ON THE

TRANSFORMATION GROUP

$$\boxed{p, \quad q, \quad r, \quad xp, \quad yq, \quad zr, \quad x^2p, \quad y^2q, \quad z^2r}$$

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By H. L. Roberts, M. A.

I.

Geometrical Peculiarities.

1. In this section we shall find the finite transformations and the path-curves of the group

$$\boxed{p, \quad q, \quad r, \quad xp, \quad yq, \quad zr, \quad x^2p, \quad y^2q, \quad z^2r}$$

where $p \equiv \dfrac{\partial f}{\partial x}$, $q \equiv \dfrac{\partial f}{\partial y}$, $r \equiv \dfrac{\partial f}{\partial z}$.

We shall also briefly discuss some of the geometrical peculiarities of the Group.

2. The most general infinitesimal transformation of the Group is

$$Uf \equiv (a_1 + 2b_1x + c_1x^2)\, p + (a_2 + 2b_2y + c_2y^2)\, q + (a_3 + 2b_3z + c_3z^2)\, r\,,$$

where a_i, b_i, c_i $(i = 1, 2, 3)$ are certain undetermined constants.

The finite transformations are given, in the usual manner, by the integration of the following simultaneous system:

$$\frac{dx_1}{a_1 + 2b_1x_1 + c_1x_1^2} = \frac{dy_1}{a_2 + 2b_2y_1 + c_2y_1^2} = \frac{dz_1}{a_3 + 2b_3z_1 + c_3z_1^2} = dt\,,$$

with the condition that $x_1 = x$, $y_1 = y$, $z_1 = z$, when $t = 0$.

Now, since the variables have already been separated in these equations, it is easily seen that whether $a_ic_i <=> b_i^2$ $(i = 1, 2, 3)$ the finite transformations have the form

$$x_1 = \frac{\alpha_1 x + \alpha_2}{\alpha_3 x + 1}, \quad y_1 = \frac{\beta_1 x + \beta_2}{\beta_3 x + 1}, \quad z_1 = \frac{\gamma_1 z + \gamma_2}{\gamma_3 z + 1},$$

where α_i, β_i, γ_i are certain constants.

3. It is evident that these transformations will transform the equations

$$x = \text{const.}, \quad y = \text{const.}, \quad z = \text{const.}$$

into

$$x_1 = \text{const.}, \quad y_1 = \text{const.}, \quad z_1 = \text{const.}$$

Hence, we observe that all the planes parallel to the yz-plane are moved parallel to themselves by the transformations of the G_9. In this case, we say these ∞^1 parallel planes form an *invariant family* of planes; that is, *the family is invariant as a whole, while the planes of the family are interchanged among each other by means of the transformations of the* G_9.

Similarly the planes parallel to the xz-plane and xy-plane form two other invariant families of planes, and we have, in all, three invariant families of parallel planes, each family consisting of ∞^1 planes.

The intersections of the planes parallel to the yz-plane and xz-plane are straight lines perpendicular to the xy-plane, and it is obvious, since the ∞^2 lines perpendicular to the xy-plane are the intersections of the planes of *invariant families* of planes, that these ∞^2 lines themselves form an *invariant family* of parallel lines. *This family of lines is invariant as a whole, while the lines of the family are interchanged among each other by means of the transformations of the* G_9.

Similarly the lines perpendicular to the yz-plane and xz-plane form two other invariant families of parallel lines, each family consisting of a line congruence.

Since the planes of each invariant family of planes are parallel, the intersections of the parallel planes of each family will be a straight line at infinity; and, as we have three invariant families of parallel planes, we shall have three such lines. It is clear, then, that *these three straight lines at infinity are absolutely invariant.*

4. To find what absolutely invariant loci exist *within a finite distance of the origin*, we equate to zero the coefficients of p, q, and r in the general transformation; for they are the increments given to x, y, and z, respectively, by the infinitesimal transformation

$$Uf \equiv (a_1 + 2b_1x + c_1x^2)\,p + (a_2 + 2b_2y + c_2y^2)\,q + (a_3 + 2b_3z + c_3z^2)\,r\,.$$

Thus we have the equations

$$a_1 + 2b_1x + c_1x^2 = 0\,,$$
$$a_2 + 2b_2y + c_2y^2 = 0\,,$$
$$a_3 + 3b_3z + c_3z^2 = 0\,,$$

which give the absolutely invariant planes

$$x = \alpha_1\,, \quad x = \beta_1\,, \quad y = \alpha_2\,, \quad y = \beta_2\,, \quad z = \alpha_3\,, \quad z = \beta_3\,,$$

where the α's and β's are the roots of the above equations, and have the values

$$\alpha_i = \frac{-b_i + \sqrt{b_i^2 - a_i c_i}}{c_i}, \quad \beta_i = \frac{-b_i - \sqrt{b_i^2 - a_i c_i}}{c_i}.$$

The intersections of these planes are, of course, invariant; hence the following points:

$$(\alpha_1, \alpha_2, \alpha_3), \quad (\alpha_1, \beta_2, \alpha_3), \quad (\beta_1, \alpha_2, \alpha_3), \quad (\beta_1, \beta_2, \alpha_3)$$
$$(\alpha_1, \alpha_2, \beta_3), \quad (\alpha_1, \beta_2, \beta_3), \quad (\beta_1, \alpha_2, \beta_3), \quad (\beta_1, \beta_2, \beta_3)$$

are absolutely *invariant points;* and are the vertices of a *parallelepipedon,* which must also be *absolutely invariant.*

If $b_i^2 < a_i c_i$, the roots of the equations become imaginary, and our absolutely invariant planes, points, and parallelepipedon, likewise, become imaginary; for the imaginary planes do not intersect in any real points or point.

If $b_i^2 = a_i c_i$, the roots of the equations are equal, and our planes now reduce to three and our parallelepipedon to an absolutely invariant point.

If $b_i = c_i = 0$, $a_1 = 1$, $a_2 = a_3 = 0$, the transformation becomes

$$Uf \equiv p.$$

In this case, the roots of the equation

$$a_1 + 2b_1 x + c_1 x^2 = 0$$

are infinite, and the planes $x =$ const. are now at infinity, while the planes $y =$ const. and $z =$ const. are still invariant. A similar discussion holds when the transformation reduces to

$$Uf \equiv q, \quad Uf \equiv r,$$

or to any mere translation.

5. The points in space are moved by the general transformation of the G_9 along ∞^2 curves, which we shall call the *Path-Curves* of the Group. We shall now find these curves, and see whether any of the absolutely invariant points just found lie on this curve congruence.

These curves, as usual, are found by integrating, in the most general manner, the equations

$$\frac{dx}{a_1 + 2b_1 x + c_1 x^2} = \frac{dy}{a_2 + 2b_2 y + c_2 y^2} = \frac{dz}{a_3 + 2b_3 z + c_3 z^2}.$$

We shall find them for the three principal cases only; that is, according as $a_i c_i < = > b_i^2$ $(i = 1, 2, 3)$.

If $a_i c_i < b_i^2$, the ∞^2 path-curves, as is readily seen, are given by the equations

$$\frac{x-\alpha_1}{x-\beta_1} = C_1\left[\frac{y-\alpha_2}{y-\beta_2}\right]^k,$$

$$\frac{x-\alpha_1}{x-\beta_1} = C_2\left[\frac{z-\alpha_3}{z-\beta_3}\right]^h,$$

where

$$\alpha_i \equiv \frac{-b_i + \sqrt{b_i^2 - a_i c_i}}{c_i}, \quad \beta_i \equiv \frac{-b_i - \sqrt{b_i^2 - a_i c_i}}{c_i},$$

$$k^2 \equiv \frac{b_1^2 - a_1 c_1}{b_2^2 - a_2 c_2}, \quad h^2 \equiv \frac{b_1^2 - a_1 c_1}{b_3^2 - a_3 c_3},$$

and C_1 and C_2 are constants of integration. Evidently the absolutely invariant points

$$P: \quad x = \alpha_1, \quad y = \alpha_2, \quad z = \alpha_3,$$

$$Q: \quad x = \beta_1, \quad y = \beta_2, \quad z = \beta_3,$$

lie on these curves, so that each curve of the congruence of path-curves passes through the points P and Q.

If $a_i c_i > b_i^2$, the ∞^2 path-curves are given by the equations

$$\tan^{-1}(\mu_1 x + \nu_1) - k\tan^{-1}(\mu_2 y + \nu_2) = \text{const.},$$

$$\tan^{-1}(\mu_1 x + \nu_1) - h\tan^{-1}(\mu_3 z + \nu_3) = \text{const.},$$

where

$$\mu_i \equiv \frac{c_i}{\sqrt{a_i c_i - b_i^2}} \quad \text{and} \quad \nu_i \equiv \frac{b_i}{\sqrt{a_i c_i - b_i^2}},$$

and k and h have the same values as above.

In this case, the absolutely invariant points become imaginary, since the imaginary planes do not intersect in any real points or point.

Finally, if $a_i c_i = b_i^2$, the ∞^2 path-curves are given by the equations

$$\frac{1}{c_1 x + b_1} - \frac{1}{c_2 y + b_2} = \text{const.},$$

$$\frac{1}{c_1 x + b_1} - \frac{1}{c_3 z + b_3} = \text{const.},$$

or

$$xy + m_1 x + n_1 y + b_1 = 0,$$

$$xz + m_2 x + n_2 z + b_2 = 0.$$

In this case, the absolutely invariant points reduce to one point, the coordinates of which are

$$x = -\frac{b_1}{c_1}, \quad y = -\frac{b_2}{c_2}, \quad z = -\frac{b_3}{c_3}.$$

This point lies on the curves just found, so that now each curve of the congruence of path-curves passes through one point.

In the first case the path-curves are usually *algebraic;* in the second, *transcendental;* and in the third, they are given as the *intersections of hyperbolic cylinders.*

Of course, other cases may arise, but they need not be further discussed.

6. We have already found the points that are absolutely invariant under the transformations of the G_9. We shall now find what "*Invariants,*" or *invariant relations* exist between *n points, three points,* and *two points;* that is, *what functions* $f(x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$ *are invariant, or what equations* $f(x_1, y_1, z_1, \ldots, x_n, y_n, z_n) = 0$, *are invariant under the* G_9.

We shall first define what is meant by an *invariant function,* and an *invariant equation.*

A *function* $f(x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$ is said to be invariant under a group of transformations in the same variables, if each increment given to the function by each transformation of the group is identically zero.

An *equation* $f(x_1, y_1, z_1, \ldots, x_n, y_n, z_n) = 0$ is said to be invariant, when the increments are either identically zero, or are zero by means of $f = 0$. The analytical condition, then, is that the expression

$$\delta f = \frac{\partial f}{\partial x_1}\,\delta x_1 + \frac{\partial f}{\partial y_1}\,\delta y_1 + \frac{\partial f}{\partial z_1}\,\delta z_1 + \ldots + \frac{\partial f}{\partial z_n}\,\delta z_n$$

shall be zero identically in case of an invariant function; and shall be zero identically or by means of $f = 0$ in case of an invariant equation $f = 0$,—where the δx_i, δy_i, δz_i must be obtained from the transformations of the Group.

7. We can now find the Invariants and invariant relations of *n points, three points,* and *two points.*

Let the coordinates of the points be x_i, y_i, z_i $(i = 1, 2, \ldots, n)$. Here we wish, first, to find what functions $f(x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$ are invariant.

In accordance with the above, we have the following conditions, which f must satisfy simultaneously:

$$\begin{aligned}
U_1 f &\equiv p_1 + p_2 + \ldots + p_n = 0\,, \\
U_2 f &\equiv q_1 + q_2 + \ldots + q_n = 0\,, \\
U_3 f &\equiv r_1 + r_2 + \ldots + r_n = 0\,, \\
U_4 f &\equiv x_1 p_1 + x_2 p_2 + \ldots + x_n p_n = 0\,, \\
U_5 f &\equiv y_1 q_1 + y_2 q_2 + \ldots + y_n q_n = 0\,, \\
U_6 f &\equiv z_1 r_1 + z_2 r_2 + \ldots + z_n r_n = 0\,, \\
U_7 f &\equiv x_1^2 p_1 + x_2^2 p_2 + \ldots + x_n^2 p_n = 0\,, \\
U_8 f &\equiv y_1^2 q_1 + y_2^2 q_2 + \ldots + y_n^2 q_n = 0\,, \\
U_9 f &\equiv z_1^2 r_1 + z_2^2 r_2 + \ldots + z_n^2 r_n = 0\,.
\end{aligned}$$

The above complete system consists of *nine* equations in $3n$ variables; and, since all the n-row determinants of the matrix of the coefficients of p_i, q_i, r_i do *not* become zero identically, there are $3(n-3)$ *solutions* of the system. From equations $U_1 f$, $U_4 f$, and $U_7 f$ we find $(n-3)$ functions $f(x_1, x_2, \ldots, x_n)$, which are solutions of the system, as none of the other equations contain x_i. In order to find these $(n-3)$ solutions, we proceed as follows (see § 13):

The solutions of equation $U_4 f$ are

$$\frac{x_2}{x_1} \equiv a_1, \quad \frac{x_3}{x_1} \equiv a_2, \quad \ldots, \quad \frac{x_n}{x_1} \equiv a_{n-1}.$$

If we introduce the quantities a_i, as new variables, into the equations $U_1 f$ and $U_2 f$,—since

$$U_1 a_1 = \frac{1}{x_1}(1-a_1), \quad U_1 a_2 = \frac{1}{x_1}(1-a_2), \quad \ldots, \quad U_1 a_{n-1} = \frac{1}{x_1}(1-a_{n-1}),$$

$$U_7 a_1 = -x_1 a_1 (1-a_1), \; U_7 a_2 = -x_1 a_2 (1-a_2), \; \ldots, \; U_7 a_{n-1} = -x_1 a_{n-1}(1-a_{n-1}),$$

we have the following equations:

$$U_{10} f \equiv (1-a_1)\frac{\partial f}{\partial a_1} + (1-a_2)\frac{\partial f}{\partial a_2} + \ldots + (1-a_{n-1})\frac{\partial f}{\partial a_{n-1}} = 0,$$

$$U_{11} f \equiv a_1(1-a_1)\frac{\partial f}{\partial a_1} + a_2(1-a_2)\frac{\partial f}{\partial a_2} + \ldots + a_{n-1}(1-a_{n-1})\frac{\partial f}{\partial a_{n-1}} = 0.$$

The solutions of equation $U_{10} f$ are

$$\frac{1-a_2}{1-a_1} \equiv \beta_1, \quad \ldots, \quad \frac{1-a_{n-1}}{1-a_1} \equiv \beta_{n-2}.$$

If we introduce the quantities β_i, as new variables, into the equation $U_{11} f$,—since

$$U_{11}\beta_1 = (1-a_1)\beta_1(\beta_1 - 1), \quad \ldots, \quad U_{11}\beta_{n-2} = (1-a_1)\beta_{n-2}(\beta_{n-2}-1),$$

we have the equation

$$U_{12} f \equiv \beta_1(\beta_1 - 1)\frac{\partial f}{\partial \beta_1} + \ldots + \beta_{n-2}(\beta_{n-2}-1)\frac{\partial f}{\partial \beta_{n-2}} = 0.$$

The solutions of this equation are

$$\frac{\beta_1 - 1}{\beta_1} \cdot \frac{\beta_2}{\beta_2 - 1} \equiv \frac{x_2 - x_3}{x_1 - x_2} \cdot \frac{x_1 - x_4}{x_2 - x_4},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\beta_1 - 1}{\beta_1} \cdot \frac{\beta_{n-2}}{\beta_{n-2} - 1} \equiv \frac{x_2 - x_3}{x_1 - x_2} \cdot \frac{x_1 - x_n}{x_2 - x_n}.$$

We can now, on account of the symmetry of the equations, write down the remaining $2(n-3)$ solutions. They are

$$f \equiv \frac{y_2 - y_3}{y_1 - y_2} \cdot \frac{y_1 - y_i}{y_2 - y_i},$$

$$f \equiv \frac{z_2 - z_3}{z_1 - z_2} \cdot \frac{z_1 - z_i}{z_2 - z_i},$$

where ($i = 4, 5, \ldots, n$).

Hence, if four points, whose x's, y's, or z's form an anharmonic ratio, are transformed by means of the transformations of the Group, the x's, y's, or z's of the new positions of the four points will form the same anharmonic ratio.

Evidently, this is equivalent to saying that the anharmonic ratio of four planes parallel to one of the coordinate planes is invariant.

For the invariant relations of *two* points, we have to find what equations $f(x_i, y_i, z_i, x_j, y_j, z_j) = 0$ satisfy all the partial differential equations, either identically or by means of $f = 0$.

It is readily seen that the equations

$$f \equiv x_i - x_j = 0\,, \quad f \equiv y_i - y_j = 0\,, \quad f \equiv z_i - z_j = 0$$

satisfy all the partial differential equations by means of $f = 0$; hence $f = 0$, in any of the above forms, is an invariant equation. This means,—when, for example, $x_1 - x_2 = 0$ is the invariant relation of two points,—that, if the two points lie in a plane parallel to the yz-plane, when they are transformed by the transformations of the Group they will still lie in a plane parallel to the yz-plane.

It may be shown that all the invariant relations of two points are of the above forms.

Similarly, for *three* points, we have the equations

$$\begin{aligned} x_i - x_j &= 0\,, & x_i - x_k &= 0\,, & x_j - x_k &= 0\,, \\ y_i - y_j &= 0\,, & y_i - y_k &= 0\,, & y_j - y_k &= 0\,, \\ z_i - z_j &= 0\,, & z_i - z_k &= 0\,, & z_j - z_k &= 0\,. \end{aligned}$$

These results are interpreted just as in the case of two points.

8. In like manner we can find the Invariants and invariant relations of a *point and a plane, two points and a plane, a plane and two points, etc.*

Writing the equation to the plane in the form

$$ux + vy + wz = 1\,,$$

and writing the transformations of the Group in the variables u, v, and w, the problem becomes to find what functions

$$f_1(x_1, y_1, z_1, u, v, w)\,, \quad f_2(x_1, y_1, z_1, x_2, y_2, z_2, u, v, w),$$
$$f_3(x_1, y_1, z_1, u_1, v_1, w_1, u_2, v_2, w_2), \text{ etc.},$$

and what equations

$$f_1 = 0\,, \quad f_2 = 0\,, \quad f_3 = 0\,, \text{ etc.},$$

are invariant under the Group.

The invariant functions, and invariant equations, are found in a manner entirely analogous to that of the preceding paragraph; and it will not be necessary to go through with the work here.

9. It will be interesting to know what will happen, if we *make some point absolutely invariant; that is, hold some point of general position as fixed.*

The plane passing through the fixed point parallel to one of the coordinate planes is invariant, since the transformations of the Group move all such planes parallel to themselves (§ 3). Two of the invariant planes of § 4 are parallel to it. Suppose these three planes are given by the equations

$$x = \gamma_1\,, \quad x = \alpha_1\,, \quad x = \beta_1\,.$$

The plane which forms an anharmonic ratio with these three planes is absolutely invariant (§ 7). Let this plane be given by the equation

$$x = \varepsilon_1\,.$$

The plane which forms an anharmonic ratio with the planes

$$x = \varepsilon_1\,, \quad x = \gamma_1\,, \quad x = \alpha_1\,,$$

is also absolutely invariant; thus there is found another absolutely invariant plane; and by this method there can be found ∞^1 absolutely invariant planes parallel to the yz-plane; ∞^1, parallel to the xz-plane; and ∞^1, parallel to the xy-plane. These three families of absolutely invariant planes intersect in ∞^3 points, which, of course, are absolutely invariant. *Hence every point in space becomes absolutely invariant, if a point of general position is held fixed.*

10. The results of the last paragraph may be shown analytically.

Let the coordinates of the fixed point, which does not lie on any of the absolutely invariant planes, be

$$x = \gamma_1\,, \quad y = \gamma_2\,, \quad z = \gamma_3\,.$$

These values of x, y, and z, by hypothesis, reduce the transformation

$$Uf \equiv (a_1 + 2b_1x + c_1x^2)\,p + (a_2 + 2b_2y + c_2y^2)\,q + (a_3 + 2b_3z + c_3z^2)\,r$$

to the identical transformation; therefore, these values must satisfy the equations

$$a_1 + 2b_1x + c_1x^2 = 0\,,$$
$$a_2 + 2b_2y + c_2y^2 = 0\,,$$
$$a_3 + 2b_3z + c_3z^2 = 0\,.$$

We, consequently, have the three following conditions, from which a_i, b_i, and c_i may be determined

$$a_i + 2b_i\gamma_i + c_i\gamma_i^2 = 0\,,$$
$$a_i + 2b_i\alpha_i + c_i\alpha_i^2 = 0\,,$$
$$a_i + 2b_i\beta_i + c_i\beta_i^2 = 0\,,$$

where α_i and β_i are the roots of the equations (§ 4).

Eliminating a_i, we have

$$c_i(\alpha_i + \gamma_i) + 2b_i = 0\,,$$
$$c_i(\alpha_i + \beta_i) + 2b_i = 0\,.$$

Eliminating b_i, we have

$$c_i(\gamma_i - \beta_i) = 0\,,$$

but since γ_i cannot equal β_i, we must have $c_i = 0$, and, consequently, $a_i = b_i = 0$. Now; if $a_i = b_i = c_i = 0$, the coefficients of p, q, and r are zero. Since these coefficients are the increments given to x, y, and z, respectively, by means of the transformation, and since they are zero, *the transformation reduces to the identical transformation.*

11. The invariant *families*, so far found, have been either planes or straight lines. We shall now show that the Group may be put into such a form that it will leave the families of all *surfaces of rotation*

$$z = \varphi(x^2 + y^2 + ax + by + c)$$

invariant.

We only need show, to this end, that the Sub-Group

$$\boxed{p, \quad q, \quad xp, \quad yq, \quad x^2p, \quad y^2q}$$

can be put into such form that it leaves the ∞^3 circles in the xy-plane invariant as a family; for the transformation zr and z^2r do not transform this plane at all, while the transformation r merely translates it.

Introduce into the Sub-Group the functions x_1 and y_1, as new variables, by means of

$$x = x_1 + iy_1\,,$$
$$y = x_1 - iy_1\,;$$

then $p \equiv \frac{\partial f}{\partial x}$ and $q \equiv \frac{\partial f}{\partial y}$ become, in these variables,

$$\frac{\partial f}{\partial x_1} + \frac{1}{i}\frac{\partial f}{\partial y_1}$$

and

$$\frac{\partial f}{\partial x_1} - \frac{1}{i}\frac{\partial f}{\partial y_1}\,,$$

or $ip_1 + q_1$ and $ip_1 - q_1$; and the transformations, as is readily seen, may be written in the forms

$$\boxed{\begin{array}{c} ip_1 + q_1\,, \quad ip_1 - q_1\,, \\ x_1q - y_1p_1 + i(x_1p_1 + y_1q_1), \quad x_1q_1 - y_1p_1 - i(x_1p_1 + y_1q_1)\,, \\ (x_1^2 - y_1^2)\,q_1 - 2x_1y_1p_1 + i\{(x_1^2 - y_1^2)\,p_1 + 2x_1y_1q_1\}, \\ (2x_1y_1p_1 - (x_1^2 - y_1^2)\,q_1 + i\{(x_1^2 - y_1^2)\,p_1 + 2x_1y_1q_1\} \end{array}}\,.$$

Hence the G_6 may be put in the form

$$\boxed{p, \quad q, \quad yp - xq, \quad xp + yq, \quad (x^2 - y^2)\,p + 2xyq, \quad 2xyp - (x^2 - y^2)\,q}\,.$$

The finite transformations of this G_6 are (§ 2)

$$x_1 = \frac{\alpha_1(x + iy) + \alpha_2}{2(\alpha_3(x + iy) + 1)} + \frac{\beta_1(x - iy) + \beta_2}{2(\beta_3(x - iy) + 1)}\,,$$
$$y_1 = \frac{1}{2i}\left[\frac{\alpha_1(x + iy) + \alpha_2}{\alpha_3(x + iy) + 1} - \frac{\beta_1(x - iy) + \beta_2}{\beta_3(x - iy) + 1}\right].$$

Let the equation to the circles in the xy-plane be

$$\Phi \equiv x_1^2 + y_1^2 + ax_1 + by_1 + c = 0\,.$$

Substituting in this equation the above values of x_1 and y_1, we get

$$\frac{\{\alpha_1(x + iy) + \alpha_2\}\{\beta_1(x - iy) + \beta_2\}}{\{\alpha_3(x + iy) + 1\}\{\beta_3(x - iy) + 1\}} + \frac{a}{2}\left\{\frac{\alpha_1(x + iy) + \alpha_2}{\alpha_3(x + iy) + 1} + \frac{\beta_1(x - iy) + \beta_2}{\beta_3(x - iy) + 1}\right\}$$
$$+ \frac{b}{2i}\left\{\frac{\alpha_1(x + iy) + \alpha_2}{\alpha_3(x + iy) + 1} - \frac{\beta_1(x - iy) + \beta_2}{\beta_3(x - iy) + 1}\right\} + c = 0\,,$$

which may be reduced to

$$x^2 + y^2 + Gx + Hy + F = 0 .$$

Hence the circles $\Phi = 0$ are invariant, as a family.

In the form

$$\boxed{\begin{gathered} p, \quad q, \quad yp - xq, \quad xp + yq, \\ (x^2 - y^2)\,p + 2xyq, \quad 2xyp - (x^2 - y^2)\,q, \\ r, \quad zr, \quad z^2r \end{gathered}}$$

the G_9 always transforms a circle in the xy-plane into a circle, since zr and z^2r do not transform that plane at all, while r translates it merely. Hence a surface which cuts that plane in a circle must be transformed into a surface which also cuts it in a circle, or the surfaces of rotation

$$z = \varphi\,(x^2 + y^2 + ax + by + c)$$

are an *invariant family* under the Group as written above.

This is readily verified from the form of the equation of surfaces of rotation

$$z = \varphi\,(x^2 + y^2 + ax + by + c) .$$

II.

Differential Invariants.

Invariant Differential Equations.

12. In this section we shall assume y and z to be functions of x; and shall proceed to find the *Differential Invariants* of the lowest order of the G_9, and then show how the Differential Invariants of higher orders may be found.

We shall also find what *ordinary differential equations* are invariant under the G_9.

13. We shall make frequent use of the following theorem from the theory of the complete system:

If $A_1f = 0, \ldots, A_rf = 0$ form a complete system in $x_1, \ldots, x_n$ $(r < n)$, the solutions of the same can be obtained in the following manner. We find the solutions $\varphi_1, \ldots, \varphi_{n-1}$ of $A_1f = 0$; and then write $A_2f = 0$ in the form

$$A_2f \equiv A_2\varphi_1 \frac{\partial f}{\partial \varphi_1} + \ldots + A_2\varphi_{n-1} \frac{\partial f}{\partial \varphi_{n-1}} = 0 .$$

If the ratios of the $A_2\varphi_k$ *are not functions of* $\varphi_1, \ldots, \varphi_{n-1}$ *alone, the equation* $A_2 f = 0$ *breaks up into several equations. We integrate one of these and introduce the corresponding solutions* $\psi_1, \ldots, \psi_{n-2}$ *into* $A_3 f = 0$. *The resulting equation*

$$A_3\psi_1 \frac{\partial f}{\partial \psi_1} + \ldots = 0$$

we handle in an analogous manner, etc. Thus we find ultimately the $(n - r)$ *solutions of the complete system.*

14. *To Find the Differential Invariant of the Lowest Order.*

Here we have to find what functions $f(x, y, z, y_1, z_1, \ldots, y_n, z_n)$ are invariant under the transformations of the Group,—where

$$y_i \equiv \frac{d^i y}{dx^i}, \quad z_i \equiv \frac{d^i z}{dx^i}.$$

Now

$$\delta f = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial y_1}\,\delta y_1 + \ldots + \frac{\partial f}{\partial z_n}\,\delta z_n \tag{1}$$

and we must obtain the increments δy_i, δz_i for each transformation of the Group.

If we have given any transformation

$$Uf \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z},$$

we find $\delta y_1, \delta z_1, \ldots, \delta y_n, \delta z_n$ as follows:

$$y_1 dx = dy\,,$$

so that

$$\delta y_1 dx + y_1 \delta dx = \delta dy\,,$$

or

$$\delta y_1 dx + y_1 d\delta x = d\delta y\,,$$

$$\therefore\ \delta y_1 = \frac{d\delta y}{dx} - y_1 \frac{d\delta x}{dx}.$$

Since $\delta x = \xi\delta t$ and $\delta y = \eta\delta t$, we have

$$\frac{\delta y_1}{\delta t} = \frac{d\eta}{dx} - y_1 \frac{d\xi}{dx} \equiv \eta_1 \text{ (say)}.$$

Similarly,

$$\frac{\delta z_1}{\delta t} = \frac{d\zeta}{dx} - z_1 \frac{d\xi}{dx} \equiv \zeta_1 \text{ (say)}.$$

In an analogous manner, we find

$$\eta_i = \frac{d\eta_{i-1}}{dx} - y_i \frac{d\xi}{dx},$$

$$\zeta_i = \frac{d\zeta_{i-1}}{dx} - z_i \frac{d\xi}{dx}.$$

We may now write the transformation in the form

$$U_n f \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} + \ldots + \eta_n \frac{\partial f}{\partial y_n} + \zeta_n \frac{\partial f}{\partial z_n};$$

and we then say that the transformation has been *extended n times*.

We obtain a complete system of nine equations in nine variables, if we find the increments δy_i and δz_i $(i = 1, 2, 3)$ for each transformation of the Group, substitute them in equation (1), and put $\delta_j f = 0$ $(j = 1, 2, \ldots, 9)$; or, what amounts to the same thing, if we extend each transformation three times, and then put each extended transformation equal to zero.

Extending each transformation of the G_9 three times, and putting each extended transformation equal to zero, we obtain the following complete system,—in which $q_i \equiv \dfrac{\partial f}{\partial y_i}$, etc.

$$p = 0, \quad q = 0, \quad r = 0,$$

$$xp - y_1q_1 - z_1r_1 - 2y_2q_2 - 2z_2r_2 - 3y_3q_3 - 3z_3r_3 = 0,$$

$$yq + y_1q_1 + y_2q_2 + y_3q_3 = 0,$$

$$zr + z_1r_1 + z_2r_2 + z_3r_3 = 0,$$

$$x^2p - 2xy_1q_1 - 2xz_1r_1 - 2(y_1 + 2xy_2)\,q_2 - 2\,(z_1 + 2xz_2)\,r_2 - 6\,(y_2 + xy_3)\,q_3 - 6\,(z_2 + xz_3)\,r_3 = 0,$$

$$y^2q + 2yy_1q_1 + 2\,(y_1^2 + yy_2)\,q_2 + 2\,(3y_1y_2 + yy_3)\,q_3 = 0,$$

$$z^2r + 2zz_1r_1 + 2\,(z_1^2 + zz_2)\,r_2 + 2\,(3z_1z_2 + zz_3)\,r_3 = 0.$$

From the first three equations, we see that the solutions, if any exist, will be free of x, y, and z; hence, we may neglect those terms containing p, q, and r. Doing this, and reducing algebraically, the system becomes

$$U_1f \equiv y_1q_1 + z_1r_1 + 2y_2q_2 + 2z_2r_2 + 3y_3q_3 + 3z_3r_3 = 0,$$
$$U_2f \equiv y_1q_1 + y_2q_2 + y_3q_3 = 0,$$
$$U_3f \equiv z_1r_1 + z_2r_2 + z_3r_3 = 0,$$
$$U_4f \equiv y_1q_2 + z_1r_2 + 3y_2q_3 + 3z_2r_3 = 0,$$
$$U_5f \equiv y_1q_2 + 3y_2q_3 = 0,$$
$$U_6f \equiv z_1r_2 + 3z_2r_2 = 0.$$

Since U_4f is the sum of U_5f and U_6f, the system reduces to one of five equations in six variables; hence there exists at least one solution.

The matrix of the coefficients of q_i and r_i is

$$\left\| \begin{array}{cccccc} y_1 & z_1 & 2y_2 & 2z_2 & 3y_3 & 3z_3 \\ y_1 & 0 & y_2 & 0 & y_3 & 0 \\ 0 & z_1 & 0 & z_2 & 0 & z_3 \\ 0 & 0 & y_1 & 0 & 3y_2 & 0 \\ 0 & 0 & 0 & z_1 & 0 & 3z_2 \end{array} \right\| .$$

All the determinants of this matrix do *not* vanish identically; hence there is *one and only one solution* of the complete system.

In order to find the solution we proceed as follows: (Cf. § 13)

The solutions common to U_5f and U_6f are

$$\frac{y_3}{y_1} - \frac{3}{2}\left[\frac{y_2}{y_1}\right]^2 \equiv \alpha\,, \quad \frac{z_3}{z_1} - \frac{3}{2}\left[\frac{z_2}{z_1}\right] \equiv \beta\,,$$

which are also solutions of U_2f and U_3f.

Introducing these solutions, as new variables, in U_1f,—since

$$U_1\alpha = 2\alpha\,, \quad U_1\beta = 2\beta\,,$$

we have the equation

$$U_7f \equiv \alpha\frac{\partial f}{\partial \alpha} + \beta\frac{\partial f}{\partial \beta} = 0\,,$$

the solution of which is

$$\frac{\beta}{\alpha} \equiv \frac{y_1^2(2z_1z_3 - 3z_2^2)}{z_1^2(2y_1y_3 - 3y_2^2)}\,.$$

Thus we have found the solution of the complete system: which is the Differential Invariant of the G_9 of the *third order*.

15. *To Find the Differential Invariants of Higher Orders.**

Suppose $\Omega\,(x, y, z, y_1, z_1, \ldots, \varphi, \varphi_1, \varphi_2, \ldots)$ to be such a function that, when φ is any Differential Invariant of the Group, Ω is also a Differential Invariant,—to find the function Ω.

We have

$$\varphi_1 dx = d\varphi\,,$$

so that

$$\delta\varphi_1 dx + \varphi_1 d\delta x = d\delta\varphi\,,$$

* Sophus Lie, Vorlesungen ueber Continuierliche Gruppen, page 670.

or

$$\delta\varphi_1 = \frac{d\delta\varphi}{dx} - \varphi_1 \frac{d\delta x}{dx};$$

but, since φ is an invariant, $\delta\varphi = 0$, and we have

$$\delta\varphi_1 = -\varphi_1 \frac{d\delta x}{dx}.$$

Now

$$\delta\Omega = \frac{\partial\Omega}{\partial x}\delta x + \ldots + \frac{\partial\Omega}{\partial y_1}\delta y_1 + \ldots + \frac{\partial\Omega}{\partial\varphi}\delta\varphi + \frac{\partial\Omega}{\partial\varphi_1}\delta\varphi_1;$$

and since $\delta\Omega$ must be zero for every transformation of the Group, we have the following complete system, from which Ω may be determined:

$$p = 0, \quad q = 0, \quad r = 0,$$

$$xp - y_1q_1 - z_1r_1 - 2y_2q_2 - 2z_2r_2 - 3y_3q_3 - \varphi_1\frac{\partial\Omega}{\partial\varphi_1} = 0,$$

$$yq + y_1q_1 + y_2q_2 + y_3q_3 = 0,$$

$$zr + z_1r_1 + z_2r_2 = 0,$$

$$x^2p - 2xy_1q_1 - 2xz_1r_1 - 2(y_1 + 2xy_2)q_2 - 2(z_1 + 2xz_2)r_2 - 6(y_2 + xy_3)q_3 - 2x\varphi_1\frac{\partial\Omega}{\partial\varphi_1} = 0,$$

$$y^2q + 2yy_1q_1 + 2(y_1^2 + yy_2)q_2 + 2(3y_1y_2 + yy_3)q_3 = 0,$$

$$z^2r + 2zz_1r_1 + 2(z_1^2 + zz_2)r_2 = 0,$$

where

$$p \equiv \frac{\partial\Omega}{\partial x}, \quad q_i \equiv \frac{\partial\Omega}{\partial y_i}, \quad r_i \equiv \frac{\partial\Omega}{\partial z_i}.$$

The first three equations show that the solutions will be independent of x, y, and z; hence, we neglect those terms containing p, q, and r.

By algebraic reduction these equations become

$$U_1\Omega \equiv y_1q_1 + z_1r_1 + 2y_2q_2 + 2z_2r_2 + 3y_3q_3 + \varphi_1\frac{\partial\Omega_1}{\partial\varphi_1} = 0,$$

$$U_2\Omega \equiv y_1q_1 + y_2q_2 + y_3q_3 = 0,$$

$$U_3\Omega \equiv z_1r_1 + z_2r_2 = 0,$$

$$U_4\Omega \equiv y_1q_2 + z_1r_2 + 3y_2q_3 = 0,$$

$$U_5\Omega \equiv y_1q_2 + 3y_2q_3 = 0,$$

$$U_6\Omega = z_1r_2 = 0.$$

We see from $U_3\Omega$ and $U_6\Omega$ that the solution will be free of z_1, z_2, and z_3. The solutions common to $U_2\Omega$ and $U_5\Omega$ are

$$\frac{y_3}{y_1}-\frac{3}{2}\left[\frac{y_2}{y_1}\right]^2\equiv\alpha\,,\ \text{and}\ \varphi_1\,.$$

Introduce these, as new variables, into $U_1\Omega$; hence, as

$$U_1\alpha=2\alpha\,,\quad U_1\varphi_1=\varphi_1\,,$$

we have the equation

$$U_7\Omega\equiv 2\alpha\frac{\partial\Omega}{\partial\alpha}+\varphi_1\frac{\partial\Omega}{\partial\varphi_1}=0\,.$$

The solution of this equation is

$$\Delta\varphi\equiv\frac{\varphi_1}{\alpha^{1/2}}\,.$$

$\Delta\varphi$ is called a "*Differential Parameter*" and has the general form

$$\Omega\,(x,\,y,\,z,\,y_1,\,z_1,\,\ldots,\,\varphi,\,\Delta\varphi)\,.$$

Since $\Delta\varphi$ is an Invariant,

$$\Delta^2\varphi\equiv\Delta\,(\Delta\varphi)\equiv\frac{\;\frac{d\Delta\varphi}{dx}\;}{\alpha^{1/2}}$$

is also a Differential Parameter, and likewise $\Delta^3\varphi$, etc., hence the most general Differential Parameter is

$$\Omega\,(x,\,y,\,z,\,y_1,\,z_1,\,\ldots,\,\varphi,\,\Delta\varphi_1,\,\Delta^2\varphi,\,\ldots)\,.$$

16. We shall now make use of the Differential Parameter to find one of the Differential Invariants of the fourth order.

Let

$$\varphi\equiv\frac{\beta}{\alpha}\,,$$

where

$$\beta\equiv\frac{z_3}{z_1}-\frac{3}{2}\left[\frac{z_2}{z_1}\right]^2,\quad\alpha\equiv\frac{y_3}{y_1}-\frac{3}{2}\left[\frac{y_2}{y_1}\right]^2.$$

Now

$$\varphi_1=\frac{d\varphi}{dx}=\frac{\beta_1\alpha-\alpha_1\beta}{\alpha^2}\,,$$

where

$$\beta_1\equiv\frac{d\beta}{dx}\,,\quad\alpha_1\equiv\frac{d\alpha}{dx}\,.$$

Hence

$$\Delta\varphi\equiv\frac{\varphi_1}{\alpha^{1/2}}\equiv\frac{\beta_1\alpha-\alpha_1\beta}{\alpha^{5/2}}$$

is a Differential Invariant of the *fourth order*.

17. *To Find the Other Differential Invariants of the Fourth Order.*

By extending each transformation of the G_9 four times, and by reducing the equation algebraically, remembering that the solutions will be free of x, y, and z, we get following complete system:

$$
\begin{aligned}
U_1f &\equiv y_1q_1 + z_1r_1 + 2y_2q_2 + 2z_2r_2 + 3y_3q_3 + 3z_3r_3 + 4y_4q_4 + 4z_4r_4 = 0\,,\\
U_2f &\equiv y_1q_1 + y_2q_2 + y_3q_3 + y_4q_4 = 0\,,\\
U_3f &\equiv z_1r_1 + z_2r_2 + z_3r_3 + z_4r_4 = 0\,,\\
U_4f &\equiv y_1q_2 + z_1r_2 + 3y_2q_3 + 3z_2r_3 + 6y_3q_4 + 6z_3r_4 = 0\,,\\
U_5f &\equiv y_1^2q_2 + 3y_1y_2q_3 + (3y_2^2 + 4y_1y_3)\,q_4 = 0\,,\\
U_6f &\equiv z_1^2r_2 + 3z_1z_2r_3 + (3z_2^2 + 4z_1z_3)\,r_4 = 0\,.
\end{aligned}
$$

This is a complete system of six equations in eight variables, and has at least two solutions.

The matrix of the coefficients of q_i, r_i is

$$
\left\|\begin{array}{cccccccc}
y_1 & z_1 & 2y_2 & 2z_2 & 3y_3 & 3z_3 & 4y_4 & 4z_4\\
y_1 & 0 & y_2 & 0 & y_3 & 0 & y_4 & 0\\
0 & z_1 & 0 & z_2 & 0 & z_3 & 0 & z_4\\
0 & 0 & y_1 & z_1 & 3y_2 & 3z_2 & 6y_3 & 6z_3\\
0 & 0 & y_1^2 & 0 & 3y_1y_2 & 0 & (3y_2^2 + 4y_1y_3) & 0\\
0 & 0 & 0 & z_1^2 & 0 & 3z_1z_2 & 0 & (3z_2^2 + 4z_1z_3)
\end{array}\right\|.
$$

All the determinants of this matrix do *not* vanish identically; hence there are *only two independent* solutions of the complete system.

In order to find the solutions of the complete system, we proceed as follows (see § 13):

The solutions common to U_2f and U_3f are

$$
\frac{y_2}{y_1} \equiv a_2\,,\quad \frac{y_3}{y_1} \equiv a_3\,,\quad \frac{y_4}{y_1} \equiv a_4\,,
$$

$$
\frac{z_2}{z_1} \equiv b_2\,,\quad \frac{z_3}{z_1} \equiv b_3\,,\quad \frac{z_4}{z_1} \equiv b_4\,.
$$

Introduce these solutions, as new variables, into U_5f and U_6f,—since

$$
U_5a_2 = y_1\,,\quad U_5a_3 = 3a_2y_1\,,\quad U_5a_4 = (3a_2^2 + 4a_3)\,y_1\,,
$$

$$
U_6b_2 = z_1\,,\quad U_6b_3 = 3b_2z_1\,,\quad U_6b_4 = (3b_2^2 + 4b_3)\,z_1\,,
$$

we have the equations

$$U_7f \equiv \frac{\partial f}{\partial a_2} + 3a_2\frac{\partial f}{\partial a_3} + (3a_2^2 + 4a_3)\frac{\partial f}{\partial a_4} = 0\,,$$

$$U_8f \equiv \frac{\partial f}{\partial b_2} + 3b_2\frac{\partial f}{\partial b_3} + (3b_2^2 + 4b_3)\frac{\partial f}{\partial b_3} = 0\,.$$

The solutions of these equations are

$$a_3 - \frac{3}{2}a_2^2 \equiv \frac{y_3}{y_1} - \frac{3}{2}\left[\frac{y_2}{y_1}\right]^2 \equiv \alpha\,,$$

$$b_3 - \frac{3}{2}b_2^2 \equiv \frac{z_3}{z_1} - \frac{3}{2}\left[\frac{z_2}{z_1}\right]^2 \equiv \beta\,,$$

$$a_4 - 4a_2a_3 + 3a_2^3 \equiv \frac{y_4}{y_1} - 4\frac{y_2y_3}{y_1^2} + 3\left[\frac{y_2}{y_1}\right]^3 \equiv \frac{d\alpha}{dx} \equiv \alpha_1\,,$$

$$b_4 - 4b_2b_3 + 3b_2^3 \equiv \frac{z_4}{z_1} - 4\frac{z_2z_3}{z_1^2} + 3\left[\frac{z_2}{z_1}\right]^3 \equiv \frac{d\beta}{dx} \equiv \beta_1\,.$$

Introduce these solutions, as new variables, into U_1f and U_4f; since

$$U_1\alpha = 2\alpha\,,\quad U_1\alpha_1 = 3\alpha_1\,,\quad U_1\beta = 2\beta\,,\quad U_1\beta_1 = 3\beta_1\,,$$
$$U_4\alpha = 0\,,\quad U_4\alpha_1 = 2\alpha\,,\quad U_4\beta = 0\,,\quad U_4\beta_1 = 2\beta\,,$$

we have the equations

$$U_9f \equiv 2\alpha\frac{\partial f}{\partial \alpha} + 2\beta\frac{\partial f}{\partial \beta} + 3\alpha_1\frac{\partial f}{\partial \alpha_1} + 3\beta_1\frac{\partial f}{\partial \beta_1} = 0\,,$$

$$U_{10}f \equiv \alpha\frac{\partial f}{\partial \alpha_1} + \beta\frac{\partial f}{\partial \beta_1} = 0\,.$$

The solutions of $U_{10}f$ are

$$\alpha,\ \beta,\ \alpha\beta_1 - \alpha_1\beta \equiv \varphi\,.$$

Finally, introduce these solutions, as new variables, into U_9f; since

$$U_9\varphi = 5\varphi\,,\ U_9\alpha = 2\alpha\,,\ U_9\beta = 2\beta\,,$$

we have the equation

$$U_{11}f \equiv 2\alpha\frac{\partial f}{\partial \alpha} = 2\beta\frac{\partial f}{\partial \beta} + 5\varphi\frac{\partial f}{\partial \varphi} = 0\,,$$

the solutions of which are

$$\frac{\beta}{\alpha},\ \frac{\varphi}{\alpha^{5/2}} \equiv \frac{\alpha\beta_1 - \alpha_1\beta}{\alpha^{5/2}},\ \frac{\varphi}{\beta^{5/2}} \equiv \frac{\alpha\beta_1 - \alpha_1\beta}{\beta^{5/2}}\,.$$

Two, and only two, of these solutions are independent; for, it is clear that the first is a function of the ratio of the last two.

18. Having found the two Differential Invariants of the fourth order, we shall now, by means of the Differential Parameter, find the Differential Invariants of the fifth, sixth, seventh, and eighth orders.

Let

$$\varphi_4 = \frac{\alpha\beta_1 - \alpha_1\beta}{\alpha^{5/2}},$$

$$\psi_4 \equiv \frac{\alpha\beta_1 - \alpha_1\beta}{\beta^{5/2}},$$

then

$$\varDelta\varphi_4 \equiv \frac{1}{\alpha^{1/2}}\frac{d\varphi_4}{dx} \equiv \frac{\alpha^2\beta_2 - \alpha\alpha_2\beta - \frac{5}{2}\alpha\alpha_1\beta_1 + \frac{5}{2}\alpha_1{}^2\beta}{\alpha^4},$$

$$\varDelta\psi_4 \equiv \frac{1}{\alpha^{1/2}}\frac{d\psi_4}{dx} \equiv \frac{\alpha\beta\beta_2 - \alpha_2\beta^2 - \frac{5}{2}\alpha\beta_1{}^2 + \frac{5}{2}\alpha_1\beta\beta_1}{\alpha^{1/2}\beta^{7/2}},$$

are independent Differential Invariants of the fifth order.

If we multiply the first by $\frac{\beta}{\alpha}$ and the second by $\frac{\alpha\beta^{7/2}}{\alpha^{9/2}}$, the results will still be independent Differential Invariants of the fifth order, since $\frac{\beta}{\alpha}$, and every function of $\frac{\beta}{\alpha}$, is an Invariant.

Adding together these results, we have

$$\frac{2\beta\beta_2 - \frac{5}{2}\beta_1{}^2}{\alpha^3} - \frac{\beta^2\left[2\alpha\alpha_2 - \frac{5}{2}\alpha_1{}^2\right]}{\alpha^5}.$$

Multiplying this by $\frac{2\alpha^3}{\beta^3}$, we get

$$\frac{4\beta\beta_2 - 5\beta_1{}^2}{\beta^3} - \frac{\alpha}{\beta}\frac{4\alpha\alpha_2 - 5\alpha_1{}^2}{\alpha^3};$$

hence the Differential Invariants of the fifth order may be written

$$\varphi_5 \equiv \frac{4\alpha\alpha_2 - 5\alpha_1{}^2}{\alpha^3}, \quad \psi_5 \equiv \frac{4\beta\beta_2 - 5\beta_1{}^2}{\beta^3},$$

the first of which, φ_5, contains y_i only, and the second, ψ_5, contains z_i only.

If we multiply the Differential Parameter, $\varDelta\varphi \equiv \frac{1}{\alpha^{1/2}}\frac{d\varphi}{dx}$ by $\left[\frac{\alpha}{\beta}\right]^{1/2}$ we get $\varDelta'\varphi \equiv \frac{1}{\beta^{1/2}}\frac{d\varphi}{dx}$, which is also a Differential Parameter, since it is a function of $\frac{\beta}{\alpha}$ and $\varDelta\varphi$ (§ 15).

We can now write down the Differential Invariants of the sixth order;

$$\varphi_6 \equiv \frac{1}{a^{1/2}}\frac{d\varphi_5}{dx} = \frac{4a^2a_3 - 18aa_1a_2 + 15a_1^3}{a^{9/2}},$$

$$\psi_6 \equiv \frac{1}{\beta^{1/2}}\frac{d\psi_5}{dx} = \frac{4\beta^2\beta_3 - 18\beta\beta_1\beta_2 + 15\beta_1^3}{\beta^{9/2}};$$

of the seventh order;

$$\varphi_7 \equiv \frac{1}{a^{1/2}}\frac{d\varphi_6}{dx} = \varphi_6\frac{d\varphi_6}{d\varphi_5},$$

$$\psi_7 \equiv \frac{1}{\beta^{1/2}}\frac{d\psi_6}{dx} = \psi_6\frac{d\psi_6}{d\psi_5};$$

finally, of the eighth order;

$$\varphi_8 \equiv \frac{1}{a^{1/2}}\frac{d\varphi_7}{dx} = \frac{1}{a^{1/2}}\frac{d\varphi_6}{dx}\frac{d\varphi_6}{d\varphi_5} + \frac{1}{a^{1/2}}\varphi_6\frac{d^2\varphi_6}{d\varphi_5^2}\frac{d\varphi_5}{dx}$$

$$\equiv \varphi_6\left[\frac{d\varphi_6}{d\varphi_5}\right]^2 + \varphi_6\frac{d^2\varphi_6}{d\varphi_5^2},$$

$$\psi_8 \equiv \frac{1}{\beta^{1/2}}\frac{d\psi_7}{dx} = \psi_6\left[\frac{d\psi_6}{d\psi_5}\right]^2 + \psi_6^2\frac{d^2\psi_6}{d\psi_5^2}.$$

The φ_i contain y_i, only, and are, therefore, invariant under the Sub-Group

$$\boxed{p, \quad q, \quad xp, \quad yq, \quad x^2p, \quad y^2q};$$

and the ψ_i contain z_i only, and are, consequently, invariant under the Sub-Group

$$\boxed{p, \quad r, \quad xp, \quad zr, \quad x^2p, \quad z^2r}.$$

19. *To Find what Differential Equations are Invariant under the* G_9.

In order to find the invariant differential equations of the nth order, which are independent of the Differential Invariants, we have to form the matrix, or determinant, of the coefficients of p, q, r, p_i, q_i $(i = 1, 2, \ldots; n)$ occurring in the simultaneous system obtained by equating to zero each transformation extended n times. Having formed the matrix, or determinant, we have to find what equations $f(x, y, z, y_i, z_i) = 0$ make all the determinants of the matrix, or all the minors of the determinant, zero simultaneously.

By examining the matrices, or determinant, it may readily be shown that there are no differential equations, of an order lower than the third, invariant under the G_9, except those obtained from the Differential Invariant of the third order.

(a). Since the Differential Invariant of the third order does not contain x, y, or z, there are no differential equations of the zero order invariant under the G_9.

(b). If we put the Differential Invariant of the third order,

$$\frac{\beta}{\alpha}=\frac{y_1^2(2z_1z_3-3z_2^2)}{z_1^2(2y_1y_3-3y_2^2)},$$

first equal to zero, and then equal to infinity, we obtain the following invariant differential equations of the first order

$$y_1=0\,,\ \frac{1}{z_1}=0\,,\ z_1=0\,,\ \frac{1}{y_1}=0\,.$$

These together with the equation $x_1=0$ (which, of course, is invariant) are the *only differential equations of the first order invariant under the G_9.*

(c). It is clear that there are *no differential equations of the second order invariant under the G_9.*

The two equations, of the second order,

$$y_2=0\,,\quad z_2=0\,,$$

are invariant under the G_9, only when they are considered *in connection with and as a consequence of $y_1=0$, and $z_1=0$.* Hence they are excluded in this discussion.

(d). If we put the Differential Invariant of the third order,

$$\frac{\beta}{\alpha}=\frac{y_1^2(2z_1z_3-3z_2^2)}{z_1^2(2y_1y_3-3y_2^2)},$$

first equal to zero and then equal to infinity, we obtain the following invariant differential equations, of the third order,

$$2z_1z_3-3z_2^2=0\,,$$

$$2y_1y_3-3y_2^2=0\,.$$

These are the only *two independent* differential equations of the third order invariant under the G_9.

(e). The invariant differential equations of higher orders are obtained by putting some arbitrary function of the Differential invariants equal to zero.

III.

Equivalence of Curves.

20. In this section we shall make use of the Differential Invariants and invariant differential equations found in the last section, in order to determine the nature of the curve-families which are composed of those curves which are "equivalent" by means of the Group.

21. Two curves are said to be "equivalent" by means of a Group, if, by means of the transformations of the Group, the one curve can be carried over into the other.

Suppose a curve is subjected to the transformations of a Group of r parameters; it will then assume ∞^r different positions, provided that it is not invariant under (or, as we sometimes say, does not "admit of") any one of the r transformations of the Group. In this case, there is generated a family consisting of ∞^r curves; and it is clear that curves, which admit of no transformation of the G_r and which are *equivalent*, must belong to such a family. If the original curve admits of q independent infinitesimal transformations of the G_r, it is readily seen that the resulting family will consist of ∞^{r-q} curves; and these ∞^{r-q} curves are, *eo ipso*, *equivalent* among each other by means of the G_r.

It is evident that such a curve-family is *invariant*, where invariant is taken in the sense that the curves of the family are interchanged among each other, while the family *as a whole* is unchanged.

This family is *defined by two independent differential equations, one of the mth order and one of the nth order, where $m + n = r - q$, which as a system is invariant under the G_9.*

22. Since the curve-family, to which the equivalent curves belong, is defined by an invariant system of differential equations, we shall, as a matter of convenience, put here those invariant systems of differential equations, which we shall need in the discussion of this subject.

The most general invariant differential equations are (see § 19):

0. O. None,

I. O. $y_1 = 0$, $z_1 = 0$, $x_1 = 0$, (see § 19, b)

II. O. None,

III O. $\psi_3 = \text{const.}$, $2z_1z_3 - 3z_1^2 = 0$, $2y_1y_3 - 3y_1^2 = 0$, (see 19, d)

IV. O. $\Omega(\psi_3, \psi_4, \varphi_4) = 0$,

$$\text{V. } O. \quad \Omega(\psi_3, \psi_4, \psi_5, \varphi_4, \varphi_5) = 0\,,$$

$$\text{VI. } O. \quad \Omega(\psi_3, \ldots, \psi_6, \varphi_4, \varphi_5, \varphi_6) = 0\,,$$

$$\text{VII. } O. \quad \Omega\left[\psi_3, \ldots, \psi_6, \frac{d\psi_6}{d\psi_5}, \varphi_4, \ldots, \frac{d\varphi_6}{d\varphi_5}\right] = 0\,,$$

$$\text{VIII. } O. \quad \Omega\left[\psi_3, \ldots, \frac{d\psi_6}{d\psi_5}, \frac{d^2\psi}{d\varphi_5^2}, \varphi_4, \ldots, \frac{d^2\varphi_6}{d\varphi_5^2}\right] = 0\,.$$

We can now write down the invariant systems of differential equations, which must consist of one differential equation of the mth order and one of the nth order, where $m + n = 9 - q$, and where q is the number of independent infinitesimal transformations of which the curve admits.

There are three invariant differential equations of the first order; but, on account of the symmetry of the Group, we may clearly choose $z_1 = 0$ as the typical invariant differential equation of the first order. Now when $z_1 = 0$, it is readily seen that $\psi_3, \psi_4, \psi_5, \psi_6, \dfrac{d\psi_6}{d\psi_5}, \dfrac{d^2\psi_6}{d\psi_5^2}$, and φ_4 become zero, so that the invariant systems containing $z_1 = 0$ are:

I. and 0. O. None,

I. and I. O. $\begin{cases} z_1 = 0\,, \\ y_1 = 0\,; \end{cases}$

I. and II. O. None,

I. and III. O. $\begin{cases} z_1 = 0, \\ 2y_1y_3 - 3y_1^2 = 0\,, \end{cases}$ (See § 19, d)

I. and IV. O. None,

I. and V. O. $\begin{cases} z_1 = 0\,, \\ \varphi_5 = \text{const.}, \end{cases}$

I. and VI. O. $\begin{cases} z_1 = 0\,, \\ \varphi_6 = f(\varphi_5)\,, \end{cases}$

I. and VII. O. $\begin{cases} z_1 = 0\,, \\ \Omega\left[\varphi_5, \varphi_6, \dfrac{d\varphi_6}{d\varphi_5}\right] = 0\,, \end{cases}$

I. and VIII. O. $\begin{cases} z_1 = 0\,, \\ \Omega\left[\varphi_5, \varphi_6, \dfrac{d\varphi_6}{d\varphi_5}, \dfrac{d^2\varphi_6}{d\varphi_5^2}\right] = 0\,. \end{cases}$

As there is no invariant differential equation of the second order, there are, evidently, no invariant systems containing a differential equation of the second order.

If $\psi_3 = \text{const.}$, it is evident that ψ_4, ψ_5, ψ_6, and φ_4 are identically zero, so that the invariant systems containing a differential equation of the third order are:

III. and III. O. $\begin{cases} 2z_1z_3 - 3z_2^2 = 0\,, \\ 2y_1y_3 - 3y_2^2 = 0\,, \end{cases}$ (See 19, d)

III. and IV. O. None,

III. and V. O. $\begin{cases} \psi_3 = \text{const.}, \\ \psi_5 = \text{const.}, \end{cases}$

III. and VI. O. $\begin{cases} \psi_3 = \text{const.}, \\ \varphi_6 = f(\varphi_5). \end{cases}$

The invariant systems containing an invariant differential equation of the fourth order are

IV. and IV. O. $\begin{cases} \Omega_1(\psi_3, \psi_4, \varphi_4) = 0\,, \\ \Omega_2(\psi_3, \psi_4, \varphi_4) = 0\,, \end{cases}$

IV. and V. O. $\begin{cases} \Omega(\psi_3, \psi_4, \varphi_4) = 0\,, \\ \Omega(\psi_3, \psi_4, \psi_5, \varphi_4, \varphi_5) = 0\,. \end{cases}$

23. We shall now show how to find the curve-families to which those curves belong that are equivalent by means of the Group of nine parameters

$$\boxed{p, \quad q, \quad r, \quad xp, \quad yq, \quad zr, \quad x^2p, \quad y^2q, \quad z^2r}\,.$$

I. *If the curve admits of no infinitesimal transformation of the G_9, it will generate a family of ∞^9 curves, when subjected to all the transformations of the G_9.*

(a). This family might be defined by an invariant system consisting of one differential equation of the *zero order* and one of the *ninth order;* but, as there is no invariant system containing a differential equation of the zero order, this case is excluded.

(b). This family may be defined by the invariant system

$$z_1 = 0$$

$$\Omega\left[\varphi_5,\ \varphi_6,\ \frac{d\varphi_6}{d\varphi_5},\ \frac{d^2\varphi_6}{d\varphi_5^2}\right] = 0\,,$$

which consists of one differential equation of the *first order* and one of the *eighth order.*

Evidently the curves of this invariant family are plane curves in the planes $z = \text{const.}$ Since zr and z^2r change nothing in the xy-plane, these ∞^8 plane curves are the same in each plane $z = \text{const.}$; hence, in the xy-plane they are defined by the invariant differential equation

$$\Omega\left[\varphi_5,\ \varphi_6,\ \frac{d\varphi_6}{d\varphi_5},\ \frac{d^2\varphi_6}{d\varphi_5^2}\right] = 0\,.$$

which is of the eighth order, and is given in terms of $y_1, \ldots, y_8$.

(c). This family might be defined by an invariant system containing one differential equation of the *second order* and one of the *seventh order;* but, as there is no invariant system containing a differential equation of the second order, this case is excluded.

(d). The family may be defined by the invariant system

$$\psi_3 = \text{const.},$$

$$\varphi_6 = f(\varphi_5)\,,$$

which consists of one differential equation of the *third order* and one of the *sixth order.*

If $\psi_3 = 0$, the family will be defined by the invariant system

$$2z_1z_3 - 3z_2^2 = 0\,,$$

$$\varphi_6 = f(\varphi_5)\,.$$

The integral of the first of these equations is

$$xz + a_1x + b_1z + c_1 = 0\,,$$

and the curves of the family will then be the intersections of the hyperbolic cylinders represented by this equation, and the cylinders given by the equation

$$\varphi_6 = f(\varphi_5).$$

If $\psi_3 = \infty$, then $\alpha \equiv \frac{y_3}{y_1} - \frac{3}{2}\left[\frac{y_2}{y_1}\right]^2 = 0$; and, consequently, φ_6 and φ_5 become identically zero; hence this case is excluded.

(e). This family may be defined by the invariant system

$$\Omega(\psi_3, \psi_4, \varphi_4) = 0,$$

$$\Omega(\psi_3, \psi_4, \psi_5, \varphi_4, \varphi_5) = 0,$$

which consists of one differential equation of the *fourth order* and one of the *fifth order*.

II. *If the curve admits of one of the infinitesimal transformations of the G_9, it will generate a family of ∞^8 curves, when subjected to the transformations of the G_9.*

(a). The first possible case is that this family may be defined by the invariant system

$$z_1 = 0,$$

$$\Omega\left[\varphi_5, \varphi_6, \frac{d\varphi_6}{d\varphi_5}\right] = 0,$$

which consists of one differential equation of the *first order* and one of the *seventh order*. As in I, (b), the curves are plane curves in the planes $z =$ const.

(b). The next possible case is that this family may be defined by the invariant system

$$\psi_3 = \text{const.},$$

$$\varphi_5 = \text{const.},$$

which consists of one differential equation of the *third order* and one of the *fifth order*.

The same reasoning holds here, as in I, (d), when ψ_3 becomes either zero or infinity.

(c). This family may be defined by the invariant system

$$\Omega_1(\psi_3, \psi_4, \varphi_4) = 0,$$

$$\Omega_2(\psi_3, \psi_4, \varphi_4) = 0,$$

which consists of two differential equations of the *fourth order*.

III. *If the curve admits of two of the infinitesimal transformations of the G_9, it will generate a family of ∞^7 curves when subjected to the transformations of the G_9.*

The only system invariant under the G_9, which will define this curve-family is

$$z_1 = 0\,,$$
$$\varphi_6 = f(\varphi_5)\,,$$

which consists of one differential equation of the *first order* and one of the *sixth order*. As in I, (b), the curves are plane curves in the planes $z =$ const.

IV. *If the curve admits of three of the infinitesimal transformations of the G_9, it will generate a family of ∞^6 curves when subjected to the transformations of the G_9.*

(a). This family may be defined by the invariant system

$$z_1 = 0\,,$$
$$\varphi_5 = \text{const.},$$

which consists of one differential equation of the first order and one of the fifth order. As in I, (b), the curves are plane curves in the plane $z =$ const.

(b). This family may be defined by the invariant system

$$2z_1z_3 - 3z_2^2 = 0\,,$$
$$2y_1y_3 - 3y_2^2 = 0\,,$$

which consists of two differential equations of the third order.

The integrals of these equations are

$$xz - a_1x + b_1z + c_1 = 0\,,$$
$$xy + a_2x + b_2y + c_2 = 0\,;$$

hence the ∞^6 curves are, in this case the intersections of the two families of hyperbolic cylinders, given by the above equations.

V. *If the curve admits of four of the infinitesimal transformations of the G_9, it will generate a family of ∞^5 curves when subjected to the transformations of the G_9.*

There is no system invariant under the G_9, that will define this family of ∞^5 curves. Hence no such invariant family exists.

VI. *If the curve admits of five of the infinitesimal transformations of the G_9, it will generate a family of ∞^4 curves when subjected to the transformations of the G_9.*

The only system invariant under the G_9, which will define this family is

$$z_1 = 0\,,$$

$$2y_1y_3 - 3y_2^2 = 0\,,$$

which consists of one differential equation of the first order and one of the third order. As in I, (b), the curves will be plane curves in the planes $z =$ const.

The integral of the equation, $2y_1y_3 - 3y_2^2 = 0$ is

$$xy + a_2x + b_2y + c_2 = 0\,;$$

hence the curves are hyperbolas in the planes $z =$ const.

VII. *If the curve admits of six of the infinitesimal transformations of the G_9, it will generate a family of ∞^3 curves when subjected to the transformations of the G_9.*

There is no system invariant under the G_9, which will define this family of ∞^3 curves. Hence no such invariant family exists.

VIII. *If the curve admits of seven of the infinitesimal transformations of the G_9, it will generate a family of ∞^2 curves when subjected to the transformations of the G_9.*

The typical system invariant under the G_9 which will define this family is

$$z_1 = 0\,,$$

$$y_1 = 0\,,$$

which consists of two differential equations of the *first order*.

In this case, it is evident that the curve-family consists of the ∞^2 straight lines perpendicular to the xz-plane.

IX. *If the curve admits of eight of the infinitesimal transformations of the G_9, it will generate a family of ∞^1 curves when subjected to the transformations of the G_9.*

There is no system invariant under the G_9 that will define this family of ∞^1 curves; consequently, no such invariant family exist.

Hence, if the two curves are equivalent by means of the transformations of the G_9, they must both belong to some of the families defined above. When the equations to the two curves are given, we can substitute in the types of differential equations defining the above invariant families, and if such differential equations are satisfied by the equations of *both* curves, the curves belong to that family and are equivalent.

24. Lie has developed a theory of integration, which may be applied to the integration of some of the above differential equations. We shall, therefore, give here two of his theorems.*

I. "If a differential equation of the mth order admits of the Group

$$\boxed{q, \quad yq, \quad y^2q, \quad p, \quad xp, \quad x^2p},$$

it is reducible to the form

$$\Omega\left[\varphi_5, \varphi_6, \ldots, \frac{d^{m-6}\varphi_6}{d\varphi_5^{m-6}}\right] = 0,$$

where

$$\varphi_5 = \frac{4aa_2 - 5a_1^2}{a^3},$$

$$\varphi_6 = \frac{4a^2a_3 - 18aa_1a_2 + 15a_1^3}{a^{9/2}},$$

and

$$a = \frac{y_3}{y_1} - \frac{3}{2}\left[\frac{y_2}{y_1}\right]^2, \quad a_1 = \frac{da}{dx}, \ldots.$$

By integrating the equation of the $(m-6)$th order, $\Omega = 0$, we get a relation

$$\varphi_6 = f(\varphi_5),$$

which is a differential equation of the third order in x and a. This equation admits of the three known infinitesimal transformations p, xp, x^2p, which, in the variables x and a, have the forms

$$\frac{\partial f}{\partial x}, \quad x\frac{\partial f}{\partial x} - 2a\frac{\partial f}{\partial a}, \quad x^2\frac{\partial f}{\partial x} + 4xa\frac{\partial f}{\partial a}.$$

In order to integrate the above differential equation of the third order, we introduce as new variables the quantities

$$u = a^{1/2}a_1, \quad \varphi_5 = 4a^{-2}a_2 - 5a^{-3}a_1^2;$$

then

$$\frac{du}{d\varphi_5} = \frac{\varphi_5 - u^2}{\varphi_6}.$$

If $W(u, \varphi_5) = \text{const.}$, is an integral of this Riccatian equation of the first order, we find, in the following manner, by mere differentiation, the other two integral equations of the differential equation $\varphi_6 = f(\varphi_5)$.

* Math. Annalen, Bd. XXXII, pp. 262, 260.

If we put

$$Uf \equiv x^2 \frac{\partial f}{\partial x} - 4xa \frac{\partial f}{\partial a} - (6xa_1 + 4a) \frac{\partial f}{\partial a_1} - (8xa_2 + 10a_1) \frac{\partial f}{\partial a_2},$$

then

$$U\varphi_1 = 0,$$

$$UW = \frac{\partial f}{\partial u} Uu = -4 \frac{\partial W}{\partial u} a^{-\frac{1}{2}},$$

$$UUW = 16 \frac{\partial^2 W}{\partial u^2} a^{-1} - 8xa^{-\frac{1}{2}} \frac{\partial W}{\partial a};$$

and since the quantities W, UW, and UUW are independent as regards u, φ_5, x, and a, by eliminating the quantities in u, and φ_5 between the three equations

$$W = \text{const.}, \quad UW = \text{const.}, \quad UUW = \text{const.}$$

we find the quantity a determined as a function of x in the form

$$a = F(x).$$

This equation is a differential equation of the third order in x and y."

II. "If a differential equation of the mth order admits of the Group q, yq, y^2q, it can be reduced to the form

$$\Omega \left[x, \ a, \ \frac{da}{dx}, \ \ldots, \ \frac{d^{m-3}a}{dx^{m-3}} \right] = 0,$$

where

$$a = \frac{y_3}{y_1} - \frac{3}{2} \left[\frac{y_2}{y_1} \right]^2 = 0.$$

If we integrate this equation of the $(m-3)$th order, we get a differential equation of the third order of the form

$$\frac{y_3}{y_1} - \frac{3}{2} \left[\frac{y_2}{y_1} \right]^2 = F(x),$$

which can be reduced to a Riccatian equation of the first order.

If we put

$$\frac{y_2}{y_1} = z,$$

we get

$$\frac{dz}{dx} = \frac{y_3}{y_1} - \left[\frac{y_2}{y_1} \right]^2,$$

or

$$\frac{dz}{dx} = \frac{1}{2} z^2 + F(x). \tag{1}$$

If $\theta(z_1x) = \text{const.}$ is an integral of this Riccatian equation, we find, in the following manner, the other two integral equations of $a = F(x)$, by mere differentiation.

When we put

$$Uf \equiv y^2 \frac{\partial f}{\partial y} + 2yy_1 \frac{\partial f}{\partial y_1} + (2yy_2 + 2y_1^2) \frac{\partial f}{\partial y_2},$$

then $U\theta =$ const., and $UU\theta =$ const. are known integrals of the equation $a = F(x)$.

It is necessary to show that θ, $U\theta$, and $UU\theta$ are independent functions of x, y, y_1 and y_2.

We have

$$U(x) = 0,$$

$$U\theta = \frac{\partial \theta}{\partial z} Uz = 2 \frac{\partial \theta}{\partial z} y_1,$$

$$UU\theta = 4 \frac{\partial^2 \theta}{\partial z^2} y_1^2 + 4 \frac{\partial \theta}{\partial z} yy_1;$$

so that θ, $U\theta$, $UU\theta$ are independent as regards y, x, y_1 and z. Hence the integration of $a = F(x)$ is made to depend upon that of the Riccatian equation (1)."

25. The equations

$$\Omega \left[\varphi_5, \varphi_6, \frac{d\varphi_6}{d\varphi_5}, \frac{d^2\varphi_6}{d\varphi_5^2} \right] = 0,$$

$$\Omega \left[\varphi_5, \varphi_6, \frac{d\varphi_6}{d\varphi_5} \right] = 0.$$

$$\varphi_6 = f(\varphi_5) = 0,$$

all admit of the Group

$$\boxed{q, \quad yq, \quad y^2q, \quad p, \quad xp, \quad x^2p};$$

hence the two theorems above may be applied to the integration of these equations as soon as we know the functions Ω and f.

26. Theorem II may be applied to the integration of the equation

$$\varphi_5 = \text{const.},$$

which may be written in the form

$$4aa_1 \frac{da_1}{da} - 5a_1^2 - k_1 a^3 = 0.$$

To integrate this equation, we put $z = a_1^2$, so that

$$\frac{dz}{dx} - \frac{5}{2}\frac{z}{a} - \frac{k_1}{2}a^2 = 0\,,$$

and, hence,

$$z = a_1^2 = k_1 a^3 + k_2 a^{5/2}.*$$

Solving this equation, we find

$$a_1 = a\sqrt{k_1 a + k_2 a^{1/2}}\,,$$

the integral of which is

$$-\frac{4}{k_2}\sqrt{\frac{k_1 a^{1/2} + k_2}{a^{1/2}}} = x + k_3\,,$$

which may be put in the form

$$a = (a^2x^2 + 2bx + c)^{-2},$$

or

$$\frac{y_3}{y_1} - \frac{3}{2}\left[\frac{y_2}{y_1}\right]^2 = (a^2x^2 + 2bx + c)^{-2}.$$

Now by writing $z = \frac{y_2}{y_1}$, we find

$$\frac{dz}{dx} = \frac{1}{2}z^2 + (a^2x + 2bx + c)^{-2}.$$

Suppose $\theta(z, x) = \text{const.}$, is an integral of this Riccatian equation of the first order. If we put

$$Uf = y^2\frac{\partial f}{\partial y} + 2yy_1\frac{\partial f}{\partial y_1} + (2yy_2 + 2y_1^2)\frac{\partial f}{\partial y_2}\,,$$

then

$$U\theta = \text{const.},\quad UU\theta = \text{const.},$$

are two known integrals of the equation

$$\frac{y_3}{y_1} - \frac{3}{2}\left[\frac{y_2}{y_1}\right]^2 = (a^2x^2 + 2bx + c)^{-2}\,;$$

and by means of these three integral equations, we can eliminate y_1 and y_2, and thus determine the complete integral sought.

27. By means of Theorem II the equation

$$2y_1y_3 - 3y_2^2 = 0$$

may be integrated.

Let $z = \frac{y_2}{y_1}$; then

$$\frac{dz}{dx} = \frac{1}{2}z^2.$$

* Page's Ordinary Diff. Eq. § 68.

Integrating this equation, we find

$$\theta(z_1x) = zx + k_1z + 2 = 0\,;$$

so that

$$U\theta = 2xy_1 + 2k_1y_1 = k_3$$

$$UU\theta = 2xyy_1 + 2k_1yy_1 = k_3$$

are two integral equations of the equation

$$2y_1y_3 - 3y_2^2 = 0\,.$$

Eliminating y_1 from the above equations we find

$$xy + a_2x + b_2y + c_2 = 0\,.$$

Similarly the integral of the equation

$$2z_1z_3 - 3z_2^2 = 0$$

is

$$xz + a_1x + b_1z + c_1 = 0\,.$$

28. In case I, (d), § 23, we have the invariant system

$$\psi_3 = \text{const.},$$

$$\varphi_6 = f(\varphi_5)\,.$$

Suppose we have integrated the second of these equations and found

$$a = F(x)\,.$$

If we substitute this value of a in the equation $\psi_3 = $ const., we have

$$\frac{z_3}{z_1} - \frac{3}{2}\left[\frac{z_2}{z_1}\right]^2 = CF(x)\,.$$

We may now apply Theorem II in order to integrate this equation.

The same reasoning applies to the integration of the invariant system of Case II, (b,) § 23.

IV.

Invariant Partial Differential Equations.

29. In this section we shall assume z to be a function of x and y, and proceed to find what functions $f(x, y, z, p, q, r, s, t, \pi, \rho, \sigma, \tau, \ldots)$ and what equations $f(x, \ldots, \tau, \ldots) = 0$ are invariant under the G_9,—where

$$p = \frac{\partial z}{\partial x}\,,\ q = \frac{\partial z}{\partial y}\,,\ r = \frac{\partial^2 z}{\partial x^2}\,,\ s = \frac{\partial^2 z}{\partial x\partial y}\,,\ t = \frac{\partial^2 z}{\partial y^2}\,,$$

$$\pi = \frac{\partial^3 z}{\partial x^3}\,,\ \rho = \frac{\partial^3 z}{\partial x^2\partial y}\,,\ \sigma = \frac{\partial^3 z}{\partial x\partial y^2}\,,\ \tau = \frac{\partial^3 z}{\partial y^3}\,.$$

30. *To Find what Functions $f(x, \ldots, \tau)$ are invariant under the G_9.*

If we extend each transformation of the Group in terms of the increments $\partial p, \partial q, \ldots, \partial\tau$, and then put each extended transformation equal to zero, we shall have a complete system consisting of nine equations in twelve variables, from which the invariant functions f may be found.

We must first, then, determine the increments $\partial p, \partial q, \ldots, \partial\tau$.

Since z is a function of x and y, we have the following identities, which must hold for all values of x and y:

$$1.\ \begin{cases} dz \equiv pdx + qdy\,, \\ \therefore d\partial z \equiv \partial pdx + \partial qdy + pd\partial x + qd\partial y\,, \end{cases}$$

$$2.\ \begin{cases} dp \equiv rdx + sdy,\ dq \equiv sdx + tdy\,, \\ \therefore d\partial p \equiv \partial rdx + \partial sdy + rd\partial x + sd\partial y\,, \\ d\partial q \equiv \partial sdx + \partial tdy + sd\partial x + td\partial y\,, \end{cases}$$

$$3.\ \begin{cases} dr \equiv \pi dx + \rho dy,\ ds \equiv \rho dx + \sigma dy,\ dt \equiv \sigma dx + \tau dy\,, \\ \therefore d\partial r \equiv \partial\pi dx + \partial\rho dy + \pi d\partial x + \rho d\partial y\,, \\ d\partial s \equiv \partial\rho dx + \partial\sigma dy + \rho d\partial x + \sigma d\partial y\,, \\ d\partial t \equiv \partial\sigma dx + \partial\tau dy + \sigma d\partial x + \tau d\partial y\,, \end{cases}$$

where ∂x, ∂y, and ∂z are the coefficients of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$, respectively, in each transformation.

In the transformation $Uf \equiv \frac{\partial f}{\partial x}$, we have $\partial x = 1$, $\partial y = \partial z = 0$, so that

$$\partial p = \partial q = \ldots = \partial\tau = 0\,.$$

Similarly the transformations $Uf \equiv \frac{\partial f}{\partial y}$ and $Uf \equiv \frac{\partial f}{\partial z}$ give

$$\partial p = \partial q = \ldots = \partial\tau = 0\,.$$

In the transformation $U \equiv x\frac{\partial f}{\partial x}$, we have $\partial x = x$, $\partial y = 0$, $\partial z = 0$. Substituting these values of $\partial x, \partial y, \partial z$ in equations 1, we have

$$0 \equiv \partial pdx + \partial qdy + pdx\,,$$

so that $\partial p = -p$, $\partial q = 0$.

If we put these values of δp and δq in equations (2), we get

$$-dp \equiv -rdx - sdy \equiv \delta rdx + \delta sdy + rdx,$$

$$0 \equiv \delta sdx + \delta tdy + sdx;$$

so that $\delta r = -2r$, $\delta s = -s$, $\delta t = 0$.

Putting these values of δr, δs, δt in equations (3), we get

$$-2dr \equiv -2(\pi dx + \rho dy) \equiv \delta\pi dx + \delta\rho dy + \pi dx,$$

$$-ds \equiv -\rho dx - \sigma dy \equiv \delta\rho dx + \delta\sigma dy + \rho dx,$$

$$0 \equiv \delta\sigma dx + \delta\tau dy + \sigma dx;$$

therefore, $\delta\pi = -3\pi$, $\delta\rho = -2\rho$, $\delta\sigma = -\sigma$, $\delta\tau = 0$.

Hence, the transformation, $Uf \equiv x\dfrac{\partial f}{\partial x}$, extended becomes

$$Uf \equiv x\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial p} - 2r\frac{\partial f}{\partial r} - s\frac{\partial f}{\partial s} - 3\pi\frac{\partial f}{\partial \pi} - 2\rho\frac{\partial f}{\partial \rho} - \sigma\frac{\partial f}{\partial \sigma}.$$

If, in a similar manner, we extend the other transformations and put each extended transformation equal to zero, we get the following complete system :

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0,$$

$$x\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial p} - 2r\frac{\partial f}{\partial r} - s\frac{\partial f}{\partial s} - 3\pi\frac{\partial f}{\partial \pi} - 2\rho\frac{\partial f}{\partial \rho} - \sigma\frac{\partial f}{\partial \sigma} = 0,$$

$$y\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial q} - s\frac{\partial f}{\partial s} - 2t\frac{\partial f}{\partial t} - \rho\frac{\partial f}{\partial \rho} - 2\sigma\frac{\partial f}{\partial \sigma} - 3\tau\frac{\partial f}{\partial \tau} = 0.$$

$$z\frac{\partial f}{\partial z} + p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial r} + s\frac{\partial f}{\partial s} + t\frac{\partial f}{\partial t} + \pi\frac{\partial f}{\partial \pi} + \sigma\frac{\partial f}{\partial \sigma} + \tau\frac{\partial f}{\partial \tau} = 0,$$

$$x^2\frac{\partial f}{\partial x} - 2xp\frac{\partial f}{\partial p} - 2(p + 2xr)\frac{\partial f}{\partial r} - 2xs\frac{\partial f}{\partial s} - 6(r + \pi x)\frac{\partial f}{\partial \pi}$$
$$- 2(s + 2\rho x)\frac{\partial f}{\partial \rho} - 2\sigma x\frac{\partial f}{\partial \sigma} = 0,$$

$$y^2\frac{\partial f}{\partial y} - 2yq\frac{\partial f}{\partial q} - 2sy\frac{\partial f}{\partial s} - 2(q + 2ty)\frac{\partial f}{\partial t} - 2\rho y\frac{\partial f}{\partial \rho} - 2(s + 2\sigma y)\frac{\partial f}{\partial \sigma}$$
$$- 6(t + y\tau)\frac{\partial f}{\partial \tau} = 0,$$

$$z^2\frac{\partial f}{\partial z} + 2zp\frac{\partial f}{\partial p} + 2zq\frac{\partial f}{\partial q} + 2(zr + p^2)\frac{\partial f}{\partial r} + 2(zs + pq)\frac{\partial f}{\partial s}$$
$$+ 2(q^2 + zt)\frac{\partial f}{\partial t} + 2(3rp + z\pi)\frac{\partial f}{\partial \pi} + 2(rq + z\rho + 2sp)\frac{\partial f}{\partial \rho}$$
$$+ 2(2sq + \sigma z + pt)\frac{\partial f}{\partial \sigma} + 3(2qt + z\tau)\frac{\partial f}{\partial \tau} = 0.$$

Since the first three equations show that the solutions will be independent of x, y, and z, we neglect those terms containing $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, and then, by algebraic reduction, we obtain

$$U_1 f \equiv p\frac{\partial f}{\partial p} + 2r\frac{\partial f}{\partial r} + s\frac{\partial f}{\partial s} + 3\pi\frac{\partial f}{\partial \pi} + 2\rho\frac{\partial f}{\partial \rho} + \sigma\frac{\partial f}{\partial \sigma} = 0\,,$$

$$U_2 f \equiv q\frac{\partial f}{\partial q} + s\frac{\partial f}{\partial s} + 2t\frac{\partial f}{\partial t} + \rho\frac{\partial f}{\partial \rho} + 2\sigma\frac{\partial f}{\partial \sigma} + 3\tau\frac{\partial f}{\partial \tau} = 0\,,$$

$$U_3 f \equiv p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial r} + s\frac{\partial f}{\partial s} + t\frac{\partial f}{\partial t} + \pi\frac{\partial f}{\partial \pi} + \rho\frac{\partial f}{\partial \rho} + \sigma\frac{\partial f}{\partial \sigma} + \tau\frac{\partial f}{\partial \tau} = 0\,,$$

$$U_4 f \equiv p\frac{\partial f}{\partial r} + 3r\frac{\partial f}{\partial \pi} + s\frac{\partial f}{\partial \rho} = 0\,,$$

$$U_5 f \equiv q\frac{\partial f}{\partial t} + s\frac{\partial f}{\partial \sigma} + 3t\frac{\partial f}{\partial \tau} = 0\,,$$

$$U_6 f \equiv p^2\frac{\partial f}{\partial r} + pq\frac{\partial f}{\partial s} + q^2\frac{\partial f}{\partial t} + 3rp\frac{\partial f}{\partial \pi} + (rq + 2sp)\frac{\partial f}{\partial \rho} + (2sq + pt)\frac{\partial f}{\partial \sigma} + 3qt\frac{\partial f}{\partial \tau} = 0\,.$$

This is a complete system of six equations in nine variables.

The matrix formed with the coefficients of $\frac{\partial f}{\partial p}, \ldots, \frac{\partial f}{\partial \tau}$, is

$$\left\Vert \begin{array}{ccccccccc} p & 0 & 2r & s & 0 & 3\pi & 2\rho & \sigma & 0 \\ 0 & q & 0 & s & 2t & 0 & \rho & 2\sigma & 3\tau \\ p & q & r & s & t & \pi & \rho & \sigma & \tau \\ 0 & 0 & p & 0 & 0 & 3r & s & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & s & 3t \\ 0 & 0 & p^2 & pq & q^2 & 3rp & (rq+2sp) & (2sq+pt) & 3qt \end{array} \right\Vert,$$

the determinants of which do *not* vanish identically, so that there *are three, and only three*, independent solutions of the complete system.

In order to find these three solutions, we proceed as follows: (See § 13.)

The solutions common to $U_4 f$ and $U_5 f$ are

$$3r^2 - 2p\pi \equiv a_1\,, \quad sr - p\rho \equiv a_2\,, \quad 3t^2 - 2q\tau \equiv a_3\,, \quad st - q\sigma \equiv a_4\,, \quad p, \quad q, \quad s.$$

If we introduce these solutions, a_i, p, q, s, as new variables into the equations $U_1 f$, $U_2 f$, $U_3 f$, and $U_6 f$,—since

$$\begin{array}{llll} U_1 a_1 = 4a_1\,, & U_1 a_2 = 3a_2\,, & U_1 a_3 = 0\,, & U_1 a_4 = a_4\,, \\ U_2 a_1 = 0\,, & U_2 a_2 = a_2\,, & U_2 a_3 = 4a_3\,, & U_2 a_4 = 3a_4\,, \\ U_3 a_1 = 2a_1\,, & U_3 a_2 = 2a_2\,, & U_3 a_3 = 2a_3\,, & U_3 a_4 = 2a_4\,, \\ U_6 a_1 = 0\,, & U_6 a_2 = -\,sp^2, & U_6 a_3 = 0\,, & U_6 a_4 = -\,sq^2, \end{array}$$

we have the following equations:

$$U_7 f \equiv 4a_1 \frac{\partial f}{\partial a_1} + 3a_2 \frac{\partial f}{\partial a_2} + a_4 \frac{\partial f}{\partial a_4} + p \frac{\partial f}{\partial p} + s \frac{\partial f}{\partial s} = 0\,,$$

$$U_8 f \equiv a_2 \frac{\partial f}{\partial a_2} + 4a_3 \frac{\partial f}{\partial a_3} + 3a_4 \frac{\partial f}{\partial a_4} + q \frac{\partial f}{\partial q} + s \frac{\partial f}{\partial s} = 0\,,$$

$$U_9 f \equiv 2a_1 \frac{\partial f}{\partial a_1} + 2a_2 \frac{\partial f}{\partial a_2} + 2a_3 \frac{\partial f}{\partial a_3} + 2a_4 \frac{\partial f}{\partial a_4} + p \frac{\partial f}{\partial p} + q \frac{\partial f}{\partial q} + s \frac{\partial f}{\partial s} = 0\,.$$

$$U_{10} f \equiv sp^2 \frac{\partial f}{\partial a_2} + sq^2 \frac{\partial f}{\partial a_4} - pq \frac{\partial f}{\partial s} = 0\,.$$

The solutions of $U_7 f$ are

$$\frac{a_1}{p^4} \equiv \beta_1\,,\quad \frac{a_2}{p^3} \equiv \beta_2\,,\quad \frac{a_4}{p} \equiv \beta_3\,,\quad \frac{s}{p} \equiv \beta_4\,,\quad a_3,\ q\,.$$

If we introduce these solutions, β_i, a_3, q, as new variables, into the equations $U_8 f$, $U_9 f$, and $U_{10} f$,—since

$$\begin{array}{llll} U_8 \beta_1 = 0\,, & U_8 \beta_2 = \beta_2\,, & U_8 \beta_3 = 3\beta_3\,, & U_8 \beta_4 = \beta_4\,, \\ U_9 \beta_1 = -\,2\beta_1\,, & U_9 \beta_2 = -\,\beta_2\,, & U_9 \beta_3 = \beta_3\,, & U_9 \beta_4 = 0\,, \\ U_{10} \beta_1 = 0\,, & U_{10} \beta_2 = \beta_4\,, & U_{10} \beta_3 = \beta_4 q^2\,, & U_{10} \beta_4 = -\,q\,, \end{array}$$

we have the following equations:

$$U_{11} f \equiv \beta_2 \frac{\partial f}{\partial \beta_2} + 3\beta_3 \frac{\partial f}{\partial \beta_3} + \beta_4 \frac{\partial f}{\partial \beta_4} + 4a_3 \frac{\partial f}{\partial a_3} + q \frac{\partial f}{\partial q} = 0\,,$$

$$U_{12} f \equiv -\,2\beta_1 \frac{\partial f}{\partial \beta_1} - \beta_2 \frac{\partial f}{\partial \beta_2} + \beta_3 \frac{\partial f}{\partial \beta_3} + 2a_3 \frac{\partial f}{\partial a_3} + q \frac{\partial f}{\partial q} = 0\,,$$

$$U_{13} f \equiv \beta_4 \frac{\partial f}{\partial \beta_2} + \beta_4 q^2 \frac{\partial f}{\partial \beta_3} - q \frac{\partial f}{\partial \beta_4} = 0\,.$$

The solutions of $U_{11} f$ are

$$\frac{\beta_2}{q} \equiv \varphi_1\,,\quad \frac{\beta_3}{q^3} - \varphi_2\,,\quad \frac{\beta_4}{q} \equiv \varphi_3\,,\quad \frac{a_3}{q^4} \equiv \varphi_4\,,\quad \beta_1\,.$$

If we introduce these solutions, β_1, φ_i, as new variables, into the equations $U_{12}f$ and $U_{13}f$,—since

$$U_{12}\varphi_1 = -2\varphi_1\,,\quad U_{12}\varphi_2 = -2\varphi_2\,,\quad U_{12}\varphi_3 = -\varphi_3\,,\quad U_{12}\varphi_4 = -2\varphi_4\,,$$
$$U_{13}\varphi_1 = \varphi_3\,,\qquad U_{13}\varphi_2 = \varphi_3\,,\qquad U_{13}\varphi_3 = -1\,,\quad U_{13}\varphi_4 = 0\,,$$

we have the following equations:

$$U_{14}f \equiv 2\beta_1\frac{\partial f}{\partial\beta_1} + 2\varphi_1\frac{\partial f}{\partial\varphi_1} + 2\varphi_2\frac{\partial f}{\partial\varphi_2} + \varphi_3\frac{\partial f}{\partial\varphi_3} + 2\varphi_4\frac{\partial f}{\partial\varphi_4} = 0\,,$$
$$U_{15}f \equiv \varphi_3\frac{\partial f}{\partial\varphi_1} + \varphi_3\frac{\partial f}{\partial\varphi_2} - \frac{\partial f}{\partial\varphi_3} = 0\,.$$

The solutions of $U_{15}f$ are

$$\varphi_1 - \varphi_2 \equiv \psi_1\,,\quad 2\varphi_1 + \varphi_3^2 \equiv \psi_2\,,\quad \beta_1\,,\quad \varphi_4\,.$$

If, finally, we introduce these solutions, ψ_i, β_1, φ_4, as new variables, into the equation $U_{14}f$,—since

$$U_{14}\psi_1 = 2\psi_1\,,\ U_{14}\psi_2 = 2\psi_2\,,$$

we have the equation

$$U_{16}f \equiv 2\beta_1\frac{\partial f}{\partial\beta_1} + 2\psi_1\frac{\partial f}{\partial\psi_1} + 2\psi_2\frac{\partial f}{\partial\psi_2} + 2\varphi_4\frac{\partial f}{\partial\varphi_4} = 0\,,$$

the solutions of which are

$$\frac{\beta_1}{\varphi_4} \equiv \frac{q^4(3r^2 - 2p\pi)}{p^4(3t^2 - 2q\tau)}\,,$$
$$\frac{\psi_1}{\varphi_4} \equiv \frac{q^3(sr - p\rho) - p^2q(st - q\sigma)}{p^3(3t^2 - 2q\tau)}\,,$$
$$\frac{\psi_2}{\varphi_4} \equiv \frac{2q^3(sr - p\rho) + pq^2s^2}{p^3(3t^2 - 2q\tau)}\,.$$

These are, therefore, the three independent solutions of the complete system.

31. *We shall now Find the Differential Parameter, when z is a Function of x and y.*

Suppose $\Omega(x, y, z, p, q, r, \ldots, \varphi, \varphi_x, \varphi_y)$ to be such a function that, when φ is an invariant function of the Group, Ω is also an invariant function,—*to find the function Ω.**

When we write $\varphi_x \equiv \dfrac{\partial\varphi}{\partial x}$, and $\varphi_y \equiv \dfrac{\partial f}{\partial y}$, we have

$$d\varphi \equiv \varphi_x dx + \varphi_y dy\,,$$

or, since φ is an invariant function

$$d\delta\varphi \equiv 0 \equiv \delta\varphi_x dx + \delta\varphi_y dy + \varphi_x d\delta x + \varphi_y d\delta y\,.$$

* Sophus Lie, Vorlesungen ueber Continuierliche Gruppen, page 670.

From these identities we can determine the increments $\partial\varphi_x$ and $\partial\varphi_y$ for each transformation of the G_9.

If we extend each transformation of the Group in terms of the increments $\partial p, \ldots, \partial\pi, \partial\varphi_x\, \partial\varphi_y$, and then put each extended transformation equal to zero we obtain the following complete system, from which Ω may be determined:

$$\frac{\partial\Omega}{\partial x} = 0, \quad \frac{\partial\Omega}{\partial y} = 0, \quad \frac{\partial\Omega}{\partial z} = 0,$$

$$x\frac{\partial\Omega}{\partial x} - p\frac{\partial\Omega}{\partial p} - 2r\frac{\partial\Omega}{\partial r} - s\frac{\partial\Omega}{\partial s} - 3\pi\frac{\partial\Omega}{\partial\pi} - \varphi_x\frac{\partial\Omega}{\partial\varphi_x} = 0,$$

$$y\frac{\partial\Omega}{\partial y} - q\frac{\partial\Omega}{\partial q} - s\frac{\partial\Omega}{\partial s} - 2t\frac{\partial\Omega}{\partial t} - \varphi_y\frac{\partial\Omega}{\partial\varphi_y} = 0,$$

$$z\frac{\partial\Omega}{\partial z} + p\frac{\partial\Omega}{\partial p} + q\frac{\partial\Omega}{\partial q} + r\frac{\partial\Omega}{\partial r} + s\frac{\partial\Omega}{\partial s} + t\frac{\partial\Omega}{\partial t} + \pi\frac{\partial\Omega}{\partial\pi} = 0,$$

$$x^2\frac{\partial\Omega}{\partial x} - 2xp\frac{\partial\Omega}{\partial p} - 2\,(p + 2xr)\frac{\partial\Omega}{\partial r} - 2xs\frac{\partial\Omega}{\partial s} - 6\,(r + \pi x)\frac{\partial\Omega}{\partial\pi}$$
$$- 2x\varphi_x\frac{\partial\Omega}{\partial\varphi_4} = 0.$$

$$y^2\frac{\partial\Omega}{\partial y} - 2yq\frac{\partial\Omega}{\partial q} - 2ys\frac{\partial\Omega}{\partial s} - 2\,(q + 2ty)\frac{\partial\Omega}{\partial t} - 2y\varphi_y\frac{\partial\Omega}{\partial\varphi_y} = 0.$$

$$z^2\frac{\partial\Omega}{\partial z} + 2zp\frac{\partial\Omega}{\partial p} + 2zq\frac{\partial\Omega}{\partial q} + 2\,(zr + p^2)\frac{\partial\Omega}{\partial r} + 2\,(q^2 + zt)\frac{\partial\Omega}{\partial t}$$
$$+ 2\,(zs + pq)\frac{\partial\Omega}{\partial s} + 2\,(3rp + z\pi)\frac{\partial\Omega}{\partial\pi} = 0.$$

By algebraic reduction this complete system may be brought to the form

$$U_1\Omega \equiv p\frac{\partial\Omega}{\partial p} + 2r\frac{\partial\Omega}{\partial r} + s\frac{\partial\Omega}{\partial s} + 3\pi\frac{\partial\Omega}{\partial\pi} + \varphi_x\frac{\partial\Omega}{\partial\varphi_x} = 0,$$

$$U_2\Omega \equiv q\frac{\partial\Omega}{\partial q} + s\frac{\partial\Omega}{\partial s} + 2t\frac{\partial\Omega}{\partial t} + \varphi_y\frac{\partial\Omega}{\partial\varphi_y} = 0,$$

$$U_3\Omega \equiv p\frac{\partial\Omega}{\partial p} + q\frac{\partial\Omega}{\partial q} + r\frac{\partial\Omega}{\partial r} + s\frac{\partial\Omega}{\partial s} + t\frac{\partial\Omega}{\partial t} + \pi\frac{\partial\Omega}{\partial\pi} = 0,$$

$$U_4\Omega \equiv p\frac{\partial\Omega}{\partial r} + 3r\frac{\partial\Omega}{\partial\pi} = 0,$$

$$U_5\Omega \equiv q\frac{\partial\Omega}{\partial t} = 0,$$

$$U_6\Omega \equiv pq\frac{\partial\Omega}{\partial s} = 0.$$

Equations $U_5\Omega$ and $U_6\Omega$ show that the solutions will be independent of s and t; hence we neglect those terms containing $\frac{\partial\Omega}{\partial s}$ and $\frac{\partial\Omega}{\partial t}$, and the complete system reduces then to one consisting of four equations in six variables.

The matrix formed with the coefficients $\frac{\partial\Omega}{\partial p}$, $\frac{\partial\Omega}{\partial q}$, $\frac{\partial\Omega}{\partial r}$, $\frac{\partial\Omega}{\partial \pi}$, $\frac{\partial\Omega}{\partial \varphi_x}$, $\frac{\partial\Omega}{\partial \varphi_y}$, is

$$\left\|\begin{matrix} p & 0 & 2r & 3\pi & \varphi_x & 0 \\ 0 & q & 0 & 0 & 0 & \varphi_y \\ p & q & r & \pi & 0 & 0 \\ 0 & 0 & p & 3r & 0 & 0 \end{matrix}\right\|,$$

all the determinants of which do *not* vanish identically, so that there are two, and only two, independent solutions of this complete system.

In order to find these solutions we proceed as follows:

The solutions common to $U_1\Omega$ and $U_2\Omega$ are

$$\frac{r}{p^2} \equiv \alpha_1\,, \quad \frac{\pi}{p^3} \equiv \alpha_2\,, \quad \frac{\varphi_x}{p} \equiv \alpha_3\,, \quad \frac{\varphi_y}{q} \equiv \alpha_4\,.$$

If we introduce these solutions, α_i, as new variables, into U_3f and U_4f,—since

$$U_3\alpha_1 = -\alpha_1\,, \quad U_3\alpha_2 = -2\alpha_2\,, \quad U_3\alpha_3 = -\alpha_3\,, \quad U_4\alpha_4 = -\alpha_4\,,$$

$$U_4\alpha_1 = \frac{1}{p}\,, \quad U_4\alpha_2 = \frac{3\alpha_1}{p}\,, \quad U_3\alpha_3 = 0\,, \quad U_4\alpha_4 = 0\,,$$

we have the following equations:

$$U_7\Omega \equiv \alpha_1\frac{\partial\Omega}{\partial\alpha_1} + 2\alpha_2\frac{\partial\Omega}{\partial\alpha_2} + \alpha_3\frac{\partial\Omega}{\partial\alpha_3} + \alpha_4\frac{\partial\Omega}{\partial\alpha_4} = 0\,,$$

$$U_8 \equiv \frac{\partial\Omega}{\partial\alpha_1} + 3\alpha_1\frac{\partial\Omega}{\partial\alpha_2} = 0\,.$$

The solutions of $U_8\Omega$ are

$$3\alpha_1^2 - 2\alpha_2 \equiv \beta\,, \ \alpha_3, \ \alpha_4.$$

If we introduce these solutions, β, α_3, α_4, as new variables, into the equation $U_7\Omega$,—since

$$U_7\beta = 2\beta\,,$$

we have the equation

$$2\beta\frac{\partial\Omega}{\partial\beta} + \alpha_3\frac{\partial\Omega}{\partial\alpha_3} + \alpha_4\frac{\partial\Omega}{\partial\alpha_4} = 0\,,$$

the solutions of which are

$$\frac{\alpha_3^2}{\beta} \equiv \frac{p^2\varphi_x^2}{3r^2 - 2p\pi} = \Delta_1\varphi\,,$$

$$\frac{\alpha_4^2}{\beta} \equiv \frac{p^4\varphi_y^2}{q^2(3r^2 - 2p\pi)} = \Delta_2\varphi\,.$$

$\Delta_1\varphi$ and $\Delta_2\varphi$ are called Differential Parameters and have the general form

$$\Omega(x, y, z, p, \ldots, \varphi, \Delta_1\varphi, \Delta_2\varphi).$$

Since $\Delta_1\varphi$ and $\Delta_2\varphi$ are invariant functions, the expressions

$$\Delta_1^2\varphi \equiv \Delta_1(\Delta_1\varphi) \equiv \frac{\partial\Delta_1\varphi}{\partial x}\left\{\frac{p^2}{3r^2 - 2p\pi}\right\},$$

$$\Delta_2(\Delta_1\varphi) \equiv \frac{\partial\Delta_1\varphi}{\partial y}\left\{\frac{p^4}{q^2(3r^2 - 2p\pi)}\right\},$$

$$\Delta_1(\Delta_2\varphi) \equiv \frac{\partial\Delta_2\varphi}{\partial x}\left\{\frac{p^2}{3r^2 - 2p\pi}\right\},$$

$$\Delta_2^2\varphi \equiv \frac{\partial\Delta_2\varphi}{\partial y}\left\{\frac{p^4}{q^2(3r^2 - 2p\pi)}\right\},$$

are also Differential Parameters; and, likewise, $\Delta_1^3\varphi$, etc.; so that the most general Differential Parameter is

$$\Omega(x, y, z, p, \ldots, \varphi, \Delta_1\varphi, \Delta_2\varphi, \Delta_1^2\varphi, \Delta_2(\Delta_1\varphi), \ldots).$$

By operating with the Differential Parameters on the invariant functions already found, we can find the invariant functions of higher orders.

32. We shall now show how the five invariant functions of the fourth order may be obtained by means of the Differential Parameter.

Let

$$\varphi_1 \equiv \frac{q^4(3r^2 - 2p\pi)}{p^4(3t^2 - 2q\tau)},$$

$$\varphi_2 \equiv \frac{q^3(sr - p\rho) - qp^2(st - q\sigma)}{p^3(3t^2 - 2q\tau)},$$

$$\varphi_3 \equiv \frac{2q^3(sr - p\rho) + s^2pq^2}{p^3(3t^2 - 2q\tau)}.$$

Hence

$$\Delta_1\varphi_1 \equiv \frac{p^2}{3r^2 - 2p\pi}\left[\frac{\partial\varphi_1}{\partial x}\right]^2 \equiv \psi_1 ,$$

$$\Delta_1\varphi_2 \equiv \frac{p^2}{3r^2 - 2p\pi}\left[\frac{\partial\varphi_2}{\partial x}\right]^2 \equiv \psi_2 ,$$

$$\Delta_1\varphi_3 \equiv \frac{p^2}{3r^2 - 2p\pi}\left[\frac{\partial\varphi_3}{\partial x}\right]^2 \equiv \varphi_3 ,$$

$$\Delta_2\varphi_1 \equiv \frac{p^4}{q^2(3r^2 - 2p\pi)}\left[\frac{\partial\varphi_1}{\partial y}\right]^2 \equiv \psi_4 ,$$

$$\Delta_2\varphi_2 \equiv \frac{p^4}{q^2(3r^2 - 2p\pi)}\left[\frac{\partial\varphi_2}{\partial y}\right]^2 \equiv \varphi_5 ,$$

are five independent functions of the fourth order invariant under the Group.

33. *To Find what Equations* $f(x, y, z, p, \ldots) = 0$ *are Invariant under the* G_9.

As in § 19, it may be shown that, by examining the matrices, or determinant, of the coefficients of p, q, r, etc., there are no partial differential equations of an order lower than the third invariant under the G_9, *except those obtained from the invariant functions of the third order.*

(a). Since none of the invariant functions, φ_1, φ_2, φ_3, contain z, there are no partial differential equations of the zero order invariant under the G_9.

(b). If we put the invariant function of the third order,

$$\varphi_1 \equiv \frac{q^4(3r^2 - 2p\pi)}{p^4(3t^2 - 2q\tau)} ,$$

first equal to zero and then equal to infinity, we obtain the following invariant partial differential equation of the first order:

$$q = 0 , \quad \frac{1}{p} = 0 , \quad p = 0 , \quad \frac{1}{q} = 0 .$$

These are the only partial differential equations of the first order invariant under the G_9.

(c). It is clear that there are no partial differential equations of the second order invariant under the G_9; for the equations of the second order,

$$r = s = t = 0 ,$$

are invariant under the G_9 only when they are considered in connection with and as a consequence of $p = 0$ and $q = 0$.

(d). The invariant partial differential equations of higher orders are obtained by putting arbitrary functions of the invariant functions equal to zero. Thus the most general invariant partial differential equation of the third order is

$$\Omega(\varphi_1, \varphi_2, \varphi_3) = 0\,.$$

(e). The most general partial differential equation of the fourth order invariant under the G_9 is

$$\Omega(\varphi_1, \varphi_2, \varphi_3, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5) = 0\,.$$

V.

Equivalence of Surfaces.

34. In this section we shall show how we may determine the nature of the families of surfaces which are composed of those surfaces which are "equivalent" by means of the transformations of the Group.

35. Two surfaces are said to be *equivalent* by means of a Group, if by means of the transformations of the Group, the one surface can be carried over into the other. As in the case of equivalent curves, equivalent surfaces belong to invariant families of surfaces, which families are defined by invariant systems of partial differential equations.

If a surface admits of no transformation of a G_n, it will generate an invariant family of ∞^n surfaces, when subjected to all the transformations of the G_n. If this family of ∞^n surfaces is given by the equation

$$z = f(x, y)\,, \tag{1}$$

where f is an analytical function, we can always write this equation in the form

$$z - z_0 = p_0(x - x_0) + \frac{r_0}{2}(x - x_0)^2 + s_0(x - x_0)(y - y_0) + \frac{t_0}{2}(y - y_0)^2 + \ldots\,, \tag{2}$$

where $z_0, p_0, \ldots, t_0, \ldots$ are the values of $z, p, \ldots, t, \ldots$, when we assign to x and y their initial values x_0 and y_0.

Since the family consists of ∞^n surfaces, n of the arbitrary constants z_0, $p_0, \ldots, t_0, \ldots$ must be connected by no relations, while the remaining arbitrary constants must be so connected by relations that they can be expressed in terms of the n arbitrary constants which are connected by no relations; that is, n *and only* n of the partial differential coefficients are connected by no relations.

Now suppose that we have an invariant system of partial differential equations, and that by means of these equations we can express all, except n of the partial differential coefficients of z with respect to x and y in terms of the remaining n partial differential coefficients; it is clear that such a system of partial differential equations will define an invariant family of exactly ∞^n surfaces, since, in this case, n of the arbitrary constants of equation (2) will be connected by no relation.

The system must consist of an infinite number of partial differential equations and it must be *completely integrable.** For equation (2) contains an infinite number of arbitrary constants, z_0, $p_0, \ldots, t_0, \ldots$ and all of these constants, except n, are determined by means of the partial differential equations in terms of the remaining n arbitrary constants.

Since the system must contain an infinite number of equations, we can suppose the system has been so arranged that by differentiating one of the partial differential equations of the system, we only obtain another partial differential equation of the system already given.

We can also suppose that the equations of the system are so arranged that beginning with the lowest order they proceed to those of higher orders; and, finally, that from the equations of the pth order we cannot eliminate all the partial differential coefficients of the pth order. It is evident, then, that such a system of partial differential equations as we have defined above will determine, from a certain point on, all the partial differential coefficients of z with respect to x and y in terms of those of *lower* orders.

If the surface admits of m of the independent infinitesimal transformations of the G_n, where $m < n$, it will generate a family of ∞^{n-m} surfaces. What we have said above in regard to the family of ∞^n surfaces is equally true of this family of ∞^{n-m}; that is, it will be defined by a completely integrable invariant system consisting of an infinite number of partial differential equations, by means of which we can determine all the higher partial differential coefficients of z with respect to x and y in terms of $(n - m)$ of the lower partial differential coefficients of z with respect to x and y.

36. We shall now show what invariant systems of partial differential equations are necessary to define the invariant families of surfaces, which are composed of those surfaces that are equivalent by means of the transformations of the Group

$$\left| \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z},\ x\frac{\partial f}{\partial x},\ y\frac{\partial f}{\partial y},\ z\frac{\partial f}{\partial z},\ x^2\frac{\partial f}{\partial x},\ y^2\frac{\partial f}{\partial y},\ z^2\frac{\partial f}{\partial z} \right| .$$

* Goursat, p. 41, Vol. II.

I. *The invariant systems that contain no partial differential equation of an order lower than the third.*

a. Suppose the invariant system contains one partial differential equation of the third order of the form (cf. § 33, d)

$$\Omega_1(\varphi_1, \varphi_2, \varphi_3) = 0 . \tag{3}$$

By means of this equation we can determine one of the partial differential coefficients of the third order in terms of the three remaining partial differential coefficients of the third order and those of lower orders; that is, in terms of nine of the partial differential coefficients. Now, since the greatest family of surfaces invariant under the G_9 consists of ∞^9 surfaces, it is evident that the greatest number of partial differential coefficients, that can be connected by *no* relations, is *nine;* so that we must have, in connection with equation (3), other equations by means of which we can determine all the higher partial differential coefficients from the fourth order on in terms of the nine partial differential coefficients of lower orders (cf. p. 68). If we differentiate equation (3) partially with respect to x and y, we have the two equations of the fourth order

$$\frac{\partial \Omega_1}{\partial x} = 0 , \quad \frac{\partial \Omega_1}{\partial y} = 0 , \tag{4}$$

which must evidently form a system which is invariant under the G_9. These two equations will determine two of the partial differential coefficients of the fourth order in terms of the nine of lower orders. In order to determine the three remaining partial differential coefficients of the fourth order, we must have three invariant equations of the fourth order (cf. 33 e)

$$\left.\begin{aligned} \Omega_2(\varphi_1, \ldots, \psi_1, \ldots, \psi_5) = 0 , \\ \Omega_3(\varphi_1, \ldots, \psi_1, \ldots, \psi_5) = 0 , \\ \Omega_4(\varphi_1, \ldots, \psi_1, \ldots, \psi_5) = 0 , \end{aligned}\right\} \tag{5}$$

no one of which can be a consequence of equation (3). By means of equations (3), (4), and (5), we can express one partial differential coefficient of the third order and the five partial differential coefficients of the fourth order in terms of the three remaining partial differential coefficients of the third order and the partial differential coefficients of lower orders.

Now by repeated partial differentiation of the above equations we obtain an infinite number of equations, belonging to a system which is invariant under the G_9. It is clear that by means of this system of equations we can determine all, except *nine*, of the partial differential coefficients of z with respect to x and y in terms of *nine* of the partial differential coefficients; hence, in this case, *nine and only nine* of the arbitrary constants of equation (2) are connected by no relations, and we see, then, that *the invariant system containing one partial differential equation of the third order and three of the fourth order will define an invariant family of exactly* ∞^9 *surfaces.*

As a similar discussion holds for all the other cases, we need only write down the remaining invariant systems.

b. If the invariant system contains two independent partial differential equations of the third order,

$$\Omega_1(\varphi_1, \varphi_2, \varphi_3) = 0,$$
$$\Omega_2(\varphi_1, \varphi_2, \varphi_3) = 0,$$

it will define an invariant family of exactly ∞^9 surfaces.

c. If the invariant system contains two partial differential equations of the third order,

$$\Omega_1(\varphi_1, \varphi_2, \varphi_3) = 0,$$
$$\Omega_2(\varphi_1, \varphi_2, \varphi_3) = 0,$$

and one partial differential equation of the fourth order,

$$\Omega_3(\varphi_1, \ldots, \psi_1, \ldots, \psi_5) = 0,$$

which is not a consequence of those of the third order, it will define an invariant family of exactly ∞^8 surfaces.

d. If the invariant system contains three independent partial differential equations of the third order,

$$\Omega_1(\varphi_1, \varphi_2, \varphi_3) = 0,$$
$$\Omega_2(\varphi_1, \varphi_2, \varphi_3) = 0,$$
$$\Omega_3(\varphi_1, \varphi_2, \varphi_3) = 0,$$

it will define an invariant family of exactly ∞^7 surfaces.

e. If the invariant system contains four independent partial differential equations of the third order,

$$\Omega_1(\varphi_1, \varphi_2, \varphi_3) = 0,$$
$$\Omega_2(\varphi_1, \varphi_2, \varphi_3) = 0,$$
$$\Omega_3(\varphi_1, \varphi_2, \varphi_3) = 0,$$
$$\Omega_4)\varphi_1, \varphi_2, \varphi_3) = 0,$$

it will define an invariant family of exactly ∞^6 surfaces.

II. *The invariant systems which contain partial differential equations of the second order and none of the first order.*

Since there are no partial differential equations of the second order invariant under the G_9, except the system $r = s = t = 0$, which is a *consequence* of the invariant system of the first order $p = q = 0$, there is no invariant system containing a partial differential equation of the second order and none of the first order. (Cf. 33, c.)

III. *The invariant systems which contain partial differential equations of the first order.*

a. If the invariant system contains the two partial differential equations of the first order, of the typical forms

$$p = 0\,,$$
$$q = 0\,,$$

it will define an invariant family of ∞^1 surfaces. Evidently, in this case, the invariant family consists of the planes

$$z = \text{const.}$$

b. If the invariant system contains one partial differential equation of the first order and one of the third order, it will define an invariant family of exactly ∞^3 surfaces.

On account of the symmetry of the G_9, we may choose

$$p = 0$$

as the typical partial differential equation of the first order; and, then, it is clear that the partial differential equation of the third order cannot be a consequence of $p = 0$.

If we put
$$\varphi_1 \equiv \frac{q^4\,(3r^2 - 2p\pi)}{p^4\,(3t^2 - 2q\tau)}$$
equal to infinity, we obtain the partial differential equation of the third order

$$3t^2 - 2q\tau = 0\,.$$

This is the only invariant partial differential equation of the third order that is not a consequence of $p = 0$, so that the invariant system is

$$p = 0$$
$$3t^2 - 2q\tau = 0\,.$$

When $p = 0$, $z = F(y)$, and the equation $3t^2 - 2q\tau = 0$ may be written

$$3\left[\frac{d^2z}{dy^2}\right]^2 - 2\,\frac{dz}{dy}\,\frac{d^3z}{dy^3} = 0\,.$$

The integral of this equation is (cf. § 27)

$$yz + \alpha z + \beta y + \gamma = 0\,.$$

Hence, in this case, the invariant family consists of the ∞^3 hyperbolic cylinders given by the above equation.

There are no other systems containing a partial differential of the first order invariant under the G_9, for there are no invariant partial differential equations of the fourth or higher orders that are not a consequence of $p = 0$.

37. We may now collect our results as follows:

A. If the invariant family consists of ∞^9 surfaces, it will be defined by

1. An invariant system containing one partial differential equation of the third order and three of the fourth order, or

2. An invariant system containing two partial differential equations of the third order.

B. If the invariant family consists of ∞^8 surfaces, it will be defined by an invariant system containing two partial differential equations of the third order and one of the fourth order.

C. If the invariant family consists of ∞^7 surfaces, it will be defined by an invariant system containing three partial differential equations of the third order.

D. If the invariant family consists of ∞^6 surfaces, it will be defined by an invariant system containing four partial differential equations of the third order.

E. There are no invariant systems that will define invariant families, consisting of ∞^5, ∞^4, or ∞^2 surfaces; hence no surface admits of exactly four, five, or seven independent infinitesimal transformations of the G_9.

F. If the invariant family consists of ∞^3 surfaces, it will be defined by an invariant system containing one partial differential of the first order and one of the third.

G. If the invariant family consists of ∞^1 surfaces, it will be defined by an invariant system containing two partial differential equations of the first order.

In this, the last section of this paper, we have shown how we may determine the nature of the invariant families of surfaces, which are composed of those surfaces that are equivalent by means of the transformations of the G_9; and if we have two surfaces that are equivalent by means of the transformations of the Group, their equations must satisfy the partial differential equations of some one of the invariant systems enumerated in cases I to III.

In the discussion of the above group I have followed the method of Sophus Lie, which can be found in his Vorlesungen ueber Continuierliche Gruppen. Where it has been deemed advisable I have given special references.