Analysis of existence, regularity and stability of solutions to wave equations with dynamic boundary conditions.

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Abstract

In this dissertation, we present an analysis of existence, smoothing properties and long-time behavior of solutions corresponding to wave equation with dynamic boundary conditions. Different damping mechanisms acting on either the interior dynamics or the boundary dynamics or both will be considered.

This leads to a consideration of a wave equation acting on a bounded 3-d domain, equipped with zero-Dirichlet boundary condtions on a portion of the boundary, coupled with another second order dynamics acting on a portion of the boundary. These are general Wentzell type of boundary conditions which describe wave equation oscillating on a tangent manifold of a lower dimension. Both interior and boundary dynamics are subject to viscoleastic and/or frictional dampings. Chapter 1.2 provides the physical motivation for the model as well as mathematical background. Then, we shall examine the regularity and stability properties of the resulting system as a function of strength and location of the dissipation. Properties such as wellposedness (chapter 2) of finite energy solutions, analyticity of the associated semigroup (chapter 4), strong and uniform stability (chapter 5) will be discussed.

The results obtained analytically are illustrated by numerical analysis. The latter shows the impact of various types of dissipation on the spectrum of the generator.

This dissertation is lovingly dedicated to Julie and Lili-Rose.

Acknowledgments and Thanks

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Chapter 1

Introduction

The purpose of this dissertation is to study a model of damped wave equations with dynamic boundary conditions. We consider the following system:

$$\begin{cases} u_{tt} + c_\Omega u_t - k_\Omega \Delta u_t - \Delta u = 0 & x \in \Omega, t > 0 \\ u(x,t) = 0 & x \in \Gamma_0,\ t > 0 \\ u_{tt} + c_\Gamma u_t + \partial_n(u + k_\Omega u_t) - k_\Gamma \boldsymbol{\Delta}_\Gamma(\alpha u_t + u) = 0 & x \in \Gamma_1,\ t > 0 \\ u(0,x) = u_0,\ u_t(0,x) = u_1 & x \in \Omega \end{cases} \tag{\textbf{GM}}$$

where $u = u(x,t), t \geq 0, x \in \Omega$ is a bounded domain in Euclidiean space with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$; Δ denotes the Laplacian operator with repect to the space variable x; ∂_n denotes the outer normal derivative; $\boldsymbol{\Delta}_\Gamma$ denotes the Laplace-Beltrami operator on the boundary Γ_1 with repect the variable x; c_Ω, c_Γ, k_Ω, k_Γ and α are non-negative constants.

1.1 Physical background & literature review

A classical and common approach to study partial differential equations, in particular wave equations, is to study them with homogeneous boundary conditions such as Dirichlet, Neumann or Robin. However, in many applications one has to also consider the dynamic behavior of the boundary which can be a wall for instance in response to the acoustic waves. This study fits into the research done during the last decades which consists in developing a mathematical theory of *dynamic boundary conditions*. It covers different sort of differential equations: elliptic equations (see e.g. [1, 29, 36, 53]), parabolic equations (see e.g. [17]) and second order hyperbolic equations (see e.g. [5]). It is now studied through the semigroup theory (see e.g. [11, 15, 19]) and also in a more abstract approach (see e.g. [47, 48]). From the mathematical point of view, these problems do not neglect acceleration terms on the boundary and were first introduced by Beale-Rosencrans, in [5, 6, 7], describing this acoustic/structure interaction by a wave equation with *acoustic boundary conditions*. Based on Beale-Rosencrans' work, several authors studied similar problems of wave equations with a dynamic behavior of the solution on the boundary, see for instance: [19, 20, 23, 25, 27, 30], we shall discuss some of these results later as a preliminary to this study. Moreover, the acoustic or dynamic boundary conditions find numerous of applications in the bio-medical domain [10, 59], as well as in applications related to stabilization and active control of large elastic structures. See [45] and references therein for some applications.

The common denominator for all these studies is the physical motivation introduced by Morse-Ingard [46]. In order to present a complete study of the model (**GM**), it is necessary to review a couple of concepts in physics to connect the physical and mathematical description of this model. We start with a non-viscous fluid, which in the absence of sound is at rest with 0-heat conductivity. We denote by ρ, $\mathtt{P}$ and $\mathtt{T}$ the uniform density, pressure and temperature respectively which become, once a wave is introduced, $\rho + \delta(x,t)$, $\mathtt{P} + \mathtt{p}(x,t), \mathtt{T} + \tau(x,t)$. The motion of fluid is induced by a change in pressure, also called the *acoustic*. The motion of the fluid is responsible for the change in density through the following formula:

$$\delta(x,t) = \rho\kappa\mathtt{p}(x,t) \tag{1.1.1}$$

where κ denotes the adiabatic compressibility. It corresponds to the variation of volume under some pressure given a constant entropy.

We now recall a basic definition which will allow us to obtain the wave equation. Consider any quantity f of the fluid at position $r = (x,y,z)$, time t and define the fluid veolocty at r and t by $\mathbf{v} = (\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z)$. Then, the total time derivative $f(r,t)$ is:

$$\begin{aligned}\frac{df}{dt}dt &= f(r + \mathbf{v}\ dt, t + dt) - f(r,t)\\ &= f(r,t) + \left(\mathbf{v}_x\frac{\partial f}{\partial x} + \mathbf{v}_y\frac{\partial f}{\partial y} + \mathbf{v}_z\frac{\partial f}{\partial z}\right)dt + \frac{\partial f}{\partial t}dt - f(x,t)\\ &= \frac{\partial f}{\partial t} + \mathbf{v}.\nabla f\end{aligned} \tag{1.1.2}$$

where $\mathbf{v}.\nabla f$ measures the rate of change of f at point r, time t caused by the fluid flow. We also introduce the notion of the flux of a fluid which is the total amount of property f passing per second across a unit cross-sectional area normal to the direction. In particular, for any property f travelling with the fluid, the flux of f is defined by:

$$\mathbf{J}(x,t) = f \times \mathbf{v} \tag{1.1.3}$$

To illustrate this concept, consider a volume element $dx\ dy\ dz$, then the flux $\mathbf{J}_x(x)$ (x-component of $\mathbf{J}(x)$) is a gain while $\mathbf{J}_x(x+dx)$ is a loss of f. Therefore, the net flux out the volume $dx\ dy\ dz$ passing across the cross-section $dy\ dz$ is

$$dy\ dz\ (-\mathbf{J}_x(x+dx) + \mathbf{J}_x(x))dx = -dy\ dz\ \frac{\partial \mathbf{J}_x}{\partial x}dx$$

Thus the total net flux out is:

$$dy\ dz\frac{\partial J_x}{\partial x}dx + dx\ dz\frac{\partial J_y}{\partial y}dy + dx\ dy\frac{\partial J_z}{\partial z}dz = \mathsf{div}\ \mathbf{J}dx\ dy\ dz \tag{1.1.4}$$

In addition, suppose that f is created at a rate $Q(r,t)$ per unit volume of fluid, then the net rate of change f within the volume $dx\ dy\ dz$ is equal to what has been created minus the net flux out:

$$\frac{\partial f}{\partial t}dx\ dy\ dz\ = Qdx\ dy\ dz\ - \mathsf{div}\ \mathbf{J}dx\ dy\ dz$$

Combining (1.1.2) and (1.1.3), we obtain the general equation of continuity:

$$\frac{df}{dt} = Q - f\mathsf{div}\ \mathbf{v} \tag{1.1.5}$$

Note that any increase of f in a region must have been brought there by fluid flow or else by specific creation $(Q(r,t))$.

The equation of continuity for the density $\rho + \delta$ is:

$$\frac{d(\rho+\delta)}{dt} = -(\rho+\delta)\mathsf{div}\ \mathbf{v}$$

$$\Rightarrow \frac{\partial\delta}{\partial t} + \mathbf{v}.\nabla\delta = -(\rho+\delta)\mathsf{div}\ \mathbf{v} \quad \text{by equation (1.1.2)}$$

Then by applying (1.1.1), we obtain the *equation of continuity* for the pressure:

$$\rho\kappa\frac{\partial\mathsf{p}}{\partial t} = -\rho(1+\kappa\mathsf{p})\mathsf{div}\ \mathbf{v} - \rho\kappa(\mathbf{v}.\nabla\mathsf{p}) \tag{1.1.6}$$

On the other hand, observe that for a mass of fluid $(\rho+\delta)dx\ dy\ dz$ the change in force in the volume $dx\ dy\ dz$ is $\mathsf{p}(x,y,z) - \mathsf{p}(x+dx, y+dy, z+dz) = -\nabla\mathsf{p}dx\ dy\ dz$. By the second's law of Newton, we obtain the *equation of motion*:

$$(\rho+\delta)dx\ dy\ dz\frac{d\mathbf{v}}{dt} = (\rho+\delta)\left[\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v}\right]dx\ dy\ dz = -\nabla\mathsf{p}dx\ dy\ dz$$

which becomes after simplification:

$$(\rho+\delta)\left[\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v}\right] = -\nabla\mathsf{p} \tag{1.1.7}$$

We choose to derive the wave equation for the velocity for matter of simplicity, however note that one could easily derive the wave equation for the pressure, the velocity potential, the height of the wave or any other property. For our need, i.e., a mathematical study of wave equations one can neglect the effects of thermal conduction or change of compressibility with pressure, thus it is enough to consider to the first order the equation of continuity (1.1.6) and the equation of motion (1.1.7):

$$\begin{cases} \rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla \mathtt{p} \\ \kappa \frac{\partial \mathtt{p}}{\partial t} = -\mathsf{div}\ \mathbf{v} \end{cases} \tag{1.1.8}$$

The first of these equations states that a velocity gradient produces a compression of the fluid; the second states that a pressure gradient produces an acceleration of the fluid.

Applying time derivative operator to the first equation and using the second to eliminate $\mathtt{p}$, we get the wave equation for the velocity:

$$\begin{cases} \kappa\rho \frac{\partial^2 \mathbf{v}}{\partial t^2} = \Delta \mathbf{v} \end{cases} \tag{1.1.9}$$

In model (**GM**), we have such a wave equation acting in the interior domain Ω (plus additional terms which we shall discuss later) inside a bounded domain. Once the incident wave strikes a surface of the boundary, the pressure of the incident wave does not make move the mass load of the surface instantanously, making the reflecting pulse unchanged. However, the impulse absorbed tends to create a damped oscillation first receding from the flow. More precisely, the acoustic pressure $\mathtt{p}$ interacts with the surface, either by forcing more fluid into its pores or by making the surface move. If we assume that the fluid can move in the direction normal to the surface, a wave motion will be induced in the material forming the surface. The relationship between the motion of two points of the surface can be determined by the wave motion inside the material as well as by the incident and reflected waves. As a consequence, imposing classical boundary conditions (Dirichlet, Neumann or Robin type) to the wave equation would not be appropriate as it would not describe such a behavior on the boundary.

Furthermore, if the different components of the surface are not strongly coupled together, it is commonly assumed that the motion of the surface or the motion of the fluid into the surface pores at a given point is only determined by the acoustic pressure at that point and not anymore by the motion of any other portion of the surface. When this is the case, we say that the surface is one of *local reaction.* For small amplitudes, one can assume that:

- the relationship between surface motion and pressure is linear,
- the pressure is proportional to the velocity normal,
- the surface motion is not affected by tangential fluid motion.

Under these assumptions *each point of the reflecting surface acts like a simple-harmonic oscillator*, thus by the Hooke's law, the acceleration is proportional to the displacement meaning that the corresponding boundary conditions must be of the form $u_{tt} + u_t + u = 0$. The first mathematical model for surface of local reaction was introduced by Beale-Rosencrans, in [7]. The authors described the acoustic/structure interaction by a wave equation with *acoustic boundary conditions*:

$$\begin{cases} u_{tt} - \Delta u = 0, x \in \Omega, t > 0 \\ u(x,t) = 0, \quad x \in \Gamma_0, \ t > 0 \\ m(x)\delta_{tt} + +d(x)\delta_t + k(x)\delta = -\rho\partial_n u_t, \ x \in \Gamma_1, \ t > 0 \\ \delta_t = \partial_n u, \ x \in \Gamma_1 \\ u(0,x) \in H^1(\Omega), \ u_t(0,x) \in L_2(\Omega), \quad x \in \Omega \\ z(0,x) \in H^1(\Omega), \ z_t(0,x) \in L_2(\Omega), \quad x \in \Omega \end{cases} \tag{1.1.10}$$

In [5, 6], they demonstrated that the problem was governed by a C_0-semigroup of contractions. In [27] (and references therein) the authors pointing out long time behavior and continuous dependance of solutions on the mass of the structure. With a more abstract approach, Mugnolo, in [47, 48] proved generation and regularity of semigroup for coupled systems of a wave equation with acoustic boundary conditions similar to (1.1.11). His work was initiated by Casarino's work (see [11]) on a heat equation with *acoustic boundary conditions*.

Inspired by model (1.1.10) which descibes the interaction between interior and boundary dynamics by a coupled system, surface of local reaction have been intensively studied for the last decade as a wave equation with *dynamic boundary conditions*:

$$\begin{cases} u_{tt} - k_\Omega \Delta u_t - \Delta u = f_1(u) \quad x \in \Omega, t > 0 \\ u(x,t) = 0 \quad x \in \Gamma_0, \ t > 0 \\ u_{tt} + \partial_n(u + k_\Omega u_t) + \rho(u_t) + f_2(u) = 0 \quad x \in \Gamma_1, \ t > 0 \\ u(0,x) = u_0, \ u_t(0,x) = u_1 \quad x \in \Omega \end{cases} \tag{1.1.11}$$

Remark 1.1.1. *What distinguishes our model* (**GM**) *from* (1.1.11) *is the presence of Laplace-Beltrami operator in the dynamic boundary condition.*

Gerbi and Said-Houari in [24, 25] studied the problem (1.1.11) with $f_2 = 0$, $f_1(u) = |u|^{p-2}u$ and a nonlinear boundary damping term of the form $\rho(u_t) = |u_t|^{m-2}$. A local existence result was obtained by combining the Faedo-Galerkin method with the contraction mapping theorem. The authors also showed that under some restrictions on the exponents m and p, there exists inital data such that the solution is global in time and decays exponentially. Graber and Said-Houari in [28] extended some of these results to more general function f_1 and f_2. They also demonstrate that for $\rho = f_2 = 0$, the model (1.1.11) was governed by an analytic semigroup on $H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1)$.

Not only wave equation with acoustic boundary conditions (Beale's model (1.1.10)) and wave equations with dynamic boundary conditions (model (**GM**) and (1.1.11)) describe similar physical phenomenons but they also coincide with the so-called *Wentzell boundary conditions* given conditions on the parameters ([23]). Such boundary conditions involve second derivatives as well as lower order terms (Robin type) and Laplace-Beltrami terms and were intensively studied by Favini, Goldstein, Gal et al., in [19, 20] and references therein, in the context of hybrid problems. Firstly, this group started to study the *Wentzell boundary conditions* with a heat equation, in [19]:

$$u_t = \Delta u \quad t \geq 0, \; in \; \Omega \tag{1.1.12}$$

$$\Delta u + \beta \partial_n u + \gamma u = 0 \qquad on \; \delta\Omega \tag{1.1.13}$$

Although Wentzell boundary conditions were usually studied in spaces of continuous functions, the authors presented a new framework involving weighted L_p spaces using both the domain and the boundary. This framework was used in most of the studies mentioned above about the model (1.1.11). It will also be used and discussed in chapter 4. Gal, from the same group, applied the Wentzell boundary conditions (equation (1.1.13)) to the wave equation and establish that the generators of the contraction C_0-semigroups governing the wave equation with GWBC and the wave equation with acoustic boundary conditions (1.1.10) only differ by an operator which is self-adjoint and bounded on the appropriate energy space ([23, theorem 4.1]). Other papers used this framework to study one-dimensional wave equation without internal damping, with Wentzell boundary conditions, see [2, 18, 60].

A fundamental extension of the initial paper [19] was made in 2010 by Favini et al. also [20]. The authors introduced a Laplace-Beltrami term in the Wentzell boundary conditions (1.1.13), which we shall now call the *General Wentzell Boundary Conditions (GWBC).* In both papers [19, 20], the authors showed that the associated semigroup was analytic on the weighted L_p space. It is one essential aspect of this study to understand how the introduction of this Laplace-Beltrami term on the boundary is affecting the general dynamic of the system.

The introduction of the Laplace-Beltrami term on the boundary is also relevant from the physical point of view. Although many surfaces react to sound waves, at least approximately, as though each portion of the surface responded to local pressure without knowledge of motion elsewhere (i.e., in the absence of Laplace-Beltrami term on the boundary), many surfaces do not react so. For instance, one can think of a plane separating two fluids or surrounded by fluids. When the surface behavior at one point depends on the behavior at neighboring points, so that the reaction is different for different incident waves, the surface can be called one of *extended reaction.* Such surfaces are of many sorts: one which behaves like membranes and one with laminated structure, in which waves are propagated parallel to the surface; one in which waves penetrate into the material of the wall. From the mathematical point of view, it means that the boundary condition are more complex than a simple harmonic oscillator, i.e., the acceleration is not proportional to the displacement anymore but to the relative displacement compared to its neighbours, in other words, such as a classical wave equation: $u_{tt} - \Delta u = 0$. Note that the Laplace operator on the boundary is not the same as the one in the interior. Thus, we denote the Laplace-Beltrami operator acting on the boundary Γ_1 on the tangential direction by $\mathbf{\Delta}_\Gamma u$.

The last aspect to describ in the model (**GM**) is the damping mechanisms. From the mathematical point of view, undamped wave equations are well-posed in an L^p-setting if and only if $p = 2$ or the space dimesion is 1 ([37]), it is well-known that the strongly damped wave equation under classical boundary conditions such as Dirichlet or Robin generates an exponentially stable analytic C_0-semigroup [12]. In fact, this results is known for a larger class of scalar problems where viscoelastic damping is given in a form of fractional powers of Laplacian [12, 13]. In this context, the study seeks to establish various results about the regularity and the stability of the semigroup. From the physical point of view, the natural physical behavior of waves propagating in a closed domain automatically provides one or several damping forces. Therefore, in order to ensure the presence of damping and thus energy dissipation in this system, we impose the following condition:

$$max\{k_\Omega, k_\Gamma\alpha, c_\Omega, c_\Gamma\} > 0 \tag{1.1.14}$$

System (**GM**) models energy dissipation through two different phenomenons. The first damping considered is the frictional one. Although friction is commonly defined as the resistance to motion which the air surrounding the body manifests, energy in the form of sound waves being sent out into the air can still be considered as friction for the energy of the system diminishes, being drained away in the form of sound, the amount depending on the radiation resistance of the medium. This resisting force depends on the velocity of the vibrator, and unless the velocity is large, it is proportional to the velocity: $c_\Omega u_t$ where the constant c_Ω is the resistance constant. The second category of damping models viscoelastic type behavior, where part of the system's energy goes into heating the medium, the amount depending on the viscosity of the medium. In such a case, the stress is proportional to the strain rate and is represented by $k_\Omega \Delta u_t$.

Analogously, boundary dissipation can be defined and is modeliezed by frictional and viscoelastic dampings terms: $c_\Gamma u_t$ and $k_\Gamma \alpha \boldsymbol{\Delta}_\Gamma u_t$ respectively.

With the above motivation in mind, we now define the core mathematical questions to be addressed in this work:

- Does there exist a unique solution to (**GM**) which is continuous in time and depends continuously on the initial conditions ?

- Does the energy of solutions to (**GM**) decay asymptotically to zero as time approaches infinity ?

- Under which assumptions does this model provide exponential stability ?

- Does the solution to (**GM**) enjoys regularity properties ?

We will answer all of these questions identifying the role of each term by setting different values for the parameters $k_\Omega,\ c_\Omega,\ k_\Gamma,\ c_\Gamma,\ \alpha$. This will provide a rigorous mathematical understanding of a wide range of possible acoustic/structure interaction submodels present in (**GM**). Because the analysis (and results to follow) are highly dependent upon parameter values, damping mechanisms, it is neither reasonnable nor beneficial to the reader to have an entire linear discussion of well-posedness, regularity and long-time behavior results. For this reason, we divided the dissertation into six chapters, each enlighting an important characteristic of system (**GM**), but often relying on previous results.

In chapter 2, we address well-posedness considerations for the model (**GM**) in the appropriate energy space. Well-posedness was previously established for few particular cases of our model such as in the absence of Laplace-Beltrami terms on the boundary ($k_\Gamma = 0$) [28]. It is one aim of this dissertation to generalize well-posedness results for (**GM**) under semigroup theory with the Lumer-Philips theorem [44].

For the following chapters, two submodels will often be distinguished from (**GM**) (i) to isolate, and thus (ii) to have a better understanding of, phenomenons caused by the viscoelastic or frictional dampings. First, we consider a wave equation with dynamic boundary conditions with only frictional damping on the interior and/or the boundary:

$$\begin{cases} u_{tt} + c_\Omega u_t - \Delta u = 0 & x \in \Omega, t > 0 \\ u(x,t) = 0 & x \in \Gamma_0,\ t > 0 \\ u_{tt} + c_\Gamma u_t + \partial_n u - \ k_\Gamma \boldsymbol{\Delta}_\Gamma u = 0 & x \in \Gamma_1,\ t > 0 \\ u(0,x) = u_0,\ u_t(0,x) = u_1 & x \in \Omega \end{cases} \qquad \textbf{(FM)}$$

In other words, the coefficients attached to the viscoelastic damping on the interior k_Ω and on the boundary α are null. It is well known that a wave equation with a frictional damping and 0-Dirichlet boundary conditions ($\Gamma_1 = \emptyset$) leads to a C_0-semigroup of contraction with exponential stability. The spectrum of this semigroup is a vertical asymptote vertical asymptote, for which all the eigenvalues have real part $-c_\Omega$, whenever $0 < c_\Omega \leq 2$, i.e., as the damping coefficient increases to 2, the vertical asymptote moves away from the imaginary axis. It can actually be shown that the stability is optimal for $c_\Omega = 2$ which often goes by critical damping. We leave to the reader the proof, and recall that for any $c_\Omega > 2$, the first conjugate eigenvalues become real, and one goes to $-\infty$ while the other one moves back to the imaginary axis. Such a situation is commonly called overdamping, which means that the exponential stability property is still holding but the decay is not as quick as for the critical damping ($c_\Omega = 2$). Note that in the literature, the friction coefficent is often defined by $c_\Omega = 2\rho$, as a consequence, the critical damping would be reached at $\rho = 1$. Among the cases we will study, figure 1.1 represents the spectrum for model (**FM**) with different values of c_Ω and c_Γ, with the additional condition $c_\Omega = c_\Gamma$, it enlights that the eigenvalues also form a vertical asymptote, as well as the presence of a critical situation also reached at $c_\Omega = c_\Gamma = 2$, represented in blue, since some eigenvalues once they hit the real axis (for $c_\Omega = c_\Gamma > 2$) move back to the imaginary axis. Indeed, figure 1.1 also shows the track (red lines from right to left) followed by each eigenvalue as the coefficients c_Ω and c_Γ increase from 1 to 3, represented in red. Note that one would get a very similar picture in the case $\Gamma_1 = \emptyset$ previously described. In addition, the presence of vertical asymptote does not suggest that the semigroup could become analytic; we will show that the semigroup in this case is also one of contraction (chapter 2) with exponential stability (chapter 5).

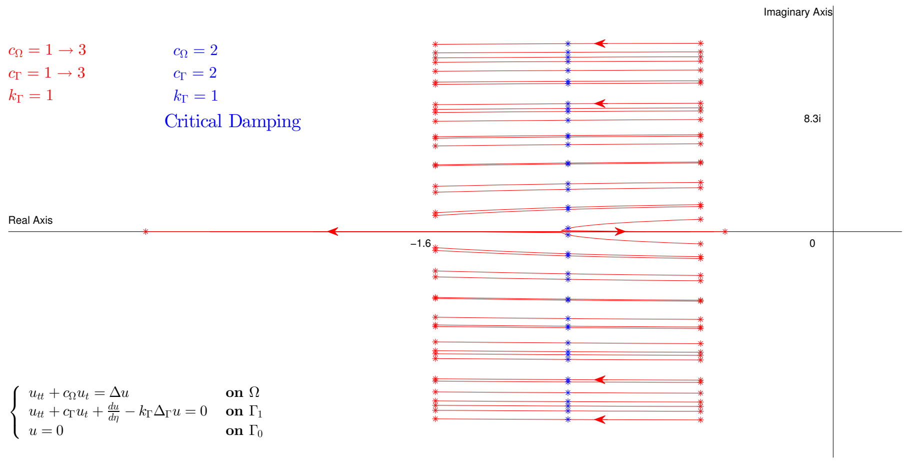


Figure 1.1: *Impact of the frictional damping on the spectrum..* Eigenvalues for the model (**FM**) ($k_\Gamma = 1$) with the damping coefficients c_Ω and c_Γ running from 1 to 3 (in red) and a critical-like damping reached at $c_\Omega = c_\Gamma = 2$ (in blue).

However, under other settings of coefficients, the exponential stability might not hold anymore. For instance, if there is no damping in the interior ($c_\Omega = 0$), one could wonder how the boundary damping would control the general dynamic of the system. Figure 1.2 represents this exact scenario. The spectrum contained two distinct components, the first one in the middle of the figure is an asymptotic-shape related to the damping force on the boundary. The second component is closed to imaginary axis and corresponds to the interior dynamics. Such a case is more complicated to treat, and the theoretical approach will help us to interpret the numercial results.

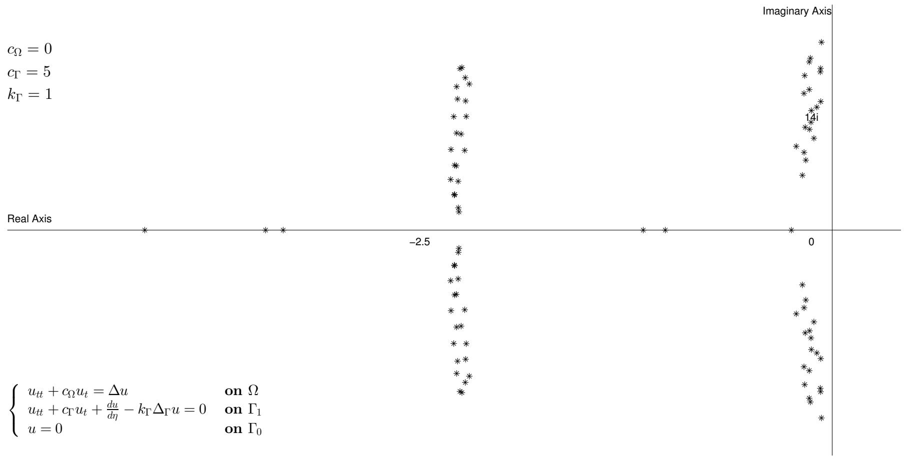


Figure 1.2: *Spectrum with only frictional boundary damping: effect of inertial boundary term u_{tt}.* Eigenvalues for the model (**FM**) in the absence of interior damping. ($c_\Omega = 0$, $c_\Gamma > 0$ and $k_\Gamma = 1$)

We are also interested in the behavior of the system (**GM**) under pure viscoelastic damping, i.e., the frictional dampings are null: $c_\Omega, c_\Gamma = 0$.

$$\begin{cases} u_{tt} - k_\Omega \Delta u_t - \Delta u = 0 & x \in \Omega, t > 0 \\ u(x,t) = 0 & x \in \Gamma_0,\ t > 0 \\ u_{tt} + \partial_n(u + k_\Omega u_t) - k_\Gamma \boldsymbol{\Delta}_\Gamma(\alpha u_t + u) = 0 & x \in \Gamma_1,\ t > 0 \\ u(0,x) = u_0,\ u_t(0,x) = u_1 & x \in \Omega \end{cases} \quad (\mathbf{VM})$$

We shall start by placing this model within existing result about the strong damped wave equation with 0-Dirichlet boundary conditions. This can be represented by the model (**VM**) under the following setting: $\Gamma_0 = \delta\Omega$, $\Gamma_1 = \emptyset$ and $k_\Omega = 1$. Chen and Triggiani in [12, Appendix A] derived an explicit formula for the eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ given by:

$$\lambda_n^{+,-} = \mu_n \left(-\frac{k_\Omega}{2} \pm \sqrt{\frac{k_\Omega^2}{4} - \mu_n^{-1}} \right)$$

where $\{\mu\}_{n=1}^{\infty}$ are the eigenvalues of Δ

The spectrum (figure 1.3) formed by these eigenvalues is a common pattern for analytic semigroup and will reappear in several scenarii of this study. The first eigenvalues are conjugate pairs running over a circle of radius $\frac{1}{k_\Omega}$ with center at $-\frac{1}{k_\Omega}+0\times i$. Then, for all n such that $k_\Omega^2 > \mu_n^{-1}$ each conjugate pair splits: λ_n^+ tends to an accumulation point at the center $-\frac{1}{k_\Omega}$ of the circle while λ_n^- goes to negative infinity as n increases. Such a spectrum can be enclosed inside a triangular sector confirming the analyticity of the associated semigroup, and the absence of eigenvalues on the imaginary axis confirms the exponential stability.

Adding dynamic boundary conditions on a portion of $\partial\Omega$ (on Γ_1) bring into question the persistence of these stability and regularity properties. Before giving the details in chapters 4 and 5, we can partially answer to this question, by considering the model (**VM**) with damping in the interior and on the boundary ($k_\Omega, k_\Gamma, \alpha > 0$), we recover a similar spectrum suggesting similar properties (analyticity and exponential stability),

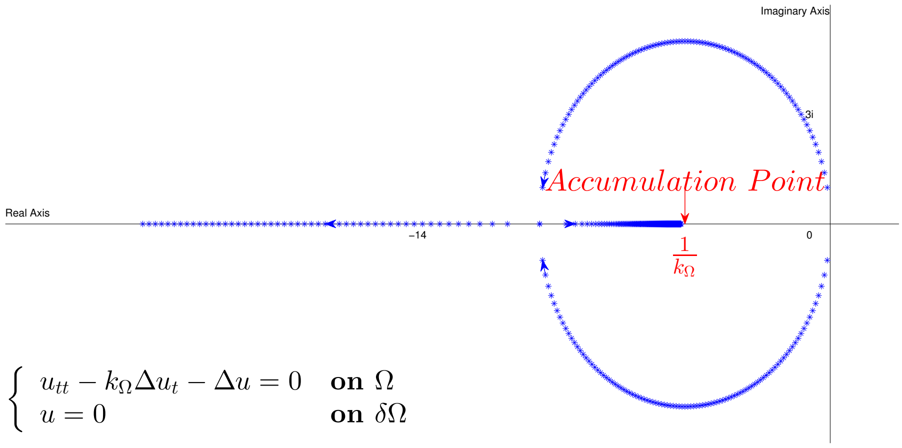


Figure 1.3: Theoretical eigenvalues for a 2-dimensional wave equation with viscoelastic damping $k_\Omega = 0.1$ and 0-Dirichlet boundary conditions.

see figure 1.4. In addition this figure illustrates how the viscoelastic damping interacts with the eigenvalues: as the damping coefficients (k_Ω and α) increase, the radius of the circle decreases while the high frequency mode hits the real axis.

Remark 1.1.2. *It is not the aim of this dissertation to establish a relationship between the interior and boundary viscoelastic dampings (k_Ω - α). Thus, whenever both coefficients are non-zero, we set them equal so that the circles formed by the eigenvalues coming from the interior and the boundary coincide providing a better readability of the figures. Otherwise, we would notice two distinct circles.*

However, when one of these two damping is dropped to 0 ($k_\Omega = 0$ or $\alpha = 0$),

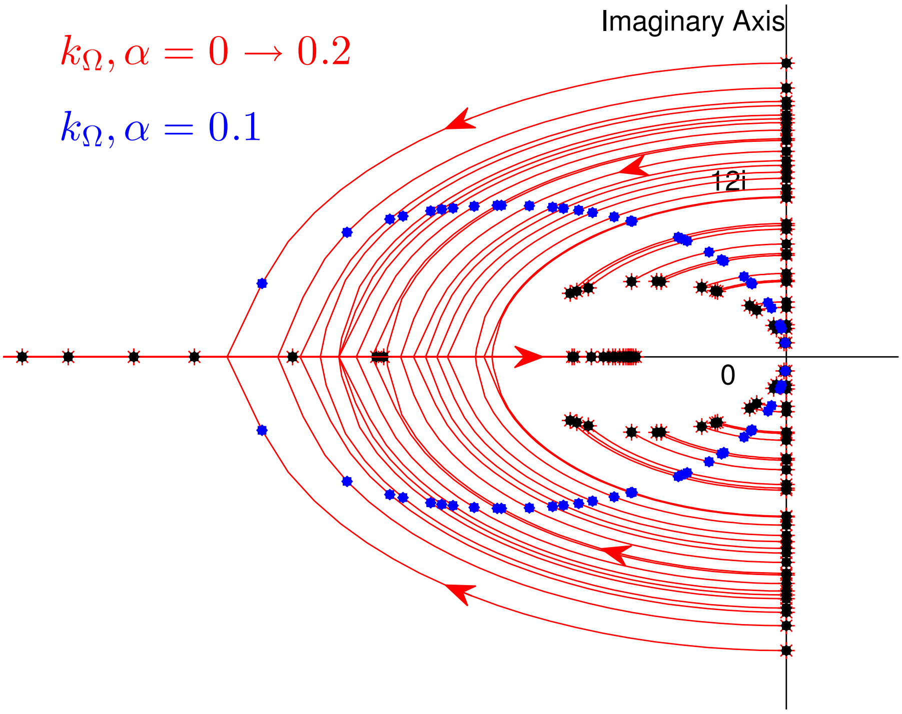


Figure 1.4: *Effect of viscoelastic damping on the spectrum.*. Eigenvalues for the model (**VM**) with viscoelastic damping both on the interior and the boundary ($k_\Omega, k_\Gamma, \alpha > 0$). In red, tracks followed by each eigenvalue as $k_\Omega, k_\Gamma \alpha = 0$ (black dots along the imaginary axis)$\to 0.2$ (black dots forming a circle). In blue, eigenvalues for the case $k_\Omega = k_\Gamma \alpha = 0.1$.

the spectrums are not easy to interpret anymore. For instance, in the absence of interior damping, we still observe a circle-shape component in the spectrum due to the damped wave equation on the boundary, but a new component appears along the imaginary axis, which is critical not only for the solution's stability but also for its regularity. Indeed, figure 1.5 suggests that the semigroup could not be analytic anymore and the exponential stability property should also be questionned. Eventually, we

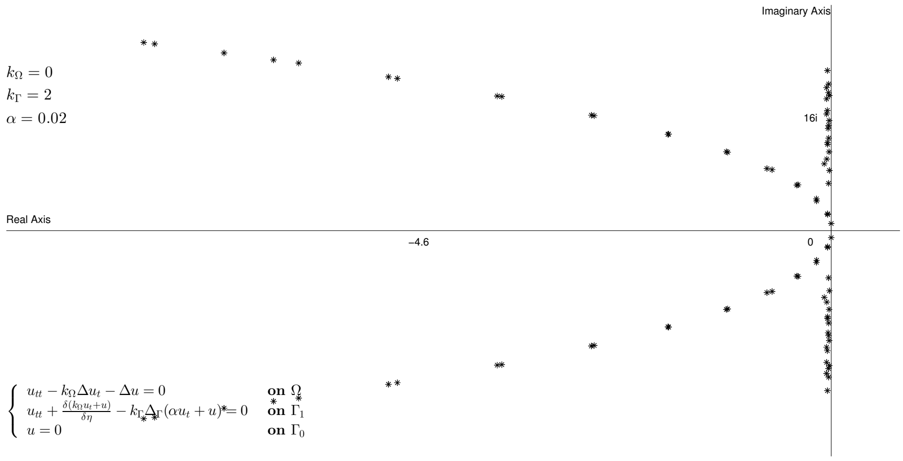


Figure 1.5: Eigenvalues for model (**VM**) with viscoelastic damping only on the boundary $k_\Gamma, \alpha > 0$ and $k_\Omega = 0$

will also study cases where we have mixed damping under our general model (**GM**), i.e. viscoelastic damping on the boundary with frictional damping in the interior and vice versa.

The four figures presented in this introduction does not cover all possible scenarii of the system (**GM**), however they provide a good overview of the results one can expect when studying the regularity and the stability of a differential equation's solution. Altough, they do not demonstrate any results of course, they add a significant benefit by illustrating the theoretical results obtained through this study. Since proving that certain properties does not hold is very challenging, these numerical approximations are also beneficial, as they suggest that the theoretical results were the best one can expect.

In chapter 3, we define and describe the numerical scheme built with Matlab, using the finite element methods. The discretization of the model (**GM**) has allowed to obtain the previous pictures as well as the other numerical observations and results in this dissertation. Finite elemnt method (FEM) is one of the most important tool in industry to study dynamic systems and the governing differential equations. The numerical framework was established following the Galerkin method on a two-dimensional plate with square elements, piecewise linear polynomials. As it was already done in the previous paragraph, the present study is written in a way that the numerical and theoretical results interact together as much as possible so that the approach to understand this model is not only from one point of view, but from both numerical and mathematical point of view. This approach to be relevant requires a precise definition of the numerical scheme, which is the purpose of the first part in chapter 3. In the second part, we derive error estimates for the models (**VM**), (**FM**). This will provide the justification for the reliability of the numerical schemes defined.

Indeed, error estimates make the link between the continuous and discretized model as they define a 'well-posedness' concept for a numerical model, by describing how far the approximate solution is from the exact solution, in other words, it measures the rate of *convergence* of the approximate solution to the exact solution. The literature for finite element methods is heterogeneous and only a few covers a wide range of differential equations with sufficient mathematical justificiation, therefore, we restrict our attention to few important references [14, 52]. Also, Thomée offers in [56] a wide range of error estimates technics for classical parabolic and hyperbolic problems such as the heat equation with 0-Dirichlet boundary conditions. Classically, convergence arguments rely on the defintion of a discrete operator projecting the exact solution onto the finite element space. Then using the projection, the convergence is reduced to the verification of the two properties:

- *the consistency*: the convergence of the projection toward the exact solution
- *the stability*: the convergence of the approximate solution to the projection which verifies that the projection does not amplify noise

In chapter 4 and 5 we study how the dynamic boundary conditions on Γ_1 affect qualitative properties of the resulting semigroup that include *regularity* (chapter 4) and *long time behavior with the analysis of various types of stability* (chapter 5). It is well known that the presence of a damping impacts these two properties. Sufficient amount of dissipation in the system may provide not only strong decays of the energy when time $t \to \infty$, but also may regularize semigroup by providing more smoothness to the solutions. The first part of chapter 4 shows that the semigroup can reach a maximum regularity if viscoelastic damping is present both in the interior and on the boundary. Then, we will see that the regularity drops to Gevrey class if the boundary wave is not strongly damped. In the last part, we demonstrate that the spectrum of the Cauchy operator associated with (**GM**) does not intersect with the imaginary axis, provided damping in the system. This spectral property allows a smooth transition to chapter 5 in which we will be interested in the stabilization of the model (**GM**). Within this framework, we will be able to assert and prove that each inital state in the $\mathcal{H}$ decays asymptotically to the zero state. We will in addition demonstrate that the solutions decay at an exponential rate, for most scenarii, as one can expect. Finally, we will study the few scenarii for which only strong stability is possible. For instance, in absence of internal damping ($k_\Omega, c_\Omega = 0$), the viscoelastic damping on the boundary ($k_\Gamma \alpha > 0$) will not be sufficient to obtain exponential stability. This general problem is, of course, not new and goes back to the fundamental work by Littman-Markus [39, 40, 42] where "hybrid " systems of elasticity were studied. It was then discovered that an addition of a lower dimensional dynamics may destabilize the system. For instance, it was shown in [38] that the hybrid system:

$$
\begin{cases}
u_{tt} - \Delta u = 0, x \in \Omega, t > 0 \\
u(x,t) = 0, \quad x \in \Gamma_0, \ t > 0 \\
mu_{tt} + \partial_n(u + k_\Omega u_t) = 0, \ x \in \Gamma_1, \ t > 0 \\
u(0,x) \in H^1(\Omega), \ u_t(0,x) \in L_2(\Omega), \quad x \in \Omega
\end{cases}
$$

with $m > 0$ is *not uniformly stable*, while the case when $m = 0$ is exponentially stable assuming suitable geometric conditions imposed on Ω [34].

Besides determining the condition under which the system (**GM**) is exponentially stable (chapter 5), we will also approximate numerically the growth bound, or in other words, how fast the solution decays, in chapter 6. More precisely, in the context of a better understanding the damping mechanisms, we will investigate which conditions is the dissipation maximum.

Whenever the model (**GM**) is parabolic ($k_\Omega > 0$), we can estimate the growth bound by determining the approximate spectral bound. However, for hyperbolic systems of dimensions greater or equal than 2, there are counterexamples ([31, 50] and references therein) where this approach does not hold anymore. In particular, in [50], the author provides a couterexample for a first order perturbation of the wave equation in two dimensions.

1.2 Preliminary Insight

Our aim is to provide a comprehensive study for the model (**GM**) of (i) wellposedness (chapter 2), (ii) regularity (chapter 4) and (iii) stability (chapter 5) of solutions under the influence of competing both interior and boundary dampings. In order to quantize the analysis we derive formal energy estimate which are the core of any analysis of a dynamical system. With respect to this energy we are interested in the well-posedness and stability of solutions $(u_{|\Omega}, u_{t|\Omega}, u_{|\Gamma_1}, u_{t|\Gamma_1})^T$ to (**GM**), along with the decay of the energy $E(t)$ of (**GM**) at $t \to \infty$.

We first introduce notational conventions which we will used throughout the dissertation.

Definition 1.2.1. $(.,.)_\Omega$ *is the inner product on* $L^2(\Omega)$ *and* $|.|_\Omega^2$ *is the corresponding norm,* $\langle .,. \rangle_{\Gamma_1}$ *is the inner product on* $L^2(\Gamma_1)$ *and* $|.|_{\Gamma_1}^2$ *is the corresponding norm; unless otherwise specified by another subscrip, e.g.* $\|.\|_{H^1_{\Gamma_0}(\Omega)}$ *is the* $H^1_{\Gamma_0}(\Omega)$ *norm.*

We begin with the definition of energy functions representing both internal and boundary structural energy of the system (**GM**).

$$
\begin{aligned}
E(t) &= E_\Omega(t) + E_\Gamma(t) \\
E_\Omega(t) &= \int_\Omega |\nabla u(t)|^2 + |u_t(t)|^2 d\Omega \\
E_{\Gamma_1}(t) &= \int_{\Gamma_1} k_\Gamma |\nabla_\Gamma u_{|\Gamma_1}(t)|^2 + |u_{t|\Gamma_1}(t)|^2 d\Gamma_1
\end{aligned}
\tag{1.2.1}
$$

Notice that there is a natural energy dissipation coming from *both* the boundary and the interior. Define the dissipation of the system (**GM**) by $D(t)$:

$$D(t) = \int_{\Omega} c_\Omega |u_t(t)|^2 + k_\Omega |\nabla u_t(t)|^2 d\Omega + \int_{\Gamma_1} c_\Gamma |u_{t|\Gamma_1}(t)|^2 + |k_\Gamma \alpha |\nabla_\Gamma u_{t|\Gamma_1}(t)|^2 d\Gamma_1 \tag{1.2.2}$$

The total energy in the system (**GM**) is classically obtained by multiplying by u_t and integrating in time, so we have the following formal energy identity:

$$\begin{aligned} E(0) &= E_\Omega(t) + E_{\Gamma_1}(t) + 2\int_0^t D(s)ds \\ &= |\nabla u(t)|_\Omega^2 + |u_t(t)|_\Omega^2 + k_\Gamma \left|\nabla_\Gamma u_{|\Gamma_1}(t)\right|_{\Gamma_1}^2 + \left|u_{t|\Gamma_1}(t)\right|_{\Gamma_1}^2 \\ &\quad + 2\int_0^t c_\Omega \left|u_t\right|_\Omega^2 + k_\Omega \left|\nabla u_t(s)\right|_{\Gamma_1}^2 + c_\Gamma \left|u_{t|\Gamma_1}(s)\right|_{\Gamma_1}^2 + k_\Gamma \alpha \left|\nabla_\Gamma u_{t|\Gamma_1}(s)\right|_{\Gamma_1}^2 ds \end{aligned} \tag{1.2.3}$$

The above energy identity (1.2.3) suggests that the energy is decreasing. However, how fast, it needs to be determined. The energy balance also suggests that there is an extra regularity in the damping. How this regularity is propagated onto the entire system is a question we aim to resolve.

Even though most studies [19, 20, 23, 25, 27, 30] use the same approach as the one we follow in this dissertation to treat similar systems, the model under consideration (**GM**) can be recast and studied in the abstract form as:

$$\begin{cases} u_{tt} &= Au(t) + Cu_t(t) \\ w_{tt} &= B_1 u(t) + B_2 u_t(t) + B_3 w(t) + B_4 w_t(t) \end{cases} \tag{1.2.4}$$

where the operators $A, C, B_i, i = 1-4$ satisfy suitable conditions and variables u and w are connected via "trace" type operator L so that $w = Lu$. In fact, this is the framework pursued by Mugnolo in [48]. Before presenting our results for (**GM**), it is essential understand the relevance of Mugnolo's results and determine whether they cover our model. A priori, the answer is not obvious as the conditions for generation and analyticity of the semigroup require a preliminary work to adapt (**GM**) into Mugnolo's approach and notation.

Firstly, the second order abstract system (1.2.4) is reformulated as an abstract second order Cauchy problem on the product space $X \times \partial X$ in the variable $U(t) \equiv (u(t), Lu(t))$ satisfying

$$U_{tt} = \mathcal{A}U(t) + \mathcal{C}U_t(t),\ t > 0 \tag{1.2.5}$$

on a product space $\mathcal{X} = X \times \partial X$ where

$$\mathcal{A} \equiv \begin{pmatrix} A & 0 \\ B_1 & B_3 \end{pmatrix}, \quad \mathcal{C} \equiv \begin{pmatrix} C & 0 \\ B_2 & B_4 \end{pmatrix}$$

are operator matrices on $\mathcal{X}$ with suitably defined domains. We also mention that a similar approach was used by Xiao et al. in [61, 62], where boundary conditions were reformulated as differential inclusions in suitable equivalence classes. The above formulation can be further reduced to first order Cauchy problem by introducing variable $\mathbf{u} \equiv (U, U_t)$ and writing

$$\mathbf{u}_t(t) = \mathbf{A}\mathbf{u}(t)$$
$$\text{where } \mathbf{A} \equiv \begin{pmatrix} 0 & I \\ \mathcal{A} & \mathcal{C} \end{pmatrix} \tag{1.2.6}$$

where the domain of $\mathbf{A}$ is:

$$\mathcal{D}(\mathbf{A}) = \{(u, x, v, y)^T \in \mathcal{D}(A) \times \mathcal{D}(B_3) \times \mathcal{D}(C) \times \mathcal{D}(B_4) : \; Lu = x, \; Lv = y\}$$

and the energy space is defined by

$$\mathbf{X} = \{(u, x)^T \in Y \times \partial Y : \; Lu = x\} \times X \times \partial X$$

A comprehensive study of well-posedness and regularity of second order evolutions (1.2.5) is given in [48] under the following standing set of general assumptions:

Assumption 1.2.2 (Assumption 2.1 - [48])**.**

1. *$Y, X, \partial Y, \partial X$ are Banach spaces such that $Y \hookrightarrow X$, $\partial Y \hookrightarrow \partial X$.*

2. *$A : D(A) \subset X \to X$, $C : D(C) \subset X \to X$ are linear opertaors.*

3. *$L : D(A) \cap D(C) \to \partial X$ is linear and surjective.*

4. *$B_1 : D(A) \to \partial X$, $B_2 : D(C) \to \partial X$ are linear operators.*

5. *$B_3 : D(B_3) \subset \partial Y \to \partial X$, $B_4 : D(B_4) \subset \partial X \to \partial X$, are linear closed operators.*

Thus, in order to compare our results for (**GM**) with these obtained in [48] we shall recast our problem within this more general framework. This is easily accomplished by setting:

$$A = -\Delta, \quad C = -k_\Omega \Delta, \quad Lu =\equiv u|_{\Gamma_1}, \text{ the trace of } u \text{ in } \Gamma_1$$

$$B_1 = -\partial_n, \quad B_2 = -k_\Omega \partial_n, \quad B_3 = -k_\Gamma \Delta_\Gamma, \quad B_4 = -k_\Gamma \alpha \Delta_\Gamma$$

with the corresponding spaces (assuming $k_\Gamma > 0$) :

$$X = L_2(\Omega), \quad Y = H^1_{\Gamma_0}(\Omega), \quad \partial X = L_2(\Gamma_1), \quad \partial Y = H^1(\Gamma_1)$$

The basic framework presented in [48] aims at proving generation and analyticity of C_0 semigroups as being equivalent having the same properties for the blocks of operators:

$$\begin{pmatrix} 0 & I \\ A & C \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & I \\ B_3 & B_4 \end{pmatrix}$$

considered on $Y \times X$ and $\partial Y \times \partial X$. (see Thm 3.3, 3.8, 4.5, 4.12 [48]).

Let $A_0 = A_{|ker(L)}$, $C_0 = C_{|ker(L)}$ and define the operators:

$$\begin{pmatrix} 0 & I_V \\ A_0 & C_0 \end{pmatrix} \text{ with domain } \mathcal{D}(A_0) \times \mathcal{D}(C_0) \tag{1.2.7}$$

$$\begin{pmatrix} 0 & I_{\partial Y} \\ B_3 & B_4 \end{pmatrix} \text{ with domain } \mathcal{D}(B_3) \times (\mathcal{D}(B_4) \cap \partial Y) \tag{1.2.8}$$

Clearly the role of the "coupling" operator L is critical. In fact, the results in [48] are categorized with respect to the properties of L as

- unbounded $Y \to \partial X$,
- bounded from $Y \to \partial X$ but unbounded $X \to \partial X$.

It is this second scenario that is relevant in our situation. The corresponding results are in section 4 [48]. However, as we shall see below, they are not applicable due to severity of assumptions imposed either on the operators $B_i, i = 1 - 4$. Before we cite and comment the results, we shall introduce more notation. Let D_λ denotes the Dirichlet map. Also, denote by $[\mathcal{D}(A)_L]$ the Banach space obtained by endowing $\mathcal{D}(A)$ with the graph norm of the closed operator $(A\ L)^T$.

Theorem A (Theorem 4.5 - Part I [48])**.** *Assume* $B_1 \in L(Y, \partial X), B_2 \in \mathcal{L}(X, \partial X)$. *Assume both* $D_\lambda B_3$ *and* $D_\lambda B_4$ *to have continuous extensions form* ∂Y *to* X *and from* ∂X *to* X*, respectively, for some* $\lambda \in \sigma(A_0)$*, where* A_0 *is defined as* A *restricted to the kernel of* L*. Then,* $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ *is closable if and only if both operator matrices in* (1.2.7) *and* (1.2.8) *are closable. In this case, the closure of* $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ *generates a* C_0*-semigroup (resp. analytic semigroup) on if and only if the closures of both operator matrices in* (1.2.7) *and* (1.2.8) *generate a* C_0*-semigroup (resp. analytic semigroup) on* $V \times X$ *and* $\partial Y \times \partial X$*, respectively.*

To wit, part I of theorem 4.5 (theorem A assumes that $B_1 \in L(Y, \partial X), B_2 \in L(X, \partial X)$. The above is never satisfied with $B_1 = \partial_n$ and the choices of spaces $X, Y, \partial X$. In addition, operators B_3, B_4 do not comply with regularity requirements postulated in Thm 4.5-unless they are bounded. Similar conclusion applies to Part

II of Thm 4.5 which discussed generation.

Theorem B (Part II - Theorem 4.12 [48]). *Assume C_0 to generate an analytic semigroup X. Assume moreover that there exists an $\epsilon \in (0,1)$ such that $[\mathcal{D}(C)_L]$ is continuously embedded in the complex interpolation space $[\mathcal{D}(C_0), X]_\epsilon$. Assume that $B_1 \in \mathcal{L}([\mathcal{D}(A)_L], \partial X)$ and $B_2 \in \mathcal{L}([\mathcal{D}(A)_L], \partial X)$. Then $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ generates an analytic C_0-semigroup on $\mathbf{X}$ if and only if both operator matrices in* (1.2.7) *and* (1.2.8) *generate analytic semigroups on $V \times X$ and $\partial Y \times \partial X$, respectively.*

Part II of that theorem 4.12 (theorem B) pertains to generation of analytic semigroups. However, here the operators B_1 and B_2 do not comply with regularity requirements unless the Dirichlet map is sufficiently smooth-as in one dimensional case. Indeed, we recall that $(D(A)_L)$ corresponds to $H^{1/2}(\Omega)$, unless the dimension of $\Omega = 1$. The main reason is that the treatment given in [48] treats the coupling operator B_1, B_2 like a perturbation -rather than a carrier of regularity. It is this second approach that is used in this dissertation where the matrix operators is not a perturbation of two blocks of operator matrices but rather perturbation of a new system which is related to Wentzell problem.

Chapter 2

Well-Posedness

In this chapter, the aim is to establish well-posedness for the semigroup governing the model (**GM**). For this purpose, we will build a suitable energy space and an operator which is a candidate for the generator. It will then remain to demonstrate the maximal dissipativity of this operator.

2.1 Preliminary operator definitions

Following the notation and some definitions from [57], we give the precise definitions of the operators which will be used in the study of the system (**GM**).

Definition 2.1.1 (The Laplacian in Ω)**.** *Let the operator $A : L^2(\Omega) \supset \mathcal{D}(A) \to L^2(\Omega)$ be defined by:*

$$Au = -\Delta u,\ cD(A) = \{u \in\ L^2(\Omega), Au \in\ L^2(\Omega), u|_{\Gamma_0} = 0, \partial_n u|_{\Gamma_1} = 0\} \tag{2.1.1}$$

Then A is self-adjoint, positive definite, and therefore the fractional powers of A are well-defined. In particular, we have the following caracterization:

$$\mathcal{D}(A^{\frac{1}{2}}) = H^1_{\Gamma_0}(\Omega) = \{z \in H^1(\Omega), z = 0 \text{ } on \text{ } \Gamma_0\} \tag{2.1.2}$$

with $\|z\|^2_{\mathcal{D}(A^{\frac{1}{2}})} = \left\|A^{\frac{1}{2}}z\right\|^2_{L^2(\Omega)} = \int_\Omega |\nabla z|^2 = \|z\|^2_{H^1_{\Gamma_0}(\Omega)}\,,\ \forall z \in \mathcal{D}(A^{\frac{1}{2}})$

where the last equatility follows from Poincare's inequality, which we recall as it will often be used.

Proposition 2.1.2 (Poincaré's inequality)**.** *Let Ω be a bounded domain. Then there exists a constant C depending on the geometry, more precisely on the thickness of the domain such that:*

$$\|u\|_{W^{1,p}(\Omega)} \leq C\,\|\nabla u\|_{L^p(\Omega)}\,,\ \forall u \in W^{1,p}_0(\Omega)$$

In order to apply this inequality, note that the function u must be zero somewhere in the domain. In our case, the boundary Γ_0 with 0-Dirichlet boundary conditions guarentees the applicability of Poincaré's inequality.

Similar to the Laplacian in Ω, we need to define the Laplace-Beltrami operator acting on the boundary Γ_1. Before, we mention some useful aspects of the tangential differential calculus from [9]. Let $b(x)$ be the positive or negative distance to the boundary Γ_1 depending on whether the point x is outside or inside the domain Ω. Then define the projection $p(x)$ of a point x onto Γ_1 as $p(x) = x - b(x)\nabla b(x)$. Given $f \in H^1(\Gamma_1)$, we define the tangential gradient ∇_Γ of the scalar function f by means of this projection as:

$$\nabla_\Gamma f = \nabla(f \circ p)(x)|_{\Gamma_1} \tag{2.1.3}$$

In a same manner we can define the tangential divergence as:

$$\mathsf{div}_{\ \Gamma_1} f = \mathsf{div}\ (f \circ p)|_{\Gamma_1} \tag{2.1.4}$$

and the Laplace-Beltrami of $f \in H^2(\Gamma_1)$ as:

$$\boldsymbol{\Delta}_\Gamma f = \mathsf{div}_{\ \Gamma_1}(\nabla_\Gamma f) = \Delta(f \circ p)|_{\Gamma_1} \tag{2.1.5}$$

We are now ready to define our tangential Laplace-Beltrami operator in our context:

Definition 2.1.3 (Laplace-Beltrami on the boundary)**.** *For all $k_\Gamma \geq 0$, set $B : L^2(\Gamma_1) \supset \mathcal{D}(B) \to L^2(\Gamma_1)$ to be:*

$$Bz = -k_\Gamma \boldsymbol{\Delta}_\Gamma z,\ \mathcal{D}(B) = \{z \in\ L^2(\Gamma_1), k_\Gamma \Delta z \in L^2(\Gamma_1)\} \tag{2.1.6}$$

with the associated norm (graph norm):

$$|u|^2_{\mathcal{D}(B^{\frac{1}{2}})} \equiv |u|^2_{\Gamma_1} + \left| B^{\frac{1}{2}} u \right|^2_{\Gamma_1} = |u|^2_{\Gamma_1} + k_\Gamma \left| \nabla_\Gamma u \right|^2_{\Gamma_1}$$

The presence of the coefficient k_Γ in this definition will be useful to study the different scenarii offered by the model (**GM**). Indeed, this definition will allow us to treat simultaneously cases in which there is no Laplace-Beltrami term on the boundary, without changing any definitions. In other words, if $k_\Gamma = 0$, the domain of $B^{\frac{1}{2}}$ is similar to $L^2(\Gamma_1)$ whereas if the coefficient k_Γ is strictly positive the domain of $B^{\frac{1}{2}}$ is similar to $H^1(\Gamma_1)$.

We now introduce classical tools to go from the boundary (Γ_1) to the interior (Ω) and vice versa.

Definition 2.1.4 (Neumann map). *Define the map N by*

$$z = Ng \Leftrightarrow \begin{cases} \Delta z = 0 & \textbf{\textit{on}}\ \Omega \\ \partial_n z|_{\Gamma_1} = g & \textbf{\textit{on}}\ \Gamma \\ z|_{\Gamma_0} = 0 & \textbf{\textit{on}}\ \Gamma_0 \end{cases} \tag{2.1.7}$$

Then, by elliptic theory, N is a bounded operator from $L^2(\Gamma_1)$ to $H^{\frac{3}{2}}(\Omega)$).

Definition 2.1.5 (Trace map γ). *Let $\gamma : H^1(\Omega) \to L^2(\Gamma_1) \subset H^{\frac{1}{2}}(\Gamma_1)$ be the restriction to Γ_1:*

$$\forall z \in H^1_{\Gamma_0}(\Omega),\ \ \gamma(z) = z|_{\Gamma_1} \tag{2.1.8}$$

Then, by [57, Lemma 2.0]:

$$N^* A z = \gamma(z)\ \forall z \in \mathcal{D}(A^{\frac{1}{2}}) \tag{2.1.9}$$

While the trace operator "requires a lost of 1/2 derivative" from the interior to the boundary, the same concept is true when one tries to estimate a boundary term with an interior term. As an illustration, we quote the following estimate, from [8, Theorem 1.6.6], which will be used in chapter 5.

Proposition 2.1.6 (Trace moment inequality). *Suppose that Ω has a Lipschitz boundary and that p is a real number in the range $1 \le p \le \infty$. Then there is a constant, C, such that:*

$$\|v\|_{L^p(\Gamma_1)} \le C \|v\|^{1-\frac{1}{p}}_{L^p(\Omega)} \|v\|^{\frac{1}{p}}_{W^{1,p}(\Omega)},\ \forall v \in W^{1,p}(\Omega) \tag{2.1.10}$$

In particular, $\forall v \in H^1(\Omega)$

$$\begin{aligned} |v|^2_{\Gamma_1} &\leq C \|v\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} \\ &\leq |\nabla v|^2_{\Omega} \end{aligned} \tag{2.1.11}$$

where the last inequality follows from Poincaré's inequality (definition 2.1.2).

2.2 Definition of the generator

Let A_0 be the laplacian operator with zero Dirichlet boundary conditions which is a positive, self-adjoint operator with domain $\mathcal{D}(A_0)$ in the Hilbert space X. It is well-known the generator of damped wave equation with classical boundary conditions, say Dirichlet, is of the form:

$$\mathbb{A} = \begin{pmatrix} 0 & I \\ -A_0 & -D \end{pmatrix} \tag{2.2.1}$$

$$\mathcal{D}(\mathbb{A}) = \mathcal{D}(A_0) \times X$$

where D is another positive, self-adjoint operator corresponding to the structural dissipation of the system. Moreover, if we let the operator D be defined as a power of the Laplacian as in [12], i.e., there exists a contant $0 < \gamma \leq 1$, and there are two constants $0 < \rho_1 < \rho_2 < \infty$ such that

$$\rho_1 A_0^\gamma \leq D \leq \rho_2 A_0^\gamma,$$

it generates an analytic C_0-semigroup of contractions for $\gamma \in [\frac{1}{2}, 1]$ and only a Gevrey C_0-semigroup of contractions for $\gamma \in (0, \frac{1}{2})$ (see [13]). We will come back to the regularity question of this problem in chapter 4.

Back to the definition of our generator, it is important to note that the system (**GM**) must be built from a wave operator similar to (2.2.1) both in the interior Ω and the boundary Γ_1. Therefore, it is natural to define the energy state (or state space) of our system as follows.

Definition 2.2.1 (Energy spaces). *Let* $U = (u, u_t, u|_{\Gamma_1}, u|_{\Gamma_1}) = (u_1, u_2, u_3, u_4)$.
The associated energy space is:

$$\mathcal{H} = \{(u_1, u_2, u_3, u_4) \in \mathcal{D}(A^{\frac{1}{2}}) \times L^2(\Omega) \times \mathcal{D}(B^{\frac{1}{2}}) \times L^2(\Gamma_1), u_1|_{\Gamma_1} = u_3\}$$

$$\text{where } \mathcal{D}(A^{\frac{1}{2}}) = H^1_{\Gamma_0}(\Omega) \text{ and } \mathcal{D}(B^{\frac{1}{2}}) \sim \begin{cases} H^1(\Gamma_1) & \text{if } k_\Gamma > 0 \\ L^2(\Gamma_1) & \text{if } k_\Gamma = 0 \end{cases} \tag{2.2.2}$$

with associated norm:

$$\|u\|^2_{\mathcal{H}} = |\nabla u_1|^2_\Omega + |u_2|^2_\Omega + \left|B^{\frac{1}{2}} u_3\right|^2_{\Gamma_1} + |u_4|^2_{\Gamma_1} \tag{2.2.3}$$

Define the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by:

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \Delta & D_\Omega & 0 & 0 \\ 0 & 0 & 0 & I \\ -\partial_n & -k_\Omega \partial_n & -B & D_{\Gamma_1} \end{pmatrix} \tag{2.2.4}$$

$$\text{where} \begin{cases} D_\Omega & = k_\Omega \Delta - c_\Omega I \\ D_{\Gamma_1} & = -\alpha B - c_\Gamma I \end{cases}$$

where we can notice the presence on the diagonal of two wave operator blocks. This domain of this operator is:

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = & \{U = [u_1, u_2, u_3, u_4]^T \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B^{\frac{1}{2}}) \times \mathcal{D}(B^{\frac{1}{2}}), \\ & \text{such that } \Delta(u_1 + k_\Omega u_2) - c_\Omega u_2 \in L^2(\Omega), \\ & \partial_n(u_1 + k_\Omega u_2) - c_\Gamma u_4 + B^{\frac{1}{2}}(B^{\frac{1}{2}} u_3 + \alpha B^{\frac{1}{2}} u_4) \in L^2(\Gamma_1), \\ & u_1|_{\Gamma_1} = N^* A u_1 = u_3,\ u_2|_{\Gamma_1} = N^* A u_2 = u_4\} \end{aligned}$$

which is densily defined in $\mathcal{H}$.

Remark 2.2.1. *The following representation will be frequently used in the calculation:*

$$-\Delta u = A(I - N\partial_n)u \tag{2.2.5}$$

Remark 2.2.2. *The following regularity properties of the elements in the domain* *result from the definition of* $\mathcal{D}(\mathcal{A})$

$$\partial_n(u_1 + k_\Omega u_2) \in [\mathcal{D}(B^{1/2})]^*, \qquad u_1, u_2 \in \mathcal{D}(A) \tag{2.2.6}$$

this in particular implies

$$\partial_n(u_1 + k_\Omega u_2) \in L_2(\Gamma_1), \ \textit{if } k_\Gamma = 0 \tag{2.2.7}$$

The above regularity resuls are stronger than classical elliptic theory which implies $\partial_n(u_1 + k_\Omega u_2) \in H^{-1/2}(\Gamma_1)$ *, where the latter is due to the fact that* $\Delta(u_1 + k_\Omega u_2) \in L_2(\Omega)$ *along with* $u_1, u_2 \in H^1(\Omega)$*. As a consequence of* (2.2.6) *one has well defined duality pairing*

$$\langle \partial_n(u_1 + k_\Omega u_2), v \rangle_{\Gamma_1}, \ \forall v \in D(B^{1/2}), \forall U \in \mathcal{D}(\mathcal{A}) \tag{2.2.8}$$

2.3 Generation of a C_0-semigroup

Within this framework, we can now state and prove our main result regarding well-posedness:

Theorem 2.3.1. *Let* $k_\Omega, c_\Omega, k_\Gamma, c_\Gamma, \alpha$ *be non negative. The operator* $\mathcal{A}$*, as given in* (2.2.4)*, generates a* C_0*-semigroup of contractions* $\{e^{\mathcal{A}t}\}_{t\geq 0}$ *on* $\mathcal{H}$*. In addition, if the damping condition* (1.1.14) *holds, i.e.* $max\{k_\Omega, k_\Gamma\alpha\} > 0$*, then the semigroup is strictly contractive.*

It follows that for any set of initial conditions in $\mathcal{H}$*, the system* (**GM**) *has a unique* *generalized solution $U(.)$ in $C([0,T],\mathcal{H})$ satisfying:*

$$\|U(t)\|_{\mathcal{H}} \leq \|U(0)\|_{\mathcal{H}} e^{\omega t} \tag{2.3.1}$$

Proof. In order to show the semigroup generation for the dynamics $\mathcal{A}$, we wish to use the Lumer-Phillips theorem, [44] and hence we must show the maximal dissipativity of $\mathcal{A}$.

Step 1: $\mathcal{A}$ is dissipative.

With the notation $-\Delta u = A(I - N\partial_n)u$, take $U = (u_1, u_2, u_3, u_4) \in \mathcal{D}(\mathcal{A})$:

$$\begin{aligned}
< \mathcal{AU}, U >_{\mathcal{H}} &= (u_2, u_1)_{\mathcal{D}(A^{\frac{1}{2}})} + (u_4, u_3)_{\mathcal{D}(B^{\frac{1}{2}})} \\
&\quad - (A(I - N\partial_n)u_1 + k_\Omega A(I - N\partial_n)u_2 + c_\Omega u_2, u_2)_\Omega \\
&\qquad\qquad \text{recalling remark 2.2.8} \\
&\quad - \langle \partial_n(u_1 + k_\Omega u_2) + Bu_3 + \alpha B u_4 + c_\Gamma u_4, N^* A u_2 \rangle_{\Gamma_1} \\
&\qquad\qquad \text{recalling equation 2.1.9} \\
&= \left(A^{\frac{1}{2}} u_2, A^{\frac{1}{2}} u_1\right)_\Omega - \left(A^{\frac{1}{2}} u_1 + k_\Omega A^{\frac{1}{2}} u_2, A^{\frac{1}{2}} u_2\right)_\Omega - c_\Omega |u_2|^2_\Omega \qquad (2.3.2) \\
&\quad + \langle \partial_n(u_1 + k_\Omega u_2), u_4 \rangle_{\Gamma_1} + \left\langle B^{\frac{1}{2}} u_4, B^{\frac{1}{2}} u_3 \right\rangle_{\Gamma_1} \\
&\quad - \langle \partial_n(u_1 + k_\Omega u_2), u_4 \rangle_{\Gamma_1} - \left\langle B^{\frac{1}{2}} u_3, B^{\frac{1}{2}} u_4 \right\rangle_{\Gamma_1} \\
&\quad - \alpha \left| B^{\frac{1}{2}} u_4 \right|^2_{\Gamma_1} - c_\Gamma |u_4|^2_{\Gamma_1} \\
&= -k_\Omega \left| A^{\frac{1}{2}} u_2 \right|^2_\Omega - c_\Omega |u_2|^2_\Omega - \alpha \left| B^{\frac{1}{2}} u_4 \right|^2_{\Gamma_1} - c_\Gamma |u_4|^2_{\Gamma_1} \\
&\leq 0
\end{aligned}$$

Therefore, $\mathcal{A}$ is ω-dissipative.

Step 2: $\mathcal{A}$ is maximal.

It remains to show that the range condition is satisfied, i.e., if $\lambda \in \mathbb{C},\ Re\lambda > 0$ and $\begin{pmatrix} f_1 & f_2 & f_3 & f_4 \end{pmatrix}^T \in \mathcal{H}$ is given, then the stationary equation:

$$(\lambda I - \mathcal{A}) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \tag{2.3.3}$$

is satisfied for some $\begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix}^T \in \mathcal{D}(\mathcal{A})$.

Note that (2.3.3) becomes:

$$\begin{cases} \lambda u_1 - u_2 = f_1 \\ \lambda u_2 + A(I - N\partial_n)u_1 + k_\Omega A(I - N\partial_n)u_2 + c_\Omega u_2 = f_2 \\ \lambda u_3 - u_4 = f_3 \\ \lambda u_4 + \partial_n(u_1 + k_\Omega u_2) + Bu_3 + \alpha Bu_4 + c_\Gamma u_4 = f_4 \end{cases} \tag{2.3.4}$$

In the first equation, note that $f_1, u_1 \in \mathcal{D}(A^{\frac{1}{2}})$, so $u_2 \in \mathcal{D}(A^{\frac{1}{2}})$. Similarly, from the third equation we have $u_4 \in \mathcal{D}(B^{\frac{1}{2}})$.

Also, from the second equation $f_2, u_2 \in L^2(\Omega)$ implies $A(I-N\partial_n)(u_1+k_\Omega u_2) \in L^2(\Omega)$.

The fourth equation, with $u_4, f_4 \in L^2(\Gamma_1)$ implies that $\partial_n(u_1 + k_\Omega u_2) \in [\mathcal{D}(B^{1/2})]'$.

The above leads to the following representation:

$$\begin{cases} u_2 = \lambda u_1 - f_1 \\ u_4 = \lambda u_3 - f_3 \\ \lambda^2 u_1 + A(I - N\partial_n)u_1 + \lambda k_\Omega A(I - N\partial_n)u_1 + \lambda c_\Omega u_1 \\ \quad = f_2 + \lambda f_1 + k_\Omega A(I - N\partial_n)f_1 + c_\Omega f_1 \\ \lambda^2 u_3 + \partial_n(u_1 + k_\Omega \lambda u_1) + Bu_3 + \lambda\alpha B u_3 + \lambda c_\Gamma u_3 \\ \quad = f_4 + \lambda f_3 + k_\Omega \partial_n f_1 + \alpha B f_3 + c_\Gamma f_3 \end{cases} \tag{2.3.5}$$

To solve the stationary problem (2.3.5), we shall use a weak formulation and Lax-Milgram theorem. Let $(v_1, v_2, v_3, v_4) \in \mathcal{D}(\mathcal{A})$, and for the time being we also take $F = (f_1, f_2, f_3, f_4)$ in $\mathcal{D}(\mathcal{A})$. Later we shall extend the argument by density to all $F \in \mathcal{H}$.

We consider the two last equations, multiply them by $\overline{v_1}$ and $\overline{v_3}$ respectively:

$$\begin{cases} (\lambda^2 u_1, v_1)_\Omega + (A(I - N\partial_n)u_1, v_1)_\Omega + \lambda k_\Omega \left(A(I - N\partial_n)u_1, v_1\right)_\Omega + \lambda c_\Omega \left(u_1, v_1\right)_\Omega \\ = (f_2 + \lambda f_1 + \lambda k_\Omega A(I - N\partial_n)f_1 + \lambda c_\Omega f_1, v_1)_\Omega \\ \langle \lambda^2 u_3, v_3\rangle_{\Gamma_1} + \langle \partial_n(u_1 + k_\Omega \lambda u_1), v_3\rangle_{\Gamma_1} + (1 + \lambda\alpha)\left\langle Bu_3, v_3\right\rangle_{\Gamma_1} + c_\Gamma \left\langle u_3, f_3\right\rangle_{\Gamma_1} \\ = \langle f_4 + \lambda f_3 + k_\Omega \partial_n f_1 + \alpha B f_3 + c_\Gamma f_3, v_3\rangle_{\Gamma_1} \end{cases} \tag{2.3.6}$$

Rewriting (2.3.6) yields:

$$\begin{cases} \lambda^2 (u_1, v_1)_\Omega + \left(A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} v_1\right)_\Omega \\ \quad + k_\Omega \lambda \left(A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} v_1\right)_\Omega + c_\Omega \lambda (u_1, v_1)_\Omega - \langle \partial_n (u_1 + k_\Omega \lambda u_1), v_3 \rangle_{\Gamma_1} \\ \quad = (f_2 + \lambda f_1, v_1)_\Omega + k_\Omega \left(A^{\frac{1}{2}} f_1, A^{\frac{1}{2}} v_1\right)_\Omega + c_\Omega (f_1, v_1)_\Omega - k_\Omega \langle \partial_n f_1, v_3 \rangle_{\Gamma_1} \\ (\lambda^2 + c_\Gamma \lambda) \langle u_3, v_3 \rangle_{\Gamma_1} + \langle \partial_n (u_1 + k_\Omega \lambda u_1), v_3 \rangle_{\Gamma_1} + (1 + \lambda \alpha) \left\langle B^{\frac{1}{2}} u_3, B^{\frac{1}{2}} v_3 \right\rangle_{\Gamma_1} \\ \quad = \langle f_4 + \lambda f_3, v_3 \rangle_{\Gamma_1} + k_\Omega \langle \partial_n f_1, v_3 \rangle_{\Gamma_1} + \alpha \left\langle B^{\frac{1}{2}} f_3, B^{\frac{1}{2}} v_3 \right\rangle_{\Gamma_1} + c_\Gamma \langle f_3, v_3 \rangle_{\Gamma_1} \end{cases} \tag{2.3.7}$$

Then combining these two equations and after simplification, we get:

$$\begin{aligned} &(\lambda^2 + c_\Omega \lambda) (u_1, v_1)_\Omega + (1 + \lambda k_\Omega) \left(A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} v_1\right)_\Omega \\ &\quad + (\lambda^2 + c_\Gamma \lambda) \langle u_3, v_3 \rangle_{\Gamma_1} + (1 + \lambda \alpha) \left\langle B^{\frac{1}{2}} u_3, B^{\frac{1}{2}} v_3 \right\rangle_{\Gamma_1} \\ &= (f_2 + \lambda f_1 + c_\Omega f_1, v_1)_\Omega + \left(k_\Omega A^{\frac{1}{2}} f_1, A^{\frac{1}{2}} v_1\right)_\Omega \\ &\quad + \langle f_4 + \lambda f_3 + c_\Gamma f_3, v_3 \rangle_{\Gamma_1} + \left\langle \alpha B^{\frac{1}{2}} f_3, v_3 \right\rangle_{\Gamma_1} \end{aligned} \tag{2.3.8}$$

This leads us to consideration of a bilinear form

$$\begin{aligned} a(u_1, u_3, v_1, v_3) \equiv (\lambda^2 + c_\Omega \lambda) (u_1, v_1)_\Omega + (1 + \lambda k_\Omega) \left(A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} v_1\right)_\Omega \\ + (\lambda^2 + c_\Gamma \lambda) \langle u_3, v_3 \rangle_{\Gamma_1} + (1 + \lambda \alpha) \left\langle B^{\frac{1}{2}} u_3, B^{\frac{1}{2}} v_3 \right\rangle_{\Gamma_1} \end{aligned}$$

defined for $u = (u_1, u_3), v = (v_1, v_3) \in \mathcal{V}$

where $\mathcal{V} \equiv \{(v_1, v_3) \in D(A^{\frac{1}{2}}) \times D(B^{\frac{1}{2}}), v_3 = v_1|_{\Gamma_1}\}$.

We are solving for the variable u the variational equation:

$$a(u, v) = F(v), \ \forall v \in \mathcal{V} \equiv D(A^{\frac{1}{2}}) \times D(B^{\frac{1}{2}})$$

where $F(v)$ be the corresponding right-hand side:

$$F(v) = (f_2 + \lambda f_1 + c_\Omega f_1, v_1)_\Omega + k_\Omega \left(A^{\frac{1}{2}} f_1, A^{\frac{1}{2}} v_1\right)_\Omega$$
$$+ \langle f_4 + \lambda f_3 + c_\Gamma f_3, v_3\rangle_{\Gamma_1} + \alpha \left\langle B^{\frac{1}{2}} f_3, B^{\frac{1}{2}} v_3\right\rangle_{\Gamma_1}$$
$$\text{with } F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$$

We have continuity of bilinear form on $\mathcal{V} \times \mathcal{V}$:

$$\begin{aligned} |a(u,v)| &\leq max\{\lambda^2 + c_\Omega|\lambda|, \lambda^2 + c_\Gamma|\lambda|, k_\Omega|\lambda|, \alpha|\lambda|, 1\} \|u\|_\mathcal{V} \|v\|_\mathcal{V} \\ |F(v)| &\leq max\{|\lambda|, c_\Omega, c_\Gamma, k_\Omega, \alpha, 1\} \|F\|_\mathcal{H} \|v\|_\mathcal{V} \end{aligned} \tag{2.3.9}$$

The bilinear is also coercive:

$$\begin{aligned} Re\ a(u,u) &= \lambda^2 (u_1, u_1)_\Omega + \left(A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} u_1\right)_\Omega + k_\Omega \lambda \left(A^{\frac{1}{2}} u_1, A^{\frac{1}{2}} u_1\right)_\Omega + c_\Omega \lambda (u_1, u_1)_\Omega \\ &+ \lambda^2 \langle u_3, u_3\rangle_{\Gamma_1} + \left\langle B^{\frac{1}{2}} u_3, B^{\frac{1}{2}} u_3\right\rangle_{\Gamma_1} + \alpha\lambda \left\langle B^{\frac{1}{2}} u_3, B^{\frac{1}{2}} u_3\right\rangle_{\Gamma_1} + c_\Gamma \lambda \langle u_3, u_3\rangle_{\Gamma_1} \\ &\geq C \left[|u_1|^2_\Omega + \left|A^{\frac{1}{2}} u_1\right|^2_\Omega + |u_3|^2_{\Gamma_1} + \left|B^{\frac{1}{2}} u_3\right|^2_{\Gamma_1} \right] \\ &\geq C \|u\|^2_\mathcal{V} \end{aligned} \tag{2.3.10}$$

Therefore $a(u,v)$ is both bounded and coercive, so by Lax Milgram for every $F \in \mathcal{H}$ there exists a unique solution $y \in \mathcal{V}$. Moreover $y = (y_1, y_3)$ satisfies the last two equations in (2.3.5).

Next we reconstruct the remaining part of the vector Y . From (2.3.5)

$$u_2 = \lambda u_1 - f_1 \in D(A^{1/2}), \quad u_4 = \lambda u_3 - f_3 \in \mathcal{D}(B^{1/2}), \forall F \in \mathcal{H} \tag{2.3.11}$$

Since $u_3 = N^* A u_1$ and $f_3 = N^* A f_1$ we conclude that $u_4 = N^* A u_2$, as required by the membership in the $\mathcal{D}(\mathcal{A})$. The remaining regularity requirements simply follow from the structure of equations in (2.3.5).

In conclusion, for all $F \in \mathcal{H}$ we obtain $U = (u_1, u_2, u_3, u_4)$ in $\mathcal{D}(\mathcal{A})$ such that $(\lambda I - \mathcal{A})U = F \in \mathcal{H}$. Thus U is our desired solution, which completes the proof for maximality.

As a consequence, $\mathcal{A}$ generates a strongly continuous semigroup of contraction $\{e^{\mathcal{A}t}\}_{t\geq 0}$. Note that this result on generation holds without any dissipation active. $\square$

Chapter 3

Finite Element Methods

Before presenting more results about the semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$, we present the finite element framework used to create the numerical scheme associated with our model (**GM**). The aim of this chapter is:

- to construct and establish the well-posedness of the numerical scheme
- to derive error estimate between the approximate and exact solutions

3.1 Construction

To begin with, we define the geometrical domain which will be meshed thereafter. Let Ω the domain be a rectangle with a hole inside (rectangular shape as well). Let Γ_1 be the exterior boundary on which the dynamic boundary conditions are applied and Γ_0 be the ineterior boundary on which the Dirichlet boundary conditions are applied as described on figure 3.1 We deliberately omit to mention the dimensions of this geometry, since we want to keep the liberty to modify them in order to possibly

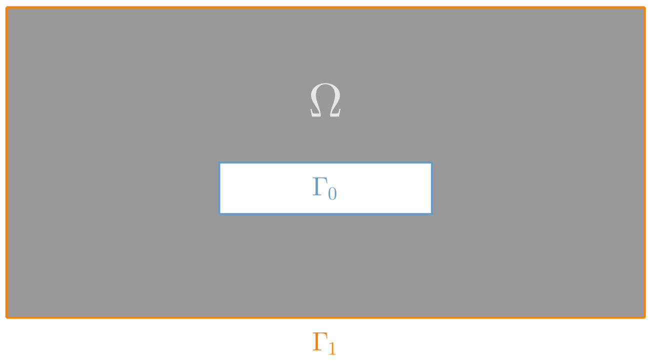


Figure 3.1: Domain Ω with active boundary Γ_1 and 0-Dirichlet boundary Γ_0

highlight phenomenon on figures. However, we assume that the lengths of the sides, which we denote by L_x and L_y are of the same order ($\frac{L_x}{L_y} \sim 1$). Moreover, we justify the choice of a two-dimensions domain and of a simple shape (rectangle) by the following reasons:

- the main goal of the dissertation is to understand the dynamics of the system (**GM**), therefore, as a first study of this model, it is not necessary to have perturbation coming from the shape of the domain and geometric considerations.

- the computation requires the inversion of large matrices (for the spectrum), thus complexifying the shape or having a higher-dimension domain would drastically increase the computation time.

Before we move to the derivation of the finite elments method, we shall introduce some classical notations and basic definitions, which can be found in any classical finite element literature. We use as main reference the textbook by Thomée [56, Chapter 1] for its accurate mathematical description and also because we will call some of his results later.

Definition 3.1.1 (General Assumption for finite element method)**.** *With Ω a simply connected open bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, let $\overline{\Omega}$ be the closure of Ω. We suppose that we are given a conforming partition $\mathcal{T}_h$ on Ω. Each element $\Omega_e \in \mathcal{T}_h$ is a closed subdomain of Ω and we assume that the usual finite element requirements are satisfied, i.e., non-overlapping and intersecting elements are disallowed. Also the union of the elements (rectangles) determines a polygonal domain Ω_h, so that it is a mesh domain contained in but not necessarly coinciding with Ω. Finally, assume that the boundaries $\partial\Omega_h$ are uniformly Lipschitz in h and with a deviation from $\partial\Omega$ bounded by Ch^r where $r \geq 2$ is an integer.*

For our model, we choose rectangular elements, each defined by four nodes (points) from $\mathcal{T}_h$. With these assumptions, it is now possible to define the space in which the numerical approximate solutions will be sought, which depends on a small parameter h representing the characteristic element size.

Assumption 3.1.2. *Let $\{S_{h,\Omega}\}_h$ be a family of finite-dimensional subspaces of $H_0^1(\Omega_h)$*

such that for small h:

$$\min_{\chi \in S_{h,\Omega}} \left\{ \|v - \chi\|_{L_2(\Omega)} + h \|(\nabla(v - \chi))\|_{L^2(\Omega)} \right\} \leq Ch^2 \|v\|_{H^2(\Omega)}$$

$$\textit{for } v \in H_0^1(\Omega) \cap H^2(\Omega), \ \forall \chi \in S_{h,\Omega}$$

If we denote by $\{P_j\}_{j=1}^{N_h}$ the interior vertices of $\mathcal{T}_h$, then a typical element of $S_{h,\Omega}$ is a continuous function on the closure $\overline{\Omega}$ of Ω, which is uniquely determined by its values at the points P_j and thus depends on N_h parameters. Let Φ_j be linear in each element of $\mathcal{T}_h$ taking value 1 at P_j but vanishing at the other vertices (pyramid functions). Then $\{\Phi_j\}_{j=1}^{N_h}$ forms a basis for $S_{h,\Omega}$ and every χ in $S_{h,\Omega}$ admits a unique representation:

$$\chi(x, y) = \sum_{j=1}^{N_h} \chi_j \Phi_j(x, y), \quad \text{with } \chi_j = \chi(P_j) \tag{3.1.1}$$

A given function v on Ω vanishing on $\partial\Omega$ may now be approximated by, for instance, its interpolant $I_h v$ in S_h, which we define as the function in $S_{h,\Omega}$ which agrees with v at the interior vertices of $\mathcal{T}_h$, i.e.,

$$I_h v(x) = \sum_{j=1}^{N_h} v(P_j)\Phi_j(x) \tag{3.1.2}$$

3.1.1 Spectral Analysis

The main aspect of this finite element model is to provide a tool displaying the eigenvalues for the model (**GM**). We display again for reference the system of equations

(**GM**):

$$\begin{cases} u_{tt} + c_\Omega u_t - k_\Omega \Delta u_t - \Delta u = 0 & x \in \Omega, t > 0 \\ u(x,t) = 0 & x \in \Gamma_0,\ t > 0 \\ u_{tt} + c_\Gamma u_t + \partial_n(u + k_\Omega u_t) - k_\Gamma \boldsymbol{\Delta}_\Gamma(\alpha u_t + u) = 0 & x \in \Gamma_1,\ t > 0 \\ u(0,x) = u_0,\ u_t(0,x) = u_1 & x \in \Omega \end{cases} \quad (\mathbf{GM})$$

Writing the solution $u(x,t)$ as:

$$u(x,t) = ue^{\lambda t} \quad (3.1.3)$$

where λ is an eigenvalue of $\mathcal{A}$, we obtain the following the characteristic equation:

$$\begin{cases} \lambda^2 u + c_\Omega \lambda u - k_\Omega \lambda \Delta u - \Delta u = 0 & \textbf{on } \Omega \\ u = 0 & \textbf{on } \Gamma_0 \\ \lambda^2 u + c_\Gamma \lambda u + \partial_n(u + k_\Omega \lambda u) - k_\Gamma \boldsymbol{\Delta}_\Gamma(\alpha \lambda u + u) = 0 & \textbf{on } \Gamma_1 \end{cases} \quad (3.1.4)$$

The reader should keep in mind that we initially define λ as the eigenvalues of the system in order to study numerically the spectrum of $\mathcal{A}$. However, once one moves to a time dependant study of this model, the equation (3.1.3) is not relevant anymore and the system (**GM**) should be transformed using a time discretization.

In order to provide the eigenvalues for this system, we need to transform the system (3.1.4) into N_h algebraic equations, where N_h is the number of nodes of the meshed domain. To be more precise, each algebraic equation will corresponds to (3.1.4) at this particular node, and will be coupled with the equations of the nodes in its neighborhood. For instance, if u_j^e is the solution of (3.1.4) at the node j, then the j^{th}

algebraic equation will have variables u_j^e and all the u_i^e's surrounding u_j^e. Rewriting the N_h algebraic equations in a matrix form:

$$F(\lambda)\begin{pmatrix} u_1^e \\ u_2^e \\ \cdots \\ u_{N_h}^e \end{pmatrix} = 0 \tag{3.1.5}$$

the values of λ for which $F(\lambda)$ is non-invertible will be the desired eigenvalues. The process of transforming (3.1.4) into (3.1.5) is called the spatial discretization.

3.1.2 Spatial discretization

Firstly, we mesh our domain Ω into equally-sized rectangles defined by 2 nodes per rectangle's side. Then, we multiply the first equation of (3.1.4) with a weight function $v_\Omega \in S_{h,\Omega}$ which is assumed to be differentiable once with respect to x and y and then integrate over the element domain Ω_e to obtain a weak formulation:

$$(\lambda^2 + c_\Omega\lambda)(u, v_\Omega)_{L^2(\Omega_e)} + (1 + k_\Omega\lambda)(\nabla u, \nabla v_\Omega)_{L^2(\Omega_e)} - (1 + k_\Omega\lambda)(\partial_n u, v_\Omega)_{L^2(\partial\Omega_e)} = 0 \tag{3.1.6}$$

In order to interpret boundary term, it is necessary to extend the definition of $S_{h,\Omega}$ to finite-dimensional subspaces of $H^1_{\Gamma_{0,h}}(\Omega_h)$ and $H^1(\Gamma_{1,h})$. Indeed, $\partial\Omega_h$ is the union of $\Gamma_{0,h}$ and $\Gamma_{1,h}$ where the elements of $S_{h,\Omega}$ vanish on Γ_0 and we define a family $\{S_{h,\Gamma_1}\}_h$ of finite-dimensional subspaces of $H^1(\Gamma_{1,h})$ such that:

$$S_h = \{\chi = (\chi_1, \chi_2), \chi_1 \in S_{h,\Omega},\ \chi_2 \in S_{h,\Gamma_1}, \chi_1|_{\Gamma_1} = \chi_2\} \tag{3.1.7}$$

and extend assumption 3.1.2 to:

Assumption 3.1.3.

$$\min_{\chi\in S_h}\left\{\|v-\chi\|_{L^2(\Omega)\times L^2(\Gamma_1)}+h\,\|(\nabla(v-\chi))\|_{L^2(\Omega)\times L^2(\Gamma_1)}\right\}\le Ch^2\,\|v\|_{H^2(\Omega)\times H^2(\Gamma_1)}$$

for $v\in[H_0^1(\Omega)\times H^1(\Gamma_1)]\cap[H^2(\Omega)\times H^2(\Gamma_1],\ \forall\chi\in S_{h,\Omega}$

Keeping the vertices $\{P_j\}_{j=1}^{N_h}$ and the conforming partition $\mathcal{T}_h$ previously defined with $S_{h,\Omega}$, a typical element of S_{h,Γ_1} is uniquely determined by its values at the points $P_j\in\mathcal{T}_h\cap\Gamma_{1,h}$. We form a basis for S_{h,Γ_1} with $\{\Phi_j^{\Gamma_1}\}_{j=1}^{N_h}$ defined by the following:

- continuous function on Γ_1
- linear in each element of $\mathcal{T}_h\cap\Gamma_{1,h}$
- equal to 1 at P_j in $\mathcal{T}_h\cap\Gamma_{1,h}$
- equal to 0 at the other vertices
- identically equal to 0 for all $(x,y)\in\Omega-\Gamma_1$

A typical element $\Gamma_{1,e}$ of $\mathcal{T}_h\cap\Gamma_1$ is a side of the rectangle-element whose nodes belong to $\mathcal{T}_h$. Then every $\chi^{\Gamma_1}(x,y)\in S_{h,\Gamma_1}$ admits a unique representation:

$$\chi^{\Gamma_1}(x,y)=\sum_{j=1}^{N_h}\chi_j^{\Gamma_1}\Phi_j^{\Gamma_1}(x,y),\quad\text{with }\chi_j^{\Gamma_1}=\chi^{\Gamma_1}(P_j)\tag{3.1.8}$$

We now multiply the third equation of (3.1.4) by a weight function $v_{|\Gamma_1}\in S_{h,\Gamma_1}$ and then integrate over the element domain $\Gamma_{1,e}$:

$$\begin{aligned}&(\lambda^2+\lambda c_\Gamma)\left(u,v_{|\Gamma_1}\right)_{L^2(\Gamma_{1,e})}+k_\Gamma(1+\alpha\lambda)\left(\nabla_\Gamma u,\nabla_\Gamma v_{|\Gamma_1}\right)_{L^2(\Gamma_{1,e})}\\&\quad=-(1+k_\Omega\lambda)\left(\partial_n u,v_{|\Gamma_1}\right)_{L^2(\Gamma_{1,e})}\end{aligned}\tag{3.1.9}$$

Figure 3.2: Typical rectangular element of size $a \times b$ and corresponding Lagrange interpolation functions ($\{\Phi_k^e\}_{k=1}^4$) in local coordinate system

Choosing $v_{|\Omega} = v_{|\Gamma_1}$ on Γ_1, the couple $(v_{|\Omega}, v_{|\Gamma_1})$ becomes an element of S_h defined (3.1.7) and we obtain the following general weak formulation, combining (3.1.6) and (3.1.9):

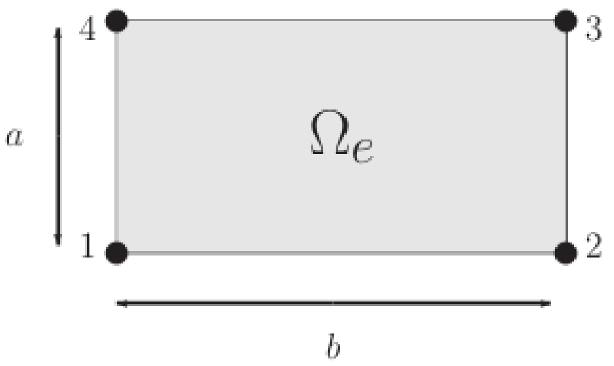
$$(\lambda^2 + c_\Omega \lambda)\,(u,v)_{L^2(\Omega_e)} + (1 + k_\Omega \lambda)\,(\nabla u, \nabla v)_{L^2(\Omega_e)} + G_{\Gamma_1} = 0$$
$$\text{where } G_{\Gamma_1} = \begin{cases} (\lambda^2 + \lambda c_\Gamma)\,(u,v)_{L^2(\Gamma_{1,e})} + k_\Gamma(1+\alpha\lambda)\,(\nabla_\Gamma u, \nabla_\Gamma v)_{L^2(\Gamma_{1,e})} \\ \quad \text{if } \exists\, \Gamma_{1,j} \in (\mathcal{T}_h \cap \Gamma_1) \text{ such that } \Omega_e \cap \Gamma_{1,j} \neq \emptyset \\ 0 \quad \text{otherwise} \end{cases} \tag{3.1.10}$$

From now, without loss of generality, we suppose that Ω_e intersects the boundary Γ_1, i.e. $G_{\Gamma_1} \neq 0$ and u is approximated by the following expressions over a typical elements Ω_e:

$$u(x,y) \approx u_h^e(x,y) = \sum_{j=1}^{N_h} (\Phi_j(x,y) + \Phi_j^{\Gamma_1}(x,y))u_j^e \tag{3.1.11}$$

where the coefficients $\{u_j^e\}_{j=1}^{N_h}$ are the N_h unknowns. It is a common procedure to choose the $\{\Phi_j\}$ to be the linear Lagrange interpolation functions in dimension 2. We recall that Φ_j takes value 1 at the point P_j is zero at the other vertices. See figure 3.2.

After substituing the finite element approximation for $(u_{|\Omega}, u_{|\Gamma_1})$ in (3.1.11) into the weak form (3.1.10), we obtain the following equation which must hold for every



$$\Phi_1^e(\zeta,\eta) = (1-\tfrac{\zeta}{a})(1-\tfrac{\eta}{b})$$

$$\Phi_2^e(\zeta,\eta) = \tfrac{\zeta}{a}(1-\tfrac{\eta}{b})$$

$$\Phi_3^e(\zeta,\eta) = (1-\tfrac{\zeta}{a})\tfrac{\eta}{b}$$

$$\Phi_4^e(\zeta,\eta) = \tfrac{\zeta}{a}\tfrac{\eta}{b}$$

admissible choice of weight function $(v_{|\Omega}, v_{|\Gamma_1}) \in S_h$:

$$\begin{aligned} 0 =& \int_{\Omega_e} (\lambda^2 + c_\Omega\lambda)\Phi_i \sum_{j=1}^{N_h} \Phi_j(x,y)u_j^e + (1+k_\Omega\lambda)\nabla\Phi_i \sum_{j=1}^{N_h} \nabla\Phi_j(x,y)u_j^e dxdy \\ &+ \int_{\Gamma_{1,e}} (\lambda^2 + \lambda c_\Gamma)\Phi_i^{\Gamma_1} \sum_{j=1}^{N_h} \Phi_j^{\Gamma_1}(x,y)u_j^e + k_\Gamma(1+\alpha\lambda)\nabla_\Gamma\Phi_i^{\Gamma_1} \sum_{j=1}^{N_h} \nabla_\Gamma\Phi_j^{\Gamma_1}(x,y)u_j^e ds \end{aligned} \tag{3.1.12}$$

For each choice of $(v_{|\Omega}, v_{|\Gamma_1})$, we obtain an algebaic relation among the $(u_{\Omega,1}, ..., u_{\Omega,N_h})$. We label the algebraic equation resulting from substitution of Φ_i (resp. $\Phi_i^{\Gamma_1}$) for v_Ω (resp. v_{Γ_1}) as the i^{th} algebraic equation, for $i = 1, ..., N_h$. Thus, the i^{th} algebraic equation is:

$$0 = \sum_{j=1}^{N_h} u_j^e \left[\begin{array}{c} \int_{\Omega_e} (\lambda^2 + c_\Omega\lambda)\Phi_j\Phi_i + (1+k_\Omega\lambda)\nabla\Phi_j\nabla\Phi_i dxdy \\ + \int_{\Gamma_{1,e}} (\lambda^2 + \lambda c_\Gamma)\Phi_j^{\Gamma_1}\Phi_i^{\Gamma_1} + k_\Gamma(1+\alpha\lambda)\nabla_\Gamma\Phi_j^{\Gamma_1}\nabla_\Gamma\Phi_i^{\Gamma_1} ds \end{array} \right] \tag{3.1.13}$$

Considering the N_h equations for the N_h unknowns, we obtain the following matrix system.

$$0 = \left[(\lambda^2 + c_\Omega \lambda)M + (1 + k_\Omega \lambda)K + (\lambda^2 + \lambda c_\Gamma)G^M + k_\Gamma(1 + \alpha\lambda)G^K\right] \begin{pmatrix} u_1^e \\ u_2^e \\ ... \\ u_{N_h}^e \end{pmatrix} \tag{3.1.14}$$

where, for $i, j = 1, 2, ..., N_h$, the entries for each matrix are given by:

$$\begin{aligned} K_{ij} &= \int_{\Omega_e} \Phi_j \Phi_i \; dxdy \quad M_{ij} &= \int_{\Omega_e} \nabla\Phi_j \nabla\Phi_i \; dxdy \\ G_{ij}^M &= \int_{\Gamma_e} \Phi_i^{\Gamma_1} \Phi_j^{\Gamma_1} \; ds \quad G_{ij}^K &= \int_{\Gamma_e} \nabla_\Gamma \Phi_i^{\Gamma_1} \nabla_\Gamma \Phi_j^{\Gamma_1} \; ds \end{aligned} \tag{3.1.15}$$

If an interior element (rectangular shape) is of dimensions $a \times b$ then the element stiffness and mass matrices for the interior domain are given by:

$$M_e = \frac{ab}{36} \begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix}$$

$$K_e = \frac{1}{6ab} \begin{pmatrix} 2(a^2+b^2) & a^2-2b^2 & -(a^2+b^2) & b^2-2a^2 \\ a^2-2b^2 & 2(a^2+b^2) & b^2-2a^2 & -(a^2+b^2) \\ -(a^2+b^2) & b^2-2a^2 & 2(a^2+b^2) & a^2-2b^2 \\ b^2-2a^2 & -(a^2+b^2) & a^2-2b^2 & 2(a^2+b^2) \end{pmatrix}$$

For the boundary matrices, recall that the elements are one dimension and correspond to one side of the rectangle element. Thus, let l be the length of this side, i.e., $l = a$

or $l = b$, then the boundary element mass and stiffness matrices are given by:

$$G_e^M = \frac{l}{6}\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad G_e^K = \frac{1}{l}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Remark 3.1.1. *We note that one can derive a similar finite element model with elements having a larger number of nodes, resulting in higher degree polynomials for the basis functions of S_h. We will see in the section dealing with the error estimate how such a change would affect the accuracy of the approximation.*

The equation (3.1.14) allows to determine approximate eigenvalues. Indeed, they correspond to the values of λ such that the determinant of $F(\lambda)$ is zero:

$$F(\lambda) = \left[(\lambda^2 + c_\Omega\lambda)M + (1 + k_\Omega\lambda)K + (\lambda^2 + \lambda c_\Gamma)G^M + k_\Gamma(1 + \alpha\lambda)G^K\right] \qquad (3.1.16)$$

3.2 Error Estimates for (VM) and (VM) models

3.2.1 Preliminaries

In this section, we shall investigate the error estimate for the finite element model developed above. The results of this section are presented in the theorems 3.2.9 and 3.2.10 for the frictional (**FM**) and viscoelastic (**VM**) models respectively. The main interest is to determine its order of convergence, i.e., the finite element error

$\|u - u_h\|$ in some norm $\|.\|$ bounded by $\mathcal{O}(h^\gamma)$, where u_h is the approximate solution, u the exact solution, and the characteristic size element h is defined as the maximal length of the sides of the partition $\mathcal{T}_h$. The error estimates derived in the following paragraphs tells us how fast the error decreases as we decrease the mesh size. We will see very quickly that the power γ is actually the degree of the polynomials from the finite-dimensional space S_h, therefore, we will derive the error estimates in the general context of polynomials of degree r, which lead to a little modification of the numerical sheme. Instead of considering 4 nodes-rectangle elements, we would define the rectangles elements with $r + 1$ nodes per side. The resulting Lagrange interpolation polynomials would be of degree r and form a basis for the space S_h now consisting of:

- 2-dimensional polynomials of degree at most r in Ω, vanishing at Γ_0
- 1-dimensional polynomials of degree at most r in Γ_1.

instead of linear functions. The procedure to determine error estimates is well-known in the literature and consists in introducing a projection of the real solution onto the finite dimensional subspace S_h, and then evaluate the error:

- between the exact solution and the projection
- between the approximate solution and the projection

A recurring theorem used in this section is Grownall's lemma, which we recall for future references:

Lemma 3.2.1 (Gronwall)**.** *Assume:*

- $u(t) \geq 0$
- $u(s)$ *continuous on* $[a, b]$
- $v(t) \leq c + \int_a^b v(s)u(s)ds$ *on* $[a, b]$

Then

$$v(t) \leq c\ exp\left[\int_a^t u(s)ds\right] \tag{3.2.1}$$

3.2.2 Definitions and Notations

As we have modified the definition of S_h, we must also extend assumption (3.1.3) to L^s-theory:

Assumption 3.2.2.

$$\min_{\chi \in S_h} \left\{ \|v - \chi\|_{L^2(\Omega)\times L^2(\Gamma_1)} + h \|(\nabla(v-\chi))\|_{L^2(\Omega)\times L^2(\Gamma_1)} \right\} \leq Ch^s \|v\|_{H^s(\Omega)\times H^s(\Gamma_1)}$$

$$\text{for } v \in [H_0^1(\Omega) \times H^1(\Gamma_1)] \cap [H^s(\Omega) \times H^s(\Gamma_1)],\ 1 \leq s \leq r,\ \forall \chi \in S_{h,\Omega}$$

Definition 3.2.3 (Operator $\mathbb{A}$ and $\mathbb{B}$)**.** *First recall the Laplace operator A, the Neumann map N and the Laplace-Beltrami operator defined respectively in* (2.1.1), (2.1.7) *and* (2.1.6)*. Then, we can define the operator $\mathbb{A} : L^2(\Omega) \times L^2(\Gamma_1) \supset \mathcal{D}(\mathbb{A}) \to L^2(\Omega) \times L^2(\Gamma_1)$ by:*

$$\mathbb{A} = \begin{pmatrix} A(I - N\partial_n) & 0 \\ \partial_n & B \end{pmatrix} \tag{3.2.2}$$

$$\mathcal{D}(\mathbb{A}) = \{[u_1, u_2]^T \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B^{\frac{1}{2}}),$$

$$\textit{such that } A(I - N\partial_n)u_1 \in L^2(\Omega),\ u_1|_{\Gamma_1} = N^*Au_1 = u_2$$

$$\partial_n u_1 + Bu_2 \in L^2(\Gamma_1)\}$$

In a similar way, we define the operator $\mathbb{B} : L^2(\Omega) \times L^2(\Gamma_1) \supset \mathcal{D}(\mathbb{A}) \to L^2(\Omega) \times L^2(\Gamma_1)$ *by:*

$$\mathbb{B} = \begin{pmatrix} k_\Omega A(I - N\partial_n) & 0 \\ k_\Omega \partial_n & \alpha B \end{pmatrix} \tag{3.2.3}$$

$$\mathcal{D}(\mathbb{B}) = \{[u_1, u_2]^T \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B^{\frac{1}{2}}),$$

$$\textit{such that } k_\Omega A(I - N\partial_n)u_1 \in L^2(\Omega),\ u_1|_{\Gamma_1} = N^*Au_1 = u_2$$

$$\partial_n u_1 + \alpha Bu_2 \in L^2(\Gamma_1)\}$$

Lemma 3.2.4. $\mathbb{B}$*, given in* (3.2.3) *is self-adjoint.*

Proof. Let $u = (u_1, u_2)$ be in $\mathcal{D}(\mathbb{B})$ and $y = (y_1, y_2)$ be in $\mathcal{D}(\mathbb{B}^*)$, then taking the inner product:

$$\begin{aligned} &(\mathbb{B}u, y)_{L^2(\Omega)\times L^2(\Gamma_1)} \\ &\quad = (k_\Omega Au_1, y_1)_\Omega - (k_\Omega AN\partial_n u_1, y_1)_\Omega + \langle k_\Omega \partial_n u_1, y_1\rangle_{\Gamma_1} + \langle \alpha Bu_1, y_1\rangle_{\Gamma_1} \\ &\quad = \left(k_\Omega A^{\frac{1}{2}}u_1, A^{\frac{1}{2}}y_1\right)_\Omega + \langle u_1, \alpha By_1\rangle_{\Gamma_1} \\ &\qquad \text{since } (AN\partial_n u_1, y_1)_\Omega = \langle \partial_n u_1, N^*Ay_1\rangle_{\Gamma_1} = \langle \partial_n u_1, y_2\rangle_{\Gamma_1} \\ &\quad = (u_1, k_\Omega Ay_1)_\Omega - (u_1, k_\Omega AN\partial_n y_1)_\Omega + \langle u_1, k_\Omega \partial_n y_1\rangle_{\Gamma_1} + \langle u_1, \alpha By_1\rangle_{\Gamma_1} \\ &\quad = (u, \mathbb{B}y)_{L^2(\Omega)\times L^2(\Gamma_1)} \end{aligned} \tag{3.2.4}$$

Therefore $\mathbb{B}$ is self-adjoint. ☐

We would like to add another convenient notation for this section. With $r \geq 1$ an integer corresponding to the maximal degree of the polynomial in S_h, define the spaces Y^s by:

$$Y^0 = L^2(\Omega) \times L^2(\Gamma_1)$$

$$Y^s = [\mathcal{D}(A^{\frac{1}{2}}) \cap H^s_{\Gamma_0}(\Omega)] \times [\mathcal{D}(B^{\frac{1}{2}}) \cap H^s(\Gamma_1)], \quad \text{for all } 1 \leq s \leq r$$

with associated norms:

$$\|U\|^2_{Y^0} = \left\|u_{|\Omega}\right\|^2_{L^2(\Omega)} + \left\|u_{|\Gamma_1}\right\|^2_{L^2(\Gamma_1)}$$

$$\|U\|^2_{Y^s} = \left\|u_{|\Omega}\right\|^2_{H^s_{\Gamma_0}(\Omega)} + k_\Gamma \left\|u_{|\Gamma_1}\right\|^2_{H^s(\Gamma_1)}, \quad \text{for all } 1 \leq s \leq r$$

This implies:

$$\begin{aligned} \|U\|^2_{Y^1} &= \left\|u_{|\Omega}\right\|^2_{H^1_{\Gamma_0}(\Omega)} + k_\Gamma \left\|u_{|\Gamma_1}\right\|^2_{H^r(\Gamma_1)} \\ &= \left\|\nabla u_{|\Omega}\right\|^2_{L^2(\Omega)} + k_\Gamma \left\|\nabla_\Gamma u_{|\Gamma_1}\right\|^2_{L^2(\Gamma_1)} \\ &= \left\|\mathbb{A}^{\frac{1}{2}} U\right\|^2_{Y^0} \end{aligned} \tag{3.2.5}$$

For $k \geq 0, \ s \geq 0$, we introduce the special convenient notation:

$$\|U\|_{Y^s,k} = \sum_{j=1}^{k} \left[\left\|D_t^j U(t)\right\|_{Y^s} + \int_0^t \left\|D_t^j U(s)\right\|_{Y^s} ds \right]$$

where $D_t = \frac{d}{dt}$.

3.2.3 Wave equation with Frictional damping (FM)

We first recall the definition of the Ritz projection, also called elliptic projection.

Definition 3.2.5 (Ritz Projection)**.** *Let $\mathbb{A} = \Delta$, the Ritz projection is defined as the orthogonal projection with respect to the inner product $(\nabla v, \nabla w)_\Omega$, so that*

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi) \text{ for } v \in H_0^1,\ \forall \chi \in S_h \tag{3.2.6}$$

Remark 3.2.1. *The Ritz projection commutes with differentiation with respect to a parameter (in our case, time).*

In our case, we would like to generalize this operator to the interior and the active boundary Γ_1. Let $U = (u|_\Omega, u|_{\Gamma_1}) \in Y^1 = H^1_{\Gamma_0}(\Omega) \times [\mathcal{D}(B^{\frac{1}{2}}) \cap H^1(\Gamma_1)]$.

Similary to the Ritz or elliptic projection, we define the projection $\mathcal{R}_h$ onto S_h as the orthogonal projection with respect to the inner product $(U, \chi)_{Y^1} = \left(\mathbb{A}^{\frac{1}{2}} U, \mathbb{A}^{\frac{1}{2}} \chi\right)_{Y^0}$, so that

$$(\mathcal{R}_h U, \chi)_{Y^1} = (U, \chi)_{Y^1}, \quad \text{for all } \chi \in S_h \tag{3.2.7}$$

To begin with, we estimate the difference between the solution and its projection on S_h and obtain a first approximation property:

Lemma 3.2.6. *Assume that* (3.2.2) *holds. Then, with $\mathcal{R}_h$ defined by* (3.2.7) *we have:*

$$\|\mathcal{R}_h U - U\|_{Y^0} + h \left\|\mathbb{A}^{\frac{1}{2}} (\mathcal{R}_h U - U)\right\|_{Y^0} \leq C h^s \|U\|_{Y^s}, \quad \text{for } 1 \leq s \leq r \tag{3.2.8}$$

Proof. Let $1 \leq s \leq r$, we will proceed in 3 steps:

Step 1: Show that $\|\mathcal{R}_h U - U\|_{Y^1} \leq C h^{s-1} \|U\|_{Y^s}$, for $1 \leq s \leq r$

Since by (3.2.7), $\mathcal{R}_h U$ is the orthogonal projection of U onto S_h with respect to the Y^1 inner product, we have by (3.2.2):

$$\begin{aligned} \|\mathcal{R}_h U - U\|_{Y^1} &\le \inf_{\chi \in S_h} \{\|U - \chi\|_{Y^1}\} \\ &\le C h^{s-1} \|U\|_{Y^s} \end{aligned} \tag{3.2.9}$$

<u>Step 2:</u> Show that $\|\mathcal{R}_h U - U\|_{Y^0} \le C h^s \|U\|_{Y^s}$, for $1 \le s \le r$

Using a duality argument, $\forall \phi \in Y^0$, let $\psi \in \left[H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right] \times \left[H^2(\Gamma_1) \cap H^1(\Gamma_1)\right]$ be a solution of

$$\begin{cases} \Delta\psi &= \phi_{|\Omega}, \text{ in } \Omega \\ -\partial_n \psi + \boldsymbol{\Delta}_\Gamma \psi &= \phi_{|\Gamma_1}, \text{ in } \Gamma_1 \\ \psi &= 0, \text{ on } \Gamma_0 \end{cases} \tag{3.2.10}$$

Then, by elliptic regularity,

$$\|\psi\|_{Y^2} \le C \|\phi\|_{Y^0} \tag{3.2.11}$$

For all $\psi_h \in S_h$, we have

$$\begin{aligned}
(\mathcal{R}_h U - U, \phi)_{Y^0} &= \big((\mathcal{R}_h U - U)_{|\Omega}, \Delta\psi\big)_{L^2(\Omega)} + \big((\mathcal{R}_h U - U)_{|\Gamma_1}, \boldsymbol{\Delta}_\Gamma \psi\big)_{L^2(\Gamma_1)} \\
&= (\mathcal{R}_h U - U, \psi)_{Y^1} \\
&= (\mathcal{R}_h U - U, \psi - \psi_h)_{Y^1} \\
&\qquad \text{since } \psi_h \in S_h,\ (3.2.8) \text{ holds and } (\mathcal{R}_h U - U, \psi_h)_{Y^1} = 0 \\
&\leq \|\mathcal{R}_h U - U\|_{Y^1} \|\psi - \psi_h\|_{Y^1} \\
&\leq C h^{s-1} \|U\|_{Y^s} \|\psi - \psi_h\|_{Y^1} \text{ by (3.2.9)} \\
&\leq C h^{s} \|U\|_{Y^s} \|\phi\|_{Y^0} \text{ by (3.2.2) with } s = 2
\end{aligned} \tag{3.2.12}$$

The desired result is obtained by choosing $\phi = \mathcal{R}_h U - U$.

Step 3: Conclusion

By the definition of $\mathcal{R}_h$, and the combination the results obtained in the two previous steps, we get (3.2.8). □

Lemma 3.2.7. *Under the assumption of lemma 3.2.6, and with k any positive integer, we have*

$$\begin{aligned}
\left\|D_t^k(\mathcal{R}_h U - U)\right\|_{Y^0} + h \left\|D_t^k \mathbb{A}^{\frac{1}{2}}(\mathcal{R}_h U - U)\right\|_{Y^0} &\leq C h^s \|U\|_{Y^s,k}\,, \quad \textit{for } 1 \leq s \leq r \\
\textit{where } \|U\|_{Y^s,k} = \sum_{j=1}^{k} \Big[\left\|D_t^j U(t)\right\|_{Y^s} &+ \int_0^t \left\|D_t^j U(s)\right\|_{Y^s} ds \Big]
\end{aligned} \tag{3.2.13}$$

Proof. Recall that $\mathcal{R}_h$ is an elliptic projection and thus commutes with differentiation with respect to time (see remark 3.2.1). Repeating the proof of lemma 3.2.6, we obtain the desired result for any positive integer k. □

Theorem 3.2.8. *Consider the initial boundary value problem:*

$$\begin{cases} U_t - \mathbb{A}U = 0 & \text{in } \Omega \times \Gamma_1 \\ U = 0 & \text{on } \Gamma_0 \\ U(.,0) = U_o & \text{on } \Omega \times \Gamma_1 \end{cases} \tag{3.2.14}$$

and the associated semidiscrete problem

$$\begin{cases} (U_{h,t}, \chi)_{Y^0} + \left(\mathbb{A}^{\frac{1}{2}}U_h, \mathbb{A}^{\frac{1}{2}}\chi\right)_{Y^0} = 0, \ \forall \chi \in S_h, \ t > 0, \\ U_h(0) = U_{h,0} \end{cases} \tag{3.2.15}$$

with $U_h(t) \in S_h$. Then, for $t \geq 0$

$$\left\| D_t^k(U_h(t) - U(t)) \right\|_{Y^0} \leq \|U_{h,0} - U_0\|_{Y^0,k} + Ch^r(\|U\|_{Y^r,k} + \int_0^T \|U_t\|_{Y^r,k}\, dt) \tag{3.2.16}$$

Proof. We start with the case $k = 0$.

A classical strategy throughout the error analysis is to write the error as a sum of two terms:

$$U_h(t) - U(t) = \theta(t) + \rho(t) \text{ where } \theta = U_h - \mathcal{R}_h U, \ \rho = \mathcal{R}_h U - U \tag{3.2.17}$$

By lemma 3.2.6,

$$\|\rho(t)\|_{Y^0} \leq Ch^r \|U(t)\|_{Y^r} \leq Ch^r \left(\|U_0\|_{Y^r} + \int_0^t \|U_t\|_{Y^r}\, ds \right) \tag{3.2.18}$$

We are left with the θ-bound, observe that:

$$\begin{aligned}(\theta_t, \chi)_{Y^0} + (\theta, \chi)_{Y^1} &= (U_{h,t}, \chi)_{Y^0} + (U_h, \chi)_{Y^1} - (\mathcal{R}_h U_t, \chi)_{Y^0} - (\mathcal{R}_h U, \chi)_{Y^1} \\ &= -(\mathcal{R}_h U_t, \chi)_{Y^0} - (\mathcal{R}_h U, \chi)_{Y^1} \\ &= -(\mathcal{R}_h U_t, \chi)_{Y^0} + (U_t, \chi)_{Y^0}, \quad \text{by (3.2.15)} \\ &= -(\rho_t, \chi)_{Y^0}\end{aligned} \tag{3.2.19}$$

where the last step was obtained since the operator $\mathcal{R}_h$ commutes with time differentiation.

But $\theta \in S_h$, so by letting $\chi = \theta$ we obtain:

$$(\theta_t, \theta)_{Y^0} + \|\theta\|_{Y^1}^2 = -(\rho_t, \theta)_{Y^0} \tag{3.2.20}$$

It follows that:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{Y^0}^2 &\leq \|\rho_t\|_{Y^0}\|\theta\|_{Y^0} \\ &\leq \|\rho_t\|_{Y^0}^2 + \|\theta\|_{Y^0}^2\end{aligned} \tag{3.2.21}$$

Integrate in time and apply Gronwall's (lemma 3.2.1):

$$\begin{aligned}\|\theta(t)\|_{Y^0}^2 &\leq \int_0^t \|\rho_t\|_{Y^0}^2 + \|\theta\|_{Y^0}^2\, ds + \|\theta(0)\|_{Y^0}^2 \\ &\leq \int_0^t \|\rho_t\|_{Y^0}^2\, ds + \|\theta(0)\|_{Y^0}^2\end{aligned} \tag{3.2.22}$$

By lemma 3.2.6, we have the two following inequalities

$$\begin{cases}\|\rho_t\|_{Y^0}^2 & \leq \|\mathcal{R}_h U_t - U_t\|_{Y^0}^2 \leq Ch^{2r}\|U_t\|_{Y^r}^2 \\ \|\theta(0)\|_{Y^0}^2 & = \|U_{h,0} - \mathcal{R}_h U_0\|_{Y^0}^2 \\ & \leq \|U_{h,0} - U_0\|_{Y^0}^2 + \|U_0 - \mathcal{R}_h U_0\|_{Y^0}^2 \\ & \leq \|U_{h,0} - U_0\|_{Y^0}^2 + Ch^{2r}\|U_0\|_{Y^r}^2\end{cases} \tag{3.2.23}$$

Using these inequalities in (3.2.21), we get:

$$\|\theta(t)\|_{Y^0} \leq \|U_{h,0} - U_0\|_{Y^0} + Ch^{2r}(\int_0^t \|U_t\|_{Y^r}\, ds + \|U_0\|_{Y^r}) \tag{3.2.24}$$

Combine (3.2.18) and (3.2.24) to get the desired result. The proof for positive k follows from the same process and the result from lemma 3.2.7 ◻

We can now turn to our main result for error estimate of the friction model:

Theorem 3.2.9. *With* $U = (u_\Omega, u_{\Gamma_1})$, *let* U *be the solution of* (**FM**) *and* U_h *be the solution of:*

$$\begin{cases} (u_{h,tt}, \chi_\Omega)_\Omega + (u_{h,t}, \chi_\Omega)_\Omega + (\nabla u_h, \nabla \chi_\Omega)_\Omega \\ \qquad \langle u_{h,tt}, \chi_{\Gamma_1}\rangle_{\Gamma_1} + \langle u_{h,t}, \chi_{\Gamma_1}\rangle_{\Gamma_1} + \langle \nabla_\Gamma u_h, \nabla_\Gamma \chi_{\Gamma_1}\rangle_{\Gamma_1} = 0 \\ U_h(0) = U_{h,0}(0), U_{h,t} = U_{h,1}(0) \end{cases} \tag{3.2.25}$$

Assume:

$$\|U_{h,0} - U_0\|_{Y^0} + h\,\|U_{h,0} - U_0\|_{Y^1} \leq Ch^r \tag{3.2.26}$$

$$\|U_{h,1} - U_1\|_{Y^0} \leq Ch^r \tag{3.2.27}$$

Then for any non-negative integer k we have:

$$\|(U_h - U)(t)\|_{Y^0} \leq C(u)h^r \tag{3.2.28}$$

Proof. First observe that:

$$\begin{cases} U_{tt} + \mathbb{C}U_t + \mathbb{A}U = 0, \quad \text{in } \Omega \times \Gamma_1, \text{ for } t \in J = [0, t) \\ U = 0, \quad \text{on } \Gamma_0, \text{ for } t \in J \\ U(0) = U_0, , U_t(0) = U_1, \quad \text{in } \Omega \times \Gamma_1 \end{cases} \tag{3.2.29}$$

is equivalent to (**FM**), with $\mathbb{C} = \begin{pmatrix} c_\Omega & 0 \\ 0 & c_\Gamma \end{pmatrix}$.

Also the semidiscrete analogue formulation to (3.2.29) is:

$$\begin{cases} (U_{h,tt}, \chi)_{Y^0} + (U_{h,t}, \chi)_{Y^0} + (U_h, \chi)_{Y^1} = 0, \text{ for } \chi \in S_h,\ t \in J \\ U_h(0) = U_{h,0}(0), U_{h,t} = U_{h,1}(0) \end{cases} \tag{3.2.30}$$

and is equivalent to (3.2.25). Therefore, it is enough to study the error between U and U_h in the context of (3.2.29) and (3.2.30). For the sake of convenience to the reader, and without loss of generality, assume that the damping matrix $\mathbb{C}$ is the identity matrix, i.e., $c_\Omega = c_\Gamma = 1$.

First, we define the operator $W : \bar{J} \to S_h$ by:

$$\begin{cases} (W_t - U_t, \chi)_{Y^0} + (W - U, \chi)_{Y^1} = 0 \\ W(0) = U_{h,0} \end{cases} \tag{3.2.31}$$

We use the common procedure to split the error $U_h - U$ into $\rho = W - U$ and $\theta = U_h - W$.

By theorem 3.2.8, we have the ρ-bound:

$$\left\| D_t^k \rho \right\|_{Y^0} \leq Ch^r \tag{3.2.32}$$

where D_t^k denotes the k^{th}-time derivative.

To get the θ-bound, observe that:

$$(\theta_{tt}, \chi)_{Y^0} + (\theta_t, \chi)_{Y^0} + (\theta, \chi)_{Y^1} = -\left(\rho_{tt}, \chi\right)_{Y^0} \tag{3.2.33}$$

Since $\theta_t \in S_h$, we can choose $\chi = \theta$:

$$\frac{1}{2}\frac{d}{dt}\left(\|\theta_t\|^2_{Y^0} + \|\theta\|^2_{Y^1}\right) + \|\theta_t\|^2_{Y^0} = -\left(\rho_{tt}, \theta_t\right)_{Y^0} \tag{3.2.34}$$

After eliminating the non-negative term $\|\theta_t\|^2_{Y^0}$, integrate in time and apply Gronwall's (lemma 3.2.1):

$$\begin{aligned}\|\theta_t(t)\|^2_{Y^0} + \|\theta(t)\|^2_{Y^1} &\leq \int_0^T \|\rho_{tt}\|^2_{Y^0} + \|\theta_t\|^2_{Y^0}\, ds + \|\theta_t(0)\|^2_{Y^0} + \|\theta(0)\|_{Y^1} \\ &\leq Ch^{2r} + \|\theta_t(0)\|^2_{Y^0} + \|\theta(0)\|_{Y^1}\end{aligned} \tag{3.2.35}$$

It remains to estimate $\|\theta_t(0)\|^2_{Y^0}$ and $\|\theta(0)\|_{Y^1}$, by (3.2.32) and (3.2.26):

$$\begin{aligned}\|\theta(0)\|_{Y^1} &\leq \|W(0) - U_0\|^2_{Y^1} + \|U_0 - U_{h,0}\|^2_{Y^1} \leq Ch^{2r} \\ \|\theta_t(0)\|_{Y^0} &\leq \|W_t(0) - U_1\|^2_{Y^0} + \|U_1 - U_{h,1}\|^2_{Y^0} \leq Ch^{2r}\end{aligned} \tag{3.2.36}$$

□

3.2.4 Wave equation with viscoelastic damping (VM)

The error estimate for our viscoelastic model (**VM**) could be obtained using the results by Thomée et al. in [35]. This paper presents intrinsically interesting error estimate, the authors studied the numerical solution of parabolic integrodifferential equations of the form:

$$u_t + Pu + \int_0^t Qu(s)ds = f(t), \text{ in } \widetilde{\Omega}, \ t \in J = (0, T], \tag{3.2.37}$$

together with homogeneous Dirichlet boundary conditions and given initial values. In their set-up, $\widetilde{\Omega}$ is a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with smooth boundary, $P(t)$

is a self-adjoint positive-definite linear elliptic partial differential operator of second order, and $Q(t,s)$ is an arbitrary second order linear partial differential operator. Acting upon the error estimate of (3.2.37), they are able to treat not only hyperbolic integrodifferential equations, but also Sobolev and viscoelasticity type equations. In particular, theorem 2.1, lemma 2.1 and theorem 5.1 from [35] lead to the error estimate for viscoelasticity type equations which we state for a general domain $\widetilde{\Omega}$:

Theorem C (Theorem 5.2 [35] - Generalized damped equation)**.** *Define the following equation with the initial boundary value problem:*

$$\begin{cases} \widetilde{u}_{tt} + P\widetilde{u}_t + Q\widetilde{u} = f & \text{in } \widetilde{\Omega}, \text{ for } t \in J \\ \widetilde{u} = 0 & \text{on } \delta\widetilde{\Omega}, \text{ for } t \in J \\ \widetilde{u}(0) = \widetilde{u}_0,\ \widetilde{u}_t(0) = \widetilde{u}_1 & \text{in } \widetilde{\Omega} \end{cases} \tag{3.2.38}$$

and its semidiscrete analogue:

$$\begin{aligned} &(\widetilde{u}_{h,tt}, \chi) + P(\widetilde{u}_{h,t}, \chi) + Q(\widetilde{u}_h, \chi) = (f(t), \chi) \text{ for } \chi \in S_h,\ t \in J \\ &\widetilde{u}_h(0) = \widetilde{u}_{h,0},\ \widetilde{u}_{h,t}(0) = u_{h,1} \end{aligned} \tag{3.2.39}$$

P is a second order self adjoint positive definite elliptic operator.

Q is a general second order differential operator.

$P(.,.)$ and $Q(.,.)$ are the associated bilinear forms.

Let $\widetilde{u}$ and $\widetilde{u}_h$ be the solutions of (3.2.38) *and* (3.2.39)*, with u smooth enough. Assume that:*

$$\|\widetilde{u}_{h,0} - \widetilde{u}_0\|_{L^2(\widetilde{\Omega})} + \|\widetilde{u}_{h,0} - \widetilde{u}_0\|_{H^1(\widetilde{\Omega})} \leq Ch^r \tag{3.2.40}$$

$$\|\widetilde{u}_{h,1} - \widetilde{u}_1\|_{L^2(\widetilde{\Omega})} \leq Ch^r \tag{3.2.41}$$

Then for any non-negative integer k we have:

$$\|(\widetilde{u}_h - \widetilde{u})(t)\|_{L^2(\widetilde{\Omega})} \leq C(\widetilde{u})h^r \tag{3.2.42}$$

The procedure to prove this theorem is very classical and follow the same process we used in the previous section to demonstrate the error estimate for the model with frictional damping (**FM**). Indeed, theorem 2.1 and lemma 2.1 from [35] allow to obtain error estimate between the exact solution and its projection onto S_h for the equation (3.2.37). Then, theorem 5.1 (in [35]) provides the error estimate for the so-called Sobolev type equations:

$$\begin{cases} Pu_t + Qu = f, & \text{in } \Omega, \text{ for } t \in J \\ u = 0, & \text{on } \delta\Omega, \text{ for } t \in J \\ u(0) = u_0, & \text{in } \Omega \end{cases}$$

The equation is used to define the projection of u onto S_h in the proof of the error estimate for the viscoelastic type equations (3.2.38) in theorem C (theorem 5.2 from [35]). Given these results, it is now possible to derive the error estimate associated with our viscoelastic model (**VM**):

Theorem 3.2.10 (Error estimates for the viscoelastic model)**.** *Suppose that* $U = (u_{|\Omega}, u_{|\Gamma_1})$ *is the solution of the viscoelastic model* (**VM**) U_h *is the solution of the* *semidiscrete problem:*

$$\begin{cases} (u_{h,tt},\chi)_\Omega + k_\Omega\,(\nabla u_{h,t},\nabla\chi)_\Omega + (\nabla u_h,\nabla\chi)_\Omega \\ \qquad \langle u_{h,tt},\chi\rangle_{\Gamma_1} + k_\Gamma\,\langle\nabla_\Gamma u_{h,t},\nabla_\Gamma\chi\rangle_{\Gamma_1} + \langle\nabla_\Gamma u_h,\nabla_\Gamma\chi\rangle_{\Gamma_1} = 0 \end{cases} \tag{3.2.43}$$

Assume:

$$\|U_{h,0}-U_0\|_{Y^0} + h\,\|U_{h,0}-U_0\|_{Y^1} \le Ch^r \tag{3.2.44}$$

$$\|U_{h,1}-U_1\|_{Y^0} \le Ch^r \tag{3.2.45}$$

Then for any non-negative integer k we have:

$$\|(U_h-U)(t)\|_{Y^0} \le C(u)h^r \tag{3.2.46}$$

Proof. We first establish the equivalence between the viscoelastic model (**VM**) and viscoelasticity type equation (3.2.38) provided:

- $\widetilde{\Omega} = \Omega\times\Gamma_1$
- define $P = \mathbb{B} = \begin{pmatrix} k_\Omega A(I-N\partial_n) & 0 \\ k_\Omega\partial_n & \alpha B\end{pmatrix}$ and $Q = \mathbb{A} = \begin{pmatrix} A(I-N\partial_n) & 0 \\ \partial_n & B\end{pmatrix}$ (definition 3.2.3)
- $\mathbb{B}$ is self-adjoint (see lemma 3.2.4)

It follows that the semidriscrete analogues are also equivalent ((3.2.43) and (3.2.39)). Therefore, we can apply Thomèe's theorem C to obtain the desired estimate. □

Chapter 4

Spectral Properties and Regularity of the Semigroup

It is well known that solutions to parabolic systems experience gains in regularity, for $t \geq 0$. Actually, under sufficiently strong damping, the dynamics of the original model (**GM**) are parabolic rather than hyperbolic. In this chapter, our aim is to classify the semigroups associated with (**GM**) by their smoothing properties from the "best" scenario: analytic regularity to the "worse" scenario: absence of regularity: hyperbolic cases, but also taking account of intermediate stage such as the Gevrey's class.

The separation of the general model (**GM**) into two submodels suddenly springs into focus in the context of regularity properties of the semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$. We note that the frictional model (**FM**) is hyperbolic while the viscoelastic model (**VM**) is parabolic (provided the presence of damping). Therefore, this chapter will concentrate on the parabolic cases of (**GM**), thus, we start with the viscoelastic model (**VM**) which we display again for reference:

$$\begin{cases} u_{tt} - k_\Omega \Delta u_t - \Delta u = 0 & x \in \Omega, t > 0 \\ u(x,t) = 0 & x \in \Gamma_0,\ t > 0 \\ u_{tt} + \partial_n(u + k_\Omega u_t) - k_\Gamma \mathbf{\Delta}_\Gamma(\alpha u_t + u) = 0 & x \in \Gamma_1,\ t > 0 \\ u(0,x) = u_0,\ u_t(0,x) = u_1 & x \in \Omega \end{cases} \tag{\textbf{VM}}$$

This model is highly interesting in the investigation of solution's regularity as it offers different "levels" of smootheness which can be identified and characterized by the bound on $\|R(i\beta, \mathcal{A})\|_{\mathcal{L}(\mathcal{H})}$ as follows:

$$\|R(i\beta, \mathcal{A})\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{f(\beta)}$$

$$\text{where } f(\beta) \sim \begin{cases} |\beta| & \text{if } \mathcal{A} \text{ generates an analytic semigroup} \\ |\beta|^\gamma, \text{ for } 0 < \gamma \leq 1 & \text{if } \mathcal{A} \text{ generates a semigroup with} \\ & \quad \text{intermediate reguliraty (Gevrey)} \\ log(|\beta|) & \text{if } \mathcal{A} \text{ generates a differentiable semigroup} \end{cases}$$

Furthermore, the function $f(\beta)$ encloses the region containing the spectrum of the generator. For instance, it is well known that analytic semigroup can be enclosed in a triangular sector (see figure 1.3 [p. 19]), i.e. by functions of the form $f(\beta) = a\beta + b$. We recall that this figure corresponds to the eigenvalues of a wave equation with viscoelastic damping and 0-Dirichlet boundary conditions (model (**VM**) with $\Gamma_1 = \emptyset$) and is governed by an analytic semigroup. It is one of the goal of this chapter to determine how the dynamic boundary conditions ($\Gamma_1 \neq \emptyset$) affect this result. Similarly, intermediate regularity semigroup's spectrum can be enclosed with functions of the form $f(\beta) = \beta^\gamma + C$ and differentiable semigroup's spectrum with functions of the form $f(\beta) = log(\beta) + C$. However, in the case of hyperbolic equations, the spectrum is not enclosable anymore and presents asymptotics components in its spectrum. Therefore the numerical spectrum provides a good illustration of regularity results, but they can also suggest that a better regularity can reached or not. In particular, the absence of viscoelastic damping on the boundary ($\alpha = 0$) gives raise to this intermediate regularity called *Gevrey regularity* as we will see in the second part of this chapter.

4.1 Analytic Semigroup

We first determine under which conditions the operator $\mathcal{A}$, defined in (2.2.4), offers a maximum regularity, i.e., when does $\mathcal{A}$ generate an analytic semigroup. We omit to mention the energy space for the moment since the analyticity can be achieved in many spaces, including $\mathcal{H}$, of course. The conditions for analyticity are related to the amount of strong damping present in the system. As we will see, the frictional dampings are transparent in the seek of analyticity, therefore one could ignore them and work only with the model (**VM**). We can now state and proof our two main theorems in this chapter. The first theorem uses a resolvent approach allowing an immediate link with the observation of the spectrum. However, it relies on the following assumption:

Assumption 4.1.1. *With $k_\Omega > 0$ $\mathcal{A}$ is exponentially stable*

Remark 4.1.1. *In chapter 5 (theorem 5.1.1), we will be able to prove the exponential stability for model* (**GM**) *with $k_\Omega > 0$ and thus one could eliminate this assumption .*

4.1.1 Resolvent approach

Theorem 4.1.2. *Suppose that assumption holds 4.1.1. Let $k_\Omega, k_\Gamma, \alpha > 0$. Suppose $|\beta| \geq 1$ so that $(i\beta - \mathcal{A})^{-1} \in \mathcal{L}(\mathcal{H})$, then with $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$*

$$\|R(i\beta; \mathcal{A})\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{|\beta|}$$

It follows that $\{e^{\mathcal{A}t}\}_{t \geq 0}$ is an analytic semigroup on $\mathcal{H}$.

Proof. We recall the definition of $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ in (2.2.4):

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \Delta & D_\Omega & 0 & 0 \\ 0 & 0 & 0 & I \\ -\partial_n & -k_\Omega \partial_n & -B & D_{\Gamma_1} \end{pmatrix}$$

$$\text{where} \begin{cases} D_\Omega & = k_\Omega \Delta - c_\Omega I \\ D_{\Gamma_1} & = -\alpha B - c_\Gamma I \end{cases}$$

Assume that $k_\Omega, k_\Gamma, \alpha > 0$. Let $\beta \in \mathbb{R}$ so that $(i\beta - \mathcal{A})^{-1} \in \mathcal{L}(\mathcal{H})$, then with $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$ and pre-image $U = (u_1, u_2, u_3, u_4)^T \in \mathcal{D}(\mathcal{A})$ we consider the resolvent equation $(i\beta - \mathcal{A})U = F$:

$$(i\beta - \mathcal{A}) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \tag{4.1.1}$$

Expanding (4.1.1), we get:

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = (i\beta - \mathcal{A})^{-1} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = (i\beta - \mathcal{A})^{-1} F \tag{4.1.2}$$

We want to show $|\beta| \, \|U\|_{\mathcal{H}} \leq C \, \|F\|_{\mathcal{H}}$.

By assumption 4.1.1, $\{e^{\mathcal{A}t}\}_{t \geq 0}$ is exponentially stable, then:

$$\|U\|_{\mathcal{H}} \leq C \, \|F\|_{\mathcal{H}} \tag{4.1.3}$$

$$\begin{cases} i\beta u_1 - u_2 = f_1 \\ i\beta u_3 - u_4 = f_3 \\ i\beta u_2 - \Delta u_1 - k_\Omega \Delta u_2 + c_\Omega u_4 = f_2 \\ i\beta u_4 + \partial_n (u_1 + k_\Omega u_2) + B u_3 + \alpha B u_4 + c_\Gamma u_4 = f_4 \end{cases} \tag{4.1.4}$$

Multiply the third equation by $\overline{u_2}$ and make use of the fourth equation:

$$\begin{aligned} & i\beta \, |u_2|_\Omega^2 + (\nabla u_1, \nabla u_2)_\Omega + k_\Omega \, |\nabla u_2|_\Omega^2 + c_\Omega \, |u_2|_\Omega^2 \\ & \quad + \langle i\beta u_4 + B u_3 + \alpha B u_4 + c_\Gamma u_4 - f_4, u_2 \rangle_{\Gamma_1} = (f_2, u_2)_\Omega \end{aligned} \tag{4.1.5}$$

Using the first equation, observe that the term $(\nabla u_1, \nabla u_2)_\Omega$ can be rewritten as:

$$\begin{aligned}(\nabla u_1, \nabla u_2)_\Omega &= (\nabla u_1, i\beta \nabla u_1 - \nabla f_1)_\Omega \\ &= -i\beta \left|\nabla u_1\right|_\Omega^2 - (\nabla u_1, \nabla f_1)_\Omega\end{aligned} \tag{4.1.6}$$

Similarly, using the third equation, we have:

$$\begin{aligned}\langle Bu_3, u_2\rangle_{\Gamma_1} &= \left\langle B^{\frac{1}{2}}u_3, B^{\frac{1}{2}}u_4\right\rangle_{\Gamma_1} \\ &= \left\langle B^{\frac{1}{2}}u_3, i\beta B^{\frac{1}{2}}u_3 - B^{\frac{1}{2}}f_3\right\rangle_{\Gamma_1} \\ &= -i\beta \left|B^{\frac{1}{2}}u_3\right|_{\Gamma_1}^2 - \left\langle B^{\frac{1}{2}}u_3, B^{\frac{1}{2}}f_3\right\rangle_{\Gamma_1}\end{aligned} \tag{4.1.7}$$

In (4.1.5), substitute $\langle Bu_3, u_2\rangle_{\Gamma_1}$ and $(\nabla u_1, \nabla u_2)_\Omega$ by (4.1.6) and (4.1.6), respectively:

$$\begin{aligned}&i\beta\left[|u_2|_\Omega^2 + |u_4|_{\Gamma_1}^2 - |\nabla u_1|_\Omega^2 - \left|B^{\frac{1}{2}}u_3\right|_{\Gamma_1}^2\right] + k_\Omega \left|\nabla u_2\right|_\Omega^2 + c_\Omega \left|u_2\right|_\Omega^2 + \alpha \left|B^{\frac{1}{2}}u_4\right|_{\Gamma_1}^2 + c_\Gamma \left|u_4\right|_{\Gamma_1}^2 \\ &\quad = (f_2, u_2)_\Omega + (f_4, u_4)_\Omega + (\nabla u_1, \nabla f_1)_\Omega + \left\langle B^{\frac{1}{2}}u_3, B^{\frac{1}{2}}f_3\right\rangle_{\Gamma_1}\end{aligned} \tag{4.1.8}$$

Taking the real part, one gets:

$$\begin{aligned}&k_\Omega \left|\nabla u_2\right|_\Omega^2 + c_\Omega \left|u_2\right|_\Omega^2 + \alpha \left|B^{\frac{1}{2}}u_4\right|_{\Gamma_1}^2 + c_\Gamma \left|u_4\right|_{\Gamma_1}^2 \\ &\quad \leq \left|\, (f_2, u_2)_\Omega + (f_4, u_4)_\Omega + (\nabla u_1, \nabla f_1)_\Omega + \left\langle B^{\frac{1}{2}}u_3, B^{\frac{1}{2}}f_3\right\rangle_{\Gamma_1} \right| \\ &\quad \leq C \left\|F\right\|_{\mathcal{H}} \left\|U\right\|_{\mathcal{H}} \\ &\quad \leq C \left\|F\right\|_{\mathcal{H}}^2, \text{ by (4.1.3)}\end{aligned} \tag{4.1.9}$$

Using the first two equations in (4.1.4) and applying estimate (4.1.9):

$$\begin{cases} \beta|\nabla u_1|_\Omega & \leq |\nabla u_2|_\Omega + |\nabla f_1|_\Omega \leq \frac{C}{k_\Omega} \left\|F\right\|_{\mathcal{H}} \\ \beta|B^{\frac{1}{2}}u_3|_{\Gamma_1} & \leq |B^{\frac{1}{2}}u_4|_{\Gamma_1} + |B^{\frac{1}{2}}f_3|_{\Gamma_1} \leq \frac{C}{\alpha} \left\|F\right\|_{\mathcal{H}} \end{cases} \tag{4.1.10}$$

Let $K = min\{k_\Omega, \alpha\}$, then, with estimate (4.1.9), (4.1.10) becomes:

$$\beta\left[|\nabla u_1|_\Omega + |B^{\frac{1}{2}}u_3|_{\Gamma_1}\right] \leq \frac{C}{K}\|F\|_{\mathcal{H}} \tag{4.1.11}$$

Note that the presence of K on the denominator forces both damping coefficients (k_Ω, α) to be non-zero.

Back to (4.1.8), multiply by β and take the imaginary part:

$$\begin{aligned}\beta^2\left[|u_2|_\Omega^2 + |u_4|_{\Gamma_1}^2\right] &\leq \beta^2\left[|\nabla u_1|_\Omega^2 + \left|B^{\frac{1}{2}}u_3\right|_{\Gamma_1}^2\right] \\ &+ |f_2|_\Omega|\beta u_2|_\Omega + |\nabla f_1|_\Omega|\beta\nabla u_1|_\Omega + |B^{\frac{1}{2}}f_3|_{\Gamma_1}|\beta B^{\frac{1}{2}}u_3|_{\Gamma_1} + |f_4|_{\Gamma_1}|\beta u_4|_{\Gamma_1}\end{aligned} \tag{4.1.12}$$

Splitting:

$$\begin{cases} |f_2|_\Omega|\beta u_2|_\Omega & \leq C_\epsilon\,|f_2|_\Omega^2 + \epsilon\,|\beta u_2|_\Omega^2 \\ |f_4|_{\Gamma_1}|\beta u_4|_{\Gamma_1} & \leq C_\epsilon\,|f_4|_{\Gamma_1}^2 + \epsilon\,|\beta u_4|_{\Gamma_1}^2 \end{cases} \tag{4.1.13}$$

Using (4.1.10) to estimate the first terms on the right-hand side of (4.1.12):

$$\beta^2\left[|u_2|_\Omega^2 + |u_4|_{\Gamma_1}^2\right] \leq C(\frac{1}{K}+1)\,\|F\|_{\mathcal{H}}^2 \tag{4.1.14}$$

Combine (4.1.11) with (4.1.14)

$$\begin{aligned}|\beta|^2\left[|\nabla u_1|_\Omega^2 + |u_2|_\Omega^2 + \left|B^{\frac{1}{2}}u_3\right|_{\Gamma_1}^2 + |u_4|_{\Gamma_1}^2\right] &= |\beta|^2\,\|R(i\beta - \mathcal{A})\|_{\mathcal{L}(\mathcal{H})}^2 \\ &\leq C(\frac{1}{K}+1)\,\|F\|_{\mathcal{H}}^2\end{aligned} \tag{4.1.15}$$

It follows that $\{e^{\mathcal{A}t}\}_{t\geq 0}$ is analytic on $\mathcal{H}$, provided $k_\Omega, k_\Gamma, \alpha > 0$. □

4.1.2 Perturbation approach via Wentzell semigroup

As we mentionned before, our problem is also related to the Wentzell problem [19, 20, 23]. By transforming our model (**GM**) into a heat equation with General Wentzell

Boundary Condtions, we are able not only to recover the previous result but also to extend it to the space:

$$\mathcal{H}_p = \{(u_1, u_2, u_3, u_4) \in W^{1,p}_{\Gamma_0}(\Omega) \times L^p(\Omega) \times W^{1,p}(\Gamma_1) \times L^p(\Gamma_1), u_1|_{\Gamma_1} = u_3\}$$

for all $1 < p < \infty$. Again, it is necessary to impose some conditions on the damping coefficients. However, this theorem allows us to relax the assumption 4.1.1 from theorem 4.1.2 and thus obtain a stronger result.

Assumption 4.1.3. *The strong damping in the interior is strictly positive ($k_\Omega > 0$) and one of the following holds:*

- $k_\Gamma \alpha > 0$
- $k_\Gamma = 0$

We note that this theorem is more general than 4.1.2.

Theorem 4.1.4. *Suppose that assumption 4.1.3 holds.*
Consider $\mathcal{A}$ given by in (2.2.4), then $\mathcal{A}$ generates an analytic C_0-semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ on $\mathcal{H}_p$, for $1 < p < \infty$.

First of all, we recall our general model (**GM**) with $k_\Omega, \alpha > 0$:

$$\begin{cases} u_{tt} + c_\Omega u_t - k_\Omega \Delta u_t - \Delta u = 0 & x \in \Omega, t > 0 \\ u(x,t) = 0 & x \in \Gamma_0,\ t > 0 \\ u_{tt} + c_\Gamma u_t + \partial_n(u + k_\Omega u_t) - k_\Gamma \boldsymbol{\Delta}_\Gamma(\alpha u_t + u) = 0 & x \in \Gamma_1,\ t > 0 \\ u(0,x) = u_0,\ u_t(0,x) = u_1 & x \in \Omega \end{cases} \qquad (\mathbf{GM})$$

Among the possible scenarii offered by (**GM**) two are parcicularly relevant for this Wentzell approach:

Scenario 1: $k_\Omega > 0, k_\Gamma = 0$ which corresponds to the absence of Laplace Beltrami operator on the boundary. In that case the result has been known (see [28, Theorem 3.5 (ii)] and references therein), however our more general proof will also yield the expected conclusion.

Scenario 2. $k_\Omega, k_\Gamma, \alpha > 0$ corresponds to the presence of Laplace Beltrami operator with additional strong damping on the boundary $\alpha > 0$. It is an alternate proof to the previous theorem.

Our proof is based on connecting the problem under consideration to Wentzell semigroups studied in [19, 20] and a private communication with J. Goldstein [26]. To this end we introduce the new variable:

$$z = u + k_\Omega u_t \text{ on } \Omega \text{ and } \Gamma_1 \tag{4.1.16}$$

We recall from the definition 2.1.3 that $B = -k_\Gamma \mathbf{\Delta}_\Gamma$, then the two first equations of (**GM**) are equivalent to:

$$\begin{cases} u_t + \frac{u}{k_\Omega} = \frac{z}{k_\Omega} & \textbf{in } \Omega \\ u_{t|\Gamma_1} + \frac{u_{|\Gamma_1}}{k_\Omega} = \frac{z_{|\Gamma_1}}{k_\Omega} & \textbf{in } \Gamma_1 \\ \frac{z_t}{k_\Omega} = \Delta z + \frac{z-u}{k_\Omega^2} - c_\Omega \frac{z-u}{k_\Omega} & \textbf{in } \Omega \\ \text{ with } \Delta z + \partial_n z - \frac{k_\Gamma \alpha}{k_\Omega} \mathbf{\Delta}_\Gamma z - k_\Gamma (1 - \frac{\alpha}{k_\Omega}) \mathbf{\Delta}_\Gamma u - c_\Gamma \frac{z-u}{k_\Omega} = 0 & \textbf{in } \Gamma_1 \end{cases} \tag{4.1.17}$$

Note that the first two equations are simply an ordinary differential equation, so we can let

$$U \equiv (u, u|_{\Gamma_1}) \in X_u = \{(u|_\Omega, u|_{\Gamma_1}) \in W^{1,p}_{\Gamma_0}(\Omega) \times W^{1,p}(\Gamma_1), \gamma(u) = u_{|\Gamma_1}\}$$

with the associated norm:

$$\|U\|^2_{X_u} = |\nabla u|^2_\Omega \, L^p(\Omega)^2 + \left\| B^{\frac{1}{2}} u_{\Gamma_1} \right\|^2_{L^p(\Gamma_1)}$$

Thus X_u is a closed subspace of $W^{1,p}(\Omega) \times W^{1,p}(\Gamma_1)$.

Consider the abstract ODE

$$V_t + \frac{1}{k_\Omega} V = 0, V(0) \in X_u$$

and denote by $\mathcal{T}(t)$ the governing semigroup. This semigroup is obviously analytic on X_u and is given by $\mathcal{T}(t)V = e^{-\frac{1}{k_\Omega}t}V$.

$$\begin{aligned}
\|\mathcal{T}(t)U_0\|_{X_u} &= \left\|\mathcal{T}(t)u_{0|\Omega}\right\|_{W^{1,p}_{\Gamma_0}(\Omega)} + \left\|\mathcal{T}(t)u_{0|\Gamma_1}\right\|_{W^{1,p}(\Gamma_1)} \\
&\leq e^{-\frac{1}{k_\Omega}t} \left[\left\|u_{0|\Omega}\right\|_{W^{1,p}_{\Gamma_0}(\Omega)} + \left\|u_{0|\Gamma_1}\right\|_{W^{1,p}(\Gamma_1)} \right] \\
&\leq e^{-\frac{1}{k_\Omega}t} \|U_0\|_{X_u}
\end{aligned}$$

Since $\|V_t(t)\|_{X_u} \leq C \|V_0\|_{X_u}$, then $\mathcal{T}(t)$ is an analytic C_0-semigroup on X_u .

Note that the solution to the first two equations of 4.1.17 can be written as a perturbation:

$$U_t + \frac{1}{k_\Omega} U = \frac{1}{k_\Omega} Z, \quad U(0) = U_0 \qquad \text{where } Z \equiv (z_{|\Omega}, \gamma(z))$$

Thus, by the variation of parameter formula we obtain:

$$U(t) = \mathcal{T}(t)U_0 + \int_0^t \mathcal{T}(t-s)Z(s)ds \tag{4.1.18}$$

The caveat is that the above equation is considered on X_u which means that with $z \in L_p(\Omega)$ one obtains unbounded forcing in (4.1.18). However, the analyticity of the semigroup will help in handling this part.

Before we move back to the system (4.1.17), we need to do some preliminary work on the z-dynamics by presenting the definitions from Favini et al. work in [20, 19]

Definition 4.1.5 (X_p)**.** *Identify every $z \in C(\bar{\Omega})$ with $z = (z|_{\Omega}, z|_{\Gamma_1})$ and define X_p to be the completion of $C(\bar{\Omega})$ in the norm:*

$$|||z|||_p := \left(\int_{\Omega} |u|_{\Omega}|^p \, dx + \int_{\Gamma_1} |u|_{\Gamma_1}|^p \, dS \right)^{\frac{1}{p}} \tag{4.1.19}$$

for $1 < p < \infty$.

Remark 4.1.2. *In general, a member of X_p is $H = (f, g)$, where $f \in L^p(\Omega)$, $g \in L^p(\Gamma_1)$. Note that f may not have a trace on Γ_1, and even if f does, this trace needs not equal g.*

Also, define the formal Laplacian operator $\mathbb{A}$ with General Wentzell Boundary Conditions (GWBC) by:

$$\mathbb{A}u = \sum_{i,j=1}^{N} \partial_i (a_{ij}(x) \partial_j u) \quad \text{in } \Omega \tag{4.1.20}$$

$$\mathbb{A}z + \beta \partial_n z + \eta z - \zeta \beta \boldsymbol{\Delta}_{\Gamma} z = 0 \quad \text{on } \delta\Omega \tag{4.1.21}$$

In [19, Theorem 3.1], the authors showed that the heat equation with GWBC and $\zeta = 0$ was governed by an analytic semigroup in X_p for $p = 2$. Then, in [20], Favini has offered two extensions of this first result:

- Theorem 3.2 [20] states that the heat equation with GWBC is governed by an analytic semigroup in X_p for $p = 2$, i.e., the Laplace-Beltrami term on the boundary is added.

- Theorem 3.3 [20] states that the heat equation with GWBC and $\zeta = 0$ is governed by an analytic semigroup in X_p for $1 < p < \infty$, i.e., the analyticity is no longer restricted to L_2-theory.

Therefore, the use of theorem 3.2 should allow us to recover analyticity in $\mathcal{H}$ as we demonstrated with the resolvent approach (see theorem 4.1.2) relaxing the condition on k_Γ. On the other hand, the use of theorem 3.3 should allow us to achieve analyticity in $\mathcal{H}_p$ under the additional condition: $k_\Gamma = 0$. Furthermore, J. Goldstein and M. Pierre are about to publish the following result in [26], which is an extention of [20, Theorem 3.3]:

Theorem D (Goldstein - Pierre [26])**.** *The closure G_p of the realization $\mathbb{A}_p$ of $\mathbb{A}$ in X_p with domain*

$$\mathcal{D}(\mathbb{A}_p) = \{z = (z|_\Omega, z|_{\delta\Omega}) \in \mathcal{D}(\mathbb{A}_2) \cap X_p \ z|_{\delta\Omega} \textit{ satisfies } (4.1.21)\}$$

is analytic on X_p for $1 < p < \infty$.

Set $\beta = k_\Omega,\ \eta = (c_\Gamma - c_\Omega),\ \zeta\beta = k_\Gamma\alpha$ and rewrite (4.1.17) using the operator $\mathbb{A} = k_\Omega\Delta$:

$$\begin{cases} u_t + \frac{u}{k_\Omega} = \frac{z}{k_\Omega} & \textbf{in } \Omega \\ \gamma(u_t) + \gamma(\frac{u}{k_\Omega}) = \gamma(\frac{z}{k_\Omega}) & \textbf{in } \Gamma_1 \\ z_t = \mathbb{A}z + (\frac{1}{k_\Omega} - c_\Omega)(z-u) & \textbf{in } \Omega \\ \quad \text{with } \mathbb{A}z + k_\Omega\partial_n z + (c_\Gamma - c_\Omega)z - k_\Gamma\alpha\mathbf{\Delta}_\Gamma z = k_\Gamma(k_\Omega - \alpha)\mathbf{\Delta}_\Gamma u + (c_\Gamma - c_\Omega)u & \textbf{in } \Gamma_1 \end{cases} \tag{4.1.22}$$

Observe that the two last equations in (4.1.22) corresponds to the heat equation (4.1.20) with General Wentzell Boundary Conditions (4.1.21) perturbed on Ω by $\frac{z}{k_\Omega} - \frac{u}{k_\Omega}$ and on Γ_1 by $-k_\Gamma(k_\Omega - \alpha)\mathbf{\Delta}_\Gamma u|_{\Gamma_1} - (c_\Gamma - c_\Omega)u|_{\Gamma_1}$.

Remark 4.1.3. *Note that the fourth equation can be rewritten as an evolution equation in $\gamma(z)$:*

$$z_t = -k_\Omega\partial_n z + k_\Gamma\alpha\mathbf{\Delta}_\Gamma z + (\frac{1}{k_\Omega} - c_\Gamma)(z-u) + k_\Gamma(k_\Omega - \alpha)\mathbf{\Delta}_\Gamma u \tag{4.1.23}$$

This way of writing makes a connection with evolution equations governed by dynamic boundary conditions.

We define the space $\mathcal{H}_{u,z}$ by the cross product between X_u and X_p.

Remark 4.1.4. *For $p = 2$, the space $\mathcal{H}_{u,z}$ and $\mathcal{H}$ are equivalent after reordering the components.*

To show the analyticity of (4.1.22) on $\mathcal{H}_{u,z} = X_u \times X_p$, we will proceed in two steps: first, we show that the system without the Laplace-Beltrami perturbed terms from Γ_1 is analytic. Note that lemma 4.1.6 corresponds to the case where the strong damping in the interior and the boundary are matching ($k_\Gamma \alpha = k_\Omega$). Then, we will show that the analyticity with the perturbation on Γ_1 is preserved (Lemma 4.1.7)

The system given in (4.1.22) can be written in a compact form as

$$\begin{cases} U_t + \frac{1}{k_\Omega} U &= \frac{1}{k_\Omega} Z \\ Z_t + G_p Z &= K(Z-U) + P(\gamma(u)) \end{cases} \tag{4.1.24}$$

where $P(\gamma(u)) \equiv \begin{pmatrix} 0 \\ k_\Gamma(k_\Omega - \alpha)\mathbf{\Delta}_\Gamma \gamma(u) \end{pmatrix}$ and $K = \begin{pmatrix} (\frac{1}{k_\Omega} - c_\Omega) & 0 \\ 0 & (\frac{1}{k_\Omega} - c_\Gamma) \end{pmatrix}$

We begin the analysis with unperturbed system. We shall show that the associated semigroup inherits analyticity properties from Wentzell semigroup.

Lemma 4.1.6. *Given the system:*

$$\begin{cases} u_t + \frac{u}{k_\Omega} = \frac{z}{k_\Omega} & \textbf{\textit{in}}\ \Omega \\ \gamma(u_t) + \gamma(\frac{u}{k_\Omega}) = \gamma(\frac{z}{k_\Omega}) & \textbf{\textit{in}}\ \Gamma_1 \\ z_t = \mathbb{A}z + (\frac{1}{k_\Omega} - c_\Omega)(z-u) & \textbf{\textit{in}}\ \Omega \\ \quad \textit{with}\ \mathbb{A}z + k_\Omega \partial_n z - k_\Gamma \alpha \mathbf{\Delta}_\Gamma z = (c_\Gamma - c_\Omega)u & \textbf{\textit{in}}\ \Gamma_1 \end{cases} \tag{4.1.25}$$

where $(u|\Omega, u|_{\Gamma_1}, z|_\Omega, z|_{\Gamma_1})^T \in H^1_{\Gamma_0}(\Omega) \times \mathcal{D}(B^{1/2}) \times \mathcal{D}(G_p^{\frac{1}{2}})$. *Then the following inequality holds,*

$$\|U_t(t)\|_{X_u} + \|Z_t(t)\|_{X_p} \leq C_0 \|U_0\|_{X_u} + \frac{C_1}{t} \|Z_0\|_{X_p} \tag{4.1.26}$$

It follows that the semigroup associated generates an analytic C_0-semigroup on $\mathcal{H}_{u,z}$.

Proof. By theorem (D), the domain $\mathcal{D}(G_p^{\frac{1}{2}})$ is contained in $W^{1,p}_{\Gamma_0}(\Omega) \times W^{1-\frac{1}{p},p}(\Gamma_1)$ and we have the following relationship, using interpolation theorem:

$$\|Z(t)\|^p_{X_u} = \|z(t)\|^p_{W^{1,p}(\Omega)} \leq \left\| G_p^{\frac{1}{2}} Z(t) \right\|^p_{X_p} \leq \left(\frac{1}{\sqrt{t}} \|Z_0\|_{X_p} \right)^p \tag{4.1.27}$$

By the variation of paramters, the first two equations of (4.1.25) have a solution $U(t)$ in X_u such that:

$$\begin{aligned} \|U(t)\|_{X_u} &\leq e^{-\frac{1}{k_\Omega}t} \|U_0\|_{X_u} + \int_0^t e^{-\frac{1}{k_\Omega}(t-s)} \|Z(s)\|_{X_u} \, ds \\ &\leq C \|U_0\|_{X_u} + C \int_0^t e^{-\frac{1}{k_\Omega}(t-s)} \left\| G_2^{\frac{1}{2}} Z(s) \right\|_{X_p} ds \\ &\leq C \|U_0\|_{X_u} + C \int_0^t e^{-\frac{1}{k_\Omega}(t-s)} \frac{1}{\sqrt{s}} \|Z_0\|_{X_p} \, ds \\ &\leq C \left[\|U_0\|_{X_u} + \|Z_0\|_{X_p} \right] \quad \text{since } e^{-\frac{1}{k_\Omega}(t-s)} \frac{1}{\sqrt{s}} \in L^1(0,T) \end{aligned} \tag{4.1.28}$$

Note that the abstract form of the system (4.1.25) is defined in (4.1.24) where the term $P(\gamma(u))$ is omitted. Then the desired estimate is obtained by combining (4.1.27) and (4.1.28):

$$\begin{aligned} \|U_t(t)\|_{X_u} + \|Z_t(t)\|_{X_p} &\leq \frac{1}{k_\Omega} \|Z(t) - U(t)\|_{X_u} + \|K(Z(t) - U(t))\|_{X_p} + \|G_2 Z(t)\|_{X_p} \\ &\leq C \left[\|U(t)\|_{X_u} + \|Z(t)\|_{X_u} \right] + \|G_2 Z(t)\|_{X_p} \\ &\leq C \left[\|U_0\|_{X_u} + (1 + \frac{1}{\sqrt{t}}) \|Z_0\|_{X_p} \right] + \frac{1}{t} \|Z_0\|_{X_p} \end{aligned} \tag{4.1.29}$$

which completes the proof of equation (4.1.26), implying analyticity of the corresponding semigroup on $\mathcal{H}_{u,z}$. □

Let $\mathbb{S}$ be the generator of the analytic semigroup governing model (4.1.25) on the space $\mathcal{H}_{u,z} = X_u \times X_p$ with the domain:

$$\mathcal{D}(\mathbb{S}) = \{V = (u|_\Omega, u|_{\Gamma_1}, z|_\Omega, z|_{\Gamma_1})^T \in W^{1,p}_{\Gamma_0}(\Omega) \times W^{1,p}(\Gamma_1) \times \mathcal{D}(G_p^{\frac{1}{2}})\}$$

Then the original system (4.1.22) with the fourth equation replaced by (4.1.23) is equivalent to:

$$V_t = (\mathbb{S} + \mathbb{P})V \tag{4.1.30}$$

where

$$\mathbb{S} = \begin{pmatrix} -J & J \\ K & K+L \end{pmatrix} \text{ with } J = \frac{1}{k_\Omega}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ L = \begin{pmatrix} G_2 & 0 \\ -k_\Omega \partial_n & k_\Gamma \alpha \boldsymbol{\Delta}_\Gamma \end{pmatrix}$$

$$\mathbb{P} = \begin{pmatrix} 0_{2\times2} & 0_{2\times2} \\ M & 0_{2\times2} \end{pmatrix} \text{ with } M = \begin{pmatrix} 0 & 0 \\ 0 & k_\Gamma(k_\Omega - \alpha)\boldsymbol{\Delta}_\Gamma \end{pmatrix}$$

Using a perturbation argument, it remains to show that the semigroup generaetd by $\mathbb{S} + \mathbb{P}$ is analytic, which is the purpose of the following lemma.

Lemma 4.1.7. *The semigroup generated by $\mathbb{S} + \mathbb{P}$ is analytic on $\mathcal{H}_p$.*

Proof. Without loss of generality, assume that $k_\Gamma = 0$, otherwise $\mathbb{P} = 0$.

Let $V = (U, Z) = (u|_\Omega, u|_{\Gamma_1}, z|_\Omega, z|_{\Gamma_1}) \in W^{1,p}_{\Gamma_0}(\Omega) \times W^{1,p}(\Gamma_1) \times X_p$, with $V_0 = (u_{0|\Omega}, u_{0|\Gamma_1}, z_{0|\Omega}, z_{0|\Gamma_1})$ be solution of (4.1.30), then

$$V(t) = e^{\mathbb{S}t}V_0 + \int_0^t e^{\mathbb{S}(t-\tau)}\mathbb{P}V(\tau)d\tau \tag{4.1.31}$$

Taking the Laplace Transform

$$V(\lambda) = R(\lambda, \mathbb{S})V_0 + R(\lambda, \mathbb{S})\mathbb{P}V(\lambda) \tag{4.1.32}$$

leads to the following relation:

$$[I - R(\lambda, \mathbb{S})\mathbb{P}]\, V(\lambda) = R(\lambda, \mathbb{S})V_0 \tag{4.1.33}$$

Thus, $\mathbb{S} + \mathbb{P}$ generates an analytic C_0-semigroup if and only if $[I - R(\lambda, \mathbb{S})\mathbb{P}]$ is invertible. This last statement follows from:

$$V(\lambda) = [I - R(\lambda, \mathbb{S})\mathbb{P}]^{-1} R(\lambda, \mathbb{S})V(0) \tag{4.1.34}$$

and the estimate:

$$\|R(\lambda, \mathbb{S})\|_{\mathcal{L}(\mathcal{H}_{u,z})} \le C|\lambda|^{-1} \tag{4.1.35}$$

Thus it is enough to check $\|R(\lambda, \mathbb{S})\mathbb{P}V\|_{\mathcal{H}_{u,z}} \le \frac{1}{2}$, for $|\lambda|$ large:

$$\begin{aligned} \|R(\lambda, \mathbb{S})\mathbb{P}V\|_{\mathcal{H}_{u,z}} &= \left\|R(\lambda, \mathbb{S})\mathbb{S}^{\frac{1}{2}}\mathbb{S}^{-\frac{1}{2}}\mathbb{P}V\right\|_{\mathcal{H}_{u,z}} \\ &\le \left\|\mathbb{S}^{\frac{1}{2}} R(\lambda, \mathbb{S})\right\|_{\mathcal{L}(\mathcal{H}_p)} \left\|\mathbb{S}^{-\frac{1}{2}}\mathbb{P}V\right\|_{\mathcal{H}_{u,z}} \\ &\le \frac{1}{\sqrt{\lambda}} \left\|\mathbb{S}^{-\frac{1}{2}}\mathbb{P}V\right\|_{\mathcal{H}_{u,z}} \end{aligned} \tag{4.1.36}$$

since $\mathbb{S}$ generates an analytic semigroup on $\mathcal{H}_{u,z}$.

It remains to bound the second term: $\left\|\mathbb{S}^{-\frac{1}{2}}\mathbb{P}V\right\|_{\mathcal{H}_{u,z}}$.

For all $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in \mathcal{D}(\mathbb{S}^{\frac{1}{2}}) \subset W^{1,p}_{\Gamma_0}(\Omega) \times W^{1,p}(\Gamma_1) \times \mathcal{D}(G_p^{\frac{1}{2}})$, define $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) = [\mathbb{S}^{-\frac{1}{2}}]^*\phi \in \mathcal{D}(\mathbb{S}^{\frac{1}{2}}) \subset W^{1,p}_{\Gamma_0}(\Omega) \times W^{1,p}(\Gamma_1) \times \mathcal{D}(G_p^{\frac{1}{2}})$, It follows:

$$\left\|\mathbb{S}^{\frac{1}{2}}\varphi\right\|_{\mathcal{H}_{u,z}} \le C \|\phi\|_{\mathcal{H}_{u,z}}$$

Then, it remains to estimate $\left(\mathbb{S}^{-\frac{1}{2}}\mathbb{P}V, \phi\right)_{\mathcal{H}_{u,z}}$:

$$\begin{aligned}\left(\mathbb{S}^{-\frac{1}{2}}\mathbb{P}V, \phi\right)_{\mathcal{H}_{u,z}} &= (\mathbb{P}V, \varphi_4)_{\mathcal{H}_{u,z}} \\ &\leq (k_\Gamma(k_\Omega - \alpha)\nabla_\Gamma u|_{\Gamma_1}, \nabla_\Gamma \varphi_4)_{L^p(\Gamma_1)} \\ &\leq C \, \|u|_{\Gamma_1}\|_{W^{1,p}(\Gamma_1)} \, \|\varphi_4\|_{W^{1,p}(\Gamma_1)} \\ &\leq C \, \|U\|_{\mathcal{H}_{u,z}} \, \|\phi\|_{\mathcal{H}_{u,z}}\end{aligned} \tag{4.1.37}$$

By choosing $\lambda > r^2$ with r sufficiently large:

$$\|R(\lambda, \mathbb{S})\mathbb{P}V\|_{\mathcal{H}_{u,z}} \leq \frac{C}{r} \|V\|_{\mathcal{H}_{u,z}} \Rightarrow \|[I - R(\lambda, \mathbb{S})\mathbb{P}]\|_{\mathcal{L}(\mathcal{H}_{u,z})} \geq \frac{1}{2}$$

□

Therefore, $\mathcal{A}$ generates an analytic semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ on $\mathcal{H}_p$, provided one of the following condition:

- $k_\Omega, \alpha > 0$
- No additional condition on $k_\Gamma, c_\Omega, c_\Gamma$, i.e., they are all non-negative

We conclude this section by illustrating these results with the 4.1 [p. 91] representing the spectrum of (**GM**) with $k_\Omega > 0$ and all other coefficients set to zero ($c_\Omega = c_\Gamma = k_\Gamma = \alpha = 0$). We observe that the eigenvalues describe a circle centered at the point X such that $ReX = -1/k_\Omega$ similarly to the 1.3 [p. 19] corresponding to a strong damped wave equation with Dirichlet boundary. We also recall 1.4 [p. 20] showing the spectrum of (**GM**) with $k_\Omega, k_\Gamma, \alpha > 0$ as the values of k_Ω and k_Γ change. For both cases eigenvalues obey to the same pattern. Hence, we can in these two cases enclosed the eigenvalues into a triangular sector confirming the analyticity of the semigroup governing these systems.

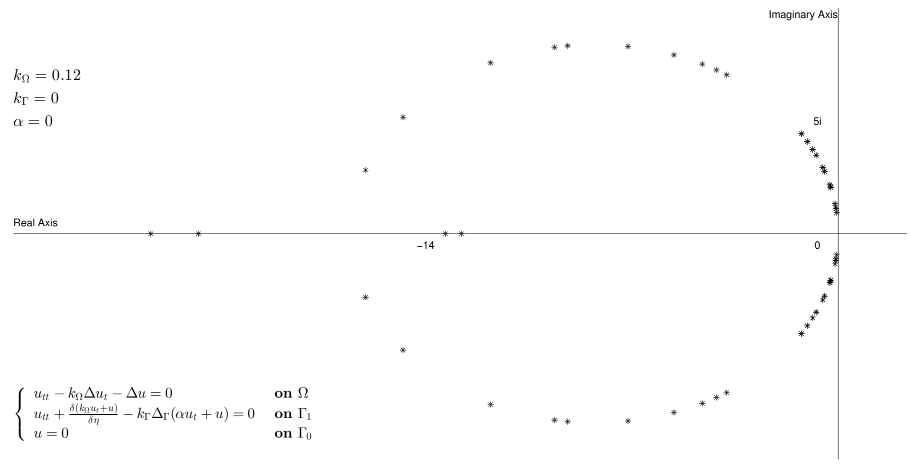


Figure 4.1: *Spectrum of a strongly damped wave equation with DBC without Laplace Beltrami term on the boundary..* Eigenvalues of (**VM**) with only interior damping ($k_\Gamma, k_\Omega > 0$ and $\alpha = 0$).

4.2 Gevrey Semigroup

Now the analyticity is established for (**GM**) under some conditions, it is a primary concern to determine how the regularity of the system is affected when we relax the conditions (assumption 4.1.3). We start by keeping the strong damping in the interior, i.e. $k_\Omega > 0$ but we now impose $\alpha = 0$. Therefore, is analyticity achieved with strong damping *only* in the interior ?

Without loss of generality, we assume that $k_\Gamma > 0$, otherwise the semigroup is analytic by the previous section. The first important observation is that the two proofs presented in the previous section failed to demonstrate the analyticity in such a case since there is no control of the elastic energy $\boldsymbol{\Delta}_\Gamma u$ on Γ_1. Indeed, with the resolvent approach the absence of strong damping decimate the control of $\beta \left|u_4\right|^2_{\Gamma_1}$ (see equation (4.1.13)). As a Wentzell problem, the analyticity also failed as the absence of $\boldsymbol{\Delta}_\Gamma u_t$ terms on the boundary implies that we can not operate to the change of variable for the term $\boldsymbol{\Delta}_\Gamma u$. It follows that after the change of variable, the corresponding heat equation with general Wentzell boundary condition is of the form:

$$\begin{aligned} &z_t = \mathbb{A}z + (\frac{1}{k_\Omega} - c_\Omega)(z-u) \ \ \textbf{in } \Omega \\ &\qquad \text{with } \mathbb{A}z + k_\Omega \partial_n z + (c_\Gamma - c_\Omega)z = k_\Gamma k_\Omega \boldsymbol{\Delta}_\Gamma u + (c_\Gamma - c_\Omega)u \ \ \textbf{in } \Gamma_1 \end{aligned} \tag{4.2.1}$$

On the left hand side, we observe that there is no more Laplace-Beltrami term, thus, preventing from the control of $\boldsymbol{\Delta}_\Gamma u$ on the right hand side. We will only partially answer to the initial question by claiming the expected result based on the observation of the numerical spectrum, see figure 4.2.

We observe that the absence of damping on the boundary has a non-negligeable impact on the spectrum of (**GM**), see figures 1.4 [p. 20], 4.1 [p. 91] for the comparaison. Indeed, the circle shape is still present and corresponds to the analytic behavior of

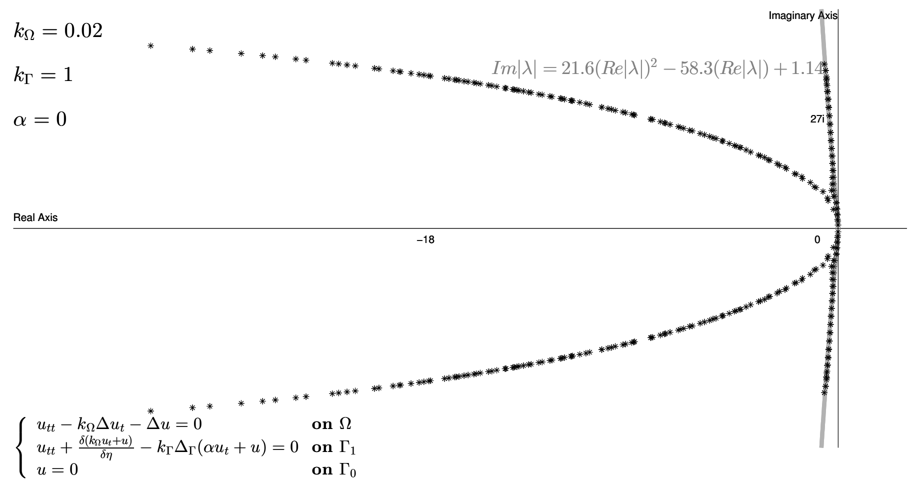


Figure 4.2: *Spectrum for a Gevrey semigroup.* Eigenvalues of $(\mathbf{VM})$ with $k_\Omega > 0$ and $k_\Gamma, \alpha = 0$.

the interior, however, a new parabolic-like component appears and indicate that the semigroup governing this partiuclar system is not analytic anymore. We point out that the parabola is of the form $|Im\lambda| = a - b \times |Re\lambda|^2$. Nevertheless, this component shows that the system keep some smoothing property from paraboliticy. We expect the semigroup to be in a class between differentiability and analyticity, and related to the theory of Gevrey Semigroups [54]. As for characterization of analyticity, Gevrey's regularity is described in terms of the bounds on all derivatives of the semigroup:

Definition 4.2.1 (Gevrey Semigroup)**.** *Let $T(t)$ be a strongly continuous semigroup on a Banach space X and let $\delta > 1$. We say that $T(t)$ is of Gevrey class δ for $t > t_0$*

if $T(t)$ is infinitely differentiable for $t \in (t_0, \infty)$ and for every compact $\mathcal{K} \subset (t_0, \infty)$, and each $\theta > 0$, there exists a constant $C = C_{\theta,\mathcal{K}}$ such that

$$\left\| T^{(n)}(t) \right\| \leq C\theta^n (n!)^\delta, \ \forall t \in \mathcal{K} \text{ and } n \in \{0, 1, 2, ...\}$$

Besides other characterizations, the author provided some sufficient conditions for semigroups to be of Gevrey class.

Theorem E (Taylor's Dissertation: [54])**.** *Let $T(t)$ be a strongly continuous semigroup satisfying $\|T(t)\| \leq Me^{\omega t}$. Assume one of the following holds:*

- *Suppose that for some γ satisfying $0 < \gamma \leq 1$:*

$$\limsup_{\beta \to \infty} |\beta|^\gamma \|R(i\beta; \mathcal{A})\| = C < \infty \tag{4.2.2}$$

- *Suppose that:*

$$\lim_{t \downarrow 0} t^\delta \|T'(t)\| = 0 \tag{4.2.3}$$

Then $T(t)$ is of Gevrey class δ for $t > 0$ (for every $\delta > \frac{1}{\gamma}$ for the first item).

The resolvent characterization relies on the contour shape formed by the eigenvalues whose imaginary parts tend to infinity. More precisely, the eigenvalues describ of polynomial-type curve for which the degree of the polynomial is δ. Back to our problem, we identify on 4.2 [p. 93] such a parabolic component which suggests that the semigroup should of Gevrey class $\delta = 2$. This remark has not yet been proved, but the picture suggests further investigation in this direction should lead to a positive conclusion.

It is also important to note from Theorem E that for $\gamma = 1$ the semigroup is analytic. This class of semigroup offers an intermediate level between analytic semigroup and differentiable semigroup.

Remark 4.2.1. *While a differentiable semigroup is not stable under bounded perturbation, as it is demonstrated by Renardy in [51], if $B \in \mathcal{H}$ and one of the characterizations from theorem E then not only $\mathcal{A}$ is Gevrey but also $\mathcal{A} + B$. This can be seen by using a similar argument as for perturbation of analytic semigroup by bounded operator (see [49, Theorem III.3.2.1]).*

One could summarize the connection between these different classes of semigroup with the following tree:

$$Analytic \to Theorem\ E \to Gevrey \to Differentiable$$

S. Taylor and W. Littman used Gevrey's regularity to study smoothing properties in the context of plate and Schrodinger equations [41, 43, 55]. We also refer the reader to [32, 33, 38, 43] for other applications of Gevrey regularity.

The semigroup $\mathcal{A}$ defined with $k_\Omega, k_\Gamma > 0, \alpha = 0$ is not analytic but numerics suggest that it is Gevrey, thus we leave the following open question to the reader: is the semigroup $\mathcal{A}$ defined with $k_\Omega, k_\Gamma > 0, \alpha = 0$ Gevrey ?

We do not go further in the investigation of regularity for this model. We will come back to the study of (**GM**) in the absence of viscoelastic damping in the interior ($k_\Omega = 0$), in chapter 5. In fact, without interior damping no regularity result is expected but more interestingly the long-time behavior of the solution is greatly affected as we shall see soon (see figure 5.2 [p. 120]).

4.3 Spectral analysis

As it was observed on all the figures presented so far, our model (**GM**) seems to have an important spectral property: the absence of spectrum on the imaginary axis. We shall investigate the conditions under which the property does hold. A classical strategy in demonstrating this strategy is to verify that the intersection of the imaginary axis with each spectrum is empty, i.e., the point spectrum $\sigma_p(\mathcal{A})$, the continuous spectrum $\sigma_c(\mathcal{A})$ and the residual spectrum $\sigma_r(\mathcal{A})$. We refer the reader to the paper [4] for similar technics. Furthermore, the verification for the continuous and residual spectrums can be achieved using the following properties.

- *If $\lambda \in \sigma_c(\mathcal{A})$, then $\mathcal{A} - \lambda$ does not have a closed range.* [22, Problem 2.54 p. 128]

- *If the eigenvalue $\lambda \in \mathbb{C}$ is in the residual spectrum of $\mathcal{A}$, then $\lambda \in \sigma_p(\mathcal{A}^*)$*[22, p. 127]:

Thus, before moving to the main result of this section, it is necessary to compute the adjoint of the operator $\mathcal{A}$ in order to verify the property for the residual spectrum. We will use the formal characterization the Laplacian which we recall: $-\Delta u = A(I - N\partial_n)u$ (see remark 2.2.1) and we display again for reference the operator $\mathcal{A}$ defined in (2.2.4) taking account of this Laplacian characterization.

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ -A(I - N\partial_n) & D_\Omega & 0 & 0 \\ 0 & 0 & 0 & I \\ -\partial_n & -k_\Omega \partial_n & -B & D_{\Gamma_1} \end{pmatrix}$$

$$\text{where} \begin{cases} D_\Omega & = -k_\Omega A(I - N\partial_n) - c_\Omega I \\ D_{\Gamma_1} & = -\alpha B - c_\Gamma I \end{cases}$$

Lemma 4.3.1 (Adjoint of $\mathcal{A}$)**.** *With $\mathcal{A}$ as defined in* (2.2.4) *and using the notation $-\Delta u = A(I - N\partial_n)u$, the adjoint $\mathcal{A}^*$ is given to be*

$$\mathcal{A}^* = \begin{pmatrix} 0 & -I & 0 & 0 \\ A(I - N\partial_n) & D_\Omega & 0 & 0 \\ 0 & 0 & 0 & -I \\ \partial_n & -k_\Omega \partial_n & B & D_{\Gamma_1} \end{pmatrix} \tag{4.3.1}$$

$$\textit{where} \begin{cases} D_\Omega & = -k_\Omega A(I - N\partial_n) - c_\Omega I \\ D_{\Gamma_1} & = -\alpha B - c_\Gamma I \end{cases}$$

$$\mathcal{D}(\mathcal{A}^*) = \{[u_1, u_2, u_3, u_4]^T \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B^{\frac{1}{2}}) \times \mathcal{D}(B^{\frac{1}{2}}),$$

$$\textit{such that } A(I - N\partial_n)(u_1 - k_\Omega u_2) - c_\Omega u_2 \in L^2(\Omega),$$

$$\partial_n(u_1 - k_\Omega u_2) + B^{\frac{1}{2}}(B^{\frac{1}{2}}u_3 - \alpha B^{\frac{1}{2}}u_4) - c_\Gamma u_4 \in L^2(\Gamma_1),$$

$$u_1|_{\Gamma_1} = N^*Au_1 = u_3 \textit{ and } u_2|_{\Gamma_1} = N^*Au_2 = u_4\}$$

Proof. Let

$$S = \{\{[y_1, y_2, y_3, y_4]^T \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B^{\frac{1}{2}}) \times \mathcal{D}(B^{\frac{1}{2}}),$$

$$\text{such that } A(I - N\partial_n)(y_1 - k_\Omega y_2) - c_\Omega y_2 \in L^2(\Omega),$$

$$\partial_n(y_1 - k_\Omega y_2) + B^{\frac{1}{2}}(B^{\frac{1}{2}}y_3 - \alpha B^{\frac{1}{2}}y_4) - c_\Gamma y_4 \in L^2(\Gamma_1),$$

$$y_1|_{\Gamma_1} = N^*Ay_1 = yu_3 \text{ and } y_2|_{\Gamma_1} = N^*Ay_2 = y_4\}$$

Step 1: Show that S is a subset of $\mathcal{D}(\mathcal{A}^*)$ and that there exists Λ such that $\mathcal{A}^*|_S = \Lambda$

Then $\forall U \in \mathcal{D}(\mathcal{A}),\ Y = [y_1, y_2, y_3, y_4]^T \in S$:

$$\begin{aligned}
&(\mathcal{A}U, Y)_{\mathcal{H}\subset\mathcal{D}(A^{\frac{1}{2}})\times L^2(\Omega)\times\mathcal{D}(B^{\frac{1}{2}})\times L^2(\Gamma_1)} = \\
&\left(A^{\frac{1}{2}}u_2, A^{\frac{1}{2}}y_1\right)_{L^2(\Omega)} - (A(u_1 - N\partial_n u_1), y_2)_{[\mathcal{D}(A^{\frac{1}{2}})]^*\times\mathcal{D}(A^{\frac{1}{2}})} \\
&- k_\Omega\,(A(u_2 - N\partial_n u_2), y_2)_{[\mathcal{D}(A^{\frac{1}{2}})]^*\times\mathcal{D}(A^{\frac{1}{2}})} - c_\Omega\,(u_2, y_2)_{L^2(\Omega)} \\
&+ \alpha\left(B^{\frac{1}{2}}u_4, B^{\frac{1}{2}}y_3\right)_{L^2(\Gamma_1)} - (\partial_n u_1, y_4)_{L^2(\Gamma_1)} - k_\Omega\,(\partial_n u_2, y_4)_{L^2(\Gamma_1)} \\
&- (Bu_3, y_4)_{[\mathcal{D}(B^{\frac{1}{2}})]^*\times\mathcal{D}(B^{\frac{1}{2}})} - \alpha\,(Bu_4, y_4)_{[\mathcal{D}(B^{\frac{1}{2}})]^*\times\mathcal{D}(B^{\frac{1}{2}})} - c_\Gamma\,(u_4, y_4)_{L^2(\Gamma_1)}
\end{aligned} \tag{4.3.2}$$

The goal is to identify an operator Λ such that $(\mathcal{A}U, Y)_{\mathcal{H}} = (U, \Lambda Y)_{\mathcal{H}}$, in other words, we reconstruct $U = (A^{\frac{1}{2}}u_1, u_2, B^{\frac{1}{2}}u_3, u_4)^T$ with the first component of the inner products in (4.3.2).

$$
\begin{aligned}
(\mathcal{A}U,Y)_{\mathcal{H}} =& (u_2, A(y_1 - N\partial_n y_1))_{\mathcal{D}(A^{\frac{1}{2}})\times[\mathcal{D}(A^{\frac{1}{2}})]^*} + (u_4, \partial_n y_1)_{L^2(\Gamma_1)} \\
& - \left(A^{\frac{1}{2}}u_1, A^{\frac{1}{2}}y_2\right)_{L^2(\Omega)} + (\partial_n u_1, N^* A y_2)_{L^2(\Gamma_1)} \\
& - k_\Omega \left(A^{\frac{1}{2}}u_2, A^{\frac{1}{2}}y_2\right)_{L^2(\Omega)} + k_\Omega (\partial_n u_2, N^* A y_4)_{L^2(\Gamma_1)} - c_\Omega (u_2, y_2)_{L^2(\Omega)} \\
& + (u_4, By_3)_{\mathcal{D}(B^{\frac{1}{2}})\times[\mathcal{D}(B^{\frac{1}{2}})]^*} - (\partial_n u_1, y_4)_{L^2(\Gamma_1)} - k_\Omega (\partial_n u_2, y_4)_{L^2(\Gamma_1)} \\
& - \left(B^{\frac{1}{2}}u_3, B^{\frac{1}{2}}y_4\right)_{L^2(\Gamma_1)} - \alpha (u_4, By_4)_{[\mathcal{D}(B^{\frac{1}{2}})]^*\times\mathcal{D}(B^{\frac{1}{2}})} - c_\Gamma (u_4, y_4)_{L^2(\Gamma_1)} \\
=& (u_2, A(y_1 - N\partial_n y_1))_{\mathcal{D}(A^{\frac{1}{2}})\times[\mathcal{D}(A^{\frac{1}{2}})]^*} - \left(A^{\frac{1}{2}}u_1, A^{\frac{1}{2}}y_2\right)_{L^2(\Omega)} \\
& - k_\Omega (u_2, A(y_2 - N\partial_n y_2))_{\mathcal{D}(A^{\frac{1}{2}})\times[\mathcal{D}(A^{\frac{1}{2}})]^*} - c_\Omega (u_2, y_2)_{L^2(\Omega)} \\
& + (u_4, \partial_n y_1)_{L^2(\Gamma_1)} - k_\Omega (u_2, \partial_n y_2)_{L^2(\Gamma_1)} + (u_4, By_3)_{\mathcal{D}(B^{\frac{1}{2}})\times[\mathcal{D}(B^{\frac{1}{2}})]^*} \\
& - \left(B^{\frac{1}{2}}u_3, B^{\frac{1}{2}}y_4\right)_{L^2(\Gamma_1)} - \alpha (u_4, By_4)_{\mathcal{D}(B^{\frac{1}{2}})\times[\mathcal{D}(B^{\frac{1}{2}})]^*} - c_\Gamma (u_4, y_4)_{L^2(\Gamma_1)} \\
=& (u, \Lambda y)_{\mathcal{H}}
\end{aligned}
\tag{4.3.3}
$$

where Λ is defined in its matrix form by:

$$
\Lambda = \begin{pmatrix} 0 & -I & 0 & 0 \\ A(I - N\partial_n) & D_\Omega & 0 & 0 \\ 0 & 0 & 0 & -I \\ \partial_n & -k_\Omega\partial_n & B & D_{\Gamma_1} \end{pmatrix}
$$

$$
\text{with} \begin{cases} D_\Omega &= -k_\Omega A(I - N\partial_n) - c_\Omega I \\ D_{\Gamma_1} &= -\alpha B - c_\Gamma I \end{cases}
$$

Therefore, $S \subseteq \mathcal{D}(\mathcal{A}^*)$ and $\mathcal{A}^*|_S = \Lambda$

Step 2: Show that $\mathcal{D}(\mathcal{A}^*) \subseteq S$.

Suppose that $\lambda \in \mathbb{C}$ with $Re\lambda > 0$, to show the opposite containment, it is enough to verify that $\forall F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$ there exists $Y = (y_1, y_2, y_3, y_4)^T \in \mathcal{D}(\mathcal{A}^*)$ such that

$$\lambda Y - \mathcal{A}^* Y = F \in \mathcal{H} \tag{4.3.4}$$

$$\begin{cases} \lambda y_1 + y_2 = f_1 \\ \lambda y_2 - A(I - N\partial_n)y_1 + k_\Omega A(I - N\partial_n)y_2 + c_\Omega y_2 = f_2 \\ \lambda y_3 + y_4 = f_3 \\ \lambda y_4 - \partial_n y_1 + k_\Omega \partial_n y_2 - By_3 + \alpha B y_4 + c_\Gamma y_4 = f_4 \end{cases} \tag{4.3.5}$$

Use the first and third equation to operate change of variables for y_2 and y_4:

$$\begin{cases} y_2 = f_1 - \lambda y_1 \\ y_4 = f_3 - \lambda y_3 \\ -\lambda^2 y_1 - A(I - N\partial_n)y_1 - \lambda k_\Omega A(I - N\partial_n)y_1 - \lambda c_\Omega y_1 \\ \quad = f_2 - [\lambda + k_\Omega A(I - N\partial_n) + c_\Omega]\, f_1 \\ -\lambda^2 y_3 - \partial_n y_1 - \lambda k_\Omega \partial_n y_1 - By_3 - \alpha B y_3 - c_\Gamma y_3 \\ \quad = f_4 - k_\Omega \partial_n f_1 - (\lambda + \alpha B + c_\Gamma) f_3 \end{cases} \tag{4.3.6}$$

To solve the stationnary problem (4.3.6), we shall use a weak formulation and Lax-Milgram theorem. Let $(v_1, v_2, v_3, v_4)^T \in \mathcal{D}(\mathcal{A}^*)$ and for the time being let $F = (f_1, f_2, f_3, f_4) \in \mathcal{D}(\mathcal{A}^*)$. Later we shall extend the argument by density to all $F \in \mathcal{H}$.

We consider the two last equations of (4.3.6), multiply them by $\overline{v_1}$ and $\overline{v_3}$, respectively:

$$\begin{aligned}
&-\left(\lambda^2 y_1, v_1\right)_\Omega - (1+\lambda k_\Omega)\left(A(I-N\partial_n)y_1, v_1\right)_\Omega - \lambda c_\Omega \left(y_1, v_1\right)_\Omega \\
&\quad - \left\langle \lambda^2 y_3, v_3\right\rangle_{\Gamma_1} - (1+\lambda k_\Omega)\left\langle \partial_n y_1, v_3\right\rangle_{\Gamma_1} - (1+\lambda\alpha)\left\langle B y_3, v_3\right\rangle_{\Gamma_1} - \lambda c_\Gamma \left\langle y_3, v_3\right\rangle_{\Gamma_1} \\
&= \left(f_2 - \lambda f_1 - k_\Omega A(I-N\partial_n) f_1 - c_\Omega f_1, v_1\right)_\Omega \\
&\quad + \left\langle f_4 - \lambda f_3 - k_\Omega \partial_n f_1 - \alpha B f_3 - c_\Gamma f_3, v_3\right\rangle_{\Gamma_1}
\end{aligned} \tag{4.3.7}$$

Rewriting, we obtain:

$$\begin{aligned}
&-\left(\lambda^2 y_1, v_1\right)_\Omega - (1+\lambda k_\Omega)\left(A^{\frac{1}{2}} y_1, A^{\frac{1}{2}} v_1\right)_\Omega - \lambda c_\Omega \left(y_1, v_1\right)_\Omega \\
&- \left\langle \lambda^2 y_3, v_3\right\rangle_{\Gamma_1} - (1+\lambda\alpha)\left\langle B^{\frac{1}{2}} y_3, B^{\frac{1}{2}} v_3\right\rangle_{\Gamma_1} - \lambda c_\Gamma \left\langle y_3, v_3\right\rangle_{\Gamma_1} \\
&\quad = \left(f_2 - \lambda f_1, v_1\right)_\Omega - k_\Omega \left(A^{\frac{1}{2}} f_1, A^{\frac{1}{2}} v_1\right)_\Omega - c_\Omega \left(f_1, v_1\right)_\Omega \\
&\quad + \left\langle f_4 - \lambda f_3, v_3\right\rangle_{\Gamma_1} - \alpha \left\langle B^{\frac{1}{2}} f_3, B^{\frac{1}{2}} v_3\right\rangle_{\Gamma_1} - c_\Gamma \left\langle f_3, v_3\right\rangle_{\Gamma_1}
\end{aligned} \tag{4.3.8}$$

This leads us to consideration of a bilinear form

$$\begin{aligned}
a(y_1, y_3, v_1, v_3) \equiv &-\left(\lambda^2 y_1, v_1\right)_\Omega - (1+\lambda k_\Omega)\left(A^{\frac{1}{2}} y_1, A^{\frac{1}{2}} v_1\right)_\Omega - \lambda c_\Omega \left(y_1, v_1\right)_\Omega \\
&- \left\langle \lambda^2 y_3, v_3\right\rangle_{\Gamma_1} - (1+\lambda\alpha)\left\langle B^{\frac{1}{2}} y_3, B^{\frac{1}{2}} v_3\right\rangle_{\Gamma_1} - \lambda c_\Gamma \left\langle y_3, v_3\right\rangle_{\Gamma_1}
\end{aligned}$$

defined for $y = (y_1, y_3), v = (v_1, v_3) \in \mathcal{V} \equiv \{(v_1, v_3) \in D(A^{\frac{1}{2}}) \times D(B^{\frac{1}{2}}), v_3 = v_1|_{\Gamma_1}\}$.

We are solving for the variable u the variational equation:

$$a(y, v) = F(v), \forall v \in \mathcal{V} \equiv D(A^{\frac{1}{2}}) \times D(B^{\frac{1}{2}})$$

where $F(v)$ be the corresponding right-hand side (in equation (4.3.8)):

$$\begin{aligned} F(v) &= (f_2 - \lambda f_1, v_1)_\Omega - k_\Omega \left(A^{\frac{1}{2}} f_1, A^{\frac{1}{2}} v_1 \right)_\Omega - c_\Omega (f_1, v_1)_\Omega \\ &+ \langle f_4 - \lambda f_3, v_3 \rangle_{\Gamma_1} - \alpha \left\langle B^{\frac{1}{2}} f_3, B^{\frac{1}{2}} v_3 \right\rangle_{\Gamma_1} - c_\Gamma \langle f_3, v_3 \rangle_{\Gamma_1} \\ &\text{with } F = (f_1, f_2, f_3, f_4) \in \mathcal{H} \end{aligned}$$

We have continuity of bilinear form on $\mathcal{V} \times \mathcal{V}$:

$$\begin{aligned} |a(y, v)| &\le C \|y\|_\mathcal{V} \|v\|_\mathcal{V} \\ |F(V)| &\le C \|F\|_\mathcal{H} \|v\|_\mathcal{V} \end{aligned} \tag{4.3.9}$$

The bilinear form is coercive:

$$\begin{aligned} Re\ a(u, u) &= \lambda^2 |y_1|^2_\Omega + (1 + \lambda k_\Omega) \left| A^{\frac{1}{2}} y_1 \right|^2_\Omega + \lambda c_\Omega |y_1|^2_\Omega \\ &+ \lambda^2 |y_3|^2_{\Gamma_1} + (1 + \lambda \alpha) \left| B^{\frac{1}{2}} y_3 \right|^2_{\Gamma_1} + \lambda c_\Gamma |y_3|^2_{\Gamma_1} \\ &\ge C \|y\|^2_\mathcal{H} \end{aligned} \tag{4.3.10}$$

Therefore $a(u, v)$ is both bounded and coercive, so by Lax Milgram for every $F \in \mathcal{H}$ there exists a unique solution $u \in \mathcal{V}$. Moreover $u = (u_1, u_3)$ satisfies the last two equations in ((4.3.6)).

Next we reconstruct the remaining part of the vector U . From ((4.3.6))

$$y_2 = \lambda y_1 - f_1 \in D(A^{\frac{1}{2}}), \quad y_4 = \lambda y_3 - f_3 \in \mathcal{D}(B^{\frac{1}{2}}), \forall F \in \mathcal{H} \tag{4.3.11}$$

Since $y_3 = N^* A y_1$ and $f_3 = N^* A f_1$ we conclude that $y_4 = N^* A y_2$, as required by the membership in the $\mathcal{D}(\mathcal{A}^*)$. The remaining regularity requirements simply follow from the structure of equations in (4.3.6).

In conclusion, for all $F \in \mathcal{H}$ we obtain $Y = (y_1, y_2, y_3, y_4)$ in $\mathcal{D}(\mathcal{A}^*)$ such that $(\lambda I - \mathcal{A})Y = F \in \mathcal{H}$. Thus Y is our desired solution and we conclude that $Y \in S$. Therefore, $\mathcal{D}(\mathcal{A}^*) = S$ and $\mathcal{A}^* = \Lambda$. □

With the adjoint computed, we can now determine whether the residual spectrum $\sigma_r(\mathcal{A})$ intersect the imaginary axis. It is well-known that an undamped wave equation gathered its eigenvalues on the imaginary axis, thus it appears necessary to impose some damping in order to verify that the spectrum is on the left of the imaginary axis.

Theorem 4.3.2. *Suppose that the damping condition* (1.1.14) *holds, i.e.* $max\{k_\Omega, k_\Gamma\alpha, c_\Omega, c_\Gamma\} > 0$. *With* $\mathcal{A}$ *defined in* (2.2.4), $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

Proof. Consider the spectrum of $\mathcal{A}$: $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_r(\mathcal{A}) \cup \sigma_c(\mathcal{A})$ where $\sigma_p(\mathcal{A})$, $\sigma_r(\mathcal{A})$ and $\sigma_c(\mathcal{A})$ denotes respectively the point spectrum, the residual spectrum and the continuous spectrum of $\mathcal{A}$; and show that it does not contain the imaginary axis denoted by $i\mathbb{R}$. We will proceed in three steps proving the spectral property for each spectrum, starting with the continuous spectrum.

Step 1: $\sigma_c(\mathcal{A}) \cap i\mathbb{R} = \emptyset$

We recall that, *if* $\lambda \in \sigma_c(\mathcal{A})$, *then* $\mathcal{A} - \lambda$ *does not have a closed range* (see [22, Problem 2.54 p. 128]).

With $\mathcal{A}$ as given in (2.2.4), assume that $\lambda = ir \in \sigma_c(\mathcal{A})$ with $r \neq 0$.

$\forall f = [f_1, f_2, f_3, f_4]^T \in \mathcal{H}$, suppose that $u = [u1, u2, u3, u4] \in \mathcal{D}(\mathcal{A})$ such that:

$$(ir - \mathcal{A}) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \tag{4.3.12}$$

which is equivalent to:

$$\begin{cases} iru_1 - u_2 = f_1 \\ iru_3 - u_4 = f_3 \\ iru_2 + A(u_1 - N\partial_n u_1) + k_\Omega A(u_2 - N\partial_n u_2) + c_\Omega u_2 = f_2 \\ iru_4 + \partial_n u_1 + k_\Omega \partial_n u_2 + Bu_3 + \alpha Bu_4 + c_\Gamma u_4 = f_4 \end{cases} \tag{4.3.13}$$

The two last equations can be rewritten, using the first to and $u_3 = N^* A u_1$ as follows:

$$\begin{cases} -r^2 u_1 + (1 + k_\Omega ir)Au_1 - AN(\partial_n u_1 + k_\Omega \partial_n u_2) + c_\Omega iru_1 = (ir + k_\Omega A + c_\Omega)f_1 + f_2 \\ \partial_n u_1 + k_\Omega \partial_n u_2 = f_4 + r^2 N^* Au_1 + irN^* Af_1 - BN^* Au_1 \\ \qquad -\alpha irBN^* Au_1 + \alpha BN^* Af_1 - c_\Gamma irN^* Au_1 + c_\Gamma N^* Af_1 \end{cases} \tag{4.3.14}$$

where we used the fact that $N^* Az = \gamma(z)$ from equation (2.1.9). Then, in (4.3.14), substitute the second equation into the first one.

$$\begin{aligned} &\left[-r^2 + (1 + k_\Omega ir)A + c_\Omega ir - r^2 ANN^* A + (1 + \alpha ir)ANBN^* A + c_\Gamma irANN^* A\right] u_1 \\ &\quad = (ir + k_\Omega A + c_\Omega)f_1 + f_2 + AN(ir + \alpha B + c_\Gamma)f_3 + ANf_4 \end{aligned} \tag{4.3.15}$$

Let $V = \mathcal{D}(A^{\frac{1}{2}}) \cap \mathcal{D}(B^{\frac{1}{2}})$ with associated norm:

$$\|u\|_V = \left|A^{\frac{1}{2}}u\right|_{\Omega}^2 + \left|B^{\frac{1}{2}}u\right|_{\Gamma_1}^2 \tag{4.3.16}$$

Consider the left hand side of (4.3.15) and define the operators $T, M, K : V \to V^*$, by

$$\begin{cases} T &= M + K \\ M &= A + ANBN^*A + ir(k_\Omega A + \alpha ANBN^*A) \\ K &= -r^2(I + ANN^*A) + c_\Omega irI + c_\Gamma irANN^*A \end{cases} \tag{4.3.17}$$

Also, using the right hand side of (4.3.15), define $F : \mathcal{H} \subset \mathcal{D}(A^{\frac{1}{2}}) \times L^2(\Omega) \times \mathcal{D}(B^{\frac{1}{2}}) \times L^2(\Gamma_1) \to V^*$ by:

$$F = \Big((ir + k_\Omega A), \qquad I, \qquad AN(ir + \alpha B), \qquad AN\Big) \tag{4.3.18}$$

Observe that, by letting $v \in V^*$ the third component of F is well-defined:

$$\begin{aligned} (\alpha ANBf_3, v)_{[\mathcal{D}(B^{\frac{1}{2}})]^* \times \mathcal{D}(B^{\frac{1}{2}})} &= \alpha \left(B^{\frac{1}{2}}f_3, B^{\frac{1}{2}}N^*Av\right)_{L^2(\Gamma_1)} \\ &\leq \alpha \left|B^{\frac{1}{2}}f_3\right|_{\Gamma_1}^2 \left|B^{\frac{1}{2}}v_{|\Gamma_1}\right|_{\Gamma_1}^2 \end{aligned} \tag{4.3.19}$$

Since $f_3 \in \mathcal{D}(B^{\frac{1}{2}})$, then $\alpha ANBf_3 \in [\mathcal{D}(A^{\frac{1}{2}})]^*$.

Similary, the other components of F can be bounded appropriately. Hereby, F is well defined. Therefore, we obtain:

$$|F(V)| \leq C\,\|F\|_{\mathcal{H}}\,\|v\|_{V^*} \tag{4.3.20}$$

It remains to show that T is invertible, we will proceed in three steps.

<u>Step 1-a:</u> Compactness of K

By definition of A, N and N^*A, K is compact from $\mathcal{D}(A^{\frac{1}{2}})$ into its dual.

Step 1-b: M is boundedly invertible.

Given M, $u \in V$ and $v \in V^*$, define its bilinear form $M(.,.)$, by:

$$M(u,v) = (1 + k_\Omega ir)\left(A^{\frac{1}{2}}u, A^{\frac{1}{2}}v\right)_\Omega + (1 + \alpha ir)\left\langle B^{\frac{1}{2}}N^*Au, B^{\frac{1}{2}}N^*Av\right\rangle_{\Gamma_1} \tag{4.3.21}$$

Then $M(.,.)$ is a coercive and bounded bilinear form, setting $K_M = max\{1, k_\Omega r + \alpha r\}$:

$$\begin{aligned} |M(u,v)| &\leq K_M \left[\left|A^{\frac{1}{2}}u\right|_\Omega^2 \left|A^{\frac{1}{2}}v\right|_\Omega^2 + \left|B^{\frac{1}{2}}N^*Au\right|_{\Gamma_1}^2 \left|B^{\frac{1}{2}}N^*Av\right|_{\Gamma_1}^2\right] \\ &\leq C\left\|u\right\|_V \left\|v\right\|_V \quad \text{by (4.3.16)} \end{aligned} \tag{4.3.22}$$

$$\begin{aligned} Re|M(u,u)| &= \left|A^{\frac{1}{2}}u\right|_\Omega^2 + \left|B^{\frac{1}{2}}N^*Au\right|_{\Gamma_1}^2 + k_\Omega r\left|A^{\frac{1}{2}}u\right|_\Omega^2 + \alpha r\left|B^{\frac{1}{2}}N^*Au\right|_{\Gamma_1}^2 \\ &\geq C(\left|A^{\frac{1}{2}}u\right|_\Omega^2 + \left|B^{\frac{1}{2}}u_{\Gamma_1}\right|_{\Gamma_1}^2) = C\left\|u\right\|_V \end{aligned} \tag{4.3.23}$$

Therefore by Lax-Milgram, the operator M is boundedly invertible. By the Fredholm's alternative, we deduce the desired invertibility of T provided that T is injective.

Step 1-c: T is injective.

Suppose that $Tu_1 = 0$, assume that $r \neq 0$ and take the duality pairing with respect to $v = u_1 \in V^*$, then:

$$
\begin{aligned}
0 &= ((M+K)u_1, v)_{[\mathcal{D}(A^{\frac{1}{2}})]^* \times \mathcal{D}(A^{\frac{1}{2}})} \\
&= ((A + ANBN^*A)u_1, v)_{[\mathcal{D}(A^{\frac{1}{2}})]^* \times \mathcal{D}(A^{\frac{1}{2}})} + ir\,((k_\Omega A + \alpha ANBN^*A)u_1, v)_{[\mathcal{D}(A^{\frac{1}{2}})]^* \times \mathcal{D}(A^{\frac{1}{2}})} \\
&\quad - r^2\,((I + ANN^*A)u_1, v)_{[\mathcal{D}(A^{\frac{1}{2}})]^* \times \mathcal{D}(A^{\frac{1}{2}})} + ir\,((c_\Omega I + c_\Gamma ANN^*A)u_1, v)_{[\mathcal{D}(A^{\frac{1}{2}})]^* \times \mathcal{D}(A^{\frac{1}{2}})} \\
&= -r^2\,|u_1|^2_\Omega + \left|A^{\frac{1}{2}}u_1\right|^2_\Omega - r^2\,|N^*Au_1|^2_{\Gamma_1} + \left|B^{\frac{1}{2}}N^*Au_1\right|^2_{\Gamma_1} \\
&\quad + ir\left[c_\Omega\,|u_1|^2_\Omega + k_\Omega\left|A^{\frac{1}{2}}u_1\right|^2_\Omega + c_\Gamma\,|N^*Au_1|^2_{\Gamma_1} + \alpha\left|B^{\frac{1}{2}}N^*Au_1\right|^2_{\Gamma_1}\right]
\end{aligned}
\tag{4.3.24}
$$

Taking the imaginary part in (4.3.24) implies $u_1 = 0$. Thus T is injective. Note that the injectivity does not necessarly hold if we do not impose the damping condition (1.1.14).

Thus, T is invertible which achieves the proof of the first step: $\sigma_c(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

Step 2: $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$

With $\mathcal{A}$ as given in (2.2.4), if for $r \in \mathbb{R}$ and $r \neq 0$, there exists $u = [u1, u2, u3, u4]^T \in \mathcal{D}(\mathcal{A})$ such that:

$$
\mathcal{A}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = ir\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{4.3.25}
$$

which is equivalent to:

$$\begin{cases} u_2 = iru_1 \\ u_4 = iru_3 \\ iru_2 + A(u_1 - N\partial_n u_1) + k_\Omega A(u_2 - N\partial_n u_2) + c_\Omega u_2 = 0 \\ iru_4 + \partial_n u_1 + k_\Omega \partial_n u_2 + Bu_3 + \alpha Bu_4 + c_\Gamma u_4 = 0 \end{cases} \tag{4.3.26}$$

Observe that (4.3.26) is equivalent to $Tu_1 = 0$. Then, in step 1, we have already shown that $u_1 = 0$ which achieves the proof for step 2.

Step 3: $\sigma_r(\mathcal{A}) \cap i\mathbb{R} = \emptyset$

First we recall that, *If the eigenvalue* $\lambda \in \mathbb{C}$ *is in the residual spectrum of* $\mathcal{A}$*, then* $\lambda \in \sigma_p(\mathcal{A}^*)$ (see [22, p. 127]).

With $\mathcal{A}^*$ as given in (4.3.1), if for $r \in \mathbb{R}$ and $r \neq 0$, there exists $u = [u1, u2, u3, u4]^T \in \mathcal{D}(\mathcal{A}^*)$ such that:

$$\mathcal{A}^* \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = ir \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{4.3.27}$$

which is equivalent to:

$$\begin{cases} -u_2 = iru_1 \\ -u_4 = iru_3 \\ Au_1 - AN\partial_n u_1 - k_\Omega Au_2 + k_\Omega AN\partial_n u_2 - c_\Omega u_2 = iru_2 \\ \partial_n u_1 - k_\Omega \partial_n u_2 + Bu_3 - \alpha Bu_4 - c_\Gamma u_4 = iru_4 \end{cases} \quad (4.3.28)$$

Proceeding as in (4.3.14) and (4.3.15), then (4.3.28) can also be written as $Tu_1 = 0$, which again implies that $u_1 = 0$, by step 1.

Therefore, the residual spectrum does not intersect the imaginary axis which completes the proof of theorem 4.3.2 □

Chapter 5

Stabilization and Uniform Decay Rates

In the previous section we classified the strongly continuous semigroups according to their regularity properties, this section will be devoted to their asymptotic behavior. In other words, we would like to know the behavior of the semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ for large $t > 0$.

The spectral property from theorem 4.3.2 already provides us a result about the stability of the model (**GM**). Indeed, one can immediately claim the strong stability provided the damping condition (1.1.14) in the system using the well-known and useful result by Arendt-Batty.

Theorem F (Stability Theorem 2.4 in [3])**.** *Let $\mathcal{T}(t)$ be a bounded C_0-semigroup with generator A. Assume that $\sigma_r(A) \cap i\mathbb{R} = \emptyset$. If $\sigma(A) \cap i\mathbb{R}$ is countable, then $\mathcal{T}(t)$ is strongly stable. This is to say; for every $u \in \mathcal{H}$*

$$\left\|e^{\mathcal{A}t}u\right\|_{\mathcal{H}} \to 0, \quad t \to \infty$$

In addition, for analytic semigroup, the spectral property from theorem 4.3.2

provide exponential stability:

Theorem G (Theorem 4.4.3 - [49])**.** *Let A be the generator of an analytic semigroup $\mathcal{T}(t)$. If*

$$sup\{Re\lambda : \lambda \in \sigma(A)\} < 0 \tag{5.0.1}$$

then there are constants $M \geq 1$ and $\mu > 0$ such that $\|\mathcal{T}(t)\|_{\leq} Me^{-\mu t}$, i.e. $\mathcal{T}(t)$ is exponentially stable.

We also note that with the resolvent approach in theorem 4.1.2, analyticity was obtained provided the exponential stability of the semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ (see assumption 4.1.1), thus in order to relax this assumption, it is necessary to demonstrate the exponential stability not being a consequence of the analyticity which is one of the purpose of theorem 5.1.1. The following diagram (figure 5)recapitulates the connexions between the theorems.

Although the exponential stability is reached in the case of analytic semigroup for the model (**GM**), which relies on the positivity of the interior viscoelastic damping ($k_\Omega > 0$) and either $k_\Gamma = 0$ or $k_\Gamma\alpha > 0$ (assumption 4.1.3), it remains to investigate if it can also be reached with strong damping only in the interior or on the boundary, or with only frictional damping. This is the second purpose of theorem 5.1.1 in which we will consider energy methods to determine the conditions under which the model (**GM**) is exponentially stable.

Once the conditions of exponential stability are identified for the model (**GM**), the

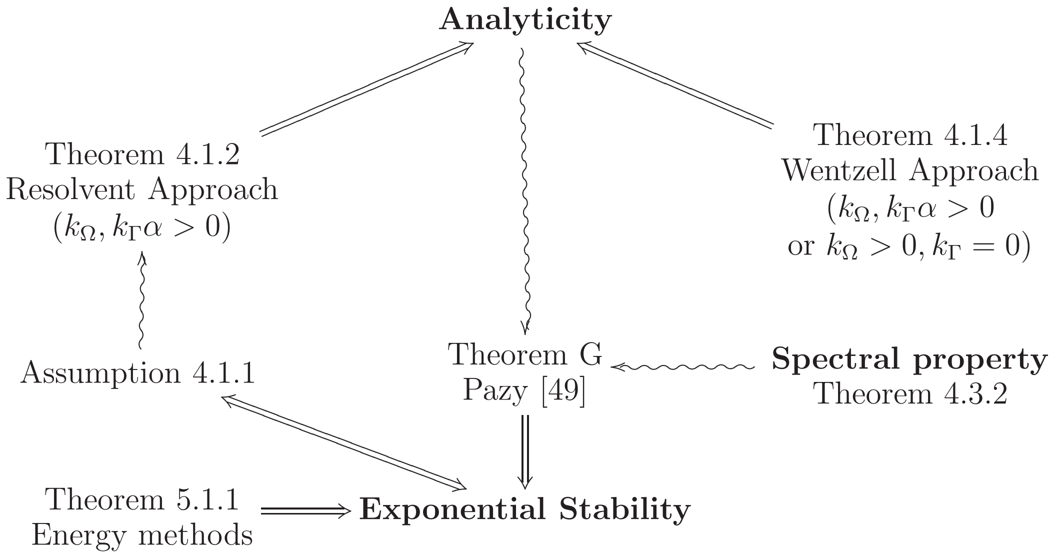


Figure 5.1: Diagram: Relationship between exponential stability and analyticity

aim of this chapter is also to highlight an interesting phenomenon first discovered by Littman and Markus in [40],[39] for the Scole model enlighting exponential stability failed in the absence of interior damping in the presence of inertial terms on the boundary.

5.1 Exponential Stability

We start by describing the conditions under which the exponential stability property will holds for model (**GM**):

1. Viscoelastic damping in the interior

$$k_\Omega > 0 \tag{5.1.1}$$

2. No viscoelastic damping in the interior

   a Frictional damping in the interior and on the boundary

$$c_\Omega, c_\Gamma > 0 \tag{5.1.2}$$

   b Frictional damping in the interior and viscoelastic damping on the boundary

$$c_\Omega, k_\Gamma \alpha > 0 \tag{5.1.3}$$

We are now ready to present the main result of this chapter:

Theorem 5.1.1. *Consider the operator $\mathcal{A}$ given in* (2.2.4). *Suppose that* (5.1.1), (5.1.2) *or* (5.1.3) *holds. Then the C_0-semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ is exponentially stable, i.e.,*

$$\exists C, \omega \geq 0 \text{ such that } \left\|e^{\mathcal{A}t}u\right\|_{\mathcal{H}} \leq Ce^{-\omega t}\left\|u\right\|_{\mathcal{H}}, \quad t > 0$$

Proof. To begin with, let's display the model (**GM**):

$$\begin{cases} u_{tt} + c_\Omega u_t - k_\Omega \Delta u_t - \Delta u = 0 & x \in \Omega, t > 0 \\ u(x,t) = 0 & x \in \Gamma_0,\ t > 0 \\ u_{tt} + c_\Gamma u_t + \partial_n(u + k_\Omega u_t) - k_\Gamma \boldsymbol{\Delta}_\Gamma(\alpha u_t + u) = 0 & x \in \Gamma_1,\ t > 0 \\ u(0,x) = u_0,\ u_t(0,x) = u_1 & x \in \Omega \end{cases} \tag{\textbf{GM}}$$

By [49, Theorem 4.4.1], to show the exponential stability of $\{e^{\mathcal{A}t}\}_{t\geq 0}$ it is enough to show that

$$\int_0^T \left\|e^{\mathcal{A}t}U\right\|_{\mathcal{H}}^2 ds < \infty \tag{5.1.4}$$

where $\mathcal{A}$ is given in (2.2.4) and we display again for reference the energy space $\mathcal{H}$ from (2.2.2)

$$\mathcal{H} = \{(u_1, u_2, u_3, u_4) \in \mathcal{D}(A^{\frac{1}{2}}) \times L^2(\Omega) \times \mathcal{D}(B^{\frac{1}{2}}) \times L^2(\Gamma_1), u_1|_{\Gamma_1} = u_3\}$$

$$\text{where } \mathcal{D}(A^{\frac{1}{2}}) = H^1_{\Gamma_0}(\Omega) \text{ and } \mathcal{D}(B^{\frac{1}{2}}) \sim \begin{cases} H^1(\Gamma) & \text{if } k_\Gamma > 0 \\ L^2(\Gamma) & \text{if } k_\Gamma = 0 \end{cases}$$

We recall that $B = -k_\Gamma \mathbf{\Delta}_\Gamma z$ (see definition 2.1.3).

Multiply by u_t the first equation of (**GM**):

$$(u_{tt}, u_t)_\Omega + k_\Omega |\nabla u_t|^2_\Omega + c_\Omega |u_t|^2_\Omega + (\nabla u, \nabla u_t)_\Omega$$
$$+ \langle u_{tt}, u_t \rangle_{\Gamma_1} + k_\Gamma \alpha |\nabla_\Gamma u_t|^2_{\Gamma_1} + c_\Gamma |u_t|^2_{\Gamma_1} + k_\Gamma (\nabla_\Gamma u, \nabla_\Gamma u_t)_\Omega = 0$$

Therefore, after rearranging the terms, we get:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(|u_t|^2_\Omega + |\nabla u|^2_\Omega + |u_t|^2_{\Gamma_1} + k_\Gamma |\nabla_\Gamma u|^2_{\Gamma_1}\right) \\ &\quad = -k_\Omega |\nabla u_t|^2_\Omega - c_\Omega |u_t|^2_\Omega - k_\Gamma \alpha |\nabla_\Gamma u_t|^2_{\Gamma_1} - c_\Gamma |u_t|^2_{\Gamma_1} \end{aligned} \tag{5.1.5}$$

Let $E_p(t) = |\nabla u|^2_\Omega + k_\Gamma |\nabla_\Gamma u|^2_{\Gamma_1}$, $E_k(t) = |u_t|^2_\Omega + |u_t|^2_{\Gamma_1}$ be the potential and kinetic energy respectively, then define the energy of this system as the summation of the potential and kinetic energies:

$$E(t) = E_p(t) + E_k(t)$$

Note that the potential energy may only consists of potential interior energy if $k_\Gamma = 0$, in which case the third component in the energy space is in $L^2(\Gamma_1)$ following our defintion of B. Using these definitions, we obtain the following energy equality after integrating in time (5.1.5):

$$E(0) = E(t) + 2\int_0^t k_\Omega \left|\nabla u_t\right|_\Omega^2 + c_\Omega \left|u_t\right|_\Omega^2 + k_\Gamma \alpha \left|\nabla_\Gamma u_t\right|_{\Gamma_1}^2 + c_\Gamma \left|u_t\right|_{\Gamma_1}^2 \, ds \tag{5.1.6}$$

Now multiply by u the first equation of (**GM**) and again integrate in space:

$$\begin{aligned} &(u_{tt}, u)_\Omega + k_\Omega \left(\nabla u_t, \nabla u\right)_\Omega + c_\Omega \left(u_t, \nabla u\right)_\Omega + \left|\nabla u\right|_\Omega^2 \\ &\quad + \langle u_{tt}, u\rangle_{\Gamma_1} + k_\Gamma \alpha \left\langle \nabla_\Gamma u_t, \nabla_\Gamma u\right\rangle_{\Gamma_1} + c_\Gamma \left\langle u_t, u\right\rangle_{\Gamma_1} + k_\Gamma \left|\nabla_\Gamma u\right|_\Omega^2 = 0 \end{aligned} \tag{5.1.7}$$

Integrate in time, and use integration by parts:

$$\begin{aligned} &\int_0^t - \left|u_t\right|_\Omega^2 - \left|u_t\right|_{\Gamma_1}^2 + \left|\nabla u\right|_\Omega^2 + k_\Gamma \left|\nabla_\Gamma u\right|_{\Gamma_1}^2 \, ds = (u_t, u)_\Omega \, |_t^0 + \langle u_t, u\rangle_{\Gamma_1} \, |_t^0 \\ &\quad + \int_t^0 \frac{1}{2}\frac{d}{dt}\left(k_\Omega \left|\nabla u\right|_\Omega^2 + c_\Omega \left|u\right|_\Omega^2 + k_\Gamma \alpha \left|\nabla_\Gamma u\right|_{\Gamma_1}^2 + c_\Gamma \left|u\right|_{\Gamma_1}^2\right) ds \end{aligned} \tag{5.1.8}$$

In (5.1.8), we identify the potential energy on the left hand side and we define F to be the right hand side:

$$F = \left[\frac{1}{2}(k_\Omega \left|\nabla u\right|_\Omega^2 + c_\Omega \left|u\right|_\Omega^2 + k_\Gamma \alpha \left|\nabla_\Gamma u\right|_{\Gamma_1}^2 + c_\Gamma \left|u\right|_{\Gamma_1}^2) + (u_t, u)_\Omega + \langle u_t, u\rangle_{\Gamma_1}\right]\Bigg|_t^0 \tag{5.1.9}$$

Thus,

$$\int_0^t E_p(s) ds \leq \int_0^t \left|u_t\right|_\Omega^2 + \left|u_t\right|_{\Gamma_1}^2 \, ds + F \tag{5.1.10}$$

Then, add $\int_0^t E_k(s)ds$ on both sides of (5.1.10):

$$\int_0^t \left\|e^{\mathcal{A}t} U\right\|_{\mathcal{H}}^2 ds = \int_0^t E_p(s) + E_k(s) ds \leq 2\int_0^t \left|u_t\right|_\Omega^2 + \left|u_t\right|_{\Gamma_1}^2 \, ds + F \tag{5.1.11}$$

Thus, it remains to bound the right hand side (5.1.11).

Before, we recall Poincare's inequality from proposition 2.1.2

$$\left|u_t\right|_\Omega^2 \leq C_p \left|\nabla u_t\right|_\Omega^2, \text{ because } u = 0 \text{ on } \Gamma_0$$

where C_p is the Poincaré's constant. Also, recall the trace moment inequality from proposition 2.1.6 (see [8]):

$$\begin{aligned} |u_t|^2_{\Gamma_1} &\leq C|\nabla u_t|^{\frac{1}{2}}_{\Omega}|u_t|^{\frac{1}{2}}_{\Omega} \\ &\leq C_m \, |\nabla u_t|^2_{\Omega} \end{aligned} \tag{5.1.12}$$

where $C_m = C \times C_p$ is the trace moment constant. These two constants will often be used in the following estimates and their presence will show the application of the corresponding inequality.

Also define K as the maximum of the damping coefficients and 1, i.e.,

$$K = max\{k_\Omega, c_\Omega, k_\Gamma \alpha, c_\Gamma, 1\}$$

Therefore, the term F defined in (5.1.9) can be bounded as follows:

$$\begin{aligned} F &\leq K \left(|\nabla u(0)|^2_\Omega + C_p \, |\nabla u(0)|^2_\Omega + |\nabla_\Gamma u(0)|^2_{\Gamma_1} + C_m \, |\nabla u(0)|^2_\Omega\right) \\ &\quad + 2 \, |u(0)|^2_\Omega + 2 \, |u(0)|^2_{\Gamma_1} + 2 \, |u_t(0)|^2_\Omega + 2 \, |u_t(0)|^2_{\Gamma_1} \\ &\leq K \left(E_p(0) + C_{p,m} E_p(0)\right) + 2 \left(C_p \, |\nabla u(0)|^2_\Omega + C_m \, |\nabla u(0)|^2_\Omega + E_k(0)\right) \\ &\leq \left(K + C_{p,m} K + 2C_{p,m}\right) E_p(0) + 2E_k(0) \\ &\leq C_{K,p,m} E(0) \end{aligned} \tag{5.1.13}$$

It remains to bound the integral $\int_0^t |u_t|^2_\Omega + |u_t|^2_{\Gamma_1} \, ds$ in (5.1.10) and this will be achieved using the energy identity (5.1.6).

Part 1: $k_\Omega > 0$

By the energy identity(5.1.6), we get:

$$\begin{aligned} \int_0^t |u_t|_\Omega^2 + |u_t|_{\Gamma_1}^2 \, ds &\leq \int_0^t C_p \, |\nabla u_t|_\Omega^2 + C_m \, |\nabla u_t|_\Omega^2 \, ds \\ &\leq \frac{C_{p,m}}{2k_\Omega} E(0) \end{aligned} \tag{5.1.14}$$

Note that depending on the positivity of the coefficients c_Ω, c_Γ and $k_\Gamma \alpha$, it is possible to get a sharper estimate.

<u>Part 2:</u> $k_\Omega = 0$

We note that in this configuration, it is necessary to have $c_\Omega > 0$ otherwise it is impossible to estimate the term $|u_t|_\Omega^2$.

<u>Case 1:</u> $c_\Gamma > 0$

Define $c = min\{c_\Omega, c_\Gamma\}$, then $\frac{c_\Omega}{c}, \frac{c_\Gamma}{c} \geq 1$. By the energy identity(5.1.6), we get:

$$\begin{aligned} \int_0^t |u_t|_\Omega^2 + |u_t|_{\Gamma_1}^2 \, ds &\leq \int_0^t \frac{c_\Omega}{c} \, |u_t|_\Omega^2 + \frac{c_\Gamma}{c} \, |u_t|_{\Gamma_1}^2 \, ds \\ &\leq \frac{1}{c} \int_0^t c_\Omega \, |u_t|_\Omega^2 + c_\Gamma \, |u_t|_{\Gamma_1}^2 \, ds \\ &\leq \frac{1}{2c} E(0) \end{aligned} \tag{5.1.15}$$

Note that this estimate suggests a relationship between the frictional dampings: $c_\Omega = c_\Gamma$.

<u>Case 2:</u> $c_\Gamma = 0$

When c_Γ and k_Ω equal zero, then the control on $|u_t|_{\Gamma_1}^2$ must be achieved using $|\nabla u_t|_{\Gamma_1}^2$. Since this term is associated with $k_\Gamma \alpha$, we must impose the strict positivity of $k_\Gamma \alpha$. Indeed, one could apply the Poincaré's inequality on the boundary:

$|u_t|^2_{\Gamma_1} \le C_{p,\Gamma_1} |\nabla_\Gamma u_t|^2_{\Gamma_1}$; without loss of generality we assume that $C_{p,\Gamma_1} \ge 1$. Define $c = min\{k_\Gamma \alpha, c_\Omega\}$, it follows that the estimate of $|u_t|^2_\Omega + |u_t|^2_{\Gamma_1}$ is given by:

$$\begin{aligned}\int_0^t |u_t|^2_\Omega + |u_t|^2_{\Gamma_1} \, ds &\le C_{p,\Gamma_1} \int_0^t \frac{c_\Omega}{c} |u_t|^2_\Omega + \frac{k_\Gamma \alpha}{c} |\nabla_\Gamma u_t|^2_{\Gamma_1} \, ds \\ &\le \frac{C_{p,\Gamma_1}}{2c} E(0)\end{aligned} \tag{5.1.16}$$

From estimates (5.1.14), (5.1.15) and (5.1.16), it follows that the left hand side of (5.1.11) is finite:

$$\int_0^t \left\| e^{\mathcal{A}t} U \right\|^2_{\mathcal{H}} ds \le 2 \int_0^t |u_t|^2_\Omega + |u_t|^2_{\Gamma_1} \, ds + F < \infty \tag{5.1.17}$$

As a consequence, by [49, Theorem 4.4.1], the semigroup $\{e^{\mathcal{A}t}\}_{t\ge0}$ is exponentially stable, provided the appropriate conditions on the damping coefficients, i.e., one of the following must hold:

- Viscoelastic damping in the interior ($k_\Omega > 0$), (5.1.1)
- Frictional damping in the interior and on the boundary ($c_\Omega, c_\Gamma > 0$), (5.1.2)
- Frictional damping in the interior and viscoelastic on the boundary ($c_\Omega, k_\Gamma \alpha$), (5.1.3)

□

5.2 Strong Stability

We recall that, provided some damping in the system (**GM**), $i\mathbb{R}$ is not a subset of the spectrum of $\mathcal{A}$ (theorem 4.3.2), thus the strongly continuous semigroup $\{e^{\mathcal{A}t}\}_{t\ge0}$, governing (**GM**) generated in theorem 2.3.1, is strongly stable by the stability theorem from Arendt-Batty theorem ([3, Theorem 2.4] cited in theorem F [p. 110]). In this section, we examine the cases where $\{e^{\mathcal{A}t}\}_{t\geq 0}$ is not exponentially stable under theorem 5.1.1. It is not our aim to demonstrate that this property does not hold but to provide the reader with arguments suggesting that only strong stability holds. With assumptions (5.1.1), (5.1.2) and (5.1.3) from exponential stability theorem 5.1.1, we can identify precisely the cases where exponential stability is not expected.

Firstly, in the absence of interior damping ($c_\Omega = k_\Omega = 0$) our model (**GM**) ressembles the Scole model (5.2.1) studied in the late 80's by Littman and Markus in [39, 40]:

$$\begin{cases} w_{tt} - w_{xxxx} = 0 & x \in \Omega, t > 0 \\ w(0,t) = 0,\ w_x(0,t) = 0 & \text{(clamped at } x = 0) \\ \mu_1 w_{tt}(1,t) - w_{xxx}(1,t) = f_1 & \text{(linked at } x = 1) \\ \mu_2 w_{xtt}(1,t) + w_{xx}(1,t) = f_2 & x \in \Omega \end{cases} \tag{5.2.1}$$

After demonstrating asymptotic stability (see [40, section 4]), the authors examine the rate of decay of a solution $w(x,t)$ under dissipative feedback boundary damping:

$$\begin{cases} f_1 &= w_t(1,t) \\ f_2 &= w_{xt}(1,t) \end{cases} \tag{5.2.2}$$

and prove the existence of solutions with arbitrarily slow decay towards zero([40, theorem 5.3]). The important common facts between the Scole model and (**GM**) explaining the non-exponential decay are the absence of damping in the interior and the presence of inertial terms on the boundary. While boundary stabilization usually leads to exponential stability, this is so for models without inertial terms on the boundary.

Similarly to the figure 1.5 [p. 21], figure 5.2 shows the spectrum of model (**GM**) without interior damping but with both frictional and viscoelastic damping on the boundary. We observe the circle described by eigenvalues coming from the parabolic behavior of the boundary ($k_\Gamma \alpha > 0$) along with a vertical component tending to the imaginary axis as the eigenvalue's imaginary part goes to infinity, which is characteristic of 'non-exponential / strong' stability.

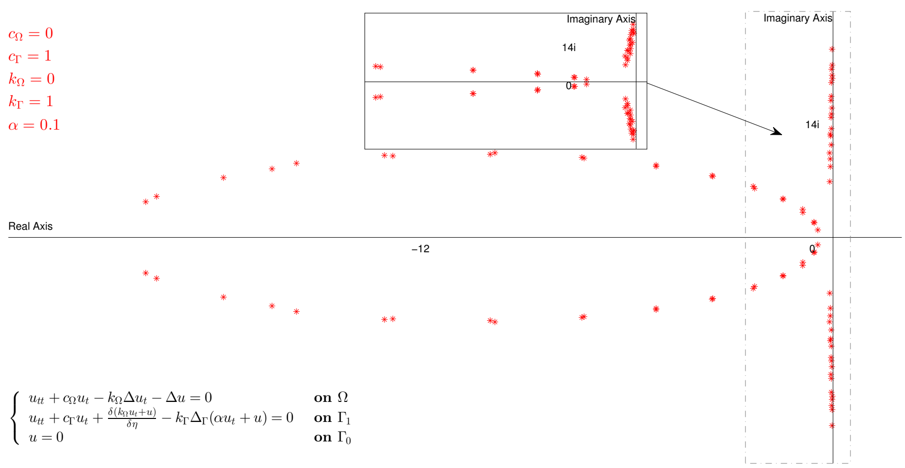


Figure 5.2: *Spectrum of a non-exponentially / strongly stable semigroup I.* Eigenvalues for the model (**GM**) without interior damping ($c_\Omega = 0$ and $k_\Gamma \alpha, c_\Gamma > 0$).

Secondly, if the interior damping is only frictional ($k_\Omega = 0$ and $c_\Omega > 0$), assumptions (5.1.2) and (5.1.3) suggests the necessity of boundary damping to achieve exponential stability. Hereby, if both boundary dampings are null which is covered by two cases: $c_\Gamma = \alpha = 0$ and $c_\Gamma = k_\Gamma = 0$, the resulting spectrums are very similar (figure 5.3). For this reason, we only show the second case.

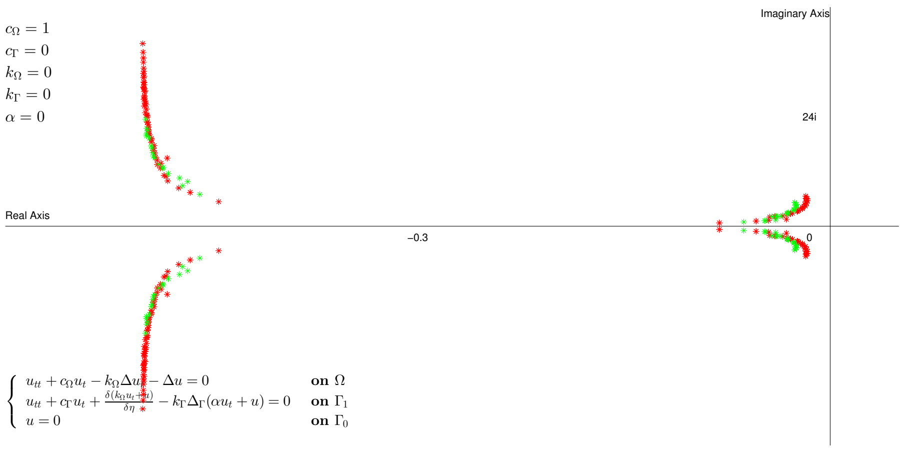


Figure 5.3: *Spectrum of a non-exponentially / strongly stable semigroup II.* Eigenvalues for the model (**FM**) without interior damping ($c_\Omega > 0$ and all other coefficients set to zero.

Given the same geometrical conditions, figure 5.3 shows two spectrums for (**GM**) with only interior frictional damping ($c_\Omega = 1$ and $k_\Omega, k_\Gamma, \alpha, c_\Gamma = 0$) but with different meshing: the red spectrum (81 nodes meshing) corresponds to a finer meshing than the green one (36 nodes meshing), i.e., as we tend to the continuous case the right component streches horizontally and tends to the imaginary axis suggesting again 'non-exponential / strong' stability. We also observe a well-known behavior: eigenvalues tend to form a vertical asymptote characteristic to the frictional damping (see figure 1.1 [p. 16]). We note that numerically this scenario is not obvious to interpret and larger computation capacity would be useful.

In order to provide more insight about this phenomenon we address a final consideration with the following system:

$$\begin{cases} u_{tt} + c_\Omega u_t - \Delta u = 0 & x \in \Omega, t > 0 \\ u(x,t) = 0 & x \in \Gamma_0,\ t > 0 \\ m u_{tt} + c_\Gamma u_t + \partial_n u = 0 & x \in \Gamma_1,\ t > 0 \\ u(0,x) = u_0,\ u_t(0,x) = u_1 & x \in \Omega \end{cases} \qquad (\mathbf{STG})$$

where m is the mass coefficient on the boundary Γ_1. Note that (**STG**) is equivalent to (**GM**) by setting $k_\Omega, k_\Gamma = 0, m = 1$.

First of all, let's observe how the inertial term on the boundary, controlled by m modifies the spectrum of (**STG**). We start with the set-up from the previous figure (figure 5.3 - $m = 1, c_\Omega = 1, c_\Gamma, k_\Gamma = 0$), represented in red in figure 5.4. As m tends to zero, we notice that the spectrum gets closer to the vertical asymptote in green corresponding to the limit case $m = 0$, i.e., a damped wave equation with 0-Dirichlet and Neumann boundary conditions. The blue and black spectrums are associated

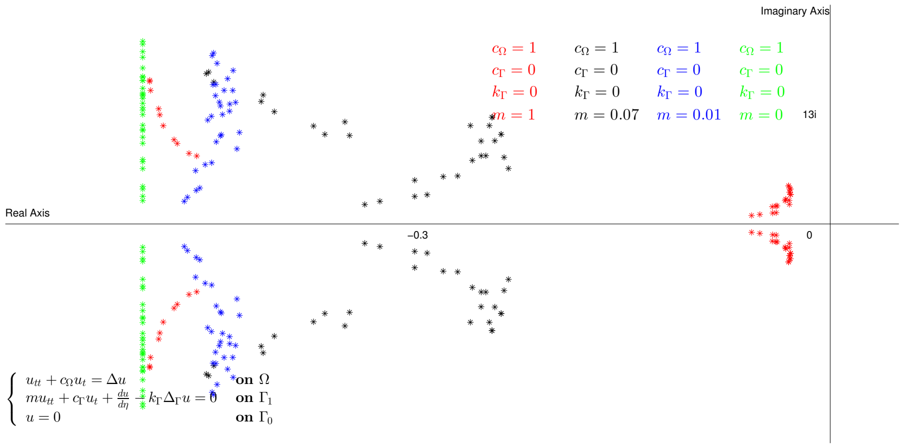


Figure 5.4: *Effect of the boundary mass coefficient on the spectrum of an damped wave equation with DBC.* Eigenvalues for the model (**STG**) with $c_\Omega = 1$, $c_\Gamma = 0$ and $m = 1$ (in red), 0.07 (in blue), 0.01 (in black), 0 (in green)

with intermediate values of the mass coefficient m, i.e., $m = 0.01$ and $m = 0.07$ respectively.

After illustrating this loss of stability, we would like to go back the other direction to recover the exponential stability keeping the inertial term on the boundary ($m = 1$). We noticed in the proof of exponential stability (theorem 5.1.1) first the necessicity of boundary damping to control the boundary inertial term, and second the relationship between c_Ω and c_Γ in the energy estimate (5.1.15).

With the introduction of the coefficient m on the boundary, figure 5.5 illustrates what should be the interplay between the mass coefficient m and the damping coefficients c_Ω and c_Γ. Given a value for the interior damping c_Ω and the mass coefficient m, the question we address is the following: how much boundary damping is necessary to obtain an ideal spectrum ? By ideal spectrum, we mean a spectrum with only a vertical asymptote as for the damped wave equation with Dirichlet and/or Neumann boundary conditions. Let $c_\Omega = 0.5$ and $m = 3$, then figure 5.5 shows that the ideal spectrum is attained for $c_\Gamma = 1.5$ (black spectrum). The spectrums for $c_\Gamma = 0$ and $c_\Gamma = 2.5$ are respectively plotted in red and blue. The relationship enlighted by the experience, which also holds for any other couples (c_Ω, m) such that $c_\Omega, m > 0$, is the following:

$$c_\Gamma = c_\Omega \times m \tag{5.2.3}$$

This observation suggests that the dynamics driving this problem are strongly coupled with the dynamics of the Wentzell problem. Indeed, for all scenarii where the equation (5.2.3) holds the change of variable $z = u + c_\Omega u_t$ allows to treat (**STG**) as a heat equation with unperturbed Wentzell boundary conditions:

$$mz + \partial_n u = 0$$

Remark 5.2.1. *This change of variable is very similar to the one we used in the proof for analyticity with the Wentzell approach (theorem 4.1.4 - equation* (4.1.16)*).*

Although we only used the Wentzell approach to study analyticity, the dynamics governing the wave equation with dynamic boundary conditions (**GM**) are highly

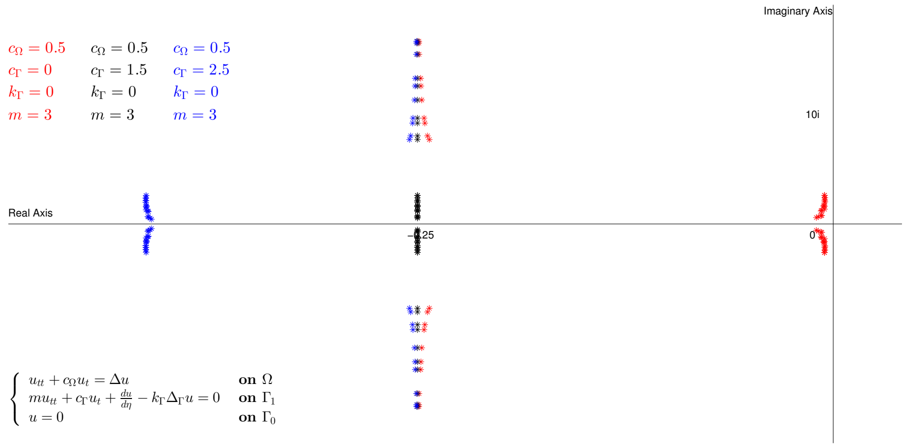


Figure 5.5: *Effect of the boundary frictional damping.* Eigenvalues for the model (**STG**) with $c_\Omega = 0.5$, $m = 3$ and $c_\Gamma = 0 \to 2.5$

related to the one of the Wentzell problem and other properties for the Wentzell problems about stability, decay rates for instance should be applicable to this type of problems.

Chapter 6

Optimal growth bound

The aim of this chapter is to summarize the results obtained during this dissertation and provide an interesting interaction between the theoretical results about analyticity (chapter 4) and exponential stability (chapter 5), and the numerical scheme (chapter 3). Firstly, we recall that for a strongly continuous semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ generated by $\mathcal{A}$, we call:

$$\omega_0 := \omega_0(e^{\mathcal{A}t}) = inf\{\omega > 0 : \exists M \geq 1 \text{ such that } \left\|e^{\mathcal{A}t}\right\| < Me^{\omega t},\ \forall t > 0\} \tag{6.0.1}$$

its growth bound. Moreover, if it is a semigroup of contraction, the setting $\omega_0 = 0$ and $M = 1$ is possible. If the stability is exponential then the growth bound is negative. However, it is often complicated to calculate the growth bound directly. It is a well-established procedure to calculate or to estimate the spectrum of the generator $\mathcal{A}$ and to try to relate the location of the spectrum to the asymptotic behaviour of the solution. For this purpose, we introduce the notion of spectral bound:

$$s(\mathcal{A}) = sup\{Re(\lambda) : \lambda \in \sigma(\mathcal{A})\} \tag{6.0.2}$$

In general we only have:

$$s(\mathcal{A}) \leq \omega_0(e^{\mathcal{A}t}) \tag{6.0.3}$$

In some cases, whenever there is a suitable spectral mapping theorem for the semigroup, i.e.,

$$e^{t\sigma(\mathcal{A})} = \sigma(e^{\mathcal{A}t}) - \{0\}, \quad t \geq 0$$

the principle of *linear stability* holds:

$$s(\mathcal{A}) = \omega_0(e^{\mathcal{A}t}) \tag{6.0.4}$$

This principle holds not only for finite-dimensional cases, but also for a wide variety of semigroups, in particular analytic semigroup. Indeed, in [16], the authors proved the following:

Theorem H (Corollary IV.3.11 and Corollary IV.2.4 in [16])**.** *For a uniformly continuous semigroup ([16, Corollary IV.2.4]) or an eventually norm continuous semigroup ([16, Corollary IV.3.11]),* $\{e^{\mathcal{A}t}\}_{t\geq 0}$ *and its generator* $\mathcal{A}$*, one has* $s(\mathcal{A}) = \omega_0(e^{\mathcal{A}t})$*.*

More precisely, one characterization of norm-continuous semigroup of our interest is ([16, p. 115]):

Proposition 6.0.1. *Let* $\mathcal{A}$ *be the generator of a uniformly exponentially stable strongly continuous semigroup* $\{e^{\mathcal{A}t}\}_{t\geq 0}$ *on a Hilbert space* $\mathcal{H}$*. Then* $\{e^{\mathcal{A}t}\}_{t\geq 0}$ *is immediately norm-continuous if and only if*

$$\lim_{|\beta|\to\infty} \|R(i\beta, \mathcal{A})\| = 0$$

From this definition, it follows that whenever the operator $\mathcal{A}$ governing (**GM**) is analytic or Gevrey (see chapter 4), it is immediately norm-continuous. Indeed, the exponential stability always holds in these two cases since the interior viscoelastic damping is strictly positive ($k_\Omega > 0$). See theorem 5.1.1 [p. 113].

It is known that "more damping" does not mean more decays. The phenomenon of overdamping is well known both in practice (engineering) and in theory (mathematics). This motivates our interest in studying the balance between the competing damping mechanisms in the interior Ω and on the boundary Γ_1 and thus determine the optimal damping. Indeed, for a given system, the optimal damping extracts as much energy as possible from the system, meaning that the growth bound is as small as possible. In [21], [58, Chapter 13, 21], the version of linear stability is expressed by the so-called *spectral absissa criterion* which requires the minimization of:

$$\min_i |Re(\lambda_i)| \tag{6.0.5}$$

where λ_i are the eigenvalues of the associated system. In fact, we use this criterion to approximate the spectral bound and thus the growth bound.

6.1 Impact of each damping mechanism

To begin with, the impact of the viscoelastic coefficients $k_\Omega, k_\Gamma\alpha$ have been well described already and could be summarized by:

- the presence of viscoelastic damping in the interior guarantees the semigroup to be analytic or Gevrey, implying smoothing properties for the solution.

- the presence of viscoelastic damping *both* in the interior and the boundary makes the semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ analytic on $\mathcal{H}_p$.

- the coefficients k_Ω and $k_\Gamma \alpha$ are inversely proportional to the radius of the circle formed by the eigenvalues (see figure 1.4 [p. 20])

Although, the growth bound can be estimated if $k_\Omega > 0$, we have already observed (see figures 1.3 [p. 19] and 1.4 [p. 20]) the variation of the viscoelastic coefficients does not modify the spectral bound, thus the growth bound, since the eigenvalues form a circle almost tangent to the imaginary axis, implying a constant spectral bound for any values of the viscoelastic coefficients.

Moreover, we have already noticed, in 1.1 [p. 16], that in the absence of viscoelastic dampings $(k_\Omega, \alpha = 0)$, the variation of the frictional coefficients causes the shift of the vertical asymptote formed by the eigenvalues associated with the hyperbolic system (**FM**). Although the semigroup governing this system is not norm-continuous, implying that the investigation of the growth bound by the spectral bound is not relevant (see equation (6.0.3)), the understanding of the impact of c_Ω and c_Γ on the eigenvalues' behavior in the absence of the viscoelastic dampings $(k_\Omega, \alpha = 0)$ may help to determine the optimal growth bound in the presence of viscoelastic dampings

$(k_\Omega, \alpha > 0)$.

We refer the reader to the proof of exponential stability (theorem 5.1.1 in Part 2, case 2), where we mentioned that the relationship $c_\Omega = c_\Gamma$ lowers the estimate on $\int_0^s \left\| e^{\mathcal{A}t} u \right\|_{\mathcal{H}}^2 ds$. This observation can be linked to figure 6.1 as well. Given a value

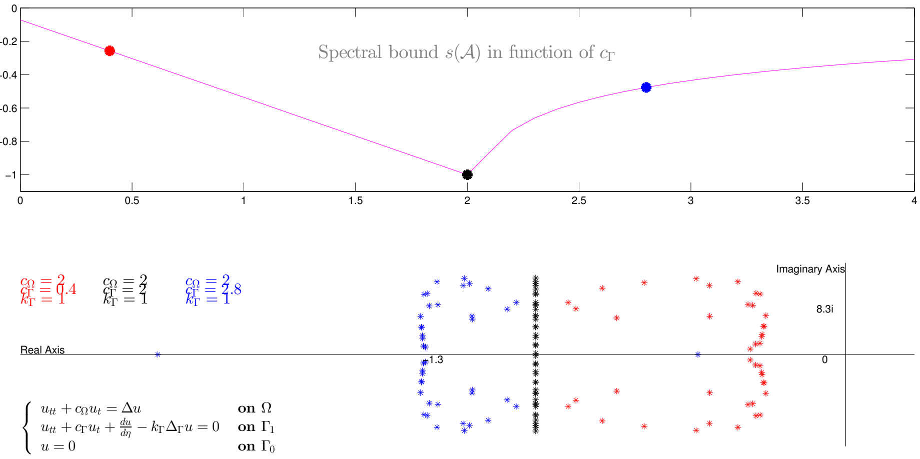


Figure 6.1: *Optimal spectral bound for a hyperbolic system.* Upper graph: optimal spectral bound in function of c_Γ. Lower graph: eigenvalues for the model (**FM**) with a fixed c_Ω and c_Γ running from 0 to $2c_\Omega$, $c_\Gamma = 0.4$ (red), 2 (black), 2.8 (blue).

of the interior frictional damping c_Ω, the coefficient c_Γ runs from 0 to $2c_\Omega$. On figure 6.1, the lower graph represents the spectrum for 3 different values of c_Ω. The red spectrum is associated with a small value of the boundary damping ($c_\Gamma = 0.2$) compared to the interior damping ($c_\Omega = 2$); as c_Γ gets closer to c_Ω, we observed that the eigenvalues tend to form a vertical asymptote (black spectrum on the lower graph) making the spectral bound reaches its minimum ($s(\mathcal{A}) = -1$). The upper graph represents the spectral bound in function of c_Γ, enlighting the spectral bound for the three spectrums with their respective colors. The optimal spectral bound is attained at $c_\Omega = c_\Gamma$. As the boundary damping c_Γ keeps increasing the first lower modes (eigenvalues with small imaginary part's absolute value) hit the real axis, then one tends to negative infinity while its conjugate moves back to the imaginary axis, implying a smaller spectral bound and thus a possible overdamping. We recall that the the semigroup governing this system is not norm continuous (hyperbolic system), hereby, the spectral bound is not a criterion to determine overdamping.

6.2 Optimal growth bound

Consider (**GM**) with all coefficients strictly positive ($c_\Omega, c_\Gamma, k_\Omega, k_\Gamma\alpha > 0$), we expect to get the optimal growth bound for this parabolic problem by adding the same amount of frictional damping in the interior and on the boundary in our model (**GM**) reminding some possible overdamping if c_Ω and c_Γ are too large (see figure 1.1 [p. 16]). Our choice for the values of k_Ω and $k_\Gamma\alpha$ do not have a physical interpretation. Without loss of generality, assume that $k_\Omega = \alpha = 0.1$ and $k_\Gamma = 1$, the consequence in the seek of optimal growth bound is not affected by this assumption however this allows a better readability of the figures. The figure 6.2 represents, on the lower graph, the spectrum of (**GM**) for:

- $c_\Omega = c_\Gamma = 0.1$ in red
- $c_\Omega = c_\Gamma = 1.9$ in black
- $c_\Omega = c_\Gamma = 2.8$ in blue

The upper graph shows the spectral bound in function of c_Ω and c_Γ under the assumption that $c_\Omega = c_\Gamma$. The spectral bound occurs for $c_\Omega = c_\Gamma \sim 1.9$. The spectrum is shifted to the left (eigenvalues' real part decreases) as the frictional coefficients increase to 1.9.

While most of the eigenvalues are still shifted to the left as the frictional dampings get larger than 1.8, we remark that purely real eigenvalues appear on the right of the circle and tend to the imaginary axis making the spectral bound increases, suggesting overdamping. The figure 6.3 enlights the exact same pattern in the absence of boundary viscoelastic damping. We recall that the semigroup governing this system is supposedly Gevrey. The circles described by the eigenvalues are not fully displayed in order to observe the shift of the parabola. On the blue spectrum ($c_\Omega = c_\Gamma = 3.6$), we observe the appearance of an eigenvalue, denoted by the big dot in blue, moving back to the imaginary axis inducing the increase of the growth bound.

Similarly to the pure frictional case ($k_\Omega = \alpha = 0$ - see figure 1.1 [p. 16]), the absolute value of the eigenvalues' imaginary part are squizzed, which is observed on figure 6.2

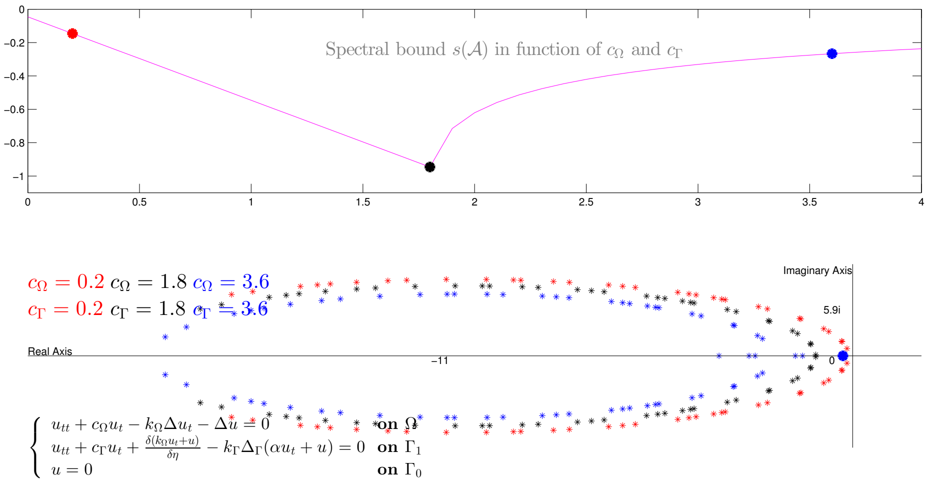


Figure 6.2: *Optimal growth bound for an analytic semigroup.* Upper graph: growth bound in function of $c_\Omega = c_\Gamma = 0 \to 4$. Lower graph: eigenvalues for the model (**GM**) with $k_\Omega = k_\Gamma \alpha = 0.1$ and $c_\Omega = c_\Gamma = 0.4$ (red), 1.7 (black), 2.8 (blue).

by the shrink of the circle's radius. An important fact is that the center of the circle remains at the same location, in the present case at $-10 + 0 \times i$, which is $-\frac{1}{k_\Omega} = -\frac{1}{\alpha}$. Thus, the center of circle is still uniquely determined by the viscoelastic dampings while the radius of the circle is affected by the presence of frictional damping. More importantly the shifting property of the frictional damping is preserved making possible the determination of the optimal growth bound. For both cases, the optimal growth bound is reached for $c_\Omega = c_\Gamma \sim 1.8$ and $s(\mathcal{A}) = \omega_0(\mathcal{A}) = 1$.

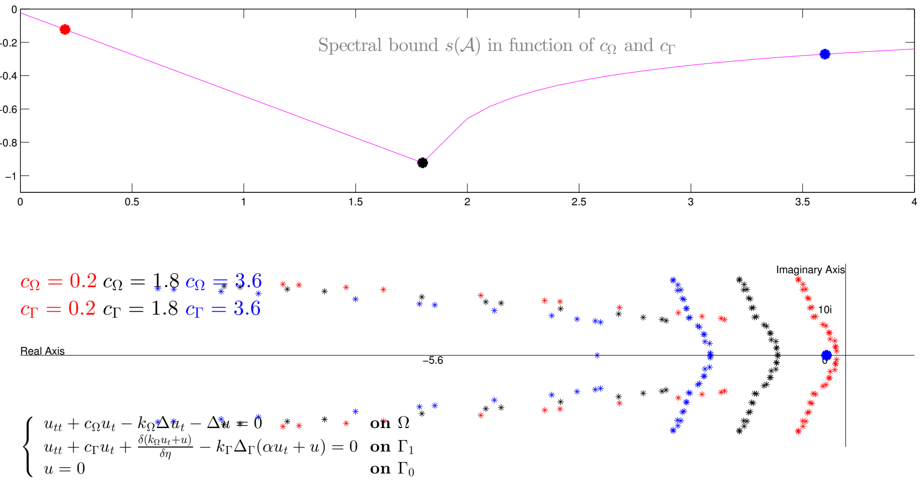


Figure 6.3: *Optimal growth bound for a Gevrey semigroup.* Upper graph: growth bound in function of $c_\Omega = c_\Gamma = 0 \to 4$. Lower graph: eigenvalues for the model (**GM**) with $k_\Omega = 0.05, k_\Gamma = 1, \alpha = 0$ and $c_\Omega = c_\Gamma = 0.2$ (red), 1.8 (black), 3.6 (blue).

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