}\Big|_{s=0} \sigma'(t) \cdot s(\sigma(t))^{-1}) = L_{\sigma(t)^{-1}*}\sigma'(t)\,, \end{aligned}$$

and thus, $\sigma \cdot a$ is $\mathcal{C}^1$, horizontal, and $\ell(a \cdot \sigma) = \ell(\sigma)$. As before, there is a one-to-one correspondence between horizontal paths connecting g_1 to g_2 and those connecting $g_1 \cdot a$ to $g_2 \cdot a$ which preserves the lengths of the paths. This proves the second point.

Showing d is a metric involves classical methods for metrics defined by minimizing over paths. We only need to observe that constant paths $\sigma \equiv g \in G_{CM}$ are horizontal with length 0 to see $d(g, g) = 0$. And if $\phi : [0,1] \to [0,1]$ be $\phi(t) = 1 - t$, then we may reverse a horizontal path σ by taking $\sigma \circ \phi$, which is again horizontal (as are all smooth parametrizations of σ), and $\sigma \circ \phi(0) = \sigma(1)$, $\sigma \circ \phi(1) = \sigma(0)$. Hence, for all $g_1, g_2 \in G_{CM}$, $d(g_1, g_2) = d(g_2, g_1)$.

And if $\sigma_1, \sigma_2 : [0,1] \to H \times Z$ are both horizontal where $\sigma_1(1) = \sigma_2(0)$, then we may define the concatenation as

$$\sigma_1 \sim \sigma_2(t) = \begin{cases} \sigma_1(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \sigma_2(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}.$$

Then let $\psi : [0,1] \to [0,1]$ be a smooth, increasing function such that $\psi'(\frac{1}{2}) = 0$. Then $(\sigma := \sigma_1 \sim \sigma_2) \circ \psi$ is $\mathcal{C}^1$, will again be horizontal, and

$$(\sigma(0) = \sigma_1(0) \qquad \sigma\left(\frac{1}{2}\right) = \sigma_1(1) = \sigma_2(0) \qquad \sigma(1) = \sigma_2(1)\,,$$

which proves the triangle inequality.

□

We will soon transition to defining d by minimizing over paths of finite energy. Recall from Section 2.1.4 that, for any Hilbert space K, we define $\mathcal{H}_0([0,1],K)$ as the set of finite-energy paths $A : [0,1] \to K$ satisfying $A(0) = 0$ with $\|A\|_{\mathcal{H}_0([0,1],K)} = \|A'\|_{L^2([0,1],K)} < \infty$. We will primarily consider $\mathcal{H}_0([0,1],H)$, which will later be abbreviated to $\mathcal{H}_0$, but we will also discuss $\mathcal{H}_0([0,1],\mathbb{R})$.

There is an explicit description of the paths that lie in $\mathcal{C}^1_h$, which was made use of in [BGM13].

Theorem 6.2. *Every $\sigma \in \mathcal{C}^1_h$ with $\sigma(0) = 0$ takes on the form*

$$\sigma(t) \;=\; \left(A(t), \frac{1}{2}\int_0^t \omega(A(s), A'(s))ds\right).$$

Define $\nu(A,B) = \frac{1}{2}\int_0^1 \omega(A(s), B'(s))ds$, and $\nu(A) = \nu(A,A)$. We may realize d as

$$d(e,(h,z)) \;=\; \inf\left\{\|A\|_{\mathcal{H}_0} \mid A \in \mathcal{H}_0\,,\; A(1) = h\,,\; \nu(A) = z\right\}.$$

Proof. If $\sigma = (A,B) : [0,1] \to H \times Z$ is in $\mathcal{C}^1$, then

$$L_{\sigma(t)^{-1}*}(\sigma'(t)) \;=\; \left(A'(t)\,,\; B'(t) - \frac{1}{2}\omega(A(t), A'(t))\right),$$

so the assumption that $L_{\sigma(t)^{-1}*}\sigma'(t) \in H \times 0$ implies that $B(t) = \frac{1}{2}\int_0^t \omega(A(s), A'(s))ds$. Note that any $\mathcal{C}^1$ path has finite energy, so $A \in \mathcal{H}_0([0,1],H)$. And we may calculate

$$\ell(\sigma) = \int_0^1 \|L_{\sigma(t)^{-1}*}(\sigma'(t))\|_{H\times Z}dt = \int_0^1 \|A(t)\|_H dt = \ell(A)\,.$$

Hölder's inequality shows that $\ell(A) \leq \|A\|_{\mathcal{H}_0}$, and that $\ell(A) = \|A\|_{\mathcal{H}_0}$ when A is parametrized by arclength. Moreover, reparametrizing a path neither changes its endpoints nor whether it's horizontal. Hence, minimizing $\mathcal{C}^1$-paths over the length is equivalent to minimizing finite-energy paths over the energy.

□

We will now study convergence properties of the map ν. As noted in Section 2.1.4, we may define the "integral" map $\mathcal{I} : L^2([0,1],\mathbb{R}) \to \mathcal{H}_0([0,1],\mathbb{R})$ as $\mathcal{I}f(t) = \int_0^t f(s)ds$. Then $\mathcal{I}$ can be viewed as mapping to $L^2([0,1],\mathbb{R})$, and is bounded linear. In fact, we have

Lemma 6.3. *The bilinear map $\mathcal{Z} : L^2([0,1],\mathbb{R}) \times L^2([0,1],\mathbb{R}) \to L^2([0,1],\mathbb{R})$ defined as $(A_1, A_2) \mapsto \mathcal{I}A_1(\cdot)A_2(\cdot)$ (integrate the first input, then apply pointwise multiplication) is weakly Hilbert-Schmidt.*

Proof. Let $f \in L^2([0,1],\mathbb{R})$. Then use $\{\sqrt{2}\sin(2\pi kt), \sqrt{2}\cos(2\pi kt)\}_{k\in\mathbb{N}} \cup \{1\} = \{\mathfrak{e}_k\}_{k\in\mathbb{Z}}$ as a basis of $L^2([0,1],\mathbb{R})$. Then

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\left|\left\langle f(t), \mathcal{I}(\mathfrak{e}_k)\,\mathfrak{e}_\ell\right\rangle_{L^2([0,1],\mathbb{R})}\right|^2 \\
&\quad= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\left|\left\langle f(t)\mathcal{I}\mathfrak{e}_k, \mathfrak{e}_\ell\right\rangle_{L^2([0,1],\mathbb{R})}\right|^2 \\
&\quad= \sum_{k=-\infty}^{\infty}\left\|f(t)\mathcal{I}(\mathfrak{e}_k)\right\|^2_{L^2([0,1],\mathbb{R})} \\
&\quad= \sum_{k=1}^{\infty}\frac{1}{2\pi^2k^2}\left\|f(t)(\cos(2\pi kt)-1)\right\|^2_{L^2([0,1],\mathbb{R})} \\
&\qquad\qquad+ \sum_{k=1}^{\infty}\frac{1}{2\pi^2k^2}\left\|f(t)\sin(2\pi kt)\right\|^2_{L^2([0,1],\mathbb{R})} + \left\|tf(t)\right\|^2_{L^2([0,1],\mathbb{R})} \\
&\quad\leq \sum_{k=1}^{\infty}\frac{5}{2\pi^2k^2}\|f\|^2_{L^2([0,1],\mathbb{R})} + \|f\|^2_{L^2([0,1],\mathbb{R})} \\
&\quad\leq K\|f\|^2_{L^2([0,1],\mathbb{R})}
\end{aligned}$$

for some K, which proves the claim. □

Lemma 6.4. *The bilinear map $S_2 : \mathcal{H}_0([0,1],\mathbb{R}) \times \mathcal{H}_0([0,1],\mathbb{R}) \to \mathbb{R}$ defined as $S_2(A,B) = \int_0^1 A(s)B'(s)ds$ is Hilbert-Schmidt.*

Proof. Recall that $\mathcal{I} : L^2([0,1],\mathbb{R}) \to \mathcal{H}_0([0,1],\mathbb{R})$ is an isomorphism. Then the map

$$(A,B) \mapsto \left(t \mapsto \int_0^t A(s)B'(s)ds\right) \tag{2.5}$$

is precisely $\mathcal{Z}$ under the identification $L^2([0,1],\mathbb{R}) \xrightarrow{\mathcal{I}} \mathcal{H}_0([0,1],\mathbb{R})$.

The map S_2 is the map in (2.5) composed with point-evaluation at 1, which is continuous on $\mathcal{H}_0$-spaces, as noted in Section 2.1.4. This implies that S_2 is weakly Hilbert-Schmidt, or equivalently Hilbert-Schmidt, since $\mathbb{R}$ is trivially finite-dimensional.

□

Theorem 6.5. *The bilinear map $\nu : \mathcal{H}([0,1],H)^{\times 2} \to Z$ is weakly Hilbert-Schmidt, and extends to $\mathcal{H}([0,1],H_1)^{\otimes 2}$.*

Proof. Recall from Section 2.1.4 that $\mathcal{H}([0,1],H) \cong \mathcal{H}([0,1],\mathbb{R}) \otimes H$. Then, if we realize ν as being defined on $(\mathcal{H}([0,1],\mathbb{R}) \otimes H)^{\times 2}$, we may write, for simple tensors $f_1 \otimes v_1, f_2 \otimes v_2 \in L^2([0,1],\mathbb{R}) \otimes H$,

$$\begin{aligned}
\nu(f_1 \otimes v_1, f_2 \otimes v_2) &= \nu(f_1 v_1, f_2 v_2) \\
&= \frac{1}{2}\int_0^1 \omega(f_1(s)v_1, f_2'(s)v_2)ds \\
&= \frac{1}{2}\left(\int_0^1 f_1(s)f_2'(s)ds\right)\omega(v_1,v_2) \\
&= \frac{1}{2}S_2(f_1,f_2)\,\omega(v_1,v_2)\,.
\end{aligned}$$

Since both S_2 and ω are weakly Hilbert-Schmidt, we can conclude that the map $\frac{1}{2}S_2 \otimes \omega : \mathcal{H}([0,1],\mathbb{R})^{\times 2} \otimes H^{\times 2} \to \mathbb{R} \otimes Z \cong Z$ has an extension to $\mathcal{H}([0,1],\mathbb{R})^{\otimes 2} \otimes H^{\otimes 2} \cong \mathcal{H}([0,1],H)^{\otimes 2}$. Then by the equation above, ν has a continuous extension as well, or equivalently, ν is weakly Hilbert-Schmidt.

□

6.2 More on the Topology

While not strictly necessary for our proofs, this subsection will discuss some topological considerations for G_{CM}. The main proof of [BGM13] relied on equating the topology of G_{CM} with that induced by the norm $\|\cdot\|_{H\times Z} = \sqrt{\|\cdot\|_H^2 + \|\cdot\|_Z^2}$. It will be shown in this section that this equivalence does not generally hold. We will still leverage the same estimates, but will ultimately require a different argument to prove Theorem 6.16.

The following 2 results offer some understanding the topology of G_{CM}.

Theorem 6.6. *Define $d^Z : Z \times Z \to \mathbb{R}$ as $d^Z(z_1,z_2) = d((0,z_1),(0,z_2))$. The set G_{CM} is equal to $\{(h,z) \in H \times Z \,:\, d^Z(0,z) < \infty\}$. Moreover, the topologies on G_{CM} determined by d and $\|\cdot - \cdot\|_H + d^Z(\cdot,\cdot)$ are homeomorphic (but note that the metrics are not necessarily equivalent).*

Proof. We will use estimates to bound each metric by a function in terms of the other, which will justify both the set and the topological equivalence. Fix $(h_0, z_0) \in G_{CM}$. For $(h, z) \in G_{CM}$, by properties of d in Proposition 6.1, and using Lemma 6.12 in the next section, and the fact that ω is anti-symmetric,

$$\begin{aligned} d((h_0, z_0), (h, z)) &= d(e, (h, z) \cdot (h_0, z_0)^{-1}) \\ &\leq d(e, (h - h_0, 0)) + d(e, (0, z - z_0)) + d\left(e, \frac{1}{2}\omega(h_0, h - h_0)\right) \\ &\leq \|h - h_0\|_H + d^Z(z, z_0) + \frac{\sqrt{\pi}}{2}\sqrt{\|h_0\|_H}\sqrt{\|h - h_0\|_H} \,. \end{aligned}$$

This proves that one direction is continuous. On the other hand, let A be a path that satisfies $A(0) = 0$, $A(1) = h - h_0$, and $\nu(A, A) = z - z_0 - \frac{1}{2}\omega(h_0, h)$. Then its length is no longer than the shortest path that starts at h_0 and ends at h, which has a length of $\|h - h_0\|_H$. This proves

$$\|h - h_0\|_H \leq d(e, (h, z) \cdot (h_0, z_0)^{-1}) = d((h_0, z_0), (h, z)) \,.$$

Then, using the triangle inequality and translation-invariance of d,

$$\begin{aligned} d^Z(z_0, z) &= d(e, (0, z - z_0)) \\ &\leq d(e, (0, z - z_0) \cdot (h, 0) \cdot (h_0, 0)^{-1}) + d(e, (h, 0) \cdot (h_0, 0)^{-1}) \\ &\leq d((h_0, z_0), (h, z)) + \|h - h_0\|_H + d\left(e, \left(0, \frac{1}{2}\omega(h_0, h)\right)\right) \\ &\leq 2d((h_0, z_0), (h, z)) + \frac{\sqrt{\pi}}{2}\sqrt{\|h_0\|_H}\sqrt{\|h - h_0\|_H} \\ &\leq 2d((h_0, z_0), (h, z)) + \frac{\sqrt{\pi}}{2}\sqrt{\|h_0\|_H}\sqrt{d((h_0, z_0), (h, z))} \,, \end{aligned}$$

and therefore

$$\|h - h_0\|_H + d^Z(z, z_0) \leq 3d((h_0, z_0), (h, z)) + \sqrt{\pi}\sqrt{\|h_0\|_H}\sqrt{d((h_0, z_0), (h, z))} \,,$$

which completes the proof.

□

Lemma 6.7. *The inclusion* $(G_{CM}, d) \to (H \times Z, \sqrt{\|\cdot\|_H^2 + \|\cdot\|_Z^2})$ *is continuous.*

Proof. Let $(h, z) \in G_{CM}$, so let $A \in \mathcal{H}_0([0, 1], H)$ be a path such that $A(0) = 0$, $A(1) = h$, and $\nu(A) = z$. Note that

$$\|h\|_H \leq \ell(A) \leq \|A\|_{\mathcal{H}_0} \,.$$

Since ν is a continuous, bilinear map, then we have

$$\|h\|_H^2 + \|z\|_Z^2 \;=\; \|h\|_H^2 + \|\nu(A,A)\|_Z^2 \;\le\; \|A\|_{\mathcal{H}_0}^2 + K\|A\|_{\mathcal{H}_0}^4$$

for some constant K. By taking the infimum over all such paths A, we may deduce

$$\sqrt{\|h\|_H^2 + \|z\|_Z^2} \;\le\; d(e,(h,z)) + \sqrt{K} d(e,(h,z))^2 \,.$$

This proves continuity at e. For any $(h_0, z_0) \in G_{CM}$, by using Theorem 6.6, we may write that there exists K such that $d((h_0,z_0),(h,z)) \le K d(e,(h-h_0, z-z_0))$, which completes the proof.

□

Before continuing, we will prove a formula that will be used in a calculation in Example 6.10.

Lemma 6.8. *If $A \in \mathcal{H}_0([0,1],H)$ is a loop (meaning $A(1)=0$), then A takes on the form*

$$A(t) \;=\; \sum_{j,k=1}^{\infty} \left(\alpha_{j,k} \frac{1}{\sqrt{2}\pi k} \big(\cos(2\pi k t) - 1\big) e_j + \beta_{j,k} \frac{1}{\sqrt{2}\pi k} \sin(2\pi k t) e_j \right), \qquad (2.6)$$

for which we have

$$\nu(A,A) \;=\; \frac{1}{2\pi} \sum_{i,j,k=1}^{\infty} \frac{1}{k} \alpha_{i,k}\beta_{j,k}\omega(e_i,e_j)\,,$$

which converges in $\|\cdot\|_Z$.

Proof. Consider that, since $\mathcal{H}_0([0,1],H) \;\cong\; \mathcal{H}_0([0,1],\mathbb{R}) \otimes H$, we have a basis of $\mathcal{H}_0([0,1],H)$ given as

$$\left\{ \frac{1}{\sqrt{2}\pi k} \big(\cos(2\pi k t) - 1\big) e_j \,,\; \frac{1}{\sqrt{2}\pi k} \sin(2\pi k t) e_j \right\}_{j,k\in\mathbb{N}} \;\cup\; \big\{ t e_j \big\}_{j\in\mathbb{N}} \,,$$

where $\{e_j\}_{j\in\mathbb{N}}$ is a basis of H. Then for any $A \in \mathcal{H}_0([0,1],H)$, if $A(1)=0$, then there exist square-summable constants $\{\alpha_{j,k}, \beta_{j,k}\}_{j,k\in\mathbb{N}}$ such that (2.6) holds.

Then, recalling that $\{\cos(2\pi k), \sin(2\pi k)\}_{k\in\mathbb{N}} \cup \{1\}$ is an orthogonal set in $L^2([0,1],\mathbb{R})$, we have

$$
\begin{aligned}
\nu(A,A) &= \frac{1}{2}\int_0^1 \omega(A(t),A'(t))dt \\
&= \sum_{i,j,k,\ell=1}^{\infty} \frac{1}{2\pi k}\int_0^1 \omega\Big(\alpha_{i,k}\big(\cos(2\pi kt)-1\big)e_i + \beta_{i,k}\sin(2\pi kt)e_i\,, \\
&\qquad\qquad \alpha_{j,\ell}\big(-\sin(2\pi\ell t)\big)e_j + \beta_{j,\ell}\cos(2\pi\ell t)e_j\Big)dt \\
&= \sum_{k=1}^{\infty}\frac{1}{2\pi k}\sum_{i,j=1}^{\infty}\alpha_{i,k}\beta_{j,k}\Big(\int_0^1\cos(2\pi kt)^2dt\Big)\omega(e_i,e_j) \\
&\qquad\qquad -\alpha_{j,k}\beta_{i,k}\Big(\int_0^1\sin(2\pi kt)^2dt\Big)\omega(e_i,e_j) \\
&= \frac{1}{2\pi}\sum_{i,j,k=1}^{\infty}\frac{1}{k}\alpha_{i,k}\beta_{j,k}\omega(e_i,e_j)\,.
\end{aligned}
$$

Note that one may use the fact that $\omega : H\times H \to Z$ is continuous bilinear, along with the Cauchy-Schwartz inequality, to show that this series always converges in $\|\cdot\|_Z$.

□

Remark 6.9. In [BGM13], recall that it was assumed $\dim(C) < \infty$. In that case, we have that $\omega : H\times H \to C$ is Hilbert-Schmidt, and there exists a constant K such that $d(e,(h,c)) \le \|h\|_H + K\sqrt{\|c\|_C}$. However, we now have tools that demonstrate that this inequality cannot hold in infinite dimensions. Indeed, if we maintain that $\omega : H\times H \to C$ is Hilbert-Schmidt, then the proof of Theorem 6.5 can be adapted to show $\nu : \mathcal{H}_0([0,1],H)\times\mathcal{H}_0([0,1],H) \to C$ is Hilbert-Schmidt (since S_2 is Hilbert-Schmidt and $\nu = \frac{1}{2}S_2\otimes\omega$), and in particular the extension $\widetilde{\nu}$ must be Hilbert-Schmidt and thus compact, so it is not surjective. This implies that there exists $c\in C$ such that $\nu(A,A)\ne c$ for all $A\in\mathcal{H}_0([0,1],H)$, so that $d(e,(0,c)) = \infty$. This certainly contradicts $d(e,(0,c)) \le K\sqrt{\|c\|_C}$.

In Example 6.10 below, we present an explicit calculation for horizontal distance that shows that we cannot expect this to be remedied even if $\|\cdot\|_C$ is replaced with $\|\cdot\|_Z$.

Example 6.10. In this example, we will consider Example 5.6 and use Theorem 6.6 to compute the set G_{CM} and determine its topology. Recall that in Example 5.6 we define $\omega : H\times H\to Z$ as

$$
\omega\left(\sum_{i=1}^{\infty}\alpha_i e_i\,,\ \sum_{j=1}^{\infty}\beta_j e_j\right) = \sum_{\ell=1}^{\infty}\Big(\alpha_{2\ell-1}\beta_{2\ell}-\alpha_{2\ell}\beta_{2\ell-1}\Big)f_\ell\,.
$$

Suppose that $A \in \mathcal{H}_0([0,1],H)$ is a loop, so that it takes on the form (2.6). Applying the definition of ω in Example 5.6 to the formula for $\nu(A,A)$ in Lemma 6.8 yields

$$\nu(A,A) \;=\; \frac{1}{2\pi}\sum_{j,k=1}^{\infty}\frac{1}{k}\big(\alpha_{2j-1,k}\beta_{2j,k}-\alpha_{2j,k}\beta_{2j-1,k}\big)f_j$$

Now, specifically for Example 5.6, given such loop A, we will prove that there exists another path $A_0 \in \mathcal{H}_0([0,1],H)$ that satisfies the following properties:

1. $A_0(1) = 0$ (so that A_0 is also a loop).
2. $\nu(A_0, A_0) = \nu(A,A)$.
3. $\|A_0\|_{\mathcal{H}_0([0,1],H)} \le \|A\|_{\mathcal{H}_0([0,1],H)}$.
4. A_0 is of the form $A_0(t) = (1-\cos(2\pi t))\sum_{j=1}^{\infty} c_j e_{2j+1} + \sin(2\pi t)\sum_{j=1}^{\infty} c_j e_{2j}$ for some square-summable coefficients $\{c_j\}_{j\in\mathbb{N}}$, and in particular A_0 is a circle.

Indeed, set $p_j = \sum_{k=1}^{\infty}\frac{1}{k}(\alpha_{2j-1,k}\beta_{2j,k}-\alpha_{2j,k}\beta_{2j-1,k})$. Then define

$$A_0(t) \;=\; \frac{1}{\sqrt{2\pi}}\bigg(\sum_{j=1}^{\infty}\sqrt{|p_j|}e_{2j-1}\bigg)\big(\cos(2\pi t)-1\big) \\ \qquad + \frac{1}{\sqrt{2\pi}}\bigg(\sum_{j=1}^{\infty}\operatorname{sgn}(p_j)\sqrt{|p_j|}e_{2j}\bigg)\sin(2\pi t)\,.$$

Then $A_0(1) = 0$, and

$$\nu(A_0,A_0) \;=\; \frac{1}{2\pi}\omega\Bigg(\sum_{i=1}^{\infty}\sqrt{|p_i|}e_{2i-1}\,,\;\sum_{j=1}^{\infty}\operatorname{sgn}(p_j)\sqrt{|p_j|}e_{2j}\Bigg) \\ \;=\; \frac{1}{2\pi}\sum_{j=1}^{\infty}\operatorname{sgn}(p_j)|p_j|\,f_j \;=\; \frac{1}{2\pi}\sum_{j=1}^{\infty}p_jf_j \;=\; \nu(A,A)\,,$$

while, using the Cauchy-Schwartz inequality,

$$\|A_0\|^2_{\mathcal{H}_0([0,1],H)} \;=\; 2\sum_{j=1}^{\infty}|p_j| \;\le\; 2\sum_{j,k=1}^{\infty}\frac{1}{k}\big(|\alpha_{2j-1,k}\beta_{2j,k}|+|\alpha_{2j,k}\beta_{2j-1,k}|\big) \\ \;\le\; \sum_{j,k=1}^{\infty}\frac{1}{k}\big(\alpha_{j,k}^2+\beta_{j,k}^2\big) \;\le\; \|A\|^2_{\mathcal{H}_0([0,1],H)}\,.$$

This proves the 4 points above. This implies that length-minimizing paths are circles, which is consistent with finite-dimensional Heisenberg group (see, for example, [HZ15]).

We now will compute G_{CM} for Example 5.6 and its topology. Suppose that $z = \sum_{\ell=1}^{\infty} \gamma_\ell f_\ell \in Z$. Then $d(e,(0,z)) < \infty$ if and only if there exists a path $A \in \mathcal{H}_0([0,1],H)$ satisfying $A(0) = 1$ and $\nu(A,A) = z$, and the calculation above justifies that we may replace A with the path $A_0 \in \mathcal{H}_0([0,1],H)$ given as

$$A_0(t) \;=\; \frac{1}{\sqrt{2\pi}}\big(\cos(2\pi t) - 1\big)h + \frac{1}{\sqrt{2\pi}}\sin(2\pi t)v\,,$$

where

$$h \;=\; \sqrt{2\pi}\sum_{j=1}^{\infty}\sqrt{|\gamma_j|}e_{2j-1} \qquad\qquad v \;=\; \sqrt{2\pi}\sum_{j=1}^{\infty}\operatorname{sgn}(\gamma_j)\,\sqrt{|\gamma_j|}e_{2j}\,.$$

For this path A_0, we have that

$$\|A_0\|^2_{\mathcal{H}_0([0,1],H)} \;=\; 4\pi\sum_{\ell=1}^{\infty}|\gamma_j|\,.$$

Thus, $(0,z) \in G_{CM}$ if and only if the path A_0 actually lies in $\mathcal{H}_0([0,1],H)$, which is equivalent to $\sum_{\ell=1}^{\infty}|\gamma_\ell| < \infty$. If this holds, then we have

$$d(e,(0,z)) \;=\; \|A_0\|_{\mathcal{H}_0([0,1],H)} \;=\; 2\sqrt{\pi}\sqrt{\sum_{\ell=1}^{\infty}|\gamma_\ell|}\,.$$

By Theorem 6.6, we see that $G_{CM} = \{(h,z) \in H \times Z \;:\; \sum_{\ell=1}^{\infty}\langle|z,f_\ell\rangle_Z| < \infty\}$, and further that, as topological spaces, $G_{CM} \cong \ell^2 \times \ell^1$.

Example 6.11. In this example, we will show that it is in fact possible that G_{CM} has a nicer-behaved topology. Recall Example 5.9. While this has been labelled as a "nonexample" for the main theorem (because $\|\omega\|_{H\otimes Z} = \infty$), we may still perform a revealing horizontal distance calculation. There, for $\{e_j\}_{j\geq 0}$ a basis for H and $\{f_\ell\}_{\ell\geq 1}$ a basis for Z, we define ω as

$$\omega\left(\sum_{i=0}^{\infty}\alpha_i e_i\,,\,\sum_{j=0}^{\infty}\beta_j e_j\right) \;=\; \sum_{\ell=1}^{\infty}\Big(\alpha_0\beta_\ell - \alpha_\ell\beta_0\Big)f_\ell\,.$$

Then given any $z = \sum_{\ell=1}^{\infty}\gamma_\ell f_\ell$, we may define the path $A(t) = \frac{1}{\sqrt{\pi}}\big((1-\cos(2\pi t))\sqrt{\|z\|_Z}e_0+$

$\sin(2\pi t)\frac{1}{\sqrt{\|z\|_Z}}\left(\sum_{k=1}^\infty \gamma_k e_k\right)$. Then A is a circle of radius $\sqrt{\frac{\|z\|_Z}{\pi}}$, and

$$\begin{aligned}\nu(A,A) &= \omega\left(\sqrt{\|z\|_Z}e_0\,,\ \frac{1}{\sqrt{\|z\|_Z}}\sum_{k=1}^\infty \gamma_k e_k\right)\\ &= \sum_{k=1}^\infty \gamma_k \omega(e_0,e_k) \;=\; \sum_{k=1}^\infty \gamma_k f_k \;=\; z\,,\end{aligned}$$

while

$$d(e,(0,z)) \;\le\; \ell(A) \;=\; 2\pi\sqrt{\frac{\|z\|_Z}{\pi}} \;=\; 2\sqrt{\pi}\sqrt[4]{\sum_{\ell=1}^\infty \gamma_\ell^2}\,.$$

Using Theorem 6.6, we may deduce that G_{CM} consists of $H\times Z$. In fact, we may combine this estimate with Lemma 6.7 to deduce that $H\times Z$ and G_{CM} are topologically equivalent.

6.3 Convergence of horizontal distance

The calculation in Example 6.10 tells us $H\times Z \to G_{CM}$ is not necessarily continuous, but the lemma below does give us partial control. An argument showing essentially the same result can be found in [GM13][Proposition 2.17].

Lemma 6.12. *$d(e,(0,\omega(h,v))) \;\le\; 2\sqrt{\pi}\sqrt{\|h\|_H\|v\|_H}$.*

Proof. Define $h',v'\in H$ as $h'=h$, $v'=v-\frac{\langle h,v\rangle_H}{\langle h,h\rangle_H}h$. Then define $h''=h'\cdot\sqrt{\frac{\|v'\|_H}{\|h'\|_H}}$ and $v''=v'\cdot\sqrt{\frac{\|h'\|_H}{\|v'\|_H}}$. Then it can be seen that $\omega(h'',v'')=\omega(h',v')=\omega(h,v)$, while $\|h''\|_H\|v''\|_H=\|h'\|_H\|v'\|_H\le\|h\|_H\|v\|_H$. Moreover, $\|h''\|_H=\|v''\|_H$ and $h''\perp v''$.

Define the path $A:[0,1]\to H$ as $A(t)=\frac{1}{\sqrt{\pi}}\Big((\cos(2\pi t)-1)h''+\sin(2\pi t)v''\Big)$. Then $A(0)=A(1)=0$, while the calculation below (or Lemma 6.8) shows

$$\begin{aligned}\nu(A,A) &= \int_0^1 \omega\Big((\cos(2\pi t)-1)h''+\sin(2\pi t)v''\,,\ -\sin(2\pi t)h''+\cos(2\pi t)v''\Big)dt\\ &= \omega(h'',v'')\int_0^1(\cos^2(2\pi t)+\sin^2(2\pi t))dt \;=\; \omega(h'',v'') \;=\; \omega(h,v)\,.\end{aligned}$$

And since A is a circle in H, we know

$$\ell(A) \;=\; 2\pi\cdot\left\|\frac{1}{\sqrt{\pi}}h''\right\|_H \;=\; 2\sqrt{\pi}\sqrt{\|h''\|_H\|v''\|_H} \;\le\; 2\sqrt{\pi}\sqrt{\|h\|_H\|v\|_H}\,.$$

The desired inequality follows. □

Recall that, for any $P \in Proj(W) \cup \{I_H\}$, $Q \in Proj(C)$, we may define the group $(G^{P,Q}, \cdot_Q)$, where $G^{P,Q} = PH \times QZ$. Then for $(h_1, z_1), (h_2, z_2) \in G^{P,Q}$, define $d^{P,Q}((h_1, z_1), (h_2, z_2))$ to be the horizontal distance via horizontal paths of $G^{P,Q}$, that is,

$$d^{P,Q}(e,(h,z)) \;=\; \inf\Big\{\ell(A) \;\Big|\; A \in \mathcal{H}_0([0,1], PH)\,,\; A(0) = 0\,,\; A(1) = h\,, \\ \frac{1}{2}\int_0^1 Q\omega(A(s), A'(s))ds = z\Big\}.$$

It is useful to apply properties of Bochner integrals to deduce $\frac{1}{2}\int_0^1 Q\omega(A(s), A'(s))ds = Q\frac{1}{2}\int_0^1 \omega(A(s), A'(s)) = Q\nu(A, A)$. Also note that, if $P \neq I_H$ and $\omega(PH \times PH) \supseteq QZ$, then Lemma 6.12 implies $d^{P,Q}(e, g) < \infty$ for all $g \in G^{P,Q}$, though Lemma 6.13 below covers $P = I_H$.

Lemma 6.13. *For $P \in Proj(W) \cup \{I_H\}$ and $Q \in Proj(C)$ where* $\mathrm{span}(Q\omega(PH \times PH)) = QZ$, $d^{P,Q}(e, g) \leq \|h\|_H + K(Q)\sqrt{\|z\|_Z}$.

Remark 6.14. As previously remarked, we cannot expect the constant K to be independent of Q, as this would show G_{CM} is homeomorphic to $H \times Z$, which is certainly not true in Example 6.10.

Proof. By assumption, we may choose a basis of QZ of the form $\{Q\omega(a_\ell, b_\ell)\}_{\ell=1}^m$. Then if $z \in QZ$ is $z = \sum_{\ell=1}^m \alpha_\ell Q\omega(a_\ell, b_\ell)$, then, using Lemma 6.12,

$$\begin{aligned} d^{P,Q}(e,(0,z)) \;&\leq\; \sum_{\ell=1}^m \sqrt{|\alpha_\ell|}\, d^{P,Q}(e,(0,\omega(a_\ell,b_\ell))) \;\leq\; 2\sqrt{\pi}\sum_{\ell=1}^m \sqrt{|\alpha_\ell|}\sqrt{\|a_\ell\|_H\|b_\ell\|_H} \\ &\leq\; 2\sqrt{\pi}\max_{1\leq\ell\leq m}\left(\sqrt{\|a_\ell\|_H\|b_\ell\|_H}\right)\sum_{\ell=1}^m\sqrt{|\alpha_\ell|} \;\leq\; K\sqrt{\|z\|_Z} \end{aligned}$$

for some constant K. Then

$$d^{P,Q}(e,(h,z)) \;\leq\; d(e,(h,0)) + d(e,(0,z)) \;\leq\; \|h\|_H + K\sqrt{\|z\|_Z}\,.$$

□

A version of Theorem 6.15 was shown in [BGM13, Lemma 3.25], using an equivalent of Lemma 6.13 above. Here, we present an alternate proof that leverages ν.

As a side note, if ν was merely determined to be a continuous bilinear map, then the following proof would be invalid. If $B : H \times H \to Z$ is a bilinear map, and x_n weakly converges to x, then we cannot conclude $B(x_n, x_n)$ weakly converges to $B(x, x)$. Indeed, if we define $B(x, y) = \langle x, y\rangle_H$, then $B(e_n, e_n) = 1$, but e_n weakly converges to 0 while $B(0,0) = 0 \neq 1$. However, if B extends to a continuous linear map $\widetilde{B} : H \otimes H \to Z$, then we can consider the sequence $x_n \otimes x_n$, show that it weakly converges, then realize that $B(x_n, x_n) = \widetilde{B}(x_n \otimes x_n)$ must weakly converge. Thus, in the arguments below, we are making use of the weakly Hilbert-Schmidt assumption.

For the sake of brevity, for the remainder of this section, we will abbreviate $\mathcal{H}_0([0,1], H)$ as $\mathcal{H}_0$, and $\mathcal{H}_0([0,1], PH)$ as $P\mathcal{H}_0$.

Theorem 6.15. *For any $(P_n)_{n\in\mathbb{N}} \in \mathrm{Proj}(W)^{\uparrow}$ and fixed $Q \in \mathrm{Proj}(C)$ with $\operatorname{span}(Q\omega(P_{n_0}H \times P_{n_0}H)) = QZ$ and $g = (h, z) \in P_{n_0}H \times QZ$ for some $n_0 \in \mathbb{N}$ (and thus for all $n \geq n_0$), then*

$$d^{P_n,Q}(e, g) \xrightarrow[n\to\infty]{} d^{I_H,Q}(e, g)\,.$$

And for more arbitrary $g \in H \times QZ$,

$$d^{I_H,Q}(e, \pi^{P_r,Q}g) \xrightarrow[r\to\infty]{} d^{I_H,Q}(e, g)\,.$$

Proof. First, we remark that, for $n \geq n_0$, $d^{P_n,Q}(e, g) < \infty$ by Lemma 6.13. Since $P_n\mathcal{H}_0 \subseteq \mathcal{H}_0$,

$$\Big\{A \in \mathcal{H}_0 \;:\; A(1) = h,\; Q\nu(A, A) = z\Big\} \supseteq \Big\{A \in P_n\mathcal{H}_0 \;:\; A(1) = h,\; Q\nu(A, A) = z\Big\} \neq \emptyset\,.$$

which implies $d^{I_H,Q}(e, g) \leq d^{P_n,Q}(e, g) < \infty$. In fact, it can be deduced that $d^{P_n,Q}(e, g)$ is decreasing in n. For $m \in \mathbb{N}$, choose $A_m \in \mathcal{H}_0$ such that $A_m(1) = h$, $Q\nu(A_m, A_m) = z$, and $|\|A_m\|_{\mathcal{H}_0} - d^{I_H,Q}(e, g)| < \frac{1}{m}$. Then we consider the sequence of endpoints of "projected" paths

$$g_n \;=\; \Big(P_nh, Q\nu(P_nA_m, P_nA_m)\Big)\,.$$

Then we have $d^{P_n,Q}(e, g_n) \leq \|P_nA_m\|_{\mathcal{H}_0}$. Using this and applying Lemma 6.13, for $n \geq n_0$,

$$\begin{aligned} d^{I_H,Q}(e, g) &\leq d^{P_n,Q}(e, g) \leq d^{P_n,Q}(e, g_n) + d^{P_n,Q}(g_n, g) \\ &\leq d^{P_n,Q}(e, g_n) + d^{P_n,Q}\big(e, (0, Q\nu(A_m, A_m) - Q\nu(P_nA_m, P_nA_m))\big) \\ &\leq \|P_nA_m\|_{\mathcal{H}_0} + K\sqrt{\big\|Q\nu(A_m, A_m) - Q\nu(P_nA_m, P_nA_m)\big\|_Z} \\ &\leq \|A_m\|_{\mathcal{H}_0} + K\sqrt{\big\|\widetilde{Q\nu}(A_m \otimes A_m - P_nA_m \otimes P_nA_m)\big\|_Z} \\ &\leq d^{I_H,Q}(e, g) + \frac{1}{m} + K\sqrt{\big\|\widetilde{Q\nu}(A_m \otimes A_m - P_nA_m \otimes P_nA_m)\big\|_Z}\,, \end{aligned}$$

where $\widetilde{Q\nu} : \mathcal{H}_0 \otimes \mathcal{H}_0 \to Z$ is the extension of $Q\nu$. Then we know that as $n \to \infty$, $P_nA_m \otimes P_nA_m \to A_m \otimes A_m$ in $\mathcal{H}_0 \otimes \mathcal{H}_0$, so by the continuity of $\widetilde{Q\nu}$, we must have

$$d^{I_H,Q}(e, g) \;\leq\; \lim_{n\to\infty} d^{P_n,Q}(e, g) \;\leq\; d^{I_H,Q}(e, g) + \frac{1}{m}\,.$$

To be true for all m, we must have, $d^{P_n,Q}(e,g) \xrightarrow{n\to\infty} d^{I_H,Q}(e,g)$.

The second statement is justified by the reverse triangle inequality and Theorem 6.6, or more simply with Lemma 6.12. Using the latter approach, for $g=(h,z)$, we have

$$\begin{aligned}\left|d^{I_H,Q}(e,(P_rh,z)) - d^{I_H,Q}(e,(h,z))\right| \\ &\le d^{I_H,Q}((P_rh,z),(h,z)) \\ &= d^{I_H,Q}\Big(e,\big(h-P_rh,-\frac{1}{2}Q\omega(h,P_rh)\big)\Big) \\ &\le \|h-P_rh\|_H + \sqrt{2\pi}\sqrt{\|h\|_H}\sqrt{\|h-P_rh\|_H}\,.\end{aligned}$$

□

Theorem 6.16. *For any* $(Q_m)_m \in \mathit{Proj}(C)^{\uparrow}$ *with* $\operatorname{span}(\omega(H\times H)) \supseteq Q_mZ$ *for all* m*, for any* $g \in G_{CM}$*,*

$$d^{I_H,Q_m}(e,\pi^{I_H,Q_m}g) \xrightarrow[m\to\infty]{} d(e,g)\,.$$

Proof. Let $g=(h,z)\in G_{CM}$. Consider that, for any path A, if $Q,\widehat{Q} \in \operatorname{Proj}(C)\cup\{I_Z\}$ satisfy $QZ \subseteq \widehat{Q}Z$, then $\widehat{Q}\nu(A,A) = \widehat{Q}z$ implies $Q\nu(A,A) = Q\widehat{Q}\nu(A,A) = Qz$, which implies that

$$\begin{aligned}&\Big\{A\in\mathcal{H}_0 \;:\; A(0)=0,\; A(1)=h,\; Q\nu(A,A)=Qz\Big\} \\ &\qquad\supseteq \Big\{A\in\mathcal{H}_0 \;:\; A(0)=0,\; A(1)=h,\; \widehat{Q}\nu(A,A)=\widehat{Q}z\Big\}\,,\end{aligned}$$

so we have $d^{I_H,Q}(e,\pi^{I_H,Q}g) \le d^{I_H,\widehat{Q}}(e,\pi^{I_H,\widehat{Q}}g)$, so $d^{I_H,Q_m}(e,\pi^{I_H,Q_m}g)$ is increasing in m. We may also deduce $d^{I_H,Q_m}(e,\pi^{I_H,Q_m}g) \le d(e,g)$ for all m, so that we have $\lim_{m\to\infty} d^{I_H,Q_m}(e,\pi^{I_H,Q_m}g) \le d(e,g)$.

For every m, choose the path $A_m \in \mathcal{H}_0$ such that $A_m(0)=0$, $A_m(1)=h$, $Q_m\nu(A_m,A_m)=Q_mz$, and $\left|d^{I_H,Q_m}(e,\pi^{I_H,Q_m}g) - \|A_m\|_{\mathcal{H}_0}\right| \le \frac{1}{m}$. Then, in particular, the sequence A_m satisfies the bound $\|A_m\|_{\mathcal{H}_0} \le d(e,g)+\frac{1}{m}$. By the Banach-Alaoglu theorem, there exists a subsequence A_{m_k} that weakly converges to some $A\in\mathcal{H}$ with $\|A\|_{\mathcal{H}_0}\le d(e,g)$.

The remainder of this proof will show that $A(0)=0$, $A(1)=h$, $\nu(A,A)=z$, and $d(e,g)=\|A\|_{\mathcal{H}_0} \le \lim_{m\to\infty} d^{I_H,Q_m}(e,\pi^{I_H,Q_m}g)$. First note that, as discussed in Section 2.1.4, the evaluation maps $A\mapsto A(0)$ and $A\mapsto A(1)$ are linear and continuous on $\mathcal{H}_0$, so we must have $A(1)=\lim_{k\to\infty} A_{m_k}(1) = \lim_{k\to\infty} h = h$, and likewise $A(0)=0$.

Furthermore, note that, for any simple tensor $B\otimes C \in \mathcal{H}_0\otimes\mathcal{H}_0$, $\langle A_{m_k}\otimes A_{m_k}, B\otimes C\rangle_{\mathcal{H}_0\otimes\mathcal{H}_0} = \langle A_{m_k},B\rangle_{\mathcal{H}_0}\langle A_{m_k},C\rangle_{\mathcal{H}_0}$. This allows us to conclude that if $a\in\mathcal{H}_0\otimes\mathcal{H}_0$ is a finite sum of simple tensors, then $\langle A_{m_k} \otimes A_{m_k}, a\rangle_{\mathcal{H}_0\otimes\mathcal{H}_0} \to \langle A \otimes A, a\rangle_{\mathcal{H}_0\otimes\mathcal{H}_0}$. Since the set of such tensors forms a dense set in $\mathcal{H}_0\otimes\mathcal{H}_0$, and since $A_{m_k}\otimes A_{m_k}$ is a bounded sequence, this convergence must hold for all tensors in $\mathcal{H}_0 \otimes \mathcal{H}_0$ by Proposition 2.2. Thus, $A_{m_k} \otimes A_{m_k}$ is weakly convergent to $A \otimes A$. By the continuity of $\widetilde{\nu}$, we know that $\nu(A_{m_k}, A_{m_k})$ weakly converges to $\nu(A, A)$.

Next, we know that, for all m, $Q_m\nu(A_m, A_m) = Q_m z$. Fix $m_0 \in \mathbb{N}$ and $x \in Q_{m_0}Z$. If $m \geq m_0$, then $Q_m x = x$, and for all $v \in Z$, $\langle v, x\rangle_Z = \langle v, Q_m x\rangle_Z = \langle Q_m v, x\rangle_Z$, so that

$$\big\langle \nu(A_m, A_m), x\big\rangle_Z \;=\; \big\langle Q_m\nu(A_m, A_m), x\big\rangle_Z \;=\; \langle Q_m z, x\rangle_Z \;=\; \langle z, x\rangle_Z\,,$$

so $\langle\nu(A_m, A_m), x\rangle_Z$ converges to $\langle z, x\rangle$ for every $x \in Q_{m_0}Z$ for any m_0. Since $\bigcup_{m_0=1}^{\infty} Q_{m_0}Z$ is dense in Z, and since the continuity of $\widetilde{\nu}$ implies the boundedness of $\nu(A_m, A_m)$, we may conclude $\nu(A_m, A_m)$ weakly converges to z.

We have shown that $\nu(A_{m_k}, A_{m_k})$ weakly converges to $\nu(A, A)$, but at the same time must weakly converge to z. Hence, $\nu(A, A) = z$. Thus, $A \in \mathcal{H}_0$ is such that $\|A\|_{\mathcal{H}_0} \leq d(e, g)$, $A(0) = 0$, $A(1) = h$, and $\nu(A, A) = z$. Therefore, $d(e, g) = \|A\|_{\mathcal{H}_0}$, so that

$$\begin{aligned} d(e,g) \;&=\; \|A\|_{\mathcal{H}_0} \leq \liminf_{k\to\infty} \|A_{m_k}\|_{\mathcal{H}_0} \\ &\leq\; \lim_{k\to\infty}\left(d^{I_H, Q_{m_k}}(e,g) + \frac{1}{m_k}\right) \;=\; \lim_{m\to\infty} d^{I_H,Q_m}(e,g)\,, \end{aligned}$$

which proves the claim. □

Remark 6.17. In proving this theorem, we have revealed an interesting fact: for any $g \in G_{CM}$, there exists a path A such that $d(e, g) = \|A\|_{\mathcal{H}^2}$. This can be proven more directly by considering approximations $A_m \in \mathcal{H}_0$ with $A_m(0) = 0$, $A_m(1) = h$, $\nu(A, A) = z$ and $\big|\|A\|_{\mathcal{H}_0} - d(e, g)\big| < \frac{1}{m}$, without considering finite-dimensional projections. It is interesting that one can prove the existence of (energy- or) length-minimizing paths in the infinite-dimensional context. Such results are usually in finite-dimensional situations and rely on compactness arguments, though they also prove smoothness properties of such paths. Proving the existence of such paths in our case relied on the weak-compactness exhibited by separable Hilbert spaces.

7 The distribution: definition and convergence of finite-dimensional projections.

The goal of this section is to first define the hypoelliptic heat kernel distribution on $G = W \times C$, and flesh out the consequences of (A2.4). Secondly, we will show a sense in which finite-dimensional approximations, denoted $g_t^{P_n,Q_m}$, converge to g_t that will be suitable for proving Theorem 8.4.

7.1 The distribution

We first provide a definition for the distribution, which will be inline with Section 3. First off, note that, if B_t is a Brownian motion on W and $P \in \mathrm{Proj}(W)$, then PB_t is a Brownian motion on PH. Next, knowing that H is a Lie algebra with Lie bracket $[\cdot,\cdot]$, for $P \in \mathrm{Proj}(W)$, we let $\mathfrak{g}^P = PH \times \mathrm{span}(\omega(PH, PH))$, Lie subalgebra generated by PH, equal to its own Lie group $G^P = \mathfrak{g}^P$. As finite-dimensional Lie groups, the groups G^P have a notion of Brownian motion, which can be realized as the solution to the Stratonovich differential equation

$$g_t^P \;=\; L_{g_t^P *}\delta PB_t \qquad\qquad g_0^P \;=\; e\,.$$

And we may refer to Theorem 2.6 and Section 2.3.3 to derive the following expression for Brownian motion on G^P as the Stratonovich integral

$$\begin{aligned} g_T^P \;&=\; \left(PB_T\,,\, \frac{1}{2}\int_{0\le t_1\le t_2\le T} \omega(\delta PB_{t_1}, \delta PB_{t_2})\right) \\ &=\; \left(PB_T\,,\, \frac{1}{2}\int_0^T \omega(PB_t, \delta PB_t)\right). \end{aligned}$$

where we may view the process $(\omega(PB_t, P\cdot) : \Omega \times PH \to Z)_{t\ge 0}$ as being adapted to the filtration determined by $(B_t)_{t\ge 0}$. Using this perspective, for the remained of this section, we will often write $\int_0^T \omega(PB_t,\cdot)\delta PB_t$ instead of $\int_0^T \omega(PB_t, \delta PB_t)$ to more closely resemble notation used for classical stochastic calculus.

We say that G is a *(simply connected graded step-2 nilpotent) abstract Wiener Lie group* if

> For any $t > 0$, there exists a G-valued random variable g_t such that, given an increasing sequence of finite-rank projections $\{P_m\}_{m\in\mathbb{N}} \in \mathrm{Proj}(W)^{\uparrow}$, for every $f \in G^*$, $f(g_t^{P_m}) \to f(g_t)$ in probability. (A2.4)

It can be quickly shown that this convergence naturally happens when C is a Hilbert space for which the inclusion $Z \hookrightarrow C$ is Hilbert-Schmidt. This is shown in [DG08], where the argument will suffice even if $\dim(C) = \infty$; the key property is that the composition $\omega : H \times H \to Z \hookrightarrow C$ would be Hilbert-Schmidt, and

$$\mathbb{E}\left\| \int_0^T \omega((P_n - P_m)B_t, \cdot)d(P_n - P_m)B_t \right\|_C^2 = \frac{1}{2}T^2 \sum_{i,j=m+1}^{n} \sum_{\ell=1}^{\infty} \langle c_\ell, \omega(e_i, e_j)\rangle_C^2 .$$

This is consistent with the definition in Section 3, in which $X = \mathfrak{g}_{CM}$ and $X_H = H$. Note that, as discussed in Section 3, the limit in (A2.4) occurring for some $t > 0$ is equivalent to it occurring for all $t > 0$. Also, this definition will necessarily satisfy (A1.2′) due to (A2.1) and the traceless aspect of ω; see Theorem 7.1 and its proof below.

As has been assumed in [BGM13; DEM16; GM13], we have the following

Theorem 7.1. *Letting δB_s denote the Stratonovich differential and dB_s the Itô differential, we have that for any $P \in \mathrm{Proj}(W)$,*

$$\int_0^T \omega(PB_t, \cdot)\delta PB_t = \int_0^T \omega(PB_t, \cdot)dPB_t .$$

Proof. Let $\{e_i\}_{1\le i\le r}$ be a basis of PH. Then note that if $(b_t)_{t\ge 0}, (b'_t)_{t\ge 0}$ are standard, independent Brownian motions on $\mathbb{R}$, then we have

$$\int_0^T b_t\, \delta b_t = \int_0^T b_t\, db_t + \frac{1}{2}t \qquad\qquad \int_0^T b_t\, \delta b'_t = \int_0^T b_t\, db'_t .$$

This suffices to justify

$$\int_0^T \omega(PB_t, \cdot)\delta PB_t = \int_0^T \omega(PB_t, \cdot)dPB_t + \frac{1}{2}\sum_{i=1}^{r} \omega(e_i, e_i) .$$

But since ω is anti-symmetric, $\omega(h, h) = 0$ for all $h \in H$. This proves the claim.

□

Theorem 7.1 justifies using the following notation:

$$g_T = \left(B_T\, , \int_0^T \omega(B_t, \cdot)dB_t \right) .$$

7.2 The action of $\exp(\mathfrak{g}_{CM})$ on G

As noted in Section 4, we do not[5] assume that ω has a continuous extension to $\omega : W \times W \to C$. Instead, our assumptions suffice to show that we have a measurable action of $\exp(\mathfrak{g}_{CM})$ on $G \to G$.

Considering the $\|\omega\|_{H\otimes Z}$ constant, we see that $\omega(h, \cdot) : H \to Z$ is Hilbert-Schmidt. Then using the theory of measurable linear maps on Hilbert spaces, (discussed in Section 2.2, or consider [Bog14; Zha82]), this extends to a measurable map $\omega(h, \cdot) : W \to Z$, which can be defined as

$$\omega(h, w) \;=\; \lim_{m\to\infty} \omega(h, P_n w)\,,$$

where $(P_n)_{n\in\mathbb{N}} \in \mathrm{Proj}(W)^{\uparrow}$, and the limit converges almost surely in $\|\cdot\|_Z$. Or alternatively, we can write

$$\omega(h, w) \;=\; \sum_{\ell=1}^{\infty} \langle \omega(h, \cdot)^* f_\ell, w\rangle_H\,,$$

again converging almost surely in $\|\cdot\|_Z$.

Using this measurable extension, for $(h, z) \in H \times Z = \exp(\mathfrak{g}_{CM})$, we can define $L_{(h,z)} : W \times C \to W \times C$ as the measurable extension of left multiplication by (h, z), meaning

$$L_{(h,z)}(w, c) = \left(h + w\,,\; z + c + \frac{1}{2}\omega(h, w)\right).$$

We will henceforth denote this as $(h, z) \cdot (w, c)$. Similarly, note that we may likewise discuss the right action $R_{(h,z)}(w, c) = (w, c) \cdot (h, z)$.

7.3 The Fourier transform

In [DEM16], there were formulae derived related to the Fourier transform of g_T, which allowed for interpretation of the distribution of $M_T := \int_0^T \omega(B_t, \cdot)dB_t$. The

[5] As a brief aside, there are situations in which we may talk about the existence of a measurable bilinear extension ω on $W \times C$, and hence measurable group operation, without assuming $\|\omega\|_{H\otimes Z} < \infty$. If C is assumed to be a Hilbert space, then $\omega : H \times H \to C$ is Hilbert-Schmidt, which suffices to show ω has an L^2-extension to $W \times W \to C$. If $W \times C$ is a (step-2 graded) path space, as in Example 3.5, then $W \times C$ has a continuous (pointwise) bracket that corresponds to a continuous bilinear definition of ω. Both of these instances are examples of when (C, Z) is also an abstract Wiener space, for which another approach is possible. In [Car78], it is shown that one may take the tensor product of abstract Wiener spaces to arrive at a new abstract Wiener space. In particular, the injective tensor product $W \otimes_\epsilon W$ is an abstract Wiener space, whose Cameron-Martin subspace is the Hilbert space tensor product $H \otimes H$. Since our assumptions tell us that we have a continuous linear extension $\tilde{\omega} : H \otimes H \to Z$, then we should have a measurable-linear extension to $\tilde{\omega} : W \otimes_\epsilon W \to C$, which may have a measurable restriction to $W \times W$.

purpose of this section is to highlight the fact that such formulae carry over to when C is infinite-dimensional. We know that ω is weakly Hilbert-Schmidt, which means $\langle z, \omega(\cdot,\cdot)\rangle_Z$ is Hilbert-Schmidt. Then, as shown in [DEM16], $\langle z, \int_0^T \omega(B_t,\cdot)dB_t\rangle_Z = \int_0^T \langle z, \omega(B_t,\cdot)\rangle_Z dB_t$ is an L^2-limit of the random variables $\int_0^T \langle z, \omega(P_n B_t,\cdot)\rangle_Z dP_n B_t$. Furthermore, we have the following result [DEM16, Theorem 2.1]:

Theorem 7.2. *For any bounded measurable $f : \mathbb{R} \to \mathbb{C}$,*

$$\mathbb{E}[f(B_T)e^{i\int_0^T \langle z,\omega(B_t,\cdot)\rangle_Z dB_t}] \;=\; \mathbb{E}[f(B_T)e^{-\frac{1}{2}\int_0^T |\langle z,\omega(B_t,\cdot)\rangle_Z|^2 dt}]\,.$$

This is equivalent to saying that, conditioned on B_T, $\int_0^T \langle z, \omega(B_t,\cdot)\rangle_Z dB_t$ has the same distribution as $\int_0^T \langle z, \omega(B_t,\cdot)\rangle_Z dB'_t$, where B'_t is a Brownian motion on W independent of B. This is also equivalent to saying $\int_0^T \langle z, \omega(B_t,\cdot)\rangle_Z dB_t$ is distributed as a "stochastic" Gaussian with random covariance determined by the function $\rho_T : Z \to Z$ given as

$$\rho_T \;=\; \int_0^T \omega(B_t,\cdot)\omega(B_t,\cdot)^* dt\,.$$

More precisely, we may write

$$\langle z_1, \rho_T z_2\rangle_Z \;=\; \int_0^T \langle \omega(B_t,\cdot)^* z_1, \omega(B_t,\cdot)^* z_2\rangle_H dt\,,$$

where the map $W \ni w \mapsto \omega(w,\cdot)^* z \in H$ is the measurable-linear extension of $H \ni h \mapsto \omega(h,\cdot)^* z \in H$, which is guaranteed to exist since $\omega(h,\cdot) : H \to Z$ is Hilbert-Schmidt, as is its adjoint $\omega(h,\cdot)^* : Z \to H$. Indeed, the formula in the theorem above can be rewritten as

$$\mathbb{E}[f(B_T)e^{i\langle z,M_T\rangle}] \;=\; \mathbb{E}[f(B_T)e^{-\frac{1}{2}\langle z,\rho_T z\rangle_Z}]\,.$$

Note that this formula determines the Fourier transform of $g_T = (B_T, M_T)$, sine every $\alpha \in C^*$ can be realized as the continuous extension of $\langle z,\cdot\rangle_Z$ for some $z \in Z$. Thus, an alternate definition of the heat kernel measure on G could be a distribution whose Fourier transform is given above.

7.4 Convergence of finite-dimensional projections

Recall that, as introduced in Section 5.3, $G^{P,Q}$ is a (finite-dimensional) Lie group with group multiplication $\cdot_Q : G^{P,Q} \times G^{P,Q} \to G^{P,Q}$ defined as

$$(h_1, z_1) \cdot_Q (h_2, z_2) \;=\; \Big(h_1 + h_2, z_1 + z_2 + \frac{1}{2}Q\omega(h_1,h_2)\Big)\,.$$

Using the reasoning from the previous section, such groups also have Brownian motion, which can be realized as the expression

$$g_t^{P,Q} \;=\; \left(PB_t, \int_0^t Q\omega(PB_s,\cdot)dPB_s\right).$$

The primary objective of this section is to prove to what extent we have $g_T^{P_n,Q_m}$ converging to g_T for fixed time T, or more generally, for any $(h,z) \in H\times Z$, $g_T^{P_n,Q_m} \cdot_{Q_m} \pi^{P_n,Q_m}(h,z)$ converging to $g_T\cdot(h,z)$. In particular, we will show that, for any bounded continuous cylinder function f,

$$\lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}[f(g_T^{P_n,Q_m} \cdot_{Q_m} \pi^{P_n,Q_m}(h,z))] \;=\; \mathbb{E}[f(g_T\cdot(h,z))]\,.$$

First, we recall a lemma from [DEM16].

Lemma 7.3. *Let $\mathcal{M}_T$ denote the continuous, square-integrable $\mathbb{R}$-valued martingales defined on Ω up to time T, for which we define*

$$\|M\|_{\mathcal{M}_T} \;:=\; \mathbb{E}\sup_{t\in[0,T]}|M_t|^2\,.$$

Then the map

$$\mathcal{L}_f(H,H) \ni A \mapsto \int_0^{\cdot} \langle AB_t,\cdot\rangle_H dB_t \in \mathcal{M}_T$$

is defined for $A \in \mathcal{L}_f(H,H)$, the set of finite-rank linear operators on H, using classical stochastic calculus, and is continuous with respect to the Hilbert-Schmidt norm. As such, it extends to a continuous linear map on $HS(H,H)$.

Consequently, we may deduce that if $A_n \to A$ in $HS(H,H)$, then $\int_0^T \langle A_nB_t,\cdot\rangle_H dB_t \to \int_0^T \langle AB_t,\cdot\rangle_H dB_t$ in $L^2(\Omega,\mathbb{R})$.

As a remark, the entirety of the proof of Lemma 7.3 can be encapsulated in the inequalities below, the first from Doob's maximal inequality, the second from the Itô isometry:

$$\begin{aligned}\sup_{t\in[0,T]}\left|\int_0^t \langle AB_s,\cdot\rangle_H dB_s\right| \;&\le\; 4\,\mathbb{E}\left|\int_0^T \langle AB_s,\cdot\rangle_H dB_s\right| \\ &=\; 4\int_0^T \mathbb{E}\|AB_s\|_H^2 ds \;=\; 4T\|A\|^2_{HS(H,H)}\,.\end{aligned}$$

Theorem 7.4. *For $(P_n)_{n\in\mathbb{N}} \in Proj(W)^{\uparrow}$ and fixed $Q \in Proj(C)$ and $b \in Z$, we have that*

$$\left\langle b, \int_0^T Q\omega(P_nB_t,\cdot)dP_nB_t\right\rangle_Z \xrightarrow{n\to\infty} \left\langle b, \int_0^T Q\omega(B_t,\cdot)dB_t\right\rangle_Z$$

in $L^2(\Omega, \mathbb{R})$. *Furthermore, for* $Q_m \in Proj(Z)^{\uparrow}$*, we have that*

$$\left\langle b, \int_0^T Q_m \omega(B_t, \cdot) dB_t \right\rangle_Z \xrightarrow{m\to\infty} \left\langle b, \int_0^T \omega(B_t, \cdot) dB_t \right\rangle_Z$$

in $L^2(\Omega, \mathbb{R})$.

Proof. Define $\mathfrak{w}_b : H \to H$ as[6] $\mathfrak{w}_b(h) := \omega(h, \cdot)^* b$. Observe that

$$\begin{aligned} \left\langle b, \int_0^T Q\omega(PB_t, P\cdot) dB_t \right\rangle_Z &= \int_0^T \langle Qb, \omega(PB_t, P\cdot)\rangle_Z dB_t \\ &= \int_0^T \langle P\mathfrak{w}_{Qb}(PB_t), \cdot\rangle_H dB_t \,. \end{aligned}$$

Thus, the crux of the proof, as is essentially demonstrated in [DEM16], lies in showing that $\mathfrak{w}_b \in HS(H, H)$, and further that it is well-approximated by $P_n \circ \mathfrak{w}_{Q_m b} \circ P_n$. Indeed,

$$\begin{aligned} \|\mathfrak{w}_b\|^2_{HS(H,H)} &= \sum_{i,j=1}^{\infty} \langle \mathfrak{w}_b(e_i), e_j\rangle_H^2 = \sum_{i,j=1}^{\infty} \langle \omega(e_i, \cdot)^* b, e_j\rangle_H^2 \\ &= \sum_{i,j=1}^{\infty} \langle b, \omega(e_i, e_j)\rangle_H^2 \le \|\omega\|^2_{H\otimes H} \|b\|^2_Z \,. \end{aligned}$$

So, for any $P_n \in \text{Proj}(W)^{\uparrow}$ and $Q \in \text{Proj}(C)$, choose an orthonormal basis $\{e_i\}_{i\in\mathbb{N}} \subseteq H$ such that $P_n H = \text{span}\{e_1, \ldots, e_n\}$, so that we have

$$\begin{aligned} \|\mathfrak{w}_{Qb} - P_n \circ \mathfrak{w}_{Qb} \circ P_n\|_{HS(H,H)} &= \sum_{i,j=1}^{\infty} \langle \mathfrak{w}_{Qb}(e_i) - P_n \mathfrak{w}_{Qb}(P_n e_i), e_j\rangle_H^2 \\ &= \sum_{i,j=n+1}^{\infty} \langle Qb, \omega(e_i, e_j)\rangle_Z^2 \,, \end{aligned}$$

which converges to 0 as $n \to \infty$. By Lemma 7.3, this means that

$$\int_0^T \langle P_n \mathfrak{w}_{Qb}(P_n B_t), \cdot\rangle_Z dB_t \to \int_0^T \langle \mathfrak{w}_{Qb}(B_t), \cdot\rangle_Z dB_t \,,$$

or equivalently,

$$\left\langle b, \int_0^T Q\omega(P_n B_t, \cdot) dP_n B_t \right\rangle_Z \to \left\langle b, \int_0^T Q\omega(B_t, \cdot) dB_t \right\rangle_Z .$$

[6] To ensure the notation is clear, as used in the previous subsections, for a fixed $h \in H$, we regard $\omega(h, \cdot)^* : Z \to H$ to be the adjoint of $\omega(h, \cdot) : H \to Z$

Additionally, for $Q_m \in \text{Proj}(C)^\uparrow$,

$$\begin{aligned}\|\mathfrak{w}_b - \mathfrak{w}_{Q_m b}\|_{HS(H,H)} &= \sum_{i,j=1}^{\infty} \langle \mathfrak{w}_b(e_i) - \mathfrak{w}_{Q_m b}(e_i), e_j\rangle_H^2 \\ &= \sum_{i,j=1}^{\infty} \langle b - Q_m b, \omega(e_i, e_j)\rangle_Z^2 \;\le\; \|\omega\|_{H\otimes H}^2 \|b - Q_m b\|_Z^2 \,,\end{aligned}$$

which also converges to 0 as $m \to \infty$, which gives the second claimed convergence.

□

Theorem 7.5. *Let f be a bounded continuous cylinder function. Then for any $P_n \in$ Proj$(W)^\uparrow$ and fixed $Q \in$ Proj(C),*

$$\mathbb{E}[f(g_T^{P_n,Q} \cdot_Q \pi^{P_n,Q}(h,z))] \xrightarrow{n\to\infty} \mathbb{E}[f(g_T^{I_W,Q} \cdot_Q \pi^{I_W,Q}(h,z))]\,.$$

Also, we may hold $g_T^{I_W,Q}$ fixed and still have convergence, so

$$\mathbb{E}[f(g_T^{I_W,Q} \cdot_Q \pi^{P_n,Q}(h,z))] \xrightarrow{n\to\infty} \mathbb{E}[f(g_T^{I_W,Q} \cdot_Q \pi^{I_W,Q}(h,z))]\,,$$

and for $(Q_m)_m \in$ Proj$(C)^\uparrow$,

$$\mathbb{E}[f(g_T^{I_W,Q_m} \cdot_{Q_m} \pi^{I_W,Q_m}(h,z))] \xrightarrow{m\to\infty} \mathbb{E}[f(g_T \cdot (h,z))]\,.$$

Proof. For $g = (w,c) \in W \times C$, $a \in H$, $b \in Z$, we simplify notation by writing $\langle a, g\rangle_H = \langle a, w\rangle_H$ and $\langle b, g\rangle_Z = \langle b, c\rangle_Z$. Note that we may write

$$\begin{aligned}&g_T^{P_n,Q} \cdot_Q \pi^{P_n,Q}(h,z) \\ &\quad= \Big(P_n B_T + P_n h\,,\, \frac{1}{2}\int_0^T Q\omega(P_n B_t, P_n \cdot)dB_t + Qz + \frac{1}{2}Q\omega(P_n B_T, P_n h)\Big)\,,\end{aligned}$$

so we begin by showing that the image of the expression above under the maps $\langle a, \cdot\rangle_H$ and $\langle b, \cdot\rangle_Z$, for some $a \in H$, $b \in Z$, will converge weakly in n (in fact, we show that they converge in $L^2(\Omega, \mathbb{R})$).

For $a \in H$, Using the embedding $H \to L^2(\Omega, \mathbb{R})$, we have that $\langle a, P_n B_T\rangle_H = \langle P_n a, B_T\rangle_H \to \langle a, B_T\rangle_H$ (in $L^2(\Omega, \mathbb{R})$). By Theorem 7.4, for $b \in Z$,

$$\Big\langle b, \int_0^T Q\omega(P_n B_t, P_n \cdot)dB_t\Big\rangle_Z \to \Big\langle b, \int_0^T Q\omega(B_t, \cdot)dB_t\Big\rangle_Z\,.$$

And we have

$$
\begin{aligned}
&\mathbb{E}\Big|\langle b, Q\omega(B_T,h)\rangle_Z - \langle b, Q\omega(P_nB_t,P_nh)\rangle_Z\Big|^2 \\
&\quad = \mathbb{E}\Big|\langle b, Q\omega((I-P_n)B_T,h)\rangle_Z \;-\; \langle b, Q\omega(P_nB_T,(I-P_n)h)\rangle_Z\Big|^2 \\
&\quad \le 2\mathbb{E}\Big|\langle b, Q\omega((I-P_n)B_T,h)\rangle_Z\Big|^2 + 2\mathbb{E}\Big|\langle b, Q\omega(P_nB_T,(I-P_n)h)\rangle_Z\Big|^2 \\
&\quad = 2\sum_{i=n+1}^{\infty}\langle b, Q\omega(e_i,h)\rangle_Z^2 + 2\sum_{i=1}^{n}\langle b, Q\omega(e_i,(I-P_n)h)\rangle_Z^2 \\
&\quad \le 2\sum_{i=n+1}^{\infty}\langle \omega(\cdot,h)^*Qb, e_i\rangle_H^2 + 2\sum_{i=1}^{n}\left\langle b\,,\; Q\omega\left(e_i, \frac{(I-P_n)h}{\|(I-P_n)h\|_H}\right)\right\rangle_Z^2 \|(I-P_n)h\|_H^2 \\
&\quad \le 2\|(I-P_n)\omega(\cdot,h)^*Qb\|_H^2 + 2\|\omega\|_{H\otimes H}^2\|Qb\|_Z^2\|(I-P_n)h\|_H^2\,,
\end{aligned}
$$

where the above converges to 0. Thus, $\langle b, Q\omega(P_nB_T,P_nh)\rangle_Z \to \langle b, Q\omega(B_T,h)\rangle_Z$ in $L^2(\Omega,\mathbb{R})$. Therefore, we may conclude that

$$
\langle a, P_nB_T + P_nh\rangle_H \xrightarrow{n\to\infty} \langle a, B_T + h\rangle_H
$$

$$
\begin{aligned}
&\left\langle b, \frac{1}{2}\int_0^T Q\omega(P_nB_t, P_n\cdot)dB_t + Qz + \frac{1}{2}Q\omega(P_nB_T,P_nh)\right\rangle_Z \\
&\qquad\qquad\qquad \xrightarrow{n\to\infty} \left\langle b, \frac{1}{2}\int_0^T Q\omega(B_t,\cdot)dB_t + Qz + \frac{1}{2}Q\omega(B_T,h)\right\rangle_Z.
\end{aligned}
$$

Or, in other words, $\langle a, g_T^{P_n,Q} \cdot_Q \pi^{P_n,Q}(h,z)\rangle_H \xrightarrow{n\to\infty} \langle a, g_T^{I_W,Q} \cdot_Q \pi^{I_W,Q}(h,z)\rangle_H$ and $\langle b, g_T^{P_n,Q} \cdot_Q \pi^{P_n,Q}(h,z))\rangle_Z \xrightarrow{n\to\infty} \langle b, g_T^{I_W,Q} \cdot_Q \pi^{I_W,Q}(h,z))\rangle_Z$. Then if f is a bounded continuous cylinder function, then it can be written as

$$
f(\cdot) = F(\langle a_1,\cdot\rangle_H,\ldots,\langle a_k,\cdot\rangle_H,\langle b_1,\cdot\rangle_Z,\ldots,\langle b_\ell,\cdot\rangle_Z)
$$

for some bounded continuous $F:\mathbb{R}^{k+\ell}\to\mathbb{C}$, $a_1,\ldots,a_k\in H$, $b_1,\ldots,b_\ell\in Z$. By the above, we may deduce $(\langle a_1, g_T^{P_n,Q} \cdot_Q \pi^{P_n,Q}(h,z)\rangle_H,\ldots,\langle b_\ell, g_T^{P_n,Q} \cdot_Q \pi^{P_n,Q}(h,z)\rangle_Z)$ converges in $L^2(\Omega,\mathbb{R}^{k+\ell})$, which implies $\mathbb{E}[f(g_T^{P_n,Q} \cdot_Q \pi^{P_n,Q}(h,z))] \xrightarrow{n\to\infty} \mathbb{E}[f(g_T^{I_W,Q} \cdot_Q \pi^{I_W,Q}(h,z))$.

The proof for the second convergence pans out identically. For the third convergence, Theorem 7.4 tells us $\left\langle b, \int_0^T Q_m\omega(B_t,\cdot)dB_t\right\rangle_Z$ converges in $L^2(\Omega,\mathbb{R})$ to $\left\langle b, \int_0^T \omega(B_t,\cdot)dB_t\right\rangle_Z$, and we may compute

$$
\begin{aligned}
\mathbb{E}\Big|\langle b, \omega(B_t,h)\rangle_Z - \langle b, Q_m\omega(B_t,h)\rangle_Z\Big|^2 &= \sum_{i=1}^{\infty}\langle b, (I-Q_m)\omega(e_i,h)\rangle_Z^2 \\
&\le \|\omega\|_{H\otimes H}^2\|(I-Q_m)b\|_Z^2\|h\|_H^2\,.
\end{aligned}
$$

Then $\langle b, g_T^{I_W,Q_m} \cdot_{Q_m} \pi^{I_W,Q_m}(h,z)) \rangle_Z \xrightarrow{m\to\infty} \langle b, g_T \cdot (h,z) \rangle_Z$. As before, this implies that, for any bounded continuous cylinder function f, $\mathbb{E}[f(g_T^{I_W,Q_m} \cdot_{Q_m} \pi^{I_W,Q_m})] \xrightarrow{m\to\infty} \mathbb{E}[f(g_T \cdot (h,z))]$. $\square$

8 Dimension-independent generalized curvature-dimension inequality

We will review the differential operators that go into curvature-dimension inequalities, proving that G satisfies such an inequality, and concluding with Theorem 8.4.

8.1 Defining the differential operators

We now redefine the differential operators involved in deriving generalized curvature-dimension bounds. The bounds, inequalities, and derivatives need only be considered for finite dimensions. As such, for the next 2 sections, we will fix projections $P \in \mathrm{Proj}(W)$, $Q \in \mathrm{Proj}(C)$ such that $\mathrm{span}(Q\omega(PH \times PH)) = QZ$, and describe P- and Q-dependent inequalities for functions defined on $G^{P,Q}$.

As described in Section 2.3.2, for any $F \in \mathcal{C}^\infty(G^{P,Q})$ (meaning F a smooth function $F : G^{P,Q} \to \mathbb{R}$) and element $(h, z) \in PH \times QZ$, we define

$$F'(w,c)(h,z) \;=\; \partial_{(h,z)}F(w,c) \;=\; \frac{d}{dt}\Big|_{t=0} F((w,c)+t(h,z))\,,$$

and

$$F''(w,c)((h_1,z_1)\otimes(h_2,z_2)) \;=\; \partial_{(h_1,z_1)}\partial_{(h_2,z_2)}F(w,c)\,.$$

Recall that each group $G^{P,Q}$ has a Q-dependent product, namely $\cdot_Q$, so our definitions of "left-invariant" derivatives will also be Q-dependent. With this in mind, for $(h, z) \in PH \times QZ$, for this section, we will let $\widetilde{(h,z)}$ denote the unique (Q-dependent) left-invariant differential operator satisfying $\widetilde{(h,z)}F(e) = \partial_{(h,z)}F(e)$. More precisely, we have

$$\widetilde{(h,z)}F(w,c) \;=\; \partial_{(h,z+\frac{1}{2}Q\omega(w,h))}F(w,c)\,.$$

We now create notation for the differential operators introduced in (2.3). For any $F \in \mathcal{C}^\infty(G^{P,Q})$, and given a basis $\{e_i\}_{i=1}^n$ of PH, define the (P- and Q-dependent) left-invariant Laplacian $\Delta F \in \mathcal{C}^\infty(G^{P,Q})$ as

$$\Delta F(x) \;=\; \sum_{i=1}^n \left[\widetilde{(e_j,0)}^2 F\right](x)\,.$$

For any $F_1, F_2 \in \mathcal{C}^\infty(G^{P,Q})$, define

$$\Gamma(F_1,F_2)(x) \;=\; \sum_{j=1}^n \Big(\widetilde{(e_i,0)}F_1\Big)(x)\Big(\widetilde{(e_i,0)}F_2\Big)(x)\,.$$

We remark that these are both independent of basis chosen. Then define

$$\Gamma_2(F_1, F_2)(x) \;=\; \frac{1}{2}\Big(\Delta\Gamma(F_1, F_2) - \Gamma(F_1, \Delta F_2) - \Gamma(F_2, \Delta F_1)\Big)\,.$$

We will also abbreviate $\Gamma(F) = \Gamma(F, F)$ and $\Gamma_2(F) = \Gamma_2(F, F)$. We may similarly define, for a basis $\{f_\ell\}_{\ell=1}^m$ of QZ,

$$\begin{aligned}
\Gamma^Z(F_1, F_2) &= \sum_{\ell=1}^m \left(\big(\widetilde{(0, f_\ell)}F_1\big)\big(\widetilde{(0, f_\ell)}F_2\big)\right) \\
\Gamma_2^Z(F_1, F_2) &= \frac{1}{2}\Big(\Delta\Gamma^Z(F_1, F_2) - \Gamma^Z(F_1, \Delta F_2) - \Gamma^Z(F_2, \Delta F_1)\Big)\,,
\end{aligned}$$

which again will be independent of basis.

8.2 Generalized curvature-dimension bounds

The generalized bounds will be in terms of $\lfloor \omega \rfloor_{H\otimes H}$ and $\|\omega\|_{H\otimes Z}$. To make good use of these, we will state a useful interpretation of these constants. Firstly, given a sequence of numbers $(b_\ell)_{\ell=1}^\infty$, we may deduce the following bound from the discussion in Section 5.2:

$$\lfloor \omega \rfloor_{H\otimes H}^2 \sum_{\ell=1}^\infty b_\ell^2 \;\le\; \sum_{i,j=1}^\infty \left(\sum_{\ell=1}^\infty \langle \omega(e_i, e_j), f_\ell\rangle_Z b_\ell\right)^2 \;\le\; \|\omega\|_{H\otimes H}^2 \sum_{\ell=1}^\infty b_\ell^2\,.$$

Secondly, consider the trilinear map $\langle \omega(\cdot,\cdot), \cdot\rangle_Z : H \times H \times Z \to \mathbb{R}$. Recall that the assumption $\|\omega\|_{H\otimes Z} < \infty$ implies that this map extends to a bilinear map $H \times H \otimes Z$. Then, given sequences $(a_{i,\ell})_{i,\ell}$ and $(b_j)_j$,

$$\left|\sum_{i,j,\ell=1}^\infty \langle \omega(e_i, e_j), f_\ell\rangle_Z a_{i,\ell} b_j\right| \;\le\; \|\omega\|_{H\otimes Z}\sqrt{\sum_{i,\ell=1}^\infty a_{i,\ell}^2}\sqrt{\sum_{j=1}^\infty b_j^2}\,.$$

If we assumed $\omega : H \times H \to Z$ was Hilbert-Schmidt, then one could write an inequality for triply-indexed sequences $a_{i,j,\ell}$, namely

$$\left|\sum_{i,j,\ell=1}^\infty \langle \omega(e_i, e_j), f_\ell\rangle_Z a_{i,j,\ell}\right| \;\le\; \|\omega\|_{H\otimes H\otimes Z}\sqrt{\sum_{i,j,\ell=1}^\infty a_{i,j,\ell}^2}\,,$$

and this is what is essentially leveraged to derive the bounds in [BGM13]. However, due to allowing $\dim(C) = \infty$, such an inequality is not available to us. Nevertheless, this section is devoted to showing that sufficient bounds still hold.

Lemma 8.1. *For any $F \in C^\infty(G^{P,Q})$,*

$$\lfloor Q\omega \rfloor^2_{PH\otimes PH} \Gamma^Z(F) \leq \sum_{i,j=1}^n \left((0, \widetilde{Q\omega(e_i, e_j)}) F \right)^2 \leq \|Q\omega\|^2_{PH\otimes PH} \Gamma^Z(F) \,.$$

Proof.

$$\begin{aligned} &\sum_{i,j=1}^n \left((0, \widetilde{Q\omega(e_i, e_j)}) F \right)^2 \\ &\qquad = \sum_{i,j=1}^n \Big(\sum_{\ell=1}^m \langle Q\omega(e_i, e_j), f_\ell \rangle_Z \widetilde{(0, f_\ell)} F \Big)^2 \\ &\qquad \geq \lfloor Q\omega \rfloor^2_{PH\otimes PH} \sum_{\ell=1}^m \left(\widetilde{(0, f_\ell)} F \right)^2 = \lfloor Q\omega \rfloor^2_{PH\otimes PH} \Gamma^Z(F) \,. \end{aligned}$$

Thus, we have shown the lower bound. The proof for the upper bound is identical. □

Lemma 8.2. *For any smooth $F \in C^\infty(G^{P,Q})$,*

$$\Gamma_2^Z(F) = \sum_{j=1}^n \sum_{\ell=1}^m \left(\widetilde{(e_j, 0)} \widetilde{(0, f_\ell)} F \right)^2$$

and

$$\sum_{i,j=1}^n \left(\widetilde{(e_i, 0)} Q (0, \widetilde{\omega(e_i, e_j)}) F \right) \left(\widetilde{(e_j, 0)} F \right) \leq \|Q\omega\|_{PH\otimes QZ} \sqrt{\Gamma(F)} \sqrt{\Gamma_2^Z(F)} \,.$$

Proof. The first equality can be derived from the definition of $\Gamma_2^Z(F)$ and the fact that $\widetilde{(h, 0)} \widetilde{(0, z)} F = \widetilde{(0, z)} \widetilde{(h, 0)} F$ for any $h \in H$, $z \in Z$, which is shown in detail in [BGM13].

For the inequality,

$$\begin{aligned} &\sum_{i,j=1}^n \left(\widetilde{(e_i, 0)} (0, \widetilde{Q\omega(e_i, e_j)}) F \right) \left(\widetilde{(e_j, 0)} F \right) \\ &\quad = \sum_{i,j=1}^n \sum_{\ell=1}^m \langle Q\omega(e_i, e_j), f_\ell \rangle_Z \left(\widetilde{(e_i, 0)} \widetilde{(0, f_\ell)} F \right) \left(\widetilde{(e_j, 0)} F \right) \\ &\quad \leq \|Q\omega\|_{PH\otimes QZ} \sqrt{\sum_{j=1}^n \left(\widetilde{(e_j, 0)} F \right)^2} \sqrt{\sum_{j=1}^n \sum_{\ell=1}^m \left(\widetilde{(e_i, 0)} \widetilde{(0, f_\ell)} F \right)^2} \\ &\quad = \|Q\omega\|_{PH\otimes QZ} \sqrt{\Gamma(F)} \sqrt{\Gamma_2^Z(F)} \,. \end{aligned}$$

□

Theorem 8.3. *For any $\nu > 0$ and any $F \in C^\infty(G^{P,Q})$,*

$$\Gamma_2(F) + \nu\Gamma_2^Z(F) \geq \frac{\lfloor Q\omega \rfloor^2_{PH\otimes PH}}{4}\Gamma^Z(f) - \frac{\|Q\omega\|^2_{PH\otimes QZ}}{\nu}\Gamma(F)\,.$$

Proof. As shown in [BGM13],

$$\begin{aligned}\sum_{i,j=1}^n \left(\widetilde{(e_i,0)}\widetilde{(e_j,0)}F\right)^2 &= \sum_{i,j=1}^n \Bigg(\frac{1}{2}\left(\widetilde{(e_i,0)}\widetilde{(e_j,0)}F + \widetilde{(e_j,0)}\widetilde{(e_i,0)}F\right) \\ &\qquad + \frac{1}{2}\left(\widetilde{(e_i,0)}\widetilde{(e_j,0)}F - \widetilde{(e_j,0)}\widetilde{(e_i,0)}F\right)\Bigg)^2 \\ &= \|\nabla^2_{PH}F\|^2 + \frac{1}{4}\sum_{i,j=1}^n \left(\widetilde{(0,Q\omega(e_i,e_j))}F\right)^2,\end{aligned}$$

where

$$\nabla^2_{PH}F := \frac{1}{2}\sum_{i,j=1}^n \widetilde{(e_i,0)}\widetilde{(e_j,0)}F + \widetilde{(e_j,0)}\widetilde{(e_i,0)}F$$

denotes the "symmetrized Hessian." Then by Lemma 8.1,

$$\sum_{i,j=1}^n \left(\widetilde{(e_i,0)}\widetilde{(e_j,0)}F\right)^2 \geq \frac{1}{4}\sum_{i,j=1}^n \left(\widetilde{(0,Q\omega(e_i,e_j))}F\right)^2 \geq \frac{\lfloor Q\omega \rfloor^2_{PH\otimes PH}}{4}\Gamma^Z(F)\,.$$

And by Lemma 8.2,

$$\begin{aligned}\left|\sum_{i,j=1}^n \left(\widetilde{(e_i,0)}\widetilde{(0,Q\omega(e_i,e_j))}F\right)\widetilde{(e_j,0)}F\right| &\leq \|Q\omega\|_{PH\otimes QZ}\sqrt{\Gamma(F)}\sqrt{\Gamma_2^Z(F)} \\ &\leq \frac{\|Q\omega\|^2_{PH\otimes QZ}}{\nu}\Gamma(F) + \nu\Gamma_2^Z(F)\,.\end{aligned}$$

Then

$$\begin{aligned}\Gamma_2(F) &= \frac{1}{2}\Big(\Delta\Gamma(F) - 2\Gamma(F,\Delta F)\Big) \\ &= \sum_{i,j=1}^n \left(\widetilde{(e_i,0)}\widetilde{(e_j,0)}F\right)^2 + \sum_{i,j=1}^n \left(\widetilde{(e_i,0)}\widetilde{(0,Q\omega(e_i,e_j))}F\right)\widetilde{(e_j,0)}F \\ &\geq \frac{\lfloor Q\omega \rfloor^2_{PH\otimes PH}}{4}\Gamma^Z(F) - \frac{\|Q\omega\|^2_{PH\otimes QZ}}{\nu}\Gamma(F) - \nu\Gamma_2^Z(F)\,,\end{aligned}$$

which proves the claim.

□

We also have the following relation:

$$\Gamma(F, \Gamma^Z(F)) = \Gamma^Z(F, \Gamma(F)), \tag{2.7}$$

which is immediately satisfied under our assumptions since every $\widetilde{(h,0)}$ commutes with every differential operator $\widetilde{(0,z)}$. As shown in [BGM13] and described in Theorem 5.1, we have that (2.7) and Theorem 8.3 ultimately allows us to conclude that, for any $P \in \text{Proj}(W)$, $Q \in \text{Proj}(C)$, any $F \in C^\infty(G^{P,Q})$, and any $g \in G^{P,Q}$,

$$\int_{G^{P,Q}} |F(x \cdot_Q g)| d\nu_T^{P,Q}(x) \\ \leq \|F\|_{L^{q'}(G^{P,Q}, \nu_T^{P,Q})} \exp\left(\left(1 + \frac{8\|Q\omega\|^2_{PH\otimes QZ}}{\lfloor Q\omega \rceil^2_{PH\otimes PH}}\right) \frac{(1+q)d^{P,Q}(e,g)^2}{2T} \right). \tag{2.8}$$

8.3 Quasi-invariance

Theorem 8.4. *For all $g \in G_{CM}$, ν_T is quasi-invariant with respect to the measurable extensions of left- and right-multiplication, L_g and $R_g : G \to G$ respectively. And for all $p \in (1, \infty)$, the Radon-Nikodyn derivative satisfies*[7]

$$\left\| \frac{d(\nu_T \circ R_g)}{d\nu_T} \right\|_{L^p(G,\nu_T)} \leq \exp\left(\left(1 + \frac{8\|\omega\|^2_{H\otimes Z}}{\lfloor \omega \rceil^2_{H\otimes H}}\right) \frac{(1+p)d(e,g)^2}{2T} \right),$$

$$\left\| \frac{d(\nu_T \circ L_g)}{d\nu_T} \right\|_{L^p(G,\nu_T)} \leq \exp\left(\left(1 + \frac{8\|\omega\|^2_{H\otimes Z}}{\lfloor \omega \rceil^2_{H\otimes H}}\right) \frac{(1+p)d(e,g)^2}{2T} \right).$$

Proof. Let $g = (h, z) \in G_{CM}$. Then choose any $(P_n)_{n\in\mathbb{N}} \in \text{Proj}(W)^\uparrow$, $(Q_m)_{m\in\mathbb{N}} \in \text{Proj}(C)^\uparrow$. Consider that, for every $m \in \mathbb{N}$, there exists an n_m such that $\text{span}(Q_m\omega(P_{n_m}H \times P_{n_m}H)) = Q_m Z$. This is because, by our Hörmander condition, $\widetilde{\omega}(H \otimes H) = Z$, so $Q_m\widetilde{\omega}(H \otimes H) = Q_m Z$. Since $\bigcup_{n\in\mathbb{N}} P_n H$ is dense in H, we must have $(S_n)_{n\in\mathbb{N}} := \big(Q_m\widetilde{\omega}(P_n H \otimes P_n H)\big)_{n\in\mathbb{N}}$ is an increasing sequence of subspaces whose union is dense in $Q_m Z$. Since $Q_m Z$ is finite-dimensional, we must have $S_{n_m} = Q_m Z$ for some $n_m \in \mathbb{N}$.

Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let f be a bounded continuous cylinder function on G, which implies that $f \circ \iota^{P,Q} \in \mathcal{C}^\infty(G^{P,Q})$ for any $P \in \text{Proj}(W)$, $Q \in \text{Proj}(C)$. Then the finite-dimensional estimate (2.8) tells us that, for any $m \in \mathbb{N}$, $r \geq n_m$, and $n \geq r$, $\pi^{P_r,Q_m} g \in G^{P_n,Q_m}$, and

[7]We provide a final reminder that these bounds will differ from those in [BGM13] due to our definition of ν_T.

$$\int_{G^{P_n,Q_m}} \left|\left(f\circ\iota^{P_n,Q_m}\right)\left(x\cdot_{Q_m}\pi^{P_r,Q_m}g\right)\right|d\nu_T^{P_n,Q_m}(x)$$
$$\leq \|f\circ\iota^{P_n,Q_m}\|_{L^p(G^{P_n,Q_m},\nu_T^{P_n,Q_m})}$$
$$\times\exp\left(\left(1+\frac{8\|Q_m\omega\|^2_{P_nH\otimes Q_mZ}}{\lfloor Q_m\omega\rfloor^2_{P_nH\otimes P_nH}}\right)\frac{(1+q)d^{P_n,Q_m}(e,\pi^{P_r,Q_m}g)^2}{2T}\right). \quad (2.9)$$

We will use (2.9) to arrive at an infinite-dimensional estimate by first taking the limit as $n\to\infty$, then $r\to\infty$, and lastly $m\to\infty$. To take these limits, we apply Theorem 7.5 for $\|f(x\cdot_{Q_m}\pi^{P_r,Q_m}g)\|_{L^{q'}(G^{P_n,Q_m},\nu_T^{P_n,Q_m})}$ and $\|f\|_{L^p(G^{P_n,Q_m},\nu_T^{P_n,Q_m})}$, Theorem 6.15 and Theorem 6.16 for $d^{P_n,Q_m}(e,\pi^{P_r,Q_m}g)$, and Lemma 5.4 for $\lfloor Q_m\omega\rfloor^2_{P_nH\otimes P_nH}$. We also remark that $\|Q_m\omega\|_{P_nH\otimes Q_mZ}\leq\|\omega\|_{H\otimes Z}$, and that our assumptions and Lemma 5.2 guarantee $\lfloor Q_m\omega\rfloor^2_{P_nH\otimes P_nH}>0$. Putting it all together, this results in

$$\int_G \left|f(x\cdot g)\right|d\nu_T(x) \;\leq\; \|f\|_{L^p(G,\nu_T)}\exp\left(\left(1+\frac{8\|\omega\|^2_{H\otimes Z}}{\lfloor\omega\rfloor^2_{H\otimes H}}\right)\frac{(1+q)d(e,g)^2}{2T}\right).$$

The remainder of this argument is identical to the reasoning provided in [BGM13], included here for convenience. Since this holds for all bounded continuous cylinder functions $f:G\to\mathbb{R}$, it must hold for all $f\in L^p(G,\nu_T)$ by density (see, for example, [Dri10, Theorem 39.7]). Then, for $g\in G_{CM}$, the linear functional ϕ_g defined on bounded continuous cylinder functions as

$$\phi_g(f) \;=\; \int_G f(x\cdot g)d\nu_T(x) \;=\; \int_G (f\circ R_g)(x)d\nu_T(x) \;=\; \int_G f(x)d(\nu_T\circ R_{g^{-1}})(x) \quad (2.10)$$

has a continuous extension to $\phi_g: L^p(G,\nu_T)\to\mathbb{R}$, still defined by (2.10), and satisfies the bound

$$\left|\phi_g(f)\right| \;\leq\; \|f\|_{L^p(G,\nu_T)}\exp\left(\left(1+\frac{8\|\omega\|^2_{H\otimes Z}}{\lfloor\omega\rfloor^2_{H\otimes H}}\right)\frac{(1+q)d(e,g)^2}{2T}\right).$$

Then the Riesz representation theorem tells us $L^p(G,\nu_T)^*\cong L^q(G,\nu_T)$, so there exists $J^r_g\in L^q(G,\nu_T)$ such that

$$\phi_g(f) \;=\; \int_G f(x)J^r_g(x)d\nu_T(x)$$

that satisfies

$$\|J^r_g\|_{L^q(G,\nu_T)} \;=\; \|\phi\|_{L^p(G,\nu_T)^*} \;\leq\; \exp\left(\left(1+\frac{8\|\omega\|^2_{H\otimes Z}}{\lfloor\omega\rfloor^2_{H\otimes H}}\right)\frac{(1+q)d(e,g)^2}{2T}\right).$$

Therefore, noting that $d(e,g) = d(e,g^{-1})$, we have that $\frac{d(\nu_T \circ R_g)}{d\nu_T}$ exists, equals J_g^r, and satisfies the supposed L^q bound. We may arrive at the bound for $\frac{d(\nu_T \circ L_g)}{d\nu_T}$ by using nearly identical analysis on expressions resembling $\int_G |f(g \cdot x)| d\nu_T(x)$. Alternatively, one can use right-translation invariance and make the observation that ν_T is invariant under the map $G \ni x \mapsto -x \in G$, and $\int_G f(-(g \cdot x)) d\nu_T(-x) = \int_G f(g^{-1} \cdot x) d\nu_T(x)$.

□

Chapter 3

A Taylor isomorphism theorem

9 Introduction

In this chapter, we will discuss spaces of measurable holomorphic functions on infinite-dimensional that are L^2 with respect to the heat kernel measure. Naturally, given the nature of this result, we remark to the reader that, unlike Chapter 2, one may have to apply "complexification" to most of the foundational concepts, like dual spaces, Hilbert space tensor products, abstract Wiener space, and Lie groups.

9.1 A brief history of Taylor isomorphisms on Lie groups

Below, we paraphrase introductory information provided in [Cec08] and [DGS09a]. Also see [GM96] for more overview. Let $f : \mathbb{C}^n \to \mathbb{C}$ be holomorphic. Then we may reconstruct f from its Taylor coefficients using

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j_1,\ldots,j_k=1}^{\infty} \big(\partial_{z_1} \ldots \partial_{z_k} f\big)(0)\, z_{j_1} \ldots z_{j_k}\,.$$

Let μ_t denote the standard Gaussian measure on $\mathbb{C}$. Then if f is also assumed to be in $L^2(\mathbb{C}^n)$ (with respect to μ_t, then the monomials $\big\{\frac{t^k}{k!} z_{j_1} \ldots z_{j_k}\big\}_{k\in\mathbb{Z},1\le j_m\le n}$ constitute an orthonormal set in $L^2(\mathbb{C}^n)$, from which we may deduce that the series above converges in L^2 and

$$\|f\|^2_{L^2(\mathbb{C}^n)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j_1,\ldots,j_k=1}^{\infty} \Big|\big(\partial_{z_1} \ldots \partial_{z_k} f\big)(0)\Big|^2. \tag{3.1}$$

Next, given any complex (finite-dimensional or infinite-dimensional separable) Hilbert space H, let $T(H) = \bigoplus_{k=0}^{\infty} H^{\otimes k}$ be the tensor algebra on H, and $T(H)'$ its algebraic dual. Then, given a basis $\{e_j\}_{j\in\mathbb{N}} \subseteq H$, we may define a norm on $T(H)'$ as

$$\|\alpha\|^2_{T(H)'_t} := \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j_1,\ldots,j_k=1}^{\infty} \big|\alpha(e_{j_1} \otimes \ldots \otimes e_{j_k})\big|^2,$$

which is independent of basis chosen. Then (3.1) above demonstrates that the "Taylor map" that sends f to the symmetric form $\alpha \in T(H)'$ determined by

$$\alpha(v_1 \otimes \ldots \otimes v_k) \;:=\; \partial_{v_1} \ldots \partial_{v_k} f(0)$$

is an isometry with closed image with respect to the norm $\|\cdot\|_{T_t(H)'}$. This is the essence of the Taylor isomorphisms first introduced by Fock [Foc28], later clarified by Segal [Seg56; Seg62] and Bargmann [Bar67].

Such an isomorphism has been proven to hold in the Lie group context. Given a complex n-dimensional Lie group G with Lie algebra $\mathfrak{g}$, a Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ with complex orthonormal set $\{e_j\}_{1 \le j \le n}$ naturally corresponds to a real inner product under which $\{e_j, ie_j\}_{1 \le j \le n}$ is an orthonormal set, which determines a heat kernel distribution on G. By considering holomorphic functions on G and replacing the linear derivatives ∂_v with the left-invariant derivatives $\widetilde{v} f(g) := f(L_{g*} v) = \frac{d}{dt}\big|_{t=0} f(g \cdot tv)$, one can arrive at a similar isomorphism. There were a series of papers that dealt with increasingly general assumptions, starting with [Hal94] for compact complex Lie groups, eventually culminating in [DG97] for general complex Lie groups with elliptic heat kernels. It wasn't until [DGS09a] that this was done for the subelliptic heat kernels, which removed the positive-definiteness from the Hermitian inner product. In this setting, one requires the Hörmander condition to be satisfied; see Section 9.4.1 for further discussion. A precise restatement of results from [DGS09a] for this context will be provided in Section 11.1.3. Related is [DGS09b], in which an alternate proof of the surjectivity of the Taylor map is shown, which remarkably extends to infinite dimensions as shown in Theorem 11.9.

Taylor isomorphisms of this nature have also been proven in a number of infinite-dimensional group settings, including [Gor00a; Gor00b; Gor02; DG10] (all nondegenerate). There are 2 papers that discuss infinite-dimensional results that are most relevant to our context. The first is [Cec08], which proved a Taylor isomorphism for the path space of a simply connected graded complex nilpotent Lie group, but again for nondegenerate heat kernels. The second is [GM13], which, to the author's knowledge, is the only piece that shows a Taylor isomorphism for the subelliptic infinite-dimensional setting aside from this work. However, it restricted itself to Heisenberg-like groups, meaning step-2 nilpotent, and furthermore was "semi-infinite" in the sense that the center was assumed finite-dimensional.

This work completes the train of thought. Many of these methods can be adjusted to elliptic settings, and path space will be a single example in our setting, so this work can be viewed of as an extension of [Cec08]. On the other hand, one may consider this work as taking [GM13] and simultaneously accounting for higher-step and infinitely-many "vertical directions." Therefore, our setup can be seen as encompassing both of these settings. The road to achieving the Taylor isomorphism is paved with substantial original ideas and proofs.

9.2 Overview of assumptions

We present here an overview of our setup, with the details present in Section 10. Similarly to Chapter 2, in order to describe the Taylor isomorphism, we must introduce 3 spaces:

1. G, an *abstract Wiener nilpotent Lie group* with heat kernel distributed element g_t, satisfying (A3.3).

2. $\mathfrak{g}_{CM}$, the *Cameron-Martin subalgebra*, with Lie bracket $[\cdot, \cdot]$, satisfying (A3.1) and (A3.2).

3. G_{CM}, the *Cameron-Martin subgroup*, with right-invariant metric d.

We first introduce $\mathfrak{g}_{CM} = H_1 \oplus \ldots \oplus H_N$, a complex Hilbert space and graded nilpotent Lie algebra with Lie bracket $[\cdot, \cdot] : \mathfrak{g}_{CM} \times \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$. The graded structure imposes that $[H_m, H_n] \subseteq H_{m+n}$ (where, for convenience, we define $H_n = 0$ for $n > N$). We also require the following technical assumption: for every $1 \le n < N$, where $\{e_{n,j}\}_{j \in \Lambda_n}$ is a basis of H_n,

$$\sup_{\|h\|_{H_{n+1}}=1} \sum_{j \in \Lambda_1, k \in \Lambda_n} \left| \langle [e_{1,j}, e_{n,k}], h \rangle_{H_{n+1}} \right|^2 \;<\; \infty \tag{A3.1}$$

$$\inf_{\|h\|_{H_{n+1}}=1} \sum_{j \in \Lambda_1, k \in \Lambda_n} \left| \langle [e_{1,j}, e_{n,k}], h \rangle_{H_{n+1}} \right|^2 \;>\; 0\,. \tag{A3.2}$$

These assumptions have interesting consequences; see Section 9.4.1 for further discussion.

The bracket gives rise to a group operation $\cdot : \mathfrak{g}_{CM} \times \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$, defined by the Baker-Campbell-Hausdorff formula (which is, in this case, a finite sum, see Section 2.3). In this way, we can regard $\mathfrak{g}_{CM}$ as a graded nilpotent Lie group, sometimes called $\exp(\mathfrak{g}_{CM})$ when emphasizing the group structure. For $g \in \exp(\mathfrak{g}_{CM})$, let $L_g : \exp(\mathfrak{g}_{CM}) \to \exp(\mathfrak{g}_{CM})$ denote the left multiplication map $x \mapsto g \cdot x$, which naturally has as derivative $L_{g*} : \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$ (as in Section 2.3.2). Let $\mathcal{AC}$ denote the set of absolutely continuous paths $\sigma : [0,1] \to \exp(\mathfrak{g}_{CM})$, on which we may define the length $\ell(\sigma) = \int_0^1 \|L_{\sigma(t)^{-1}*}\sigma'(t)\|_{\mathfrak{g}_{CM}} dt$. We say σ is *horizontal* if $L_{\sigma(t)^{-1}*}\sigma'(t) \in H_1 = H_1 \times 0 \times \ldots \times 0$ for all $t \in [0,1]$, and we denote the set of such paths as $\mathcal{AC}_h$. Then for any $h \in \exp(\mathfrak{g}_{CM})$, we define the *horizontal distance* from the origin as

$$d(e,g) \;=\; \inf\left\{ \ell(\sigma) \;\middle|\; \sigma \in \mathcal{AC}_h\,,\ \sigma(0) = e\,,\ \sigma(1) = g \right\}.$$

Then we define $G_{CM} = \{g \in \exp(\mathfrak{g}_{CM}) \mid d(e,g) < \infty\}$.

We now define the finite-dimensional subgroups that will serve as approximations to G. For a finite-rank projection $P : \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$, define

$$G^P = PH_1 \oplus \text{span}(([PH_1, PH_1]) \oplus \ldots \oplus \text{span}([PH_1, \ldots, [PH_1, PH_1] \ldots]) \subseteq G_{CM} .$$

This will be a group/algebra under the restricted multiplication/bracket, and by definition will satisfy the Hörmander condition. We will naturally identify $\mathfrak{g}^P = G^P$, and denote the canonical inclusion map as $\iota^P : G^P \to G$.

Finally, we let $G = W_1 \times \ldots \times W_N$ be an *infinite-dimensional complex graded nilpotent abstract Wiener Lie group*, which is an example of the notion that is introduced in Section 3. First, we assume that each H_n is a dense subset of W_n, and that (W_1, H_1) constitutes an abstract Wiener space. Given a Brownian motion $(B_t)_{t\geq 0}$ on (W_1, H_1), we may consider G^P-valued Brownian motion $(g_t^P)_{t\geq 0}$ as the solution to the Stratonovich stochastic differential equation

$$\delta g_t^P = L_{g_t^P *} \delta P B_t \qquad g_0^P = e .$$

Then the primary assumption on G is the following.

For some $t > 0$, there exists a G-valued random variable g_t such that, for every $f \in G^*$, there exists an increasing sequence of finite-rank projections $\{P_m\}_{m\in\mathbb{N}} \in \text{Proj}(W)^\uparrow$ such that $f(g_t^{P_m}) \to f(g_t)$ in probability. (A3.3)

The distribution of g_t is called the *heat kernel distribution*). We will discuss this condition and demonstrate some conclusions in Section 10.3, and we will provide examples in Section 10.4.2. Note that G will indeed satisfy the definition in Section 3 for a nilpotent abstract Wiener Lie group, using $X = \mathfrak{g}_{CM}$ and $X_H = H_1$.

9.3 Statement of theorems

Similar to other infinite-dimensional Taylor isomorphisms, ours spans 3 different function spaces, which will now be described.

Let $\mathcal{P}$ denote the set of continuous holomorphic cylinder polynomials $G \to \mathbb{C}$, which are linear combinations of monomials of the form

$$f_1(\cdot) \ldots f_k(\cdot)$$

for some (complex-linear) $f_1, \ldots, f_k \in G^*$. Then define $\mathcal{H}L_t^2(G)$ as the closure of $\mathcal{P}$ with respect to $L^2(G)$, equipped with the L^2 inner product. Though, as remarked in Section 11.1, these functions are not, strictly speaking, holomorphic.

Our second space will be denoted $\mathcal{H}L_t^2(G_{CM})$, which can roughly be thought of as the "L^2 holomorphic functions on G_{CM}." However, this exact space will not be defined until Section 11.5.3. Standing in its place for most of this work will be $\mathcal{H}L^2_t(\bigcup_P G^P)$, the set of functions $f : \bigcup_P G^P \to \mathbb{C}$ for which $f \circ \iota^P : G^P \to \mathbb{C}$ is in $\mathcal{H}L^2_t(G^P)$, the L^2 holomorphic functions on G^P, under the norm

$$\sup_{P \in Proj(W_1)} \|f \circ \iota^P\|_{\mathcal{H}L^2_t(G^P)} .$$

In the event that $\bigcup_P G^P = G_{CM}$, there is no distinction between $\mathcal{H}L^2_t(\bigcup_P G^P)$ and $\mathcal{H}L^2_t(G_{CM})$. Otherwise, it will be proven that every function in $\mathcal{H}L^2_t(\bigcup_P G^P)$ has a unique natural extension to G_{CM}, defined in Theorem 11.23, where the set of such extensions will constitute $\mathcal{H}L^2_t(G_{CM})$.

Our third and final space will be an infinite-dimensional noncommutative Fock space. Starting with the tensor algebra $T(\mathfrak{g}_{CM})$ with algebraic dual $T(\mathfrak{g}_{CM})'$, let $J(\mathfrak{g}_{CM})$ the 2-sided ideal of $T(\mathfrak{g}_{CM})$ generated by $v \otimes w - w \otimes v - [v, w]$. Then let $J(\mathfrak{g}_{CM})^0$ be the backwards anihilator of $J(\mathfrak{g}_{CM})$, that is,

$$J^0(\mathfrak{g}_{CM}) \;=\; \{\alpha \in T(\mathfrak{g}_{CM})' \;:\; \langle \alpha, v\rangle = 0 \,\forall v \in J(\mathfrak{g}_{CM})\} .$$

Then, for $\alpha \in J^0(\mathfrak{g}_{CM})$, and a basis $\{e_j\}_{j\in\mathbb{N}} \subseteq H_1$, define the norm

$$\|\alpha\|_{J^0_t(\mathfrak{g}_{CM})} \;=\; \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j_1,\ldots,j_k=1}^{\infty} |\langle \alpha, e_{j_1} \otimes \ldots \otimes e_{j_k}\rangle|^2 .$$

Then we may define $J^0_t(\mathfrak{g}_{CM}) = \{\alpha \in J^0(\mathfrak{g}_{CM}) \;:\; \|\alpha\|_{J^0_t(\mathfrak{g}_{CM})} < \infty\}$.

The subelliptic Taylor isomorphism, which is stated in its final form in Theorem 11.23, will take the form of the composition of the maps $\mathcal{R}$ and $\mathcal{T}$, illustrated below.

$$\mathcal{H}L^2_t(G) \xrightarrow{\;\mathcal{R}\;} \mathcal{H}L^2_t(G_{CM}) \xrightarrow{\;\mathcal{T}\;} J^0_t(\mathfrak{g}_{CM}) .$$

The first map, $\mathcal{R}$, will be an extension of restriction, $\mathcal{P} \ni f \mapsto f|_{G_{CM}} \in \mathcal{H}L^2_t(G_{CM})$. The second map, $\mathcal{T}$, will be an extension of the Taylor map, so that $\mathcal{T}f(h_1 \otimes \ldots \otimes h_k) = \widetilde{h}_1 \ldots \widetilde{h}_k f(e)$. Both of these maps will be isometric isomorphisms. Furthermore, the composition $\mathcal{T} \circ \mathcal{R}$ is unitary.

There are other facts that will be proven along the way. For example, in Section 11.5, we will provide expressions for inverting the maps $\mathcal{R}$ and $\mathcal{T}$, including a way in which we have Taylor expansions for functions in $\mathcal{H}L^2_t(G)$ and $\mathcal{H}L^2_t(G_{CM})$.

9.4 Further discussion

9.4.1 The Hörmander condition

Recall from Section 2.3.3 that, given a (real finite-dimensional) Lie group G and a heat kernel with corresponding real inner-product q, and an associated real basis

$\{x_j\}_{1\le j\le n} \subseteq \mathfrak{g}$ orthonormal with respect to q, the Hörmander condition is satisfied when

$$\text{span}\{x_{j_1}\ ,\ [x_{j_1}, x_{j_2}]\ ,\ [[x_{j_1}, x_{j_2}], x_{j_3}]\ ,\ \ldots\}_{j_1,\ldots,j_n=1}^n \ =\ \mathfrak{g}\,,$$

and a result by Hörmander [Hör67] implies that the corresponding heat kernel density is strictly positive and smooth.

Throughout this work, as stated above, we make use of a generalized Hörmander condition (referred to as *our Hörmander condition*). As explained in Section 10.1 (using methods in Proposition 2.4), (A3.1) implies that the restrictions of the Lie bracket $[\cdot,\cdot]|_{H_1\times H_n} : H_1 \times H_n \to H_{n+1}$ have continuous extensions to the Hilbert space tensor product $[\cdot]|_{H_1\otimes H_n} : H_1 \otimes H_n \to H_{n+1}$, and (A3.2) implies that these extensions are surjective. In Theorem 10.1, it will be proven that this is equivalent to the surjectivity of the extensions of the compositions $[\ldots[\cdot,\cdot],\ldots,\cdot] : H_1^n \to H_n$ to $H_1^{\otimes n} \mapsto H_n$. This certainly resembles the Hörmander condition in finite-dimensions, as it is consistent with the idea that H_1 generates $\mathfrak{g}_{CM}$ as a Lie algebra. It will be seen that this condition serves a similar role to that the Hörmander condition played in [DGS09a] and [GM13], like showing that $\|\cdot\|_{J_t^0(\mathfrak{g}_{CM})}$ is a genuine norm.

9.4.2 The Cameron-Martin subalgebra and subgroup

The comments regarding $\mathfrak{g}_{CM}$ and G_{CM} made in Section 4.2.2 can be once again made here. Our Hörmander condition allows us to recognize that $\mathfrak{g}_{CM}$ is the Lie algebra generated by H_1, the Cameron-Martin subspace of W_1. Meanwhile, the generalized Baker-Campbell-Hausdorff formula (Theorem 2.6, or [Str87]) suggests that G_{CM} can be thought of as the group generated by H_1. But, in general, $\mathfrak{g}_{CM} \neq G_{CM}$ (see Example 6.10). However, there will be some discussion in which we regard elements in $\mathfrak{g}_{CM}$ as corresponding to left-invariant vector fields on G_{CM}, see Section 11.3. Of course, the Taylor map itself presents a connection between these spaces by showing that $\mathcal{H}L_t^2(G_{CM})$ and $J_t^0(\mathfrak{g}_{CM})$ are isomorphic.

9.4.3 Further directions

One could consider a "partially degenerate" diffusion, where the diffusion is partially generated by some directions on higher-step strata. Also, with the exception of surjectivity, many of these results should hold if the graded structure is removed, though it may take some reworking to prove theorems regarding horizontal distance approximations. To mimic the argument presented here, it would be critical to show that the weak-limit of horizontal paths is again horizontal, though this could perhaps be shown by justifying that horizontal paths correspond to paths whose line integral against a "vertical-valued vector field" is always 0, and showing this criterion is closed under weak limits.

Removing the nilpotency assumption opens many more questions. In addition to requiring new approaches to defining the heat kernel measure on G and taming an even less well-behaved horizontal distance on G_{CM}, one will have to propose an even more general infinite-dimensional Hörmander condition for $\mathfrak{g}_{CM}$. One possibility is the following: there exist constants $0 < c \leq C < \infty$ such that, given an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of H_1, for any $h \in \mathfrak{g}_{CM}$,

$$c\|h\|^2_{\mathfrak{g}_{CM}} \;\leq\; \sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=1}^{\infty} \left| \big\langle [e_{j_1}, \ldots, [e_{j_{n-1}}, e_{j_n}] \ldots], h \big\rangle_{\mathfrak{g}_{CM}} \right|^2 \;\leq\; C\|h\|^2_{\mathfrak{g}_{CM}}$$

This is equivalent to assuming that the continuous linear map between Hilbert spaces

$$\begin{array}{rcl} \widetilde{\bigoplus}_{n=1}^{\infty} H_1^{\otimes n} & \longrightarrow & \mathfrak{g}_{CM} \\ (h_1\ ,\ h_2 \otimes h_3\ ,\ h_4 \otimes h_5 \otimes h_6, \ldots) & \longmapsto & h_1 + [h_2, h_3] + [h_4, [h_5, h_6]] + \ldots \end{array}$$

is continuous and surjective, where $\widetilde{\bigoplus}_{n=1}^{\infty} H_1^{\otimes n}$ denotes the Hilbert space direct sum, or ℓ^2 sum, of the $H_1^{\otimes n}$s[1].

[1]The sum $\widetilde{\bigoplus}_{n=1}^{\infty} H_1^{\otimes n}$ is necessarily a Hilbert space, in contrast to our definition of the tensor algebra $T(H_1) = \bigoplus_{n=0}^{\infty} H_1^{\otimes n}$ used in this work, which uses an algebraic direct sum, as described in Section 11.1.

10 Brackets, Geometry, Stochastics

This section is devoted to listing our assumptions and providing examples, providing far more details than those in Section 9.2. Firstly, in Section 10.1, we introduce the Cameron-Martin subalgebra $\mathfrak{g}_{CM}$, on which the Lie bracket has structured assumptions, along with discussion of our Hörmander condition. Secondly, in Section 10.2, we will define G_{CM}, the Cameron-Martin subgroup, which carries the geometric information of our setup. Lastly, in Section 10.3, we will describe G the ambient space on which our stochastic process will exist.

10.1 Brackets: the Cameron-Martin Subalgebra

We let $\mathfrak{g}_{CM}$ be a complex Hilbert space with Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}_{CM}}$, and is also a complex nilpotent Lie algebra, meaning that it has a nilpotent Lie bracket $[\cdot, \cdot] : \mathfrak{g}_{CM} \times \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$, as defined in Section 2.3 that is complex bilinear.

We assume that $\mathfrak{g}_{CM}$ is *graded*, meaning that it has an orthogonal decomposition $\mathfrak{g}_{CM} = H_1 \oplus \ldots \oplus H_N$ in which $[H_m, H_n] \subseteq H_{m+n}$ (we say $H_n = 0$ for $n > N$). We denote the restrictions of $[\cdot, \cdot]$ as $[\cdot, \cdot]_{m,n} := [\cdot, \cdot]|_{H_m \times H_n} : H_m \times H_n \to H_{m+n}$, and will sometimes refer to them as (m, n) brackets. For $a = (a_1, \ldots, a_N), b = (b_1, \ldots, b_N) \in \mathfrak{g}_{CM}$, we may compile the (m, n) brackets into the full bracket $[\cdot, \cdot]$ as

$$[a, b] \;=\; \sum_{m,n=1}^{N} [a_m, b_n]_{m,n} \,.$$

The rest of this section will prove facts and equivalences of certain conditions on $[\cdot, \cdot]$. We will summarize our full set of assumptions of $[\cdot, \cdot]$ at the end of this section. Of critical relevance is the notion of a multilinear map being *weakly Hilbert-Schmidt*, as defined in Proposition 2.4. The extension of a bracket $[\cdot, \cdot]$ to the tensor product will sometimes simply be denoted as $[\cdot]$.

Theorem 10.1. *Assuming that the $(1, n)$-brackets $[\cdot, \cdot]_{1,n} : H_1 \times H_n \to H_{n+1}$ are weakly Hilbert-Schmidt (and thus have an extension to $H_1 \otimes H_n$) for all $1 \le n \le N$, then*

1. *the iterated $(1, n)$-brackets $[\cdot, \ldots, [\cdot, \cdot]_{1,1} \ldots]_{1,n-1} : H_1 \times \ldots \times H_1 \to H_n$ are weakly Hilbert-Schmidt.*

2. *all the $(1, n)$-brackets have surjective extensions if and only if all the iterated $(1, n)$-brackets have surjective extensions.*

Proof. If $B : K_1 \times K_2 \to K_3$ and $C : K_4 \times K_3 \to K_5$ are weakly Hilbert-Schmidt bilinear maps on Hilbert spaces, then consider the composition of maps determined by

$$K_4 \otimes K_1 \otimes K_2 \ni a \otimes b \otimes c \ \mapsto \ a \otimes B(b,c) \ \mapsto \ C(a, B(b,c)) \in K_5\,,$$

from which we see that, using $\widetilde{\cdot}$ to denote extensions to tensor products, $\widetilde{C(\cdot, B(\cdot,\cdot))} = \widetilde{C} \circ (I \otimes \widetilde{B})$. Then this composition will be continuous on the tensor product $K_4 \otimes K_1 \otimes K_2$.

Then observe that the extension of $[\cdot, \dots, [\cdot,\cdot]_{1,1} \dots]_{1,n-1}$ to $H_1^{\otimes n}$ can be realized as

$$[\cdot]_{1,n-1} \circ (I \otimes [\cdot]_{1,n-2}) \circ \dots \circ (I \otimes \dots \otimes I \otimes [\cdot]_{1,1})\,.$$

We note that a function f is surjective if and only if $I \otimes f$ is surjective. This fact, along with induction, will justify both points.

□

Theorem 10.2. *Assuming that the iterated 1-brackets* $[\cdot, \dots, [\cdot,\cdot]_{1,1} \dots]_{1,n-1} : H_1 \times \dots \times H_1 \to H_n$ *are weakly Hilbert-Schmidt with surjective extension for all* $2 \le n \le N$*, then we have*

1. *all* (m,n)*-brackets* $[\cdot,\cdot]_{m,n} : H_m \times H_n \to H_{n+m}$ *(and iterations thereof) are weakly Hilbert-Schmidt.*
2. *the compiled bracket* $[\cdot,\cdot] : H \times H \to H$ *(and iterations thereof) is weakly Hilbert-Schmidt.*

Proof. Note that the previous proof justifies that all types of iterated brackets will be weakly Hilbert-Schmidt, provided that we prove the (m,n) brackets and compiled bracket are weakly Hilbert-Schmidt.

First consider that, since the bracket $[\cdot,\cdot]$ is assumed to satisfy anti-symmetry and the Jacobi property, then its extension $[\cdot]$ to $\mathfrak{g}_{CM} \otimes \mathfrak{g}_{CM}$ must satisfy corresponding properties. With this in mind, for $a \in H_m, b \in H_n$, if $\left[\sum_{j=1}^\infty c_j \otimes d_j\right]_{1,m-1} = a$, then by the Jacobi identity,

$$\begin{aligned}
[a,b]_{m,n} &= \left[\left[\sum_{j=1}^\infty c_j \otimes d_j\right]_{1,m-1}, b\right]_{m,n} \\
&= -\sum_{j=1}^\infty \left[[d_j, b]_{m-1,n}, c_j\right]_{m+n-1,1} - \sum_{j=1}^\infty \left[[b, c_j]_{n,1}, d_j\right]_{n+1,m-1} \\
&= \sum_{j=1}^\infty \left[c_j, [d_j, b]_{m-1,n}\right]_{1,m+n-1} - \sum_{j=1}^\infty \left[d_j, [c_j, b]_{1,n}\right]_{m-1,n+1},
\end{aligned}$$

which can be realized as a sum of a composition of brackets of lower index. Indeed, set $R : H_1 \otimes H_{m-1} \otimes H_n \to H_{m-1} \otimes H_1 \otimes H_n$ to be the canonical isomorphism, and let $\phi_n : H_n \to H_1 \otimes H_{n-1}$ be a one-sided inverse of $[\cdot]_{1,n-1}$ (so that $[\cdot]_{1,n-1} \circ \phi_n = I_{H_n}$, existence is guaranteed by the assumed surjectivity; we provide more details for similar maps in the proof of Theorem 11.2). Then taking $\sum_j c_j \otimes d_j = \phi_n(a)$, the equation above means

$$[\cdot,\cdot]_{m,n} = [\cdot]_{1,m+n-1} \circ (I \otimes [\cdot]_{m-1,n}) \circ \big(\phi_n \otimes I\big) \\ - [\cdot]_{m-1,n+1} \circ (I \otimes [\cdot]_{1,n}) \circ R \circ \big(\phi_n \otimes I\big)\,.$$

So ultimately, $[\cdot,\cdot]_{1,n}$ and $[\cdot,\cdot]_{m-1,n}$ being weakly Hilbert-Schmidt for all n implies that $[\cdot,\cdot]_{m,n}$ is weakly Hilbert-Schmidt for all n, so the first point follows by induction. The second point follows by the fact that the compiled bracket is a sum of brackets of the form $[\cdot,\cdot]_{m,n}$.

□

For the remainder of this paper, we will assume that $(\mathfrak{g}_{CM}, [\cdot,\cdot])$ is a complex graded nilpotent Lie algebra, and that the $(1,n)$-brackets $[\cdot,\cdot]_{1,n}$ satisfy the following assumptions: for all $1 \leq n < N$, we assume $\{e_{n,j}\}_{j\in\Lambda_n}$ is a basis for H_n, and

$$\sup_{\|h\|_{H_{n+1}}=1} \sum_{j\in\Lambda_1, k\in\Lambda_n} \left|\langle [e_{1,j}, e_{n,k}]_{1,n}, h\rangle_{H_{n+1}}\right|^2 < \infty \tag{A3.1}$$

$$\inf_{\|h\|_{H_{n+1}}=1} \sum_{j\in\Lambda_1, k\in\Lambda_n} \left|\langle [e_{1,j}, e_{n,k}]_{1,n}, h\rangle_{H_{n+1}}\right|^2 > 0\,. \tag{A3.2}$$

Using results from Proposition 2.4, we see that (A3.1) is equivalent to the $(1,n)$ brackets being weakly Hilbert-Schmidt, and (A3.2) tells us that the extension to the tensor product is surjective onto H_{n+1}. The results in this section demonstrate that this is equivalent to assuming that every iterated $(1,n)$-bracket $[\cdot,\ldots,[\cdot,\cdot]_{1,1},\ldots]_{1,n}$ is weakly Hilbert-Schmidt with surjective extension onto H_{n+1}, or equivalently that the iterated $(1,n)$ brackets have similar finite, nonzero constants.

10.2 Geometry: group structures and the Cameron-Martin subgroup

One goal of this section is to describe the different group structures the lie in $\mathfrak{g}_{CM}$, including $\exp(\mathfrak{g}_{CM})$ and finite-dimensional approximations $\exp(\mathfrak{g}^P) = G^P$. Most of this section will be dedicated to defining and describing G_{CM}, which satisfies $G^P \subseteq G_{CM} \subseteq \exp(\mathfrak{g}_{CM})$, and is the critical geometric object of study in this work. Importantly, in this section, we will prove Theorem 10.11 , which shows that the geometric structure of G_{CM} is well-approximated by the finite-dimensional subgroups G^P.

Many of the definitions, notions, and theorems will be similar to those in Section 6, but both the increased step and the nature of the result we must prove requires different methods. Thus, we will not rely on Section 6, and we instead provide Section 10.2 as an essentially self-contained section.

10.2.1 Introducing group structures

First off, recall from Section 2.3 that we may view the Lie algebra $\mathfrak{g}_{CM}$ as being equal to its own corresponding simply connected Lie group $\exp(\mathfrak{g}_{CM})$ by defining a group operation through the Baker-Campbell-Hausdorff formula, (1.2).

Let $Proj(H_1)$ denote the set of finite-rank projections $P : H_1 \to H_1$. Then for $P \in Proj(H_1)$, we define

$$\mathfrak{g}^P = PH_1 \oplus \operatorname{span}([PH_1, PH_1]) \oplus \ldots \oplus \operatorname{span}([PH_1, \ldots, [PH_1, PH_1] \ldots]) .$$

In other words, $\mathfrak{g}^P$ is the Lie algebra generated by PH_1. This too can be realized as a simply connected nilpotent Lie group, written as $\exp(\mathfrak{g}^P)$, or more often G^P, to emphasize the group structure. By definition, this group will satisfy the Hörmander condition, as mentioned in Section 9.4.1.

Let $\mathcal{C}^1 = \mathcal{C}^1([0,1], \exp(\mathfrak{g}_{CM}))$ donote the set of continuously differentiable paths in $\exp(\mathfrak{g}_{CM}) = \mathfrak{g}_{CM}$, on which we may define the length $\ell(\sigma) = \int_0^1 \|L_{\sigma(t)^{-1}*}\sigma'(t)\|_{\mathfrak{g}_{CM}} dt$. We say σ is *horizontal* (or "admissible" as it appears in sub-Riemannian geometry literature) if $L_{\sigma(t)^{-1}*}\sigma'(t) \in H_1 = H_1 \times 0 \times \ldots \times 0$ for all $t \in [0,1]$, and we denote the set of horizontal $\mathcal{C}^1$ paths as $\mathcal{C}^1_h$.

For $g_1, g_2 \in \mathfrak{g}_{CM}$, we define the horizontal distance as

$$d(g_1, g_2) \;=\; \inf\left\{\ell(\sigma) \mid \sigma \in \mathcal{C}^1_h,\ \sigma(0) = g_1,\ \sigma(1) = g_2\right\}.$$

Then we define G_{CM} as

$$G_{CM} \;:=\; \{g \in \mathfrak{g}_{CM} \;:\; d(e, g) < \infty\}.$$

Then we have

Proposition 10.3.

1. *We let* $\delta_\alpha : G \to G$ *denote the natural dilation on* $\exp(\mathfrak{g}_{CM})$*, so for* $g = (g_1, \ldots, g_N) \in \exp(\mathfrak{g}_{CM})$*,* $\delta_\alpha(g) = (\alpha g_1, \alpha^2 g_2, \ldots, \alpha^N g_N)$*. Then, for any* $g = (g_1, \ldots, g_N) \in G_{CM}$*,* $\alpha \in \mathbb{C}$*,*

$$d(e, \delta_\alpha g) \;=\; d(e, (\alpha g_1, \alpha^2 g_2, \ldots, \alpha^N g_N)) \;=\; |\alpha|\, d(e, g)\,.$$

2. *For all* $g_1, g_2, a \in G_{CM}$*,* $d(g_1, g_2) = d(g_1 \cdot a, g_2 \cdot a)$*.*

3. d *is a metric on* G_{CM}*.*

Proof (sketch). A curve is horizontal if and only if

$$L_{\sigma(t)^{-1}*}\sigma'(t) \;=\; \frac{d}{ds}\bigg|_{s=0}\Big(-\sigma(t) \cdot (s\,\sigma'(t))\Big) \;\in\; H_1\,.$$

Then Proposition 10.3 will follow from the fact that many operations on horizontal curves will still be horizontal. For the first point, we may pointwise-apply δ_α to produce a horizontal curve whose length is $|\alpha|$-times longer. For the second, we may pointwise multiply a horizontal curve by an element in G_{CM} to produce one of equal length. We may also "reverse" a horizontal path and concatenate 2 horizontal paths (perhaps after a $\mathcal{C}^1$ reparametrization, if necessary) to deduce the symmetric and transitive properties of d. The reader may review the proof of Proposition 6.1 for more explicit calculations of this sort.

□

Define d^P as the horizontal distance on G^P, meaning

$$d^P(h_1, h_2) \;=\; \inf\{\ell(\sigma) \mid \sigma[0,1] \to G^P \text{ horizontal, } \sigma(0) = h_1,\ \sigma(1) = h_2\}\,.$$

That d^P is always finite on G^P will follow by the references above. However, using the graded Lie group structure, we will show that this holds with "bare hands."

Proposition 10.4. *For all* $P \in Proj(W_1)$*,* d^P *is finite on* G^P*. In particular, we have* $G^P \subseteq G_{CM}$*.*

Proof (sketch). First note that elements of the form $h = (h, 0, \ldots, 0) \in PH_1 \times 0 \times \ldots \times 0$ have $d^P(e, h) = \|h\|_{H_1}$ by using the horizontal straight-line path from the origin, that is, $\sigma(t) = th$. We also note that, using the properties in Proposition 10.3, for all $g_1, g_2 \in \mathfrak{g}_{CM}$, $d^P(e, g_2 \cdot g_1) \le d^P(e, g_1) + d^P(g_1, g_2 \cdot g_1) = d^P(e, g_1) + d^P(e, g_2)$.

Thus, the proof reduces to showing that all elements in G^P can be written as a finite product of elements in PH_1. This is ultimately a consequence of PH_1 generating G^P as a Lie algebra.

Indeed, let $h = (h_1, \ldots, h_N)$. Suppose that we have a finite product $g_1 \cdot \ldots \cdot g_k = (h_1, \ldots, h_{n-1}, z_n, \ldots, z_N) \in G^P$, for some $z_j \in H_j$ (we see from the preceding paragraph that this is true for $n = 2$). Then, by the definition of G^P, $h_n - z_n \in \operatorname{span}([\ldots[PH_1, PH_1], \ldots, PH_1])$, so there exists a finite collection $\{a_{m,j}\}_{m\in\Lambda, 1\leq j\leq n} \subseteq PH_1$ where $h_n - z_n = \sum_{m\in\Lambda}[\ldots[a_{m,1}, a_{m,2}], \ldots, a_{m,n}]$. Then $a_{m,1} \cdot a_{m,2} \cdot a_{m,1}^{-1} \cdot a_{m,2}^{-1} = (0, [a_{m_1}, a_{m_2}], \ldots)$. Similarly, we may take a product in $\{a_{m,j}, a_{m,j}^{-1}\}_{m\in\Lambda, 1\leq j\leq n}$ to produce an element of the form $g = (0, \ldots, 0, h_n - z_n, \ldots)$. Then $g_1 \cdot \ldots \cdot g_k \cdot g = (h_1, \ldots, h_{n-1}, h_n, z'_{n+1}, \ldots, z'_N)$ is also a product of elements in PH_1, for some $z'_j \in H_j$. We may iteratively repeat this procedure to produce a product in PH_1 equal to our initial h.

□

We can immediately deduce that the inclusions $(G^P, d^P) \hookrightarrow (G_{CM}, d)$ are continuous, since $d(h_1, h_2) \leq d^P(h_1, h_2)$.

10.2.2 Weakly Hilbert-Schmidt integral maps

A few of the results and proofs will closely resemble those in Section 6, but we will repeat them here for convenience. We define the "integral" map $\mathcal{I} : L^2([0,1], \mathbb{C}) \to L^2([0,1], \mathbb{C})$ as $\mathcal{I}f(t) = \int_0^t f(s)ds$. Then $\mathcal{I}$ indeed maps to $L^2([0,1], \mathbb{C})$ and is bounded linear (and in fact Hilbert-Schmidt).

Lemma 10.5. *The bilinear map $\mathcal{Z} : L^2([0,1], \mathbb{C}) \times L^2([0,1], \mathbb{C}) \to L^2([0,1], \mathbb{C})$ defined as $\mathcal{Z}(A_1, A_2)(t) = \mathcal{I}A_1(t)A_2(t)$ is weakly Hilbert-Schmidt.*

Proof. Let $f \in L^2([0,1], H)$. Then use $\{e^{2\pi ikt}\}_{k\in\mathbb{Z}}$ as an orthonormal basis of $L^2([0,1], \mathbb{C})$. Then

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\left|\left\langle f(t),\mathcal{I}(e^{2\pi ikt})\,e^{2\pi i\ell t}\right\rangle_{L^2([0,1],\mathbb{C})}\right|^2\\
&\qquad= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\left|\left\langle f(t)\overline{\mathcal{I}(e^{2\pi ikt})},e^{2\pi i\ell t}\right\rangle_{L^2([0,1],\mathbb{C})}\right|^2\\
&\qquad= \sum_{k=-\infty}^{\infty}\left\|f(t)\overline{\mathcal{I}(e^{2\pi ikt})}\right\|^2_{L^2([0,1],\mathbb{C})}\\
&\qquad= \sum_{k\neq 0}\frac{1}{k^2}\left\|f(t)(e^{-2\pi ikt}-1)\right\|^2_{L^2([0,1],\mathbb{C})}+\left\|tf(t)\right\|^2_{L^2([0,1],\mathbb{C})}\\
&\qquad\leq \sum_{k=-\infty}^{\infty}\frac{4}{k^2}\|f\|^2_{L^2([0,1],\mathbb{C})}+\|f\|_{L^2([0,1],\mathbb{C})}\\
&\qquad\leq K\|f\|^2_{L^2([0,1],\mathbb{C})}\,,
\end{aligned}
$$

which proves the claim. □

Recall that we define $\mathcal{H}_0([0,1],\mathbb{C})$ in Section 2.1.4 as the set of finite-energy paths in $\mathbb{C}$, meaning functions $f:[0,1]\to\mathbb{C}$ such that $\int_0^1|f'(t)|^2dt<\infty$, which has a natural norm and inner product structure.

Lemma 10.6. *The bilinear map $S_2:\mathcal{H}_0([0,1],\mathbb{C})\times\mathcal{H}_0([0,1],\mathbb{C})\to\mathcal{H}_0([0,1],\mathbb{C})$ defined as $S_2(A,B)(t)=\int_0^t A(s)B'(s)ds$ is weakly Hilbert-Schmidt.*

Proof. By the definition of $\langle\cdot,\cdot\rangle_{\mathcal{H}_0([0,1],\mathbb{C})}$, we see that $\mathcal{I}:L^2([0,1],\mathbb{C})\to\mathcal{H}_0([0,1],\mathbb{C})$ is an isomorphism. Then the map S_2 is precisely $\mathcal{Z}$ from Lemma 10.5 under the identification $L^2([0,1],\mathbb{C})\xrightarrow{\mathcal{I}}\mathcal{H}_0([0,1],\mathbb{C})$. □

Lemma 10.7. *For a multilinear map $M:K_1\otimes\ldots\otimes K_n\to K$, we may define the continuous map $\mathcal{M}:\mathcal{H}_0([0,1],K_1)\otimes\ldots\otimes\mathcal{H}_0([0,1],K_n)\to\mathcal{H}_0([0,1],K)$ on simple tensors as*

$$
\mathcal{M}(A_1\otimes\ldots\otimes A_n)(t)\;=\;\int_{\Delta_t^n}M(A_1'(s_1),\ldots,A_n'(s_n))ds\,, \tag{3.2}
$$

where $\Delta_t^n=\{s\in[0,1]^n\;:\;0<s_1<\ldots<s_n<t\}$

Proof. First note that, by Lemma 10.6, we have the weakly Hilbert-Schmidt map $S_2:\mathcal{H}_0([0,1],\mathbb{C})^{\otimes 2}\to\mathcal{H}_0([0,1],\mathbb{C})$ given by $S_2(f_1,f_2)(t)=\int_0^t f_1(s_2)f_2'(s_2)ds_2=\int_0^t\int_0^{s_2}f_1'(s_1)f_2'(s_2)ds_1ds_2=\int_{\Delta_t^2}f_1'(s_1)f_2'(s_2)ds$. Then we inductively define

$$
S_k:\mathcal{H}_0([0,1],\mathbb{C})^k\to\mathcal{H}_0([0,1],\mathbb{C})
$$

as

$$S_k(f_1,\dots,f_k)(t) \;:=\; S_2\big(S_{k-1}(f_1,\dots,f_{k-1})\,,\; f_k\big)(t)$$
$$\;=\; \int_0^t S_{k-1}(f_1,\dots,f_{k-1})(s) f_k'(s)ds\,.$$

Then letting $\widetilde{\cdot}$ denote the extension to the tensor product, we see that if S_{k-1} has an extension $\widetilde{S_{k-1}}$, then so does S_k, since $S_k(f_1,\dots,f_k) = \widetilde{S}_2 \circ (\widetilde{S_{k-1}} \otimes I)(f_1 \otimes \dots, \otimes f_k)$, from which we apply induction to deduce that S_k must extend continuously to $\mathcal{H}_0([0,1],\mathbb{C})^{\otimes k}$ for all $2 \le k \le n$. Note that we can also write

$$S_k(f_1,\dots,f_k)(t) \;=\; \int_0^t \left(\int_{\Delta_{s_k}^{k-1}} (f_1'(s_1)\dots f_{k-1}'(s_{k-1}))ds_1 \dots ds_{k-1} \right) f_k'(s_k)ds_k$$
$$\;=\; \int_{\Delta_t^k} f_1'(s_1)\dots f_k'(s_k)ds\,.$$

Now define $\mathcal{M} : \mathcal{H}_0([0,1],\mathbb{C})^{\otimes n} \otimes (K_1 \otimes \dots \otimes K_n) \to \mathcal{H}_0([0,1],\mathbb{C}) \otimes K$ as $\mathcal{M} = S_n \otimes M$. Then, for $f_1,\dots,f_n \in \mathcal{H}_0([0,1],\mathbb{C})$ and $x_k \in K_k$, expressions of the form $f_1 \otimes \dots \otimes f_n \otimes x_1 \otimes \dots \otimes x_n$ constitute simple tensors in $\mathcal{H}_0([0,1],\mathbb{C})^{\otimes n} \otimes K_1 \otimes \dots \otimes K_n$, and we have

$$\mathcal{M}(f_1 \otimes \dots \otimes f_n \otimes x_1 \otimes \dots \otimes x_n) \;=\; \left(\int_{\Delta_t^n} f_1'(s_1)\dots f_n'(s_n)ds \right) M(x_1,\dots,x_n)$$
$$\;=\; \int_{\Delta_t^n} M\Big((x_1f_1)'(s_1)\dots(x_nf_n)'(s_n)\Big)ds\,. \tag{3.3}$$

As discussed in Section 2.1.4, $\mathcal{H}_0([0,1],K) \cong \mathcal{H}_0([0,1],\mathbb{C}) \otimes K$ and $\mathcal{H}_0([0,1],K_n) \cong \mathcal{H}_0([0,1],\mathbb{C}) \otimes K_n$ for all n. Then under this identification, we may realize $\mathcal{M} : \mathcal{H}_0([0,1],K_1) \otimes \dots \otimes \mathcal{H}_0([0,1],K_n) \to \mathcal{H}_0([0,1],K)$, and by (3.3), our definition of $\mathcal{M}$ will satisfy (3.2). □

10.2.3 Horizontal paths

Theorem 10.8. *A horizontal path $\sigma \in \mathcal{C}_h^1$ with $\sigma(0) = e$ necessarily takes on the form, for some $\mathcal{C}^1$ path $A : [0,1] \to H_1$,*

$$\sigma(t) \;=\; \sum_{n=1}^N \sum_{\sigma \in \mathcal{S}_n} c_\sigma \int_{\Delta_T^n} [\dots[A'(t_{\sigma(1)}), A'(t_{\sigma(2)})], \dots, A'(t_{\sigma(n)})]\,, \tag{3.4}$$

where the coefficients c_σ are defined in Theorem 2.6. Moreover, $\ell(\sigma) = \int_0^1 \|A'(s)\|_{H_1} ds$.

Proof. The horizontal criterion imposes $L_{\sigma(t)^{-1}*}\sigma'(t) \in H_1$. And for $g \in \mathfrak{g}_{CM}$ and $x \in \mathfrak{g}_{CM}$, recall from Section 2.3 that

$$L_{g*}x \;=\; c_1 x + c_2[g,x] + c_3[g,[g,x]] + \dots$$

Then, using the graded structure of $\mathfrak{g}_{CM}$, we can deduce that $L_{\sigma(t)^{-1}*}\sigma'(t) = c_1\sigma'(t) + c_2[-\sigma(t),\sigma'(t)] + \dots$ has, as its H_1 component, $\sigma_1'(t)$. By the horizontal condition, we must have $L_{\sigma(t)^{-1}*}\sigma'(t) = \sigma_1'(t)$.

Now set $A = \sigma_1$, which is necessarily $\mathcal{C}^1$. Then $L_{\sigma(t)^{-1}*}\sigma'(t) = A'(t)$, or $\sigma'(t) = L_{\sigma(t)}A'(t)$. Then by Theorem 2.6, we must have that (3.4) holds.

Furthermore, $\ell(\sigma) = \int_0^1 \|L_{\sigma(s)^{-1}*}\sigma'(s)\|_{\mathfrak{g}_{CM}} ds = \int_0^1 \|A'(s)\|_{H_1} ds$.

□

Lemma 10.7 and Theorem 10.8 will come together to give us the following.

Corollary 10.9. *There exist multilinear maps $\nu_k : \mathcal{H}_0([0,1],H_1)^k \to \mathcal{H}_0([0,1],H_k)$ such that every horizontal path $\sigma : [0,1] \to \mathfrak{g}_{CM}$ with $\sigma(0) = e$ can be expressed as*

$$\begin{aligned}\sigma(t) \;=\; \nu(A)(t) \;:=&\; \Big(\nu_1(A)(t), \nu_2(A,A)(t), \dots, \nu_N(A,\dots,A)(t)\Big) \\ &\in\; H_1 \times H_2 \times \dots \times H_N \;=\; \mathfrak{g}_{CM} \;=\; \exp(\mathfrak{g}_{CM}),\end{aligned}$$

where $\nu_1 A = A$, and each ν_k is weakly Hilbert-Schmidt and extends to $\mathcal{H}_0([0,1],H_1)^{\otimes k}$.

And we may realize d as

$$\begin{aligned} d(e,h) \;&=\; \inf\Big\{\ell(A) \mid A : [0,1] \to H_1 \text{ is } C^1,\; A(0)=0,\; \nu A(1) = h\Big\} \\ &=\; \inf\Big\{\|A\|_{\mathcal{H}_0([0,1],H_1)} \mid A \in \mathcal{H}_0([0,1],H_1),\;\; \nu A(1) = h\Big\}. \end{aligned}$$

Proof. Using (3.4) and keeping the graded structure in mind, we need only assign

$$\nu_n(A_1,\dots,A_n)(t) := \sum_{\sigma\in\mathcal{S}_n} c_\sigma \int_{\Delta_t^n} [\dots[A_1'(t_{\sigma(1)}), A_2'(t_{\sigma(2)})],\dots, A_n'(t_{\sigma(n)})]\,.$$

Then Theorem 10.8 tells us that any horizontal path $\sigma : [0,1] \to \mathfrak{g}_{CM}$ can be realized as $\sigma = (\nu_1(A),\dots,\nu_N(A,\dots,A))$ when $A = \sigma_1$, and Lemma 10.7 tells us that each ν_n is weakly Hilbert-Schmidt.

As for the last remark, we know that, by Holder's inequality, $\ell(\sigma) = \int_0^1 \|A'(s)\|_{H_1} ds \le \sqrt{\int_0^1 \|A'(s)\|_{H_1}^2 ds}$. We also have that these are equal when A is parametrized by arclength. Such a reparemetrization does not change the endpoints of νA, which can be observed by demonstrating that this holds for S_2. Indeed, for any increasing $\mathcal{C}^1$ bijection $\phi : [0,1] \to [0,1]$,

$$\begin{aligned} S_2(A\circ\phi, B\circ\phi)(t) &= \int_0^t A(\phi(s))B'(\phi(s))\phi'(s)ds \\ &= \int_0^{\phi(t)} A(u)B'(u)du \;=\; \big(S_2(A,B)\circ\phi\big)(t)\,. \end{aligned}$$

Lastly, consider that the set of $\mathcal{C}^1$ paths in H_1 is dense in $\mathcal{H}_0([0,1],H_1)$ (in the same way that the set of continuous functions are dense in $L^2([0,1],H)$). This completes the proof.

□

As remarked in Section 6, nice continuity properties are not always exhibited by general multilinear maps, but the components of ν are bounded linear maps on tensor powers of H_1. This allows us to conclude that ν will have certain continuity properties that bounded linear maps enjoy, such as respecting weak convergence. The next lemma uses this (and only this) to illustrate 3 specific properties that will be used in Theorem 10.11. This is just a matter of proving properties of linear maps; there is nothing else special about ν being used here.

For this lemma and the theorem to follow, we will abbreviate $\mathcal{H}_0 := \mathcal{H}_0([0,1],H_1)$.

Lemma 10.10 (Continuity of ν)**.**

1. *If A_m converges weakly to A in $\mathcal{H}_0$, then $\nu A_m(1)$ converges weakly to $\nu A(1)$ in $\mathfrak{g}_{CM}$.*
2. *If A_m converges in norm to A in $\mathcal{H}_0$, then $\nu A_m(1)$ converges in norm to $\nu A(1)$ in $\mathfrak{g}_{CM}$.*
3. *If $\|A_m - B_m\|_{\mathcal{H}_0} \xrightarrow{m\to\infty} 0$ and $\nu A_m(1)$ weakly converges to some $g \in \mathfrak{g}_{CM}$, then $\nu B_m(1)$ must also weakly converge to g.*

Proof. For the first point, consider that the set of finite sums of simple tensors in $H_1^{\otimes n}$ forms a dense set, and that for any simple tensor $\alpha \in H_1^{\otimes n}$, $\langle \alpha, (A_m)^{\otimes n}\rangle_{H_1^{\otimes n}} \xrightarrow{m\to\infty} \langle \alpha, A^{\otimes n}\rangle_{H_1^{\otimes n}}$. And $(A_m)^{\otimes n}$ is bounded in $H_1^{\otimes n}$. Then, by Proposition 2.2, we must have $(A_m)^{\otimes n} \to A^{\otimes n}$ weakly in $H_1^{\otimes n}$. Since each $\nu_n : \mathcal{H}_0^{\otimes n}([0,1],H_1) \to \mathcal{H}_0([0,1],H_n)$ is linear continuous, and as is point evaluation $\mathcal{H}_0([0,1],H_n) \ni f \mapsto f(t) \in \mathbb{C}$ (see Section 2.1.4), then each $\nu_n(A_m^{\otimes n})$ converges weakly to $\nu_n(A^{\otimes n})$, so that νA_m converges weakly to νA, and $\nu A_m(1)$ converges weakly to $\nu A(1)$.

The proof of the second point is similar. If $A_m \to A$, then

$$\begin{aligned}
&\|A_m^{\otimes n} - A^{\otimes n}\|_{\mathcal{H}_0^{\otimes n}} \\
&\qquad \le \|A_m^{\otimes n} - A_m^{\otimes (n-1)} \otimes A\|_{\mathcal{H}_0^{\otimes n}} + \|A_m^{\otimes (n-1)} \otimes A - A^{\otimes n}\|_{\mathcal{H}_0^{\otimes n}} \\
&\qquad = \|A_m\|_{\mathcal{H}_0}^{n-1} \|A_m - A\|_{\mathcal{H}_0} + \|A_m^{\otimes (n-1)} - A^{\otimes (n-1)}\|_{\mathcal{H}_0^{\otimes (n-1)}} \|A\|_{\mathcal{H}_0} .
\end{aligned}$$

so by remarking that the sequence A_m is bounded and applying induction, we may deduce $A_m^{\otimes n} \to A^{\otimes n}$, from which we can conclude $\nu A_m(1)$ converges in norm to $\nu A(1)$.

For the third point, if $\|A_m - B_m\|_{\mathcal{H}_0} \to 0$, then, arguing similarly to the above, $\|(A_m)^{\otimes n} - (B_m)^{\otimes n}\|_{\mathcal{H}_0^{\otimes n}} \to 0$ for all $1 \le n \le N$, so by the continuity of ν, $\|\nu A_m(1) - \nu B_m(1)\|_{\mathfrak{g}_{CM}} \to 0$. Then for any $f \in \mathfrak{g}_{CM}^*$, $|f(\nu A_m(1)) - f(\nu B_m(1))| \le \|f\|_{\mathfrak{g}_{CM}^*} \|\nu A_m(1) - \nu B_m(1)\|_{\mathfrak{g}_{CM}} \to 0$. Thus, the convergence of $f(\nu A_m(1))$ implies the convergence of $f(\nu B_m(1))$, so $\nu B_m(1)$ must weakly converge.

□

Theorem 10.11. *Let $g \in G_{CM}$. Then there exists a sequence $(P_m)_m \in Proj(W_1)^{\uparrow}$ and $g_m \in G^{P_m}$ such that*

1. $g_m \xrightarrow{m\to\infty} g$ *in* $\|\cdot\|_{\mathfrak{g}_{CM}}$.
2. $d^{P_m}(e, g_m) \xrightarrow{m\to\infty} d(e, g)$.

Proof. Recall that we abbreviate $\mathcal{H}_0([0,1], H_1)$ as $\mathcal{H}_0$, and we also write $\mathcal{H}_0([0,1], PH_1)$ as $P\mathcal{H}_0$ for $P \in Proj(W_1)$. Since $g \in G_{CM}$, we may choose $A_m \in \mathcal{H}_0$ such that $\nu A_m(1) = g$, and such that

$$d(e, g) \le \|A_m\|_{\mathcal{H}_0} \le d(e, g) + \frac{1}{m} .$$

For every $m \in \mathbb{N}$, choose $P_m \in Proj(W_1)$ such that $(P_m)_{m\in\mathbb{N}} \in Proj(W_1)^{\uparrow}$ and $\|P_m A_m - A_m\|_{\mathcal{H}_0} < \frac{1}{m}$. Now for each m, choose $B_m \in P_m\mathcal{H}_0$ such that $\nu B_m(1) = \nu P_m A_m(1)$, and

$$d^{P_m}(e, \nu P_m A_m(1)) \le \|B_m\|_{\mathcal{H}_0} \le d^{P_m}(e, \nu P_m A_m(1)) + \frac{1}{m} .$$

Now observe that

$$\begin{aligned}
\sup_{m\in\mathbb{N}} \|B_m\|_{\mathcal{H}_0} &\le \sup_{m\in\mathbb{N}} d^{P_m}(e, \nu P_m A_m(1)) + 1 \le \sup_{m\in\mathbb{N}} \|P_m A_m\|_{\mathcal{H}_0} + 1 \\
&\le \sup_{m\in\mathbb{N}} \|A_m\|_{\mathcal{H}_0} + 1 \le d(e, g) + 2 .
\end{aligned}$$

Thus, by weak compactness, we may choose a subsequence $(B_{n_m})_m \in \mathcal{H}_0$ weakly converging to some $B \in \mathcal{H}_0$.

By Lemma 10.10, point-evaluation respects weak convergence, so that (regarding the limits below as weak limits in $\mathfrak{g}_{CM}$)

$$\nu B(1) \;=\; \operatorname*{wlim}_{m\to\infty} \nu B_{n_m}(1) \;=\; \operatorname*{wlim}_{m\to\infty} \nu P_{n_m} A_{n_m}(1)\,.$$

And $\|P_{n_m}A_{n_m} - A_{n_m}\|_{\mathcal{H}_0} \le \frac{1}{n_m} \xrightarrow{m\to\infty} 0$, so again using Lemma 10.10,

$$\operatorname*{wlim}_{m\to\infty} \nu P_{n_m} A_{n_m}(1) \;=\; \operatorname*{wlim}_{m\to\infty} \nu A_{n_m}(1) \;=\; g\,.$$

Then $\nu B(1) = g$, and

$$\begin{aligned} d(e,g) \;&\le\; \|B\|_{\mathcal{H}_0} \;\le\; \liminf_{m\to\infty} \|B_{n_m}\|_{\mathcal{H}_0} \;\le\; \liminf_{m\to\infty}\, d^{P_{n_m}}(e, \nu P_{n_m} A_{n_m}(1)) \\ &\le\; \limsup_{m\to\infty}\, d^{P_{n_m}}(e, \nu P_{n_m} A_{n_m}(1)) \;\le\; \limsup_{m\to\infty} \|P_{n_m} A_{n_m}\|_{\mathcal{H}_0} \\ &\le\; \limsup_{m\to\infty} \|A_{n_m}\|_{\mathcal{H}_0} \;=\; d(e,g)\,. \end{aligned}$$

Thus, $d^{P_{n_m}}(e, \nu P_{n_m} A_{n_m}(1)) \xrightarrow{m\to\infty} d(e,g)$. Also, we may similarly say that

$$d(e,g) \;\le\; \|B\|_{\mathcal{H}_0} \;\le\; \liminf_{m\to\infty} \|B_{n_m}\|_{\mathcal{H}_0} \;\le\; \limsup_{m\to\infty} \|B_{n_m}\|_{\mathcal{H}_0} \;\le\; d(e,g)\,.$$

Then we know that $\|B_{n_m}\|_{\mathcal{H}_0} \to \|B\|_{\mathcal{H}_0}$, meaning that B is actually the norm limit of $(B_{n_m})_m$, so $\nu(B_{n_m})(1)$ converges in $\|\cdot\|_{\mathfrak{g}_{CM}}$ to $\nu(B)(1) = g$. Thus, by setting $g_m = \nu B_{n_m}(1) = \nu P_{n_m} A_{n_m}(1)$, we are done.

□

Remark 10.12. If $g \in G^P$, then we can say something stronger: for any $P_m \in Proj(W_1)$ where $P_1 = P$, $d^{P_m}(e, \nu P_m A(1)) \downarrow d(e, \nu A(1))$ (no subsequence necessary). This can be proven with a slightly modified argument, roughly as follows. Choose a sequence of paths $A_m \in P\mathcal{H}$ so that $\nu A_m(0) = e$, $\nu A_m(1) = g$, and $d^{P_m}(e,g) \le \|A_m\|_{\mathcal{H}} \le d^{P_m}(e,g) + \frac{1}{m}$. Now choose a weakly-convergent subsequence A_{n_m}, weakly converging to some $A \in \mathcal{H}$. Then, arguing as before, one gets $\nu A(1) = g$ and a chain of inequalities that lets us conclude $\lim_{m\to\infty} d^{P_{n_m}}(e,g) = d(e,g)$. But $d^{P_m}(e,g)$ is a decreasing sequence, so we must have $\lim_{m\to\infty} d^{P_m}(e,g) = d(e,g)$.

10.3 Stochastics: Infinite-dimensional heat kernel measure

This section is devoted to providing the definition and properties for an infinite-dimensional heat kernel distribution on a Banach space G, inline with the definition provided in Section 3.

We now make a remark that parallels one made at the start of Section 7. It is natural to assume that G itself is a simply connected nilpotent Lie group with a group structure, perhaps by means of a continuous bilinear bracket $[\cdot,\cdot] : G \times G \to G$. Alternatively, it is possible to assume that the bracket is merely "measurable bilinear," in which $[x,\cdot] : G \to G$ is measurable for all $x \in G$, or even just $x \in \mathfrak{g}_{CM}$, all while the restriction of the bracket to $\mathfrak{g}_{CM}$ has more structure. In this result, we will take the most abstract stance possible: we do not assume that $[\cdot,\cdot]$ has a well-defined extension to G whatsoever. Just as with abstract Wiener spaces, the structure on $\mathfrak{g}_{CM}$ will determine the probabalistic structure on G, and this is all that will be needed.

10.3.1 Definition

Let G be a complex Banach space, in which $\mathfrak{g}_{CM} = H_1 \oplus \ldots \oplus H_N \subseteq G$ is continuously and densely included. For $1 \le n \le N$, we let $W_n := \overline{H_n}^{\|\cdot\|_G}$. In this way, we may write $G = W_1 + \ldots + W_N$, and $W_n \cap W_m = \{0\}$ for $n \neq m$.

We further assume that (W_1, H_1) is an abstract Wiener space. From this, we know we have a "flat" W_1-valued (or, if you prefer, $\mathfrak{w}_1$-valued) Brownian motion B_t, as described in Section 2.2. Then, letting $Proj(W_1)$ denote the set of (complex-linear) finite-rank projections of H_1 that extend continuously to W_1 (and thus to G), then, for $P \in Proj(W_1)$, PB_t is a PH_1-valued Brownian motion. We may define G^P-valued Brownian motion, g_t^P, as the solution to the stochastic differential equation

$$dg_t^P = L_{g_t^P *} \delta PB_t \qquad\qquad g_0 = e\,,$$

where δ denotes the Stratonovich stochastic integral. As described by Theorem 2.6 and Section 2.3.3, the solution is given by

$$g_t^P = \sum_{n=1}^{N} \sum_{\sigma \in S_n} \left((-1)^{e(\sigma)} \Big/ n^2 \begin{bmatrix} n-1 \\ e(\sigma) \end{bmatrix} \right) \times \int_{\Delta_T^n} [\ldots[\delta PB_{s_{\sigma(1)}}, \delta PB_{s_{\sigma(2)}}], \ldots, \delta PB_{s_{\sigma(n)}}]\,, \quad (3.5)$$

where δ corresponds to the Stratonovich stochastic integral. We will henceforth refer to the constants as c_σ. Here, by the graded structure, each $n = \{1, \ldots, N\}$ corresponds to a different stratum.

We say that G is an *(infinite-dimensional complex graded simply connected) nilpotent abstract Wiener Lie Group* if the following holds.

$$\text{For some } t>0, \text{ there exists a } G\text{-valued random variable } g_t \text{ such that, for every } f \in G^*, \text{ there exists an increasing sequence of finite-rank projections } \{P_m\}_{m\in\mathbb{N}} \in \operatorname{Proj}(W)^{\uparrow} \text{ such that } f(g_t^{P_m}) \to f(g_t) \text{ in probability.} \tag{A3.3}$$

We will regard g_t as equalling

$$g_t = \sum_{n=1}^{N} \sum_{\sigma\in\mathcal{S}_n} c_\sigma \int_{\Delta_t^n} [\dots[\delta B_{s_{\sigma(1)}}, \delta B_{s_{\sigma(2)}}], \dots, \delta B_{s_{\sigma(n)}}]\,.$$

Note that this G indeed satisfies the definition for a nilpotent abstract Wiener Lie group from Section 3, where $X = \mathfrak{g}_{CM}$ and $X_H = H_1$. Also, recall from Section 3 that this limit occurring for some $t > 0$ is equivalent to it occurring for all $t > 0$. Furthermore, the traceless nature of the iterated brackets, along with (A3.1), will imply that (A1.2′) holds; see Lemma 10.14, Theorem 10.15 and their proofs.

Before proving deeper properties of the heat kernel measure, we present a fact analogous to a result in abstract Wiener space. See also [GM13, Proposition 2.30] and [Mel21, Proposition 4.7].

Proposition 10.13. $\mathbb{P}(g_t \in \mathfrak{g}_{CM}) = 0$.

Proof. The random variable $g_t = ((g_t)_1, \dots, (g_t)_N) \in W_1 \times \dots \times W_N$ satisfies $(g_t)_1 = B_t$, a Brownian motion on W_1. Then we can write $\mathbb{P}(g_t \in \mathfrak{g}_{CM}) \le \mathbb{P}(B_t \in H_1) = 0$. □

10.3.2 Convergence of linear functionals

This section will use the complex structure to prove that holomorphic linear functions in G^* must lie in $L^p(G)$ for all $p \in [1,\infty)$ with respect to the distribution of g_t, and in fact the convergence $f(g_t^{P_m})$ to $f(g_t)$ occurs in L^p for all $p \in [1,\infty)$. Below, we state and prove a helpful lemma that uses the complex structure. Indeed, this is a direct consequence of assuming that B_t is a complex Brownian motion, and that iterated brackets $[\dots[\cdot,\cdot],\dots,\cdot]$ are complex multilinear.

Lemma 10.14. *The following Stratonovich and Itô integrals are almost-surely equal:*

$$\int_{\Delta_t^n} \delta PB_{t_1} \otimes \dots \otimes \delta PB_{t_n} = \int_{\Delta_t^n} dPB_{t_1} \otimes \dots \otimes dPB_{t_n}\,. \tag{3.6}$$

Proof. Note that this result is essentially proven as Theorem 6.26 in [Dri15]. We will make use of formulae from Section 3.

Let $\{f_\ell\}_{1\le\ell\le 2r}$ be a real basis of PH_1. We define $\mathcal{J}^m_n = \{\alpha \in \{1,2\}^m \mid \sum_{k=1}^m \alpha_k = n\}$, and then let

$$dPX_t^{\alpha,j} = \begin{cases} dPB_t & \text{if } \alpha_j = 1 \\ \sum_{\ell=1}^{2r} f_\ell \otimes f_\ell & \text{if } \alpha_j = 2 \end{cases}$$

(which is independent of the basis chosen). Then

$$\int_{\Delta_t^n} \delta PB_{s_1} \otimes \ldots \otimes \delta PB_{s_n} = \sum_{m=n/2}^{n} \frac{1}{2^{n-m}} \sum_{\alpha\in\mathcal{J}^n_m} \int_{\Delta_t^n} dPX_{s_1}^{\alpha,1} \otimes \ldots \otimes dPX_{s_n}^{\alpha,n}. \quad (3.7)$$

Say $\{f_\ell\}_{1\le\ell\le 2r} = \{e_j\}_{1\le j\le r} \cup \{ie_j\}_{1\le j\le r}$, where $\{e_j\}_{1\le j\le r}$ is a complex basis of PH_1. Then

$$\sum_{\ell=1}^{2r} f_\ell \otimes f_\ell = \sum_{j=1}^{r} e_j \otimes e_j + \sum_{j=1}^{r} ie_j \otimes ie_j = 0\,.$$

Then (3.7) need only be summed for $m = n$, which yields (3.6).

□

As a result of Lemma 10.14,

$$g_t^P = \sum_{n=1}^{N} \sum_{\sigma\in\mathcal{S}_n} c_\sigma \int_{\Delta_t^n} [\ldots[dPB_{s_{\sigma(1)}}, dPB_{s_{\sigma(2)}}], \ldots, dPB_{s_{\sigma(n)}}]\,.$$

In this way, we may view

$$g_t = \sum_{n=1}^{N} \sum_{\sigma\in\mathcal{S}_n} c_\sigma \int_{\Delta_t^n} [\ldots[dB_{s_{\sigma(1)}}, dB_{s_{\sigma(2)}}], \ldots, dB_{s_{\sigma(n)}}]\,.$$

Theorem 10.15. *Define*

$$\Phi^P_{t,k} := \int_{\Delta_t^k} dPB_{s_1} \otimes \ldots \otimes dPB_{s_k}$$

and let $\alpha \in H_1^{*\otimes k}$. *Then* $\mathbb{E}|\alpha(\Phi^P_{t,k})|^2 = \frac{t^k}{k!}\|\alpha\|^2_{PH_1^{*\otimes k}}$.

Proof. This is a proof by induction. First note that for $k = 1$, $\Phi^P_{t,1} = PB_t$, so if $\alpha \in H_1^*$, then $\mathbb{E}|\alpha(\Phi^P_{t,1})|^2 = t\|\alpha\|_{PH_1^*}$. Next, we continue inductively, so suppose the statement holds for $k-1$. If $\alpha \in H_1^{\otimes k}$, then $\langle \alpha, (\cdot) \otimes h \rangle \in PH_1^{*\otimes k-1}$ for any $h_1 \in H_1$. Let $\{e_j\}_{1 \le j \le r}$ be a basis of PH_1. Then

$$\begin{aligned}
\mathbb{E}|\alpha(\Phi_{t,k}^P)|^2 &= \mathbb{E}\left| \int_0^t \alpha\big(\Phi_{s_k,k-1}^P \otimes dPB_{s_k}\big) \right|^2 \\
&= \int_0^t \sum_{j=1}^d \mathbb{E}\left[\left| \alpha\Big(\Phi_{s_k,k-1}^P \otimes \frac{1}{\sqrt{2}} e_j\Big) \right|^2 + \left| \alpha\Big(\Phi_{s_k,k-1}^P \otimes \frac{i}{\sqrt{2}} e_j\Big) \right|^2 \right] ds_k \\
&= \int_0^t \sum_{j=1}^d \mathbb{E}|\alpha(\Phi_{s_k,k-1}^P \otimes e_j)|^2 ds_k \\
&= \sum_{j=1}^d \int_0^t \frac{s_k^{k-1}}{(k-1)!} \|\alpha(\cdot \otimes e_j)\|_{PH_1^{*\otimes k-1}}^2 ds_k \\
&= \frac{t^k}{k!} \|\alpha\|_{PH_1^{*\otimes k}}^2 .
\end{aligned}$$

□

Theorem 10.16. *For every k, $p \in [1, \infty)$, $\alpha \in H_1^{*\otimes k}$, and $(P_m)_{m\in\mathbb{N}} \in Proj(W_1)^{\uparrow}$, $\alpha(\Phi_{t,k}^{P_m})$ converges in L^p as $m \to \infty$. The limit is independent of the choice of $(P_m)_{m\in\mathbb{N}}$, and we denote the limit as $\alpha(\Phi_{t,k}) = \int_{\Delta_t^k} \alpha(dB_{t_1} \otimes \ldots \otimes dB_{t_k})$. And $\mathbb{E}|\alpha(\Phi_{t,k})|^2 = \frac{t^k}{k!}\|\alpha\|_{H_1^{*\otimes k}}^2$.*

Proof. We begin by first showing convergence in L^2. If $\alpha \in H_1^{*\otimes k}$, $\{e_j\}_{1\le j \le r_m}$ a basis of $P_m H_1$ for every $m \in \mathbb{N}$, and $m < n$, then by Theorem 10.15,

$$\begin{aligned}
&\mathbb{E}|\alpha(\Phi_{t,k}^{P_n}) - \alpha(\Phi_{t,k}^{P_m})|^2 \\
&\quad = \mathbb{E}|\alpha \circ P_n^{\otimes k}(\Phi_{t,k}^{P_n}) - \alpha \circ P_m^{\otimes k}(\Phi_{t,k}^{P_m})|^2 \\
&\quad = \mathbb{E}|\alpha \circ (P_n^{\otimes k} - P_m^{\otimes k})(\Phi_{t,k}^{P_n})|^2 \\
&\quad = \frac{t^k}{k!}\|\alpha \circ (P_n^{\otimes k} - P_m^{\otimes k})\|_{H_1^{*\otimes k}}^2 \\
&\quad = \frac{t^k}{k!}\left(\sum_{j_1,\ldots,j_k=1}^{r_n} |\alpha(e_{j_1} \otimes \ldots \otimes e_{j_k})|^2 - \sum_{j_1,\ldots,j_k=1}^{r_m} |\alpha(e_{j_1} \otimes \ldots \otimes e_{j_k})|^2 \right),
\end{aligned}$$

which is small for sufficiently large n, m. So $\alpha(\Phi_{t,k}^{P_m})$ is Cauchy in L^2, and hence must converge to some $\alpha(\Phi_{t,k})$. It is also here that we remark that this limit is independent of the sequence $(P_m)_m$ chosen. Indeed, if $(P'_m)_m \in Proj(W_1)^{\uparrow}$ is another sequence, then choose $(P''_m)_m \in Proj(W_1)^{\uparrow}$ such that $P_m H_1 + P'_m H_1 \subseteq P''_m H_1$. Then

$$\begin{aligned}
\mathbb{E}\big|\alpha(\Phi_{t,k}^{P_m}) - \alpha(\Phi_{t,k}^{P'_m})\big|^2 &= \mathbb{E}\big|\alpha \circ (P_m^{\otimes k} - P_m'^{\otimes k})(\Phi_{t,k}^{P''_m})\big|^2 \\
&= \frac{t^k}{k!}\|\alpha \circ (P_m^{\otimes k} - P_m'^{\otimes k})\|_{H_1^{*\otimes k}}^2 ,
\end{aligned}$$

which necessarily goes to 0 as $m \to \infty$, which implies $\alpha(\Phi_{t,k}^{P'_m})$ has the same limit as $\alpha(\Phi_{t,k}^{P_m})$.

For general L^p convergence, it can be seen that $\alpha(\Phi_{t,k}^{P})$ necessarily has a finite-degree chaos expansion. It is a theorem of Nelson ([Nel73, Lemma 2] that, for every $j \in \mathbb{N}$, there exists a constant c_j such that $\mathbb{E}[|\alpha(\Phi_{t,k}^{P})||^{2j}] \leq c_j(\mathbb{E}[|\alpha(\Phi_{t,k}^{P})|^2])^j$. Then we may deduce that, for any $p \in [1, \infty)$,

$$\begin{aligned}\mathbb{E}\left|\alpha(\Phi_{t,k}^{P_n}) - \alpha(\Phi_{t,k}^{P_m})\right|^p &\leq \mathbb{E}\left|\alpha(\Phi_{t,k}^{P_n}) - \alpha(\Phi_{t,k}^{P_m})\right|^{\lceil p \rceil} \\ &= \mathbb{E}\left|\alpha \circ (P_n^{\otimes k} - P_m^{\otimes k})(\Phi_{t,k}^{P_n})\right|^{\lceil p \rceil} \\ &\leq c_{\lceil p \rceil}\left(\mathbb{E}\left|\alpha \circ (P_n^{\otimes k} - P_m^{\otimes k})(\Phi_{t,k}^{P_n})\right|^2\right)^{\lceil p \rceil},\end{aligned}$$

so that the series must converge in L^p.

Lastly, $\left|\mathbb{E}|\alpha(\Phi_{t,k})|^2 - \mathbb{E}|\alpha(\Phi_{t,k}^{P_m})|^2\right| \leq \mathbb{E}|\alpha(\Phi_{t,k}) - \alpha(\Phi_{t,k}^{P_m})|^2$, which justifies that $\mathbb{E}|\alpha(\Phi_{t,k})|^2 = \lim_{m\to\infty} \mathbb{E}|\alpha(\Phi_{t,k}^{P_m})|^2 = \frac{t^k}{k!}\|\alpha\|^2_{H_1^{*\otimes k}}$. □

Lemma 10.17. *For all $f \in G^*$, $p \in [1, \infty)$, and all $(P_m)_m \in Proj(W_1)^{\uparrow}$, $f(g_t^{P_m})$ converges in L^p to $f(g_t)$.*

Proof. Choose $(P_m)_{m\in\mathbb{N}} \in Proj(W_1)^{\uparrow}$ such that $g_t^{P_m} \to g_t$ in probability. Then

$$\begin{aligned} f(g_t^{P_m}) &= \sum_{n=1}^{N} \sum_{\sigma \in S_n} c_\sigma \int_{\Delta_t^n} f([\dots[dP_m B_{s_{\sigma(1)}}, dP_m B_{s_{\sigma(2)}}], \dots, dP_m B_{s_{\sigma(n)}}]) \\ &= \sum_{n=1}^{N} \sum_{\sigma \in S_n} c_\sigma\, f \circ [\dots[\cdot,\cdot]_{1,1}, \dots,]_{n-1,1} \circ \widetilde{\sigma}(\Phi_{t,n}^{P_m}), \end{aligned}$$

where for any permutation $\sigma \in \mathcal{S}_n$, we let $\widetilde{\sigma}$ be the natural action on $H_1^{\otimes k}$ that satisfies $\widetilde{\sigma}(h_1 \otimes \dots \otimes h_n) = h_{\sigma(1)} \otimes \dots \otimes h_{\sigma(n)}$. Then, for each n, $f \circ [\dots[\cdot,\cdot]_{1,1}, \dots,]_{n-1,1} \circ \widetilde{\sigma} : H_1^{\otimes n} \to \mathbb{C}$ is in $H_1^{*\otimes n}$. On the one hand, Theorem 10.16 implies that for each n, $\langle f \circ [\dots[\cdot,\cdot]_{1,1}, \dots,]_{n-1,1} \circ \widetilde{\sigma}, \Phi_{t,n}^{P_m}\rangle$ converges to the random variable $\langle f \circ [\dots[\cdot,\cdot]_{1,1}, \dots,]_{n-1,1} \circ \widetilde{\sigma}, \Phi_{t,n}\rangle$ in L^p for all $p \in [1, \infty)$, so as a finite sum of such random variables, $f(g_t^{P_m})$ must also converge in L^p. On the other hand, since f is continuous on G, $f(g_t^{P_m})$ converges to $f(g_t)$ in probability, and therefore must converge to $f(g_t)$ in L^p.

We remark that the convergence will hold for an arbitrary choice of $(P_m)_{m\in\mathbb{N}} \in Proj(W_1)^{\uparrow}$ because the convergence in Theorem 10.16 allowed for this.

□

10.4 Examples

We begin with Section 10.4.1, which provides examples of the "skeleton structure" $\mathfrak{g}_{CM}$ that satisfy the weakly and surjectivity assumptions. Then in Section 10.4.2, we will provide example structures on G that guarantee the existence of the random variable g_t and its limiting properties.

10.4.1 Examples of $\mathfrak{g}_{CM}$

Example 10.18 (Step-2 graded complex)**.** Suppose $H_1 \times H_2$ is a step-2 graded complex nilpotent Lie group. Then the bracket structure is completely determined by an antisymmetric complex-multilinear weakly Hilbert-Schmidt map $\omega : H_1 \times H_1 \to H_2$ defined as $\omega(h_1, h_2) = [h_1, h_2]$. As long as $\widetilde{\omega} : H_1 \otimes H_1 \to H_2$ is surjective (or replacing H_2 with $im\widetilde{\omega}$ if not), we have $\mathfrak{g}_{CM} = H_1 \times H_2$. This is the (complex equivalent of the) topic of study in Chapter 2.

It is worth pointing out that, while the assumptions of Chapter 2 actually contradicted the step-2 path space construction ($\|\omega\|_{H \otimes Z}$ is not satisfied: see Example 5.11), it does satisfy the assumptions in Chapter 3 for the Taylor isomorphism (see Example 10.21).

Example 10.19 (Infinite product)**.** This example can be compared to Example 5.7. Here, we consider the infinite product of step N Lie groups. We suppose that we have an infinite sequence of finite-dimensional simply connected graded complex nilpotent Lie groups $G^{(m)} = G_1^{(m)} \oplus \ldots \oplus G_N^{(m)}$ where each $G_n^{(m)}$ has as orthonormal basis $\{e_{n,j}^{(m)}\}_{j \in \Lambda_n^{(m)}} \subseteq G_n^{(m)}$, so that each $G^{(m)}$ has as orthonormal basis $\bigcup_{1 \leq n \leq N} \{e_{n,j}^{(m)}\}_{j \in \Lambda_n^{(m)}}$, and each $G^{(m)}$ is considered equal to its Lie algebra $\mathfrak{g}^{(m)}$, equipped with a bracket $[\cdot, \cdot]^{(m)}$. Note that the "step" of each $G^{(m)}$ is uniformly bounded by N. Then their infinite orthogonal product $\mathfrak{g}_{CM} = \bigoplus_{m=1}^{\infty} G^{(m)}$ can be considered an infinite-dimensional Lie algebra with bracket $[a, b] = \sum_{m=1}^{\infty} [a^{(m)}, b^{(m)}]^{(m)}$.

In addition to requesting that each $G^{(m)}$ satisfies the Hörmander condition, we must request that it is done uniformly. More precisely, we require that, for every $m \in \mathbb{N}$, $1 \leq n \leq N$, there exist constants $c_n^{(m)}, C_n^{(m)}$ such that, for all $h \in G_n^{(m)}$,

$$c_n^{(m)} \|h\|_{G_n^{(m)}}^2 \leq \sum_{i \in \Lambda_1^{(m)}, j \in \Lambda_{n-1}^{(m)}} \left| \langle [e_{1,i}^{(m)}, e_{n-1,j}^{(m)}]_{1,n-1}^{(m)}, h \rangle_{G_n^{(m)}} \right|^2 \leq C_m^{(m)} \|h\|_{G_n^{(m)}}^2 ,$$

in which, for all n, $c_n := \inf_{m \in \mathbb{N}} c_n^{(m)} > 0$ and $C_n := \sup_{m \in \mathbb{N}} C_n^{(m)} < \infty$. Indeed, if this is satisfied, then for any $h = (h^{(m)})_{m \in \mathbb{N}} \in \bigoplus_{m=1}^{\infty} G_n^{(m)}$, the nth step of $\mathfrak{g}_{CM}$, using the orthogonality of the $G^{(m)}$s, we can calculate

$$\begin{aligned}\sum_{m_1,m_2=1}^{\infty} \sum_{i\in\Lambda_1^{(m_1)},j\in\Lambda_{n-1}^{(m_2)}} & \left|\langle [e_{1,i}^{(m_1)}, e_{n-1,i}^{(m_2)}]_{1,n-1}\ ,\ h\rangle_{\mathfrak{g}_{CM}}\right|^2 \\ &= \sum_{m=1}^{\infty} \sum_{i\in\Lambda_1^{(m)},j\in\Lambda_{n-1}^{(m)}} \left|\langle [e_{1,i}^{(m)}, e_{n-1,i}^{(m)}]_{1,n-1}\ ,\ h^{(m)}\rangle_{G^{(m)}}\right|^2 \\ &\leq \sum_{m=1}^{\infty} C_n^{(m)} \|h^{(m)}\|^2_{G^{(m)}} \ = \ C_n\|h\|^2_{\mathfrak{g}_{CM}}\,,\end{aligned}$$

and similarly

$$\sum_{m_1,m_2=1}^{\infty} \sum_{i\in\Lambda_1^{(m_1)},j\in\Lambda_{n-1}^{(m_2)}} \left|\langle [e_{1,i}^{(m_1)}, e_{n-1,i}^{(m_2)}]_{1,n-1}\ ,\ h\rangle_{\mathfrak{g}_{CM}}\right|^2 \ \geq\ c_n\|h\|^2_{\mathfrak{g}_{CM}}\,.$$

It is also not impossible to define and discuss infinite-dimensional semi-direct products.

10.4.2 Examples of G

Example 10.20 (Complex Hilbert Space)**.** This example can be compared to Example 3.4, though with formulae and criteria made easier due to the complex assumption (which gives, in particular, the equivalence of the Stratonovich and Itô integrals). Assume that G is a Hilbert space with orthogonal decomposition $G = W_1 \times \ldots \times W_n$, and suppose further that each $H_n \hookrightarrow W_n$ is Hilbert-Schmidt. As a consequence, every composition $[\ldots[\cdot,\cdot]_{1,1},\ldots,\cdot]_{1,n-1} : H_1^{\otimes k} \to W_n$ is (in fact, considering our Hörmander condition, these criteria are equivalent).

For any orthonormal basis $\{w_{n,j}\}_{j\in\Lambda_n}$ of W_n, Theorem 10.15 gives that

$$\begin{aligned}\mathbb{E}\|g_t\|^2_G &= \sum_{n=1}^{N} \mathbb{E}\big\|\big(g_t\big)_n\big\|^2_{W_n} \\ &= \sum_{n=1}^{N}\sum_{j\in\Lambda_n} \mathbb{E}\big|\langle w_{n,j}, [\ldots[\cdot,\cdot]_{1,1},\ldots,\cdot]_{1,n-1}\rangle_{W_n}(\Phi_{t,k})\big|^2 \\ &= \sum_{n=1}^{N}\sum_{j\in\Lambda_n} \frac{t^n}{n!}\big\|\langle w_{n,j}, [\ldots[\cdot,\cdot]_{1,1},\ldots,\cdot]_{1,n-1}\rangle_{W_n}\big\|^2_{H_1^{*\otimes n}} \\ &= \sum_{n=1}^{N} \frac{t^n}{n!}\big\|[\ldots[\cdot,\cdot]_{1,1},\ldots,\cdot]_{1,n-1}\big\|^2_{HS(H_1,W_n)}\,.\end{aligned}$$

From this estimate, it can be easily argued that $g_t^{P_m}$ converges to g_t in L^2 with respect to $\|\cdot\|_G$. In fact, as remarked in [Mel21], Doob's maximal inequality implies

$$\mathbb{E}\max_{0\le t\le T}\|g_t^{P_m}\|_G^2 \le 4\mathbb{E}\|g_T^{P_m}\|_G^2\,,$$

so the L^2 martingales $(g_t^{P_m})_{0\le t\le T}$ converge to an L^2 G-valued martingale $(g_t)_{0\le t\le T}$.

Example 10.21 (Hypoelliptic path space)**.** This example can be compared to Example 3.5, again being simpler due to the complex assumption of this chapter. Let $V = V_1 \times \ldots \times V_N$ be a finite-dimensional graded nilpotent complex Lie group. Let $\mathcal{W}_0([0,1],V) = \{f : [0,1] \to V \mid f \text{ continuous}, f(0) = 0\}$, and $\mathcal{H}_0([0,1],V_n)$ the usual set of based finite-energy paths in V_n, as defined in Section 2.1.4. For now, we will set

$$\begin{aligned} G \;=\; \mathfrak{g} \;&=\; \mathcal{W}_0([0,1],V) \\ H \;&=\; \mathcal{H}_0([0,1],V) \;=\; \mathcal{H}_0([0,1],V_1)\oplus\ldots\oplus\mathcal{H}_0([0,1],V_N) \\ H_1 \;&=\; \mathcal{H}_0([0,1],V_1) \end{aligned}$$

(one may expect that $\mathfrak{g}_{CM} = H$, but we will refrain from discussing this for now). From here, we may define brackets on G via $[f,g](t) = [f(t),g(t)]$, which is consistent with the multiplication $f\cdot g(t) = f(t)\cdot g(t)$.

Let $(b_t)_{t\ge 0}$ denote $\mathcal{W}_0([0,1],V_1)$-valued Brownian motion corresponding to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_0([0,1],V_1)}$ by way of the classical Wiener space definition. Then there exists a group-valued heat kernel distributed element $g_T \in \mathcal{W}_0([0,1],V)$. The existence of the path space process has been worked out in much greater generality in [CD08]. There, it is shown that $(g_t)_{t\ge 0}$ can be realized as an L^2 martingale.

We have that g_T satisfies, for all $\tau\in[0,1]$,

$$\begin{aligned} g_T(\tau) \;&=\; \sum_{n=1}^{N}\sum_{\sigma\in\mathcal{S}_n} c_\sigma \int_{\Delta_T^n} [\ldots[\delta b_{s_{\sigma(1)}}(\tau),\delta b_{s_{\sigma(2)}}(\tau)],\ldots,\delta b_{s_{\sigma(n)}}(\tau)] \\ &=\; \sum_{n=1}^{N}\sum_{\sigma\in\mathcal{S}_n} c_\sigma \int_{\Delta_T^n} [\ldots[db_{s_{\sigma(1)}}(\tau),db_{s_{\sigma(2)}}(\tau)],\ldots,db_{s_{\sigma(n)}}(\tau)]\,, \end{aligned}$$

where the Stratonovich and Itô differentials are taken with respect to $s\mapsto b_s(\tau)$.

This also satisfies our assumption for being a nilpotent abstract Wiener Lie group, as it follows from Theorem 3.6. Note that a shorter proof could be provided, since the "real traces" of complex multilinear functionals are 0 (this is the key property applied in the proof of Lemma 10.14). Then for $f\in\mathcal{H}_0([0,1],V)^*$, we would only have to show the bound

$$\max_{1\le n\le N}\max_{\sigma\in\mathcal{S}_n}\left\| f\circ[\ldots[\cdot,\cdot],\ldots,\cdot]\circ\widetilde{\sigma}\right\|_{\mathcal{H}_0([0,1],V_1)^*} \;\le\; C\|f\|_{\mathcal{H}_0([0,1],V_1)}$$

(from opposed to using the "tracial norm").

We now turn to defining $\mathfrak{g}_{CM}$, which will not be as straightforward as one might hope, as we require our Hörmander condition to hold. Recall from Section 2.1.4 that $\mathcal{H}_0([0,1],V_n) \cong \mathcal{H}_0([0,1],\mathbb{C}) \otimes V_n$, so that if $\{\mathfrak{e}_k\}_{k\in\mathbb{Z}} = \{\frac{1}{2\pi ik}(e^{2\pi ikt}-1)\}_{k\neq 0} \cup \{t\}$ is a basis of $\mathcal{H}_0([0,1],\mathbb{C})$ and $\{v_{n,k}\}_{1\leq k\leq \dim(V_n)}$ is a basis of V_n, then $\{\mathfrak{e}_k v_{n,j}\}_{1\leq j\leq \dim(V),k\in\mathbb{Z}}$ constitutes a basis of $\mathcal{H}_0([0,1],V_n)$. If we define $\mathfrak{m}_n : \mathcal{H}_0([0,1],\mathbb{C})^n \to \mathcal{H}_0([0,1],\mathbb{C})$ as n-fold multiplication, then for $f \in \mathcal{H}_0([0,1],\mathbb{C})$ and $v \in V_n$,

$$\begin{aligned}
&\sum_{k_1,\ldots,k_n\in\mathbb{Z}} \sum_{j_1,\ldots,j_n=1}^{\dim(V_n)} \left| \left\langle fv, [\ldots[\mathfrak{e}_{k_1}v_{1,j_1}, \mathfrak{e}_{k_2}v_{j_2}],\ldots,\mathfrak{e}_{k_n}v_{j_n}] \right\rangle_{\mathcal{H}_0([0,1],V_n)} \right|^2 \\
&= \sum_{k_1,\ldots,k_n\in\mathbb{Z}} \left| \left\langle f, \mathfrak{m}_n(\mathfrak{e}_{k_1},\ldots,\mathfrak{e}_{k_n}) \right\rangle_{\mathcal{H}_0([0,1],\mathbb{C})} \right|^2 \sum_{j_1,\ldots,j_n=1}^{\dim(V_n)} \left| \left\langle v, [v_{j_1},v_{j_2}],\ldots,v_{j_n}] \right\rangle_{V_n} \right|^2 . \quad (3.8)
\end{aligned}$$

By its finite-dimensional nature, the latter sum will be bounded above and below by a constant times $\|v\|_V^2$. The former, however, will fail to have such a lower bound, and will pose issues for satisfying the Hörmander condition.

Proposition 10.22. *The multiplication map* $\mathfrak{m}_2 : \mathcal{H}_0([0,1],\mathbb{C}) \times \mathcal{H}_0([0,1],\mathbb{C}) \to \mathcal{H}_0([0,1],\mathbb{C})$ *is weakly Hilbert-Schmidt, but its extension is not surjective.*

Proof. For this proof, we will abbreviate $\mathcal{H}_0 := \mathcal{H}_0([0,1],\mathbb{C})$. For $h \in \mathcal{H}_0$,

$$\left| \left\langle \mathfrak{m}_2(t,t), h \right\rangle_{\mathcal{H}_0} \right|^2 = \left| 2\int_0^1 t\overline{h'(t)}dt \right|^2 \leq 4\int_0^1 |th'(t)|^2 dt \leq 4\|h\|_{\mathcal{H}_0}^2$$

and

$$\begin{aligned}
\sum_{k\neq 0} \left| \left\langle \mathfrak{m}_2(t,\mathfrak{e}_k), h \right\rangle_{\mathcal{H}_0} \right|^2 &= \sum_{k\neq 0} \left| \int_0^1 t\mathfrak{e}_k'(t)\overline{h'(t)}dt + \int_0^1 \mathfrak{e}_k(t)\overline{h'(t)}dt \right|^2 \\
&\leq 2\int_0^1 |th'(t)|^2 dt + \frac{1}{2\pi^2k^2}\int_0^1 |e^{2\pi ikt}-1|^2|h'(t)|^2 dt \\
&\leq 2\|h\|_{\mathcal{H}_0}^2 + \frac{1}{3}\|h\|_{\mathcal{H}_0}^2 ,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k,\ell\neq 0} \left| \left\langle \mathfrak{m}_2(\mathfrak{e}_k,\mathfrak{e}_\ell), h \right\rangle_{\mathcal{H}_0} \right|^2 &= \sum_{k,\ell\neq 0} \left| \frac{-1}{4\pi^2 k\ell} \left\langle e^{2\pi i(k+\ell)t} - e^{2\pi ikt} - e^{2\pi i\ell t} + 1, h \right\rangle_{\mathcal{H}_0} \right|^2 \\
&\leq \sum_{k,\ell\neq 0} \frac{1}{4\pi^2} \left| \frac{k+\ell}{k\ell} \langle \mathfrak{e}_{k+\ell}, h\rangle_{\mathcal{H}_0} - \frac{1}{\ell}\langle \mathfrak{e}_k, h\rangle_{\mathcal{H}_0} - \frac{1}{k}\langle \mathfrak{e}_\ell, h\rangle_{\mathcal{H}_0} \right|^2 \\
&\leq \frac{1}{\pi^2} \sum_{k,\ell\neq 0} \frac{1}{k^2}|\langle \mathfrak{e}_{k+\ell}, h\rangle_{\mathcal{H}_0}|^2 + \frac{1}{\ell^2}|\langle \mathfrak{e}_{k+\ell}, h\rangle_{\mathcal{H}_0}|^2 + \frac{1}{\ell^2}|\langle \mathfrak{e}_k, h\rangle_{\mathcal{H}_0}|^2 + \frac{1}{k^2}|\langle \mathfrak{e}_\ell, h\rangle_{\mathcal{H}_0}|^2 .
\end{aligned}$$

We note that $\sum_{\ell\neq 0}|\langle h,\mathfrak{e}_{k+\ell}\rangle_{\mathcal{H}_0}|^2 \leq \sum_{\ell\in\mathbb{Z}}|\langle h,\mathfrak{e}_{k+\ell}\rangle|^2 = \|h\|^2_{\mathcal{H}_0}$ and $\sum_{k\geq 1}\frac{1}{k^2} = \frac{\pi^2}{6}$, which justifies the sum above to be bounded by a constant times $\|h\|^2_{\mathcal{H}_0}$. Hence, $\mathfrak{m}$ is weakly Hilbert-Schmidt.

However, we must now show that the extension is not surjective. To show this, we will show that the image is contained in a dense proper subspace $\mathcal{K}$. Define a norm $\|f\|^2_{\mathcal{K}} := \|f\|^2_{\mathcal{H}_0} + \left(\lim_{t\to 0}\frac{|f(t)|}{t}\right)^2$, and set $\mathcal{K} = \{f\in\mathcal{H}_0 : \|f\|_{\mathcal{K}} < \infty\}$, on which we may define the inner product $\langle f,g\rangle_{\mathcal{K}} = \langle f,g\rangle_{\mathcal{H}_0} + \left(\lim_{t\to 0}\frac{f(t)}{t}\right)\left(\lim_{t\to 0}\frac{\overline{g(t)}}{t}\right)$.

Then consider that, for any $f\in\mathcal{H}_0$, similar to the calculation in (1.1),

$$|f(t)| = \Big|\int_0^1 f'(t)dt\Big| \leq \Big(\int_0^t |f'(t)|^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^t 1^2 dt\Big)^{\frac{1}{2}} = \|f' 1_{[0,t]}\|_{L^2}\sqrt{t}\,,$$

which is not only bounded by $\sqrt{t}\|f\|_{\mathcal{H}_0}$, but we also see $\frac{|f(t)|}{\sqrt{t}}$ is converging to 0 as $t\to 0$. Then for any $f,g\in\mathcal{H}_0$, $\lim_{t\to 0}\frac{|f(t)g(t)|}{t} = 0$. And for $h\in\mathcal{K}$, $\langle\mathfrak{m}_2(f,g),h\rangle_{\mathcal{K}} = \langle\mathfrak{m}_2(f,g),h\rangle_{\mathcal{H}_0} + (0)$. Then $\mathfrak{m}_2 : \mathcal{H}_0\times\mathcal{H}_0 \to \mathcal{K}$ is also weakly , meaning that the image of the extension is contained in (a closed subspace of) $\mathcal{K}$, which is the dense proper subset of $\mathcal{H}_0$ of functions with finite Lipschitz constant at $t = 0$. □

The key problem lies in the fact that multiplying finite-energy paths with the condition $f(0) = 0$ results in a function with improved regularity at $t = 0$. Indeed, $\{\mathfrak{e}_k\}_{k\neq 0}\cup\{t,1\}$ is a basis for

$$\mathcal{H}([0,1],\mathbb{C}) = \mathcal{H}_0([0,1],\mathbb{C})\oplus\mathbb{C} = \left\{f:[0,1]\to\mathbb{C} : \int_0^1 |f'(t)|^2 dt < \infty\right\}$$

with

$$\langle f,g,\rangle_{\mathcal{H}([0,1],\mathbb{C})} := \int_0^1 f'(t)\overline{g'(t)}dt + f(0)\overline{g(0)}\,,$$

and $|\langle\mathfrak{m}(1,1),h\rangle_{\mathcal{H}}|^2 + |\langle\mathfrak{m}_2(1,t),h\rangle_{\mathcal{H}}|^2 + \sum_{k\neq 0}|\langle\mathfrak{m}_2(1,\mathfrak{e}_k),h\rangle|^2 = \|h\|^2_{\mathcal{H}}$ Therefore the proof above can be adjusted to show that $\mathfrak{m}_2 : \mathcal{H}([0,1],\mathbb{C})\times\mathcal{H}([0,1],\mathbb{C}) \to \mathcal{H}([0,1],\mathbb{C})$ is both weakly Hilbert-Schmidt *and* has surjective extension. This suggests that we could instead use the set of *unbased* finite-energy paths $\mathcal{H}([0,1],V_1)\cong\mathcal{H}([0,1],\mathbb{C})\otimes V_1\cong\mathcal{H}_0([0,1],\mathbb{C})\otimes\mathbb{C}\otimes V_1$ as our generator for path space diffusion. However, this would change the process and dynamics, as it would correspond to a path space diffusion in which the starting point of a path is randomly chosen.

Now we describe another remedy that preserves the diffusion. Define $\mathcal{M}_1([0,1],\mathbb{C}) = \mathcal{H}_0([0,1],\mathbb{C})$ and $\mathcal{M}_n([0,1],\mathbb{C}) := \mathrm{im}(\widetilde{\mathfrak{m}_n})$ for $2\leq n\leq N$, and give it an inner product such that $\widetilde{\mathfrak{m}_n} : \mathcal{H}_0([0,1],\mathbb{C})^{\otimes n}/\ker(\widetilde{\mathfrak{m}_n}) \to \mathcal{M}_n([0,1],\mathbb{C})$ is unitary. Then define $\mathcal{M}_n([0,1],V_n) = \mathcal{M}_n([0,1],\mathbb{C})\otimes V_n \subseteq \mathcal{H}_0([0,1],V_n)$.

Using (3.8), it will be the case that $[\cdot,\cdot]_{1,n} : \mathcal{M}_1([0,1],V_1) \times \mathcal{M}_n([0,1],V_n) \to \mathcal{M}_{n+1}([0,1],V_{n+1})$ is weakly Hilbert-Schmidt and surjective. So we define

$$\mathfrak{g}_{CM} \;=\; \mathcal{M}_1([0,1],V_1) \times \ldots \times \mathcal{M}_N([0,1],V_N)\,.$$

Thus, in one sense, rather than changing the diffusion, we have modified our analysis of the diffusion to fit the assumptions.

11 The Taylor Isomorphism

11.1 Setup

11.1.1 $\mathcal{H}L^2_t(G)$

Let $\mathcal{P}$ denote the set of continuous holomorphic cylinder polynomials $f : G \to \mathbb{C}$, which are finite sums of monomials of the form $f_1 \cdot \ldots \cdot f_k$ where each $f_k \in G^*$. Consider the following.

Corollary 11.1. *For any $f \in \mathcal{P}$, $(P_m)_m \in Proj(W_1)^{\uparrow}$, and $p \in [1, \infty)$, $f(g_t^{P_m})$ converges in L^p to $f(g_t)$. In particular, we may say $\mathcal{P} \subseteq L^2(G, \nu_t)$.*

Proof. If f is of degree 1, then the conclusion follows by Theorem 10.15. Now suppose that $f_1, f_2 : G \to \mathbb{C}$ are holomorphic polynomials such that, for $j \in \{1, 2\}$, $f_j(g_t^{P_m})$ converges to $f_j(g_t)$ in L^p for $p \in [1, \infty)$. Then

$$\Big(\mathbb{E}|f_1(g_t)f_2(g_t)|^p\Big)^{1/p} \leq \Big(\mathbb{E}|f_1(g_t)|^{2p}\Big)^{1/2p}\Big(\mathbb{E}|f_2(g_t)|^{2p}\Big)^{1/2p},$$

so the product $f_1(g_t)f_2(g_t)$ is in L^p for all $p \in [1, \infty)$ as well, and

$$\begin{aligned}
&\Big(\mathbb{E}|f_1(g_t) \cdot f_2(g_t) - f_1(g_t^{P_m}) \cdot f_2(g_t^{P_m})|^p\Big)^{1/p} \\
&\quad \leq \Big(\mathbb{E}|(f_1(g_t) - f_1(g_t^{P_m}))f_2(g_t)|^p\Big)^{1/p} + \Big(\mathbb{E}|f_1(g_t^{P_m})(f_2(g_t) - f_2(g_t^{P_m}))|^p\Big)^{1/p} \\
&\quad \leq \Big(\mathbb{E}|f_1(g_t) - f_1(g_t^{P_m})|^{2p}\Big)^{1/2p}\Big(\mathbb{E}|f_2(g_t)|^{2p}\Big)^{1/2p} \\
&\qquad\quad + \Big(\mathbb{E}|f_1(g_t^{P_m})|^{2p}\Big)^{1/2p}\Big(\mathbb{E}|f_2(g_t) - f_2(g_t^{P_m})|^{2p}\Big)^{1/2p},
\end{aligned}$$

from which it is seen that $f_1(g_t^{P_m}) \cdot f_2(g_t^{P_m})$ converges to $f_1(g_t) \cdot f_2(g_t)$ in L^p. Iteratively applying this tells us that, for any monomial $f : G \to \mathbb{C}$ defined as $f = \prod_{j=1}^k f_j$ for $f_j \in G^*$, $f(g_t^{P_m})$ converges in L^p to $f(g_t)$. Then the same must be true of finite sums, which concludes the proof.

∎

We may now define $\mathcal{H}L^2_t(G)$ as the L^2-closure of $\mathcal{P}$. As G is a complex Banach space, there is a notion of holomorphic functions $f : G \to \mathbb{C}$. The definition in [HP74, Definition 3.17.2] is that a function $f : G \to \mathbb{C}$ is holomorphic if f is locally bounded, meaning that for all $x \in G$, there exists $r > 0$ such that $\sup\{|f(y)| : \|x - y\| < r\} < \infty$, and if f is complex Gâteaux differentiable on G, meaning that for every $x \in G$ and $v \in G$, the map $\mathbb{C} \ni h \mapsto f(x + hv) \in \mathbb{C}$ is complex differentiable at $\lambda = 0$.

However, it should be noted that functions in $\mathcal{H}L_t^2(G)$ do not, strictly speaking, satisfy these criteria, in particular the locally bounded condition. Indeed, even in the commutative case, we can consider the measurable-linear extension of $\langle \cdot, h\rangle_H$, where $h \in H_1 \setminus W_1^*$, which will not be continuous. Instead, the elements could perhaps be called "measurable holomorphic," and should be compared to the holomorphic Wiener functions of an abstract Wiener space, as described in [Sug97].

In [DG10] and [GM13], it is mentioned that it is not known if one may take the L^2-closure of holomorphic cylinder functions and arrive at the same set. This continues to be unanswered, but Corollary 11.1 suggests that we may regard $\mathcal{H}L_t^2(G)$ as the L^2-closure of the set of holomorphic functions $f : G \to \mathbb{C}$ such that $f(g_t^{P_m})$ converges in L^2 to a random variable. As noted in [DG10], after proving the Taylor isomorphism, we will be able to say something more general: $\mathcal{H}L_t^2(G)$ is equal to the L^2-closure of holomorphic functions for which $\sup_{m\in\mathbb{N}} \mathbb{E}|f(g_t^{P_m})|^2 < \infty$. It could perhaps be said that such a holomorphic function is in $L^2(G)$ "for the right reasons."

11.1.2 Noncommutative Fock space: defining $J_t^0(\mathfrak{g}_{CM})$

For any Hilbert space H, let $T(H) = H \oplus H^{\otimes 2} \oplus \ldots$ be the tensor algebra of H, by which we mean the set of finite sums of elements of the Hilbert space tensor powers of H. Given a basis $\{e_j\}_{j\in\Lambda}$ of H, elements of $T(H)$ can be expressed as $\sum_{k=0}^K \sum_{j_1,\ldots,j_k\in\Lambda} \alpha_{k,j_1,\ldots,j_k} e_{j_1} \otimes \ldots \otimes e_{j_k}$, in which the sum is finite over k, and for each k the coefficients $(\alpha_{k,j_1,\ldots,j_k})_{j_1,\ldots,j_k\in\Lambda}$ are square-sumamble. For such an element, we may refer to K as its *rank*.

Let $J(H)$ be the 2-sided ideal of $T(H)$ generated by $v\otimes w - w\otimes v - [v, w]$. Then let $T(H)'$ denote the algebraic dual of $T(H)$, and let $J^0(H)$ be the backwards anihilator of $J(H)$, that is,

$$J^0(H) = \{\alpha \in T(H)' : \langle \alpha, v\rangle = 0 \,\forall v \in J(H)\}.$$

Then, for $\alpha, \beta \in J^0(\mathfrak{g}_{CM})$, and a basis $\{e_j\}_{j\in\mathbb{N}} \subseteq H_1$, define the complex inner product

$$\begin{aligned}\langle \alpha, \beta\rangle_{J_t^0(\mathfrak{g}_{CM})} &:= \sum_{k=0}^\infty \frac{t^k}{k!} \langle \alpha, \beta\rangle_{\mathfrak{g}_{CM}^{*\otimes k}} \\ &= \sum_{k=0}^\infty \frac{t^k}{k!} \sum_{i_1,\ldots,i_k=1}^\infty \alpha(e_{i_1}\otimes\ldots\otimes e_{i_k})\overline{\beta(e_{i_1}\otimes\ldots\otimes e_{i_k})},\end{aligned}$$

with an associated norm $\|\cdot\|_{J_t^0(\mathfrak{g}_{CM})}$. We now define $J_t^0(\mathfrak{g}_{CM}) = \{\alpha \in J^0(\mathfrak{g}_{CM}) : \|\alpha\|_{J_t^0(\mathfrak{g}_{CM})} < \infty\}$. If $P \in Proj(W_1)$ with basis $\{e_j\}_{1\le j\le r}$, and if $\alpha \in J^0(\mathfrak{g}^P)$, we will similarly define

$$\|\alpha\|_{J_t^0(\mathfrak{g}^P)}^2 = \sum_{k=0}^\infty \frac{t^k}{k!} \sum_{j_1,\ldots,j_k=1}^r |\alpha(e_{j_1}\otimes\ldots\otimes e_{j_k})|^2$$

and set $J^0_t(\mathfrak{g}^P) = \{\alpha \in J^0(\mathfrak{g}^P) \,:\, \|\alpha\|_{J^0_t(\mathfrak{g}^P)} < \infty\}$.

We remark that we may view $J(\mathfrak{g}^P) \subseteq J(\mathfrak{g}_{CM})$, and in turn a natural inclusion $J^0(\mathfrak{g}_{CM}) \hookrightarrow J^0(\mathfrak{g}^P)$ defined via restriction. In fact, by noting $\|\alpha\|_{J^0_t(\mathfrak{g}^P)} \leq \|\alpha\|_{J^0_t(\mathfrak{g}_{CM})}$, we can conclude that we have the continuous inclusion $J^0_t(\mathfrak{g}_{CM}) \hookrightarrow J^0_t(\mathfrak{g}^P)$.

Theorem 11.2, stated and proven below, is a direct consequence of our Hörmander condition. This is directly analogous to [DGS09a, Lemma 2.12] and [GM13, Lemma 3.4]. Similarly to both of these works, we will prove the existence of a homomorphism $\Psi : T(\mathfrak{g}_{CM}) \to T(H_1)$ with nice properties. We define

$$(J^0)_0 := \{b \in T(\mathfrak{g}_{CM}) \,:\, \langle \alpha, b\rangle = 0 \text{ for all } \alpha \in J^0(\mathfrak{g}_{CM}\}\,.$$

Importantly, $(J^0)_0$ remains a 2-sided ideal, $(J^0)_0 \supseteq J(\mathfrak{g}_{CM})$, and $(J^0)_0$ will correspond to a certain type of closure of $J(\mathfrak{g}_{CM})$ that includes "finite-rank infinite sums" of elements in $T(\mathfrak{g}_{CM})$, as demonstrated in the proof below.

The proof of Theorem 11.8 uses the "lower constants" to compare $\|\alpha\|_{J^0_t(\mathfrak{g}_{CM})}$ with quantities testing α in non-H_1 directions, as in (3.12), and it is possible to prove this theorem with this estimate as well. Still, we will prove Theorem 11.2 using Ψ to show, after appropriate modification, methods used in previous contexts will continue to work in this one.

Theorem 11.2. *$\|\cdot\|_{J^0_t(\mathfrak{g}_{CM})}$ is a norm, so that $J^0_t(\mathfrak{g}_{CM})$ is a Hilbert space.*

Proof. For $2 \leq n \leq N$, we assume that $[\cdot]_{1,n-1} : H_1 \otimes H_{n-1} \to H_n$ is surjective. Then let $\phi_n : H_n \to H_1 \otimes H_{n-1}$ be the a right-inverse of $[\cdot]_{1,n-1}$, for example the inverse of the bijection $[\cdot] : \ker([\cdot]_{1,n-1})^\perp \to H_n$. Then let $\widetilde{s}_n$ be the cannonical "swap" $H_1 \otimes H_n \to H_n \otimes H_1$ that satisfies, for $h_1 \in H_1$, $h_n \in H_n$, $\widetilde{s}_n(h_1 \otimes h_n) = h_n \otimes h_1$. Then we define $\psi_n : H_n \to H_1 \otimes H_{n-1} + H_{n-1} \otimes H_n$ as $\psi_n = \phi_n - \widetilde{s}_n \circ \phi_n$. Then consider that, for $h \in H_n$,

$$\frac{1}{2}[\cdot] \circ \psi_n(h) \;=\; \frac{1}{2}\Big([\cdot]_{1,n-1} \circ \phi_n\Big)(h) - \frac{1}{2}\Big([\cdot]_{n-1,1} \circ \widetilde{s}_n \circ \phi_n\Big)(h) \;=\; h\,.$$

It can be seen that $\mathrm{im}(\psi_n) \subseteq H_1 \otimes H_{n-1} + H_{n-1} \otimes H_1 \subseteq \mathfrak{g}_{CM}^{\otimes 2}$ are all alternating tensors, and as such can be written as (potentially infinite) sums of elements of the form $a \otimes b - b \otimes a$.

Next, I claim that, for any $h \in H_n$, $\psi_n(h) - h \in (J^0)_0$. To show this, if $\psi_n(h) = \sum_{k=1}^\infty (a_k \otimes b_k - b_k \otimes a_k)$, which converges in $\mathfrak{g}_{CM}^{\otimes 2}$, then

$$h \;=\; \frac{1}{2}[\cdot] \circ \psi_n(h) \;=\; \frac{1}{2}\sum_{k=1}^\infty \big([a_k, b_k] - [b_k, a_k]\big) \;=\; \sum_{k=1}^\infty [a_k, b_k]\,,$$

from which it can be seen that

$$\psi_n(h_n) - h_n = \sum_{k=1}^{\infty} \Big(a_k \otimes b_k - b_k \otimes a_k - [a_k, b_k] \Big) .$$

Then we know that, for any $\alpha \in J^0(\mathfrak{g}_{CM})$, the partial sums of this expression will evaluate to 0, or more precisely, for $K \in \mathbb{N}$,

$$\alpha\left(\sum_{k=1}^{K} \Big(a_k \otimes b_k - b_k \otimes a_k - [a_k, b_k] \Big) \right) = 0$$

and $\alpha \in T(\mathfrak{g}_{CM})'$ means in particular that α is continuous on the Hilbert space $\mathfrak{g}_{CM} \oplus \mathfrak{g}_{CM}^{\otimes 2}$. This, along with the assumed continuity of $[\cdot]_{1,n-1} : H_1 \otimes H_{n-1} \to H_n$, proves the claim.

Define $S_n := \bigoplus_{k=1}^n H_k \subseteq \mathfrak{g}_{CM}$. For each n, we will apply 2 extensions to $\psi_n : H_n \to S_{n-1}^{\otimes 2}$. The first is by extending to $\psi_n : S_n \to S_{n-1} \oplus S_{n-1}^{\otimes 2}$ by declaring $\psi_n(h) = h$ for all $h \in S_{n-1}$. We remark that, by iteratively tensoring the map ψ_n with itself, we also have continuous maps $\psi_n^{\otimes k} : S_n^{\otimes k} \to (S_{n-1} \oplus S_{n-1}^{\otimes 2})^{\otimes k}$ for all $k \in \mathbb{N}$. Then the second extension is to an algebra homomorphism $\Psi_n : T(S_n) \to T(S_{n-1})$ by declaring, for $\alpha = \sum_{k=0}^K (\alpha)_k \in T(S_n)$, $\Psi_n(\alpha) = \sum_{k=0}^K \psi_n^{\otimes k}\big((\alpha)_k\big)$ (where $\psi_n^{\otimes 0} : \mathbb{C} \to \mathbb{C}$ is merely the identity).

Now define $\Psi = \Psi_2 \circ \ldots \circ \Psi_N : T(\mathfrak{g}_{CM}) \to T(H_1)$. Using the fact that $(J^0)_0$ is an ideal, we still have that, for all $a \in T(S_n)$, $\Psi_n(a) - a \in (J^0)_0$. As a consequence, for all $a \in T(\mathfrak{g}_{CM})$, $\Psi(a) - a \in (J^0)_0$. In particular, we have that $ker\Psi \subseteq (J^0)_0$.

Finally, for any $\alpha \in J^0(\mathfrak{g}_{CM})$, if $\|\alpha\|_{J_t^0(\mathfrak{g}_{CM})} = 0$, then $\alpha|_{T(H_1)} \equiv 0$. We then have that, for all $a \in T(\mathfrak{g}_{CM})$, $\langle \alpha, a - \Psi(a) \rangle = 0$, so that $\alpha = \alpha \circ \Psi$. Since Ψ maps into H_1, $\alpha \circ \Psi = \alpha|_{T(H_1)} \circ \Psi = 0$. Thus, $\| \cdot \|_{J_t^0(\mathfrak{g}_{CM})}$ is a norm. ☐

Remark 11.3. With a bit more effort, one can prove a theorem analogous to Theorem 2.7 and Corollary 2.14 of [DGS09a], which states that the following are equivalent: (1) our Hörmander condition holds, (2) $\|\cdot\|_{J_t^0(\mathfrak{g}_{CM})}$ is a norm, (3) $T(\mathfrak{g}_{CM}) = T(H_1) \oplus (J^0)_0$.

11.1.3 The finite-dimensional Taylor isomorphism theorems

As defined in Section 10.2, every G^P is a finite-dimensional complex graded nilpotent Lie group that satisfies the Hörmander condition. We have that $\mathcal{H}L_t^2(G^P) := \mathcal{L}^2(G^P, \nu_t^P, \mathbb{C}) \cap \mathcal{H}(G^P)$ is a closed subset of $\mathcal{L}_t^2(G^P)$ (see, for example, [Dri15, Corollary 3.3]).

Given holomorphic $f : G^P \to \mathbb{C}$, we define the derivative $\widehat{f}(e) \in T(\mathfrak{g}^P)^*$ as $\widehat{f}(e)(h_1 \otimes \ldots \otimes h_k) = \widetilde{h}_1 \ldots \widetilde{h}_k f(e)$. Then we have the finite-dimensional Taylor isomorphism below.

Theorem 11.4 ([DGS09a, Theorem 6.1])**.** *The map* $\mathcal{H}L^2_t(G^P) \to J^0_t(\mathfrak{g}^P)$ *defined as* $f \mapsto \widehat{f}(e)$ *is unitary.*

And the theorem below is a result that is shown in [DGS09a] along the way.

Theorem 11.5 ([DGS09a, Corollary 5.15])**.** *For all* $g \in G^P$*, the map* $\mathcal{H}L^2_t(G) \to \mathbb{C}$ *defined by point evaluation* $f \mapsto f(g)$ *is continuous, and*

$$|f(g)| \le \|\widehat{f}(e)\|_{J^0_t(G^P)} e^{d^P(e,g)^2/2t}.$$

11.2 The restriction map

We will now begin defining the first of the 2 maps that make up the Taylor isomorphism, namely the restriction map $\mathcal{R}$. Any continuous holomorphic cylinder polynomial $f \in \mathcal{P}$ has a natural restriction, namely $f|_{G_{CM}}$, and the goal of this section is to prove that we may naturally associate any $f \in \mathcal{H}L^2_t(G)$ with a function $\mathcal{R}f : G_{CM} \to \mathbb{C}$ with holomorphic properties. This is in the spirit of the "skeleton map" as in abstract Wiener space, as discussed in [Sug94a; Sug94b].

As discussed in Section 9.2, we define $\mathcal{H}L^2_t(\bigcup_{P\in Proj(W_1)} G^P)$ as the set of functions $f : \bigcup_{P\in Proj(W_1)} G^P \to \mathbb{C}$ such that, for all $P \in Proj(W_1)$, $f \circ \iota^P \in \mathcal{H}L^2_t(G^P)$ and $\sup_{P\in Proj(W_1)} \|f\circ\iota^P\|_{\mathcal{H}L^2_t(G^P)} < \infty$ (recall that $\iota^P : G^P \to G$ is the canonical inclusion). We will henceforth abbreviate this notation as $\mathcal{H}L^2_t(\bigcup_P G^P)$ and $\sup_P \|f\circ\iota^P\|_{\mathcal{H}L^2_t(G^P)}$.

We begin with a lemma.

Lemma 11.6. *For* $f \in \mathcal{P}$*,* $f \circ \iota^P$ *is always in* $\mathcal{H}L^2_t(G^P)$*, and* $\|f \circ \iota^{P_m}\|_{\mathcal{H}L^2_t(G^{P_m})}$ *increases to* $\sup_P \|f \circ \iota^P\|_{\mathcal{H}L^2_t(G^P)}$*, which equals* $\|f\|_{\mathcal{H}^2_t(G)}$*.*

Proof. We see that, if $f \in \mathcal{P}$, then $f \circ \iota^P$ is a holomorphic polynomial on G^P. Moreover, by Corollary 11.1, $f \in L^2(G^P)$, and $\mathbb{E}|f(g_t^{P_m})|^2 \to \mathbb{E}|f(g_t)|^2$, so that $\sup_P \|f \circ \iota^P\|_{\mathcal{H}L^2_t(G^P)} \ge \lim_{m\to\infty} \|f \circ \iota^{P_m}\|_{\mathcal{H}L^2_t(G^P)} = \|f\|_{\mathcal{H}L^2_t(G)}$.

Next, say $\{e_j\}_{1\le j\le r_m}$ is a basis of $P_m H_1$ for all m. Consider that, by the finite-dimensional Taylor isometry, Theorem 11.4,

$$\|f \circ \iota^{P_m}\|^2_{\mathcal{H}L^2_t(G^{P_m})} = \|\widehat{f \circ \iota^{P_m}}(e)\|^2_{J^0_t(\mathfrak{g}^{P_m}_{CM})} = \sum_{k=0}^{\infty} \sum_{j_1,\dots,j_k=1}^{r_m} \frac{t^k}{k!} |\widetilde{e_{j_1}} \dots \widetilde{e_{j_k}} f(e)|^2 .$$

Then

$$\begin{aligned}\|f\circ\iota^{P_m}\|^2_{\mathcal{H}L^2_t(G^{P_m})} &= \sum_{k=0}^{\infty}\sum_{j_1,\dots,j_k=1}^{r_m}\frac{t^k}{k!}|\widetilde{e_{j_1}}\dots\widetilde{e_{j_k}}f(e)|^2\\ &\le \sum_{k=0}^{\infty}\sum_{j_1,\dots,j_k=1}^{r_{m+1}}\frac{t^k}{k!}|\widetilde{e_{j_1}}\dots\widetilde{e_{j_k}}f(e)|^2\\ &= \|f\circ\iota^{P_{m+1}}\|^2_{\mathcal{H}L^2_t(G^{P_{m+1}})}.\end{aligned}$$

Thus, $\|f\circ\iota^{P_m}\|_{\mathcal{H}L^2_t(G^{P_m})}$ is increasing. It can be concluded that, for any $P\in Proj(W_1)$, and for any sequence (P_m) where $P\in\{P_m\}_m\in Proj(W_1)^{\uparrow}$, we have $\|f\circ\iota^P\|_{\mathcal{H}L^2_t(G^P)}\le\lim_{m\to\infty}\|f\circ\iota^{P_m}\|_{\mathcal{H}L^2_t(G^{P_m})}=\|f\|_{\mathcal{H}L^2_t(G)}$ for all P, which implies that $\sup_P\|f\circ\iota^P\|_{\mathcal{H}L^2_t(G^P)}\le\|f\|_{\mathcal{H}L^2_t(G)}$. Therefore, $\sup_P\|f\circ\iota^P\|_{\mathcal{H}L^2_t(G^P)}=\|f\|_{\mathcal{H}L^2_t(G)}$.

□

The next theorem proves the existence of the restriction map.

Theorem 11.7. *To every $f\in\mathcal{H}L^2_t(G)$, we may associate a function $\mathcal{R}f:G_{CM}\to\mathbb{C}$ such that*

- *for all $f\in\mathcal{P}$, $\mathcal{R}f=f|_{G_{CM}}$.*
- *for $f\in\mathcal{H}L^2_t(G)$ and $g\in G_{CM}$, $|\mathcal{R}f(g)|\le\|f\|_{L^2(\nu_t)}e^{d(e,g)^2/2t}$.*
- *for every $P\in Proj(W_1)$, $\mathcal{R}f\circ\iota^P\in\mathcal{H}L^2_t(G^P)$, and $\sup_P\|\mathcal{R}f\circ\iota^P\|_{\mathcal{H}L^2_t(G^P)}=\|f\|_{\mathcal{H}L^2_t(G)}$. In other words, $\mathcal{R}f|_{\bigcup_P G^P}\in\mathcal{H}L^2_t(\bigcup_P G^P)$.*

Proof. Let $f\in\mathcal{P}$. By Theorem 11.5, for all $P\in Proj(W_1)$, for all $g\in G^P$, $|f(g)|\le\|f\circ\iota^P\|_{\mathcal{H}L^2_t(G^P)}e^{d^P(e,g)^2/2t}$. Now take any $g\in G_{CM}$. By Theorem 10.11, there exists a sequence $P_m\in Proj(W_1)^{\uparrow}$ and $g_m\in G^{P_m}$ such that $g_m\to g$ in $\|\cdot\|_{\mathfrak{g}_{CM}}$ (and hence in $\|\cdot\|_G$), and $d^{P_m}(e,g_m)\to d(e,g)$. Then, by the continuity of f, we may take the limit of the inequality $|f(g_m)|\le\|f\|_{\mathcal{H}L^2_t(G^{P_m})}e^{d^{P_m}(e,g_m)^2/2t}$ to get $|f(g)|\le\|f\|_{\mathcal{H}L^2_t(G)}e^{d(e,g)^2/2t}$. Then for $g\in G_{CM}$, the evaluation map $\mathcal{R}_g:\mathcal{P}\to\mathbb{C}$ defined as $\mathcal{R}_gf=f(g)$ satisfies $|\mathcal{R}_gf|\le\|f\|_{\mathcal{H}L^2_t(G)}e^{d(e,g)^2/2t}$. Then we conclude that there is a continuous linear extension $\mathcal{R}_g:\mathcal{H}L^2_t(G)\to\mathbb{C}$, for which the same bound applies.

Now for any $f\in\mathcal{H}L^2_t(G)$, we define $\mathcal{R}f:G_{CM}\to\mathbb{C}$ as $\mathcal{R}f(g)=\mathcal{R}_gf$. This definition will immediately satisfy the first 2 points. For the third, we must show that, for arbitrary $f\in\mathcal{H}L^2_t(G)$, the map $\mathcal{R}f$ has holomorphic properties. To do this, we will construct an alternate restriction map that factors through $\mathcal{H}L^2_t(G^P)$. Consider that we may apply the restriction $\mathcal{P}\ni f\mapsto f\circ\iota^P\in\mathcal{H}L^2_t(G^P)$ and use the bound from Lemma 11.6, $\|f\circ\iota^P\|_{\mathcal{H}L^2_t(G^P)}\le\|f\|_{\mathcal{H}L^2_t(G)}$, and extend to get a continuous map

$\mathcal{R}^P : \mathcal{H}L_t^2(G) \mapsto \mathcal{H}L_t^2(G^P)$ with the bound $\|\mathcal{R}^P f\|_{\mathcal{H}L_t^2(G^P)} \leq \|f\|_{\mathcal{H}L_t^2(G)}$. Furthermore, by Theorem 11.5, for any $f \in \mathcal{H}L_t^2(G)$, and $g \in G^P$,

$$|\mathcal{R}^P f(g)| \leq \|\mathcal{R}^P f\|_{\mathcal{H}L_t^2(G^P)} e^{d^P(e,g)^2/2t} \leq \|f\|_{\mathcal{H}L_t^2(G)} e^{d^P(e,g)^2/2t}. \tag{3.9}$$

Then for $P \in Proj(W_1)$ and $g \in G^P$, we define $(\mathcal{R}^P)_g : \mathcal{H}L_t^2(G) \to \mathbb{C}$ as the composition $f \mapsto \mathcal{R}^P f \mapsto \mathcal{R}^P f(g)$, where (3.9) above justifies continuity.

Consider that, for any $g \in G^P$, $\mathcal{R}_g$ and $(\mathcal{R}^P)_g$ coincide on $\mathcal{P}$. Indeed, if $f \in \mathcal{P}$, $(\mathcal{R}^P)_g f = f|_{G^P}(g) = f(g) = \mathcal{R}_g f$. This implies that $(\mathcal{R}^P)_g f = \mathcal{R}_g f$ for $f \in \mathcal{H}L_t^2(G)$, so that $\mathcal{R}f \circ \iota^P = \mathcal{R}^P f$ for all $P \in Proj(W_1)$, so we must have $\mathcal{R}f \circ \iota^P \in \mathcal{H}L_t^2(G^P)$. Moreover, $\sup_P \|\mathcal{R}f \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)} = \sup_{P \in Proj(W_1)} \|\mathcal{R}^P f\|_{\mathcal{H}L_t^2(G^P)} \leq \|f\|_{\mathcal{H}L_t^2(G)}$.

Lastly, for $f \in \mathcal{H}L_t^2(G)$, let $(f_m)_{m \in \mathbb{N}}$ be a sequence in $\mathcal{P}$ be such that $f_m \to f$ in $\mathcal{H}L_t^2(G)$. Then

$$\begin{aligned}\left|\sup_P \|\mathcal{R}f \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)} - \sup_P \|\mathcal{R}f_m \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)}\right| \\ \leq \sup_P \|\mathcal{R}f \circ \iota^P - \mathcal{R}f_m \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)} \\ \leq \|f - f_m\|_{\mathcal{H}L_t^2(G)} \to 0\end{aligned}$$

and thus, since $\|f_m\|_{\mathcal{H}L_t^2(G)} = \sup_P \|\mathcal{R}f_m \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)}$ for all m, we must have $\|f\|_{\mathcal{H}L_t^2(G)} = \sup_P \|\mathcal{R}f \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)}$, which proves the final claim.

□

We now remind the reader that, as noted in the introduction, while we know that $\bigcup_P G^P \subseteq G_{CM}$, it cannot be deduced whether $\bigcup_P G^P = G_{CM}$ in every case. However, Section 11.5.3, we will show that every function $F \in \mathcal{H}L_t^2(\bigcup_P G^P)$ has a natural extension to G_{CM}, which will correspond to the G_{CM}-wide definition that $\mathcal{R}$ provides.

11.3 The Taylor map

Recall that, for $f : G^P \to \mathbb{C}$, we define $\widehat{f}(e) : T(\mathfrak{g}^P) \to \mathbb{C}$ as $\widehat{f}(e)(h_1 \otimes \ldots \otimes h_n) := \widetilde{h_1} \ldots \widetilde{h_n} f(e)$. By [DGS09a], we know $\widehat{f}(e) \in J_t^0(\mathfrak{g}^P)$. In the following proof, we will first show that if $f \in \mathcal{H}L_t^2(\bigcup_P G^P)$, then $\widehat{f}(e)$ is defined on $T(H_1)$, then show that it has an extension to $T(\mathfrak{g}_{CM})$. In doing so, we will demonstrate that, in a certain sense, we can take derivatives of functions in $\mathcal{H}L_t^2(\bigcup_P G^P)$ in directions that do not even exist in G_{CM}. This is can done by realizing that, for example, a first order derivative in H_N is equivalent to a (potentially infinite) sum of Nth-order derivatives in H_1 (such an operator should perhaps be merely thought of as a "derivation[2]" for functions defined on $\bigcup_P G^P$). The convergence of this expression will come from the assumption $\sup_P \|f \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)} < \infty$, while the existence of such an expression will be achieved by using the Hörmander condition. Equivalently, one could easily extend $\widehat{f}(e)$ to $T(\mathfrak{g}_{CM})$ by merely defining $\widehat{f}(e)(h_n) := \widehat{f}(e)(\Psi(h_n))$, where Ψ is defined in the proof of Theorem 11.2, but this would still leave the task of proving that this is an honest extension of $f \circ \iota^P \in T(\mathfrak{g}^P)$, as well as that this lies in $J^0(\mathfrak{g}_{CM})$. Instead, we will more carefully define the extension, denoted $\mathcal{T}f$, and these facts will be proven. In particular it will be a consequence (instead of an assumption) that $\mathcal{T}f \circ \Psi = \mathcal{T}f$.

Theorem 11.8. *Let $F \in \mathcal{H}L_t^2(\bigcup_P G^P)$. The multilinear map $\mathcal{T}^{(k)}F : H_1^k \to \mathbb{C}$ defined as $\mathcal{T}^{(k)}F(h_1, \ldots, h_k) = \widetilde{h}_1 \ldots \widetilde{h}_k(F \circ \iota^P)(e)$ where $PH_1 = \operatorname{span}(h_1, \ldots, h_k)$ makes sense and uniquely determines a map $\mathcal{T}F \in T(\mathfrak{g}_{CM})'$ satisfying, for $h_1, \ldots, h_k \in H_1$, $\mathcal{T}F(h_1 \otimes \ldots \otimes h_k) = \mathcal{T}^{(k)}F(h_1, \ldots, h_k)$. Additionally, $\mathcal{T}F$ is an extension of $\widehat{F \circ \iota^P}(e)$ for all $P \in Proj(W_1)$. In other words, $\widehat{F \circ \iota^P}(e) = \mathcal{T}F|_{T(\mathfrak{g}^P)}$.*

And lastly, for every F, $\mathcal{T}F \in J_t^0(\mathfrak{g}_{CM})$, and the map $\mathcal{T} : \mathcal{H}L_t^2(\bigcup_P G^P) \to J_t^0(\mathfrak{g}_{CM})$ is isometric, so that $\|\mathcal{T}F\|_{J_t^0(\mathfrak{g}_{CM})} = \sup_P \|F \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)}$.

Proof. For any set $h_1, \ldots, h_k$, let $P : H_1 \to H_1$ be the finite-rank projection onto $\operatorname{span}(h_1, \ldots, h_k)$. Then for $h \in PH_1$, $g \in G^P$, since G^P is a subgroup of G, we have

$$\widetilde{h}(F \circ \iota^P)(g) = \frac{d}{dt}\Big|_{t=0} F(\iota^P(g \cdot (th))) = \frac{d}{dt}\Big|_{t=0} F(g \cdot (th)) = ((\widetilde{h}F) \circ \iota^P)(g) \,. \tag{3.10}$$

Hence, $\widetilde{h}(F \circ \iota^P) = (\widetilde{h}F) \circ \iota^P$. We may deduce that $\widetilde{h}_1 \ldots \widetilde{h}_k F(e)$ exists and equals $\widetilde{h}_1 \ldots \widetilde{h}_k(F \circ \iota^P)(e)$. So we may indeed define the multilinear map $T^{(k)}F : H_1^k \to \mathbb{C}$.

Recall that, by the finite-dimensional isometry, Theorem 11.4, $\|F \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)} = \|\widehat{F \circ \iota^P}(e)\|_{J_t^0(\mathfrak{g}^P)}$. Then observe that, for any $P \in Proj(W_1)$, with basis $\{e_j\}_{1 \le j \le r}$ of PH_1, and for any $\ell \in \mathbb{N}$,

$$\begin{aligned}
\sum_{j_1,\ldots,j_\ell=1}^{r} |\mathcal{T}^{(\ell)}F(e_{j_1}, \ldots, e_{j_\ell})|^2 &= \sum_{j_1,\ldots,j_\ell=1}^{r} |\widetilde{e_{j_1}} \ldots \widetilde{e_{j_\ell}} F(e)|^2 \\
&\le \frac{\ell!}{t^\ell} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j_1,\ldots,j_k=1}^{r} |\widetilde{e_{j_1}} \ldots \widetilde{e_{j_k}} F(e)|^2 \\
&= \frac{\ell!}{t^\ell} \|F \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)} \le \frac{\ell!}{t^\ell} \sup_P \|F \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)} \,.
\end{aligned}$$

[2] Moreover, the proof of Theorem 11.8 may provide justification that $\mathfrak{g}_{CM}$ could be considered the "set of left-invariant vector fields" of G_{CM}, making it fitting of being considered its corresponding Lie algebra; see Section 9.4.2 for more discussion.

Then for every $\ell \in \mathbb{N}$, $\widehat{F}^{(\ell)}(e)$ is Hilbert-Schmidt and extends to $H_1^{\otimes \ell}$. Thus, we may regard $\mathcal{T}F = (\mathcal{T}^{(\ell)}F)_{\ell\in\mathbb{N}} \in T(H_1)'$.

We must show that $\mathcal{T}F$ has an extension to $T(\mathfrak{g}_{CM})$. Firstly, note that each iterated 1-bracket $[\cdot, \ldots, [\cdot,\cdot]_{1,1}, \ldots]_{1,n-1} : H_1^{\otimes n} \to H_n$ is surjective. Then if $(P_m)_{m\in\mathbb{N}} \in Proj(W_1)^{\uparrow}$, it follows that $\bigcup_{m\in\mathbb{N}}[\ldots[P_mH_1, P_mH_1]_{1,1}, \ldots, P_mH_1]_{1,n-1}$ is dense in H_n, which implies that $\bigcup_{m\in\mathbb{N}} \mathfrak{g}^{P_m}$ is dense in $\mathfrak{g}_{CM}$. Thus, for each H_n, we may choose a basis $\{e_{n,j}\}_{j\in\Lambda_n}$ such that $\{e_{1,j}\}_{1\le j\le r_m} = \{e_j\}_{1\le j\le r_m}$ is a basis of P_mH_1, and each $e_{n,j} \in \mathfrak{g}^{P_m}$ for some $m \in \mathbb{N}$.

Then recall that, by our Hörmander condition and Proposition 2.4, there exists a constant c_n such that, for all $v \in H_n$,

$$\sum_{j_1,\ldots,j_n=1}^{\infty} \left|\langle [e_{j_1}, \ldots, [e_{j_{n-1}}, e_{j_n}]\ldots]\,,\, v\rangle_{H_n}\right|^2 \ge c_n \|v\|_{H_n}^2 .$$

Then consider the following calculation[3].

$$\begin{aligned}
\sum_{r=1}^{\infty} |\widetilde{e_{n,r}}F(e)|^2 &= \left\| \sum_{r=1}^{\infty} \big(\widetilde{e_{n,r}}F(e)\big) e_{n,r} \right\|_{H_n}^2 \\
&\le \frac{1}{c_n} \sum_{j_1,\ldots,j_n=1}^{\infty} \left| \left\langle [e_{j_1}, \ldots, [e_{j_{n-1}}, e_{j_n}]\ldots]\,,\, \sum_{r=1}^{\infty} \big(\widetilde{e_{n,r}}F(e)\big) e_{n,r} \right\rangle_{H_n} \right|^2 \\
&= \frac{1}{c_n} \sum_{j_1,\ldots,j_n=1}^{\infty} \left| \sum_{r=1}^{\infty} \big\langle [e_{j_1}, \ldots, [e_{j_{n-1}}, e_{j_n}]\ldots], e_{n,r} \big\rangle_{H_n} \big(\widetilde{e_{n,r}}F(e)\big) \right|^2 \\
&= \frac{1}{c_n} \sum_{j_1,\ldots,j_n=1}^{\infty} \left| \widetilde{[e_{j_1}, \ldots, [e_{j_{n-1}}, e_{j_n}]\ldots]}F(e) \right|^2 \\
&\le \frac{K}{c_n} \sum_{j_1,\ldots,j_n=1}^{\infty} |\widetilde{e_{j_1}} \ldots \widetilde{e_{j_n}} F(e)|^2
\end{aligned} \tag{3.11}$$

for some $K \in \mathbb{R}$, where we remark that the final inequality follows from the fact that $\widetilde{[e_{j_1}, \ldots, [e_{j_{n-1}}, e_{j_n}]\ldots]}F(e)$ is a sum of 2^n terms of the form $\widetilde{e_{j_{\sigma(1)}}} \ldots \widetilde{e_{j_{\sigma(n)}}} F(e)$ over permutations $\sigma \in S_n$. This shows that the map $H_n \ni h \mapsto \widetilde{h}F(e) \in \mathbb{C}$ is Hilbert-Schmidt.

Moreover, for surjective continuous maps $M_1 : H_1^{\otimes n_1} \to K_1$ and $M_2 : H_1^{\otimes n_2} \to K_2$, the tensor product $M_1 \otimes M_2 : H_1^{\otimes(n_1+n_2)} \to K_1 \otimes K_2$ is also surjective, and by

[3]Note that, by the choice of the bases $\{e_{n,r}\}_{j\in\Lambda_n}$, the expression in the third line is a finite sum in r (for each choice of $j_1, \ldots, j_k$), and importantly that there is no question of its convergence in r.

Proposition 2.4 will have a constant c such that, for all $z \in K_1 \otimes K_2$,

$$\sum_{j_1,\ldots,j_{n_1+n_2}=1}^{\infty} \left| \left\langle M_1(e_{j_1},\ldots,e_{j_{n_1}}) \otimes M_2(e_{j_{n_1+1}},\ldots,e_{j_{n_1+n_2}}\ ,\ z \right\rangle_{K_1\otimes K_2} \right|^2 \ \geq\ c\|z\|^2_{K_1\otimes K_2}$$

Then applying this to iterated brackets, we'll have that such a lower constant c exists for arbitrary tensor products of iterated brackets of the form

$$\Big([\cdot,\ldots,[\cdot,\cdot]_{1,1}\ldots]_{1,n_1-1} \Big) \otimes \ldots \otimes \Big([\cdot,\ldots,[\cdot,\cdot]_{1,1}\ldots]_{1,n_k-1} \Big) : \\ H_1^{\otimes n_1} \otimes \ldots \otimes H_1^{\otimes n_k} \to H_{n_1} \otimes \ldots \otimes H_{n_k}\,,$$

and in fact $c = c_{n_1} \cdot \ldots c_{n_k}$. Then, using similar techniques to (3.11), we have

$$\begin{aligned}
&\sum_{r_1,\ldots,r_k=1}^{\infty} |\widetilde{e_{n_1,r_1}} \ldots \widetilde{e_{n_k,r_k}} F(e)|^2 \\
&= \left\| \sum_{r_1,\ldots,r_k=1}^{\infty} \left(\widetilde{e_{n_1,r_1}} \ldots \widetilde{e_{n_k,r_k}} F(e) \right) e_{n_1,r_1} \otimes \ldots \otimes e_{n_k,r_k} \right\|^2_{H_{n_1}\otimes\ldots\otimes H_{n_k}} \\
&\leq \frac{1}{c} \sum_{\substack{j_{1,1},\ldots,j_{1,n_1}=1 \\ \vdots \\ j_{k,1},\ldots,j_{k,n_k}=1}}^{\infty} \Bigg| \Bigg\langle [e_{j_{1,1}},\ldots,[e_{j_{1,n_1-1}},e_{j_{1,n_1}}]\ldots] \otimes \ldots \otimes [e_{j_{k,1}},\ldots,[e_{j_{k,n_k-1}},e_{j_{k,n_k}}]\ldots]\,, \\
&\qquad \sum_{r_1,\ldots,r_k=1}^{\infty} \left(\widetilde{e_{n_1,r_1}} \ldots \widetilde{e_{n_k,r_k}} F(e) \right) e_{n_1,r_1} \otimes \ldots \otimes e_{n_k,r_k} \Bigg\rangle_{H_{n_1}\otimes\ldots\otimes H_{n_k}} \Bigg|^2 \\
&= \frac{1}{c} \sum_{\substack{j_{1,1},\ldots,j_{1,n_1}=1 \\ \vdots \\ j_{k,1},\ldots,j_{k,n_k}=1}}^{\infty} \Bigg| \sum_{r_1,\ldots,r_k=1}^{\infty} \left\langle [e_{j_{1,1}},\ldots,[e_{j_{1,n_1-1}},e_{j_{1,n_1}}]\ldots], e_{n_1,r_1} \right\rangle_{H_{n_1}} \cdots \\
&\qquad \left\langle [e_{j_{k,1}},\ldots,[e_{j_{k,n_k-1}},e_{j_{k,n_k}}]\ldots], e_{n_k,r_k} \right\rangle_{H_{n_k}} \left(\widetilde{e_{n_1,r_1}} \ldots \widetilde{e_{n_k,r_k}} F(e) \right) \Bigg|^2 \\
&\leq \frac{K}{c} \sum_{\substack{j_{1,1},\ldots,j_{1,n_1}=1 \\ \vdots \\ j_{k,1},\ldots,j_{k,n_k}=1}}^{\infty} \left| \left(\widetilde{e_{j_{1,1}}} \ldots \widetilde{e_{j_{1,n_1}}} \right) \ldots \left(\widetilde{e_{j_{k,1}}} \ldots \widetilde{e_{j_{k,n_k}}} \right) F(e) \right|^2
\end{aligned} \tag{3.12}$$

for some K. This proves that the map $H_{n_1} \times \ldots \times H_{n_k} \ni (h_1,\ldots,h_n) \mapsto \widetilde{h_1} \ldots \widetilde{h_n} F(e) \in \mathbb{C}$ is weakly Hilbert-Schmidt, and thus has a unique extension to $H_{n_1} \otimes \ldots \otimes H_{n_k}$.

In this way, since $\mathfrak{g}_{CM} = H_1 \oplus \ldots \oplus H_N$, we may conclude that we have an extension of $\mathcal{T}F$ to $\mathfrak{g}_{CM}^{\otimes k}$ for all k, and thus to $T(\mathfrak{g}_{CM})$. We will denote this extension as $\mathcal{T}F \in T(\mathfrak{g}_{CM})'$.

We now justify that $\mathcal{T}F \in J^0(\mathfrak{g}_{CM})$. We know that if $a, b \in \mathfrak{g}^P$ for some $P \in Proj(W_1)$, then $\mathcal{T}F$ is an extension of $\widehat{F \circ \iota^P}(e) \in J^0(\mathfrak{g}^P)$, so we know $\mathcal{T}F(a \otimes b - b \otimes a - [a,b]) = 0$. Now let $a, b \in \mathfrak{g}_{CM}$ be arbitrary. Since $\{\mathfrak{g}^P\}_{P \in Proj(W_1)}$ is dense in $\mathfrak{g}_{CM}$, we may choose sequences $a_m, b_m \in \bigcup_{P \in Proj(W_1)} \mathfrak{g}^P$ such that $a_m \to a$ and $b_m \to b$ in $\|\cdot\|_{\mathfrak{g}_{CM}}$. By the continuity of $\mathcal{T}F$ on $T(\mathfrak{g}_{CM})$, knowing that $\mathcal{T}F(a_m \otimes b_m - b_m \otimes a_m - [a_m, b_m]) = 0$ for all m allows us to conclude $\mathcal{T}F(a \otimes b - b \otimes a - [a,b]) = 0$. Using a similar limiting technique, we may tensor an element of the form $a \otimes b - b \otimes a - [a,b]$ with any other element in $T(\mathfrak{g}_{CM})$ (on the left *or* the right) and still conclude that $\mathcal{T}F$ would evaluate to be 0. We would also get the same result for sums of such elements. This suffices to prove $\mathcal{T}F(J(\mathfrak{g}_{CM})) = 0$, so that $\mathcal{T}F \in J^0(\mathfrak{g}_{CM})$.

The proof will be complete if we show that $\|\mathcal{T}F\|_{J_t^0(\mathfrak{g}_{CM})}$ is finite and equals $\sup_P \|F \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)}$. Again let $\{e_j\}_{j \in \mathbb{N}}$ be a basis of H_1 such that $\{e_j\}_{1 \le j \le r}$ is a basis of PH_1. Then

$$\begin{aligned}
\|\widehat{F \circ \iota^P}(e)\|_{J_t^0(\mathfrak{g}^P)}^2 &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j_1,\ldots,j_k=1}^{r} |\mathcal{T}F(e_{j_1} \otimes \ldots \otimes e_{j_k})|^2 \\
&\le \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j_1,\ldots,j_k=1}^{\infty} |\mathcal{T}F(e_{j_1} \otimes \ldots \otimes e_{j_k})|^2 = \|\mathcal{T}F\|_{J_t^0(\mathfrak{g}_{CM})}^2, \quad (3.13)
\end{aligned}$$

so $\sup_P \|\widehat{F \circ \iota^P}\|_{J_t^0(\mathfrak{g}^P)} \le \|\mathcal{T}F\|_{J_t^0(\mathfrak{g}_{CM})}$. And for any sequence $(P_m)_{m \in \mathbb{N}} \in Proj(W_1)^{\uparrow}$, and corresponding bases $\{e_j\}_{1 \le j \le r_m}$,

$$\begin{aligned}
\sup_P \|\widehat{F \circ \iota^P}(e)\|_{J_t^0(\mathfrak{g}^P)} &\ge \lim_{m\to\infty} \|\widehat{F \circ \iota^{P_m}}(e)\|_{J_t^0(\mathfrak{g}^{P_m})} \\
&= \lim_{m\to\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j_1,\ldots,j_k=1}^{r_m} |\mathcal{T}F(e_{j_1} \otimes \ldots \otimes e_{j_k})|^2 \\
&= \|\mathcal{T}F\|_{J_t^0(\mathfrak{g}_{CM})}. \quad (3.14)
\end{aligned}$$

Hence, $\sup_P \|F \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)} = \|\mathcal{T}F\|_{J_t^0(\mathfrak{g}_{CM})}$. This completes the proof. □

11.4 Surjectivity

At this point, we will begin to make substantial use of the graded structure, more so than previous results presented. We will first work to show Lemma 11.10, which states that, for a certain dense collection of $\alpha \in J_t^0(\mathfrak{g}_{CM})$, there exists an $F_\alpha \in \mathcal{H}L_t^2(\bigcup_P G^P)$ such that $\mathcal{T} F_\alpha(e) = \alpha$.

The next theorem, Theorem 11.9, originally appeared in [DGS09b, Lemma 3-5] for (finite-dimensional) graded nilpotent Lie groups, and has been applied to other infinite-dimensional settings in [GM13, Lemma 3.17] and [Cec08, Theorem 41]. As remarked in these papers, this proof does not rely on any finite-dimensional aspects; it is more a matter of "rank." It has been included for the sake of completion, and with a few details added for the sake of clarity.

We say that an element $\alpha \in T(\mathfrak{g}_{CM})'$ is of *finite rank* if there exists a $K \in \mathbb{N}$ such that, for all $k \geq K$, and collections $h_1, \ldots, h_k \in \mathfrak{g}_{CM}$, $\alpha(h_1 \otimes \ldots \otimes h_k) = 0$. While one may tell that the elements of $J_t^0(\mathfrak{g}_{CM})$ can be approximated by finite-rank elements in $T(\mathfrak{g}_{CM})'$, it's far less obvious that the finite-rank approximations lie in $J^0(\mathfrak{g}_{CM})$ themselves.

Theorem 11.9. *The finite-rank elements of $J_t^0(\mathfrak{g}_{CM})$ are dense in $J_t^0(\mathfrak{g}_{CM})$.*

Proof. For $\theta \in [0, 2\pi)$, let $\delta_\theta : \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$ be the natural dilation on $\mathfrak{g}_{CM}$ by $e^{i\theta}$, meaning that, for $h = (h_1, \ldots, h_N) \in \mathfrak{g}_{CM}$, $\delta_\theta(h_1, \ldots, h_N) = (e^{i\theta} h_1, \ldots, e^{iN\theta} h_N)$. Then define $\Gamma_\theta : T(\mathfrak{g}_{CM}) \to T(\mathfrak{g}_{CM})$ to be the unique extension to an algebra homomorphism satisfying $\Gamma_\theta(h_1 \otimes \ldots \otimes h_N) = \delta_\theta(h_1) \otimes \ldots \otimes \delta_\theta(h_N)$.

Consider that, for any $a \in H_m, b \in H_n$, we have

$$\begin{aligned}
\Gamma_\theta(a \otimes b - b \otimes a - [a, b]) &= e^{i(n+m)\theta}(a \otimes b - b \otimes a - [a, b]) \\
&= (e^{im\theta} a) \otimes (e^{in\theta} b) - (e^{in\theta} b) \otimes (e^{im\theta} a) - [e^{im\theta} a, e^{in\theta} b] \\
&= (\Gamma_\theta a) \otimes (\Gamma_\theta b) - (\Gamma_\theta b) \otimes (\Gamma_\theta a) - [\Gamma_\theta a, \Gamma_\theta b]
\end{aligned}$$

which means $\Gamma_\theta(J(\mathfrak{g}_{CM})) \subseteq J(\mathfrak{g}_{CM})$. Then for $\alpha \in J^0(\mathfrak{g}_{CM})$, we see that $\alpha \circ \Gamma_\theta \in J^0(\mathfrak{g}_{CM})$. Furthermore, for any $h_1, \ldots, h_k \in \mathfrak{g}_{CM}$, $|\alpha \circ \Gamma_\theta(h_1 \otimes \ldots \otimes h_k)| = |\alpha(h_1 \otimes \ldots \otimes h_k)|$, which implies that $\|\alpha \circ \Gamma_\theta\|_{J_t^0(\mathfrak{g}_{CM})} = \|\alpha\|_{J_t^0(\mathfrak{g}_{CM})}$. Hence, $\alpha \circ \Gamma_\theta \in J_t^0(\mathfrak{g}_{CM})$.

Now[4] consider that, for $\theta, \theta' \in [0, 2\pi)$,

$$\begin{aligned}\|\alpha \circ \Gamma_\theta - &\alpha \circ \Gamma_{\theta'}\|^2_{J_t^0(\mathfrak{g}_{CM})} \\ &= \sum_{k=0}^\infty \frac{t^k}{k!} \sum_{j_1,\ldots,j_k=1}^\infty |\alpha(\Gamma_\theta(e_{j_1} \otimes \ldots \otimes e_{j_k})) - \alpha(\Gamma_{\theta'}(e_{j_1} \otimes \ldots \otimes e_{j_k}))|^2 \\ &= \sum_{k=0}^\infty \frac{t^k}{k!} |e^{ik\theta} - e^{ik\theta'}|^2 \sum_{j_1,\ldots,j_k=1}^\infty |\alpha(e_{j_1} \otimes \ldots \otimes e_{j_k})|^2 .\end{aligned}$$

Knowing that $|e^{ik\theta} - e^{ik\theta'}|^2$ is bounded by 4 for all k and $\to 0$ as $\theta' \to \theta$ for each k is sufficient to deduce $\|\alpha \circ \Gamma_\theta - \alpha \circ \Gamma_{\theta'}\|_{J_t^0(\mathfrak{g}_{CM})} \to 0$ as $\theta' \to \theta$, which means that the map $\theta \mapsto \alpha \circ \Gamma_\theta \in J_t^0(\mathfrak{g}_{CM})$ is continuous.

Now let $\mathcal{F}_n(\theta) = \frac{1}{2\pi n} \sum_{k=0}^{n-1} \sum_{\ell=-k}^{k} e^{i\ell\theta}$, known as the Fejer kernel, which has the properties that $\mathcal{F}_n(\theta) \geq 0$ for all θ, $\int_{-\pi}^{\pi} \mathcal{F}_n(\theta) d\theta = 1$, and for any continuous $f : [-\pi, \pi] \to \mathbb{C}$, $\int_{-\pi}^{\pi} \mathcal{F}_n(\theta) f(\theta) d\theta \to f(0)$.

Knowing that each $\mathcal{F}_n$ is bounded allows us to define $\alpha_n := \int_{-\pi}^{\pi} \mathcal{F}_n(\theta)(\alpha \circ \Gamma_\theta) d\theta \in J_t^0(\mathfrak{g}_{CM})$ as a Bochner integral. Then we claim that each α_n is finite-rank and converges to α in $\|\cdot\|_{J_t^0(\mathfrak{g}_{CM})}$.

Observe that, for any simple tensor $\beta \in H_{r_1} \otimes \ldots \otimes H_{r_k}$, $\Gamma_\theta \beta = e^{ir\theta}\beta$, where $r = \sum_{j=1}^k r_j$. Then for any n,

$$\alpha_n(\beta) = \int_{-\pi}^{\pi} \mathcal{F}_n(\theta)\, (\alpha \circ \Gamma_\theta)(\beta)\, d\theta = \left(\int_{-\pi}^{\pi} \mathcal{F}_n(\theta) e^{ir\theta} d\theta \right) \alpha(\beta) .$$

Since each $\mathcal{F}_n$ is a finite trigonometric polynomial, we know that for sufficiently large r, $\alpha_n(\beta) = 0$, so for such large r, $\alpha_n(\mathfrak{g}_{CM}^{\otimes r}) = 0$, meaning α_n is of finite rank.

Lastly, observe that

$$\begin{aligned}\|\alpha - \alpha_n\|_{J_t^0(\mathfrak{g}_{CM})} &= \left\| \int_{-\pi}^{\pi} \mathcal{F}_n(\theta) \alpha d\theta - \int_{-\pi}^{\pi} \mathcal{F}_n(\theta)(\alpha \circ \Gamma_\theta) d\theta \right\|_{J_t^0(\mathfrak{g}_{CM})} \\ &\leq \int_{-\pi}^{\pi} \mathcal{F}_n(\theta) \|\alpha - \alpha \circ \Gamma_\theta\|_{J_t^0(\mathfrak{g}_{CM})} d\theta \xrightarrow{n\to\infty} \alpha .\end{aligned}$$

□

[4] We remark that, in [DGS09b], there is a statement about $\alpha \circ \Gamma_\theta$ being "clearly" continuous in θ with respect to $\|\cdot\|_{J_t^0(\mathfrak{g}_{CM})}$, which also appeared in [GM13]. However, it's easy to make such a statement by relying on incorrect reasoning. Just because every $\alpha|_{\mathfrak{g}_{CM}^{*\otimes k}} \circ \delta_\theta^{\otimes k}$ has this continuity does not mean that one has continuity in $J_t^0(\mathfrak{g}_{CM})$. The argument below provides more details, and resembles the argument in [Cec08].

The following lemma statement and proof closely resemble those of [Cec08, Theorem 44], and some similar reasoning can also be found in [DGS09b, Lemma 3-6], [GM13, Lemma 3.19], [DG10, Theorem 6.10].

Lemma 11.10. *For $\alpha \in J_t^0(\mathfrak{g}_{CM})$ of finite rank K, the function $F_\alpha : \mathfrak{g}_{CM} \to \mathbb{C}$ defined as*

$$F_\alpha(g) := \sum_{k=1}^{K} \frac{1}{k!} \langle \alpha, g^{\otimes k} \rangle$$

satisfies $F_\alpha \circ \iota^P \in \mathcal{H}L_t^2(G^P)$ for all $P \in Proj(W_1)$, and $\sup_P \|F_\alpha \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)} = \|\alpha\|_{J_t^0(\mathfrak{g}_{CM})}$. Thus, we may regard $F_\alpha \in \mathcal{H}L_t^2(\bigcup_P G^P)$. Furthermore, all left-invariant derivatives $\widetilde{h_1} \dots \widetilde{h_\ell} F_\alpha(e)$ exist, and $\widehat{F_\alpha}(e) = \mathcal{T} F_\alpha = \alpha$.

Proof. We know that $\alpha : \mathfrak{g}_{CM}^{\otimes k} \to \mathbb{C}$ is continuous, so by the continuity of $g \mapsto g^{\otimes k}$, we have that $\alpha(\cdot^{\otimes k}) : \mathfrak{g}_{CM} \to \mathbb{C}$ is continuous as well. Furthermore, for $P \in Proj(W_1)$, each $\alpha(\cdot^{\otimes k})|_{G^P} : G^P \to \mathbb{C}$ is a holomorphic polynomial on G^P. To show this explicitly, if $\{b_j\}_{j\in\mathbb{N}}$ is any basis for $\mathfrak{g}_{CM}$ where $\{b_j\}_{1\le j\le r}$ is a basis for G^P, then

$$\begin{aligned} \alpha(g^{\otimes k}) &= \sum_{j_1,\dots,j_k=1}^{\infty} \alpha_{j_1,\dots,j_k} \langle g^{\otimes k}, b_{j_1} \otimes \dots \otimes b_{j_k} \rangle_{\mathfrak{g}_{CM}^{\otimes k}} \\ &= \sum_{j_1,\dots,j_k=1}^{\infty} \alpha_{j_1,\dots,j_k} \langle g, b_{j_1} \rangle_{\mathfrak{g}_{CM}} \dots \langle g, b_{j_k} \rangle_{\mathfrak{g}_{CM}} \end{aligned}$$

for some constants $(\alpha_{j_1,\dots,j_k})_{k\in\mathbb{N}\cup 0, j_1,\dots,j_k\in\mathbb{N}}$. Then, for $g \in G^P$,

$$\alpha(g^{\otimes k}) = \sum_{j_1,\dots,j_k=1}^{r} \alpha_{j_1,\dots,j_k} \langle g, b_{j_1} \rangle_{\mathfrak{g}_{CM}} \dots \langle g, b_{j_k} \rangle_{\mathfrak{g}_{CM}}$$

We may then conclude $F_\alpha|_{G^P}$ is a holomorphic polynomial on G^P. Now fix $h_1, \dots, h_\ell \in \mathfrak{g}_{CM}$, and let G_0 be any finite-dimensional Lie subgroup containing every h_j, and choose the basis $\{b_j\}_{j\in\mathbb{N}}$ such that $\{b_j\}_{1\le j\le r}$ is a basis for G_0. Then, in the likeness of (3.10), all holomorphic functions $F : \mathfrak{g}_{CM} \to \mathbb{C}$ and $h \in G_0$ satisfy $\widetilde{h}(F|_{G_0}) = (\widetilde{h}F)|_{G_0}$. Thus, we may deduce $\widetilde{h_1} \dots \widetilde{h_k} F_\alpha(e)$ exisis, since it equals $\widetilde{h_1} \dots \widetilde{h_k}(F_\alpha|_{G_0})(e)$.

Next, consider that, for $h \in \mathfrak{g}_{CM}$,

$$\langle \widehat{F_\alpha}(e), h^{\otimes k} \rangle = \frac{d^k}{dt^k}\Big|_{t=0} F_\alpha(th) = \frac{d^k}{dt^k}\Big|_{t=0} \frac{1}{k!} \alpha(t^k h^{\otimes k}) = \alpha(h^{\otimes k}).$$

Then for $h \in \mathfrak{g}^P$, $\widehat{F_\alpha \circ \iota^P}(e)(h^{\otimes k}) = \alpha(h^{\otimes k})$. Note that $\mathrm{span}\{h^{\otimes k}\}_{h\in\mathfrak{g}^P, k\in\mathbb{N}} = \mathcal{S}(\mathfrak{g}^P)$, the set of symmetric tensors in $T(\mathfrak{g}^P)$, by the Poincare-Birkoff-Witt theorem ([Var84, Lemma 3.3.3] or [Hum78, Corollary E]), $T(\mathfrak{g}^P) = \mathcal{S}(\mathfrak{g}^P) \oplus J(\mathfrak{g}^P)$. Knowing that

$\widehat{F_\alpha \circ \iota^P}(e)$ and α coincide on $\mathcal{S}(\mathfrak{g}^P)$ and vanish on $J(\mathfrak{g}^P)$ allows us to conclude $\widehat{F_\alpha \circ \iota^P}(e) = \alpha|_{T(\mathfrak{g}^P)}$. Also, we see that, arguing similarly to the proof in Theorem 11.8, in particular (3.13) and (3.14),

$$\sup_P \|F_\alpha \circ \iota^P\|_{\mathcal{H}L^2_t(G^P)} = \sup_P \|\alpha|_{\mathfrak{g}^P}\|_{J^0_t(\mathfrak{g}^P)} = \lim_{m\to\infty} \|\alpha|_{\mathfrak{g}^{P_m}}\|_{J^0_t(\mathfrak{g}^{P_m})} = \|\alpha\|_{J^0_t(\mathfrak{g}_{CM})}.$$

So $F_\alpha \in \mathcal{H}L^2_t(\bigcup_P G^P)$. Then for all $k \in \mathbb{N}$, $\widehat{F_\alpha}(e)$ and $\mathcal{T}F_\alpha$ agree on all simple tensors in $(\bigcup_P \mathfrak{g}^P)^{\otimes k}$. This suffices to say $\widehat{F_\alpha}(e)$ has an extension to $T(\mathfrak{g}_{CM})$, which must agree with $\mathcal{T}F_\alpha$. The same can be said of $\widehat{F_\alpha}(e)$ and α, which completes the proof.

□

In order to complete the proof, we will show that any $F \in \mathcal{H}L^2_t(\bigcup_P G^P)$ for which $\mathcal{T}F \in J^0_t(\mathfrak{g}_{CM})$ is of finite rank can be approximated by elements of $\mathcal{P}$. To do this, we will now define a new type of projection.

Recall that, for $1 \le n \le N$, we define $W_n := \overline{H_n}^{\|\cdot\|_G}$. Then for each n, we have a dense inclusion $W_n^* \subseteq H_n^* \cong H_n$, so we may choose an orthonormal basis $\{e_{n,j}\}_{j\in\Lambda_n}$ of H_n that lies in W_n^*. Then we let $(Q_n^{(m)})_{m\in\mathbb{N}}$ be the corresponding sequence of finite-rank projections $H_n \to \operatorname{span}\{e_{n,1}, \ldots, e_{n,m}\}$, so that $Q^{(m)} = Q_1^{(m)} \oplus \ldots \oplus Q_N^{(m)} : \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$ is a sequence of finite rank projections that converges to $I_{\mathfrak{g}_{CM}}$ in the strong operator topology. Then, by choice of basis, each $Q_n^{(m)}$ extends to a continuous linear finite-rank map $Q_n^{(m)} : W_n \to \operatorname{span}\{e_{n,1}, \ldots, e_{n,m}\}$, so that $Q^{(m)}$ itself extends to a continuous linear finite-rank map defined on G. We will use $Proj(G)$ to denote the set of finite-rank projections of the form $Q^{(m)}$, and $Proj(G)^{\uparrow}$ to denote the set of sequences $(Q^{(m)})_{m\in\mathbb{N}}$ constructed in this way.

We will complete our proof of surjectivity through comparisons between left-invariant derivatives and linear derivatives, in a similar fashion to [DG10; Cec08; GM13]. However, these previous works have followed this line of reasoning: for any $Q \in Proj(G)$, $\widetilde{h}_1 \ldots \widetilde{h}_k(f - f \circ Q)(e) = \sum_{j=\lfloor k/N \rfloor}^{k} f^{(j)}(e)(M^Q_{k,j}(h_1, \ldots, h_k))$ for some $M^Q_{k,j} : H_1^k \to T(\mathfrak{g}_{CM})$ (see $V^k_{\mathcal{P}}(h_1, \ldots, h_k, g)$ in [Cec08, Proposition 53], or $(h_1, \ldots, h_k)^\theta_P$ in [GM13, Proposition 4.4]). The challenge is to then argue that this expression converges to 0 in the appropriate manner as $Q \to I_G$. This has shown to be messy even for step 2 examples, and requires substantial notation and calculation for higher step cases[5]. Here, we will apply a different philosophy: $\widetilde{h}_1 \ldots \widetilde{h}_k f(e) = \sum_{j=\lfloor k/N \rfloor}^{k} f^{(j)}(e)(M_{k,j}(h_1, \ldots, h_k))$, where the $M_{k,j}$ do not depend on the function f and are weakly maps. This, combined with control on the linear derivatives, will suffice to carry out our convergence calculations.

[5] We remark that the argument for the corresponding result in [Cec08] constituted 25 pages, which is approximately half of the publication.

As a remark, one can deduce the existence of these expressions by using propositions 53 and 56 of [Cec08] (treating $P = I$ and using the provided recursive formula), but we derive them here for the sake of completion, to be clear that they exist outside of the context of path space, and most importantly to emphasize that these expressions are weakly Hilbert-Schmidt under our assumptions.

To justify the next 2 theorems, let us work out the calculation for the first few derivatives, assuming that G is of step $N = 3$. Let $h \in \mathfrak{g}_{CM}$, $g \in \mathfrak{g}_{CM}$, and say $\phi : \mathfrak{g}_{CM} \to V$ is smooth for any Hilbert space V. Then

$$\widetilde{h}_g \phi \;=\; \widetilde{h}\phi(g) \;=\; \phi'(g)(L_{g*}h) \;=\; \phi'(g)\bigg(h + \frac{1}{2}[g,h] + \frac{1}{12}[g,[g,h]]\bigg)\,.$$

For $f : \mathfrak{g}_{CM} \to \mathbb{C}$, we wish to derive a similar expression for $\widetilde{h}_1 \dots \widetilde{h}_k f(g)$. To do so, we will regard $\widetilde{h}_{\cdot} : \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$ by writing

$$\widetilde{h}_g \;=\; h + \frac{1}{2}[g,h] + \frac{1}{12}[g,[g,h]]\,.$$

So

$$\begin{aligned}
\widetilde{h}_1\widetilde{h}_{2g} := \widetilde{h}_{1g}(\widetilde{h}_{2\cdot}) \;&=\; \widetilde{h}_{1g}\bigg(h_2 + \frac{1}{2}[\cdot,h_2] + \frac{1}{12}[\cdot,[\cdot,h_2]]\bigg) \\
&=\; \frac{1}{2}[\widetilde{h}_{1g},h_2] + \frac{1}{12}[\widetilde{h}_{1g},[g,h_2]] + \frac{1}{12}[g,[\widetilde{h}_{1g},h_2]] \\
&=\; \frac{1}{2}\bigg[h_1 + \frac{1}{2}[g,h_1] + \frac{1}{12}[g,[g,h_1]]\,,\; h_2\bigg] \\
&\qquad + \frac{1}{12}\bigg[h_1 + \frac{1}{2}[g,h_1] + \frac{1}{12}[g,[g,h_1]],\, \Big[g,h_2\Big]\bigg] \\
&\qquad + \frac{1}{12}\bigg[g,\, \Big[h_1 + \frac{1}{2}[g,h_1] + \frac{1}{12}[g,[g,h_1]],h_2\Big]\bigg]\,.
\end{aligned}$$

Using this and going further, we may derive

$$\begin{aligned}
\widetilde{h}_1 f(g) \;&=\; f'(g)(\widetilde{h}_g) \;=\; f'(g)\bigg(h_1 + \frac{1}{2}[g,h_1] + \frac{1}{12}[g,[g,h_1]]\bigg) \\
\widetilde{h}_1\widetilde{h}_2 f(g) \;&=\; f''(g)(\widetilde{h}_{1g},\widetilde{h}_{2g}) \;+\; f'(g)(\widetilde{h}_{1g}(\widetilde{h}_{2\cdot})) \\
&=\; f''(g)(h_1 + c_2[g,h_1] + c_3[g,[g,h_1]]\,,\; h_2 + c_2[g,h_2] + c_3[g,[g,h_2]]) \\
&\qquad + f'(g)(c_2[\widetilde{h}_{1g},h_2] + c_3[\widetilde{h}_{1g},[g,h_2]] + c_3[g,[\widetilde{h}_{1g},h_2]]) \\
\widetilde{h}_1\widetilde{h}_2\widetilde{h}_3 f(g) \;&=\; f'''(g)(\widetilde{h}_{1g},\widetilde{h}_{2g},\widetilde{h}_{3g}) + f''(g)(\widetilde{h}_{1g}(\widetilde{h}_{2\cdot}),\widetilde{h}_{3g}) + f''(g)(\widetilde{h}_{2g}\,,\;\widetilde{h}_{1g}(\widetilde{h}_{3\cdot})) \\
&\qquad + f''(g)(\widetilde{h}_{1g},\widetilde{h}_{2g}(\widetilde{h}_{3\cdot})) + f'(g)(\widetilde{h}_{1g}(\widetilde{h}_2\widetilde{h}_{3\cdot}))\,.
\end{aligned}$$

Easier to digest is rewriting these expressions into tensor notation and evaluating them at the identity. This produces

$$\begin{aligned}
\widetilde{h}_1 f(e) &= f'(g)(h_1) \\
\widetilde{h}_1\widetilde{h}_2 f(e) &= f''(e)(h_1\otimes h_2) + f'(g)(\frac{1}{2}[h_1,h_2]) \\
\widetilde{h}_1\widetilde{h}_2\widetilde{h}_3 f(e) &= f'''(e)(h_1\otimes h_2\otimes h_3) \\
&\quad + f''(e)(\frac{1}{2}[h_1,h_2]\otimes h_3) + f''(e)(h_2\otimes \frac{1}{2}[h_1,h_3]) + f''(e)(h_1\otimes\frac{1}{2}[h_2,h_3]) \\
&\quad + f'(e)(\frac{1}{4}[[h_1,h_2],h_3] + \frac{1}{12}[h_2,[h_1,h_3]] + \frac{1}{12}[h_1,[h_2,h_3]])\,.
\end{aligned}$$

Observe that every expression is in terms of brackets, tensors, and sums of g, h_1, and h_2. Importantly, this expression is a sum of derivatives composed with weakly Hilbert-Schmidt maps. I claim that all expressions of the form $\widetilde{h}_1\ldots\widetilde{h}_k f(e)$ take on such a form, and this is argued precisely in the next 2 lemmas.

We use the phrase *iteration of brackets and tensors* to mean compositions and tensors of maps of the form $\widetilde{\sigma}^{\otimes a} : \mathfrak{g}_{CM}^{\otimes a} \to \mathfrak{g}_{CM}^{\otimes a}$ where $\sigma \in \mathcal{S}_a$ is some permutation that acts naturally on $\mathfrak{g}_{CM}^{\otimes a}$, and $I_{\mathfrak{g}_{CM}}^{\otimes a} \otimes [\cdot] \otimes I_{\mathfrak{g}_{CM}}^{\otimes b} : \mathfrak{g}_{CM}^{\otimes a+b+2} \to \mathfrak{g}_{CM}^{\otimes a+b+1}$. In the spirit of Proposition 2.4, we may apply this term to a multilinear map $M : \mathfrak{g}_{CM}^k \to \mathfrak{g}_{CM}^{\otimes j}$ by considering its extension to $\mathfrak{g}^{\otimes k}$. In particular, iterated brackets $[\cdot,\ldots,[\cdot,\cdot],\ldots]$, other compositions like $[[\cdot,\cdot],[\cdot,\cdot]]$, and extensions and tensor products thereof are all examples. By applying the Jacobi identity, one can see that every iteration of brackets and tensors mapping $\mathfrak{g}_{CM}^n \to \mathfrak{g}_{CM}^{\otimes j}$ can be rewritten as a finite sum of maps (composed with permutations $\widetilde{\sigma} : \mathfrak{g}_{CM}^{\otimes n} \to \mathfrak{g}_{CM}^{\otimes n}$) of the form

$$(h_1,\ldots,h_n) \mapsto [h_1,\ldots,[h_{n_1-1},h_{n_1}]\ldots]\otimes\ldots\otimes[h_{n-n_j+1},\ldots,[h_{n-1},h_n]\ldots] \quad (3.15)$$

where $\sum_{i=1}^j n_i = n$.

Lemma 11.11. *Let $f : \mathfrak{g}_{CM} \to \mathbb{C}$ be smooth. Then there exist multilinear functions $M_{k,j}^{(\ell)} : H_1^k \times \mathfrak{g}_{CM}^\ell \to \mathfrak{g}_{CM}^{\otimes j}$ such that, for any finite collection $h^1,\ldots,h^k \in H_1$,*

$$(\widetilde{h}_1\ldots\widetilde{h}_k f)(g) = \sum_{j=1}^{k}\sum_{\ell=0}^{N} f^{(j)}(M_{k,j}^{(\ell)}(h_1,\ldots,h_k,\underbrace{g,\ldots,g}_{\ell \text{ times}})) \quad (3.16)$$

where, in fact, each $M_{k,j}^{(\ell)}$ is a sum of iterations of brakets and tensors. In particular, we have that each $M_{k,j}^{(\ell)}$ extends to a continuous linear map $H_1^{\otimes k}\otimes\mathfrak{g}_{CM}^{\otimes \ell} \to \mathfrak{g}_{CM}^{\otimes j}$. Moreover, for $j < \lfloor k/N \rfloor$, $M_{k,j}^{(\ell)} \equiv 0$ (so the sum above can be assumed to start at $j = \lfloor k/N\rfloor$).

Proof. We begin by defining

$$M^{(0)}_{1,1}(h) := h \qquad M^{(1)}_{1,1}(h,g_1) := \frac{1}{2}[g_1,h] \qquad M^{(2)}_{1,1}(h,g_1,g_2) := \frac{1}{12}[g_1,[g_2,h]] \qquad \dots$$

such that

$$\widetilde{h}_g \;=\; h + \frac{1}{2}[g,h] + \frac{1}{12}[g,[g,h]] + \dots \;=\; M^{(0)}_{1,1}(h) + M^{(1)}_{1,1}(h,g) + M^{(2)}_{1,1}(h,g,g) + \dots \tag{3.17}$$

which implies

$$\widetilde{h}f(g) \;=\; \sum_{\ell=0}^{N} f^{(1)}(g)\Big(M^{(\ell)}_{1,1}(h,g,\dots,g)\Big)$$

so the statement holds for $k=1$.

We will now inductively define the $M^{(\ell)}_{k,j}$ terms. For now, let $M : H_1^k \times \mathfrak{g}_{CM}^{\ell} \to \mathfrak{g}_{CM}^{\otimes j}$ be an arbitrary multilinear continuous function. To make the notation clear, for the rest of this proof, we will use the notation $M(h_1,\dots,h_k,\cdot,\dots,\cdot)$ to refer to the map

$$\mathfrak{g}_{CM} \ni g \;\mapsto\; M(h_1,\dots,h_k,g,\dots,g) \in \mathfrak{g}_{CM}^{\otimes j}$$

(that is, every instance of the symbol "$\cdot$" is equal to one another). For such an M,

$$\widetilde{h}_g(M(h_1,\dots,h_k,\cdot,\dots,\cdot)) \;=\; \sum_{m=1}^{\ell} M(h_1,\dots,h_k,\underbrace{g,\dots,g}_{m-1},\widetilde{h}_g,\underbrace{g,\dots,g}_{\ell-m})\,. \tag{3.18}$$

Substituting (3.17) into (3.18) and applying multilinearity, we see that if the multilinear map M can be expressed as a sum of iterations of tensors and brackets, then the map

$$H_1^{k+1} \times \mathfrak{g}_{CM}^{\ell+1} \ni (h_1,\dots,h_k,h,g_1,\dots,g_\ell,g) \\ \mapsto \widetilde{h}_g(M(h_1,\dots,h_k,\cdot,\dots,\cdot))(g_1,\dots,g_\ell)$$

can also be expressed as a sum of iterations of tensors and brackets.

Now observe that[6]

$$\begin{aligned} &\widetilde{h}_g\Bigg(\sum_{j=1}^{k}\sum_{\ell=0}^{N} f^{(j)}(\cdot)\Big(M^{(\ell)}_{k,j}(h_1,\dots,h_k,\cdot,\dots,\cdot)\Big)\Bigg) \\ &\qquad = \sum_{j=1}^{k}\sum_{\ell=0}^{N} f^{(j+1)}(g)\Big(\widetilde{h}_g \otimes M^{(\ell)}_{k,j}(h_1,\dots,h_k,g,\dots,g)\Big) \\ &\qquad\qquad + \sum_{j=1}^{k}\sum_{\ell=1}^{N}\sum_{m=1}^{\ell} f^{(j)}(g)\Big(M^{(\ell)}_{k,j}(h_1,\dots,h_k,\underbrace{g,\dots,}_{m-1}\widetilde{h}_g,\underbrace{\dots,g}_{\ell-m})\Big) \end{aligned} \tag{3.19}$$

[6]Note that the ℓ index on the final sum starts at 1 now, because the derivative of a constant is 0

which is comprised of sums of iterations of tensors and brackets $h_1, \dots, h_k, h, g$ that are substituted into linear derivatives of f at g. It can then be deduced that there exists an inductive definition of $M^{(\ell)}_{k,j}$ such that (3.19) is satisfied, and each is sum of iterations of tensors and brackets, and therefore weakly Hilbert-Schmidt.

For a concrete definition, we wish to satisfy the inductive relation

$$\sum_{j=1}^{k+1}\sum_{\ell=0}^{N} f^{(j)}(g)\Big(M^{(\ell)}_{k+1,j}(h_1,\dots,h_k,h,g,\dots,g)\Big) = \widetilde{h}_g\left(\sum_{j=1}^{k}\sum_{\ell=0}^{N} f^{(j)}(\cdot)\Big(M^{(\ell)}_{k,j}(h_1,\dots,h_k,\cdot,\dots,\cdot)\Big)\right). \quad (3.20)$$

Then (3.19) and (3.20) will both be true if the following is satisfied, for every fixed $1 \le j \le k+1$.

$$\sum_{\ell=0}^{N} M^{(\ell)}_{k+1,j}(h_1,\dots,h_k,h,g,\dots,g) = 1_{j\neq 1}\sum_{\ell=0}^{N}\widetilde{h}_g \otimes M^{(\ell)}_{k,j-1}(h_1,\dots,h_k,g,\dots,g) + 1_{j\neq k+1}\sum_{\ell=1}^{N}\sum_{m=1}^{\ell} M^{(\ell)}_{k,j}(h_1,\dots,h_k,\underbrace{g,\dots,}_{m-1}\widetilde{h}_g,\underbrace{\dots,g}_{\ell-m})$$

To satisfy this, we need only set, for $1 \le j \le k+1$ and $0 \le \ell \le N$,

$$\begin{aligned} M^{(\ell)}_{k+1,j}&(h_1,\dots,h_k,h,g_1,\dots,g_\ell) \\ &= 1_{j\neq 1}\sum_{\substack{\ell_1+\ell_2=\ell\\ \ell_1\ge 0,\ell_2\ge 0}} M^{(\ell_1)}_{1,1}(h,g_1,\dots,g_{\ell_1}) \otimes M^{(\ell_2)}_{k,j-1}(h_1,\dots,h_k,g_{\ell_1+1},\dots,g_\ell) \\ &\quad + 1_{j\neq k+1}\sum_{\substack{\ell_1+\ell_2=\ell+1\\ \ell_1\ge 1,\ell_2\ge 0}}\sum_{m=1}^{\ell_1} M^{(\ell_1)}_{k,j}\Big(h_1,\dots,h_k,g_1,\dots,g_{m-1}, \\ &\qquad M^{(\ell_2)}_{1,1}(h,g_m,\dots,g_{m+\ell_2}),g_{m+\ell_2+1},\dots,g_\ell\Big) \end{aligned}$$

which will ensure each $M^{(\ell)}_{k,j} : H_1^k \times \mathfrak{g}^\ell_{CM} \to \mathfrak{g}^{\otimes j}_{CM}$ is again a sum of iterations of brackets and tensors, and in particular weakly Hilbert-Schmidt, and (3.16) will hold.

For the final statement, suppose that $M : \mathfrak{g}^{\otimes\eta}_{CM} \to \mathfrak{g}_{CM}$ is an iteration of brackets and tensors. Then, considering (3.15) with $j = 1$, M can be rewritten as a sum of permutations of the extension of the η-fold iterated bracket

$$(g_1,\dots,g_\eta) \mapsto [g_1,\dots,[g_{\eta-1},g_\eta]\dots].$$

Since $\mathfrak{g}_{CM}$ is assumed nilpotent of step N, then if $\eta > N$, then $M \equiv 0$. Similarly, suppose $M : \mathfrak{g}_{CM}^{\otimes \eta} \to \mathfrak{g}_{CM}^{\otimes j}$ is an iteration of brackets and tensors. Then once again M can be rewritten as a finite sum of maps of the form in (3.15), a j-fold tensor product of iterations of brackets. Suppose $\eta > Nj$. By the pigeon-hole principle, for each summand, one factor in the tensor product must consist of an (at least) $(N+1)$-fold composition of brackets, which as argued above must be identically 0, which forces $M \equiv 0$. Therefore, since $M_{k,j}^{(\ell)} : \mathfrak{g}_{CM}^{\otimes(k+\ell)} \to \mathfrak{g}_{CM}^{\otimes j}$, if $\lfloor k/N \rfloor > j$, then $k > Nj$, and for all $0 \leq \ell \leq N$, $k + \ell > Nj$, so that $M_{k,j}^{(\ell)} \equiv 0$.

□

The following lemma can be deduced from Lemma 11.11.

Lemma 11.12. *There exist multilinear functions $M_{k,j} : H_1^k \to \mathfrak{g}_{CM}^{\otimes j}$ such that*

$$(\widetilde{h}_1 \dots \widetilde{h}_k f)(e) = \sum_{j=\lfloor k/N \rfloor}^{k} f^{(j)}(M_{k,j}(h_1, \dots, h_k))$$

where, in fact, each $M_{k,j}$ is a sum of iterations of tensors and brakets in H_1. In particular, we have that each $M_{k,j}$ extends to a continuous map $H_1^{\otimes k} \to \mathfrak{g}_{CM}^{\otimes j}$.

Proof. We evaluate the derivatives at $g = e$ to get the expressions in this theorem. In particular, we have that for all $k \in \mathbb{N}$, $1 \leq j \leq k$, $M_{k,j} := M_{k,j}^{(0)}$.

□

We are now able to prove our surjectivity theorem.

Theorem 11.13. *For $F : \mathfrak{g}_{CM} \to \mathbb{C}$ for which $\widehat{F}(e) = \alpha \in J_t^0(\mathfrak{g}_{CM})$ is of finite rank, there exists a sequence $F_m \in \mathcal{P}$ such that $\|\widehat{F_m}(e) - \widehat{F}(e)\|_{J_t^0(\mathfrak{g}_{CM})} \to 0$. As a consequence, $\mathcal{T} \circ \mathcal{R} : \mathcal{H}L_t^2(G) \to J_t^0(\mathfrak{g}_{CM})$ is surjective.*

Proof. Let $\alpha \in J_t^0(\mathfrak{g}_{CM})$ be of finite rank. By Lemma 11.10, we may choose $F_\alpha : \mathfrak{g}_{CM} \to \mathbb{C}$ such that $\widehat{F_\alpha}(e) = \mathcal{T}F_\alpha = \alpha$.

Before going further, we prove the following claim: the linear derivatives at e, $F_\alpha^{(k)}(e) : \mathfrak{g}_{CM}^k \to \mathbb{C}$, all exist, and in fact $\|F_\alpha^{(k)}(e)\|_{HS(\mathfrak{g}_{CM}^k, \mathbb{C})} < \infty$ for all k, and for $k \geq K$, $F_\alpha^{(k)} \equiv 0$. To argue this, first note that $F'_\alpha(e)(v) = \alpha(v)$, and that $\|F'_\alpha(e)\|_{HS(\mathfrak{g}_{CM}, \mathbb{C})} = \sum_{j=1}^\infty |\alpha(b_j)|^2 = \|\alpha|_{\mathfrak{g}_{CM}}\|_{\mathfrak{g}_{CM}^*}^2 < \infty$.

More generally, if $f : \mathfrak{g}_{CM}^{\otimes \ell} \to \mathbb{C}$ is defined as $f(g) = \langle \beta, g^{\otimes \ell} \rangle$, then the product rule yields that $f'(g)(v) = \sum_{j=1}^\ell \langle \beta, g^{\otimes j-1} \otimes v \otimes g^{\otimes \ell - j} \rangle$. One can repeatedly apply this formula and see, for all $g \in \mathfrak{g}_{CM}$,

$$F_\alpha^{(k)}(g)(v_1, \ldots, v_k) \;=\; \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \alpha(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)})\,. \tag{3.21}$$

Or alternatively, one may observe that the expressions in (3.21) are symmetric and agree when $v_1 = \ldots = v_k$ because, arguing as in the proof of Lemma 11.10,

$$F_\alpha^{(k)}(e)(v^{\otimes k}) \;=\; \frac{d^k}{dt^k}\bigg|_{t=0} F_\alpha(tv) \;=\; \frac{d^k}{dt^k}\bigg|_{t=0} \frac{1}{k!}\alpha(t^k v^{\otimes k}) \;=\; \alpha(v^{\otimes k})\,.$$

Then $\|F_\alpha^{(k)}(e)\|^2_{HS(\mathfrak{g}_{CM}^k, \mathbb{C})} = \|\alpha|_{\mathfrak{g}_{CM}^{\otimes k}}\|^2_{\mathfrak{g}_{CM}^{*\otimes k}} < \infty$. Also, we should see that for $k > K$, $F_\alpha^{(k)} \equiv 0$.

Let $(Q_m)_{m\in\mathbb{N}} \in Proj(G)^{\uparrow}$ be a sequence of finite-rank projections Q_m that each map onto $\operatorname{span}\{e_{n,j}\}_{1\le n\le N, 1\le j\le m}$, where each $\{e_{n,j}\}_{1\le j\le \Lambda_n}$ is a basis of H_n. Then we may write

$$F_\alpha(g) \;=\; \sum_{k=0}^{K} \sum_{n_1,\ldots,n_k=1}^{N} \sum_{j_1,\ldots,j_k=1}^{\infty} \alpha_{(n_1,j_1),\ldots,(n_k,j_k)} \langle g^{\otimes k}, e_{n_1,j_1} \otimes \ldots \otimes e_{n_k,j_k}\rangle_{\mathfrak{g}_{CM}^{\otimes k}}$$

for some square-summable coefficients $\alpha_{(n_1,j_1),\ldots,(n_k,j_k)}$ (where the limit is legitimate pointwise for $g \in \mathfrak{g}_{CM}$), so that we may define

$$\begin{aligned} F_{\alpha,m}(g) \;&=\; F_\alpha \circ Q_m(g) \\ &=\; \sum_{k=0}^{K} \sum_{n_1,\ldots,n_k=1}^{N} \sum_{j_1,\ldots,j_k=1}^{m} \alpha_{(n_1,j_1),\ldots,(n_k,j_k)} \langle g^{\otimes k}, e_{n_1,j_1} \otimes \ldots \otimes e_{n_k,j_k}\rangle_{\mathfrak{g}_{CM}^{\otimes k}}\,. \end{aligned}$$

Then by the choice of the bases, each $\langle \cdot, e_{n,j}\rangle_{H_n} : \mathfrak{g}_{CM} \to \mathbb{C}$ can be extended to a continuous complex linear map on G, or equivalently, is an element of G^* restricted to $\mathfrak{g}_{CM}$. In the same way, any $\langle \cdot^{\otimes k} e_{n_1,j_1} \otimes \ldots \otimes e_{n_k,j_k}\rangle_{\mathfrak{g}_{CM}^{\otimes k}} = \langle \cdot, e_{n_1,j_1}\rangle_{H_{n_1}} \ldots \langle \cdot, e_{n_k,j_k}\rangle_{H_{n_k}}$ can be realized as a continuous holomorphic cylinder polynomial on G, so we can say the same of $F_{\alpha,m} \in \mathcal{P}$.

We now must show that $\|\widehat{F_\alpha}(e) - \widehat{F_{\alpha,m}}(e)\|_{J_t^0(\mathfrak{g}_{CM})} \xrightarrow{m} 0$. To show this, consider the following estimate on an arbitrary holomorphic $f : \mathfrak{g}_{CM} \to \mathbb{C}$, where we set $e_j := e_{1,j}$ and use the expressions in Lemma 11.12.

$$\sum_{j_1,\ldots,j_k=1}^{\infty} |\widetilde{e_{j_1}}\ldots\widetilde{e_{j_k}} f(e)|^2 = \sum_{j_1,\ldots,j_k=1}^{\infty} \left| \sum_{\ell=\lfloor k/N \rfloor}^{k} f^{(\ell)}(M_{k,\ell}(e_{j_1},\ldots,e_{j_k})) \right|^2$$
$$\le C \sum_{j_1,\ldots,j_k=1}^{\infty} \sum_{\ell=\lfloor k/N \rfloor}^{k} \left| f^{(\ell)}(M_{k,\ell}(e_{j_1}\otimes\ldots\otimes e_{j_k})) \right|^2$$
$$= C \sum_{\ell=\lfloor k/N \rfloor}^{k} \|f^{(\ell)} \circ M_{k,\ell}\|^2_{HS(H_1^{\otimes k},\mathbb{C})}$$
$$\le C \sum_{\ell=\lfloor k/N \rfloor}^{k} \|M_{k,\ell}\|^2_{H_1^{\otimes k},\mathfrak{g}_{CM}^{\otimes \ell}} \|f^{(\ell)}\|^2_{HS(\mathfrak{g}_{CM}^{\otimes \ell},\mathbb{C})},$$

where the final inequality comes from the fact that the composition of a Hilbert-Schmidt operator with a linear operator is again Hilbert-Schmidt (Proposition 2.3). Then, applying this estimate to $F_\alpha - F_{\alpha,m} = F_\alpha - F_\alpha \circ Q_m$ and setting $C' := C \max_{1\le\ell\le k} \|M_{k,\ell}\|^2_{H_1^{\otimes k},\mathfrak{g}_{CM}^{\otimes \ell}}$, we get

$$\sum_{j_i=1}^{\infty} |\widetilde{e_{j_1}}\ldots\widetilde{e_{j_k}}(F_\alpha - F_\alpha\circ Q_m)(e)|^2 \le C' \sum_{\ell=1}^{k} \|F_\alpha^{(\ell)}(e) - (F_\alpha\circ Q_m)^{(\ell)}(e)\|_{HS(H_1^{\otimes \ell},\mathbb{C})}$$
$$= C' \sum_{\ell=1}^{k} \|F_\alpha^{(\ell)}(e)\circ(I^{\otimes \ell} - Q_m^{\otimes \ell})\|_{HS(H_1^{\otimes \ell},\mathbb{C})},$$

where, for each ℓ, $\|f^{(\ell)}(e)\circ(I^{\otimes\ell} - Q_m^{\otimes\ell})\|_{HS(H_1^{\otimes\ell},\mathbb{C})} \to 0$ as $m\to\infty$.

Also, consider that each $F_{\alpha,m}$ is a polynomial of degree at most K. Then if $k > NK$ and $\ell \ge \lfloor k/N \rfloor$, then $\ell \ge k/N > K$, so that $(F_\alpha - F_{\alpha,m})^{(\ell)} \equiv 0$. Then, for such k, we have

$$\sum_{j_1,\ldots,j_k=1}^{\infty} |\widetilde{e_{j_1}}\ldots\widetilde{e_{j_k}}(F_\alpha - F_{\alpha,m})(e)|^2$$
$$= \sum_{j_1,\ldots,j_k=1}^{\infty} \left| \sum_{\ell=\lfloor k/N \rfloor}^{k} (F_\alpha - F_{\alpha,m})^{(\ell)}(M_{k,\ell}(e_{j_1},\ldots,e_{j_k})) \right|^2 = 0.$$

Then, at last, we have

$$\begin{aligned}
\|\widehat{F_\alpha}(e) - \widehat{F_{\alpha,m}}(e)\|_{J_t^0(\mathfrak{g}_{CM})} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j_1,\dots,j_k=1}^{\infty} \left| \widetilde{e_{j_1}} \dots \widetilde{e_{j_k}} (F_\alpha - F_{\alpha,m})(e) \right|^2 \\
&= \sum_{k=0}^{K} \frac{t^k}{k!} \sum_{j_1,\dots,j_k=1}^{\infty} \left| \widetilde{e_{j_1}} \dots \widetilde{e_{j_k}} (F_\alpha - F_{\alpha,m})(e) \right|^2 \\
&\xrightarrow{m\to\infty} 0\,.
\end{aligned}$$

By Theorem 11.9, the set of finite rank α is dense in $J_t^0(\mathfrak{g}_{CM})$, so the argument above shows the image of $\mathcal{T} \circ \mathcal{R}$ is dense in $J_t^0(\mathfrak{g}_{CM})$. Since $\mathcal{T} \circ \mathcal{R}$ is an isometry, we can conclude that $\mathcal{T} \circ \mathcal{R}$ is surjective. □

11.5 Inverse formulae and closing gaps

In this section, we will prove further results regarding Taylor expansions of functions in $\mathcal{H}L_t^2(G)$.

The following is a result that appears in [DGS09a, Proposition 5.13]. A briefer outline of the proof is available in [Dri15, Proposition 6.13]. Though we remark that the notation in [DGS09a; Dri15] starts with a horizontal path $g : [0,1] \to \mathfrak{g}_{CM}$ with $g(0) = e$, and then defines $A(t) = \int_0^t L_{g(s)^{-1}*} g'(s) ds$. For such an A, we must have $L_{g(t)*} A'(t) = g'(t)$, so by Theorem 2.6, we may conclude $g(t) = \nu A(t)$. Thus, we may state Theorem 11.14 in terms of our notation.

Theorem 11.14 (Finite-dimensional Taylor expansion along a horizontal curve)**.** *For any holomorphic function $f : G^P \to \mathbb{C}$, and any path $A \in \mathcal{H}_0([0,1], PH_1)$,*

$$f(\nu A(1)) \;=\; \sum_{k=0}^{\infty} \int_{\Delta_t^k} \widehat{f}(e)(A'(s_1) \otimes \ldots \otimes A'(s_k)) ds$$

which converges absolutely.

We also have the stochastic version, [Dri15, Theorem 6.24].

Theorem 11.15 (Finite-dimensional stochastic Taylor expansion)**.** *For any holomorphic $f : G^P \to \mathbb{C}$,*

$$f(g_t^P) \;=\; \sum_{k=0}^{\infty} \int_{\Delta_t^k} \widehat{f}(e)(dPB_{s_1} \otimes \ldots \otimes dPB_{s_k})$$

which converges almost-surely.

As noted in [DGS09a, Remark 5.14] and [Dri15, Remarks 6.14 and 6.27], the convergence in Theorem 11.14 and Theorem 11.15 does not require f to be in $L^2(G^P)$. Indeed, f is merely assumed holomorphic, and it is with holomorphic properties alone that these theorems are proved. On the other hand, it is possible to produce natural proofs of these theorems when $f \in \mathcal{H}L_t^2(G^P)$, which use the L^2 properties of f, and such proofs readily extend to infinite dimensions. It is entirely reasonable to resort to these methods, as we are, in part, making statements about "measurable holomorphic" functions on G. Of these 2 theorems, only the second has made an appearance in infinite dimensions, as [DG10, Theorem 1.9].

We will prove analogues of these 2 theorems for our case, as a Taylor expansion theorem, Theorem 11.17, and a stochastic Taylor expansion theorem, Theorem 11.20. Our proofs will very closely resemble the methods in [DG10], and in fact the proof of Theorem 11.20 is nearly identical.

While not necessary for the Taylor isomorphism, these results are not without their uses. Theorem 11.17 will allow us to make a proper formulation of $\mathcal{H}L^2_t(G_{CM})$. Theorem 11.20 will allow us to describe an inverse of $\mathcal{R}$. Both theorems will allow us to write the inverses of $\mathcal{T}$ and $\mathcal{T}\circ\mathcal{R}$, after their surjectivity is proven. We caution the reader that these Taylor expansion theorems on their own do not prove surjectivity, as they at best describe an inverse on a subset of $J^0_t(\mathfrak{g}_{CM})$, namely the image of $\mathcal{H}L^2_t(G)$. Only after proving surjectivity will we know that they are everywhere defined.

The final subsection, Section 11.5.3 will be devoted to discussing the restriction map $\mathcal{R}$, namely its inverse, and at last defining $\mathcal{H}L^2_t(G_{CM})$, providing multiple descriptions and using it to state the final version of the Taylor isomorphism result.

11.5.1 Taylor expansion along a curve

For any $F = \mathcal{R}f : G_{CM} \to \mathbb{C}$, we wish to have a series expansion for F that can be described in terms of $\mathcal{T}F$. If the step of G is 1, meaning that $G \cong \mathbb{C}^{\mathbb{N}}$ is commutative, then an $F \in \mathcal{H}L^2_t(G_{CM}) = \mathcal{H}L^2_t(H_1)$ would have a Taylor series of the form

$$F(g) \;=\; \sum_{k=0}^{\infty} \mathcal{T}F(g^{\otimes k})\,.$$

However, given the noncommutative structure of G, we must instead develop an infinite-dimensional version of Theorem 11.14 and prove that we have a "Taylor expansion along a curve."

Given $k \in \mathbb{N}$, $A \in \mathcal{H}_0([0,1], H_1)$, we define $\Psi_{t,k}(A) = \int_{\Delta^k_t} A'(s_1)\otimes \ldots \otimes A'(s_k)ds \in H_1^{\otimes k}$, and say $\Psi_{t,0}(A) = 1$. We can immediately see that this is well-defined via Bochner integrals, by the estimate

$$\int_{\Delta^n_t} \|A'(s_1)\|_{\mathcal{H}_0([0,1],H_1)} \ldots \|A'(s_n)\|_{\mathcal{H}_0([0,1],H_1)} ds \;\le\; \frac{1}{k!}\bigg(\int_0^1 \|A(s)\|_{\mathcal{H}_0([0,1],H_1)}\bigg)^k,$$

though in Lemma 10.7, we show that the assignment $\mathcal{H}_0([0,1],H_1)^k \ni (A,\ldots,A) \mapsto \Psi^P_{t,k}(A) \in (H_1)^{\otimes k}$ is weakly Hilbert-Schmidt.

Lemma 11.16. *Given $A \in \mathcal{H}_0([0,1],H_1)$, $\alpha \in J^0_t(\mathfrak{g}_{CM})$, and $\tau \in [0,1]$, $\sum_{k=0}^{\infty} \alpha(\Psi_{\tau,k}(A))$ converges absolutely. Furthermore, the map $J^0_t(\mathfrak{g}_{CM}) \ni \alpha \mapsto \sum_{k=0}^{\infty} \alpha(\Psi_{\tau,k}(A)) \in \mathbb{C}$ is continuous.*

Proof. For any $K \in \mathbb{N}$, we may write $\sum_{k=0}^{K} \Psi_{\tau,k}(A) \in T(\mathfrak{g}_{CM})$. Then via the assignment $\alpha \mapsto \alpha\big(\sum_{k=0}^{K} \Psi_{\tau,k}(A)\big)$, $\sum_{k=0}^{K} \Psi_{\tau,k}(A)$ is a linear map on $J^0_t(\mathfrak{g}_{CM})$, and we claim it is continuous (so an element of $J_t^0(\mathfrak{g}_{CM})^*$), because

$$\begin{aligned}
&\left|\alpha\Big(\sum_{k=K_1}^{K_2}\Psi_{\tau,k}(A)\Big)\right|^2 \\
&\leq \left(\sum_{k=K_1}^{K_2}\big|\alpha(\Psi_{\tau,k}(A))\big|\right)^2 \\
&\leq \left(\sum_{k=K_1}^{K_2}\sum_{j_1,\ldots,j_k=1}^{\infty}\sqrt{\frac{t^k}{k!}}\sqrt{\frac{k!}{t^k}}\big|\alpha(e_{j_1}\otimes\ldots\otimes e_{j_k})\langle\Psi_{\tau,k}(A),e_{j_1}\otimes\ldots\otimes e_{j_k}\rangle_{H_1^{\otimes k}}\big|\right)^2 \\
&\leq \|\alpha\|^2_{J_t^0(\mathfrak{g}_{CM})}\sum_{k=K_1}^{K_2}\frac{k!}{t^k}\sum_{j_1,\ldots,j_k=1}^{\infty}\big|\langle\Psi_{\tau,k}(A),e_{j_1}\otimes\ldots\otimes e_{j_k}\rangle_{H_1^{\otimes k}}\big|^2 \\
&= \|\alpha\|^2_{J_t^0(\mathfrak{g}_{CM})}\sum_{k=K_1}^{K_2}\frac{k!}{t^k}\sum_{j_1,\ldots,j_k=1}^{\infty}\left|\left\langle\int_{\Delta_\tau^k}A'(s_1)\otimes\ldots\otimes A'(s_k)ds\ ,\ e_{j_1}\otimes\ldots\otimes e_{j_k}\right\rangle_{H_1^{\otimes k}}\right|^2 \\
&\leq \|\alpha\|^2_{J_t^0(\mathfrak{g}_{CM})}\sum_{k=K_1}^{K_2}\frac{k!}{t^k}\sum_{j_1,\ldots,j_k=1}^{\infty}\int_{\Delta_1^k}|\langle A'(s_1),e_{j_1}\rangle_{H_1}|^2\ldots|\langle A'(s_k),e_{j_k}\rangle_{H_1}|^2 \\
&= \|\alpha\|^2_{J_t^0(\mathfrak{g}_{CM})}\sum_{k=K_1}^{K_2}\frac{1}{t^k}\left(\int_0^1\|A'(s)\|^2_{H_1}ds\right)^k = \|\alpha\|^2_{J_t^0(\mathfrak{g}_{CM})}\sum_{k=K_1}^{K_2}\frac{1}{t^k}\|A\|^{2k}_{\mathcal{H}([0,1],H_1)}\,.
\end{aligned}$$

Knowing that $\sum_{k=0}^{\infty}\frac{1}{t^k}\|A\|^{2k}_{\mathcal{H}([0,1],H_1)} = e^{\|A\|^2_{\mathcal{H}([0,1],H_1)}/t}$, this calculation simultaneously shows that $\sum_{k=0}^{\infty}\alpha(\Psi_{\tau,k})$ converges absolutely for any given $\alpha\in J_t^0(\mathfrak{g}_{CM})$, and that $\sum_{k=0}^{K}\Psi_{\tau,k}(A)$ is a Cauchy sequence in $J_t^0(\mathfrak{g}_{CM})^*$, and thus converges to an element in $J_t^0(\mathfrak{g}_{CM})^*$, which is the map determined by $\alpha\mapsto\sum_{k=0}^{\infty}\langle\alpha,\Psi_{\tau,k}\rangle$. This completes the proof.

□

Theorem 11.17. *For $f\in\mathcal{H}L_t^2(G)$, $A\in\mathcal{H}_0([0,1],H_1)$, and $t\in[0,1]$,*

$$\mathcal{R}f(\nu A(t)) = \sum_{k=0}^{\infty}\mathcal{T}\mathcal{R}f(\Psi_{t,k}(A)) = \sum_{k=0}^{\infty}\mathcal{T}\mathcal{R}f\left(\int_{\Delta_t^k}A'(s_1)\otimes\ldots\otimes A'(s_k)ds\right)$$

which converges absolutely.

Proof. Let $f\in\mathcal{P}$. Recall from Theorem 10.8 that $\nu PA(t)$ satisfies the differential equation $(\nu PA)'(t)=L_{\nu PA(t)*}PA'(t)$, so that

$$\frac{d}{ds}\Big(f(\nu PA(s))\Big)\Big|_{s=t} = f'\big(\nu PA(t)\big)\big(L_{\nu PA(t)*}PA'(t)\big) = \Big(\widetilde{PA'(t)}f\Big)(\nu PA(t))\,.$$

Then, by the (finite-dimensional) fundamental theorem of calculus,

$$f(\nu PA(t)) = f(e)+\int_0^t \left(\widetilde{PA'(s)}f\right)(\nu PA(s))ds = f(e)+\int_0^t \left\langle \widehat{f}(\nu PA(s)), PA'(s)\right\rangle ds\,.$$

Then consider that, by applying the fundamental theorem of calculus again,

$$\begin{aligned}\left(\widetilde{PA'(s)}f\right)(\nu PA(s)) \;&=\; \left(\widetilde{PA'(s)}f\right)(e) + \int_0^s \widetilde{PA'(r)}\widetilde{PA'(s)}f(\nu PA(r))dr \\ &=\; \left\langle \widehat{f}(e), PA'(s)\right\rangle + \int_0^s \left\langle \widehat{f}(\nu PA(r)), PA'(r)\otimes PA'(s)\right\rangle dr\,,\end{aligned}$$

from which we may write

$$f(\nu PA(t)) \;=\; f(e) + \int_0^t \langle \widehat{f}(e), PA'(s)\rangle ds + \int_0^t\int_0^s \langle \widehat{f}(\nu A(r)), PA'(r)\otimes PA'(s)\rangle drds\,.$$

Then consider that $f\in\mathcal{P}$ implies that $\widehat{f}(e)(H_1^{\otimes k}) = 0$ for $k\geq K$ for some K, as argued in the proof of Theorem 11.13. Then we may iteratively repeat this application of the fundamental theorem of calculus, which will terminate at the series

$$f(\nu PA(t)) \;=\; f(e) + \sum_{k=1}^K \int_{\Delta_t^k} \widehat{f}(e)\Big(PA'(s_1)\otimes\ldots\otimes PA'(s_k)\Big)ds \;=\; \sum_{k=0}^K \widehat{f}(e)(\Psi_{t,k}^P(A))$$

Now consider that, by Lemma 10.10, $\nu P_m A(1)\to\nu A(1)$ in $\|\cdot\|_{\mathfrak{g}_{CM}}$ (and hence in $\|\cdot\|_G$), so that $f(\nu P_m A(1))\to f(\nu A(1))$. On the other hand, by Lemma 10.7, we know that $A_1\otimes\ldots\otimes A_k \mapsto \int_{\Delta^k} A_1'(s_1)\otimes\ldots\otimes A_k'(s_k)ds \in \mathcal{H}_0([0,1],H_1^{\otimes k})$ is continuous, so that $\Psi_{\cdot,k}:\mathcal{H}_0([0,1],H_1)\to\mathcal{H}_0([0,1],H_1^{\otimes k})$ is continuous, so that $\Psi_{t,k}(P_mA)\to\Psi_{t,k}(A)$ in $H_1^{\otimes k}$. Then, by taking the limit of both sides, we have that for any $f\in\mathcal{P}$ and $A\in\mathcal{H}_0([0,1],H_1)$,

$$f(\nu A(t)) \;=\; \sum_{k=0}^\infty \widehat{f}(e)\big(\Psi_{t,k}(A)\big)\,.$$

Now let $f\in\mathcal{H}L_t^2(G)$ be arbitrary. Choose $f_m\in\mathcal{P}$ such that $f_m\to f$ in $\mathcal{H}L_t^2(G)$. Then by the isometry properties of Theorem 11.7 and Theorem 11.8,

$$\|\mathcal{T}\mathcal{R}f - \mathcal{T}\mathcal{R}f_m\|_{J_t^0(\mathfrak{g}_{CM})} \;=\; \sup_P \|\mathcal{R}(f-f_m)\circ\iota^P\|_{\mathcal{H}L_t^2(G^P)} \;=\; \|f-f_m\|_{\mathcal{H}L_t^2(G)}\to 0\,.$$

Then Lemma 11.16 tells us $\alpha\mapsto\sum_{k=0}^\infty \alpha(\Psi_{t,k}(A))$ is continuous, which allows us to write

$$\begin{aligned}\mathcal{R}f(\nu A(t)) \;&=\; \lim_{m\to\infty}\mathcal{R}f_m(\nu A(t)) \;=\; \lim_{m\to\infty}\sum_{k=0}^\infty \widehat{f_m}(e)\big(\Psi_{t,k}(A)\big)\\ &=\; \lim_{m\to\infty}\sum_{k=0}^\infty \mathcal{T}\mathcal{R}f_m\big(\Psi_{t,k}(A)\big) \;=\; \sum_{k=0}^\infty \mathcal{T}\mathcal{R}f\big(\Psi_{t,k}(A)\big)\,.\end{aligned}$$

□

11.5.2 The stochastic Taylor expansion

Here, we will prove an infinite-dimensional version of Theorem 11.15. We will almost exactly imitate the proof of [DG10, Theorem 1.9] with no special considerations. This is because, despite assuming G is step 2 nilpotent with an elliptic diffusion, much of the analysis only needs to occur on H_1, like computing $J_t^0(\mathfrak{g}_{CM})$-norms. This calculation will also bear some resemblance to the proof of Theorem 11.17.

Recall from section Section 10.3 that we define $\Phi_{t,k}^P = \int_{\Delta_t^k} dPB_{t_1} \otimes \ldots \otimes dPB_{t_k}$. Let $\alpha \in T(H_1)'$. Then by Theorem 10.16, we know that, for each k, $\alpha(\Phi_{t,k}^{P_m})$ converges in L^2 to a random variable, which we denote as $\alpha(\Phi_{t,k})$, and $\mathbb{E}|\alpha(\Phi_{t,k})|^2 = \frac{t^k}{k!}\|\alpha\|^2_{H_1^{*\otimes k}}$. We now generalize these results. We begin with a generalization of Theorem 10.15.

Lemma 11.18. *Suppose $\alpha \in H^{*\otimes k}$, $\beta \in H^{*\otimes \ell}$. Then*

$$\mathbb{E}\big[\langle \alpha, \Phi_{t,k}\rangle \overline{\langle \beta, \Phi_{t,\ell}\rangle}\big] = \begin{cases} \frac{t^k}{k!}\langle \alpha, \beta\rangle_{H_1^{*\otimes k}} & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases} .$$

Proof. Let $P \in Proj(W_1)$. Theorem 10.15 tells us that if $k = \ell$, then $\mathbb{E}|\alpha(\Phi_{t,k}^P)|^2 = \frac{t^k}{k!}\|\alpha\|_{PH_1^{\otimes k}}$. By polarization, we may deduce that if $k = \ell$, then $\mathbb{E}\big[\alpha(\Phi_{t,k}^P)\overline{\beta(\Phi_{t,k}^P)}\big] = \frac{t^k}{k!}\langle \alpha, \beta\rangle_{PH_1^{*\otimes k}}$.

Now suppose that $k \neq \ell$. Then $\mathbb{E}[\alpha(\Phi_{t,k}^P)\overline{\beta(\Phi_{t,\ell}^P)}] = 0$ comes from the orthogonality of iterated Itô integrals. This can be deduced by considering the iterated Itô integral approach to chaos, which was first introduced in [Itô51]. Alternatively, this orthogonality is directly stated in [DØP09, Proposition 1.4].

We may now set $P = P_m$ for some $(P_m)_{m\in\mathbb{N}} \in Proj(W_1)^\uparrow$ and take the limit in m to reach the desired conclusion.

□

Lemma 11.19. *Let $\alpha \in J_t^0(\mathfrak{g}_{CM})$. Then the sum $\sum_{k=0}^\infty \alpha(\Phi_{t,k})$ converges in L^2. And the assignment $J_t^0(\mathfrak{g}_{CM}) \ni \alpha \mapsto \sum_{k=0}^\infty \alpha(\Phi_{t,k}) \in L^2$ is continuous.*

Proof. Using the formula from Lemma 11.18,

$$\mathbb{E}\Bigg|\sum_{k=K_1}^{K_2} \alpha(\Phi_{t,k})\Bigg|^2 = \sum_{k=K_1}^{K_2} \mathbb{E}|\alpha(\Phi_{t,k})|^2 = \sum_{k=K_1}^{K_2} \frac{t^k}{k!}\|\alpha\|^2_{H_1^{*\otimes k}} .$$

Knowing that $\sum_{k=0}^{\infty} \frac{t^k}{k!} \|\alpha\|^2_{H_1^{*\otimes k}} = \|\alpha\|^2_{J_t^0(\mathfrak{g}_{CM})} < \infty$, we see that the sequence $\sum_{k=0}^{K} \alpha(\Phi_{t,k})$ is Cauchy in L^2, and hence converges to a random variable, and that this must satisfy $\mathbb{E}\big|\sum_{k=0}^{\infty} \alpha(\Phi_{t,k})\big|^2 = \|\alpha\|^2_{J_t^0(\mathfrak{g}_{CM})}$. □

Theorem 11.20. *For any $f \in \mathcal{H}L_t^2(G)$, we have*

$$f(g_t) \;=\; \sum_{k=0}^{\infty} \mathcal{TR}f\big(\Phi_{t,k}\big) \;=\; \sum_{k=0}^{\infty} \int_{\Delta_t^k} \mathcal{TR}f\big(dB_{t_1} \otimes \ldots \otimes dB_{t_k}\big)$$

Proof. We show this in nearly an identical fashion to Theorem 11.17, but using stochastic calculus. First suppose that $f \in \mathcal{P}$. Then, by applying Itô's formula, noting that the Itô and Stratonovich integrals coincide here,

$$f(g_t^P) = f(e) + \int_0^t \widehat{f}(g_s^P)(dPB_s)$$

and, in general, using the same methods as in the proof of Theorem 11.17 and stochastic calculus, we may iteratively apply Itô's formula to get

$$f(g_t^P) = \sum_{k=0}^{K} \int_{\Delta_t^k} \widehat{f}(e)(dPB_{s_1} \otimes \ldots \otimes dPB_{s_k}) = \sum_{k=0}^{K} \widehat{f}(e)(\Phi_{t,k}^P)\,,$$

where K is chosen such that $\widehat{f}(e)|_{H_1^{\otimes k}} = 0$ for all $k \geq K$.

Now consider that, by Corollary 11.1, $f(g_t^{P_m}) \to f(g_t)$ in L^2. On the other hand, by Theorem 10.16, we see that $\sum_{k=0}^{K} \widehat{f}(e)(\Phi_{t,k}^{P_m}) \to \sum_{k=0}^{K} \widehat{f}(e)(\Phi_{t,k})$ in L^2. Thus, almost surely, we may write

$$f(g_t) \;=\; \sum_{k=0}^{K} \widehat{f}(e)(\Phi_{t,k}) \;=\; \sum_{k=0}^{\infty} \widehat{f}(e)(\Phi_{t,k})\,.$$

Now let $f \in \mathcal{H}L_t^2(G)$ be arbitrary. Then choose $f_m \in \mathcal{P}$ such that $f_m \to f$ in $\mathcal{H}L_t^2(G)$. Then $f_m(g_t)$ converges to $f(g_t)$ in L^2. But by the isometric properties in Theorem 11.7 and Theorem 11.8, $\|\mathcal{TR}f - \mathcal{TR}f_m\|_{J_t^0(\mathfrak{g}_{CM})} = \|f - f_m\|_{\mathcal{H}L_t^2(G)} \to 0$. So using the continuity described in Lemma 11.19, we see that

$$f(g_t) \;=\; \lim_{m\to\infty} f_m(g_t) \;=\; \lim_{m\to\infty} \sum_{k=0}^{\infty} \mathcal{TR}f_m(\Phi_{t,k}) \;=\; \sum_{k=0}^{\infty} \mathcal{TR}f(\Phi_{t,k})\,.$$

□

11.5.3 On the restriction map and the restricted functions

We will first describe the inverse of the restriction map, just as Theorem 11.17 and Theorem 11.20 describe inverses of $\mathcal{T}$ and $\mathcal{T}\circ\mathcal{R}$. Then we will explicitly write the statement to the Taylor isomorphism on G, which will also explain how to define $\mathcal{H}L_t^2(G_{CM})$ when $\bigcup_P G^P \neq G_{CM}$.

Theorem 11.21. *For any $f \in \mathcal{H}L_t^2(G)$, $\big(\mathcal{R}f(g^{P_m})\big)_{m\in\mathbb{N}}$ converges in L^2, and $f(g_t) = \lim_{m\to\infty}\mathcal{R}f(g^{P_m})$.*

Proof. By Theorem 11.20, we have that

$$f(g_t) \;=\; \sum_{k=0}^{\infty}\mathcal{T}\mathcal{R}f(\Phi_{t,k}) \;=\; \sum_{k=0}^{\infty}\int_{\Delta_t^k}\mathcal{T}\mathcal{R}f\big(dB_{s_1}\otimes\ldots\otimes dB_{s_k}\big)\,.$$

On the other hand, for any $P\in Proj(W_1)$, our methods suffice to prove the finite-dimensional counterpart (or one may directly use the formula in Theorem 11.15):

$$\mathcal{R}f(g_t^P) \;=\; \sum_{k=0}^{\infty}\mathcal{T}\mathcal{R}f(\Phi_{t,k}^P) \;=\; \sum_{k=0}^{\infty}\int_{\Delta_t^k}\mathcal{T}\mathcal{R}f\big(dPB_{s_1}\otimes\ldots\otimes dPB_{s_k}\big)\,.$$

Then, by once again applying the exhibited orthogonality from Lemma 11.18, and otherwise using methods similar to those in the proof of Theorem 10.16, if $\{e_j\}_{1\le j\le r_m}$ is a basis of P_mH_1, then

$$\begin{aligned}
&\mathbb{E}\Big|f(g_t)-\mathcal{R}f(g_t^{P_m})\Big|^2\\
&\quad=\sum_{k=0}^{\infty}\mathbb{E}\Big|\mathcal{T}\mathcal{R}f(\Phi_{t,k})-\mathcal{T}\mathcal{R}f(\Phi_{t,k}^{P_m})\Big|^2\\
&\quad=\sum_{k=0}^{\infty}\mathbb{E}\big|\mathcal{T}\mathcal{R}f\circ(I_H^{\otimes k}-P_m^{\otimes k})(\Phi_{t,k})\big|^2\\
&\quad=\sum_{k=0}^{\infty}\frac{t^k}{k!}\Bigg(\sum_{j_1,\ldots,j_k=1}^{\infty}\big|\mathcal{T}\mathcal{R}f(e_{j_1}\otimes\ldots\otimes e_{j_k})\big|^2-\sum_{j_1,\ldots,j_k=1}^{r_m}\big|\mathcal{T}\mathcal{R}f(e_{j_1}\otimes\ldots\otimes e_{j_k})\big|^2\Bigg)
\end{aligned}$$

then $\|\mathcal{T}\mathcal{R}f\|_{J_t^0(\mathfrak{g}_{CM})}<\infty$ implies that the expression above converges to 0 in m, which proves the claim.

□

Theorem 11.22. *For every $F\in\mathcal{H}L_t^2(\bigcup_P G^P)$, there exists an extension $\widetilde{F}: G_{CM}\to\mathbb{C}$ that satisfies the following.*

- *If $\mathcal{R}f(g) = F(g)$ for all $g \in \bigcup_P G^P$, then $\mathcal{R}f(g) = \widetilde{F}(g)$ for all $g \in G_{CM}$.*
- *If $A \in \mathcal{H}_0([0,1], H_1)$, then*

$$\widetilde{F}(\nu A(1)) = \sum_{k=0}^{\infty} \int_{\Delta_1^k} \mathcal{T}F(A'(s_1) \otimes \ldots \otimes A'(s_k))ds\,.$$

In particular, $\widetilde{F}$ can be calculated from the pullback $\mathcal{R}^{-1}F$ or from the push forward $\mathcal{T}F$.

Proof. Let $F \in \mathcal{H}L_t^2(\bigcup_P G^P)$. Then by Theorem 11.7 and Theorem 11.13, $\mathcal{R} : \mathcal{H}L_t^2(G) \to \mathcal{H}L_t^2(\bigcup_P G^P)$ is a bijection, so there exists a unique $f = \mathcal{R}^{-1}F \in \mathcal{H}L_t^2(G)$ such that $\mathcal{R}f = F$ on $\bigcup_P G^P$. But recall from Theorem 11.7 that $\mathcal{R}f$ (or more precisely $g \mapsto \mathcal{R}_g f$) is actually defined on G_{CM}. Hence, by defining $\widetilde{F} = \mathcal{R}f = \mathcal{R}\mathcal{R}^{-1}F$, the first point is satisfied.

The second bullet is a consequence of applying Theorem 11.17 to $\widetilde{F} = \mathcal{R}f$.

□

Theorem 11.23. *We define $\mathcal{H}L_t^2(G_{CM})$ in the following way.*

- *If $\bigcup_P G^P = G_{CM}$, then we define $\mathcal{H}L_t^2(G_{CM})$ to be $\mathcal{H}L_t^2(\bigcup_P G^P)$.*
- *In the event that $\bigcup_P G^P \neq G_{CM}$, we let $\mathcal{H}L_t^2(G_{CM})$ to be the set of extensions of elements in $\mathcal{H}L_t^2(\bigcup_P G^P)$ to G_{CM}, as described in Theorem 11.22.*

In any case, $\mathcal{H}L_t^2(G_{CM})$ is a Banach space identical to $\mathcal{H}L_t^2(\bigcup_P G^P)$ when equipped with the norm $\|f\|_{\mathcal{H}L_t^2(G_{CM})} := \sup_P \|f \circ \iota^P\|_{\mathcal{H}L_t^2(G^P)}$. For this set, we have the restriction map $\mathcal{R} : \mathcal{H}L_t^2(G) \to \mathcal{H}L_t^2(G_{CM})$ and Taylor map $\mathcal{T} : \mathcal{H}L_t^2(G_{CM}) \to J_t^0(\mathfrak{g}_{CM})$, both of which are isometric isomorphisms, and for which the composition $\mathcal{T} \circ \mathcal{R}$ is unitary.

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