# Global Well-Posedness and Exponential Stability for a Nonlinear Thermoelastic Kirchhoff-Love Plate System

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A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia
August, 2017

# Abstract

We study an initial-boundary-value problem for a quasilinear thermoelastic plate of Kirchhoff & Love-type with parabolic heat conduction due to Fourier, mechanically simply supported and held at the reference temperature on the boundary. For this problem, we show the short-time existence and uniqueness of classical solutions under appropriate regularity and compatibility assumptions on the data. Further, we use barrier techniques to prove the global existence and exponential stability of solutions under a smallness condition on the initial data. It is the first result of this kind established for a quasilinear non-parabolic thermoelastic Kirchhoff & Love plate in multiple dimensions.

# Acknowledgment

First and foremost, I would like to express my heartfelt appreciation to my advisor, Dr. Irena Lasiecka, for her guidance, inspiration, criticism, and patience throughout my Ph.D program. She is a perfect model to me as a mathematician and as a person.

I would like to express my deep gratitude to Dr. Roberto Triggiani. His immense knowledge and shape judgment have always been encouraging. Working with him and my advisor was one of my best life experiences, and I will always be grateful to them.

I would like to thank Dr. Zoran Grujic for his inspirational teaching both inside and outside classroom.

A special thank you to my academic sibling Dr. Michael Pokojovy, whom I enjoy so much to work with.

I would like to express my sincere gratitude to Dr. Georg Hetzer for leading me into math research. I am very thankful for all the help he offered.

I would like to acknowledge Dr. Wenxian Shen and many other great teachers of mine, who inspired me into the path of a serious mathematician as well as a respectful teacher.

I would like to thank my family for their tremendous support throughout the years. 亲爱的爸爸妈妈，感谢你们一直以来的支持和理解。这篇论文是对你们多年养育和教导的恩情的回报。无论我取得怎样的成绩，里面永远都有你们的一份。亲爱的姐姐，从小到大，你一直是我最好的朋友，也是最理解我的人。你总能在我疑惑的时刻给我指明方向。未来，希望我们能继续共同前行。

Last but not least, I would like to thank all my friends who stood behind me. It has been a long journey, and I am grateful that you guys were there with me.

# Contents

# Chapter 1

# Introduction

## 1.1 Literature Review

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a smooth bounded domain representing the mid-plane of a thermoelastic plate. With $w$ and $\theta$ denoting the vertical deflection and an appropriately weighted thermal moment with respect to the plate thickness, both depending on a scaled time variable $t > 0$ and the space variable $(x_1, x_2)$ or $(x_1, x_2, x_3) \in \Omega$, the nonlinear Kirchhoff & Love thermoelastic plate system reads as

$$w_{tt} - \gamma \triangle w_{tt} + \triangle^2 w + \alpha \triangle \theta + b \triangle \left( (\triangle w)^3 \right) = 0 \quad \text{in } (0, \infty) \times \Omega, \tag{1.1a}$$

$$\beta \theta_t - \eta \triangle \theta + \sigma \theta - \alpha \triangle w_t = 0 \quad \text{in } (0, \infty) \times \Omega \tag{1.1b}$$

along with the boundary conditions (hinged mechanical/Dirichlet thermal)

$$w = \triangle w = \theta = 0 \text{ in } (0, \infty) \times \partial\Omega \tag{1.1c}$$

and the initial conditions

$$w(0, \cdot) = w^0, \quad w_t(0, \cdot) = w^1, \quad \theta(0, \cdot) = \theta^0 \text{ in } \Omega. \tag{1.1d}$$

Here, $\alpha, \beta, \gamma, \eta, \sigma, a$ are positive constants. For thin plates, $\gamma$ behaves like $h^3$ as $h \to 0$ (cf. [13, Equation (2.16), p. 13]), where $h$ stands for the uniform thickness of the plate, and is, therefore, neglected in some literature. In Chapter 2, we present a short physical deduction of Equations (1.1a)–(1.1d). In particular, the nonlinearity in (1.1a) arises from a nonlinear material response law (2.4) and (2.11), which also motives a treatment of the local well-posedness for a more general system in Chapter 3.

After the thermoelastic Kirchhoff-Love plate systems were introduced, there have been many papers in the past three decades devoted to this field. In the 90's, a lot of work was done for the linear thermoelastic plate theory. For instance, Kim in [11] studied the one-dimensional case with homogeneous Hinged/Dirichlet boundary conditions and proved the exponential decay of the energy. Lasiecka and Triggiani, through a series of papers, achieved the analyticity of the s.c. contraction semigroup when $\gamma = 0$ under five different types of boundary conditions, including the challenging Free B.C. [17, p. 202–203]. Furthermore, they proved the lack of analyticity in certain cases when $\gamma > 0$, which gave the initial guidance to the work presented in this thesis (see their book [18]).

Meanwhile, the nonlinear thermoelastic plate was also studied in various settings. Lasiecka et al. [14] studied a quasilinear PDE system similar to (1.1a)–(1.1d) in a smooth, bounded domain $\Omega$ of $\mathbb{R}^d$ with $d \leq 3$ given by a Kirchhoff & Love plate with parabolic heat conduction

$$w_{tt} + \triangle^2 w - \triangle\theta + b\triangle\big((\triangle w)^3\big) = 0 \text{ in } (0,T)\times\Omega, \tag{1.2a}$$

$$\theta_t - \triangle\theta + \triangle w_t = 0 \text{ in } (0,T)\times\Omega \tag{1.2b}$$

together with boundary conditions (1.1c) and initial conditions (1.1d) for an arbitrary $T > 0$. For the initial-boundary-value problem (1.2a)–(1.2b), (1.1c)–(1.1d), they proved the global existence of weak solutions $(w,\theta)$ and their uniform decay in the norm for $\{w, w_t, \theta\}$ of

$$\Big(W^{1,\infty}\big(0,T;L^2(\Omega)\big)\cap L^\infty(0,T;W^{2,4}(\Omega))\Big)\times L^\infty\big(0,T;W^{1,2}(\Omega)\big).$$

The existence proof was based on a Galerkin approximation and compactness theorems, while the exponential stability was obtained with the aid of energy techniques.

In their monograph [3], Chueshov and Lasiecka give an extensive study on the von Kármán plate system both in pure elastic and thermoelastic cases. With $w\colon \Omega\to\mathbb{R}$ denoting the vertical displacement and $v\colon\Omega\to\mathbb{R}$ standing for the Airy stress function of a plate with its mid-plane occupying in the reference configuration a domain $\Omega\subset\mathbb{R}^2$, the pure elastic version of Kármán plate system reads as

$$u_{tt} - \alpha\triangle u_{tt} + \triangle^2 u - [u, v + F_0] + Lu = p \text{ in } (0,\infty)\times\Omega, \tag{1.3a}$$

$$\triangle^2 v + [u,u] = 0 \text{ in } (0,\infty)\times\Omega, \tag{1.3b}$$

where $[v,w] := v_{x_1x_1}w_{x_2x_2} + v_{x_2x_2}w_{x_1x_1} - 2v_{x_1x_2}w_{x_1x_2}$, $L$ is a first-order differential operator and $F_0, p\colon\Omega\to\mathbb{R}$ are given "force" functions. Imposing standard initial conditions, under various sets of boundary conditions, Chueshov and Lasiecka proved that Equations (1.3a)–(1.3b) possess a unique generalized, weak or strong solution depending on the data regularity. The proof was based on a nonlinear Galerkin-type approximation. Further, they studied the semiflow associated with the solution to Equations (1.3a)–(1.3b), in particular, they analyzed its long-time behavior and the existence of attracting sets. Various damping mechanisms, thermoelastic effects, structurally coupled systems such as acoustic chambers or gas flow past a plate were studied. An extremely detailed and comprehensive literature overview was also given.

Denk et al. [4] considered a linearization of (1.2a)–(1.2b), which corresponds to letting $b \equiv 0$, in a bounded or exterior $C^4$-domain of $\mathbb{R}^d$ for $d \geq 2$ subject to the initial conditions from Equation (1.1d) and the clamped boundary conditions

$$w = \partial_\nu w = \theta = 0 \text{ on } (0, T) \times \partial\Omega, \tag{1.4}$$

where $\partial_\nu = (\nabla \cdot)^T \nu$ and $\nu$ denotes the outer unit normal vector to $\Omega$ on $\partial\Omega$. By proving a resolvent estimate both in the whole space and in the half-space and employing localization techniques, they showed that the $C_0$-semigroup for $(w, w_t, \theta)$ on the space

$$W_D^{2,p}(\Omega) \times L^p(\Omega) \times L^p(\Omega) \text{ with } W_D^{2,p}(\Omega) = \{u \in W^{2,p}(\Omega) \,|\, u = \partial_\nu u = 0 \text{ on } \partial\Omega\}$$

is analytic. In case $\Omega$ is bounded, they also proved an exponential stability result for the semigroup.

Lasiecka and Wilke [15] presented an $L^p$-space treatment of Equations (1.2a)–(1.2b) ($\gamma = 0$), (1.1c)–(1.1d) in bounded $C^2$-domains $\Omega$ of $\mathbb{R}^d$. By proving the maximal $L^p$-regularity for the linearized problem, they adopted the classical approach to prove the existence and uniqueness of strong solutions satisfying

$$(\triangle w, w_t, \theta) \in \Big(L^p_\mu\big(0,T; W^{2,p}(\Omega)\big) \cap W^{1,p}_\mu\big(0,T; L^p(\Omega)\big) \cap BUC\big(0,T; W^{2\mu-2/p,p}(\Omega)\big)\Big)^3$$

for $p > 1 + \frac{d}{2}$, where $L^p_\mu(\Omega)$ is the space of strongly measurable functions $u$ for which $t \mapsto t^{1-\mu}u(t)$ lies in $L^p(\Omega)$ and $W^{1,p}_\mu(\Omega)$ stands for the space of weakly differentiable functions from $L^p_\mu(\Omega)$ whose first-order weak derivatives also lie in $L^p_\mu(\Omega)$. For $T \leq \infty$, they showed a global strong solvability result for sufficiently small initial data in the interpolation space

$$(\triangle w^0, w^1, \theta^0) \in \Big(\big(L^p(\Omega), W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\big)_{\mu-1/p,p}\Big)^3.$$

They pointed out that similar arguments can be used to obtain a short-time existence for arbitrarily large initial data. Finally, they studied the first- and higher-order differentiability as well as analyticity of solutions under appropriate assumptions on the data.

Recently, Denk and Schnaubelt [5] considered a structurally damped elastic plate equation

$$w_{tt} + \triangle^2 w - \rho \triangle w_t = f \text{ in } (0,\infty) \times \Omega \tag{1.5}$$

in a domain $\Omega \subset \mathbb{R}^d$, being either the whole space, a half-space or a bounded $C^4$-domain, subject to inhomogeneous Dirichlet-Neumann boundary conditions

$$w = g_0, \quad \partial_\nu w = g_1 \text{ on } (0,\infty) \times \Omega$$

and the initial conditions

$$w(0, \cdot) = w^0, \quad w_t(0, \cdot) = w^1 \text{ in } \Omega,$$

with the data coming from appropriate $L^p$-Sobolev spaces for $p \in (1, \infty)\backslash\{3/2, 3\}$. By showing the $\mathcal{R}$-sectoriality of the operator driving the flow $t \mapsto \big(w(t), w_t(t)\big)$ both in the whole space and the half-space scenarios, they proved the $L^p$-maximum regularity for the generator on any finite time horizon $T > 0$. In case of bounded $C^4$-domains, a standard localization technique was adopted to deduce the maximum $L^p$-regularity for any time horizon $T \in (0, \infty]$.

## 1.2 Main Results

In this thesis, I study the quasilinear PDE system associated with Equations (1.1a)–(1.1d). In contrast to earlier works, to deal with a quasilinear system without maximal $L^p$-regularity property, it is technically beneficial to look for classical rather than weak or strong solutions. The necessity of studying smooth solutions results in a much higher complexity of the existence and uniqueness proof as it has to be carried out at a higher energy level, which, in turn, is based on a Kato-type approximation procedure rather than a Galerkin scheme.

In this section, we state the main results on the well-posedness and long-time behavior of Equations (1.1a)–(1.1d). While the local result assumes smoothness of the boundary of the domain, regularity of the initial data and nonlinearities as well as certain compatibility conditions, the global results rely additionally and critically on some further smallness assumption on the initial data. Recall $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is a bounded domain throughout this paper. In addition, as mentioned above, even though the local well-posedness results below are according to Equations (1.1a)–(1.1d), the proof is however done on a more general system (Equations (3.1a)–(3.1b) in Chapter 3) for both mathematical and modeling reasons.

**Definition 1.1.** *Let $s \geq 2$. By a classical solution to Equations (1.1a)–(1.1d) on $[0,T]$ at the energy level $s$, we understand a function pair $(w,\theta)\colon [0,T]\times\bar{\Omega} \to \mathbb{R}\times\mathbb{R}$ satisfying*

$$w \in \Big(\bigcap_{m=0}^{s-1} C^m\big([0,T], H^{s+2-m}(\Omega)\cap H_0^1(\Omega)\big)\Big) \cap C^s\big([0,T], H^2(\Omega)\cap H_0^1(\Omega)\big),$$
$$\theta \in \Big(\bigcap_{k=0}^{s-2} C^k\big([0,T], H^{s+1-k}(\Omega)\cap H_0^1(\Omega)\big)\Big) \cap C^{s-1}\big([0,T], H_0^1(\Omega)\big)$$

*such that it satisfies pointwise Equations (1.1a)–(1.1d). Classical solutions on $[0,T)$ and $[0,\infty)$ are defined correspondingly.*

**Definition 1.2.** *Let $w^m$, $m \geq 2$, and $\theta^k$, $k \geq 1$, denote the "initial values" for $\partial_t^m w$ and $\partial_t^k \theta$ formally and recursively computed in terms of $w^0, w^1$ and $\theta^0$ based on Equations (3.5a)–(3.5d) (cf. [9, p. 96])*

To proceed with our well-posedness result, we first perform a short calculation on the nonlinear term in (1.1a) to achieve an equivalent equation

$$w_{tt} - \gamma\triangle w_{tt} + \left[1 + 3b(\triangle w)^2\right]\triangle^2 w + \alpha\triangle\theta + 6b(\triangle w)|\nabla\triangle w|^2 = 0 \quad \text{in } (0,\infty) \qquad (1.6)$$

Now, we require the following assumptions.

**Assumption 1.3.** *Let $s \geq 3$ be an integer and let $\partial\Omega \in C^s$.*

1. *Let the initial data satisfy the regularity*

$$w^0, \triangle w^0 \in H^s(\Omega) \cap H_0^1(\Omega), \qquad w^1, \triangle w^1 \in H^{s-1}(\Omega) \cap H_0^1(\Omega),$$

$$\theta^0 \in H^{s+1}(\Omega) \cap H_0^1(\Omega)$$

*as well as compatibility conditions*

$$w^m, \triangle w^m \in H^{s-m}(\Omega) \cap H_0^1(\Omega) \text{ for } m = 2, \dots, s-1 \text{ and } w^s \in H^2(\Omega) \cap H_0^1(\Omega),$$

$$\theta^k \in H^{s+1-k}(\Omega) \cap H_0^1(\Omega) \text{ for } k = 1, \dots, s-2 \text{ and } \theta^{s-1} \in H_0^1(\Omega).$$

2. *Further, assume the "initial ellipticity" condition for $[1 + 3b(\triangle w^0)^2] \triangle^2 w$, i.e.,*

$$\min_{x \in \bar{\Omega}} \big[1 + 3b(\triangle w^0)^2\big] > 0,$$

*where $\triangle w^0 \in C^0(\bar{\Omega})$ by virtue of Sobolev's imbedding theorem.*

Now, we can formulate our local well-posedness result.

**Theorem 1.4** (Local Well-Posedness)**.** *If Assumption 1.3 is satisfied for some $s \geq 3$, Equations (1.1a)–(1.1d) possess a unique classical solution $(w, \theta)$ at the energy level $s$ on a maximal interval $[0, T_{\max}) \neq \emptyset$ additionally satisfying*

$$\partial_t^{s-1}\theta \in L^2\big(0, T; H^2(\Omega) \cap H_0^1(\Omega)\big) \text{ and } \partial_t^s \theta \in L^2\big(0, T; L^2(\Omega)\big)$$

*along with*

$$\min_{x \in \bar{\Omega}} \big[1 + 3b(\triangle w(t,x))^2\big] > 0. \text{ for any } t \in [0, T_{\max}).$$

*Unless* $T_{\max} = \infty$, *we have* $\min_{x\in\bar\Omega} a\big(\triangle w(t,x)\big) \to 0$ *as* $\nearrow T_{\max}$ *or/and*

$$\sum_{k=0}^{s} \big\|\partial_t^k w(t,\cdot)\big\|^2_{H^{s+2-k}(\Omega)} + \sum_{k=0}^{s-2} \big\|\partial_t^k \theta(t,\cdot)\big\|^2_{H^{s+1-k}(\Omega)} + \big\|\partial_t^{s-1}\theta(t,\cdot)\big\|^2_{H^1(\Omega)} \to \infty$$

*as* $t \nearrow T_{\max}$.

Let $\|(w,\theta)\|_{\mathcal{Z}_s\times\mathcal{T}_s} \equiv \|(\partial_t^{\le s} w, \partial_t^{\le s-1}\theta)\|^2_{L^2(\Omega)}$, with $\partial_t^{\le k} = (1, \partial_t, \dots, \partial_t^k)$, denote the standard norm associated with the solution space in Definition 1.1 (a rigorous definition is given in Equation (4.2)). We now present our global results.

**Theorem 1.5** (Global Well-Posedness)**.** *Let Assumption 1.3 be satisfied for* $s = 3$*. Then, there exists a positive number* $\epsilon$ *(defined in Theorem 4.5 of Chapter 4) such that for any initial data* $(w^0, w^1, \theta^0)$ *satisfying*

$$\|(w^0, w^1, \theta^0)\|_{\mathcal{Z}_s\times\mathcal{T}_s} \equiv \|(w^0, w^1, \dots, w^s, \theta^0, \dots, \theta^{s-1})\|^2_{L^2(\Omega)} < \epsilon, \tag{1.7}$$

*(which roughly means the smallness of* $\|w^0\|^2_{H^5(\Omega)} + \|w^1\|^2_{H^4(\Omega)} + \|\theta^0\|^2_{H^4(\Omega)}$ *when* $s = 3$*), the unique local solution of system from Theorem 1.4 exists globally, i.e.,* $T_{\max} = \infty$*.*

**Theorem 1.6** (Exponential Stability)**.** *When* $s = 3$*, under the conditions of Theorem 1.5 and assuming additionally*

$$\|(w^0, w^1, \theta^0)\|_{\mathcal{Z}_s\times\mathcal{T}_s} \equiv \|(w^0, w^1, \dots, w^s, \theta^0, \dots, \theta^{s-1})\|^2_{L^2(\Omega)} < \tilde\epsilon \tag{1.8}$$

*for some small positive* $\tilde\epsilon$ *(to be defined in Corollary 4.6 of Chapter 4), there exist positive constants* $C$ *and* $k$ *such that*

$$\big\|(\partial_t^{\le s} w, \partial_t^{\le s-1}\theta)(t,\cdot)\big\|_{\mathcal{Z}_s\times\mathcal{T}_s} \le Ce^{-kt}\|(w^0, w^1, \theta^0)\|_{\mathcal{Z}_s\times\mathcal{T}_s} \quad \textit{for any } t \ge 0.$$

This thesis is structured as follows. After the short introduction in Chapter 1, we present in Chapter 2 a brief physical deduction of the Kirchhoff & Love plate from Equations (1.1a)–(1.1d). In Chapter 3, an existence and uniqueness result for Equations (1.1a)–(1.1d) in the class of classical solutions is shown. The long-time behavior of Equations (1.1a)–(1.1d) is studied in Chapter 4. Under a collection of smallness assumptions on the initial data, the global existence and uniqueness of solution is proved using energy estimates and the barrier method. Further, this global solution is shown to decay at an exponential rate to the zero equilibrium state. Finally, in the Appendix, we present a well-posedness theory along with higher energy estimates for a linear wave equation with time- and space-dependent coefficients as well as the homogeneous isotropic heat equation.

# Chapter 2

# Model Description

Consider a prismatic solid plate of uniform thickness $h > 0$ and constant material density $\rho > 0$ occupying in a reference configuration the domain $\mathcal{B}_h := \Omega \times (-\frac{h}{2}, \frac{h}{2})$ of $\mathbb{R}^3$, where $\Omega \subset \mathbb{R}^2$ is bounded. The underlying material is assumed to be elastically and thermally isotropic. Further, we restrict ourselves to the case of infinitesimal thermoelasticity with both stresses/strains and temperature gradient/heat flux being small. Additionally, we assume the strains linearly decompose into elastic and thermal ones. Despite of these linearity assumptions, a nonlinear (hypo)elastic law will be postulated allowing for materials with genuinely nonlinear response such as rubber, liquid crystal elastomers, etc. Figure 2.1 below (adopted from [22, Chapter 1]) displays a prismatic plate together with its mid-plane in the reference configuration.

We start by interpreting the plate as a 3D body. Let $\mathbf{U} = (U_1, U_2, U_3)^T$ be the displacement vector in Lagrangian coordinates, $T$ stand for the absolute temperature and $\mathbf{q} = (q_1, q_2, q_3)^T$ be the associated heat flux. Denote by $T_0 > 0$ a reference temperature for which the body occupies the reference configuration and is free of

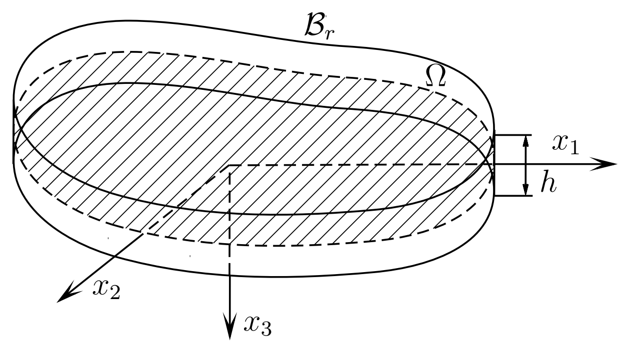


Figure 2.1: Prismatic plate

any stresses or strains. Further, let $S$ denote the entropy and

$$\boldsymbol{\sigma} = (\sigma_{ij})_{i=1,2,3}^{j=1,2,3} \quad \text{and} \quad \boldsymbol{\varepsilon} = \tfrac{1}{2}\big(\nabla\mathbf{U} + (\nabla\mathbf{U})^T\big) \tag{2.1}$$

stand for the first Piola & Kirchhoff stress tensor and the infinitesimal Cauchy strain tensor. In contrast to the theory of Finite Elasticity, the latter relation in Equation (2.1) ignores the so-called geometric nonlinearity. Parenthetically, replacing this linearization with its original quadratic version [20, Equation (17a)] and following the streamlines of [20] would lead to a fully nonlinear hypoelastic plate model. As the geometric nonlinearity is topologically of lower order compared to the one originating from the nonlinear elastic response in Equation (2.4), the former one was neglected for the sake of simplicity.

We assume the total stress tensor decomposes into elastic and thermal stresses according to

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{elast}} - \boldsymbol{\sigma}^{\text{therm}}. \tag{2.2}$$

In the absence of external body forces and heat sources, the momentum and energy balance equations (cf. [1, p. 142] and [13, Chapter 1]) read then as

$$\rho \mathbf{U_{tt}} + \operatorname{div} \boldsymbol{\sigma} = 0 \text{ in } (0, \infty) \times \mathcal{B}_h, \tag{2.3a}$$

$$T S_t + \operatorname{div} \mathbf{q} = 0 \text{ in } (0, \infty) \times \mathcal{B}_h. \tag{2.3b}$$

Similar to Ilyushin [8, p. 42], we define the elastic strain intensity $\varepsilon_{\text{int}}$ as a properly scaled second invariant of the elastic strain deviator tensor by means of

$$\varepsilon_{\text{int}}^{\text{elast}} = \tfrac{\sqrt{2}}{3}\Big( (\operatorname{tr} \boldsymbol{\varepsilon}^{\text{elast}})^2 - \operatorname{tr}\big( (\boldsymbol{\varepsilon}^{\text{elast}})^2 \big) \Big).$$

Similarly, we can define the elastic stress intensity via

$$\sigma_{\text{int}}^{\text{elast}} = \tfrac{\sqrt{2}}{3}\Big( (\operatorname{tr} \boldsymbol{\sigma}^{\text{elast}})^2 - \operatorname{tr}\big( (\boldsymbol{\sigma}^{\text{elast}})^2 \big) \Big).$$

Within the classical hypoelasticity, a relation between these two quantities needs to be postulated. Here, we consider a general material law given by

$$\sigma_{\text{int}} = \kappa(\varepsilon_{\text{int}}), \tag{2.4}$$

which generalizes power-law-type materials

$$\sigma_{\text{int}} = a\varepsilon_{\text{int}} - b\varepsilon_{\text{int}}^m \text{ for } a > 0, b \in \mathbb{R} \text{ and } m > 1 \tag{2.5}$$

considered by Ambartsumian et al. [1, Equation (6)]. The response function $\kappa(\cdot)$ is often referred to as a stress-strain curve and determined experimentally. Ignoring for simplicity the existence of yield and fracture points, natural assumptions on $\kappa(\cdot)$ are $\kappa(s) \geq 0$ for $s \geq 0$ and $\kappa(s) = 0$ if and only if $s = 0$. Obviously, if $b > 0$ –

which is the case for copper (cf [1, p. 144]) – Ambartsumian et al.'s [1] power-law response functions in Equation (2.5) do not satisfy these conditions since $\kappa(s) \to -\infty$ as $s \to \infty$. Hence, they are only meaningful in a neighborhood of zero.

For the thermal stresses and strains, we select a linear material law

$$\boldsymbol{\sigma}^{\text{therm}} = \tfrac{E}{1-2\mu} \boldsymbol{\varepsilon}^{\text{therm}}, \tag{2.6}$$

where $E$ and $\mu$ play the role of Young's modulus and Poisson's ratio and can be reconstructed from the Hooke's law resulting from linearizing equation (2.4) around zero.

With $\tau = T - T_0$ denoting the relative temperature, the thermal linearity and isotropy assumptions imply

$$\boldsymbol{\varepsilon}^{\text{therm}} = \alpha \tau \mathbf{I}_{3\times 3}, \tag{2.7}$$

where $\alpha > 0$ is the thermal expansion coefficient (cf. [13, p. 29]). According to Nowacki [21, Chapter 1], a linear approximation for the entropy reads as

$$S = \gamma \operatorname{tr}\left(\varepsilon^{\text{elast}}\right) + \tfrac{\rho c}{T_0} \tau, \tag{2.8}$$

where $c > 0$ is the heat capacity and $\gamma = \frac{E\alpha}{1-2\mu}$. Plugging Equations (2.2), (2.6), (2.7) and (2.8) into Equations (2.3a)–(2.3b) and linearizing with respect to $\tau$ around zero, we get

$$\rho \mathbf{U}_{tt} + \operatorname{div} \boldsymbol{\sigma}^{\text{elastic}} + \gamma \nabla \tau = 0 \text{ in } (0, \infty) \times \mathcal{B}_h, \tag{2.9a}$$

$$\rho c \tau_t - \lambda_0 \triangle \tau + \gamma T_0 \operatorname{tr}\left(\boldsymbol{\varepsilon}_t^{\text{elast}}\right) = 0 \text{ in } (0, \infty) \times \mathcal{B}_h. \tag{2.9b}$$

Together with Equation (2.4), Equations (2.9a)–(2.9b) constitute the PDE system of 3D thermoelasticity. In the following, we exploit these equations to deduce our thermoelastic plate model.

As it is typical for most plate theories, we postulate the hypothesis of undeformable normals, i.e., the linear filaments being perpendicular to the mid-plane before deformation should also remain linear after the deformation. Since we are interested in obtaining a Kirchhoff & Love-type plate model, we additionally assume these deformed filaments remain perpendicular to the deformed mid-plane. The in-plane displacements are assumed negligible. Mathematically, these structural assumptions can be written as

$$\begin{aligned} U_1(x_1,x_2,x_3) &= -x_3 w_{x_1}(x_1,x_2), \quad U_2(x_1,x_2,x_3) = -x_3 w_{x_2}(x_1,x_2), \\ U_3(x_1,x_2,x_3) &= \qquad w(x_1,x_2), \end{aligned} \tag{2.10}$$

where $w$ is referred to as the bending component or the vertical displacement. Thus, the elastic behavior of our plate can fully be described merely by $w$. Figure 2.2 is self-describing and illustrates these structural assumptions.

As for the thermal part of the system, a properly weighted momentum of the relative temperature $\tau$ with respect to $x_3$ given by

$$\theta(x_1,x_2) = \frac{12\alpha}{h^3}\int_{-h/2}^{h/2} x_3 \tau(x_1,x_2,x_3)\mathrm{d}x_3$$

will play a crucial role. Proceeding as Lagnese and Lions [13, pp. 29–31], Equation (2.9b) can be reduced to

$$\rho c\theta_t - \lambda_0 \triangle\theta + \tfrac{12\lambda_0}{\rho c h^2}\big(\tfrac{h\lambda_1}{2}+1\big)\theta + \tfrac{\alpha\gamma}{\rho c}\triangle w_t = 0 \text{ in } (0,\infty)\times\Omega,$$

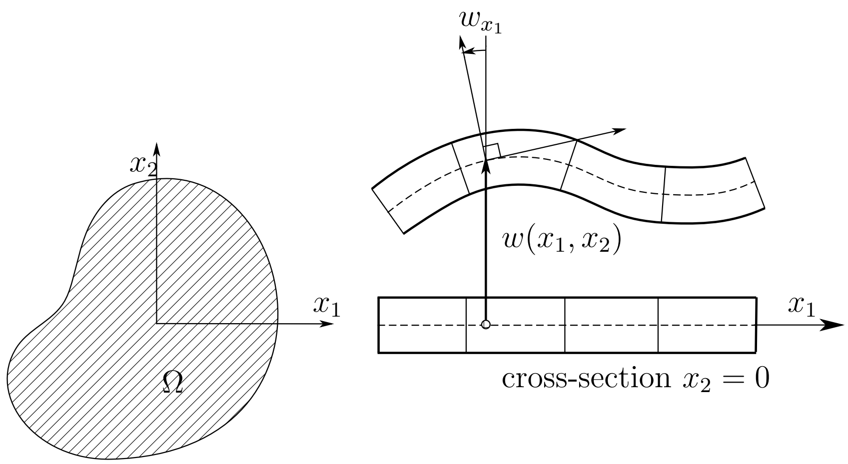


Figure 2.2: Mid-plane of a plate as well as plate cross-sections $x_2 = 0$ before and after the deformation

where $\lambda_1 \geq 0$ is the parameter from the Newton's cooling law applied to the lower and upper faces of the plate.

Returning to the elastic part and assuming for a moment the material response $\kappa(\cdot)$ from Equation (2.4) is an analytic function possessing a Taylor expansion with the vanishing constant term

$$\kappa(s) = \sum_{n=1}^{\infty} a_n s^n \text{ for some } a_n \in \mathbb{R},$$

we combine the approaches of Ambartsumian et al. [1] and Lagnese & Lions [13, Chapter 1] to deduce

$$\rho h w_{tt} - \tfrac{\rho h^3}{12} \triangle w_{tt} + \triangle K(\triangle w) + D \tfrac{1+\mu}{2} \triangle \theta = 0 \text{ in } (0, \infty) \times \Omega,$$

where $D = \frac{Eh^3}{12(1-\mu^2)}$ denotes the flexural rigidity and $K(\cdot)$ is obtained from $\kappa(\cdot)$ by means of

$$K(s) = \sum_{n=1}^{\infty} 2\big(\tfrac{2}{\sqrt{3}}\big)^{n+1} \tfrac{h^{n+2}}{n+2} a_n s^n.$$

Obviously, $K(\cdot)$ is also analytic and its Taylor series has the same absolute convergence region as $\kappa(\cdot)$. Taking into account

$$\begin{aligned} K(s) &= \sum_{n=1}^{\infty} 2\big(\tfrac{2}{\sqrt{3}}\big)^{n+1} \tfrac{h^{n+2}}{n+2} a_n s^n = s^{-2} \sum_{n=1}^{\infty} 2\big(\tfrac{2}{\sqrt{3}}\big)^{n+1} \tfrac{h^{n+2}}{n+2} a_n s^{n+2} \\ &= s^{-2} \sum_{m=1}^{\infty} 2\big(\tfrac{2}{\sqrt{3}}\big)^{-1} a_n \frac{\big(\tfrac{2}{\sqrt{3}}hs\big)^{n+2}}{n+2} = h^2\big(\tfrac{2}{\sqrt{3}}hs\big)^{-2}\big(\tfrac{4}{\sqrt{3}}\big) \sum_{n=1}^{\infty} a_n \frac{\big(\tfrac{2}{\sqrt{3}}hs\big)^{n+2}}{n+2} \\ &= \tfrac{4h^2}{\sqrt{3}}\big(\tfrac{2}{\sqrt{3}}hs\big)^{-2} \sum_{n=1}^{\infty} a_n \frac{\big(\tfrac{2}{\sqrt{3}}hs\big)^{n+2}}{n+2} = \tfrac{4h^2}{\sqrt{3}} \sum_{n=1}^{\infty} a_n \big[I\big((\cdot)^n\big)\big]\big(\tfrac{2}{\sqrt{3}}hs\big) \end{aligned}$$

with the linear operator

$$\big(If\big)(s) = s^{-2} \int_0^s \xi f(\xi) \mathrm{d}\xi \text{ for } s \in \mathbb{R}\backslash\{0\},$$

the function $K(\cdot)$ can equivalently be written as

$$K(s) = \tfrac{4h^2}{\sqrt{3}} [I\kappa]\big(\tfrac{2}{\sqrt{3}}hs\big) \text{ for } s \in \mathbb{R}\backslash\{0\}. \tag{2.11}$$

By density and continuity, $I(\cdot)$ can uniquely be extended to a mapping from the set of continuous functions differentiable and vanishing at 0 with the following norm being bounded

$$\|f\| = \max\Big\{ \sup_{x\in\mathbb{R}} |f(x)|, |f'(0)| \Big\}$$

into itself.

Summarizing, our thermoelastic plate system reads as

$$\rho h w_{tt} - \tfrac{\rho h^3}{12} \triangle w_{tt} + \triangle K(\triangle w) + D \tfrac{1+\mu}{2} \triangle \theta = 0 \text{ in } (0, \infty) \times \Omega, \tag{2.12a}$$

$$\rho c \partial_t \theta - \lambda_0 \triangle \theta + \tfrac{12\lambda_0}{\rho c h^2} \big( \tfrac{h\lambda_1}{2} + 1 \big) \theta + \tfrac{\alpha\gamma\lambda_0}{\rho c} \triangle w_t = 0 \text{ in } (0, \infty) \times \Omega. \tag{2.12b}$$

In contrast to [1], the $\triangle w_{tt}$-term is not neglected here allowing for an adequate description of thicker plates than those accounted for by the standard theory. Various boundary conditions can be adopted. We refer to [2, Chapter 2], [8, Chapter 4] and [13, Chapter 1] for further details. Here, we consider a simply supported plate held at the reference temperature at the boundary:

$$w = \triangle w = \theta = 0 \text{ in } (0, \infty) \times \partial\Omega.$$

# Chapter 3

# Proof of Theorem 1.4: Local Well-Posedness

As mentioned in the Introduction, the local well-posedness result, Theorem 1.4, will be proved on a more general system below.

$$w_{tt} - \gamma \triangle w_{tt} + a(-\triangle w)\triangle^2 w + \alpha \triangle \theta = f(-\triangle w, -\nabla \triangle w) \quad \text{in } (0,\infty)\times\Omega, \tag{3.1a}$$

$$\beta\theta_t - \eta\triangle\theta + \sigma\theta - \alpha\triangle w_t = 0 \quad \text{in } (0,\infty)\times\Omega \tag{3.1b}$$

along with the boundary conditions (hinged mechanical/Dirichlet thermal)

$$w = \triangle w = \theta = 0 \text{ in } (0,\infty)\times\Omega \tag{3.1c}$$

and the initial conditions

$$w(0,\cdot) = w^0, \quad w_t(0,\cdot) = w^1, \quad \theta(0,\cdot) = \theta^0 \text{ in } \Omega. \tag{3.1d}$$

It is clear that Equations (1.1a)–(1.1d) is a special case of (3.1a)–(3.1d). Indeed, if we consider the following specific functions:

$$a(z) = 1 + 3bz^2 \qquad \text{and} \qquad f(z, \nabla z) = 6bz|\nabla z|^2 \tag{3.2}$$

where (as before) the operator $A$ denotes the negative Dirichlet-Laplacian (cf. Equation (3.3)) and $z = -\triangle w = Aw$. This choice is motivated by [6], [14] and [15] (cf. Equation (1.2a)), where the same cubic nonlinearity

$$\begin{aligned}\triangle\big(K(\triangle w)\big) = \triangle\big(w + (\triangle w)^3\big) &= \triangle^2 w + 3(\triangle w)^2\triangle^2 w + 6(\triangle w)|\nabla\triangle w| \\ &= Az + 3z^2 Az - 6z|\nabla z|^2\end{aligned}$$

originating (up to positive constants) from the material response function $\kappa(s) = s+s^3$ according to Equation (2.11) is considered. This choice of $\kappa(\cdot)$ is inasmuch physically meaningful as it corresponds (up to physical constants) to nonlinear power-type materials considered by Ambartsumian et al. [1, Equation (6)] while ignoring geometric nonlinearity by adopting the infinitesimal Cauchy's stress tensor. In contrast to Ambartsumian et al.'s example of cubic nonlinearity $\kappa(s) = s - s^3$ on [1, p. 144] characterizing the elastic response of copper (and violating the positivity condition as $s \to \infty$), for the sake of consistency with earlier works [6], [14] and [15], we let $\kappa(s) = s + s^3$. It should though be pointed out that the sign of the nonlinear term in $\kappa(\cdot)$ and $K(\cdot)$ is not essential for our approach since we are interested in small classical solutions. Parenthetically, it should be mentioned that cubic terms may also result from the geometric nonlinearity as observed in [20]. In contrast to $\triangle K(\triangle w)$ coming from the elastic response, the geometric nonlinearity is a lower-order term reading as div $\big(N(\nabla w)\nabla w\big)$ with a matrix function $N(\cdot)$ being a second-order polynomial in $\nabla w$.

To facilitate the analytical treatment of (1.1a)–(1.1d), we first reduce the order in space from four to two. To this end, let $A$ denote the $L^2(\Omega)$-realization of the negative Dirichlet-Laplacian, i.e.,

$$A := -\triangle, \qquad D(A) := \big\{u \in H_0^1(\Omega) \,|\, \triangle u \in L^2(\Omega)\big\} = H^2(\Omega) \cap H_0^1(\Omega). \tag{3.3}$$

Assuming $\partial\Omega$ is of class $C^2$, the elliptic regularity theory yields $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Moreover, $A$ is an isomorphism between $D(A)$ and $L^2(\Omega)$, $A^{-1}$ is a compact self-adjoint operator and $(-\infty, 0]$ is contained both in the resolvent set of $A$ and $A^{-1}$. Letting

$$z := Aw = -\triangle w, \tag{3.4}$$

which lead to $w_{tt} = A^{-1} z_{tt}$. Apply these identities and we rewrite Equations (3.1a)–(3.1d) as an initial-boundary value problem for a system of partial differential equations given by

$$\big(A^{-1} + \gamma\big) z_{tt} + a(z)Az - \alpha A\theta = f(z, \nabla z) \quad \text{in } (0,\infty) \times \Omega, \tag{3.5a}$$

$$\beta\theta_t + \eta A\theta + \sigma\theta + \alpha z_t = 0 \quad \text{in } (0,\infty) \times \Omega, \tag{3.5b}$$

$$z = \theta = 0 \quad \text{in } (0,\infty) \times \partial\Omega, \tag{3.5c}$$

$$z(0,\cdot) = z^0, \quad z_t(0,\cdot) = z^1, \quad \theta(0,\cdot) = \theta^0 \quad \text{in } \Omega, \tag{3.5d}$$

where $z^0 := -\triangle w^0$ and $z^1 := -\triangle w^1$. Note that, for any $s \geq 0$, the operator $A^{-1} + \gamma$ restricted onto $H^s(\Omega)$ is an automorphism of $H^s(\Omega)$. Therefore, Definition 1.1 is equivalent to the following one in the new variable $z$:

**Definition 3.1.** *Let $s \geq 2$. By a classical solution to Equations (3.5a)–(3.5d) on $[0, T]$ at the energy level $s$, we understand a function pair $(z, \theta)\colon [0, T] \times \bar{\Omega} \to \mathbb{R} \times \mathbb{R}$*

*satisfying*

$$z \in \Big( \bigcap_{m=0}^{s-1} C^m\big([0,T], H^{s-m}(\Omega) \cap H_0^1(\Omega)\big)\Big) \cap C^s\big([0,T], L^2(\Omega)\big),$$
$$\theta \in \Big( \bigcap_{k=0}^{s-2} C^k\big([0,T], H^{s+1-k}(\Omega) \cap H_0^1(\Omega)\big)\Big) \cap C^{s-1}\big([0,T], H_0^1(\Omega)\big)$$

*such that it satisfies pointwise Equations (3.5a)–(3.5d). Classical solutions on* $[0,T)$ *and* $[0,\infty)$ *are defined correspondingly.*

**Remark 3.1.** *The choice* $s = 2$ *in Definition 3.1 is standard for the linear situation, i.e., when* $a(\cdot)$ *is constant and the function* $f(\cdot,\cdot)$ *is linear. In this case, by virtue of the standard semigroup theory, for any initial data* $(z^0, z^1, \theta^0) \in \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times H_0^1(\Omega) \times \big(H^3(\Omega) \cap H_0^1(\Omega)\big)$ *with* $\triangle\theta^0 \in H_0^1(\Omega)$*, there exists a unique classical solution at the energy level* $s = 2$*.*

*On the contrary, if* $a(\cdot)$ *and* $f(\cdot,\cdot)$ *are both genuinely nonlinear, one usually can not expect of obtaining a classical solution for the initial data at the energy level* $s = 2$ *(cf. [10, Remark 14.4]). Therefore, taking a higher energy level is inevitable to obtain classical solutions in the general nonlinear case. Unfortunately, this not only amounts to putting an additional Sobolev regularity assumption on the initial data and smoothness conditions on* $a(\cdot)$ *and* $f(\cdot,\cdot)$*, but also makes it necessary to postulate appropriate compatibility conditions.*

To better understand the nature of compatibility conditions, we make the following observation. Assuming there exists a classical solution at an energy level $s \geq 2$, we can use the smoothness in $t = 0$ and Equations (3.5a)–(3.5b) to compute

$$\begin{aligned} z_{tt} &= \left(A^{-1} + \gamma\right)^{-1}\Big(f(z, \nabla z) - a(z)Az + \alpha A\theta\Big), \\ \theta_t &= -\tfrac{1}{\beta}\left(\eta A\theta + \sigma\theta + \alpha z_t\right). \end{aligned} \tag{3.6}$$

Evaluating these equations at $t = 0$, we obtain

$$\begin{aligned} z_{tt}(0, \cdot) &= \left(A^{-1} + \gamma\right)^{-1}\Big(f(z^0, \nabla z^0) - a(z^0)Az^0 + \alpha A\theta^0\Big), \\ \theta_t(0, \cdot) &= -\tfrac{1}{\beta}\left(\eta A\theta^0 + \sigma\theta^0 + \alpha z^1\right). \end{aligned}$$

Assuming both $a(\cdot)$ and $f(\cdot, \cdot)$ are sufficiently smooth, we can differentiate Equation (3.6) with respect to $t$ and repeat the procedure to explicitly evaluate $\partial_t^m z(0, \cdot)$ or $\partial_t^k \theta$ for $m = 2, \ldots, s$ or $k = 1, \ldots, s-1$, respectively. Thus, Definition 1.2 and Assumption 1.3 are equivalent to the following ones:

**Definition 3.2.** *Let $z^m$, $m \geq 2$, and $\theta^k$, $k \geq 1$, denote the "initial values" for $\partial_t^m z$ and $\partial_t^k \theta$ formally and recursively computed in terms of $z^0, z^1$ and $\theta^0$ based on Equations (3.5a)–(3.5d) (cf. [9, p. 96]).*

**Assumption 3.3.** *Let $s \geq 3$ be an integer and let $\partial\Omega \in C^s$.*

1. *Let $a \in C^{s-1}(\mathbb{R}, \mathbb{R})$.*
2. *Let $f \in C^{s-1}(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$.*
3. *Let the initial data satisfy the regularity*

$$z^0 \in H^s(\Omega) \cap H_0^1(\Omega), \quad z^1 \in H^{s-1}(\Omega) \cap H_0^1(\Omega), \quad \theta^0 \in H^{s+1}(\Omega) \cap H_0^1(\Omega)$$

*as well as compatibility conditions*

$$z^m \in H^{s-m}(\Omega) \cap H_0^1(\Omega) \text{ for } m = 2, \dots, s-1 \text{ and } z^s \in L^2(\Omega),$$

$$\theta^k \in H^{s+1-k}(\Omega) \cap H_0^1(\Omega) \text{ for } k = 1, \dots, s-2 \text{ and } \theta^{s-1} \in H_0^1(\Omega).$$

4. *Further, assume the "initial ellipticity" condition for $a(z^0)A$, i.e.,*

$$\min_{x \in \bar{\Omega}} a\big(z_0(x)\big) > 0, \text{ where } z^0 \in C^0(\bar{\Omega}) \text{ by virtue of Sobolev's imbedding theorem.}$$

We can reformulate our local well-posedness result Theorem 1.4 in terms of $z$ as follows:

**Theorem 3.4.** *If Assumption 3.3 is satisfied for some $s \geq 3$, Equations (3.5a)–(3.5d) possess a unique classical solution $(z, \theta)$ at the energy level $s$ on a maximal interval $[0, T_{\max}) \neq \emptyset$ additionally satisfying*

$$\partial_t^{s-1}\theta \in L^2\big(0, T; H^2(\Omega) \cap H_0^1(\Omega)\big) \text{ and } \partial_t^s\theta \in L^2\big(0, T; L^2(\Omega)\big)$$

*along with*

$$\min_{x \in \bar{\Omega}} a\big(z(t, x)\big) > 0 \text{ for any } t \in [0, T_{\max}).$$

*Unless $T_{\max} = \infty$, we have*

$$\min_{x \in \bar{\Omega}} a\big(z(t, x)\big) \to 0 \text{ as } t \nearrow T_{\max} \tag{3.7}$$

*or/and*

$$\sum_{k=0}^{s} \big\|\partial_t^k z(t, \cdot)\big\|_{H^{s-k}(\Omega)}^2 + \sum_{k=0}^{s-2} \big\|\partial_t^k \theta(t, \cdot)\big\|_{H^{s+1-k}(\Omega)}^2 + \big\|\partial_t^{s-1}\theta(t, \cdot)\big\|_{H^1(\Omega)}^2 \to \infty \text{ as } t \nearrow T_{\max}. \tag{3.8}$$

*Proof.* First, exploiting the second Hilbert's resolvent identity

$$\big(A^{-1}+\gamma\big)^{-1} = \tfrac{1}{\gamma} - \tfrac{1}{\gamma}A^{-1}\big(A^{-1}+\gamma\big)^{-1},$$

we rewrite Equations (3.5a)–(3.5d) as

$$z_{tt} + \tfrac{1}{\gamma}a(z)Az - \tfrac{\alpha}{\gamma}A\theta = \quad F(z,\theta) \qquad \text{in } (0,\infty)\times\Omega, \tag{3.9a}$$

$$\theta_t + \tfrac{\eta}{\beta}A\theta = -\tfrac{1}{\beta}\big(\alpha z_t + \sigma\theta\big) \qquad \text{in } (0,\infty)\times\Omega, \tag{3.9b}$$

$$z = \theta = 0 \qquad \text{on } (0,\infty)\times\partial\Omega, \tag{3.9c}$$

$$z(0,\cdot) = z^0, \quad z_t(t,\cdot) = z^1, \quad \theta(0,\cdot) = \theta^0 \qquad \text{in } \Omega, \tag{3.9d}$$

where the nonlinear operator $F$ is given by

$$F(z,\theta) = \tfrac{1}{\gamma}(1-K)f(z,\nabla z) + \tfrac{1}{\gamma}K\big(a(z)Az\big) - \tfrac{\alpha}{\gamma}KA\theta$$

with the compact linear operator

$$K := A^{-1}\big(A^{-1}+\gamma\big)^{-1}$$

continuously mapping $H^s(\Omega)$ to $H^{s+2}(\Omega)\cap H^1_0(\Omega)$ for any $s \geq 0$ (cf. proof of Theorem **A**.12). Now, Equations (3.9a)–(3.9d) are a pseudo-differential perturbation of a second-order hyperbolic-parabolic system constituted by a quasi-linear wave equation coupled to a linear heat equation.

*Step 1: Modify the nonlinearity* $a(\cdot)$. Since no global positivity assumption is imposed on the nonlinearity $a(\cdot)$, the ellipticity condition for $a(z)A$ can be violated at any time $t > 0$. To (preliminarily) rule out this possible degeneracy, the following construction is performed.

Taking into account the continuity of $z^0$ and the connectedness of $\Omega$, we have

$$z^0(\bar{\Omega}) = \big[\min_{x\in\bar{\Omega}} z^0(x), \max_{x\in\bar{\Omega}} z^0(x)\big] =: J_0. \tag{3.10}$$

By Assumption 3.3.4, $a(\cdot)$ is strictly positive on $J_0$. Consider an arbitrary *closed* set $J$ such that

$$J_0 \subset \operatorname{int}(J) \text{ and } a(z) > 0 \text{ for } z \in J, \tag{3.11}$$

which must exist due to the continuity of $a(\cdot)$. By standard continuation arguments, there exists a $C^s$-function $\hat{a}(\cdot)$ such that

$$\hat{a}(\zeta) = a(\zeta) \text{ for } \zeta \in J \text{ and } \inf_{\zeta\notin J} \hat{a}(\zeta) > 0.$$

Now, we replace Equation (3.9a) with

$$z_{tt} + \tfrac{1}{\gamma}\hat{a}(z)Az - \tfrac{\alpha}{\gamma}A\theta = F(z,\theta) \quad \text{ in } (0,\infty)\times\Omega \tag{3.12}$$

and first consider Equations (3.12), (3.9b)–(3.9d). To solve this new problem, we transform it to a fixed-point problem and use the Banach fixed-point theorem. Our proof will be reminiscent of that one by Jiang and Racke [9, Theorem 5.2] carried out for the quasilinear system of thermoelasticity.

*Step 2: Define the fixed-point mapping.* Here and in the sequel, $H^0_0(\Omega) \equiv H^0(\Omega) := L^2(\Omega)$. For $N > 0$ and $T > 0$, let $X(N,T)$ denote the set of all regular distributions $(z,\theta)$ such that $(z,\theta)$ together with their weak derivatives satisfy the regularity conditions

$$\partial_t^m z \in C^0\big([0,T], H^{s-m}(\Omega)\big) \text{ for } m = 0, 1, \ldots, s,$$

$$\partial_t^k \theta \in C^0\big([0,T], H^{s+1-k}(\Omega)\big) \text{ for } k = 0, 1, \ldots, s-2, \quad \partial_t^{s-1}\theta \in C^0\big([0,T], H_0^1(\Omega)\big),$$

$$\partial_t^{s-1}\theta \in L^2\big(0,T; H^2(\Omega) \cap H_0^1(\Omega)\big) \text{ and } \partial_t^s \theta \in L^2\big(0,T; L^2(\Omega)\big)$$

the boundary

$$\partial_t^m z = \partial_t^k \theta = 0 \text{ in } [0,T] \times \partial\Omega \text{ for } m, k = 0, 1, \ldots, s-1$$

and the initial conditions

$$\partial_t^m z(0,\cdot) = z^m \text{ for } m = 0, 1, \ldots, s \text{ and } \partial_t^k \theta(0,\cdot) = \theta^k \text{ for } k = 0, 1, \ldots, s-1 \text{ in } \Omega \tag{3.13}$$

as well as the energy inequality

$$\max_{0 \le t \le T} \|\bar{D}^s z(t,\cdot)\|_{L^2(\Omega)}^2 + \sum_{k=0}^{s-2} \max_{0 \le t \le T} \|\partial_t^k \theta(t,\cdot)\|_{H^{s+1-k}(\Omega)}^2 + \max_{0 \le t \le T} \|\partial_t^{s-1}\theta(t,\cdot)\|_{H^1(\Omega)}^2 + \int_0^T \big(\|\triangle \partial_t^{s-1}\theta(t,\cdot)\|_{L^2(\Omega)}^2 + \|\partial_t^s \theta(t,\cdot)\|_{L^2(\Omega)}^2\big)\mathrm{d}t \le N^2. \tag{3.14}$$

Here, for $n \ge 0$, we let

$$\bar{D}^n := \big((\partial_t, \nabla)^\alpha \,|\, 0 \le |\alpha| \le n\big).$$

For any $T_0 > 0$ and sufficiently large $N > 0$, the set $X(N,T)$ is not empty for any $T \in (0, T_0]$. Indeed, if $N$ is sufficiently large, any pair $(z, \theta)$ of Taylor polynomials

$$z(t,\cdot) = \sum_{k=0}^{s} \frac{z^k t^k}{k!} + P_z(t,\cdot)t^{s+1}, \quad \theta(t,\cdot) = \sum_{k=0}^{s-1} \frac{\theta^k t^k}{k!} + P_\theta(t,\cdot)t^s,$$

is contained in $X(N,T)$, where $P_z$, $P_\theta$ are arbitrary $C_0^\infty(\Omega)$-valued polynomials w.r.t. $t$.

For $(\bar{z},\bar{\theta}) \in X(N,T)$, consider the linear operator $\mathscr{F}$ mapping $(\bar{z},\bar{\theta})$ to a function pair $(z,\theta)$ such that $\theta$ is the unique classical solution to the linear heat equation

$$\begin{aligned} \theta_t(t,x) - \tfrac{\eta}{\beta}\triangle\theta(t,x) = \bar{g}(t,x) \quad & \text{for } (t,x) \in (0,T)\times\Omega, \\ \theta = 0 \quad & \text{for } (t,x) \in (0,T)\times\partial\Omega, \\ \theta(0,\cdot) = \theta^0(x) \quad & \text{for } x \in \Omega \end{aligned} \tag{3.15}$$

with

$$\bar{g}(t,x) = -\tfrac{1}{\beta}\big(\alpha\bar{z}_t(t,x) + \sigma\bar{\theta}(t,x)\big) \text{ for } (t,x) \in [0,T]\times\bar{\Omega} \tag{3.16}$$

and, subsequently, define $z$ to be the unique classical solution to the linear wave equation

$$\begin{aligned} z_{tt}(t,x) - \bar{a}_{ij}(t,x)\triangle z(t,x) = \bar{f}(t,x) \quad & \text{for } (t,x) \in (0,T)\times\Omega, \\ z(t,x) = 0 \quad & \text{for } (t,x) \in (0,T)\times\partial\Omega, \\ z(0,x) = z^0(x), \quad z_t(0,x) = z^1(x) \quad & \text{for } x \in \Omega \end{aligned} \tag{3.17}$$

with

$$\begin{aligned} &\bar{a}_{ij}(t,x) := \tfrac{1}{\gamma}\hat{a}\big(\bar{z}(t,x)\big)\delta_{ij} \text{ and} \\ &\bar{f}(t,x) := \tfrac{1}{\gamma}\big((1-K)f(\bar{z},\nabla\bar{z})\big)(t,x) + \tfrac{1}{\gamma}\big(K\big(\hat{a}(\bar{z})A\bar{z}\big)\big)(t,x) - \tfrac{\alpha}{\gamma}\big((1-K)A\theta\big)(t,x) \end{aligned} \tag{3.18}$$

for $(t,x) \in [0,T]\times\Omega$. Note that the right-hand side $\bar{f}$ depends on $A\theta$ and not $A\bar{\theta}$ as the standard procedure would suggest.

We prove $\mathscr{F}$ is well-defined. By the definition of $\bar{g}$ in Equation (3.16) and the regularity of $(\bar{z}, \bar{g}) \in X(N, T)$, we trivially have

$$\partial_t^k \bar{g} \in C^0\big([0,T], H^{s-1-k}(\Omega)\big) \text{ for } k = 0, 1, \ldots, s-1.$$

By virtue of Theorem **A**.12, Equation (3.15) possesses a unique classical solution $\theta$ satisfying

$$\partial_t^k \theta \in C^0\big([0,T], H^{s+1-k}(\Omega) \cap H_0^1(\Omega)\big) \text{ for } k = 0, 1, \ldots, s-2,$$

$$\partial_t^{s-1} \theta \in C^0\big([0,T], H_0^1(\Omega)\big) \cap L^2\big(0,T; H^2(\Omega) \cap H_0^1(\Omega)\big) \text{ and } \partial_t^s \theta \in L^2\big(0,T; L^2(\Omega)\big).$$

Now, taking into account the regularity of $\bar{z}$ and $\theta$, exploiting Assumption 3.3 and applying Sobolev's imbedding theorem, we can verify that Assumption **A**.9 is satisfied with

$$\gamma_i = \max_{0 \le t \le T} \bar{\gamma}_i\big(\|\bar{z}(t,\cdot)\|_{H^{s-1}(\Omega)}\big) \text{ for } i = 0, 1, \tag{3.19}$$

where $\gamma_0, \gamma_1 \colon [0,\infty) \to (0,\infty)$ are continuous functions. Here, we used the Sobolev imbedding $\nabla \bar{z}(t,\cdot) \in H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ along with the estimate

$$\big\|K\big(\hat{a}(\bar{z})A\bar{z}\big)\big\|_{H^{m+2}(\Omega)} \le C\big\|\hat{a}(\bar{z})A\bar{z}\big\|_{H^m(\Omega)} \text{ for } m = 0, 1, \ldots, s-2.$$

Here and in the following, $C > 0$ denotes a generic constant. Hence, by Theorem **A**.10, Equation (3.17) possesses a unique classical solution

$$z \in \bigcap_{m=0}^{s-1} C^m\big([0,T], H^{s-m}(\Omega) \cap H_0^1(\Omega)\big) \cap C^s\big([0,T], L^2(\Omega)\big)$$

implying $(z, \theta) \in X(N, T)$. Therefore, the mapping $\mathscr{F}$ is well-defined.

*Step 3: Show the self-mapping property.* We prove that $\mathscr{F}$ maps $X(N,T)$ into itself provided $N$ is sufficiently large and $T$ is sufficiently small. We define

$$E_0(T) := \sum_{m=0}^{s} \|z^m\|^2_{H^{s-m}(\Omega)} + \sum_{k=0}^{s-2} \|\theta^k\|^2_{H^{s+1-k}(\Omega)} + \|\theta^{s-1}\|^2_{H^1(\Omega)}.$$

Recalling the definition of $\bar{g}$ in Equation (3.16), applying Theorem **A**.12 and using Equation (3.14), we can estimate

$$\begin{aligned}\sum_{k=0}^{s-2} \max_{0\le t\le T} \|\partial_t^k \theta(t,\cdot)\|^2_{H^{s+1-k}(\Omega)} + \max_{0\le t\le T} \|\partial_t^{s-1}\theta(t,\cdot)\|^2_{H^1(\Omega)} \\ + \int_0^T \big(\|\triangle \partial_t^{s-1}\theta(t,\cdot)\|^2_{L^2(\Omega)} + \|\partial_t^s \theta(t,\cdot)\|^2_{L^2(\Omega)}\big)\mathrm{d}t \le CN^2 + CE_0.\end{aligned} \tag{3.20}$$

Further, taking into account Equations (3.13), (3.14) and (3.18) and applying Sobolev imbedding theorem and [9, Theorem B.6], we obtain

$$\int_0^T \|\partial_t^{s-1}\bar{f}(t,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}t \le C(N)(1+T) \tag{3.21}$$

and

$$\begin{aligned}&\sum_{m=0}^{s-2} \max_{0\le t\le T} \|\partial_t^m \bar{f}(t,\cdot)\|^2_{H^{s-2-m}(\Omega)} \\ \le &\sum_{m=0}^{s-2} \max_{0\le t\le T} \Big\|\partial_t^m \big(\tfrac{1}{\gamma}(1-K)f(\bar{z},\nabla\bar{z}) + \tfrac{1}{\gamma}K\big(\hat{a}(\bar{z})A\bar{z}\big) - \tfrac{\alpha}{\gamma}(1-K)A\theta\big)(t,\cdot)\Big\|^2_{H^{s-2-m}(\Omega)} \\ \le &\sum_{m=0}^{s-2} \Big\|\partial_t^m \big(\tfrac{1}{\gamma}(1-K)f(\bar{z},\nabla\bar{z}) + \tfrac{1}{\gamma}K\big(\hat{a}(\bar{z})A\bar{z}\big) - \tfrac{\alpha}{\gamma}(1-K)A\theta\big)(0,\cdot)\Big\|^2_{H^{s-2-m}(\Omega)} \\ + &\sum_{m=0}^{s-2} \int_0^T \Big\|\partial_t^m \big(\tfrac{1}{\gamma}(1-K)f(\bar{z},\nabla\bar{z}) + \tfrac{1}{\gamma}K\big(\hat{a}(\bar{z})A\bar{z}\big) - \tfrac{\alpha}{\gamma}(1-K)A\theta\big)(t,\cdot)\Big\|^2_{H^{s-2-m}(\Omega)}\mathrm{d}t \\ \le &C(E_0) + C(N)(1+T),\end{aligned} \tag{3.22}$$

where the fundamental theorem of calculus was employed. Plugging Equations (3.21) and (3.22) into the energy estimate in Theorem **A**.10, we obtain

$$\max_{0\leq t\leq T} \|\bar{D}^s z(t,\cdot)\|_{L^2(\Omega)}^2 \leq \bar{K}(E_0,\gamma_0,\gamma_1)\zeta(N,T) \tag{3.23}$$

with positive constants $\gamma_0, \gamma_1$ defined in Equation (3.19), a positive constant $K$ being a continuous function of its variables and

$$\zeta(N,T) = \Big(1 + C(N)T^{1/2}\sum_{i=0}^{5} T^{i/2}\Big) \exp\big(T^{1/2}C(N)(1+T^{1/2}+T+T^{3/2})\big).$$

Combining the estimates in Equations (3.20) and (3.23), we obtain

$$\begin{aligned}\max_{0\leq t\leq T}\|\bar{D}^s z(t,\cdot)\|_{L^2(\Omega)}^2 &+ \sum_{k=0}^{s-2}\max_{0\leq t\leq T}\|\partial_t^k\theta(t,\cdot)\|_{H^{s+1-k}(\Omega)}^2 + \max_{0\leq t\leq T}\|\partial_t^{s-1}\theta(t,\cdot)\|_{H^1(\Omega)}^2 \\ &+ \int_0^T \big(\|\triangle\partial_t^{s-1}\theta(t,\cdot)\|_{L^2(\Omega)}^2 + \|\partial_t^s\theta(t,\cdot)\|_{L^2(\Omega)}^2\big)\mathrm{d}t \leq \bar{K}(E_0,\gamma_0,\gamma_1)\zeta(N,T),\end{aligned} \tag{3.24}$$

possibly, with an increased constant $\bar{K}$.

We now select $N$ such that

$$N^2 \leq \tfrac{1}{2}\bar{K}(E_0,\gamma_0,\gamma_1).$$

Due to continuity of $\zeta(N,\cdot)$ in $T=0$ and $\zeta(N_0,0)=1$, there exists $T>0$ such that $\zeta\big(N,(0,T]\big)\subset[1,2]$. Hence, the estimate in Equation (3.24) is satisfied with $N^2$ on its right-hand side. Therefore, $(z,\theta)\in X(N,T)$ and $\mathscr{F}$ maps $X(N,T)$ into itself.

*Step 4: Prove the contraction property.* We consider the metric space

$$Y := \Big\{(z,\theta)\,\big|\, z, z_t, \nabla z \in L^\infty\big(0,T;L^2(\Omega)\big) \text{ and } \theta\in L^\infty\big(0,T;H^1(\Omega)\big)\Big\}$$

equipped with the metric

$$\rho\big((z,\theta),(\bar{z},\bar{\theta})\big) = \Big(\operatorname*{ess\,sup}_{0\le t\le T} \big\|\bar{D}^1\big(z-\bar{z}\big)(t,\cdot)\big\|^2_{L^2(\Omega)} + \operatorname*{ess\,sup}_{0\le t\le T} \big\|(\theta-\bar{\theta})(t,\cdot)\big\|^2_{H^1(\Omega)}\Big)^{1/2}$$

for $(z,\theta),(\bar{z},\bar{\theta}) \in Y$. Obviously, $Y$ is complete. Further, $X(N,T) \subset Y$. Moreover, $X(N,T)$ is closed in $Y$. Indeed, consider a sequence $\big((z_n,\theta_n)\big)_{n\in\mathbb{N}} \subset X(N,T)$ such that it is a Cauchy sequence in $Y$ and, thus, converges to some $(z,\theta) \in Y$. With the uniform energy bound in Equation (3.14) being valid for $\big((z_n,\theta_n)\big)_{n\in\mathbb{N}}$, it must possess a subsequence which weakly-$*$ converges to some element $(z^*,\theta^*) \subset X(N,T)$ in respective topologies. Since strong and weak-$*$ limits coincide, we have $(z,\theta) = (z^*,\theta^*) \in X(N,T)$.

We now prove that $\mathscr{F}\colon X(N,T) \to X(N,T)$ is a contraction mapping w.r.t. $\rho$. For $(\bar{z},\bar{\theta}),(\bar{z}^*,\bar{z}^*) \in X(N,T)$, let $(z,\theta) := \mathscr{F}\big((\bar{z},\bar{\theta})\big)$, $(z^*,\theta^*) := \mathscr{F}\big((\bar{z}^*,\bar{\theta}^*)\big)$. With $(\bar{z},\bar{\theta})$, $(\bar{z}^*,\bar{\theta}^*)$, $(z,\theta)$, $(z^*,\theta^*)$ all lying in $X(N,T)$, Equation (3.14) along with Sobolev imbedding theorem imply

$$\operatorname*{ess\,sup}_{0\le t\le T} \big\|\big(\bar{D}^1(\bar{z},\bar{z}^*,z,z^*)\big)(t,\cdot)\big\|_{L^\infty(\Omega)} \le CN. \tag{3.25}$$

Recalling Equations (3.9a)–(3.9b), we can easily see that $(\tilde{z},\tilde{\theta}) := (z-z^*,\theta-\theta^*)$ satisfies

$$\tilde{z}_{tt} + \tfrac{1}{\gamma}\hat{a}(z)A\tilde{z} - \tfrac{\alpha}{\gamma}A\tilde{\theta} = F(\bar{z},\bar{\theta}) - F(\bar{z}^*,\bar{\theta}^*) - \big(\hat{a}(\bar{z}) - \hat{a}(\bar{z}^*)\big)Az^*, \tag{3.26}$$

$$\tilde{\theta}_t + \tfrac{\eta}{\beta}A\tilde{\theta} = -\tfrac{\alpha}{\beta}(\bar{z}_t - \bar{z}^*_t) - \tfrac{\sigma}{\beta}(\bar{\theta}-\bar{\theta}^*) \tag{3.27}$$

in $(0, T) \times \Omega$. Further, we have

$$\tilde{z} = \tilde{\theta} = 0 \qquad \text{on } (0, \infty) \times \partial\Omega, \tag{3.28}$$

$$\tilde{z}(0, \cdot) \equiv 0, \quad \tilde{z}_t(t, \cdot) \equiv 0, \quad \tilde{\theta}(0, \cdot) \equiv 0 \qquad \text{in } \Omega. \tag{3.29}$$

Multiplying Equation (3.27) in $L^2(\Omega)$ with $A\tilde{\theta}$, using Young's inequality and integrating w.r.t. $t$, we obtain

$$\begin{aligned}
\|A^{1/2}\tilde{\theta}(t, \cdot)\|^2_{L^2(\Omega)} + \tfrac{\eta}{\beta} \int_0^t \|A\tilde{\theta}(\tau, \cdot)\|^2_{L^2(\Omega)} \\
\leq \varepsilon \int_0^t \|A\tilde{\theta}(\tau, \cdot)\|^2_{L^2(\Omega)} \mathrm{d}\tau \\
+ C_\varepsilon T \operatorname*{ess\,sup}_{0 \leq \tau \leq t} \Big( \big\|\bar{D}^1\big(\bar{z} - \bar{z}^*\big)(\tau, \cdot)\big\|^2_{L^2(\Omega)} + \big\|(\bar{\theta} - \bar{\theta}^*)(\tau, \cdot)\big\|^2_{L^2(\Omega)} \Big).
\end{aligned}$$

Hence, by Poincare-Friedrichs' inequality, selecting $\varepsilon > 0$ sufficiently small, we obtain

$$\begin{aligned}
\|\tilde{\theta}(t, \cdot)\|^2_{H^1(\Omega)} \leq -\tfrac{\eta}{2\beta} \int_0^t \|A\tilde{\theta}(\tau, \cdot)\|^2_{L^2(\Omega)} \mathrm{d}\tau \\
+ CT \operatorname*{ess\,sup}_{0 \leq \tau \leq t} \Big( \big\|\bar{D}^1\big(\bar{z} - \bar{z}^*\big)(\tau, \cdot)\big\|^2_{L^2(\Omega)} + \big\|(\bar{\theta} - \bar{\theta}^*)(\tau, \cdot)\big\|^2_{L^2(\Omega)} \Big)
\end{aligned} \tag{3.30}$$

Similarly, multiplying Equations (3.26) in $L^2\big(0, T; L^2(\Omega)\big)$ with $\tilde{z}_t$, applying Green's formula, using chain and product rules, taking into account Equation (3.28), exploiting the local Lipschitz continuity of $\hat{a}(\cdot)$ and $f(\cdot, \cdot)$, using Equations (3.14) and (3.30) as well as exploiting Young's and Poincaré-Friedrichs' inequalities, we can estimate for any $t \in [0, T]$

$$\begin{aligned}
\big\|\bar{D}^1\tilde{z}(t, \cdot)\big\|^2_{L^2(\Omega)} \leq \tfrac{\eta}{2\beta} \int_0^t \|A\tilde{\theta}(\tau, \cdot)\|^2_{L^2(\Omega)} \mathrm{d}\tau + CT \operatorname*{ess\,sup}_{0 \leq \tau \leq t} \big\|\big(\bar{\theta} - \bar{\theta}^*\big)(\tau, \cdot)\big\|^2_{L^2(\Omega)} \\
+ C(N)\Big((1 + T^{-1/2}) \int_0^t \big\|\bar{D}^1\tilde{z}(\tau, \cdot)\big\|^2_{L^2(\Omega)} \mathrm{d}\tau \\
+ T^{1/2}(1 + T) \operatorname*{ess\,sup}_{0 \leq \tau \leq t} \big\|\bar{D}^1\big(\bar{z} - \bar{z}^*\big)(\tau, \cdot)\big\|^2_{L^2(\Omega)}\Big).
\end{aligned} \tag{3.31}$$

Adding up Equations (3.30)–(3.31), using Gronwall's inequality, taking into account Equation (3.29) and selecting $T$ sufficiently small, we can estimate

$$\rho\big((z,\theta),(z^*,\theta^*)\big) \leq \lambda\rho\big((\bar{z},\bar{\theta}),(\bar{z}^*,\bar{\theta}^*)\big)$$

for some $\lambda \in (0,1)$. Hence, $\mathscr{F}$ is a contraction on $X(N,T)$ in the metric of space $Y$. With $X(N,T)$ being closed, Banach fixed-point theorem implies $\mathscr{F}$ has a unique fixed point $(z,\theta) \in X(N,T)$. Finally, due to the smoothness of $(z,\theta)$, we can easily verify $(z,\theta)$ is a unique classical solution to Equations (3.12), (3.9b)–(3.9d) at the energy level $s$.

*Step 5: Continuation to the maximal interval.* Observing that $z(T,\cdot), z_t(T,\cdot)$ and $\theta(T,\cdot)$ satisfy the regularity and compatibility assumptions and carrying out the standard continuation argument, we obtain a maximal interval $[0,T_J^*)$ for which the classical solution (uniquely) exists. Due to the interval's maximality, unless $T_J^* = \infty$, we have

$$\sum_{k=0}^{s} \big\|\partial_t^k z(t,\cdot)\big\|_{H^{s-k}(\Omega)}^2 + \sum_{k=0}^{s-2} \big\|\partial_t^k \theta(t,\cdot)\big\|_{H^{s+1-k}(\Omega)}^2 + \big\|\partial_t^{s-1}\theta(t,\cdot)\big\|_{H^1(\Omega)}^2 \to \infty \text{ as } t \nearrow T_J^*. \tag{3.32}$$

*Step 6: Returning to the original system.* By virtue of Sobolev's imbedding theorem, the function $a \circ z$ is continuous on $[0,T_J^*) \times \bar{\Omega}$. Hence, the number

$$T_{\max,J} := \begin{cases} T_J^*, & \text{if } \hat{a}\circ z \equiv a \circ z \text{ in } [0,T^*) \times \bar{\Omega}, \\ \min\big\{t \in [0,T^*) \,\big|\, a(t,x) \notin \operatorname{int}(J)\big\}, & \text{otherwise} \end{cases}$$

is well-defined and positive by Equation (3.11). Denote by $(z_J,\theta_J)$ the unique classical solution to (3.12), (3.9b)–(3.9d) restricted onto $[0,T_{\max,J})$. Consider now an increasing sequence $(J_n)_{n\in\mathbb{N}}$ of closed sets satisfying Equation (3.11) such that

$$T_{\max,J_n} \nearrow T_{\max} := \sup\left\{T_{\max,J} \,\middle|\, J \text{ satisfies Equation (3.11)}\right\} \text{ as } n \to \infty. \tag{3.33}$$

By construction, $(z_{J_n}, \theta_{J_n})$ solves the original problem (3.9a)–(3.9d) on $[0, T_{\max,J_n})$ and

$$(z_{J_m}, \theta_{J_m}) \equiv (z_{J_n}, \theta_{J_n}) \text{ on } [0, T_{\max,J_m}) \text{ for } m, n \in \mathbb{N} \text{ with } m \leq n.$$

Hence, letting for $t \in [0, T_{\max})$,

$$(z, \theta)(t) := (z_{J_n}, \theta_{J_n})(t) \text{ for any } n \in \mathbb{N} \text{ such that } T_{\max,J_n} > t,$$

we observe $(z, \theta)$ uniquely defines a classical solution to (3.9a)–(3.9d) on $[0, T_{\max})$. Moreover, unless $T_{\max} = \infty$, we have Equation (3.8) and/or Equation (3.7). Indeed, if neither was the case, we could redefine $J_0$ from Equation (3.10) via

$$J_0 := \big[\min_{x\in\bar{\Omega}} a(z(T_{\max}, x)), \max_{x\in\bar{\Omega}} a(z(T_{\max}, x))\big]$$

and repeat Step 5 to obtain a classical solution $(z_J, \theta_J)$ existing beyond $T_{\max}$, which would contradict Equation (3.33). The overall uniqueness follows similar to Step 4. □

**Remark 3.2.** *Equation (3.8) is equivalent to*

$$\big\|z(t,\cdot)\big\|^2_{H^s(\Omega)} + \big\|z(t,\cdot)\big\|^2_{H^{s-1}(\Omega)} \to \infty \text{ as } t \nearrow T_{\max}.$$

*Indeed, arguing by contradiction, if the norms in Equation (3.8) are bounded, Equations (3.9a), (3.5b) suggest the derivatives of $z$ and $\theta$ as well as $\theta$ itself are bounded in respective topologies, which contradicts the maximality of $T_{\max}$.*

# Chapter 4

# Proof of Theorem 1.5 and 1.6: Global Well-Posedness and Exponential Stability

In this chapter, we will show the global well-posedness and the exponential stability of the local solution to Equations (1.1a)–(1.1d) (or, equivalently, (3.5a)–(3.5d)) established in Theorem 3.4 provided the initial data are 'small.'

Recall that the system we study in $(z, \theta)$ is

$$\big(A^{-1} + \gamma\big) z_{tt} + Az - \alpha A\theta = -3z^2 Az + 6z|\nabla z|^2 \qquad \text{in } (0,\infty)\times\Omega, \quad (4.1\text{a})$$

$$\beta\theta_t + \eta A\theta + \sigma\theta + \alpha z_t = 0 \qquad \text{in } (0,\infty)\times\Omega, \quad (4.1\text{b})$$

$$z = \theta = 0 \qquad \text{in } (0,\infty)\times\partial\Omega, \quad (4.1\text{c})$$

$$z(0,\cdot) = z^0, \quad z_t(0,\cdot) = z^1, \quad \theta(0,\cdot) = \theta^0 \qquad \text{in } \Omega. \quad (4.1\text{d})$$

Under Assumption 3.3, Theorem 3.4 establishes the local existence of a unique classical solution to Equations (4.1a)–(4.1d) at the energy level $s \geq 3$:

$$
\begin{aligned}
z &\in \Big( \bigcap_{m=0}^{s-1} C^m\big([0, T_{\max}), H^{s-m}(\Omega) \cap H_0^1(\Omega)\big)\Big) \cap C^s\big([0, T_{\max}), L^2(\Omega)\big) \\
&\qquad =: C^0\big([0, T_{\max}), \mathcal{Z}_s\big), \\
\theta &\in \Big( \bigcap_{k=0}^{s-2} C^k\big([0, T_{\max}), H^{s+1-k}(\Omega) \cap H_0^1(\Omega)\big)\Big) \cap C^{s-1}\big([0, T_{\max}), H^1(\Omega)\big) \\
&\qquad =: C^0\big([0, T_{\max}), \mathcal{T}_s\big),
\end{aligned}
\tag{4.2}
$$

where $[0, T_{\max})$ is the maximal existence interval (in time) with $T_{\max} \leq \infty$. Unless $T_{\max} = \infty$, either the solution norm explodes or the hyperbolicity of Equation (4.1a) is violated at $T_{\max}$, the latter of which can never be the case for the specific form of nonlinearity in our model.

For the solution pair $(z, \theta)$, we introduce the squared norm functionals $E_k(t)$ for $k = 1, 2, 3$ and $0 \leq t < T_{\max}$

$$
E_1(t) := \tfrac{1}{2}\big\|A^{-\frac{1}{2}} z_t(t,\cdot)\big\|_{L^2(\Omega)}^2 + \tfrac{\gamma}{2}\big\|z_t(t,\cdot)\big\|_{L^2(\Omega)}^2 + \tfrac{1}{2}\big\|A^{\frac{1}{2}} z(t,\cdot)\big\|_{L^2(\Omega)}^2 + \tfrac{1}{2}\big\|A^{\frac{1}{2}} \theta(t,\cdot)\big\|_{L^2(\Omega)}^2
\tag{4.3}
$$

$$
E_2(t) := \tfrac{1}{2}\big\|z_t(t,\cdot)\big\|_{L^2(\Omega)}^2 + \tfrac{\gamma}{2}\big\|A^{\frac{1}{2}} z_t(t,\cdot)\big\|_{L^2(\Omega)}^2 + \tfrac{1}{2}\big\|A z(t,\cdot)\big\|_{L^2(\Omega)}^2 + \tfrac{1}{2}\big\|A \theta(t,\cdot)\big\|_{L^2(\Omega)}^2,
\tag{4.4}
$$

$$
E_3(t) := \tfrac{1}{2}\big\|z_{tt}(t,\cdot)\big\|_{L^2(\Omega)}^2 + \tfrac{\gamma}{2}\big\|A^{\frac{1}{2}} z_{tt}(t,\cdot)\big\|_{L^2(\Omega)}^2 + \tfrac{1}{2}\big\|A z_t(t,\cdot)\big\|_{L^2(\Omega)}^2 + \tfrac{1}{2}\big\|A \theta_t(t,\cdot)\big\|_{L^2(\Omega)}^2
\tag{4.5}
$$

and define the natural energy at level $s = 3$ by means of

$$
X(t) \equiv \big\|(z, z_t, z_{tt}, \theta, \theta_t)(t,\cdot)\big\|_X^2 := E_2(t) + E_3(t).
\tag{4.6}
$$

Note that $E_1(t)$ represents the basic 'natural' energy of the system. For the sake of brevity – and slightly abusing the notation – we will write in the following:

$$\big\|(z,\theta)\big\|_X \text{ instead of } \big\|(z,z_t,z_{tt},\theta,\theta_t)\big\|_X$$

and

$$\big\|(z,\theta)\big\|_{\mathcal{Z}_s\times\mathcal{T}_s} \text{ instead of } \big\|(\partial_t^{\leq s}z,\partial_t^{\leq s-1}\theta)\big\|_X$$

where $\partial_t^{\leq k} := (1,\partial_t,\ldots,\partial_t^k)$.

**Remark 4.1.** *In order to prove the system is globally well-posed, we first seek for an* a priori *estimate for the solution and then prove $T_{\max}=\infty$. To capture the essential decay of the energy, we work with $\|(z,\theta)\|_X$, instead of $\|(z,\theta)\|_{\mathcal{Z}_s\times\mathcal{T}_s}$, for most part of this chapter. Although all main results in this chapter are presented in terms of $X(t)$, their equivalence with the statements in Section 1.2, given the smallness of the initial data, is shown in Lemma 4.7.*

We start by an observation that, for small data, $z$ has one extra order of hidden regularity in space encoded in the definition of $E_2$.

**Lemma 4.1** ($z$-energy boost)**.** *For any $t\in(0,T_{\max})$, if $(z,\theta)$ satisfies*

$$E_2(t) < \epsilon_1 := \tfrac{1}{2\sqrt{C'}} \text{ for some constant } C'>0 \text{ (defined in (4.11))}, \tag{4.7}$$

*there holds*

$$\|z(t)\|^2_{H^3(\Omega)} \leq C\big(X(t)+X^3(t)\big) \text{ for some } C>0.$$

*Proof.* From Equation (4.1b), $-\eta A^{3/2}\theta = \alpha A^{1/2} z_t + \beta A^{1/2}\theta_t + \sigma A^{1/2}\theta$. Hence,

$$\big\|A^{3/2}\theta\big\|^2_{L^2(\Omega)} \leq C\big(\big\|A^{1/2}z_t\big\|^2_{L^2(\Omega)} + \big\|A^{1/2}\theta_t\big\|^2_{L^2(\Omega)} + \big\|A^{1/2}\theta\big\|^2_{L^2(\Omega)}\big) \leq C\big(E_2(t) + E_3(t)\big). \tag{4.8}$$

Using Hölder's inequality

$$\|a \cdot b\|^2_{L^2(\Omega)} \leq \|a\|^2_{L^6(\Omega)} \cdot \|b\|^2_{L^3(\Omega)}$$

and Sobolev imbedding theorem $H^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^4(\Omega) \hookrightarrow L^3(\Omega)$, we arrive at:

$$\begin{aligned} &\big\|A^{1/2}(6z|\nabla z|^2)\big\|^2_{L^2(\Omega)} \\ &\qquad \leq C\big\|A^{1/2}z \cdot |\nabla z|^2\big\|^2_{L^2(\Omega)} + C\big\|z(\nabla z \cdot \nabla A^{1/2}z)\big\|^2_{L^2(\Omega)} \\ &\qquad \leq C\|A^{1/2}z\|^2_{L^6(\Omega)}\|\nabla z|\|^4_{L^6(\Omega)} + C'\|Az\|^2_{{}_{L^2(\Omega)}}\|A^{3/2}z\|^2_{L^2(\Omega)}\big\|Az\big\|^2_{L^2(\Omega)} \\ &\qquad \leq CE_2^3(t) + C'E_2^2(t)\|A^{3/2}z\|^2_{L^2(\Omega)} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \big\|A^{1/2}(1+3z^2) \cdot Az\big\|^2_{L^2(\Omega)} &\leq C\big\|zA^{1/2}z \cdot Az\big\|^2_{L^2(\Omega)} \\ &\leq C\|z\|^2_{H^2(\Omega)}\|A^{1/2}z\|^2_{H^2(\Omega)}\big\|Az\big\|^2_{L^2(\Omega)} \\ &\leq C'\|Az\|^4_{L^2(\Omega)}\|A^{3/2}z\|^2_{L^2(\Omega)} \\ &\leq C'E_2^2(t)\|A^{3/2}z\|^2_{L^2(\Omega)} \text{ for some } C, C' > 0. \end{aligned} \tag{4.10}$$

Now, to estimate $\|A^{3/2}z\|^2_{L^2(\Omega)}$, we successively transform Equation (4.1a) to obtain:

$$\begin{aligned}
(1+3z^2)Az &= -\left[A^{-1}z_{tt}+\gamma z_{tt}-\alpha A\theta-6z|\nabla z|^2\right],\\
A^{1/2}[(1+3z^2)Az] &= -A^{1/2}\left[A^{-1}z_{tt}+\gamma z_{tt}-\alpha A\theta-6z|\nabla z|^2\right],\\
A^{1/2}(1+3z^2)\cdot Az+(1+3z^2)A^{3/2}z &= -A^{1/2}\left[A^{-1}z_{tt}+\gamma z_{tt}-\alpha A\theta-6z|\nabla z|^2\right],\\
(1+3z^2)A^{3/2}z &= -A^{1/2}\left[A^{-1}z_{tt}+\gamma z_{tt}-\alpha A\theta-6z|\nabla z|^2\right],\\
&\quad -A^{1/2}(1+3z^2)\cdot Az,\\
A^{3/2}z &= -\tfrac{1}{1+3z^2}\Big[A^{-1/2}z_{tt}+A^{1/2}\gamma z_{tt}-\alpha A^{3/2}\theta\\
&\quad -A^{1/2}(6z|\nabla z|^2)+A^{1/2}(1+3z^2)\cdot Az\Big].
\end{aligned}$$

Taking into account $\frac{1}{1+3z^2}\leq 1$, Equations (4.8), (4.9) and (4.10) yield

$$\begin{aligned}
\big\|A^{3/2}z\big\|^2_{L^2(\Omega)} &\leq \big\|\tfrac{1}{1+3z^2}\big\|^2_{L^\infty(\Omega)}\Big[\big\|A^{-1/2}z_{tt}\big\|_{L^2(\Omega)}+\gamma\big\|A^{1/2}z_{tt}\big\|_{L^2(\Omega)}+\alpha\big\|A^{3/2}\theta\big\|_{L^2(\Omega)}\\
&\quad+\big\|A^{1/2}(6z|\nabla z|^2)\big\|_{L^2(\Omega)}+\big\|A^{1/2}(1+3z^2)\cdot Az\big\|_{L^2(\Omega)}\Big]^2\\
&\leq C\Big[\big\|A^{-1/2}z_{tt}\big\|^2_{L^2(\Omega)}+\big\|A^{1/2}\gamma z_{tt}\big\|^2_{L^2(\Omega)}+\big\|A^{3/2}\theta\big\|^2_{L^2(\Omega)}\\
&\quad+\big\|A^{1/2}(6z|\nabla z|^2)\big\|^2_{L^2(\Omega)}+\big\|A^{1/2}(1+3z^2)\cdot Az\big\|^2_{L^2(\Omega)}\Big]\\
&\leq C\Big[E_3(t)+E_3(t)+[E_2(t)+E_3(t)]\\
&\quad+CE_2^3(t)+C'E_2^2(t)\|A^{3/2}z\|^2_{L^2(\Omega)}+C'E_2^2(t)\|A^{3/2}z\|^2_{L^2(\Omega)}\Big]. \qquad (4.11)
\end{aligned}$$

Hence,

$$\big\|A^{3/2}z\big\|^2_{L^2(\Omega)}\leq\tfrac{C}{1-2E_2^2(t)C'}\big[E_2(t)+E_3(t)+E_2^3(t)\big].$$

By the assumption in Equation (4.7), $1 - 2E_2^2(t)C' > \frac{1}{2}$ and, therefore,

$$\left\|A^{3/2}z\right\|_{L^2(\Omega)}^2 \leq C\big[E_2(t) + E_3(t) + E_2^3(t)\big] \leq C\big[X(t) + X^3(t)\big],$$

which finishes the proof. □

**Lemma 4.2** (*A priori* estimate)**.** *Let Assumption 3.3 and the smallness assumption of $E_2(t)$ in Lemma 4.1 be satisfied for some $s = 3$, and a positive time $T$ such that $0 < T < T_{\max}$ with $T_{\max} > 0$ denoting the maximal existence time from Theorem 3.4 and let*

$$X(0) < 1. \tag{4.12}$$

*Then,*

$$X_s(T) + \int_0^T X_s(t)\mathrm{d}t \leq C_1 X_s(0) + C_2 \sum_{i\in I} X_s^{\alpha_i}(T) + C_3 \sum_{j\in J} \int_0^T X_s^{\beta_j}(t)\mathrm{d}t \tag{4.13}$$

*where $C_k \geq 0$, $k = 1, 2, 3$ are constants, both $I, J \subset \mathbb{N}$ are finite sets, $\alpha_i > 1$ for any $i \in I$ and $\beta_j > 1$ for any $j \in J$.*

*Proof.* The proof mainly consists of energy estimates at the energy levels $s = 1, 2$ and 3. Estimates for higher energy spaces ($s \geq 4$) follow similarly. Throughout this proof, $\langle \cdot, \cdot \rangle$ denotes the standard $L^2(\Omega)$-inner product. Moreover, without loss of generality, we assume

$$X(t) < 1 \text{ for } t \in [0, T]. \tag{4.14}$$

If not, by continuity of $X(t)$, we can find a smaller $T$ so that (4.14) holds true. Equations (4.12) and (4.14) are critical in Step 4.2 of this proof. The dimension of the domain is also important by virtue of, for instance, Equation (4.36).

*Step 1: Level 1 energy identity.* Multiplying Equation (4.1a) with $z_t$ in $L^2(\Omega)$ and integrating by parts, we get

$$\tfrac{1}{2}\partial_t\big\|A^{-1/2}z_t\big\|^2_{L^2(\Omega)} + \tfrac{\gamma}{2}\partial_t\big\|z_t\big\|^2_{L^2(\Omega)} + \tfrac{1}{2}\partial_t\big\|A^{1/2}z\big\|^2_{L^2(\Omega)} - \alpha\left\langle A\theta, z_t\right\rangle, = \langle F, z_t\rangle \quad (4.15)$$

where $F(z, \nabla z, Az) = -3z^2Az + 6z|\nabla z|^2$ from (3.2). Similar actions on (4.1b) multiplied by $A\theta$ lead to

$$\tfrac{\beta}{2}\partial_t\big\|A^{1/2}\theta\big\|^2_{L^2(\Omega)} + \eta\big\|A\theta\big\|^2_{L^2(\Omega)} + \sigma\big\|A^{1/2}\theta\big\|^2_{L^2(\Omega)} + \alpha\left\langle z_t, A\theta\right\rangle = 0. \quad (4.16)$$

From Equations (4.15) and (4.16), we get the $E_1$-identity:

$$E_1(T) + \int_0^T \left(\eta\big\|A\theta\big\|^2_{L^2(\Omega)} + \sigma\big\|A^{1/2}\theta\big\|^2_{L^2(\Omega)}\right)\mathrm{d}t = E_1(0) + \int_0^T \langle F, z_t\rangle\,\mathrm{d}t. \quad (4.17)$$

*Step 2: Level 2 energy estimate.* Recalling from Equation (4.4)

$$E_2(t) = \tfrac{1}{2}\big\|z_t(t,\cdot)\big\|^2_{L^2(\Omega)} + \tfrac{\gamma}{2}\big\|A^{1/2}z_t(t,\cdot)\big\|^2_{L^2(\Omega)} + \tfrac{1}{2}\big\|Az(t,\cdot)\big\|^2_{L^2(\Omega)} + \tfrac{1}{2}\big\|A\theta(t,\cdot)\big\|^2_{L^2(\Omega)}$$

and using Equation (4.17) along with Poincaré-Friedrichs inequality, we get:

$$E_1(T) \le E_1(0) + \int_0^T \langle F, z_t\rangle\,\mathrm{d}t \le C\left[E_2(0) + \int_0^T \langle F, z_t\rangle\,\mathrm{d}t\right]. \quad (4.18)$$

Similarly,

$$\int_0^T \|A^{1/2}\theta\|_{L^2(\Omega)}^2 \mathrm{d}t \leq C \int_0^T \|A\theta\|_{L^2(\Omega)}^2 \mathrm{d}t \leq C \left[ E_2(0) + \int_0^T \langle F, z_t \rangle \, \mathrm{d}t \right], \tag{4.19}$$

$$\int_0^T \|A\theta\|_{L^2(\Omega)}^2 \mathrm{d}t \leq C \int_0^T \|A\theta\|_{L^2(\Omega)}^2 \mathrm{d}t \leq C \left[ E_2(0) + \int_0^T \langle F, z_t \rangle \, \mathrm{d}t \right], \tag{4.20}$$

$$\max \left\{ \|z_t(t,\cdot)\|_{L^2(\Omega)}^2, \|A^{1/2} z(t,\cdot)\|_{L^2(\Omega)}^2 \right\} \leq C E_1(T) \leq C \left[ E_2(0) + \int_0^T \langle F, z_t \rangle \, \mathrm{d}t \right]. \tag{4.21}$$

In order to estimate $E_2(t)$ and $\int_0^T E_2(t)\mathrm{d}t$, we employ a set of higher energy multipliers. We first start by multiplying Equation (4.1a) with $Az_t$ and recalling (4.18) to observe that

$$\begin{aligned} &\tfrac{1}{2}\|z_t(T)\|_{L^2(\Omega)}^2 + \tfrac{\gamma}{2}\|A^{1/2} z_t(T)\|_{L^2(\Omega)}^2 + \tfrac{1}{2}\|Az(T)\|_{L^2(\Omega)}^2 - \alpha \int_0^T \langle A\theta, Az_t \rangle \, \mathrm{d}t \\ &\leq C \left[ E_2(0) + \int_0^T \langle F, z_t \rangle \, \mathrm{d}t \right] + \int_0^T \langle F, Az_t \rangle \, \mathrm{d}t. \end{aligned} \tag{4.22}$$

Second, in order to estimate $\|A\theta(T)\|_{L^2(\Omega)}^2$, we multiply (4.1b) by $A\theta_t$ to get

$$\begin{aligned} &\frac{\eta}{2}\|A\theta(T)\|_{L^2(\Omega)}^2 + \frac{\sigma}{2}\|A^{1/2}\theta(T)\|_{L^2(\Omega)}^2 + \beta \int_0^T \|A^{1/2}\theta_t\|_{L^2(\Omega)}^2 \\ &= \frac{\eta}{2}\|A\theta(0)\|_{L^2(\Omega)}^2 + \frac{\sigma}{2}\|A^{1/2}\theta(0)\|_{L^2(\Omega)}^2 - \alpha \int_0^T \langle A^{1/2} z_t, A^{1/2}\theta_t \rangle. \end{aligned} \tag{4.23}$$

We apply Young's inequality to the inner product term, we get the following estimate

$$\begin{aligned} &\frac{\eta}{2}\|A\theta(T)\|_{L^2(\Omega)}^2 + \beta \int_0^T \|A^{1/2}\theta_t\|_{L^2(\Omega)}^2 \\ &\leq E_2(0) + \frac{\sigma}{2}\|A^{1/2}\theta(T)\|_{L^2(\Omega)}^2 + \alpha\frac{\beta}{\alpha} \int_0^T \|A^{1/2}\theta_t\|_{L^2(\Omega)}^2 + C \int_0^T \|A^{1/2} z_t\|_{L^2(\Omega)}^2 \end{aligned} \tag{4.24}$$

After performing cancellations, employing (4.18), and rescaling, the estimate becomes

$$C\left\|A\theta(T)\right\|_{L^2(\Omega)}^2 \le C\left[E_2(0)+\int_0^T \langle F, z_t\rangle\,\mathrm{d}t\right]+\epsilon\int_0^T \left\|A^{1/2}z_t\right\|_{L^2(\Omega)}^2. \tag{4.25}$$

Last, by Equation (4.1a), $z_{tt} = -BAz + \alpha BA\theta + BF$, where $B = (A^{-1}+\gamma)^{-1}$. Multiplying Equation (4.1b) by $Az_t$, we get

$$\begin{aligned}-\eta\int_0^T \langle A\theta, Az_t\rangle\,\mathrm{d}t &= \alpha\int_0^T \left\|A^{1/2}z_t\right\|_{L^2(\Omega)}^2\mathrm{d}t+\beta\left\langle\theta, Az_t\right\rangle\Big|_0^T+\beta\int_0^T\langle A\theta, BAz\rangle\,\mathrm{d}t\\ &\quad-\alpha\beta\int_0^T\langle A\theta, BA\theta\rangle\,\mathrm{d}t-\beta\int_0^T\langle A\theta, BF\rangle\,\mathrm{d}t+\sigma\int_0^T\langle\theta, Az_t\rangle\,\mathrm{d}t.\end{aligned} \tag{4.26}$$

Recall from the proof of Theorem 3.4 that $B$ is a continuous operator. Therefore, $\|B\| \le C_\gamma$. Adding (4.25) and a multiple of Equation (4.26) to (4.22), we have:

$$\begin{aligned}E_2(T)+\tfrac{\alpha^2}{\eta}\int_0^T \left\|A^{1/2}z_t\right\|_{L^2(\Omega)}^2\mathrm{d}t &\le C\left[E_2(0)+\int_0^T\langle F, z_t\rangle\,\mathrm{d}t\right]+\int_0^T\langle F, Az_t\rangle\,\mathrm{d}t\\ &+C_\epsilon\left\|A^{1/2}\theta(T)\right\|_{L^2(\Omega)}^2+\epsilon\left\|A^{1/2}z_t(T)\right\|_{L^2(\Omega)}^2\\ &+C\int_0^T\left\|A\theta\right\|_{L^2(\Omega)}^2\mathrm{d}t+C_\epsilon\int_0^T\left\|A^{1/2}\theta\right\|_{L^2(\Omega)}^2\mathrm{d}t+\epsilon\int_0^T\left\|Az\right\|_{L^2(\Omega)}^2\mathrm{d}t\\ &+C_\epsilon\int_0^T\left\|\theta\right\|_{L^2(\Omega)}^2\mathrm{d}t+\epsilon\int_0^T\left\|Az\right\|_{L^2(\Omega)}^2\mathrm{d}t+\tfrac{\alpha\beta}{\eta}\int_0^T\langle A\theta, BF\rangle\,\mathrm{d}t.\end{aligned} \tag{4.27}$$

After merging respective terms and applying Equations (4.18) and (4.19), (4.27) becomes

$$(1-\epsilon)E_2(T)+\tfrac{\alpha^2}{\eta}\int_0^T \left\|A^{1/2}z_t\right\|^2_{L^2(\Omega)}\mathrm{d}t$$
$$\le C\left[E_2(0)+\int_0^T \langle F, z_t\rangle\,\mathrm{d}t+\int_0^T \langle F, Az_t\rangle\,\mathrm{d}t+\int_0^T \langle BF, A\theta\rangle\right]\mathrm{d}t$$
$$+2\epsilon\int_0^T E_2(t)\mathrm{d}t,\ \text{ where } C=C(\epsilon,\alpha,\beta,\gamma,\eta,\sigma). \tag{4.28}$$

Third, to estimate $\int_0^T \|Az\|^2_{L^2(\Omega)}\mathrm{d}t$, we multiply(4.1a) by $Az$ and, again, use (4.18) to get

$$\begin{aligned}\int_0^T \|Az\|^2_{L^2(\Omega)}\mathrm{d}t \le \frac{\alpha^2}{2\eta C_{\gamma,\Omega}}\left|\int_0^T \langle F, Az\rangle\,\mathrm{d}t\right|+\epsilon'\int_0^T E_2(t)\mathrm{d}t+\epsilon' E_2(T)\\ +C_{\epsilon'}\left[E_2(0)+\int_0^T \langle F, z_t\rangle\,\mathrm{d}t\right]+\frac{\alpha^2}{2\eta}\int_0^T \left\|A^{1/2}z_t\right\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned} \tag{4.29}$$

Finally, after combining Equations (4.19), (4.28) and (4.29), we arrive at

$$\begin{aligned}E_2(T)+C_1\int_0^T E_2(t)\mathrm{d}t \le C_2E_2(0)+C_3\Bigg\{\left|\int_0^T \langle F, z_t\rangle\,\mathrm{d}t\right|+\left|\int_0^T \langle F, Az\rangle\,\mathrm{d}t\right|\\ +\left|\int_0^T \langle F, Az_t\rangle\,\mathrm{d}t\right|+\left|\int_0^T \langle BF, A\theta\rangle\,\mathrm{d}t\right|\Bigg\}.\end{aligned} \tag{4.30}$$

*Step 3: Level 3 energy estimate.* The 3rd level energy space ($E_3$) is one order higher in time than the 2nd level space ($E_2$). Hence, after differentiating Equation (4.1a)–(4.1b) in time

$$A^{-1}z_{ttt}+\gamma z_{ttt}+Az_t-\alpha A\theta_t=\partial_t F(z,\nabla z,\triangle z) \qquad \text{in } (0,\infty)\times\Omega, \tag{4.31a}$$

$$\beta\theta_{tt}+\eta A\theta_t-\sigma\theta_t+\alpha z_{tt}=0 \qquad \text{in } (0,\infty)\times\Omega, \tag{4.31b}$$

$$z=z_t=\theta=\theta_t=0 \qquad \text{in } (0,\infty)\times\partial\Omega, \tag{4.31c}$$

a procedure similar to Step 2 can be employed. Denote the right-hand side of (4.31a) by

$$G(z) \equiv \partial_t F = \partial_t[-3z^2 Az + 6z|\nabla z|^2] = -6zz_t Az - 3z^2 Az_t + 6z_t|\nabla z|^2 + 12z(\nabla z \cdot \nabla z_t) \tag{4.32}$$

and calculate

$$\begin{aligned} \partial_t G(z) = &-6z_t^2 Az - 6zz_{tt}Az - 12zz_t Az_t - 3z^2 Az_{tt} \\ &+ 6z_{tt}|\nabla z|^2 + 24z_t(\nabla z \cdot \nabla z_t) + 12z|\nabla z_t|^2 + 12z(\nabla z_t \cdot \nabla z_{tt}). \end{aligned} \tag{4.33}$$

Letting $\widetilde{z} = z_t$ and $\widetilde{\theta} = \theta_t$, we have

$$E_3(t) = \tfrac{1}{2}\left\|\widetilde{z}_t\right\|^2_{L^2(\Omega)} + \tfrac{\gamma}{2}\left\|A^{1/2}\widetilde{z}_t\right\|^2_{L^2(\Omega)} + \tfrac{1}{2}\left\|A\widetilde{z}\right\|^2_{L^2(\Omega)} + \tfrac{1}{2}\left\|A\widetilde{\theta}\right\|^2_{L^2(\Omega)}.$$

Therefore, in a fashion similar to Equation (4.30), we get the following 3rd level energy estimate:

$$\begin{aligned} E_3(T) + C_1\int_0^T E_3(t)\mathrm{d}t \le C_2E_3(0) + C_3\Bigg\{&\left|\int_0^T \langle G(z), \widetilde{z}_t\rangle \,\mathrm{d}t\right| + \left|\int_0^T \langle G(z), A\widetilde{z}\rangle \,\mathrm{d}t\right| \\ &+ \left|\int_0^T \langle G(z), A\widetilde{z}_t\rangle \,\mathrm{d}t\right| + \left|\int_0^T \left\langle BG(z), A\widetilde{\theta}\right\rangle \mathrm{d}t\right|\Bigg\}. \end{aligned} \tag{4.34}$$

Recalling $X(t) = E_2(t) + E_3(t)$, combine (4.30) and (4.34):

$$\begin{aligned} X(T) + C_1\int_0^T X(t)\mathrm{d}t \le C_2X(0) + C_3\Bigg\{&\left|\int_0^T \langle F, z_t\rangle \,\mathrm{d}t\right| + \left|\int_0^T \langle F, Az\rangle \,\mathrm{d}t\right| \\ &+ \left|\int_0^T \langle F, Az_t\rangle \,\mathrm{d}t\right| + \left|\int_0^T \langle BF, A\theta\rangle \,\mathrm{d}t\right| + \left|\int_0^T \langle G, \widetilde{z}_t\rangle \,\mathrm{d}t\right| \\ &+ \left|\int_0^T \langle G, A\widetilde{z}\rangle \,\mathrm{d}t\right| + \left|\int_0^T \langle G, A\widetilde{z}_t\rangle \,\mathrm{d}t\right| + \left|\int_0^T \left\langle BG, A\widetilde{\theta}\right\rangle \mathrm{d}t\right|\Bigg\}. \end{aligned} \tag{4.35}$$

*Step 4:* We now need to estimate the integrals on the right-hand side (r.h.s.) of Equation (4.35) (eight terms in the brackets) to get (4.13) for $s = 3$. We will be using the fact that

$$H^2(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{and} \quad H^2(\Omega) \hookrightarrow W^{1,4}(\Omega) \qquad \text{for } d \in \{2,3\}. \tag{4.36}$$

*Step 4.1: The first four terms on the r.h.s. of* (4.35). The embeddings in Equation (4.36) together with Young's inequality lead to an estimate of the first term:

$$\begin{aligned} \left|\int_0^T \langle F, z_t\rangle \, \mathrm{d}t\right| &\leq \left|\int_0^T \left\langle 3z^2 Az, z_t\right\rangle \mathrm{d}t\right| + \left|\int_0^T \left\langle 6z|\nabla z|^2, z_t\right\rangle \mathrm{d}t\right| \\ &\leq C_\epsilon \int_0^T X^3(t)\mathrm{d}t + \epsilon \int_0^T X(t)\mathrm{d}t. \end{aligned} \tag{4.37}$$

Here, we noted $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$. In general, $W^{2,p}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ for $p > \frac{4d}{4+d}$. Since $d = 2$ or $3$, we chose $p = 2$. Similar arguments apply to the next two terms with the following inequalities:

$$\left|\int_0^T \langle F, Az\rangle \, \mathrm{d}t\right| \leq C \int_0^T \|Az\|^3_{L^2(\Omega)} \|Az\|_{L^2(\Omega)} \mathrm{d}t \leq C \int_0^T X^2(t)\mathrm{d}t, \tag{4.38}$$

$$\left|\int_0^T \langle F, Az_t\rangle \, \mathrm{d}t\right| \leq C_\epsilon \int_0^T X^3(t)\mathrm{d}t + \epsilon \int_0^T X(t)\mathrm{d}t. \tag{4.39}$$

Again, by a similar argument, using the continuity of the operator $B$ and Equation (4.37), we can estimate the $4^{\text{th}}$ term:

$$\begin{aligned} \left|\int_0^T \langle BF, A\theta\rangle \, \mathrm{d}t\right| &\leq C \left|\int_0^T \left\|B(3z^2 Az + 6z|\nabla z|^2)\right\|^2_{L^2(\Omega)} \mathrm{d}t\right| + C \left|\int_0^T \|A\theta\|^2_{L^2(\Omega)}\right| \\ &\leq C \int_0^T \|Az\|^6_{L^2(\Omega)} \mathrm{d}t + C \left[E_2(0) + \left|\int_0^T \langle F, z_t\rangle \, \mathrm{d}t\right|\right] \\ &\leq CX(0) + C_\epsilon \int_0^T X^3(t)\mathrm{d}t + \epsilon \int_0^T X(t)\mathrm{d}t. \end{aligned} \tag{4.40}$$

*Step 4.2: The highest order terms on the r.h.s. of* (4.35). In order to estimate the remaining four terms containing $G$ (cf. (4.32)), we rewrite $G = G_1 + G_2$, where

$$G_1 = -6zz_tAz - 3z^2Az_t + 6z_t|\nabla z|^2 \quad \text{and} \quad G_2 = 12z(\nabla z \cdot \nabla z_t).$$

Therefore, the 7th term can be bounded as follows (after two integrations by parts):

$$\begin{aligned}
\left|\int_0^T \langle G, A\widetilde{z}_t\rangle \,\mathrm{d}t\right| &\leq \left|\int_0^T \langle G_1(z), Az_{tt}\rangle \,\mathrm{d}t\right| + \left|\int_0^T \langle G_2(z), Az_{tt}\rangle \,\mathrm{d}t\right| \\
&\leq \left|\langle G_1(z), Az_t\rangle \Big|_0^T\right| + \left|\int_0^T \langle \partial_t G_1(z), Az_t\rangle \,\mathrm{d}t\right| + \left|\int_0^T \langle G_2(z), Az_{tt}\rangle \,\mathrm{d}t\right| \\
&\leq \tfrac{1}{2}\big\|G_1(z(0))\big\|^2_{L^2(\Omega)} + \tfrac{1}{2}\big\|Az_t(0)\big\|^2_{L^2(\Omega)} + C_\epsilon\big\|G_1(z(T))\big\|^2_{L^2(\Omega)} \\
&\quad + \epsilon\big\|Az_t(T)\big\|^2_{L^2(\Omega)} + \left|\int_0^T \langle \partial_t G_1(z), Az_t\rangle \,\mathrm{d}t\right| + \left|\int_0^T \langle G_2(z), Az_{tt}\rangle \,\mathrm{d}t\right|.
\end{aligned} \tag{4.41}$$

*Step 4.2.1: The first four terms on the r.h.s. of* (4.41).

$$\begin{aligned}
&\big\|G_1(z(0))\big\|^2_{L^2(\Omega)} \\
&\leq 6\big\|z(0)z_t(0)Az(0)\big\|^2_{L^2(\Omega)} + 3\big\|z(0)^2Az_t(0)\big\|^2_{L^2(\Omega)} + 6\big\|z_t(0)|\nabla z(0)|^2\big\|^2_{L^2(\Omega)} \\
&\leq C\Big\{\|z(0)\|^6_{H^2(\Omega)} + \|z_t(0)\|^6_{H^2(\Omega)} + \big\|Az(0)\big\|^6_{L^2(\Omega)} + \|z(0)\|^8_{H^2(\Omega)} \\
&+ \big\|Az_t(0)\big\|^4_{L^2(\Omega)} + \|z_t(0)\|^4_{H^2(\Omega)} + \|z(0)\|^8_{H^2(\Omega)}\Big\} \\
&\leq C\Big\{\|z(0)\|^2_{H^2(\Omega)} + \|z_t(0)\|^2_{H^2(\Omega)} + \big\|Az(0)\big\|^2_{L^2(\Omega)} + \|z(0)\|^2_{H^2(\Omega)} + \big\|Az_t(0)\big\|^2_{L^2(\Omega)} \\
&+ \|z_t(0)\|^2_{H^2(\Omega)} + \|z(0)\|^2_{H^2(\Omega)}\Big\} \\
&\leq CX(0).
\end{aligned} \tag{4.42}$$

Here, we used the 'smallness' assumption $X(0) < 1$ from Equation (4.12). An argument similar to Equation (4.42) yields

$$\begin{aligned} C_\epsilon \big\|G_1(z(T))\big\|^2_{L^2(\Omega)} &\le C_\epsilon \big\|(-6zz_tAz - 3z^2Az_t + 6z_t|\nabla z|^2)(T)\big\|^2_{L^2(\Omega)} \\ &\le C_\epsilon \Big\{\|z(T)\|^6_{H^2(\Omega)} + \|z_t(T)\|^6_{H^2(\Omega)} + \big\|Az(T)\big\|^6_{L^2(\Omega)} \\ &+ \|z(T)\|^8_{H^2(\Omega)} + \big\|Az_t(T)\big\|^4_{L^2(\Omega)} + \|z_t(T)\|^4_{H^2(\Omega)} + \|z(T)\|^8_{H^2(\Omega)}\Big\} \\ &\le \ C_\epsilon\big[X^4(T) + X^6(T) + X^8(T)\big]. \end{aligned} \tag{4.43}$$

Trivially,

$$\tfrac{1}{2}\big\|Az_t(0)\big\|^2_{L^2(\Omega)} \le X(0) \quad \text{and} \quad \epsilon\big\|Az_t(T)\big\|^2_{L^2(\Omega)} \le \epsilon X(T). \tag{4.44}$$

*Step 4.2.2: The $5^{th}$ terms on the r.h.s. of* (4.41). Estimating $\left|\int_0^T \langle \partial_t G_1(z), Az_t\rangle \,\mathrm{d}t\right|$ with $G_1 = -6zz_tAz - 3z^2Az_t + 6z_t|\nabla z|^2$ and

$$\partial_t G_1(z) = -6z_t^2Az - 6zz_{tt}Az - 12zz_tAz_t - 3z^2Az_{tt} + 6z_{tt}|\nabla z|^2 + 12z_t(\nabla z\cdot\nabla z_t) \tag{4.45}$$

amounts to dealing with each of the respective six terms.

(a) First, we use Young's inequality to write

$$\left|\int_0^T \big\langle z_t^2 Az, Az_t\big\rangle\right| \le C\int_0^T \|z_t\|^2_{H^2(\Omega)}\left(\big\|Az\big\|^2_{L^2(\Omega)} + \big\|Az_t\big\|^2_{L^2(\Omega)}\right) \le C\int_0^T X^2(t). \tag{4.46}$$

(b) By Hölder's inequality, choose $p = 3, q = 3/2$, we get

$$\|a\cdot b\|^2_{L^2(\Omega)} \le \|a\|^2_{L^6(\Omega)}\cdot\|b\|^2_{L^3(\Omega)}. \tag{4.47}$$

In bounded domains of $\mathbb{R}^d$ for $d = 2, 3$, we have $H^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^4(\Omega) \hookrightarrow L^3(\Omega)$, i.e.,

$$\|a\|_{L^6(\Omega)} \le C\|a\|_{H^1(\Omega)}, \qquad \|a\|_{L^4(\Omega)} \le C\|a\|_{H^1(\Omega)}, \qquad \|a\|_{L^3(\Omega)} \le C\|a\|_{H^1(\Omega)}.$$

Hence,

$$\begin{aligned}
\left|\int_0^T \langle z z_{tt} A z, A z_t\rangle \,\mathrm{d}t\right| &\leq C\int_0^T \|z\|_{H^2(\Omega)} \left(\|z_{tt}\|^2_{H^1(\Omega)}\cdot\|Az\|^2_{H^1(\Omega)} + \left\|Az_t\right\|^2_{L^2(\Omega)}\right)\mathrm{d}t \\
&\leq C\int_0^T \left(\|Az\|_{L^2(\Omega)}\|A^{1/2}z_{tt}\|^2_{L^2(\Omega)}\|A^{3/2}z\|^2_{L^2(\Omega)} + \|Az\|_{L^2(\Omega)}\left\|Az_t\right\|^2_{L^2(\Omega)}\right)\mathrm{d}t \\
&\leq C\int_0^T \|Az\|^3_{L^2(\Omega)}\mathrm{d}t + C\int_0^T \|A^{1/2}z_{tt}\|^6_{L^2(\Omega)}\mathrm{d}t + C\int_0^T \|A^{3/2}z\|^6_{L^2(\Omega)}\mathrm{d}t \\
&+ \epsilon\int_0^T \|Az\|^2_{L^2(\Omega)}\mathrm{d}t + C_\epsilon\int_0^T \left\|Az_t\right\|^4_{L^2(\Omega)}\mathrm{d}t \qquad (4.48)\\
&\leq C\int_0^T \|Az\|^3_{L^2(\Omega)}\mathrm{d}t + C\int_0^T \|A^{1/2}z_{tt}\|^6_{L^2(\Omega)}\mathrm{d}t + C\Big[X(t) + X^3(t)\Big]^3\mathrm{d}t \\
&+ \epsilon\int_0^T \|Az\|^2_{L^2(\Omega)}\mathrm{d}t + C_\epsilon\int_0^T \left\|Az_t\right\|^4_{L^2(\Omega)}\mathrm{d}t \\
&\leq C\int_0^T X^{3/2}(t)\mathrm{d}t + C\int_0^T X^3(t)\mathrm{d}t + \epsilon\int_0^T X(t)\mathrm{d}t + C_\epsilon\int_0^T X^2(t)\mathrm{d}t.
\end{aligned}$$

Here, we used (4.12) and assumption in Equation (4.14) implying $X^k(t) \leq X(t)$ for $k \geq 1$.

(c) By Hölder's inequality,

$$\begin{aligned}
\left|\int_0^T \langle z z_t A z_t, A z_t\rangle \,\mathrm{d}t\right| &\leq C\int_0^T \|z\|_{L^\infty(\Omega)}\|z_t\|_{L^\infty(\Omega)}\left\|Az_t\right\|^2_{L^2(\Omega)}\mathrm{d}t \\
&\leq \epsilon\int_0^T X(t)\mathrm{d}t + C_\epsilon\int_0^T X^3(t)\mathrm{d}t.
\end{aligned} \qquad (4.49)$$

(d) Since

$$\partial_t\left\langle z^2 A z_t, A z_t\right\rangle = \langle 2 z z_t A z_t, A z_t\rangle + \left\langle 2 z^2 A z_t, A z_{tt}\right\rangle,$$

we have

$$\begin{aligned}
\left|\int_0^T \left\langle z^2 Az_{tt}, Az_t\right\rangle \mathrm{d}t\right| &= \left|\tfrac{1}{2}\int_0^T \partial_t \left\langle z^2 Az_t, Az_t\right\rangle \mathrm{d}t - \int_0^T \left\langle zz_t Az_t, Az_t\right\rangle \mathrm{d}t\right| \\
&\leq C\left|\left\langle z^2 Az_t, Az_t\right\rangle (T)\right| + C\left|\left\langle z^2 Az_t, Az_t\right\rangle (0)\right| + C\left|\int_0^T \left\langle zz_t Az_t, Az_t\right\rangle \mathrm{d}t\right| \\
&\leq C\|z(T)\|^2_{H^2(\Omega)}\left\|Az_t(T)\right\|^2_{L^2(\Omega)} + C\|z(0)\|^2_{H^2(\Omega)}\left\|Az_t(0)\right\|^2_{L^2(\Omega)} \\
&+ C\int_0^T \|z\|_{L^\infty(\Omega)}\|z_t\|_{L^\infty(\Omega)}\left\|Az_t\right\|^2_{L^2(\Omega)}\mathrm{d}t \qquad (4.50)\\
&\leq CX^2(0) + CX^2(T) + C\int_0^T \|Az\|_2\left\|Az_t\right\|^3_{L^2(\Omega)}\mathrm{d}t \\
&\leq CX(0) + CX^2(T) + \epsilon\int_0^T X(t)\mathrm{d}t + C_\epsilon\int_0^T X^3(t)\mathrm{d}t.
\end{aligned}$$

Here, $X^2(0) \leq X(0)$ by Equation (4.12).

(e) Again, using Equation (4.47), we get

$$\begin{aligned}
\left|\int_0^T \left\langle z_{tt}|\nabla z|^2, Az_t\right\rangle \mathrm{d}t\right| &\leq C_\epsilon \int_0^T \|z_{tt}\|^2_{L^6(\Omega)}\||\nabla z|^2\|^2_{L^3(\Omega)} + \epsilon\int_0^T \left\|Az_t\right\|^2_{L^2(\Omega)}\mathrm{d}t \\
&\leq C_\epsilon\int_0^T X^2(t)\mathrm{d}t + C_\epsilon\int_0^T X^4(t)\mathrm{d}t + \epsilon\int_0^T X(t)\mathrm{d}t.
\end{aligned} \tag{4.51}$$

(f) Similarly,

$$\left|\int_0^T \langle z_t(\nabla z, \nabla z_t), Az_t\rangle \,\mathrm{d}t\right| \leq C\int_0^T X^{3/2}(t)\mathrm{d}t + C\int_0^T X^{3/2}(t)\mathrm{d}t + C\int_0^T X^3(t)\mathrm{d}t. \tag{4.52}$$

Now, collecting Equations (4.46)–(4.52) and recalling (4.45), we get

$$\begin{aligned}
\left|\int_0^T \langle \partial_t G_1(z), Az_t\rangle \,\mathrm{d}t\right| \leq &CX(0) + CX^2(T) + \epsilon\int_0^T X(t)\mathrm{d}t \\
&+ C_\epsilon\int_0^T \left[X^{3/2}(t) + X^2(t) + X^3(t) + X^4(t)\right] \mathrm{d}t.
\end{aligned} \tag{4.53}$$

*Step 4.2.3: The 6th (last) term on the r.h.s. of* (4.41). The estimate is produced in a similar fashion to Equation (4.52):

$$\begin{aligned}\left|\int_0^T \langle G_2(z), Az_{tt}\rangle \,\mathrm{d}t\right| &\le C\left|\int_0^T \|z\|_{H^2(\Omega)}\||\nabla z|\|_{H^2(\Omega)}\left\langle Az_t, A^{1/2}z_{tt}\right\rangle \mathrm{d}t\right| \\ &\le C\int_0^T X^{3/2}(t)\mathrm{d}t + C\int_0^T X^{3/2}(t)\mathrm{d}t + C\int_0^T X^3(t)\mathrm{d}t.\end{aligned} \tag{4.54}$$

*Step 4.3: The 5th and 6th term on the r.h.s. of* (4.35). These are lower-order terms compared to those from Step 4.2. Hence, we skip the details and just state the final results:

$$\left|\int_0^T \langle G, \widetilde{z}_t\rangle \,\mathrm{d}t\right| \le \epsilon\int_0^T X(t)\mathrm{d}t + C_\epsilon\int_0^T \left[X^{3/2}(t) + X^3(t) + X^4(t) + X^6(t)\right]\mathrm{d}t, \tag{4.55}$$

$$\left|\int_0^T \langle G, A\widetilde{z}\rangle \,\mathrm{d}t\right| \le \epsilon\int_0^T X(t)\mathrm{d}t + C_\epsilon\int_0^T \left[X^{3/2}(t) + X^3(t) + X^4(t) + X^6(t)\right]\mathrm{d}t. \tag{4.56}$$

*Step 4.4: The 8th (last) term on the r.h.s. of* (4.35). By an argument similar to Equation (4.19), we get

$$\int_0^T \left\|A\theta_t\right\|^2_{L^2(\Omega)}\mathrm{d}t \le E_3(0) + \int_0^T \langle G, z_{tt}\rangle \,\mathrm{d}t. \tag{4.57}$$

Therefore,

$$\begin{aligned}&\left|\int_0^T \left\langle BG, A\widetilde{\theta}\right\rangle \mathrm{d}t\right| \le C\int_0^T \left\|G\right\|^2_{L^2(\Omega)} + CE_3(0) + C\int_0^T \langle G, z_{tt}\rangle \,\mathrm{d}t \\ &\le \ \epsilon\int_0^T X(t)\mathrm{d}t + CX(0) + C_\epsilon\int_0^T \left[X^{3/2}(t) + X^2(t) + X^3(t) + X^4(t) + X^6(t)\right]\mathrm{d}t.\end{aligned} \tag{4.58}$$

*Step 5:* Plugging Equations (4.37)–(4.40), (4.54)–(4.56) and (4.58) into (4.35), we finally estimate

$$\begin{aligned}(1-\epsilon)X(T) + (C_1 - 8\epsilon)\int_0^T X(t)\mathrm{d}t \\ \leq C_\epsilon X(0) + C_\epsilon \int_0^T \left[X^{3/2}(t) + X^2(t) + X^3(t) + X^4(t) + X^6(t)\right]\mathrm{d}t \\ + C_\epsilon \left[X^2(T) + X^4(T) + X^6(T) + X^8(T)\right],\end{aligned}$$

that is,

$$\begin{aligned}X(T) + \int_0^T X(t)\mathrm{d}t \\ \leq C_1 X(0) + C_2\left[X^2(T) + X^4(T) + X^6(T) + X^8(T)\right]\mathrm{d}t \\ + C_3 \int_0^T \left[X^{3/2}(t) + X^2(t) + X^3(t) + X^4(t) + X^6(t)\right],\end{aligned} \tag{4.59}$$

which finishes the proof. □

**Remark 4.2.** *With Equation* (4.59) *at hand, we can now apply the standard 'barrier method' (cf. [7, Lemma 5.1, p 485]) to deduce the globality of the local solution, whose existence is guaranteed by Theorem 3.4 (or Theorem 1.4)* – not *in the energy space (endowed with* $\|\cdot\|_X$*), but in the phase space (endowed with* $\|\cdot\|_{\mathcal{Z}_s\times\mathcal{T}_s}$*) instead. Apparently,* $\max_{0\leq t\leq T}\|\cdot\|_X \leq \max_{0\leq t\leq T}\|\cdot\|_{\mathcal{Z}_3\times\mathcal{T}_3}$ *for any* $0 < T < T_{\max}$*. In Lemma 4.3, we will show a 'reverse' inequality, which is sufficient for a contradiction proof (see the proof of Theorem 4.5). After the exponential stability of the energy is established, a second lemma, i.e., Lemma 4.7, will be presented to show the equivalence over the whole time half-line* $[0,\infty)$*. In the spirit of Remark 4.1, we have:*

**Lemma 4.3** (Controlling $\max_{0\leq t\leq T}\|\cdot\|_{\mathcal{Z}_3\times\mathcal{T}_3}$ in terms of $\max_{0\leq t\leq T}\|\cdot\|_X$)**.** *Assume a classical solution* $(z,\theta)$ *to Equations (4.1a)–(4.1d) over a time interval* $[0, T_{\max})$

*satisfies the smallness condition $E_2(t) < \epsilon_1$ from Equation* (4.7)*, then, there holds for any $T \in (0, T_{\max})$:*

$$\max_{0\le t\le T} \|(z,\theta)\|^2_{\mathcal{Z}_3\times\mathcal{T}_3} \le C \max_{0\le t\le T} \big(\|(z,\theta)\|^2_X + \|(z,\theta)\|^6_X\big) \text{ for some } C = C(T) > 0.$$

*Proof.* It suffices to consider the four highest energy terms: $\|A^{3/2}z\|_{L^2(\Omega)}$, $\|z_{ttt}\|_{L^2(\Omega)}$, $\|A^2\theta\|_{L^2(\Omega)}$ and $\|A^{1/2}\theta_{tt}\|_{L^2(\Omega)}$. The first one, as shown in Lemma 4.1, is bounded by $C\big(\|(z,\theta)\|_X + \|(z,\theta)\|^3_X\big)$ for any $t \ge 0$. Using Equations (4.1a)–(4.1b), the second and the third term can be controlled by appropriate lower order terms. Indeed, applying $\partial_t$ and $A$ to Equations (4.1a) and (4.1b), respectively, and exploiting the bounded invertibility of $(A^{-1} + \gamma)$, we estimate

$$\|z_{ttt}\|_{L^2(\Omega)} \le C\big(\|Az_t\|_{L^2(\Omega)} + \|A\theta_t\|_{L^2(\Omega)} + \|F'\|_{L^2(\Omega)}\big) \le C\big(\|(z,\theta)\|_X + \|(z,\theta)\|^3_X\big), \tag{4.60}$$

$$\|A^2\theta\|_{L^2(\Omega)} \le C\big(\|A\theta_t\|_{L^2(\Omega)} + \|A\theta\|_{L^2(\Omega)} + \|Az_t\|_{L^2(\Omega)}\big) \le C\|(z,\theta)\|_X \tag{4.61}$$

for any $t \ge 0$, which remains true after passing to supremum. The last term is treated in the same fashion as in the proof of Theorem **A**.12. Estimating

$$\|z_{ttt}\|^2_{L^2(0,T;L^2(\Omega))} \le \int_0^T \big(X(t) + X^3(t)\big)\mathrm{d}t \le T\Big(1 + \max_{0\le t\le T} X^2(t)\Big) \max_{0\le t\le T} X(t) \tag{4.62}$$

via (4.60) and exploiting the maximal $L^2$-regularity of $A$ on $(0,T)$ applied to Equation (4.1b) differentiated twice in time, the desired estimate follows. □

After achieving the super-linear energy inequality (4.13) (equivalently, (4.59)) in Lemma 4.2, we are now in the position to prove the global well-posedness. Let's start with the following technical lemma.

**Lemma 4.4.** *With the same constants* $C_1, C_2, C_3, \alpha_i$*, and* $\beta_j$ *as in* (4.13) *of Lemma 4.2, all of which are independent of* $t$*, we define*

$$k(x) = x - C_2 \sum_{i \in I} x^{\alpha_i}$$

*and*

$$h(x) = x - C_3 \sum_{j \in J} x^{\beta_j}.$$

*Furthermore, given a constant* $T > 0$*, assume the following inequality holds for any* $\widetilde{T} \in [0, T]$*,*

$$k(x(\widetilde{T})) + \int_0^{\widetilde{T}} h(x(t))\ dt \leq C_1 x(0), \tag{4.63}$$

*where* $x(t)$ *is a continuous function of* $t \in [0, T]$*. Then there exists a small number* $\epsilon > 0$ *such that, if* $0 < x(0) < \frac{\epsilon}{C_1}$*, then*

$$h(x(t)) > 0 \text{ for any } t \in [0, T]$$

.

*Proof. Step 1:* To make the presentation easier, let's start by some notations and several related observations. We will make five smallness assumptions of $\epsilon$ in this step.

It is easy to see that the graphs of $k(x)$ and $h(x)$ look like the ones in Figure 4.1 and Figure 4.2.

In particular, we observe from the graphs that, since $\alpha_i > 1$ for any $i \in I$ in the definition of $k(x)$ (recall (4.13)), the (algebraic) function $k(x)$ has a unique positive

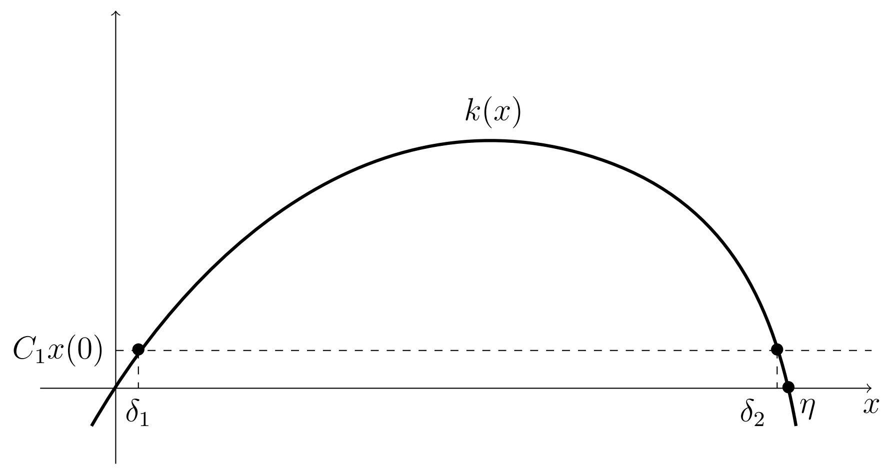


Figure 4.1: Graph of $k(x)$

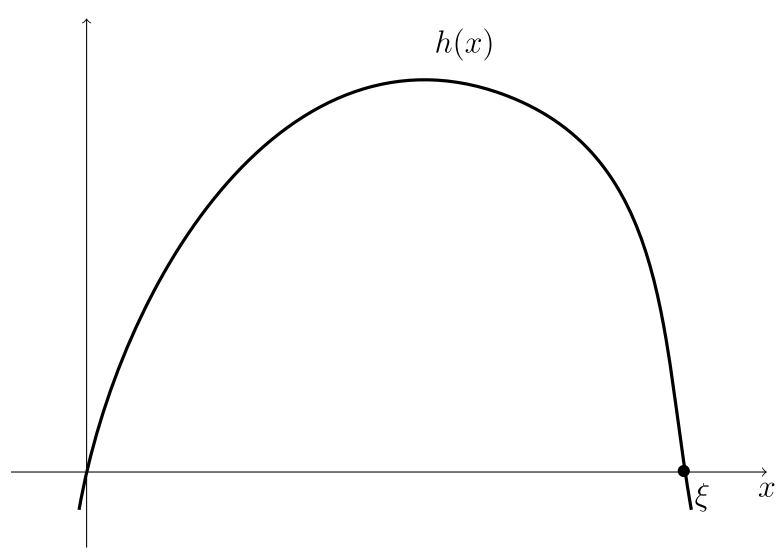


Figure 4.2: Graph of $h(x)$

solution denoted by $\eta$, and there also holds $k(x) > 0$ for $x \in (0, \eta)$. In addition, $\max_x\{k(x)\}$ is a finite positive number.

Similarly, $h(x)$ has a unique positive solution denoted by $\xi$. Besides, $h(x) > 0$ for $x \in (0, \xi)$.

*Step 2:*

$$\textbf{\textit{Smallness Assumption of }} \epsilon \textbf{\textit{ \#1:}} \quad \epsilon < \max_x\{k(x)\}, \tag{4.64}$$

which induces $C_1 x(0) < \max_x\{k(x)\}$. Under this assumption, there are two different positive solutions of the equation $k(x) = C_1 x(0)$, which we call $\delta_1$ and $\delta 2$, respectively. In addition, $k(x) < C_1 x(0) (< \epsilon)$ implies $x \in [0, \delta_1) \cup (\delta_2, \infty)$.

It is also clear that, as $\epsilon$ goes to 0, $\delta_1$ goes to 0 and $\delta_2$ goes to $\eta$, which makes the following assumptions valid.

$$\textbf{\textit{Smallness Assumption of }} \epsilon \textbf{\textit{ \#2:}} \quad \epsilon < C_1 \delta_2, \tag{4.65}$$

which induces $x(0) < \dfrac{\epsilon}{C_1} < \delta_2$.

$$\textbf{\textit{Smallness Assumption of }} \epsilon \textbf{\textit{ \#3:}} \ \epsilon \text{ is small enough to make } \delta_1 < \xi \tag{4.66}$$

$$\textbf{\textit{Smallness Assumption of }} \epsilon \textbf{\textit{ \#4:}} \ \epsilon \text{ is small enough to make } \delta_1 < \epsilon_1, \tag{4.67}$$

where $\epsilon_1$ is the condition in Lemma 4.1 in order for the $z$-energy boost hold.

*Step 3:* Now we want to prove, by contradiction, the claim that $h(x(t)) > 0$ for any $t \in [0, T]$. If not, by the continuity of $h \circ x$, let $T^*$ be the smallest number in the interval $(0, T]$ such that $h(x(T^*)) = 0$. Hence,

$$x(T^*) = \xi. \tag{4.68}$$

Here we have the last smallness assumption:

$$\textbf{\textit{Smallness Assumption of}}\ \epsilon\ \textbf{\textit{\#5:}} \quad \epsilon < C_1\xi, \tag{4.69}$$

which means $x(0) < \dfrac{\epsilon}{C_1} < \xi$. Therefore, $h(x(0)) > 0$. Since $T^*$ the smallest positive number to make $h(x(T^*)) = 0$, it suggests that $h\big[X(t)\big] \geq 0$ for any $t \in [0, T^*]$.

*Step 4:* The above inequality, together with (4.63), suggests that $k(x(t)) \leq C_1 x(0) < \epsilon$ for any $t \in [0, T^*]$. Then, by the Smallness Assumption #1, $x(t) \in [0, \delta_1] \cup [\delta_2, \infty)$ for any $t \in [0, T^*]$. Furthermore, by the Smallness Assumption #2, we can eliminate the second interval, otherwise $h(x(t))$ will be greater than $C_1 x(0)$ for some $t \in (0, T^*)$.

Hence,

$$x(t) \in [0, \delta_1] \text{ for any } t \in [0, T^*].$$

More specifically, $x(T^*) \leq \delta_1 < \xi$ by (4.66) in Smallness Assumption #3 , which contradicts (4.68). Therefore, the original claim is true that $h(x(t)) > 0$ for any $t \in [0, T]$.

□

**Theorem 4.5** (Global Well-posedness)**.** *Let Assumption 3.3 be satisfied for some* $s \geq 3$. *Then, there exists a positive number* $\epsilon$ *such that for any initial data satisfying* $X(0) < \epsilon$ *(which roughly means the smallness of* $\|z^0\|^2_{H^3(\Omega)} + \|z^1\|^2_{H^2(\Omega)} + \|\theta^0\|^2_{H^4(\Omega)}$*), the associated unique local solution of system* (4.1a)–(4.1d) *from Theorem 3.4 exists globally, namely,* $T_{\max} = \infty$.

*Proof.* We choose $\epsilon$ so that Smallness Assumptions #1–5 in Lemma 4.4 are all satisfied. Moreover, without loss of generality, assume $0 < X(0) < 1$. Indeed, if $X(0) = 0$, the only solution to (4.1a)–(4.1d) is the trivial one and, therefore, exists globally.

In order to show that $T_{\max} = \infty$, it suffices to show that the energy $X(t)$ is bounded by a constant independent of time $t$. Indeed, the contraposition is part of what have been shown for the local solution in (3.8) of Theorem 3.4.

Notice that by Smallness Assumption #4, the condition in Lemma 4.1 is satisfied at least over a non-trivial time interval $[0, T)$. We will show below that $\delta_1$ is indeed the uniform bound mentioned above. A short argument of contradiction will deduce that Lemma 4.1 truly holds for any positive time $t$. The argument is very similar as the one in Step 3 of Lemma 4.4 and is, therefore, omitted here.

For the reason stated above, the superlinear energy inequality (4.59) holds and satisfies all conditions in Lemma 4.4. Therefore,

$$k(X(T)) + \int_0^T h(X(t))\, dt \leq C_1 X(0), \tag{4.70}$$

and

$$h(X(t)) > 0 \tag{4.71}$$

for any $T \in [0, T_{\max}]$. Exploiting Equations (4.71) and (4.70), we get $k\big[X(t)\big] \leq C_1 X(0)$ for any $t \geq 0$, which leads to the global bound on $X(t)$

$$X(t) \leq \delta_1 \text{ for any } t \geq 0 \tag{4.72}$$

by a similar argument as above. Hence, after a possible rescaling, Assumption 3.3 is satisfied by $(z, z_t, \theta)(T_{\max}, \cdot)$. Therefore, Theorem 3.4 implies the solution exists on

$[T_{\max}, T')$ for some $T' > T_{\max}$, which contradicts the maximality of $[0, T_{\max})$. □

**Corollary 4.6** (Exponential Stability)**.** *Under the assumptions of Theorem 4.5 with $X(0) < \tilde{\epsilon}$ for some positive number $\tilde{\epsilon}$ (possibly smaller than the $\epsilon$ from Theorem 4.5), there exist positive constants $C$ and $k$ such that*

$$X(t) \le e^{-kt} C X(0) \text{ for } t \ge 0. \tag{4.73}$$

*Proof.* Let $k(x)$ and $h(x)$ be defined as in Lemma 4.4. Define $\hat{k}(x) = \dfrac{k(x)}{x}$ and $\hat{h}(x) = \dfrac{h(x)}{x}$. Notice that, since $\alpha_i, \beta_j > 1$, both $\hat{k}(x)$ and $\hat{h}(x)$ contain terms of non-negative power of $x$, only.

We can then rewrite Equation (4.13) again as follows:

$$X(T) \cdot \hat{k}(X(T)) + \int_0^T X(t) \cdot \hat{h}(X(t))\, dt \le C_1 X(0). \tag{4.74}$$

Choosing a bound $\epsilon_3$ on $X(0)$ small enough, we can make the global bound $\delta_1$ of $X(t)$ satisfy

$$\hat{k}(\delta_1) \ge \tfrac{1}{2} \quad \text{and} \quad \hat{h}(\delta_1) \ge \tfrac{1}{2}.$$

These lower bounds hold for any $\hat{k}(x(t))$ and $\hat{h}(x(t))$ as these two function are both decreasing with respect to $x$. Together with Equation (4.74), it implies

$$X(T) + \int_0^T X(t) \mathrm{d}t \le 2C_1 X(0), \tag{4.75}$$

which gives $X(T) \le 2C_1 X(0)$ for any $T > 0$. Now we impose the final assumption on $X(0)$. Recall the number $\epsilon$ from The Smallness Assumptions #1–5, and let

$$\tilde{\epsilon} = \min\left\{\epsilon, \tfrac{\epsilon}{2C_1}, \epsilon_3\right\}. \tag{4.76}$$

Since $X(0) < \tilde{\epsilon} \leq \frac{\epsilon}{2C_1}$, then $X(t) \leq \epsilon$ for any $t > 0$. Thus, Equation (4.75) can be extended to

$$X(T) + \int_s^T X(t)\mathrm{d}t \leq 2C_1 X(s) \tag{4.77}$$

for any $s \in (0, T]$. Hence,

$$X(t) \geq \tfrac{1}{2C_1} X(T) \text{ for } t \in [0, T]. \tag{4.78}$$

Combining Equation (4.78) with (4.75), we get $X(T) + \frac{T}{2C_1} X(T) \leq 2C_1 X(0)$. Therefore,

$$X(T) \leq \frac{1}{1 + \frac{T}{2C_1}} X(0) \text{ for any } T > 0. \tag{4.79}$$

By choosing $T$ large enough, we get

$$X(T) \leq \kappa X(0) \text{ for some } \kappa < 1. \tag{4.80}$$

Repeating the procedure on $[T, 2T]$, $[2T, 3T]$, etc., we arrive at

$$X(t) \leq \kappa^{\lceil t/T \rceil} X(0) \leq \kappa^{t/T} X(0) \leq e^{-\left(|\ln(\kappa)|/T\right)t} X(0) \text{ for } t \geq 0,$$

which finishes the proof. □

We have now proved all main results stated in Section 1.2 in the energy space associated with $\sup_{0 \leq t < \infty} X(t)$ from Equation (4.6). As announced in Remark 4.1 and stated in Lemma 4.7, the Banach space generated by the energy (supremum) $\sup_{0 \leq t < \infty} X(t)$ is isomorphic to the solution space in Equation (4.2) from our Theorem 3.4 for $s = 3$ when the initial data are sufficiently small.

**Lemma 4.7** (Equivalence of $\sup_{0\le t<\infty}\|\cdot\|_{\mathcal{Z}_3\times\mathcal{T}_3}$ and $\sup_{0\le t<\infty}\|\cdot\|_X$)**.** *If a classical solution $(z,\theta)$ to Equations (4.1a)–(4.1d) is global, satisfies the smallness condition $E_2(t)<\epsilon_1$ from Equation* (4.7) *and decays exponentially as in Equation (4.73), both norms mentioned above are equivalent:*

$$c_1 \sup_{0\le t<\infty}\|(z,\theta)\|_X \le \|(z,\theta)\|_{\mathcal{Z}_3\times\mathcal{T}_3} \le c_2 \sup_{0\le t<\infty}\|(z,\theta)\|_X \quad \text{for some } c_1 \text{ and } c_2>0. \tag{4.81}$$

*Proof.* The former inequality is trivial. For the latter one, in contrast to Lemma 4.3, all superlinear terms are linearly dominated because they are bounded by 1, and we are only left to show Equation (4.62) with a constant independent of $T$. However, due to the exponential decay of $X(t)$ (Equation (4.73)), (4.62) becomes

$$\|z_{ttt}\|^2_{L^2(0,\infty;L^2(\Omega))} \le \tilde{C}\int_0^\infty X(t)\mathrm{d}t \le \tilde{C}C\int_0^\infty e^{-kt}X(0)\mathrm{d}t \le c_2X(0) \le c_2 \sup_{0\le t<\infty} X(t). \tag{4.82}$$

With $A$'s maximal $L^2$-regularity on $(0,\infty)$, the estimate for $\sup_{0\le t<\infty}\left\|A^{1/2}\theta_{tt}\right\|_{L^2(\Omega)}$ follows. □

Since the Smallness Assumption #1–5 are satisfied in both Theorem 4.5 and Equation (4.76) of Corollary 4.6, we resubstitute $w=A^{-1}z$ and conclude with the desired results Theorem 1.5 and 1.6.

# Appendix: Existence Theory for Linear Evolution Equations

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a $C^s$-boundary $\partial\Omega$ for some $s \geq \lfloor \frac{d}{2} \rfloor + 2$ and let $T > 0$ be arbitrary, but fixed. The following well-posedness results are based on Kato's solution theory [10] for abstract time-dependent evolution equations and its improved version presented by Jiang and Racke in [9, Appendix A] as well as maximal $L^p$-regularity theory (see, e.g., [12]).

Thoughout this appendix and in the proof of Theorem 3.4, we employ the following notation. For $n \geq 0$, we define

$$\bar{D}^n := \big((\partial_t, \nabla)^\alpha \,|\, 0 \leq |\alpha| \leq n\big) \text{ and } H^0_0(\Omega) \equiv H^0(\Omega) := L^2(\Omega).$$

Let $\phi_\delta \colon \mathbb{R} \to [0, \infty)$ denote the one-dimensional Friedrichs' mollifier with a 'bandwidth' $\delta > 0$. For an $L^1$-function $z \colon [0, T] \times \Omega \to \mathbb{R}$, we let

$$z_\delta(t, \cdot) = \int_0^T \phi_\delta(t - s) z(s, \cdot) \mathrm{d}s \text{ for } t \in [0, T] \text{ in } \Omega.$$

For details on approximation properties of mollifiers, we refer to [23, Chapters 8 and

9]. The following result is known from [9, Lemma A.12].

**Lemma A.8.** *Let* $a \in C^1\big([0,T], L^\infty(\Omega)\big)$, $v \in C^0\big([0,T], L^2(\Omega)\big)$,

*and* $w \in L^2\big(0,T; H^{-1}(\Omega)\big)$. *Then, for any sufficiently small* $\varepsilon > 0$, *there holds*

$$\int_\varepsilon^{T-\varepsilon} \big\|\partial_t\big((av)_\delta(t,\cdot) - av_\delta(t,\cdot)\big)\big\|^2_{L^2(\Omega)} \mathrm{d}t \to 0 \text{ and}$$
$$\int_\varepsilon^{T-\varepsilon} \|w_\delta(t,\cdot)\|^2_{H^{-1}(\Omega)} \mathrm{d}t \to \int_\varepsilon^{T-\varepsilon} \|w(t,\cdot)\|^2_{H^{-1}(\Omega)} \mathrm{d}t \text{ as } \delta \to 0.$$

## A.1 Linear Wave Equation

We consider a general linear wave equation with time- and space-dependent coefficients:

$$z_{tt}(t,x) - \bar{a}_{ij}(t,x)\partial_{x_i}\partial_{x_j} z(t,x) = \bar{f}(t,x) \quad \text{for } (t,x) \in (0,T)\times\Omega, \tag{A.1a}$$

$$z(t,x) = 0 \quad \text{for } (t,x) \in [0,T]\times\partial\Omega, \tag{A.1b}$$

$$z(0,x) = z^0(x), \quad z_t(0,x) = z^1(x) \quad \text{for } x \in \Omega. \tag{A.1c}$$

**Assumption A.9.** *Let* $s \geq \lfloor \frac{d}{2} \rfloor + 2$ *be a fixed integer and let* $\gamma_0, \gamma_1$ *be positive numbers. Assume the following conditions are satisfied.*

1. Coefficient symmetry*:* $\bar{a}_{ij}(t,x) = \bar{a}_{ji}(t,x)$ *for* $(t,x) \in [0,T]\times\bar{\Omega}$.

2. Coefficient regularity*:* $\bar{a}_{ij} \in C^0\big([0,T]\times\bar{\Omega}\big)$ *and*

$$\partial_{x_k}\bar{a}_{ij} \in L^\infty\big(0,T; H^{s-1}(\Omega)\big), \quad \partial_t^m \bar{a}_{ij} \in L^\infty\big(0,T; H^{s-1-m}(\Omega)\big)$$

*for* $m = 1, 2, \ldots, s-1$.

*3.* Coercivity*: For $z \in H_0^1(\Omega)$ and $t \in [0,T]$,*

$$\|z\|_{H^1(\Omega)}^2 \le \gamma_0 \Big( \langle \bar{a}_{ij} \partial_{x_i} z, \partial_{x_j} z \rangle_{L^2(\Omega)} + \|z\|_{L^2(\Omega)}^2 \Big).$$

*4.* Elliptic regularity*: For $m = 0, 1, \ldots, s-2$, $z(t,\cdot) \in H_0^1(\Omega)$ and $\bar{a}_{ij}(t,\cdot)\partial_{x_i}\partial_{x_j} z(t,\cdot) \in H^m(\Omega)$ for a.e. $t \in [0,T]$ implies $u(t,\cdot) \in H^{m+2}(\Omega)$ and for a.e. $t \in [0,T]$,*

$$\|z(t,\cdot)\|_{H^m(\Omega)} \le \gamma_1 \Big( \|\bar{a}_{ij}(t,\cdot)\partial_{x_i}\partial_{x_j} z(t,\cdot)\|_{H^m(\Omega)} + \|z(t,\cdot)\|_{L^2(\Omega)} \Big).$$

*5.* Right-hand side regularity*: For $m = 0, 1, \ldots, s-2$,*

$$\partial_t^m \bar{f} \in C^0\big([0,T], H^{s-2-m}(\Omega)\big), \quad \partial_t^{s-1} \bar{f} \in L^2\big(0,T; L^2(\Omega)\big).$$

*6.* Compatibility conditions*: For $m = 0, 1, \ldots, s-1$,*

$$\bar{z}^m \in H^{s-m}(\Omega) \cap H_0^1(\Omega), \quad \bar{z}^s \in L^2(\Omega),$$

*where $\bar{z}^m$ is recursively defined by*

$$\bar{z}^0(x) = z^0(x), \quad \bar{z}^1(x) = z^1(x),$$
$$\bar{z}^m(x) = \Big( \sum_{n=0}^{m-2} \binom{m-2}{n} \partial_t^n \bar{a}_{ij} \partial_{x_i}\partial_{x_j} \bar{z}^{m-2-n} + \partial_t^{m-2} \bar{f}_i \Big)(0,x) \text{ for } m \ge 2$$

*for $x \in \Omega$.*

Note that Assumption **A**.9.2 differs from [9, Assumption A.2.1.1]. This extra regularity for $\bar{a}_{ij}$ will enable us to prove our *a priori* estimate at an energy level which is one order lower than in [9, Theorem A.13].

**Theorem A.10.** *Under Assumption* **A**.*9, the initial boundary value problem (*A.*1a)-(*A.*1c) possesses a unique classical solution, which satisfies*

$$z \in \bigcap_{m=0}^{s-1} C^m\big([0,T], H^{s-m}(\Omega) \cap H_0^1(\Omega)\big) \cap C^s\big([0,T], L^2(\Omega)\big).$$

*Moreover, for* $d \in \{2,3\}$*, letting*

$$\phi_0 = \|\bar{a}_{ij}(0,\cdot)\|_{L^\infty(\Omega)} + \|\partial_{x_k}\bar{a}_{ij}(0,\cdot)\|_{H^{s-1}(\Omega)},$$
$$\phi = \sup_{0\le t\le T}\Big(\|\bar{a}_{ij}(t,\cdot)\|_{L^\infty(\Omega)} + \|\partial_{x_k}\bar{a}_{ij}(t,\cdot)\|_{H^{s-1}(\Omega)} + \sum_{m=1}^{s-1}\|\partial_t^m\bar{a}_{ij}(t,\cdot)\|_{H^{s-1-m}(\Omega)}\Big),$$

*there exists a positive number* $K_1$*, which is a continuous function of* $\phi_0$*,* $\gamma_0$ *and* $\gamma_1$*, and a positive number* $K_2$*, which continuously depends on* $\phi$*,* $\gamma_0$ *and* $\gamma_1$*, such that*

$$\sup_{0\le t\le T}\|\bar{D}^s z(t,\cdot)\|^2_{L^2(\Omega)} \le K_1\Lambda_0 \exp\big(K_2 T^{1/2}(1+T^{1/2}+T+T^{3/2})\big),$$

*where*

$$\Lambda_0 := \sum_{m=0}^{s}\|\bar{z}\|^2_{H^{s-m}(\Omega)} + (1+T)\sup_{0\le t\le T}\big\|\bar{D}^{s-2}\bar{f}(t,\cdot)\big\|_{L^2(\Omega)} + T^{1/2}\int_0^T\|\partial_t^{s-1}\bar{f}(t,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}t.$$

*Proof.* Our proof is based on an abstract well-posedness and regularity result [9, Theorems A.3 and A.9].

*Existence and uniqueness at basic regularity level.* Similar to the proof of [9, Theorem A.11], we define for $t \in [0,T]$ a bounded linear operator

$$A(t) := \begin{pmatrix} 0 & -1 \\ -\bar{a}_{ij}(t,\cdot)\partial_{x_i}\partial_{x_j} & 0 \end{pmatrix} : Y_1 \longrightarrow X_0, \tag{A.2}$$

where the Hilbert space $X_0 := H_0^1(\Omega) \times L^2(\Omega)$ is equipped with the standard inner product induced by the product topology, whereas the inner product on the Hilbert space $Y_1 := H^2(\Omega) \cap H_0^1(\Omega)$ reads as

$$\langle V, \bar{V} \rangle_t := \big\langle \bar{a}_{ij}(t, \cdot) \partial_{x_i} z, \partial_{x_j} \bar{z} \big\rangle_{L^2(\Omega)} + \langle y, \bar{y} \rangle_{L^2(\Omega)} \tag{A.3}$$

for $V = (z, y)$ and $\bar{V} = (\bar{z}, \bar{y}) \in X_0$. Due to uniform coercivity of $\bar{a}_{ij}$ and by virtue of Poincaré-Friedrichs' inequality, each of the norms induced by $\langle \cdot, \cdot \rangle_t$ for any $t \in [0, T]$ is equivalent to the standard norm on $X_0$. With this notation, letting $V := (z, \partial_t z)$, Equations (A.1a)–(A.1c) can be rewritten as an abstract Cauchy problem

$$\partial_t V(t) + A(t) V(t) = F(t) \text{ in } (0, T), \quad V(0) = V^0 \tag{A.4}$$

with $F = (0, \bar{f})$ and $V^0 = (z^0, z^1)$.

We want to show that the triple $\big(A; X_0, Y_1\big)$ is a CD-system in sense of [9, Section A.1]. For $t \in [0, T]$, consider the elliptic problem

$$\big(A(t) + \lambda\big) V = F \text{ with } F \in X_0.$$

Recalling Assumption **A**.9.3, Lemma of Lax & Milgram implies the resolvent estimate

$$\big\| \big(A(t) + \lambda\big)^{-1} \big\|_{L(X_0)} \leq \tfrac{1}{\lambda - C} \text{ for } \lambda > \beta \text{ for some constants } \beta, C > 0, \tag{A.5}$$

where we used Assumption **A**.9.2 and Sobolev's imbedding theorem to deduce

$$a_{ij}(t, \cdot) \in W^{1,\infty}(\Omega) \text{ for any } t \in [0, T].$$

The continuity of the bilinear form follows similarly. By standard elliptic regularity theory applied to $A(t)$, which is possible because of Assumption **A**.9.2 and **A**.9.4 as well as $C^s$-smoothness of $\partial\Omega$, the maximal domain of $A(t)$ coincides with $Y_1$. Hence, the operator $A(t)$ is closed. This along with Equation (A.5) implies $(\beta,\infty) \subset \rho\big(A(t)\big)$. Therefore, $\big(A(t); t \in [0,T]\big)$ is a stable family of infinitesimal negative generators of $C_0$-semigroups on $X_0$ with stability constants $1, \beta$. Taking into account regularity conditions from Assumption **A**.9.5, we can apply [9, Theorem A.3], we get a unique classical solution

$$V \in C^0\big([0,T], Y_1\big) \cap C^1\big([0,T], X_0\big)$$

at the at basic regularity level, which is equivalent to

$$z \in C^2\big([0,T], L^2(\Omega)\big) \cap C^1\big([0,T], H_0^1(\Omega)\big) \cap C^0\big([0,T], H^2(\Omega) \cap H_0^1(\Omega)\big).$$

*Higher regularity.* For the proof of higher solution regularity, we consider the following increasing double scale $(X_j, Y_j)$ of Hilbert spaces

$$X_j = \big(H^{j+1}(\Omega) \cap H_0^1(\Omega)\big) \times H^j(\Omega) \text{ for } j \geq 1,$$

$$Y^j = \big(H^{j+1}(\Omega) \cap H_0^1(\Omega)\big) \times \big(H^j(\Omega) \cap H_0^1(\Omega)\big) \text{ for } j \geq 1, \quad Y_0 = X_0.$$

By virtue of Equation (A.2), the condition

$$\partial_t A \in \mathrm{Lip}\big([0,T], L(Y_{j+r+1}, X_j)\big) \text{ for } j = 0, \ldots, s-r-1 \text{ and } r = 0, \ldots, s-2$$

is equivalent to

$$\partial_t^r \bar{a}_{ij}(t,\cdot)\partial_{x_i}\partial_{x_j} \in \mathrm{Lip}\Big([0,T], L\big(H^{j+r+2}(\Omega) \cap H_0^1(\Omega), H^j(\Omega)\big)\Big) \tag{A.6}$$

for $j = 0, \ldots, s-r-1$ and $r = 0, \ldots, s-2$, while the latter is a direct consequence of Assumption **A**.9.2 and Sobolev imbedding theorem due to the fact $H^{\lfloor d/2 \rfloor + 1}(\Omega) \hookrightarrow$

$L^\infty(\Omega)$. Similarly, exploiting Assumption **A**.9.4, one can easily verify for $j = 0, \dots, s-2$ and $\phi \in Y_1$ and a.e. $t \in [0,T]$ that $A(t)\phi \in X_j$ implies

$$\phi \in Y_{j+1} \text{ and } \|\phi\|_{Y_{j+1}} \leq K\big(\|A(t)\phi\|_{X_j} + \|\phi\|_{X_0}\big) \text{ for some constant } K > 0,$$

which does not depend on $\phi$. Further, Assumption **A**.9.5 yields

$$\partial_t F \in C^0\big([0,T], X_{s-1-k}\big) \text{ for } k = 0, \dots, s-2 \text{ and } \partial_t^{s-1} F \in L^1(0,T;X_0).$$

Finally, Assumption **A**.9.6 implies compatibility conditions in sense of [9, Equations (A.8) and (A.9)]. Hence, applying [9, Theorem A.9] at the energy level $s-1$, we obtain additional regularity for the classical solution satisfying

$$V \in \bigcap_{m=0}^{s-1} C^m\big([0,T], Y_{s-1-m}\big).$$

Rewriting $z$ in terms of $V$, this yields the desired regularity for $z$.

*Energy estimates.* For $n = 1, \dots, s-1$, applying the $\partial_t^{n-1}$-operator to Equation (A.1a), we obtain a linear wave equation for $\partial_t^{n-1} z$ reading as

$$\partial_t^2\big(\partial_t^{n-1} z\big) - \bar{a}_{ij}\partial_{x_i}\partial_{x_j}\big(\partial_t^{n-1} z\big) = h^{n-1} \text{ in } (0,\infty) \times \Omega, \tag{A.7}$$

where we used Leibniz' rule to compute

$$h^{n-1} = \partial_t^{n-1}\bar{f} + \sum_{m=1}^{n-1} \binom{n-1}{m} \big(\partial_t^m \bar{a}_{ij}\big)\partial_{x_i}\partial_{x_j}\partial_t^{n-1-m} z. \tag{A.8}$$

Multiplying Equation (A.7) in $L^2(\Omega)$ with $\partial_t^n z$, applying Green's formula and using Young's inequality, we obtain the estimate

$$\tfrac{1}{2}\partial_t\big(\|\partial_t^n z(t,\cdot)\|^2_{L^2(\Omega)} + \|\bar{a}(t,\cdot)\nabla\partial_t^{n-1} z(t,\cdot)\|^2_{L^2(\Omega)}\big)$$
$$\leq \tfrac{1}{2}\big\|\big(\partial_{x_i}\bar{a}_{ij}(t,\cdot)\big)\partial_{x_j} z(t,\cdot)\|^2_{L^2(\Omega)} + \|\partial_t^n z(t,\cdot)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|h^{n-1}(t,\cdot)\|^2_{L^2(\Omega)}.$$

Integrating w.r.t. to $t$ over $[0,T]$, exploiting Assumption **A**.9.3 and recalling the definition of $\phi_0$, we get

$$\begin{aligned}\|\partial_t^n z(t,\cdot)\|^2_{L^2(\Omega)}+\|\partial_t^{n-1} z(t,\cdot)\|^2_{H^1(\Omega)} &\leq C(\gamma_0,\phi_0)\big(\|\bar{z}^n\|^2_{L^2(\Omega)} + \|\bar{z}^{n-1}\|^2_{H^1(\Omega)}\big)\\ &+ C(\gamma_0,\phi)\int_0^t \big(\|\partial_t^n z(\tau,\cdot)\|^2_{L^2(\Omega)} + \|\partial_t^{n-1} z(\tau,\cdot)\|^2_{H^1(\Omega)}\big)\mathrm{d}\tau \\ &+ C(\gamma_0)\int_0^t \|h^{n-1}(\tau,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\tau,\end{aligned} \tag{A.9}$$

where we used Sobolev imbedding theorem to estimate

$$\max_{0\leq t\leq T}\|\partial_{x_i}\bar{a}_{ij}(t,\cdot)\|_{L^\infty(\Omega)} \leq C\max_{0\leq t\leq T}\|\partial_{x_i}\bar{a}_{ij}(t,\cdot)\|_{H^{s-1}(\Omega)} \leq \phi.$$

Here and in the sequel, $C$ denotes a positive generic constant which does not depend on the unknown function $z$.

To derive an estimate for $\partial_t^s z$ and $\nabla\partial_t^{s-1} z$, we need to employ a molifier technique similar to [9, Section A.2]. First, we select $0 < \delta < \varepsilon < T$. Convolving Equations (A.7) for $n = s-1$ with $\phi_\delta$, we obtain for $t \in [\varepsilon, T-\varepsilon]$

$$(\partial_t^s z)_\delta - \bar{a}_{ij}\big(\partial_{x_i}\partial_{x_j}\partial_t^{s-2} z\big)_\delta = (h^{s-2})_\delta + \eta^s(\cdot,\cdot;\delta) \tag{A.10}$$

with a correction term

$$\eta^s(t,\cdot;\delta) = \big(\bar{a}_{ij}\partial_t^{s-2}\partial_{x_i}\partial_{x_j} z\big)_\delta - \bar{a}_{ij}\big(\partial_t^{s-2}\partial_{x_i}\partial_{x_j} z\big)_\delta \text{ for } t \in [0,T].$$

Differentiating Equation (A.10) w.r.t. $t$

$$\partial_t(\partial_t^s z)_\delta - \partial_t\big(\bar a_{ij}\big(\partial_{x_i}\partial_{x_j}\partial_t^{s-2}z\big)_\delta\big) = \partial_t(h^{s-2})_\delta + \partial_t\eta^s(\cdot,\cdot;\delta),$$

multiplying the resulting equation in $L^2(\Omega)$ with $\partial_t^s z_\delta$, applying Green's formula and using Young's inequality, we estimate

$$\begin{aligned}
&\tfrac{1}{2}\partial_t\big(\|(\partial_t^n z)_\delta(t,\cdot)\|^2_{L^2(\Omega)} + \|\bar a(t,\cdot)\nabla(\partial_t^{n-1}z)_\delta(t,\cdot)\|^2_{L^2(\Omega)}\big)\\
&\qquad\le \tfrac{1}{2}\big\|\big(\partial_{x_i}\bar a_{ij}(t,\cdot)\big)\partial_{x_j}z_\delta(t,\cdot)\big\|^2_{L^2(\Omega)} + \tfrac{1}{2}(2+T^{-1/2})\|(\partial_t^n z)_\delta(t,\cdot)\|^2_{L^2(\Omega)}\\
&\qquad+ \tfrac{1}{2}T^{1/2}\|\partial_t h_\delta^{s-2}(t,\cdot)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|\eta^s(t,\cdot;\delta)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{A.11}$$

Here, we exploited the fact $(\partial_t z)_\delta = \partial_t z_\delta$ if $w$ is once weakly differentiable w.r.t. $t$, $z_\delta|_{\partial\Omega} = z|_{\partial\Omega}$ and $(\partial_{x_i}z)_\delta = \partial_{x_i}z_\delta$ if $w$ is once weakly differentiable w.r.t. $x_i$. Now, integrating Equation (A.11) w.r.t. $t$ over $[\varepsilon, T-\varepsilon]$, letting $\delta$ and then $\varepsilon$ go to zero, exploiting the regularity of $z$, applying Lemma **A**.8 and using Assumption **A**.9.3 , we get

$$\begin{aligned}
&\|\partial_t^s z(t,\cdot)\|^2_{L^2(\Omega)} + \|\partial_t^{s-1}z(t,\cdot)\|^2_{H^1(\Omega)} \le C(\gamma_0,\phi_0)\big(\|\bar z^s\|^2_{L^2(\Omega)} + \|\bar z^{s-1}\|^2_{H^1(\Omega)}\big)\\
&\quad + C(\gamma_0,\phi)(1+T^{-1/2})\int_0^t \big(\|\partial_t^n z(\tau,\cdot)\|^2_{L^2(\Omega)} + \|\partial_t^{n-1}z(\tau,\cdot)\|^2_{H^1(\Omega)}\big)\mathrm{d}\tau\\
&\quad + C(\gamma_0)T^{1/2}\int_0^t \|h^{s-2}(\tau,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\tau.
\end{aligned} \tag{A.12}$$

Combining Equations (A.9) and (A.12) leads to

$$\begin{aligned}
&\sum_{n=1}^{s}\big(\|\partial_t^n z(t,\cdot)\|^2_{L^2(\Omega)} + \|\partial_t^{m-1}z(t,\cdot)\|^2_{H^1(\Omega)}\big) \le C(\gamma_0,\phi_0)\Lambda_0\\
&\quad + C(\gamma_0,\phi)(1+T^{-1/2})\int_0^t \|\bar D^s z(\tau,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\tau\\
&\quad + C(\gamma_0)\sum_{n=1}^{s-1}\int_0^t \|h^{n-1}(\tau,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\tau + C(\gamma_0)T^{1/2}\int_0^t \|\partial_t h^{s-2}(\tau,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\tau.
\end{aligned} \tag{A.13}$$

Using Sobolev imbedding theorem

$$W^{1,2}(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^4(\Omega) \text{ for } d \leq 3,$$

we can estimate

$$\begin{aligned}
\sum_{m=1}^{n-1} \|(\partial_t^m \bar{a}_{ij}) \partial_t^{n-1-m} \partial_{x_i} \partial_{x_j} z\|_{L^2(\Omega)}^2 \leq & \sum_{m=1}^{\min\{n-1,1\}} \|\partial_t^m \bar{a}_{ij}\|_{L^\infty(\Omega)}^2 \|\partial_t^{n-1-m} z\|_{H^2(\Omega)}^2 \\
& + \sum_{m=\min\{n,2\}}^{n-1} \|\partial_t^m \bar{a}_{ij}\|_{L^4(\Omega)}^2 \|\partial_t^{n-1-m} z\|_{W^{2,4}(\Omega)}^2 \\
& \leq C(\phi) \|\bar{D}^n z\|_{L^2(\Omega)}^2 + C \sum_{m=\min\{n,2\}}^{n-1} \|\partial_t^m \bar{a}_{ij}\|_{H^1(\Omega)}^2 \|\partial_t^{n-1-m} z\|_{W^{3,2}(\Omega)}^2 \\
& \leq C(\phi) \|\bar{D}^n z\|_{L^2(\Omega)}^2 + C \sum_{m=\min\{n,2\}}^{n-1} \|\partial_t^m \bar{a}_{ij}\|_{H^{s-1-m}(\Omega)}^2 \|\bar{D}^{n-m+2} z\|_{L^2(\Omega)}^2 \\
& \leq C(\phi) \|\bar{D}^{s-1} z\|_{L^2(\Omega)}^2.
\end{aligned}$$

Recalling Equation (A.8), we obtain

$$\begin{aligned}
\int_0^t \|h^{n-1}(\tau, \cdot)\|_{L^2(\Omega)}^2 \mathrm{d}\tau \leq t \max_{0 \leq \tau \leq t} \|\partial_t^{n-1} \bar{f}(\tau, \cdot)\|_{L^2(\Omega)}^2 \\
+ C(\phi) \int_0^t \|\bar{D}^{s-1} z(t, \cdot)\|_{L^2(\Omega)}^2 \mathrm{d}\tau
\end{aligned} \tag{A.14}$$

for $n = 1, \ldots, s-1$ and $t \in [0, T]$. Similarly, for $t \in [0, T]$,

$$\int_0^t \|\partial_t h^{s-2}(\tau, \cdot)\|_{L^2(\Omega)}^2 \mathrm{d}\tau \leq \int_0^t \|\partial_t^{s-1} \bar{f}(t, \cdot)\|_{L^2(\Omega)}^2 \mathrm{d}\tau + C(\phi) \int_0^t \|\bar{D}^s z(\tau, \cdot)\|_{L^2(\Omega)}^2 \mathrm{d}\tau. \tag{A.15}$$

Now, combining Equation (A.13) as well as Equations (A.14) and (A.15), we arrive at

$$\sum_{n=1}^{s}\left(\|\partial_t^n z(t,\cdot)\|^2_{L^2(\Omega)}+\|\partial_t^{n-1}z(t,\cdot)\|^2_{H^1(\Omega)}\right)$$
$$\le C(\gamma_0,\phi_0)\Lambda_0+C(\gamma_0,\phi)(1+T^{1/2}+T^{-1/2})\int_0^t\|\bar{D}^s z(\tau,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\tau \text{ for } t\in[0,T]. \tag{A.16}$$

To finish the proof, we need to establish estimates for the remaining derivatives. For $n=1,\ldots,s-1$, consider Equation (A.7). Application of the elliptic regularity (viz. Assumption **A**.9.4) with $m=s-n-1$ yields

$$\begin{aligned}\|\partial_t^{n-1}z(t,\cdot)\|^2_{H^{m+2}(\Omega)}\le\gamma_1\big(&\|\partial_t^{n+1}z(t,\cdot)\|^2_{H^m(\Omega)}+\|h^{n-1}(t,\cdot)\|^2_{H^m(\Omega)}\\&+\|\partial_t^{n-1}z(t,\cdot)\|^2_{H^m(\Omega)}\big)\text{ for } t\in[0,T].\end{aligned} \tag{A.17}$$

Using Assumption **A**.9.1, Sobolev imbedding theorem and Jensen's inequality and applying the fundamental theorem of calculus to the second term on the right-hand side of Equation (A.17), we obtain

$$\begin{aligned}&\|h^{n-1}(t,\cdot)\|^2_{H^m(\Omega)}\le\\&\quad\le C(\phi_0)\Lambda_0+CT\sum_{k=1}^{n-1}\int_0^t\left\|\partial_t\big((\partial_t^k\bar{a}_{ij})\partial_t^{n-1-k}\partial_{x_i}\partial_{x_j}z\big)\right\|^2_{H^m(\Omega)}(\tau,\cdot)\mathrm{d}\tau\\&\quad\le C(\phi_0)\Lambda_0+C(\phi)T\int_0^t\|\bar{D}^s z(\tau,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\tau.\end{aligned} \tag{A.18}$$

Note that this estimate is only true if $s\ge 3$, which is trivially satisfied due to Assumption **A**.9. Combining Equations (A.16), (A.17) and (A.18) finally yields

$$\begin{aligned}\|\bar{D}^s z(t,\cdot)\|^2_{L^2(\Omega)}&\le C(\gamma_0,\gamma_1,\phi_0)\Lambda_0\\&\quad+C(\gamma_0,\gamma_1,\phi)(1+T^{1/2}+T+T^{-1/2})\int_0^t\|\bar{D}^s z(\tau,\cdot)\|^2_{L^2(\Omega)}\mathrm{d}\tau\end{aligned}$$

for any $t\in[0,T]$. The claim is now a direct consequence of Gronwall's inequality. □

**Remark 4.3.** *It should be pointed out that our proof differs from that of Jiang and Racke [9] as we can carry it out at the energy level* $s \geq [\frac{d}{2}]+2$ *whereas Jiang and Racke [9] require* $s \geq [\frac{d}{2}]+3$*. This "improvement" is possible since Theorem* **A***.10 is applied to a quasilinear wave equation with the quasilinearity depending on the function itself and not its gradient. A comment on this issue can also be found in [10, Remark 14.4].*

## A.2 Linear Heat Equation

In this appendix section, we consider an initial-boundary-value problem with Dirichlet boundary conditions for the linear homogeneous isotropic heat equation reading as

$$\theta_t(t,x) - a\triangle\theta(t,x) = \bar{g}(t,x) \quad \text{for } (t,x) \in (0,T) \times \Omega, \tag{A.19a}$$

$$\theta(t,x) = 0 \quad \text{for } (t,x) \in [0,T] \times \partial\Omega, \tag{A.19b}$$

$$\theta(0,x) = \theta^0(x) \quad \text{for } x \in \Omega. \tag{A.19c}$$

We present a well-posedness result for Equations (A.19a)–(A.19c). In contrast to [9, Chapter A.3], our proof is based on the operator semigroup theory, in particular, the maximal $L^2$-regularity theory, which is equivalent to analyticity of the semigroup generated by Dirichlet-Laplacian. A different technique is employed here to obtain a higher solution regularity needed for the fixed-point iteration in Theorem 3.4. Besides, the topologies used for the data $(\theta^0, \bar{g})$ and the solution $\theta$ differ from those in [9, Chapter A.3].

**Assumption A.11.** *Let $s \geq 2$ and $a > 0$. Assume the following assumptions are satisfied.*

1. Right-hand side regularity: *For $k = 0, \ldots, s-1$, $\partial_t^k \bar{g} \in C^0\big([0,T], H^{s-1-k}(\Omega)\big)$. Recall $H_0^0(\Omega) \equiv H^0(\Omega) := L^2(\Omega)$.*

2. Regularity and compatibility conditions: *For $k = 0, 1, \ldots, s-2$, let*

$$\bar{\theta}^k \in H^{s+1-k}(\Omega) \cap H_0^1(\Omega) \qquad \text{and} \qquad \bar{\theta}^{s-1} \in H_0^1(\Omega),$$

*where $\bar{\theta}^k$'s are given by*

$$\bar{\theta}^l(x) = a^l \triangle^l \theta^0(x) + \sum_{n=0}^{l-1} a^n \triangle^n \theta^0(x)\, \partial_t^{l-1-n} \bar{g}(0,x) \text{ for } x \in \Omega \text{ and } l = 0, \ldots, s-1.$$

**Theorem A.12.** *Under Assumption **A**.11, the system (A.19a)–(A.19c) possesses a unique classical solution satisfying*

$$\partial_t^k \theta \in C^0\big([0,T], H^{s+1-k}(\Omega) \cap H_0^1(\Omega)\big) \text{ for } k = 0, \ldots, s-2,$$

$$\partial_t^{s-1} \theta \in C^0\big([0,T], H_0^1(\Omega)\big) \cap L^2\big(0,T; H^2(\Omega) \cap H_0^1(\Omega)\big) \text{ and } \partial_t^s \theta \in L^2\big(0,T; L^2(\Omega)\big).$$

*Moreover, there exists a constant $C > 0$ such that*

$$\sum_{k=0}^{s-2} \max_{0 \leq t \leq T} \|\partial_t^k \theta(t,\cdot)\|^2_{H^{s+1-k}(\Omega)} + \max_{0 \leq t \leq T} \|\partial_t^{s-1} \theta(t,\cdot)\|^2_{H^1(\Omega)} + \int_0^T \big(\|\triangle \partial_t^{s-1} \theta(t,\cdot)\|^2_{L^2(\Omega)} + \|\partial_t^s \theta(t,\cdot)\|^2_{L^2(\Omega)}\big) \mathrm{d}t \leq C\Theta_0,$$

*where*

$$\Theta_0 = (1+T)\Big(\sum_{k=0}^{s-2} \|\bar{\theta}^k\|^2_{H^{s+1-k}(\Omega)} + \|\bar{\theta}^{s-1}\|^2_{H^1(\Omega)} + \sum_{k=0}^{s-1} \max_{0 \leq t \leq T} \|\partial_t^k \bar{g}(t,\cdot)\|^2_{H^{s-1-k}(\Omega)} \mathrm{d}t\Big).$$

*Proof.* Let $A := \triangle_D$ denote the $L^2$-realization of the Dirichlet-Laplacian with the domain

$$D(A) := \left\{\theta \in H_0^1(\Omega) \,\middle|\, \triangle\theta \in L^2(\Omega)\right\} = H^2(\Omega) \cap H_0^1(\Omega),$$

where the latter identity follows by standard elliptic regularity theory. (Note the difference in sign over Chapters3 and 4.) Using Lax & Milgram lemma to prove the resolvent identity

$$\sup_{\lambda \in \mathbb{C}\setminus(-\infty,0]} \left\|\lambda(\lambda - A)^{-1}\right\|_{L(L^2(\Omega))} < \infty,$$

we conclude that $aA$ generates a bounded analytic semigroup of angle $\frac{\pi}{2}$ on $L^2(\Omega)$. Due to the Hilbert space structure, [12, 1.7 Corollary] further implies $aA$ has the maximal $L^p$-regularity property.

Consider the solution map $\mathscr{S}$ sending $(\tilde{\theta}^0, \tilde{g})$ to the (mild) solution $\tilde{\theta}$ of

$$\tilde{\theta}_t - aA\tilde{\theta} = \tilde{g} \text{ in } (0,T), \quad \tilde{\theta}(0,\cdot) = \tilde{\theta}^0. \tag{A.20}$$

By classic $C_0$-semigroup theory and the maximal $L^p$-regularity theory, we have:

- The mapping

  $$\mathscr{S} \colon H_0^1(\Omega) \times L^2\big(0,T;L^2(\Omega)\big) \to H^1\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^2(\Omega)\cap H_0^1(\Omega)\big) \tag{A.21}$$

  is well-defined as an isomorphism between the two spaces.

- The mapping

  $$\begin{aligned}\mathscr{S} \colon \big(H^2(\Omega)\cap H_0^1(\Omega)\big) \times C^1\big([0,T],L^2(\Omega)\big) \to \\ C^1\big([0,T],L^2(\Omega)\big) \cap C^0\big([0,T],H^2(\Omega)\cap H_0^1(\Omega)\big)\end{aligned} \tag{A.22}$$

is well-defined and continuous in respective topologies.

*Existence and uniqueness at basic level:* On the strength of Assumption **A**.11, we (in particular) have $\theta^0 \in H_0^2(\Omega) \cap H_0^1$ and $\bar{g} \in C^1\big([0,T], L^2(\Omega)\big)$. Hence, there exists a unique classical solution

$$\theta \in C^1\big([0,T], L^2(\Omega)\big) \cap C^0\big([0,T], H^2(\Omega) \cap H_0^1(\Omega)\big) \tag{A.23}$$

to Equations (A.19a)–(A.19c).

*Higher regularity in time:* We argue by induction over $k = 1, \dots, s-2$ starting at $k = 1$. Applying $\partial_t^k$ to Equation (A.19a) and using Assumption **A**.11.2, we obtain (in distributional sense)

$$\partial_t(\partial_t^k \theta) - aA(\partial_t^k \theta) = \partial_t^k \bar{g} \text{ in } (0,T). \tag{A.24}$$

This motivates to consider Equation (A.20) with

$$\tilde{\theta}^0 = \bar{\theta}^k \in H^2(\Omega) \cap H_0^1(\Omega) \text{ and } \tilde{g} = \partial_t^k \bar{g} \in C^1\big([0,T], L^2(\Omega)\big). \tag{A.25}$$

By Equation (A.22),

$$\tilde{\theta} \in C^0\big([0,T], H^2(\Omega) \cap H_0^1(\Omega)\big) \cap C^1\big([0,T], L^2(\Omega)\big).$$

We now show the function

$$\bar{\theta}(t,\cdot) = \sum_{l=0}^{k-1} \frac{t^l}{l!}\bar{\theta}^l + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \tilde{\theta}(\tau,\cdot)\mathrm{d}\tau \mathrm{d}t_{k-1} \dots \mathrm{d}t_1$$

coincides with $\partial_t^k \theta$. By construction, $\bar{\theta}$ satisfies

$$\partial_t(\partial_t^k \bar{\theta}) - aA(\partial_t^k \bar{\theta}) = \partial_t^k \bar{g} \text{ in } (0,T). \tag{A.26}$$

Subtracting Equation (A.26) from Equation (A.24), multiplying with $\partial_t^{k-1}(\bar\theta-\theta)(t,\cdot)$ in $L^2(\Omega)$ and using Green's formula, we get

$$\tfrac{1}{2}\partial_t\big\|\partial_t^{k-1}(\bar\theta-\theta)(t,\cdot)\big\|_{L^2(\Omega)} + a\big\|\nabla\partial_t^{k-1}(\bar\theta-\theta)(t,\cdot)\big\|_{L^2(\Omega)} = 0. \tag{A.27}$$

This along with the fact $\partial_t^l\bar\theta(0,\cdot)\equiv\bar\theta^l\equiv\partial_t^l\theta(0,\cdot)$ for $l=0,\dots,k-1$ enables us to use the Gronwall's inequality together with the fundamental theorem of calculus to deduce $\bar\theta\equiv\theta$. Therefore, we have shown

$$\partial_t^k\theta\equiv\partial_t^k\bar\theta\equiv\tilde\theta\in C^0\big([0,T],H^2(\Omega)\cap H_0^1(\Omega)\big)\cap C^1\big([0,T],L^2(\Omega)\big). \tag{A.28}$$

For $k=s-1$, a slightly modified argument needs to be utilized. In this case, we only have

$$\tilde\theta^0=\theta^{s-1}\in H_0^1(\Omega) \text{ and } \tilde g=\partial_t^{s-1}\bar g\in C^0\big([0,T],L^2(\Omega)\big)\hookrightarrow L^2\big(0,T;L^2(\Omega)\big)$$

to plug into Equation (A.20). Instead of Equation (A.22), we use (A.21) to infer

$$\tilde\theta\in H^1\big(0,T;L^2(\Omega)\big)\cap L^2\big(0,T;H^2(\Omega)\cap H_0^1(\Omega)\big). \tag{A.29}$$

Defining

$$\bar\theta(t,\cdot)=\sum_{l=0}^{s-2}\frac{t^l}{l!}\bar\theta^l+\int_0^t\int_0^{t_1}\cdots\int_0^{t_{s-2}}\tilde\theta(\tau,\cdot)\mathrm{d}\tau\mathrm{d}t_{s-2}\dots\mathrm{d}t_1,$$

by the same kind of argument, we have $\bar\theta\equiv\theta$ and, therefore,

$$\partial_t^{s-1}\theta\equiv\tilde\theta\in H^1\big(0,T;L^2(\Omega)\big)\cap L^2\big(0,T;H^2(\Omega)\cap H_0^1(\Omega)\big)\hookrightarrow C^0\big(0,T;H_0^1(\Omega)\big). \tag{A.30}$$

*Higher regularity in space:* For $k=0,\dots,s-2$, applying $\partial_t^k$ to Equation (A.19a), we observe

$$A\partial_t^k\theta=-\tfrac{1}{a}\partial_t^{k+1}\theta+\tfrac{1}{a}\partial_t^k\bar g. \tag{A.31}$$

Hence, using the fact $A$ is an isomorphism between $H^{s+1-k}(\Omega) \cap H_0^1(\Omega)$ and $H^{s-1-k}(\Omega)$ along with Assumption **A**.11.1 and Equations (A.28), (A.30), we inductively obtain (beginning at $k = s-2$ and going downward to $k = 0$)

$$\partial_t^k \theta \in C^0\big([0,T], H^{s+1-k}(\Omega) \cap H_0^1(\Omega)\big) \text{ for } k = 0, 1, \ldots, s-2.$$

The 'remaining' case $k = s-1$ has already been treated in the previous step so that

$$\partial_t^{s-1}\theta \in C^0\big([0,T], H_0^1(\Omega)\big) \cap H^1\big(0,T; L^2(\Omega)\big) \cap L^2\big(0,T; H^2(\Omega) \cap H_0^1(\Omega)\big).$$

*Energy estimate.* The energy estimate easily follows from the solution operator continuity. □

# Bibliography

[1] S. A. Ambartsumian, M. V. Belubekian, and M. M. Minassian. On the problem of vibrations of non-linear-elastic electroconductive plates in transverse and longitudinal magnetic fields. *International Journal of Non-Linear Mechanics*, 19(2):141–149, 1984.

[2] S. A. Ambartsumyan. *Theory Of Anisotropic Plates: Strength, Stability, Vibrations*, volume II of *Progress in Material Science Series*. Technomic Publishing Co. Inc., Stamford, CT, 1970.

[3] I. Chueshov and I. Lasiecka. *Von Karman Evolution Equations: Well-Posedness and Long Time Dynamics*. Springer Science & Business Media, New York – Dordrecht – Heidelberg – London, 2010, 766 pages.

[4] R. Denk, R. Racke, and Y. Shibata. $L_p$ theory for the linear thermoelastic plate equations in bounded and exterior domains. *Advances in Differential Equations*, 14(7/8):685–715, 2009.

[5] R. Denk and R. Schnaubelt. A structurally damped plate equation with Dirichlet-Neumann boundary conditions. *Journal of Differential Equations*, 259(4):1323–1353, 2015.

[6] D. Hasanyan, N. Hovakimyan, A. J. Sasane, and V. Stepanyan. Analysis of nonlinear thermoelastic plate equations. *43rd IEEE Conference on Decision and Control (CDC)*, pages 1514–1519, IEEE, 2004.

[7] Mihaela Ignatova, Igor Kukavica, Irena Lasiecka, and Amjad Tuffaha. On well-posedness and small data global existence for an interface damped free boundary fluid–structure model. *Nonlinearity*, 27(3):467, 2014.

[8] A. A. Ilyushin. *Plastičnost'. Uprugo-Plastičeskije Deformacii.* Klassičeskij Universitetskij Učebnik. Logos, Moscow, 2004.

[9] S. Jiang and R. Racke. *Evolution Equations in Thermoelasticity*, volume 112 of *Monographs and Surveys in Pure and Applied Mathematics.* Chapman & Hall/CRC, Boca Raton, London, New York, Washington D.C., 2000.

[10] T. Kato. *Abstract Differential Equations and Nonlinear Mixed Problems.* Fermi Lectures. Scuola Normale Superiore, Pisa, 1985.

[11] J. Kim. On the Energy Decay of a Linear Thermoelastic Bar and Plate. *SIAM J. Math. Anal.* 23(4):889–899, 1992.

[12] P. C. Kunstmann and L. Weis. *Maximal $L_p$-Regularity for Parabolic Equations, Fourier Multiplier Theorems and $H^\infty$-Functional Calculus.* Functional Analytic Methods for Evolution Equations. Springer, Berlin – Heidelberg, 2004.

[13] J. Lagnese and J. L. Lions. *Modelling, Analysis and Control of Thin Plates*, volume 6 of *Recherches en Mathématiques Appliquées.* Mason, Paris, 1988.

[14] I. Lasiecka, S. Maad, and A. Sasane. Existence and exponential decay of solutions to a quasilinear thermoelastic plate system. *Nonlinear Differential Equations and Applications*, 15:689–715, 2008.

[15] I. Lasiecka and M. Wilke. Maximal regularity and global existence of solutions to a quasilinear thermoelastic plate system. *Discrete and Continuous Dynamical Systems A*, 33:5189–5202, 2013.

[16] I. Lasiecka, M. Pokojovy, and X. Wan. Global Existence and Stability for a Nonlinear Thermoelastic Kirchhoff-Love Plate. *Nonlinear Analysis: Real World Applications*, 38: 184-221, 2017.

[17] I. Lasiecka and R. Triggiani. Analyticity, and lack thereof, of thermo-elastic semigroups. *ESAIM: Proceedings*, 4:199–222. EDP Sciences, 1998.

[18] I. Lasiecka and R. Triggiani. *Control theory for partial differential equations: Volume 1, Abstract parabolic systems: Continuous and approximation theories.* Cambridge University Press, 2000.

[19] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Volume 2, Abstract Hyperbolic-like Systems Over a Finite Time Horizon: Continuous and Approximation Theories.* Cambridge University Press, 2000.

[20] L. Librescu, D. Hasanyan, Z. Qin, and D. R. Ambur. Nonlinear magnetothermoelasticity of anisotropic plates immersed in a magnetic field. *Journal of Thermal Stresses*, 26(11-12): 1277-1304, 2003.

[21] W. Nowacki. *Thermoelasticity.* Pergamon Press, Oxford – New York – Toronto, etc., 2nd ed. edition, 1986.

[22] M. Pokojovy. *Zur Theorie wärmeleitender Reissner-Mindlin Platten.* PhD thesis. University of Konstanz, 2011.

[23] T. Schuster, B. Kaltenbacher, B. Hofmann, and K. S. Kazimierski. *Regularization Methods in Banach Spaces*, volume 10 of *Radon Series on Computational and Applied Mathematics.* Walter de Gruyter, Berlin/Boston, 2012.