Castelnuovo-Mumford Regularity And Edge Ideals

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To My Mother

Abstract

In this dissertation we study the homological algebra of the monomial ideals with a special emphasis on the topics of the Castenuovo-Mumford regularity and the powers of edge ideals of finite simple graphs. The main problem of this dissertation is to find optimal bounds for the regularity of powers of edge ideals. To do this, we prove the existence of a very special order of the minimal monomial generators of powers of the edge ideal. Using this order and some short exact sequence techniques we prove that the regularity of a power of an edge ideal can be bounded by the maximum of the regularities of the edge ideals of some very closely related graphs, and as corollaries we show that for various classes of graphs the higher powers of edge ideals have linear minimal free resolutions. One of these corollaries partially answers a case of a conjecture proposed by Eran Nevo and Irena Peeva. In the process of this study we introduce a new notion called even connectedness in finite simple graphs and derive various results related to it. In particular, we show that this behaves particularly nicely in the case of bipartite graphs and prove some results related to regularity of powers of edge ideals of bipartite graphs. We also study path ideals of finite simple graphs in the same spirit and show that various classes of path ideals also have linear minimal free resolution. Using similar techniques we also study the Cohen-Macaulayness of bipartite edge ideals and prove a new characterization for it.

Yatha sikha mayuranam

Naganam manayo yatha

Tadvadvedangasastranam

Ganitam murdhani sthitam

“Like the crowning crest of a peacock and the shining gem in the cobras hood, mathematics is the supreme Vedanga Sastra”.

Yajurveda, Circa 600 BC

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Introduction

The main themes of this thesis are the Castelnuovo-Mumford regularity and the edge ideals of finite simple graphs. The Castelnuovo-Mumford regularity (or simply regularity) of a graded module over a graded ring is a *measure of complexity* in the sense that an ideal generated by higher-degree polynomials is more complex. Homogeneous ideals in polynomial rings with low regularities are known to have "simple minimal free resolutions". This motivates mathematicians to find classes of homogeneous ideals with regularities bounded by certain values. Monomial ideals are examples of homogeneous ideals that come with lots of combinatorial data and it becomes interesting to interpret the algebraic properties in terms of the combinatorial properties. A significant portion of this work is devoted to the study of the interplay between them.

As a result of this study, we introduce a new technique for bounding the regularity of the powers of edge ideals and use that technique to find various new upper bounds. We also find some new upper bounds for the regularity of some special classes of path ideals using a somewhat similar method. After these we study the Cohen-Macaulayness of the bipartite edge ideals and derive a new characterization.

Chapter 1 deals with the basic preliminaries of free resolutions of finitely generated multigraded modules over polynomial rings, introduces some combinatorial notions related to the monomial ideals, and states some well-known results about regularity. Some examples are computed to illustrate these.

In Chapter 2 we study the basic properties of the Castelnuovo-Mumford regularity, especially its behaviour with respect to the short exact sequences. This chapter builds the framework for the theory developed in the next two chapters. At the end of this chapter, we prove two new theorems about the regularity of the edge ideals.

Chapter 3 is devoted to the study of the regularity of powers of edge ideals. We introduce a new notion called *even connection* in this chapter and prove various results related to that. One of these partially answers a question asked by Irena Peeva and Eran Nevo by proving all higher powers of the edge ideals of the gap free and cricket free graphs have linear minimal free resolutions. In another result we find an upper bound for the regularities of the higher powers of the edge ideals of the gap free graph with a fixed regularity r in terms of r. Our main result in this chapter is the following result regarding minimal free resolution:

Theorem: If G is a gap-free and cricket-free graph with edge ideal $I(G)$, then for every $s \geq 2$ the ideal $I(G)^s$ has a linear minimal free resolution.

In Chapter 4 we study the regularities of the path ideals in a somewhat similar way. However, the theory seems to be much more difficult. Here too we prove some new results of the same flavour as in Chapter 3. In particular, we prove that the higher path ideals of a gap-free, claw-free and whiskered K_4-free graph have linear minimal free resolutions. Our main result in this chapter is:

Theorem: Let G be a finite simple graph with t-path ideal I_t for $t \geq 3$. If G is gap free and claw free and $I_t \neq 0$, then I_t has a linear minimal free resolution for $t = 3, 4, 5, 6$. If G is gap free, claw free and whiskered K_4 free and $I_t \neq 0$ then I_t has linear minimal free resolution for all $t \geq 3$.

Chapter 5 is devoted to the study of the Cohen-Macaulay bipartite edge ideals. Here we first give a new proof an existing characterization of the Cohen-Macaulay bipartite edge ideals. The most interesting aspect of our proof is that unlike the other proofs it never uses Hall's Marriage theorem or any equivalent form of it. It simply uses the fact that a Cohen-Macaulay quotient is unmixed and connected at codimension one. After this we also prove a new characterization of this class of graphs. Our main result in this chapter is the following:

Theorem: Let G be a bipartite graph with edge ideal I and size of each partition n. Then I is Cohen-Macaulay if and only if there exists exactly n edges $e_1, ..., e_n$, such that $(I^2 : e_i)$ is Cohen-Macaulay, for $i \neq j$, e_i and e_j are disjoint and for any other edge e, $(I^2 : e)$ is not Cohen-Macaulay.

Finally we conclude this thesis in Chapter 6 by mentioning some of the ongoing works on each topic discussed earlier.

Contents

Chapter 1

Preliminaries

In this chapter we collect some basic facts about the free resolutions of the finitely generated multigraded modules over polynomial rings and the monomial ideals related to finite simple graphs. These will be used in the subsequent chapters. All along we'll assume $S = \mathbb{K}[x_1, \ldots, x_n]$, a polynomial ring in n variables over an arbitrary field $\mathbb{K}$.

1.1 Free Resolutions

Let M be a multigraded module over S, that is $\bigoplus_{i\in\mathbb{Z}^n} M_i$ such that for every degree d monomial α and for every $s \in M_i$, the element αs belongs to M_{i+d}. It is known that M can be successively approximated by free modules. Formally speaking there exists an exact sequence of minimal possible length called a minimal free resolution of M:

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{.d_p} \mathbb{F}_{p-1} \cdots \xrightarrow{.d_2} \mathbb{F}_1 \xrightarrow{.d_1} \mathbb{F}_0 \xrightarrow{.d_0} M \longrightarrow 0 \qquad (*)$$

Here $\mathbb{F}_i = \bigoplus S(\sigma^{-1})^{\beta_{i\sigma}}$, where $S(\sigma^{-1})$ denotes the free S-module generated in the degree σ for some monomial σ. Here $\beta_{i\sigma}$s are positive integers that are called the multigraded Betti numbers of M. For every j, $\beta_{ij} = \sum_{\{\sigma||\sigma|=j\}} \beta_{i\sigma}$ is called the ijth standard graded Betti number of M. Three very important homological invariants that are related to these numbers are the Castelnuovo-Mumford regularity, or simply regularity, the depth and the projective dimension, denoted by reg(M), depth(M) and pd(M) respectively:

$$\operatorname{reg}(M) = \max\{|\sigma| - i|\beta_{i\sigma} \neq 0\}$$

$$\operatorname{depth}(M) = \inf\{i|\operatorname{Ext}^i(\mathbb{K}, M) \neq 0\}$$

$$\operatorname{pd}(M) = \max\{i|\text{there is a } \sigma, \beta_{i\sigma} \neq 0\}$$

After introducing these we shall define three important notions.

Definition 1.1.1. *If all the entries of the matrices corresponding to the d_is in $(*)$ are either 0 or some variable then M is said to have a* linear minimal free resolution*. The linear minimal free resolution is the case of minimum possible regularity.*

Definition 1.1.2. *If all the entries of the matrices corresponding to the d_is in $(*)$ are either 0 or some variable for all $i \leq t$ then M is said to have a* t-linear minimal free resolution*.*

Definition 1.1.3. *If depth(M) is same as its Krull dimension then M is said to be Cohen-Macaulay.*

The following is a very important theorem:

Theorem 1.1.4 (Auslander-Buchbaum)**.** *Let R be a commutative noetherian local ring with unity. If M is a finitely generated R-module with finite projective dimension then depth(M)+ pd(M)=depth(R)*

We now illustrate these concepts with few examples.

Example 1.1.5. Let $M = \frac{\mathbb{Q}[x_1,\ldots,x_5]}{(x_1x_2,x_2x_3,x_3x_4,x_4x_5,x_5x_1)}$. Then the minimal free resolution of M is:

$$0 \longrightarrow \mathbb{F}_3 \xrightarrow{.d_3} \mathbb{F}_2 \xrightarrow{.d_2} \mathbb{F}_1 \xrightarrow{.d_1} \mathbb{F}_0 \xrightarrow{.d_0} M \longrightarrow 0$$

Here:

$$\beta_{0\sigma} = 1 \text{ if } \sigma = 1, \text{ and } \beta_{0\sigma} = 0 \text{ otherwise}$$

$$\beta_{1\sigma} = 1 \text{ if } \sigma = x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1, \text{and } \beta_{1\sigma} = 0 \text{ otherwise}$$

$$\beta_{2\sigma} = 1 \text{ if } \sigma = x_1x_2x_3, x_2x_3x_4, x_1x_2x_5, x_1x_4x_5, x_3x_4x_5, \text{and } \beta_{2\sigma} = 0 \text{ otherwise}$$

$$\beta_{3\sigma} = 1 \text{ if } \sigma = x_1x_2x_3x_4x_5, \text{ and } \beta_{3\sigma} = 0 \text{ otherwise}$$

Consequently the regularity is 2, and projective dimension is 3. Hence by the Auslander-Buchbaum theorem the depth of M is 2. As its Krull dimension is 2 it is Cohen-Macaulay.

Example 1.1.6. Let $M = \frac{\mathbb{Q}[x_1,\ldots,x_6]}{(x_1x_2,x_2x_3,x_3x_4,x_4x_5,x_5x_6)}$. Then the minimal free resolution of M is:

$$0 \longrightarrow \mathbb{F}_4 \xrightarrow{.d_4} \mathbb{F}_3 \xrightarrow{.d_3} \mathbb{F}_2 \xrightarrow{.d_2} \mathbb{F}_1 \xrightarrow{.d_1} \mathbb{F}_0 \xrightarrow{.d_0} M \longrightarrow 0$$

Here:

$$\beta_{0\sigma} = 1 \text{ if } \sigma = 1, \text{ and } \beta_{0\sigma} = 0 \text{ otherwise}$$

$$\beta_{1\sigma} = 1 \text{ if } \sigma = x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, \text{and } \beta_{1\sigma} = 0 \text{ otherwise}$$

$$\beta_{2\sigma} = 1 \text{ if } \sigma = x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_6, x_1x_2x_4x_5, x_1x_2x_5x_6, x_2x_3x_5x_6,$$

$$\text{and } \beta_{2\sigma} = 0 \text{ otherwise}$$

$$\beta_{3\sigma} = 1 \text{ if } \sigma = x_1x_2x_3x_4x_5, x_1x_2x_3x_5x_6, x_1x_2x_4x_5x_6, x_2x_3x_4x_5x_6$$

$$\text{and } \beta_{3\sigma} = 0 \text{ otherwise}$$

$$\beta_{4\sigma} = 1 \text{ if } \sigma = x_1x_2x_3x_4x_5x_6, \text{ and } \beta_{4\sigma} = 0 \text{ otherwise}$$

So it has regularity 2 and projective dimension 4. So by the Auslander-Buchbaum theorem the depth is 2. As its Krull dimension is 3, M is not Cohen-Macaulay.

1.2 Edge Ideals And Path Ideals

Let G be a finite simple graph (that is G has no loops or multiple edges) on $x_1, \ldots, x_n$. We first recall some relevant definitions.

Definition 1.2.1. *For x_i, x_j, we let $d(x_i, x_j)$ denote the* distance *between x_i and x_j, that is the fewest number of edges that must be traversed to travel from x_i to x_j.*

Definition 1.2.2. *A subgraph $G' \subseteq G$ is called* induced *if uv is an edge of G' whenever u and v are vertices of G' and uv is an edge of G.*

Definition 1.2.3. *The* complement *of a graph G, for which we write G^c, is the graph on the same vertex set in which uv is an edge of G^c if and only if it is not an edge of G.*

Notation 1.2.4. *Let C_k denote the cycle on k vertices, and we let $K_{m,n}$ denote the complete bipartite graph with m vertices on one side, and n on the other.*

One of the most important concepts in this thesis is the next definition.

Definition 1.2.5. *Let G be a graph. We say two edges uv and xy form a* gap *in G if G does not have an edge with one endpoint in $\{u, v\}$ and the other in $\{x, y\}$. A graph without gaps is called* gap-free*. Equivalently, G is gap-free if and only if G^c contains no induced C_4.*

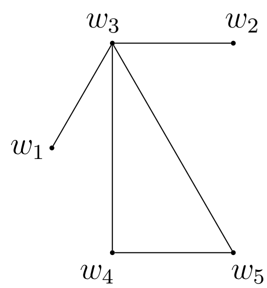
Thus, G is gap-free if and only if it does not contain two vertex-disjoint edges as an induced subgraph.

Definition 1.2.6. *Any graph isomorphic to $K_{1,3}$ is called a* claw*. Any graph isomorphic to $K_{1,n}$ is called an* n-claw*. If $n > 1$, the vertex with degree n is called the root in $K_{1,n}$. A graph without an induced claw is called* claw-free*. A graph without an induced* n-claw *is called* n-claw-free*. In both cases the vertex with degree more than one is called the root.*

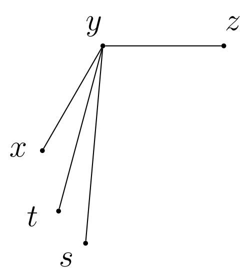
The n-claw is also called a star graph in some literature.

Definition 1.2.7. *Any graph isomorphic to the graph with set of vertices $\{w_1, w_2, w_3, w_4, w_5\}$ and set of edges $\{w_1w_3, w_2w_3, w_3w_4, w_3w_5, w_4w_5\}$ is called a* cricket*. A graph without an induced cricket is called* cricket-free*.*

The following is a cricket:



The following is a 4-claw:



Definition 1.2.8. *An edge in a graph is called a whisker if any of its vertices has degree one.*

Definition 1.2.9. *A graph with* 8 *vertices* $\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$ *and set of edges* $\{x_iy_i | \textit{for all } i\} \cup \{y_iy_j | \textit{for all } i, j\}$ *is called a whiskered-*K_4*. A graph without an induced whiskered-*K_4 *is called* whiskered-K_4 free*.*

Definition 1.2.10. *A graph is called an anticycle if its complement is a cycle.*

Observation 1.2.11. *A claw-free graph is cricket-free.*

There are various monomial ideals in S that are associated to G. Among these, the edge ideals and the path ideals along with their powers and colons have been the major focus of this thesis.

For any graph G with set the of vertices $x_1, \ldots, x_n$, let S be the polynomial ring on $x_1, \ldots, x_n$ over a field. We also denote the set of edges of G by $E(G)$.

Definition 1.2.12. *For every $t \geq 2$ the t-path ideal $I_t(G)$ is defined as follows:*

$$I_t(G) = (x_{i_1} \cdots x_{i_t} |\ \textit{for all } l, l', l \neq l', x_{i_l} \neq x_{i_{l'}},\ \textit{and for all } j, x_{i_j} x_{i_{j+1}} \in E(G))$$

Definition 1.2.13. *For $t = 2$, $I_t(G)$ is called the edge ideal of G and is denoted by $I(G)$.*

Example 1.2.14. If G is the 5-cycle on $x_1 \cdots x_5$ then

$I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$,

$I_3(G) = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2)$,

$I_4(G) = (x_1x_2x_3x_4, x_2x_3x_4x_5, x_3x_4x_5x_1, x_4x_5x_1x_2, x_5x_1x_2x_3)$,

$I_5(G) = (x_1x_2x_3x_4x_5)$ and $I_t = (0)$ for all $t \geq 6$.

G is both gap free and claw free and as G^c is also a 5-cycle, G is an anticycle too.

Chapter 2

Castelnuovo-Mumford Regularity

Castelnuovo-Mumford regularity is one of the most important homological invariants. Having lower Castelnuovo-Mumford regularity is equivalent to having “simpler” minimal free resolution in some sense. There is an ongoing stream of research to find classes of modules with minimal regularity or low regularity. In the case of monomial ideals related to graphs one is interested to find classes of graphs whose edge ideals or their powers have minimal regularity.

In the first section of this chapter we recall some of the basic results and notions related to this concept. In particular we state and prove a lemma which forms the basic framework for many proofs of this thesis. We also state some results about regularity of edge ideals and their powers.

In the last section we prove two new results that are generalizations of a result by Eran Nevo (Theorem 3.1, [N]). All along we assume that S is a polynomial ring in finitely many variables over an arbitrary field $\mathbb{K}$.

2.1 Some Known Facts

In this section we collect some known facts about Castelnuovo-Mumford regularity that will be used subsequently. We mostly skip the proofs but we provide references for the interested reader. We begin this section by recalling the definition of a k-step linear resolution (already defined in the previous chapter). This is a generalization of the notion of a linear minimal free resolution.

Definition 2.1.1. *For a finitely generated S-module M, we say that M is* k-steps linear *whenever the matrices of the minimal free resolution of M over the polynomial ring consist of linear terms up to the k^{th} step. We say that M has linear minimal free resolution if the minimal free resolution is k-steps linear for all $k \geq 1$. We say that M has a linear presentation if it has a 1-step linear minimal free resolution.*

Example 2.1.2. As we saw in the previous chapter, if $S = \mathbb{Q}[x_1, \ldots, x_5]$ and I is the edge ideal of the 5-cycle then $\frac{S}{I}$ has a 2-linear resolution which is not 3-linear.

The following is immediate from the definition of a minimal free resolution:

Observation 2.1.3. *Let $I(G)$ be the edge ideal of a graph G. Then $I(G)^s$ has a linear minimal free resolution if and only if reg$(I(G)^s) = 2s$.*

We first prove a well-known result about Castelnuovo-Mumford regularity, which we shall use repeatedly throughout this thesis.

Lemma 2.1.4. *Let M', M, and M'' be three multigraded modules over S such that the following is an exact sequence of homogeneous S-modules, that is the maps are homogeneous of degree zero:*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

Then $\operatorname{reg}(M) \leq \max\{\operatorname{reg}(M'), \operatorname{reg}(M'')\}$.

Proof. We consider the homogeneous long exact sequence of Tor modules corresponding to the given short exact sequence:

$$\cdots \longrightarrow \operatorname{Tor}_i(M', \mathbb{K}) \longrightarrow \operatorname{Tor}_i(M, \mathbb{K}) \longrightarrow \operatorname{Tor}_i(M'', \mathbb{K}) \longrightarrow \operatorname{Tor}_{i-1}(M', \mathbb{K}) \longrightarrow \cdots$$

As this sequence is both homogeneous and exact, for every monomial σ if $\operatorname{Tor}_i(M, \mathbb{K})_\sigma \neq 0$ then either $\operatorname{Tor}_i(M', \mathbb{K})_\sigma \neq 0$ or $\operatorname{Tor}_i(M'', \mathbb{K})_\sigma \neq 0$. We now observe that the result follows from the definition of regularity. □

We now state a few well-known results without proofs. We refer reader to [B1] and [DHS] for reference.

Lemma 2.1.5. *Let $I \subseteq S$ be a monomial ideal. Then for any variable x, reg$(I, x) \leq$* $\operatorname{reg}(I)$. *In particular, if v is a vertex in a graph G and $G - v$ is the graph obtained from G by deleting v then* $\operatorname{reg}(I(G - v)) \leq \operatorname{reg}((I(G))$.

The next lemma is a straight forward consequence of Lemma 2.1.1.

Lemma 2.1.6. *Let $I \subseteq S$ be a monomial ideal, and let m be a monomial of degree d. Then*

$$\operatorname{reg}(I) \leq \max\{\operatorname{reg}(I : m) + d, \operatorname{reg}(I, m)\}.$$

Moreover, if m is a variable x appearing in I, then $\operatorname{reg}(I)$ is equal to one of these terms.

The following lemma is by Dao, Huneke and Schweig shows that if m is a variable then the situation gets significantly better. It is Lemma 2.10 of [DHS].

Lemma 2.1.7 (Dao, Huneke, Schweig)**.** *Let $I \subseteq S$ be a monomial ideal, and let x be a variable appearing in I. Then*

$$\operatorname{reg}(I) \leq \max\{\operatorname{reg}(I : x) + 1, \operatorname{reg}(I, x)\}.$$

Moreover $\operatorname{reg}(I)$ is equal to one of these terms.

The following lemma is a consequence of Lemma 2.1.4. This lemma provides the framework for many proofs in this thesis. Although it is well known, due to its relevance we prove it below.

Lemma 2.1.8. *Let I and J be two homogeneous monomial ideals in S generated in degrees n_1 and n_2 respectively. Assume $J \subset I$ and n_2 is strictly greater than n_1. If the unique set of minimal monomial generators of I is $\{m_1, m_2, \ldots, m_k\}$, $A = \max\{\operatorname{reg}(J : m_1) + n_1\}$, $B = \max\{\operatorname{reg}((J, m_1, \ldots, m_l) : m_{l+1}) + n_1 | 1 \leq l \leq k-1\}$ and $C = \operatorname{reg}(I)$, then $\operatorname{reg} J \leq \max\{A, B, C\}$.*

Proof. We consider the following short exact sequence:

$$0 \longrightarrow \frac{S}{(J : m_1)}(-n_1) \xrightarrow{.m_1} \frac{S}{J} \longrightarrow \frac{S}{(J, m_1)} \longrightarrow 0$$

This gives us $\operatorname{reg}(J) \leq \max\{A, \operatorname{reg}(J, m_1)\}$. Let $J_l := ((J, m_1, \ldots, m_{l-1}) : m_l)$ for all $l \geq 2$. For all $1 \leq l \leq k-1$ we consider the exact sequence

$$0 \longrightarrow \frac{S}{(J_{l+1})}(-n_1) \xrightarrow{.m_{l+1}} \frac{S}{(J, m_1, \ldots, m_l)} \longrightarrow \frac{S}{(J, m_1, \ldots, m_{l+1})} \longrightarrow 0,$$

This gives us

$$\operatorname{reg}(J, m_1, \ldots, m_l) \leq \max\{\operatorname{reg}(J_{l+1}) + n_1, \operatorname{reg}(J, m_1, \ldots, m_{l+1})\}$$

from which $\operatorname{reg}(J) \leq \max\{A, B, C\}$ follows. □

The next well-known theorem connects regularity with local cohomology, for a proof see Chapter 4 of [E2]:

Theorem 2.1.9. *If $H^j_m(M)$ denotes the j^{th} local cohomology module of M with support m, where m is the homogeneous maximal ideal then the following holds:*

$$\operatorname{reg}(M) = max_j\{\operatorname{reg}(H^j_m(M) + j\}.$$

Finally, the following theorem due to Fröberg (See Theorem 1 of [Fro] and Theorem 1.1 of [NP]) is used repeatedly throughout this thesis:

Theorem 2.1.10 (Fröberg)**.** *The minimal free resolution of $I(G)$ is linear if and only if the complement graph G^c is chordal, that is, every induced cycle in G^c is a triangle.*

The next few results are of a similar spirit.

Theorem 2.1.11 (Herzog, Hibi, Zheng)**.** *If $I(G)$ has a linear minimal free resolution then for all $s \geq 2$, $I(G)^s$ also has a linear minimal free resolution.*

Theorem 2.1.12 (Francisco, Ha, Van Tuyl)**.** *If $I(G)^s$ has a linear resolution for an $s \geq 1$, the G^c has no induced 4-cycles.*

Theorem 2.1.13 (Nevo)**.** *Suppose G is both claw-free and G^c does not have any induced 4-cycle. Then reg $I(G) \leq 3$ and reg $I(G)^2 = 4$.*

Definition 2.1.14. *For any graph G, we write* $\operatorname{reg}(G)$ *as a shorthand for* $\operatorname{reg}(I(G))$.

Recall that the *star* of a vertex x of G, for which we write st x, is given by

$$\text{st } x = \{y \in V(G) : xy \text{ is an edge of } G\} \cup \{x\}.$$

The following lemma is Lemma 3.2 of [B1], which we shall use a lot in this work.

Lemma 2.1.15. *Let x be a vertex of G with neighbors $y_1, y_2, \ldots, y_m$. Then*

$$(I(G) : x) = (I(G - st\ x), y_1, \ldots, y_m) \text{ and } (I(G), x) = (I(G - x), x).$$

Thus, $\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G - st\ x) + 1, \operatorname{reg}(G - x)\}$*. Moreover,* $\operatorname{reg}(G)$ *is equal to one of these terms.*

The next proposition is Proposition 3.3 of [DHS] (See also [No]).

Proposition 2.1.16. *Let G be gap-free, and let x be a vertex of G of highest degree. Then $d(x, y) \leq 2$ for all vertices y of G.*

The next result is a generalization of Fröber's theorem by Eisenbud, Hulek and Popescu. It is the Theorem 2.1 of [EHP].

Theorem 2.1.17 (Eisenbud-Hulek-Popescu). *Let G be a finite simple graph with edge ideal $I(G)$. Then $I(G)$ has a p-linear resolution if and only if every induced cycle in G^c that is not a triangle has length $\geq p+3$*

Example 2.1.18. *Every induced cycle in a complement of a 5-cycle has length $\geq 2+3$. One can check from Example 1.1.5 that a 5-cycle has a 2-linear resolution.*

We finish this section with a result about Betti numbers which follows from Lemma 1.3.8 of Kummini's thesis [K2].

Lemma 2.1.19. *Let I be a squarefree monomial ideal in S and x be a variable. If $\beta_{ij}(I, x) \neq 0$ then either $\beta_{ij}(I) \neq 0$ or $\beta_{i-1j-1}(I) \neq 0$.*

In the next section we generalize Nevo's result in two different ways, using the results of this section.

2.2 Two New Results

In this section we prove two generalizations of Nevo's (Theorem 3.1, [N]) result:

Theorem 2.2.1. *Suppose G is both cricket-free and gap-free. Then* $\operatorname{reg}(G) \leq 3$.

Proof. Let x be a vertex of maximum degree. As G is gap free and cricket free, so is $G-x$. By induction, $G-x$ has regularity less than or equal to 3. Because of Theorem

2.1.10 and Lemma 2.1.15, it is enough to show that $(G - \text{st } x)^c$ has no induced cycle of length greater than or equal to 4. As G is gap free, so is $(G - \text{st } x)$; hence, $(G - \text{st } x)^c$ has no induced 4−cycle. So it is enough to show it does not have an induced cycle of length greater than or equal to 5.

Let $\{y_1, y_2, y_3, y_4, \ldots, y_n\}$ be an induced cycle $(n \geq 5)$ in $(G - \text{st } x)^c$; because of Proposition 2.1.16, there is a w such that xw and wy_1 are edges in G. As y_2y_n is an edge in G, and neither y_1y_2 nor y_1y_n are edges in G, either wy_2, wy_n or both are edges in G. If both are edges then $\{x, w, y_1, y_2, y_n\}$ forms an induced cricket.

Suppose only one of them is an edge. Without loss of generality, we may assume wy_2 is an edge. As y_3y_n is an edge in G, and G gap free, wy_3 is an edge in G; otherwise $\{x, w, y_3, y_n\}$ forms a gap in G. This makes $\{x, w, y_1, y_2, y_3\}$ an induced cricket. □

Theorem 2.2.2. *The edge ideal of a graph which is gap free and n-claw free, has regularity less than or equal to n.*

Proof. For $n = 3$, this was proved by E. Nevo and this is Theorem 3.3 of [DHS]. So we may assume $n \geq 4$. Let x be a vertex with maximum degree. Because of Lemma 2.1.15, it is enough to show $G - \text{st } x$ has regularity less than or equal to $n - 1$; as $G - x$ has regularity less than or equal to n by induction on number of vertices. Hence, it is enough to show $G - \text{st } x$ is $(n-1)$-claw free.

If $a_1, a_2, a_3, \ldots, a_n$ is a $(n-1)$-claw with root a_1 in $G - \text{st } x$ then any w in the neighborhood of x is either connected to a_1 or all of $a_2, a_3, \ldots, a_n$; otherwise if w is not connected to a_1 and a_i then xw and $a_1 a_i$ will form a gap. If a_1 is connected to all the neighbors of x, it has a degree strictly more than x, which is contradictory to the assumption that x is a vertex with maximum degree. Hence, there is a neighbor w which is not connected to a_1 but is connected to all of $a_2, a_3, \ldots, a_n$. As x is not connected to any of the a_is, $\{x, w, a_2, a_3, \ldots, a_n\}$ forms an n-claw with root w, which is contradictory to the hypothesis. $\square$

2.3 Asymptotic Behaviour Of Regularity

One question that has been studied by researchers extensively is, for a homogeneous ideal I in S, whether $\text{reg}(I^s)$ shows some asymptotic behaviour as s goes to infinity. The following important result gives some indication about what to expect:

Theorem 2.3.1 (Cutkosky-Herzog-Trung)**.** *If I is a homogeneous ideal in S with maximum degree of generator $d(I)$ then $\text{reg}(I^s)$ is asymptotically a linear function of s and there is a number e such that $\text{reg}(I^s) \leq sd(I) + e$ for all $s \geq 1$.*

Similarly:

Theorem 2.3.2 (Kodiyalam)**.** *If I is a homogeneous ideal generated in degree d then* *there exists n such that* $\operatorname{reg}(I^s)$ *equals to* $Ps + Q$ *for* $s \geq n$*, where* P *and* Q *are two constants.*

Eisenbud and Harris proved a related result in [EH].

Theorem 2.3.3 (Eisenbud-Harris). *Let* $X \subseteq \mathbb{P}^n$ *be a projective scheme with homogeneous coordinate ring* S_X*, and let* $\phi : X \to \mathbb{P}^s$ *be a linear projection whose center does not meet* X*, defined by an* $s+1$*-dimensional vector space of linear forms* V*. Let* $I \subseteq S_X$ *be the ideal generated by* V*, and let* m *be the maximal homogeneous ideal of* S_X*. The maximum of the Castelnuovo-Mumford regularities of the fibers of* ϕ *over closed points of* $\mathbb{P}^s$ *is one more than the least* ϵ *such that, for large* t*,* $m^{t+\epsilon} \subseteq I^t$*; the number* $t + \epsilon$ *is equal, for large* t*, to the Castelnuovo-Mumford regularity of* I^t*.*

For the definition of the relevant notions a proof of this result see [EH].

These results open up research to find the values of P, Q, and n. The result by Herzog-Hibi-Zheng regarding edge ideals show that if I is an edge ideal with a linear minimal free resolution then $P = 2$, $Q = 0$, and $n = 1$. In the following chapters we shall compute these values for various classes of monomial ideals. Recently similar work has been done in [BHT].

Chapter 3

Linear Resolutions Of Powers Of Edge Ideals

In this chapter we find new upper bounds for the regularity of powers of edge ideals of some classes of graphs. Our original motivation is the following question, which is the base case of the Open Problem 1.11(2) in [NP]:

Question 3.0.4. *Let $I(G)$ be the edge ideal of the graph G which does not have any induced four cycle in its complement. If $\operatorname{reg}(I(G)) \leq 3$, then is it true that for all $s \geq 2$, $I(G)^s$ has a linear minimal free resolution?*

Bounds on the regularity of edge ideals have been studied by a number of researchers (see [DHS], [A], [Fr], [HVT1]). For example, Fröberg (see [Fr]) has shown that, when $I(G)$ is the edge ideal of a graph whose complement does not have any induced cycle of size greater than or equal to four, then $I(G)$ has a linear minimal free resolution.

We are interested in finding upper bounds on the regularities of the higher powers of $I(G)$. Herzog, Hibi and Zheng have shown in [HHZ] that if $I(G)$ is the edge ideal of a graph G which has no induced cycle of length greater than or equal to four in its complement (that is $I(G)$ has a linear minimal free resolution) then for all $s \geq 2$, $I(G)^s$ has a linear minimal free resolution. Fransisco, Hà, and Van-Tuyl have further shown that if $I(G)^s$ has a linear minimal free resolution for some s, then G has no induced four cycle in its complement (Proposition 1.8 in [NP]). These two results lead us to study bounds on the regularity of powers of $I(G)$ when G has no induced four cycle in its complement. Our main result is Theorem 3.3.5 where we prove all higher powers of edge ideals of a gap free (equivalently, no induced four cycle in complement, as observed in section 2) and cricket free (defined in section 2) graph have linear minimal free resolutions, that is (to use notation of theorem 2.3.2) in this case $P = 2$, $Q = 0$ and $n = 2$. More precisely:

Theorem 3.0.5. *For any gap free and cricket free graph G and for all $s \geq 2$, $\mathrm{reg}(I(G)^s) = 2s$ and as a consequence $I(G)^s$ has a linear minimal free resolution.*

This partially answers Question 3.0.3, as we proved in the previous chapter that the edge ideals of gap free and cricket free graphs have regularity less than or equal to 3 (Theorem 2.2.1). As claw free graphs are automatically cricket free, our results generalize a previous result by E. Nevo (Theorem 1.2 of [N]) that says the edge ideals of gap free and claw free graphs have regularity less than or equal to 3 and their squares have linear minimal free resolutions.

Notation 3.0.6. *Let* $(m : n)$ *stand for* $((m) : (n))$ *for monomials* m *and* n*.*

In order to prove Theorem 3.3.5, we first show that the minimal monomial generators of powers of edge ideal $I(G)$ for any finite simple graph G have a specific order that satisfies some nice properties (Lemma 3.1.11, Theorem 3.1.12). More precisely:

Theorem 3.0.7. *For each* $n \geq 1$ *there exists an ordered list* $L^{(n)}$ *of minimal monomial generators of* $I(G)^n$ *which satisfies the following property:*
For all $k \geq 1$ *and for all* $j \leq k$*, if* $(L_j^{(n)} : L_{k+1}^{(n)})$ *is not contained in* $(I(G)^{n+1} : L_{k+1}^{(n)})$ *then there exists* $i \leq k$*, such that* $(L_i^{(n)} : L_{k+1}^{(n)})$ *is generated by a variable and* $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (L_i^{(n)} : L_{k+1}^{(n)})$*.*

Using this ordering we shall prove that $\operatorname{reg}(I(G)^n)$ is bounded above by the maximum of $\operatorname{reg}(I(G)^n : e_1 \cdots e_{n-1}) + 2n - 2$ for all possible $(n-1)$-fold products of edges $e_1 \cdots e_{n-1}$ and $\operatorname{reg}(I(G)^{n-1})$ (See Theorem 3.1.13). Next we prove that the ideals

$(I(G)^n : e_1 \cdots e_{n-1})$ are quadratic monomial ideals with generators satisfying certain conditions (See Theorems 3.2.1, 3.2.5, 3.2.7). Finally, by using polarization technique we get edge ideals corresponding to these quadratic monomial ideals with same regularity (See [K2], Section 3.2 and Exercise 3.15 of [MS] for details) and using Fröberg's theorem (See Theorem 1 of [Fro] and Theorem 1.1 of [NP]) get bounds on them. As a consequence we also get a different proof of the Herzog, Hibi and Zheng's theorem mentioned above (Theorem 2.1.11).

3.1 A Special Order

In this section we show that the minimal monomial generators of powers of edge ideals can be ordered in a very specific way. This will be immensely helpful. The work of this section can be found in the section 4 of our paper [B1].

Discussion 3.1.1. *Let the set of minimal monomial generators of any ideal $J \subset S$ be denoted by* $\mathrm{Mingens}(J)$. *Let I be an arbitrary edge ideal. Set* $\mathrm{Mingens}(I) = \{L_1, L_2, \ldots, L_k\}$. *We give* $\mathrm{Mingens}(I)$ *the following order: $L_1 > L_2 > \ldots > L_k$. We will put an order on* $\mathrm{Mingens}(I^n)$ *for all integers $n \geq 2$ as follows: For $n > 1$, we say $M > N$ for $M, N \in$* $\mathrm{Mingens}(I^n)$ *if there exists an expression $L_1^{a_1} L_2^{a_2} \cdots L_k^{a_k} = M$*

such that for all expressions $L_1^{b_1} \cdots L_k^{b_k} = N$, *we have* $(a_1, \ldots, a_k) >_{lex} (b_1, ..., b_k)$. *If* $(a_1, ..., a_k) \geq_{lex} (c_1, \ldots, c_k)$ *for all* $(c_1, \ldots, c_k)$ *such that* $L_1^{c_1} \cdots L_k^{c_k} = M$ *then* $L_1^{a_1} L_2^{a_2} \cdots L_k^{a_k}$ *is called a maximal expression of* M. *Let* $L^{(n)}$ *be the totally ordered set of minimal monomial generators of* I^n, *ordered in the way discussed above.*

Definition 3.1.2. *If* m_1 *is a minimal monomial generator of* I^k *and* m_2 *is a minimal monomial generator of* I^n *where* $n > k$, *we say* m_1 *divides* m_2 *as an edge and use the notation* $m_1|^{edge}m_2$, *if there exists* m_3, *a minimal monomial generator of* I^{n-k} *with* $m_2 = m_1 m_3$.

Example 3.1.3. *If* $I = (ab, bc, ad, bd)$ *then* $ab|^{edge}ab^2d$ *as* $bd = \frac{ab^2d}{ab}$ *is a minimal monomial generator of* I *but* $ab \nmid^{edge} abcd$ *as* $cd = \frac{abcd}{ab}$ *is not a minimal monomial generator of* I.

Discussion 3.1.4. *We have the following for the list* $L^{(n)}$ *created above:*

1. $L^{(1)} = L := \{L_1 > \ldots > L_k\}$

2. For any minimal monomial generator m *of* I^n, $n \geq 2$, *the maximal expression of* m, *is an expression of* m *as a product of* n *elements of* L, $m = L_{i_1} L_{i_2} \cdots L_{i_n}$, *where:*

a. i_1 is the minimum integer such that $L_{i_1}|^{edge}m$

b. For all $l \geq 1$, i_{l+1} is the minimal integer such that $L_{i_{l+1}}|^{edge}\frac{m}{L_{i_1}\cdots L_{i_l}}$. For any edge cd we say cd is a part of the maximal expression of m if $cd = L_{i_k}$ for some k.

This expression is unique by the construction.

3. For two minimal monomial generators m_1, m_2 with maximal expressions $m_1 = L_{i_1}\cdots L_{i_n}$ and $m_2 = L_{j_1}\cdots L_{j_n}$, we have $m_1 >_{lex} m_2$ if for the minimum integer l such that $i_l \neq j_l$, $i_l < j_l$.

4. If L_i and L_j are two generators of I with $i < j$, then we say "L_j comes after L_i" or "L_i comes before L_j".

Example 3.1.5. *Let $I = (ab, bc, ad, bd)$. Let $L^{(1)} = \{ab > bc > ad > bd\}$. Then $L^{(2)} = \{a^2b^2 > ab^2c > a^2bd > ab^2d > b^2c^2 > abcd > b^2cd > a^2d^2 > abd^2 > b^2d^2\}$.*

Definition 3.1.6. *If $L_i = ab$ is an edge, that is a minimal monomial generator of I, and m is a minimal monomial generator of I^n, $n \geq 2$, then we say m belongs to ab, or m belongs to L_i, if i is the least integer such that $L_i|^{edge}m$.*

Example 3.1.7. *Let $I = (ab, bc, ad, bd)$ with $L = L^{(1)} = \{ab > bc > ad > bd\}$. Then*

$abcd$ belongs to $L_2 = bc$ as $ab \nmid^{edge} abcd$ and $bc|^{edge}abcd$ and ab^2d belongs to $L_1 = ab$ as $ab|^{edge}ab^2d$.

We record several easy observations that we need in the sequel.

Observation 3.1.8. *For two minimal monomial generators m_1, m_2, if m_1 belongs to an edge L_i and m_2 belongs to another edge L_j with $i < j$, then $m_1 >_{lex} m_2$.*

Observation 3.1.9. *For two minimal monomial generators m_1, m_2 of I^n which both belong to an edge L_i, we see that $m_1 >_{lex} m_2$ if and only if $\frac{m_1}{L_i} >_{lex} \frac{m_2}{L_i}$.*

Observation 3.1.10. *Suppose m is a minimal monomial generator of I^n, $n \geq 2$, and gh is an edge which is a part of the maximal expression of m. Write $m = ghm'$. For any minimal monomial generator m'' of I^{n-1} such that $m'' >_{lex} m'$, then $ghm'' >_{lex} m$.*

Proof. Let $L = \{L_1 > L_2 > \ldots > L_k\}$. Let $gh = L_j$ for some j. Let $m'' = L_1^{a_1} L_2^{a_2} \cdots L_k^{a_k}$ be the maximal expression of m'' and $m' = L_1^{b_1} L_2^{b_2} \cdots L_k^{b_k}$ be the maximal expression of m'. As gh is part of the maximal expression of m, the maximal expression of m is $L_1^{b_1}\cdots L_j^{b_j+1}\cdots L_k^{b_k}$. As by assumption $(a_1,\ldots,a_j,\ldots a_k) >_{\text{lex}} (b_1,\ldots,b_j,\ldots,b_k)$, we have $(a_1,\ldots,a_j+1,\ldots,a_k) >_{\text{lex}} (b_1,\ldots,b_j+1,\ldots,b_k)$. Now $L_1^{a_1}\cdots L_j^{a_j+1}\cdots L_k^{a_k}$ is an expression for ghm''. Hence $ghm'' >_{\text{lex}} ghm' = m$. $\square$

The next lemma is the most important technical result of this thesis.

Lemma 3.1.11. *For all $k \geq 1$ and for all $j \leq k$, if $(L_j^{(n)} : L_{k+1}^{(n)})$ is not contained in $(I^{n+1} : L_{k+1}^{(n)})$ and $L_j^{(n)}$ belongs to an edge that comes before the edge $L_{k+1}^{(n)}$ belongs to, then there exists $i \leq k$, such that $(L_i^{(n)} : L_{k+1}^{(n)})$ is generated by a variable, $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (L_i^{(n)} : L_{k+1}^{(n)})$ and $L_i^{(n)}$ belongs to an edge that comes before or equal to the edge $L_j^{(n)}$ belongs to.*

Proof. We prove the Lemma by induction on n. We recall that for two monomials m_1 and m_2, $(m_1 : m_2) = (\frac{m_1}{\gcd(m_1,m_2)})$. This is going to be used in several places.

If $n = 1$, $(L_j : L_{k+1})$ is either (L_j), in which case $(L_j : L_{k+1}) \subseteq (I^2 : L_{k+1})$ or it is generated by a variable in which case we take $L_i = L_j$. Hence the lemma is true for $n = 1$.

Suppose the result is true for $n-1$. Let $L_j^{(n)}$ belong to ab, so that $L_j^{(n)} = abM_1$

where $M_1 \in L^{(n-1)}$. By assumption $L_{k+1}^{(n)}$ belongs to an edge which comes after ab in L. If neither a nor b divide $L_{k+1}^{(n)}$ then $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (ab) \subseteq (I^{n+1} : L_{k+1}^{(n)})$ which is contrary to our assumption.

Without loss of generality we assume $a | L_{k+1}^{(n)}$. As $L_{k+1}^{(n)}$ is a product of edges,there exists an edge ac with $ac|^{\text{edge}} L_{k+1}$, where ac is a part of the maximal expression of $L_{k+1}^{(n)}$. So, $L_{k+1}^{(n)} = acM_2$ for some $M_2 \in L^{(n-1)}$ which is the remaining part of the maximal expression. Now $ab \nmid^{\text{edge}} L_{k+1}^{(n)}$ as $L_{k+1}^{(n)}$ belongs to an edge that comes after ab. Hence $b \neq c$.

If $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (b)$, then we take $L_i^{(n)} = abM_2$. Clearly $L_i^{(n)}$ belongs to ab or some edge that comes before ab. Also, $(L_i^{(n)} : L_{k+1}^{(n)}) = (abM_2 : acM_2) = (b)$. Hence $L_i^{(n)}$ has all the required properties.

If $(L_j^{(n)} : L_{k+1}^{(n)})$ is not contained in (b), then there is a variable d such that bd is an edge and $bd|^{\text{edge}} M_2$ and bd is a part of maximal expression of M_2. Let $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (f)$ where f is a variable. If $(L_j^{(n)} : L_{k+1}^{(n)}) = (f)$ then we take $L_i^{(n)} = L_j^{(n)}$. This has all the required properties.

So let us assume $(L_j^{(n)} : L_{k+1}^{(n)}) = (M_1 b : M_2 c) \subsetneq (f)$. Let $(L_j^{(n)} : L_{k+1}^{(n)}) = (fm)$ where m is a monomial which is not 1. So there is an edge fg such that $fg|^{\text{edge}} M_1$

and fg is part of the maximal expression of M_1. If $g \nmid M_2c$ then $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (fg) \subseteq (I^{n+1} : L_{k+1}^{(n)})$ which contradicts our assumption. So $g|M_2c$.

If $g = c$ then either $f = d$, that is $fcab = bdac$ or $(fcab : bdac) = (f)$. In the first case $L_{k+1} = acM_2 = acbd\frac{M_2}{bd} = fcab\frac{M_2}{bd}$. Now $bd|^{\text{edge}}M_2$, so $ab|^{\text{edge}}L_{k+1}^{(n)}$ which is a contradiction. In the second case we take $L_i^{(n)} = (fc)(ab)\frac{L_{k+1}^{(n)}}{bdac}$. Clearly $L_i^{(n)}$ belongs to ab or a some edge that comes before ab and $(L_i^{(n)} : L_{k+1}^{(n)}) = (f)$, which contains $(L_j^{(n)} : L_{k+1}^{(n)})$. Hence $L_i^{(n)}$ has the required properties.

Now let us assume $g \neq c$. So there is an edge gh such that $gh|^{\text{edge}}M_2$, such that gh is a part of the maximal expression of M_2. Let $\frac{M_1}{fg} = N_1$ and $\frac{M_2}{gh} = N_2$. As $(L_j^{(n)} : L_{k+1}^{(n)}) = (fm)$, $fgabN_1|fmghacN_2$. So $abN_1|hmacN_2$. So $(hm) \subset (abN_1 : acN_2)$. We observe that $(abN_1 : acN_2)$ is either (m) or (hm). For if $m'|m$ then $abN_1|hm'acN_2$ implies $fgabN_1|fm'ghacN_2$ implies $fm|fm'$ implies $m = m'$.

If $(N_1ab : N_2ac) = (m)$ then $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (m) = (abN_1 : acN_2)$. Now both abN_1 and acN_2 are in $L^{(n-1)}$. As abN_1 belongs to ab and acN_2 belongs to some edge which comes after ab, $abN_1 >_{\text{lex}} acN_2$. By induction either $(abN_1 : acN_2) \subseteq (I^n : acN_2)$ or there exists M_0 in $L^{(n-1)}$, $M_0 >_{\text{lex}} acN_2$, $(abN_1 : acN_2) \subseteq (M_0 : acN_2)$, $(M_0 : acN_2)$ is generated by a variable and M_0 belongs to an edge that comes before or equal to ab. In the first case $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (abN_1 : acN_2) \subseteq (I^n : acN_2) \subset (I^{n+1} : ghacN_2) =$

$(I^{n+1} : L_{k+1}^{(n)})$, which is a contradiction. In the second case write $L_i^{(n)} = ghM_0$. We know that $L_i^{(n)} >_{\text{lex}} L_{k+1}^{(n)}$ as M_0 belongs to an edge that comes before or equal to ab. Also $(L_i^{(n)} : L_{k+1}^{(n)}) = (M_0 : acN_2)$, $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (m) = (abN_1 : acN_2) \subseteq (M_0 : acN_2)$ and $(M_0 : acN_2)$ is generated by a variable.

Now let us assume $(abN_1 : acN_2) = (hm)$. As $abN_1 >_{\text{lex}} acN_2$, by induction either $(abN_1 : acN_2) \subseteq (I^n : acN_2)$ or there exists M_0' in $L^{(n-1)}$, $M_0' >_{\text{lex}} acN_2$, with $(abN_1 : acN_2) \subseteq (M_0' : acN_2)$, $(M_0' : acN_2)$ is generated by a variable, and M_0' belongs to an edge that comes before or equal to ab. In the first case $hmacN_2 \in I^n$, so $fmghacN_2 = fgmhacN_2 \in I^{n+1}$. So $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (I^{n+1} : L_{k+1}^{(n)})$, which is a contradiction. In the second case if $(M_0' : acN_2) \neq (h)$ then let $L_i^{(n)} = ghM_0'$. As M_0' belongs to an edge that comes before or equal to ab, $L_i^{(n)} >_{\text{lex}} L_{k+1}^{(n)}$. Also $(L_i^{(n)} : L_{k+1}^{(n)}) = (M_0' : acN_2)$ which contains $(L_j^{(n)} : L_{k+1}^{(n)})$ and is generated by a variable. If $(M_0' : acN_2) = (h)$ we take $L_i^{(n)} = fgM_0'$. By same reasoning $L_i^{(n)} >_{\text{lex}} L_{k+1}^{(n)}$. As $L_i^{(n)}$ can not be same as $L_{k+1}^{(n)}$ we observe $(L_i^{(n)} : L_{k+1}^{(n)}) = (f)$. So this $L_i^{(n)}$ has all the required properties. This completes the proof.

□

The next theorem results from the previous lemma and provides a very strong tool to study the regularity of the powers of the edge ideals. For this theorem we continue with the notation from the previous lemma.

Theorem 3.1.12. *For all $k \geq 1$ and for all $j \leq k$, if $(L_j^{(n)} : L_{k+1}^{(n)})$ is not contained in*

$(I^{n+1} : L_{k+1}^{(n)})$ then there exists $i \leq k$, such that $(L_i^{(n)} : L_{k+1}^{(n)})$ is generated by a variable and $(L_j^{(n)} : L_{k+1}^{(n)}) \subseteq (L_i^{(n)} : L_{k+1}^{(n)})$.

Proof. We have $L_j^{(n)} = mm_1$ and $L_{k+1}^{(n)} = mm_2$ where $m \in \text{Mingens}(I^k)$ and $m_1, m_2 \in \text{Mingens}(I^{n-k})$ with m_1 belongs to an edge that comes strictly before the edge m_2 belongs. We observe $(L_j^{(n)} : L_{k+1}^{(n)}) = (m_1 : m_2)$ and $(I^{n-k+1} : m_2) \subseteq (I^{n+1} : mm_2)$. With these two observations the theorem follows from Lemma 3.1.11. This finishes the proof. □

The next theorem gives us a framework for proving upper bounds of regularity of powers of edge ideals.

Theorem 3.1.13. *For any finite simple graph G and any $s \geq 1$, let the set of minimal monomial generators of $I(G)^s$ be $\{m_1, \ldots, m_k\}$, then*

$$\text{reg}(I(G)^{s+1}) \leq \max\{\text{reg}(I(G)^{s+1} : m_l) + 2s, 1 \leq l \leq k, \text{reg}(I(G)^s)\}$$

Proof. Minimal monomial generators of $I(G)^s$ forms the ordered list $L^{(s)}$ from the Lemma 3.1.11. So by Lemma 2.1.8,

$$\text{reg}(I(G)^{s+1}) \leq \max\{A, B, C\}$$

Where

$$A = \max\{\text{reg}(I(G)^{s+1} : L_1^{(s)}) + 2s\}$$

$$B = \max\{\mathrm{reg}(((I(G)^{s+1}, L_1^{(s)}, \dots, L_l^{(s)}) : L_{l+1}^{(s)}) + 2s | 1 \leq l \leq k-1\}$$

$$C = \mathrm{reg}(I(G)^s).$$

In light of Theorem 3.1.12, $((I(G)^{s+1}, L_1^{(s)}, \dots, L_l^{(s)}) : L_{l+1}^{(s)})$ is the same as $((I(G)^{s+1} : L_{l+1}^{(s)}), \text{some variables})$. So by Lemma 2.1.5

$$\mathrm{reg}((I(G)^{s+1}, L_1^{(s)}, \dots, L_l^{(s)}) : L_{l+1}^{(s)}) \leq \mathrm{reg}((I(G)^{s+1} : L_{l+1}^{(s)}),$$

and the theorem follows. □

As a corollary to the above theorem we get the following important result:

Corollary 3.1.14. *If for all $s \geq 1$ and for all minimal monomial generator m of $I(G)^s$, $\mathrm{reg}(I(G)^{s+1} : m) \leq 2$ and $\mathrm{reg}(I(G)) \leq 4$ then for all $s \geq 1, \mathrm{reg}(I(G)^{s+1}) = 2s+2$; as a consequence $I(G)^{s+1}$ has a linear minimal free resolution.*

Proof. We observe that under the condition if $\mathrm{reg}(I(G)^s) \leq 2s+2$ then $\mathrm{reg}(I(G)^{s+1}) \leq 2s+2$ too. Now $\mathrm{reg}(I(G)) \leq 4$ implies $\mathrm{reg}(I(G)^2) \leq 4$. By induction assume $\mathrm{reg}\, I(G)^k \leq 2k$. As $2k < 2k+2$, $\mathrm{reg}\, I(G)^k \leq 2k+2$. Hence $\mathrm{reg}\, I(G)^{k+1} \leq 2k+2$. This proves the corollary. □

3.2 Even-Connection In Simple Graphs

In this section we introduce the notion of even connection. The main goal is to carefully analyse the ideal $(I(G)^{s+1} : e_1 \cdots e_s)$ for an arbitrary s-fold product of edges (i.e. for $i \neq j$, $e_i = e_j$ is a possibility) and give a combinatorial description. Now any

s-fold product can be written as product of s edges in various ways. In this section we fix a presentation and work with respect to that. We first prove that these ideals are generated in degree two for any graph G.

Theorem 3.2.1. *For any graph G and for any s-fold product $e_1 \cdots e_s$ of edges in G (with the possibility of e_i being same as e_j as an edge for $i \neq j$), the ideal $(I(G)^{s+1} : e_1 \cdots e_s)$ is generated by monomials of degree two.*

Proof. We prove this using induction on s. For $s = 0$ the result is clear as $(I(G) : (1)) = I(G)$, which is generated by monomials of degree two. Now let us assume the theorem is true till $s - 1$.

Let m be a minimal monomial generator of $(I(G)^{s+1} : e_1 \cdots e_s)$. Then $e_1 \cdots e_s m$ is divisible by an $s+1$-fold product of edges. By degree consideration m can not have degree 1. If m has degree greater than or equal to 3 then again by a degree consideration for some i, $e_i = pq$ such that $e_1 \cdots e_{i-1} q e_{i+1} \cdots e_s m$ is divisible by an $s+1$-fold product of edges. Without loss of generality we may assume $e_1 = pq$ and there is an $s+1$-fold product $f_1 \cdots f_{s+1}$ such that $f_1 \cdots f_{s+1} | q e_2 \cdots e_s m$.

If $q | f_1 \cdots f_{s+1}$, without loss of generality we may assume $f_1 = p'q$. So $p'q f_2 \cdots f_{s+1} | q e_2 \cdots e_s m$. Hence $f_2 \cdots f_{s+1} | e_2 \cdots e_s m$. If q does not divide $f_1 \cdots f_{s+1}$ then $f_1 \cdots f_{s+1} | e_2 \cdots e_s m$ and hence $f_2 \cdots f_{s+1} | e_2 \cdots e_s m$. In both cases $m \in (I(G)^s :$

$e_2 \cdots e_s)$.

Now $(I(G)^s : e_2 \cdots e_s) \subset (I(G)^{s+1} : e_1 \cdots e_s)$ and m is a minimal monomial generator of $(I(G)^{s+1} : e_1 \cdots e_s)$. So m has to be a minimal monomial generator of $(I(G)^s : e_2 \cdots e_s)$. Hence by induction m has degree two, which is a contradiction to the assumption that m has degree greater than or equal to three. Hence m has to have degree two. $\square$

To analyse the generators of $(I(G)^{s+1} : e_1 \cdots e_s)$, we introduce the notion of *even-connectedness* with respect to s-fold products.

Definition 3.2.2. *Two vertices u and v (u may be same as v) are said to be even-connected with respect to an s-fold product $e_1 \cdots e_s$ if there is a path $p_0 p_1 \cdots p_{2k+1}$, $k \geq 1$ in G such that:*

1. $p_0 = u, p_{2k+1} = v$.

2. For all $0 \leq l \leq k-1$, $p_{2l+1}p_{2l+2} = e_i$ for some i.

3. For all i,

$$|\{l \geq 0 | p_{2l+1}p_{2l+2} = e_i\}| \leq |\{j | e_j = e_i\}|$$

4. For all $0 \leq r \leq 2k$, $p_r p_{r+1}$ is an edge in G.

If these properties are satisfied then $p_0, \ldots, p_{2k+1}$ is said to be an even-connection between u and v with respect to $e_1 \cdots e_s$.

Example 3.2.3. *Let* $I(G) = (xy, xu, yv, yw, wz, zv)$ *and* $e_1 = xy$, $e_2 = wz$ *then* u, x, y, w, z, v *is an even-connection between* u *and* v *with respect to* e_1e_2.

The following observation is an immediate consequence of the definition:

Observation 3.2.4. *If* $u = p_0, \ldots, p_{2k+1} = v$ *is an even-connection with respect to some* s*-fold product* $e_1 \cdots e_s$, *then for any* $j' \geq j \geq 0$, *any neighbor* x *of* p_{2j+1} *and any neighbor* y *of* $p_{2j'+2}$ *are even connected with respect to* $e_1 \cdots e_s$.

The next theorem also easily follows from the definition.

Theorem 3.2.5. *If* $u = p_0, \ldots, p_{2k+1} = v$ *is an even-connection with respect to some* s*-fold product* $e_1 \cdots e_s$ *the* $uv \in (I(G)^{s+1} : e_1 \cdots e_s)$.

Proof. By condition 2 and 3 of the definition, $e_1 \cdots e_s = p_1 \cdots p_{2k}.e_{j_1} \cdots e_{j_{s-k}}$, for some $\{j_1, j_2, \ldots, j_{s-k}\} \subset \{1, \ldots, s\}$ and by condition 1 and 4 of definition $up_1 \cdots p_{2k}v$ is a $k+1$-fold product of edges in G. Hence $uve_1 \cdots e_s$ is an $s+1$-fold product of edges in G and the result follows. $\square$

Although we fix a representation for all s-fold product and work with respect to that representation, it is worth noting that our definition of even-connectedness is independent of the representation we choose in the following sense:

Theorem 3.2.6. *If $f_1 \cdots f_s = e_1 \cdots e_s$ are two different representations of same s-fold product as product of edges and u and v are even-connected with respect to $e_1 \cdots e_s$, then u and v are even-connected with respect to $f_1 \cdots f_s$.*

Proof. Let $u = p_0, \ldots, p_{2k+1} = v$ be an even-connection between u and v with respect to $e_1 \cdots e_s$. We shall construct an even-connection $q_0, \ldots, q_{2r+1}$ between u and v with respect to $f_1 \cdots f_s$.

Let i be minimal such that $p_{2i+1}p_{2i+2}$ is not equal to any edge $f_1, \ldots, f_s$. Let $q_0 = p_0, \ldots, q_{2i+1} = p_{2i+1}$. We have $(up_1)(p_2p_3)\cdots(p_{2k}v)e_{t_1}\cdots e_{t_{s-k}} = (uv)f_1 \cdots f_s$. Then $p_{2i+1}(p_{2i+2}p_{2i+3})\cdots(p_{2k}v)e_{t_1}\cdots e_{t_{s-k}} = vf_{j_1}\cdots f_{j_{s-i}}$. If $v = p_{2i+1}$ we are done. Otherwise p_{2i+1} divides one of the fs; without loss of generality let $f_{j_1} = p_{2i+1}q_{2i+2}$. If vq_{2i+2} is an edge in G, we are done by taking $q_{2i+3} = v$. Otherwise we have $vq_{2i+2}f_{j_2}\cdots f_{s-i}$ is an $s-i$-fold product of edges $g_1 \cdots g_{s-i}$, where without loss of generality $g_1 = q_{2i+2}q_{2i+3}$ and $f_{j_2} = q_{2i+3}q_{2i+4}$. After selecting (without loss of generality) $g_l = q_{2i+2l}q_{2i+2l+1}$ and $f_{j_{l+1}} = q_{2i+2l+1}q_{2i+2l+2}$, we select $q_{2i+2l+3}$ inductively. If $vq_{2i+2l+2}$ is an edge in G, we are done by choosing $q_{2i+2l+3} = v$. Other wise, $g_{l+1}\cdots g_{s-i} = vq_{2i+2l+2}f_{j_{l+2}}\cdots f_{j_{s-i}}$. If v is connected to $q_{2i+2l+2k}$ for some k in G then we are done by choosing $q_{2i+2l+2k+1} = v$. If not then $g_1 \cdots g_{s-i} = vg_1g_2\cdots g_{s-i-1}q_{2i+2s-2}$; but this will force $g_{s-i} = q_{2i+2s-2}v$, contradicting the fact that v is not connected to $q_{2i+2l+2k}$ for any k.

The conditions $1, 2, 4$ of the definition are automatically satisfied by our construction. Condition 3 is satisfied because each $q_{2i+1}q_{2i+2}$ is f_{r_i} for some integer r_i and $q_{2i+3}q_{2i+4}$ is some $f_{r_{i+1}}$ where $r_{i+1} \notin \{r_1, \ldots, r_i\}$. $\square$

We now observe that all edges of G belong to $(I(G)^{s+1} : e_1 \cdots e_s)$. If uv, u may be equal to v, belongs to $(I(G)^{s+1} : e_1 \cdots e_s)$ and uv is not an edge, then we prove that u and v has to be even-connected with respect to the s-fold product $e_1 \cdots e_s$. The conditions $1, 2, 3, 4$ are satisfied by the way of construction.

Theorem 3.2.7. *Every generator uv (u may be equal to v) of $(I(G)^{s+1} : e_1 \cdots e_s)$ is either an edge of G or even-connected with respect to $e_1 \cdots e_s$, for $s \geq 1$.*

Proof. Suppose uv is not an edge and u and v are not even-connected. Now $uve_1 \cdots e_s = f_0 \cdots f_s$ is an $s+1$-fold product of edges, where $f_0 = up_0$ such that there is an edge $e_{i_0} = p_0q_1, 1 \leq i_0 \leq s$. After selecting $f_j = q_jp_j$ and $e_{i_j} = p_jq_{j+1}$, $1 \leq i_j \leq s$ and all i_j are different, we select f_{j+1} and $e_{i_{j+1}}$ inductively. q_{j+1} is part of an edge $q_{j+1}p_{j+1}$ in the $s+1$ fold product $f_0 \cdots f_s$. We choose $f_{j+1} = q_{j+1}p_{j+1}$. Now as u and v are not even-connected p_{j+1} is not v. So it is part of an edge amongst the remaining e_i s. So there exists $e_{i_{j+1}} = p_{j+1}q_{j+2}$, $i_{j+1} \in \{1, .., s\} \setminus \{i_1, \ldots, i_j\}$. Now as u and v are not even-connected, $v \neq p_k$ for any k. We observe $f_0 \cdots f_s = u(p_0q_1)(p_1q_2) \cdots (p_{s-1}q_s)p_s = uve_1 \cdots e_s$. By construction $(p_0q_1)(p_1q_2) \cdots (p_{s-1}q_s) = e_1 \cdots e_s$. This forces $p_s = v$, which is a contradiction. $\square$

Example 3.2.8. *Let* $I(G) = (xy, xu, xv, xz, yz, yw)$*. Then* $(I(G)^2 : xy) = I(G) + (z^2, uz, vz, wz, uw, vw)$*. Here* z *is even-connected to itself and* u, v, w *with respect to* xy*; also* u, w *and* v, w *are even-connected with respect to* xy*.*

We observe that $(I(G)^{s+1} : e_1 \cdots e_s)$ need not be square free as there is a possibility that some vertex u is even-connected to itself with respect to $e_1 \cdots e_s$. So we polarize $(I(G)^{s+1} : e_1 \cdots e_s)$ to get a square free quadratic monomial ideal (i.e. an edge ideal) $(I(G)^{s+1} : e_1 \cdots e_s)^{\text{pol}}$. For details of polarization we refer to [9], Section 3.2 of [MS] and Exercise 3.15 of [10]. Here we just recall the definition and one theorem which states a quadratic monomial ideal and its polarization have same regularity.

Definition 3.2.9. *For any quadratic monomial ideal* I *in* $K[x_1, \dots, x_n]$*,* I^{pol} *is a square free quadratic monomial ideal in* $K[x_1, \dots, x_n, x'_1, \dots, x'_n]$ *where* $I^{\text{pol}} =< x_i x_j, x_k x'_k | x_i x_j \in I, x_k^2 \in I >$*.*

The following theorem, which we state without proof is a special case of Proposition 1.3.4 of [K2], we also refer to section 3.2 and exercise 3.15 of [MS].

Theorem 3.2.10. $\text{reg}(I^{\text{pol}}) = \text{reg}(I)$.

Clearly by Theorems 3.2.1, 3.2.5, 3.2.7 and 3.2.10, $(I(G)^{s+1} : e_1 \cdots e_s)^{\text{pol}}$ is an edge ideal with the same regularity as $\text{reg}(I(G)^{s+1} : e_1 \cdots e_s)$. We describe the graph associated to this edge ideal in the following Lemma:

Lemma 3.2.11. $(I(G)^{s+1} : e_1 \cdots e_s)^{\mathrm{pol}}$ *is the edge ideal of a new graph* G' *which has:*

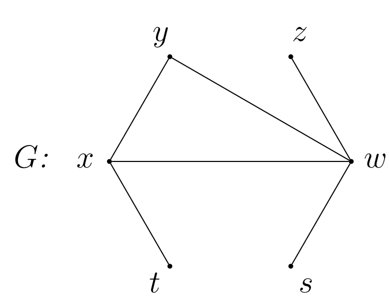
1. All vertices and edges of G*.*

2. Any two vertices $u, v, u \neq v$ *of* G *that are even-connected with respect to* $e_1 \cdots e_s$ *are connected by an edge in* G'*.*

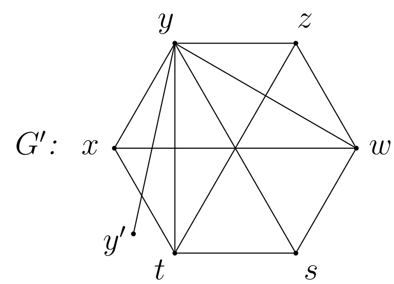
3. For every vertex u *which is even connected to itself with respect to* $e_1 \cdots e_s$*, there is a new vertex* u' *which is connected to* u *by an edge and not connected to any other vertex (so* uu' *is a whisker).*

Proof. By Theorem 3.2.7, every generator uv (u may be equal to v) of $(I(G)^{s+1} : e_1 \cdots e_s)$ is either an edge of G or even-connected with respect to $e_1 \cdots e_s$, for $s \geq 1$. If it is an edge in G, it satisfies condition 1; if it is an even-connection with $u \neq v$ it satisfies condition 2; if it is an even-connection with $u = v$, then by definition of polarization there will be a whisker u' on u in G' and hence it will satisfy condition 3. Conversely edges described by the conditions 1,2 and 3 belong to G' by Theorems 3.2.5 and 3.2.7. $\square$

Example 3.2.12. *Let* G *be the following graph:*



Then the graph G' *associated to* $(I(G)^2 : xw)^{\mathrm{pol}}$ *is the following:*



3.3 New Results

In this section we give some new bounds on $\text{reg}(I(G)^s)$ for certain classes of gap free graphs G. First we prove several lemmas that will be useful to get our main results.

Lemma 3.3.1. *Suppose $u = p_0, \ldots, p_{2k+1} = v$ is an even-connection between u and v and $z = q_0, \ldots, q_{2l+1} = w$ is an even connection between z and w, both with respect to $e_1 \cdots e_s$. If for some i and j, $p_{2i+1}p_{2i+2}$ and $q_{2j+1}q_{2j+2}$ has a common vertex in G then u is even-connected to either z or w with respect to $e_1 \cdots e_s$ and v is even-connected to either z or w with respect to $e_1 \cdots e_s$.*

Proof. We prove it for u, and the proof for v follows by symmetry. Let i be the smallest integer such that there is j with the required property. If $p_{2i+1} = q_{2j+1}$ then $u = p_0, \ldots, p_{2i+1} = q_{2j+1}, q_{2j+2}, q_{2j+3}, \ldots, q_{2l+1} = w$ gives an even-connection between u and w with respect to $e_1 \cdots e_s$ (conditions 1,2 and 4 are automatically satisfied and condition 3 is satisfied as i is the smallest integer such that there is a j). Similar if $p_{2i+1} = q_{2j+2}$ then $u = p_0, ..., p_{2i+1} = q_{2j+2}, q_{2j+1}, q_{2j}, \ldots, q_0 = z$

gives an even-connection between u and z with respect to $e_1 \cdots e_s$; if p_{2i+1} is not same as either q_{2j+1} or q_{2j+2} and $p_{2j+2} = q_{2j+1}$ then $u = p_0, \ldots, p_{2i+1}, p_{2j+2} = q_{2j+1}, q_{2j+2}, q_{2j+1}, q_{2j}, \ldots, q_0 = z$ gives an even-connection between u and z with respect to $e_1 \cdots e_s$; if p_{2i+1} is not same as either q_{2j+1} or q_{2j+2} and $p_{2j+2} = q_{2j+2}$ then $u = p_0, \ldots, p_{2i+1}, p_{2j+2} = q_{2j+2}, q_{2j+1}, q_{2j+2}, , \ldots, q_{2l+1} = w$ gives an even-connection between u and w with respect to $e_1 \cdots e_s$; in each of these cases conditions 1,2 and 4 are satisfied automatically and condition 3 is satisfied as i is the smallest integer with the property. This covers all the cases. $\square$

The next two lemmas are results about gap free graphs:

Lemma 3.3.2. *If G is gap free so is the graph G' associated to $(I(G)^{s+1} : e_1 \cdots e_s)^{\mathrm{pol}}$, for every s-fold product $e_1 \cdots e_s$.*

Proof. There are three possibilities of gap formation in G' :

1. Between two edges from G.
2. Between two edges that are not edges in G.
3. Between two edges where one of them is an edge in G another is not.

No two edges in G can form a gap in G as G is gap free. So they can't form an edge in G' as in G' no edge of G is being deleted.

For the second case suppose uv and zw are even-connected with respect to $e_1 \cdots e_s$ and neither uv nor zw is an edge in G. Without loss of generality we may assume $\gcd(uv, zw) = 1$ as there is no question of gap formation otherwise. Let $u = p_0, \ldots, p_{2k+1} = v$ be an even-connection between u, v with respect to $e_1 \cdots e_s$ and let $z = q_0, \ldots, q_{2l+1} = w$ be an even-connection between z, w with respect to $e_1 \cdots e_s$. In light of Lemma 3.3.1, we may assume for no i, j, $p_i = q_j$. If $u = q_1$ then $zu = zq_1$ is an edge in G and if $z = p_1$ then $uz = up_1$ is an edge in G, so there is nothing to prove. Otherwise as up_1 and zq_1 are edges in G and G is gap free there are four possibilities:

a. u is connected to z in G, in which case uv (or uu' in case $u = v$) and zw (or zz' in case $z = w$) can't form a gap, as in that case uz is an edge in G' too.

b. p_1 is connected to z, in which case $z, p_1, \ldots, p_{2k+1} = v$ is an even-connection between z and v in G so zv is an edge in G' hence uv (or uu' if $u = v$) and zw (or zz' if $z = w$) can't form a gap.

c. p_1 is connected to q_1, in which case $v = p_{2k+1}, p_{2k}, \ldots, p_1, q_1, q_2, \ldots, q_{2l+1} = w$ gives an even-connection between v and w, and vw is an edge in G'.

d. q_1 is connected to u, in which case $u, q_1, \ldots, q_{2l+1} = w$ is an even-connection between u and w in G so uw is an edge in G' hence uv (or uu' if $u = v$) and zw (or zz' if $z = w$) can't form a gap.

In the third case, u, v are even-connected with respect to $e_1 \cdots e_s$ and zw is an edge in G and uv is not an edge in G. Like before, we may assume $\gcd(uv, zw) = 1$. Let $u = p_0, \ldots, p_{2k+1} = v$ be an even-connection between u, v with respect to $e_1 \cdots e_s$. If $z = p_1$ then $uz = up_1$ is an edge in G and if $w = p_1$ then $uw = up_1$ is an edge in G, so there is nothing to prove in these cases. Otherwise as up_1 and zw are edges in G and G is gap free there are four choices:

a. u is connected to z, in which case uv (or uu' in case $u = v$) and zw can't form a gap as in that case uz is an edge G' too.

b. p_1 is connected to z, in which case $z, p_1, \ldots, p_{2k+1} = v$ is an even-connection between z and v in G so zv is an edge in G' hence uv (or uu' if $u = v$) and zw can't form a gap.

c. p_1 is connected to w, in which case $v = p_{2k+1}, p_{2k}, \ldots, p_1, w$ is an even-connection; hence uv and zw can not form a gap.

d. w is connected to u, in which case uw is an edge in G, hence in G'.

This finishes the proof. □

Lemma 3.3.3. *Suppose G is gap free. If $w_1, \ldots, w_n$ is an anticycle in the graph G' defined by $(I(G)^{s+1} : e_1 \cdots e_s)$ for some $s \geq 1$ and for $n \geq 5$, then $w_1, ..., w_n$ is an anticycle in G.*

Proof. First of all, whiskers on any vertex can not be part of any anticycle of length ≥ 5 as they only have degree 1. Observe that it is enough to prove that for all i, j, w_i, w_{i+j} are never even-connected with respect to $e_1 \cdots e_s$. Suppose on the contrary such i, j exists. Without loss of generality we may choose j to be minimal such that for some i, w_i and w_{i+j} are even-connected with respect to $e_1 \cdots e_s$. Observe that $j \geq 2$ as $w_i w_{i+1}$ can't be connected in an anticycle. Without loss of generality we may further assume w_1 and w_{1+j} are even-connected with respect to $e_1 \cdots e_s$ via $w_1 = p_0, p_1, ..., p_{2k+1} = w_{1+j}$. Now observe w_{2+j} is not connected to p_1 by an edge in G as that will force w_{1+j} and w_{2+j} to be connected in G' by observation 6.4 leading to a contradiction. So there exists a smallest $l \geq 0$, $2 + j \leq n - l \leq n$ such that w_{n-l} is not connected to p_1 by an edge in G. If $l = 0$, then w_n is not connected to p_1 by an edge in G and if $l > 0$ then w_{n-l} is not connected to p_1 by an edge to p_1 in G and $w_n, w_{n-1}, \ldots, w_{n-l+1}$ are connected to p_1 by an edge in G

Next, we look at the edge $w_2 w_{n-l}$ in G'. If w_2 is connected to p_1 in G then $w_2, p_1, \ldots, p_{2k+1} = w_{1+j}$ will be an even connection that will violate the minimality of j. If w_2 is connected to p_2 in G then by Observation 3.2.4 $w_1 w_2$ has to be an edge in G', which will contradict the fact $w_1, \ldots, w_n$ is an anticycle. We observe w_{n-l} can't be connected to p_1 by selection. If w_{n-l} is connected to p_2 and $l = 0$ then by Observation 3.2.4 w_1 and w_n have to be connected to each other in G'. If w_{n-l} is connected to p_2 and $l > 0$ then by Observation 3.2.4 w_{n-l+1} and w_{n-l} have to be connected to each other in G'. Both cases lead to a contradiction as $w_1, \ldots, w_n$ is an anticycle, so w_2 and w_{n-l} are not connected to each other in G and neither of them are connected to p_1 or p_2 (and hence w_2, w_{n-l}, p_1, p_2 are four distinct vertices). As $p_1 p_2$ is an edge in

G, w_2w_{n-l} can not be an edge in G; otherwise they will form a gap. So w_2 and w_{n-l} are even-connected with respect to $e_1 \cdots e_s$. Let $w_2 = q_0, \ldots, q_{2r+1} = w_{n-l}$ be an even connection between w_2 and w_{n-l} with respect to $e_1 \cdots e_s$.

If for some $t_1, t_2 \geq 0$, $p_{2t_1+1}p_{2t_1+2}$ and $q_{2t_2+1}q_{2t_2+2}$ are the same edges of G then by Lemma 3.3.1, w_2 has to be even connected to either w_1 or w_{1+j}. The first case is not possible as $w_1..w_n$ is an anticycle and the second case is not possible by the minimality of j. So for no $t_1, t_2 \geq 0$, $p_{2t_1+1}p_{2t_1+2}$ and $q_{2t_2+1}q_{2t_2+2}$ are the same edges of G. So we look at $w_{n-l}q_{2r}$ and p_1p_2. Observe that p_1 is not connected to w_{n-l} because of the selection. If w_{n-l} is connected to p_2 and $l = 0$ then by Observation 6.4 w_1 and w_n have to be connected to each other in G'. If w_{n-l} is connected to p_2 and $l > 0$ then by Observation 3.2.4 w_{n-l+1} and w_{n-l} have to be connected to each other in G'. Both cases lead to a contradiction as $w_1, \ldots, w_n$ is an anticycle. So p_2 is not connected to w_{n-l} in G. If p_1 is connected to q_{2r} then w_2 and w_{1+j} will be even-connected with respect to $e_1 \cdots e_s$ violating the minimality of j. If p_2 is connected to q_{2r} then w_1 and w_2 will be even-connected and hence connected in G'.

Hence for no i, j are w_i and w_{i+j} even-connected with respect to $e_1 \cdots e_s$. So $w_1, \ldots, w_n$ is an anticycle in G. $\square$

Using this lemma we get the following theorem of Herzog, Hibi and Zheng (Theorem 1.2 of [NP]) as a corollary:

Theorem 3.3.4. *If $I(G)$ has linear resolution, then for all $s \geq 2$, $I(G)^s$ has regularity $2s$. In other words $I(G)^s$ has a linear minimal free resolution.*

Proof. As $I(G)$ has a linear resolution, it is gap free and hence the polarizations of all $(I(G)^{s+1} : e_1 \cdots e_s)$ are gap free and any anticycle of length ≥ 5 in the polarization of $(I(G)^{s+1} : e_1 \cdots e_s)$ is an anticycle of G. But as $I(G)$ has linear resolution G does not have an any anticycle. Hence $\operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s)^{\text{pol}} = 2$ for all $e_1 \cdots e_s$. Hence we have $\operatorname{reg}(I(G)^{s+1}) = 2s + 2$. □

Next we prove that for any gap free and cricket free graph G, and for all $s \geq 2$, $\operatorname{reg}(I(G)^s) = 2s$. This result is our main new result in this paper. This answers Question 1.1 partially. This also generalizes Nevo's result (Theorem 1.2 of [12]) that for any gap free and claw free graph G, $\operatorname{reg} I(G)^2 = 4$.

Theorem 3.3.5. *For any gap free and cricket free graph G and for all $s \geq 2$, $\operatorname{reg}(I(G)^s) = 2s$.*

Proof. In light of Theorem 2.1.10, Theorem 3.1.13, Lemma 3.3.3, it is enough to show the polarization of $(I(G)^{s+1} : e_1 \cdots e_s)$ does not have any anticycle $w_1....w_n$ for $n \geq 5, s \geq 1$, for every s-fold product $e_1 \cdots e_s$.

Suppose $w_1, \ldots, w_n$, $n \geq 5$, is an anticycle in the polarization of $(I^{s+1} : e_1 \cdots e_s)$ and $e_1 = xy$. By Lemma 3.3.3 $w_1, \ldots, w_n$ is also an anticycle of G. Either w_1 or w_3 is a neighbor of x or neighbor of y else w_1w_3 and e_1 forms a gap in G, a contradiction. Without loss of generality, we may assume w_1 is a neighbor of x. Now neither w_2 nor w_n can be x as they are not connected to w_1; also neither of them are y as if say $y = w_2$ then w_nxyw_1 is an even connection hence w_1w_n is an edge in G', a contradiction to the assumption on anticycle; similar thing happens if $y = w_n$. By Observation 3.2.4 every neighbor of y is connected to every neighbor of x in G'. As neither w_1w_n, nor w_1w_2 is an edge in G', neither of w_2 and w_n are neighbors of y in G. So one of them has to be neighbor of x, as G is gap free. Again, without loss of generality, we may assume w_2 is a neighbor of x. Next we consider w_3w_n. As w_1 and w_2 are neighbors of x and neither w_1w_n nor w_2w_3 are edges in G', by Observation 3.2.4 neither w_3 nor w_n can be neighbor of y. Neither w_3 nor w_n can be x as they are w_2w_3 and w_1w_n are not edges in G'. If $w_3 = y$, as w_1w_3 is an edge in G, w_1, being a neighbor of y, has to be connected to w_2, which is a neighbor of x in G' by Observation 3.2.4. That will force w_1w_2 to be an edge in G', which is a contradiction. Similarly if $w_n = y$, w_3 being a neighbor of y has to be connected to w_2 in G' leading to a contradiction. Then either w_3 or w_n of them has to be a neighbor of x. Without loss of generality we may assume w_3 is a neighbor of x. Notice that y is not connected to w_1 in G as that will force w_2, a neighbor of x to be connected to w_1 in G' leading to a contradiction. Hence $\{y, w_2, x, w_1, w_3\}$ forms a cricket. $\square$

Next we prove that for any gap free graph G with $\operatorname{reg}(I(G)) = r$, the $\operatorname{reg}(I(G)^s)$ is bounded above by $2s+r-1$. But to do that we need a lemma about "longest" connections. Observe that if G' is the graph associated to the polarization of $(I(G)^{s+1} : e_1 \cdots e_s)$, for some s-fold product, and u,v are even-connected with respect to $u = p_0, \ldots, p_{2k+1} = v$, then uv is not only an edge in G' but also an edge in the graph $(G' - \{y_1, ...y_l\})$ for any set of points $y_1, \ldots, y_l$ as long as $u, v \notin \{y_1, \ldots, y_l\}$. We further emphasize that some of the p_is can also belong to $\{y_1, \ldots, y_l\}$ as long as they are not same as u or v.

Lemma 3.3.6. *Let G' be the graph associated to the polarization of $(I(G)^{s+1} : e_1 \cdots e_s)$ for some s-fold product. Let us assume u,v are even-connected with respect to $u = p_0, \ldots, p_{2k+1} = v$. Suppose for some set of vertices $\{y_1, \ldots, y_l\}$ we have $u, v \notin \{y_1, \ldots, y_l\}$. Let us also assume for any other even-connection $u' = p'_0, \ldots, p'_{2k'+1} = v'$ such that $u', v' \notin \{y_1, \ldots, y_l\}$ we have $k' \leq k$. Then $(G' - \{y_1, \ldots, y_l\} - \operatorname{st} u)$ is $G'' \cup \{$isolated whisker vertices$\}$, where G'' is a subgraph of G obtained by deleting vertices.*

Proof. For the set of points $\{y_1, \ldots, y_l\}$, uv is an edge in $(G' - \{y_1, \ldots, y_l\})$ such that $u, v \notin \{y_1, \ldots, y_l\}$ are even-connected with respect to $e_1 \cdots e_s$ via $u = p_0, p_1, p_2, ..$ $.., p_{2k+1} = v$. We also have that k is maximum over all such even-connected edges in $(G' - \{y_1, \ldots, y_l\})$. Let $u'v'$ be any edge in $(G' - \{y_1, \ldots, y_l\})$ such that $u', v' \notin \{y_1, \ldots, y_l\}$ and they are even-connected with respect to $e_1 \cdots e_s$ via $u' = x_0, x_1, x_2, \ldots, x_{2k'+1} = v'$. If for any j, j', $p_{2j+1}p_{2j+2}$ and $x_{2j'+1}x_{2j'+2}$ form the same edge in G then by Lemma 3.3.1, either u' or v' will be not a vertex in $(G' - \{y_1, \ldots, y_l\} - \operatorname{st} u)$. Now observe, if for any j, j', $p_{2j+1}p_{2j+2}$ and $x_{2j'+1}x_{2j'+2}$ do not form same edge in G then either x_1 or x_2 has to be connected to p_1 or p_2 to avoid x_1x_2 and p_1p_2 forming a gap. If any of them (for example x_1) is connected to p_1 in G that will make $\{v' = x_{2k'+1}, x_{2k'}, \ldots, x_1, p_1, \ldots, p_{2k+1}\}$ a longer connection violating the maximality of k. A similar thing happens if x_2 is connected to p_1 in G. So either of them has to be connected to p_2. If x_1 is connected to p_2 in G then u is connected to v' in G' as $u, p_1, p_2, x_1, \ldots, x_{2k'+1} = v'$ will be an even-connection. Similarly if x_2 is connected to p_2 then u is connected to u' in G' as $u, p_1, p_2, x_2, x_1, u'$ will be an even-connection. In both the cases either u' or v' will not be a vertex in $(G' - \{y_1, \ldots, y_l\} - \operatorname{st} u)$. This proves that any edge in $(G' - \{y_1, \ldots, y_l\} - \operatorname{st} u)$ is an edge in G. Hence the Lemma follows. □

Using Lemma 3.3.6 we prove the next theorem which guarantees that the gap between the regularity of powers of edge ideals of gap free graphs and the regularity of monomial ideals generated in the same degree and having a linear resolution, can not be arbitrarily large:

Theorem 3.3.7. *For any gap free graph G with $\operatorname{reg}(I(G)) = r$ and any $s \geq 2$ the $\operatorname{reg}(I(G)^s)$ is bounded above by $2s + r - 1$.*

Proof. Let G' be the graph associated to the polarization of $(I(G)^{s+1} : e_1 \cdots e_s)$. We have $\operatorname{reg}(G') \leq \max\{\operatorname{reg}(G' - \operatorname{st} x) + 1, \operatorname{reg}(G' - x)\}$, for each x. We choose u_1 and v_1 even connected by $u_1 = p_0, \ldots, p_{2k_1+1} = v_1$ such that k_1 is maximum. By Lemma

3.3.6 $(G' - \operatorname{st} u_1)$ is a subgraph of G obtained by vertex deletion along with some isolated whisker vertices. As isolated vertices do not affect the regularity of edge ideal, $\operatorname{reg}((G' - \operatorname{st} u_1) \leq r$.

Next we we delete a vertex u_2 from $(G' - u_1)$ which is even-connected to another vertex v_2 via $u_2 = q_0, \ldots, q_{2k_2+1} = v_2$ with k_2 maximum. Again by Lemma 6.18 $(G' - u_1 - \operatorname{st} u_2)$ is a subgraph obtained from $G - u_1$ by deletion of vertices along with some whisker vertices. Hence $\operatorname{reg}(G' - u_1 - \operatorname{st} u_2) \leq r$. We keep selecting $u_1, u_2, ...$ and apply Lemma 3.3.6. As we are in a finite set-up, for some $l, (G' - u_1, \ldots, u_l)$ itself is a subgraph of G obtained by repeated vertex deletion along with some isolated whisker vertices and $reg(G') \leq r + 1$. Therefore, by induction the result follows. $\square$

3.4 A Worked Out Example

In this section we work out an example with the help of Macaulay 2 to illustrate the proof of the Theorem 3.3.5. We know that a 5-cycle is a gap free and cricket free graph. In this example we show that the second and the third power of its edge ideal have linear resolutions.

Example 3.4.1. Let $S = \mathbb{Q}[a, b, c, d, e]$ and $I = (ab, bc, cd, de, ea)$. We calculate the regularities using Macaulay 2; all other computations are elementary and can be done by hand . If we take $\{ab, bc, cd, de, ea\}$ to be the ordered list of generators of I then one can check that the ordered set of generators of I^2 that satisfies the condition of Theorem 3.1.12 is

$$\{a^2b^2, ab^2c, abcd, abde, a^2be, b^2c^2, bc^2d, bcde, bcea, c^2d^2, cd^2e, cdea, d^2e^2, de^2a, e^2a^2\}$$

Now one can check, $\operatorname{reg}(I) = 3$

$(I^2 : ab) = (de, ce, ae, cd, bc, ab)$, and its regularity is 2.

$((I^2 + ab)) : bc) = (a, de, cd, bc)$, and its regularity is 2.

$((I^2 + ab + bc) : cd) = (b, de, ae, cd)$, and its regularity is 2.

$((I^2 + ab + bc + cd) : de) = (c, de, ae, ab)$, and its regularity is 2.

$((I^2 + ab + bc + cd + de) : ea) = (d, b, ae)$, and its regularity is 2.

So we have $\operatorname{reg}(I^2) \leq \max\{4, 4, 4, 4, 4, 3\} = 4$

As I^2 is generated in degree 4, this forces that $\operatorname{reg}(I^2) = 4$ which proves that it has linear resolution.

Now we focus into I^3. We observe that,

$(I^3 : a^2b^2) = (de, ce, ae, cd, bc, ab)$, and its regularity is 2,

$(I^3 + a^2b^2 : ab^2c) = (a, de, ce, cd, bc)$, and its regularity is 2,

$(I^3 + a^2b^2 + ab^2c : abcd) = (b, e^2, de, ce, ae, cd)$, and its regularity is 2,

$(I^3 + a^2b^2 + ab^2c + abcd : abde) = (c, de, ae, ab)$, and its regularity is 2,

$(I^3 + a^2b^2 + ab^2c + abcd + abde : a^2be) = (d, b, ce, ae)$, and its regularity is 2,

$(I^3 + a^2b^2 + ab^2c + abcd + abde + a^2be : b^2c^2) = (a, de, cd, bc)$,

and its regularity is 2,

$(I^3 + a^2b^2 + ab^2c + abcd + abde + a^2be + b^2c^2 : bc^2d) = (b, a, de, cd)$,

and its regularity is 2,

$(I^3 + a^2b^2 + ab^2c + abcd + abde + a^2be + b^2c^2 + bc^2d : bcde) = (c, a, de)$,

and its regularity is 2,

$(I^3 + a^2b^2 + ab^2c + abcd + abde + a^2be + b^2c^2 + bc^2d + bcde : bcea) = (d, b, a)$,

and its regularity is 1,

$(I^3 + a^2b^2 + ab^2c + abcd + abde + a^2be + b^2c^2 + bccd + bcde + bcea : c^2d^2)$

$= (b, de, ae, cd)$, and its regularity is 2.

$(I^3 + a^2b^2 + ab^2c + abcd + abde + a^2be + b^2c^2 + bc^2d + bcde + bcea + c^2d^2 : cd^2e)$

$= (c, b, de, ae)$, and its regularity is 2,

$(I^3 + a^2b^2 + ab^2c + abcd + abd + a^2be + b^2c^2 +$

$bc^2d + bcde + bcea + c^2d^2 + cd^2e : cdea) = (d, b, ae)$,and its regularity is 2

$(I^3 + a^2b^2 + ab^2c + abcd + abde + a^2be + b^2c^2 + bc^2d$

$+ bcde + bcea + c^2d^2 + cd^2e + cdea : d^2e^2) = (c, de, ae, ab)$, and its regularity is 2,

$(I^3 + a^2b^2 + ab^2c + abcd + abde + a^2be + b^2c^2 + bc^2d +$

$bcde + bcea + c^2d^2 + cd^2e + cdea + d^2e^2 : de^2a) = (d, c, b, ae)$,

and its regularity is 2,

$(I^3 + a^2b^2 + ab^2c + abcd + abde + a^2be + b^2c^2 +$

$bc^2d + bcde + bcea + c^2d^2 + cd^2e + cdea + d^2e^2 + de^2a : e^2a^2) = (d, b, ae)$,

and its regularity is 2.

These shows that $\operatorname{reg}(I^3) \leq \max\{6, 5, 5\} = 6$

As I^3 is generated in degree six this forces $\operatorname{reg}(I^3) = 6$ and as a result I^3 has linear minimal free resolution.

Chapter 4

Path Ideals

In this chapter we study the regularity of path ideals and find several upper bounds for them. After their introduction in [CD], path ideals have been studied by various researchers (e.g. [AS1], [AS2], [BHK], [KO]). Examples indicate that for various classes of graphs "small regularity" for edge ideals forces the higher path ideals to have small regularity. We prove various results of that type in this chapter. Our approach is similar to that of previous chapter however the situation is somewhat simpler for path ideals. As we shall see in Section 2 of this chapter, we don't need any special ordering of the generators. Of course we prove our result for some particular classes of path ideals and one way to approach the more general classes is to investigate whether there exists ordering of minimal generators which "behaves nicely" (in the spirit of Theorems 3.1.12 and 3.1.13) with respect to short exact sequences.

All along we assume that G is a gap free graph whose t-path ideal is denoted by I_t for all $t \geq 3$ and whose edge ideal is denoted by I. This chapter mainly consists of the work done in [B2].

4.1 3-Path Ideals And 4-Path Ideals

Our main result in this chapter is that for gap free, claw free and whiskered-K_4 free graphs G, $I_t(G)$ has a linear minimal free resolution for all $t \geq 3$. Before going into the investigation of I_t for general t we restrict ourselves to cases $t = 3$ and 4. We prove various different results in these two cases.

We first study I_3 and prove a bound for regularity of I_3 in terms of regularity of I. The following lemma is the first step toward that result.

Lemma 4.1.1. *If $e = uv$ is a generator of I and $I_3 \neq 0$ then $(I_3 : e)$ is generated in degree one. As a consequence it is a prime ideal generated by variables.*

Proof. Let m be a minimal monomial generator of $(I_3 : e)$. So there exists $a, b, c \in V(G)$ with $ab, bc \in E(G)$ such that $abc|uvm$. If $\{a, b, c\} \cap \{u, v\} = \emptyset$ then $abc|m$. As G is gap free one of ua, va, ub, vb is an edge in G. If ua is an edge then ae is a minimal monomial generator of I_3. Hence $m = a$ as m is minimal. If ub is an edge then be is a minimal monomial generator of I_3 and $m = b$. By symmetry we conclude that m has degree one in the remaining two cases too.

Now we assume that $\{a, b, c\} \cap \{u, v\} \neq \emptyset$. First let us assume $u = b$. As $a \neq c$, v can't be equal to both a and c. If $v \neq a$ then $a|m$ and ae is a 3-path making $a = m$ by minimality of m. If $u = b$ and $v \neq c$ then $c|m$ and ce is a 3-path hence $m = c$. If $u = a, v \neq b$ then $be \in I_3$ and $b|m$. Hence by similar argument $m = b$. If

$u = a, v = b$ then $ce \in I_3$ and $c|m$. Again by similar argument $m = c$. By symmetry m is a variable in the other cases too. This completes the proof. □

We illustrate this in the case of 5-cycle:

Example 4.1.2. Let $S = \mathbb{Q}[x_1, \ldots, x_5]$ and $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$. We observe that, $I_3 = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2)$

$(I_3 : x_1x_2) = (x_5, x_3), (I_3 : x_2x_3) = (x_4, x_1), (I_3 : x_2x_4) = (x_5, x_2),$

$(I_3 : x_4x_5) = (x_3, x_1), (I_3 : x_5x_1) = (x_4, x_2).$

Next we prove our bound for the regularity of I_3 in terms of the regularity of I. We note that as a consequence of this Theorem it follows that if I has regularity less than or equal to 3 then I_3 has a linear minimal free resolution.

Theorem 4.1.3. *If $I_3 \neq 0$ and reg$(I) = r$ then reg$(I_3) \leq$ max$\{r, 3\}$.*

Proof. Notice that for any two different edges $e = ab, f = cd$ with no common vertices, $(e : f) = (e)$. As G is gap free at least one of the vertices of e forms an edge with a vertex of f. Without loss of generality we can assume ac is an edge. However we observe that in this case $(a) \subseteq (I_3 : f)$.

In case e and f have a common vertex, $(e : f)$ is generated by a variable. So it follows from the previous lemma that for different edges $e_1, ..., e_k$, $(I_3, e_1, ..., e_{k-1}) : (e_k)$ is J where J is an ideal generated by some variables.

In light of these we observe that the result follows due to Theorem 2.1.8. $\square$

We continue with the 5-cycle example:

Example 4.1.4. Let $S = \mathbb{Q}[x_1, \ldots, x_5]$ and $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$. Here $I_3 = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2)$. One can show that (using Macaulay 2 for example) $\operatorname{reg}(I_3) = 3$. We know that in this case $\operatorname{reg}(I) = 3$. So in this case $\operatorname{reg}(I_3) \leq \max\{3, \operatorname{reg}(I)\}$. In particular I_3 has a linear minimal free resolution.

We devote rest of this section to the study of I_4. We bound $\operatorname{reg}(I_4)$ in terms of $\operatorname{reg}(I)$ in two different cases. To achieve this, we first prove a useful lemma. This lemma gives a description of $(I_4 : e)$ where e is an edge in G, in a way similar to our description of $(I^2 : e)$ in terms of even-connections. Like $(I^2 : e)$, $(I_4 : e)$ is a quadratic monomial ideal too. In fact we shall prove that it is a squarefree quadratic monomial ideal.

Lemma 4.1.5. *Let us assume $I_4 \neq 0$. For any edge $e = xy$, $(I_4 : e)$ is a squarefree quadratic monomial ideal whose minimal monomial generators are the edges of G which do not share a common vertex with e and the square free quadratic monomials uv such that ux and vy are edges in G with $\{u, v\} \cap \{x, y\} = \emptyset$.*

Proof. Clearly any minimal generator has to have degree at least two. Any edge that has no vertex in common with e is a generator of $(I_4 : e)$ by the fact that G is gap free. For any square free quadratic monomials uv such that ux and vy are edges in G with $\{u, v\} \cap \{x, y\} = \emptyset$, $uxyv$ forms a 4-path and hence $uv \in (I_4 : e)$; uv has to be a generator by degree consideration. This proves one containment.

To prove the other, let m be a minimal monomial generator of $(I_4 : e)$. So there is a 4-path $f = abcd$ with ab, bc, cd edges in G such that $f|mxy$. Now f is squarefree. If $x|m$ then clearly $f|\frac{m}{x}e$. Then $\frac{m}{x} \in (I_4 : e)$. This clearly violates the minimality of m. A similar thing happens if $y|m$. So we may assume that m is not divisible by x or y. If m is not divisible by an edge that does not have a common vertex with e then m is not divisible by any edge (as neither x nor y divides m). Now at least two among a, b, c, d divide m. If any three of them divide m then m will be divisible by an edge which is a contradiction. So m is divisible by exactly two of them. As a consequence $xy|abcd$. If $x = a$ then $y = c$ otherwise m will be divisible by an edge. In this case we take $u = b$ and $v = d$. As uv is a generator of $(I_4 : e)$ by degree consideration $m = uv$. Similarly if $x = b$ then y is either c or d otherwise m will be divisible by an edge. In both cases we take $u = a$; in the first case we take $v = d$ and in the second case we take $v = c$. Again by degree consideration $m = uv$ in bot the cases. The existence of such u and v in all other cases follows by symmetry. This completes the proof. $\square$

Example 4.1.6. If G is the 5-cycle on $x_1 \cdots x_5$ then

$I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$ and

$I_4(G) = (x_1x_2x_3x_4, x_2x_3x_4x_5, x_3x_4x_5x_1, x_4x_5x_1x_2, x_5x_1x_2x_3)$

We compute $(I_4 : e)$ for x_1x_2 to illustrate the previous lemma. Note that colon with any other edge can be computed simply by symmetry. $(I_4 : x_1x_2) = (x_4x_5, x_3x_5, x_3x_4)$, x_4x_5 and x_3x_4 are edges in G who do not share a vertex with x_1x_2 and x_3x_5 is a generator of the second kind namely x_3 is a neighbor of x_2 and x_5 is a neighbor of x_1.

Notation 4.1.7. *As $(I_4 : e)$ is a square free quadratic monomial ideal, it is an edge ideal and we denote the corresponding graph by G'. Let e be xy, X be the set of all neighbors of x other than y and Y be the set of all neighbors of y other than x. By construction, $V(G')$ is a subset of $V(G)$* NOT *containing either x or y and the set edges of G', $E(G')$ consists of two types of elements:*

1. Any edge in G that does not contain either x or y.

2. Every squarefree quadratic uv with $u \in X, v \in Y$.

We shall call these second type of generators the new edges*.*

The next two lemmas show that the induced cycles of length greater than or equal to four of G'^c are also induced cycles of G^c. The first of them is similar to the Lemma 3.3.2.

Lemma 4.1.8. *If G' is the graph associated to $(I_4 : e)$ then G' is gap free. In particular $(I_4 : e)$ has a linear presentation.*

Proof. We first observe that two edges in G' can't form a gap in G' if both of them are also edges in G. This holds because by definition of G' if ab is an edge in G and both a and b are vertices in G' then ab is an edge in G'. If ab is an edge in G that remains an edge in G', it cannot form a gap in G' with any new edge. This holds due to the following reason: as G is gap free either a or b is neighbor of either x or y. If a is a neighbor of x then by definition of G' it is connected to every element in Y so ab does not form a gap with any new age. The other cases follows by symmetry.

It only remains to show that two new edges also can't form a gap. If uv and $u'v'$ are two new edges in G' with u, u' neighbor of x in G and v, v' neighbors of y in G we observe uv' is an edge in G' and hence we conclude no two new edges can form a gap. This finishes the proof. $\square$

The next lemma is similar to Lemma 3.3.3.

Lemma 4.1.9. *If G' is the graph associated to $(I_4 : e)$ then any induced cycle of length greater than or equal to five in G'^c is an induced cycle in G^c.*

Proof. We show that if $w_1...w_n$ is an induced cycle in G'^c with $n \geq 5$ then it is an induced cycle in G^c too. Clearly as $V(G')$ does not contain x or y none of the variables $w_1, ..., w_n$ can be x or y. Observe that it is enough to prove that for all i, j, w_i, w_{i+j} is not an edge in $E(G') \setminus E(G)$. For this, it is enough to prove that there is no i, j, such that either $w_i \in X$ and $w_{i+j} \in Y$, or $w_i \in Y$ and $w_{i+j} \in X$. Suppose on the contrary such i, j exists. Without loss of generality we may choose j to be minimal with this property. Observe that $j \geq 2$ as $w_i w_{i+1}$ can't be connected in an anticycle. Without loss of generality we may further assume $w_1 \in X$ and $w_{1+j} \in Y$. Now observe w_{2+j} is not connected to x by an edge in G as that will force w_{1+j} and w_{2+j} to be connected in G' leading to a contradiction. So there exists a smallest $l \geq 0$, $2+j \leq n-l \leq n$ such that w_{n-l} is not connected to x by an edge in G. If $l = 0$, then w_n is not connected to x by an edge in G and if $l > 0$ then w_{n-l} is not connected to x by an edge in G and $w_n, w_{n-1}, .., w_{n-l+1}$ are connected to x by edges in G.

Next, we look at the edge w_2w_{n-l} in G'. If w_2 is connected to x in G then as w_{1+j} is connected to y that will violate the minimality of j. If w_2 is connected to y in G then w_1w_2 has to be an edge in G', which will contradict the fact $w_1....w_n$ is an anticycle. We observe w_{n-l} can not be connected to x by selection. If w_{n-l} is connected to y and $l = 0$ then w_1 and w_n have to be connected to each other in G'. If w_{n-l} is connected to y and $l > 0$ then w_{n-l+1} and w_{n-l} have to be connected to each other in G'. Both cases lead to a contradiction as $w_1....w_n$ is an anticycle. As xy is an edge in G, w_2w_{n-l} can not be an edge in G; otherwise they will form a gap. So w_2 and w_{n-l} are not connected to each other in G and neither of them are connected to x or y (w_2, w_{n-l}, x, y are four distinct vertices). So w_2w_{n-l} is not an edge in G' and this gives a contradiction. Hence $w_1...w_n$ is an induced cycle in G^c. $\square$

We now prove our main results about the regularity of 4-path ideals. The first one is comparable to the Theorem 2.1.11.

Theorem 4.1.10. *Let $I_4 \neq 0$. If I has a minimal free resolution which is linear up* *to step $p \geq 2$ then so does I_4. In particular if I has a linear resolution then so does I_4.*

Proof. Let e be any edge in G and G' be the graph associated with $(I_4 : e)$. By the previous lemma and the Theorem 2.1.17, G'^c does not have an induced cycle of length less than $p+3$ that is not a triangle. Hence we conclude that if I has linear minimal free resolution up to step p so does $(I_4 : e)$.

Next we observe as G is gap free, for any two different edges e and f in G, who do not share a common vertex $(f : e) = (f) \subseteq (I_4 : e)$. Hence either $(f : e)$ is generated by a variable or it is contained in $(I_4 : e)$. So for different edges $e_1, ..., e_k, e$ of G, $(I_4, e_1, ..., e_k) : (e)$ is $(I_4 : e) + J$ where J is an ideal generated by some variables.

Assume that $E(G) = \{e_1, ..., e_l\}$. Consider the following short exact sequences:

$$0 \longrightarrow \frac{S}{(I_4 : e_1)}(-2) \xrightarrow{.e_1} \frac{S}{I_4} \longrightarrow \frac{S}{(I_4, e_1)} \longrightarrow 0$$

$$0 \longrightarrow \frac{S}{((I_4, e_1) : (e_2))}(-2) \xrightarrow{.e_2} \frac{S}{(I_4, e_1)} \longrightarrow \frac{S}{(I_4, e_1, e_2)} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow \frac{S}{((I_4, e_1, ..., e_{l-1}) : (e_l))}(-2) \xrightarrow{.e_l} \frac{S}{(I_4, e_1, ..., e_{l-1})} \longrightarrow \frac{S}{I} \longrightarrow 0$$

In light of the observations made in previous paragraphs, Lemmas 4.1.8, 4.1.9, 2.1.8, 2.1.15 and 2.1.19 and Theorem 2.1.17, if I has linear resolution upto step p then so does I_4. Hence the result follows. □

Our next theorem is similar to Theorem 3.3.5 for powers of edge ideals.

Theorem 4.1.11. *If G is gap free and cricket free then I_4 has a linear minimal free resolution.*

Proof. We observe that since G is gap free, for any two different edges e and f in G, who does not share a common vertex $(f : e) = (f) \subseteq (I_4 : e)$. Hence either $(f : e)$ is generated by a variable or it is contained in $(I_4 : f)$. So for different edges $e_1, ..., e_k, e$ of G, $(I_4, e_1, ..., e_k) : (e)$ is $(I_4 : e) + J$ where J is an ideal generated by some variables. Hence in light of Lemmas 2.1.8, 2.1.5 and Theorem 2.1.10 it is enough to show that for every edge e the $\operatorname{reg}(I_4 : e) \leq 2$ that is if G' is the graph associated with $(I_4 : e)$ then G'^c is chordal.

We know from Lemma 3.4 that G' is gap free. If $w_1....w_n$ is an induced cycle in G'^c with $n \geq 5$, then it is also an induced cycle in G^c by Lemma 3.5. Then either w_1 or w_3 is a neighbor of x or neighbor of y else w_1w_3 and e forms a gap in G, a contradiction. Without loss of generality, we may assume w_1 is a neighbor of x. Now every neighbor of y is connected to every neighbor of x in G' if they are not same . As neither w_1w_n, nor w_1w_2 is an edge in G', neither w_2 nor w_n are neighbors of y in

G. So one of them has to be neighbor of x, as G is gap free. Again, without loss of generality, we may assume w_2 is a neighbor of x. Next we consider w_3w_n. As w_1 and w_2 are neighbors of x and neither w_1w_n nor w_2w_3 are edges in G', so neither w_3 nor w_n can be neighbor of y. Then either w_3 or w_n has to be a neighbor of x. Without loss of generality we may assume w_3 is a neighbor of x. Notice that y is not connected to w_1 in G as that will force w_2, a neighbor of x, to be connected to w_1 in G' leading to a contradiction. Hence $\{y, w_2, x, w_1, w_3\}$ forms a cricket leading to contradiction.

Hence by Theorem 2.1.10 reg $(I_4 : e) = 2$ and our result follows from Lemma 2.1.8. $\square$

We finish this section by explaining our last theorem by an example.

Example 4.1.12. *A five cycle is both gap free and cricket free so by previous theorem one expects the 4-path ideal to have linear resolution. One can check (by Macaulay 2, for example) that the 4-path ideal has regularity 4; that is, it has a linear minimal free resolution.*

4.2 Main Results

In this section we study general path ideals and prove our main result of this chapter. Our main result says that all path ideals of a gap free and claw free graph have linear minimal free resolutions. One observes that this result is similar to our result about linear resolutions of powers of gap free and cricket free edge ideals. However in case of powers of edge ideals we needed a special ordering on the generators to prove our result. For path ideal case no such order is required.

We first prove two very useful lemmas that will help us to prove our main theorem.

Lemma 4.2.1. *Let G be gap free, claw free and whiskered-K_4 free and $I_t \neq 0$ for some $t \geq 6$. If $e \neq f$ are two generators of I_t then either $(e : f)$ is generated by a variable or $(e : f) \subseteq (I_{t+1} : f)$. We get the same conclusion for all gap free and claw free graphs for $t = 3, 4, 5$.*

Proof. Assume $(e : f)$ is not generated by a variable. That means $m = \frac{e}{\gcd(e,f)}$, which is the generator of $(e : f)$, is a monomial of degree greater than or equal to 2. We also have for any $m'|m$ with $m' \neq m$, e does not divide $m'f$. Let $f = x_1 \cdots x_t$ and $e = y_1 \cdots y_t$. First we show that if a is a variable such that $a|m$ and ax_i is an edge in G for any $i \in \{1, 2, t-1, t\}$, then $af \in I_{t+1}$ as G is claw free. This is clear if ax_1 or ax_t is an edge as in that case $ax_1 \cdots x_t$ or $ax_t \cdots x_1$ will be in I_{t+1}. If ax_2 is an edge then for $ax_2x_1x_3$ to avoid being a claw either ax_1 or ax_3 or x_1x_3 is an edge. In the first case it is again clear. In the second case $af \in I_{t+1}$ as $x_1x_2ax_3 \cdots x_t$ forms a $t+1$ path. In third case $ax_2x_1x_3 \cdots x_t$ forms a $t+1$ path. The other cases follow by symmetry. In all these cases $(e : f) \subseteq (a) \subseteq (I_{t+1} : a)$.

If there is an edge $h|m$ then as G is gap free considering x_1x_2 and h, we get that there is a variable a dividing m such that ax_1 or ax_2 is an edge and hence we are done by arguments of previous paragraph; also if there exists a variable a dividing m such that for some i both ax_i and ax_{i+1} are edges in G then $x_1 \cdots x_i a x_{i+1} \cdots x_t$ is a generator of I_{t+1} and $(e : f) \subseteq (a) \subseteq (I_{t+1} : f)$.

Now we may assume that neither of the above holds. We have $y_1 \cdots y_t | m x_1 \cdots x_t$. If degree of m is α then $y_1 \cdots y_t = m x_{i_1} \cdots x_{i_{t-\alpha}}$ for some x variables. As $\alpha \geq 2$, m is not divisible by an edge and variables of m are part of a t-path (namely e) we have two variables a, b dividing m and two indices i, j (with the possibility that $i = j$) such that ax_i and bx_j are edges.

If a and b both connected to x_i, for x_{i-1}, x_i, a, b to avoid being a claw we must have either ax_{i-1} or bx_{i-1} is an edge in G contradicting the assumption; as no edge divides m, ab can not be an edge. So we may assume that a and b are not connected to the same x, in particular we have $i \neq j$. If $t \leq 5$ then this forces $\{i, j\} \cap \{1, 2, t-1, t\} \neq \emptyset$ and we are done by assumption. So let us assume $t \geq 6$ and $\{i, j\} \cap \{1, 2, t-1, t\} = \emptyset$.

Without loss of generality we assume $i \geq j$. Also as the graph is claw free, considering a, x_i, x_{i-1}, x_{i+1} we conclude that $x_{i-1}x_{i+1}$ is an edge. This follows because by assumption a is not connected to two consecutive edges. Similarly $x_{j-1}x_{j+1}$ is an edge.

Now consider ax_i and x_1x_2. As G is gap free and by assumption a is not connected to x_1 or x_2, we have either x_1x_i or x_2x_i is an edge. If x_1x_i is an edge then $ax_ix_1x_2\cdots x_{i-1}x_{i+1}\cdots x_t$ forms a $t+1$ path and $a \in (I_{t+1} : f)$. Hence $(e : f) \subseteq (I_{t+1} : f)$. So we may assume that is not the case, so x_2x_i is an edge. By symmetry we may assume $x_2x_j, x_ix_{t-1}, x_jx_{t-1}$ are edges and x_1x_j, x_ix_t, x_jx_t are not edges.

As we are in a gap free graph, both a and b are not connected to same x and ab is not an edge, we have x_ix_j is an edge. Otherwise ax_i and bx_j will form a gap.

Finally we observe that if x_1x_t is an edge then $ax_ix_2x_3\cdots x_{i-1}x_{i+1}\cdots x_tx_1$ is a $t+1$ path and $a \in (I_{t+1} : f)$ and hence $(e : f) \subseteq (I_{t+1} : f)$. So we may assume this is not the case. If x_1x_{t-1} is an edge, then we consider $x_1x_{t-1}x_tx_i$. This forms a claw. Hence we may assume x_1x_{t-1} is not an edge. Similarly, x_2x_t is not an edge. As we are in a gap free graph this forces x_2x_{t-1} to be an edge; otherwise x_1x_2 and $x_{t-1}x_t$ forms a gap.

Now we consider the induced subgraph on $\{a, b, x_1, x_2, x_i, x_j, x_{t-1}, x_t\}$. The set of edges of this induced subgraph is

$$\{x_1x_2, x_{t-1}x_t, ax_i, bx_j, x_2x_i, x_2x_j, x_2x_{t-1}, x_ix_j, x_ix_{t-1}, x_jx_{t-1}\}.$$

This forms a whiskered-K_4, which gives a contradiction.

$\square$

We explain this lemma in the next example for a 5-cycle.

Example 4.2.2. Let $S = \mathbb{Q}[x_1, \ldots, x_n]$ and $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5)$. In this case we know from previous examples that

$I_3 = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2)$

and $I_4(G) = (x_1x_2x_3x_4, x_2x_3x_4x_5, x_3x_4x_5x_1, x_4x_5x_1x_2, x_5x_1x_2x_3)$. We know that a 5-cycle is both gap free and claw free. We observe that

$(x_1x_2x_3 : x_2x_3x_4) = (x_1)$, which is an ideal generated by a variable

$(x_1x_2x_3 : x_3x_4x_5) = (x_1x_2) \subseteq (I_4 : x_3x_4x_5)$, as $x_2x_3x_4x_5$ is 4-path,

$(x_1x_2x_3 : x_4x_5x_1) = (x_2x_3) \subseteq (I_4 : x_4x_5x_1)$, as $x_3x_4x_5x_1$ is a 4-path

and $(x_1x_2x_3 : x_5x_1x_2) = (x_3)$, which is an ideal generated by variables.

The other cases follow by symmetry.

Lemma 4.2.3. *Let G be gap free, claw free, whiskered K_4 free and $I_{t+1} \neq 0$. If f is a generator of I_t for any $t \geq 6$ then $(I_{t+1} : f)$ is generated by variables. If $t = 3, 4, 5$ then the same conclusion holds for every gap free and claw free graph.*

Proof. Let m be a minimal monomial generator of $(I_{t+1} : f)$ which is not a variable. So $mf \in I_{t+1}$. Hence there is a $t+1$ path e such that $e|fm$. As e is squarefree and m is minimal, we can assume $\gcd(f, m) = 1$.

Let $f = x_1 \cdots x_t$ and $e = y_1 \cdots y_{t+1}$. First observe that if a is a variable such that $a|m$ and ax_i is an edge in G for any $i \in \{1, 2, t-1, t\}$. Then $af \in I_{t+1}$ as G is claw free. This is clear if ax_1 or ax_t is an edge as in that case $ax_1 \cdots x_t$ or $ax_t \cdots x_1$ will be in I_{t+1}. If ax_2 is an edge then for $ax_2x_1x_3$ to avoid being a claw either ax_1 or ax_3 or x_1x_3 is an edge. In first case it is again clear. In the second case $af \in I_{t+1}$ as $x_1x_2ax_3\cdots x_t$ forms a $t+1$ path. In third case $ax_2x_1x_3\cdots x_t$ forms a $t+1$ path. The other cases follow by symmetry. In all these cases $(m) \subseteq (a) \subseteq (I_{t+1} : f)$ and by minimality of m we have $m = a$

If there is an edge $h|m$ then as G is gap free considering x_1x_2 and h we get that there is a variable a dividing m such that ax_1 or ax_2 is an edge. Hence we are done by arguments of previous paragraph. We also observe that if there exists a variable a dividing m such that for some i both ax_i and ax_{i+1} are edges in G, then $x_1\cdots x_iax_{i+1}\cdots x_t$ is a generator of I_{t+1} and $(m) \subseteq (a) \subseteq (I_{t+1} : f)$ and $m = a$ by minimality.

Now we may assume that neither of the above holds. We have $y_1\cdots y_{t+1}|mx_1\cdots x_t$. If degree of m is α then $y_1\cdots y_{t+1} = mx_{i_1}\cdots x_{i_{t+1-\alpha}}$ for some x variables. As $\alpha \geq 2$, m is not divisible by an edge and variables of m are part of a t+1-path (namely e) we have two variables a, b dividing m and two indices i, j (with the possibility that $i = j$). such that ax_i and bx_j are edges.

If a and b both connected to x_i, for x_{i-1}, x_i, a, b to avoid being a claw we must have either ax_{i-1} or bx_{i-1} is an edge in G contradicting the assumption; as no edge divides m, ab can't be an edge. So we may assume that a and b are not connected to same x, in particular we have $i \neq j$. If $t \leq 5$ then this forces $\{i, j\} \cap \{1, 2, t-1, t\} \neq \emptyset$ and we're done by assumption. So let us assume $t \geq 6$ and $\{i, j\} \cap \{1, 2, t-1, t\} = \emptyset$.

As the graph is claw free, considering a, x_i, x_{i-1}, x_{i+1} we conclude that $x_{i-1}x_{i+1}$ are edges. This follows because by assumption a is not connected to two consecutive edges. Similarly $x_{j-1}x_{j+1}$ is an edge.

Now consider ax_i and x_1x_2. As G is gap free and by assumption a is not connected to x_1 or x_2, we have either x_1x_i or x_2x_i is an edge. If x_1x_i is an edge then $ax_ix_1x_2 \cdots x_{i-1}x_{i+1} \cdots x_t$ forms a $t+1$ path and $a \in (I_{t+1} : f)$. Hence $m = a$. So we may assume that is not the case, so x_2x_i is an edge. By symmetry we may assume $x_2x_j, x_ix_{t-1}, x_jx_{t-1}$ are edges and x_1x_j, x_ix_t, x_jx_t are not edges.

As we are in a gap free graph, both a and b are not connected to same x and ab is not an edge, we have x_ix_j is an edge. Otherwise ax_i and bx_j will form a gap.

Finally we observe that if x_1x_t is an edge then $ax_ix_2x_3 \cdots x_{i-1}x_{i+1} \cdots x_tx_1$ is a $t+1$ path and $a \in (I_{t+1} : f)$ and hence $m = a$. So we may assume this is not the case. If x_1x_{t-1} is an edge, then we consider $x_1x_{t-1}x_tx_i$. This forms a claw. Hence we may assume x_1x_{t-1} is not an edge. Similarly, x_2x_t is not an edge. As we're in a gap free graph this forces x_2x_{t-1} to be an edge; otherwise x_1x_2 and $x_{t-1}x_t$ forms a gap.

Now we consider the induced sub graph on $\{a, b, x_1, x_2, x_i, x_j, x_{t-1}x_t\}$. The set of edges of this induced subgraph is

$$\{x_1x_2, x_{t-1}x_t, ax_i, bx_j, x_2x_i, x_2x_j, x_2x_{t-1}, x_ix_j, x_ix_{t-1}, x_jx_{t-1}\}.$$

This forms a whiskered K_4, which gives a contradiction.

$\square$

We continue our illustration via the 5-cycle example.

Example 4.2.4. As before let $S = \mathbb{Q}[x_1, \ldots, x_n]$ and $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5)$. In this case we know from previous examples that

$I_3 = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2)$ and

$I_4 = (x_1x_2x_3x_4, x_2x_3x_4x_5, x_3x_4x_5x_1, x_4x_5x_1x_2, x_5x_1x_2x_3)$. By symmetry it is enough to compute the colon ideal for one 3-path. We simply observe that

$(I_4 : x_1x_2x_3) = (x_4, x_5)$, which is an ideal generated by variables.

The main theorem follows from these two lemmas.

Theorem 4.2.5. *If G is gap free and claw free and $I_t \neq 0$ then I_t has linear minimal free resolution for $t = 3, 4, 5, 6$. If G is gap free, claw free and whiskered K_4 free and $I_t \neq 0$ then I_t has linear minimal free resolution for all $t \geq 3$.*

Proof. For $t = 3$ this follows from the Theorem 4.1.3 and the Theorem 2.2.1 as a claw free graph is automatically cricket free. Let us assume by induction the result holds for $(t-1)$ for some $t \geq 4$. If $m_1, ..., m_k$ are k different monomials representing $(t-1)$-paths then by the previous two lemmas $((I_t, m_1, ..., m_{k-1}) : (m_k))$ is an ideal generated by variables and hence has regularity 1. The result now follows from the Lemma 2.1.8. $\square$

For the sake of completion we finish this chapter with the following example,

Example 4.2.6. One checks using Macaulay 2, that for a 5-cycle, $\operatorname{reg}(I_t) = t$ for $t = 3, 4, 5$.

Chapter 5

Cohen-Macaulay Bipartite Graphs

The relationship between the combinatorics of a bipartite graph and the homological algebra of the corresponding edge ideal is known to be very deep and studied extensively by various mathematicians (see for example, [HH], [K1], [K2], [MV], [Vi1]). Among other nice properties, the bipartite graphs with Cohen-Macaulay edge ideals are known to have perfect matching. For this reason people are interested to find characterizations for the Cohen-Macaulay bipartite graphs. There are different characterizations of the Cohen-Macaulay bipartite graphs and most of them use Hall's Marriage theorem (or one of its equivalent forms like König's theorem) for their proofs. Observing these we got curious to know whether one can prove a characterization without using any of those theorems.

In the first section of this chapter we give an elementary proof of the characterization of Cohen-Macaulay bipartite graphs by Herzog and Hibi (see [HH]). It is worth noticing that our proof of the characterization by Herzog and Hibi does not use any strong graph theoretic results like the Marriage theorem or König's theorem. We use short exact sequences and the fact that a Cohen-Macaulay quotient is unmixed and connected in codimension one.

In the second section we prove a new characterization for Cohen-Macaulay bipartite graphs. For this too, we do not use Hall's theorem. We use the description of $(I^2 : e)$ by even-connections and Herzog-Hibi's characterization. It has been brought to our attention by R. H. Villarreal that Theorem 5.2.3 (which is crucial tool for proving our characterization) follows from Theorem 2.9 (e) of [MRV] and Corollaries 8 and 9 of [ZN] and both of these works have been done prior to our work.

Many people have tried to characterize the bipartite graphs with related properties. For example, R. H. Villarreal has characterized unmixed bipartite graphs in [Vi1]. One natural higher degree generalization of the results in this section will be characterizing the similar properties for the edge ideals of the t-uniform, t-partite hypergraphs (see the Sections 1 and 2 of [KM] for relevant definitions) in the similar way. However this seems to be a much harder problem than the bipartite case, even for t=3. Even characterizing the unmixed 3-uniform, 3-partite hypergraphs looks formidable. A more general question will be to characterize all such hypergraphs whose edge ideals satisfy Serre's S_i condition. One important step in our proof is to show that for the bipartite graphs, unmixed and connected at codimension one is equivalent to being Cohen-Macaulay. This is of course false in general but whether this is true for the t-uniform t-partite hypergraphs or some subclasses of them is not known and investigating that may shed more light into this area. Cohen-Macaulayness and unmixedness are connected to linear resolutions and linear presentations respectively via the so-called Alexander duality (see the Section 2 of [DHS] and [ER] for the definitions and relevant discussions). In light of these it will be interesting to explore the utility of the techniques that are useful in the study of regularity. We pose all these in the next question.

Question 5.0.7. *1. Characterize Cohen-Macaulay t-uniform, t-partite hypergraphs for $t \geq 3$.*

2. Characterize unmixed t-uniform, t-partite hypergraphs for $t \geq 3$.

3. Characterize t-uniform, t-partite hypergraphs whose edge ideals satify Serre's condition S_i for $t \geq 3$

4. Characterize the t-uniform, t-partite hypergraphs for which unmixed and connected at codimension one implies Cohen-Macaulay.

Throughout this chapter we assume that G is a connected bipartite graph. We refer the reader to [W] for the elementary properties of bipartite graphs.

5.1 A New Proof Of The Herzog-Hibi Characterization

We first state and prove the following theorem, which was originally proved by Herzog and Hibi in [HH]. We're grateful to our advisor Professor Craig Huneke for suggesting the main idea of this proof. For this proof we use the notion of *connected in codimension one*.

Theorem 5.1.1. *Let G be a bipartite graph with bipartition $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_{n'}\}$. Then $I(G)$ is Cohen-Macaulay if and only if $n = n'$ and there exists an enumeration of x-variables and y-variables with the following three properties:*

a. $x_i y_i \in I(G)$.

b. $x_i y_j \in I(G) \implies i \leq j$

c. $x_i y_j \in I(G), x_j y_k \in I(G) \implies x_i y_k \in I(G)$.

Proof. We first prove the if part.

Consider:

$$0 \to \frac{S}{(I(G) : x_1)} \to \frac{S}{I(G)} \to \frac{S}{(I(G), x_1)} \to 0$$

Notice that $(I(G), x_1) = (I(G'), x_1)$ where G' is the graph obtained by deleting x_1 and y_1 from G. Clearly G' satisfies all the conditions and hence $I(G')$ is Cohen-Macaulay of dimension $n - 1$ by induction. So $(I(G), x_1)$ is Cohen-Macaulay of dimension n. Let $\{y_1, y_{i_2}, ..., y_{i_k}\}$ be the degree one generators of $(I(G) : x_1)$ for some $i_1, ..., i_k$. Let $x_{i_j} y_l \in I(G)$. As $x_1 y_{i_j} \in I(G)$ by the condition c, $x_1 y_l \in I(G)$ and hence $l \in \{1, i_2, ..., i_k\}$. So $(I(G) : x_1) = (I(G'), y_1, ..., y_{i_k})$, where G' is the graph obtained from G by deleting $x_1, y_1, x_{i_2}, y_{i_2}, ..., x_{i_k}, y_{i_k}$. But by induction $I(G')$ is Cohen-Macaulay of dimension $n - k$. Hence $(I(G) : x_1)$ is Cohen-Macaulay of dimension n. So by the Depth Lemma and the fact that the Krull dimension of $\frac{S}{I(G)}$ is less than or equal to that of $\frac{S}{(I(G):x_1)}$, we conclude that $I(G)$ is Cohen Macaulay of dimension n.

To prove the converse we first observe that $n = n'$ as Cohen-Macaulay implies unmixed and both $(x_1, ..., x_n)$ and $(y_1, ..., y_{n'})$ are minimal primes. Next we prove the existence of conditions a and b by induction. The condition c will follow from the Cohen-Macaulayness.

First observe that Cohen Macaulay implies unmixed and connected in codimension 1. Let $s \subset \{1, ..., n\}$. Define $y^s = \Pi_{i \in s} y_i$ and $x^s = \Pi_{i \in s} x_i$. Given $s \subset \{1, ..., n\}$, define $T_s = \{j | x_j$ is not connected to any $y_i, i \in s\}$, and let $u_s = y^s x^{T_s}$. Note that $u^s \notin I(G)$.

We now consider the ideals $(I(G) : u^s)$. Using this we prove the existence of an order of the required type. We actually prove that any y with minimum degree can serve as y_1.

Let $s = \{1, ..., n\}$. Then $(I : u^s) = (x_1, ..., x_n)$. So ht $I = n$ as I is unmixed. Clearly $(I : u^s) = (x_{j_1}, ..., x_{j_t}, y_{l_1}, ..., y_{l_{t'}})$ where x_{j_i} connected to some y in s and y_{l_k} is connected to any x not connected to any y in s. Hence $\mathrm{ht}(I : u^s) = n$ and $t + t' = n$.

Choose y_i with minimum degree. Without loss of generality we may assume $i = 1$. Let $x_1, ..., x_t$ be neighbors of y_1. Then there exist exactly $n - t$ y's that are connected to other x's as $x_1, .., x_t$ and these y's form a prime ideal containing I which is minimal. After relabelling $y_1, ..., y_t$ are only connected to $x_1, ..., x_t$. As t is minimal the induced subgraph on $x_1, ..., x_t, y_1, ..., y_t$ forms a complete bipartite graph.

So if any minimal prime P of I does not contain some x_i between 1 to t then it has to contain every y_i between 1 to t is there. As I is unmixed and connected codimension one this forces $t = 1$ and y_1 is only connected to x_1 as otherwise there is no path from $(x_1, ..., x_n)$ to $(y_1, ..., y_n)$ in codimension one; this can be seen in the following way: for any path in codimension one between $(x_1, ..., , x_n) = P_0, ..., P_l = (y_1, ..., y_n)$; let for some $l' < l$, $P_0, ..., P_{l'}$ contains all of $x_1, ..., x_t$. As $\mathrm{ht}(P_i + P_{i+1}) = \mathrm{ht}(P_i) + 1$ and the variables $x_{t+1}, ..., x_n$ are onely connected to $y_{t+1}, ..., y_n$, we observe that the prime ideals $P_0, ..., P_{l'}$ do not contain $y_1, ..., y_t$. Now $P_{l'+1}$ misses atleast one of $x_1, ..., x_t$; hence it has to contain all $y_1, ..., y_t$. So $\mathrm{ht}((P_{l'} + P_{l'+1}) \geq \mathrm{ht}(P_{l'}) + t$. This gives a contradiction.

Now consider (I, x_1). Any minimal prime of (I, x_1) is a minimal prime of I, so (I, x_1) is unmixed. We now show that (I, x_1) is connected at codimension one. Any minimal prime of I has to contain either x_1 or y_1; as it is minimal it can not have both as y_1 is only connected to x_1. As I is connected at codimension one, between any two minimal primes of (I, x_1) there is a path in codimension one of minimal primes of I. If any prime appearing in that path has y_1 simply changing it into x_1 we get a path in condimension one of minimal primes of (I, x_1). This shows that (I, x_1) is connected in codimension one. If G' is the graph obtained from G by deleting x_1

then $I(G')$ is Cohen-Macaulay by induction. So there exists an ordering $\{x_2, ..., x_n\}$ and $\{y_2, ..., y_n\}$ with the required property. As y_1 is only connected to x_1 the result follows.

To prove that condition c holds, take x_i, y_j and x_j, y_k in $E(G)$ such that i, j, k are distinct. Assume that $x_i y_k$ is not an edge. Then there is a minimal prime P that does not contain either x_i or y_k as the ideal generated by all x-variables except x_i and all y-variables except y_k is a prime ideal that contains I and does not contain x_i or y_k. Now because G is unmixed, height of this prime has to be n. Since x_i and y_k are not on P, we get that y_j and x_j are both in P. As P contains at least one of x_m or y_m for all m, one observes that height of P is strictly bigger than n, which is a contradiction. $\square$

We illustrate this theorem via following example.

Example 5.1.2. Let $S = \mathbb{Q}[a, b, c, x, y, z]$ and $I = (ax, ay, az, by, bz, cz)$. Clearly I is a bipartite edge ideal. Using Macaulay 2 we observe that dimension$(\frac{S}{I})$=3 and pd$(\frac{S}{I}) = 3$. So by the Auslander-Buchbaum theorem depth$(\frac{S}{I}) = 3$ and hence $\frac{S}{I}$ is Cohen-Macaulay. Now observe that we can rename the variables by $a = x_1, b = x_2, c = x_3, x = y_1, y = y_2, z = y_3$ and this new enumeration has the property prescribed by the theorem.

5.2 A New Characterization

In this section we prove a new characterization for Cohen-Macaulay bipartite graphs using the even-connection description of $(I^2 : e)$ for an edge e. To do that we first prove a lemma which describes the nature of $(I^2 : e)$ in a bipartite graph.

Lemma 5.2.1. *Let G be a bipartite graph with partitions $\{x_1, ..., x_k\}$ and $\{y_1, ..., y_l\}$ and edge ideal I. For any edge x_iy_j in G, $(I^2 : x_iy_j) = I + (x_my_n | x_my_j, x_iy_n \in E(G))$*

Proof. The proof follows from Theorem 3.2.7 and Definition 3.2.2 and the fact that bipartite graphs do not have any triangles. □

We illustrate this with the following example.

Example 5.2.2. Let $S = \mathbb{Q}[a, b, c, x, y, z]$ and $I = (ax, ay, az, by, bz, cz)$. Clearly I is a bipartite edge ideal with bipartition $\{a, b, c\}$ and $\{x, y, z\}$. We observe that $(I^2 : ay) = I + (bx)$, which is exactly what the lemma says.

Our next result leads to our new characterization of Cohen-Macaulay bipartite graphs. From now on we call two edges e and f disjoint if they share no common vertices.

Theorem 5.2.3. *Let G be a bipartite graph with edge ideal I and size of each partition n. Then I is Cohen-Macaulay if and only if there exist n pairwise disjoint edges $e_1, ..., e_n$ such that $(I^2 : e_i) = I$ and for any other edge e, $(I^2 : e) \neq I$.*

Proof. If I is Cohen-Macaulay, we have orderings $x_1, ..., x_n$ and $y_1, ..., y_n$ of the vertices of G which satisfy the conditions of previous theorem. Condition c implies for all i, $I^2 : x_iy_i = I$ and conditions a and b implies for $i \neq j$ $(I^2 : x_iy_j) \neq I$.

Now suppose there exist $e_1 = x_1y_1, ..., e_n = x_ny_n$ with the condition. First we show that if G_i is the induced subgraph obtained by deleting x_i and y_i then the edge ideal J_i related to G_i is satisfies the condition. Without loss of generality, we prove this for G_1. Clearly $(J_1^2 : e_i) = J_1$ for $e_2, ..., e_n$. Suppose there exists an edge $x_iy_j, i \neq j$ such that $(J_1^2 : x_iy_j) = J_1$. Without loss of generality we may assume $i = 2, j = 3$. As $(I^2 : x_2y_3) \neq I$ and x_1y_1 is an edge we can conclude that there exists a minimal generator of $(I^2 : x_2y_3)$ which is an edge that is either of the form x_1y_l or x_my_1. Again without loss of generality we may assume it is of the form x_1y_l as the proof for the other follows simply by interchanging roles of x and y. So x_1y_3 and x_2y_l are edges in G. As $(J_1^2 : x_2y_3) = J_1$ we conclude x_3y_2 is an edge in G. As $(I^2 : x_3y_3) = I$ we observe that x_1y_2 has to be an edge in G. So $l \neq 2, 3$. Without loss of generality we may assume $l = 4$. Now $(I^2 : x_2y_2) = I$ so x_3y_4 has to be an edge in G. Again $(I^2 : x_3y_3) = I$ hence x_1y_4 is an edge in G contradicting the assumption. So we may assume for all i the edge ideal of the graph obtained by deleting x_i and y_i satisfies the condition.

Now by induction we may assume the result holds for $n - 1$. Pick $e_i = x_iy_i$

such that y_i has minimum degree. Let G' be the induced subgraph on vertices other than x_i, y_i with edge ideal I'. As I' satisfies the condition it is Cohen-Macaulay by induction. Without loss of generality we may assume $i = 1$ and ordering that gives ordering of previous theorem for I' is $x_2, ..., x_n, y_2, ..., y_n$. As y_2 has degree one in G' it can have at most degree 2 in G. If x_1y_2 is not an edge, due to minimality degree of y_1 is at most 1. If x_1y_2 is an edge in G and x_iy_1 is an edge in G for $i > 2$, as $(I^2 : x_1y_1) = I$, we have x_iy_2 is an edge in G and hence in G' contradicting the assumption. If x_1y_2 and x_2y_1 both are edges in G then, x_2y_1 also satisfies the hypothesis as x_1 has to be connected to any neighbour of x_2 as x_1y_2 is an edge and x_2y_2 satisfies the hypothesis, leading to contradiction. Hence no x_i for $i > 1$ is connected to y_1. This guarantees that conditions a and b of Theorem 5.0.8 are satisfied. The condition c is satisfied as for all i, $(I^2 : x_iy_i) = I$. $\square$

We illustrate using our previous example which is known to be Cohen-Macaulay.

Example 5.2.4. Let $S = \mathbb{Q}[a, b, c, x, y, z]$ and $I = (ax, ay, az, by, bz, cz)$. Clearly I is a bipartite edge ideal with bipartition $\{a, b, c\}$ and $\{x, y, z\}$. We observe that

$(I^2 : ax) = (I^2 : by) = (I^2 : cz) = I$

$(I^2 : ay) = I + (bx), (I^2 : az) = (cz, bz, az, cy, by, ay, cx, bx, ax),$

$(I^2 : bz) = I + (cy)$

So there are exactly 3 edges e such that $(I^2 : e) = I$

The following theorem is the main result of this section. We give a characterization of Cohen-Macaulay bipartite edge ideals.

Theorem 5.2.5. *Let G be a bipartite graph with edge ideal I and size of each partition n. Then I is Cohen-Macaulay if and only if the there exist n pairwise disjoint edges $e_1, ..., e_n$, such that $(I^2 : e)$ is Cohen-Macaulay and for any other edge e, $(I^2 : e)$ is not Cohen-Macaulay.*

Proof. To prove the if part, we pick y with minimum degree and call it y_1 and the corresponding edge e_1. If degree of y_1 more than one then degree of any other vertex is more than one; as $(I^2 : e_1)$ is Cohen-Macaulay this will be a contradiction. So y_1 has degree one. Hence $(I^2 : e_1) = I$ and I is Cohen-Macaulay.

For the only if part let $e_1, ..., e_n$ be the ordering prescribed by the Herzog-Hibi characterization. All we need to show $J = (I^2 : x_i y_j)$ is not Cohen-Macaulay for $i > j$. This follows as $(J^2 : e) = J$ for $e = x_j y_i$ (which is a minimal monomial generator of J) as well as for $e_1, ..., e_n$. To see this first we show that $(J^2 : e_k) = J$ for all k. Here at every step we use the description of colon ideal provided by Lemma 5.2.1. If $x_l y_m$ is a minimal monomial generator of $(J^2 : e_k)$ which is not in J then $x_l y_k$ and $x_k y_m$ are in J. Both of them can not belong to I as from $(I^2 : e_k) = I$ that will imply $x_l y_m$ belongs to I and as a result will belong to J, contradicting the assumption. Without loss of generality assume $x_k y_m$ does not belong to I. Then $x_k y_j$ and $x_i y_m$ is in I. If $x_l y_k$ does not belong to I then $x_l y_j$ and $x_i y_k$ belong to I. If $x_l y_k$ is in I as $x_k y_j$ is in I and $(I^2 : e_k) = I$ we have $x_l y_j$ is in I. In either case we have $x_l y_j$ and $x_i y_m$ belong to I. Hence $x_l y_m$ belongs to J contradicting our assumption.

Next we show that $(J^2 : x_j y_i) = J$. Here too we use Lemma 5.2.1 heavily. If $x_l y_k$ is a minimal monomial generator of $(J^2 : x_j y_i)$ which is not in J then $x_j y_k$ and $x_l y_i$ is in J. As $x_j y_k$ is in J it is either in I or y_k is a neighbor of x_i in G. If $x_j y_k$ is in I as $(I^2 : x_j y_j) = I$ we have $x_i y_k$ is in I. By symmetry $x_l y_j$ is in I. Hence $x_l y_k$ is in J contrary to the assumption. Hence J is not Cohen-Macaulay. □

We illustrate this theorem using our previous example which is known to be Cohen-Macaulay.

Example 5.2.6. Let $S = \mathbb{Q}[a, b, c, x, y, z]$ and $I = (ax, ay, az, by, bz, cz)$. Clearly I is a bipartite edge ideal with bipartition $\{a, b, c\}$ and $\{x, y, z\}$. We observe that $(I^2 : ax) = (I^2 : by) = (I^2 : cz) = I$, so all of them are Cohen-Macaulay.
$\operatorname{depth}(I^2 : ay) = \operatorname{depth}(I^2 : az) = (I^2 : bz) = 4$, so none of them are Cohen-Macaulay.
So there are exactly 3 edges e such that $(I^2 : e)$ is Cohen-Macaulay

The next theorem gives insight into the associated graded ring of a Cohen-Macaulay bipartite edge ideal. The proof of this theorem uses the description of the colon via *even-connection.*

Theorem 5.2.7. *Let I be Cohen-Macaulay bipartite edge ideal with ordering $e_1, ..., e_n$. Then for all i and for all k, $(I^k : e_i) = I^{k-1}$. Hence e_is are non zero divisors in the associated graded ring of I.*

Proof. Let e_i be $x_i y_i$. Let $f \in (I^k : e_i) \subset (I^{k-1} : e_i)$ be a minimal monomial generator of $(I^k : e_i)$. By induction $(I^{k-1} : e_i) = I^{k-2}$. So $f = gh_1....h_{k-2}$ where h_js are minimal monomial generators of I and g any monomial. So $e_ih_1....h_{k-2}g \in I^k$. As f is a minimal monomial generator, without loss of generality we may assume g is of degree 2 and $e_ih_1..h_{k-2}g$ is a minimal monomial generator of I^k. Let g be x_ky_l. If g is an edge we are done. Otherwise by Theorem 3.2.7, x_k and y_l are even connected with respect to $e_ih_1...h_{k-2}$. If x_iy_l is an edge and for some j, m, p, x_my_i is an edge and $h_j = x_my_p$, then by third condition of Cohen-Macaulayness in Herzog-Hibi theoremx_my_l is an edge and hence proceeding inductively we show g is an edge. This observation along with the third condition of Cohen-Macaulayness in Herzog-Hibi theorem proves that x_k and y_l are even connected with respect to $h_1....h_{k-2}$ and hence we get the result. □

We illustrate this theorem using the ideal of our previous example for $k = 3, 4$.

Example 5.2.8. Let $S = \mathbb{Q}[a, b, c, x, y, z]$ and $I = (ax, ay, az, by, bz, cz)$.

One can check using Macaulay 2,

$(I^3 : ax) = (I^3 : by) = (I^3 : cz) = I^2$, (Checking $J == I^2$ returns TRUE in Macaulay 2 for all these ideals)

and, $(I^4 : ax) = (I^4 : by) = (I^4 : cz) = I^3$, (Checking $J == I^3$ returns TRUE in Macaulay 2 for all these ideals).

Chapter 6

Some Open Questions

In this chapter we discuss some open questions and further directions of research related to the topics covered in this thesis. These can be broadly divided into two groups, questions related to Castelnuovo-Mumford regularity of ideals related to finite simple graphs and questions related to homological algebra of edge ideals of even-connections. Many mathematicians have studied the questions of the first type in recent years and many interesting results have come up. As an example we cite the recent works of Bayerslan, Hà, and Trung ([BHT]) or that of Hà, Trung, and Trung (HTT). Although far from getting a complete picture (or even understanding what that means in this context), our understanding of the connection between the combinatorics of the graph and the regularity of powers of edge ideals is getting better. From the above stated works and also from the work done in this thesis it appears that one way to better this understanding is to study the related colon ideals. This leads to questions of a second kind. As we saw earlier even connections contain lot of information about these ideals. In these works we showed some relation between even connections and the edge ideal itself; however there are many directions in which research can be pursued and hopefully more results about regularity of powers of edge ideals can be produced. Apart from this connection with regularity, even-connections provide interesting classes of edge ideals and we know almost nothing about their algebraic properties, for example primary decomposition, depth, dimension, etc. Research in these directions is expected to provide more results, as well as new questions. In the subsequent sections we discuss some questions involving these two themes.

6.1 Some Open Questions About Regularity

In this section we mention some open questions about regularity that are related to this work. We already mentioned in a previous chapter the question by Nevo and Peeva about regularity three edge ideals. A general version of that question is stated in [NP], Open Problem 1.11:

Question 6.1.1 (Nevo-Peeva). *1. Is it true that G^c has no induced four cycle if and only if $I(G)^s$ has linear resolutions for large enough s?*

2. If G^c has no induced four cycle, is it true that:

$$\operatorname{reg}(I(G)^{s+1}) \leq max\{2s+2, \operatorname{reg}(I(G)^s)+1\}$$

.

We know that the first part of this question has a positive answer in some cases. We prove it for gap free and cricket free graphs in this thesis and it is also known to be true for chordal graphs. One notes that $\operatorname{reg}(I(G)^{s+1})$ is always less than or equal to $2s+2$ as $I(G)$ is generated in degree 2. As we answered a part of this using even-connection techniques and in [BHT] a similar problem has also been solved using even-connection, one expects that more research about properties of even-connection will help to answer this question (more on this in the next section).

Various open problems related to regularity are stated in section 6 of [H]. We discuss some of those that are closely related to this work. Just like Fröberg's theorem classifies all regularity 2 finite simple graphs, it is tempting to try to classify all finite simple graphs with any given regularity. For various topological reasons this question is known to be very difficult. There is no known progress even in the next simplest case, which is the Problem 6.3 of [H]:

Question 6.1.2 (Hà)**.** *Classify all finite simple graphs of regularity 3.*

It is interesting to note that in [GR], the authors characterize all regularity 3 bipartite graphs.

As we saw in this work, the inequality $\operatorname{reg}(I) \leq \max\{\operatorname{reg}(I : x) + 1, \operatorname{reg}(I, x)\}$, for a monomial ideal I and variable x, is ubiquitous in this area. There are many things that are unknown regarding this inequality. Problem 6.5 of [H] addresses one of these:

Question 6.1.3 (Hà). *Classify monomial ideals I and variables x such that the above mentioned inequality is an equality.*

In the same direction we ask the following question:

Question 6.1.4. *Classify all monomial ideals I and x such that the* $\operatorname{reg}(I)$

1. Is Equal to the max$\{\operatorname{reg}(I : x) + 1, \operatorname{reg}(I, x)\}$.

2. Is Equal to the min$\{\operatorname{reg}(I : x) + 1, \operatorname{reg}(I, x)\}$.

3. Is equal to $\operatorname{reg}(I : x) + 1$.

4. Is equal to $\operatorname{reg}(I, x)$.

5. Is equal to both (and as a consequence both of them have same value).

6. Strictly greater than $\operatorname{reg}(I, x)$

We also ask:

Question 6.1.5. *Classify all monomial ideals I and x such that:*

1. $\operatorname{reg}(I : x) + 1 \geq \operatorname{reg}(I, x)$.

2. $\operatorname{reg}(I : x) + 1 \leq \operatorname{reg}(I, x)$.

3.$\operatorname{reg}(I : x) + 1 > \operatorname{reg}(I, x)$.

4.$\operatorname{reg}(I : x) + 1 < \operatorname{reg}(I, x)$.

One philosophical point should be made about all of the above three questions. It is not clear that what we mean by "classify". One may think of some purely algebraic classification or some combinatorial classification. Even in case of combinatorial classification there can be more than one classification as there can be both hypergraph and simplicial complex structure associated to them and the relation between these two is far from being clear. In fact none of these are known in the case of edge ideals or path ideals where we can ask a more concrete question:

Question 6.1.6. *Do the classifications asked in the previous three questions for finite simple graphs and their edge ideals as well as various path ideals.*

We want to mention that a very interesting reduction technique that can be useful for these questions was explored in Lemma 4.6 of [DHS] which shows that one can sequentially eliminate some vertices to achieve a subgraph which has some of the desired properties. We expect that a closer inquiry into this technique might shed some light to this direction.

As edge ideals are simply the base cases of path ideals one is tempted to ask similar questions regarding general path ideals too. Comparing to the edge ideals, much less is known about general path ideals. In fact one does not know the answer to the following questions, which we partially answer in this thesis:

Question 6.1.7. *Let G be a finite simple graph, 1. Is it true that if G is chordal then every path ideal of G has linear minimal free resolution?*

2. Can one classify all finite simple graphs with linear t-path ideals?

3. Is it true that if $I_t(G)$ has linear resolution then so does $I_{t+1}(G)$ for all t?

We answered this question partially in this thesis by proving that all path ideals of gap free and claw free graphs have linear minimal free resolutions. Our proof uses the fact that $(I_t(G) : f)$ is "very well behaved" for claw free and gap free graphs where f is a $(t-1)$-path. One can hope that further investigation about the properties of these colons will be helpful for research in this direction.

The main result of [AB] shows that the 4-cycle condition in Question 6.1.1 is essential. The work done in [Co] shows that the kind of result we're expecting for edge ideals fails completely for general monomial ideals. For example, the Examples 3.1, 3.2, and 3.3 show the existence of regularity 3 monomial ideals I, where $\operatorname{reg}(I^2) > 6$. In light of these it is difficult to expect a generalization of Question 6.1.1 or anything similar for general classes of monomial ideals. However even for edge ideals the answer to the following straightforward question seems to be unknown:

Question 6.1.8. *For $s \geq 1$, is it true that $\operatorname{reg}(I^{s+1}) \geq \operatorname{reg}(I^s)$ for an edge ideal I?*

Finally as the Cohen-Macaulayness of a squarefree monomial ideal is related to the linear resolution of its Alexander dual (see Definition 2.2 and Theorem 2.7 of [DHS]), we mention a problem involving the Cohen-Macaulayness of path ideals. A question which seems to be of interest is the Cohen-Macaulayness of the path ideals and its relation with the edge ideals. In general neither of them implies the other, which is explained in the following example:

Example 6.1.9. *Let $S = K[x, y, z]$. If $I = (xy, xz)$ then it is an edge ideal which* *is not Cohen-Macaulay but the corresponding three path ideal $J = (xyz)$ is definitely Cohen-Macaulay. On the other hand let $S' = K[x, y, z, w]$. If $I' = (xy, xw, zw)$ then it is an edge ideal which is Cohen-Macaulay but the corresponding 3-path ideal (xyw, xzw) is not Cohen-Macaulay.*

However it seems interesting to find classes of graphs where there is a relation between the two.

Question 6.1.10. *For which classes of graphs does Cohen-Macaulayness of edge ideals imply Cohen-Macaulayness of path ideals or vice versa?*

One way to approach this problem seems to be to understand the relation between the corresponding minimal vertex covers, which leads to the following open-ended question about which not much is known:

Question 6.1.11. *Can one find classes of G such that there is some nice relation between primary decompositions of various path ideals?*

In the next section we state some questions related to even-connections which, apart from being interesting in their own right, are expected to shed light on many questions mentioned in this section.

6.2 Some Open Questions About Even-Connections

In this final section of this thesis we state some open question regarding even-connections. For this section we introduce the following notation: let G be a finite simple graph with edge ideal I, $e = e_1 \cdots e_{s-1}$ be an $(s-1)$-fold product of edges, G_e be the corresponding graph after even-connection (defined in chapter 3) and polarization and I_e be the edge ideal of G_e. In this thesis (also in [AB] and [BHT]) we see examples where I_e has "nice properties" for every e. However we don't know in general much about the relation between algebra of G and G_e, so we ask the following open-ended question:

Question 6.2.1. *What is the relation between the Betti numbers of I and I_e?*

Various more specific questions can be asked:

Question 6.2.2. *Classify G and e such that* $\operatorname{reg}(I) \geq \operatorname{reg}(I_e)$.

Question 6.2.3. *What are the relations between dimension, depth, and projective dimension of I and I_e?*

Other than these general questions one can ask various questions about even-connections of special classes of graphs. For example, Lemma 5.1 of [BHT] shows that in the case of cycles the graphs coming from even-connections have some special properties which helps them to derive a formula for regularity of all powers of edge ideals of cycles. One can look for other classes of graphs for which similar properties hold. In this thesis we saw that for gap free and cricket free graphs G and for all e, G_e is chordal. In light of this we ask the following question:

Question 6.2.4. *Can one classify all graphs G such that G_e is chordal for all e?*

In [AB] it is proved that G_e is chordal for all e if G is a regularity 3 bipartite graphs using the characterization of regularity 3 bipartite graphs found in [GR]. The work done in [AB] and Macaulay 2 calculations motivate the following questions:

Question 6.2.5. *Is it true that if G bipartite and regularity of I is r then $\operatorname{reg}(I_e) \leq r$? Under what condition $\operatorname{reg}(I_e) < r$?*

Question 6.2.6. *If G is bipartite with $\operatorname{reg}(I) = 4$, what nice properties do the graphs G_e have?*

Finally in the chapter on Cohen-Macaulay bipartite graph, we saw that if G is a Cohen-Macaulay bipartite graph of dimension n, then there are exactly n edges f such that $(I^2 : f)$ is Cohen-Macaulay. In light of this we ask the following:

Question 6.2.7. *1. If G is a bipartite graph such that I satisfies Serre's S_i condition then for how many edges f does $(I^2 : f)$ have the same property?*

2. Under the same condition as above, for how many edges f does $(I^2 : f)$ satisfy

Serre's S_j condition for a fixed $j \neq i$?

3. Can anything be said regarding the converses of either of the above two?

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