On Higher Turán Inequalities for the Plane Partitions, Ellipsoidal T-Designs, and j-Inversion

Badri Vishal Pandey
Lucknow, India

M.Math, Indian Statistical Institute Bangalore, 2019
B.Math. Indian Statistical Institute Bangalore, 2017

A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia

Peter Humphries

Ken Ono (Advisor)

You Qi

Jianhui Zhou

Acknowledgements

I will try my best to express my sincerest gratitude as concisely as possible. If you are reading this and even though I haven't listed your name explicitly, please know that I am deeply grateful to you.

I am extremely grateful to my adviser Ken Ono for his fantastic guidance throughout my time at University of Virginia.

I would like to thank University of Virginia and NSF for supporting me financially. University of Virginia has provided a wonderful environment for learning and for research, and I want to thank the professors there who have taught me. I would also like to thank my fellow grad students for being so friendly and helping me for last 3-4 years. I would like to give special thanks to Will Craig and Sarasij Maitra who helped me when I first came to USA and provided a place to stay. They both have been amazing friends with whom I could discuss about not only Mathematics but also general matters in life. I will always be grateful to Alejandro De Las Penas Castano for teaching me Mathematica, discussing mathematics and for helping so many times.

I would like to show my gratitude to the ESL and VISAS program by Institute of

World Languages at UVa who helped me better my knowledge of American English and culture. I also made new friends there and had a lot of fun.

I would like to thanks my past roommates Pavan Ajjarapu and Ruchir Shah, and many friends made during my stay at Copeley housing apartments, Shivani Dewan, Archana Narayanan, Veena Shah to name a few, for making great memories in my first year.

A couple of friends from my undergrad school who came to USA same year as I are still in contact with me. I would like to thank Prabhanka Deka, Tejaswi Tripathi and Karthik Vasu for keeping in touch, talking on a regular basis and exploring amazing places in USA together. They were great help during pandemic as well.

Finally, I would like to thank my family for always supporting me in every decision of my life. Their support and smile have always made me confident enough to be able to live on my own for so long. I have lived away from my family for last 8-9 years but have never felt lonely for once. They talk to me everyday. My mother is the most important person in my life. I would like to dedicate my thesis to her.

Abstract

This thesis is about combinatorics and number theory. More precisely, the content of our thesis is on *Higer Turán inequalities for plane partitions*, introduction of a generalization of spherical t-designs, which we call *ellipsoidal t-design* [Pan22b], and providing *inversion formulae for j-function around elliptic points* [DLPCP22], out of which the later work is a joint work with, my fellow PhD student at University of Virginia, Alejandro De Las Penas Castano.

Plane partition function is a 2-dimensional analog of partition function, where you study the number of ways a number can be written (in a nice order) in an array. Here we study the roots of the doubly infinite family of Jensen polynomials $J_{\mathrm{PL}}^{d,n}(x)$ associated to MacMahon's plane partition function $\mathrm{PL}(n)$. Recently, Ono, Pujahari, and Rolen [OPR22] proved that $\mathrm{PL}(n)$ is log-concave for all $n \geq 12$, which is equivalent to the polynomials $J_{\mathrm{PL}}^{2,n}(x)$ having real roots. Moreover, they proved, for each $d \geq 2$, that the $J_{\mathrm{PL}}^{d,n}(x)$ have all real roots for sufficiently large n. Here we make their result effective. Namely, if $N_{\mathrm{PL}}(d)$ is the minimal integer such that $J_{\mathrm{PL}}^{d,n}(x)$ has

all real roots for all $n \geq N_{\mathrm{PL}}(d)$, then we show that

$$N_{\mathrm{PL}}(d) \leq 279928 \cdot d(d-1) \cdot \left(6d^3 \cdot (22.2)^{\frac{3(d-1)}{2}}\right)^{2d} e^{\frac{\Gamma(2d^2)}{(2\pi)^{2d+2}}}.$$

Moreover, using the ideas that led to the above inequality, we explicitly prove that $N_{\mathrm{PL}}(3) = 26, N_{\mathrm{PL}}(4) = 46, N_{\mathrm{PL}}(5) = 73, N_{\mathrm{PL}}(6) = 102$ and $N_{\mathrm{PL}}(7) = 136$.

A spherical t-design is is a finite set of points on a sphere such that integration of a polynomial of degree less than or equal to t is same as averaging over this set. In recent work, Miezaki introduced the notion of a *spherical T-design* in $\mathbb{R}^2$, where T is a potentially infinite set. As an example, he offered the $\mathbb{Z}^2$-lattice points with fixed integer norm (a.k.a. shells). These shells are *maximal* spherical T-designs, where $T = \mathbb{Z}^+ \setminus 4\mathbb{Z}^+$. We generalize the notion of a spherical T-design to special ellipses, and extend Miezaki's work to the norm form shells for rings of integers of imaginary quadratic fields with class number 1.

One of the most fundamental results in the theory of elliptic functions is the inversion formulas for j-function around infinity. Recently, Hong, Mertens, Ono, and Zhang [HMOZar] proved a conjecture of Căldăraru, He, and Huang [CHH21] that expresses the Taylor series of the modular j-function around the elliptic points i and $\rho = e^{\pi i/3}$ as rational functions arising from the signature 2 and 3 cases of Ramanujan's theory of elliptic functions to alternative bases. We extend these results and give inversion formulas for the j-function around i and ρ arising from Gauss' hypergeometric functions and Ramanujan's theory in signatures 4 and 6.

Contents

Chapter 1

Introduction

In this thesis, I present original results on higher Turán inequalities for plane patition function, an application of the modular forms in the study of a combinatorial object called ellipsoidal T-designs, and the classical problem of inverting the modular j-function.

1.1 Plane partitions and Turán inequalities

The theory of partitions is ubiquitous not only in Mathematics but in the nature itself. It is one of the very few branches of mathematics that can be appreciated by anyone who has little more than a lively interest in the subject. Its applications can be found wherever discrete objects are to be counted or classified, whether in the molecular and the atomic study of matter, study of small black holes in String theory, in the theory of numbers, representation theory or in the combinatorial problems of all sort. The theory of partitions has brought interest of many great mathematicians in the past- Cayley, Euler, Gauss, Hardy, Jacobi, Littlewood, Rademacher, Ramanujan, Schur, Sylvester- to name a few.

Our interest is in the study of a very special type of partition function called plane patitions. Plane partition function is 2-dimensional analog of partitions function, and hence a natural starting point for our discussion is the partition function itself.

A *partition* of a positive integer n is a non-increasing finite sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k > 0)$ such that $\sum_{i=1}^{k} \lambda_i = n$. The *partition function* $p(n)$ counts the number of such partitions of n. As an obligatory example, we have $p(4) = 5$ since

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

We have a generating function for $p(n)$ due to Euler [Eul97] given by

$$\sum_{n=0}^{\infty} p(n)q^n := \prod_{n=1}^{\infty}(1 - q^n)^{-1} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 \cdots . \qquad (1.1.1)$$

Euler proved a recurrence relation among many other beautiful results using this generating function and Euler's pentagonal number theorem. One can program a computer to calculate partition numbers using this recurrence relation but almost nothing can be inferred about its analytic properties. A natural question is if one can get a closed formula for $p(n)$?

In the words of George Andrews [And84], one of the crowning achievements not only in the theory of partitions but in all of mathematics is the an exact formula for $p(n)$, an achievement undertaken and mostly completed by G. H. Hardy and S.

Ramanujan[1] [HR18] and perfected by H. Rademacher [Rad38]. This unbelievable formula for $p(n)$ is given by an infinite series involving π, square roots, complex roots of unity, and derivatives of hyperbolic functions. Here we present an asymptotic formula, which is coming from the first term of this infinite series,

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

as $n \to \infty$. Not only is the formula remarkable, but the tools developed during this proof is equally remarkable. The circle method was one key tool that since its introduction has played an instrumental role in many other fields. We will talk about a modification of this method called Wright's circle method in Section 2.1.1.

Even though plane partitions are not as well studied as partitions, quite a bit is known. A plane partition of size n is a 2-dimensional array of positive integers $\pi := (\pi_{i,j})$ such that $\sum_{i,j} \pi_{i,j} = n$, in which the rows and columns are non-increasing. Below is a 3-d rendering of a plane partition for $n = 30$.

5	5	4	3
4	3	2	
3	1		

$\longrightarrow$

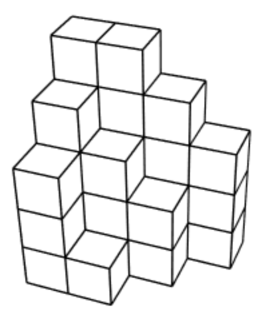


Figure 1.1: A plane partition for $n = 30$

[1] The story of the Hardy and Ramanujan collaboration on this formula is an amazing read.

The plane partition function $\mathrm{PL}(n)$ counts the number of plane partitions of size n. The generating function for $\mathrm{PL}(n)$ is due to MacMahon [Mac04] who proved that

$$f(x) = \sum_{n=0}^{\infty} \mathrm{PL}(n)x^n := \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^n} = 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + 48x^6 + \cdots . \tag{1.1.2}$$

This function is of great importance in physics. It appears prominently in connection with the enumeration of small black holes in string theory, as $f(x)$ is the generating function (for example, see Appendix E of [DDMP05]) for the number of BPS bound states between a $D6$ brane and $D0$ branes on $\mathbb{C}^3$.

In 1930, Wright [Wri31] adapted the "circle method" of Hardy and Ramanujan to prove an amazing asymptotic formulae for $\mathrm{PL}(n)$. His asymptotic formula is rather remarkable because unlike partitions, the plane partition generating function is not "modular" and so it becomes quite complex when you apply circle method. His theorem gives the following asymptotics:

$$\mathrm{PL}(n) \sim \frac{(2^{25}\zeta(3)^7)^{\frac{1}{36}} e^{\zeta'(-1)}}{\sqrt{12\pi} n^{\frac{25}{36}}} \exp\left(\sqrt[3]{\frac{27\zeta(3)n^2}{4}}\right).$$

In a recent work, Ono, Pujahari and Rolen [OPR22] made his formulae effective. They showed that there are infinitely many formulae, one for each non-negative r, where for large n, the implied error terms are smaller with larger choice of r. We differ these formulae to Chapter 2. Although bigger choice of r makes approximation better but there is a dependence of error part on r, so we will have good approximation only if n is big enough. In our work, we specialize these asymptotic formulae in a different form, make results effective in this form and then make the right choice of the parameter r.

1.1.1 Turán inequalities and Jensen polynomials

The Turán inequalities and the higher order Turán inequalities arise in the study of the Maclaurin coefficients of real entire functions in the Laguerre-Pólya class. A sequence $\alpha : \mathbb{N} \to \mathbb{R}$ is said to satisfy Turán inequality or to be log concave at n if

$$\alpha(n)^2 \geq \alpha(n-1)\alpha(n+1).$$

It is said to satisfy higher order Turán inequality at n if

$$4(\alpha(n)^2-\alpha(n-1)\alpha(n+1))(\alpha(n+1)^2-\alpha(n)\alpha(n+2))-(a(n)a(n+1)-a(n-1)a(n+2))^2 \geq 0.$$

DeSalvo and Pak [DP15] considered $p(n)$ and showed that this sequence satisfy Turán inequality for all $n \geq 25$. Chen, Jia and Wang [CJW19] proved that $p(n)$ also satisfy higher Turán inequalities for all $n \geq 95$. There are many more higher order Turán inequalities for a sequence that are encapsulated by Jensen polynomials associated to it.

Given a sequence $\alpha : \mathbb{N} \to \mathbb{R}$ and positive integers d and n, the associated *Jensen polynomial of degree d and shift n* is defined as

$$J_\alpha^{d,n}(x) := \sum_{j=0}^{d} \binom{d}{j} \alpha(n+j)x^j. \tag{1.1.3}$$

Notice that in the case of degree $d = 2$, we have that

$$J_\alpha^{2,n-1}(x) = \alpha(n-1) + 2\alpha(n)x + \alpha(n+1)x^2,$$

whose roots are

$$\frac{-\alpha(n) \pm \sqrt{\alpha(n)^2 - \alpha(n-1)\alpha(n+1)}}{\alpha(n+1)}.$$

In particular, α is log-concave or satisfy Turán inequality at n if and only if the roots of $J_\alpha^{2,n-1}(x)$ are real. Similarly it can be shown that α satisfy higher order Turán inequality at n if $J_\alpha^{3,n}(x)$ has real distinct roots. In general a real sequence is said to satisfy the degree d Turán inequality at n if $J_\alpha^{d,n}(x)$ is hyperbolic, where a polynomial with real coefficients is called *hyperbolic* if all of its zeros are real.

The significance of hyperbolicity for higher degrees was recognized by the works of Jensen and Pólya in connection to Riemann hypothesis. Building on some unpublished works of Jensen, Pólya [P2́7] proved that the Riemann hypothesis (RH) is equivalent to the hyperbolicity of all Jensen polynomials for the Taylor coefficients of the Riemann Xi-function at $s = 1/2$. More precisely, he showed that the RH is equivalent to the hyperbolicity of all Jensen polynomials associated with the sequence of Taylor coefficients $\gamma = \{\gamma_n\}$ defined by

$$(-1+4z^2)\Lambda\left(\frac{1}{2}+z\right) =: \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} z^{2n},$$

where $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Recently, Griffin, Ono, Rolen, and Zagier [GORZ19] established the hyperbolicity of this sequence for all but finitely many of the Jensen polynomials of each degree. In a subsequent paper, Griffin, Ono, Rolen, Thorner, Tripp, and Wagner [GOR$^+$22] made this result effective. Namely, they showed that there exists a constant c such that $J_\gamma^{d,n}(n)$ is hyperbolic whenever we have $n \geq ce^d$. It is widely believed that RH is true which in turn implies that all $J_{\gamma}^{d,n}(x)$ should be hyperbolic for all d and n. They also established that this is the case for atleast $d \leq 9 \times 10^{24}$. These recent results have reignited mathematician's interest in the study of Jensen polynomials of other sequences.

From our previous discussion, an obvious such sequence is $\{p(n)\}$. Inspired by the results on Turán and higher Turán inequalities for $p(n)$, Chen, Jia and Wang [CJW19] conjectured, for every degree $d \geq 1$, that there is a minimal integer $N_p(d)$ such that $J_p^{d,n}(x)$ is hyperbolic for all $n \geq N_p(d)$. Griffin et. al. [GORZ19] proved their conjecture by showing that Jensen polynomials associated to partition function of each degree are hyperbolic for all sufficiently large shift n. Larson and Wagner [LW19] proved an effective form of this theorem by giving a upper bound for $N_p(d)$. Namely, they showed that $N_p(d) \leq (3d)^{24d}(50d)^{3d^2}$. Extending beyond the work of Chen et. al., they also proved that $N_p(4) = 206$ and $N_p(5) = 381$.

1.1.2 Main results

Our sequence of interest is $\{\mathrm{PL}(n)\}$. Heim, Neuhauser and Tröger [HNT21] undertook the study of the plane partitions in analogy with the hyperbolicity results of $p(n)$. They proved many inequalities satisfied by $\mathrm{PL}(n)$ including proving that $\mathrm{PL}(n)$ is log-concave for sufficiently large n. They also conjectured the bound to be 12. Ono, Pujahari and Rolen [OPR22] proved this conjecture. In addition, they also proved that for each degree d, the Jensen polynomials are hyperbolic for sufficiently large shift n. To prove this, they perfected the strong asymptotic formulae for the plane partition function proved by Wright. These asymptotic formulae satisfies conditions required by Theorems 3 and 6 of [GORZ19], which implies that the limiting behavior of $J_{\mathrm{PL}}^{d,n}(x)$ as $n \to \infty$ can be modeled by *Hermite polynomials* which are known to be hyperbolic. This proves their theorem (for details see Chapter 2).

In our work, we make their result effective. More precisely, for any d, suppose that $N_{\mathrm{PL}}(d)$ is the minimal integer for which every Jensen polynomials of degree d are hyperbolic for all shifts $n \geq N_{\mathrm{PL}}(d)$. Then we give an upper bound on $N_{\mathrm{PL}}(d)$.

Theorem 1.1. *For a positive integer $d \geq 4$, we have*

$$N_{\mathrm{PL}}(d) \leq 279928 \cdot d(d-1) \cdot \left(6d^3 \cdot (22.2)^{\frac{3(d-1)}{2}}\right)^{2d} e^{\frac{\Gamma(2d^2)}{(2\pi)^{2d+2}}}.$$

Moreover, by working explicitly with the expression that arises in the proof of Theorem 1.1, we are able to compute $N_{\mathrm{PL}}(d)$ for the cases $d = 3, 4, 5, 6$ and 7.

Theorem 1.2. *We have that $N_{\mathrm{PL}}(3) = 26, N_{\mathrm{PL}}(4) = 46, N_{\mathrm{PL}}(5) = 73, N_{\mathrm{PL}}(6) = 102$ and $N_{\mathrm{PL}}(7) = 136$.*

d	8	9	10	11	12	13	14	15	16	17	18	19	20
$N_{\mathrm{PL}}(d)$	173	215	260	307	359	414	472	533	596	662	731	803	873

Table:Conjectural value for $N_{\mathrm{PL}}(d)$ for small d.

Remark 1.3. (1) It is quite surprising that $N_{\mathrm{PL}}(d)$ for smaller values of d is smaller than corresponding values of $N_p(d)$ since plane partition function has much more complex asymptotic formula than partition function.

1.2 Modular forms and Ellipsoidal T-designs

Our goal for this section is to introduce the concept of Ellipsoidal T-design and its connection with classical spherical t-designs and finally state our result on it.

1.2.1 Spherical t-design

Spherical t-designs were introduced in 1977 by Delsarte, Goethals and Seidel [DGS77], and they have played an important role in algebra, combinatorics, number theory and quantum mechanics (for background see [Ban84], [BOT15], [CFL11], [HTH05], [Sek92], [Mie13]). A spherical t-design is a nonempty finite set of points on the unit sphere with the property that the average value of any real polynomial of degree $\leq t$ over this set equals the average value over the sphere. Namely, if S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ centered at the origin, then a finite nonempty subset $X \subset S^{n-1}$ is a spherical t-design if

$$\frac{1}{|X|}\sum_{x\in X} P(x) = \frac{1}{\mathrm{Vol}(S^{n-1})}\int_{S^{n-1}} P(x)d\sigma(x) \tag{1.2.1}$$

for all polynomials $P(x)$ of degree $\leq t$. The right-hand side of (1.2.1) is the usual surface integral over S^{n-1}. In general, a finite nonempty subset X of $S_{n-1}(r)$, the sphere of radius r centered at the origin, is a spherical t-design if $\frac{1}{r}X$ satisfies (1.2.1). Since a spherical t-design is also a spherical t'-design for all $t' \leq t$, we say that X has *strength* t if it is the maximum of all such numbers.

Examples 1.4. Any subset of a sphere with antipodal points (i.e. $x \in X \implies -x \in X$) is spherical 1-design.

Delsarte, Goethals and Seidel developed a very simple criterion for determining spherical t-designs. This criterion involves *homogeneous harmonic* polynomials of bounded degree. A polynomial in n variables is *harmonic* if it is annihilated by the Laplacian operator $\Delta := \sum_{i=1}^{n} \partial^2/\partial x_i^2$, and they showed [DGS77] that $X \subset S^{n-1}$ is a spherical t-design if

$$\sum_{x \in X} P(x) = 0 \tag{1.2.2}$$

for all homogeneous harmonic polynomials $P(x)$ of nonzero degree $\leq t$. This criterion is a consequence of two results from harmonic analysis. The first result is the mean value property for harmonic functions [ABR92, p. 5], which implies that the integral of a harmonic polynomial over a sphere centered at the origin vanishes, combined with the fact that homogeneous polynomials of fixed degree are spanned by certain harmonic polynomials [ABR92, Th. 5.7].

1.2.2 Ellipsoidal t-designs

In view of this framework, it is natural to ask whether there are generalizations of spherical t-designs to other curves, surfaces and varieties. Here we consider certain *ellipsoids*[2] in dimension two. To be precise, for square-free $D \geq 1$ we define the norm r ellipses

$$C_D(r) := \begin{cases} \{(x,y) \in \mathbb{R}^2 : x^2 + Dy^2 = r\} & \text{if } D \equiv 1, 2 \pmod 4, \\ \left\{(x,y) \in \mathbb{R}^2 : x^2 + xy + \frac{1+D}{4} y^2 = r\right\} & \text{if } D \equiv 3 \pmod 4. \end{cases} \tag{1.2.3}$$

Remark 1.5. These ellipses arise from certain imaginary quadratic orders.

For $D \equiv 1, 2 \pmod 4$, we say that a finite nonempty subset $X \subset C_D(r)$ is an *ellipsoidal t-design* if

$$\frac{1}{|X|} \sum_{(x,y) \in X} P(x,y) = \frac{\sqrt{D}}{2\pi} \int_{C_D(r)} \frac{P(x,y)}{\sqrt{x^2 + y^2 D^2}} d\sigma(x,y) \tag{1.2.4}$$

for all polynomials $P(x,y)$ of degree $\leq t$ over $\mathbb{R}$. For $D \equiv 3 \pmod 4$, instead we require

$$\frac{1}{|X|} \sum_{(x,y) \in X} P(x,y) = \frac{\sqrt{D}}{\pi} \int_{C_D(r)} \frac{P(x,y)}{\sqrt{20x^2 + (D^2 + 2D + 5)y^2 + (20 + 4D)xy}} d\sigma(x,y). \tag{1.2.5}$$

Here the right-hand sides are line integrals. As in the case of spherical t-designs, every ellipsoidal t-design is also an ellipsoidal t'-design for all $t' \leq t$, and the maximum of all such t's is called the *strength* of X. These definitions coincide with the notion of a spherical t-design when $D = 1$.

Example 1.6. Any subset of an ellipse with antipodal points is ellipsodal 1-design.

[2] We do not use the term *ellipse* to avoid possible confusion that might arise with the term *elliptical*.

In analogy to Delsarte, Goethals and Seidel, we have a natural criterion for confirming ellipsoidal t-designs. To this end, we consider the 2-dimensional real vector space

$$H_{D,j}^{\mathbb{R}}[x,y] := \begin{cases} \left\langle \mathrm{Re}\left(x+\sqrt{-D}y\right)^j, \mathrm{Im}\left(x+\sqrt{-D}y\right)^j \right\rangle & \text{if } D \equiv 1,2 \pmod 4, \\ \left\langle \mathrm{Re}\left(x+\frac{1+\sqrt{-D}}{2}y\right)^j, \mathrm{Im}\left(x+\frac{1+\sqrt{-D}}{2}y\right)^j \right\rangle & \text{if } D \equiv 3 \pmod 4. \end{cases} \tag{1.2.6}$$

In terms of these vector spaces of polynomials, we have the following ellipsoidal t-design criterion.

Theorem 1.7. *A finite nonempty set $X \subset C_D(r)$ is an ellipsoidal t-design if*

$$\sum_{x \in X} P(x,y) = 0$$

for all $P(x,y) \in H_{D,j}^{\mathbb{R}}[x,y]$ for all $0 < j \leq t$.

Remark 1.8. 1) Observe that if $X \subset S^1$ is a spherical t-design, then $Y = \left\{(x, y/\sqrt{D}) | (x,y) \in X\right\} \subset C_D$ (resp. $Y = \left\{(x + y/\sqrt{D}, 2y/\sqrt{D} | (x,y) \in X\right\} \subset C_D$) is an ellipsoidal t-design for $D \equiv 1,2 \pmod 4$ (resp. $D \equiv 3 \pmod 4$). Therefore, the existence of a spherical t-design implies the existence of a corresponding ellipsoidal t-design. In fact, there is a one-to-one correspondence between spherical t-designs and ellipsoidal t-designs. However, the proof of Theorem 1.7 is not a direct consequence because care is required for justifying the role of the vector spaces $H_{D,j}^{\mathbb{R}}[x,y]$.

2)Since there is one-to-one correspondence between spherical and ellipsoidal t-designs, we get a lower bound [DGS77, pg 2] on the size of ellipsoidal t-design X,

$$|X| \geq t+1.$$

1.2.3 Ellipsoidal T-design

Recently, Miezaki in [Mie13] introduced a generalization of the notion of spherical t-designs. Instead of restricting to polynomials of degree $\leq t$, he considered harmonic polynomials of degree $j \in T \subset \mathbb{N}$, where T is a potentially infinite set. The main theorem from [Mie13] gives infinitely many spherical T-designs for $T := \mathbb{Z}^+ \setminus 4\mathbb{Z}^+$ in dimension two. Namely, he considered norm r shells, integer points on $x^2 + y^2 = r$ for fixed $r \in \mathbb{Z}^+$. He showed that these r-shells are spherical T-designs. Moreover, these sets have strength T, meaning that (1.2.2) fails if any multiple of 4 is added to T. His proof makes use of theta functions arising from complex multiplication by $\mathbb{Z}[i]$.

We generalize Miezaki's work to ellipsoidal T-designs. We call $X \subset C_D$ an *ellipsoidal T-design* if the condition in Theorem 1.7 is satisfied for all polynomials in $H^{\mathbb{R}}_{D,j}[x,y]$ with $j \in T$. We say X has strength T if it is maximal among such sets. For each square-free positive integer D, let $\mathcal{O}_D$ be the ring of integers of $\mathbb{Q}(\sqrt{-D})$. In particular, this means that

$$\mathcal{O}_D = \begin{cases} \mathbb{Z}\left[\sqrt{-D}\right] & \text{if } D \equiv 1,2 \pmod 4, \\ \mathbb{Z}\left[\frac{1+\sqrt{-D}}{2}\right] & \text{if } D \equiv 3 \pmod 4. \end{cases} \tag{1.2.7}$$

We consider $D \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$, the square-free positive integers for which $\mathcal{O}_D$ has class number 1. To make this precise, we define the *norm r shells* in $C_D(r)$ by

$$\Lambda_D^r := \mathcal{O}_D \cap C_D(r). \tag{1.2.8}$$

Generalizing Miezaki's work for $D = 1$, we obtain the following theorem.

Theorem 1.9. *If $D \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$, then every non-empty shell Λ_D^r is an ellipsoidal T_D design with strength T_D, where*

$$T_D := \begin{cases} \mathbb{Z}^+ \setminus 4\mathbb{Z}^+ & \text{if } D = 1, \\ \mathbb{Z}^+ \setminus 6\mathbb{Z}^+ & \text{if } D = 3, \\ \mathbb{Z}^+ \setminus 2\mathbb{Z}^+ & \text{otherwise.} \end{cases}$$

Remark 1.10. The method used here seems to be well-poised only for the dimension 2 cases. It would be interesting to obtain higher dimensional analogues.

Example 1.11. We consider $D = 3$, and $r = 691$. Then we have

$$\begin{aligned} \Lambda_3^{691} =& \{(11, 19), (-11, -19), (19, 11), (-19, -11), (11, -30), (-11, 30), (30, -19), \\ & (-30, 19), (30, -11), (-30, 11), (19, -30), (-19, 30)\}. \end{aligned}$$

We consider the polynomial $P(x, y) = 2x^2 + 3462xy + 1729y^2 \in H_{3,2}^{\mathbb{R}}[x, y]$, and we find that $\sum_{(x,y)\in\Lambda_3^{691}} P(x, y) = 0$ which shows that Λ_3^{691} is an elliptical 2-design and $2 \in T_3$. On the other hand, Theorem 1.9 implies that Λ_3^{691} is not an ellipsoidal 6-design.To see this we choose $Q(x, y) = 2x^2 + 6x^5y - 15x^4y^2 - 40x^3y^3 - 15x^2y^4 + 6xy^5 + 2y^6 \in H_{3,6}^{\mathbb{R}}(x, y)$, and we find that $\sum_{(x,y)\in\Lambda_3^{691}} Q(x, y) = -4818834696 \neq 0$.

1.3 Inversion of j-function around elliptic points

Recently, Hong, Mertens, Ono, and Zhang [HMOZar] proved a conjecture of Căldăraru, He, and Huang [CHH21] that expresses the Taylor series of the modular j-function around the elliptic points i and $\rho = e^{\pi i/3}$ as rational functions arising from the signature 2 and 3 cases of Ramanujan's theory of elliptic functions to alternative bases. We extend these results and give inversion formulas for the j-function around i and ρ arising from Gauss' hypergeometric functions and Ramanujan's theory in signatures 4 and 6.

1.3.1 Klein j-function

The Klein j-function

$$j(\tau) := \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots \qquad (q = e^{2\pi i\tau},\ \tau \in \mathbb{H})$$

is a modular function on the full modular group $\mathrm{SL}_2(\mathbb{Z})$. It is of great importance to number theory. In the theory of elliptic curves, the j-function parametrizes isomorphism classes of elliptic curves over $\mathbb{C}$. In Class Field Theory, its values at CM points, the so called singular moduli, generate Hilbert Class Fields of imaginary quadratic extensions. Another famous example of its importance is the observation that the Fourier coefficients of the j-function encode the graded dimensions of the infinite dimensional graded algebra of the Monster group. This observation led to the Monstrous Moonshine conjecture and its eventual proof by Borcherds [Bor92].

The j-function defines a bijective holomorphic function from $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ to $\mathbb{C}$. In particular, the j-function has an inverse function. Due to its central role in the theories stated above, among many others, it is natural to seek explicit formulas for its inverse map. Work in this direction began with Ramanujan giving striking formulas for $1/\pi$ which gave rise to Ramanujan's theory of elliptic functions to alternative basis (cf. [Ram14], [BBG95], and [BC99]). This theory produced several explicit formulas which express $j(\tau)$ as a rational function in t, where the parameters τ and t are related via Gauss' hypergeometric functions (see Section 2.4).

1.3.2 Main results

More recently, a conjecture of Căldăraru, He, and Huang [CHH21] cast new light on the problem of inverting the j-function. In contrast to Ramanujan's Theory which uses the Fourier expansion of the j-function around the cusp $i\infty$ [BC99], their conjecture is about the Taylor series expansion around the elliptic points i and $\rho := e^{\pi i/3}$. They conjecture that these Taylor series, when specialized at the normalized flat coordinate of the corresponding moduli space of versal deformations of elliptic curves, are the rational functions that appear in the classical hypergeometric inversion formulae for the j-function. Shortly after, Hong, Mertens, Ono, and Zhang [HMOZar] proved their conjecture.

To state their results more precisely, let $\mathbb{H}$ be upper half plane, and $\mathbb{D}$ be the unit disc. For $\tau_* \in \{i, \rho\}$, we have the uniformizing map $S_{\tau_*} : \mathbb{H} \to \mathbb{D}$ and its inverse

$S_{\tau_*}^{-1} : \mathbb{D} \to \mathbb{H}$ defined by

$$S_{\tau_*}(\tau) = \frac{\tau - \tau_*}{\tau - \overline{\tau_*}} \quad \text{and} \quad S_{\tau_*}^{-1}(w) = \frac{\tau_* - \overline{\tau_*} w}{1 - w}. \tag{1.3.1}$$

Next, we renormalize the conformal map

$$s_{\tau_*}^{-1}(w) := S_{\tau_*}^{-1}\left(\frac{w}{2\pi\Omega_\rho^2}\right), \tag{1.3.2}$$

where Ω_{τ_*} is the standard Chowla-Selberg periods (for example, see (96) of Section 6.3 of [Zag08]) defined by

$$\Omega_i := \frac{1}{\sqrt{8\pi}} \frac{\Gamma(1/4)}{\Gamma(3/4)} \quad \text{and} \quad \Omega_\rho := \frac{1}{\sqrt{6\pi}} \left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)^{3/2}. \tag{1.3.3}$$

With these uniformizing maps, the Taylor series of the j-function around i and ρ are defined as:

$$\begin{aligned} j(s_i^{-1}(w)) &= 1728 + 20736w^2 + 105984w^4 + \frac{1594112}{5}w^6 + \cdots \\ j(s_\rho^{-1}(w)) &= 13824w^3 - 39744w^6 + \frac{1920024}{35}w^9 - \frac{1736613}{35}w^{12} + \cdots. \end{aligned}$$

These formulas follow from the theory of Taylor coefficients of modular forms (see for example Section 5.4 of [CS17]).

Hong, Mertens, Ono, and Zhang considered the two distinguished power series $c_i(t)$ and $c_\rho(t)$ of Căldăraru, He, and Huang (see Section 2.1 of [HMOZar]) whose first few terms are

$$\begin{aligned} c_i(t) &= t + t^3 + \frac{32}{15}t^5 + \frac{17}{3}t^7 + \frac{1054}{63}t^9 + \frac{368}{7}t^{11} + \frac{4652300}{27027}t^{13} + \cdots \quad (|t| < 1/2) \\ c_\rho(t) &= t - \frac{1}{3}t^4 + \frac{103}{315}t^7 - \frac{169}{405}t^{10} + \frac{522169}{868725}t^{13} - \frac{186119}{200475}t^{16} + \cdots \quad (|t| < 1). \end{aligned}$$

They proved that surprisingly the Taylor series of the j-function around i and ρ evaluated at $c_i(t)$ and $c_\rho(t)$ respectively, turn out to be rational functions in t. Namely, they proved that

$$j(s_i^{-1}(c_i(t)) = 64\frac{(3+4t)^3}{(1-4t^2)^2} \quad \text{and} \quad j\left(\frac{s_\rho^{-1}(c_\rho(t))+1}{3}\right) = 27t^3\left(\frac{8-t^3}{1+t^3}\right)^3.$$

Their proof used the theory of hypergeometric functions of signatures 2 and 3. Namely, they realized c_i and c_ρ as quotients of hypergeoemetric functions.

In view of this, it is natural to ask whether there are other examples of this phenomenon. More precisely, does the theory of hypergeometric functions in signature 4 and 6 yield other inversion formulas for the j-function around the elliptic points i and ρ? We answer this here in joint work with Castano.

Let us define

$$C_i(t) := t \cdot \frac{{}_2F_1\left(\frac{3}{4},\frac{3}{4};\frac{3}{2};4t^2\right)}{{}_2F_1\left(\frac{1}{4},\frac{1}{4};\frac{1}{2};4t^2\right)} \quad \text{and} \quad C_\rho(t) := \frac{t^2}{2}\frac{{}_2F_1\left(\frac{5}{6},\frac{5}{6};\frac{5}{3};-2t^3\right)}{{}_2F_1\left(\frac{1}{6},\frac{1}{6};\frac{1}{3};-2t^3\right)}, \tag{1.3.4}$$

where ${}_2F_1$ is Gauss' hypergeometric function (see Section 2.4). Then we have the following theorems:

Theorem 1.12. *If $|t| < 1/2$, then we have*

$$j\left(\frac{s_i^{-1}(C_i(t))+1}{2}\right) = 64\frac{(16t^2-3)^3}{4t^2-1}.$$

Theorem 1.13. *If $|t| < 1/\sqrt[3]{2}$, then we have*

$$j\left(s_\rho^{-1}(C_\rho(t))\right) = -\frac{1728t^6}{2t^3+1}.$$

Remarks 1.14. (1) The above formulas can be thought of as inversion formulas of the j-function around i and ρ because when specialized to $t = 0$, the above formulas reduce to the classic identities

$$j(i) = 1728 \quad \text{and} \quad j(\rho) = 0.$$

(2) By analytic continuation, we can extend the domain where the formulas in Theorems 1.12 and 1.13 are valid to wherever $C_i(t)$ and $C_\rho(t)$ are defined, see the comments at the end of section 5.2. However, for explicit computations, we require $|t| < 1/2$ and $|t| < 1/\sqrt[3]{2}$ in order to compute the power series expansions of C_i and C_ρ respectively.

(3) These results tell us that finding an approximate solution to $j(\tau) = \alpha$ boils down to solving a degree six polynomial equation, see Section 5.3 for examples.

Chapter 2

Background

In this chapter we record all the necessary background needed for the results obtained in this thesis. Namely, we write down asymptotic formulae for plane partition function and recall Hankel determinants and their relation with hyperbolicity of a polynomial. In the theory of modular forms, we recall besic definitions, Hecke operators, Hecke characters, theta series and necessary result regarding them. Finally, we recall modular j-function, hypergeometric function and some classical inversion formulas for j-function around infinity which is derived by the Ramanujan's theory of elliptic functions to alternate bases.

2.1 Plane partition function

2.1.1 Asymptotic formula of $\mathrm{PL}(n)$

In [OPR22], Ono, Pujahari and Rolen obtained very strong asymptotic formulas for $\mathrm{PL}(n)$. In fact, there are infinitely many formulas, one for each positive integer r, where for large n, the implied error terms are smaller with larger choice of r. To make this precise, we need two constants

$$A := \zeta(3) \approx 1.20206..., \quad \text{and} \quad c := 2\int_0^\infty \frac{y \log y}{e^{2\pi y}-1}dy = \zeta'(-1) \approx -0.16542.... \tag{2.1.1}$$

Furthermore, for non-negative integers s and m, define coefficients $c_{s,m}(n)$ by

$$\sum_{n=0}^{\infty} c_{s,m}(n)y^n := \frac{(1+y)^{2s+2m+\frac{13}{12}}}{(3+2y)^{m+\frac{1}{2}}}. \tag{2.1.2}$$

We define an important parameter using these coefficients,

$$b_{s,m} := c_{s,m}(2m). \tag{2.1.3}$$

The asymptotic formulas are given in terms of special numbers $\beta_0, \beta_1, ...$ defined by

$$\sum_{n=0}^{\infty} \beta_s y^s := \exp\left(-\sum_{i=1}^{\infty} \alpha_i y^i\right), \tag{2.1.4}$$

where

$$\alpha_s := \frac{2\Gamma(2s+2)\zeta(2s)\zeta(2s+2)}{s(2\pi)^{4s+2}}. \tag{2.1.5}$$

Also, to reduce the complexity of error terms, for non-negative r, Ono et. al. defined

$$n_r := \min\left\{n \geq 1 \,:\, 0.056 \cdot \sum_{s=1}^{r+1} \left(\frac{s \cdot A^{\frac{1}{3}}}{2^{\frac{7}{6}} n^{\frac{1}{3}}}\right)^{2s} \left(\frac{\pi^2 n^{\frac{1}{3}}}{(2A)^{\frac{1}{3}} s} + 2\right) < 1\right\}, \tag{2.1.6}$$

and

$$l_r := \min\left\{n \geq 1 \,:\, 2^{r+4}\pi^3 \alpha_{r+2} \left(\frac{2A}{n}\right)^{\frac{2r+4}{3}} + 5e^{-4.7(\frac{n}{2A})^{1/3}} < \frac{1}{2}\right\}. \tag{2.1.7}$$

The explicit bounds on the error terms are given in terms of $\mathcal{X}_r(n), \mathcal{Y}_r(n), \mathcal{Z}_r(n)$. To define $\mathcal{X}_r(n)$, and $\mathcal{Y}_r(n)$, we let

$$C_r := 2\max_{|z|=1}\left\{\left|e^{-\sum_{s=1}^{r+1}\alpha_s z^s}\right|\right\}. \tag{2.1.8}$$

We require one additional parameter to define $\mathcal{Z}_r$. First, we define

$$\chi_s(t) := \frac{v^{2s+\frac{25}{12}}\sqrt{2v+1}}{2\pi(v^2+v+1)}, \tag{2.1.9}$$

where $t^2 = 3 - 2v - v^{-2}$. Using this we define the parameter

$$D_r := \frac{1}{(2r+4)!} \max\left\{\max\left\{\left|\chi_s^{(2r+4)}(t)\right|\right\}_{t\in\mathbb{R}}\right\}_{s=0}^{r+1}. \tag{2.1.10}$$

Now we define $\mathcal{X}_r(n), \mathcal{Y}_r(n)$, and $\mathcal{Z}_r(n)$ by

$$\mathcal{X}_r(n) := e^{c+AN_n^2} 2^{r+\frac{49}{24}} C_r N_n^{-2r-\frac{49}{12}}, \tag{2.1.11}$$

$$\begin{aligned}\mathcal{Y}_r(n) := \Bigg| & e^{c+AN_n^2}\left(2^{r+5}\pi^3\alpha_{r+2}N_n^{-2r-4} + 10e^{-4.7N_n}\right) \\ & \times \left(2^{r+\frac{49}{24}} C_r N_n^{-2r-\frac{49}{12}} + \sum_{s=0}^{r+1} 2^{s+\frac{1}{24}} \beta_s N_n^{-2s-\frac{1}{12}}\right)\Bigg|,\end{aligned} \tag{2.1.12}$$

and

$$\mathcal{Z}_r(n) := e^c \left(D_r \cdot \Gamma\left(r+\frac{5}{2}\right)(AN_n^2)^{-r-\frac{5}{2}} e^{3AN_n^2} + 0.64 \cdot 2^{r+1} e^{2AN_n^2}\right) \sum_{s=0}^{r+1} \beta_s N_n^{-2s-\frac{13}{12}}. \tag{2.1.13}$$

where $N_n := \left(\frac{n}{2A}\right)^{1/3}$. With the notation above, Ono et. al. proved the following theorem.

Theorem 2.1 ([OPR22], Theorem 1.3)**.** *If $r \in \mathbb{Z}^+$, then for every integer $n \geq \max(n_r, l_r, 87)$, then we have that*

$$\mathrm{PL}(n) = \frac{e^{c+3AN_n^2}}{2\pi} \sum_{s=0}^{r+1}\sum_{m=0}^{r+1} \frac{(-1)^m \beta_s b_{s,m} \Gamma(m+\frac{1}{2})}{A^{m+\frac{1}{2}} N_n^{2s+2m+\frac{25}{12}}} + E_r^{\mathrm{maj}}(n) + E^{\mathrm{min}}(n), \tag{2.1.14}$$

where

$$|E^{\min}(n)| \leq \exp\left(\left(3A - \frac{2}{5}\right) N_n^2\right), \tag{2.1.15}$$

and

$$|E_r^{\mathrm{maj}}(n)| \leq \frac{(\mathcal{X}_r(n) + \mathcal{Y}_r(n))e^{2AN_n^2}}{N_n\pi} + |\mathcal{Z}_r(n)|. \tag{2.1.16}$$

Remark 2.2. The proof presented here is due to Ono, Pujahari, Rolen [OPR22] who made Wright's [Wri31] proof for asymptotic for $\mathrm{PL}(n)$ effective. Wright's strategy for obtaining asymptotics for $\mathrm{PL}(n)$ is a modification of the classical "circle method". The quantity $E^{\min}(n)$ arises from minor arc integrals, and $E_r^{\mathrm{maj}}(n)$ arises from major arc integrals.

Proof. Consider the circle

$$C_{N_n} := \left\{x \in \mathbb{C} : |x| = e^{-\frac{1}{N_n}}\right\}. \tag{2.1.17}$$

Throughout, we let θ_x denote the principal branch of $\mathrm{Arg}(x)$, we divide C_{N_n} into two arcs: "major arc" C'_{N_n}, consisting of x such that $|\theta_x| < \frac{1}{N_n}$, and "minor arc" C''_{N_n}, its compliment.

Using the generating function (1.1.2), Cauchy's integral formula gives us that

$$\mathrm{PL}(n) = J(n) + E^{\min}(n), \tag{2.1.18}$$

where

$$J(n) = \frac{1}{2\pi i}\int_{C'_{N_n}} \frac{f(x)}{x^{n+1}}dx \quad \text{and} \quad E^{\min}(n) = \frac{1}{2\pi i}\int_{C''_{N_n}} \frac{f(x)}{x^{n+1}}dx \tag{2.1.19}$$

with the usual counter-clockwise orientation for integration. Now we analyze both integrals separately. The asymptotic arises from $J(n)$, and $E^{\min}(n)$ gives us a small error term. This justifies the nomenclature “major” and “minor”.

Explicit bounds over the minor arcs

To bound $E^{\min}(n)$, Ono et. al. used a different method from that of Wright. Instead of working directly with $f(x)$, they used its logarithmic derivative, which is the generating function of the sum of square of divisors. Using this interpretation, they make connection with work of Zagier [Zag06], and then effectively bounded $E^{\min}(n)$ using Euler-Maclaurin summation and calculus.

Proposition 2.3. *For all $n \geq 87$, we have $\left|E^{\min}(n)\right| \leq e^{\left(3A-\frac{2}{5}\right)N_n^2}$.*

The idea is to give a bound on $f(x)$ when $x \in C''_{N_n}$ which just depends on n.

Lemmata for Proposition 2.3

Lemma 2.4. *If $x \in C''_{N_n}$, then we have* $\log f(|x|) \leq AN_n^2 + 033N_n - 0.5$.

Proof of Lemma 2.4. We start by noticing that when you take a log of (1.1.2), the product becomes summation, then using expansion of $\log(1-x)$ gives us a double summation which one can interchange since the series is absolutely convergent, and summing over inner parameter gives an arithmetico-geometric series which in turn gives

$$\log f(|x|) = \sum_{m=1}^{\infty} \frac{|x|^m}{m(1-|x|^m)^2}.$$

For simplicity we let $t = 1/N_n$ and $q := |x| = e^{-t}$. Differentiating above gives us the Lambert series

$$L(q) := q\frac{d}{dq}\sum_{m=1}^{\infty} \frac{q^m}{m(1-q^m)^2} = \sum_{m=1}^{\infty} \frac{q^m(1+q^m)}{(1-q^m)^3}. \tag{2.1.20}$$

Using elementary calculus we have $X(1+X)/(1+X)^3 = \sum_{k=1}^{\infty} k^2 X^k$, so we get

$$L(q) = g_3(q) := \sum_{m=1}^{\infty} \frac{m^2 q^m}{1-q^m},$$

where $g_3(q)$ is the same as in Zagier's work [Zag06]. Notice that $q\frac{d}{dq} = -\frac{d}{dt}$, so we have $\frac{d}{dt}\log f(e^{-t}) = -g_3(e^{-t})$. Integrating this relation gives us

$$\begin{aligned} \log f(|x|) = \log f(e^{-t}) &= \int_t^1 g_3(e^{-z})dz + \log f(e^{-1}) \\ &\leq \left|\int_t^1 g_3(e^{-z})dz\right| + 1.04, \end{aligned} \tag{2.1.21}$$

since $\log f(e^{-1}) \approx 1.036$. To estimate integral we make use of Zagier's work [Zag06] on generating functions of arbitrary divisor power sums. He considers (see p. 15 of [Zag06]) the function $g_3(e^{-z})$ as $z \searrow 0$. In the $k = 3$ case he applies Proposition 3 of [Zag06] with $F(t) := \frac{t^2}{e^t - 1}$ to obtain an asymptotic expansion for $g_3(e^{-t}) = \frac{1}{t^2}\sum_{m\geq 1} F(mt)$.

To obtain an estimate with explicitly bounded error, we analyze the proof of Proposition 3 of [Zag06]. Euler-Maclaurin summation formula gives, for each $k \geq 1$, an exact formula for $g_3(e^{-t})$ where k controls the number of terms in the asymptotic expansion, with an integral. We consider $k=1$ case, in which case we have

$$\sum_{m\geq 1} F(mt) = \frac{1}{t}\int_0^\infty F(z)dz + \frac{(-1)^0 B_1 F(0) t^0}{1!} + (-t)^0 \int_0^\infty \frac{F'(z)\bar{B}_1(z)}{1!}dz,$$

where $B_1 = -1/2$ is the first Bernoulli number, and $\bar{B}_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$ is a periodization of the first Bernoulli polynomial $B_1(x) = x - \frac{1}{2}$. Notice that $F(t)$ has a removable singularity at $t=0$ with limiting value $F(0)=0$. Moreover, Zagier computed that

$$\int_0^\infty F(z)dz = \int_0^\infty t^{k-1}\left(e^{-t} + e^{-2t} + + \cdots\right)dt = (3-1)!\zeta(3) = 2A.$$

Combining everything we get

$$\begin{aligned}
g_3(e^{-t}) &= \frac{2A}{t^3} - \frac{1}{t^2}\int_0^\infty F'(z)(z - \lfloor z \rfloor - 1/2)dz. \\
&\leq \frac{2A}{t^3} + \frac{1}{2t^2}\int_0^\infty F'(z)dz = \frac{2A}{t^3} + \frac{1}{2t^2}\int_0^\infty \left|\frac{(ze^z - 2e^z + 2)z}{(e^z-1)^2}\right| dz \\
&\leq \frac{2A}{t^3} + \frac{0.33}{t^2},
\end{aligned}$$

where last inequality is due to the fact that $\int_0^\infty F'(z)dz \approx 0.6471$. Therefore, (2.1.21) gives

$$\log f(|x|) \leq \int_t^1 \left(\frac{2A}{z^3} + \frac{0.33}{z^2}\right)dz + 1.04 \leq \frac{A}{z^2} + \frac{0.33}{t} - 0.5.$$

We are done since $t = 1/N_n$. $\square$

We remind the reader that our goal is to get a bound on $|\log f(x)|$. Note that

$$
\begin{aligned}
|\log f(x)| &= \sum_{m=1}^{\infty} \frac{x^m}{m(1-x^m)^2} \leq \frac{|x|}{|1-x|^2} + \sum_{m=2}^{\infty} \frac{|x|^m}{m(1-|x|^m)^2} \\
&\leq \sum_{m=1}^{\infty} \frac{|x|^m}{m(1-|x|^m)^2} - \left(\frac{|x|}{(1-|x|)^2} - \frac{|x|}{|1-x|^2} \right) \\
&= \log f(|x|) - \left(\frac{|x|}{(1-|x|)^2} - \frac{|x|}{|1-x|^2} \right).
\end{aligned} \tag{2.1.22}
$$

We give an upper bound on the expression in brackets on the right hand-side.

Lemma 2.5. *If $x \in C''_{N_n}$, then we have*

$$
\frac{|x|}{(1-|x|)^2} - \frac{|x|}{|1-x|^2} \geq \frac{N_n^2}{2} - \frac{1}{12}.
$$

Proof of Lemma 2.5. Note that

$$
\begin{aligned}
\frac{|x|}{(1-|x|)^2} - \frac{|x|}{|1-x|^2} &= \frac{|x|}{(1-|x|)^2} \left(1 - \left(\frac{1-|x|}{|1-x|} \right)^2 \right) \\
&\leq \frac{|x|}{(1-|x|)^2} \left(1 - \left(\frac{1-e^{-1/N_n}}{|1-e^{-1/N_n} e^{i/N_n}|} \right)^2 \right).
\end{aligned} \tag{2.1.23}
$$

For last inequality, we note that every point in C''_{N_n} has argument in the interval $[\frac{1}{N_n}, 2\pi - \frac{1}{N_n}]$. Geometrically, we have that the closest a point x on this arc can get to the point $(1,0)$ in the plane is when the argument of x is one of the extreme points.

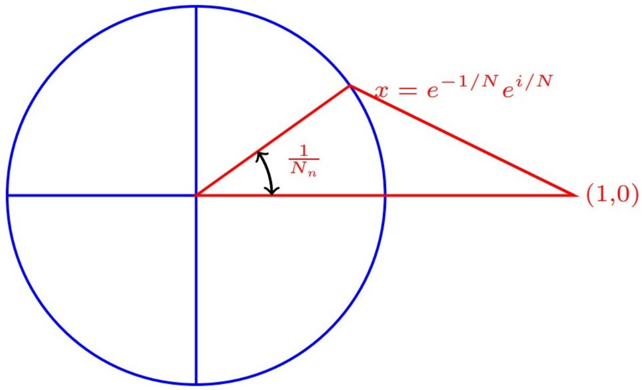


Figure 2.1: Triangle used to evaluate $|1-x|$.

We find an upper bound on right hand side of (2.1.23). From the diagram we have

$$\left|1-e^{-1/N_n}e^{i/N_n}\right| = \sqrt{1+e^{-2/N_n}-2e^{-1/N_n}\cos(1/N_n)} = \sqrt{1+e^{-2t}-2e^{-t}\cos(t)}.$$

Thus, we get

$$h(t) := \frac{1-e^{-1/N_n}}{|1-e^{-1/N_n}e^{i/N_n}|} = \frac{1-e^{-t}}{\sqrt{1+e^{-2t}-2e^{-t}\cos(t)}} = \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{48}t^2 + \cdots .$$

We have $t = 1/N_n = (2A/n)^{\frac{1}{3}} \leq (2A)^{\frac{1}{3}} \leq 1.34$. On the interval $[0, 1.34]$, the maximum absolute value of $h''(t)$ is $\sqrt{2}/24$. Hence, by Taylor's theorem, we get

$$h(t) = \frac{1}{\sqrt{2}} + O_{\leq}\left(\frac{\sqrt{2}}{48}t^2\right),$$

where $O_{\leq}(\cdot)$ means that the expression is bounded by $\cdot$ in absolute value (i.e. we can choose the constant associated to $O(\cdot)$ equal to 1). Hence, for $x \in C''_{N_n}$, we have

$$\frac{1-|x|}{|1-x|} = \frac{1}{\sqrt{2}} + O_{\leq}\left(\frac{\sqrt{2}}{48N_n^2}\right). \tag{2.1.24}$$

We also have that

$$\frac{|x|}{(1-|x|)^2} = \frac{e^{-t}}{(1-e^{-t})^2} = t^{-2} - \frac{1}{12} + \frac{1}{240}t^2 + \cdots > N_n^2 - \frac{1}{12}$$

Combining everything gives

$$\frac{|x|}{(1-|x|)^2} - \frac{|x|}{|1-x|^2} \geq \left(N_n^2 - \frac{1}{12}\right)\left(1 - \left(\frac{1}{\sqrt{2}} + O_{\leq}\left(\frac{\sqrt{2}}{48N_n^2}\right)\right)^2\right) \geq N_n^2 - \frac{1}{12}.$$

□

Proof of Proposition 2.3. From (2.1.22), we have

$$|\log f(x)| \leq AN_n^2 + 033N_n - 0.5 - N_n^2 - \frac{1}{12} \leq \left(A - \frac{1}{2}\right)N_n^2 + 0.33N_n.$$

Hence we have

$$\left|\frac{f(x)}{x^{n+1}}\right| \leq e^{\left(A-\frac{1}{2}\right)N_n^2 + 0.33N_n + (n+1)N_n} = e^{\left(3A-\frac{1}{2}\right)N_n^2 + 0.33N_n + \frac{1}{N_n}}.$$

Combining above with the definition of $E^{\min}(n)$ (2.1.19) we get

$$\begin{aligned}\left|E^{\min}(n)\right| \leq \frac{1}{2\pi i}\int_{C''_{N_n}}\left|\frac{f(x)}{x^{n+1}}\right| dx &\leq \frac{e^{\left(3A-\frac{1}{2}\right)N_n^2 + 0.33N_n + \frac{1}{N_n}}}{2\pi}\int_{C''_{N_n}} 1 \cdot dx \\ &\leq e^{\left(3A-\frac{1}{2}\right)N_n^2 + 0.33N_n}.\end{aligned}$$

The claimed inequality for $n \geq 87$ follows by analyzing this last expression. □

Explicit major arc formulas

The size of $\mathrm{PL}(n)$ is given by the major arc integral $J(n)$. We get a formula for each positive integer r, which holds for sufficiently large n. As we increase r, the estimates become better.

Proposition 2.6. *If $r \in \mathbb{Z}^+$, then for every $n \geq \max(l_r, n_r, 55)$ we have*

$$J(n) = \frac{e^{c+3AN_n^2}}{2\pi} \sum_{s=0}^{r+1} \sum_{m=0}^{r+1} \frac{(-1)^m \beta_s b_{s,m} \Gamma(m+\frac{1}{2})}{A^{m+\frac{1}{2}} N_n^{2s+2m+\frac{25}{12}}} + E_r^{\mathrm{maj}}(n),$$

where

$$|E_r^{\max}(n)| \leq \frac{(\mathcal{X}_r(n) + \mathcal{Y}_r(n))e^{2AN_n^2}}{N_n \pi} + |\mathcal{Z}_r(n)| \ \ (\textit{see}\,(2.1.16)).$$

Lemmata for Proposition 2.6

The idea is to use Cauchy's integral formula to get an asymptotic formula for the generating function for $\mathrm{PL}(n)$ and use that to obtain the eventual formula for $J(n)$. We start with some basic setup. For $x \in C'_{N_n}$, we define

$$z = \log\left(\frac{1}{x}\right) = \frac{1}{N_n} - i\nu =: \rho e^{i\phi}, \tag{2.1.25}$$

and

$$w := \Re\left(\frac{\pi}{2z}\right) = \frac{\pi \cos(\phi)}{2\rho}. \tag{2.1.26}$$

Note that $v < \frac{1}{N_n}$, and hence

$$\rho = \sqrt{N_n^{-2} + \nu^2} \in \left[\frac{1}{N_n}, \frac{\sqrt{2}}{N_n}\right]. \tag{2.1.27}$$

Furthermore, we have

$$|\phi| = |\arctan(\nu N_n)| \leq \arctan(1) = \frac{\pi}{4}. \tag{2.1.28}$$

Using the identity

$$\int_0^\infty t \log\left(1 - e^{-tz}\right) dt = -\frac{A}{z^2},$$

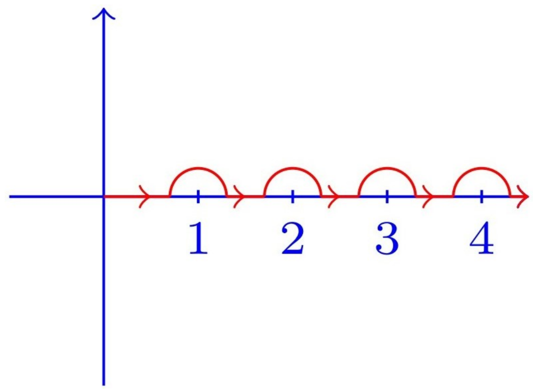


Figure 2.2: The path of integration Γ.

we get

$$\begin{aligned}
-\log f(x) + \frac{A}{z^2} &= -\log f(x) - \int_0^\infty t \log\left(1 - e^{-tz}\right) dt \\
&= \sum_{m=1}^{\infty} m \log\left(1 - e^{-mz}\right) - \int_0^\infty t \log\left(1 - e^{-tz}\right) dt \\
&= \sum_{m=1}^{\infty} m \log\left(1 - e^{-mz}\right) + \int_\Gamma \left(\frac{t \log\left(1 - e^{-tz}\right)}{e^{2\pi i t} - 1} - \frac{t \log\left(1 - e^{-tz}\right)}{1 - e^{-2\pi i t}} \right) dt,
\end{aligned}$$

where Γ is the path from 0 to ∞ which travels along the real axis, apart from sufficiently small semicircles at the positive integers above the real axis to avoid the poles of the integrand (see Fig. 2.2). (Here one has to be careful near 0 but it is easy to check that it is a removable singularity with value 0.) Let Γ' be the reflection of Γ about x-axis. Then using the Residue Theorem (note that the residue of $\frac{t \log\left(1-e^{-tz}\right)}{e^{2\pi i t}-1}$) at $t = m \in \mathbb{Z}$ is $m \log\left(1 - e^{-mz}\right)/(2\pi i)$ we get

$$-\log f(x) + \frac{A}{z^2} = \int_{\Gamma'} \frac{t \log\left(1 - e^{-tz}\right)}{e^{2\pi i t} - 1} dt - \int_\Gamma \frac{t \log\left(1 - e^{-tz}\right)}{1 - e^{-2\pi i t}} dt. \qquad (2.1.29)$$

We study the above two integrals separately.

Lemma 2.7. *Assuming the notation and hypotheses above, we have*

$$\int_\Gamma \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt = \int_0^{iw} \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt + O_\leq\left(35N_n^2 e^{-\pi^2 N_n}\right),$$
$$\int_{\Gamma'} \frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}dt = \int_0^{-iw} \frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}dt + O_\leq\left(N_n^2 e^{-\pi^2 N_n}\right).$$

Proof of Lemma 2.7. Start by noticing that the integrands above have simple poles at integer points and when $1-e^{-tz}=0$ i.e. $t=2\pi ik/z$ for all $k\in\mathbb{Z}^+$. In particular, there is no pole in the region between Γ, imaginary axis, and the line parallel to real-axis from the point iw. So we get

$$\int_\Gamma \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt = \int_0^{iw} \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt + \int_{iw}^{iw+\infty} \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt, \tag{2.1.30}$$

where second integral is along the horizontal line (i.e. parallel to real-axis) starting at iw. Note that

$$\left|\frac{1}{1-e^{-2\pi i(iw+x)}}\right| \leq \left|\frac{1}{1-|e^{-2\pi i(iw+x)}|}\right| = \left|\frac{1}{1-e^{\pi^2\cos\phi/\rho}}\right|.$$

Using (2.1.27), this is bounded by

$$\left|\frac{1}{(1-e^{\pi^2 N_n})}\right| = \left|\frac{e^{-\pi^2 N_n}}{(e^{-\pi^2 N_n}-1)}\right| \leq \left|\frac{e^{-\pi^2 N_n}}{(e^{-\pi^2 N_1}-1)}\right| \leq 1.007e^{-\pi^2 N_n}.$$

Therefore, we have

$$\begin{aligned}\left|\int_{iw}^{iw+\infty} \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt\right| &\leq 1.007e^{-\pi^2 N_n}\int_{iw}^{iw+\infty}\left|t\log\left(1-e^{-tz}\right)\right|dt \\ &= 1.007e^{-\pi^2 N_n}\int_L \left|v\log\left(1-e^{-v}\right)\right|dv =: U(n),\end{aligned} \tag{2.1.31}$$

where we made the change of variable $v=tz$ and L is an array from $\frac{\pi i}{2}e^{i\phi}\cos\phi$ at an angle ϕ.

We split the integral in (2.1.31) into two pieces. This is one of the places where proof in [OPR22] differs from [Wri31] to obtain effective estimates. Throughout, we let $v_t := iwz + te^{i\phi}$, so that $L = \{v_t : t \in [0, \infty)\}$. Note that

$$wz = \frac{\pi \cos \phi}{2\rho} \cdot \rho e^{i\phi} = \frac{\pi \cos \phi e^{i\phi}}{2}. \tag{2.1.32}$$

Using this we estimate the piece $|\log(1 - e^{v_t})|$ in the integrand of (2.1.31). We need following inequality which can be easily checked:

$$|\log(1 + y)| \le -\log(1 - |y|), \quad \text{for all } |y| < 1. \tag{2.1.33}$$

We will break up the line L into a compact piece L_1 and a remaining piece L_2, where we can utilize this bound. To see where it applies, we compute

$$\left|e^{-v_t}\right| = e^{-\Re(v_t)} = e^{w\Im(z) - \Re(te^{i\phi})} = e^{-w\nu - t\cos\phi} = e^{-\cos\phi\left(\frac{\pi\nu}{2\rho} + t\right)},$$

which is less than or equal to one if and only if $\cos\phi\left(\frac{\pi\nu}{2\rho} + t\right) \ge 0$. Using (2.1.27) and (2.1.28), and the fact that $\nu \ge 1/N_n$, we have

$$\cos\phi\left(\frac{\pi\nu}{2\rho} + t\right) > \frac{\sqrt{2}}{2}\left(\frac{\pi}{2\sqrt{2}} + t\right) = -\frac{\pi}{4} + \frac{t}{\sqrt{2}},$$

which gives us

$$\left|e^{-v_t}\right| \le e^{\frac{\pi}{4} - \frac{t}{\sqrt{2}}}. \tag{2.1.34}$$

Hence, we let

$$L_1 := \left\{v_t : t \in [0, \pi/2\sqrt{2} + 1)\right\}, \quad L_2 := \left\{v_t : t \in [\pi/2\sqrt{2} + 1, \infty)\right\},$$

then we can use (2.1.33) to estimate the integrand on L_2. For $v_t \in L_2$, we have

$$\left|v_t \log\left(1-e^{-v_t}\right)\right| \leq -|v_t| \cdot \log\left(1-\left|e^{-v_t}\right|\right) \leq |v_t| \log\left(1-e^{\frac{\pi}{4}-\frac{t}{\sqrt{2}}}\right).$$

Note that $|v_t| \leq |\cos\phi| \frac{\pi}{2} + t \leq \frac{\pi}{2} + t$, so using the fact that $-\log(1-x) < x/(1-x)$ for real $0 \neq x < 1$, we find that

$$\left|v_t \log\left(1-e^{-v_t}\right)\right| \leq \frac{\left(\frac{\pi}{2}+t\right) e^{\frac{\pi}{4}-\frac{t}{\sqrt{2}}}}{1-e^{\frac{\pi}{4}-\frac{t}{\sqrt{2}}}}.$$

Since L_1 is compact, we then find the following estimate for (2.1.31) (note that when we integrate on L_2, since we are integrating absolute values, the change of variables in the differential goes away as it has absolute value 1):

$$\begin{aligned}
U(n) &= 1.007 N_n^2 e^{-\pi_n^N} \left(\int_{L_1} \left|v \log\left(1-e^{-v}\right)\right| dv + \int_{L_2} \left|v \log\left(1-e^{-v}\right)\right| dv\right) \\
&\leq 1.007 N_n^2 e^{-\pi_n^N} \left(\frac{\pi}{2\sqrt{2}}+1\right) \cdot \max\left\{|v_t| \cdot \left|\log\left(1-e^{-v}\right)\right| : t \in [0, \pi/2\sqrt{2}+1)\right\} \\
&\quad + 1.007 N_n^2 e^{-\pi_n^N} \int_{\frac{\pi}{2\sqrt{2}}}^{\infty} \frac{\left(\frac{\pi}{2}+t\right) e^{\frac{\pi}{4}-\frac{t}{\sqrt{2}}}}{1-e^{\frac{\pi}{4}-\frac{t}{\sqrt{2}}}} dv \\
&\leq 1.007 N_n^2 e^{-\pi_n^N} \left(\frac{\pi}{2\sqrt{2}}+1\right) \cdot \max\left\{|v_t| \cdot \left|\log\left(1-e^{-v}\right)\right| : t \in [0, \pi/2\sqrt{2}+1)\right\} \\
&\quad + 4.8 N_n^2 e^{-\pi^2 N_n}.
\end{aligned}$$

Now we estimate first part above. We start with $\log\left(1-e^{-v_t}\right)$. We begin by recalling that for complex y, the principal branch of the logarithm is given by $\log y = \log(|y|) + i\text{Arg}(y)$. Thus we have

$$\left|\log\left(1-e^{-v_t}\right)\right| \leq \left|\log\left|1-e^{-v_t}\right|\right| + \pi. \tag{2.1.35}$$

To bound the logarithm, we find the maximum and minimum on the interval $t \in [0, \pi/2\sqrt{2}+1)$. The only critical point of $|1-e^{-v_t}|$ is at $\cosh(i\phi)$, thus the potential extrema of $|1-e^{-v_t}|$ are at $t = 0, \pi/2\sqrt{2}$, and $\cosh(i\phi)$. That is, the maximum of $\log|1-e^{-v_t}|$ is bounded by

$$\max\left\{\left|1-e^{-v_0}\right|, \left|1-e^{-v_{\pi/2\sqrt{2}}}\right|, \left|1-e^{-v_{\cosh(i\phi)}}\right|\right\}$$

Using the fact that $\phi \in [-\pi/4, \pi/4]$, one gets

$$0.75 \leq \left|1-e^{-v_0}\right| \leq 1.85, \;\; 0.9 \leq \left|1-e^{-v_{\pi/2\sqrt{2}}}\right| \leq 1.4, \;\; 0.7 \leq \left|1-e^{-v_{\cosh(i\phi)}}\right| \leq 1.45.$$

So (2.1.35) implies that $|\log(1-e^{-v_t})| \leq 3.76$. To bound v_t on this interval, using (2.1.32) and the triangle inequality, we find that $|v_t| \leq \frac{\pi\cos\phi}{2} + t \leq \frac{\pi}{2} + \frac{\pi}{2\sqrt{2}} = 3.68....$ Therefore, we conclude that

$$\begin{aligned} &1.007 N_n^2 e^{-\pi_n^N}\left(\frac{\pi}{2\sqrt{2}}+1\right)\cdot\max\left\{|v_t|\cdot\left|\log\left(1-e^{-v}\right)\right| : t\in[0,\pi/2\sqrt{2}+1)\right\} \\ &\leq 30 N_n^2 e^{-\pi^2 N_n}. \end{aligned}$$

As a consequence, we obtain

$$U(n) \leq 35 N_n^2 e^{-\pi^2 N_n}.$$

Combining everything we have proved the first part of the lemma which is that

$$\int_\Gamma \frac{t\log(1-e^{-tz})}{1-e^{-2\pi i t}}dt = \int_0^{iw}\frac{t\log(1-e^{-tz})}{1-e^{-2\pi i t}}dt + O_{\leq}\left(35 N_n^2 e^{-\pi^2 N_n}\right). \tag{2.1.36}$$

The second claim in the lemma follows by arguing as above (after suitable sign changes in the integrand) using the path of integration along Γ'. Namely, we get

$$\int_{\Gamma'} \frac{t\log(1-e^{-tz})}{e^{2\pi i t}-1}dt = \int_0^{-iw}\frac{t\log(1-e^{-tz})}{e^{2\pi i t}-1}dt + O_{\leq}\left(N_n^2 e^{-\pi^2 N_n}\right). \tag{2.1.37}$$

□

So far we have proved that

$$-\log f(x)+\frac{A}{z^2}=\int_0^{iw}\frac{t\log\left(1-e^{-tz}\right)}{e^{2\pi it}-1}dt-\int_0^{-iw}\frac{t\log\left(1-e^{-tz}\right)}{1-e^{-2\pi it}}dt+O_{\leq}\left(36N_n^2e^{-\pi^2N_n}\right). \tag{2.1.38}$$

Now we need to evaluate first two integrals. For that we need the following lemma:

Lemma 2.8. *Assuming the notation and hypotheses above, the following are true:*

(1) If n is a positive integer, then we have

$$\mathcal{I}_1:=2\int_0^{w}\frac{y\log(yz)}{e^{2\pi y}-1}dy=c+\frac{\log y}{12}+O_{\leq}\left(3e^{-4.7N_n}\right).$$

(2) If $n\geq n_r$ is an integer, then we have

$$\mathcal{I}_2:=-2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^{w}\frac{y^{2s+1}}{e^{2\pi y}-1}dy=-\sum_{s=1}^{r+1}\alpha_s z^{2s}+O_{\leq}\left(e^{-\frac{\pi^2N_n}{2}}\right),$$

where α_s is defined by (2.1.5).

(3) If n is a positive integer, then we have

$$\mathcal{I}_3:=-2\sum_{s\geq r+2}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^{w}\frac{y^{2s+1}}{e^{2\pi y}-1}dy\leq 2^{r+4}\pi^3\alpha_{r+2}N_n^{-2r-4}.$$

Proof of Lemma 2.8. We estimate these three quantities one-by-one. For $\mathcal{I}_1$, we have

$$\begin{aligned}\mathcal{I}_1&=2\int_0^{\infty}\frac{y\log(yz)}{e^{2\pi y}-1}dy-2\int_w^{\infty}\frac{y\log(yz)}{e^{2\pi y}-1}dy\\&=2\int_0^{\infty}\frac{y\log y}{e^{2\pi y}-1}dy+2\log z\int_0^{\infty}\frac{y}{e^{2\pi y}-1}dy-2\int_w^{\infty}\frac{y\log(yz)}{e^{2\pi y}-1}dy\\&=c+\frac{1}{12}\log z-2\int_w^{\infty}\frac{y\log(yz)}{e^{2\pi y}-1}dy,\end{aligned}$$

since

$$\int_0^\infty \frac{y}{e^{2\pi y}-1}dy = \frac{1}{4}B_2 = \frac{1}{24},$$

where B_2 is the second Bernoulli number also remember the definition of c (2.1.1). To evaluate the last integral above, we first note that $w = \frac{\pi\cos\phi}{2\rho} \geq \frac{\pi\sqrt{2}}{4\rho} \geq \frac{\pi N_n}{4} \geq 0.58$, where we used (2.1.28) and (2.1.27) and that $n \geq 1$. By direct manipulation, for $y \geq 0.58$, we have

$$\frac{1}{e^{2\pi y}-1} \leq 1.1e^{-2\pi y}. \tag{2.1.39}$$

Then on the interval $w \geq 0.58$, we find that

$$\begin{aligned}\left|-2\int_w^\infty \frac{y\log(yz)}{e^{2\pi y}-1}dy\right| &\leq 2.2\left|-2\int_w^\infty y\log(yz)e^{-2\pi y}dy\right| \\ &\leq \frac{2.2}{4\pi^2}e^{-2\pi w}\left|1-e^{2\pi w}\mathrm{Ei}(-2\pi w)+(1+2\pi w)\log(wz)\right|,\end{aligned}$$

where $\mathrm{Ei}(x) := -\int_{-x}^\infty \frac{e^{-t}}{t}dt$. Straightforward manipulation then gives

$$\left|-2\int_w^\infty \frac{y\log(yz)}{e^{2\pi y}-1}dy\right| \leq e^{-2\pi w}\left(1+0.056(1+2\pi w)(|\log\rho|+\pi)\right).$$

If $n \geq 7$ then $\sqrt{2}/N_n < 1$, and so $w \geq \pi N_n/4$. Therefore, we have

$$\left|-2\int_w^\infty \frac{y\log(yz)}{e^{2\pi y}-1}dy\right| \leq e^{-\frac{\pi^2 N_n}{2}}\left(1+0.056\left(1+\frac{\pi^2 N_n}{2}\right)(|\log N_n|+\pi)\right) \leq 3e^{-4.7N_n}. \tag{2.1.40}$$

By direct computation for $1 \leq n \leq 6$ we find that this holds in general, and hence we have

$$\mathcal{I}_1 = c + \frac{1}{12}\log z + O_{\leq}\left(3e^{-4.7N_n}\right) \tag{2.1.41}$$

Now we look at $\mathcal{I}_2$. Using (2.1.39) and the integral representation, we get

$$\begin{aligned}
\mathcal{I}_2 &= -2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^\infty \frac{y^{2s+1}}{e^{2\pi y}-1dy} + 2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_w^\infty \frac{y^{2s+1}}{e^{2\pi y}-1}dy \\
&= -2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^\infty \frac{y^{2s+1}}{e^{2\pi y}-1}dy + O_\leq\left(2.2\cdot\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_w^\infty y^{2s+1}e^{-2\pi y}\right) \\
&= -2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\left(\frac{\Gamma(2+2s)\zeta(2+2s)}{(2\pi)^{2+2s}}\right) \\
&\qquad O_\leq\left(2.2\cdot\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\left(\frac{\Gamma(2+2s;2\pi w)}{(2\pi)^{2+2s}}\right)\right),
\end{aligned}$$

where $\Gamma(a;x) := \int_x^\infty t^{a-1}e^{-t}dt$ is the *incomplete Gamma function.* Using (2.1.5), this becomes

$$\mathcal{I}_2 = -\sum_{s=1}^{r+1}\alpha_s z^{2s} + O_\leq\left(2.2\cdot\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\left(\frac{\Gamma(2+2s;2\pi w)}{(2\pi)^{2+2s}}\right)\right).$$

To estimate this, we use Theorem 1.1 of [Pin20] and Lemma 2.2 of [BKRT21] which shows, for $a>2$, that

$$\Gamma(a;x) \leq \frac{(x-b_a)^a - x^a}{a\cdot b_a}e^{-x} \leq (x-b_a)^{a-1},$$

where $b_a := \Gamma(a+1)^{\frac{1}{a-1}}$. Combined with the Bernoulli number formula for $\zeta(s)$ at positive even integers, this gives

$$\begin{aligned}
&\left|2.2\cdot\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\left(\frac{\Gamma(2+2s;2\pi w)}{(2\pi)^{2+2s}}\right)\right| \\
&\qquad \leq \frac{1.1e^{-2\pi w}}{4\pi^2}\sum_{s=1}^{r+1}\frac{\rho^{2s}B_{2s}}{s(2\pi)^{2s}(2s)!}\left(2\pi w + ((2s+1)!)^{\frac{1}{2s+1}}\right)^{2s+1} \\
&\qquad \leq \frac{1.1e^{-2\pi w}}{4\pi^2}\sum_{s=1}^{r+1}\frac{\rho^{2s}B_{2s}}{s(2\pi)^{2s}(2s)!}\left(2\pi w + 2s\right)^{2s+1}.
\end{aligned}$$

Using the Bernoulli number upper bound from (24.9.8) of [DLMF], recalling that $w \geq \pi N_n/4$, and noting that $w = \frac{\pi \cos\phi}{2\rho} \leq \frac{\pi}{2\rho} < \frac{\pi N_n}{2}$, this is bounded by

$$\begin{aligned}
0.056 e^{-2\pi w} \sum_{s=1}^{r+1} \frac{\rho^{2s}}{s(2\pi)^{4s}} \left(2\pi w + 2s\right)^{2s+1} \\
&= 0.056 e^{-2\pi w} \sum_{s=1}^{r+1} \rho^{2s} \left(\frac{1}{2\pi} + \frac{s}{2\pi^2}\right)^{2s} \left(\frac{2\pi w}{s} + 2\right) \\
&\leq 0.056 e^{-\frac{\pi^2 N_n}{2}} \sum_{s=1}^{r+1} \left(\frac{s\sqrt{2}}{4N_n}\right)^{2s} \left(\frac{\pi^2 N_n}{s} + 2\right) = n_r \cdot e^{-\frac{\pi^2 N_n}{2}}.
\end{aligned}$$

Therefore, if $n \geq n_r$, then we obtain the claimed inequality for $\mathcal{I}_2$.

Finally, we turn to the bound for $\mathcal{I}_3$, which we recall is

$$\mathcal{I}_3 := -2 \sum_{s \geq r+2} \frac{\zeta(2s) z^{2s}}{s(2\pi)^{2s}} \int_0^w \frac{y^{2s+1}}{e^{2\pi y} - 1} dy.$$

As $\zeta(2r+4) \geq \zeta(2s)$ for all $s \geq r+4$, we have

$$|\mathcal{I}_3| \leq \frac{\zeta(2r+4)}{r+2} \frac{\rho^{2r+4}}{(2\pi)^{2r+4}} \int_0^w \frac{y^{2r+5}}{e^{2\pi y} - 1} \sum_{m=0}^{\infty} \left(\frac{y\rho}{2\pi}\right)^{2s} dy.$$

Therefore, using (2.1.5) we find that

$$\begin{aligned}
|\mathcal{I}_3| &\leq \frac{\zeta(2r+4)}{r+2} \frac{\rho^{2r+4}}{(2\pi)^{2r+4}} \int_0^w \frac{y^{2r+5}}{\left(e^{2\pi y} - 1\right)\left(1 - \left(\frac{y\rho}{2\pi}\right)^2\right)} dy \\
&< \frac{\zeta(2r+4)}{r+2} \frac{\rho^{2r+4}}{(2\pi)^{2r+4}} \int_0^w \frac{y^{2r+5}}{e^{2\pi y} - 1} dy \\
&< \frac{\Gamma(2r+6)\zeta(2r+4)\zeta(2r+6)\rho^{2r+4}}{(r+2)(2\pi)^{2r+7}} = 4\pi^3 \alpha_{r+2} \rho^{2r+4}.
\end{aligned}$$

Since we have $\rho \leq \frac{\sqrt{2}}{N_n}$, this proves the inequality for $\mathcal{I}_3$. □

Proof of Proposition 2.6. We begin by combining two integrals in (2.1.38) to get

$$\log\left(f(x)\right) = \frac{A}{z^2} + \int_0^w \frac{y \log\left(2 \sin\left(\frac{yz}{2}\right)\right)}{e^{2\pi y} - 1} dy + O_{\leq}\left(36 N_n^2 e^{-\pi^2 N_n}\right).$$

For calculating the integral and use the Lemma 2.8, we employ the identity

$$\sin(\tau) = \tau \cdot \prod_{n\geq 1}\left(1 - \frac{\tau^2}{n^2\pi^2}\right).$$

This implies that

$$\log(\sin(\tau)) = \log(\tau) + \sum_{s=1}^{\infty}\sum_{m=1}^{\infty}\frac{\tau^{2s}}{s\cdot m^{2s}\pi^{2s}} = \log(\tau) - \sum_{s=1}^{\infty}\frac{\zeta(2s)\tau^{2s}}{s\pi^{2s}}.$$

Hence, for $r \geq 1$, the integral above transforms into

$$\begin{aligned}
&\int_0^w \frac{y\log\left(2\sin\left(\frac{yz}{2}\right)\right)}{e^{2\pi y}-1}dy\\
&= 2\int_0^w \frac{y\log(yz)}{e^{2\pi y}-1}dy - 2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^w \frac{y^{2s+1}}{e^{2\pi y}-1}dy - 2\sum_{s\geq r+2}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^w \frac{y^{2s+1}}{e^{2\pi y}-1}dy\\
&= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}$$

This combined with Lemma 2.8 gives us

$$\log\left(f(x)\right) = \frac{A}{z^2}+c+\frac{\log(z)}{12}-\sum_{s=1}^{r+1}\alpha_s z^{2s}+O_{\leq}\left(2^{r+4}\pi^3\alpha_{r+2}N_n^{-2r-4}+5e^{-4.7N_n}\right). \quad (2.1.42)$$

To get $f(x)$ from this, we make use of a complex-analytic version for the remainder terms of the Taylor series of $f(x)$, which is required as our estimates make use of the expressions involving β_s which is defined, at (2.1.4), to be the $r+2$-th Taylor coefficient of

$$g_r(z) := e^{-\sum_{s=1}^{r+1}\alpha_s z^s} = \sum_{s=0}^{r+1}\beta_s z^s + R_r(z),$$

where $R_r(z)$ is the tail of the Taylor series. For convenience, we assume that $n \geq 55$, which guarantees that $|z| = \rho \leq \sqrt{2}/N_n \leq \frac{1}{2}$. The standard complex Taylor series remainder estimate (for example, see Theorem B.21 of [Kna16] with $R = 1$) gives

$$|R_r(z)| \leq \frac{\max_{|z|=1}\{|g_r(z)|\}\,|z|^{r+2}}{1-|z|} \leq C_r|z|^{r+2},$$

where

$$C_r := 2\max_{|z|=1}\left(\left|e^{-\sum_{s=1}^{r+1}\alpha_s z^s}\right|\right). \tag{2.1.43}$$

By replacing z with z^2 (this is allowed since we demanded that $|z| < 1$, and so $|z^2| < 1$ is still in the range of validity for the remainder estimate), we obtain

$$g_r(z^2) := e^{-\sum_{s=1}^{r+1}\alpha_s z^{2s}} = \sum_{s=0}^{r+1}\beta_s z^s + O_{\leq}\left(C_r|z|^{2r+4}\right).$$

Therefore, by exponentiating (2.1.42), for $n \geq 55$ we obtain

$$\begin{aligned} f(x) = e^c z^{\frac{1}{12}} e^{\frac{A}{z^2}} \cdot \left(\sum_{s=0}^{r+1}\beta_s z^{2s} + O_{\leq}\left(C_r|z|^{2r+4}\right)\right) \\ \times O_{\leq}\left(\exp\left(2^{r+4}\pi^3\alpha_{r+2}N_n^{-2r-4} + 5e^{-4.7N_n}\right)\right). \end{aligned}$$

To address the error term on the far right above, we assume that $n \geq l_r$, which by (2.1.7) gives

$$2^{r+4}\pi^3\alpha_{r+2}N_n^{-2r-4} + 5e^{-4.7N_n} \leq \frac{1}{2} < 1 \text{ and } \frac{1}{1-\left(2^{r+4}\pi^3\alpha_{r+2}N_n^{-2r-4} + 5e^{-4.7N_n}\right)} \leq 2.$$

Thanks to (4.5.11) of [DLMF], which states that $e^x < 1 + x/(1-x)$ for $x < 1$, this gives

$$\begin{aligned} f(x) &= e^c z^{\frac{1}{12}} e^{\frac{A}{z^2}} \cdot \left(\sum_{s=0}^{r+1}\beta_s z^{2s} + O_{\leq}\left(C_r|z|^{2r+4}\right)\right) \\ &\quad \times O_{\leq}\left(1 + 2^{r+5}\pi^3\alpha_{r+2}N_n^{-2r-4} + 10e^{-4.7N_n}\right) \\ &= M(x) + \mathcal{X}_r(n) + \mathcal{Y}_r(n), \end{aligned}$$

where we let $M(x) := e^{c+\frac{A}{z^2}} \sum_{s=0}^{r+1} \beta_s z^{2s+\frac{1}{12}}$, and $\mathcal{X}_r(n)$ and $\mathcal{Y}_r(n)$ are defined by (2.1.11) and (2.1.12) respectively. This encodes the compilation of error on C'_{N_n} using the facts that $\left|e^{\frac{A}{z^2}}\right| \leq e^{\left|\frac{A}{\rho^2}\right|} \leq e^{AN_n^2}$ and $|z| = \rho \leq \sqrt{2}N_n^{-1}$. Now, recalling that $n = 2AN_n^3$, we obtain

$$\begin{aligned} J(n) &= \frac{1}{2\pi i}\int_{\frac{1-i}{N_n}}^{\frac{1+i}{N_n}} f(e^{-z})e^{2AN_n^3 z}dz = \frac{1}{2\pi i}\int_{\frac{1-i}{N_n}}^{\frac{1+i}{N_n}} (M(x) + \mathcal{X}_r(n) + \mathcal{Y}_r(n))dz \\ &= \frac{e^c}{2\pi i}\int_{\frac{1-i}{N_n}}^{\frac{1+i}{N_n}} \left(\sum_{s=0}^{r+1} \beta_s z^{2s+\frac{1}{12}}\right) e^{\frac{A}{z^2}+2AN_n^3 z}dz + O_{\leq}\left(\frac{(\mathcal{X}_r(n) + \mathcal{Y}_r(n)) \cdot e^{2AN_n^3}}{N_n\pi}\right), \end{aligned}$$

where we used that on the path of integration, $\left|e^{2AN_n^3 z}\right| = e^{2AN_n^3}$, and that the length is $2N_n^{-1}$. We now let $v = N_n z$, and define

$$P_s := \frac{1}{2\pi i}\int_{1-i}^{1+i} v^{2s+\frac{1}{12}} \exp\left(AN_n^2\left(2v + \frac{1}{v^2}\right)\right) dv,$$

which give us

$$J(n) = e^c \sum_{s=0}^{r+1} \frac{\beta_s P_s}{N_n^{2s+\frac{13}{12}}} + O_{\leq}\left(\frac{(\mathcal{X}_r(n) + \mathcal{Y}_r(n)) \cdot e^{2AN_n^3}}{N_n\pi}\right). \tag{2.1.44}$$

To complete the proof we require an explicit version of Wright's expansion of P_s, that he obtained via the method of steepest descent. Here we explain that in detail. We first let $\mathcal{C}$ be the plane curve defined by the equation $(x^2+y^2)^2 = x$, together with the labeled points $E = (2^{-2/3}, 2^{-2/3}), D = (2^{-2/3}, -2^{-2/3})$ on $\mathbb{C}$ and the point $O = (0,0), G = (1,1)$ and $F = (1,-1)$ in the plane. This is illustrated in Fig. 2.3. Note that if $\xi_s(v) := (2\pi i)^{-1} v^{2v+1/12} \exp(AN_n^2(2v + 1/v^2))$, the since $|v| \leq \sqrt{2}$, $x \leq 1$ on OG and OF, then we have

$$|\xi_s(v)| = \frac{|v|^{2s+\frac{1}{2}}}{2\pi} e^{2AN_n^2 x} \leq \frac{2^{s+\frac{1}{4}} e^{2AN_n^2}}{2\pi}, \quad (v \in OF \cup OG).$$

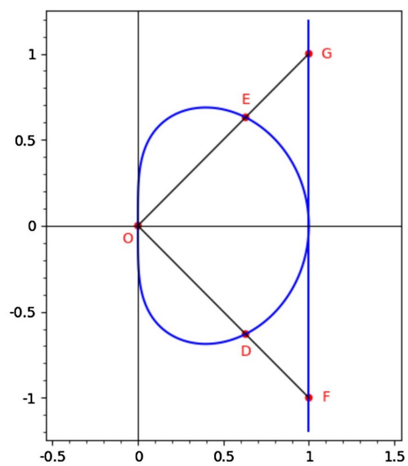


Figure 2.3: The curve $\mathcal{C}$ and points E,D,O,G,F.

One can show that

$$P_s = \int_{\mathcal{C}} \xi_s(v)dv + O_{\leq}\left(\left|\int_D^F \xi_s(v)dv\right| + \left|\int_G^E \xi_s(v)dv\right| + \left|\int_E^O \xi_s(v)dv\right| + \left|\int_O^D \xi_s(v)dv\right|\right) \tag{2.1.45}$$

$$= \int_{\mathcal{C}} \xi_s(v)dv + O_{\leq}\left(\frac{2^{s+\frac{1}{4}}e^{2AN_n^2}}{2\pi}\left(|DF| + |GE| + |EO| + |OD|\right)\right)$$

$$= \int_{\mathcal{C}} \xi_s(v)dv + O_{\leq}\left(0.64 \cdot 2^s e^{2AN_n^2}\right).$$

Making the change of variables $t^2 = 3 - 2v - v^{-2}$ 2 to estimate the integral $\int_{\mathcal{C}} \xi_s(v)dv$, we have

$$\int_{\mathcal{C}} \xi_s(v)vd = e^{3AN_n^2} \int_{\mathbb{R}} \chi_s(t)e^{-AN_n^2 t^2}dt,$$

where we recall that $\chi_s(t)$ is defined at (2.1.9) by

$$\chi_s(t) := \frac{v^{2s+\frac{37}{12}} \cdot t}{2\pi i(1 - v^3)} = \frac{v^{2s+\frac{24}{12}}\sqrt{2v+1}}{2\pi(v^2 + v + 1)},$$

which is a smooth function on $\mathcal{C}$. We can write its complex Taylor polynomial as

$$\chi_s(t) = \sum_{m=0}^{2r+3} a_{s,m} t^m + O_{\leq}\left(D_r |t|^{2r+4}\right),$$

where

$$D_r := \frac{1}{(2r+4)!} \max\left\{\max\left\{\left|\chi_s^{(2r+4)}(t)\right|\right\}_{t\in\mathbb{R}}\right\}_{s=0}^{r+1}, \tag{2.1.46}$$

and

$$a_{2m} = \frac{1}{2\pi i}\int_{0^+} \chi_s(t)\frac{dt}{t^{2m+1}},$$

the integral being taken along a small loop around $t = 0$. Then we have

$$\begin{aligned} a_{2m} &= -\frac{1}{4\pi^2}\int_{1^+} \frac{v^{2s+\frac{1}{12}}}{t^{2m+1}} dv \\ &= -\frac{(-1)^m}{4\pi^2 i}\int_{1^+} \frac{v^{2s+\frac{1}{12}+2m+1}}{(v-1)^{2m+1}(2v+1)^{m+\frac{1}{2}}} dv. \end{aligned}$$

Putting $v = 1 + y$, we have

$$\begin{aligned} a_{2m} &= -\frac{(-1)^m}{4\pi^2 i}\int_{1^+} \frac{(1+y)^{2s+\frac{1}{12}+2m+1}}{y^{2m+1}(3+2y)^{m+\frac{1}{2}}} dv \\ &= \frac{(-1)^m b_{s,m}}{2\pi}, \end{aligned}$$

where $b_{s,m}$ is defined at (2.1.4). It should be noted that D_r is explicitly computable, as $\chi_s(t)$ is a smooth function on the compact curve $\mathcal{C}$. Thus, we find that

$$\begin{aligned} \int_{\mathcal{C}} \xi_s(v) dv &= e^{3AN_n^2} \sum_{m=0}^{2r+3} \int_{\mathbb{R}} a_{s,m} t^m e^{-AN_n^2 t^2} dt + O_{\leq}\left(D_r \cdot e^{3AN_n^2}\int_{\mathbb{R}} |t|^{2r+4} e^{-AN_n^2 t^2} dt\right) \\ &= e^{3AN_n^2} \sum_{m=0}^{2r+3} \int_{\mathbb{R}} a_{s,m} t^m e^{-AN_n^2 t^2} dt + O_{\leq}\left(D_r \cdot \Gamma\left(r+\frac{5}{2}\right)(AN_n^2)^{-r-\frac{5}{2}} e^{3AN_n^2}\right). \end{aligned}$$

Putting this in (2.1.45) gives

$$\begin{aligned}
P_s &= e^{3AN_n^2}\sum_{m=0}^{r+1}\frac{a_{s,2m}\Gamma\left(m+\frac{1}{2}\right)}{(AN_n^2)^{m+\frac{1}{2}}} \\
&\quad + O_{\leq}\left(D_r\cdot\Gamma\left(r+\frac{5}{2}\right)(AN_n^2)^{-r-\frac{5}{2}}e^{3AN_n^2}+0.64\cdot 2^s e^{2AN_n^2}\right) \\
&= e^{3AN_n^2}\sum_{m=0}^{r+1}\frac{(-1)^m b_{s,m}\Gamma\left(m+\frac{1}{2}\right)}{(2\pi AN_n^2)^{m+\frac{1}{2}}} \\
&\quad + O_{\leq}\left(D_r\cdot\Gamma\left(r+\frac{5}{2}\right)(AN_n^2)^{-r-\frac{5}{2}}e^{3AN_n^2}+0.64\cdot 2^s e^{2AN_n^2}\right).
\end{aligned}$$

Plugging this into (2.1.44) gives

$$\begin{aligned}
J(n) = \frac{e^{c+3AN_n^2}}{2\pi}\sum_{s=0}^{r+1}\sum_{m=0}^{r+1}\frac{(-1)^m\beta_s b_{s,m}\Gamma(m+\frac{1}{2})}{A^{m+\frac{1}{2}}N_n^{2s+2m+\frac{25}{12}}} + O_{\leq}\Bigg(\frac{(\mathcal{X}_r(n)+\mathcal{Y}_r(n))\cdot e^{2AN_n^3}}{N_n\pi} \\
+\left|e^c\left(D_r\cdot\Gamma\left(r+\frac{5}{2}\right)(AN_n^2)^{-r-\frac{5}{2}}e^{3AN_n^2}+0.64\cdot 2^s e^{2AN_n^2}\right)\sum_{s=0}^{r+1}\beta_s N_n^{-2s-\frac{13}{12}}\right|\Bigg).
\end{aligned} \tag{2.1.47}$$

Therefore, the proof is complete by letting

$$|E_r^{\mathrm{maj}}(n)| \leq \frac{(\mathcal{X}_r(n)+\mathcal{Y}_r(n))e^{2AN_n^2}}{N_n\pi}+|\mathcal{Z}_r(n)|,$$

where $\mathcal{Z}_r(n)$ is defined by (2.1.13). □

Putting everything into (2.1.18) gives us the asymptotic formula as mentioned in Theorem 2.1. □

2.2 Hankel Determinant

For a given real polynomial $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$, let $S_k := \sum_{i=1}^{d} \lambda_i^k$ be the kth power sum of real roots. Then take

$$\Delta_m(P(x)) := \begin{vmatrix} S_0 & S_1 & \cdots & S_{m-1} \\ S_1 & S_2 & \cdots & S_m \\ S_2 & S_3 & \cdots & S_{m+1} \\ \vdots & \vdots & \vdots & \vdots \\ S_{m-1} & S_m & \cdots & S_{2m-2} \end{vmatrix} = \sum_{i_1<i_2<\cdots<i_m} \prod_{a<b} (\lambda_{i_a} - \lambda_{i_b})^2. \tag{2.2.1}$$

For convenience, we define

$$D_{d,m}(P(x)) = D_{d,m}(a_0, a_1, \cdots, a_d) := a_d^{2m-2} \cdot \Delta_m(P(x)), \tag{2.2.2}$$

so that $D_{d,d}(a_0, a_1, \cdots, a_d)$ is the discriminant of $P(x)$, and $D_{d,m}(a_0, a_1, \cdots, a_d)$ is a homogeneous polynomial of degree $2m-2$ in the coefficients a_i. A theorem of Hermite [Obr03] says the hyperbolicity of $P(x)$ is implied by the positivity of the so-called "Hankel determinants" $D_{d,m}(a_0, a_1, \cdots, a_d)$ for all $2 \le m \le d$.

We will prove Theorems 1.1 and 1.2 by showing that

$$\begin{aligned} \mathcal{D}_{d,\mathrm{PL},m}(n) &:= D_{d,m}\left(\frac{J_{\mathrm{PL}}^{d,n}(x)}{\mathrm{PL}(n)}\right) \qquad (2.2.3)\\ &= D_{d,m}\left(1, \binom{d}{1}\frac{\mathrm{PL}(n+1)}{\mathrm{PL}(n)}, \binom{d}{2}\frac{\mathrm{PL}(n+2)}{\mathrm{PL}(n)}, \cdots, \binom{d}{d}\frac{\mathrm{PL}(n+d)}{\mathrm{PL}(n)}\right) > 0, \end{aligned}$$

for each $m = 2, \cdots, d$ and all n greater than $N_{\mathrm{PL}}(d)$ mentioned in the theorem. Note that the limit $\lim_{n\to\infty} \frac{\mathrm{PL}(n+j)}{\mathrm{PL}(n)} = 1$ for a fixed j, which implies that $\lim_{n\to\infty} J_{\mathrm{PL}}^{d,n}(x)/\mathrm{PL}(n) =$

$(x+1)^d$. This implies that $\mathcal{D}_{d,\mathrm{PL},m}$ approaches 0 in the limit as $n \to \infty$, which a priori, makes the sign of $\mathcal{D}_{d,\mathrm{PL},m}(n)$ difficult to determine.

However, we can determined the rate at which $\mathcal{D}_{d,\mathrm{PL},m}(n)$ approaches 0 and the coefficient of the leading term using the results in [OPR22] and [GORZ19]. More precisely, we have that

$$\lim_{n\to\infty} \frac{1}{\delta(n)^{m(m-1)}} \Delta_m \left(\frac{J_{\mathrm{PL}}^{d,n}(x)}{\mathrm{PL}(n)} \right) \tag{2.2.4}$$
$$= \lim_{n\to\infty} \Delta_m \left(\frac{J_{\mathrm{PL}}^{d,m}\left(\delta(n)x - e^{-\sqrt{3A}w(n)}\right)}{\mathrm{PL}(n)} \right) = \Delta_m(H_d(x)).$$

If we do a change of variable using (3.1.1) and use $\mathcal{D}_{d,\mathrm{PL},m}$ notation, then this translates to

$$\lim_{w\to 0} \frac{1}{w^{2m(m-1)}} \mathcal{D}_{d,\mathrm{PL},m}(n) = \left(\frac{\sqrt{3A}}{2} \right)^{m(m-1)} \Delta_m(H_d(x)). \tag{2.2.5}$$

Since Hermite polynomials have distinct real roots, the right hand side is a positive constant. We will exploit this fact by using $(2m(m-1)+1)$-Taylor polynomial of $\mathcal{D}_{d,\mathrm{PL},m}(n)$ around 0. The constant term in the left hand side will be a constant multiple of $\Delta_m(H_d(x))$. We will then find explicit bounds for the remaining terms that are tending to zero.

2.3 Classical Modular forms

Modular forms are ubiquitous in mathematics. It plays an important role in number theory, representation theory, combinatorics, and mathematical physics to name a few. A recent example of their relevance in mathematics is Maryna Viazovska's Fields medal winning solution to the sphere packing problem in dimension 8 and 24.

We will only recall facts about integer weight modular forms since they are enough for our work. (For background, please see [CS17], [Kob12], [Miy06])

2.3.1 Modular group

Let $\mathbb{H}$ denote the upper half complex plane and let $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ denote the full modular group, the set of 2×2 integer matrices with determinant 1. Γ acts on $\mathbb{H}$ by linear fractional transformations, given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ acts on z as

$$gz := \frac{az+b}{cz+d},$$

since we have

$$\Im(gz) = \Im\frac{az+b}{cz+d} = \Im\frac{az+b)(c\bar{z}+d}{|cz+d|^2} = \frac{\Im(adz+bc\bar{z})}{|cz+d|^2} = \frac{\Im(z)}{|cz+d|^2}.$$

Γ is generated by

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and they act on $\mathbb{H}$ by $z \to z+1$ and $z \to -1/z$ respectively. We also need to define the congruence subgroups of Γ :

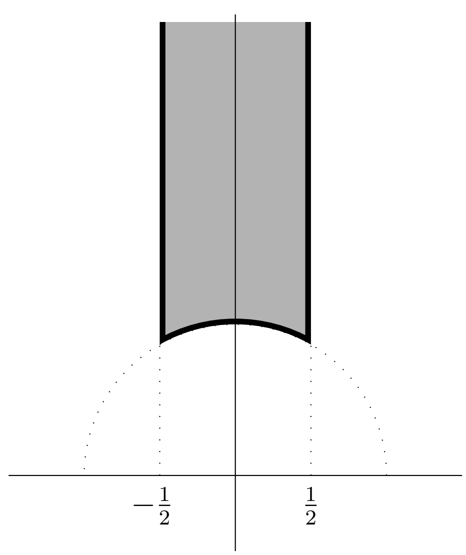
$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod N \right\}, \tag{2.3.1}$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod N \right\}, \tag{2.3.2}$$

for every $N \in \mathbb{N}$. Any subgroup of Γ containing one of these subgroups is called *congruence subgroup of level N*.

Whenever a group acts on a set, it divides it into equivalence classes, where two points are said to be in same equivalence class if there is an element in the group that takes one point to the other. For the action of Γ on $\mathbb{H}$, we get a fundamental domain, a set of points all Γ non-equivalent points,

$$F := \left\{ z \in \mathbb{H} : -\frac{1}{2} \leq \Re(z) < \frac{1}{2} \text{ and } |z| \geq 1 \right\}. \tag{2.3.3}$$



The action of a congruent subgroup naturally extends to $\mathbb{Q} \cup i\infty$. A *cusp* of a congruent subgroup Γ' is an equivalent class of $\mathbb{Q} \cup i\infty$ under its action.

2.3.2 Modular forms

We start with Dirichlet characters.

Definition 2.9 (Dirichlet character)**.** *A complex-valued arithmetic function $\chi : \mathbb{Z} \to \mathbb{C}$ is called a Dirichlet character modulo a positive integer N if for all $a, b \in \mathbb{Z}$ the following are true :*

- $\chi(ab) = \chi(a)\chi(b)$,
- $\chi(a) = 0 \iff \gcd(a, N) \neq 0$,
- $\chi(a + N) = \chi(a)$.

Now we define modular form.

Definition 2.10 (Modular form)**.** *Let $f(z)$ be a holomorphic function on $\mathbb{H}$, χ a Dirichlet character modulo N and let k a positive integer. We say that $f(z)$ is a modular form of weight k on a congruent subgroup Γ' of level N if*

- $f(gz) = \chi(d)(cz + d)^k f(z)$ *for all* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$,
- $f(z)$ *grows at most polynomially as* $\Im(z) \to \infty$, *and analogues condition holds at other cusp of* Γ'.

The set of all such modular forms is denoted by $M_k(\Gamma', \chi)$.

Since $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma'$, first condition implies that it has a Fourier expansion of the form $f(z) = \sum_{n=0}^{\infty} a_f(n) q_N^n; q_N := e^{2\pi i z/N}$. In particular, modular forms over $\Gamma_1(N)$ have a Fourier expansion of the form $f(z) = \sum_{n=0}^{\infty} a_f(n) q^n$; $q := e^{2\pi i z}$. A modular form that vanishes at cusps is called *cusp form*. Set of all such forms on Γ' of level N and character χ is denoted by $S_k(\Gamma', \chi)$

Remark 2.11. (1) Our definition of a modular form is the same as holomorphic modular form in most text books. In general one can relax the holomorphicity condition to meromorphicity condition.

(2) Both $M_k(\Gamma', \chi)$ and $S_k(\Gamma', \chi)$ are finite dimensional spaces.

Examples 2.12. (1) Modular forms can often be constructed by averaging some function over the action of modular group. The simplest examples arise from

$$E_k(z) := \frac{1}{2\zeta(k)} \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(mz+n)^k},$$

the Eisenstein series. For even $k > 2$, this is a modular form of weight k with respect to the full modular group. Its Fourier series is given as follows:

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k is the k-th Bernoulli number and $\sigma_{k-1}(n)$ is the $(k-1)$th divisor sum of n given by :

$$\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}.$$

(2) It turns out that E_4 and E_6 generate all modular forms on full modular group as an isobaric polynomial ring over $\mathbb{C}$. This means that any weight k modular form can be written as a polynomial in E_4 and E_6 such that each monomial $a_{i,j}E_4^iE_6^j$ has weight k i.e. $4i+6j=k$. One of the most fundamental and maybe the prototypical example of a cusp form is the Δ function defined as:

$$\Delta(z) := E_4(z)^3 - E_6(z)^2.$$

As one can easily see from the q-expansion of E_k that there is constant term in the Fourier expansion of Δ. It has an Euler product form given by:

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n := q\prod_{n\geq 1}(1-q^n)^{24},$$

where $\tau(n)$ is the well known Ramanujan τ-function.

2.3.3 Hecke operators

$M_k(\Gamma', \chi)$ and $S_k(\Gamma', \chi)$ form vector spaces over $\mathbb{C}$ and one can define many types of linear operators on them. For our work we need the Hecke operators which are an infinite family of operators, one for each positive integer n coprime to the level of Γ'. They are the most important operators when studying modular forms.

There are many ways to define Hecke operators. Here we define it by direct formulas, for its action on a modular form, which is best for our purposes.

Definition 2.13. *Suppose that $f(z) = \sum_{m=0}^{\infty} a_f(n)q^n \in M_k(\Gamma_0(N), \chi)$. For each positive integer n coprime to N, the n-th Hecke operator, denoted as T_n, acts on $f(z)$*

by :

$$T_n f := \sum_{m=0}^{\infty} b(m) q^n, \tag{2.3.4}$$

where

$$b(m) := \sum_{d|\gcd(m,n), d>0} \chi(d) d^{k-1} a\left(\frac{mn}{d^2}\right). \tag{2.3.5}$$

We define $a(m/n) = 0$ when $m/n \notin \mathbb{Z}$.

It should be noted that these operators commute with each other i.e. $T_n T_m = T_m T_n$.

For a congruence subgroup Γ', one can define an inner product on $S_k(\Gamma', \chi)$. For $f, g \in S_k(\Gamma', \chi)$, we define the inner product as

$$\langle f, g \rangle_{\Gamma'} = \frac{1}{[\Gamma : \Gamma']} \int_{\Gamma' \backslash \mathbb{H}} f(\tau) \overline{g(\tau)} y^k \frac{dxdy}{y^2}, \tag{2.3.6}$$

where x and y are real and imaginary parts of τ. The normalizing factor ensures that this inner product is independent of Γ'. It turns out that $T(n)$ are χ-Hermitian i.e. $\langle T(n)f, g \rangle = \chi(n) \langle f, T(n)g \rangle$. Because of this we can obtain simultaneous eigenvectors called *Hecke eigenforms* or simply *eigenform.* The coefficients of these Hecke eigenforms act multiplicatively when *normalized* (re-scaling $a_1 = 1$ which can be done since it turns out that $a_1 \neq 0$). We summarize these facts in the following proposition.

Proposition 2.14. *Suppose that $f(z) = \sum_{m=0}^{\infty} a_f(n) q^n \in M_k(\Gamma_0(N), \chi)$ is a normalized Hecke eigenform. Then the following is true:*

(1) If $\gcd(n, m) = 1$ then

$$a_f(nm) = a_f(n) a_f(m)$$

(2) For primes $p \nmid N$ and $\alpha > 0$, we have

$$a_f(p^\alpha) = a_f(p)a_f(p^{\alpha-1}) - \chi(p)p^j a_f(p^{\alpha-2})$$

(3) For p prime and $\alpha > 0$, we have

$$a_f(p^\alpha) = a_f(p)^\alpha \pmod{p}.$$

Remark 2.15. All the coefficients $a_f(p)$ are algebraic integers in a fixed number field so (3) makes sense.

Next we consider a very special type of modular forms which arise from orders of imaginary quadratic extensions.

2.3.4 Hecke Grössencharacters and Hecke L-functions

Hecke Grössencharacters are a natural generalization of Dirichlet characters to other number fields in the sense that one can think of Dirichlet characters as homomorphisms defined over $\mathbb{Z}$ which is the ring of integer of $\mathbb{Q}$. This was introduced by Erich Hecke to construct a class of L-functions larger than Dirichlet L-functions, and a natural setting for the Dedekind zeta-functions and certain others which have functional equations analogous to that of the Riemann zeta-function.

Let K be a number field of degree n over $\mathbb{Q}$ with $k_1, k_2, \cdots k_n$ embeddings in $\mathbb{C}$, r_1 real and $2r_2$ imaginary ones. Let $\mathcal{O}_K$ be its ring of integer. We have that $n = r_1 + 2r_2$ and we assume that first r_1 embeddings are real and next r_2 are imaginary with following next being conjugates of these r_2 (for $r_1+1 \leq i \leq r_2$, $k_{i+r_2} = \overline{k_i}$). For a non-zero integral ideal $\mathfrak{f}$ of $\mathcal{O}_K$, we put

$$I(\mathfrak{f}) := \{\mathfrak{a} \text{ integral ideal } : (\mathfrak{a}, \mathfrak{f}) = 1\}$$

$$P(\mathfrak{f}) := \{(a) : a \in \mathcal{O}_K \text{ and } a \equiv 1 (\text{mod}^* \mathfrak{f})\},$$

where $\alpha \equiv 1 \pmod{^*\mathfrak{f}}$ means $\nu_{\mathfrak{p}}(\alpha - 1) \geq \nu_{\mathfrak{p}}(\mathfrak{f})$ for all $\mathfrak{p}|\mathfrak{f}$.

Definition 2.16. *Let $\xi : I(\mathfrak{f}) \to \mathbb{C}^\times$ be a group homomorphism. We say that ξ is a* Hecke Character (or a Grösencharacter) *modulo $\mathfrak{f}$ if there exists a group homomorphism $\xi_\infty : K^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ such that we have*

$$\xi(\alpha\mathcal{O}_D) = \xi_\infty(\alpha) := \prod_{v=1}^{r_1+r_2} \left(\frac{k_v(\mathfrak{a})}{|k_v(\mathfrak{a})|}\right)^{u_v} |k_v(\mathfrak{a})|^{i\nu_v} \text{ for all } \alpha \in K^\times \text{ such that } \alpha \equiv 1 \pmod{^*\mathfrak{f}},$$

with real numbers $u_v, \nu_v (1 \leq v \leq r_1 + r_2)$ such that

(i) $u_v \in \begin{cases} \{0,1\} & \text{if } v \leq r_1, \\ \mathbb{Z} & \text{if } v \geq r_1, \end{cases}$

(ii) $\sum_{v=1}^{r_1+r_2} \nu_v = 0,$

where $\alpha \equiv 1 \pmod{^*\mathfrak{f}}$ means $\nu_{\mathfrak{p}}(\alpha - 1) \geq \nu_{\mathfrak{p}}(\mathfrak{f})$ for all $\mathfrak{p}|\mathfrak{f}$.

We have the following immediate lemma about Hecke characters.

Lemma 2.17. *Let ξ be a Hecke character modulo $\mathfrak{f}$, and for any $\alpha \in K^\times$ coprime to $\mathfrak{f}$ set $\xi_f(\alpha) = \xi(\alpha\mathcal{O}_D)/\xi_\infty(\alpha)$. Then ξ_f induces an ordinary character on the finite abelian group $(\mathcal{O}_D/\mathfrak{f})^\times$, so that $\xi(\alpha\mathcal{O}_D) = \xi_f(\alpha)\xi_\infty(\alpha)$.*

Remark 2.18. (1) The above definition is true for any general number field.

(2) As usual we will set $\xi(\mathfrak{a}) = 0$ if $\mathfrak{a} \notin I(\mathfrak{f})$ and likewise $\xi_f(\alpha) = 0$ if α is not coprime to $\mathfrak{f}$. The charcters ξ_f and ξ_∞ are called the finite and infinite parts of ξ, respectively.

We define *Hecke L-function* for ξ by

$$L(s,\xi) := \sum_{0\neq \mathfrak{a} \text{ integral ideal}} \xi(\mathfrak{a})\mathcal{N}(a)^{-s} = \prod_{\mathfrak{p} \text{ prime ideal}} (1-\xi(\mathfrak{p})\mathcal{N}^{-s})^{-1}, \tag{2.3.7}$$

where the Euler product formula is valid for $\Re(s) > 1$. Hecke obtained the functional equation for any Heeke L-function by generalizing the proof for the Riemann zeta-function. We can get a functional equation for Hecke L-functions similar to Riemann zeta function. To write that we define Gauss sum of a Hecke character as

$$W(\xi) := \frac{\xi_\infty(b)}{\xi(\mathfrak{c})} \sum_{\bar{\mathfrak{a}} \in \mathfrak{c}/\mathfrak{f}\mathfrak{c}} \xi_f(\mathfrak{a}) e^{2\pi i \operatorname{tr}(\mathfrak{a}/(b))}, \tag{2.3.8}$$

where $\mathfrak{c}$ is any integral ideal such that $\mathfrak{D}\mathfrak{f}\mathfrak{c} = (b)$ is principle where $\mathfrak{D}$ is the different ideal. The value $W(\xi)$ is independent of the choice of $\mathfrak{c}, b$ and set of representatives of $\mathfrak{c}/\mathfrak{f}\mathfrak{c}$.

To state the functional equation for Hecke L-function $L(s,\xi)$,we put

$$\Lambda(s,\xi) := \left(\frac{2^{r_1}|\Delta_K|\mathcal{N}(\mathfrak{f})}{(2\pi)^n}\right)^{s/2} \prod_{v=1}^{r_1+r_2} \Gamma\left(\frac{n_v s + |u_v| + i\nu_v}{2}\right) L(s,\xi), \tag{2.3.9}$$

where $n_v = 1$ or 2 depending on $v \leq r_1$ or $v > r_1$ respectively. Now the functional equation for a Hecke L-function is as follows.

Theorem 2.19. *With notations as above, we have the following:*

(1) $\Lambda(s,\xi)$ is analytically continued to a meromorphic function on the whole s-plane,

and satisfies the functional equation

$$\Lambda(1-s,\xi) = T(\xi)\Lambda(s,\overline{\xi}),$$

where

$$T(\xi) = 2^{iv} i^{-u} W(\xi)/\mathcal{N}(\mathfrak{f})^{1/2},$$
$$u = \sum_{v=1}^{r_1+r_2} u_v,\ v = \sum_{v=1}^{r_1+r_2} \nu_v.$$

(2) If ξ is trivial then $\Lambda(s,\xi)$ is holomorphic except for simple poles at $s = 0,1$; otherwise $\Lambda(s,\xi)$ is entire. The function $\Lambda(s,\xi)$ is bounded on any set of the form

$$\{s \in \mathbb{C} : a \leq \Re(s) \leq b,\ \Im(s) \geq c\} \quad (a < b, c > 0).$$

(3) $L(s,\xi)$ is entire if ξ is non-trivial. If ξ is trivial, then $L(s,\xi)$ is holomorphic except for a simple pole at $s = 1$ with residue

$$\frac{2^{r_1+r_2}\pi^{r_2} R \cdot h(K)}{w\sqrt{|\Delta_K|}},$$

where w is the number of roots of unity contained in K and R is the regulator of K.

Given a formal q-series $f = \sum_{n=0}^{\infty} a_f(n)q^n$, one can attach an L-function to it defined by

$$L(s,f) := \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s},$$

and for each positive integer N,

$$\Lambda_N(s,f) := \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s) L(s,f).$$

Thses L-function have nice analytic properties and a lot about the cusp form can be deduced using this.

Next we present a theorem due to Weil which tells us when a q series is modular form by looking at its L-function.

Theorem 2.20 (Weil)**.** *Let k and N be positive integers, and χ a Dirichlet character modulo N such that $\chi(-1) = (-1)^k$. For two sequences $\{a_n\}_0^\infty$ and $\{b_n\}_0^\infty$ of complex numbers such that $a_n = O(n^v)$, $b_n = O(n^v)$ $(v > 0)$, put*

$$f(z) := \sum_{n=0}^{\infty} a_n e^{2\pi i z},$$

and

$$g(z) := \sum_{n=0}^{\infty} b_n e^{2\pi i z} \quad (z \in \mathbb{H}).$$

Then $f(z) \in M_k(N, \chi)$ and $g(z) \in M_k(N, \overline{\chi})$ if the following two conditions are satisfied:

(1) Both $\Lambda_N(s, f)$ and $\Lambda_N(s, g)$ can be analytically continued to the whole s-plane, satisfy the functional equation

$$\Lambda_N(s, f) = i^k \Lambda(k - s, g),$$

and the function

$$\Lambda_N(s, f) + \frac{a_0}{s} + \frac{a_1}{k - s}$$

is holomorphic on the whole s-plane and bounded on any vertical strip.

(2) For any primitive dirichlet character χ with conductor prime m_χ coprime to N,

and $\Lambda_N(s, f_\chi)$ *and* $\Lambda_N(s, g_{\overline{\chi}})$ *(*f_χ *is the twist of* f *by* χ *meaning* n*-th Fourier coefficient of* f_χ *is* $a_n\chi(n)$*) can be holomorphically continued to the whole* s*-plane, bounded on any vertical strip, and satisfies the functional equation:*

$$\Lambda_N(s, f_\chi) = i^k C_\chi \Lambda_N(k - s, g_{\overline{\chi}})$$

for some constant C_χ*. Moreover, if* $L(s, f)$ *is absolutely convergent at* $s = k - \delta$ *for* $\delta > 0$*, then* $f(z)$ *and* $g(z)$ *are cusp forms.*

We also need the following lemma.

Lemma 2.21. *Let* $\chi_d(*) = \left(\frac{d}{*}\right)$ *be the Kronecker symbol. We have the following:*

(1) $W(\chi_d) = \begin{cases} \sqrt{d} & \text{if } d > 0, \\ i\sqrt{|d|} & \text{if } d < 0 \end{cases},$

(2) If $(d, p) = 1$*, then we have* $W(\chi \circ \mathcal{N}_K) = \chi_d(p)\chi(|d|)W(\chi)^2$*, for any primitive Dirichlet character* χ *modulo* p*.*

Proof. Using Theorem 2.19 for $K = \mathbb{Q}(\sqrt{d})$ and character χ_d, since we have

$$\zeta_K(s) = \zeta(s)L(s, \chi_d),$$

where ζ is the Riemann zeta function and ζ_K is the Dedekind zeta function of K, we get $T(\chi_d) = 1$ by comparing the functional equations of both sides. Since $\chi_d(-1) = 1$ or -1 depending on K is real or imaginary extension respectively, this implies (1). Since

$$L(s, \chi \circ \mathcal{N}_K) = L(s, \chi)L(s, \chi\chi_d),$$

we obtain

$$T(\chi \circ \mathcal{N}_K) = T(\chi)T(\chi\chi_d).$$

This together with the property of Gauss sum that $W(\chi\chi') = \chi(m_{\chi'})\chi'(m_\chi)W(\chi)W(\chi')$, for Dirichlet characters χ and χ' with conductor m_χ and $m_{\chi'}$ respectively, and (1) gives us (2). $\square$

Assume $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic extension with discriminant Δ_D, $\mathcal{O}_D$ its ring of integers defined by (1.2.7) and $\mathfrak{f}$ an integral ideal of K. We let $I(\mathfrak{f})$ denote the group of all fractional ideals coprime to $\mathfrak{f}$.

For imaginary quadratic extensions, $\xi_\infty(\alpha) = (\alpha/|\alpha|)^u$ for some integer u. Since $\alpha\mathcal{O}_D = \beta\mathcal{O}_D$ if and only if $\alpha\beta^{-1}$ is a unit of K, a necessary (and sufficient) condition for this definition to make sense is that $\xi_\infty(\varepsilon) = 1$ for all units $\varepsilon \equiv 1 \pmod{^*\mathfrak{f}}$. For example, when $\Delta_D > 4$ the only units are $\varepsilon = \pm 1$, so if $\mathfrak{f}|2\mathcal{O}_D$, the condition is that u must be even, and otherwise there is no condition.

Now we define a family of q series which are essential for our purposes. We will show that these series are modular forms.

Definition 2.22. *If K be as above and ξ ia Hecke character modulo $\mathfrak{f}$ of trivial finite type and infinite type $\xi_\infty(\alpha) = (\alpha/|\alpha|)^{k-1}$, we set*

$$\Theta_{\xi,k}(z) := \sum_{\mathfrak{a} \text{ integral}} \xi(\mathfrak{a})\mathcal{N}(\mathfrak{a})^{(k-1)/2} q^{\mathcal{N}(\mathfrak{a})}.$$

In the case of the above theta function, the attached L-functios is called a *Hecke* L-function which is defined as:

Definition 2.23. *The Hecke L-function attached to Θ_ξ is defined as*

$$L_{\xi,k}(s) := L(s,\Theta_\xi) = \sum_{\mathfrak{a} \text{ integral}} \xi(\mathfrak{a})\mathcal{N}(\mathfrak{a})^{(k-1)/2}\frac{1}{\mathcal{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \frac{1}{1-\xi(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s+(k-1)/2}}.$$

Theorem 2.24. *If $k \geq 1$ and ξ be primitive with conductor $\mathcal{O}_D$, then the function $\Theta_\xi(z) \in M_k(\Gamma_0(|D|),\chi_D)$, where $\chi_D(\cdot) = \left(\frac{-D}{\cdot}\right)$. In addition, if $k \geq 2$, then $\Theta_\xi(z)$ is a cusp form.*

Proof. We will employ Theorem 2.20 to prove this. By definition, we have $L_{\xi,k}(s) = L(s-(k-1)/2,\chi)$. Let χ be a primitive Dirichlet character of prime conductor p coprime to N. Then since $L_{\Theta_\chi,k}(s) = L(s-(k-1)/2,\xi(\chi\circ\mathcal{N}_K))$, we see from Theorem 2.19 and Lemma 2.21 that

$$\begin{aligned}
\Lambda_N(s,\Theta_\chi) &= \left(\frac{p\sqrt{N}}{2\pi}\right)^s \Gamma(s)L\left(s-(k-1)/2,\xi(\chi\circ\mathcal{N}_K)\right) \\
&= \left(\frac{p\sqrt{N}}{2\pi}\right)^{\frac{k-1}{2}} \Lambda\left(s-(k-1)/2,\xi(\chi\circ\mathcal{N}_K)\right) \\
&= \left(\frac{p\sqrt{N}}{2\pi}\right)^{\frac{k-1}{2}} i^{k-1}\frac{\xi((p))\chi_D(p)\chi(N)W(\xi)W(\chi)^2}{p\mathcal{N}_K(\mathfrak{f})^{1/2}} \\
&\times \Lambda\left((k+1)/2-s,\overline{\xi}(\overline{\chi}\circ\mathcal{N}_K)\right) \\
&= i^k C_\chi \Lambda_N\left(k-s,g_{\overline{\chi}}\right),
\end{aligned}$$

where

$$C_\chi = \frac{\chi_D(p)\chi(N)W(\chi)^2}{p} = \chi_D(p)\chi(-N)\frac{W(\chi)}{W(\overline{\chi})},$$

$$g(z) = i^{-2k+1}\frac{W(\chi)}{\mathcal{N}_K(\mathfrak{f})^{1/2}}\sum_{\mathfrak{a}\text{ integral}} \overline{\xi}(\mathfrak{a})\mathcal{N}_K(\mathfrak{a})^{\frac{k-1}{2}}e^{2\pi i\mathcal{N}_K(\mathfrak{a})z}.$$

Therefore, Theorem 2.20 tells us that $\Theta_\xi \in M_k(N, \chi_D)$ and it's a cusp form if $k \geq 2$, since $L_{\xi,k}(s)$ is convergent for $\Re(s) > (k+1)/2$. $\square$

We have the following amazing theorem to tell when a modular form is a Hecke eigenform by looking at its L-function.

Theorem 2.25. *Let $0 \neq f(z) = \sum_{n=0}^{\infty} a_f(n) q^n \in M_k(\Gamma_0(N), \chi)$ with $a_f(1) \neq 0$. Then if the L-function of f has an Euler product then $f(n)$ is an eigenform.*

Proof. Assume that

$$L(s, f) = a_f(1) \prod_{p \text{ prime}} (1 - t(p)p^{-s} + \chi(p)p^{k-1-2s})^{-1},$$

is the Euler product. Put $t(n) = a_f(n)/a_f(1)$ for positive integers n. Then we see

$$\sum_{n=1}^{\infty} t(n) n^{-s} = \prod_{p \text{ prime}} (1 - t(p)p^{-s} + \chi(p)p^{k-1-2s})^{-1}.$$

One easily gets that

$$t(m)t(n) = \sum_{0<d|(m,n)} d^{k-1} \chi(d) t\left(\frac{mn}{d^2}\right).$$

Multiplying both sides by $a_f(l)$, we get

$$t(m)a_f(n) = \sum_{0<d|(m,n)} d^{k-1} \chi(d) a_f\left(\frac{mn}{d^2}\right).$$

The right-hand side is equal to the n-th Fourier coefficient of $f|T(m)$. Let $b(0)$ be the constant term of $f|T(m)$ then

$$(f|T(m))(z) - t(m)f(z) = b(0) - c(0).$$

Since $f|T(m) - t(m)f$ is an element of $S_k(\Gamma_0(N), \chi)$, we get $b(0) = c(0)$. Thus

$$f|T(m) = t(m)f.$$

$\square$

Due to above theorem and Definition 2.23 we have the following.

Theorem 2.26. *Θ_ξ is a Hecke eigenform.*

2.4 Gaussian Hypergeometric functions

Here we recall the necessary facts about hypergeometric functions. Hypergeometric functions have been studied from very long time. The term "hypergeometric series" was first used by John Wallis in his 1655 book Arithmetica Infinitorum. They were also studied by Leonhard Euler, but the first full systematic treatment was given by Carl Friedrich Gauss (1813). In nineteenth century, they were studied by Kummer and Riemann as well.

Definition 2.27. *Let $a, b \in \mathbb{R}$ and $c \in \mathbb{R}\backslash\mathbb{Z}^-$. The Gaussian hypergeometric function is defined as:*

$${}_2F_1\left(a, b; c; z\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

for $|z| < 1$, where $(s)_n$ is the Pochhammer symbol defined as

$$(s)_n = s(s+1)\cdots(s+n-1).$$

Examples 2.28.

(1)

$$
\begin{aligned}
{}_2F_1(1,1;2;-z) &= \sum_{n=0}^{\infty} \frac{(1)_n(1)_n}{(2)_n} \frac{(-z)^n}{n!} = \sum_{n=0}^{\infty} \frac{n!n!}{(n+1)!} \frac{(-z)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)} (-z)^n = \frac{\log(1+z)}{z}.
\end{aligned}
$$

(2)

$$
\begin{aligned}
{}_2F_1(a,b;b;z) &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(b)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} (a)_n \frac{z^n}{n!} \\
&= (1-z)^{-a}.
\end{aligned}
$$

Notation. For a positive integer r, we define the following special hypergeometric function

$$
\lambda_r(z) := {}_2F_1\left(\frac{1}{r}, 1-\frac{1}{r}; 1, z\right). \tag{2.4.1}
$$

Remarks 2.29. (1) The hypergeometric function ${}_2F_1\left(a,b;c;z\right)$ is a solution of Euler's hypergeometric differential equation

$$
z(1-z)\frac{d^2w}{dz^2} + [c-(a+b+1)z]\frac{dw}{dz} - abw = 0, \tag{2.4.2}
$$

which has three regular singular points: $0, 1$ and ∞.

(2) The above series converges absolutely and uniformly on compact sets in the unit disk $|z| < 1$ and using its integral representation it can be extended analytically to the region $\mathbb{C} \setminus [1,\infty)$, i.e., for $|\mathrm{Arg}(1-z)| < \pi$ (see for example [EMOT55]). On the line $[1,\infty)$, we extend the definition as:

$$
{}_2F_1\left(a,b;c;x\right) := \lim_{\varepsilon \to 0^+} {}_2F_1\left(a,b;c;x+i\varepsilon\right).
$$

Note that this makes ${}_2F_1$ well-defined on the whole complex plane, analytic on the region $|\mathrm{Arg}(1-z)| < \pi$ but discontinuous on the line $[1,\infty)$. Furthermore, the difference between the principal branches on the two sides of the branch cut is:

$$\begin{aligned}&\lim_{\varepsilon\to 0^+} {}_2F_1\left(a,b;c;x+i\varepsilon\right) - \lim_{\delta\to 0^+} {}_2F_1\left(a,b;c;x-i\delta\right)\\ &= \frac{2\pi i\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}(1-x)^{c-a-b}{}_2F_1\left(c-a,c-b;c-a-b+1;1-x\right)\end{aligned}$$

(see Section 15.2 of [DLMF])

(3) Note that in the definition of ${}_2F_1$, the arguments a and b are symmetric so we will swap them as the situation requires without additional comments.

We require the following classical hypergeometric transformation law which gives the solution of a hypergeometric function around ∞ in terms of solutions of hypergeometric functions around other two singularities, namely 0 and 1.

Proposition 2.30. *(Equation 15.10.33 of [DLMF])For $0 < |\mathrm{Arg}(1-z)| < \pi$, we have*

$$\begin{aligned}{}_2F_1\left(a,b;c;z\right) &= \frac{\Gamma(1-b)\Gamma(c)}{\Gamma(a-b+1)\Gamma(c-a)}\left(\frac{1}{z}\right)^a {}_2F_1\left(a-c+1,a;a-b+1;\frac{1}{z}\right)\\ &+\frac{\Gamma(1-b)\Gamma(c)}{\Gamma(a)\Gamma(c-a-b+1)}\left(1-\frac{1}{z}\right)^{c-a-b}\left(-\frac{1}{z}\right)^b {}_2F_1\left(c-a,1-a;c-a-b+1;1-\frac{1}{z}\right).\end{aligned}$$

Remark 2.31. In view of Remark 2.29, the above proposition can be extended to $x > 1$ in the following manner:

$$\begin{aligned}{}_2F_1\left(a,b;c;x\right) &= \frac{\Gamma(1-b)\Gamma(c)}{\Gamma(a-b+1)\Gamma(c-a)}\left(\frac{1}{x}\right)^a \lim_{\delta\to 0^+} {}_2F_1\left(a-c+1,a;a-b+1;\frac{1}{x}-i\delta\right)\\ &+\frac{\Gamma(1-b)\Gamma(c)}{\Gamma(a)\Gamma(c-a-b+1)}\left(1-\frac{1}{x}\right)^{c-a-b}\left(-\frac{1}{x}\right)^b {}_2F_1\left(c-a,1-a;c-a-b+1;1-\frac{1}{x}\right).\end{aligned}$$

2.5 Ramanujan's theory of elliptic functions to alternate bases

In his famous paper [Ram14], Ramanujan offered several beautiful series representations for $1/\pi$. He first stated three formulas, one of which is

$$\frac{1}{\pi}=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(6n+1)\left(\frac{1}{2}\right)_n^3}{(n!)^3 4^n}.$$

He then remarked that "There are corresponding theories in which q is replaced by one or other of the functions

$$q_1 := e^{-\pi K_1'/K_1},\ q_2 := e^{-2\pi K_2'/K_2\sqrt{3}},\ q_3 := e^{-2\pi K_3'/K_3},$$

where

$$K_1 = {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;k^2\right),\ K_2 = {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;k^2\right),\ K_3 = {}_2F_1\left(\frac{1}{6},\frac{5}{6};1;k^2\right)."$$

Here $K_j' = K_j(k')$, where $k' = \sqrt{1-k^2}$, and $0 \le k \le 1$. In the classical theory, the hypergeometric functions above are replaced by ${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right)$. 2). Ramanujan then offered 16 further formulas for $1/\pi$ that arise from these alternative theories, but he provides no details for his proofs. Ramanujan's formulas for $1/\pi$ were established in 1987 by J.M. and P.B. Borwein [BB87] and they also made remarkable advances towards Ramanujan's Theories. This alternate theory was further developed by Berndt, Bhargava and Garvan [BBG95]. Here we present some important results needed for our work.

Proposition 2.32. *(Theorem 9.5-6 of [BBG95]) If $\tau \in \mathbb{H}$ and γ satisfies*

$$\tau_4 = \frac{i}{\sqrt{2}} \frac{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-\gamma\right)}{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \gamma\right)},$$

then we have

$$j(\tau_4) = \frac{64(1+3\gamma)^3}{\gamma(\gamma-1)^2}. \tag{2.5.1}$$

Remark 2.33. In [BBG95], $j(\tau)$ is not explicitly calculated, but Theorems 9.5 and 9.6 of [BBG95] give formulas for $E_4(\tau)$ and $E_6(\tau)$, respectively, from which the formula for $j(\tau)$ is immediately deduced.

Proposition 2.34. *(Theorem 11.4-5 of [BBG95]) Let $\gamma \in \mathbb{C}$ and $\beta = -\gamma^3/2$. If we define*

$$\tau_6 = i \cdot \frac{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1, \beta\right)},$$

then we have

$$j(\tau_6) = \frac{1728}{1-(1-2\beta)^2} = \frac{-1728}{\gamma^3(2+\gamma^3)}. \tag{2.5.2}$$

The idea of the proof is to get a relation with classical setting when

$$\tau_2(\beta) := \frac{i}{2} \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, \beta\right)} = \frac{i}{2} \frac{\lambda_2(1-\beta)}{\lambda_2(\beta)}.$$

If we let $q = e^{2\pi i \tau_2}$, in which case Ramanujan gave [AB12, p. 126]

$$E_4(2\tau_2) = \lambda_2(\beta)^4(1-\beta+\beta^2) \tag{2.5.3}$$
$$E_6(2\tau_2) = \lambda_2(\beta)^6(1+\beta)\left(1-\frac{1}{2}\beta\right)(1-2\beta).$$

Using these results we prove Proposition 2.32. Proof of Proposition 2.34 is similar.

Proof of Proposition 2.32. We start with two identities from Ramanujan's second notebook [AB09, p. 94-95]

$$
\begin{aligned}
{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{2\beta}{1+\beta}\right) &= \sqrt{1+\beta}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\beta^2\right)\\
{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1-\beta}{1+\beta}\right) &= \sqrt{\frac{1}{2}(1+\beta)}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;1-\beta^2\right).
\end{aligned}
$$

Replacing x by $\sqrt{x}$ above gives us that

$$
\tau_4 = \frac{i}{\sqrt{2}}\frac{{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\beta\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1-\sqrt{\beta}}{1+\sqrt{\beta}}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right)} = 2\tau_2\left(\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right).
$$

Using (2.5.3), we get that

$$
\begin{aligned}
E_4(\tau_4) &= E_4\left(2\tau_2\left(\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right)\right) \qquad (2.5.4)\\
&= \lambda_2\left(\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right)^4\left(1-\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}+\left(\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right)^2\right)\\
&= \frac{1}{(1+\sqrt{\beta})^2}\lambda_2\left(\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right)^4(1+3\beta),
\end{aligned}
$$

and

$$
\begin{aligned}
E_6(\tau_4) &= E_6\left(2\tau_2\left(\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right)\right) \qquad (2.5.5)\\
&= \lambda_2\left(\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right)^6\left(1+\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right)\left(1-\frac{1}{2}\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right)\\
&\quad\times\left(1-2\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right)\\
&= \frac{1}{(1+\sqrt{\beta})^3}\lambda_2\left(\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right)^6(1-9\beta).
\end{aligned}
$$

Combining above two gives us

$$\begin{aligned} j(\tau) &= 1728 \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2} \\ &= \frac{64(1+3\beta)^3}{\beta(\beta-1)^2}, \end{aligned}$$

which proves the proposition. $\square$

Chapter 3

Higher Turán Inequalities for plane partitions

This chapter is dedicated towards proving Theorem 1.1 and Theorem 1.2 about higher Turán inequlities for plane partition. We start with giving a form of asymptotic formula for $\mathrm{PL}(n)$ which is useful for our purposes.

3.1 Asymptotic formula for $\mathrm{PL}(n)$

To obtain our results, we must make asymptotic formula for $\mathrm{PL}(n)$ explicit, and then make good choices of the parameters for our application. To this end, we make the following change of variables:

$$w(n) := \frac{2^{1/3}}{\sqrt{3}A^{1/6}n^{1/3}} \quad \text{and} \quad \delta(n) := \frac{\sqrt{3A}}{2}w(n)^2. \tag{3.1.1}$$

For our purpose, we restrict to the case when $w \in [0, \varepsilon_{r,d}]$, where

$$\varepsilon_{r,d} := d^{-2d}2^{-d(d-1)} \cdot \left(\frac{4e}{\sqrt{3A}}\right)^{-d(d-1)} \left(e^{\frac{\Gamma(2d^2)}{(2\pi)^{2d^2+2}}}6^{2d-2}(6A)^r(r+1)\right)^{-\frac{1}{3}} \tag{3.1.2}$$
$$\times \left(0.1485 \cdot 2^{14}(3A)^3\pi^3\right)^{-\frac{1}{3}},$$

corresponding to our eventual bound on $N_{\mathrm{PL}}(d)$ for right choice of r (depending on d), since we also want to give an upper bound on errors for $w \in [0, \varepsilon_{r,d}]$.

Theorem 3.1. *If $r \in \mathbb{Z}^+$ and $w := w(n)$, then for every $w \in [0, \varepsilon_{r,d}]$ we have*

$$\mathrm{PL}(n) = \widehat{\mathrm{PL}}_r(w) + E_r(w),$$

where

$$\begin{aligned}\widehat{\mathrm{PL}}_r(w) :=& \frac{e^{c+\frac{1}{w^2}} w^{\frac{25}{12}}}{2\pi} \sum_{s=0}^{r+1} \sum_{m=0}^{r+1} f_{s,m} w^{2s+2m} \\ :=& \frac{e^{c+\frac{1}{w^2}} w^{\frac{25}{12}}}{2\pi} \sum_{s=0}^{r+1} \sum_{m=0}^{r+1} (-1)^m \beta_s b_{s,m} \Gamma\left(m + \frac{1}{2}\right) 3^{s+m+\frac{25}{24}} A^{s+\frac{13}{24}} w^{2s+2m},\end{aligned} \tag{3.1.3}$$

and

$$|E_r(w)| \leq e^{c+\frac{1}{w^2}} \cdot C_r \cdot 2^{r+8+\frac{1}{24}} \pi^2 (3A)^{r+\frac{61}{24}} (r+2) \cdot w^{2r+5+\frac{1}{12}}. \tag{3.1.4}$$

Remark 3.2. We stress that $\widehat{\mathrm{PL}}_r(w(n)) \sim \mathrm{PL}(n)$ as $n \to \infty$.

Proof. First let's convert $\mathcal{X}_r, \mathcal{Y}_r$ and $\mathcal{Z}_r$ from n to w using (3.1.1).

$$\begin{aligned}\widehat{\mathcal{X}}_r(w) := \mathcal{X}_r\left(\frac{2}{3^{3/2}\sqrt{A} w^3}\right) &= e^{c+\frac{1}{3w^2}} 2^{r+\frac{49}{24}} C_r (3A)^{r+\frac{49}{24}} w^{2r+4+\frac{1}{12}} \\ &\leq e^{c+\frac{1}{3w^2}} \cdot C_r \cdot 2^{r+6+\frac{1}{24}} \pi^3 (3A)^{r+\frac{49}{24}} (r+2) \cdot w^{2r+4+\frac{1}{12}}.\end{aligned} \tag{3.1.5}$$

where by (2.1.8), we have

$$C_r \leq 2 \cdot e^{\max\{0.02, \frac{(r+1)}{2}\alpha_{r+1}\}} = \begin{cases} 2 \cdot e^{\frac{(r+1)}{2}\alpha_{r+1}} & \text{if } r \geq 22 \\ 2e^{0.02} & \text{otherwise,} \end{cases} \tag{3.1.6}$$

since $\alpha_{r+1} \geq \alpha_r$ for all $r > 21$ and $10^{-3} < \alpha_r < 10^{-17}$ for $r \leq 21$. We also have

$$\begin{aligned}
\widehat{\mathcal{Y}}_r(w) &:= \mathcal{Y}_r\left(\frac{2}{3^{3/2}\sqrt{A}w^3}\right) \qquad (3.1.7)\\
&= \Bigg| e^{c+\frac{1}{3w^2}}\left(2^{r+5}\pi^3\alpha_{r+2}(3A)^{r+2}w^{2r+4} + 10e^{-4.7\frac{1}{\sqrt{3A}w}}\right)\\
&\quad \times \left(2^{r+\frac{49}{24}}C_r(3A)^{r+\frac{49}{24}}w^{2r+\frac{49}{12}} + \sum_{s=0}^{r+1} 2^{s+\frac{1}{24}}\beta_s(3A)^{s+\frac{1}{24}}w^{2s+\frac{1}{12}}\right)\Bigg|\\
&\leq 2\cdot 2\cdot e^{c+\frac{1}{3w^2}}2^{r+5+\frac{1}{24}}\pi^3(3A)^{r+\frac{49}{24}}\alpha_{r+2}(r+2)w^{2r+4+\frac{1}{12}}\\
&\leq e^{c+\frac{1}{3w^2}}\cdot C_r\cdot 2^{r+6+\frac{1}{24}}\pi^3(3A)^{r+\frac{49}{24}}(r+2)\cdot w^{2r+4+\frac{1}{12}},
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\mathcal{Z}}_r(w) &:= \mathcal{Z}_r\left(\frac{2}{3^{3/2}\sqrt{A}w^3}\right) \qquad (3.1.8)\\
&= e^c\left(D_r\cdot\Gamma\left(r+\frac{5}{2}\right)(3w^2)^{r+\frac{5}{2}}e^{\frac{1}{w^2}} + 0.64\cdot 2^{r+1}e^{\frac{2}{3w^2}}\right)\sum_{s=0}^{r+1}\beta_s(3A)^{s+\frac{13}{24}}w^{2s+\frac{13}{12}}\\
&\leq e^{c+\frac{1}{w^2}}\cdot 2\cdot D_r\Gamma\left(r+\frac{5}{2}\right)3^{r+\frac{5}{2}+\frac{13}{24}}A^{\frac{13}{24}}(r+2)w^{2r+5+\frac{13}{12}}.
\end{aligned}$$

We investigate D_r. To this end we recall that

$$\chi_s(t) = \frac{v^{2s+\frac{25}{12}}\sqrt{2v+1}}{2\pi(v^2+v+1)},$$

where $t^2 = 3 - 2v - v^{-2}$. First, since we have that

$$\frac{dv}{dt} = \frac{tv^3}{1-v^3} = i\frac{v^2\sqrt{2v+1}}{v^2+v+1}, \qquad (3.1.9)$$

we get

$$\begin{aligned}\frac{d}{dt}v^k =& i\frac{kv^{k+1}\sqrt{2v+1}}{v^2+v+1}; && (3.1.10)\\ \frac{d}{dt}(2v+1)^k =& i\frac{(2k)(2v+1)^{k-1/2}v^2}{v^2+v+1}; && (3.1.11)\\ \frac{d}{dt}(v^2+v+1)^{-k} =& i\frac{(-k)v^2(2v+1)^{3/2}}{(v^2+v+1)^{k+2}}. && (3.1.12)\end{aligned}$$

The parameterization of the curve traced by v is given by $x \pm \sqrt{\sqrt{x}-x^2}, x \in [0,1]$ (see the proof of asymptotic formula for $\mathrm{PL}(n)$). Using Mathematica, one checks that

$$|v| \leq 1; \qquad 1 \leq |2v+1| \leq 3; \qquad 0.9621 \leq \left|v^2+v+1\right| \leq 3.$$

One can differentiate $\chi_s(t) = fgh$ with respect to t using the product rule (where $f = v^{k_1}, g = (2v+1)^{k_2}$ and $h = (v^2+v+1)^{k_3}$) and the composition rule and using (3.1.9)-(3.1.12), a simple induction shows that the n-th derivative of $\chi_s(t)$ has 3^n terms of the form $c_{k_1,k_2,k_3} \cdot \frac{v^{k_1}(2v+1)^{k_2}}{(v^2+v+1)^{k_3}}$. The values of c_{k_1,k_2,k_3} are bounded above by $\prod_{k=0}^{n}\left(2s+2+\frac{25}{12}+k\right)$, and maximizing the possible powers of each of the $v, 2v+1, v^2+v+1$, one gets that

$$\left|\frac{d^n}{dt^n}\chi_s(t)\right| \leq \frac{3^n 3^{\frac{3n}{2}+\frac{1}{2}}}{2\pi(0.9621)^{2n}}\prod_{k=0}^{n}\left(2s+2+\frac{25}{12}+k\right).$$

Hence, by the definition of D_r, we have that

$$D_r \leq \frac{3^{5r+10+\frac{1}{2}}}{2\pi(2r+4)!(0.9621)^{4r+8}}\prod_{k=0}^{2r+4}\left(2r+2+\frac{25}{12}+k\right). \qquad (3.1.13)$$

So we get that

$$\widehat{\mathcal{Z}}_r(w) \leq e^{c+\frac{1}{w^2}} \cdot 2 \cdot \frac{2^{2r+4}3^{5r+10+\frac{1}{2}}}{(0.9621)^{4r+8}} \Gamma\left(r+\frac{5}{2}\right) 3^{r+\frac{5}{2}+\frac{13}{24}} A^{\frac{13}{24}} (r+2) w^{2r+5+\frac{13}{12}} \tag{3.1.14}$$

$$\leq e^{c+\frac{1}{w^2}} \cdot 2 \cdot \frac{2^{2r+4}3^{5r+10+\frac{1}{2}}}{(0.9621)^{4r+8}} \Gamma\left(r+\frac{5}{2}\right) 3^{r+\frac{5}{2}+\frac{13}{24}} A^{\frac{13}{24}} (r+2) \cdot \varepsilon_{r,d} \cdot w^{2r+5+\frac{1}{12}}$$

$$\leq e^{c+\frac{1}{w^2}} \cdot C_r \cdot 2^{r+6+\frac{1}{24}} \pi^2 (3A)^{r+\frac{61}{24}} (r+2) \cdot w^{2r+5+\frac{1}{12}},$$

where the last inequality comes after substituting the value $\varepsilon_{r,d}$ and comparing with $\widehat{\mathcal{X}}_r$. We also have

$$\left|E^{\min}(n)\right| \leq e^{c+\frac{1}{w^2}} \cdot C_r \cdot 2^{r+6+\frac{1}{24}} \pi^2 (3A)^{r+\frac{61}{24}} (r+2) \cdot w^{2r+5+\frac{1}{12}}.$$

Everywhere above we are using the fact that w is small enough so that the dominant term is the term with the smallest power of w. This gives us

$$\begin{aligned} |E_r(w)| &\leq \left|E_r^{\text{maj}}(n) + E^{\min}(n)\right| \\ &\leq \left|\frac{(\widehat{\mathcal{X}}_r(w) + \widehat{\mathcal{Y}}_r(w))e^{2AN_n^2}}{N_n\pi}\right| + \left|\widehat{\mathcal{Z}}_r(w)\right| + \exp\left(\left(3A - \frac{2}{5}\right)\frac{1}{3Aw^2}\right) \\ &\leq 4 \cdot e^{c+\frac{1}{w^2}} \cdot C_r \cdot 2^{r+6+\frac{1}{24}} \pi^2 (3A)^{r+\frac{61}{24}} (r+2) \cdot w^{2r+5+\frac{1}{12}}. \end{aligned}$$

□

3.1.1 Approximation of ratios of plane partition

In this subsection, we give approximations for the ratios of the plane partitions with the error function. For each non-negative integers r and j, we define approximation function for $\frac{\mathrm{PL}(n+j)}{\mathrm{PL}(n)}$ by

$$R_r(j,w) := \frac{\widehat{\mathrm{PL}}_r\left(\frac{w}{\left(1+\frac{3^{3/2}\sqrt{A}}{2}jw^3\right)^{\frac{1}{3}}}\right)}{\widehat{\mathrm{PL}}_r(w)} \sim \frac{\mathrm{PL}(n+j)}{\mathrm{PL}(n)}. \tag{3.1.15}$$

In order to state precisely how well $R_r(j,w)$ approximates $\mathrm{PL}(n+j)/\mathrm{PL}(n)$, let's define

$$L_r(w) := \frac{E_r(w)}{\widehat{\mathrm{PL}}_r(w)} \leq \frac{\sqrt{3A}}{0.1485} \cdot C_r \cdot 2^{r+9+\frac{1}{24}}\pi^3(3A)^{r+1}(r+2) \cdot w^{2r+3}. \tag{3.1.16}$$

Then we have the following lemma.

Lemma 3.3. *For all $n \geq 1$, we have*

$$\left|\frac{\mathrm{PL}(n+j)}{\mathrm{PL}(n)} - R_r(j,w)\right| \leq R_r(j,w)\left|\frac{2L_r(w)}{1-L_r(w)}\right|.$$

Proof. We have that $\widehat{E}_r(w) = \mathrm{PL}(n) - \widehat{\mathrm{PL}}_r(w) = E_r^{\mathrm{maj}}(n) + E^{\mathrm{min}}(n)$. By direct calculations, we have

$$\begin{aligned}
\left|\frac{\mathrm{PL}(n+j)}{\mathrm{PL}(n)} - R_r(j,w)\right| &= \left|\frac{\mathrm{PL}(n+j)}{\mathrm{PL}(n)} - \frac{\widehat{\mathrm{PL}}_r\left(w(n+j)\right)}{\widehat{\mathrm{PL}}_r(w(n))}\right| \\
&= \frac{\widehat{\mathrm{PL}}_r\left(w(n+j)\right)}{\widehat{\mathrm{PL}}_r(w(n))}\left|\frac{1+\frac{\widehat{E}_r(w(n+j))}{\widehat{\mathrm{PL}}_r(w(n+j))}}{1+\frac{\widehat{E}_r(w(n))}{\widehat{\mathrm{PL}}_r(w(n))}} - 1\right| \\
&= R_r(j,w)\left|\frac{\frac{\widehat{E}_r(w+j)}{\widehat{\mathrm{PL}}_r(w(n+j))} - \frac{\widehat{E}_r(w(n))}{\widehat{\mathrm{PL}}_r(w(n))}}{1+\frac{\widehat{E}_r(w(n))}{\widehat{\mathrm{PL}}_r(w(n))}}\right| \leq R_r(j,w)\left|\frac{2L_r(w)}{1-L_r(w)}\right|.
\end{aligned}$$

□

To study the behavior of $\mathrm{PL}(n+j)/\mathrm{PL}(n)$ for large n, we want to study $R_r(j,w)$ near $w=0$. To this end, let $A_{r,s}(j,w)$ be a degree $s-1$ Taylor polynomial of $R_r(j,w)$. Applying Lemma 3.3 and Taylor's Theorem, we immediately obtain the following.

Lemma 3.4. *Let $n \geq 1$ and $w \in [0, \varepsilon]$ for some $0 < \varepsilon \leq \frac{2^{1/3}}{\sqrt{3}A^{1/6}}$. Then we have that*

$$\frac{\mathrm{PL}(n+j)}{\mathrm{PL}(n)} = A_{r,s}(j,w) + E_{r,s}(j,w)w^s,$$

where

$$|E_{r,s}(j,w)| \leq \frac{1}{s!} \cdot \sup_{x\in[0,\varepsilon]} \left|R_r^{(s)}(j,x)\right| + \sup_{x\in[0,\varepsilon]} \left|R_r(j,x)\frac{2L_r(x)}{x^s(1-L_r(x))}\right|. \tag{3.1.17}$$

In view of (3.1.17), for each choice of s, and $2r+3 \geq s$ we have that $E_{r,s}(j,w)$ is bounded. From here onward we make the choice of

$$s = 2d(d-1)+1 \text{ and } r = d(d-1). \tag{3.1.18}$$

We also denote

$$\varepsilon := \varepsilon_{d(d-1)}. \tag{3.1.19}$$

It is easy to see that

$$0 \leq |L_r(w)| < \frac{1}{2}, \quad w \in [0,\varepsilon], \tag{3.1.20}$$

and so we get

$$\left|\frac{\mathrm{PL}(n+j)}{\mathrm{PL}(n)} - R_r(j,w)\right| \leq R_r(j,w)\left|4L_r(w)\right|. \tag{3.1.21}$$

To use Lemma 3.4 effectively, we need to obtain a bound on derivatives of $R_r(j,w)$.

3.1.2 Bound on nth derivative of $R_r(j,w)$

The polynomial $\mathcal{D}_{d,\mathrm{PL},m}(n)$ is homogeneous of degree $2m-2$ in the coefficients of $J_{\mathrm{PL}}^{d,n}(X)/\mathrm{PL}(n)$ and homogeneous of degree $m(m-1)$ in its roots. So, it has the form

$$\mathcal{D}_{d,\mathrm{PL},m}(n) = \sum_{i_1+i_2+\cdots+i_{2m-2}=m(m-1)} A_{i_1,i_2,\cdots,i_{2m-2}} \cdot \prod_{k=1}^{2m-2} \binom{d}{i_k} \frac{\mathrm{PL}(n+d-i_k)}{\mathrm{PL}(n)} \tag{3.1.22}$$

for some constants $A_{i_1,i_2,\cdots,i_{2m-2}}$. To bound the errors when we expand in terms of w, we find bounds on the derivatives $R_r^{(s)}(j,w)$ for w in the interval $[0,\varepsilon]$. For convenience, let $t = t(j) := \frac{3^{3/2}\sqrt{A}}{2}j$.

Lemma 3.5. *Assume that $w \in [0,\varepsilon]$ with ε as above. Then for each $n \geq 1$, we have that*

$$\left|R_r^{(n)}(j,w)\right| \leq n!\binom{n+3}{3}e^{g(w)}e^{(3tw^2+3tw)n} \cdot t^{\frac{7n}{3}} \cdot \left((r+1)^2 \cdot 6213 \cdot \alpha_{\frac{n}{2}}\Gamma\left(n+\frac{13}{12}+1\right)\right)^n,$$

where $g(w) = \frac{(1+tw^3)^{2/3}-1}{w^2}$.

Proof. The idea of the proof is to use the product rule to split up $R_r(j,w)$ into four more manageable parts and use Faà di Bruno's formula for iterated applications of the chain rule to evaluate each part as needed. This formula says that for differentiable functions $f(x)$ and $g(x)$, we have

$$\frac{d^n}{dx^n}f(g(x)) = \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{m_1!m_2!\cdots m_n!} f^{(m_1+m_2+\cdots+m_n)}(g(x)) \prod_{j=1}^{n}\left(\frac{g^{(j)}(x)}{j!}\right)^{m_j}. \tag{3.1.23}$$

First we define

$$\begin{aligned} A :=& A(t,w) = e^{\frac{t^2w^4+2tw}{\left((1+tw^3)^{2/3}+\frac{1}{2}\right)^2+\frac{3}{4}}}, \\ B :=& B(t,w) = \frac{1}{(1+tw^3)^{25/36}}, \\ C :=& \tilde{C}_r(t,w) = \sum_{s=0}^{r+1}\sum_{m=0}^{r+1} f_{s,m}\left(\frac{w}{(1+tw^3)^{1/3}}\right)^{2s+2m}, \\ D :=& D(t,w) = \frac{1}{\sum_{s=0}^{r+1}\sum_{m=0}^{r+1} f_{s,m}w^{2s+2m}}. \end{aligned}$$

Then we have that

$$R_r^{(m)}(j,w) = \sum_{m_1+m_2+m_3+m_4=m} \frac{m!}{m_1!m_2!m_3!m_4!} \left(\frac{d^{m_1}A}{dw^{m_1}}\right) \left(\frac{d^{m_2}B}{dw^{m_2}}\right) \left(\frac{d^{m_3}C}{dw^{m_3}}\right) \left(\frac{d^{m_4}D}{dw^{m_4}}\right). \tag{3.1.24}$$

We will prove the bound on n-th derivative of A in full detail, and others will follow similarly. We use (3.1.23) with $f(x) = e^x$ and $g(w) = \frac{t^2w^4+2tw}{\left((1+tw^3)^{2/3}+\frac{1}{2}\right)^2+\frac{3}{4}}$, and find that

$$\frac{d^nA}{dw^n} = \frac{d^n}{dw^n} f(g(w)) = \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{m_1!m_2!\cdots m_n!} e^{g(w)} \prod_{k=1}^{n} \left(\frac{g^{(k)}(w)}{k!}\right)^{m_k}. \tag{3.1.25}$$

We write $g(w) = g_1(w) \cdot g_2(w)$, where $g_1(w) = t^2w^4+2tw$, and $g_2(w) = \frac{1}{\left((1+tw^3)^{2/3}+\frac{1}{2}\right)^2+\frac{3}{4}}$.

We find that

$$\left|g_1^{(n)}(w)\right| \leq 4!t^2, \tag{3.1.26}$$

and again using (3.1.23), we obtain

$$\begin{aligned}\left|g_2^{(n)}(w)\right| \leq &\sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{m_1!m_2!\cdots m_n!} \frac{(\sum m_i)!}{\left(\left((1+tw^3)^{2/3}+\frac{1}{2}\right)^2+\frac{3}{4}\right)^{\sum m_i}} \\ &\times \prod_{k=1}^{n}\left(2^{k+1}\left(\left(1+tw^3\right)^{\frac{2}{3}}+\frac{1}{2}\right)e^{(3tw^2+3tw)k}t^{\frac{k}{3}}\right)^{m_k} \\ \leq &\sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{m_1!m_2!\cdots m_n!} \frac{(\sum m_i)!}{\left(\left((1+tw^3)^{2/3}+\frac{1}{2}\right)^2+\frac{3}{4}\right)^{\sum m_i}} \\ &\times 2^{\sum im_i+m_i}\left(\left(1+tw^3\right)^{\frac{2}{3}}+\frac{1}{2}\right)^{\sum m_i} e^{(3tw^2+3tw)\sum im_i}\cdot t^{\frac{\sum im_i}{3}} \\ \leq &n!2^{2n}e^{(3tw^2+3tw)n}t^{\frac{n}{3}}\cdot \sum_{m_1+2m_2+\cdots+nm_n=n}\frac{(\sum m_i)}{m_1!m_2!\cdots m_n!} \\ \leq &n!2^{3n}e^{(3tw^2+3tw)n}t^{\frac{n}{3}},\end{aligned} \tag{3.1.27}$$

where we use the fact that the sum $\sum_{m_1+2m_2+\cdots+nm_n=n} 1$ is counting the number of ordered partitions of n. This gives us that

$$\begin{aligned}\left|g^{(n)}(w)\right| \leq &\sum_{m_1+m_2=n}\frac{n!}{m_1!m_2!}\cdot\frac{d^{m_1}g_1}{dw^{m_1}}\cdot\frac{d^{m_2}g_2}{dw^{m_2}} \\ \leq &\sum_{m_1+m_2=n}\frac{n!}{m_1!m_2!}(4!t^2)m_2!2^{3m_2}e^{(3tw^2+3tw)m_2}t^{\frac{m_2}{3}} \\ \leq &4!n!2^{3m}t^{2+\frac{n}{3}}e^{(3tw^2+3tw)n}\sum_{\substack{m_1+m_2=n\\ m_1\leq 4}}\frac{1}{m_1!} \\ \leq &n!2^{3n}41\cdot t^{2+\frac{n}{3}}e^{(3tw^2+3tw)n}.\end{aligned} \tag{3.1.28}$$

So, combining (3.1.25)−(3.1.28), we obtain

$$\left|\frac{d^nA}{dw^n}\right| \leq n!2^{3n}(41)^n e^{g(w)} \cdot t^{2n+\frac{n}{3}} e^{(3tw^2+3tw)n} \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{(m_1+m_2\cdots m_n)!}{m_1!m_2!\cdots m_n} \tag{3.1.29}$$

$$\leq e^{g(w)} n! e^{(3tw^2+3tw)n} 2^{4n}(41)^n t^{\frac{7}{3}n}.$$

Next, it can be shown that

$$\left|\frac{d^nB}{dw^n}\right| \leq n! e^{(3tw^2+3tw)n} t^{\frac{n}{3}}. \tag{3.1.30}$$

Now, we give bounds on C and D. First, we let $W(w) = \frac{w}{(1+tw^3)^{1/3}}$, then we have

$$\left|W^{(n)}(w)\right| \leq w\frac{d^n}{dw^n}\left(\frac{1}{(1+tw^3)^{1/3}}\right) + n\frac{d^{n-1}}{dw^{n-1}}\left(\frac{1}{(1+tw^3)^{1/3}}\right). \tag{3.1.31}$$

Direct calculation gives

$$\begin{aligned}\left|\frac{d^n}{dw^n}\left(\frac{1}{(1+tw^3)^{1/3}}\right)\right| \leq & \left| \sum_{m_1+2m_2+3m_3=n} \frac{n!}{m_1!m_2!m_3!}\binom{-1/3}{m_1+m_2+m_3} \right. \\ & \left. \times \frac{(3tw^2)^{m_2}(3tw)^{m_2}(t)^{m_3}}{(1+tw^3)^{\frac{1}{3}+m_1+m_2+m_3}} \right| \\ \leq & n! e^{3tw^2+3tw} t^{\frac{n}{3}}.\end{aligned}$$

Since $|w| < 1$, this implies that

$$\left|W^{(n)}(w)\right| \leq 2n! e^{3tw^2+3tw} t^{\frac{n}{3}}. \tag{3.1.32}$$

The definition of C gives

$$\begin{aligned}\left|\frac{d^nC}{dw^n}\right| \leq & \left| \sum_{\substack{s=0\\2s+2m\geq n}}^{r+1} \sum_{m=0}^{r+1} f_{s,m} \frac{(2s+2m)!}{n!} \right. \\ & \left. \times \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{m_1!m_2!\cdots m_n!} W^{2s+2m-n} \prod_{k=1}^{n} \left(2e^{3tw^2+3tw} t^{\frac{k}{3}}\right)^{m_k} \right| \\ \leq & 2^{2n} n! e^{(3tw^2+3tw)n} t^{\frac{n}{3}} (r+1)^2 \sum_{2s+2m=n} |f_{s,m}| .\end{aligned} \tag{3.1.33}$$

Here in the second inequality, we use that for $w \in [0, \varepsilon]$, the sum on right hand side is dominated by constant term. Now we look into C and D, which we need to bound $f_{s,m}$. First notice that $|\beta_s| \leq 1$ for $s \leq 54$, and for $s \geq 55$, we have

$$\begin{aligned}|\beta_s| = & \left| \frac{1}{s!} \sum_{m_1+2m_2+\cdots sm_s=s} \frac{s!}{m_1!m_2!\cdots m_s!} \prod_{k=1}^{s} (-\alpha_k)^{m_k} \right| \\ = & \left| -\alpha_s + \sum_{m_1+2m_2+\cdots(s-1)m_{s-1}=s} \frac{s!}{m_1!m_2!\cdots m_s!} \prod_{k=1}^{s-1} (-\alpha_k)^{m_k} \right| \leq \alpha^s,\end{aligned} \tag{3.1.34}$$

since there is an alternating signs in each terms and α_s dominates. Also, we have

$$\begin{aligned}|b_{s,m}| = & \left| \frac{1}{(2m)!} \frac{d^{2m}}{dy^{2m}} \left(\frac{(1+y)^{2s+2m+\frac{13}{12}}}{(3+2y)^{\frac{1}{2}}} \right) \Bigg|_{y=0} \right| \\ \leq & \frac{1}{(2m)!} \sum_{m_1+m_2=2m} \frac{(m_1+m_2)!}{m_1!m_2!} \Gamma\left(2s+2m-m_1+\frac{13}{12}+1\right) \Gamma\left(-\frac{1}{2}-m_2+1\right) \\ & \times \left(\frac{3}{2}\right)^{m_2} \sqrt{3} \\ \leq & \frac{1}{(2m)!} \Gamma\left(2s+2m+\frac{13}{12}+1\right) \Gamma\left(\frac{1}{2}\right) \sqrt{3} \sum_{m_1+m_2=2m} \frac{(2m)!}{m_1!m_2!} \left(\frac{3}{2}\right)^{m_2} \\ \leq & \frac{1}{(2m)!} \sqrt{3} \cdot \Gamma\left(2s+2m+\frac{13}{12}+1\right) \cdot \Gamma\left(\frac{1}{2}\right) \cdot \left(1+\frac{3}{2}\right)^{2m} \\ \leq & \frac{1}{(2m)!} \sqrt{3} \cdot \frac{5^{2m}}{2^{2m}} \cdot \Gamma\left(2s+2m+\frac{13}{12}+1\right) \cdot \Gamma\left(\frac{1}{2}\right) .\end{aligned} \tag{3.1.35}$$

So, we get that

$$\sum_{2s+2m=n} |f_{s,m}| \leq 5 \cdot 3^{\frac{n}{2}+\frac{25}{24}+\frac{1}{2}} \cdot \Gamma\left(n+\frac{13}{12}+1\right)\sqrt{\pi}A^{\frac{13}{24}}\alpha^{n}_{\frac{n}{2}}\left(\frac{25}{4}\right)^{n/2}. \tag{3.1.36}$$

Combining these facts we obtain

$$\left|\frac{d^n C}{dw^n}\right| \leq 2^{2n} n! e^{(3tw^2+3tw)n} t^{\frac{n}{3}}(r+1)^2\left(\frac{75}{A}\right)^{\frac{n}{2}} 3^{\frac{37}{24}} 5\Gamma\left(n+\frac{13}{12}+1\right)\sqrt{\pi}A^{\frac{13}{12}}\alpha^{n}_{\frac{n}{2}}. \tag{3.1.37}$$

Using a similar argument and the fact that $|D| \leq \frac{1}{0.1485}$ when $w \in [0, \varepsilon]$ using Mathematica , we get that

$$\left|\frac{d^n D}{dw^n}\right| \leq \frac{n!}{0.1485}\left(\frac{(r+1)^2 \cdot \sqrt{75} \cdot \sqrt{\pi}3^{\frac{37}{24}} \cdot 5 \cdot A^{\frac{13}{24}} \cdot 2\alpha_{\frac{n}{2}}}{0.1485}\right)^n \Gamma\left(n+\frac{13}{12}+1\right)^n. \tag{3.1.38}$$

Therefore, thanks to (3.1.24) we obtain

$$
\begin{aligned}
\left|R_r^{(n)}(j,w)\right| \leq & \sum_{m_1+\cdots+m_4=n} \frac{m!}{m_1!\cdots m_4!} e^{g(w)} m_1! e^{(3tw^2+3tw)m_1} 2^{4m_1}(41)^{m_1} t^{\frac{7}{3}m_1} \cdot m_2! \qquad (3.1.39)\\
& \times e^{(3tw^2+3tw)m_2} t^{\frac{m_2}{3}} 2^{2m_3} m_3! e^{(3tw^2+3tw)m_3} t^{\frac{m_3}{3}} (r+1)^2 \left(\frac{75}{A}\right)^{\frac{m_3}{2}} 3^{\frac{37}{24}}\\
& \times 5\cdot\Gamma\left(m_3+\frac{13}{12}+1\right)\sqrt{\pi}A^{\frac{13}{12}}\alpha_{\frac{m_3}{2}}^{m_3}\frac{m_4!}{0.1485}\cdot\Gamma\left(m_4+\frac{13}{12}+1\right)^{m_4}\\
& \times\left(\frac{(r+1)^2\cdot\sqrt{75}\cdot\sqrt{\pi}3^{\frac{37}{24}}\cdot 5\cdot A^{\frac{13}{24}}\cdot 2\alpha_{\frac{m_4}{2}}}{0.1485}\right)^{m_4}\\
\leq & n! e^{g(w)} e^{(3tw^2+3tw)n}\cdot t^{\frac{7n}{3}}\cdot\left(\frac{(r+1)^2\cdot\sqrt{75}\cdot\sqrt{\pi}3^{\frac{37}{24}}\cdot 5\cdot A^{\frac{13}{24}}\cdot 2\alpha_{\frac{n}{2}}}{0.1485}\right)^n\\
& \times\Gamma\left(n+\frac{13}{12}+1\right)^n\cdot\sum_{m_1+\cdots+m_4=n}1\\
\leq & n!\binom{n+3}{3} e^{g(w)} e^{(3tw^2+3tw)n}\cdot t^{\frac{7n}{3}}\cdot\left(\frac{(r+1)^2\cdot\sqrt{75}\cdot\sqrt{\pi}3^{\frac{37}{24}}\cdot 5\cdot A^{\frac{13}{24}}\cdot 2\alpha_{\frac{n}{2}}}{0.1485}\right.\\
& \left.\Gamma\left(n+\frac{13}{12}+1\right)\right)^n\\
\leq & n!\binom{n+3}{3} e^{g(w)} e^{(3tw^2+3tw)n}\cdot t^{\frac{7n}{3}}\cdot\left((r+1)^2\cdot 6213\cdot\alpha_{\frac{n}{2}}\cdot\Gamma\left(n+\frac{13}{12}+1\right)\right)^n.
\end{aligned}
$$

□

Thanks to (3.1.22), we want to estimate products of ratios of plane partition function values. Given $\underline{i} = (i_1, i_2, \cdots, i_{2m-2})$ with $i_1 + i_2 + \cdots + i_{2m-2} = m(m-1)$, let $T_{d,\mathrm{PL},m}(\underline{i}; w)$ be the degree $2m(m-1)$ Taylor polynomial of $\prod_{k=1}^{2m-2} R_r(d - i_k, w)$.

Lemma 3.6. *If $w \in [0, \varepsilon]$, then we have that*

$$\prod_{k=1}^{2m-2} \frac{\mathrm{PL}(n+d-i_k)}{\mathrm{PL}(n)} = T_{d,\mathrm{PL},m}(\underline{i}; w) + E_{d,\mathrm{PL},m}(\underline{i}; w) w^{2m(m-1)+1},$$

where

$$|E_{d,\mathrm{PL},m}(\underline{i};w)| \leq 2\cdot\left(e^{\frac{\Gamma(2d^2)}{(2\pi)^{2d^2+2}}}6^{2d-2}(6A)^r(r+1)(2d-2)\right)^{\frac{12}{13}}\left(0.1485\cdot 2^{12}(3A)^3\pi^3\right)^{\frac{12}{13}}.$$

Proof. By Lemma 3.4, we can write that

$$\prod_{k=1}^{2m-2}\frac{\mathrm{PL}(n+d-i_k)}{\mathrm{PL}(n)} = \prod_{k=1}^{2m-2}R_r(d-i_k,w)(1+U_{r,k}(w)) = \prod_{k=1}^{2m-2}R_r(d-i_k,w)+U_r(w),$$

where

$$\begin{aligned}
|U_r(w)| \leq& \prod_{k=1}^{2m-2}R_r(d-i_k,w)\left(\left(1+\frac{2L_r(w)}{1-L_r(w)}\right)^{2m-2}-1\right)\\
\leq& 2^{2m-2}\cdot(2m-2)\cdot 3^{2m-2}\cdot\left|\frac{2L_r(w)}{1-L_r(w)}\right| \leq 2^{2m-2}\cdot(2m-2)\cdot 3^{2m-2}\left|4\cdot L_r(w)\right|\\
\leq& 2^{2m-2}\cdot(2m-2)\cdot 3^{2m-2}16\frac{\sqrt{3A}}{0.1485}\cdot C_r\cdot 2^{r+7+\frac{1}{24}}\pi^3(3A)^{r+\frac{49}{24}}(r+2)w^{2r+3}.
\end{aligned}$$

Here we use that $|R_r(d-i_k,w)| \leq 2$ throughout and (3.1.16) in the last inequality.

If we choose $s = 2m(m-1)+1$, then we have that

$$\left|\frac{U_r(w)}{w^s}\right| \leq 2^{2m-2}\cdot(2m-2)\cdot 3^{2m-2}16\frac{\sqrt{3A}}{0.1485}\cdot C_r\cdot 2^{r+7+\frac{1}{24}}\pi^3(3A)^{r+\frac{49}{24}}(r+2)\cdot\varepsilon^2 \tag{3.1.40}$$

$$\begin{aligned}
&\leq 6^{2d-2}\cdot(2d-2)\cdot\frac{\sqrt{3A}}{0.1485}\cdot C_r\cdot 2^{r+11+\frac{1}{24}}\pi^3(3A)^{r+\frac{49}{24}}(r+2)\\
&\times\left(e^{\frac{\Gamma(2d^2)}{(2\pi)^{2d^2+2}}}6^{2d-2}(6A)^r(r+1)\right)^{-\frac{2}{3}}\left(72^{12}(3A)^3\right)^{-\frac{2}{3}}\\
&\leq\left(e^{\frac{\Gamma(2d^2)}{(2\pi)^{2d^2+2}}}6^{2d-2}(6A)^r(r+1)(2d-2)\right)^{\frac{1}{3}}\left(0.1485\cdot 2^{12}(3A)^3\pi^3\right)^{\frac{1}{3}},
\end{aligned}$$

where we get the last inequality from (2.1.8). On the other hand, using the product rule and Lemma 3.5 we obtain

$$\begin{aligned}
&\frac{1}{s!}\left|\frac{d^s}{dw^s}\prod_{k=1}^{2m-2} R_r(d-i_k,w)\right| \qquad (3.1.41)\\
&\quad\le e^{(2m-2)g(\varepsilon)}\left(e^{3t\varepsilon+3t\varepsilon^2}t^{\frac{7}{3}}(r+1)^2\cdot 6213\cdot\alpha_{\frac{s}{2}}\Gamma\left(s+\frac{13}{12}+1\right)\right)^s\\
&\quad\times\sum_{n_1+n_2+\cdots+n_{2m-2}=m(m-1)}\binom{n_1+3}{3}\binom{n_2+3}{3}\cdots\binom{n_{2m-2}+3}{3}.
\end{aligned}$$

The largest term in the sum on the right hand side occurs if each n_i is equal, which in turn is bounded by replacing each n_i with $m\ge\frac{m(m-1)}{2m-2}$. Counting the number of terms, we see that the sum is bounded above by

$$\begin{aligned}
&\sum_{n_1+n_2+\cdots+n_{2m-2}=2m(m-1)}\binom{n_1+3}{3}\binom{n_2+3}{3}\cdots\binom{n_{2m-2}+3}{3}\\
&\qquad\le\binom{m+4}{3}^{2m-2}\binom{2(m-1)(m+1)}{2m-3}\\
&\qquad\le\left(\frac{5}{2}m^3\right)^{2m-2}(2m^2)^{2m-2}=(5m^5)^{2m-2}.
\end{aligned}$$

This shows that

$$\begin{aligned}
&\left|\frac{d^s}{dw^s}\prod_{k=1}^{2m-2}R_r(d-i_k,w)-T_{d,\mathrm{PL},m}(\underline{i};w)\right|\\
&\quad\le e^{(2m-2)g(\varepsilon)}\left(e^{3t\varepsilon+3t\varepsilon^2}t^{\frac{7}{3}}(r+1)^2\cdot 6213\cdot\alpha_{\frac{s}{2}}\right)^s\\
&\qquad\qquad\times\Gamma\left(s+\frac{13}{12}+1\right)^s\cdot(5m^5)^{2m-2}\cdot w^s+\\
&\quad+\left(e^{\frac{\Gamma(2d^2)}{(2\pi)^{2d^2+2}}}6^{2d-2}(6A)^r(r+1)(2d-2)\right)^{\frac{1}{3}}\left(0.1485\cdot 2^{12}(3A)^3\pi^3\right)^{\frac{1}{3}}w^s\\
&\quad\le 2\cdot\left(e^{\frac{\Gamma(2d^2)}{(2\pi)^{2d^2+2}}}6^{2d-2}(6A)^r(r+1)(2d-2)\right)^{\frac{1}{3}}\left(0.1485\cdot 2^{12}(3A)^3\pi^3\right)^{\frac{1}{3}}w^s,
\end{aligned}$$

where the above inequality follows by noticing that the second part of the sum is larger of the two. This is true since the second function has exponential growth rate and the first one has polynomial growth in the factorial, so we just need to check when second part becomes bigger than first one, which happens when $d \geq 4$. $\square$

In order to finish bounding the monomials in (3.1.22), we need the following result proved in a similar way as [LW19, Lemma 4.3].

Lemma 3.7. *Suppose $0 \leq m \leq d$ and $i_1 + i_2 + \cdots + i_{2m-2} = m(m-1)$ for positive integers i_k. Then we have that*

$$\left| \left(\frac{2}{\sqrt{3A}} \right)^{m(m-1)} \prod_{k=1}^{2m-2} \binom{d}{i_k} \right| \leq \left(\frac{4e}{\sqrt{3A}} \right)^{d(d-1)}.$$

Proof. The product $\prod_{k=1}^{2m-2} \binom{d}{i_k}$ is maximized when all of i_k are equal and equal to $\frac{m}{2}$. Using standard bounds on binomial coefficients, we therefore have that

$$\left(\frac{2}{\sqrt{3A}} \right)^{m(m-1)} \prod_{k=1}^{2m-2} \binom{d}{i_k} \leq \left(\frac{4ed}{\sqrt{3A}m} \right)^{m(m-1)} \leq \left(\frac{4e}{\sqrt{3A}} \right)^{d(d-1)},$$

since the right hand side is maximized when $m = d$. $\square$

We need one more lemma which gives the necessary bounds on the coefficients $A_{i_1,i_2,\cdots,i_{2m-2}}$ to achieve the required bound on $\mathcal{D}_{d,\mathrm{PL},m}(n)$.

Lemma 3.8 ([LW19], Lemma 4.4)**.** *If $n > \frac{2}{\sqrt{3A}\varepsilon^3}$ and $A_{i_1,i_2,\cdots,i_{2m-2}}$ is as in* (3.1.22), *then we have*

$$\sum_{i_1,i_2,\cdots,i_{2m-2}} |A_{i_1,i_2,\cdots,i_{2m-2}}| \leq d^{2d} \cdot 2^{d(d-1)}.$$

Proof. By the Newton-Girard identities, the power sums S_k in the matrix in (2.2.1) can be written as a sum of at most

$$S_k := k \sum_{r_1+2r_2+\cdots+kr_k=k} \frac{(r_1+r_2+\cdots+r_k-1)!}{r_1!\cdot r_2!\cdots r_k!} \leq k2^{k-1}$$

monomials in the coefficients of our polynomial. The determinant of the matrix in (2.2.1) is made up of a sum of at most $m!$ monomials of the form

$$\prod_{l=1}^{m} S_{i_l} \quad \text{where } i_1+i_2+\cdots+i_m = m(m-1).$$

Plugging in the elementary symmetric functions for each S_{i_l} in this product and expanding will express each of these "S-monomial" as a sum of at most

$$\prod_{l=1}^{m} i_l 2^{i_l-1} \leq (m-1)^m 2^{m(m-1)}$$

monomials in the coefficients. To obtain $\mathcal{D}_{d,\mathrm{PL},m}(n)$ from this, we must multiply by $\left(\frac{\mathrm{PL}(n+d)}{\mathrm{PL}(n)}\right)^{2m-2}$. Since n is so large, we easily have $\frac{\mathrm{PL}(n+d)}{\mathrm{PL}(n)} < 2$. Multiplying together the factors discussed above gives the result. $\square$

Because the limiting behavior of $J_{\mathrm{PL}}^{d,n}(x)$ is modeled by Hermite polynomials, we need the following lemma.

Lemma 3.9 ([LW19], Lemma 4.5)**.** *For each $m \leq d$, we have that $\Delta_m(H_d(x)) \geq 1$.*

Proof. We have that

$$\Delta_m(H_d(x)) = \sum_{i_1<i_2<\cdots<\ss_m} \prod_{a<b} (\lambda_a-\lambda_b)^2,$$

so by the inequality of the arithmetic and geometric mean

$$\Delta_m(H_d(x)) \geq \binom{d}{m} \prod_{i_1<i_2<\cdots<ß_m} \left(\prod_{a<b} (\lambda_a - \lambda_b)^2\right)^{\frac{1}{\binom{d}{m}}} = \binom{d}{m} \left(\prod_{j<k} (\lambda_j - \lambda_k)^{2\frac{d-2}{m-2}}\right)^{\frac{1}{\binom{d}{m}}}$$
$$= \binom{d}{m} \Delta_d(H_d(x))^{\frac{m(m-1)}{d(d-1)}}.$$

By Theorem 6.71 of [C.40], and the fact that $a_d(H_d(x)) = 2^d$, we have

$$\Delta_d(H_d(x)) = \frac{\mathrm{Disc}(H_d(x))}{2^{d(d-1)}} = 2^{-\frac{d(d-1)}{2}} \prod_{v=1}^{d} v^v \geq 1,$$

so the result follows. □

Now, proving Theorem 1.1 is just a matter of collecting and bounding all of the higher order terms from expanding $\mathcal{D}_{d,\mathrm{PL},m}(n)$ in terms of w. .

3.1.3 Proof of Theorem 1.1

Suppose that $n \geq \frac{2}{\varepsilon\sqrt{27A}}$ so that $w \in [0, \varepsilon]$. By (3.1.22), we have that

$$\frac{\mathcal{D}_{d,\mathrm{PL},m}(n)}{w^{2m(m-1)}} = \sum_{i_1+\cdots+i_{2m-2}=2m(m-1)} \frac{A_{i_1,\cdots,i_{2m-2}}}{w^{2m(m-1)}} \cdot \prod_{k=1}^{2m-2} \binom{d}{i_k} \Big(T_{d,\mathrm{PL},m}(\underline{i}; w)$$
$$+ E_{d,\mathrm{PL},m}(\underline{i}; w) w^{2m(m-1)+1}\Big)$$
$$= \left(\frac{\sqrt{3A}}{2}\right)^{m(m-1)} \Delta_m(H_d(x)) + w \cdot \mathcal{E}_{d,\mathrm{PL},m}(w),$$

where by Lemmas 3.6, 3.7, and 3.8 and the choice of ε, we have that

$$\left(\frac{2}{\sqrt{3A}}\right)^{m(m-1)} \cdot |\mathcal{E}_{d,\mathrm{PL},m}(w)| \cdot w \leq d^{2d} 2^{d(d-1)} \cdot \left(\frac{4e}{\sqrt{3A}}\right)^{d(d-1)} \cdot$$
$$2 \cdot \left(e^{\frac{\Gamma(2d^2)}{(2\pi)^{2d^2+2}}} 6^{2d-2} (6A)^r (r+1)(2d-2)\right)^{\frac{1}{3}} \left(0.1485 \cdot 2^{12} (3A)^3 \pi^3\right)^{\frac{1}{3}} \cdot \varepsilon \leq 1.$$

Since $\Delta_m(H_d(x)) \geq 1$, it follows that $\mathcal{D}_{d,\mathrm{PL},m}(n) > 0$ and therefore $J_{\mathrm{PL}}^{d,m}(x)$ is hyperbolic. We use (3.1.1) to get the upper bound on $N_{\mathrm{PL}}(d)$. $\square$

3.1.4 Proof of Theorem 1.2

We now prove Theorem 1.2 by bounding the error terms that accumulate from approximating $\mathrm{PL}(n+j)/\mathrm{PL}(n)$ by the $(s-1)$th Taylor polynomial $A_{r,s}(j,w)$ of $R_r(j,w)$ using Lemma 3.4, in the polynomial expression for $\mathcal{D}_{d,\mathrm{PL},m}(n(w))$. This gives us that there exists an ε such that $\mathcal{D}_{d,\mathrm{PL},m}(n(w)) \geq 0$ for all $w \in [0,\varepsilon]$ (i.e. $n \geq n_\varepsilon$) which in turn allows us to reduce to checking finitely many cases.

Using the Newton-Girard identities to write the power sum of the roots in terms of elementary symmetric function, one can generate symbolic expressions for the polynomials $D_{d,m}(a_0, a_1, \cdots, a_d)$ in terms of $a_0, a_1, \cdots, a_d$. To obtain $\mathcal{D}_{d,\mathrm{PL},m}(n)$, we substitute

$$\binom{d}{j}\left(A_{r,s}(j,w) + E_j w^s\right)$$

in for a_j in these polynomials, introducing E_j as a variable. For example, when $d = 3, r = 5$ and $s = 10$, we have that

$$D_{3,2}(a_0, a_1, a_2, a_3) = 2a_2^2 - 6a_1 a_3.$$

So we get that

$$\mathcal{D}_{3,\mathrm{PL},2}(n) = 18\left(A_{5,10}(w) + E_2 w^{10}\right)^2 - 18\left(A_{5,10}(w) + E_1 w^{10}\right)\left(A_{5,10}(w) + E_3 w^{10}\right).$$

This gives rise to a polynomial expression in w whose coefficients are polynomials in E_j. It turns out that all the coefficients less than $k = 2m(m-1)$ vanish in the expression. So diving by w^k, gives an expression of the form

$$\mathcal{D}_{d,\mathrm{PL},m}(w) = c_0 + c_1 w + c_2(E_1, E_2, \cdots, E_d)w^2 + \cdots + c_{(2m-2)s-k}(E_1, E_2, \cdots, E_d)w^{(2m-2)s-k}, \tag{3.1.42}$$

for each $2 \leq m \leq d$, where c_0 and c_1 are positive constants.

We use Mathematica [Inc] to calculate the upper bound on $E_j = E_{r,s}(j, w)$ for $w \in [0, \varepsilon]$ using Lemma 3.4, where we choose

$r = 5, 7, 10, 10, 10,\ s = 10, 12, 18, 18, 20$ and $\varepsilon = 0.051, 0.032, 0.06, 0.03, 0.02$

for $d = 3, 4, 5, 6, 7$ respectively.

With the help of Mathematica again, we minimize (3.1.42) using these bounds and it turns out that in each case the minimum is positive for all $2 \leq m \leq d$, which proves the hyperbolicity of $J_{\mathrm{PL}}^{d,n}(x)$ for $d = 3, 4, 5, 6$, and 7 for all $n \geq n_\varepsilon$.

To get this n_ε, we use the condition that $w \leq \varepsilon$ and the relation between w and n given by (3.1.1). This gives us that $N_{\mathrm{PL}}(3) \leq 2647$, $N_{\mathrm{PL}}(4) \leq 10714, N_{\mathrm{PL}}(5) \leq 1626, N_{\mathrm{PL}}(6) \leq 13003$ and $N_{\mathrm{PL}}(7) \leq 43883$. Checking the finite number of remaining possible counter examples directly now proves the theorem. Annotated Mathematica code to implement the full procedure described above is at [Pan22a].

Chapter 4

Ellipsoidal T-designs

This chapter is dedicated towards developing the theory of ellipsoidal designs and proving Theorem 1.9.

4.1 Criterion for ellipsoidal t-Design

In this section we prove Theorem 1.7, criterion for confirming ellipsoidal t-designs. Throughout this section we assume that $D \geq 1$ is square-free and $j \geq 1$.

To prove that Theorem 1.7 is indeed a criterion for confirming ellipsoidal t-designs, we first need to show that the spaces $H^{\mathbb{R}}_{D,k}[x,y]$, for $0 < k \leq j$, generate all the polynomials of degree $\leq j$ when restricted to $C_D(r)$. It suffices to show this for $P^{\mathbb{R}}_j[x,y]$, the set of homogeneous polynomials of degree j.

Lemma 4.1. *If $D \geq 1$ is square-free and $j \geq 1$, then the following are true:*

1) If $D \equiv 1, 2 \mod 4$, then we have

$$P^{\mathbb{R}}_j[x,y] = \bigoplus_{k=0}^{\lfloor j/2 \rfloor} \left(x^2 + Dy^2\right)^k H^{\mathbb{R}}_{D,j-2k}[x,y].$$

2) If $D \equiv 3 \mod 4$*, then we have*

$$P_j^{\mathbb{R}}[x,y] = \bigoplus_{k=0}^{\lfloor j/2 \rfloor} \left(x^2 + xy + \frac{1+D}{4} y^2 \right)^k H_{D,j-2k}^{\mathbb{R}}[x,y].$$

Proof. The lemma is well known for homogeneous harmonic polynomials (for example, see [ABR92, Thm 5.7]). Namely, if $H_k^{\mathbb{R}}[x,y]$ is the set of homogeneous harmonic polynomials of degree k then

$$P_j^{\mathbb{R}}(x,y) = \bigoplus_{k=0}^{\lfloor j/2 \rfloor} \left(x^2 + y^2 \right)^k H_{j-2k}^{\mathbb{R}}[x,y].$$

We extend it to general D. It is well known that $H_j^{\mathbb{R}}[x,y] = \langle \mathrm{Re}(x+iy)^j, \mathrm{Im}(x+iy)^j \rangle$, and so if we do the change of variable for $D \equiv 1, 2 \mod 4$ (resp. $D \equiv 3 \mod 4$), $x' = x$,$y' = \sqrt{D}y$ (resp. $x' = x + y/2$,$y' = 2y/\sqrt{D}$), then $H_{j-2}^{\mathbb{R}}(x', y') = \langle \mathrm{Re}(x'+iy')^j, \mathrm{Im}(x'+iy')^j \rangle$ gives

$$P_j^{\mathbb{R}}[x',y'] = \bigoplus_{k=0}^{\lfloor j/2 \rfloor} \left(x'^2 + y'^2 \right)^k H_{j-2k}^{\mathbb{R}}[x',y'].$$

Therefore, if $D \equiv 1, 2 \mod 4$, then we have

$$P_j^{\mathbb{R}}(x,y) = \bigoplus_{k=0}^{\lfloor j/2 \rfloor} \left(x^2 + Dy^2 \right)^k H_{D,j-2k}^{\mathbb{R}}[x,y].$$

If $D \equiv 3 \mod 4$, then we have

$$P_j^{\mathbb{R}}(x,y) = \bigoplus_{k=0}^{\lfloor j/2 \rfloor} \left(x^2 + xy + \frac{1+D}{4} y^2 \right)^k H_{D,j-2k}^{\mathbb{R}}[x,y].$$

$\square$

We now prove Theorem 1.7.

Proof of Theorem 1.7. Lemma 4.1 shows that the set of polynomials when restricted to C_D are generated by the spaces $H^{\mathbb{R}}_{D,j}[x,y]$ since $x^2+Dy^2=r$ $\Big($resp., $x^2+xy+\frac{1+D}{4}y^2=r\Big)$ on $C_D(r)$. Therefore, it suffices to show that if $P(x,y)\in H^{\mathbb{R}}_{D,j}[x,y]$, then the following are true:

1) If $D\equiv 1,2 \mod 4$, then we have

$$\int_{C_D(r)} \frac{P(x,y)}{\sqrt{x^2/D^2+y^2}}d\sigma(x,y)=0.$$

2) If $D\equiv 3 \mod 4$, then we have

$$\int_{C_D(r)} \frac{P(x,y)}{\sqrt{20x^2+(D^2+2D+5)y^2+(20+4D)xy}}d\sigma(x,y)=0.$$

As $H^{\mathbb{R}}_{D,j}[x,y]$ is a vector space, it is enough to show these claims for basis vectors. Since $X\subset C_D(r)$ is an ellipsoidal t-design if and only if $\frac{1}{r}\subset C_D(1)$ is an ellipsoidal t-design, it's enough to consider $r=1$. For $D\equiv 1,2 \pmod 4$, $H^{\mathbb{R}}_{D,j}[x,y]=\langle \mathrm{Re}(x+\sqrt{-D}y)^j, \mathrm{Im}(x+\sqrt{-D}y)^j\rangle$. By the parametrization of $C_D(1): x^2+Dy^2=1$ as $\gamma:=\{(\cos\theta,\sin\theta/\sqrt{D})|0\le\theta\le 2\pi\}$, we have

$$\begin{aligned}
&\int_{C_D(1)} \frac{\mathrm{Re}(x+\sqrt{-D}y)^j}{\sqrt{x^2/D^2+y^2}}d\sigma(x,y)\\
&\qquad=\int_0^{2\pi}\frac{\mathrm{Re}(\cos\theta+\sqrt{-D}(\sin\theta/\sqrt{D}))^j}{\sqrt{\cos\theta^2/D^2+\sin\theta^2/D}}\sqrt{\sin\theta^2+\cos\theta^2/D}d\theta\\
&\qquad=\sqrt{D}\int_0^{2\pi}\mathrm{Re}(\cos\theta+i\sin\theta)^j d\theta=\sqrt{D}\int_{S^1}\mathrm{Re}(x+iy)^j dz=0.
\end{aligned}$$

Since $\mathrm{Re}(x+iy)^j$ is harmonic, the last integral over S^1 is 0.

A similar argument shows that

$$\int_{C_D(1)} \frac{\mathrm{Im}(x+\sqrt{-D}y)^j}{\sqrt{x^2/D^2+y^2}}d\sigma(x,y)=0.$$

If $D \equiv 3 \pmod 4$, $H^{\mathbb{R}}_{D,j}[x,y] = \langle \mathrm{Re}(x+\frac{1+\sqrt{-D}}{2}y)^j, \mathrm{Im}(x+\frac{1+\sqrt{-D}}{2}y)^j \rangle$. By the parametrization of $C_D(1): x^2+xy+\frac{1+D}{4}y^2 = 1$ as $\gamma := \{(\cos\theta - \sin\theta/\sqrt{D}, 2\sin\theta/\sqrt{D}) : 0 \le \theta \le 2\pi\}$, we have

$$\begin{aligned}
&\int_{C_D(1)} \frac{\mathrm{Re}(x+(1+\sqrt{-D})y/2)^j}{\sqrt{20x^2+(D^2+2D+5)y^2+(20+4D)xy}} d\sigma(x,y) \\
=&\int_0^{2\pi} \frac{\mathrm{Re}(\cos\theta - \sin\theta/\sqrt{D} + (1+\sqrt{-D}\sin\theta/\sqrt{D})^j}{\sqrt{4D\sin\theta^2+20\cos\theta^2+8\sqrt{D}\sin\theta\cos\theta}} \\
&\quad \times \sqrt{\sin\theta^2+5\cos\theta^2/D+2\sin\theta\cos\theta/\sqrt{D}} d\theta \\
=&\frac{1}{2\sqrt{D}}\int_0^{2\pi} \mathrm{Re}(\cos\theta + i\sin\theta)^j d\theta = \frac{1}{2\sqrt{D}}\int_{S^1} \mathrm{Re}(x+iy)^j dz = 0.
\end{aligned}$$

A similar argument shows that

$$\int_{C_D(1)} \frac{P(x)}{\sqrt{20x^2+(D^2+2D+5)y^2+(20+4D)xy}} d\sigma(x,y) = 0.$$

$\square$

4.2 Ellipsoidal T-Designs

Here we prove Theorem 1.9, the construction of ellipsoidal T-designs arising from the ring of integers of imaginary quadratic fields with class number 1. We use the theory of theta functions with complex multiplication. Throughout, we shall assume that $D \in \{1,2,3,7,11,19,43,67,163\}$.

4.2.1 Theta functions

Given an n-dimensional lattice Λ and a polynomial $P(x)$ of degree j in n variables, the theta function of $P(x)$ over the lattice Λ is defined by the Fourier series (note $q := e^{2\pi i z}$)

$$\Theta(\Lambda, P; z) := \sum_{x\in\Lambda} P(x)q^{N(x)} = \Theta(\Lambda, P; z) = \sum_{n=0}^{\infty} a(\Lambda, P, n)q^n, \tag{4.2.1}$$

where $N(x)$ is the standard norm in $\mathbb{R}^n$. The theta functions for $\Lambda_D = \mathcal{O}_D$ play an important role in the study of ellipsoidal T-designs. Namely, if $\Theta(\Lambda_D, P; z) = \sum_{r=0}^{\infty} a(\Lambda_D, P, r)q^r$, then

$$a(\Lambda_D, P, r) = \sum_{(x,y)\in\Lambda_D^r} P(x, y). \tag{4.2.2}$$

The theta function $\Theta(\Lambda_D, P; z) \in \mathcal{M}_k(\Gamma_0(4D), \chi)$, the space of holomorphic modular forms with weight $k = j + 1$ and nebentypus $\chi(A) = (\frac{-D}{d})$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ [Iwa97, Thm 10.8]. Moreover, $\Theta(\Lambda_D, P; z)$ is a cusp form when $j > 0$.

To ease the study of these theta function, it is convenient to introduce the following the polynomials for each $j \geq 1$:

$$R_{D,j}(x, y) := \begin{cases} \mathrm{Re}\left(x + \sqrt{-D}y\right)^j & \text{if } D \equiv 1, 2 \pmod 4, \\ \mathrm{Re}\left(x + \frac{1+\sqrt{-D}}{2}y\right)^j & \text{if } D \equiv 3 \pmod 4, \end{cases} \tag{4.2.3}$$

and

$$I_{D,j}(x, y) := \begin{cases} \mathrm{Im}\left(x + \sqrt{-D}y\right)^j & \text{if } D \equiv 1, 2 \pmod 4, \\ \mathrm{Im}\left(x + \frac{1+\sqrt{-D}}{2}y\right)^j & \text{if } D \equiv 3 \pmod 4. \end{cases} \tag{4.2.4}$$

By definition, we have that $H^{\mathbb{R}}_{D,j}[x,y] = \langle R_{D,j}(x,y), I_{D,j}(x,y)\rangle$. Moreover $\Theta(\Lambda_D, R_{D,j}; z)$ and $\Theta(\Lambda_D, I_{D,j}; z)$ are cusp forms. Theorem 1.7 together with the discussion above gives the following lemma which transforms the problem of determining ellipsoidal T-designs into the vanishing of certain coefficients of special theta functions.

Lemma 4.2. *The norm r shell $\Lambda_D^r = \Lambda_D \cap C_D(r)$ is an ellipsoidal T-design if and only if $a(\Lambda_D, R_{D,j}, r) = 0$ and $a(\Lambda_D, I_{D,j}, r) = 0$ for all $j \in T$.*

We require some standard facts from the theory of newforms. Since $\mathcal{O}_D$ has class number 1, each *Hecke character* mod $\mathcal{O}_D$ is defined by its values on principal ideals. Let $(\alpha) \subset \mathcal{O}_D$ be a principal ideal. Let u_D be the number of units in $\mathcal{O}_D$, namely

$$u_D := \begin{cases} 4 & \text{if } D = 1, \\ 6 & \text{if } D = 3, \\ 2 & \text{otherwise.} \end{cases} \tag{4.2.5}$$

For each positive $j_D \equiv 0 \pmod{u_D}$, define Hecke characters mod $\mathcal{O}_D$ by:

$$\zeta_{j_D}((\alpha)) = \left(\frac{\alpha}{|\alpha|}\right)^{j_D}$$

Then by Theorem 2.24, we have the following well known lemma about the modular form

$$f_{j_D}(\zeta_{j_D}; z) := \begin{cases} \Theta\left(\Lambda_D, \left(x + \sqrt{-D}y\right)^j; z\right) & \text{if } D \equiv 1, 2 \pmod 4, \\ \Theta\left(\Lambda_D, \left(x + \frac{1+\sqrt{-D}}{2}y\right)^j; z\right) & \text{if } D \equiv 3 \pmod 4. \end{cases}$$

Thanks to Theorem 2.24 and 2.26, we have the following lemma.

Lemma 4.3. *Assuming the notations above, we have*

$$f_{j_D}(\zeta_{j_D};z) = \sum_{(\alpha)\subset\mathcal{O}_D} \zeta_{j_D}((\alpha))N(\alpha)^{j/2}q^{N(\alpha)} \in \mathcal{S}_{k_D}(\Gamma_0(N),\chi),$$

the space of cusp forms of weight $k_D = j_D + 1$ with nebentypus $\chi \pmod{N}$. Here $N := |\Delta_{\mathcal{O}_D}|$, the absolute value of the discriminant of $\mathcal{O}_D$. Moreover, $f_{j_D}(\zeta_{j_D};z)$ is a newform.

4.2.2 Other Propositions and Lemmas

Recall that $\Lambda_D^r = C_D(r)\cap\mathcal{O}_D$. Using well known facts about the positive definite binary quadratic forms corresponding to class number 1 norm forms, we have the following lemma.

Lemma 4.4. *Suppose r is a positive integer. Then Λ_D^r is nonempty if and only if $ord_p(r)$ is even for every prime $p \nmid r$ for which Λ_D^p is nonempty.*

Rewriting (4.2.2), we have

$$a(\Lambda_D, P, r) = \sum_{(x,y)\in\Lambda_D^r} P(x,y). \tag{4.2.6}$$

Lemma 4.2 implies that Λ_D^r is an ellipsoidal T-design if and only if $a(\Lambda_D, R_{D,j}, r)$ and $a(\Lambda_D, I_{D,j}, r)$ vanish for all $j\in T$. Since Λ_D^r is antipodal (*i.e.* $-\Lambda_D^r = \Lambda_D^r$ for all r), $a(\Lambda_D, R_{D,j}, r)$ and $a(\Lambda_D, I_{D,j}, r)$ are 0 for all $j\in\mathbb{Z}^+\setminus 2\mathbb{Z}^+$. Therefore, we have that following proposition.

Proposition 4.5. *Suppose $r\in\mathbb{Z}^+$ such that Λ_D^r is nonempty. Then Λ_D^r is an ellipsoidal $\mathbb{Z}^+\setminus 2\mathbb{Z}^+$-design.*

Our objective is to find maximal set T_D for which Λ_D^r is ellipsoidal T-design. By proposition above we have that $\mathbb{Z}^+ \setminus 2\mathbb{Z}^+ \subset T_D$. So we only look for all even j which can be in T_D.

Proposition 4.6. *Suppose $j \equiv 0 \pmod 2$, and $r \in \mathbb{Z}^+$. Then the following are true:*

1) We have that $a(\Lambda_D, I_{D,j}, r) = 0$.

2) We have that $a(\Lambda_D, R_{D,j}, r) = \begin{cases} \displaystyle\sum_{(x_0,y_0)\in\Lambda_D^r} \left(x+\sqrt{-D}y\right)^j & \text{if } D \equiv 1,2 \pmod 4, \\ \displaystyle\sum_{(x_0,y_0)\in\Lambda_D^r} \left(x+\frac{1+\sqrt{-D}}{2}y\right)^j & \text{if } D \equiv 3 \pmod 4. \end{cases}$

Proof. Part *(2)* is an obvious consequence of part *(1)*. So it is enough to prove part *(1)*. The idea is to show that points in Λ_D^r occur in pairs on which value of $I_{D,j}$ cancel. If $D \equiv 1,2 \pmod 4$, then $I_{D,j} = \mathrm{Im}(x+\sqrt{-D}y)^j$. In this case $(a,b),(a,-b) \in \Lambda_D^r$ such that $I_{D,j}(a,b) + I_{D,j}(a,-b) = 0$. This is true because each term of $I_{D,j}(x,y)$ has odd power in both the variables x, y. If $D \equiv 3 \pmod 4$, then $I_{D,j} = \mathrm{Im}((x+\frac{1}{2}y)+\frac{\sqrt{-D}}{2}y)^j$. In this case $(a,b),(a+b,-b) \in \Lambda_D^j$ such that $I_{D,j}(a,b) + I_{D,j}(a+b,-b) = 0$. This is because each term of $I_{D,j}(x,y)$ has odd power in $x+y/2, y$. $\square$

We notice that if $(x_0,y_0) \in \mathcal{O}_D$, then we have

$$\sum_{\alpha_D \in \mathcal{O}_D : |\alpha_D|=1} R_{D,j}(\alpha_D(x_0,y_0)) = R_{D,j}(x_0,y_0) \sum_{\alpha_D \in \mathcal{O}_D : |\alpha_D|=1} \alpha_D^j. \tag{4.2.7}$$

Proposition 4.7. *If $r \geq 1$, $1 \leq j \not\equiv 0 \pmod{u_D}$, and Λ_D^r nonempty, then $a(\Lambda_D^r, R_{D,j}, r) =$ 0*

Proof. The idea is that if $(x_0,y_0) \in \Lambda_D^r$ then $\alpha_D(x_0,y_0) \in \Lambda_D^r$ where α_D is a unit in $\mathcal{O}_D$. Therefore enough to show that the sum in RHS of (4.2.7) is 0. For $D = 1$, number of units in $\mathcal{O}_D$, $u_D = 4$ which are $\{1, -1, i, -i\}$. We have $1^j + (-1)^j + i^j + (-i)^j = 0$. For $D = 3$, number of units in $\mathcal{O}_D$, $u_D = 6$ which are $\{\pm 1, \frac{\pm 1 \pm \sqrt{-3}}{2}\}$. A brute force calculation shows the result. For other D, the number of units in $\mathcal{O}_D$, $u_D = 2$ which are $\{1, -1\}$. For all j odd, $(1)^j + (-1)^j = 0$ $\square$

From here on we will only consider the theta function $\Theta\left(\Lambda_D, \frac{1}{u_D} R_{D,j}; z\right)$ so let's give its coefficients a shorthand.

$$\Theta\left(\Lambda_D, \frac{1}{u_D} R_{D,j}; z\right) = \sum_{r=0}^{\infty} a(D, j, r) q^r. \tag{4.2.8}$$

Proposition 4.6, together with Lemma 4.3, gives us that if $j \equiv 0 \pmod{u_D}$, then the theta function $\Theta\left(\Lambda_D, \frac{1}{u_D} R_{D,j}; z\right) \in \mathcal{S}_{j+1}(\Gamma_0(N), \chi)$ is a Hecke eigenform. So Proposition 2.14 gives us the following lemma.

Lemma 4.8. *Suppose $j \in u_D \mathbb{Z}^+$. Then the following is true:*

1) If $\gcd(r_1, r_2) = 1$ *then*

$$a(D, j, r_1 r_2) = a(D, j, r_1) a(D, j, r_2).$$

2) For p prime and $\alpha > 0$, we have

$$a\left(D, j, p^{\alpha}\right) = a\left(D, j, p\right) a\left(D, j, p^{\alpha-1}\right) - \chi(p) p^j a\left(D, j, p^{\alpha-2}\right).$$

3) For p prime and $\alpha > 0$, we have

$$a(D, j, p^{\alpha}) = a(D, j, p)^{\alpha} \pmod{p}.$$

Suppose p be a prime such that Λ_D^p be nonempty. Let $(x_p, y_p) \in \Lambda_D^p$ and $j \equiv 0 \pmod{u_D}$. When $p = D$ then it ramifies in $\mathcal{O}_D$ and there are exactly u_D points in Λ_D^p. From (4.2.7) we have $a(D, j, p) = R_{D,j}(x_p, y_p)$. If $p \neq D$ then it's unramified and we get exactly $2u_D$ solutions. In this case $a(D, j, p) = 2R_{D,j}(x_p, y_p)$.

Lemma 4.9. *Suppose $j \in u_D\mathbb{Z}^+$ and p be an odd prime such that Λ_D^p is nonempty. Let $(x_p, y_p) \in \Lambda_D^p$ then $R_{D,j}(x_p, y_p) \not\equiv 0 \pmod{p}$. In particular, $a(D, j, p)$ is non-zero.*

Proof. We will consider two cases, $D \equiv 1, 2 \pmod 4$ and $D \equiv 3 \pmod 4$. Proof is essentially same in both the cases.

If $D \equiv 1, 2 \pmod 4$ then $p = x_p^2 + Dy_p^2$, in particular $x_p \not\equiv 0 \pmod{p}$. we consider the binomial expansion

$$\begin{aligned}
R_{D,j}(x_p, y_p) &= \operatorname{Re}\left(x_p + \sqrt{-D}y_p\right)^j \\
&= \frac{1}{2}\sum_{n=0}^{j/2}\binom{j}{2n}x_p^{j-2n}(-1)^n\left(Dy_p^2\right)^n = \frac{1}{2}\sum_{n=0}^{j/2}\binom{j}{2n}x_p^{j-2n}(-1)^n\left(p - x_p^2\right)^n \\
&\equiv \frac{1}{2}x_p^j\sum_{n=0}^{j/2}\binom{j}{2n} \equiv 2^{j-2}x_p^j \not\equiv 0 \pmod{p}.
\end{aligned}$$

If $D \equiv 1, 2 \pmod 4$ then $p = (x_p + y_p/2)^2 + Dy_p^2/4$, in particular $x_p + y_p/2 \not\equiv 0 \pmod{p}$. we consider the binomial expansion

$$\begin{aligned}
R_{D,j}(x_p, y_p) &= \operatorname{Re}\left(x_p + y_p/2 + \sqrt{-D}y_p/2\right)^j = \frac{1}{2}\sum_{n=0}^{j/2}\binom{j}{2n}\left(x_p + \frac{y_p}{2}\right)^{j-2n}(-1)^n\left(\frac{Dy_p^2}{4}\right)^n \\
&= \frac{1}{2}\sum_{n=0}^{j/2}\binom{j}{2n}\left(x_p + \frac{y_p}{2}\right)^{j-2n}(-1)^n\left(p - \left(x_p + \frac{y_p}{2}\right)^2\right)^n \\
&\equiv \frac{1}{2}\left(x_p + \frac{y_p}{2}\right)^j\sum_{n=0}^{j/2}\binom{j}{2n} \equiv 2^{j-2}\left(x_p + \frac{y_p}{2}\right)^j \not\equiv 0 \pmod{p}.
\end{aligned}$$

$\square$

Proposition 4.10. *For prime 2, Λ_D^2 is nonempty only for $D = 1, 2, 7$. In this case $a(D, j, 2)$ does not vanish for all $j \in 2\mathbb{Z}^+$. Moreover, we have that $a(7, j, 2) \equiv 1$* (mod 2).

Proof. For $D = 1, 2$, $2|\Delta_{\mathcal{O}_D}(= -4D)$ so the ideal (2) is ramified in $\mathcal{O}_D$, in particular there are elements of norm 2. For $D \in \{3, 7, 11, 19, 43, 67, 163\}$, $2 \nmid \Delta_{\mathcal{O}_D}(= -D)$. So the ideal (2) is unramified in $\mathcal{O}_D$. Here we need to check whether 2 splits or not. We have the condition that 2 splits if and only if $-D \equiv 1 \pmod 8$. Only $D = 7$ satisfies the condition.

A brute force calculation shows that $a(1, j, 2) = (1+i)^j \neq 0$, $a(2, j, 2) = i^j 2^{j+1} \neq 0$, and $a(7, j, 2) = 4\mathrm{Re}\left(\frac{1+\sqrt{-7}}{2}\right)^j \neq 0$.

We prove that $a(7, j, 2) \equiv 1 \pmod 2$ using induction on even j. First, note that $a(7, 2, 2) = -3 \equiv 1 \pmod 2$. Now we assume that $a(7, j, 2) \equiv 1 \pmod 2$, which implies that $\mathrm{Re}\left(\frac{1+\sqrt{-7}}{2}\right)^j = (2k + 1)/2$ for some k. The norm of $\left(\frac{1+\sqrt{-7}}{2}\right)^j$ is even, so we get that $\mathrm{Im}\left(\frac{1+\sqrt{-7}}{2}\right)^j = \sqrt{7}(2k' + 1)/2$ for some k'. An easy calculation shows that $a(7, j + 2, 2) = -3\mathrm{Re}\left(\frac{1+\sqrt{-7}}{2}\right)^j - \sqrt{7}\mathrm{Im}\left(\frac{1+\sqrt{-7}}{2}\right)^j \equiv 1 \pmod 2$. $\square$

4.2.3 Proof of Theorem 1.9

Proposition 4.5, 4.6 and 4.7 together imply that $a(\Lambda_D, R_{D,j}, r)$ and $a(\Lambda_D, I_{D,j}, r)$ vanish for all $j \not\equiv 0 \pmod{u_D}$, which implies that every nonempty shell Λ_D^r is an ellipsoidal T_D-design (remember that $T_D = \mathbb{Z}^+ \setminus u_D\mathbb{Z}^+$).

Now we prove the maximality of T_D. We show that $a(D,j,r) \neq 0$ (note that $a(D,j,r) = \frac{1}{u_D}a(\Lambda_D, R_{D,j}, r))$ for all $j \notin T_D$ and Λ_D^r nonempty. By Lemma 4.8, it is enough to take r to be a prime power. Suppose p be a prime and $\alpha \geq 1$ be such that $\Lambda_D^{p^a} \neq \phi$. There are two cases possible, either Λ_D^p is empty, or it is not. First suppose Λ_D^p is nonempty. If p is 2 then $a(D,j,2) \neq 0$ by Proposition 4.10. By part *(2)* of Lemma 4.8, we have that $a(D,j,2^\alpha) = a(D,j,2)^\alpha \neq 0$ for $D = 1,2$ since $\chi(2) = 0$. When $D = 7$ then part *(3)* of Lemma 4.8, we have $a(7,j,2^\alpha) \neq 0$. If p is an odd prime, then Lemma 4.9 implies that $a(D,j,p) \neq 0$. Now using part *(3)* of Lemma 4.8 again, we have $a(D,j,p^\alpha) \neq 0$. Suppose Λ_D^p is empty then $a(D,j,p) = 0$ and Lemma 4.4 implies α is even. Now by part *(2)* of Lemma 4.9, we get $a(D,j,p^\alpha) = p^{j\alpha/2} \neq 0$ (note that this case includes 2 too). So we get that $a(D,j,p^\alpha) \neq 0$ whenever $\Lambda_D^{p^\alpha}$ is nonempty.

Chapter 5

Inversion of j-function around elliptic points

In this chapter we prove Theorems 1.12 and 1.13 which give inversion formulae for j-function around elliptic points.

5.1 Proof of Theorem 1.12

The following calculations depend on the argument of t. For the moment, we assume that $0 < \mathrm{Arg}(t) < \frac{\pi}{2}$. Apply Proposition 2.30 to $a = b = \frac{3}{4}$, $c = \frac{3}{2}$ and $z = 4t^2$ to ${}_2F_1\left(\frac{3}{4}, \frac{3}{4}; \frac{3}{2}; 4t^2\right)$, using the functional equation $s\Gamma(s) = \Gamma(s+1)$, and the lambda notation in (2.4.1) we get

$$
\begin{aligned}
&{}_2F_1\left(\frac{3}{4}, \frac{3}{4}; \frac{3}{2}; 4t^2\right) && (5.1.1)\\
&= \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}\left(\frac{1}{4t^2}\right)^{3/4}\left[{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{4t^2}\right) + e^{3\pi i/4}{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{1}{4t^2}\right)\right]\\
&= \frac{\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}\left(\frac{1}{4t^2}\right)^{3/4}\left[\lambda_4\left(\frac{1}{4t^2}\right) + e^{3\pi i/4}\lambda_4\left(1 - \frac{1}{4t^2}\right)\right]. && (5.1.2)
\end{aligned}
$$

Similarly, we obtain

$$ {}_2F_1\left(\frac{1}{4},\frac{1}{4};\frac{1}{2};4t^2\right)=\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)}\left(\frac{1}{4t^2}\right)^{1/4}\left[\lambda_4\left(\frac{1}{4t^2}\right)+e^{\pi i/4}\lambda_4\left(1-\frac{1}{4t^2}\right)\right]. \tag{5.1.3} $$

Next, we divide (5.1.2) by (5.1.3) and use the formula for Ω_i to get

$$ \frac{t}{2\pi\Omega_i^2}\frac{{}_2F_1\left(\frac{3}{4},\frac{3}{4};\frac{3}{2};4t^2\right)}{{}_2F_1\left(\frac{1}{4},\frac{1}{4};\frac{1}{2};4t^2\right)}=\frac{\lambda_4\left(\frac{1}{4t^2}\right)+e^{3\pi i/4}\lambda_4\left(1-\frac{1}{4t^2}\right)}{\lambda_4\left(\frac{1}{4t^2}\right)+e^{\pi i/4}\lambda_4\left(1-\frac{1}{4t^2}\right)}. $$

Therefore we have

$$ \frac{C_i(t)}{2\pi\Omega_i^2}=\frac{\lambda_4\left(\frac{1}{4t^2}\right)+e^{3\pi i/4}\lambda_4\left(1-\frac{1}{4t^2}\right)}{\lambda_4\left(\frac{1}{4t^2}\right)+e^{\pi i/4}\lambda_4\left(1-\frac{1}{4t^2}\right)}. $$

Taking $\gamma=1-1/4t^2$ and τ as in Proposition **??** we get

$$ \frac{C_i(t)}{2\pi\Omega_i^2}=\frac{-i\sqrt{2}\tau+e^{3\pi i/4}}{-i\sqrt{2}\tau+e^{\pi i/4}}=\frac{(2\tau-1)-i}{(2\tau-1)-\bar{i}}=S_i(2\tau-1). \tag{5.1.4} $$

Therefore, we obtain

$$ \frac{s_i^{-1}(C_i(t))+1}{2}=\tau, $$

and the inversion formula in Proposition 2.32 gives us

$$ j\left(\frac{s_i^{-1}(C_i(t))+1}{2}\right)=j(\tau)=\frac{64\left(1+3\left(1-\frac{1}{4t^2}\right)\right)^3}{\left(1-\frac{1}{4t^2}\right)\left(\left(1-\frac{1}{4t^2}\right)-1\right)^2}=\frac{64(16t^2-3)^3}{4t^2-1}. $$

For the case when $\mathrm{Arg}(t)\notin[0,\pi/2)$, then equation (5.1.4) has the form

$$ \frac{C_i(t)}{2\pi\Omega_i^2}=(-1)^a\frac{(2\tau+(-1)^b)-i}{(2\tau+(-1)^b)-\bar{i}}. $$

The values of a and b as a function of the argument of t are given by the following table:

$\mathrm{Arg}(t)$	$[0,\pi/2)$	$[\pi/2,\pi)$	$[-\pi/2,0)$	$[-\pi,\pi/2)$
a	0	1	1	0
b	1	0	1	0
$\dfrac{C_i(t)}{2\pi\Omega_i^2}$	$\frac{(2\tau-1)-i}{(2\tau-1)-\bar{i}}$	$\frac{-\frac{1}{2\tau+1}-i}{-\frac{1}{2\tau+1}-\bar{i}}$	$\frac{-\frac{1}{2\tau-1}-i}{-\frac{1}{2\tau-1}-\bar{i}}$	$\frac{(2\tau+1)-i}{(2\tau+1)-\bar{i}}$
$\frac{s_i^{-1}(C_i(t))+1}{2}$	τ	$\frac{\tau-1}{2\tau-1}$	$\frac{\tau}{2\tau+1}$	$\tau+1$

Clearly, all possible values of $\dfrac{s_i^{-1}(C_i(t))+1}{2}$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent and thus their j-values are invariant.

5.2 Proof of Theorem 1.13

The following calculations depend on the argument of t. For the moment, we assume that $0<\mathrm{Arg}(t)<\frac{\pi}{3}$. If we apply Proposition 2.30 to ${}_2F_1\left(\frac{5}{6},\frac{5}{6};\frac{5}{3};-2t^3\right)$, using the functional equation $s\Gamma(s)=\Gamma(s+1)$ and the lambda notation in (2.4.1) we get

$$ {}_2F_1\left(\frac{5}{6},\frac{5}{6};\frac{5}{3};-2t^3\right)=\frac{\frac{2}{3}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{5}{6}\right)}\left(\frac{1}{2t^3}\right)^{5/6}\left[e^{5\pi i/6}\lambda_6(1/2t^3)+\lambda_6(1+1/2t^3)\right] \tag{5.2.1}$$

Similarly, if we apply Proposition 2.30 to ${}_2F_1\left(\frac{1}{6},\frac{1}{6};\frac{1}{3};-2t^3\right)$ to get

$$ {}_2F_1\left(\frac{1}{6},\frac{1}{6};\frac{1}{3};-2t^3\right)=\frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{6}\right)}\left(\frac{1}{2t^3}\right)^{1/6}\left[e^{\pi i/6}\lambda_4(-1/2t^3)+\lambda_6(1+1/2t^3)\right] \tag{5.2.2}$$

We divide (5.2.1) by (5.2.2), use Legendre's Duplication formula

$$\sqrt{\pi}\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma\left(s+\tfrac{1}{2}\right),$$

and the definition of Ω_ρ, to obtain

$$\frac{C_\rho(t)}{2\pi\Omega_\rho^2} = \frac{e^{5\pi i/6}\lambda_6(-1/2t^3) + \lambda_6(1+1/2t^3)}{e^{\pi i/6}\lambda_6(-1/2t^3) + \lambda_6(1+1/2t^3)}$$

Taking $\gamma = 1/t$ and τ as in Proposition **??** we get

$$\frac{C_\rho(t)}{2\pi\Omega_\rho^2} = \frac{e^{5\pi i/6} - i\tau}{e^{\pi i/6} - i\tau} = \frac{\tau-\rho}{\tau-\overline{\rho}} = S_\rho(\tau). \tag{5.2.3}$$

Therefore we obtain

$$s_\rho^{-1}(C_\rho(t)) = \tau,$$

and the inversion formula in Proposition 2.34 gives

$$j\left(s_\rho^{-1}(C_\rho(t))\right) = j(\tau) = -\frac{1728t^6}{(2t^3+1)}.$$

For arbitrary arguments of t, (5.2.3) has the following three forms:

$\mathrm{Arg}(t)$	$(-\frac{2\pi}{3}, -\frac{\pi}{3}) \cup (0, \pi/3) \cup (\frac{2\pi}{3}, \pi)$	$(-\pi, -\frac{2\pi}{3}] \cup (-\frac{\pi}{3}, 0] \cup (\frac{\pi}{3}, \frac{2\pi}{3}]$	$\{-\frac{\pi}{3}, \frac{\pi}{3}, \pi\}$
$\frac{C_\rho(t)}{2\pi\Omega_\rho^2}$	$\frac{\tau-\rho}{\tau-\overline{\rho}}$	$\frac{\tau+1-\rho}{\tau+1-\overline{\rho}}$	$\frac{-\frac{1}{\tau-1}-\rho}{-\frac{1}{\tau-1}-\overline{\rho}}$
$s_\rho^{-1}(C_\rho(t))$	τ	$\tau+1$	$-\frac{1}{\tau-1}$

Recall that for the case $\mathrm{Arg}(t) = \pi/3, \pi, -\pi/3$, we have to use Remark 2.31. Clearly, all possible values of $s_\rho^{-1}(C_\rho(t)$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent and thus their j-values are invariant.

Some comments

Notice that in both the proofs above we did not require any conditions on $|t|$. However, we do require two conditions on t, namely t must be in the domains where the inversion formulas in Propositions 2.32 and 2.34 are valid, and whenever $C_i(t)$ and $C_\rho(t)$ are well defined. The latter happens for $t \neq \pm 1/2$, and $t \neq (1/\sqrt[3]{2})\rho^a$ for $a = 1, 3, 5$, respectively. The former happens when $\tau = \tau(t) \in \mathbb{H}$ which, by equations (5.1.4) and (5.2.3), is equivalent to $C_i(t)/2\pi\Omega_i^2 \in \mathbb{D}$ and $C_\rho(t)/2\pi\Omega_\rho^2 \in \mathbb{D}$ for Theorems 1.12 and 1.13 respectively. This can be verified easily using the Maximum Modulus Principle on $|t| < 1/2$ and $|t| < 1/\sqrt[3]{2}$ respectively.

Furthermore, from the definition of ${}_2F_1$, $C_i(t)$ and $C_\rho(t)$ are discontinuous on the rays

$$\{u(-1)^a \mid u \geq 1/2, a = 0, 1\} \quad \text{and} \quad \{u\rho^a \mid u \geq 1/\sqrt[3]{2}, a = 1, 3, 5\}$$

respectively, but, as apparent from the Tables above $j((s_i^{-1}C_i(t)+1)/2)$ and $j(s_\rho^{-1}C_\rho(t))$ become continuous (and thus analytic) functions of t which can be proved using Remark 2.31. Therefore, Theorems 1.12 and 1.13 are valid for all t except for two and three points respectively.

5.3 Examples

Here we offer some examples.

Example 5.1. It is well known that $j(\sqrt{-2}) = 8000$. We verify this using Theorem

1.12. First we solve the degree six equation in t

$$64\frac{(16t^2-3)^3}{4t^2-1} = 8000.$$

One solution is

$$t_0 = \frac{i}{4\sqrt{2}}\sqrt{5\sqrt{2}-1} = i \cdot 0.4355695915...,$$

so that $|t_0| < 1/2$. We approximate $C_i(t_0)$ using the first 3000 terms of its power series expansion to get:

$$C_i(t_0) = i \cdot 0.375476877103748...$$

Thus

$$\tau_0 := \frac{s_i^{-1}(C_i(t_0))+1}{2} = 0.333333333333333... + i \cdot 0.471404520791031...$$

Notice that $\tau_0 \neq \sqrt{-2}$, but they are $\mathrm{SL}_2(\mathbb{Z})$-equivalent. Indeed, we have

$$-\frac{1}{\tau_0} + 1 = i \cdot 1.414213562373095... \approx \sqrt{-2}.$$

In fact, the above approximation is correct up to 364 decimal places.

Example 5.2. Now we use Theorem 1.13 to verify $j((1+\sqrt{-7})/2) = -3375$. We solve the degree 6 equation

$$-\frac{1728t^6}{2t^3+1} = -3375.$$

The solutions satisfy

$$t^3 = \frac{5}{64}(25 \pm 3\sqrt{105}),$$

so we take the real cubic root of $\frac{5}{64}(25-3\sqrt{105})$, which is

$$t_0 = \sqrt[3]{\frac{5}{64}(25-3\sqrt{105})} = -0.765459354046599...$$

Since $|t_0| < 1/\sqrt[3]{2} \approx 0.793700...$, we can approximate $C_\rho(t_0)$ using the first 3000 terms of the power series expansion :

$$C_\rho(t_0) = 0.538697866211295...$$

Thus

$$\tau_0 := s_\rho^{-1}(C_\rho(t_0)) = 0.500000000000000... + i \cdot 1.322875655532295... \approx \frac{1+\sqrt{-7}}{2}.$$

In fact, the above approximation is correct up to 145 decimal places.

Example 5.3. To illustrate the general inversion process, we try to find τ_0 such that $j(\tau_0) = -50,000$. We solve the degree 6 equation

$$-\frac{1728t^6}{2t^3+1} = -50,000.$$

The solutions satisfy

$$t^3 = \frac{25}{108}(125-\sqrt{16165}),$$

so we take the real cubic root of $\frac{25}{108}(125-\sqrt{16165})$, which is

$$t_0 = \sqrt[3]{\frac{25}{108}(125-\sqrt{16165})} = -0.791446942386710....$$

Notice that $|t_0| < 1/\sqrt[3]{2} \approx 0.793700...$, but it is very close to the upper bound, which means that we have to use more terms in the power series of $C_\rho(t)$ to get a reasonable approximation. Using the first 5000 terms yields

$$C_\rho(t_0) = 0.855243324301038...$$

and

$$\tau_0 = 0.500000000000000... + i \cdot 1.724359831532281....$$

We find that

$$j(\tau_0) = -49,999.9999999999999996....$$

In fact, the above approximation is correct up to 16 decimal places when we just use the first 5000 terms of power series.

Bibliography

[AB09] G.E. Andrews and B.C. Berndt. *Ramanujan's Lost Notebook: Part II*. Number pt. 2. Springer New York, 2009.

[AB12] G.E. Andrews and B.C. Berndt. *Ramanujan's Lost Notebook: Part III*. Number pt. 3. Springer New York, 2012.

[ABR92] S. Axler, P. Bourdon, and W. Ramey. *Harmonic Function Theory*, volume 2197-5612. Springer New York, NY, 1992.

[And84] G. E. Andrews. *The Theory of Partitions*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1984.

[Ban84] E. Bannai. Spherical t-designs which are orbits of finite groups. *Journal of the Mathematical Society of Japan*, 36(2):341 – 354, 1984.

[BB87] J. M. Borwein and P. B. Borwein. *Pi and the AGM*. John Wiley, New York, 1987.

[BBG95] B. Berndt, S. Bhargava, and F. Garvan. Ramanujan's theories of elliptic functions to alternative bases. *Transactions of the American Mathematical Society*, 347, 08 1995.

[BC99] B. C. Berndt and H. H. Chan. Ramanujan and the modular j-invariant. *Canadian Mathematical Bulletin*, 42(4):427–440, 1999.

[BKRT21] K. Bringmann, B. Kane, L. Rolen, and Z. Tripp. Fractional partitions and conjectures of chern-fu-tang and heim-neuhauser. *Transactions of the American Mathematical Society, Series B*, 8:615–634, 07 2021.

[Bor92] R. E. Borcherds. Monstrous moonshine and monstrous lie superalgebras. *Inventiones mathematicae*, 109, 1992.

[BOT15] E. Bannai, T. Okuda, and M. Tagami. Spherical designs of harmonic index t. *Journal of Approximation Theory*, 195:1–18, 2015. Special Issue Dedicated to Dick Askey on the occasion of his 80th birthday.

[C.40] E. T. C. Orthogonal polynomials. by G. Szegö. pp. ix, 401. 6. 1939. american mathematical society colloquium publications, 23. (american mathematical society, new york). *The Mathematical Gazette*, 24(258):66–67, 1940.

[CFL11] X. Chen, A. Frommer, and B. Lang. Computational existence proofs for spherical t-designs. *Numerische Mathematik*, 117:1–18, 2011. Special Issue Dedicated to Dick Askey on the occasion of his 80th birthday.

[CHH21] A. Căldăraru, Y. He, and S. Huang. Moonshine at landau-ginzburg points, 2021.

[CJW19] W. Y. C. Chen, D. X. Q. Jia, and L. X. W. Wang. Higher order turán inequalities for the partition function. *Trans. Amer. Math. Soc. 372 , 2143-2165*, (2019).

[CS17] H. Cohen and F. Strömberg. *Modular Forms*. Graduate Studies in Mathematics. American Mathematical Society, 2017.

[DDMP05] A. Dabholkar, F. Denef, G. W Moore, and B. Pioline. Precision counting of small black holes. *Journal of High Energy Physics*, 2005(10):096–096, oct 2005.

[DGS77] P. Delsarte, J. M. Goethals, and J. J. Seidel. Spherical codes and designs. *Geometriae Dedicata*, 6:363–388, 1977.

[DLMF] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.4 of 2022-01-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[DLPCP22] A. De Las Penas Castano and B. V. Pandey. Inversion formulas for the j-function around elliptic points. *Archiv der Mathematik*, 119:359–369, 2022.

[DP15] S. DeSalvo and I. Pak. Log-concavity of the partition function. *The Ramanujan Journal*, 38, 2015.

[EMOT55] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. Higher transcendental functions. vol. iii. based, in part, on notes left by harry bateman. mcgraw-hill book company, inc., new york-toronto-london, 1955.

[Eul97] L. Euler. *Introductio in analysin infinitorum*. Number Bd. 1 in Nineteenth Century Collections Online (NCCO): Science, Technology, and Medicine: 1780-1925. Bernuset, Delamolliere, Falque, 1797.

[GOR+22] M. J. Griffin, K. Ono, L. Rolen, J. Thorner, Z. Tripp, and I. Wagner. Jensen polynomials for the riemann xi-function. *Advances in Mathematics*, 397:108186, 2022.

[GORZ19] M. Griffin, K. Ono, L. Rolen, and D. Zagier. Jensen polynomials for the riemann zeta function and other sequences. *Proceedings of the National Academy of Sciences*, 116(23):11103–11110, 2019.

[HMOZar] L. Hong, M. H. Mertens, K Ono, and S. Zhang. Proof of the elliptic expansion moonshine conjecture of Căldăraru, He, and Huang. *Proceedings of the American Mathematical Society*, to appear.

[HNT21] B. Heim, M. Neuhauser, and R. Tröger. Inequalities for plane partitions. *arxiv, https://arxiv.org/abs/2109.15145*, 2021.

[HR18] G. H. Hardy and S. Ramanujan. Asymptotic formulaÃŠ in combinatory analysis. *Proceedings of the London Mathematical Society*, s2-17(1):75–115, 1918.

[HTH05] A. Hayashi, Hashimoto T., and M. Horibe. Reexamination of optimal quantum state estimation of pure states. *Phys. Rev. A*, 73, 2005.

[Inc] Wolfram Research Inc. Mathematica, Version 12.1.0. Champaign, IL, 2021.

[Iwa97] H. Iwaniec. *Topics in Classical Automorphic Forms*, volume 17. Graduate Studies in Mathematics AMS, 1997.

[Kna16] Anthony W Knapp. Appendix b. elementary complex analysis. In *Basic Real Analysis*, pages 631–717. Project Euclid (distributor), 2016.

[Kob12] N.I. Koblitz. *Introduction to Elliptic Curves and Modular Forms*. Graduate Texts in Mathematics. Springer New York, 2012.

[LW19] H. Larson and I. Wagner. Hyperbolicity of the partition jensen polynomials. *Research in Number Theory*, 5, 2019.

[Mac04] P. A. MacMahon. *Combinatory analysis. Vol. I, II (bound in one volume).* Dover Publications, Inc., Mineola, NY, 2004.

[Mie13] T. Miezaki. On a generalization of spherical designs. *Discrete Mathematics*, 313(4):375–380, 2013.

[Miy06] T. Miyake. *Modular forms.* Translated from the 1976 Japanese original by Yoshitaka Maeda. Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006.

[Obr03] N. Obrechkoff. Zeros of polynomials. *Publ. Bulg. Acad. Sci, Sofia, 1963, English translation (by I. Dimovski and P. Rusev) published by The Marin Drinov Academic Publishing House, Sofia*, 2003.

[OPR22] K. Ono, S. Pujahari, and L. Rolen. Turán inequalities for the plane partition function. *Advances in Mathematics*, 2022.

[Ṕ27] G. Pólya. *Uber die algebraisch-Funktionentheoretische Untersuchungen von J.L.W.V. Jensen.* A.F. Hst, 1927.

[Pan22a] B. V. Pandey. Mathematica code for calculations of Higher Turán inequalities for plane partition, 2022.

[Pan22b] B. V. Pandey. Modular forms and ellipsoidal T-designs. *The Ramanujan Journal*, 58:1245–1257, 2022.

[Pin20] I. Pinelis. Exact lower and upper bounds on the incomplete gamma function. *Mathematical Inequalities & Applications*, 4:1261–1278, 2020.

[Rad38] H. Rademacher. On the partition function p(n). *Proceedings of the London Mathematical Society*, s2-43(1):241–254, 1938.

[Ram14] S. Ramanujan. Modular equations and approximations of π. *Quarterly Journal of Mathematics*, pages 350–372, 1914.

[Sek92] G. Seki. On some non-rigid spherical t-designs. *Memoirs of the Faculty of Science, Kyushu University. Series A, Mathematics*, 46(1):169–178, 1992.

[Wri31] E. M. Wright. Asymptotic partition formulae, i: Plane partitions. *The Quarterly Journal of Mathematics, Volume os-2, Issue 1*, pages 177–189, (1931).

[Zag06] Don Zagier. Appendix: The Mellin transformation and other useful analytic techniques. In *Quantum Field Theory I: Basics in Mathematics and Physics*, pages 307–323. Springer, Heidelberg, 2006. author: Zeidler, Eberhard.

[Zag08] D. Zagier. *Elliptic Modular Forms and Their Applications*, pages 1–103. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.