K-theoretic Catalan Functions

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Abstract

The framework of Catalan functions provided new proof methods for resolving conjectures about k-Schur functions, which serve as Schubert representatives for the homology of the affine Grassmannian for SL_{k+1}. We prove that the K-theoretic refinement, K-k-Schur functions, are part of a family of inhomogenous symmetric functions whose top homogeneous components are Catalan functions. Lam-Schilling-Shimozono identified the K-k-Schur functions as Schubert representatives for K-homology of the affine Grassmannian for SL_{k+1}. Our perspective reveals that the K-k-Schur functions satisfy a shift invariance property, and we deduce positivity of their branching coefficients from a positivity result of Baldwin and Kumar. We further show that a slight adjustment of our formulation for K-k-Schur functions produces a second shift-invariant basis satisfying a rectangle factorization property and that conjecturally has positive branching. Building on work of Ikeda-Iwao-Maeno, we conjecture that this second basis gives the images of the Lenart-Maeno quantum Grothendieck polynomials under a K-theoretic analog of the Peterson isomorphism. Finally, as a potential application to other affine settings, we provide conjectural descriptions for the Schubert homology representatives of the affine Grassmannian for Sp_{2n}.

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Chapter 1

Introduction

In the nineteenth century, mathematician Hermann Schubert was interested in solving enumerative geometric problems about how various subvarieties of a projective variety might intersect. Schubert's ideas served as a precursor to incredibly important mathematical theories, such as the theory of characteristic classes. In 1900, Hilbert recognized that further developing Schubert's ideas could lead to new mathematical developments and made Schubert's work the subject of his 15th problem (out of the 23 celebrated "Hilbert problems"):

> The problem consists in this: To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him. [Hil02]

Since then, Schubert's original questions have been reframed in terms of understanding the cohomology ring structure constants of the Grassmannian. From a combinatorial point of view, these structure constants also manifest as a subset of the multiplication structure constants of certain symmetric polynomials, known as Schur functions. Subsequently, the scope of Schubert calculus has been broadened, now generally interpreted to mean the study of the structure of generalized cohomology theories of various homogeneous spaces, which includes finding polynomial representatives whose structure constants match those of the cohomology theory. Schubert calculus, in this sense, has been and continues to be an active area of mathematical research. This work focuses primarily on symmetric function representatives for the K-theory and K-homology of a space called the "affine Grassmannian".

1.1 Symmetric functions with geometric meaning

The $\mathbb{Z}$-algebra of symmetric functions, Λ, has homogeneous components Λ^n whose dimension is equal to the number of integer partitions of n. Thus, bases of Λ are indexed by partitions, the set of which we denote Par.

Question 1.1.1. Given a basis of symmetric functions $\{f_\lambda\}_{\lambda\in\mathrm{Par}}$, some questions one may ask are as follows.

(a) Is there a direct formula for f_λ in terms of other symmetric functions?

(b) What are the multiplication structure constants for $\{f_\lambda\}_{\lambda\in\mathrm{Par}}$?

While, in general, one can define any number of symmetric function bases, there exist distinguished bases that have geometric or representation theoretic significance. In these instances, the multiplication structure constants are typically non-negative and encode some geometric or representation theoretic meaning. Perhaps the most famous of such bases is the basis of Schur functions, s_λ, which has geometric meaning for the Grassmannian and for which many answers to Questions 1.1.1(a) and (b) are known.

From the study of Macdonald polynomials, [LLM03] discovered k-Schur functions, denoted $s_\lambda^{(k)}$ where $k \in \mathbb{Z}_{>0}$ and λ is a partition such that $\lambda_1 \leq k$. These k-Schur functions conjecturally serve as a basis for the k-bounded subspace of symmetric functions, $\Lambda_{(k)} = \mathbb{Z}[s_1, s_2, \ldots, s_k]$. The work of [LM03] gave an alternative formulation of k-Schur functions, known to be a basis of $\Lambda_{(k)}$ and conjecturally equivalent to the definition of [LLM03].

In [Lam08], the k-Schur functions of [LM03] (when the parameter t is specialized to 1) were shown to have geometric significance for the "affine Grassmannian", an infinite dimensional space associated to SL_{k+1}, denoted $\mathrm{Gr} = \mathrm{Gr}_{\mathrm{SL}_{k+1}}$. Specifically, [Lam08] shows $s_\lambda^{(k)}$ is the image of a distinguished basis of $H_*(\mathrm{Gr})$ under a Hopf isomorphism between $H_*(\mathrm{Gr})$ and $\Lambda_{(k)}$. We also note that k-Schur functions serve as a refinement Schur functions, illustrated by the fact that $s_\lambda^{(k)} \to s_\lambda$ as $k \to \infty$.

Question 1.1.2. What are the coefficients $b_{\lambda\mu}$ in the expansion $s_\lambda^{(k)} = \sum_\mu b_{\lambda\mu} s_\mu^{(k+1)}$?

These coefficients, known as "branching coefficients", remained a mystery for many years. In the ungraded case, [LLMS10, LLMS13] gives a combinatorial method to describe the $b_{\lambda\mu}$, but this failed to generalize to the case with a generic t parameter. More recently, the coefficients $b_{\lambda\mu}$ with a generic t parameter were described in [BMPS19] via a simpler combinatorial formula. [BMPS19] accomplished this via a technical but fundamental result called "shift invariance", linking the branching coefficients with the multiplicative structure constants of the basis dual to $s_\lambda^{(k)}$ under the Hall inner product. This, in turn, was only possible by giving a "raising operator" formula for $s_\lambda^{(k)}$ and the proof that this formula of [BMPS19] matches the definition of $s_\lambda^{(k)}$ used in [LM03, Lam08, LLMS10] constitutes a large part of [BMPS19]. Instrumental to showing these definitions coincide, [BMPS19] shows that the k-Schur basis is identified with a subfamily of symmetric functions called *Catalan functions*. Catalan functions came out of the study of Euler characteristics of vector bundles on the flag variety [Bro94, SW00, Che10, Pan10]. These functions are defined by a raising operator formula and are indexed by pairs (Ψ, γ), where Ψ is one of Catalan many upper order ideals in the set of positive $A_{\ell-1}$ roots, Δ_ℓ^+, and $\gamma \in \mathbb{Z}^\ell$.

The power of Catalan functions comes from the fact that they provide a "calculus" that interpolates between various bases of symmetric functions. In particular, Catalan functions are equipped with numerous recurrence relations among themselves, providing a framework to gradually transform a given Catalan function into other Catalan functions. This approach stands in contrast to many other proofs in the theory of symmetric functions that require more direct analysis of transition matrices between symmetric function bases.

In Chapter 2, we provide details of answers to Questions 1.1.1 and 1.1.2 for Schur and k-Schur functions while simultaneously introducing background about Catalan functions necessary for later chapters.

1.2 Survey of results

Just as k-Schur functions are Schubert homology representatives of the affine Grassmannian, over the last several decades, a K-theoretic counterpart to has been emerging. The K-homology $K_*(\mathrm{Gr})$ is also Hopf isomorphic to $\Lambda_{(k)}$ [LSS10b], and Schubert representatives are now given by a basis of inhomogeneous symmetric functions called *K-k-Schur functions*, $g_\lambda^{(k)} \in \Lambda_{(k)}$. They satisfy an elegant Pieri rule and are conjecturally surrounded with positivity properties. Foremost is the following branching property, analogous to Question 1.1.2, which we prove in Theorem 3.7.2.

Conjecture 1.2.1 ([LSS10b, Conjecture 7.20(3)], [Mor12, Conjecture 44])**.** *For any partition λ with $\lambda_1 \leq k$,*

$$g_\lambda^{(k)} = \sum_\mu a_{\lambda\mu}\, g_\mu^{(k+1)} \qquad \textit{satisfies } (-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}. \tag{1.1}$$

Previous geometric and algebraic descriptions of K-k-Schur functions had failed to yield proofs of Conjecture 1.2.1. We overcome this with an explicit raising operator formula for $g_\lambda^{(k)}$, which enables us to settle and to derive new properties of the basis (Theorem 3.7.1, Corollaries 3.7.3, 3.7.4).

In Chapter 3, we prove this raising operator formula by connecting it to the Pieri rule for $g_\lambda^{(k)}$ through careful analysis of intermediate raising operator objects between $g_\lambda^{(k)}$ and $g_{1^r} g_\lambda^{(k)}$, for $g_{1^r} \in \Lambda$ a *dual stable Grothendieck polynomial* discussed below. This powerful approach to Schubert calculus was initiated in [Tam11, BKT15, BKT17], further leveraged in [And19, AF18]. We advance this program using methods of [BMPS19]. Specifically, we extend the Catalan functions to an inhomogeneous family of symmetric functions using additional information from a multiset M supported on $\{1,\ldots,\ell\}$. These functions, $K(\Psi, M, \gamma)$, are called Katalan functions. In this way, we endow K-k-Schur functions with a calculus that interpolates between various bases of symmetric functions as [BMPS19] did for k-Schur functions. Computer experimentation leads us to propose natural conditions for Schur positive expansions of Katalan functions, as well as positive expansions (up to predictable sign) in the basis of dual stable Grothendieck polynomials $\{g_\lambda\}$, Hall-dual to the basis of Fomin-Kirillov stable these conjectures in § 4.6.

We prove that the K-k-Schur functions are a distinguished subfamily of Katalan functions. The simplicity of our formula reveals that, as with k-Schur functions, the K-k-Schur basis satisfies *shift invariance*:

$$G^{\perp}_{1^\ell}\, g^{(k+1)}_{\lambda+1^\ell} = g^{(k)}_{\lambda}\,. \tag{1.2}$$

This remarkable property implies that the branching coefficients of (1.1) are none other than a subset of dual Pieri coefficients. From this, a positivity result of Baldwin and Kumar [BK17] enables us to prove several conjectures about K-k-Schur functions, including positive branching.

In Chapter 4, we take up another application of the Katalan formulation for K-k-Schur functions involving the quantum K-theory ring, a deformation of the Grothendieck ring of coherent sheaves on Fl_n studied by Givental and Lee [GL03]. Kirillov and Maeno [KM] proposed a presentation $\mathcal{QK}(Fl_n)$ for $QK(Fl_n)$ which was recently established by Anderson-Chen-Tseng [ACT17]. Lenart and Maeno [LM06] introduced *quantum Grothendieck polynomials* $\mathfrak{G}^Q_w$ as potential representatives for the Schubert basis, just confirmed in [LNS20].

Using Ruijsenaars's relativistic Toda lattice, Ikeda-Iwao-Maeno [IIM20] produced an explicit ring isomorphism Φ between localizations of $K_*(\mathrm{Gr})$ and $\mathcal{QK}(Fl_{k+1})$ and conjectured that the images of quantum Grothendieck polynomials expand unitriangularly into K-k-Schur functions with coefficients having predictable sign; building on this work, Ikeda conjectured a precise description for the images. Kato [Kat18] also considers related ideas in general type.

Conjecture 1.2.2 ([IIM20, Conjecture 1.8],[Ike20]). *For $w \in S_{k+1}$,*

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{d \in \mathrm{Des}(w)} g_{(k+1-d)^d}}\,, \qquad \text{for } \tilde{g}_w := (1 - G_1^\perp)\Bigg(\sum_{\mu_1 \le k,\, w_\mu \le w_\lambda} g_\mu^{(k)} \Bigg) \in \Lambda_{(k)}\,, \quad (1.3)$$

where $\lambda = \theta(w)^{\omega_k}$ is a partition with $\lambda_1 \le k$, defined in §4.3, w_λ denotes the minimal coset representative of S_{k+1} in $\widehat{S}_{k+1}$ associated to λ (see §2.4.2), and $\le$ denotes Bruhat order on $\widehat{S}_{k+1}$.

To give geometric context for this conjecture, under the Hopf algebra isomorphism $\Lambda_{(k)} \to K_*(\mathrm{Gr})$, the sum $\sum_{\mu_1 \le k,\, w_\mu \le w_\lambda} g_\mu^{(k)}$ maps to the class of the structure sheaf of the Schubert variety $X_{w_\lambda} \subseteq \mathrm{Gr}$, whereas $g_\lambda^{(k)}$ maps to the class of the ideal sheaf of the boundary ∂X_{w_λ}; see [LLMS18, Theorem 1] and [LSS10b, Theorems 5.4 and 7.17(1)].

We conjecture an explicit operator formula for the $\tilde{g}_w$'s by realizing them as a subfamily of Katalan functions; it requires only a slight adjustment to our Katalan description of K-k-Schur functions. We show our conjectured representatives satisfy certain properties that should be satisfied by the $\tilde{g}_w$'s (Proposition 4.4.4 and Theorem 4.4.5), but fall short of proving that our explicit formula is, in fact, the correct one in general. However, we are able to verify Conjecture 1.2.2 for Grassmannian permutations, completing the proof strategy of [IIM20]. We do so by establishing the following missing ingredient, which is an immediate consequence of the Katalan formulation for K-k-Schur functions. See Corollary 3.7.3.

Conjecture 1.2.3 ([Mor12]). *For a partition λ where $\lambda_1 + \ell(\lambda) - 1 \le k$, $g_\lambda^{(k)} = g_\lambda$.*

Finally, in Chapter 5, we explore potential applications of raising operator techniques techniques to Schubert representatives of the affine Grassmannian of Sp_{2n}, referred to as *type C k-Schur functions*, by presenting promising conjectures in the same vein as the raising operator formulas for $s_\lambda^{(k)}$ and $g_\lambda^{(k)}$, as well as providing a conjectural weight generating function description of the type C k-Schur functions over certain tableaux.

Chapter 2

Background

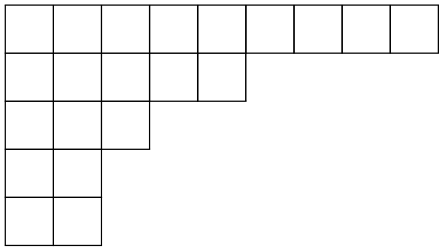
2.1 Schubert calculus

Let $X = \mathrm{Gr}(m,n) = \{W \subseteq \mathbb{C}^{m+n} \mid \dim W = m\}$ be the *Grassmannian* of m-planes in $\mathbb{C}^{m+n}$. Then, there exist multiple ways to endow X with a topological structure (see e.g. [Bri05, Man01]). For example, for $\mathrm{GL}_k(\mathbb{C})$ the Lie group of invertible k by k matrices, we have $\mathrm{Gr}(m,n) \cong GL_{m+n}(\mathbb{C})/P$ where P is a maximal parabolic subgroup of $\mathrm{GL}_{m+n}(\mathbb{C})$ fixing the subspace $\langle e_1, \ldots, e_m\rangle$. We can also view the Grassmannian as a projective variety via the Plücker embedding. Furthermore, the Grassmannian is known to have a nice cellular decomposition with the following indexing data.

Definition 2.1.1. For any $\ell \in \mathbb{Z}_{>0}$, we say $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}^{\ell}_{\geq 0}$ is a *partition* if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$. We denote the set of all such partitions as Par_ℓ. Furthermore, we say λ is a *partition of* r if $|\lambda| = \lambda_1 + \cdots + \lambda_\ell = r$. Finally, we say the *length* of λ, denoted $\ell(\lambda)$, is the number of non-zero parts of λ.

Definition 2.1.2. The (English style) Young diagram of a partition λ is a finite collection of boxes arranged in left-justified rows, with the row lengths in non-increasing order from top to bottom.

We often think of a partition λ interchangeably with its diagram.

$\lambda = (9,5,3,2,2) \leftrightarrow$ 

Definition 2.1.3. Given partitions λ, μ, we say $\lambda \supseteq \mu$ if the diagram of λ contains the diagram of μ.

Definition 2.1.4. Fix a complete flag of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_i \subsetneq \cdots \subsetneq V_{m+n} = \mathbb{C}^{m+n}$$

such that $\dim V_i = i$ for all $0 \le i \le m+n$. Then, if λ is a partition such that $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$, we define the *Schubert cell*

$$\Omega_\lambda = \{W \in \mathrm{Gr}(m,n) \mid \dim(W \cap V_j) = i \text{ if } n+i-\lambda_i \le j \le n+i-\lambda_{i+1}\}. \tag{2.1}$$

Furthermore, we define the *Schubert variety* $X_\lambda = \overline{\Omega_\lambda}$ to be the closure of Ω_λ.

Proposition 2.1.5. *We have the following facts.*

(a) $\mathrm{Gr}(m,n)$ *decomposes as a disjoint union of Schubert cells. That is,*

$$\mathrm{Gr}(m,n) = \bigsqcup_{\lambda \in \mathrm{Par}_m,\, \lambda_1 \le n} \Omega_\lambda \tag{2.2}$$

(b) $X_\lambda = \bigsqcup_{\mu \supseteq \lambda} \Omega_\mu$ *is a topologically closed cell of dimension* $2(mn - |\lambda|)$.

(c) $X_\lambda \supseteq X_\mu$ *if and only if* $\lambda \subseteq \mu$.

Given two Schubert varieties X_λ and X_μ, mathematicians of the 19th century (including Hermann Schubert) were interested in the structure of $X_\lambda \cap X_\mu$. Modern mathematics translates this problem into one of cohomology theory. Let $H^*(X) = H^*(X;\mathbb{Z})$ be the singular cohomology of X with coefficients in $\mathbb{Z}$. Then, we have the following general algebraic geometric fact.

Lemma 2.1.6 ([Ful97, Appendix B])**.** *Given irreducible algebraic subvarieties Y and Y' of a compact and connected complex variety X with codimensions c and c', respectively, we have*

$$Y \cap Y' = \bigcap_{i \in I} Z_i$$

for Z_i some irreducible subvarieties of X and, furthermore, if this intersection is transverse (which implies Z_i has codimension $c + c'$),

$$[Y] \smile [Y'] = \sum_{i \in I} [Z_i] \in H^{2c+2c'}(X)$$

In the case of the Grassmannian, we can make use of our Schubert cell decomposition. For any partition $\lambda \in \mathrm{Par}^m$ with $\lambda_1 \leq n$, let $\sigma_\lambda = [X_\lambda]^*$ be the Poincaré dual of the homology class of X_λ. We call such classes *Schubert classes*. Furthermore, σ_λ is independent of the choice of reference flag used to define X_λ.

Proposition 2.1.7. *For $X = \mathrm{Gr}(m, n)$, we have*

(a) $\sigma_\lambda \in H^{2|\lambda|}(X)$.

(b) $H^*(X) = \bigoplus_{\lambda \in \mathrm{Par}^m,\, \lambda_1 \leq n} \mathbb{Z}\sigma_\lambda$ *as $\mathbb{Z}$-modules.*

Thus, to understand the ring structure of $H^*(X)$, one wishes to understand the constants $\sigma_\lambda \smile \sigma_\mu = \sum_\nu c^\nu_{\lambda\mu} \sigma_\nu$ where $c^\nu_{\lambda\mu} = \#\{X_\lambda \cap X_\mu \cap X_{\nu^\vee}\}$ for $\nu^\vee$ the complement of ν in (n^m), that is $\nu^\vee = (n - \nu_m, n - \nu_{m-1}, \ldots, n - \nu_1)$. To examine the $c^\nu_{\lambda\mu}$ further, we will turn our attention to the ring of symmetric functions.

2.2 Symmetric polynomials and symmetric functions

The rings of symmetric polynomials and symmetric functions are endowed with a rich combinatorial structure. Indeed, the ring of symmetric functions shall be the primary setting for the results presented in this work. The facts in this subsection can all be found in [Mac15, I] with proof.

2.2.1 The ring of symmetric polynomials

For any $m \in \mathbb{Z}_{>0}$, consider the polynomial ring $\mathbb{Z}[x_1, \ldots, x_m]$. There is a degree preserving S_m-action on $\mathbb{Z}[x_1, \ldots, x_m]$ given by $\sigma.f(x_1, \ldots, x_m) = f(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$ for any $f \in \mathbb{Z}[x_1, \ldots, x_m]$, $\sigma \in S_m$. Then, let the *ring of symmetric polynomials* on m indeterminates be given by

$$\Lambda(\mathbf{x}_m) = \{f \in \mathbb{Z}[x_1, \ldots, x_m] \mid \sigma.f = f, \forall \sigma \in S_m\} .$$

By construction, $\Lambda(\mathbf{x}_m)$ is a subring of $\mathbb{Z}[x_1, \ldots, x_m]$ and a $\mathbb{Z}$-algebra. For any $\alpha \in \mathbb{Z}^m_{\geq 0}$, let $x^\alpha = \prod_{i=1}^m x_i^{\alpha_i} \in \mathbb{Z}[x_1, \ldots, x_m]$. Furthermore, for $\alpha \in \mathbb{Z}^m_{\geq 0}$, let $\mathrm{sort}(\alpha)$ be the partition obtained by sorting α into a weakly decreasing sequence. Then, for any partition λ and any $\alpha \in \mathbb{Z}^\ell_{\geq 0}$, we will say $\lambda \sim \alpha$ if $\mathrm{sort}(\alpha) = \lambda$. Using this notation, we can define the *monomial symmetric polynomials*

$$m_\lambda = m_\lambda(x_1, \ldots, x_m) = \sum_{\alpha \in \mathbb{Z}^m_{\geq 0}, \alpha \sim \lambda} x^\alpha \in \Lambda(\mathbf{x}_m) .$$

It is straightforward to deduce that the set $\{m_\lambda(x_1, \ldots, x_m)\}_{\lambda \in \mathrm{Par}_m}$ forms a basis of $\Lambda(\mathbf{x}_m)$. For $r \in \mathbb{Z}_{>0}$, consider also $h_r = h_r(x_1, \ldots, x_m) = \sum_{i_1 \leq \cdots \leq i_r} x_{i_1} \cdots x_{i_m}$ and $e_r = e_r(x_1, \ldots, x_m) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_m}$ are both elements in $\Lambda(\mathbf{x}_m)$. Then, if we set $h_0 = e_0 = 1$ and $h_s = e_s = 0$ for all $s < 0$, we can define $h_\alpha = h_{\alpha_1} \cdots h_{\alpha_\ell}$ and $e_\alpha = e_{\alpha_1} \cdots e_{\alpha_\ell}$ for any $\alpha \in \mathbb{Z}^\ell$.

Proposition 2.2.1. *For $m \in \mathbb{Z}_{>0}$, we have the following facts about $\Lambda(\mathbf{x}_m)$.*

(a) $\Lambda(\mathbf{x}_m)$ *is generated by* $\{h_1, \ldots, h_m\}$ *and by* $\{e_1, \ldots, e_m\}$*. That is,* $\Lambda(\mathbf{x}_m) = \mathbb{Z}[h_1, \ldots, h_m] = \mathbb{Z}[e_1, \ldots, e_m]$.

(b) The sets $\{h_\lambda\}_{\lambda \in \mathrm{Par}_m}$ *and* $\{e_\lambda\}_{\lambda \in \mathrm{Par}_m}$ *form bases of* $\Lambda(\mathbf{x}_m)$.

Thus, we now have the tools necessary to make an initial connection between $H^*(\mathrm{Gr}(m, n))$ and $\Lambda(\mathbf{x}_m)$ as follows.

Proposition 2.2.2. *There exists a surjection of $\mathbb{Z}$-algebras* $\phi_{m,n} \colon \Lambda(\mathbf{x}_m) \to H^*(\mathrm{Gr}(m, n))$ *that sends* $\phi_{m,n}(h_i) = \sigma_{(i,0,\ldots,0)}$ *for* $1 \leq i \leq m$.

Next, we will see there exists a family of symmetric polynomials $\{s_\lambda\}_{\lambda \in \mathrm{Par}_m} \subseteq \Lambda(\mathbf{x}_m)$ such that $\phi_{m,n}(s_\lambda) = \sigma_\lambda$ if $\lambda_1 \leq n$.

2.2.2 Schur polynomials

Definition 2.2.3. For $\gamma \in \mathbb{Z}^\ell$, we define the *Schur polynomial*

$$s_\gamma = \det(h_{\gamma_i + j - i}) \,. \tag{2.3}$$

There are, in fact, many definitions of Schur polynomials, but this definition allows us to define s_γ for any integer vector $\gamma \in \mathbb{Z}^\ell$. Many other definitions are limited to γ being a partition.

Proposition 2.2.4. *We have the following results about Schur functions.*

(a) $s_r = h_r$ *for all* $r \in \mathbb{Z}$.

(b) $\{s_\lambda\}_{\lambda \in \mathrm{Par}^m}$ *forms a basis of* Λ_m.

(c) For any $\gamma \in \mathbb{Z}^m$,

$$s_\gamma = \begin{cases} \mathrm{sign}(\gamma + \rho) s_{\mathrm{sort}(\gamma+\rho)-\rho} & \text{if } \gamma + \rho \text{ has non-negative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\rho = (m-1, m-2, \ldots, 0)$, $\mathrm{sort}(\beta)$ is the weakly decreasing sequence obtained by sorting β, and $\mathrm{sign}(\beta)$ denotes the sign of the shortest length permutation taking β to $\mathrm{sort}(\beta)$.

Schur polynomials arise in a variety of contexts, including as the type A irreducible characters in the Weyl Character Formula. However, for Schubert calculus, their central importance is given by the following result.

Proposition 2.2.5. *For $\phi_{m,n} \colon \Lambda(\mathbf{x}_m) \to H^*(\mathrm{Gr}(m,n))$ as in Proposition 2.2.2 and $\lambda \in \mathrm{Par}_m$,*

$$\phi_{m,n}(s_\lambda) = \begin{cases} \sigma_\lambda & \text{if } \lambda_1 \leq n\,, \\ 0 & \text{otherwise.} \end{cases}$$

2.2.3 Symmetric Functions

Note that there is always a surjection $\Lambda(\mathbf{x}_{m+1}) \twoheadrightarrow \Lambda(\mathbf{x}_m)$ given by specializing $x_{m+1} = 0$. Because of this fact, it is often useful to work with symmetric polynomials in infinitely many variables, Λ, also called *symmetric functions*.

Definition 2.2.6. Let Λ be the $\mathbb{Z}$-algebra of symmetric functions in an infinite alphabet of variables $X = x_1, x_2, \ldots$ with coefficients in $\mathbb{Z}$, i.e., Λ is the subalgebra of $\mathbb{Z}[\![x_1, x_2, \ldots]\!]$ such that, for all $f(X) \in \Lambda$,

(a) $f(X)$ is invariant under any permutations of the indeterminates and

(b) the degrees of every monomial in $f(X)$ are bounded.

Note, [Mac15, I.2] gives an equivalent description of Λ using inverse limits. Following [Mac15], for any $\gamma \in \mathbb{Z}^\ell$, we will let $e_\gamma = e_\gamma(X) = e_{\gamma_1}(X) \cdots e_{\gamma_\ell}(X)$ be the elementary symmetric functions, $h_\gamma = h_\gamma(X) = h_{\gamma_1}(X) \cdots h_{\gamma_\ell}(X)$ be the complete homogeneous symmetric functions, and $s_\gamma = s_\gamma(X)$ be the Schur functions. For any partition λ, we also let $m_\lambda = m_\lambda(X)$ be the monomial symmetric functions. Using this notation, the algebra of symmetric functions has some useful features.

Proposition 2.2.7. *Let λ be a partition.*

(a) There is an involutory $\mathbb{Z}$-algebra automorphism $\omega\colon \Lambda \to \Lambda$ given by any of the following formulas

$$\omega\, e_r = h_r, \quad \omega\, h_r = e_r, \quad \omega\, s_\lambda = s_{\lambda'}\,, \tag{2.4}$$

for λ' the transpose of λ.

(b) There is a symmetric bilinear inner product $\langle -, - \rangle\colon \Lambda \times \Lambda \to \mathbb{Z}$ defined by either

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}\,, \quad \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}\,, \tag{2.5}$$

for

$$\delta_{\lambda,\mu} = \begin{cases} 1 & \text{if } \lambda = \mu\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

We call this inner product the Hall inner product.

2.2.4 Schur functions

The basis of Schur functions indexed by partitions is unitriangularly related to the basis of complete homogeneous functions in the following sense.

Definition 2.2.8. For partitions λ, μ of d, we say $\lambda \leq \mu$ in *dominance order* if

$$\lambda_1 + \cdots + \lambda_r \leq \mu_1 + \cdots + \mu_r \text{ for all } r\,, \tag{2.7}$$

where the sum includes trailing zeros for $r > \ell(\lambda)$ or $r > \ell(\mu)$.

Lemma 2.2.9. *For any partition $\lambda \in \mathrm{Par}_\ell$,*

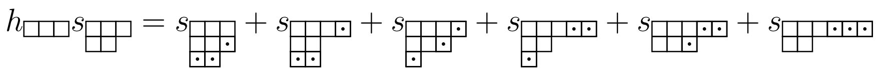
$$s_\lambda = h_\lambda + \sum_{\mu \in \mathrm{Par}_\ell, \mu \geq \lambda} a_{\lambda\mu} h_\mu \quad \text{for } a_{\lambda\mu} \in \mathbb{Z}\,. \tag{2.8}$$

Proposition 2.2.10 (Pieri rule)**.** *Let λ be a partition. Then, for any $r \in \mathbb{Z}_{\geq 0}$,*

$$h_r s_\lambda = \sum_{\substack{\mu \in \mathrm{Par}, |\mu| = |\lambda| + r \\ \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq \mu_{\ell+1}}} s_\mu \, . \tag{2.9}$$

Often, the condition in the summation in (2.9) is stated "$\mu = \lambda +$ a horizontal r-strip."

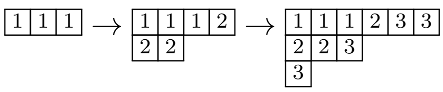
Example 2.2.11. For example, with $\lambda = (3, 2)$ and $r = 3$, added boxes are filled with $\bullet$'s below.



The Pieri rule will play a central role in the work that follows. In particular, the Pieri rule, along with the unitriangularity condition, uniquely determine the Schur functions.

Definition 2.2.12. A *semistandard Young tableau* T is a sequence of partitions $(\lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(r)})$ such that $\lambda^{(i)}/\lambda^{(i-1)}$ has no more than 1 box in any column. We say T has shape $\lambda^{(r)}/\lambda^{(0)}$ and depict T by filling the boxes of $\lambda^{(i)}/\lambda^{(i-1)}$ with the letter i. We say T has weight $(w_1, \ldots, w_r)$ where $w_i = |\lambda^{(i)}/\lambda^{(i-1)}|$. Finally, let $\mathrm{SSYT}(\lambda)$ be the set of all semistandard tableaux of shape λ.

Example 2.2.13.



Definition 2.2.14. For partitions λ, μ, the *Kostka number* $K_{\lambda\mu}$ is the number of semistandard tableaux of shape λ and weight μ.

Thus, if we iterate the Pieri rule, we get

$$h_\mu = h_{\mu_1} \cdots h_{\mu_\ell} = \sum_\lambda K_{\lambda\mu} s_\lambda$$

Furthermore, since $\langle h_\mu, s_\lambda \rangle = K_{\lambda\mu}$, we then recover $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$ by (2.5). Thus, we get the following proposition.

Proposition 2.2.15. *For any partition λ, we have*

$$s_\lambda(X) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(T)} \quad \textit{where } x^{\mathrm{wt}(T)} = \prod_{i \in T} x_i \,. \tag{2.10}$$

In other words, s_λ is the "weight generating function" of semistandard tableaux.

Thus, we can conclude that the Pieri rule uniquely determines the Schur functions with some initial condition.

Corollary 2.2.16. *The Schur functions are the unique basis of Λ satisfying* (2.8) *and* (2.9).

Corollary 2.2.16 serves as an incredibly useful tool for showing correspondences with Schur functions. For instance, the proof of Proposition 2.2.5 relies on showing that σ_λ satisfies the Pieri rule (up to a quotient). In other words, Proposition 2.2.5 follows from Corollary 2.2.16 and the fact that, for $0 \leq r \leq n$ and $\lambda \in \mathrm{Par}_m$ satisfying $\lambda_1 \leq n$,

$$\sigma_r \sigma_\lambda = \sum_{\substack{\mu \in \mathrm{Par}_m, |\mu| = |\lambda| + r \\ n \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_m \geq \lambda_m}} \sigma_\mu \,.$$

This technique will be central to the program of Chapter 3.

2.2.5 Schubert Calculus Revisited

Now, via the surjection of $\mathbb{Z}$-algebras $\phi_{m,n} \colon \Lambda(\mathbf{x}_m) \to H^*(\mathrm{Gr}(m,n))$ from Corollary 2.2.16 sending s_λ to either σ_λ or 0, we can translate the study of the structure constants of Schur functions to the study of structure constants of Schubert classes in the following sense:

$$s_\lambda s_\mu = \sum_{\nu \in \mathrm{Par}_m, \nu_1 \leq n} c^\nu_{\lambda\mu} s_\nu + \sum_{\nu \in \mathrm{Par} \setminus \{\nu \in \mathrm{Par}_m | \nu_1 \leq n\}} c^\nu_{\lambda\mu} s_\nu \mapsto \sigma_\lambda \sigma_\mu = \sum_{\nu \in \mathrm{Par}_m, \nu_1 \leq n} c^\nu_{\lambda\mu} \sigma_\nu \,.$$

Thus, the structure coefficients $c^\nu_{\lambda\mu}$ of Schur functions with λ, μ, ν inside an $m \times n$ rectangle satisfy $c^\nu_{\lambda\mu} = \#\{X_\lambda \cap X_\mu \cap X_{\nu^\vee}\}$. Moreover, due to the ubiquity of Schur functions, the coefficients $c^{\nu}_{\lambda\mu}$, known as *Littlewood-Richardson coefficients*, are well-studied and have many established combinatorial formulas (see, e.g. [Ste02, vL01]). In this way, the geometric study of intersections of Schubert varieties is translated into the combinatorial study of the multiplication of Schur functions. From our perspective, this illustrates one of the primary goals of Schubert calculus: to translate geometric intersection problems to combinatorial problems involving classes of polynomials.

2.2.6 Positivity

Due to the correspondence of Schur functions with Schubert classes of the Grassmannian, it is clear that the Littlewood-Richardson coefficients must be non-negative because they count a non-negative quantity: the number of points in a triple intersection. In this situation, we would say $s_\lambda s_\mu$ is "Schur positive" for any partitions λ, μ.

Definition 2.2.17. Given a basis of symmetric functions $\mathcal{B} = \{f_\lambda\}_{\lambda \in \mathrm{Par}}$, we say a symmetric function $g = \sum_\mu a_\mu f_\mu \in \Lambda$ *expands positively in* $\mathcal{B}$ or g is $\mathcal{B}$*-positive* if $a_\mu \in \mathbb{Z}_{\geq 0}$ for all partitions μ.

In particular, Schur-positivity is an interesting property due to the geometric significance of Schur functions, but also due to representation theoretic considerations since Schur polynomials also given irreducible SL_ℓ characters and Schur functions correspond to irreducible S_n-characters under the Frobenius characteristic.

2.3 Catalan functions

2.3.1 Raising operators

Raising operators were first informally introduced by Young to act on partitions [You32]. Informally, we will think of raising operators as acting on integer vectors as follows: for $i < j$, a raising operator R_{ij} acts on $\gamma \in \mathbb{Z}^\ell$ via $R_{ij}\gamma = \gamma + \epsilon_i - \epsilon_j$ for ϵ_i the unit coordinate vector with a 1 in entry i and 0's elsewhere. We extend this notion to an action on a symmetric function $R_{ij}f_\gamma = f_{\gamma+\epsilon_i-\epsilon_j}$, but this does not give a well-defined operator on Λ, e.g. $R_{12}s_{12} = s_{21} \neq 0$ but $s_{12} = 0$. Thus, care must be taken when working with raising operator identities so that they act on subscripts γ rather than the f_γ.

For the sake of demonstrating how to formally work with raising operators, we reproduce the proof of the following result.

Proposition 2.3.1 ([Mac15, I (3.4")]). *For $\gamma \in \mathbb{Z}^\ell$,*

$$s_\gamma = \prod_{1\le i<j\le \ell}(1 - R_{ij})h_\gamma$$

Proof. In $\mathbb{Z}[z_1^{\pm 1}, \dots, z_\ell^{\pm 1}]$, we have for $\rho = (\ell-1, \cdots, 1, 0)$,

$$\sum_{w\in S_n} \operatorname{sign}(w) z^{\gamma+\rho-w\rho} = z^{\gamma+\rho}\prod_{i<j}(z_i^{-1} - z_j^{-1}) = \prod_{i<j}\left(1 - \frac{z_i}{z_j}\right) z^\gamma \tag{2.11}$$

Then, if we apply the linear map $z^\beta \mapsto h_\beta$ to (2.11), we recover Definition 2.2.3 on the lefthand side and the desired identity on the righthand side. □

2.3.2 Root ideals

Let $\Delta_\ell^+ = \Delta^+ := \{(i,j) \mid 1 \le i < j \le \ell\}$ be the set of labels for the positive roots of the root system of type $A_{\ell-1}$, which we will often refer to as roots for brevity. For $\alpha = (i,j) \in \Delta_\ell^+$, we will write $\varepsilon_\alpha = \epsilon_i - \epsilon_j \in \mathbb{Z}^\ell$ for the corresponding positive root.

Definition 2.3.2. A *root ideal* Ψ is an upper order ideal of the poset Δ^+ with partial order given by $(a,b) \le (c,d)$ when $a \ge c$ and $b \le d$.

Note, we will also work with the complement $\Delta^+ \setminus \Psi$, a lower order ideal of Δ^+. We depict a root ideal Ψ with the boxes of Ψ shaded as in Figure 2-1.

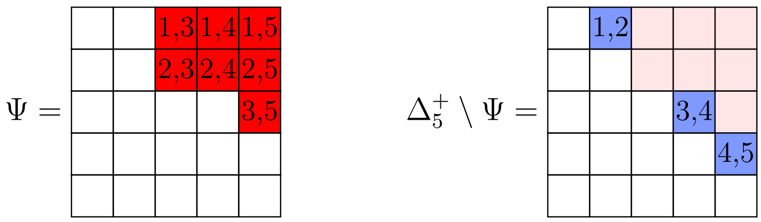


Figure 2-1: A root ideal $\Psi = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,5)\} \subseteq \Delta_5^+$ and its complement $\Delta_5^+ \setminus \Psi = \{(1,2),(3,4),(4,5)\}$.

2.3.3 Parameterless Catalan functions

The work of [BMPS19] defines a large family of symmetric functions called "Catalan functions", building off of the work of [Pan10, Che10].

Definition 2.3.3. A *(parameterless) Catalan function*, indexed by a pair (Ψ, γ) consisting of a root ideal $\Psi \subseteq \Delta_\ell^+$ and a weight $\gamma \in \mathbb{Z}^\ell$, is defined by

$$H(\Psi;\gamma) = \prod_{(i,j)\in\Delta^+\setminus\Psi} (1 - R_{ij}) h_\gamma\,, \tag{2.12}$$

where the raising operator R_{ij} acts on subscripts by $R_{ij}h_\gamma = h_{\gamma+\epsilon_i-\epsilon_j}$ and ϵ_i is the unit vector with a 1 in position i and 0's elsewhere.

Example 2.3.4. Let $\ell = 4, \Psi = \{(1,3),(1,4)\}$ and $\gamma = 2111$. Then, $\Delta_4^+ \setminus \Psi = \{(1,2),(2,3),(2,4),(3,4)\}$ and we compute $H(\Psi;\gamma)$ as follows.

$$\begin{aligned}(1-R_{12})(1-R_{23})(1-R_{24})(1-R_{34})h_{2111} &= h_{2111}\\ &\quad - h_{3011} - h_{2201} - h_{2210} - h_{2120}\\ &\quad + h_{3101} + h_{3110} + h_{3020} + h_{2300} + h_{2210} + h_{222\text{-}1}\\ &\quad - h_{3200} - h_{3110} - h_{312\text{-}1} - h_{231\text{-}1}\\ &\quad + h_{321\text{-}1}\\ &= h_{2111} - 2h_{221} + h_{32}\\ &= s_{2111} + s_{311}\end{aligned}$$

Raising operators are somewhat informal, but we will give a precise definition of a generalization of Catalan functions in Section 3.4.

In [BMPS19], Catalan functions are defined with an extra t parameter, but it is necessary to specialize $t = 1$ for applications to Schubert calculus. With this specialization, Catalan functions give a method of interpolation between the complete homogeneous symmetric functions and the Schur functions. In fact, we have the following extreme cases of Catalan functions.

Proposition 2.3.5. *Let $\gamma \in \mathbb{Z}^\ell$.*

(a) $H(\varnothing; \gamma) = s_\gamma$.

(b) $H(\Delta_\ell^+; \gamma) = h_\gamma$.

Catalan functions provide a technique for finding relationships between families of symmetric functions. In particular, Catalan functions provide a framework for expanding a Catalan function into other families of Catalan functions, such as Schur functions. Furthermore, a large class of Catalan functions is Schur positive.

Theorem 2.3.6 ([BMP20, Theorem 2.18])**.** *For any root ideal $\Psi \subseteq \Delta_\ell^+$ and partition $\mu \in \mathrm{Par}_\ell$, the Catalan function $H(\Psi; \mu)$ is Schur positive.*

In fact, [BMP20, Theorem 2.18] gives an explicit combinatorial formula for the Schur positive expansion in terms of katabolism, but we will not take this up here. For us, the key consequence is that Catalan functions give recurrence relations among a Schur positive family of symmetric functions.

2.3.4 Root ideal combinatorics and recurrences among Catalan functions

We will not directly use any results from this subsection, but present them for intuition for the K-theoretic refinements presented in § 3.4.

Definition 2.3.7. Let $\Psi \subseteq \Delta^+$ be a root ideal. We say $\alpha \in \Psi$ is a *removable root* of Ψ when $\Psi \setminus \alpha$ is a root ideal and a root $\beta \in \Delta^+ \setminus \Psi$ is *addable to* Ψ if $\Psi \cup \beta$ is a root ideal.

Then, we have the following recurrence relations among Catalan functions.

Proposition 2.3.8 (Root Expansions [BMPS19, Propositon 5.6])**.** *Let $\Psi \subseteq \Delta^+$ be a root ideal and $\mu \in \mathbb{Z}^\ell$. Then,*

(a) for any addable root β of Ψ,

$$H(\Psi;\mu) = H(\Psi \cup \beta;\mu) - H(\Psi \cup \beta;\mu + \varepsilon_\beta)\,;$$

(b) for any removable root α of Ψ,

$$H(\Psi;\mu) = H(\Psi \setminus \alpha;\mu) + H(\Psi;\mu + \varepsilon_\alpha)\,.$$

The symmetric group S_ℓ acts on subsets $\Psi \subseteq [\ell] \times [\ell]$ given by $s_i\Psi = \{(s_i(a), s_i(b) \mid (a,b) \in \Psi)\}$. Then, Catalan functions also exhibit the following relation, generalizing Proposition 2.2.4.

Proposition 2.3.9 (Straightening Rule [BMPS19, Lemma 6.1])**.** *Let $\Psi \subseteq \Delta_\ell^+$ be a root ideal such that $s_i\Psi = \Psi$. Then, for any $\gamma \in \mathbb{Z}^\ell$,*

$$H(\Psi;\gamma) + H(\Psi;\epsilon_{i+1} - \epsilon_i + s_i\gamma) = 0\,.$$

2.4 k-Schur functions

Throughout this section, fix $k \in \mathbb{Z}_{>0}$ and $\ell \in \mathbb{Z}_{\geq 0}$ throughout. Let $\mathrm{Par}_\ell^k = \{(\mu_1, \ldots, \mu_\ell) \in \mathbb{Z}^\ell \mid k \geq \mu_1 \geq \cdots \geq \mu_\ell \geq 0\}$ denote the set of partitions contained in the $\ell \times k$ rectangle and let Par^k be the set of partitions μ with $\mu_1 \leq k$. In this case, the *length* $\ell(\mu)$ is always the number of nonzero parts of μ.

In [BMPS19], the *k-Schur functions* $\{s_\mu^{(k)}\}_{\mu\in\mathrm{Par}^k}$ were identified with a subfamily of Catalan functions.

Definition 2.4.1. For $k,\ell \geq 1$ and $\mu \in \mathrm{Par}_\ell^k$, $s_\mu^{(k)} = H(\Delta^k(\mu);\mu)$ where

$$\Delta^k(\mu) = \{(i,j) \in \Delta_\ell^+ \mid k - \mu_i + i < j\}\,. \tag{2.13}$$

[BMPS19] connected this description of k-Schur functions to the combinatorial definition of $s_\lambda^{(k)}$ in [LLMS10] by showing they have the same dual Pieri rule. However, from the Catalan function description, [BMPS19] were able to settle long-standing conjectures about the t-generalizations of k-Schur functions, although we will only work with the $t = 1$ specialization throughout.

The following properties are immediate from Definition 2.4.1 of a k-Schur function.

Proposition 2.4.2. *For $k \geq 1$ and $\lambda \in \mathrm{Par}^k$,*

(a) $s_\lambda^{(k)} \in \Lambda_{(k)} = \mathbb{Z}[h_1,\ldots,h_k]$ *and* $\{s_\lambda^{(k)}\}_{\lambda\in\mathrm{Par}^k}$ *forms a basis of* $\Lambda_{(k)}$.

(b) $s_\lambda^{(k)} = s_\lambda$ *if* $\ell(\lambda) + \lambda_1 - 1 \geq k$.

One major discovery of [BMPS19] was the property of Shift Invariance.

Proposition 2.4.3 (Shift Invariance [BMPS19, (2.6)]). *For $\lambda \in \mathrm{Par}_\ell^k$ and $e_\ell^\perp : \Lambda \to \Lambda$ the dual of multiplication by e_ℓ under the Hall inner product,*

$$s_\lambda^{(k)} = e_\ell^\perp s_{\lambda+1^\ell}^{(k+1)}\,. \tag{2.14}$$

This property connects the expansion of $s_\lambda^{(k)}$ into the $\{s_\mu^{(k+1)}\}_{\mu\in\mathrm{Par}^{k+1}}$ basis to the combinatorially understood dual Pieri rule, discussed in 2.4.4. In particular, we have the following corollary.

Corollary 2.4.4 ([BMPS19, Theorem 2.6] (weakened)). *For $k \geq 1$ and $\lambda \in \mathrm{Par}_\ell^k$,*

$$s_\lambda^{(k)} = e_\ell^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_{\mu\in\mathrm{Par}_\ell^{k+1}} b_{\lambda\mu} s_\mu^{(k+1)} \tag{2.15}$$

where $b_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ are given by the dual Pieri rule for k-Schur functions, discussed in 2.4.4.

2.4.1 Affine symmetric group

The *affine symmetric group*, $\widehat{S}_n$, is the group generated by generators $\langle s_i \mid i \in \mathbb{Z}/n\mathbb{Z}\rangle$ subject to the relations

$$s_i^2 = id, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \text{ for } i - j \not\equiv 0, \pm 1$$

with all indices considered modulo n. From this presentation, it is immediate that the affine symmetric group is a Coxeter group and thus general notions of reduced word, length, and Bruhat order apply. For the sake of completeness, we will define these properties in this special case.

Definition 2.4.5. (a) The *length* $\ell(w)$ of $w \in \widehat{S}_n$ is the minimum m such that $w = s_{i_1} s_{i_2} \cdots s_{i_m}$ for some $i_j \in I$; any expression for w with $\ell(w)$ generators is said to be *reduced*.

(b) Let $S_n \leq \widehat{S}_n$ be generated by $\langle s_1, \ldots, s_{n-1}\rangle$. Then, the set of *affine Grassmannian elements*, $\widehat{S}_n^0$, is the set of the minimal length coset representatives of $\widehat{S}_n/S_n$. Equivalently, $w \in \widehat{S}_n^0$ if every reduced expression of w has rightmost letter s_0.

(c) (Bruhat order) For $w, v \in \widehat{S}_n$, we say $w \leq v$ if every reduced expression of v contains a subexpression that is a reduced expression of w.

We also note that S_n carries all the same relevant definitions of reduced expression, length, and Bruhat order by being embedded as $\langle s_1, \ldots, s_n\rangle \leq \widehat{S}_n$. Also, unlike $\widehat{S}_n$, the symmetric group is finite. Thus, S_n has unique maximal length element $w_0 = s_1(s_2 s_1) \cdots (s_{n-1} s_{n-2} \cdots s_2 s_1)$.

2.4.2 Affine combinatorics

For a fixed $k \geq 1$, there are some fundamental bijections between various combinatorial objects that underly the combinatorics of k-Schur functions. Broadly speaking, we have

$$\{\text{affine Grassmannian words of } \widehat{S}_{k+1}\} \leftrightarrow \{(k+1)\text{-cores}\} \leftrightarrow \mathrm{Par}^k \ .$$

Definition 2.4.6. We define the following relevant notions.

(a) For any cell in (the diagram of) a partition, the *hook length* counts the number of cells below it in its column and the number of cells weakly to its right in its row. Explicitly, recalling that λ' is the transpose of λ, we say cell (i, j) has hook length $\lambda_i + \lambda'_j - i - j + 1$.

(b) An n-core is a partition such that none of the cells have hook length n. We use $\mathcal{C}^n$ to denote the collection of n-cores.

Proposition 2.4.7 ([LM05])**.** *We have the following bijections.*

(a) There is a bijection $\mathfrak{w} \colon \mathrm{Par}^k \to \widehat{S}^0_{k+1}$ *given by* $\lambda \mapsto w_\lambda$ *for*

$$w_\lambda = (s_{\lambda_\ell - \ell} \cdots s_{-\ell+1}) \cdots (s_{\lambda_2 - 2} \cdots s_{-1})(s_{\lambda_1 - 1} \cdots s_0)$$

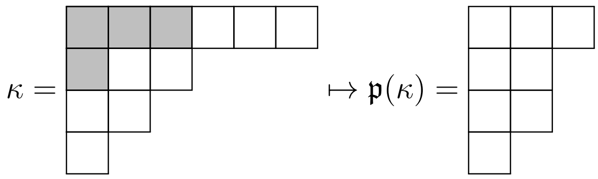
where $\ell = \ell(\lambda)$ *(see* [LM05, §8.2]*).*

(b) ([LM05, Theorem 7]*) There is a bijection* $\mathfrak{p} \colon \mathcal{C}^{k+1} \to \mathrm{Par}^k$ *where* $\mathfrak{p}(\kappa) = \lambda$ *is the partition whose* r*-th row,* λ_r*, is the number of cells in the* r*-th row of* κ *with hook length* $\leq k$*.*

Example 2.4.8. We have the following examples of the bijections for $k = 3$.

(a) For $\lambda = 3221$, we have $\mathfrak{w}(\lambda) = w_{3221} = s_1\, s_3 s_2\, s_0 s_3\, s_2 s_1 s_0$.

(b) For $\kappa = 6321$, we have $\mathfrak{p}(\kappa) = 3221$.



2.4.3 Schubert representatives

As with Schur functions, a Schubert calculus interpretation of k-Schur functions is known. Specifically, by [Lam08], the k-Schur functions serve as the homology Schubert representatives for the "affine Grassmannian" of SL_{k+1}.

Definition 2.4.9. Fix $n > 2$. Let $F = \mathbb{C}((t))$ and $O = \mathbb{C}[\![t]\!]$. Then, $\mathrm{Gr} = \mathrm{Gr}_{\mathrm{SL}_n} = \mathrm{SL}_n(F)/\mathrm{SL}_n(O)$ is the *affine Grassmannian.*

Then, just as in classical Schubert calculus, the affine Grassmannian exhibits a Schubert cell decomposition

$$\mathrm{Gr} = \bigsqcup_{w \in \widehat{S}_n^0} \Omega_w = \bigcup_{w \in \widehat{S}_n^0} X_w \,, \tag{2.16}$$

where $X_w = \overline{\Omega_w}$ are Schubert varieties. In fact, this result follows from the Bruhat decomposition of $\mathrm{SL}_n(F)$ and works for general Kac-Moody groups. We will let $\sigma_w \in H_*(\mathrm{Gr})$ be the Schubert class in homology. Note that $H^*(\mathrm{Gr})$ and $\Lambda_{(k)}$ are both Hopf algebras. Then, we have the following theorem, analogous to the classical Schubert calculus setting.

Theorem 2.4.10 ([Lam08, Theorem 7.1])**.** *There is a Hopf algebra isomorphism $\theta\colon H_*(\mathrm{Gr}_{\mathrm{SL}_{k+1}}) \to \Lambda_{(k)}$; the homology Schubert basis element σ_{w_λ} satisfies $\theta(\sigma_{w_\lambda}) = s_\lambda^{(k)}$, for $\lambda \in \mathrm{Par}^k$.*

2.4.4 k-Schur combinatorics

Just as with Schur functions, k-Schur functions have a number of useful and well-known combinatorial formulas, far beyond what we will list here. However, one notable fact is the (vertical) Pieri rule for k-Schur functions. To state it, we first need the following combinatorial definitions.

Definition 2.4.11. An element $w \in \widehat{S}_{k+1}$ is *cyclically increasing* if it can be written as $w = s_{i_1} s_{i_2} \cdots s_{i_m}$, for distinct indices i_j such that an index i never occurs to the east of an $i+1$ (modulo $k+1$).

Proposition 2.4.12 (Pieri rule [Lam08]). *The $\{s_\lambda^{(k)}\}_{\lambda \in \mathrm{Par}^k}$ for a basis for $\Lambda_{(k)}$ and satisfy the following Pieri rule for all $r \in [k]$:*

$$s_{1^r} s_\lambda^{(k)} = \sum_{\substack{u \in \widehat{S}_{k+1} \text{ cyclically increasing} \\ \ell(u)=r \\ u w_\lambda = w;\, w \in \widehat{S}^0_{k+1} \\ \ell(w) = \ell(w_\lambda)+r}} s^{(k)}_{\mathfrak{w}^{-1}(w)}\,. \tag{2.17}$$

Moreover, the $\{s_\lambda^{(k)}\}_{\lambda \in \mathrm{Par}^k}$ are the unique elements of $\Lambda_{(k)}$ satisfying (2.17) for all $r \in [k]$.

Proposition 2.4.12 should be thought of as the analogue to Proposition 2.2.10 and Corollary 2.2.16 for Schur functions. Thus, in principle, (2.17) could be taken as the definition of $s_\lambda^{(k)}$, and this was a historical definition. However, [BMPS19] showed that (2.17) and Definition 2.4.1 are equivalent since both families of functions satisfy the same dual Pieri rule, which is a combinatorial expression for $e_r^\perp s_\lambda^{(k)}$ in terms of k-Schur functions (Proposition 2.4.15, below). In principle, one could show Definition 2.4.1 satisfies (2.17) directly, which will be a consequence of our results in Chapter 3, but proving the equivalence via the dual Pieri rule suited the purposes of [BMPS19] better with the presence of the generic t parameter.

Definition 2.4.13 ([BMPS19, §2]). We define the following notions on $k+1$-cores.

(a) A *strong cover* $\tau \Longrightarrow \kappa$ is a pair of $k+1$-cores such that $\tau \subseteq \kappa$ and $|\mathfrak{p}(\tau)|+1 = |\mathfrak{p}(\kappa)|$.

(b) A *strong marked cover* $\tau \overset{r}{\Longrightarrow} \kappa$ is a strong cover $\tau \Longrightarrow \kappa$ together with a positive integer r which is allowed to be the smallest row index of any connected component of the skew shape κ/τ.

(c) For $\eta = (\eta_1, \eta_2, \dots) \in \mathbb{Z}^\infty_{\geq 0}$ with $m = |\eta| := \sum_i \eta_i$ finite. A *strong marked tableau* T

of weight η is a sequence of strong marked covers

$$\kappa^{(0)} \xrightarrow{r_1} \kappa^{(1)} \xrightarrow{r_1} \cdots \xrightarrow{r_m} \kappa^{(m)}$$

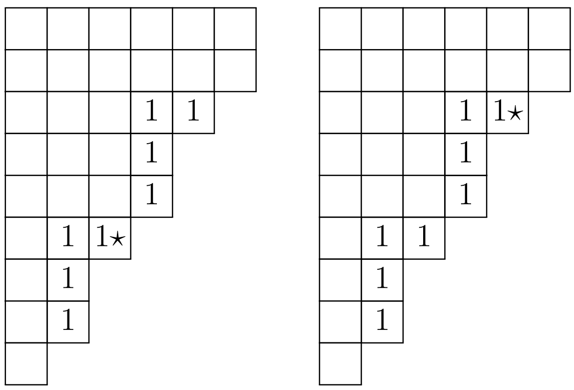
such that $r_{v_i+1} \geq r_{v_1+2} \geq \cdots \geq r_{v_i+\eta_i}$ for all $i \geq 1$, where $v_i = \eta_1 + \cdots + \eta_{i-1}$.

(d) A *vertical strong marked tableau* is defined the same way except we require $r_{v_1+1} < \cdots < r_{v_i+\eta_i}$ to be strictly increasing rather than weakly decreasing.

(e) We say $\text{inside}(T) = \mathfrak{p}(\kappa^{(0)})$ and $\text{outside}(T) = \mathfrak{p}(\kappa^{(m)})$.

(f) The set of strong marked tableaux (resp. vertical strong marked tableaux) T of weight η with $\text{outside}(T) = \mu$ is denoted $\text{SMT}^k_\eta(\mu)$ (resp. $\text{VSMT}^k_\eta(\mu)$).

Example 2.4.14 ([BMPS19, Example 2.11]). For $k = 4$, let $\tau = 663331111$ and $\kappa = 665443221$. Then, $\mathfrak{p}(\tau) = 332221111$ and $\mathfrak{p}(\kappa) = 222222221$. Then, $\tau \Longrightarrow \kappa$ is a strong cover and it has two distinct markings:



$\tau \xrightarrow{6} \kappa$ $\qquad$ $\tau \xrightarrow{3} \kappa$

Proposition 2.4.15 (Dual (Vertical) Pieri Rule [LLMS10]). *For $k \geq 1, d \leq k$, and $\lambda \in \text{Par}^k$,*

$$e_d^{\perp} s_\lambda^{(k)}(x) = \sum_{T \in \text{VSMT}^k_{(d)}(\lambda)} s^{(k)}_{\text{inside}(T)}. \tag{2.18}$$

Note, via some algebraic manipulations involving Newton's identities, this is equivalent to giving a weight generating function for $s_\lambda^{(k)}$, but we will not take this up here.

Then, since $s_\lambda^{(k)} = e_\ell^\perp s_{\lambda+1^\ell}^{(k+1)}$ by (2.14), we get the following immediate corollary to Proposition 2.4.15.

Corollary 2.4.16 (Branching Rule [BMPS19, Theorem 2.6])**.** *For $k \geq 1$ and $\lambda \in \mathrm{Par}_\ell^k$, we have*

$$s_\lambda^{(k)} = \sum_{T \in \mathrm{VSMT}_{(\ell)}^{k+1}(\lambda+1^\ell)} s_{\mathrm{inside}(T)}^{(k+1)} .$$

Then, since $s_\mu^{(k)} = s_\mu$ for all k sufficiently large by Proposition 2.4.2(b), we arrive at the Schur expansion of a $s_\lambda^{(k)}$ by noting that $s_\lambda^{(k)} = (e_\ell^\perp)^m s_{\lambda+m^\ell}^{(k+m)}$.

Corollary 2.4.17 (Schur expansion of k-Schur functions [BMPS19, Theorem 2.7])**.** *For $k \geq 1$ and $\lambda \in \mathrm{Par}_\ell^k$, set $m = \max(|\lambda| - k, 0)$. Then, the Schur expansion of $s_\lambda^{(k)}$ is given by*

$$s_\lambda^{(k)} = \sum_{T \in \mathrm{VSMT}_{(\ell^m)}^{k+m}(\lambda+m^\ell)} s_{\mathrm{inside}(T)} . \tag{2.19}$$

Prior to the work of [BMPS19], the branching coefficients were known to be positive for parameterless k-Schur functions for geometric reasons reflecting the fact that the image of a Schubert class is a positive sum of Schubert classes under an inclusion $H^*(\mathrm{Gr}_{\mathrm{SL}_{k+1}}) \to H^*(\mathrm{Gr}_{\mathrm{SL}_{k+2}})$ [Lam11]. However, the argument of [LLMS13] does not extend to the general case for k-Schur functions with parameter t. The raising operator proof techniques of [BMPS19] will also be better suited for our purposes in Chapter 3.

Chapter 3

Katalan functions and K-k-Schur functions

In this chapter, we present a K-theoretic refinement to the material presented in Chapter 2. Generalizing the cohomology of the Grassmannian, Schubert representatives for the K-theory and K-homology of the Grassmannian are well-known. Furthermore, some partial progress was made on the K-theory and K-homology of the affine Grassmannian, primarily due to work in [LSS10b, Mor12]. This chapter resolves conjectures made in [LSS10b, Mor12] by providing an explicit formula for the K-homology representatives of the affine Grassmannian, called K-k-Schur functions, and fitting them into a broader family of symmetric functions we call Katalan functions. Most of the material in this chapter was originally presented in [BMS20].

3.1 Grothendieck polynomials and their duals

Grothendieck polynomials indexed by a permutation $w \in S_n$ were first defined by Lascoux and Schutzenberger. We will give a brief treatment of them in this generality in § 4.2.1. For now, we will concern ourselves only with those indexed by "Grassmannian permutations", which are in bijection with partitions. In [Buc02], it is shown that these Grothendieck polynomials are given as a weight generating function over "set-valued tableaux." We use the definition as stated in [Mor12].

Definition 3.1.1. (a) Given a partition λ, a *set-valued tableau* is a filling of the boxes of λ with sets such that any set X west of a set Y satisfies $\max X \leq \min Y$ and any set X north of a set Y satisfies $\max X < \min Y$.

(b) The weight of a set-valued tableau T is the integer vector $\mathrm{wt}(T) = (w_1, w_2, \ldots)$ is such that w_i counts the number of occurrences i in T.

(c) The *Grothendieck polynomial* indexed by a partition λ is given by the formula

$$G_\lambda = G_\lambda(X) = \sum_{\substack{T \text{ set valued} \\ \mathrm{shape}(T)=\lambda}} (-1)^{|\,\mathrm{wt}(T)|+|\lambda|} x^{\mathrm{wt}(T)} \tag{3.1}$$

The formula (3.1) for Grothendieck polynomials is technically infinite series, and thus we must consider the G_λ as elements of the graded completion of Λ, given by $\hat{\Lambda} = \prod_{\lambda \in \mathrm{Par}} \mathbb{Z} m_\lambda$. This presents a number of subtleties, but throughout this chapter, we will only ever use the following identity of Grothendieck polynomials following from the set-valued tableaux description. For any $m \in \mathbb{Z}_{>0}$,

$$G_{1^m} = \sum_{i \geq 0} (-1)^i \binom{m+i-1}{m-1} e_{m+i}\,. \tag{3.2}$$

We now move on to a discussion of dual Grothendieck polynomials. For $m, r \in \mathbb{Z}$, define

$$k_m^{(r)} = \sum_{i=0}^{m} \binom{r+i-1}{i} h_{m-i}\,, \tag{3.3}$$

where $\binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!}$ and $\binom{n}{0} = 1$ for $n \in \mathbb{Z}, i \in \mathbb{Z}_{\geq 1}$; thus, note that $k_m^{(0)} = h_m$ and $k_m^{(r)} = 0$ when $m < 0$. For $\gamma \in \mathbb{Z}^\ell$, we let $k_\gamma = k_{\gamma_1}^{(0)} k_{\gamma_2}^{(1)} \cdots k_{\gamma_\ell}^{(\ell-1)}$ and

$$g_\gamma = \det(k_{\gamma_i+j-i}^{(i-1)})_{1 \leq i,j \leq \ell}\,. \tag{3.4}$$

When γ is a partition, these are the *dual stable Grothendieck polynomials*, first studied implicitly in [Len00] and determinantally formulated in [LN14]. Note the following properties of these symmetric functions.

Proposition 3.1.2. *Let* $\lambda \in \mathrm{Par}$*. Then, we have*

$$k_\lambda = h_\lambda + \sum_{\substack{\mu \in \mathrm{Par} \\ |\mu|<|\lambda|}} a_{\lambda\mu} h_\mu \tag{3.5}$$

for some coefficients $a_{\lambda\mu} \in \mathbb{Z}$*. Similarly,*

$$g_\lambda = s_\lambda + \sum_{\substack{\mu \in \mathrm{Par} \\ |\mu|<\lambda}} b_{\lambda\mu} s_\mu \tag{3.6}$$

for some coefficients $b_{\lambda\mu} \in \mathbb{Z}$*. Thus,* $\{k_\mu\}_{\mu\in\mathrm{Par}}$ *and* $\{g_\mu\}_{\mu\in\mathrm{Par}}$ *are both bases of* Λ*.*

Furthermore, we can give a raising operator formula for dual Grothendieck polynomials.

Proposition 3.1.3. *For* $\gamma \in \mathbb{Z}^\ell$*,*

$$g_\gamma = \prod_{i<j} (1 - R_{ij}) k_\gamma \tag{3.7}$$

Proof. If we apply the map $z^\beta \mapsto k_\beta$ to (2.11), we recover (3.4) on the lefthand side and the desired identity on the righthand side. □

Remark 3.1.4. The dual Grothendieck polynomials are the Schubert representatives for the K-homology of the Grassmannian. To make this precise, see, e.g., [LP07].

Due to this geometric significance, as well as the resemblance of (3.4) to (2.3) and that the top homogeneous degree component of g_λ is s_λ, we can think of g_λ as an inhomogeneous analogue of s_λ. Using the formal definition of raising operators, we will give a straightening relation (analogous to Proposition 2.2.4(c)) for g_γ in Proposition 3.4.5.

3.2 Positivity and alternating properties

In the K-theoretic setting, it becomes natural to examine positivity that is signed by degree.

Definition 3.2.1. Given a basis $\mathcal{B} = \{f_\mu\}_{\mu \in \mathrm{Par}}$ of Λ, we say a symmetric function $A = \sum_\mu a_\mu f_\mu \in \mathcal{B}$ with highest degree d is *$\mathcal{B}$-alternating* if $(-1)^{d-|\mu|} a_\mu \in \mathbb{Z}_{\geq 0}$ for all $\mu \in \mathrm{Par}$.

Proposition 3.2.2. *For any $\lambda \in \mathrm{Par}$, we have the following.*

(a) s_λ is g_μ-alternating.

(b) g_λ is Schur positive, i.e., the coefficients $b_{\lambda\mu}$ in (3.6) lie in $\mathbb{Z}_{>0}$.

In fact, the expansions of Schur functions into dual Grothendieck polynomials and vice-versa are governed by the combinatorics of "elegant fillings". See, e.g., [Len00, LP07].

3.3 K-k-Schur functions and affine Grothendieck polynomials

After the discovery that k-Schur functions serve as the Schubert basis for the homology of the affine Grassmannian, [LSS10b, Mor12] simultaneously explored the Schubert representatives for the K-theory and K-homology of the affine Grassmannian. In this setting, the role of the affine symmetric group is replaced by the 0-Hecke algebra.

Definition 3.3.1. The *0-Hecke algebra* H_{k+1} is the free $\mathbb{Z}$-algebra generated by $\{T_i \mid i \in \mathbb{Z}/(k+1)\mathbb{Z}\}$ with the same relations as $\widehat{S}_{k+1}$ (see § 2.4.1) except $T_i^2 = -T_i$ in place of $s_i^2 = id$. It has a $\mathbb{Z}$-basis $\{T_w \mid w \in \widehat{S}_{k+1}\}$, where $T_w = T_{i_1} T_{i_2} \cdots T_{i_m}$ for any reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_m}$.

Definition 3.3.2. For $k \in \mathbb{Z}_{>0}$ and $\lambda \in \mathrm{Par}^k$, we define the *affine Grothendieck polynomial* $G_\lambda^{(k)}$ as follows. For any $\mu \in \mathrm{Par}_a$, the coefficient of m_μ in $G_\lambda^{(k)}$ is equal to $(-1)^{|\mu|-|\lambda|}$

times the number of factorizations $T_{w_\lambda} = \pm T_{u_1} \cdots T_{u_a}$ in H_{k+1} for cyclically decreasing words $u_1, \ldots, u_a$ of lengths $\mu_1, \ldots, \mu_a$.

Note that $G_\lambda^{(k)}$, like G_λ, is an infinite series and thus must be thought of as living in the graded completion of $\hat{\Lambda}^{(k)}$, denoted $\hat{\Lambda}^{(k)}$.

Proposition 3.3.3 ([LSS10b, Theorem 7.17(3)]). *There is a dual Hopf algebra isomorphism $K^*(\mathrm{Gr}) \to \hat{\Lambda}^{(k)}$ sending the class of the structure sheaf of a Schubert variety indexed by w_λ to $G_\lambda^{(k)}$.*

Furthermore, [Mor12] gives a weight generating function description of $G_\lambda^{(k)}$ as a sum over "affine set-valued tableaux", simultaneously generalizing strong marked tableaux and set-valued tableaux. Then, examining the dual of $G_\lambda^{(k)}$ under the Hall inner product, the work of [LSS10b, Mor12] together yields the following theorem.

Theorem 3.3.4. *There is a Hopf algebra isomorphism $\Theta\colon K_*(\mathrm{Gr}_{\mathrm{SL}_{k+1}}) \to \Lambda_{(k)}$; the K-homology Schubert basis element $\xi^0_{w_\lambda}$ has image denoted $g_\lambda^{(k)} = \Theta(\xi^0_{w_\lambda})$, for $\lambda \in \mathrm{Par}^k$. The $\{g_\lambda^{(k)}\}_{\lambda\in\mathrm{Par}^k}$ form a basis for $\Lambda_{(k)}$ and satisfy the following Pieri rule for all $r \in \{0, 1, \ldots, k\}$:*

$$g_{1^r} g_\lambda^{(k)} = \sum_{\substack{u\in\widehat{S}_{k+1} \text{ cyclically increasing} \\ \ell(u)=r \\ T_uT_{w_\lambda}=\pm T_w;\, w\in\widehat{S}^0_{k+1}}} (-1)^{\ell(w_\lambda)+r-\ell(w)} g^{(k)}_{\mathfrak{w}^{-1}(w)} \,. \tag{3.8}$$

Moreover, the $\{g_\lambda^{(k)}\}_{\lambda\in\mathrm{Par}^k}$ are the unique elements of $\Lambda_{(k)}$ satisfying (3.8) for all $r \in [k]$.

Note, unlike with k-Schur functions, the dual Pieri rule (analogous to Proposition 2.4.15) and a weight generating function description of $g_\lambda^{(k)}$ remain unknown. Furthermore, (3.8) is (essentially) the only symmetric function theoretic definition of $g_\lambda^{(k)}$ presented in [LSS10b, Mor12], but it is an indirect formulation. This served as an obstacle to proving many other conjectured properties about $g_\lambda^{(k)}$ analogous to ones for k-Schur functions, such as the positivity of branching coefficients (see Corollary 2.4.16 for the k-Schur case).

Conjecture 3.3.5 ([LSS10b, Conjecture 7.20(3)], [Mor12, Conjecture 44])**.** *For any partition* $\lambda \in \mathrm{Par}^k$*,*

$$g_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} g_\mu^{(k+1)} \text{ satisfies } (-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}\,. \tag{3.9}$$

Just as k-Schur functions become Schur functions as k gets sufficiently large, K-k-Schur functions become dual Grothendieck polynomials for k sufficiently large.

Proposition 3.3.6 ([Mor12])**.** *For* $k \geq 1$ *and* $\lambda \in \mathrm{Par}^k$ *such that* $k \geq |\lambda|$*,* $g_\lambda^{(k)} = g_\lambda$*.*

However, this bound is not as tight as the one for k-Schur functions in Proposition 2.4.2. We will prove a tighter bound than Proposition 3.3.6 for K-k-Schur functions in Corollary 3.7.3. More generally, the aim of this chapter is to show that there exists a direct raising operator description of $g_\lambda^{(k)}$, which can be leveraged to resolve the conjecture above.

Later, we will make use of the following technical lemma.

Lemma 3.3.7. *For any* $k \in \mathbb{Z}_{>0}$*,* $m \in \mathbb{Z}_{\geq 0}$*, we have* $G_{1^m}^{(k)} = G_{1^m}$*.*

Proof. We have $w_{1^m} = s_{-m+1} \cdots s_{-1} s_0$ with indices taken modulo $k+1$. Since no braid or commutations relations can be applied to this word, the only factorizations of $T_{w_{1^m}}$ of the above form are

$$T_{w_{1^m}} = \pm T_{-m+1}^{a_m} \cdots T_{-1}^{a_2} T_0^{a_1}\,, \quad a_i \geq 1\,,$$

each u_i being a simple reflection. Thus the coefficient of m_μ in $G_{1^m}^{(k)}$ is 0 unless $\mu = (1^a)$ for $a \geq m$. For each such a, there are exactly $\binom{a-1}{m-1}$ possible factorizations. Therefore

$$G_{1^m}^{(k)} = \sum_{a \geq m} (-1)^{a-m} \binom{a-1}{m-1} m_{1^a} = \sum_{i \geq 0} (-1)^i \binom{m+i-1}{m-1} e_{m+i} = G_{1^m}\,,$$

where the last equality is a well-known formula for G_{1^m}. □

3.4 Basic properties of K-theoretic Catalan functions

We use the notation $[a, b]$ for $\{i \in \mathbb{Z} \mid a \leq i \leq b\}$ and $[n] = [1, n]$. Throughout, we say a *multiset* M *on* $[\ell]$ is a multiset whose support is contained in $[\ell]$; its multiplicity function is denoted $m_M \colon [\ell] \to \mathbb{Z}_{\geq 0}$.

Definition 3.4.1. Given a root ideal $\Psi \subseteq \Delta_\ell^+$, a multiset on $[\ell]$, and a weight $\gamma \in \mathbb{Z}^\ell$, we define the *Katalan function*

$$K(\Psi; M; \gamma) = \prod_{j \in M} (1 - L_j) \prod_{(i,j) \in \Psi} (1 - R_{ij}) g_\gamma \,. \tag{3.10}$$

As discussed in Section 2.3.1, raising operators are informal and are not well-defined operators on Λ despite their name. The formal interpretation of Definition 3.4.1 is as follows: set $\mathbb{A} = \mathbb{Z}[\![\frac{z_1}{z_2}, \ldots, \frac{z_{\ell-1}}{z_\ell}]\!][z_1^{\pm 1}, \ldots, z_\ell^{\pm 1}]$, an arbitrary element of which has the form $\sum_{\gamma \in \mathbb{Z}^\ell} c_\gamma \mathbf{z}^\gamma$ where the support $\{\gamma \in \mathbb{Z}^\ell \mid c_\gamma \neq 0\}$ is contained in $Q^+ + F$ for some finite subset $F \subseteq \mathbb{Z}^\ell$, where $Q^+ := \mathbb{Z}_{\geq 0}\{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{\ell-1} - \epsilon_\ell\} \subseteq \mathbb{Z}^\ell$. For a root ideal $\Psi \subseteq \Delta_\ell^+$, multiset M on $[\ell]$, and $\gamma \in \mathbb{Z}^\ell$,

$$K(\Psi; M; \gamma) = g\left(\prod_{(i,j) \in \Psi} \left(1 - \frac{z_i}{z_j}\right)^{-1} \prod_{j \in M} \left(1 - \frac{1}{z_j}\right) \mathbf{z}^\gamma \right), \tag{3.11}$$

where $g \colon \mathbb{A} \to \mathbb{Z}[h_1, h_2, \ldots]$ is defined by $\sum_{\gamma \in \mathbb{Z}^\ell} c_\gamma \mathbf{z}^\gamma \mapsto \sum_{\gamma \in \mathbb{Z}^\ell} c_\gamma g_\gamma$; note that by (3.4), $g_\gamma = 0$ when $\gamma_i < i - \ell$, and hence $\sum_\gamma c_\gamma g_\gamma$ has finitely many nonzero terms and so indeed lies in $\mathbb{Z}[h_1, h_2, \ldots]$.

Further, defining $\kappa \colon \mathbb{A} \to \mathbb{Z}[h_1, h_2, \ldots]$ by $\sum_{\gamma \in \mathbb{Z}^\ell} c_\gamma \mathbf{z}^\gamma \mapsto \sum_{\gamma \in \mathbb{Z}^\ell} c_\gamma k_\gamma$, it follows from (3.4) that

$$g(f) = \kappa\Big(\prod_{1 \leq i < j \leq \ell} \left(1 - \frac{z_i}{z_j}\right) \cdot f \Big) \tag{3.12}$$

for all $f \in \mathbb{A}$. This yield the following proposition.

Proposition 3.4.2. *For a root ideal $\Psi \subseteq \Delta_\ell^+$, a multiset M with $\operatorname{supp}(M) \subseteq \{1, \ldots, \ell\}$, and $\gamma \in \mathbb{Z}^\ell$,*

$$K(\Psi; M; \gamma) = \prod_{j \in M} (1 - L_j) \prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - R_{ij})\, k_\gamma\,. \tag{3.13}$$

Using formulas (3.10) and (3.13), we can look at extremal cases of Katalan functions.

Proposition 3.4.3. *Let $\gamma \in \mathbb{Z}^\ell$.*

(a) The Katalan functions contain the family of Catalan functions: $K(\Psi; \Delta_\ell^+; \gamma) = H(\Psi; \gamma)$ for any root ideal $\Psi \subseteq \Delta_\ell^+$. In particular, $K(\varnothing; \Delta_\ell^+; \gamma) = s_\gamma$ and $K(\Delta_\ell^+; \Delta_\ell^+; \gamma) = h_\gamma$.

(b) $K(\varnothing; \varnothing; \gamma) = g_\gamma$.

(c) $K(\Delta_\ell^+; \varnothing; \gamma) = k_\gamma$.

Proof. Statement (b) is immediate from Definition 3.4.1 and (c) is immediate from Proposition 3.4.2. To prove (a), for $m, r \in \mathbb{Z}$, we note that, by Pascal's formula,

$$k_{m-1}^{(r)} + k_m^{(r-1)} = \sum_{i=0}^{m} \left[\binom{r+i-2}{i-1} + \binom{r+i-2}{i} \right] h_{m-i} = \sum_{i=0}^{m} \binom{r+i-1}{i} h_{m-i} = k_m^{(r)}\,. \tag{3.14}$$

Therefore, $\prod_{(i,j)\in\Delta^+}(1-L_j)k_\gamma = h_\gamma$ and thus (a) follows from Proposition 3.4.2 and (2.12). ☐

Although Katalan functions are defined for arbitrary multisets, in this chapter we mainly work with those where the associated multiset comes from a root ideal $\mathcal{L} \subseteq \Delta_\ell^+$ via the function

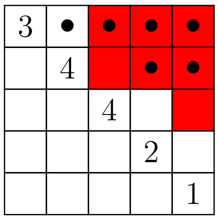
$$L(\mathcal{L}) = \bigsqcup_{(i,j) \in \mathcal{L}} \{j\}\,. \tag{3.15}$$

In this scenario, we use the shorthand $K(\Psi; \mathcal{L}; \gamma) = K(\Psi; L(\mathcal{L}); \gamma)$.

Given a root ideal $\Psi \subseteq \Delta_\ell^+$, a multiset M on $[\ell]$, and $\gamma \in \mathbb{Z}^\ell$, we represent the Katalan function $K(\Psi; M; \gamma)$ by the $\ell \times \ell$ grid of boxes (labeled by matrix-style coordinates) with the boxes of Ψ shaded, $m_M(a)$ $\bullet$'s in column a (assuming $m_M(a) < a$), and the entries of γ written along the diagonal.

Example 3.4.4. Let $\Psi = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,5)\} \subseteq \Delta_5^+$, $M = \{2,3,4,4,5,5\}$, and $\gamma = (3,4,4,2,1)$. Then, $K(\Psi;M;\gamma)$ is depicted by:

$$K(\Psi;M;\gamma) =$$



The symmetric group S_ℓ acts on the ring $\mathbb{A}$ by permuting the z_i. In particular, the simple reflections $s_1, \ldots, s_{\ell-1}$ act by $s_i\big(\sum_\gamma c_\gamma \mathbf{z}^\gamma\big) = \sum_\gamma c_\gamma \mathbf{z}^{s_i\gamma}$, where

$$s_i\gamma = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \gamma_i, \gamma_{i+2}, \ldots).$$

We also consider an action of S_ℓ on subsets $\Psi \subseteq [\ell] \times [\ell]$ defined by $s_i\Psi = \{(s_i(a), s_i(b)) \mid (a,b) \in \Psi\}$, and an action on multisets M on $[\ell]$ with s_iM defined by its multiplicity function $m_{s_iM}(a) = m_M(s_i(a))$ for all $a \in [\ell]$.

Proposition 3.4.5. *For any $\gamma \in \mathbb{Z}^\ell$, $g_\gamma - g_{\gamma-\epsilon_{i+1}} = g_{s_i\gamma-\epsilon_i} - g_{s_i\gamma+\epsilon_{i+1}-\epsilon_i}$. Hence, we have the operator identity*

$$g \circ \left(1 - \frac{1}{z_{i+1}}\right)\left(1 + \frac{z_{i+1}}{z_i}s_i\right) = 0.$$

Proof. Using the definition $g_\gamma = \det(k^{(i-1)}_{\gamma_i+j-i})_{1\le i,j\le \ell}$, we can write $g_\gamma - g_{\gamma-\epsilon_{i+1}} = \det(A)$, for A the matrix whose $i+1$-st row is $(k^{(i)}_{\gamma_{i+1}+j-i-1} - k^{(i)}_{\gamma_{i+1}+j-i-2})_{j\in[\ell]}$ and whose other rows agree with the matrix defining g_γ; similarly, we can write $g_{s_i\gamma+\epsilon_{i+1}-\epsilon_i} - g_{s_i\gamma-\epsilon_i} = \det(A')$. Simplifying the $i+1$-st rows of A and A' using (3.14), we see that A and A' differ by swapping their i and $i+1$-st rows. The result follows. $\square$

We give a Katalan function analogue to the Catalan function straightening relation Proposition 2.3.9.

Lemma 3.4.6. *Let $\Psi \subseteq \Delta^+$ be any root ideal and M on $[\ell]$ be any multiset such that*

(a) $s_i\Psi = \Psi$ *and*

(b) $m_M(i+1) = m_M(i) + 1$.

Then, for any $\gamma \in \mathbb{Z}^\ell$,

$$K(\Psi; M; \gamma) + K(\Psi; M; s_i\gamma - \epsilon_i + \epsilon_{i+1}) = 0\,.$$

Proof. The map g from (3.11) allows us to express $K(\Psi; M; \gamma) + K(\Psi; M; s_i\gamma - \epsilon_i + \epsilon_{i+1})$ as

$$g \circ \left(1 - \frac{1}{z_{i+1}}\right) \prod_{(a,b)\in\Psi} \left(1 - \frac{z_a}{z_b}\right)^{-1} \prod_{b\in M\setminus\{i+1\}} \left(1 - \frac{1}{z_b}\right) \left(1 + \frac{z_{i+1}}{z_i} s_i\right) (\mathbf{z}^\gamma)\,.$$

Since $s_i\Psi = \Psi$ and $s_i(M \setminus \{i+1\}) = M \setminus \{i+1\}$, the operator s_i commutes with multiplication by $\prod_{(a,b)\in\Psi}(1 - \frac{z_a}{z_b})^{-1} \prod_{b\in M\setminus\{i+1\}}(1 - \frac{1}{z_b})$, hence so does the operator $1 + \frac{z_{i+1}}{z_i} s_i$. Therefore, $K(\Psi; M; \gamma) + K(\Psi; M; s_i\gamma - \epsilon_i + \epsilon_{i+1})$ equals

$$g \circ \left(1 - \frac{1}{z_{i+1}}\right) \left(1 + \frac{z_{i+1}}{z_i} s_i\right) \prod_{(a,b)\in\Psi} \left(1 - \frac{z_a}{z_b}\right)^{-1} \prod_{b\in M\setminus\{i+1\}} \left(1 - \frac{1}{z_b}\right) (\mathbf{z}^\gamma)\,, \tag{3.16}$$

which vanishes by Proposition 3.4.5. □

Lemma 3.4.7. *Given a root ideal $\Psi \subseteq \Delta^+_{\ell+1}$, a multiset M on $[\ell+1]$, and $\gamma \in \mathbb{Z}^\ell$, we have that*

$$K(\Psi; M; (\gamma, 0)) = K(\hat{\Psi}; \hat{M}; \gamma)\,,$$

where $\hat{\Psi} := \{(i,j) \in \Psi \mid 1 \le i < j \le \ell\}$ and $\hat{M} := \{j \in M \mid 1 \le j \le \ell\}$.

Proof. Proposition 3.4.2 implies that

$$K(\Psi; M; (\gamma, 0)) = \prod_{j\in\hat{M}} (1 - L_j) \prod_{(i,j)\in\Delta^+_\ell\setminus\hat{\Psi}} (1 - R_{ij}) \prod_{j=1}^{m_M(\ell+1)} (1 - L_{\ell+1}) \prod_{(h,\ell+1)\in\Delta^+_{\ell+1}\setminus\Psi} (1 - R_{h,\ell+1})\, k_{(\gamma,0)}$$

$$= \prod_{j \in \hat{M}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \hat{\Psi}} (1 - R_{ij})\, k_\gamma\,,$$

since $k_0^{(\ell)} = 1$ and $k_m^{(\ell)} = 0$ for $m < 0$. □

Remark 3.4.8. In light of Lemma 3.4.7, we sometimes abuse notation by saying that, for $\ell' \geq \ell$, root ideal $\Psi \subseteq \Delta_{\ell'}^+$, multiset M on $[\ell']$, and $\gamma \in \mathbb{Z}^\ell$,

$$K(\Psi; M; \gamma) := K(\hat{\Psi}; \hat{M}; \gamma)\,.$$

Lemma 3.4.9. *For $r \geq 0, s \geq 1$, and $\gamma \in \mathbb{Z}^s$,*

$$\prod_{j=r+1}^{r+s} (1 - L_j)^r k_{(0^r,\gamma)} = k_\gamma\,.$$

Proof. We note that

$$k_a^{(b-r)} = k_a^{(b-r+1)} - k_{a-1}^{(b-r+1)} = \cdots = \sum_{i=0}^{r} (-1)^i \binom{r}{i} k_{a-i}^{(b)}$$

by iterating (3.14). Then,

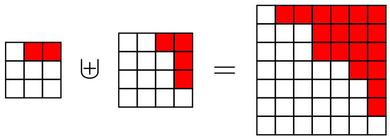
$$k_\gamma = k_{0^r} k_{\gamma_1}^{(0)} \cdots k_{\gamma_s}^{(s-1)} = k_{0^r} \left(\sum_{i_1=0}^{r} (-1)^{i_1} \binom{r}{i_1} k_{\gamma_1 - i_1}^{(r)} \right) \cdots \left(\sum_{i_s=0}^{r} (-1)^{i_s} \binom{r}{i_s} k_{\gamma_s - i_s}^{(r+s-1)} \right)$$
$$= \prod_{j=r+1}^{r+s} (1 - L_j)^r k_{(0^r,\gamma)}\,.$$

□

Definition 3.4.10. Given root ideals $\Psi \subseteq \Delta_\ell^+$ and $\Psi' \subseteq \Delta_{\ell'}^+$, we define the root ideal $\Psi \uplus \Psi' \subseteq \Delta_{\ell+\ell'}^+$ to be the result of placing Ψ and Ψ' catty-corner and including the full $\ell \times \ell'$ rectangle of roots in between. Equivalently, $\Psi \uplus \Psi'$ is determined by

$$\Delta_{\ell+\ell'}^+ \setminus (\Psi \uplus \Psi') = (\Delta_\ell^+ \setminus \Psi) \ \sqcup\ \{(i + \ell, j + \ell) \mid (i,j) \in \Delta_{\ell'}^+ \setminus \Psi\}\,.$$

For example,



Lemma 3.4.11. *Given $\lambda \in \mathbb{Z}^\ell, \mu \in \mathbb{Z}^{\ell'}$, root ideals $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$, and root ideals $\Psi', \mathcal{L}' \subseteq \Delta_{\ell'}^+$, we have*

$$K(\Psi; \mathcal{L}; \lambda) K(\Psi'; \mathcal{L}'; \mu) = K(\Psi \uplus \Psi'; \mathcal{L} \uplus \mathcal{L}'; \lambda\mu)\,,$$

where $\lambda\mu = (\lambda_1, \ldots, \lambda_\ell, \mu_1, \ldots, \mu_{\ell'})$.

Proof. By Proposition 3.4.2,

$$K(\Psi \uplus \Psi'; \mathcal{L} \uplus \mathcal{L}'; \lambda\mu) = \prod_{(i,j)\in\mathcal{L}\uplus\mathcal{L}'} (1 - L_j) \prod_{(i,j)\in\Delta^+_{\ell+\ell'}\setminus\Psi\uplus\Psi'} (1 - R_{ij}) k_{\lambda\mu}\,.$$

However, since $\Delta^+_{\ell+\ell'} \setminus \Psi \uplus \Psi'$ has no roots in $\{(r,s) \mid 1 \le r \le \ell, \ell+1 \le s \le \ell+\ell'\}$,

$$K(\Psi \uplus \Psi'; \mathcal{L} \uplus \mathcal{L}'; \lambda\mu) = \prod_{(i,j)\in\mathcal{L}\uplus\mathcal{L}'} (1 - L_j) \prod_{(i,j)\in\Delta^+_{\ell}\setminus\Psi} (1 - R_{ij}) \prod_{(i,j)\in\Delta^+_{\ell'}\setminus\Psi'} (1 - R_{i+\ell,j+\ell}) k_{\lambda\mu}\,.$$

By definition of $\mathcal{L}\uplus\mathcal{L}'$, $\prod_{(i,j)\in\mathcal{L}\uplus\mathcal{L}'}(1-L_j) = \prod_{(i,j)\in\mathcal{L}}(1-L_j)\prod_{(i,j)\in\mathcal{L}'}(1-L_{\ell+j})\prod_{j=\ell+1}^{\ell+\ell'}(1-L_j)^\ell$. Noting $k_{\lambda\mu} = k_\lambda k_{(0^\ell,\mu)}$, we thus have

$$\begin{aligned} K(\Psi \uplus \Psi'; \mathcal{L} \uplus \mathcal{L}'; \lambda\mu) = &\prod_{(i,j)\in\mathcal{L}} (1 - L_j) \prod_{(i,j)\in\Delta^+_{\ell}\setminus\Psi} (1 - R_{ij})\, k_\lambda \\ &\times \prod_{j=\ell+1}^{\ell+\ell'} (1 - L_j)^\ell \prod_{(i,j)\in\mathcal{L}'} (1 - L_{\ell+j}) \prod_{(i,j)\in\Delta^+_{\ell'}\setminus\Psi'} (1 - R_{i+\ell,j+\ell}) k_{(0^\ell,\mu)}\,. \end{aligned}$$

The first line is $K(\Psi; \mathcal{L}; \mu)$. To see the second line is $K(\Psi'; \mathcal{L}'; \mu)$, expand $\prod_{(i,j)\in\mathcal{L}'}(1 - L_{\ell+j})\prod_{(i,j)\in\Delta^+_{\ell'}\setminus\Psi'}(1-R_{i+\ell,j+\ell})k_{(0^\ell,\mu)} = \sum_\gamma k_{(0^\ell,\gamma)}$, and note for each summand, $\prod_{j=\ell+1}^{\ell+\ell'}(1-L_j)^\ell k_{(0^\ell,\gamma)} = k_\gamma$ by Lemma 3.4.9. □

We now present Katalan function analogues of the Catalan function root expansion identities in Proposition 2.3.8.

Proposition 3.4.12. *Let $\Psi \subseteq \Delta^+$ be a root ideal, M on $[\ell]$ be a multiset, and $\mu \in \mathbb{Z}^\ell$. Then,*

(a) for any addable root β of Ψ,

$$K(\Psi; M; \mu) = K(\Psi \cup \beta; M; \mu) - K(\Psi \cup \beta; M; \mu + \varepsilon_\beta)\,;$$

(b) for any removable root α of Ψ,

$$K(\Psi; M; \mu) = K(\Psi \setminus \alpha; M; \mu) + K(\Psi; M; \mu + \varepsilon_\alpha)\,;$$

(c) for any $y \in M$,

$$K(\Psi; M; \mu) = K(\Psi; M \setminus y; \mu) - K(\Psi; M \setminus y; \mu - \epsilon_y)\,;$$

(d) for any $y \in [\ell]$,

$$K(\Psi; M; \mu) = K(\Psi; M \sqcup y; \mu) + K(\Psi; M; \mu - \epsilon_y)\,.$$

Proof. The first identity follows directly from Proposition 3.4.2:

$$K(\Psi; M; \mu) = \prod_{j \in M} (1 - L_j) \prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - R_{ij}) k_\mu = \prod_{j \in M} (1 - L_j) \prod_{(i,j) \in \Delta^+ \setminus (\Psi \cup \beta)} (1 - R_{ij})(k_\mu - k_{\mu + \varepsilon_\beta})\,.$$

Part (b) is then obtained by applying (a) with $\Psi = \Psi \setminus \alpha$ and $\beta = \alpha$. A similar computation gives (c):

$$K(\Psi; M; \mu) = \prod_{j \in M} (1 - L_j) \prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - R_{ij}) k_\mu = \prod_{j \in M \setminus y} (1 - L_j) \prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - R_{ij})(k_\mu - k_{\mu - \epsilon_y})\,,$$

and (d) is obtained by applying (c) with $M \sqcup \{y\}$ in place of M. □

These root expansions give rise to other powerful identities, derived by their successive application.

Lemma 3.4.13. *Let $\Psi \subseteq \Delta_\ell^+$, M be a multiset on $[\ell]$, and $\mu \in \mathbb{Z}^\ell$ with $\mu_\ell = 1$. If $\ell \in M$ and Ψ has a removable root $\alpha = (x, \ell)$ for some x, then*

$$K(\Psi; M; \mu) = K(\Psi \setminus \alpha; M \setminus \ell; \mu) + K(\hat{\Psi}; \hat{M} \sqcup x; (\mu_1, \dots, \mu_{\ell-1}) + \epsilon_x)\,,$$

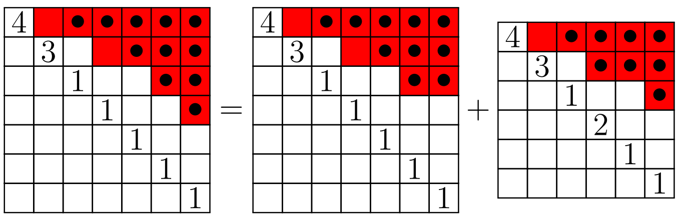
where $\hat{\Psi} = \{(i,j) \in \Psi \mid j < \ell\}$ and $\hat{M} = \{j \in M \mid j < \ell\}$.

Proof. By Proposition 3.4.12, we expand first on the removable root $\alpha = (x, \ell)$ of Ψ and then on $\ell \in M$, to obtain

$$\begin{aligned} K(\Psi; M; \mu) &= K(\Psi \setminus \alpha; M; \mu) + K(\Psi; M; \mu + \varepsilon_\alpha) \\ &= K(\Psi \setminus \alpha; M \setminus \ell; \mu) - K(\Psi \setminus \alpha; M \setminus \ell; \mu - \epsilon_\ell) + K(\Psi; M; \mu + \varepsilon_\alpha)\,. \end{aligned}$$

Lemma 3.4.7 allows the substitution of $K(\Psi \setminus \alpha; M \setminus \ell; \mu - \epsilon_\ell) = K(\hat{\Psi}; \hat{M}; \hat{\mu})$ for $\hat{\mu} = (\mu_1, \dots, \mu_{\ell-1})$, as well as $K(\Psi; M; \mu + \varepsilon_\alpha) = K(\hat{\Psi}; \hat{M}; \hat{\mu} + \epsilon_x)$. Proposition 3.4.12(c) on column x then gives $-K(\hat{\Psi}; \hat{M}; \hat{\mu}) + K(\hat{\Psi}; \hat{M}; \hat{\mu} + \epsilon_x) = K(\hat{\Psi}; \hat{M} \sqcup x; \hat{\mu} + \epsilon_x)$. □

Example 3.4.14. We apply Lemma 3.4.13 to the following scenario, with $\ell = 7$ and root $\alpha = (4, 7)$:



3.5 Mirror lemmas and straightening relations

Let $\Psi \subseteq \Delta_\ell^+$ be a root ideal and $x \in [\ell]$. If there is a removable root (x, j) of Ψ, then define $\mathrm{down}_\Psi(x) = j$; otherwise, $\mathrm{down}_\Psi(x)$ is undefined. Similarly, if there is a removable root (i, x) of Ψ, then define $\mathrm{up}_\Psi(x) = i$; otherwise, $\mathrm{up}_\Psi(x)$ is undefined. The *bounce graph* of a root ideal $\Psi \subseteq \Delta_\ell^+$ is the graph on the vertex set $[\ell]$ with edges $(r, \mathrm{down}_\Psi(r))$ for each

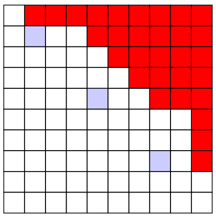
$r \in [\ell]$ such that $\text{down}_\Psi(r)$ is defined. The bounce graph of Ψ is a disjoint union of paths called *bounce paths of* Ψ.

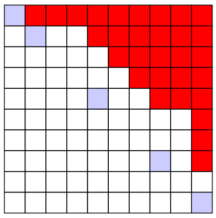
For each vertex $r \in [\ell]$, distinguish $\text{top}_\Psi(r)$ to be the minimum element of the bounce path of Ψ containing r. For $a, b \in [\ell]$ in the same bounce path of Ψ with $a \leq b$, we define

$$\text{path}_\Psi(a, b) = \{a, \text{down}_\Psi(a), \text{down}^2_\Psi(a), \ldots, b\},$$

i.e., the set of indices in this path lying between a and b. We also set $\text{uppath}_\Psi(r)$ to be $\text{path}_\Psi(\text{top}_\Psi(r), r)$ for any $r \in [\ell]$.

Example 3.5.1. A path and uppath for the root ideal Ψ are given below:





$\text{path}_\Psi(2, 8) = \{2, 5, 8\}$ $\qquad$ $\text{uppath}_\Psi(10) = \{10, 8, 5, 2, 1\}$

Definition 3.5.2. Let $d \in \mathbb{Z}_{>0}$. For a root ideal Ψ, we say there is

a wall in rows $r, r+1, \ldots, r+d$ if rows $r, \ldots, r+d$ of Ψ have the same length,

a ceiling in columns $c, c+1, \ldots, c+d$ if columns $c, \ldots, c+d$ of Ψ have the same length,

a mirror in rows $r, r+1$ if Ψ has removable roots (r, c), $(r+1, c+1)$ for some $c > r+1$.

Example 3.5.3. In Example 3.5.1, the root ideal Ψ has a ceiling in columns $2, 3, 4$, and in columns $8, 9$, a wall in rows $6, 7, 8$, and in rows $9, 10$, and a mirror in rows $2, 3$, in rows $3, 4$, and in rows $4, 5$.

Lemma 3.5.4. *Suppose a root ideal* $\Psi \subseteq \Delta^+_\ell$*, a multiset* M *on* $[\ell]$*,* $\mu \in \mathbb{Z}^\ell$*, and* $z \in [\ell - 1]$ *satisfy*

(a) Ψ *has a ceiling in columns* $z, z+1$*;*

(b) Ψ *has a wall in rows* $z, z+1$*;*

(c) $\mu_z = \mu_{z+1} - 1$.

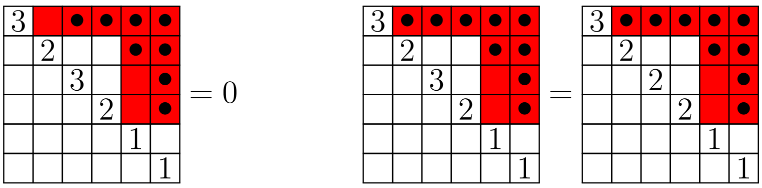
If $m_M(z+1) = m_M(z) + 1$*, then* $K(\Psi; M; \mu) = 0$*. If* $m_M(z) = m_M(z+1)$*, then* $K(\Psi; M; \mu) = K(\Psi; M; \mu - \epsilon_{z+1})$*,*

Proof. Conditions (a) and (b) are equivalent to $s_z\Psi = \Psi$ and condition (c) implies $\mu = s_z\mu - \epsilon_z + \epsilon_{z+1}$. Thus, if $m_M(z+1) = m_M(z)+1$, the result follows from Lemma 3.4.6. If $m_M(z+1) = m_M(z)$, Lemma 3.4.12(d) implies that

$$K(\Psi; M; \mu) = K(\Psi; M \sqcup \{z+1\}; \mu) + K(\Psi; M; \mu - \epsilon_{z+1}).$$

By the preceding case, $K(\Psi; M \sqcup \{z+1\}; \mu)$ vanishes. □

Example 3.5.5. For $z = 2$, Lemma 3.5.4 applies in the following two situations:



Lemma 3.5.6 (Mirror Lemma). *Let* $\Psi \subseteq \Delta_\ell^+$ *be a root ideal,* M *a multiset on* $[\ell]$*,* $\mu \in \mathbb{Z}^\ell$*, and* $1 \le y \le z < \ell$ *be indices in the same bounce path of* Ψ *satisfying*

(a) Ψ *has a ceiling in columns* $y, y+1$*;*

(b) Ψ *has a mirror in rows* $x, x+1$ *for all* $x \in \mathrm{path}_\Psi(y, \mathrm{up}_\Psi(z))$*;*

(c) Ψ *has a wall in rows* $z, z+1$*;*

(d) $m_M(x+1) = m_M(x) + 1$ *for all* $x \in \mathrm{path}_\Psi(\mathrm{down}_\Psi(y), z)$*;*

(e) $\mu_x = \mu_{x+1}$ *for all* $x \in \mathrm{path}_\Psi(y, \mathrm{up}_\Psi(z))$*;*

(f) $\mu_z = \mu_{z+1} - 1$.

If $m_M(y+1) = m_M(y)+1$*, then* $K(\Psi;M;\mu) = 0$*. If* $m_M(y+1) = m_M(y)$*, then* $K(\Psi;M;\mu) = K(\Psi;M;\mu-\epsilon_{z+1})$*.*

Proof. We proceed by induction on $z-y$, with Lemma 3.5.4 giving the base case $z=y$. Assume $z>y$. Condition (b) implies that $\mathrm{up}_\Psi(z+1) = \mathrm{up}_\Psi(z)+1$ and thus the root $\beta = (\mathrm{up}_\Psi(z+1), z)$ is addable to Ψ. Proposition 3.4.12(a) thus implies that $K(\Psi;M;\mu) = K(\Psi\cup\beta;M;\mu) - K(\Psi\cup\beta;M;\mu+\varepsilon_\beta)$. The root ideal $\Psi\cup\beta$ has a ceiling in columns $z, z+1$ and so $K(\Psi\cup\beta;M;\mu) = 0$ by Lemma 3.5.4. Therefore,

$$K(\Psi;M;\mu) = -K(\Psi\cup\beta;M;\mu+\varepsilon_\beta)\,.$$

Because there is a wall in rows $\mathrm{up}_\Psi(z), \mathrm{up}_\Psi(z+1)$ of the root ideal $\Psi\cup\beta$, $K(\Psi\cup\beta;M;\mu+\varepsilon_\beta)$ can be addressed by induction: when $m_M(y+1) = m_M(y)+1$, $K(\Psi\cup\beta;M;\mu+\varepsilon_\beta) = 0$ implies the vanishing of $K(\Psi;M;\mu)$, and otherwise, $K(\Psi\cup\beta;M;\mu+\varepsilon_\beta) = K(\Psi\cup\beta;M;\mu+\varepsilon_\beta-\epsilon_{\mathrm{up}_\Psi(z)+1}) = K(\Psi\cup\beta;M;\mu-\epsilon_z)$ gives $K(\Psi;M;\mu) = -K(\Psi\cup\beta;M;\mu-\epsilon_z)$. We then use Lemma 3.4.6 with $i=z$ to find

$$K(\Psi;M;\mu) = -K(\Psi\cup\beta;M;\mu-\epsilon_z) = K(\Psi\cup\beta;M;\mu-\epsilon_{z+1})\,.$$

Now expand the righthand side on the removable root $\beta\in\Psi\cup\beta$ with Proposition 3.4.12(b) to obtain

$$K(\Psi;M;\mu) = K(\Psi\cup\beta;M;\mu-\epsilon_{z+1}) = K(\Psi;M;\mu-\epsilon_{z+1}) + K(\Psi\cup\beta;M;\mu-\epsilon_{z+1}+\varepsilon_\beta)\,.$$

Finally, $K(\Psi\cup\beta;M;\mu-\epsilon_{z+1}+\varepsilon_\beta)$ vanishes by Lemma 3.5.4 since $\Psi\cup\beta$ has a wall in rows $z, z+1$ and a ceiling in columns $z, z+1$ and $\mu-\epsilon_{z+1}+\varepsilon_\beta$ satisfies the necessary conditions. □

Lemma 3.5.7. *Suppose a root ideal* $\Psi\subseteq\Delta_\ell^+$*, multiset* M *on* $[\ell]$*,* $\gamma\in\mathbb{Z}^\ell$*, and* $j\in[\ell]$ *satisfy*

(a) Ψ *has a removable root* (i,j) *in column* j*;*

(b) Ψ *has a ceiling in columns* $j, j+1$ *and a wall in rows* $j, j+1$*;*

(c) $m_M(j+1) = m_M(j) + 1$*;*

(d) $\gamma_j = \gamma_{j+1}$*.*

Then, $K(\Psi; M; \gamma) = K(\Psi \setminus (i,j); M; \gamma)$*.*

Proof. A root expansion on the removable root (i,j) with Proposition 3.4.12 gives

$$K(\Psi; M; \gamma) = K(\Psi \setminus (i,j); M; \gamma) + K(\Psi; M; \gamma + \epsilon_i - \epsilon_j)\,,$$

and the second summand vanishes by Lemma 3.5.4 with $z = j$. □

Lemma 3.5.8. *Suppose a root ideal* $\Psi \subseteq \Delta_\ell^+$*, multiset* M *on* $[\ell]$*,* $\gamma \in \mathbb{Z}^\ell$*, and* $j \in [\ell]$ *satisfy*

(a) $j \in M$*;*

(b) Ψ *has a ceiling in columns* $j, j+1$ *and a wall in rows* $j, j+1$*;*

(c) $m_M(j+1) = m_M(j)$*;*

(d) $\gamma_j = \gamma_{j+1}$*.*

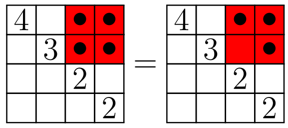
Then, $K(\Psi; M; \gamma) = K(\Psi; M \setminus j; \gamma)$*. If, in addition,* Ψ *has a removable root* (i,j) *in column* j*, then* $K(\Psi; M; \gamma) = K(\Psi \setminus (i,j); M \setminus j; \gamma)$*.*

Proof. We expand on $j \in M$ with Proposition 3.4.12 to obtain

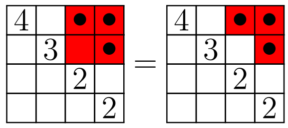
$$K(\Psi; M; \gamma) = K(\Psi; M \setminus j; \gamma) - K(\Psi; M \setminus j; \gamma - \epsilon_j)\,,$$

and note that $K(\Psi; M \setminus j; \gamma - \epsilon_j) = 0$ by Lemma 3.5.4. If, in addition, (i,j) is removable from Ψ, the second equality holds since $K(\Psi; M \setminus j; \gamma)$ satisfies the conditions of Lemma 3.5.7. □

Example 3.5.9. By the first equality of Lemma 3.5.8,



By Lemma 3.5.7,



Combining both these equalities is an application of the second equality of Lemma 3.5.8.

Lemma 3.5.10 (Mirror Straightening Lemma)**.** *Let $\Psi \subseteq \Delta_\ell^+$ be a root ideal, M a multiset on $[\ell]$, and $\mu \in \mathbb{Z}^\ell$. Let $1 \leq y \leq z < \ell$ be indices in the same bounce path of Ψ satisfying*

(a) $m_M(y) = m_M(y+1)$*;*

(b) *Ψ has an addable root $\alpha = (\mathrm{up}_\Psi(y+1), y)$ and a removable root $\beta = (\mathrm{up}_\Psi(y+1), y+1)$;*

(c) *Ψ has a mirror in rows $x, x+1$ for all $x \in \mathrm{path}_\Psi(y, \mathrm{up}_\Psi(z))$;*

(d) *Ψ has a wall in rows $z, z+1$;*

(e) *$m_M(x+1) = m_M(x) + 1$ for all $x \in \mathrm{path}_\Psi(\mathrm{down}_\Psi(y), z)$;*

(f) *$\mu_x = \mu_{x+1}$ for all $x \in \mathrm{path}_\Psi(y, \mathrm{up}_\Psi(z))$, and $\mu_z = \mu_{z+1} - 1$.*

Then,

$$K(\Psi; M; \mu) = K(\Psi \cup \alpha; M \sqcup (y+1); \mu + \epsilon_{\mathrm{up}_\Psi(y+1)} - \epsilon_{z+1}) + K(\Psi; M; \mu - \epsilon_{z+1}) .$$

Proof. First consider the case $z = y$. We have $K(\Psi; M; \mu) = K(\Psi; M \sqcup (y+1); \mu) + K(\Psi; M; \mu - \epsilon_{z+1})$ by Proposition 3.4.12(d), and must prove that

$$K(\Psi; M \sqcup (y+1); \mu) = K(\Psi \cup \alpha; M \sqcup (y+1); \mu + \epsilon_{\mathrm{up}_\Psi(z+1)} - \epsilon_{z+1}) . \tag{3.17}$$

Since $\alpha = (\text{up}_\Psi(z+1), z)$ is addable to Ψ, we expand with Proposition 3.4.12 to obtain

$$K(\Psi; M \sqcup (y+1); \mu) = K(\Psi \cup \alpha; M \sqcup (y+1); \mu) - K(\Psi \cup \alpha; M \sqcup (y+1); \mu + \varepsilon_\alpha)\,.$$

Conditions (b) and (d) imply that $\Psi \cup \alpha$ has a ceiling in columns $y, y+1$ and a wall in rows $y, y+1$, and (a) gives that $M \sqcup (y+1)$ has one more occurrence of $y+1$ than y. Therefore, since $\mu_z = \mu_{z+1} - 1$, Lemma 3.4.6 with $i = y = z$ applies and straightens the term

$$-K(\Psi \cup \alpha; M \sqcup (y+1); \mu + \varepsilon_\alpha) = K(\Psi \cup \alpha; M \sqcup (y+1); \mu + \epsilon_{\text{up}_\Psi(z+1)} - \epsilon_{z+1})\,.$$

For the same reasons, Lemma 3.5.4 applies to the other term, giving $K(\Psi \cup \alpha; M \sqcup (y+1); \mu) = 0$. Thus (3.17) is proved.

Proceed by induction for $z - y > 0$. Given Ψ has a mirror in rows $w = \text{up}_\Psi(z)$ and $w+1$, the root $\gamma = (w+1, z)$ is addable to Ψ and expanding on it using Proposition 3.4.12 yields

$$K(\Psi; M; \mu) = K(\Psi \cup \gamma; M; \mu) - K(\Psi \cup \gamma; M; \mu + \varepsilon_\gamma)\,.$$

Since $\Psi \cup \gamma$ has a ceiling in columns $z, z+1$, with conditions (d) and (e), Lemma 3.4.6 straightens the term

$$-K(\Psi \cup \gamma; M; \mu + \varepsilon_\gamma) = K(\Psi \cup \gamma; M; \mu + \epsilon_{w+1} - \epsilon_{z+1})\,.$$

The same conditions imply that $K(\Psi \cup \gamma; M; \mu) = 0$ by Lemma 3.5.4. Therefore,

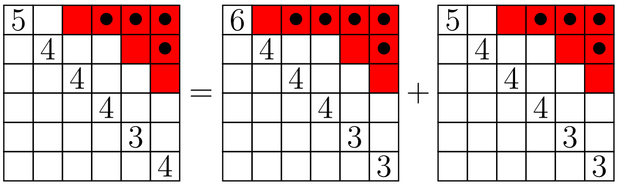
$$K(\Psi; M; \mu) = K(\Psi \cup \gamma; M; \mu + \epsilon_{w+1} - \epsilon_{z+1})\,.$$

Since $\Psi \cup \gamma$ has a wall in rows w and $w+1$, and $\nu = \mu + \epsilon_{w+1} - \epsilon_{z+1}$ satisfies $\nu_w = \nu_{w+1} - 1$, we can apply the induction hypothesis with $z = w$ to the righthand side and obtain

$$K(\Psi; M; \mu) = K(\Psi \cup \{\gamma, \alpha\}; M \sqcup (y+1); \mu + \epsilon_{\mathrm{up}_\Psi(y+1)} - \epsilon_{z+1}) + K(\Psi \cup \gamma; M; \mu - \epsilon_{z+1})\,.$$

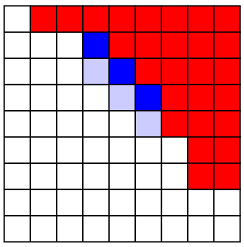
Lemma 3.5.7 enables us to remove γ from both terms, proving the claim. □

Example 3.5.11. The following is an example of an application of Lemma 3.5.10 with $y = 2, z = 5$.



For $1 \le x < y \le z \le \ell$, define the *diagonal* $D^z_{x,y} = \{(i,j) \mid j - i = y - x\,,\, y \le j \le z\} \subseteq \Delta^+_\ell$.

Example 3.5.12. In the following, $D^6_{3,4}$ is the light blue (removable) diagonal and $D^6_{2,4}$ is depicted in dark blue.



Lemma 3.5.13 (Diagonal Removal Lemma)**.** *Let $\Psi \subseteq \Delta^+_\ell$ be a root ideal, M a multiset on $[\ell]$, $\gamma \in \mathbb{Z}^\ell$, and integers $1 \le x < y \le z \le \ell$ be such that*

(a) Ψ has a ceiling in columns $z-1, z$ and every root of $D^{z-1}_{x,y} \subseteq \Psi$ is removable from Ψ;

(b) $L(D^{z-1}_{x,y}) \subseteq M$ and $m_M(z) = m_M(z-1) = m_M(z-2)+1 = \cdots = m_M(y)+z-1-y$;

(c) Ψ has a wall in rows $y, y+1, \ldots, z$;

(d) $\gamma_y = \cdots = \gamma_z$.

Then,

$$K(\Psi; M; \gamma) = K(\Psi'; M'; \gamma)$$

where $\Psi' = \Psi \setminus D_{x,y}^{z-1}$ *and* $M' = M \setminus L(D_{x,y}^{z-1})$.

Proof. Let $\beta^0, \beta^1, \dots, \beta^{z-y-1}$ be the roots of the diagonal $D_{x,y}^{z-1}$ from lowest to highest, i.e., $\beta^j = (a_j, b_j)$ with $a_j = z + x - y - j - 1$ and $b_j = z - j - 1$. Define $\Psi^{j+1} = \Psi^j \setminus \{\beta^j\}$ and $M^{j+1} = M^j \setminus \{b_j\}$, starting with $\Psi^0 = \Psi$ and $M^0 = M$; thus $\Psi^{z-y-1} = \Psi'$ and $M^{z-y-1} = M'$. By condition (a) for $j = 0$ and by construction for $j > 0$, β^j is a removable root of Ψ^j, and Ψ^j has a ceiling in columns $b_j, b_j + 1$. Similarly, (b) implies that $b_j \in M^j$ and $m_{M^j}(b_j + 1) = m_{M^j}(b_j)$. Therefore, using also (c) and (d), we can repeatedly apply Lemma 3.5.8 to obtain

$$K(\Psi; M; \gamma) = K(\Psi^1; M^1; \gamma) = K(\Psi^2; M^2; \gamma) = \cdots = K(\Psi^{z-y-1}; M^{z-y-1}; \gamma)\,. \qquad \square$$

3.6 K-k-Schur functions are Katalan functions

Recall from (2.13) that $\Delta^k(\mu) = \{(i, j) \in \Delta_\ell^+ \mid k - \mu_i + i < j\}$.

Definition 3.6.1. For $\lambda \in \mathrm{Par}_\ell^k$, define the *$k$-Schur Katalan function* by

$$\mathfrak{g}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)\,.$$

We show that the k-Schur Katalan functions are the K-k-Schur functions. This operator formula is considerably more direct and explicit than any previously known description of the K-k-Schur functions and readily resolves several outstanding conjectures, including positive branching.

$\Delta^k(\mu)$ and $\mathfrak{g}_\mu^{(k)}$ are defined for $\mu \in \mathrm{Par}_\ell^k$, but the definition can be extended to any $\mu \in \mathbb{Z}_{\le k}^\ell$ such that $\mu_i \ge \mu_{i+1} - 1$ for all $i \in [\ell - 1]$. Several useful properties are satisfied by these k-Schur root ideals, immediate from their construction, which will be used throughout this section.

Remark 3.6.2. Let $\lambda \in \mathrm{Par}^k_m$, $\Psi = \Delta^k(\lambda)$, and $\mathcal{L} = \Delta^{k+1}(\lambda)$. Let z be the lowest nonempty row of Ψ.

(a) (Wall-free) For $x \in [z]$, Ψ does not have a wall in rows $x, x+1$. Hence, for all $x \in [m-1]$, either Ψ has a ceiling in columns $x, x+1$ or has removable roots (y, x) and $(y+1, x+1)$. In the latter case, if $y \neq x-1$, then Ψ has a mirror in rows $y, y+1$.

(b) (Equal weight mirrors) For $x \in [z-1]$, Ψ has a mirror in rows $x, x+1$ if and only if $\mu_x = \mu_{x+1} < k$.

(c) (Wall-free lowering ideal) For $x \in [m-1]$, $\mathrm{up}_\Psi(x)$ exists $\iff m_{L(\mathcal{L})}(x) = m_{L(\mathcal{L})}(x+1) - 1$. Otherwise, $m_{L(\mathcal{L})}(x) = m_{L(\mathcal{L})}(x+1)$.

(d) (Adjustable end) Let $S \subseteq \mathbb{Z}_{\geq m+2}$ satisfying $\max(S) - \min(S) \leq k-1$ if it is nonempty. Set $\mu = \lambda + \epsilon_S \in \mathbb{Z}^\ell$ for $\ell = \max(S \cup \{m\})$. Then $\Delta^k(\mu) = \Delta^k((\lambda, 0^{\ell-m}))$ and $\Delta^{k+1}(\mu) = \Delta^{k+1}((\lambda, 0^{\ell-m}))$, hence (a)–(c) apply with data $\ell, \mu, \Delta^k(\mu), \Delta^{k+1}(\mu)$ in place of $m, \lambda, \Psi, \mathcal{L}$.

Here and throughout the remainder of the paper, for $\lambda \in \mathrm{Par}^k_\ell$ and $\alpha \in \mathbb{Z}^j$ with $j \geq \ell$, we define $\lambda + \alpha = (\lambda, 0^{j-\ell}) + \alpha$.

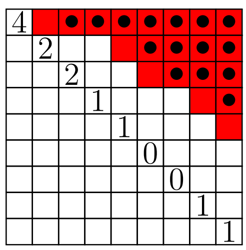
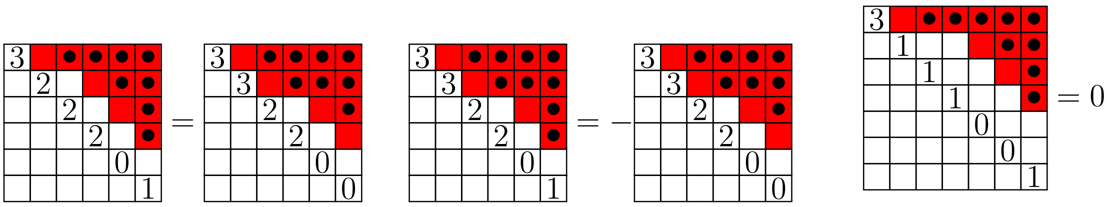


Figure 3-1: $K(\Psi; \mathcal{L}; \mu)$ with $\mu = 422110011$, $\Psi = \Delta^4(\mu)$ shown in red, and $\mathcal{L} = \Delta^5(\mu)$ superimposed as $\bullet$'s as in Example 3.4.4. The nonzero row lengths of Ψ and $\mathcal{L}$ decrease by at least one from top to bottom (illustrating (a) and (c)), and there are mirrors in rows $2, 3$ and in rows $4, 5$ corresponding to $\mu_2 = \mu_3$ and $\mu_4 = \mu_5$ (illustrating (b)).

Proposition 3.6.3 (Pieri straightening)**.** *Let $\lambda \in \mathrm{Par}^k_m$ and $S \subseteq \mathbb{Z}_{\geq m+2}$ nonempty with* $\max(S) - \min(S) \leq k-1$*. Set $\mu = \lambda + \epsilon_S$, $\Psi = \Delta^k(\mu)$, $M = L(\Delta^{k+1}(\mu))$, and $j = \min(S)$. There holds*

$$K(\Psi; M \sqcup S; \mu) = \begin{cases} K(\Delta^k(\nu); L(\Delta^{k+1}(\nu)) \sqcup (S \setminus j); \nu) & y = \mathrm{top}_\Psi(j-1) > \mathrm{top}_\Psi(j) \\ -K(\Psi; M \sqcup (S \setminus j); \mu - \epsilon_j) & \mathrm{top}_\Psi(j) > \mathrm{top}_\Psi(j-1) + 1 \\ 0 & \mathrm{top}_\Psi(j) = \mathrm{top}_\Psi(j-1) + 1 \end{cases} \tag{3.18}$$

where $\nu := \mu + \epsilon_{\mathrm{up}_\Psi(y+1)} - \epsilon_j$ in the first case.



(a) $j = 6, S = \{6\}$ $y = \mathrm{top}_\Psi(5) = 3 > \mathrm{top}_\Psi(6) = 1$

(b) $j = 6, S = \{6\}$ $\mathrm{top}_\Psi(6) = 4 > 2 = \mathrm{top}_\Psi(5) + 1$

(c) $j = 7, S = \{7\}$ $\mathrm{top}_\Psi(7) = 4 = \mathrm{top}_\Psi(6) + 1$

Figure 3-2: Examples of the three cases of Proposition 3.6.3 for $k = 3$.

Proof. First, apply Proposition 3.4.12(c) to $j \in S$ to obtain

$$K(\Psi; M \sqcup S; \mu) = K(\Psi; M \sqcup (S \setminus j); \mu) - K(\Psi; M \sqcup (S \setminus j); \mu - \epsilon_j). \tag{3.19}$$

Note that $\mu_{j-1} = \mu_j - 1 = 0$ since $\mu = \lambda + \epsilon_S$, $j = \min(S) \geq m+2$, and $\lambda \in \mathrm{Par}^k_m$. Also, note throughout that, since $\mu_{j-1} = 0$, then $(j-1, j) \notin \Psi$ and thus $\mathrm{uppath}_\Psi(j) \cap \mathrm{uppath}_\Psi(j-1) = \varnothing$.

If $y = \mathrm{top}_\Psi(j-1) > \mathrm{top}_\Psi(j)$, then $\mathrm{up}_\Psi(y)$ does not exist but $\mathrm{up}_\Psi(y+1)$ does, so Ψ does not have a ceiling in columns $y, y+1$. Thus, Remark 3.6.2 gives the conditions for Mirror Straightening Lemma 3.5.10 applied with $z = j-1$ to $K(\Psi; M \sqcup (S \setminus j); \mu)$

in (3.19), giving

$$K(\Psi; M\sqcup(S\setminus j); \mu) = K(\Psi\cup\alpha; M\sqcup(y+1)\sqcup(S\setminus j); \mu+\epsilon_{\mathrm{up}_\Psi(y+1)}-\epsilon_j)+K(\Psi; M\sqcup(S\setminus j); \mu-\epsilon_j)\,,$$

where $\alpha = (\mathrm{up}_\Psi(y+1), y)$. Therefore,

$$K(\Psi; M \sqcup S; \mu) = K(\Psi \cup \alpha; M \sqcup (y+1) \sqcup (S \setminus j); \mu + \epsilon_{\mathrm{up}_\Psi(y+1)} - \epsilon_j)\,.$$

Using $\Psi \cup \alpha = \Delta^k(\nu)$ and $M \sqcup (y+1) = L(\Delta^{k+1}(\nu))$, the top case of (3.18) follows.

If $\mathrm{top}_\Psi(j) > \mathrm{top}_\Psi(j-1)+1$, then Remark 3.6.2 gives the conditions to apply Mirror Lemma 3.5.6 with $z = j-1$; note that, in this case, there is no removable root in column $\mathrm{top}_\Psi(j)$ of Ψ by definition of top, but there is a removable root of Ψ in column $\mathrm{top}_\Psi(j)-1$, so Ψ has a ceiling in these columns. In addition, $m_M(\mathrm{top}_\Psi(j))-1 = m_M(\mathrm{top}_\Psi(j)-1)$ by Remark 3.6.2(c), so it is the first statement in Mirror Lemma 3.5.6 that applies. Hence the term $K(\Psi; M \sqcup (S \setminus j); \mu)$ in (3.19) vanishes, as desired.

If $\mathrm{top}_\Psi(j) = \mathrm{top}_\Psi(j-1)+1$, then there are no removable roots in columns $\mathrm{top}_\Psi(j-1), \mathrm{top}_\Psi(j)$ of Ψ by definition of top, so there is a ceiling in columns $\mathrm{top}_\Psi(j-1), \mathrm{top}_\Psi(j)$. Remark 3.6.2 gives the conditions to apply Mirror Lemma 3.5.6. Since $m_{L(\mathcal{L})}(\mathrm{top}_\Psi(j-1)) = m_{L(\mathcal{L})}(\mathrm{top}_\Psi(j))$ by Remark 3.6.2(c), we obtain $K(\Psi; M \sqcup (S \setminus j); \mu) = K(\Psi; M \sqcup (S \setminus j); \mu - \epsilon_j)$, and thus the right side of (3.19) is zero, as desired. □

3.6.1 Katalan multiplication via root expansions

Recall that $D^z_{x,y} \subseteq \Delta^+$ denotes the diagonal occupying columns y to z, starting in row x. For $1 \le x < y \le z$, a succession of diagonals, each occupying columns y to z, forms a *staircase*, $E^{z,h}_{x,y} = D^z_{x,y} \cup D^z_{x+1,y} \cup \cdots \cup D^z_{x+h-1,y}$. In Example 3.5.12, $E^{6,2}_{2,4}$ is the union of light and dark blue cells.

Lemma 3.6.4. *For $\ell \ge 1$ and $r \ge 0$, consider a root ideal $\Psi \subseteq \Delta^+_{\ell+r}$ and a multiset M on $[\ell+r]$. Let $x, h \ge 0$ with $x+r+h-2 \le \ell$ be such that* *(a)* $E_h := E_{x,\ell+1}^{\ell+r,h} \subseteq \Psi$*;*

(b) $\Psi' = \Psi \setminus E_h$ *is a root ideal;*

(c) $m_M(\ell+1) \geq h$ *and* $m_M(\ell+r) = m_M(\ell+r-1)+1 = \cdots = m_M(\ell+1)+r-1$.

Then, for $\gamma \in \mathbb{Z}^\ell$ *and* $M' = M \setminus L(E_h)$*,*

$$K(\Psi; M; (\gamma, 1^r)) = \sum_{a=0}^{r} \sum_{\substack{\mu=\gamma+\epsilon_S+\epsilon_{S'} \\ S\subseteq\{x+r-a,\ldots,x+r+h-2\} \\ |S|=a \\ S'=\{\ell+1,\ldots,\ell+r-a\}}} K(\Psi'; M' \sqcup S; \mu)\,.$$

where each summand is understood to be truncated in the manner of Remark 3.4.8.

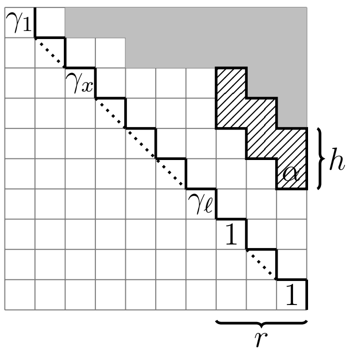


Figure 3-3: Schematic of the setup for Lemma 3.6.4 where Ψ' are the roots in light gray, E_h is the diagonally shaded region, and $\Psi = \Psi' \cup E_h$.

Proof. If $r = 0$ or $h = 0$, E_h is the empty set and the equality holds trivially. We proceed by induction on $r + h$ with $r, h > 0$. Noting that $\alpha = (x + r + h - 2, \ell + r)$ is the only root in the lowest row of E_h, it is removable from Ψ by (b). Thus, Lemma 3.4.13 implies

$$K(\Psi; M; (\gamma, 1^r)) = K(\Psi\setminus\alpha; M\setminus(\ell+r); (\gamma, 1^r)) + K(\hat{\Psi}; \hat{M}\sqcup(x+r+h-2); (\gamma, 1^{r-1})+\epsilon_{x+r+h-2})\,.$$

We shall apply Diagonal Removal Lemma 3.5.13 with $x = x+h-1, y = \ell+1, z = \ell+r$ to the first term on the righthand side; indeed, $\Psi \setminus \alpha$ has a ceiling in $\ell+r-1, \ell+r$ and (c) implies $M \setminus (\ell+r)$ has the same number of occurrences of $\ell+r-1, \ell+r$. Furthermore, since Ψ has no roots lower than α, $\Psi \setminus \alpha$ has a wall in rows $x+r+h-2, \ldots, \ell+r$ (recall that $x+r+h-2 \leq \ell$). By definition, $D^{\ell+r}_{x+h-1,\ell+1} = E_h \setminus E_{h-1}$ is the lowest diagonal of E_h and thus every root of $D^{\ell+r-1}_{x+h-1,\ell+1} = D^{\ell+r}_{x+h-1,\ell+1} \setminus \alpha$ is removable from $\Psi \setminus \alpha$. Therefore,

$$\begin{aligned} K(\Psi; M; (\gamma, 1^r)) &= K(\Psi' \cup E_{h-1}; M' \sqcup L(E_{h-1}); (\gamma, 1^r)) \\ &\quad + K(\hat{\Psi}; \hat{M} \sqcup (x+r+h-2); (\gamma, 1^{r-1}) + \epsilon_{x+r+h-2}) \,. \end{aligned}$$

The inductive hypothesis applied to the first term with $h = h-1$ and applied to the second term with $r = r-1$ gives

$$\begin{aligned} K(\Psi; M; (\gamma, 1^r)) = \sum_{a=0}^{r} &\sum_{\substack{\mu = \gamma + \epsilon_T + \epsilon_{T'} \\ T \subseteq \{x+r-a, \ldots, x+r+h-3\} \\ |T| = a \\ T' = \{\ell+1, \ldots, \ell+r-a\}}} K(\Psi'; M' \sqcup T; \mu) \\ + \sum_{a=0}^{r-1} &\sum_{\substack{\mu = \gamma + \epsilon_{x+r+h-2} + \epsilon_T + \epsilon_{T'} \\ T \subseteq \{x+r-1-a, \ldots, x+r+h-3\} \\ |T| = a \\ T' = \{\ell+1, \ldots, \ell+r-1-a\}}} K(\Psi'; M' \sqcup (T \cup \{x+r+h-2\}); \mu) \end{aligned}$$

Reindexing the second sum to go from 1 to r readily shows that we recover the desired sum, with the first sum corresponding to $x+r+h-2 \notin S$ and the second to $x+r+h-2 \in S$. □

Proposition 3.6.5 (Unstraightened Pieri Rule). *For $\lambda \in \mathrm{Par}^k_{\ell-k-1}$ and $0 \leq r \leq k$,*

$$g_{1^r} \mathfrak{g}^{(k)}_\lambda = \sum_{a=0}^{r} \sum_{\substack{\mu = \lambda + \epsilon_S + \epsilon_{S'} \\ S \subseteq \{\ell-k+1+r-a, \ldots, \ell\} \\ |S| = a \\ S' = \{\ell+1, \ldots, \ell+r-a\}}} K(\Delta^k(\mu); L(\Delta^{k+1}(\mu)) \sqcup (S \cup S'); \mu) \,.$$

Proof. For $\lambda \in \mathrm{Par}^k_{\ell-k-1}$, Definition 3.6.1 and Lemma 3.4.7 give

$$\mathfrak{g}^{(k)}_\lambda = K(\Delta^k((\lambda, 0^{k+1})); \Delta^{k+1}((\lambda, 0^{k+1})); (\lambda, 0^{k+1})) \,.$$

Since $g_{1^r} = K(\varnothing_r; \varnothing_r; 1^r)$ by Proposition 3.4.3(b) where $\varnothing_r \subseteq \Delta_r^+$ denotes the empty root ideal of length r, the concatenation rule of Lemma 3.4.11 implies that

$$g_{1^r}\mathfrak{g}_\lambda^{(k)} = K(\Psi, M, (\lambda, 0^{k+1}, 1^r))\,,$$

for $\Psi = \Delta^k(\lambda, 0^{k+1}) \uplus \varnothing_r$ and $M = L(\Delta^{k+1}(\lambda, 0^{k+1}) \uplus \varnothing_r)$.

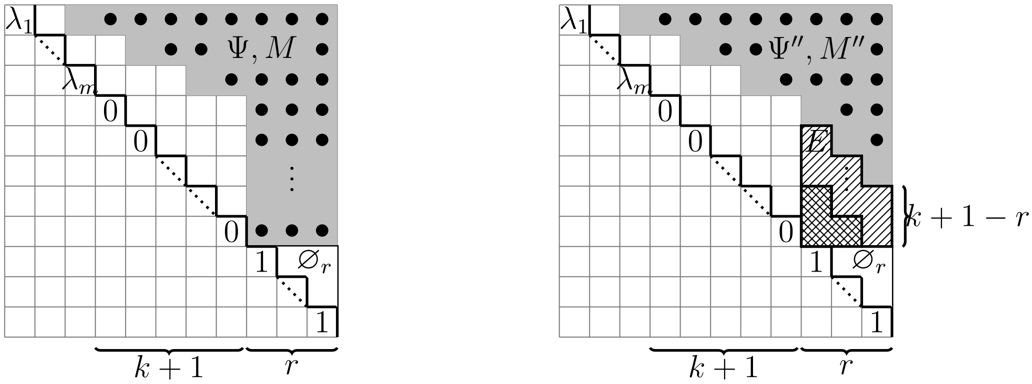


Figure 3-4: The schematic on the left represents $\Psi = \Delta^k(\lambda, 0^{k+1}) \uplus \varnothing_r$ and $M = L(\Delta^{k+1}(\lambda, 0^{k+1}) \uplus \varnothing_r)$. On the right, Ψ'' and M'' are the solid grey region and $\bullet$'s, respectively, and the crosshatched region is $\Psi \backslash \Psi' = \Psi \setminus (\Psi'' \cup E)$. Here, $m = \ell - k - 1$.

Let $E = E_{\ell-k+1,\ell+1}^{\ell+r,k+1-r}$ and set

$$\Psi'' = \Delta^k(\lambda, 0^{k+1}, 1^r) \quad \text{and} \quad \Psi' = \Psi'' \cup E\,;$$
$$M'' = L(\Delta^{k+1}(\lambda, 0^{k+1}, 1^r)) \quad \text{and} \quad M' = M'' \sqcup \{\ell+1, \ldots, \ell+r\} \sqcup L(E).$$

Observe that $\Psi \setminus \Psi' = D_{\ell,\ell+1}^{\ell+1} \cup D_{\ell-1,\ell+1}^{\ell+2} \cup \cdots \cup D_{\ell-r+2,\ell+1}^{\ell+r-1}$ and $M \setminus M' = L(\Psi \setminus \Psi')$. We remove these diagonals from Ψ by iteratively applying Diagonal Removal Lemma 3.5.13 until

$$g_{1^r}\,\mathfrak{g}_\lambda^{(k)} = K(\Psi, M, (\lambda, 0^{k+1}, 1^r)) = K(\Psi'; M'; (\lambda, 0^{k+1}, 1^r))\,.$$

We can then apply Lemma 3.6.4 with $x = \ell - k + 1, h = k + 1 - r, \Psi = \Psi'$ and $M = M'$

to get

$$K(\Psi';M';(\lambda,0^{k+1},1^r)) = \sum_{a=0}^{r} \sum_{\substack{\mu=\lambda+\epsilon_S+\epsilon_{S'} \\ S\subseteq\{\ell+r-k+1-a,\dots,\ell\} \\ |S|=a \\ S'=\{\ell+1,\dots,\ell+r-a\}}} K(\Psi'';(M'\setminus L(E))\sqcup S;\mu)\,.$$

Since $M'\setminus L(E) = M''\sqcup\{\ell+1,\dots,\ell+r\}$, we have for each summand $K(\Psi'';(M'\setminus L(E))\sqcup S;\mu) = K(\Psi'';M''\sqcup(S\cup S');\mu)$ by Remark 3.4.8. Further, since $\mu=\lambda+\epsilon_{S\cup S'}$ with $\max(S\cup S')-\min(S\cup S')\le k-1$, this is equal to $K(\Delta^k(\mu);L(\Delta^{k+1}(\mu))\sqcup(S\cup S');\mu)$ by Remark 3.6.2(d) and Remark 3.4.8. □

Lemma 3.6.6. *For $\ell\ge 1$ and $0\le r\le k$, the map*

$$\mathrm{rm}\colon \bigsqcup_{a=0}^{r} \bigsqcup_{\substack{S\subseteq\{\ell-k+1+r-a,\dots,\ell\} \\ |S|=a \\ S'=\{\ell+1,\dots,\ell+r-a\}}} \{S\cup S'\} \to \big\{R\subseteq\mathbb{Z}/(k+1)\mathbb{Z} \,:\, |R|=r\big\}$$

given by $S\cup S'\mapsto\{\overline{-s}\mid s\in S\cup S'\}$ is a bijection, where $\overline{z}$ denotes the image of z in $\mathbb{Z}/(k+1)\mathbb{Z}$.

Proof. For each $0\le a\le r$, $S\cup S'$ is a subset of the k consecutive entries $\{\ell-k+1+r-a,\dots,\ell+r-a\}$ and $|S\cup S'|=r$. Thus, rm is well-defined and one-to-one. Given $R\subseteq\mathbb{Z}/(k+1)\mathbb{Z}$ with $|R|=r$, to construct its preimage $S\cup S'$, consider the largest b such that $\{\overline{-(\ell+1)},\dots,\overline{-(\ell+b)}\}\subseteq R$ or set $b=0$ if $\overline{-(\ell+1)}\notin R$. Then $S\cup S'=f_{\ell,b}(R)$, for the map $f_{\ell,b}\colon\mathbb{Z}/(k+1)\mathbb{Z}\to\mathbb{Z}$ given by

$$\begin{aligned} \overline{-(\ell+i)} &\mapsto \ell+i \quad \text{for } 1\le i\le b \\ \overline{-(\ell+b+j)} &\mapsto \ell+b+j-k-1 \quad \text{for } 1\le j\le k+1-b. \end{aligned}$$ □

Combining Proposition 3.6.5 and Lemma 3.6.6 yields the following result.

Corollary 3.6.7. *For $\lambda \in \mathrm{Par}_\ell^k$ and $0 \le r \le k$,*

$$g_{1^r}\mathfrak{g}_\lambda^{(k)} = \sum_{\substack{R\subseteq \mathbb{Z}/(k+1)\mathbb{Z} \\ |R|=r}} K(\Delta^k(\lambda+\epsilon_A); L(\Delta^{k+1}(\lambda+\epsilon_A)) \sqcup A; \lambda+\epsilon_A)$$

where $A = \mathrm{rm}^{-1}(R)$.

3.6.2 Root ideal to core dictionary

Recall that the diagram of a partition λ is the subset of cells $\{(r,c) \in \mathbb{Z}_{\ge 1} \times \mathbb{Z}_{\ge 1} : c \le \lambda_r\}$ in the plane, drawn in English (matrix-style) notation so that rows (resp. columns) are increasing from north to south (resp. west to east). Each cell in a diagram has a *hook length* which counts the number of cells below it in its column and weakly to its right in its row. An *n-core* is a partition with no cell of hook length n. We use $\mathcal{C}^{k+1}$ to denote the collection of $k+1$-cores. Recall from Proposition 2.4.7 that we have the bijection,

$$\mathfrak{p} : \mathcal{C}^{k+1} \to \mathrm{Par}^k,$$

where $\mathfrak{p}(\kappa) = \lambda$ is the partition whose r-th row, λ_r, is the number of cells in the r-th row of κ with hook length $\le k$. Let $\mathfrak{c} = \mathfrak{p}^{-1}$. The *content* of a cell $(r,c) \in \mathbb{Z} \times \mathbb{Z}$ is $c - r$ and its $k+1$-*residue* is $\overline{c-r} \in \mathbb{Z}/(k+1)\mathbb{Z}$.

Given a $k+1$-core κ, define the row residue map

$$\mathrm{r} \colon \mathbb{Z}_{\ge 1} \to \mathbb{Z}/(k+1)\mathbb{Z}, \quad a \mapsto \overline{\kappa_a - a},$$

so that $\mathrm{r}(a)$ is the $k+1$-residue of the cell (a, κ_a) (if $a \le \ell(\kappa)$, this lies on the eastern border of κ but we also allow $a > \ell(\kappa)$ and understand $\kappa_a = 0$ in this case). We use the following lemma, obtained by taking [BMPS19, Proposition 8.2(b)] modulo $k+1$.

Lemma 3.6.8. *Let $\lambda \in \mathrm{Par}_\ell^k$ and $\Psi = \Delta^k(\lambda)$. If $\mathrm{up}_\Psi(x)$ is defined, then $\mathrm{r}(\mathrm{up}_\Psi(x)) = \mathrm{r}(x)$.*

Proposition 3.6.9. *Let $\lambda \in \mathrm{Par}_\ell^k$ and $\kappa = \mathfrak{c}(\lambda)$. The root ideal $\Delta^k(\lambda)$ has at most $k+1$ distinct bounce paths and cells (a, κ_a) and (b, κ_b) have the same $k+1$-residue if and only if a and b are in the same bounce path.*

Proof. Let $\Psi = \Delta^k(\lambda, 0^{k+1})$. By construction, Ψ has no roots in rows $[\ell+1, \ell+k+1]$, implying that each of $\ell+1, \dots, \ell+k+1$ lies in a distinct bounce path, $B_1, \dots, B_{k+1}$, respectively. Since $\mathrm{down}_\Psi(x)$ exists for all $x \in [\ell]$, $B_1, \dots, B_{k+1}$ are the *only* bounce paths in Ψ. Now, for $i \in [k+1]$, the $k+1$-residue of $(\ell+i, \kappa_{\ell+i})$ is

$$\mathrm{r}(\ell+i) = \overline{\kappa_{\ell+i} - \ell - i} = \overline{0 - (\ell+i)}\,.$$

Thus the residues $\mathrm{r}(\ell+1), \dots, \mathrm{r}(\ell+k+1)$ are distinct and so, by Lemma 3.6.8, $\mathrm{r}(a) = \mathrm{r}(i+\ell)$ for all $a \in B_i$. Therefore, $\mathrm{r}(a) = \mathrm{r}(b)$ if and only if a and b lie in the same bounce path. Because the bounce path of $x \in [\ell]$ in $\Delta^k(\lambda)$ is a (possibly empty) truncation of its bounce path in Ψ, the claim follows. □

Given a partition κ, an *addable* i-corner is a cell $(r, c) \notin \kappa$ of $k+1$-residue i such that $\kappa \cup \{(r, c)\}$ is a partition; a *removable* i-corner is a cell $(r, c) \in \kappa$ of $k+1$-residue i such that $\kappa \setminus \{(r, c)\}$ is a partition.

Proposition 3.6.10 (K-k-Schur root ideal to core dictionary)**.** *Let $\lambda \in \mathrm{Par}_j^k$ with $\lambda_{j-1} = \lambda_j = 0$. Set $i = \overline{-j+1}$. Then the bounce paths of $\Psi = \Delta^k(\lambda)$ are related to the $k+1$-core $\kappa = \mathfrak{c}(\lambda)$ as follows. Also, (a)–(c) below hold more generally with root ideal $\Delta^k(\lambda + \epsilon_S)$ in place of Ψ, for any $S \subseteq \mathbb{Z}_{\geq j}$.*

(a) $y = \mathrm{top}_\Psi(j-1) > \mathrm{top}_\Psi(j)$ *if and only if the lowest addable i-corner of κ lies in row* $a = \mathrm{up}_\Psi(y+1)$,

(b) $\mathrm{top}_\Psi(j) > \mathrm{top}_\Psi(j-1) + 1$ *if and only if κ has a removable i-corner,*

(c) $\mathrm{top}_\Psi(j) = \mathrm{top}_\Psi(j-1) + 1$ *if and only if κ has neither a removable i-corner nor addable i-corner.*

Proof. Let r be the row residue map of κ. Noting $\mathrm{r}(j) = i-1$ and $\mathrm{r}(j-1) = i$, by Proposition 3.6.9, the set of row indices $\{z \in [j] \mid \mathrm{r}(z) = i-1\} = \mathrm{uppath}_\Psi(j)$ and $\{z \in [j] \mid \mathrm{r}(z) = i\} = \mathrm{uppath}_\Psi(j-1)$. Using this, we have

$$\kappa \text{ has an addable } i\text{-corner in row } z \in [j] \iff \mathrm{r}(z-1) \neq i \text{ and } \mathrm{r}(z) = i-1$$

$$\iff z-1 \notin \mathrm{uppath}_\Psi(j-1) \text{ and } z \in \mathrm{uppath}_\Psi(j)\,; \tag{3.20}$$

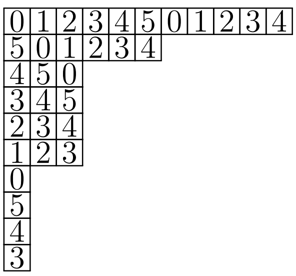
$$\kappa \text{ has a removable } i\text{-corner in row } z-1 \in [j] \iff \mathrm{r}(z-1) = i \text{ and } \mathrm{r}(z) \neq i-1$$

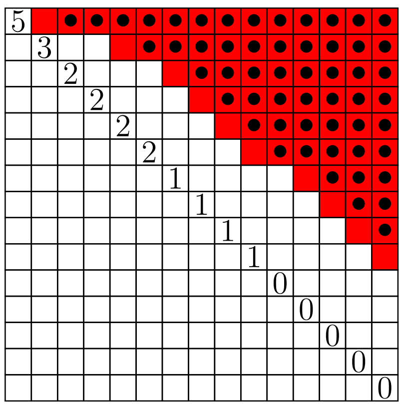
$$\iff z-1 \in \mathrm{uppath}_\Psi(j-1) \text{ and } z \notin \mathrm{uppath}_\Psi(j). \tag{3.21}$$

For $y = \max\{\mathrm{top}_\Psi(j-1), \mathrm{top}_\Psi(j)-1\}$, since j and $j-1$ cannot be in the same bouncepath, Ψ has a mirror in rows $x, x+1$ for $x \in \mathrm{uppath}_\Psi(\mathrm{up}_\Psi(j-1))$ such that $x \geq y$ by Remark 3.6.2(a). Thus, the bounce paths $\mathrm{uppath}_\Psi(j-1)$ and $\mathrm{uppath}_\Psi(j)$ have one of the following forms: (a) $j-1, j_2-1, j_3-1, \ldots, y$ and $j, j_2, j_3, \ldots, y+1, a, \ldots$; (b) $j-1, j_2-1, \ldots, y, b, \ldots$ and $j, j_2, \ldots, y+1$; or (c) $j-1, j_2-1, \ldots, y$ and $j, j_2, \ldots, y+1$. The result now follows from (3.20)–(3.21). Note that the more general statement holds simply because $\mathrm{uppath}_{\Delta^k(\lambda+\epsilon_S)}(j-1) = \mathrm{uppath}_\Psi(j-1)$ and $\mathrm{uppath}_{\Delta^k(\lambda+\epsilon_S)}(j) = \mathrm{uppath}_\Psi(j)$. $\square$

Example 3.6.11. For $k = 5$ and $\lambda = 532222111100000$, set $\Psi = \Delta^5(\lambda)$ and $\mathcal{L} = \Delta^6(\lambda)$. Then,

$\kappa = \mathfrak{c}(\lambda) =$





$= K(\Psi; \mathcal{L}; \lambda)\,,$

where we have filled the cells of κ with their $k+1$-residues. Note that, for example, $\overline{4} = \mathrm{r}(1) = \mathrm{r}(2) = \mathrm{r}(5) = \mathrm{r}(9) = \mathrm{r}(14)$ illustrating Lemma 3.6.8. We can also observe examples of all three cases of Proposition 3.6.10. For (a), let $j = 14$. Then, $\mathrm{top}_\Psi(14) = 1 < 4 = \mathrm{top}_\Psi(13)$ and the lowest addable corner of residue $i = \overline{-14+1} = \overline{5}$ is in row 2 of κ. For (b), let $j = 15$. Then, $\mathrm{top}_\Psi(15) = 6 > 1+1 = \mathrm{top}_\Psi(14)+1$ and κ has a removable corner of residue $i = \overline{-15+1} = \overline{4}$. Finally, for (c), let $j = 13$. Then, $\mathrm{top}_\Psi(13) = 4 = \mathrm{top}_\Psi(12)+1$ and κ has neither a removable nor an addable corner of residue $\overline{-13+1} = \overline{0}$.

Lemma 3.6.12 ([LM05, Proposition 22, §8.1])**.** *Let κ be a $k+1$-core and $\lambda = \mathfrak{p}(\kappa)$. Then $s_i w_\lambda \in \widehat{S}^0_{k+1}$ if and only if κ has an addable or removable i-corner. Moreover, κ has an addable i-corner if and only if $s_i w_\lambda = w_{\lambda+\epsilon_a} \in \widehat{S}^0_{k+1}$, where a is the row index of the lowest addable i-corner of κ. The core κ has a removable i-corner if and only if $s_i w_\lambda = w_{\lambda-\epsilon_a} \in \widehat{S}^0_{k+1}$, where a is the row index of the lowest removable i-corner of κ.*

3.6.3 Proof of the vertical Pieri rule

Definition 3.6.13. Given a root ideal $\Psi \subseteq \Delta^+_\ell$ and weight $\gamma \in \mathbb{Z}^\ell$, define

$$\mathrm{maxband}(\Psi,\gamma) = \max\{\gamma_i + \mathrm{nr}(\Psi)_i : i \in [\ell]\}, \text{ for } \mathrm{nr}(\Psi)_i := \big|\{j \in \{i+1,\dots,\ell\} : (i,j) \notin \Psi\}\big|. \tag{3.22}$$

Proposition 3.6.14. *For a root ideal $\Psi \subseteq \Delta^+_\ell$, a multiset M on $[\ell]$, and $\gamma \in \mathbb{Z}^\ell$ satisfying maxband$(\Psi,\gamma) \le k$, there holds $K(\Psi;M;\gamma) \in \Lambda_{(k)}$.*

Proof. Consider that, by definition,

$$K(\Psi;M;\gamma) = \sum_{A\subseteq M} (-1)^{|A|} H(\Psi;\gamma-\epsilon_A)\,,$$

where the summation is over all sub-multisets A of M. Since $\mathrm{maxband}(\Psi,\gamma-\epsilon_A) \le k$, each summand $H(\Psi;\gamma-\epsilon_A) \in \Lambda_{(k)}$ by [BMPS20, Proposition 1.4]. □

Proposition 3.6.15. *The set $\{\mathfrak{g}^{(k)}_\lambda\}_{\lambda\in\mathrm{Par}^k}$ forms a basis for $\Lambda_{(k)}$. Moreover, it is unitriangularly related to the k-Schur basis, i.e., $\mathfrak{g}^{(k)}_\lambda = s^{(k)}_\lambda + \sum_{|\mu|<|\lambda|} a_{\lambda\mu} s^{(k)}_\mu$ for $a_{\lambda\mu} \in \mathbb{Z}$.*

Proof. By Proposition 3.6.14, $\mathfrak{g}_\lambda^{(k)}$ lies in $\Lambda_{(k)}$ and so can be written in terms of the k-Schur basis of $\Lambda_{(k)}$; this expansion has the stated form since the highest degree term of $K(\Psi; M; \gamma)$ is $H(\Psi;\gamma)$ irrespective of M. Hence, the transition matrix from $\{\mathfrak{g}_\lambda^{(k)}\}$ to $\{s_\mu^{(k)}\}$ is unitriangular and thus the former is a basis. □

Recall from Section 2.4.1 that $w_\lambda \in \widehat{S}^0_{k+1}$ is the minimal coset representative corresponding to $\lambda \in \mathrm{Par}^k$. For any $\lambda \in \mathrm{Par}^k$, set $\mathfrak{g}_{w_\lambda}^{(k)} = \mathfrak{g}_\lambda^{(k)}$, so that the basis $\{\mathfrak{g}_\lambda^{(k)}\}_{\lambda\in\mathrm{Par}^k}$ can also be written $\{\mathfrak{g}_v^{(k)}\}_{v\in\widehat{S}^0_{k+1}}$. Recall that H_{k+1} denotes the 0-Hecke algebra of $\widehat{S}_{k+1}$ with generators $\{T_i \mid i \in \{0,1,\ldots,k\}\}$ (Definition 3.3.1).

Proposition 3.6.16. *The rule*

$$T_i \cdot \mathfrak{g}_v^{(k)} = \begin{cases} \mathfrak{g}_{s_i v}^{(k)} & \ell(s_i v) > \ell(v) \text{ and } s_i v \in \widehat{S}^0_{k+1}\,, \\ -\mathfrak{g}_v^{(k)} & \ell(s_i v) < \ell(v)\,, \\ 0 & s_i v \notin \widehat{S}^0_{k+1}\,, \end{cases} \tag{3.23}$$

for $i \in \{0,1,\ldots,k\}$ and $v \in \widehat{S}^0_{k+1}$, determines an action of H_{k+1} on $\Lambda_{(k)}$.

Note that the three cases are mutually exclusive since $\ell(s_i v) < \ell(v)$ implies $s_i v \in \widehat{S}^0_{k+1}$.

Proof. Consider $e = \sum_{w\in S_{k+1}} T_w \in H_{k+1}$ and note that for $i \in [k]$, $T_i e = 0$ and so $T_u e = 0$ for any $u \in \widehat{S}_{k+1} \setminus \widehat{S}^0_{k+1}$. Recalling that $\{T_w\}_{w\in\widehat{S}_{k+1}}$ is a $\mathbb{Z}$-basis of H_{k+1}, it follows that the left module $M = H_{k+1}e$ has $\mathbb{Z}$-basis $\{T_v e\}_{v\in\widehat{S}^0_{k+1}}$. We then check that the $\mathbb{Z}$-linear map $M \to \Lambda_{(k)}$ given by $T_v e \mapsto \mathfrak{g}_v^{(k)}$ is an H_{k+1}-module isomorphism by computing

$$T_i \cdot T_v e = T_i T_v e = \begin{cases} T_{s_i v} e & \ell(s_i v) > \ell(v) \text{ and } s_i v \in \widehat{S}^0_{k+1}\,, \\ -T_v e & \ell(s_i v) < \ell(v)\,, \\ T_{s_i v} e = 0 & \ell(s_i v) > \ell(v) \text{ and } s_i v \notin \widehat{S}^0_{k+1}\,. \end{cases}$$

□

Lemma 3.6.17. *For $\lambda \in \mathrm{Par}^k_m$, $0 \le r \le k$, and $S = \{a_1 < a_2 < \cdots < a_r\} \subseteq \mathbb{Z}_{\ge m+2}$ with $a_r - a_1 \le k-1$,*

$$K(\Delta^k(\mu); L(\Delta^{k+1}(\mu)) \sqcup S; \mu) = T_{i_r} \cdots T_{i_1} \mathfrak{g}^{(k)}_{w_\lambda},$$

where $\mu = \lambda + \epsilon_S$ and $i_z := \overline{-a_z + 1}$ for $z \in [r]$.

Proof. If $|S| = 0$, then the claim holds by definition of $\mathfrak{g}^{(k)}_{w_\lambda}$. Proceed by induction, with $|S| = r > 0$. Set $\kappa = \mathfrak{c}(\lambda)$, $\Psi = \Delta^k(\mu)$, and $M = L(\Delta^{k+1}(\mu))$. Let $j = a_1 = \min(S)$, and note $i_1 = \overline{-j+1}$.

First suppose $y = \mathrm{top}_\Psi(j-1) > \mathrm{top}_\Psi(j)$. Then Proposition 3.6.3 implies

$$K(\Psi; M \sqcup S; \mu) = K(\Delta^k(\nu); L(\Delta^{k+1}(\nu)) \sqcup (S \setminus j); \nu),$$

for $\nu := \mu + \epsilon_a - \epsilon_j$, where $a = \mathrm{up}_\Psi(y+1)$. Since $\nu = (\lambda + \epsilon_a) + \epsilon_{S \setminus \{j\}}$, induction gives

$$K(\Delta^k(\nu); L(\Delta^{k+1}(\nu)) \sqcup (S \setminus j); \nu) = T_{i_r} \cdots T_{i_2} \mathfrak{g}^{(k)}_{\lambda+\epsilon_a}.$$

By Proposition 3.6.10(a), the lowest addable i_1-corner of κ lies in row a. Therefore, $w_{\lambda+\epsilon_a} = s_{i_1} w_\lambda$ by Lemma 3.6.12. Then, by Proposition 3.6.16 and the fact that $\ell(w_\lambda) = |\lambda|$, we have $\mathfrak{g}^{(k)}_{\lambda+\epsilon_a} = \mathfrak{g}^{(k)}_{s_{i_1} w_\lambda} = T_{i_1} \mathfrak{g}^{(k)}_\lambda$.

Next suppose $\mathrm{top}_\Psi(j) > \mathrm{top}_\Psi(j-1) + 1$. Proposition 3.6.3 yields

$$K(\Psi; M \sqcup S; \mu) = -K(\Psi; M \sqcup (S \setminus \{j\}); \lambda + \epsilon_{S \setminus \{j\}}).$$

Rewriting using Remark 3.6.2(d) with $\nu = \lambda + \epsilon_{S\setminus\{j\}}$, and then applying induction yields

$$-K(\Psi; M \sqcup (S \setminus \{j\}); \lambda + \epsilon_{S\setminus\{j\}}) = -K(\Delta^k(\nu); L(\Delta^{k+1}(\nu)) \sqcup (S \setminus \{j\}); \nu) = -T_{i_r} \cdots T_{i_2} \mathfrak{g}^{(k)}_\lambda.$$

By Proposition 3.6.10(b), κ has a removable i_1-corner, so $-\mathfrak{g}^{(k)}_\lambda = T_{i_1} \mathfrak{g}^{(k)}_\lambda$ by Lemma 3.6.12 and Proposition 3.6.16.

Finally, suppose $\mathrm{top}_\Psi(j) = \mathrm{top}_\Psi(j-1) + 1$. Proposition 3.6.3 yields $K(\Psi; M \sqcup S; \mu) =$

0. By Proposition 3.6.10(c), κ has neither an addable nor a removable i_1-corner, so $T_{i_r} \cdots T_{i_2} T_{i_1} \mathfrak{g}_\lambda^{(k)} = T_{i_r} \cdots T_{i_2} \big(T_{i_1} \mathfrak{g}_\lambda^{(k)} \big) = 0$ by Lemma 3.6.12 and Proposition 3.6.16. $\square$

We can now show the $\mathfrak{g}_\lambda^{(k)}$ satisfy the Pieri rule (3.8), thereby showing $\mathfrak{g}_\lambda^{(k)}$ is equal to the K-k-Schur function $g_\lambda^{(k)}$.

Theorem 3.6.18. *For $0 \leq r \leq k$ and $\lambda \in \mathrm{Par}^k$,*

$$g_{1^r} \mathfrak{g}_\lambda^{(k)} = \sum_{\substack{u \in \widehat{S}_{k+1} \text{ cyclically increasing} \\ \ell(u) = r \\ T_u T_{w_\lambda} = \pm T_w;\, w \in \tilde{S}^0_{k+1}}} (-1)^{\ell(w_\lambda) + r - \ell(w)} \mathfrak{g}_w^{(k)} \,.$$

Proof. Corollary 3.6.7 gives

$$g_{1^r} \mathfrak{g}_\lambda^{(k)} = \sum_{\substack{R \subseteq \mathbb{Z}/(k+1)\mathbb{Z} \\ |R| = r}} K(\Delta^k(\mu); L(\Delta^{k+1}(\mu)) \sqcup A; \mu) \,,$$

where $\mu = \lambda + \epsilon_A$ for $A = \mathrm{rm}^{-1}(R)$. The result then follows by applying Lemma 3.6.17 to each summand to get

$$g_{1^r} \mathfrak{g}_\lambda^{(k)} = \sum_{s_{i_r} \cdots s_{i_1} \text{ cyclically increasing}} T_{i_r} \cdots T_{i_1} \mathfrak{g}_{w_\lambda}^{(k)}$$

and then using Proposition 3.6.16. $\square$

Theorem 3.6.19. *For any $\lambda \in \mathrm{Par}^k$, $\mathfrak{g}_\lambda^{(k)} = g_\lambda^{(k)}$. Thus, the k-Schur Katalan functions are representatives for the Schubert basis of the K-homology of the affine Grassmannian of SL_{k+1}.*

3.7 K-k-Schur properties

The foremost application of the Katalan function formulation for K-k-Schur functions is the ease with which shift invariance (3.24) follows.

Theorem 3.7.1 (Shift Invariance). *For $\lambda \in \mathrm{Par}_\ell^k$,*

$$G_{1^\ell}^\perp \, \mathfrak{g}_{\lambda+1^\ell}^{(k+1)} = \mathfrak{g}_\lambda^{(k)} \quad \textit{where} \quad G_{1^\ell} = \sum_{i\geq 0} (-1)^i \binom{\ell-1+i}{\ell-1} e_{\ell+i}\,. \tag{3.24}$$

Hence by Theorem 3.6.19, $G_{1^\ell}^\perp \, g_{\lambda+1^\ell}^{(k+1)} = g_\lambda^{(k)}$ as well.

Proof. We use that $e_s^\perp h_m = h_m e_s^\perp + h_{m-1} e_{s-1}^\perp$ from [GP92, Equation 5.37] to deduce

$$e_s^\perp k_m^{(r)} = \sum_{i=0}^{m} \binom{r+i-1}{i} (h_{m-i} e_s^\perp + h_{m-i-1} e_{s-1}^\perp) = k_m^{(r)} e_s^\perp + k_{m-1}^{(r)} e_{s-1}^\perp\,.$$

Using that $e_i^\perp(1) = 0$ for $i > 0$, this applies to the formulation for Katalan functions in Proposition 3.4.2, giving that, for $s \geq 0$, $\Psi \subseteq \Delta^+$ a root ideal, M a multiset with $\mathrm{supp}(M) \subseteq \{1, \dots, \ell\}$, and $\gamma \in \mathbb{Z}^\ell$,

$$e_s^\perp K(\Psi; M; \gamma) = \sum_{S \subseteq [\ell],\ |S| = s} K(\Psi; M; \gamma - \epsilon_S)\,,$$

where $\epsilon_S = \sum_{i \in S} \epsilon_i$. In particular, $e_\ell^\perp K(\Psi; M; \gamma + 1^\ell) = K(\Psi; M; \gamma)$. Now for $\lambda \in \mathrm{Par}_\ell^k$, noting that $\Delta^m(\lambda + 1^\ell) = \Delta^{m-1}(\lambda)$ for any $m \geq k+1$, we obtain

$$e_\ell^\perp \mathfrak{g}_{\lambda+1^\ell}^{(k+1)} = e_\ell^\perp K(\Delta^{k+1}(\lambda+1^\ell); \Delta^{k+2}(\lambda+1^\ell); \lambda+1^\ell) = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda) = \mathfrak{g}_\lambda^{(k)}\,.$$

Therefore, $e_\ell^\perp \mathfrak{g}_{\lambda+1^\ell}^{(k+1)} = \mathfrak{g}_\lambda^{(k)}$. Since $e_s^\perp K(\Psi; \mathcal{L}; \lambda) = 0$ for $s > \ell$, we can replace $e_\ell^\perp$ by $G_{1^\ell}^\perp$. □

Shift invariance implies that K-k-Schur branching coefficients are a subset of the Pieri coefficients for affine stable Grothendieck polynomials, settling Conjecture 1.2.1.

Theorem 3.7.2. *For any $\lambda \in \mathrm{Par}^k$,*

$$g_\lambda^{(k)} = \sum_{\mu \in \mathrm{Par}^{k+1}} a_{\lambda\mu}\, g_\mu^{(k+1)} \quad \textit{where } (-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}. \tag{3.25}$$

Proof. Fix $\ell = \ell(\lambda)$. For $\mu \in \mathrm{Par}^{k+1}$, Baldwin and Kumar [BK17] proved that

$$G^{(k+1)}_{1^\ell} G^{(k+1)}_{\mu} = \sum_{\gamma} c_{\gamma\mu}\, G^{(k+1)}_{\gamma} \text{ satisfy } (-1)^{|\gamma|-\ell-|\mu|} c_{\gamma\mu} \in \mathbb{Z}_{\geq 0}\,. \tag{3.26}$$

Since $\langle g^{(k+1)}_{\alpha}, G^{(k+1)}_{\beta} \rangle = \delta_{\alpha\beta}$ for $\alpha, \beta \in \mathrm{Par}^{k+1}$, from (3.26) we obtain

$$c_{\gamma\mu} = \Big\langle g^{(k+1)}_{\gamma}, \sum_{\beta} c_{\beta\mu}\, G^{(k+1)}_{\beta} \Big\rangle = \Big\langle g^{(k+1)}_{\gamma}, G^{(k+1)}_{1^\ell} G^{(k+1)}_{\mu} \Big\rangle = \Big\langle (G^{(k+1)}_{1^\ell})^{\perp} g^{(k+1)}_{\gamma}, G^{(k+1)}_{\mu} \Big\rangle\,.$$

Therefore, for $\gamma = \lambda + 1^\ell$,

$$\sum_{\mu} c_{\gamma\mu} g^{(k+1)}_{\mu} = (G^{(k+1)}_{1^\ell})^{\perp} g^{(k+1)}_{\gamma} = g^{(k)}_{\lambda},$$

where we can apply Theorem 3.7.1 (shift-invariance) to the second equality because $G^{(k+1)}_{1^\ell} = G_{1^\ell}$ by Lemma 3.3.7.We thus have that $a_{\lambda\mu} = c_{\lambda+1^\ell,\mu}$, and the result follows from (3.26). □

Other properties of K-k-Schur functions are readily apparent from the Katalan/raising operator description. For example, the following property was conjectured in [Mor12]; while seemingly simple, it was not apparent from previous descriptions and is the missing ingredient for resolving conjectures in [LSS10b, Mor12, IIM20].

Corollary 3.7.3. *For $\mu \in \mathrm{Par}^k_\ell$ with $\mu_1 + \ell - 1 \leq k$, $g^{(k)}_{\mu} = g_{\mu}$.*

Proof. Since $\Delta^k(\mu) = \varnothing = \Delta^{k+1}(\mu)$ when $k - \mu_1 + 1 \geq \ell$, the result follows from Definition 3.6.1. □

By iterating branching to obtain an expansion for $g^{(k)}_{\lambda}$ in terms of $g^{(a)}_{\mu}$ for large enough a so that Corollary 3.7.3 applies to every term, we establish [Mor12, Conjecture 46] as well.

Corollary 3.7.4. *For* $\lambda \in \mathrm{Par}^k$,

$$g_\lambda^{(k)} = \sum_\mu b_{\lambda\mu}\, g_\mu \qquad \textit{where } (-1)^{|\lambda|-|\mu|} b_{\lambda\mu} \in \mathbb{Z}_{\geq 0}.$$

Chapter 4

Applications to the K-theoretic Peterson isomorphism

In this chapter, we turn our attention to the K-theoretic Peterson isomorphism, which is a map from the quantum K-theory of the flag variety to the K-homology of the affine Grassmannian. In the (co)homological setting, the Peterson isomorphism was first presented in lectures by Peterson in MIT in 1997 and published in [LS10]. In this case, the Schubert representatives of the quantum cohomology of the flag variety are sent to a k-Schur function, up to predictable localization. The K-theoretic situation is slightly more intricate and is the motivation for this chapter.

4.1 Schubert calculus of the flag variety

4.1.1 The flag variety

We start with the classically studied flag variety or flag manifold.

Definition 4.1.1. (a) Given a complex vector space V of dimension n, a *(complete)* *flag* in V is a sequence of subspaces

$$A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n = V\,, \quad \dim_{\mathbb{C}} A_i = i$$

Note that a flag on V induces a $\mathbb{C}$-basis $\{b_1, \ldots, b_n\}$ on V by picking basis vector $b_i \in A_i/A_{i-1}$ (take $A_0 = \{0\}$).

(b) Let $Fl(V)$ be the collection of all flags of V, called the *flag variety* or *flag manifold.*

Remark 4.1.2. We can give $Fl(V)$ the structure of a projective variety via

$$Fl(V) \subseteq \prod_{i=0}^{n} \mathrm{Gr}_i(V) \subseteq \prod_{i=0}^{n} \mathbb{P}(\wedge^i V) \subseteq \mathbb{P}^N$$

One also observers that GL_n acts on $Fl(V)$ (for $\dim V = n$) and, since the standard Borel subgroup B of upper triangular matrices of GL_n stabilizes the coordinate flag, $0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq V$, we can identify $Fl(V) = GL_n/B$.

Then, we give the following definitions of Schubert cells.

Definition 4.1.3. For $w \in S_n$,

(a) We define the flag

$$F_w := 0 \subseteq \langle e_{w(1)} \rangle \subseteq \langle e_{w(1)}, e_{w(2)} \rangle \subseteq \cdots \subseteq V$$

(b) We define *Schubert cells* in $Fl(V)$ by $C_w := BF_w = BwB/B$.

(c) We define *Schubert varieties* in $Fl(V)$ by $X_w = \overline{C_w}$.

(d) Let $m = \dim_{\mathbb{R}} Fl(V) = n(n-1)$ for $\dim V = n$. Given Schubert variety $X_w \subseteq Fl(V)$, we define its *Schubert class* $\sigma_w \in H^{m-2\ell(w)}$ to be the Poincare dual to the representative in homology of X_w.

Proposition 4.1.4. *For $V = \mathbb{C}^n$, $H^*(Fl(V)) \cong \bigoplus_{w \in S_n} \mathbb{Z}\sigma_w$ as a $\mathbb{Z}$-module.*

As with the Grassmannian case, we wish to understand the ring structure of $H^*(Fl(V))$ and we do so by finding polynomial representatives for σ_w in the form of Schubert polynomials.

4.1.2 Schubert polynomials

Definition 4.1.5. For fixed $n \in \mathbb{Z}_{>0}$, define the *divided difference operator* for $i \in [n-1]$ as the operator ∂_i on $\mathbb{Z}[x_1, \dots, x_n]$ via the formula, for $f \in \mathbb{Z}[x_1, \dots, x_n]$,

$$(\partial_i f)(x_1, \dots, x_n) = \frac{f(\dots, x_i, x_{i+1}, \dots) - f(\dots, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}},$$

which we abbreviate as $\partial_i = (x_i - x_{i+1})^{-1}(1 - s_i)$.

Proposition 4.1.6. *The divided difference operators satisfy the relations*

$$\partial_i^2 = 0 \quad \partial_i\partial_j = \partial_j\partial_i \text{ if } |i-j| > 1 \quad \partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}\,.$$

Thus, for $w \in S_n$ with reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, we can define $\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}$. By Proposition 4.1.6, ∂_w is independent of choice of reduced expression for w and thus is well-defined.

Definition 4.1.7. For fixed $n \in \mathbb{Z}_{>0}$, let $w_0 \in S_n$ be the longest element. Then, for any $w \in S_n$, we define

$$\mathfrak{S}_w = \partial_{w^{-1}w_0} x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1\,. \tag{4.1}$$

We call $\mathfrak{S}_w$ a *Schubert polynomial.*

Proposition 4.1.8. *We have the following special cases of Schubert polynomials.*

(a) $\mathfrak{S}_{id} = 1$,

(b) For $1 \le i \le n$, $\mathfrak{S}_{s_i} = x_1 + \cdots + x_i$.

Thus, $x_1 = \mathfrak{S}_1$ and $x_i = \mathfrak{S}_{s_i} - \mathfrak{S}_{s_{i-1}}$ for $1 < i < n$. Furthermore, we have the following multiplication rule.

Proposition 4.1.9 (Monk's rule)**.** *For $w \in S_n$ and $1 \leq r < n$, we have that*

$$\mathfrak{S}_{s_r}\mathfrak{S}_w = \sum_{\substack{i \leq r < j \\ \ell(wt_{ij}) = \ell(w)+1}} \mathfrak{S}_{wt_{ij}} , \tag{4.2}$$

where $t_{ij} \in S_n$ is the transposition that swaps i and j.

For fixed n, let L_n be a $\mathbb{Z}$-submodule of $\mathbb{Z}[x_1, \ldots, x_n]$ generated by $L_n = \langle x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} \mid 0 \leq i_j \leq n-j \rangle$. We have that $\mathfrak{S}_w \in L_n$. Then, for $I_n = \langle h_i(x_1, \ldots, x_n) \mid i \geq 1 \rangle$, it is immediate that $L_n \cong \mathbb{Z}[x_1, \ldots, x_n]/I_n$.

Proposition 4.1.10. *Let $I = (h_i(x) \mid 1 \leq i) \subseteq \mathbb{Z}[x_1, \ldots, x_n]$. Then, there is a ring isomorphism $H^*(Fl_n) \to \mathbb{Z}[x_1, \ldots, x_n]/I$ sending*

$$\sigma_w \mapsto \mathfrak{S}_w . \tag{4.3}$$

Thus, we see the Schubert polynomials are the Schubert representatives for the cohomology of the flag variety.

4.2 Grothendieck polynomials and quantum Grothendieck polynomials

4.2.1 Grothendieck polynomials

Definition 4.2.1. For fixed $n \in \mathbb{Z}_{>0}$, define operators π_i on $\mathbb{Z}[x_1, \ldots, x_n]$ by the formula $\pi_i = \partial_i(1 - x_{i+1})$.

As with ∂_w, the definition $\pi_w = \pi_{s_{i_1}} \cdots \pi_{s_{i_\ell}}$ for any reduced expression $s_{i_1} \cdots s_{i_\ell} = w \in S_n$ is well-defined.

Definition 4.2.2. For any $w \in S_n$, define the *Grothendieck polynomial* $\mathfrak{G}_w$ by

$$\mathfrak{G}_w = \pi_{w^{-1}w_0} x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 .$$

Example 4.2.3. We have the following examples of Grothendieck polynomials when $n = 3$ with permutations written in one-line notation.

$$\mathfrak{G}_{321}(x) = x_1^2 x_2$$
$$\mathfrak{G}_{312}(x) = x_1^2 \quad \mathfrak{G}_{231}(x) = x_1 x_2$$
$$\mathfrak{G}_{213}(x) = x_1 \quad \mathfrak{G}_{132}(x) = x_1 + x_2 - x_1 x_2$$
$$\mathfrak{G}_{123}(x) = 1$$

In general, $\mathfrak{G}_w = \mathfrak{S}_w +$ higher degree terms. The Grothendieck polynomials play a similar role for the K-theory of Fl_n as the Schubert polynomials play for its cohomology.

Proposition 4.2.4 ([Bri02, Lemma 2]). *Let $I = (h_i(x) \mid 1 \leq i) \subseteq \mathbb{Z}[x_1, \ldots, x_n]$. Then, there is a ring isomorphism $K^0(Fl(V)) \cong \mathbb{Z}[x_1, \ldots, x_n]/I$ sending*

$$[\mathcal{O}_{X_w}] \mapsto \mathfrak{G}_w \tag{4.4}$$

where $[\mathcal{O}_{X_w}]$ is the class of the structure sheaf $\mathcal{O}_{X_w}$ of X_w in $K^0(Fl(V))$.

4.2.2 Quantum Grothendieck polynomials

Lenart and Maeno defined [LM06, Definition 3.18] the *quantum Grothendieck polynomials* $\{\mathfrak{G}_w^q(x_1, \ldots, x_{k+1}, q_1, \ldots, q_k)\}_{w \in S_{k+1}}$ as the image of the ordinary Grothendieck polynomials $\{\mathfrak{G}_w\}_{w \in S_{k+1}}$ under a quantization map. The $\mathfrak{G}_w^q$'s specialize to the $\mathfrak{G}_w$ at $q_1 = \cdots q_k = 0$. Let $L_n^q = \mathbb{Z}[q_1, \ldots, q_{n-1}] \otimes L_n$ for L_n defined as in § 4.1.2. In order to define the quantization map, we give the following definitions. However, we will not use them for any proofs.

Definition 4.2.5 ([LM06, (3.1) and (3.8)]). Let $0 \le p \le k \le n$.

(a) Define the polynomials $F_p^k \in \mathbb{Z}[q_1, \dots, q_n][x_1, \dots, x_n]$ by $F_0^k = 1$ and, for $p \ge 1$, by

$$F_p^k = \sum_{\substack{I \subseteq [k] \\ |I|=p}} \prod_{i \in I} (1 - x_i) \prod_{\substack{i \in I \\ i+1 \notin I}} (1 - q_i) \,.$$

(b) Define the polynomials $\hat{E}_p^k$ by

$$\hat{E}_p^k = \sum_{i=0}^{p} (-1)^i \binom{k-i}{p-i} F_i^k \,.$$

(c) Define $\hat{E}_{p_1,\dots,p_{n-1}} = \hat{E}_{p_1}^1 \cdots \hat{E}_{p_{n-1}}^{n-1}$.

Note, whenever the condition $0 \le p \le k$ is violated, we let $F_p^k = 0$.

Then, it turns out that $\hat{E}_{p_1,\dots,p_{n-1}} \in L_n^q$ and we can define the following map.

Definition 4.2.6 ([LM06, Definition 3.14]). For this definition only, let $e_p^k = e_p(x_1, \dots, x_k)$ and $e_{p_1 \dots p_m} = e_{p_1}^1 \cdots e_{p_m}^m$. Then, we define quantization map $\hat{Q}\colon L_n^q \to L_n^q$ to be the $\mathbb{Z}[q_1, \dots, q_n]$-linear map given by

$$\hat{Q}(e_{p_1 \dots p_{n-1}}) = \hat{E}_{p_1,\dots,p_{n-1}} \,,$$

where $0 \le p_i \le i$.

Definition 4.2.7. The *quantum Grothendieck polynomial* $\mathfrak{G}_w^q$ for $w \in S_n$ is

$$\mathfrak{G}_w^q = \hat{Q}(\mathfrak{G}_w) \,.$$

Proposition 4.2.8 ([LM06, Proposition 3.22(2)]). *For any $w \in S_n$, specializing the q variables in $\mathfrak{G}_w^q$ recovers $\mathfrak{G}_w$. In other words, $\mathfrak{G}_w^q|_{q_i=0} = \mathfrak{G}_w$.*

The quantum K-theory ring $QK(Fl_n)$ can be identified with a quotient of $\mathbb{C}[z_1, \ldots, x_n, q_1, \ldots, q_{n-1}]$ by [KM, ACT17]; see, e.g., [IIM20, §1.1–1.2] which includes an explicit description of the defining ideal. Then, we have the following result.

Proposition 4.2.9 ([LNS20]). *$\{\mathfrak{G}^q_w\}_{w\in S_n}$ serves as a Schubert basis for $QK(Fl_n)$.*

4.3 The Peterson isomorphism

A *k-rectangle* is a partition of the form $R_i := (k+1-i)^i$ for $i \in [k]$. Define $\sigma_i = \sum_{\mu\subseteq R_i} g_\mu$ for $i \in [k]$, and set $\sigma_0 = \sigma_{k+1} = g_{R_0} = g_{R_{k+1}} = 1$. Ikeda, Iwao, and Maeno give the following description of a K-theoretic version of the Peterson isomorphism [IIM20, Theorem 1.5]:

$$\Phi\colon QK(Fl_{k+1})[Q_1^{-1}, \ldots, Q_k^{-1}] \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{(k)}[g_{R_1}^{-1}, \ldots, g_{R_k}^{-1}, \sigma_1^{-1}, \ldots, \sigma_k^{-1}]$$
$$z_i \mapsto \frac{g_{R_i}\sigma_{i-1}}{g_{R_{i-1}}\sigma_i}, \quad Q_i \mapsto \frac{g_{R_{i-1}}g_{R_{i+1}}}{g_{R_i}^2}.$$

Following [IIM20, §5.4] in this section, we work with $\{\mathfrak{G}^Q_w(z_1, \ldots, z_{k+1}, Q_1, \ldots, Q_k)\}_{w\in S_{k+1}} \subseteq QK(Fl_{k+1})$ which differs from the $\mathfrak{G}^q_w$'s by the change of variables $z_i = 1 - x_i$ for all $i \in [k+1]$ and $Q_i = q_i$ for $i \in [k]$.

The images $\Phi(\mathfrak{G}^Q_w)$ are described in terms of a map $\theta\colon S_{k+1} \to \mathrm{Par}^k$. For $w = w_1 \cdots w_{k+1} \in S_{k+1}$ in one-line notation, the *descent set* of w is $\mathrm{Des}(w) = \{i : w_i > w_{i+1}\}$, and its *inversion sequence* $\mathrm{Inv}(w) \in \mathbb{Z}^k_{\geq 0}$ is given by $\mathrm{Inv}_i(w) = \big|\{j > i : w_i > w_j\}\big|$. Define an injection $\zeta : S_{k+1} \to \mathrm{Par}^k$ by letting column i of $\zeta(w)$ be

$$\binom{k+1-i}{2} + \mathrm{Inv}_i(w_0 w)\,, \tag{4.5}$$

for all $i \in [k]$, where w_0 denotes the longest element of S_{k+1}. An element of Par^k is *irreducible* if it has at most $k-i$ parts of size i, or equivalently, it contains no k-rectangle as a subsequence. For any $\mu \in \mathrm{Par}^k$, define the unique irreducible partition $\mu_\downarrow$ by deleting from μ the k-rectangles it contains as a subsequence. Set $\theta(w) = \zeta(w)_\downarrow$. By [BMPS20, Lemma 7.3], the map θ is the same as the map λ from [LS12, §6], [IIM20, §7.1].

The k-conjugate involution on Par^k introduced in [LM05] can be described as follows: for $\mu \in \mathrm{Par}^k$, its k-conjugate is $\mu^{\omega_k} = \mathfrak{w}^{-1} \circ \tau \circ \mathfrak{w}(\mu)$, for $\tau\colon \widehat{S}_{k+1} \to \widehat{S}_{k+1}$ the automorphism given by $s_i \mapsto s_{k+1-i}$. Note that for μ contained in a k-rectangle, μ^{ω_k} is equal to the (ordinary) conjugate partition μ' of μ.

Ikeda conjectured that the image $\Phi(\mathfrak{G}_w^Q)$ is in fact not best described with K-k-Schur functions, but instead proposed [Ike20] the functions $\tilde{g}_w = (1-G_1^{\perp})\Big(\sum_{\mu \in \mathrm{Par}^k,\, w_\mu \le w_\lambda} g_\mu^{(k)}\Big)$. More precisely, we have the following conjecture.

Conjecture 4.3.1 ([IIM20, Conjecture 1.8], [Ike20])**.** *For $w \in S_{k+1}$,*

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{d \in \mathrm{Des}(w)} g_{(k+1-d)^d}}\,, \qquad \text{for } \tilde{g}_w := (1-G_1^{\perp})\Bigg(\sum_{\mu_1 \le k,\, w_\mu \le w_\lambda} g_\mu^{(k)}\Bigg) \in \Lambda_{(k)}\,, \quad (4.6)$$

where w_λ denotes the minimal coset representative of S_{k+1} in $\widehat{S}_{k+1}$ associated to λ (see §2.4.2), and $\le$ denotes Bruhat order on $\widehat{S}_{k+1}$.

4.4 Candidates for the image

We conjecture the following explicit raising operator formula for Ikeda's functions.

Definition 4.4.1. For $\lambda \in \mathrm{Par}_\ell^k$, the *closed k-Schur Katalan function* is

$$\tilde{\mathfrak{g}}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)\,.$$

Conjecture 4.4.2. *Let $w \in S_{k+1}$ and $\mu \in \mathrm{Par}_\ell^k$ be arbitrary and set $\lambda = \theta(w)^{\omega_k}$. Then*

(a) $\tilde{\mathfrak{g}}_\lambda^{(k)} = \tilde{g}_w$,

(b)

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{\mathfrak{g}}_\lambda^{(k)}}{\prod_{d \in \mathrm{Des}(w)} g_{R_d}}\,,$$

(c) *(alternating dual Pieri rule) the coefficients in* $G^{\perp}_{1^m}\tilde{\mathfrak{g}}^{(k)}_{\mu} = \sum_{\nu} c_{\mu\nu}\tilde{\mathfrak{g}}^{(k)}_{\nu}$ *satisfy* $(-1)^{|\mu|-|\nu|}c_{\mu\nu} \in \mathbb{Z}_{\geq 0}$,

(d) *(k-branching) the coefficients in* $\tilde{\mathfrak{g}}^{(k)}_{\mu} = \sum_{\nu} a_{\mu\nu}\tilde{\mathfrak{g}}^{(k+1)}_{\nu}$ *satisfy* $(-1)^{|\mu|-|\nu|}a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$,

(e) *(K-k-Schur alternating) the coefficients in* $\tilde{\mathfrak{g}}^{(k)}_{\mu} = \sum_{\nu} b_{\mu\nu}\, g^{(k)}_{\nu}$ *satisfy* $(-1)^{|\mu|-|\nu|}b_{\mu\nu} \in \mathbb{Z}_{\geq 0}$,

Example 4.4.3. Let us directly verify Conjecture 4.4.2 (b) for $k = 2$ and $w = 213$ (one-line notation), using the definition of Φ. The quantum Grothendieck is $\mathfrak{G}^Q_w = 1 - z_1 + z_1 Q_1$. Thus using $g_{R_1} = h_2$, $g_{R_2} = h_1^2 - h_2 + h_1$, $\sigma_1 = h_2 + h_1 + 1$, and $\sigma_2 = h_1^2 - h_2 + 2h_1 + 1$,

$$\Phi(\mathfrak{G}^Q_w) = 1 - \frac{g_{R_1}}{\sigma_1} + \frac{g_{R_1}}{\sigma_1}\frac{g_{R_2}}{g^2_{R_1}} = \frac{(h_2 + h_1 + 1)h_2 - h_2^2 + h_1^2 - h_2 + h_1}{h_2(h_2 + h_1 + 1)} = \frac{h_1}{h_2} = \frac{\tilde{\mathfrak{g}}^{(2)}_{(1)}}{g_{R_1}}.$$

This is the desired conclusion as $\zeta(w)' = (2, 1)$, $\theta(w) = (1) = \theta(w)^{\omega_2}$, and $\mathrm{Des}(w) = \{1\}$.

For $k = 4$ and $v = 13254 \in S_5$, we use $\mathrm{Inv}(w_0 v) = (4, 2, 2, 0)$ to find $\zeta(v)' = (10, 5, 3, 0)$ and $\theta(v) = (3, 2, 2, 1)$. We have $\theta(v)^{\omega_4} = (3, 2, 1, 1, 1)$ and $\mathrm{Des}(v) = \{2, 4\}$, so Conjecture 4.4.2 (b) states that

$$\Phi(\mathfrak{G}^Q_v) = \frac{\tilde{\mathfrak{g}}^{(4)}_{(3,2,1,1,1)}}{g_{R_2}\, g_{R_4}}\,,$$

as can be confirmed in Sage.

We also point out the following straightforward properties of $\tilde{\mathfrak{g}}^{(k)}_{\lambda}$.

Proposition 4.4.4. *The closed k-Schur Katalan functions* $\{\tilde{\mathfrak{g}}^{(k)}_{\lambda}\}_{\lambda \in \mathrm{Par}^k}$

(a) *form a basis for* $\Lambda_{(k)}$*;*

(b) *are unitriangularly related to K-k-Schur functions*

$$\tilde{\mathfrak{g}}^{(k)}_{\lambda} = g^{(k)}_{\lambda} + \sum_{\nu \in \mathrm{Par}^k : |\nu| < |\lambda|} b_{\lambda\nu}\, g^{(k)}_{\nu}\,; \tag{4.7}$$

(c) *satisfy shift invariance*

$$G^{\perp}_{1^{\ell}}\tilde{\mathfrak{g}}^{(k+1)}_{\lambda+1^{\ell}} = \tilde{\mathfrak{g}}^{(k)}_{\lambda}\,;$$

(d) *simplify as* $\tilde{\mathfrak{g}}^{(k)}_\lambda = g_\lambda$ *for* λ *contained in a* k*-rectangle, i.e.,* $\lambda \in \mathrm{Par}^k_\ell$ *with* $\lambda_1+\ell-1 \le k$.

Proof. Property (c) is proved just as in Theorem 3.7.1, and (d) just as in Corollary 3.7.3. For (a)–(b), the similar result for the $\mathfrak{g}^{(k)}_\lambda$'s in Proposition 3.6.15 easily adapts to this setting. □

Furthermore, we show that closed K-k-Schur functions satisfy the following relation that would follow from Conjecture 4.4.2(a) since it is known by [Tak19, Theorem 1.4] that Ikeda's $\tilde{g}_w$ functions satisfy the k-rectangle property.

Theorem 4.4.5 (k-rectangle property)**.** *For* $d \in [k]$, $g_{R_d}\,\tilde{\mathfrak{g}}^{(k)}_\mu = \tilde{\mathfrak{g}}^{(k)}_{\mu\cup R_d}$*, where* $\mu \cup R_d$ *is the partition made by concatenating the parts of* μ *and those of* R_d *and then sorting.*

The proof of this theorem will be carried out by § 4.5.2.

It is worth pointing out that having the Katalan formulations for (closed) K-k-Schur functions readily enables us to complete the proof of Conjecture 4.3.1 for Grassmannian permutations outlined in [IIM20, Theorem 1.7].

Proposition 4.4.6. *Conjecture 4.3.1 holds for* $w \in S_{k+1}$ *with* $\mathrm{Des}(w) = \{d\}$*. In fact, in this case, we have*

$$\Phi(\mathfrak{G}^Q_w) = \frac{g_{\theta(w)'}}{g_{R_d}} = \frac{\tilde{g}_w}{g_{R_d}} = \frac{\tilde{\mathfrak{g}}^{(k)}_{\theta(w)'}}{g_{R_d}} = \frac{g^{(k)}_{\theta(w)'}}{g_{R_d}}\,. \tag{4.8}$$

Proof. The first equality of (4.8) is established in Theorem 1.7 and Lemma 7.1 of [IIM20]. The partition $\lambda = \theta(w)^{\omega_k} = \theta(w)'$ lies in a k-rectangle by [BMPS20, Lemma 7.5]. Thus, by Corollary 3.7.3 and Proposition 4.4.4 (d), $g_{\theta(w)'} = g^{(k)}_{\theta(w)'} = \tilde{\mathfrak{g}}^{(k)}_{\theta(w)'}$. It remains to prove $g_{\theta(w)'} = \tilde{g}_w$. Using again Corollary 3.7.3 on the definition of $\tilde{g}_w$ in (4.6) gives

$$\tilde{g}_w = (1 - G_1^\perp)\big(\textstyle\sum_{w_\mu \le w_\lambda} g_\mu\big) = (1 - G_1^\perp)\big(\textstyle\sum_{\mu\subseteq\lambda} g_\mu\big) = g_\lambda,$$

where the second equality follows using [LM05, Proposition 40] in addition to the fact that μ is equal to the $(k+1)$-core of μ for μ lying in a k-rectangle (cores are discussed in §3.6.2), and the last equality holds by the following result of Takigiku [Tak18]: the map $1 - G_1^{\perp}\colon \Lambda \to \Lambda$ is a ring automorphism with inverse $F\colon h_i \mapsto \sum_{j \le i} h_j$ and satisfies $F(g_\nu) = \sum_{\mu \subseteq \nu} g_\nu$ for all ν. □

Another conjecture of [IIM20] about the image of the quantum Grothendieck polynomials is that

$$\Phi(\mathfrak{G}^Q_{w_0}) = \frac{\prod_{i=1}^{k-1} g_{(k-i)^i}}{g_{R_1} \cdots g_{R_k}} . \tag{4.9}$$

We prove the corresponding result for the closed k-Schur Katalan functions:

Proposition 4.4.7. *For w_0 the longest permutation in S_{k+1} and $\lambda = \theta(w_0)^{\omega_k}$, $\tilde{\mathfrak{g}}^{(k)}_\lambda = \prod_{i=1}^{k-1} g_{(k-i)^i}$.*

Proof. From $\mathrm{Inv}(w_0 w_0) = 0^k$ we have $\zeta(w_0)' = \left(\binom{k}{2}, \dots, \binom{1}{2}\right) = \theta(w_0)'$, and thus $\theta(w_0) = \cup_{i=1}^{k-1} (k-i)^i$. The proposition states that $\prod_{i=1}^{k-1} g_{(k-i)^i} = \tilde{\mathfrak{g}}^{(k)}_{\theta(w_0)^{\omega_k}}$, but we will first prove

$$\prod_{i=1}^{k-1} g_{(k-i)^i} = \tilde{\mathfrak{g}}^{(k)}_{\theta(w_0)} . \tag{4.10}$$

Consider that, by Proposition 3.4.3 and Lemma 3.4.11,

$$\prod_{i=1}^{k-1} g_{(k-i)^i} = \prod_{i=i}^{k-1} K(\varnothing_i; \varnothing_i; (k-i)^i) = K(\uplus_{i=1}^{k-1} \varnothing_i; \uplus_{i=1}^{k-1} \varnothing_i; \cup_{i=1}^{k-1} (k-i)^i)$$

where $\varnothing_i \subseteq \Delta_i^+$ denotes the empty root ideal of length i and $\uplus_{i=1}^{k-1} \varnothing_i = \varnothing_1 \uplus \varnothing_2 \uplus \cdots \uplus \varnothing_{k-1}$. Set $\gamma = \cup_{i=1}^{k-1} (k-i)^i$. We now proceed iteratively on $i = 1, \dots, k-1$ with $\Psi^i := \Delta^{(k)}(\cup_{j=1}^{i} (k-j)^j) \uplus (\uplus_{j=i+1}^{k-1} \varnothing_j)$. For fixed i, let $a = 1 + 2 + \cdots + i = \binom{i+1}{2}$. Note that Ψ^i has a ceiling in columns $a+1, \dots a+i+1$, a wall in rows $a+1, \dots, a+i+1$, and $\gamma_{a+1} = \cdots = \gamma_{a+i+1} = k-i-1$. Now, we can apply Diagonal Removal Lemma 3.5.13 to $K(\Psi^i; \Psi^i; \gamma)$ iteratively with $x = a - d$, $y = a+1$, and $z = a+1+d$ for $0 \le d < i$ to get, for $D_d = D^{a+1+d}_{a-d,a+1}$ and $\Psi^i_d := \Psi^i \setminus (D_0 \cup \cdots \cup D_d)$,

$$K(\Psi^i;\Psi^i;\gamma) = K(\Psi^i_0;\Psi^i_0;\gamma) = K(\Psi^i_1;\Psi^i_1;\gamma) = \cdots = K(\Psi^i_{i-1};\Psi^i_{i-1};\gamma) = K(\Psi^{i+1};\Psi^{i+1};\gamma)\,, \tag{4.11}$$

where the last equality follows since $\Psi^i \setminus (D_0 \cup \cdots \cup D_{i-1})$ has i nonroots in rows $a-i+1,\ldots,a$ and is thus equal to Ψ^{i+1}. Then, (4.10) follows by applying (4.11) iteratively since $\Psi^1 = \uplus_{j=1}^k \varnothing_j$ and $\Psi^{k-1} = \Delta^k(\gamma)$.

By [Mor12, §8], there is an involution $\Omega\colon \Lambda_{(k)} \to \Lambda_{(k)}$ defined by $\Omega(h_r) = g_{1^r}$, and it satisfies $\Omega(g^{(k)}_\mu) = g^{(k)}_{\mu^{\omega_k}}$ for all $\mu \in \mathrm{Par}^k$. Thus, by Theorem 3.6.19, $\Omega(\mathfrak{g}^{(k)}_\mu) = \mathfrak{g}^{(k)}_{\mu^{\omega_k}}$ as well. Applying Ω to both sides of (4.10) yields

$$\prod_{i=1}^{k-1} \Omega(g_{(k-i)^i}) = \Omega(\tilde{\mathfrak{g}}^{(k)}_{\theta(w_0)})\,. \tag{4.12}$$

The left side is in fact equal to $\prod_{i=1}^{k-1} g_{((k-i)^i)^{\omega_k}} = \prod_{i=1}^{k-1} g_{(i)^{k-i}} = \prod_{i=1}^{k-1} g_{(k-i)^i}$, where we have used that $g_\mu = g^{(k)}_\mu$ and $\mu^{\omega_k} = \mu'$ for any μ contained in a k-rectangle, by Corollary 3.7.3 and [LM05, Remark 10]. Thus, $\tilde{\mathfrak{g}}^{(k)}_{\theta(w_0)}$ is fixed by Ω. However, we also know by Proposition 4.4.4(b) that

$$\mathfrak{g}^{(k)}_{\theta(w_0)} + \sum_{\substack{\mu\in\mathrm{Par}^k\\ |\mu|<|\theta(w_0)|}} a_\mu \mathfrak{g}^{(k)}_\mu = \tilde{\mathfrak{g}}^{(k)}_{\theta(w_0)} = \Omega(\tilde{\mathfrak{g}}^{(k)}_{\theta(w_0)}) = \mathfrak{g}^{(k)}_{\theta(w_0)^{\omega_k}} + \sum_{\substack{\mu\in\mathrm{Par}^k\\ |\mu|<|\theta(w_0)|}} a_\mu \mathfrak{g}^{(k)}_{\mu^{\omega_k}}\,, \text{ for some } a_\mu \in \mathbb{Z}\,.$$

Looking at only the top degree, this tells us $s^{(k)}_{\theta(w_0)} = s^{(k)}_{\theta(w_0)^{\omega_k}}$ and thus $\theta(w_0) = \theta(w_0)^{\omega_k}$. Therefore, $\tilde{\mathfrak{g}}_{\theta(w_0)} = \tilde{\mathfrak{g}}_{\theta(w_0)^{\omega_k}}$. Applying this to (4.10) completes the proof. □

4.5 Proof of the k-rectangle property

4.5.1 Generalizing from root ideals

While root ideals are convenient for positivity results, in this section, we make use of the fact that a Katalan function is defined for any subset of positive roots of Δ_ℓ^+. In other words, for any subset $S \subseteq \Delta_\ell^+$, multiset M on $[\ell]$, and $\gamma \in \mathbb{Z}^\ell$, we define

$$K(S; M; \gamma) = \prod_{(i,j) \in \Delta_\ell^+ / S} (1 - R_{ij}) \prod_{j \in M} (1 - L_j) k_\gamma \,. \tag{4.13}$$

In this situation, we say a root $\beta \in \Delta_\ell^+$ is addable to S if $\beta \notin S$ and a root $\alpha \in S$ is always removable. Furthermore, if every root in row $r + d$ of $S \subseteq \Delta_\ell^+$ for some $d \in \mathbb{Z}_{>0}$ satisfies

$$(r + d, y) \in S \Longrightarrow (x, y) \in S \text{ for all } x \in [r, r + d] \text{ and } (r + d, z) \in S \text{ for all } z \geq y \,,$$

we say S has a wall in rows $r, r + 1, \ldots, r + d$. Similarly, if, for every root in column c of $S \subseteq \Delta_\ell^+$ satisfies

$$(x, c) \in S \Longrightarrow (x, y) \in S \text{ for all } y \in [c, c + d] \text{ and } (z, c) \in S \text{ for all } z \leq x \,,$$

we say S has a ceiling in columns $c, c + 1, \ldots, c + d$. These notions coincide with Definition 3.5.2 when S is a root ideal. We also give the following definitions of up and down.

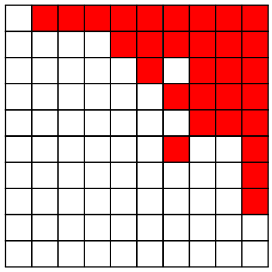
Definition 4.5.1. Let $S \subseteq \Delta_\ell^+$ be a subset of positive roots and $x \in [\ell]$.

(a) Let $j = \min\{c \mid (x, c) \in S\}$. If j is defined and $(r, j) \notin S$ for all $r > x$, we say $\mathrm{down}_S(x) = j$; otherwise, $\mathrm{down}_S(x)$ is undefined.

(b) Let $i = \max\{r \mid (r, x) \in S\}$. If i is defined and $(i, c) \notin S$ for all $c < x$, we say $\mathrm{up}_S(x) = i$; otherwise, $\mathrm{up}_S(x)$ is undefined.

These notions coincide with the definitions of up and down given at the beginning of § 3.5 when S is a root ideal. Philosophically, these definitions are based on the idea that up and down make sense when S is "locally" a root ideal.

Example 4.5.2. For S as below, $\mathrm{down}_S(1) = 2$, $\mathrm{down}_S(2) = 5$, $\mathrm{down}_S(5) = 8$, and $\mathrm{down}_S(8) = 10$. Furthermore, $\mathrm{down}_S(3) = 6$, but $\mathrm{down}_S(4)$ is undefined.

$S =$ 

S has a wall in rows $7, 8$ and a ceiling in columns $2, 3, 4$ as well as columns $8, 9$.

Furthermore, we also generalize the following results to Katalan functions $K(S; M; \gamma)$.

Lemma 4.5.3. *(a) Lemma 3.4.6 holds when Ψ is replaced by an arbitrary subset $S \subseteq \Delta_\ell^+$ satisfying $s_i S = S$.*

(b) All the identities of Proposition 3.4.12 hold when Ψ is replaced by an arbitrary subset $S \subseteq \Delta_\ell^+$.

(c) Lemma 3.5.4 holds when Ψ is replaced by an arbitrary subset $S \subseteq \Delta_\ell^+$.

(d) Lemmas 3.5.7 and 3.5.8 hold when Ψ is replaced by an arbitrary subset $S \subseteq \Delta_\ell^+$.

Proof. The proofs of all these results do not use any properties of Ψ being a root ideal. □

We also will need the following notation to state another useful lemma generalizing Lemma 3.4.11.

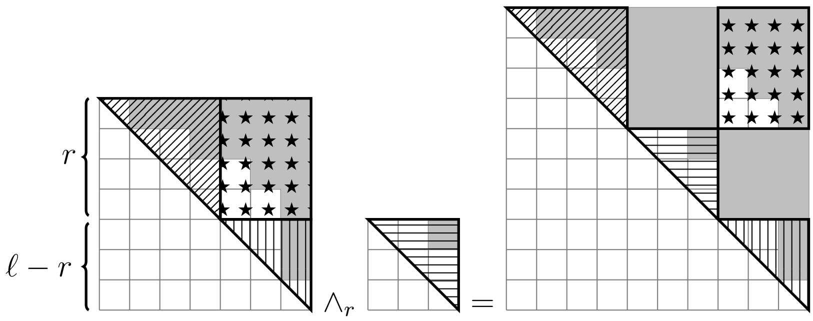
Definition 4.5.4. (a) Given a subset $S \subseteq \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ and $a \leq b$, let $S|_{[a,b]} \subseteq \Delta_{b-a+1}^+$ be the root ideal given by the roots of S in $[a, b] \times [a, b]$. More precisely,

$$S|_{[a,b]} = \{(i - a + 1, j - a + 1) \in \Delta_{b-a+1}^+ \mid (i, j) \in S \cap ([a, b] \times [a, b])\}.$$

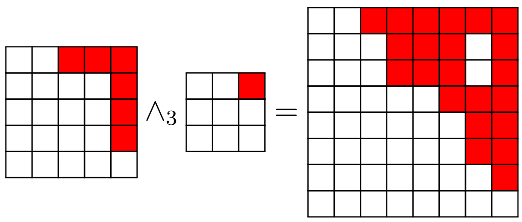
(b) Consider two subsets of roots $S \subseteq \Delta_\ell^+$ and $T \subseteq \Delta_m^+$. For any $r \in [\ell]$, we define $S \wedge_r T \subseteq \Delta_{\ell+m}^+$ to be a generalization of $\uplus$ where we slice S into 3 regions and position them around T to form a new root subset. Precisely,

$$S \wedge_r T = (S|_{[1,r]} \uplus T \uplus S|_{[r+1,\ell]}) \setminus \{(i, m+j) \in \Delta_{\ell+m}^+ \mid (i,j) \in \Delta_\ell^+ \setminus S, i \leq r, \text{ and } j > r\}.$$

The schematic below gives a visual guide for the operation $\wedge_r$.



Example 4.5.5. Consider the following example with $\ell = 5$, $m = 3$, $r = 3$.



Lemma 4.5.6. *Consider root subsets (not necessarily ideals) $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$, $\Psi', \mathcal{L}' \subseteq \Delta_{\ell'}^+$, as well as $\lambda \in \mathbb{Z}^\ell$ and $\mu \in \mathbb{Z}^{\ell'}$. Then, for any $r \in [\ell]$, we have*

$$K(\Psi; \mathcal{L}; \lambda) \times K(\Psi'; \mathcal{L}'; \mu) = K(\Psi \wedge_r \Psi'; \mathcal{L} \wedge_r \mathcal{L}'; (\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_{\ell'}, \lambda_{r+1}, \dots, \lambda_\ell))$$

Proof. Let $\nu = (\lambda_1, \dots, \lambda_r)$ and $\eta = (\lambda_{r+1}, \dots, \lambda_\ell)$. Furthermore, set $\Theta = \{(i, \ell' + j) \mid (i,j) \in \Delta_\ell^+ \setminus S, i \leq r, \text{ and } j > r\}$ and define the following raising operator expressions.

$$\Gamma_{(1)} = \prod_{(i,j) \in \Delta^+ \setminus \Psi|_{[1,r]}} (1 - R_{ij}) \prod_{j \in L(\mathcal{L}|_{[1,r]})} (1 - L_j)$$

$$\prod_{(i,j)\in\Delta^+\setminus\Psi|_{[r+1,\ell]}} (1-R_{\ell'+i,\ell'+j}) \prod_{j\in L(\mathcal{L}|_{[r+1,\ell]})} (1-L_{\ell'+j})$$

$$\prod_{(i,j)\in\Theta} (1-R_{ij}) \prod_{j\in L(\Delta^+\setminus\Theta)} (1-L_j)\,,$$

$$\Gamma_{(2)} = \prod_{(i,j)\in\Delta^+\setminus\Psi'} (1-R_{r+i,r+j}) \prod_{j\in L(\mathcal{L}')} (1-L_{r+j})\,.$$

Then, Definition 4.5.4 gives that

$$K(\Psi\wedge_r\Psi';\mathcal{L}\wedge_r\mathcal{L}';\nu\mu\eta) = \Gamma_{(1)}\Gamma_{(2)} \prod_{j=r+1}^{r+\ell'} (1-L_j)^r \prod_{j=r+\ell'+1}^{\ell+\ell'} (1-L_j)^{\ell'} k_{\nu\mu\eta}\,.$$

Careful consideration of the root sets contributing to $\Gamma_{(1)}$ shows that $\Gamma_{(1)}$ does not change the value of μ and a similar analysis shows that $\Gamma_{(2)}$ only changes the value of μ, leaving ν and η untouched. Hence,

$$K(\Psi\wedge_r\Psi';\mathcal{L}\wedge_r\mathcal{L}';\nu\mu\eta) = \left(\prod_{j=r+\ell'+1}^{\ell+\ell'} (1-L_j)^{\ell'}\,\Gamma_{(1)}\,k_{\nu\,0^{\ell'}\eta}\right)\times\left(\prod_{j=r+1}^{r+\ell'} (1-L_j)^r\,\Gamma_{(2)}\,k_{0^r\mu\,0^{\ell-r}}\right). \tag{4.14}$$

Let $c_{\gamma,\gamma'}\in\mathbb{Z}$ be such that $\Gamma_{(1)}k_{\nu\,0^{\ell'}\theta} = \sum_{\gamma\gamma'} c_{\gamma,\gamma'}k_{\gamma\,0^{\ell'}\gamma'}$. In particular, all but finitely many $c_{\gamma,\gamma'}=0$. Now, using (3.14), the first parenthesized factor of (4.14) satisfies

$$\begin{aligned}\prod_{j=r+\ell'+1}^{\ell+\ell'} (1-L_j)^{\ell'}\,\Gamma_{(1)}\,k_{\nu 0^{\ell'}\eta} &= \prod_{j=r+\ell'+1}^{\ell+\ell'} (1-L_j)^{\ell'}\left(\sum_{\gamma,\gamma'} c_{\gamma,\gamma'}k_{\gamma 0^{\ell'}\gamma'}\right)\\ &= \sum_{\gamma,\gamma'} c_{\gamma,\gamma'}k_{\gamma\gamma'} = \prod_{(i,j)\in\Delta_\ell^+\setminus\Psi} (1-R_{ij}) \prod_{j\in L(\mathcal{L})} (1-L_j)k_{\nu\eta} \quad = K(\Psi;\mathcal{L};\lambda)\,.\end{aligned}$$

A similar analysis shows the second factor of (4.14) is $K(\Psi';\mathcal{L}';\mu)$. □

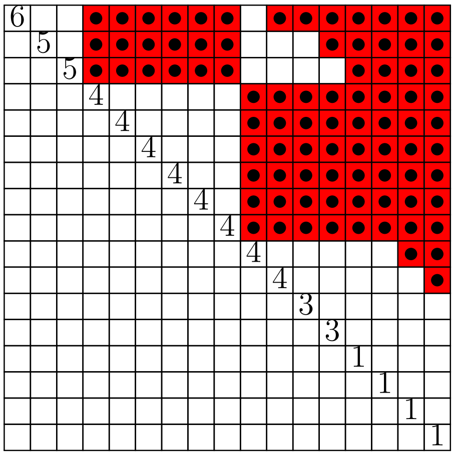
Corollary 4.5.7. *Let $k,\ell\geq 1$, $\mu\in\mathrm{Par}_\ell^k$ and $d\in[k]$. Furthermore, let r be the number such that $\mu_r > k+1-d$ but $\mu_{r+1}\leq k+1-d$, taking $\mu_0=\infty$ and $\mu_{\ell+1}=0$. Then,*

$$\tilde{\mathfrak{g}}_\mu^{(k)}\tilde{\mathfrak{g}}_{R_d}^{(k)} = K(\Delta^k(\mu)\wedge_r\varnothing_d;\Delta^k(\mu)\wedge_r\varnothing_d;(\mu_1,\ldots,\mu_r,k+1-d,\ldots,k+1-d,\mu_{r+1},\ldots,\mu_\ell))$$

Proof. This result follows from Lemma 4.5.6 by taking $\Psi = \mathcal{L} = \Delta^k(\mu)$ and $\Psi' = \mathcal{L}' = \Delta^k(R_d)$, noting that $\Delta^k(R_d) = \varnothing_d$ by construction. □

Example 4.5.8. For $k = 9$, $\mu = 65544331111$, and $d = 6$, we get $r = 3$ and Corollary 4.5.7 gives the following.

$$\tilde{\mathfrak{g}}^{(9)}_{65544331111}\tilde{\mathfrak{g}}^{(9)}_{R_6} =$$

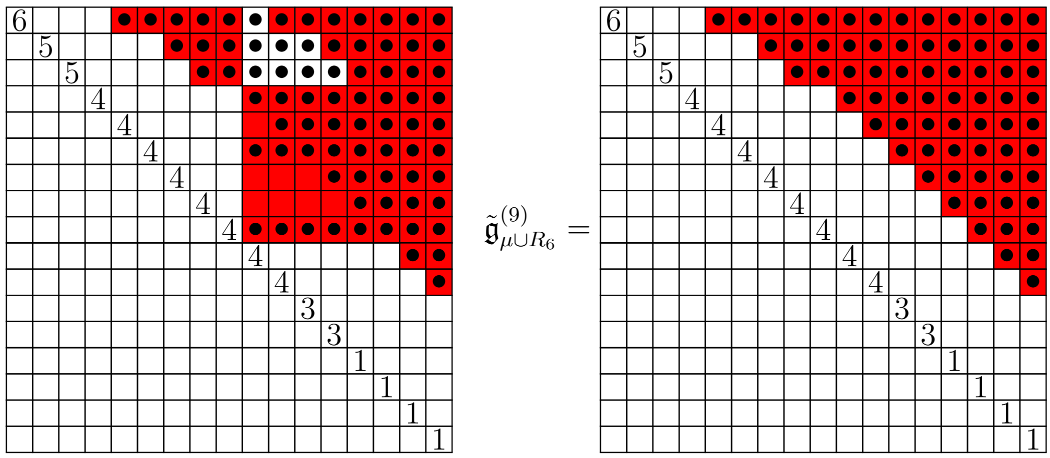


Lemma 4.5.9. *For $\mu \in \mathrm{Par}^k_\ell$, $d \in [k]$, and r such that $\mu_r > k+1-d$ but $\mu_{r+1} \leq k+1-d$, taking $\mu_0 = \infty$ and $\mu_{\ell+1} = 0$, let $\nu = (\mu_1, \ldots, \mu_r)$ and $\eta = (\mu_{r+1}, \ldots, \mu_\ell)$. Then,*

$$\tilde{\mathfrak{g}}^{(k)}_{R_d}\tilde{\mathfrak{g}}^{(k)}_{\mu} = K(\Psi; \Psi; \nu R_d \eta)\,,$$

where $\Psi = (\Delta^k(\nu R_d) \uplus \Delta^k(\eta)) \setminus \Theta$ for $\Theta = \{(i, d+j) \mid (i,j) \in \Delta^+_\ell \setminus \Delta^k(\mu), i \leq r,$ and $j > r\}$.

Proof. We start with $\tilde{\mathfrak{g}}^{(k)}_{R_d}\tilde{\mathfrak{g}}^{(k)}_{\mu} = K(\Psi'; \Psi'; \nu R_d \eta)$ for $\Psi' = \Delta^k(\lambda) \wedge_r \varnothing_d$ as in Corollary 4.5.7. We then seek to remove roots in columns $r+1, \ldots, r+d$ of Ψ' by repeated applications of Lemma 4.5.3(d) in columns $r+1, \ldots, r+d$. We do this iteratively, first starting by removing $k - \nu_r$ roots from row r. Note, by our choice of r, it must be that $(r, r+d-1) \in \Psi$ and $(r, r+d) \in \Psi$, and therefore this is always possible. Then, since $\Delta^k(\nu R_d)$ is "wall-free" (Remark 3.6.2(a)), the result follows for higher rows by similar reasoning until we arrive at Ψ. □



$$\tilde{\mathfrak{g}}^{(9)}_{\mu}\tilde{\mathfrak{g}}^{(9)}_{R_6} = \qquad \tilde{\mathfrak{g}}^{(9)}_{\mu\cup R_6} =$$

Figure 4-1: An example of Lemma 4.5.9 with $\mu = 6544331111$ on the left. Note that $\bullet$'s can be moved up and down the column without changing the result, and we have done so to illustrate the bijection of Lemma 4.5.12.

4.5.2 The proof

Throughout this subsection, let us fix $k, \ell \geq 1$, $\mu \in \mathrm{Par}^k_\ell$, $d \in [k]$, $r \in [0, \ell]$ such that $\mu_r > k+1-d$ but $\mu_{r+1} \leq k+1-d$, $\nu = (\mu_1, \dots, \mu_r)$, $\eta = (\mu_{r+1}, \dots, \mu_\ell)$, and $\Psi = (\Delta^k(\nu R_d) \uplus \Delta^k(\eta)) \setminus \Theta$ for $\Theta = \{(i, d+j) \mid (i,j) \in \Delta^+_\ell \setminus \Delta^k(\mu), i \leq r, \text{ and } j > r\}$ as in Lemma 4.5.9.

It is necessary at this point to carry out a detailed comparison of Ψ with $\Delta^k(\mu)$. Figures 4-1 and 4-2 are useful for this analysis. First, since $\mu_r > k+1-d$, we note that $\Delta^k(\mu)$ and $\Delta^k(\nu R_d)$ cannot have more than $d-1$ non-roots in each row $1, \dots, r$. This has two consequences.

Remark 4.5.10. We have the following facts for Ψ.

(a) $\Psi \setminus \Delta^k(\mu \cup R_d) \subseteq [r+1, \dots, r+d] \times [r+d+1, \min\{\ell+d, r+2d-1\}]$ and $\Delta^k(\mu \cup R_d) \setminus \Psi \subseteq [1, \dots, r] \times [r+d+1, \min\{\ell+d, r+2d-1\}]$. In words, the differences between Ψ and $\Delta^k(\mu \cup R_d)$ lie only in the columns $r+d+1, \dots, \min\{\ell+d, r+2d-1\}$.

(b) For all $i \in [r]$, $(i, r+d+j) \notin \Psi \iff (i, r+j) \notin \Psi$ for any $1 \leq j \leq \min\{d-1, \ell-r\}$.

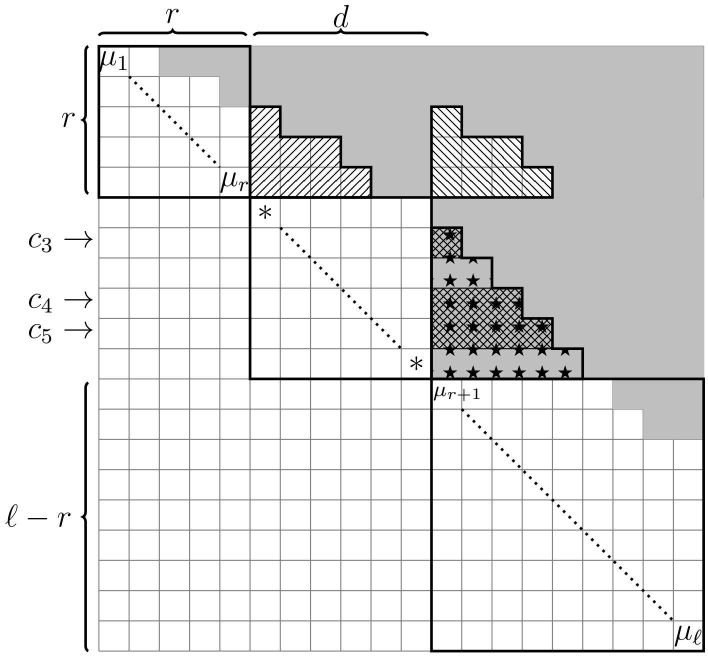


Figure 4-2: A schematic for the proof where $* = k + 1 - d$. Gray shaded roots represent Ψ and Ψ differs from $\Delta^k(\mu \cup R_d)$ by adding northwest shaded roots and removing starred roots.

Consider the set C of column indices given by $c \in C$ if Ψ does not have a ceiling in columns $c-1, c$ and $r+1 < c \leq r+d$. Note, $\mathrm{up}_\Psi(c)$ is defined for all $c \in C$. Let us label such columns by the rule $c_i \in C$ satisfies $\mathrm{up}_\Psi(c_i) = i$. By construction, c_i is such that row i has non-roots $\{(i, i+1), \ldots, (i, c_i - 1)\}$ in columns $i+1, \ldots, r+d$, but this last non-root is also equal to $(i, k - \nu_i + i)$ by construction of the k-Schur root ideal.

$$c_i - 1 = k - \nu_i + i\,. \tag{4.15}$$

Note that, since a k-Schur root ideal is wall-free, c_i is defined for all $i \in [a, r]$ where $a = \min\{i \mid k - \nu_i + i > r + 1\}$. (If a is undefined, then $C = \varnothing$.)

Example 4.5.11. In Figure 4-1, $r = 3$ and $c_1 = 5, c_2 = 7, c_3 = 8$.

Lemma 4.5.12. *For each $1 \leq i \leq r$, there is a bijection between the roots of $\Delta^k(\nu R_d \eta) \setminus \Psi$ in row i and the roots of $\Psi \setminus \Delta^k(\nu R_d \eta)$ in row c_i given by $(i, j) \mapsto (c_i, j)$.*

Example 4.5.13. The bijection is illustrated in Figure 4-1 between the boxes of the form $\blacksquare$ and boxes of the form $\boxdot$. In Figure 4-2, the bijection is between the northwest shaded roots and the crosshatch shaded roots.

Proof of Lemma 4.5.12. By construction of $\Delta^k(\nu R_d\eta)$, it must be that $\{(c_i, r+d+1), \dots, (c_i, \min\{\ell+d, c_i+d-i\})\}$ are all the roots of row c_i in $\Psi \setminus \Delta^k(\nu R_d\eta)$. Similarly, $\{(i, r+d+1), \dots, (i, \min\{\ell+d, d+k-\nu+i\})\}$ must be all the roots of row i in $\Delta^k(\nu R_d\eta) \setminus \Psi$. However, $c_i+d-i = d+k-\nu_i+i$ and so these two sets have the same cardinality. Thus, we have the desired one-to-one correspondence. □

Lemma 4.5.14. *Suppose a subset of roots $S \subseteq \Delta_\ell^+$, a multiset M on $[\ell]$, $\gamma \in \mathbb{Z}^\ell$, and $y \in [\ell]$ satisfy*

(a) S has a root in row y and $z = \mathrm{down}_S(y)$ is defined;

(b) $z \in M$;

(c) S has a ceiling in columns $y-1, y$ and a wall in rows $y-1, y$;

(d) $m_M(y-1) = m_M(y)$;

(e) $\gamma_{y-1} = \gamma_y$.

Then, we have

$$K(S; M; \gamma) = K(S \setminus (y,z); M \setminus \{z\}; \gamma)\,.$$

Proof. First we expand on $z \in M$ to get

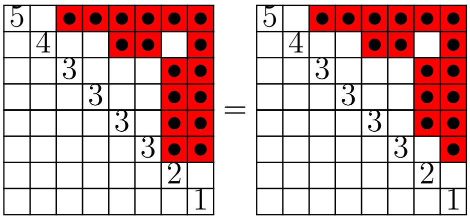
$$K(S; M; \gamma) = K(S; M \setminus \{z\}; \gamma) - K(S; M \setminus \{z\}; \gamma - \epsilon_z)\,. \tag{4.16}$$

Next, we expand on $(y,z) \in S$ in the first summand on the righthand side of (4.16) to get

$$K(S; M \setminus \{z\}; \gamma) = K(S \setminus (y,z); M \setminus \{z\}; \gamma) + K(S; M \setminus \{z\}; \gamma + \epsilon_y - \epsilon_z)\,. \tag{4.17}$$

However, $K(S; M \setminus \{z\}; \gamma + \epsilon_y - \epsilon_z) = K(S; M \setminus \{z\}; \gamma - \epsilon_z)$ by Lemma 4.5.3(c). Thus, substituting (4.17) into the righthand side of (4.16) gives the result. □

Example 4.5.15. Lemma 4.5.14 gives the following equality with $y = 6$.



Lemma 4.5.16. *Suppose a subset of roots $S \subseteq \Delta_\ell^+$, a multiset M on $[\ell]$, $\gamma \in \mathbb{Z}^\ell$, and $1 \leq x < y < z \leq \ell$ satisfy*

(a) $y = \mathrm{down}_S(x)$ and $z = \mathrm{down}_S(y)$;

(b) $(x, z) \notin S$;

(c) $(x, y-1) \notin S$ and $S \cup (x, y-1)$ has a ceiling in columns $y-1, y$;

(d) S has a wall in rows $y-1, y$;

(e) $m_M(y-1) = m_M(y) - 1$;

(f) $\gamma_{y-1} = \gamma_y$.

Then,

$$K(S; M; \gamma) = K((S \cup (x, z)) \setminus (y, z); M; \gamma)\,.$$

Proof. First, we get $K(S; M; \gamma) = K(S \cup (x, y-1); M; \gamma)$ by Lemma 4.5.3(d). Then, by Proposition 3.4.12,

$$K(S \cup (x, y-1); M; \gamma) = K(S \cup (x, y-1) \cup (x, z); M; \gamma) - K(S \cup (x, y-1) \cup (x, z); M; \gamma + \epsilon_x - \epsilon_z)\,. \tag{4.18}$$

Next, we observe using Proposition 3.4.12 that

$$K(S \cup \{(x, y-1), (x, z)\}; M; \gamma) = \\ K(S \cup \{(x, y-1), (x, z)\} \setminus (y, z); M; \gamma) + K(S \cup \{(x, y-1), (x, z)\}; M; \gamma + \epsilon_y - \epsilon_z) \,. \tag{4.19}$$

However, $K(S \cup \{(x, y-1), (x, z)\}; M; \gamma + \epsilon_y - \epsilon_z) = 0$ by Lemma 4.5.3(c). After this, we observe using Proposition 3.4.12 that

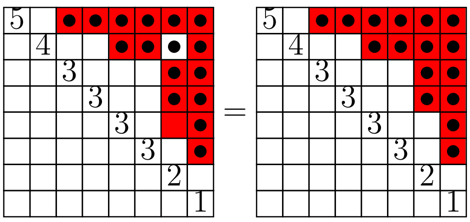
$$\begin{aligned} &K(S \cup \{(x, y-1), (x, z)\} \setminus (y, z); M; \gamma) \\ &= K(S \cup (x, z) \setminus (y, z); M; \gamma) \\ &+ K(S \cup \{(x, y-1), (x, z)\} \setminus (y, z); M; \gamma + \epsilon_x - \epsilon_{y-1}) \,. \end{aligned} \tag{4.20}$$

However, via yet another application of Proposition 3.4.12,

$$\begin{aligned} &K(S \cup \{(x, y-1), (x, z)\} \setminus (y, z); M; \gamma + \epsilon_x - \epsilon_{y-1}) \\ &= K(S \cup \{(x, y-1), (x, z)\}; M; \gamma + \epsilon_x - \epsilon_{y-1}) \\ &- K(S \cup \{(x, y-1), (x, z)\}; M; \gamma + \epsilon_x - \epsilon_{y-1} + \epsilon_y - \epsilon_z) \,, \end{aligned} \tag{4.21}$$

noting also that $K(S \cup \{(x, y-1), (x, z)\}; M; \gamma + \epsilon_x - \epsilon_{y-1}) = 0$ by Lemma 4.5.3(c) and $-K(S \cup \{(x, y-1), (x, z)\}; M; \gamma + \epsilon_x - \epsilon_{y-1} + \epsilon_y - \epsilon_z) = K(S \cup \{(x, y-1), (x, z)\}; M; \gamma + \epsilon_x - \epsilon_z)$ by Lemma 4.5.3(a). Thus, combining (4.18)–(4.21) and canceling yields the result. □

Example 4.5.17. Lemma 4.5.16 gives the following equality with $y = 5$.



Proof of the k-rectangle property. Recall notation for Ψ, ν, η was fixed at the beginning of the subsection. We start with Lemma 4.5.9, which states

$$\tilde{\mathfrak{g}}^{(k)}_{R_d}\tilde{\mathfrak{g}}^{(k)}_{\mu} = K(\Psi;\Psi;\nu R_d \eta)\,.$$

Note that $\Delta^k(\mu \cup R_d)$ is obtained from Ψ by filling in the northwest shaded roots and removing the starred roots in Figure 4-2. To do this, we will carry out a process that removes the starred roots column by column, proceeding from bottom to top of a column and removing columns left to right. Roots that are starred but not cross-hatched will be removed by applications of Lemma 4.5.14. Roots that are starred and cross-hatched will be removed by applications of Lemma 4.5.16 while simultaneously adding roots that are shaded by northwest lines. We carry this out precisely below.

Let $\Psi^{(0)} = \Psi$, $\Psi^{(s)} = \Psi^{(s-1)} \setminus \{(r+1+s, r+d+s), \ldots, (r+d, r+d+s)\} \cup \{(i, r+d+s) \mid c_i \in C, c_i \geq r+1+s\}$. Then, $\Psi^{(e)} = \Delta^k(\mu \cup R_d)$ for $e = \min(d-1, \ell - r)$. We will show

$$K(\Psi^{(s-1)};\Psi^{(s-1)};\mu \cup R_d) = K(\Psi^{(s)};\Psi^{(s)};\mu \cup R_d) \tag{4.22}$$

To do so, let $\Psi^{(s-1,0)} = \Psi^{(s-1)}$ and $\Psi^{(s-1,t)} = \Psi^{(s-1,t-1)} \setminus (r+d+1-t, r+d+s) \cup \{(i, r+d+s) \mid c_i = r+d+1-t\}$. Then, $\Psi^{(s-1,d-s)} = \Psi^{(s)}$ and so we will show

$$K(\Psi^{(s-1,t-1)};\Psi^{(s-1,t-1)};\mu \cup R_d) = K(\Psi^{(s-1,t)};\Psi^{(s-1,t)};\mu \cup R_d)\,. \tag{4.23}$$

To prove this equality, let $y = r+d+1-t$ and we examine the two possible cases.

(a) If $y-1, y$ has a ceiling, (4.23) follows from Lemma 4.5.14.

(b) If $x = \mathrm{up}_{\Psi^{(s-1,t-1)}}(y)$ is defined, (4.23) follows from Lemma 4.5.16. Note the identity holds in the lowering ideal since $L(\Psi^{(s-1,t-1)}) = L(\Psi^{(s-1,t)})$ in this case.

Then, (4.22) follows by iterating (4.23) and thus the result follows by e applications of (4.22):

$$K(\Psi^{(0)};\Psi^{(0)};\mu \cup R_d) = K(\Psi^{(1)};\Psi^{(1)};\mu \cup R_d) = \cdots = K(\Psi^{(e)};\Psi^{(e)};\mu \cup R_d)\,.$$

□

4.6 Positivity conjectures for Katalan functions

We end this section with a discussion on general positivity conjectures for Katalan functions.

Let Ψ be a root ideal and let us again return to the root ideal definition of Katalan functions given in Definition 3.4.1. Define $RC(\Psi)$ to be $\Psi \setminus \{\text{removable roots of } \Psi\}$. For a nonnegative integer a, iteratively define $RC^a(\Psi) = RC(RC^{a-1}(\Psi))$, starting from $RC^0(\Psi) = \Psi$. Maxband is defined as in Definition 3.6.13.

Conjecture 4.6.1. *For a root ideal $\Psi \subseteq \Delta_\ell^+$ and $\lambda \in \mathrm{Par}_\ell^k$ such that maxband$(\Psi, \lambda) \le k$,*

$$K(\Psi;\Psi;\lambda) = \sum_{\substack{\mu \in \mathrm{Par}_\ell^k \\ |\mu| \le |\lambda|}} a_{\lambda\mu}\, \tilde{\mathfrak{g}}_\mu^{(k)} \quad \text{for} \quad (-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\ge 0}\,. \tag{4.24}$$

For $a \in \mathbb{Z}_{\ge 0}$,

$$K(\Psi; RC^a(\Psi); \lambda) = \sum_{\substack{\mu \in \mathrm{Par}_\ell^k \\ |\mu| \le |\lambda|}} b_{\lambda\mu}\, s_\mu^{(k)} \quad \text{for} \quad b_{\lambda\mu} \in \mathbb{Z}_{\ge 0}\,. \tag{4.25}$$

Remark 4.6.2. The large k limit ($k \ge |\lambda|$) of Conjecture 4.6.1 is already quite strong: for $k \ge |\lambda| \ge |\mu|$, we have $g_\mu^{(k)} = g_\mu$ [Mor12] and $s_\mu^{(k)} = s_\mu$ [LM07], so (4.24) and (4.25) become conjectures on g_μ-alternating and Schur positivity, respectively. Conjecture (4.24) can be seen as a generalization of branching Conjecture 4.4.2(e) as setting $\Psi = \Delta^{k-1}(\lambda)$ gives $K(\Psi;\Psi;\lambda) = \tilde{\mathfrak{g}}_\lambda^{(k-1)}$. Conjecture (4.25) can be seen as a vast generalization of the conjectured k-Schur positivity of the $g_\lambda^{(k)}$'s posed in [LSS10b, Conjecture 7.20(1)].

The following conjecture also generalizes [LSS10b, Conjecture 7.20(1)] in a different direction.

Conjecture 4.6.3. *For $\lambda \in \mathrm{Par}_\ell^k$, $\Psi = \Delta^k(\lambda)$, and M a multiset on $[\ell]$ such that $m_M(i) \le$*

$m_M(i+1) \leq m_M(i) + 1$,

$$K(\Psi; M; \lambda) = \sum_{\substack{\mu \in \mathrm{Par}_\ell^k \\ |\mu| \leq |\lambda|}} c_{\lambda\mu} s_\mu^{(k)} \quad \textit{for} \quad c_{\lambda\mu} \in \mathbb{Z}_{\geq 0} \,. \tag{4.26}$$

Chapter 5

Future Work: Type C k-Schur functions

5.1 Schubert calculus of the affine Grassmannian of the symplectic group

As with the Type A situation, we first survey the classical results that provide fundamental objects of study and then provide background for the affine setting. Of crucial combinatorial importance will be the notion of a *strict partition* λ.

Definition 5.1.1. We say a partition $\lambda \in \mathrm{Par}_\ell$ of length ℓ is *strict* if $\lambda_1 > \cdots > \lambda_\ell > 0$.

The work of [Pra91] provides a Schubert calculus approach to the isotropic Grassmannian, that is the Grassmannian associated to Sp_{2n}. From the symmetric function perspective, the relevant algebra will be a subring $\Gamma \subseteq \Lambda$.

5.1.1 The subspace $\Gamma \subseteq \Lambda$

We define the following symmetric functions.

Definition 5.1.2. (a) For $r \geq 0$, let q_r be given by the generating function

$$Q(t) = \prod_i \frac{1 + tx_i}{1 - tx_i} = \sum_{r \geq 0} q_r t^r .$$

(b) For any $\gamma \in \mathbb{Z}^\ell$, let $q_\gamma = q_{\gamma_1} q_{\gamma_2} \cdots q_{\gamma_\ell}$.

(c) For λ a strict partition, let the *Schur Q-function*, Q_λ, be given by

$$Q_\lambda = \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}} q_\lambda \,, \tag{5.1}$$

where $R_{ij} q_\gamma = q_{\gamma + \epsilon_i - \epsilon_j}$.

(d) For λ a strict partition of length ℓ, let the *Schur P-function* be given by $P_\lambda = 2^{-\ell} Q_\lambda$.

Remark 5.1.3. Schur-P and Q functions were originally studied by Schur when examining the spin characters of the symmetric group. Notably, Schur-Q functions were originally formulated as a "Pfaffian" of a matrix in the q_r's, similar to the determinantal form of the Jacobi-Trudi identity. This is equivalent to (5.1). Later, it was discovered that the Schur-P and Q functions are $t = -1$ specializations of the P and Q variants of Hall-Littlewood polynomials. However, we do not take up the study of Hall-Littlewood polynomials here, and thus use more classical definitions. See, e.g., [Mac15, III.8].

Consider the subring $\Gamma = \mathbb{Z}[q_1, q_2, \cdots] \subseteq \Lambda$. It turns out the q_i's are not linearly independent, satisfying the recursion relation [Mac15, III (8.2')],

$$q_{2m} = \sum_{r=1}^{m-1} (-1)^{r-1} q_r q_{2m-r} + \frac{1}{2}(-1)^{m-1} q_m^2 \,. \tag{5.2}$$

Remark 5.1.4. There exists a well-known surjective algebra homomorphism $\theta \colon \Lambda \to \Gamma$ given by $\theta(h_n) = q_n = \theta(e_n)$ (see [Mac15, III. 8, 10]). This is essentially the same fact as (5.2) by Newton's identities.

Performing some combinatorial manipulations, one arrives at the following facts.

Proposition 5.1.5. *Let* $\mathrm{SP} = \{\lambda \in \mathrm{Par} \mid \lambda \textit{ is strict}\}$.

(a) $\{q_\lambda\}_{\lambda \in \mathrm{SP}}$ *forms a* $\mathbb{Z}$*-basis of* Γ.

(b) $\{Q_\lambda\}_{\lambda \in \mathrm{SP}}$ *forms a* $\mathbb{Z}$*-basis of* Γ.

Definition 5.1.6. Let $X = \mathrm{Sp}_{2n}(\mathbb{C})/P$ for P a maximal parabolic subgroup of $\mathrm{Sp}_{2n}(\mathbb{C})$ be the *isotropic Grassmannian.*

Proposition 5.1.7 ([Pra91, Theorem 6.17(i)])**.** *The Schur-Q functions are Schubert representatives for the isotropic Grassmannian. In other words, there is a ring surjection* $\Gamma \to H^*(X)$ *sending* $Q_\lambda \mapsto \sigma_\lambda$ *if* $\lambda \subseteq \delta_n = (n, n-1, \ldots, 2, 1)$ *or* 0 *otherwise.*

5.1.2 Affine type C Stanley symmetric functions and k-Schur functions

The work of [LSS10a] provides results on the Schubert calculus of the homology and cohomology of $\mathrm{Gr}_{\mathrm{Sp}_{2n}}$, the affine Grassmannian of Sp_{2n}. In particular, [LSS10a] identifies $H_*(\mathrm{Gr}_{\mathrm{Sp}_{2n}})$ and $H^*(\mathrm{Gr}_{\mathrm{Sp}_{2n}})$ with dual Hopf algebras of symmetric functions, $\Gamma_{(n)}$ and $\Gamma^{(n)}$, which are defined in terms of Schur P-functions and Q-functions. To this end, [LSS10a] define an affine type C Stanley symmetric function.

Definition 5.1.8. Let $\tilde{C}_n$ be the affine Weyl group of type C given by generators $\{s_0, s_1, \ldots, s_n\}$ and relations

$$s_i^2 = 1, \quad s_is_j = s_js_i \text{ if } |i-j| > 1, \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \text{ if } 1 \leq i \leq n-2,$$
$$s_0s_1s_0s_1 = s_1s_0s_1s_0, \quad s_{n-1}s_ns_{n-1}s_n = s_ns_{n-1}s_ns_{n-1}\,.$$

Furthermore, since $\tilde{C}_n$ is a Coxeter group, we define reduced expression, length, and Bruhat order as in Definition 2.4.5. For a reduced expression $u = s_{i_1}\cdots s_{i_r}$, we say $\mathrm{supp}(u) = \{i_1, \ldots, i_r\} \subseteq [0, n]$ and note that, for $w \in \tilde{C}_n$, $\mathrm{supp}(w) = \mathrm{supp}(s_{i_1}\cdots s_{i_r})$ for any choice of reduced expression $s_{i_1}\cdots s_{i_r} = w$.

Definition 5.1.9 ([LSS10a, (1.2)])**.** For $w \in \tilde{C}_n$, the affine Weyl group of type C, we define the *affine type C Stanley symmetric functions* by the generating function

$$Q_w^{(n)}(y) = \sum_{(v^1, v^2, \ldots)} \prod_i 2^{c(v^i)} y_i^{\ell(v^i)}$$

where the sum runs over the factorizations $v^1v^2\cdots = w$ of w such that $v^i \in \mathcal{Z}$ and $\ell(v^1) + \ell(v^2) + \cdots = \ell(w)$ where $\mathcal{Z}$ is the Bruhat order ideal in $\tilde{C}_n$ generated by the conjugates of the element

$$\rho_{2n} := s_1s_2\cdots s_{n-1}s_ns_{n-1}\cdots s_1s_0\,.$$

We also define $c(w)$ to be the number of connected components in the support of a word, $\mathrm{supp}(w)$ (see Example 5.1.10).

Example 5.1.10. Let $w = s_0s_2s_1s_0 = 0210 = 2010 \in \tilde{C}_2$. Then, we have (non-redundant) subwords

$$0210, 0+210, 02+10, 021+0, 02+1+0, 0+21+0, 0+2+10, 0+2+1+0$$
$$2010, 2+010, 2+01+0, 2+0+10, 2+0+1+0$$

and $\rho_4 = s_1s_2s_1s_0$. Thus, $\mathcal{Z}$ is all words which have a reduced word that is a subword of 1210, 2101, 1012, 0121, the conjugates of ρ_4. This gives

(v_i)	$(2^{c(v_i)})$	$y_i^{\ell(v_i)}$
$0+210$	$2\cdot 2$	$y_1y_2^3$
$021+0$	$2\cdot 2$	$y_1^3y_2$
$02+10$	$2^2\cdot 2$	$y_1^2y_2^2$
$02+1+0$	$2^2\cdot 2\cdot 2$	$y_1^2y_2y_3$
$0+21+0$	$2\cdot 2\cdot 2$	$y_1y_2^2y_3$
$2+01+0$	$2\cdot 2\cdot 2$	$y_1y_2^2y_3$
$0+2+10$	$2\cdot 2\cdot 2$	$y_1y_2y_3^2$
$2+0+10$	$2\cdot 2\cdot 2$	$y_1y_2y_3^2$
$0+2+1+0$	$2\cdot 2\cdot 2\cdot 2$	$y_1y_2y_3y_4$
$2+0+1+0$	$2\cdot 2\cdot 2\cdot 2$	$y_1y_2y_3y_4$

Thus, summing together all the terms, we get

$$Q^{(2)}_{0210} = 4m_{31} + 8m_{22} + 16m_{211} + 32m_{1111}\,,$$

or, in the language of [LSS10a], if we set $M_\lambda = 2^{\ell(\lambda)} m_\lambda$, we get

$$Q^{(2)}_{0210} = M_{31} + 2M_{22} + 2M_{211} + 2M_{1111}\,.$$

Definition 5.1.11. (a) Let $\Gamma_* = \mathbb{Z}[P_1, P_2, \ldots]$ and $\Gamma^* = \mathbb{Z}[Q_1, Q_2, \ldots]$.

(b) Let $[\cdot,\cdot]\colon \Gamma_* \times \Gamma^* \to \mathbb{Z}$ be such that $[P_\lambda, Q_\mu] = \delta_{\lambda\mu}$.

(c) Let $\Gamma_{(n)} := \mathbb{Z}[P_1, P_2, \ldots, P_{2n}]$.

(d) Since $\Gamma_{(n)}$ defines a Hopf subalgebra of Γ_*, we can define the dual quotient Hopf algebra $\Gamma^{(n)}$, which is induced by the surjection $\mathbb{Z}[Q_1, Q_2, \ldots] \twoheadrightarrow \Gamma^{(n)}$.

Remark 5.1.12. Note that $\Gamma_{(n)}$ is not a polynomial algebra in $P_1, P_2, \ldots, P_n$. However, $\Gamma_{(n)} = \mathbb{Z}[P_1, P_3, \ldots, P_{2n-1}]$ as a polynomial algebra.

Let $\tilde{C}^0_n$ be the *affine Grassmannian* words of $\tilde{C}_n$, that is, for $C_n = \langle s_1, \ldots, s_n\rangle \leq \tilde{C}_n$, let $\tilde{C}^0_n$ be the minimal length coset representatives of $\tilde{C}_n/C_n$, in analogy with Definition 2.4.5(b) in the type A setting. Then, [LSS10a] provide the following result on the type C affine Stanley symmetric functions.

Proposition 5.1.13 ([LSS10a, Theorem 1.2])**.** *The series $Q^{(n)}_w$ is symmetric and defines an element of $\Gamma^{(n)}$. Furthermore, the subset $\{Q^{(n)}_v \mid v \in \tilde{C}^0_n\}$ forms a basis of $\Gamma^{(n)}$ such that all product and coproduct structure constants are positive and every $Q^{(n)}_w$ for $w \in \tilde{C}_n$ is positive in this basis.*

Using that the pairing $[\cdot,\cdot]\colon \Gamma_{(*)} \times \Gamma^{(*)} \to \mathbb{Z}$ descends to one on $\Gamma_{(n)} \times \Gamma^{(n)}$, [LSS10a] define the following dual basis of type C k-Schur functions.

Definition 5.1.14. Let $\{P^{(n)}_w \mid w \in \tilde{C}^0_n\}$ be the basis dual to $\{Q^{(n)}_w \mid w \in \tilde{C}^0_n\}$ under the pairing $[\cdot,\cdot]\colon \Gamma_{(n)} \times \Gamma^{(n)} \to \mathbb{Z}$.

Thus, we formally arrive at the following Pieri rule for $P^{(n)}_w$.

Proposition 5.1.15 (Pieri Rule for $P_w^{(n)}$). [LSS10a, Theorem 1.4] *For $w \in \tilde{C}_n^0$, we have that*

$$P_i P_w^{(n)} = \sum_{v \in \mathcal{Z}_i} 2^{c(v)-1} P_{vw}^{(n)}$$

where the sum is taken over all $v \in \mathcal{Z}_i = \{u \in \mathcal{Z} \mid \ell(u) = i\}$ such that $\ell(vw) = \ell(v) + \ell(w)$ and $vw \in \tilde{C}_n^0$.

However, [LSS10a] provides no direct formulation for $P_w^{(n)}$. In the following sections, we provide two conjectural formulations for $P_w^{(n)}$.

Conjecture 5.1.16. *For $w \in \tilde{C}_n^0$,*

(a) $P_w^{(n)}$ is a weight generating function over "type C strong marked tableaux."

(b) $P_w^{(n)}$ has a raising operator formula of the form

$$\prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - R_{ij}) \prod_{(i,j) \in \Psi'} (1 + R_{ij})^{-1} \mathbb{P}_{\lambda(w)}$$

for a map $w \mapsto \lambda(w)$ sending $w \in \tilde{C}_k^0 \to \text{Par}$ (see Corollary 5.1.22), $\Psi = \Delta^{(2n)}(\lambda(w))$, and Ψ' a set of roots depending on $\lambda(w)$, and $\mathbb{P}_\mu = P_{\mu_1} \cdots P_{\mu_\ell}$. (See Conjecture 5.2.8.)

5.1.3 Type C affine combinatorics

Throughout this section, we will use French notation for the diagram of a partition, instead of English notation as in the first 4 chapters. This is in keeping with the existing literature on core combinatorics in the affine setting.

Recall from Section 3.6.2 that a cell (i, j) of a partition κ has *n-residue* $\overline{j - i} \in \mathbb{Z}/n\mathbb{Z}$. An *addable* i-corner is a cell $(r, c) \notin \kappa$ of n-residue i such that $\kappa \cup \{(r, c)\}$ is a partition; a *removable* i-corner is a cell $(r, c) \in \kappa$ of n-residue i such that $\kappa \setminus \{(r, c)\}$ is a partition. In an n-core, an i-corner cannot be both addable and removable.

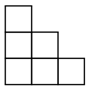
We first spell out a bijection that is stated in [LSS10a, Lemma 5.3] and was brought to their attention through private correspondence with Jennifer Morse. This bijection is also presented in [HJ12], which gives models for types BCD affine Weyl groups.

Lemma 5.1.17. *For fixed n, the elements of $\tilde{C}_n^0$ are in bijection with the set of self-conjugate (symmetric) $2n$-cores via the following procedure.*

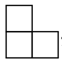
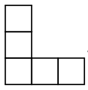
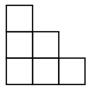
(1) Take a reduced expression of $w = s_{i_\ell} \cdots s_{i_2} s_{i_1}$.

(2) Add all corners of residue i_s and for $i_s \neq 0, n$, also add corners of residue $2n - i_s$.

Example 5.1.18. Let $n = 2 \implies 2n = 4$. Then,

$s_0 s_2 s_1 s_0 \leftrightarrow$ 

via

$s_0 \mapsto$ , $s_1 s_0 \mapsto$ , $s_2 s_1 s_0 \mapsto$ , $s_0 s_2 s_1 s_0 \mapsto$ 

Lemma 5.1.19. *There exists a bijection from self-conjugate $2n$-cores to $2n$-bounded partitions such that all parts weakly smaller than n are distinct via the following procedure.*

(1) For a $2n$-core γ, let $\mathrm{skew}(\gamma)$ denote the collection of all cells with hook length less than or equal to $2n$.

(2) Let $\mathrm{skew}^{\perp}(\gamma)$ be only the cells of $\mathrm{skew}(\gamma)$ that lie strictly below the main diagonal.

(3) Define a partition by setting,

$$\lambda(\gamma) := (\lambda_1, \ldots, \lambda_\ell), \quad \lambda_i = 1 + \text{ the number of cells in row } i \text{ of } \mathrm{skew}^{\perp}(\gamma)\,,$$

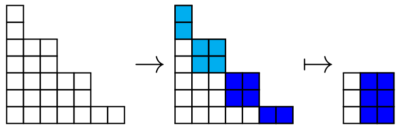
where ℓ is equal to the number of cells of the form $(i, i) \in \gamma$.

Definition 5.1.20. Let us say a partition $\lambda \in \mathrm{Par}$ is an *n-type C partition* if, for some $n \in \mathbb{Z}_{>0}$,

(a) λ is $2n$-bounded and

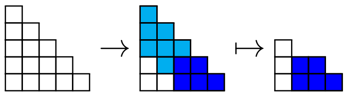
(b) all parts less than or equal to n are distinct.

Example 5.1.21. Let $n = 2$ and consider the self-conjugate $2n = 4$-core



where all the blue cells are skew(γ) and the dark blue cells are skew$^{\perp}(\gamma)$.

A second example with $n = 3$ is given by considering the 6-core



Corollary 5.1.22. *There exists a bijection between words $w \in \tilde{C}_n^0$ and n-type C partitions. Let us denote such a bijection by $w \mapsto \lambda(w)$.*

For the sake of completeness, we state some facts about our bijection.

Proposition 5.1.23. *(a)* [LSS10a, 5.6. Proof of Theorem 1.3] *The leading term of the M_μ expansion of $Q_w^{(n)}$ is $M_{\lambda(w)}$ where $M_\mu = 2^{\ell(\mu)} m_\mu$ in the monomial basis.*

(b) Let

$$\rho_i = \begin{cases} s_{i-1}s_{i-2}\cdots s_1 s_0 & \text{for } 1 \le i \le n \\ s_{2n+1-i}s_{2n+2-i}\cdots s_{n-1}s_n s_{n-1}\cdots s_1 s_0 & \text{for } n+1 \le i \le 2n\,. \end{cases}$$

Then, $\lambda(\rho_i) = (i)$, a single row partition, for all $1 \le i \le 2n$.

5.2 Two conjectured formulas

5.2.1 Type C strong marked tableaux

Definition 5.2.1. Given two symmetric $2n$ cores κ and τ such that $\kappa = u.\varnothing$ for $u \in \tilde{C}_n^0$ and $\tau = w.\varnothing$ for $w \in \tilde{C}_n^0$, we say κ *strongly covers* τ if $\tau \subseteq \kappa$ and $\ell(u) = \ell(w) + 1$. We will denote this $\tau \Rightarrow \kappa$.

Proposition 5.2.2. *(a) ([HJ12, Theorem 5.11]) For $w, u \in \tilde{C}_n$, if $w.\varnothing = \tau$ and $u.\varnothing = \kappa$, then $w \leq u \iff (\tau \subseteq \kappa)$.*

(b) If $\tau \Rightarrow \kappa$, then κ/τ is a collection of disjoint ribbons.

Proof. We prove part (b) by contradiction. Assume κ/τ contains at least one 2×2 block. Then, at least one of these blocks must have a removable corner at its northeast cell, otherwise τ would not be a partition. Let us say this removable corner has residue r. However, this means that there exists a symmetric $2n$-core γ given by removing all removable cells of residue r from κ and all removable cells of residue $2n - r$ if $r \neq 0, k$. Thus, there is an s_i such that $s_i.\gamma = \kappa$ for $i = \min\{r, 2n - r\}$. Such a process cannot add a 2×2 block since such a block has two cells of residue r, only one of which is removable. Therefore, $\tau \subseteq \gamma \subseteq \kappa$ with τ, γ, κ all distinct, thus violating the condition $\tau \Rightarrow \kappa$.

□

Definition 5.2.3. (a) Given a strong cover of $2n$-cores $\tau \Rightarrow \kappa$, we define a *strong marked cover* to have the extra data of a marking $c \in \mathbb{Z}$, where c is the content of a ribbon head in κ/τ. We denote the strong marked cover $\tau \Rightarrow_c \kappa$.

(b) We define a *strong marked strip* of size $r \leq 2n$ to be a sequence of strong marked covers

$$\tau^{(0)} \Rightarrow_{c_1} \tau^{(1)} \Rightarrow_{c_2} \cdots \Rightarrow_{c_r} \tau^{(r)}$$

such that $c_1 < c_2 < \cdots < c_r$.

(c) We define a *strong marked tableau* of shape κ and weight $(r_1, \ldots, r_s)$ to be a sequence of strong marked strips

$$
\begin{aligned}
&((\varnothing \Rightarrow_{c_{1,1}} \tau^{(1,1)} \Rightarrow_{c_{1,2}} \cdots \Rightarrow_{c_{1,r_1}} \tau^{(1,r_1)}),\\
&(\tau^{(1,r_1)} \Rightarrow_{c_{2,1}} \tau^{(2,1)} \Rightarrow_{c_{2,2}} \cdots \Rightarrow_{c_{2,r_2}} \tau^{(2,r_2)}),\\
&\ldots,\\
&(\tau^{(s-1,r_{s-1})} \Rightarrow_{c_{s,1}} \cdots \Rightarrow_{c_{s,r_s}} \tau^{(s,r_s)}))
\end{aligned}
$$

such that $\tau^{(s,r_s)} = \kappa$.

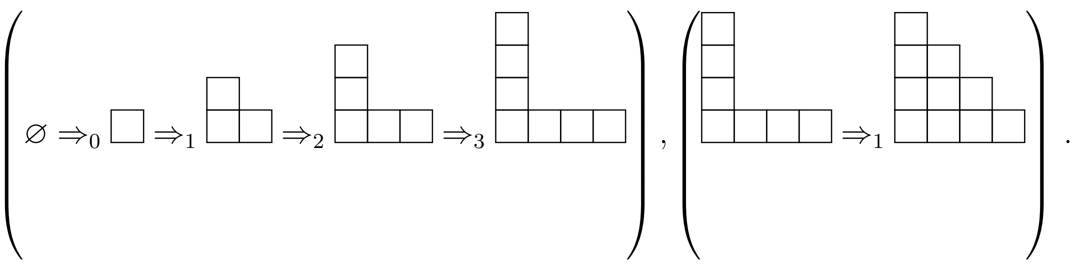
(d) We denote the set of all strong marked tableaux of shape κ as $\mathrm{CSMT}^n(\kappa)$ and set $\mathrm{wt}(\kappa) = (r_1, \ldots, r_s)$.

Remark 5.2.4. In practice, we encode a strong marked tableau as a filling of κ by boxes where

(1) we put the letter i in the boxes of $\tau^{(i,r_i)}/\tau^{(i-1,r_{i-1})}$ where we consider $\tau^{(0,r_0)} = \varnothing$ and

(2) we put markings (denoted by a $'$) on the ribbon heads from each strong cover.

Example 5.2.5. If we let $n = 2$, we have a strong marked tableau of shape $(4, 3, 2, 1)$ and weight $(4, 1)$ given by



We could encode this as

1			
1	2		
1	2	$2'$	
$1'$	$1'$	$1'$	$1'$

.

In fact, in this example, the conditions force that this is the unique strong marked tableau of this shape and weight.

We now present our conjectured formulation of $P_w^{(n)}$ in direct analogy to (2.19) for $s_\lambda^{(k)}$.

Conjecture 5.2.6. *Let κ be the symmetric $2n$ core associated to an affine Grassmannian word $w \in \tilde{C}_n^0$. Then,*

$$P_w^{(n)}(x) = \sum_{\mathsf{T} \in \mathrm{CSMT}^n(\kappa)} x^{\mathrm{wt}(\mathsf{T})} \in \mathbb{Z}[P_1, P_2, \ldots, P_{2n}] \,.$$

Prior to this conjecture, we are not aware of any other weight generating function description of $P_w^{(n)}$.

5.2.2 A raising operator formula

While a weight generating function description of k-Schur functions historically preceded a raising operator formula, the previous chapters of this work outline the power of a raising operator formula for proving results that were otherwise inaccessible from a weight generating function description. Furthermore, this work showed applications of a raising operator description for K-k-Schur functions, for which no known weight generating function description exists. In the type C setting, [BKT15, BKT17, Tam11] had success applying raising operator descriptions to the Schubert representatives of non-maximal isotropic Grassmannians, defining theta-polynomials. Here, we give a raising operator formula conjecture for $P_w^{(n)}(x)$, in direct analogy with Definition 2.4.1 for $s_\lambda^{(k)}$.

Definition 5.2.7. For a partition $\lambda \in \mathbb{Z}^\ell$, let

$$\mathbb{P}_\lambda = P_{\lambda_1} \cdots P_{\lambda_\ell} \,.$$

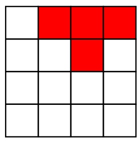
Conjecture 5.2.8. *For $n \geq 1$ and $w \in \tilde{C}_n^0$, we conjecture the following formula:*

$$P_w^{(n)} = \prod_{(i,j)\in\Delta^+\setminus\Delta^{(2n)}(\lambda(w))} (1 - R_{ij}) \prod_{(i,j)\in\Psi'(\lambda(w))} (1 + R_{ij})^{-1}\mathbb{P}_{\lambda(w)}(x)\,,$$

where $\Psi'(\lambda) = \Delta^+_{\ell(\lambda)} \setminus \{(i,j) \in \Delta^+ \mid \lambda_i + \lambda_j - 2 + (j-i) \geq 2n\}$ and $R_{ij}\mathbb{P}_\gamma = \mathbb{P}_{\gamma+\epsilon_i-\epsilon_j}$.

In the language of [BKT15, BKT17, Tam11], we might call $\Phi = \Phi(\lambda) = \{(i,j) \in \Delta^+ \mid \lambda_i + \lambda_j - 2 + (j-i) \geq 2n\}$ a "type C root ideal" since it satisfies $(i,j) \in \Phi \implies (i-1,j) \in \Phi$ by definition and, since $\lambda(w)$ is a n-type C partition, a case-by-case analysis for λ_i and λ_j shows $(i,j) \in \Phi \implies (i,j-1) \in \Phi$.

Example 5.2.9. For $n = 3$, consider 3-type C partition $(6,6,3,1)$. Then, $\Phi(\lambda) = \{(1,2),(1,3),(1,4),(2,3)\}$ as depicted below.



Now, consider the following raising operator formulas as well.

(a) For λ a strict partition of length ℓ,

$$P_\lambda = \prod_{(i,j)\in\Delta^+_\ell} (1 - R_{ij}) \prod_{(i,j)\in\Delta^+_\ell} (1 + R_{ij})^{-1}\mathbb{P}_\lambda\,.$$

(b) For $\lambda \in \mathrm{Par}_\ell$,

$$\mathbb{P}_\lambda = \prod_{(i,j)\in\varnothing} (1 - R_{ij}) \prod_{(i,j)\in\varnothing} (1 + R_{ij})^{-1}\mathbb{P}_\lambda\,.$$

(c) For $\lambda \in \mathrm{Par}_\ell$,

$$\frac{1}{2^\ell}\theta(s_\lambda) = \prod_{(i,j)\in\Delta^+_\ell} (1 - R_{ij}) \prod_{(i,j)\in\varnothing} (1 + R_{ij})^{-1}\mathbb{P}_\lambda\,,$$

for $\theta\colon \Lambda \to \Gamma$ given in Remark 5.1.4.

In light of these facts and Conjecture 5.2.8, it is tempting to consider a definition such as the following.

Definition 5.2.10. For a root ideal $\Psi \subseteq \Delta_\ell^+$, a type C root ideal $\Phi \subseteq \Delta_\ell^+$, and $\lambda \in \mathbb{Z}^\ell$, we define a *spin Catalan function* $P(\Psi;\Phi;\gamma)$ to be given by the formula

$$P(\Psi;\Phi;\gamma) = \prod_{(i,j)\in\Delta_\ell^+\setminus\Psi} (1-R_{ij}) \prod_{(i,j)\in\Delta_\ell^+\setminus\Phi} (1+R_{ij})\mathbb{P}_\lambda \,.$$

However, unlike with Catalan functions and Katalan functions, we do not have a conjecture about when a spin Catalan function, as defined above, would be positive in terms of the Schur-P functions, or even in terms of Schur functions. This suggests the definition provided above is ultimately not the one that will prove useful for proving Conjecture 5.2.8.

Remark 5.2.11. We refrain from calling these "type C Catalan functions" since the work of [Pan10] is type independent and thus reasonably already defines type C Catalan functions, which live in the character ring of the Lie group Sp_{2n}.

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