Monetary and Fiscal Policy and Debt Structure in Small Open Economies

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Abstract

In the past two decades, emerging economies have registered a reduction in the share of sovereign debt denominated in foreign currency and an extension in its maturity. First, I study how these changes affect the optimal monetary and fiscal policy. I derive analytical results using a commitment framework. A higher share of debt denominated in local currency increases the reliance on variations in inflation to hedge against fiscal stress, which in turn increases with the maturity. Additionally, it reduces the relative exposure to foreign shocks. Second, I study the differences in currency composition across countries, focusing on Latin America. In this case I use a time-consistency framework, to capture the large shares of debt denominated in foreign currency observed in the data. I find that large external debt-to-GDP ratios, long debt duration, and low inflation costs encourage more borrowing in inflation-indexed bonds and can explain the larger share of inflation-indexed debt in Uruguay compared to other Latin American countries.

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Contents

Chapter 1

Introduction

In the last two decades, emerging economies have experience significant changes in the composition of their sovereign debt, which directly affects fiscal financing, and have potential implications for monetary and fiscal policy implementation and economic growth. In this document I focus on two specific facts: the reduction in the share of sovereign debt denominated in foreign currency, and the lengthen of the maturity structure.

In the first chapter, I explore how the above-mentioned changes in sovereign debt structure affect the optimal (joint) monetary and fiscal policy. I follow Leeper and Zhou (2021), incorporating an open economy dimension. Traditionally, emerging economies have been modeled as small open economies that issue short-term, foreign-currency denominated debt (see for example Schmitt-Grohe and Uribe 2004's book), which contradicts the facts stated before. In my analysis I consider different currency compositions and maturity structures of the sovereign debt –taken as given in this chapter–.

In the second chapter, I explore empirically and quantitatively the differences in sovereign debt currency composition across emerging economies. In particular, I focus on Latin American countries, which evidence a wide variety of interesting examples, as captured in the recent book *A Monetary and Fiscal History of Latin America, 1960–2017* (2021). In this case, I make the currency composition decision endogenous, using a time-consistency framework.[1] First, I document the differences in the currency composition of the sovereign debt, distinguishing between bonds denominated in local currency, foreign currency and inflation-indexed. Second, I incorporate inflation-indexed bonds to Ottonello and Perez (2019)'s model and provide an analytical and quantitative analysis.

Let's see in more detail how changes in debt structure, as the ones experienced by emerging economies, affect optimal monetary and fiscal policy.

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[1] As explained in Chapter 3, the time-consistency framework provides a reason for emerging economies to issue bonds in foreign currency. On the one hand, the real value of bonds denominated in foreign currency cannot be altered by government's decisions. On the other hand, the government has incentives to reduce the real value of bonds denominated in local currency and inflation-indexed. As a result, the government will choose some fraction of foreign currency bonds to avoid inflationary costs and real exchange rate distortions.

Schmitt-Grohe, Stephanie and Martin Uribe (2004). "Optimal fiscal and monetary policy under sticky prices". In: *Journal of Economic Theory* 114.2, pp. 198–230. URL: https://ideas.repec.org/a/eee/jetheo/v114y2004i2p198-230.html.

Chapter 2

Optimal Monetary and Fiscal Policy under Commitment: The Role of Maturity Structure and Currency Denomination of Government Bonds

2.1 Introduction

In the last decades, emerging economies have evidence changes in the composition of their sovereign debt that challenge the way they were traditionally modelled —small open economies that issue short-term, foreign-currency denominated debt–.[1] On one hand, the share of sovereign debt denominated in foreign currency has decreased,

[1] In this chapter I focus on total sovereign debt, i.e. the sum of domestic and external sovereign debt.

leading to an increase in the share of local currency and inflation-indexed debt (see Figure 2.1). On the other hand, emerging economies have extended the average time-to-maturity of their sovereign debt (see Figure 2.2). In light of these changes, in this chapter I study how maturity and currency denomination of government debt affect optimal monetary and fiscal policy.

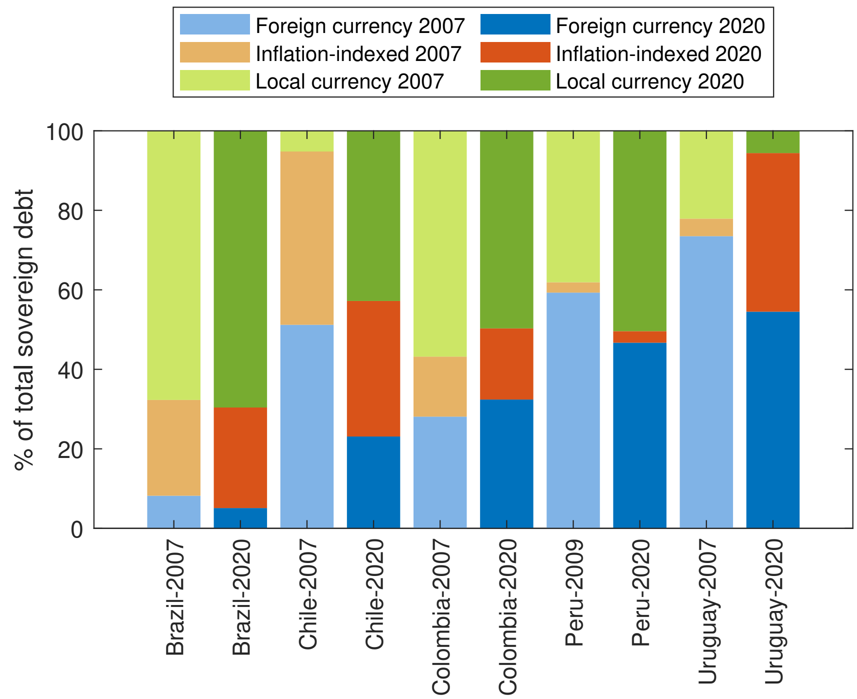

Figure 2.1: Currency composition of total government debt (domestic and external). Source: central bank and ministry of finance of each country

As discussed in the vast and growing literature on (joint) optimal monetary and fiscal policy, sovereign debt structure determines the alternatives the government has to respond to macroeconomic shocks.[2] For example, Leeper and Zhou (2021) emphasize the role of maturity structure, and P. Benigno and Woodford (2004) discuss the case of indexed debt. This literature mostly targets advanced economies and use closed economy models. In this chapter, I extend the analysis to a small open economy framework, to capture how emerging economies are affected and respond to domestic

[2]Leeper and Zhou (2021), in their introduction, provide an excellent summary of the existing papers highlighting the main drivers of their (in some cases contradicting) results.

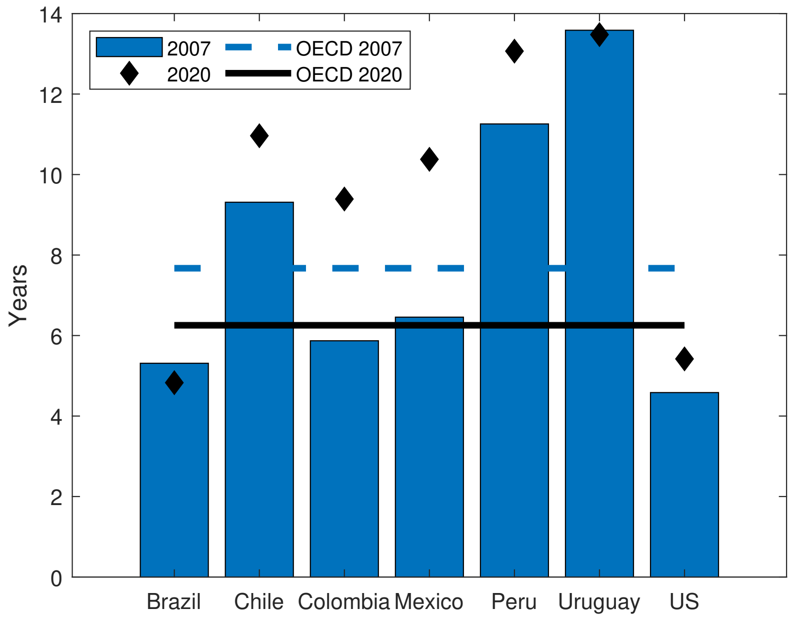

Figure 2.2: Average time-to-maturity.
Source: central bank and ministry of finance of each country and OECD calculations.

and foreign shocks under different debt compositions.

I start my analysis by assuming commitment and a timeless perspective, as in P. Benigno and Woodford (2004), and I apply their micro-founded linear-quadratic approach in order to obtain neat analytical solutions that help me characterize the properties of optimal policies. I study optimal monetary and fiscal policy under commitment in a small open economy that features complete asset markets from the household perspective, monopolistic competition in production, sticky prices, and distorting taxes as the only available source of government revenue. I consider local currency, foreign currency and inflation-indexed bonds, to capture the differences in currency denomination of government debt observed in the data (Figure 2.1). Moreover, following Leeper and Zhou (2021), I incorporate maturity structure of government debt to analyze how the observed increase in the average time-to-maturity of the sovereign bonds in emerging economies (Figure 2.2) affects optimal policy.

As in closed economies, I find that with debt denominated in local currency, the role of inflation as a fiscal shock absorber increases with the average maturity of debt and with the debt level. This role disappears when the debt is denominated in foreign currency or inflation-indexed, but real exchange rate effects persist. Regardless of the debt structure, external shocks affect the small open economy; however, the vulnerability of the small open economy to foreign disturbances is higher with debt denominated in foreign currency, and increases with the maturity.

This paper is closely related to De Paoli and G. Benigno (2010) and De Paoli and G. Benigno (2010), which also study optimal monetary and fiscal policy in a small open economy framework using P. Benigno and Woodford (2004) linear-quadratic approach. However, they consider monetary and fiscal policies separately, and they only contemplate one-period, local-currency bonds. Their focus is on terms of trade externalities, which arises because imported goods are not perfect substitutes to goods produced domestically.

The chapter proceeds as follows. I describe the small open economy model in Section 2.2, and introduce the optimal policy problem faced by the government in Section 2.3. I conduct the optimal policy analysis in Section 2.4. Section 2.5 concludes.

2.2 The Model

The model is a standard New Keynesian model for a small open economy where households have access to complete markets, augmented to include the government's budget constraint where government spending can be financed by distortionary taxation and debt. I consider different maturities (bonds with geometrically decaying coupons) and currency denomination (foreign currency, local currency and inflation-indexed) of government bonds.

2.2.1 Households

The small open economy is populated by a continuum of identical households. Each household has preferences over a composite consumption index, C_t, and hours worked in home-industry h, $N_t(h)$. Preferences are:

$$U_0 \equiv E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_{jt}) = E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \int_0^1 \frac{N_t(h)^{1+\varphi}}{1+\varphi} dh \right) \tag{2.1}$$

where σ^{-1} denotes the intertemporal elasticity of substitution, φ^{-1} parametrizes the Frisch elasticity of labor supply, and the consumption index C_t is defined by

$$C_t \equiv \left[\vartheta^{\frac{1}{\eta}} (C_{H,t})^{\frac{\eta-1}{\eta}} + (1-\vartheta)^{\frac{1}{\eta}} (C_{F,t})^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

where η is the intratemporal elasticity of substitution between domestic and foreign goods, $C_{H,t}$ and $C_{F,t}$ are the consumption sub-indices of home-produced and foreign-produced goods, respectively. The parameter determining home consumer's preferences for foreign goods, $(1-\vartheta)$, is a function of the relative size of the small open economy with respect to the rest of the world, n, and the degree of openness, α: $(1-\vartheta) = (1-n)\alpha$.

Similar preferences are specified for the rest of the world. Note that for $\alpha < 1$ the specification of ϑ and ϑ^* gives rise to home bias in consumption, as in Sutherland (2005).

The sub-indices of home-produced goods, $C_{H,t}$, and foreign-produced goods, $C_{F,t}$, are defined as:

$$C_{H,t} \equiv \left[\left(\frac{1}{n}\right)^{\frac{1}{\epsilon}} \int_0^n C_{H,t}(j)^{\frac{\epsilon-1}{\epsilon}} dj \right]^{\frac{\epsilon}{\epsilon-1}} \quad \text{and} \quad C_{F,t} \equiv \left[\left(\frac{1}{1-n}\right)^{\frac{1}{\epsilon}} \int_n^1 C_{F,t}(j)^{\frac{\epsilon-1}{\epsilon}} dj \right]^{\frac{\epsilon}{\epsilon-1}} \tag{2.2}$$

where $j \in [0, 1]$ denotes the good variety and $\epsilon > 1$ denotes the elasticity of substitution between varieties (produced within any given country). Analogous expressions hold for $\left(C^*_{H,t}\right)$ and $\left(C^*_{F,t}\right)$.

The optimal allocation of expenditures between domestic and imported goods is given by:

$$C_{H,t} = \vartheta \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t; \quad C_{F,t} = (1-\vartheta)\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t$$

where P_t is the consumer price index (CPI) given by $P_t \equiv [\vartheta(P_{H,t})^{1-\eta} + (1-\vartheta)(P_{F,t})^{1-\eta}]^{\frac{1}{1-\eta}}$, and $P_{H,t}$ is the price sub-index for home-produced goods expressed in the domestic currency and $P_{F,t}$ is the sub-index for foreign produced goods expressed in the domestic currency:

$$P_{H,t} \equiv \left[\left(\frac{1}{n}\right)\int_0^n P_{H,t}(j)^{1-\epsilon}dj\right]^{\frac{1}{1-\epsilon}} \quad \text{and} \quad P_{F,t} \equiv \left[\left(\frac{1}{1-n}\right)\int_n^1 P_{F,t}(j)^{1-\epsilon}dj\right]^{\frac{1}{1-\epsilon}} \tag{2.3}$$

Analogous expressions can be derived for $C^*_{H,t}, C^*_{F,t}, P^*_t, P^*_{H,t}$ and $P^*_{F,t}$.

I assume that the law of one price holds for each variety, so $P_{H,t}(h) = \mathcal{E}_t P^*_{H,t}(h)$ and $P_{F,t}(f) = \mathcal{E}_t P^*_{F,t}(f)$, where $\mathcal{E}_t$ is the nominal exchange rate (the price of foreign currency in terms of domestic currency).

The definition of the aggregate price levels and the price sub-indexes, together with the law of one price for each variety, imply that $P_{H,t} = \mathcal{E}_t P^*_{H,t}$ and $P_{F,t} = \mathcal{E}_t P^*_{F,t}$. However, as the definition of the CPI illustrates, home bias specification leads to deviations form purchasing power parity, i.e. $P_t \neq \mathcal{E}_t P^*_t$. For this reason, I define the real exchange rate as $\mathcal{Q}_t = \frac{\mathcal{E}_t P^*_t}{P_t}$.

To characterize the small open economy I use the definition of ϑ and ϑ^* and take the limit for $n \to 0$. The the CPI index of the rest of the world becomes $P^*_t = P^*_{F,t}$, and using the law of one price and the definition of the real exchange rate $P_{F,t} = \mathcal{Q}_t P_t$. Then, the CPI in the small open economy can be expressed as $P_t^{1-\eta} = (1-\alpha)(P_{H,t})^{1-\eta} + \alpha(\mathcal{Q}_t P_t)^{1-\eta}$.

For later use, I define the CPI inflation, $\pi_t = P_t/P_{t-1}$, as

$$\pi_t = \left[\frac{(1-\alpha)(P_{H,t})^{1-\eta} + \alpha(\mathcal{Q}_t P_t)^{1-\eta}}{(1-\alpha)(P_{H,t-1})^{1-\eta} + \alpha(\mathcal{Q}_{t-1} P_{t-1})^{1-\eta}} \right]^{\frac{1}{1-\eta}} \tag{2.4}$$

Accordingly, the household's problem implies the maximization of equation (2.1) subject to a sequence of budget constraints of the form:

$$P_t C_t + E_t \left\{ Q_{t,t+1} D_t \right\} = P_{H,t} Z_t + \int_0^1 \left[\frac{W_t(h) N_t(h)}{\mu_t^W} + P_{H,t} \Pi_t(h) \right] dh + D_{t-1} \tag{2.5}$$

$W_t(h)$ is the nominal wage in industry h, $\Pi_t(h)$ is the share of profits paid by the hth industry to the households and Z_t are lump-sum government transfer payments. $\mu_t^W \leq 1$ is an exogenous wage markup factor and common to all domestic labor markets.[3] I assume that households have access to a complete set of state contingent claims, traded domestically and internationally. D_t is the household's bond portfolio at the end of period t, which may include state-contingent claims of many sorts. The (nominal) market value of such a bundle of state-contingent claims in period t is given by $E_t \left\{ Q_{t,t+1} D_t \right\}$, where $Q_{t,t+1}$ is the stochastic discount factor for pricing arbitrary financial claims. One-period government bonds, which could be denominated in local currency, $B_t^{LC,S}$, inflation-indexed units, b_t^S, and foreign currency, $B_t^{FC,S}$, are included in D_t and pay gross interest rate R_t, r_t, and R_t^* in period $t+1$, respectively. We price these bonds, but shall impose they are in zero net supply, $\tilde{B}_t^S \equiv 0$. Additionally, households have access to long-term government debt portfolios, also included in D_t. These portfolios could be denominated in local currency, $B_t^{LC,M}$, inflation-indexed units, b_t^M, and foreign currency, $B_t^{FC,M}$, with respective prices $Q_t^{LC,M}$, q_t^M, and $Q_t^{FC,M}$. Each portfolio consists of perpetuities with coupons that decay exponentially, as in

[3] I follow P. Benigno and Woodford (2007) to include a time-varying exogenous wage markup that introduces a "pure" cost-push shock. Chari, Kehoe, and McGrattan (2009) show that many underlying setups yield an equivalent wedge between the real wage and the marginal rate of substitution.

Woodford (2001). A bond issued at date t pays ρ^{k-1} units at date $t+k$, for $k \geq 1$ and $\rho \in [0,1]$ is the coupon decay factor that parameterizes the average maturity of the debt portfolio. A consol is the special case when $\rho = 1$ and one-period debt arise when $\rho = 0$. I assume the same average maturity for all currency denominations, therefore, the duration of the composite long-term debt protfolio is $(1-\beta\rho)^{-1}$.[4]

Using the definition of the real exchange rate and assuming symmetric initial conditions across countries (i.e. zero net foreign asset holdings and an *ex ante* identical environment), we obtain the international risk sharing condition $C_t = C_t^* \mathcal{Q}_t^{\frac{1}{\sigma}}$.[5]

2.2.2 Firms

A continuum of monopolistically competitive firms produce differenciated goods. Production of good h, produced in the home economy, and of good f, produced in the foreign economy, are represented by $Y_t(h) = A_t N_t(h)$ and $Y_t^*(f) = A_t^* N_t^*(f)$, where A_t and A_t^* are exogenous aggregate technology shocks, common across firms in the respective economies. Firms h, in the home economy, and firms f, in the foreign economy, face the demand schedules $Y_t(h) = \left(\frac{P_{H,t}(h)}{P_{H,t}}\right)^{-\epsilon} Y_t$ and $Y_t(f) = \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\epsilon} Y_t^*$. With demand imperfectly price-elastic, each firm has some market power, leading to the monopolistic competition distortion in the economy.

Another distortion stems from nominal rigidities. In both economies prices are staggered, as in Calvo (1983), with a fraction of $1-\theta$ of firms permitted to choose a new price, $\breve{P}_{H,t}(h)$, each period, while the remaining forms cannot adjust their prices. Firms that can reset their price choose $\breve{P}_{H,t}(h)$ to maximize expected discounted profits, $\max E_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+1} \left[(1-\tau_{t+k})\breve{P}_{H,t}(h) Y_{t+k|t}(h) - \Psi_{t+k}(Y_{t+k|t}(h))\right]$ subject to the demand schedule $Y_{t+k|t}(h) = \left(\frac{\breve{P}_{H,t}(h)}{P_{H,t+k}}\right)^{-\epsilon} Y_{t+k}$, where $Q_{t,t+1}$ is the stochastic discount factor for the price at t of one unit of composite consumption goods at

[4] Appendix 2.A details the first-order conditions of the households.

[5] Appendix 2.A details the no-arbitrage condition between one-period and long-term bonds, and the derivation of the international risk sharing condition.

$t+k$, defined by $Q_{t,t+k} = \beta^k \frac{U_{C,t+k}}{U_{C,t}} \frac{P_t}{P_{t+k}}$. Sales revenues are taxed at rate τ_t, Ψ_t is the cost function, and $Y_{t+k|t}(h)$ is the output in period $t+k$ for a firm that last reset its price in period t.

The first-order condition for this maximization problem implies that the newly chosen price in period t, $\breve{P}_{H,t}(h)$, satisfies $\left(\breve{P}_{H,t}(h)/P_{H,t}\right)^{1+\epsilon\varphi} = \epsilon/(\epsilon-1)(K_t/J_t)$, where K_t and J_t are aggregate variables defined in Appendix 2.B.

2.2.3 Government

The government consists of monetary and fiscal authorities who face the consolidated budget constraint, expressed in real terms:

$$Q_t^{LC,M} \frac{B_t^{LC,M}}{P_t} + q_t^M b_t^M + Q_t^{FC,M} \frac{\mathcal{E}_t B_t^{FC,M}}{P_t} + Q_t^{LC,S} \frac{B_t^{LC,S}}{P_t} + q_t^S b_t^S + Q_t^{FC,S} \frac{\mathcal{E}_t B_t^{FC,S}}{P_t} + p_{H,t} S_t$$
$$= (1 + Q_{t-1}^{LC,M}) \frac{B_{t-1}^{LC,M}}{P_t} + q_{t-1}^M b_{t-1}^M + Q_{t-1}^{FC,M} \frac{\mathcal{E}_t B_{t-1}^{FC,M}}{P_t} + \frac{B_{t-1}^{LC,S}}{P_t} + b_{t-1}^S + \frac{\mathcal{E}_t B_{t-1}^{FC,S}}{P_t} \tag{2.6}$$

where S_t is the real primary budget surplus defined as $S_t = \tau_t Y_t - G_t - Z_t$. G_t is government demand for the composite good and Z_t is government transfer payments. I assume that the public sector only consumes home goods and has preferences for differentiated goods analogous to the ones of the private sector (given by equation (2.2)). I consider a fiscal regime in which both G_t and Z_t are exogenous processes and only τ_t adjusts endogenously to ensure government solvency.

An intertemporal equilibrium -or solvency- condition links the real market value of outstanding government debt to the expected present value of primary surpluses.[6] After imposing the expectations theory of the term structure, that condition is

$$\left[1 + E_t \sum_{k=0}^{\infty} \rho^{k+1} \tilde{Q}_t^S \tilde{Q}_{t+1}^S \ldots \tilde{Q}_{t+k}^S\right] \frac{\tilde{B}_{t-1}^M}{P_t} = E_t \sum_{k=0}^{\infty} \mathcal{R}_{t,t+k} S_{t+k} \tag{2.7}$$

[6] Appendix 2.C derives this condition.

where $\mathcal{R}_{t,t+k}$ is the k-period real discount factor.[7]

The price level today must be consistent with expected future monetary and fiscal policies, whether those policies are set optimally or not. When the government issues bonds in local currency ($\gamma_{LC} \neq 0$), debt maturity introduces a fresh channel for expected monetary policy –choices of short-term nominal interest rates, R_{t+k}– to affect the current price level through the government's solvency condition. When the government issues bonds in foreign currency ($\gamma_{FC} \neq 0$), the price level today is also affected by the monetary policy conducted in the rest of the world through the short-term nominal interest rate, $\hat{R}_t^*$.

Regarding the rest of the world, I assume the government issues bonds only in foreign currency, and the average maturity of its portfolio is indexed by ρ^*.

2.2.4 Equilibrium

Goods market clearing in the small open economy and in the rest of the world requires

$$Y_t = \left(\frac{1-\alpha\mathcal{Q}_t^{1-\eta}}{1-\alpha}\right)^{\frac{-\eta}{1-\eta}} \left[(1-\alpha)C_t + \alpha\left(\frac{1}{\mathcal{Q}_t}\right)^{-\eta} C_t^*\right] + G_t \quad \text{and} \quad Y_t^* = C_t^* + G_t^*,$$

Labor market clearing in the small open economy requires $\Delta_t^{\frac{1}{1+\varphi}} Y_t = A_t N_t$.[8] $\Delta_t = \int_0^1 \left(\frac{P_{H,t}(h)}{P_{H,t}}\right)^{-\epsilon(1+\varphi)} dh$ denotes the measure of price dispersion across firms and satisfies the recursive relation

$$\Delta_t = (1-\theta)\left[\frac{1-\theta\pi_{H_t}^{\epsilon-1}}{1-\theta}\right] + \theta\pi_{H,t}^{\epsilon(1+\varphi)}\Delta_{t-1}$$

Price dispersion is the source of welfare losses from inflation variability. An analogous condition characterizes the labor market clearing in the rest of the world.

[7] See Appendix 2.C for more details.

[8] Appendix 2.D derives the market clearing conditions.

Combining the budget constraint of the households (2.5) and the government (2.6), we get the small open economy aggregate resource constraint

$$P_{H,t}\left(Y_t - G_t - \frac{P_t}{P_{H,t}}C_t\right) = E_t\left\{Q_{t,t+1}D_t\right\} - \tilde{Q}_t^S\tilde{B}_t^S - \tilde{Q}_t^M\tilde{B}_t^M - \left[D_{t-1} - \tilde{B}_{t-1}^S - (1+\rho\tilde{Q}_t^M)\tilde{B}_{t-1}^M\right]$$

where the right-hand-side represents small open economy's net exports, which are equal to the net foreign assets (total assets net of government bonds).

2.3 Optimal Policy

In the optimal policy problem, the government chooses functions for the tax rate, τ_t, and the short-term nominal interest rate, R_t, taking as given the exogenous processes for technology, A_t, the wage markup, μ_t^W, government purchases, G_t, and lump-sum transfers, Z_t. The optimal Ramsey problem implies choosing optimal paths $\{Y_t, \pi_{H,t}, \tau_t, b_t, \Delta_t, J_t, K_t, Q_t^M, R_t\}$ to maximize the welfare of households given by

$$E_0\sum_{t=0}^{\infty}\beta^t U(\vec{Y}_t, \Delta_t, \xi_t)$$

subject to households and firms optimality conditions, the aggregate resource constraint, the law of motion for price dispersion, the relationship between relative prices, and the government's identity.[9]

I use $\vec{Y}_t$ to refer to the vector of endogenous variables, $\vec{Y}_t = [Y_t, C_t, p_{H,t}, \mathcal{Q}_t, \tau_t, \Delta_t]$, and ξ_t to refer to the vector of exogenous shocks, $\xi_t = [A_t, \mu_t^w, G_t, Z_t, C_t^*]$. Note that C_t^* is a function of foreign exogenous shocks $[A_t^*, \mu_t^{w*}, G_t^*, Z_t^*]$. For the cases of only local-currency bonds or inflation-indexed bonds, is enough to know C_t^*, but when the government issues foreign-currency bonds, we need to know each foreign shock separately, because foreign interest rate, R_t^*, and foreign inflation, π_t^*, will affect the

[9]Appendix 2.E details the constraints and Appendix 2.F defines the symmetric deterministic steady state.

small open economy too.

In the unconstrained Ramsey problem, the presence of expectations of variables in the constraint set makes the optimal policy time inconsistent. The Ramsey equilibrium would involve ex ante promises that appear suboptimal ex post if the government can reoptimize at a later date. To be comparable to existing work, we avoid time-inconsistency problem by adopting Woodford (2003)'s "timeless perspective". I formulate the Ramsey problem recursively as if the optimal rule had been computed in the distant past. The government commits to a time-invariant policy rule that is optimal subject to an initial pre-commitment, with the property of self-consistency.

2.3.1 Linear-quadratic approximation

Following P. Benigno and Woodford (2004), I compute a linear-quadratic approximation to the nonlinear optimal solutions. Distortionary taxes and monopolistic competition in product and labor markets make the deterministic steady state inefficient.[10] With a distorted steady state, an *ad hoc* linear-quadratic representation of the problem does not yield an accurate approximation of the optimal policy.[11] The main issue arises from the presence of a linear term in the second-order approximation to the welfare loss function. In this case, a first-order approximation to the equilibrium conditions ignores second-order terms potentially relevant to welfare.[12]

I adopt P. Benigno and Woodford (2004)'s approach because it leads to neat analytical solutions that separate the channels through which long-term debt affects

[10]The size of the steady state distortion is measured by a parameter Φ that derives a wedge between the marginal product of labor (MPN) and the marginal rate of substitution, $-\frac{U_N}{U_C} = (1-\Phi)MPN$, where $\Phi = 1 - \frac{1-\bar{\tau}}{\bar{\mu}^W}\frac{\epsilon-1}{\epsilon} > 0$, depends on the steady state tax rate, the steady state wage markup and the elasticity of substitution between differentiated goods.

[11]One way to eliminate the inefficiency of the steady state is to assume that an employment subsidy offsets the distortion from the market power of monopolistically-competitive price-setters or distorting taxes. Then the steady state with zero inflation involves an efficient level of output. We instead consider the more plausible case in which an employment subsidy is not available. Because lump-sum taxes are not available, $\Phi = 0$ is possible only when initial government debt is negative.

[12]See J. Kim and S. Kim (2003), Galí (2008) and Woodford (2011) for further discussion.

optimal allocations and it nests conventional analyses of both optimal inflation-smoothing and optimal tax-smoothing, to connect the two literatures.

Even though I conduct the second-order approximation for the general case, I will focus from now on on a special case that abstracts from terms of trade externalities. The special parameter configuration that delivers this result corresponds to $\sigma\eta = 1$.[13]

The representative household experiences welfare losses that, up to a second-order approximation, are proportional to[14]

$$\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\left(q_{\pi\alpha}\hat{\pi}_{H,t}^2 + q_{x\alpha}\hat{x}_t^2\right) \tag{2.8}$$

where $\hat{x}_t$ denotes the welfare-relevant output gap, defined as the deviation between $\hat{Y}_t$ and its efficient level $\hat{Y}^e_{\alpha,t}$, $\hat{x}_t \equiv \hat{Y}_t - \hat{Y}^e_{\alpha,t}$. Efficient output, $\hat{Y}^e_{\alpha,t}$, depends on the five fundamental shocks and is given by $\hat{Y}^e_t = q_{A\alpha}\hat{A}_t + q_{G\alpha}\hat{G}_t + q_{Z\alpha}\hat{Z}_t + q_{W\alpha}\hat{\mu}^W_t + q_{C^*}\hat{C}^*_t$.[15] The relative weight on output stabilization, $\lambda_\alpha \equiv q_{x\alpha}/q_{\pi\alpha}$, depends on model parameters defined in Appendix 2.J. All qs depend upon the degree of openness of the economy. The more open the economy, the smaller the weight on output stabilization relative to inflation. Furthermore, the dependence of the efficient output on fundamental shocks is affected by the degree of openness. In particular, the more open the economy is, the larger is the effect of foreign consumption on efficient output, capturing the larger exposure to the rest of the world. (See Figure 2.J.1)

2.3.2 Linear Constraints

Constraints on the optimization problem come from log-linear approximations to the model equations. The first constraint is the Phillips curve, which I follow P.

[13] $\sigma\eta = 1$ eliminates the terms of trade externalities, and $\sigma = \eta = 1$ implies a balance trade at all times (as in the simple case considered by Galí and Monacelli 2005).

[14] Appendices 2.G-2.J contain detailed derivations.

[15] Appendix 2.J defines parameters $q_{A\alpha}, q_{G\alpha}, q_{Z\alpha}, q_{W\alpha}$ and q_{C^*}.

Benigno and Woodford (2004) to rewrite as

$$\hat{\pi}_{H,t} = \beta E_t \hat{\pi}_{H,t+1} + \kappa \hat{x}_t + \kappa\psi(\hat{\tau}_t - \hat{\tau}^e_{\alpha,t}) \tag{2.9}$$

where $\hat{\tau}^e_{\alpha,t} \equiv -\frac{u_{\alpha,t}}{\kappa\psi}$, and $u_{\alpha,t}$ is a composite cost-push shock that depends on all exogenous disturbances, domestic and foreign, as defined in Appendix 2.L.

The functional form of the Phillips curve for the small open economy corresponds to that of the closed economy, at least as far as domestic inflation, $\hat{\pi}_{H,t}$, is concerned. When $\sigma\eta = 1$, i.e. no terms of trade externalities, the slope of the Phillips curve, κ, is identical to its closed economy version, and the degree of openness, α, affects the dynamics of the domestic inflation through the foreign shocks that make up the composite cost-push shock. Currency composition of sovereign bonds does not affect the Phillips curve. For more details see Appendix 2.L.

The household's Euler equation for domestic bonds produces a second constraint. After imposing market clearing, it becomes

$$\hat{x}_t = E_t \hat{x}_{t+1} + \gamma_{LC} \frac{s_C}{\sigma_{\alpha\gamma}} E_t \hat{\pi}_{H,t+1} - \frac{s_C}{\sigma_{\alpha\gamma}} \left(\hat{\hat{R}}_t - \hat{\hat{R}}^e_{\alpha,t} \right) \tag{2.10}$$

where s_C is the steady state consumption to GDP ratio, $\hat{\hat{R}}_t = \gamma_{LC}\hat{R}_t + \gamma_{CPI}\hat{r}_t + \gamma_{FC}\hat{R}^*_t$ and $\hat{\hat{R}}^e_{\alpha,t} \equiv \frac{\sigma_{\alpha\gamma}}{s_c} \upsilon_{\alpha,t}$ is the setting of the short-term nominal interest rate that exactly offsets the composite demand-side shock, which is defined in Appendix 2.M, as well as parameter $\sigma_{\alpha\gamma}$.

The functional form of the IS curve for the small open economy is similar to that found in the closed economy. However, two differences can be pointed out. First, the degree of openness and the currency composition of sovereign debt influence the sensitivity of the output gap to interest rate changes, $s_C/\sigma_{\alpha\gamma}$. A higher share of bonds denominated in foreign currency (or smaller share of bonds denominated in local currency), an increase in openness raises the sensitivity of the output gap to interest rate changes (through the stronger effects of the introduced terms of trade changes on demand). This effect increases with the degree of openness. Second, openness makes the composite demand-side shock depend on foreign shocks, in addition to domestic shocks, and the currency composition of the sovereign debt affects their impact. Also notice that when there are no bonds denominated in local currency, $\gamma_{LC} = 0$, then the output gap does not respond to changes in home inflation. The details are described in Appendix 2.M.

If policy makers faced only constraints (2.9) and (2.10), monetary and fiscal policy could stabilize inflation and output gap completely to achieve the first-best outcome, $\hat{\pi}_{H,t} = \hat{x}_t = 0$, by setting by setting $\hat{\tau}_t = \hat{\tau}^e_{\alpha,t}$ and $\hat{\tilde{R}}_t = \hat{\tilde{R}}^e_{\alpha,t}$. To reach this first-best outcome, policy must have access to a non-distorting source of revenues or to state-contingent debt that can adjust to ensure that the government's solvency requirements do not impose additional restrictions on achievable outcomes. Note that in the small open economy this implies responding to external shocks as well. The nature of this response will depend on both the degree of openness and the currency composition of the sovereign debt.

When non-distorting revenues are not available and the government issues bonds denominated in local currency, policy choice can, in effect, convert nominal debt into state-dependent real debt by issuing long-term nominal debt. However, when the government issues only inflation-indexed bonds or foreign currency bonds, the real value of private claims on the government at period t, b^M_{t-1} or $b^{FC,M}_{t-1}$, are predetermined variables. This means that unexpected inflation variations are no longer able to relax the intertemporal government solvency condition. Policies must be consistent with the flow government budget identity:

$$\left(\frac{\hat{\tilde{B}}_{t-1}^M}{P_{t-1}}\right) + \tilde{f}_t = \beta\left(\frac{\hat{\tilde{B}}_t^M}{P_t}\right) + (1-\beta)\frac{\bar{\tau}}{s_D}(\hat{x}_t + \hat{\tau}_t) + \gamma_{LC}\hat{\pi}_{H,t} + \beta(1-\rho)\hat{\tilde{Q}}_t^M$$
$$- \alpha\frac{\sigma}{s_C}(1-\beta)\hat{x}_t + \frac{\sigma}{s_C}\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\Delta\hat{x}_t \quad (2.11)$$

where $s_D \equiv \bar{S}/\bar{Y}$ is the steady-state surplus-output ratio. $\tilde{f}_t$, a composite fiscal shock that reflects all exogenous disturbances to the government's budget identity, is defined in Appendix 2.N. Note that in the small open economy, the composite fiscal shocks includes foreign shocks.

Naturally, the flow government budget constraint is affected by the currency composition of its debt. Equation (2.11) allows us to consider different currency compositions by varying γ. Note that domestic inflation is present in this expression only when the government issues some fraction of local currency bonds, $\gamma_{LC} \neq 0$. Additionally, openness interacts with the currency composition of the debt and affects the fiscal composite shock and the fiscal resources derived from different output gap levels.

Absence of arbitrage between short-term and long-term bonds delivers the fourth constraint on the optimal policy program

$$\beta\rho E_t\hat{\tilde{Q}}_{t+1}^M = \hat{\tilde{Q}}_t^M + \hat{\tilde{R}}_t \quad (2.12)$$

Iterating on (2.12) and applying a terminal condition yields the term structure relation $\hat{\tilde{Q}}_t^M = -E_t\sum_{k=0}^{\infty}(\beta\rho)^k\hat{\tilde{R}}_{t+k}$. When $\rho = 0$, all bonds are one period, the long-term interest rate at time t is proportional to the current short-term interest rate, so any disturbance to the long rate will also affect the current short rate. When $\rho > 0$, the long-term interest rate at time t is determined by the whole path of future short-term interest rates, making intertemporal smoothing possible. A disturbance to the long-term interest rate can be absorbed by adjusting future short-term interest rates, with no change in the current short rate. By separating current and future monetary policies, long bonds provide policy additional leverage. If the government issues only bonds in foreign currency, the interest rate structure is determined by the international interest rate, which is not controlled by the small open economy monetary policy. However, ρ define this dependence. When $\rho = 0$ the long-term interest rate in the small open economy at time t is proportional to the current short-term interest rate in the rest of the world, so any disturbance to the long rate in the rest of the world will also affect the current short rate in the small open economy. When $\rho > 0$, the long-term interest rate in the small open economy at time t is determined by the whole path of future short-term interest rates in the rest of the world.

Solving the government budget identity forward and imposing transversality and the term structure relation yields an intertemporal version of the solvency condition

$$\left(\frac{\hat{\tilde{B}}^M_{t-1}}{P_{t-1}}\right) + \tilde{F}_t = \gamma_{LC}\hat{\pi}_{H,t} - \frac{\sigma}{s_C}\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\hat{x}_{t-1} + \frac{\sigma}{s_C}\hat{x}_t$$
$$+ (1-\beta)E_t\sum_{k=0}^{\infty}\beta^k\left[b_\tau\left(\hat{\tau}_{t+k} - \hat{\tau}^e_{t+k}\right) + b_x\hat{x}_{t+k}\right]$$
$$+ E_t\sum_{k=0}^{\infty}(\beta\rho)^{k+1}\left(\hat{\tilde{R}}_{t+k} - \hat{\tilde{R}}^e_{t+k}\right) \qquad (2.13)$$

Note that condition (2.13) holds as long as the government issues some fraction of local currency bonds, $\gamma_{LC} \neq 0$. I consider other cases in Appendix 2.O.

The sum $\left(\frac{\hat{\tilde{B}}^M_{t-1}}{P_{t-1}}\right) + \tilde{F}_t$ summarizes the fiscal stress that prevents complete stabilization of inflation and the welfare-relevant output gap. Given the definitions of $\hat{\tau}^e_t$ and $\hat{\tilde{R}}^e_t$, $\tilde{F}_t$ reflects fiscal stress stemming from three conceptual distinct but related sources: the composite fiscal shocks, $\tilde{f}_t$, the composite cost-push shock, u_t (through $\hat{\tau}^e_t$), and the composite aggregate demand shock, υ_t (through $\hat{\tilde{R}}^e_t$). $\tilde{F}_t$ and parameters b_τ and

b_x are defined in Appendix 2.O. Note that in the small open economy these shocks are domestic and foreign.

With $\tilde{F}_t$ fluctuating exogenously, complete stabilization of inflation and output, $\hat{\pi}_{H,t} = \hat{x}_t = 0$, which requires $\hat{\tau}_t = \hat{\tau}_t^e$, $\hat{\tilde{R}}_t = \hat{\tilde{R}}_t^e$ will not generally satisfy (2.13) and the government would be insolvent. The additional fiscal solvency constraint prevents policy from achieving the first-best allocation. Shocks originated in the rest of the world can affect the small open economy's fiscal stress.

2.4 Optimal Policy Analysis

This section characterizes the nature of optimal policy behavior under sticky prices for different currency denominations of the sovereign debt. The flexible price solution is relegated to Appendix 2.P.

2.4.1 Sticky Prices

Under sticky prices, policy seeks paths for $\left\{\hat{\pi}_{H,t}, \hat{x}_t, \hat{\tau}_t, \hat{\tilde{R}}_t, \hat{\tilde{B}}_t^M, \hat{\tilde{Q}}_t^M\right\}$ that minimize (2.8) subject to (2.9)-(2.12). To facilitate interpretation, I express the optimality conditions for inflation and output gap in terms of the Lagrange multiplier for the term structure, L_t^q, and the flow government budget identity, L_t^b,[16]

$$q_{\pi\alpha}\hat{\pi}_{H,t} = -\frac{1-\beta}{\kappa\psi} b_\tau (L_t^b - L_{t-1}^b) - \gamma_{LC} L_t^b + \gamma_{LC}\frac{1}{\beta} L_{t-1}^q \tag{2.14}$$

$$q_{x\alpha}\hat{x}_t = \left[\left(\frac{1}{\psi} - 1\right) b_\tau - \frac{(\sigma_{\alpha\gamma} - 1)}{s_C}\right](1-\beta)L_t^b - \frac{\sigma_{\alpha\gamma}}{s_C} L_t^q + \frac{1}{\beta}\frac{\sigma_{\alpha\gamma}}{s_C} L_{t-1}^q \tag{2.15}$$

$$\beta(1-\rho)L_t^b = L_t^q - \rho L_{t-1}^q \tag{2.16}$$

$$E_t L_{t+1}^b = L_t^b \tag{2.17}$$

I solve for state-contingent paths for $\left\{\hat{\pi}_{H,t}, \hat{x}_t, \hat{\tau}_t, \hat{\tilde{R}}_t, \hat{\tilde{B}}_t^M, \hat{\tilde{Q}}_t^M, L_t^q, L_t^b\right\}$ that satisfy the

[16] The derivation of the first order conditions of this problem is detailed in Appendix 2.Q.

constraints and optimality conditions.

These conditions make it clear that debt maturity and disturbances to the government affect inflation and the output gap, and that currency composition and openness play an important role. The shadow price of the government budget identity, L_t^b, follows a martingale, according to (2.137), a property that reflects intertemporal smoothing in fiscal financing. L_t^b measures how binding the solvency constraint is on fiscal policy. The term structure multiplier, L_t^q, measures the tightness of the solvency constraint on monetary policy by linking L_t^q to a distributed lag of L_t^b with weights that decay with ρ, debt's duration

$$L_t^q = \beta(1-\rho)\sum_{k=0}^{\infty}\rho^k L_{t-k}^b \tag{2.18}$$

Maturity structure matters through its implications for fiscal financing. How much monetary policy is constrained by fiscal financing depends on the entire history of shadow prices of the government budget, L_{t-k}^b, and the degree of history dependence rises with the average maturity of government debt, ρ. Restricting attention to only one-period bonds so that $\rho = 0$, makes $L_t^q = \beta L_t^b$. This eliminates the history dependence and monetary policy through the term structure of interest rates to render monetary and fiscal policies equally constrained by fiscal solvency.[17] At the opposite extreme, consols set $\rho = 1$, so $L_t^q \equiv 0$ and current monetary policy is not constrained, regardless of how binding the government's budget has been in the past. Note that this conclusion is valid regardless the currency composition of sovereign bonds and the degree of openness of the small open economy.

2.4.2 Stabilizing Optimal Policies

I examine some special cases that sharply characterize the optimal equilibrium and

[17] This is the exercise that finds that active monetary/passive fiscal policies yield highest welfare (Schmitt-Grohé and Uribe 2007, and Kirsanova and Wren-Lewis 2012).

the stabilization roles of fiscal and monetary policy.

Only One-Period Bonds

With only one-period bonds, long-term and short-term interest rates are proportional and L_t^b and L_t^q covary perfectly. Expressions for inflation, (2.14), and output gap, (2.15), become

$$q_{\pi\alpha}\hat{\pi}_{H,t} = -\left(\frac{1-\beta}{\kappa\psi}b_\tau + \gamma_{LC}\right)\left(L_t^b - L_{t-1}^b\right) \tag{2.19}$$

$$q_{x\alpha}\hat{x}_t = \left[\left(\frac{1}{\psi}-1\right)b_\tau(1-\beta) - \beta\frac{\sigma}{s_C} - \frac{(\sigma_{\alpha\gamma}-1)}{s_C}\right]L_t^b + \frac{\sigma_{\alpha\gamma}}{s_C}L_{t-1}^b \tag{2.20}$$

Condition (2.19) implies that inflation is proportional to the forecast error in L_t^b. The degree of proportionality depends on the share of bonds denominated in local currency. Because (2.17) requires there are no forecastable variations in L_t^b, the expectation of inflation is zero and the price level follows a martingale

$$E_t\hat{\pi}_{H,t} = 0 \qquad \Rightarrow \qquad E_t\hat{P}_{H,t+1} = \hat{P}_{H,t} \tag{2.21}$$

Condition (2.20) makes the output gap a weighted average of L_t^b and L_{t-1}^b. Note that the weights are affected by the degree of openness and the currency composition of the sovereign debt, since both enter σ_{α_γ}. Taking expectations yields

$$E_t\hat{x}_{t+1} - \hat{x}_t = -\frac{1}{\lambda_\alpha}\frac{\sigma_{\alpha\gamma}}{s_C}\left(\frac{1-\beta}{\kappa\psi}b_\tau + \gamma_{LC}\right)^{-1}\hat{\pi}_{H,t} \tag{2.22}$$

so the expected change in the output gap is proportional to current inflation, and the degree of proportionality depends on the degree of openness and the currency composition of the sovereign debt. The optimal degree of output gap smoothing varies with λ_α, the relative weight on the output gap in the welfare loss function, $q_{x\alpha}/q_{\pi\alpha}$. λ_α is affected by the degree of openness of economy: as shown in Figure

2.J.1, the more open the economy is, the smaller the optimal degree of output gap smoothing. Larger λ_α delivers more output-gap smoothing. Flexible prices are a special case with $\lambda_\alpha = \infty$. Under most calibrations, λ_α is quite small, to deliver little smoothing of output gap. But the martingale property of L_t^b implies smoothing of expected future output gaps after a one-time jump. Taking expectations of (2.22) yields

$$\hat{x}_t \neq E_t \hat{x}_{t+1} = E_t \hat{x}_{t+2} = ... = E_t \hat{x}_{t+k} = ... \tag{2.23}$$

Taken together, (2.21) and (2.23) state that with only one-period debt, optimal policies smooth the price level and use fluctuations in the output gap to absorb innovations in fiscal conditions. The reason is apparent: without long-term debt, policy cannot smooth inflation across time, and surprise inflation -and the resulting price dispersion- is far more costly than variations in the output gap; it is optimal to minimize inflation variability and use output (and tax rates) as a shock absorber. In the small open economy, fluctuations in the output gap are use to absorbe both domestic and foreign shocks.

Only Consols

When the government issues only consols, fiscal stress that moves long rates does not need to change short rates contemporaneously. Inflation and output gap are now

$$q_{\pi\alpha} \hat{\pi}_{H,t} = -\frac{1-\beta}{\kappa\psi} b_\tau (L_t^b - L_{t-1}^b) - \gamma_{LC} L_t^b \tag{2.24}$$

$$q_{x\alpha} \hat{x}_t = \left[\left(\frac{1}{\psi} - 1 \right) b_\tau - \frac{(\sigma_{\alpha\gamma} - 1)}{s_C} \right] (1-\beta) L_t^b \tag{2.25}$$

Note that the degree of proportionality of the output gap to L_t^b depends on the degree of openness of the economy and the currency composition, since both enter

$\sigma_{\alpha\gamma}$. Condition (2.25) makes output gap proportional to L_t^b, so the gap inherits the martingale property of L_t^b

$$E_t\hat{x}_{t+1} = \hat{x}_t \tag{2.26}$$

Taking expectations of (2.24) and combining with (2.26), yields

$$E_t\hat{\pi}_{H,t+1} - \hat{\pi}_t = \lambda_\alpha \frac{b_\tau}{\kappa\psi}\left[\left(\frac{1}{\psi}-1\right)b_\tau - \frac{(\sigma_{\alpha\gamma}-1)}{s_C}\right]^{-1}(\hat{x}_t - \hat{x}_{t-1}) \tag{2.27}$$

Condition (2.27) implies that the expected change in inflation is proportional to the change in $\hat{x}_t$. The degree of inflation smoothing varies inversely with ψ, the coefficient on the tax rate in the Phillips curve. It also varies with the degree of openness of the economy, through λ_α and $\sigma_{\alpha\gamma}$, and the currency composition, through $\sigma_{\alpha\gamma}$.

Combining (2.26) and (2.27), we draw opposite conclusions from the case of one-period debt. With consols, intertemporal smoothing of L_t^b smooths the output gap; fluctuations in inflation absorb disturbances to fiscal needs. Now the bond price can absorb fiscal shocks: bad news about future surpluses can reduce the value of outstanding debt, leaving the real discount factor unaffected. A constant real discount factor smooths the output gap, which explains the absence of forecastable variations in the output gap. Variations in the bond price correspond to adjustments in expected inflation. The longer the duration of debt -higher ρ- the less is the required change in bond prices and future inflation for a given change in the present value of surpluses. Although with consols it is optimal to allow surprise inflation to absorb shocks, the expectation of inflation is stabilized after a one-time jump: $\hat{\pi}_{H,t} \neq E_t\hat{\pi}_{H,t+1} = E_t\hat{\pi}_{H,t+2} = ... = E_t\hat{\pi}_{H,t+k} = ...$

General Case

I briefly consider intermediate values for the average duration of debt, $0 < \rho < 1$.

Rewrite (2.14) and (2.15) using the lag-operator, $\mathbb{L}^j x_t \equiv x_{t-j}$

$$q_{\pi\alpha}\hat{\pi}_{H,t} = -\frac{1-\beta}{\kappa\psi} b_\tau (1-\mathbb{L}) L_t^b - \gamma_{LC}(1-\mathbb{L})(1-\rho\mathbb{L})^{-1} L_t^b \tag{2.28}$$

$$q_{x\alpha}\hat{x}_t = \left[\left(\frac{1}{\psi}-1\right) b_\tau - \frac{(\sigma_{\alpha\gamma}-1)}{s_C}\right](1-\beta)L_t^b - \frac{\sigma_{\alpha\gamma}}{s_C}\beta(1-\rho)\left(1-\frac{1}{\beta}\mathbb{L}\right)(1-\rho\mathbb{L})^{-1}L_t^b \tag{2.29}$$

The optimality condition for debt that requires L_t^b to be a martingale may be written as $(1-\mathbb{B})E_{t-1}L_t^b = 0$, where $\mathbb{B}$ is the backshift operator, defined as $\mathbb{B}^{-j}E_t\xi_t \equiv E_t\xi_{t+j}$.

Taking expectations of (2.28) and (2.29) and applying the backshift operator, we obtain general expressions for the k-step-ahead expectations of inflation and output gap

$$E_t\hat{\pi}_{H,t+k} = \rho^k\hat{\pi}_{H,t} + \gamma_{LC}\rho^k\alpha_\pi(L_t^b - L_{t-1}^b) \tag{2.30}$$

$$E_t\hat{x}_{t+k} = \rho^k\hat{x}_t + (1-\rho^k)\alpha_x L_t^b \tag{2.31}$$

where $\alpha_\pi = \frac{1-\beta}{\kappa\psi q_{\pi\alpha}} b_\tau$ and $\alpha_x = \frac{1-\beta}{q_{x\alpha}}\left(\frac{b_\tau}{\psi} - b_x\right)$.

Equations (2.30) and (2.31) summarize the policy problem. The first term on the right hand side comes from the welfare improvements that arise from smoothing. That both terms involve ρ^k means that longer maturity debt helps to smooth expectations of both inflation and output. The second terms bring in the government solvency dimension of optimal policy through the Lagrangian multipliers, and are therefore affected by the currency composition of the sovereign bonds. They capture the trade off between relying on variations in inflation to hedge against fiscal stress and using variations in output to absorb shocks. Maturity has opposite effects on the two variables. As maturity increases, changes in government solvency affect future inflation more strongly, while the output gap becomes less responsive. As maturity extends, it is optimal to trade off inflation for output stabilization. For any maturities short of perpetuities, $0 \leq \rho < 1$, as the forecast horizon extends, $k \to \infty$, expected inflation converges to zero whereas the expected output gap converges to $\alpha_x L_t^b$. Inflation is anchored on zero, but the output gap's "anchor" varies with the state at t. Moreover, a higher share of bonds denominated in local currency enables a higher reliance on inflation variations.

2.5 Concluding Remarks

This chapter examines the role of the currency composition and the maturity structure of the sovereign debt on the optimal monetary and fiscal policy in a small open economy. I extend P. Benigno and Woodford (2004) to a small open economy whose government can commit and finances its expenditure by collecting distortionary taxes and issuing a portfolio of debt consisting of short-term and long-term bonds, denominated in local currency, CPI inflation-indexed units and foreign currency. For analytical purposes, I abstract from terms of trade externalities.[18]

First, I obtain that the degree of openness affects the welfare loss function. The higher the openness, the smaller the relative weight of the output gap relative to home inflation. Moreover, the degree of openness changes the effects of the fundamental shocks on the efficient output, and incorporates the response to exogenous shocks, summarized by foreign consumption.

Second, the linear constraints faced by the small open economy verify some discrepancies relative to the closed economy. The Phillips curve shows a relationship between domestic inflation and output gap. The effect of domestic and foreign exogenous disturbances on the composite cost-push shock depend on the degree of openness. The functional form of the IS curve and the government solvency condition critically depends on the currency composition of the government debt.

[18] See De Paoli (2009) and De Paoli and G. Benigno (2010) for an analysis about optimal monetary and fiscal policy with terms of trade externalities.

The output gap only responds to inflation if the government issues some fraction of bonds denominated in local currency. The sensitivity of the output gap to inflation and interest rates and the effects of domestic and foreign exogenous disturbances on the composite demand-side shocks, depend on the degree of openness of the economy and the currency composition of the government debt. Finally, the larger the share of bonds denominated in local currency, the more can the government relay on inflation as a source of fiscal financing. The more open the economy, the bigger the effect of relative prices on the resources available to the government, since its primary surplus is denominated in units of home goods. In an open economy, the real value of the government debt changes with the real exchange rate. The effect of domestic and foreign exogenous disturbances on the composite fiscal shock depend on the degree of openness and the currency composition of government debt.

I leave for future research the quantitative exercises for particular emerging economies and the study of policy rules that deliver similar results to the optimal obtained in this analysis.

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Appendix

Appendix 2.A Household's optimality conditions, no-arbitrage condition, international risk sharing and uncovered interest parity

The optimality conditions for the household's problem are:

$$N_t(h)\rfloor \; \frac{W_t(h)}{P_t} = \mu_t^W C_t^{\sigma} N_t(h)^{\varphi} \tag{2.32}$$

$$\frac{B_t^{LC,S}}{P_t}\rfloor \; \beta E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \right\} = Q_t^{LC,S} \tag{2.33}$$

$$\frac{B_t^{LC,M}}{P_t}\rfloor \; \beta E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \left(1 + \rho Q_{t+1}^{LC,M}\right) \right\} = Q_t^{LC,M} \tag{2.34}$$

$$\frac{b_t^S}{P_t}\rfloor \; \beta E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \right\} = q_t^S \tag{2.35}$$

$$\frac{b_t^M}{P_t}\rfloor \; \beta E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \left(1 + \rho q_{t+1}^M\right) \right\} = q_t^M \tag{2.36}$$

$$\frac{B_t^{FC,S}}{P_t}\rfloor \; \beta E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right\} = Q_t^{FC,S} \tag{2.37}$$

$$\frac{\mathcal{E}_t B_t^{FC,M}}{P_t}\rfloor \; \beta E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \left(1 + \rho Q_{t+1}^{FC,M}\right) \right\} = Q_t^{FC,M} \tag{2.38}$$

$$\frac{D_t}{P_t}\rfloor \; \beta E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \right\} = E_t \left\{ Q_{t,t+1} \right\} \tag{2.39}$$

(2.32) corresponds to the standard intratemporal optimality condition. Rearranging terms in (2.33), (2.35), and (2.37), we obtain conventional stochastic Euler equations for the one-period domestic bonds:

$$\beta R_t E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \right\} = 1$$
$$\beta r_t E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \right\} = 1$$
$$\beta R_t^* E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right\} = 1$$

For later use, define

$$\tilde{R}_t^{-1} \equiv \gamma_{LC} \frac{1}{R_t} + \gamma_{CPI} \frac{1}{r_t} + \gamma_{FC} \frac{1}{R_t^*} \tag{2.40}$$

with

$$\gamma_{LC} \equiv \frac{B^{LC,M}}{B^{LC,M} + Pb^M + \mathcal{E}B^{FC,M}}$$
$$\gamma_{CPI} \equiv \frac{b^M}{B^{LC,M} + Pb^M + \mathcal{E}B^{FC,M}}$$
$$\gamma_{FC} \equiv \frac{\mathcal{E}B^{FC,M}}{B^{LC,M} + Pb^M + \mathcal{E}B^{FC,M}}$$

Then, we can combine the Euler equation for each type of bond in the following way:

$$\beta \tilde{R}_t E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \left(\gamma_{LC} \frac{P_t}{P_{t+1}} + \gamma_{CPI} + \gamma_{FC} \frac{P_t}{P_{t+1}} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right) \right\} = 1 \tag{2.41}$$

No-arbitrage condition between one-period and long-term bonds. Combining (2.33) and (2.34), (2.35) and (2.36), and (2.37) and (2.38) yields the no-arbitrage conditions between one-period and long-term bonds

$$
\begin{aligned}
Q_t^{LC,M} &= E_t Q_t^{LC,S}\left(1+\rho Q_{t+1}^{LC,M}\right) \\
q_t^{M} &= E_t q_t^{S}\left(1+\rho q_{t+1}^{M}\right) \\
Q_t^{FC,M} &= E_t Q_t^{FC,S}\left(1+\rho Q_{t+1}^{FC,M}\right)
\end{aligned} \tag{2.42}
$$

International risk sharing. Under the assumption of complete international financial markets, a first order condition analogous to (2.39) must hold for the representative household in the rest of the world:

$$
\beta E_t\left\{\frac{(C_{t+1}^*)^{-\sigma}}{(C_t^*)^{-\sigma}}\frac{P_t^*}{P_{t+1}^*}\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right\} = E_t\left\{Q_{t,t+1}\right\} \tag{2.43}
$$

Combining equations (2.39) and (2.43), together with the definition of the real exchange rate, it follows that

$$
C_t = \varrho^* C_t^* \mathcal{Q}_t^{\frac{1}{\sigma}} \tag{2.44}
$$

for all t, and where ϱ is a constant which will generally depend on initial conditions regarding relative net asset positions. Henceforth, and without loss of generality, I assume symmetric initial conditions (i.e. zero net foreign asset holdings and ex ante identical environment), in which case $\varrho = \varrho^* = 1$ for all i. As shown in Appendix 2.F, in the symmetric perfect foresight steady state we also have that $\bar{C} = \bar{C}^*$ and $\bar{\mathcal{Q}} = 1$.

Uncovered interest parity. Under the assumption of complete international financial markets, the equilibrium price (in terms of domestic currency) of a riskless bond denominated in foreign currency is given by $\mathcal{E}_t(R_t^*)^{-1} = E_t\{Q_{t,t+1}\mathcal{E}_{t+1}\}$. The previous pricing equation can be combined with the domestic bond pricing equation, $\tilde{R}_t^{-1} = E_t\{Q_{t,t+1}\}$ to obtain a version of the uncovered interest parity condition:

$$
E_t\left\{Q_{t,t+1}\left[\tilde{R}_t - R_t^*\left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)\right]\right\} = 0
$$

Appendix 2.B Characterization of the price setting of the firms

Firms are assumed to set prices as in Calvo (1983). Hence, a measure $1-\theta$ of randomly selected firms sets new prices each period, with an individual firm's probability of re-optimizing in any given period being independent of the time elapsed since it last reset its price. Let $\breve{P}_{H,t}(h)$ denote the price set by a firm h adjusting its price in period t. Under the Calvo price-setting structure, $P_{H,t+k}(h) = \breve{P}_{H,t}(h)$ with probability θ^k for $k = 0, 1, 2, ...$

Given the Calvo-type setup, the price index evolves according to the following law of motion

$$P_{H,t}^{1-\epsilon} = (1-\theta)\breve{P}_{H,t}(h)^{1-\epsilon} + \theta P_{H,t-1}^{1-\epsilon} \tag{2.45}$$

Firms that can reset their price choose $\breve{P}_{H,t}(h)$ to maximize the expected sum of discounted future profits by solving:

$$\text{Max}_{\breve{P}_{H,t}(h)} \; E_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+1} \left[(1-\tau_{t+k})\breve{P}_{H,t}(h) Y_{t+k|t}(h) - \Psi_{t+k}(Y_{t+k|t}(h)) \right]$$

subject to the demand schedule:

$$Y_{t+k|t}(h) = \left(\frac{\breve{P}_{H,t}(h)}{P_{H,t+k}} \right)^{-\epsilon} Y_{t+k}$$

where $Q_{t,t+1}$ is the stochastic discount factor for the price at t of one unit of composite consumption goods at $t+k$, defined by $Q_{t,t+k} = \beta^k (C_{t+k}/C_t)^{-\sigma}(P_t/P_{t+k})$ (see (2.50)). Sales revenues are taxed at rate τ_t, Ψ_t is the cost function, and $Y_{t+k|t}(h)$ is the output in period $t+k$ for a firm that last reset its price in period t.

Note that

$$\Psi_{t+k}(Y_{t+k|t}(h)) = C_{t+k}^{\sigma} \left(\frac{P_{H,t}(h)}{P_{H,t+k}}\right)^{-\epsilon\varphi} \left(\frac{Y_{t+k}}{A_{t+k}}\right)^{\varphi} P_{t+k} \left(\frac{\breve{P}_{H,t}(h)}{P_{H,t+k}}\right)^{-\epsilon} \frac{Y_{t+k}}{A_{t+k}}$$

where the industry wage is obtained from the labor supply equation (2.32), under the assumption that each of the firms in industry h (other than the firm in consideration) charges the common price $P_{H,t}(h)$. Because all firms in a given industry are assumed to adjust their prices at the same time, in equilibrium the prices of firms in a given industry are always identical. We must nonetheless define the profit function for the case in which the firm in consideration deviates from the industry price, in order to determine whether the industry price is optimal for each individual firm.

The first-order condition for this maximization problem implies that the newly chosen price in period t, $\breve{P}_{H,t}(h)$, satisfies

$$\begin{aligned} \left(\frac{\breve{P}_{H,t}(h)}{P_{H,t}}\right)^{1+\epsilon\varphi} &= \frac{\epsilon}{\epsilon-1} \frac{E_t \sum_{k=0}^{\infty} (\beta\theta)^k \mu_t^W \left(\frac{Y_{t+k}}{A_{t+k}}\right)^{\varphi+1} \left(\frac{P_{H,t+k}}{P_{H,t}}\right)^{\epsilon(1+\varphi)}}{E_t \sum_{k=0}^{\infty} (\beta\theta)^k (1-\tau_{t+k}) \frac{P_{H,t+k}}{P_{t+k}} C_{t+k}^{-\sigma} Y_{t+k} \left(\frac{P_{H,t+k}}{P_{H,t}}\right)^{\epsilon-1}} \\ &= \frac{\epsilon}{\epsilon-1} \frac{K_t}{J_t} \end{aligned} \tag{2.46}$$

where K_t and J_t are aggregate variables that satisfy the recursive relations

$$K_t = \mu_t^W \left(\frac{Y_t}{A_t}\right)^{\varphi+1} + \beta\theta E_t K_{t+1} \pi_{H,t+1}^{\epsilon(1+\varphi)} \tag{2.47}$$

$$J_t = (1-\tau_t) p_{H,t} C_t^{-\sigma} Y_t + \beta\theta E_t J_{t+1} \pi_{H,t+1}^{\epsilon-1} \tag{2.48}$$

and $\pi_{H,t} = \frac{P_{H,t}}{P_{H,t-1}}$.

Substituting equation (2.46) into the law of motion for the price index (2.45) yields a short-run aggregate-supply relation between inflation and output of the form

$$\left(\frac{1-\theta\pi_{H,t}^{\epsilon-1}}{1-\theta}\right)^{\frac{1+\epsilon\varphi}{1-\epsilon}} = \frac{\epsilon}{\epsilon-1}\frac{K_t}{J_t} \tag{2.49}$$

Same logic applies to the rest of the world.

Appendix 2.C Derivation of long-term debt price and intertemporal equilibrium condition of the government

Using (2.39), define

$$Q_{t,t+k} = \beta^k \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\frac{P_t}{P_{t+k}} \tag{2.50}$$

as the stochastic discount factor for the price at t of one unit of composite consumption goods at $t+k$. Now combine (2.33) and (2.39) to obtain

$$Q_t^{LC,S} = E_t Q_{t,t+1}$$

Then (2.34) in Appendix 2.A can be written as

$$Q_t^{LC,M} = E_t Q_{t,t+1}\left(1+\rho Q_{t+1}^{LC,M}\right) \tag{2.51}$$

Iterating (2.51) forward and imposing a terminal condition yields

$$
\begin{aligned}
Q_t^{LC,M} &= E_t\left\{Q_{t,t+1} + \rho Q_{t,t+1} Q_{t+1}^{LC,M}\right\} \\
&= E_t\left\{Q_{t,t+1} + \rho Q_{t,t+1} E_{t+1} Q_{t+1,t+2}^{LC,M} + \rho^2 Q_{t,t+1} E_{t+1}\left\{Q_{t+1,t+2} E_{t+2} Q_{t+2,t+3}\right\} + ...\right\} \\
&= Q_t^{LC,S} + \rho E_t\left\{Q_{t,t+1} Q_{t+1}^{LC,S}\right\} + \rho^2 E_t\left\{Q_{t,t+1} E_{t+1}\left\{Q_{t+1,t+2} Q_{t+2}^{LC,S}\right\}\right\} + ... \\
&= Q_t^{LC,S} + \rho E_t\left\{Q_{t,t+1} Q_{t+1}^{LC,S}\right\} + \rho^2 E_t\left\{Q_{t,t+2} Q_{t+2}^{LC,S}\right\} + ... + \rho^k E_t\left\{Q_{t,t+k} Q_{t+k}^{LC,S}\right\} + ... \\
&= E_t \sum_{k=0}^{\infty} \rho^k Q_{t,t+k} Q_{t+k}^{LC,S}
\end{aligned}
\tag{2.52}
$$

Equation (2.52) implies that the long-term debt's price is determined by the weighted average of expectations of future short-term debt's prices.

Substitute (2.50) into (2.52)

$$
\begin{aligned}
Q_t^{LC,M} &= E_t\left\{\beta \frac{U_{C,t+1}}{U_{C,t}} \frac{P_t}{P_{t+1}} + \rho\beta^2 \frac{U_{C,t+2}}{U_{C,t}} \frac{P_t}{P_{t+2}} + ... + \rho^{k-1}\beta^k \frac{U_{C,t+k}}{U_{C,t}} \frac{P_t}{P_{t+k}} + ...\right\} \\
&= E_t \sum_{k=1}^{\infty} \rho^{k-1}\beta^k \frac{U_{C,t+k}}{U_{C,t}} \frac{P_t}{P_{t+k}} \\
&= E_t \sum_{k=1}^{\infty} \rho^{k-1} Q_{t,t+k}
\end{aligned}
\tag{2.53}
$$

Condition (2.53) implies that the long-term debt price is determined by the whole path of expected future price level, discounted by consumption growth rate. The long-term debt price is negatively correlated with expected future inflation rate and consumption growth rate.

We can now rewrite (2.51) as

$$\begin{aligned} Q_t^{LC,M} &= E_t Q_{t,t+1}\left(1+\rho Q_{t+1}^{LC,M}\right) \\ &= E_t Q_{t,t+1} E_t\left(1+\rho Q_{t+1}^{LC,M}\right)+\rho Cov\left(Q_{t,t+1},Q_{t+1}^{LC,M}\right) \\ &= E_t Q_t^{LC,S}\left(1+\rho Q_{t+1}^{LC,M}\right)+\rho Cov\left(Q_{t,t+1},Q_{t+1}^{LC,M}\right) \end{aligned} \tag{2.54}$$

Using (2.53)

$$Q_{t+1}^{LC,M} = E_t \sum_{k=1}^{\infty} Q_{t+1,t+1+k} \tag{2.55}$$

$Q_{t+1}^{LC,M}$ is determined by weighted average of expected future discounted value of future stochastic discount factors. Therefore, without loss of generality, we assume $Cov\left(Q_{t,t+1},Q_{t+1}^{LC,M}\right)=0$, and (2.54) can be expressed as

$$\begin{aligned} Q_t^{LC,M} &= E_t Q_t^{LC,S}\left(1+\rho Q_{t+1}^{LC,M}\right) \\ &= E_t\left\{Q_t^{LC,S}+\rho Q_t^{LC,S} Q_{t+1}^{LC,M}\right\} \\ &= E_t\left\{Q_t^{LC,S}+\rho Q_t^{LC,S} Q_{t+1}^{LC,M}+\rho^2 Q_t^{LC,S} Q_{t+2}^{LC,M}\right\} \\ &= E_t \sum_{k=0}^{\infty} \rho^k Q_t^{LC,S} Q_{t+1}^{LC,S} ... Q_{t+k}^{LC,S} \end{aligned} \tag{2.56}$$

Same logic applies to inflation-indexed bonds and bonds denominated in foreign currency.

To simplify notation define the composite long-term government portfolio as:

$$\tilde{B}_t^M \equiv B_t^{LC,M} + E_t P_{t+1} b_t^M + E_t \mathcal{E}_{t+1} B_t^{FC,M}$$

The price of the long-term composite portfolio is then

$$\tilde{Q}_t^M \equiv \gamma_{LC,t} Q_t^{M,LC} + \gamma_{CPI,t} E_t\left\{\frac{q_t^M}{\pi_{t+1}}\right\} + \gamma_{FC,t} E_t\left\{\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right\}$$

with

$$\gamma_{LC,t} = \frac{B_t^{LC,M}}{\tilde{B}_t^M}, \qquad \gamma_{CPI,t} = \frac{E_t P_{t+1} b_t^M}{\tilde{B}_t^M}, \qquad \gamma_{FC,t} = \frac{E_t \mathcal{E}_{t+1} B_t^{FC,M}}{\tilde{B}_t^M}$$

Then,

$$(1+\tilde{\rho Q_t^M}) = \gamma_{LC,t}\left(1+\rho Q_t^{LC,M}\right) + \gamma_{CPI,t}\left(1+\rho q_t^M\right) + \gamma_{FC,t}\left(1+\rho Q_t^{FC,M}\right)$$

To derive the intertemporal equilibrium condition of the government, we iterate its period budget constraint, (2.6), and impose asset-pricing relations and the household's transversality condition (assuming that it holds for each type of bond):

$$\begin{aligned}
(1+\tilde{\rho Q_t^M})\frac{\tilde{B}_{t-1}^M}{P_t} &= \tilde{Q}_t^M \frac{\tilde{B}_t^M}{P_t} + p_{H,t} S_t \\
&= E_t \left\{ \frac{\tilde{Q}_t^M \tilde{Q}_{t+1}^M}{(1+\tilde{\rho Q_{t+1}^M})} \frac{\tilde{B}_{t+1}^M}{P_{t+1}} + \frac{\tilde{Q}_t^M}{(1+\tilde{\rho Q_{t+1}^M})} p_{H,t+1} S_{t+1} + p_{H,t} S_t \right\} \\
&= E_t \left\{ p_{H,t} S_t + \frac{\tilde{Q}_t^M}{(1+\tilde{\rho Q_{t+1}^M})} p_{H,t+1} S_{t+1} \right. \\
&\quad \left. + \frac{\tilde{Q}_t^M \tilde{Q}_{t+1}^M}{(1+\tilde{\rho Q_{t+1}^M})(1+\tilde{\rho Q_{t+2}^M})} p_{H,t+2} S_{t+2} + \ldots \right\}
\end{aligned} \tag{2.57}$$

Substituting (2.50) and (2.51), and their analogs for inflation-indexed and foreign currency bonds, into (2.57) yields

$$\begin{aligned}
(1+\tilde{\rho Q_t^M})\frac{\tilde{B}_{t-1}^M}{P_{t-1}} &= E_t \sum_{k=0}^{\infty} \beta^k \frac{U_{C,t+k}}{U_{C,t}} p_{H,t+k} S_{t+k} \\
&= E_t \sum_{k=0}^{\infty} \mathcal{R}_{t,t+k} S_{t+k}
\end{aligned} \tag{2.58}$$

where $\mathcal{R}_{t,t+k} = \beta^k \frac{U_{C,t+k}}{U_{C,t}} p_{H,t+k}$ is the k-period real discount factor. Notice that, because $p_{H,t+k}$ appears in the real discount factor, changes in relative prices affect the present value of primary surpluses.

The left-hand side of (2.58) highlights a key role of long-term debt. With only one-period debt, $\rho = 0$, the nominal value of outstanding government debt, $\tilde{B}^M_{t-1}$, is predetermined, so the only way for the monetary policy to absorb an unexpected change to the present value of primary surpluses is by surprise inflation or deflation at time t. This can only be done as long as the government issues some fraction of local currency bonds ($\gamma_{LC,t} \neq 0$). Long-term debt, $\rho > 0$, implies that the nominal value of government debt, $(1 + \tilde{\rho Q^M_t})\tilde{B}^M_{t-1}$, is no longer predetermined. Asset-pricing conditions (2.33) and (2.34), and their ananlogous for inflation-indexed and foreign currency bonds, imply the bond price, $\tilde{Q}^M_t$, depends on expected future short-term bond prices (interest rates)

$$\tilde{Q}^M_t = E_t \sum_{k=0}^{\infty} \rho^k \tilde{Q}^S_t \tilde{Q}^S_{t+1} ... \tilde{Q}^S_{t+k} \tag{2.59}$$

and solvency condition (2.58) may be written as

$$\underbrace{\left[1 + E_t \sum_{k=0}^{\infty} \rho^{k+1} \tilde{Q}^S_t \tilde{Q}^S_{t+1} ... \tilde{Q}^S_{t+k}\right]}_{\substack{\text{current and future small open economy} \\ \text{and rest of the world monetary policy}}} \frac{\tilde{B}^M_{t-1}}{P_t} = \underbrace{E_t \sum_{k=0}^{\infty} \mathcal{R}_{t,t+k} S_{t+k}}_{\text{current and future small open economy fiscal policy}} \tag{2.60}$$

Now an unexpected change to the present value of primary surpluses can be absorbed through the monetary policy by adjustments in current and future interest rates, reducing the reliance on current inflation. Note that in presence of bonds denominated in foreign currency ($\gamma_{FC,t} \neq 0$), current and futures changes in the monetary policy of the rest of the world affect the nominal value of government debt (since $\tilde{Q}^S_t$ is a wighted average of R_t^{-1}, R_t^{-1} and $(R^*_t)^{-1}$).

Equilibrium condition (2.60) reflects a fundamental symmetry between monetary and fiscal policies. The price level today must be consistent with future monetary and fiscal policies, whether those policies are set optimally or not. Debt maturity matters:

so long as the average maturity exceed one period, $\rho > 0$, expected future monetary policy –choices of short-term nominal interest rates, R_{t+k} (with $\gamma_{LC,t} \neq 0$), choices of short-term real interest rates (with $\gamma_{CPI,t} \neq 0$), and expected future monetary policy in the rest of the world-choices of short-term nominal interest rates, R^*_{t+k} (with $\gamma_{FC,t} \neq 0$)–, play a role in determining the current price level through the government's solvency condition.

Regarding the rest of the world, I assume the government issues bonds only in foreign currency, and the average maturity of its portfolio is indexed by ρ^*:

$$Q_t^{*M} \frac{B_t^{*M}}{P_t^*} + \frac{P_H^*}{P_t^*} S_t^* = \left(1 + \rho^* Q_t^{*M}\right) \frac{B_{t-1}^{*M}}{P_t^*}$$

Appendix 2.D Derivation of the equilibrium

Goods market clearing in the small open economy requires

$$\begin{aligned}
Y_t(h) &= C_{H,t}(h) + C^*_{H,t}(h) + G_t(h) \\
&= \left(\frac{P_{H,t}(h)}{P_{H,t}}\right)^{-\epsilon} C_{H,t} + \left(\frac{P^*_{H,t}(h)}{P^*_{H,t}}\right)^{-\epsilon} C^*_{H,t} + \left(\frac{P_{H,t}(h)}{P_{H,t}}\right)^{-\epsilon} G_t \\
&= \left(\frac{P_{H,t}(h)}{P_{H,t}}\right)^{-\epsilon} \vartheta \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \left(\frac{P^*_{H,t}(h)}{P^*_{H,t}}\right)^{-\epsilon} \frac{\vartheta^*(1-n)}{n} \left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta} C^*_t + \left(\frac{P_{H,t}(h)}{P_{H,t}}\right)^{-\epsilon} G_t \\
&= \left(\frac{P_{H,t}(h)}{P_{H,t}}\right)^{-\epsilon} \left\{ \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} \left[\vartheta C_t + \frac{\vartheta^*(1-n)}{n} \left(\frac{1}{\mathcal{Q}_t}\right)^{-\eta} C^*_t\right] + G_t \right\}
\end{aligned} \tag{2.61}$$

where h represents a generic good produced in the home economy.

Goods market clearing in the rest of the world requires

$$
\begin{aligned}
Y_t(f) &= C_{F,t}(f) + C^*_{F,t}(f) + G^*_t(f) \\
&= \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\epsilon} C_{F,t} + \left(\frac{P^*_{F,t}(f)}{P^*_{F,t}}\right)^{-\epsilon} C^*_{F,t} + \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\epsilon} G^*_t \\
&= \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\epsilon} \frac{(1-\vartheta)n}{1-n} \left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t + \left(\frac{P^*_{F,t}(f)}{P^*_{F,t}}\right)^{-\epsilon} (1-\vartheta^*) \left(\frac{P^*_{F,t}}{P^*_t}\right)^{-\eta} C^*_t \\
&+ \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\epsilon} G^*_t \\
&= \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\epsilon} \left\{ \left(\frac{P_{F,t}}{P_t}\right)^{-\eta} \left[\frac{(1-\vartheta)n}{1-n} C_t + (1-\vartheta^*) \left(\frac{1}{\mathcal{Q}_t}\right)^{-\eta} C^*_t \right] + G^*_t \right\}
\end{aligned} \tag{2.62}
$$

where f represents a generic good produced in the foreign economy.

I assume that the public sector in the home (foreign) economy only consumes home (foreign) goods and has preferences for differentiated goods analogous to the ones of the private sector (given by equations (2.2) and its analog in the rest of the world).

To characterize the small open economy we use the definition of ϑ and ϑ^* and take the limit for $n \to 0$

$$
\begin{aligned}
\lim_{n\to 0} 1 - \vartheta &= \lim_{n\to 0} (1-n)\alpha = \alpha \\
\lim_{n\to 0} \vartheta &= \lim_{n\to 0} 1 - (1-n)\alpha = 1 - \alpha \\
\lim_{n\to 0} 1 - \vartheta^* &= \lim_{n\to 0} 1 - \alpha n = 1
\end{aligned}
$$

so that

$$Y_t(h) = \left(\frac{P_{H,t}(h)}{P_{H,t}}\right)^{-\epsilon} \left\{ \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} \left[(1-\alpha)C_t + \alpha \left(\frac{1}{\mathcal{Q}_t}\right)^{-\eta} C_t^*\right] + G_t \right\} \tag{2.63}$$

$$\begin{aligned} Y_t(f) &= \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\epsilon} \left[\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} \left(\frac{1}{\mathcal{Q}_t}\right)^{-\eta} C_t^* + G_t^* \right] \\ &= \left(\frac{P_{F,t}(f)}{P_{F,t}}\right)^{-\epsilon} (C_t^* + G_t^*) \end{aligned} \tag{2.64}$$

Equations (2.63) and (2.64) show that changes in the rest of the world's consumption, C_t^*, affect the small open economy, but the opposite is not true. Moreover, movements in the real exchange rate, $\mathcal{Q}_t$, do not affect the total demand for goods produced in the rest of the world.

Plugging (2.63) into the definition of aggregate domestic output $Y_t \equiv \left[\int_0^1 Y_t(h)^{\frac{\epsilon-1}{\epsilon}} dh\right]^{\frac{\epsilon}{\epsilon-1}}$, we obtain

$$Y_t = \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} \left[(1-\alpha)C_t + \alpha \left(\frac{1}{\mathcal{Q}_t}\right)^{-\eta} C_t^*\right] + G_t \tag{2.65}$$

Similarly for the rest of the world:

$$Y_t^* = C_t^* + G_t^* \tag{2.66}$$

We can derive a first order log-linear approximation to (2.65) and (2.66) around the symmetric steady state:

$$\hat{Y}_t = s_C \hat{C}_t + s_G \hat{G}_t + \frac{\alpha\omega}{\sigma} s_C \frac{\hat{\mathcal{Q}}_t}{1-\alpha} \tag{2.67}$$

$$= s_C \hat{C}_t + s_G \hat{G}_t + \frac{\alpha\omega}{\sigma} s_C \hat{\mathcal{T}}_t \tag{2.68}$$

with $s_C = \bar{C}/\bar{Y}$, $s_G = \bar{G}/\bar{Y}$, $\omega \equiv \sigma\eta + (1-\alpha)(\sigma\eta - 1)$. Notice that $\sigma\eta = 1$ implies $\omega = 1$.

In the rest of the world $\hat{Y}_t^* = s_C^* \hat{C}_t^* + s_G^* \hat{G}_t^*$, then using the risk sharing condition

$$\hat{Y}_t = \frac{s_C}{s_C^*}\left(\hat{Y}_t^* - s_G^* \hat{G}_t^*\right) + s_G \hat{G}_t + \frac{1}{\sigma_\alpha}\hat{\mathcal{T}}_t$$

where $\sigma_\alpha \equiv \frac{\sigma}{(1-\alpha)+\alpha\omega} > 0$. Notice that when $\sigma\eta = 1$, $\sigma_\alpha = \sigma$.

Market clearing in labor market requires

$$\begin{aligned} N_t &= \int_0^1 N_t(h)dh \\ &= \int_0^1 \frac{Y_t(h)}{A_t} dh \\ &= \frac{Y_t}{A_t}\int_0^1 \left(\frac{P_{H,t}(h)}{P_{H,t}}\right)^{-\epsilon} dh \\ &= \frac{Y_t}{A_t}\Delta_t^{\frac{1}{1+\varphi}} \end{aligned}$$

where $\Delta_t = \int_0^1 \left(\frac{P_{H,t}(h)}{P_{H,t}}\right)^{-\epsilon(1+\varphi)} dh$ denotes the measure of price dispersion across firms and satisfies the recursive relation

$$\Delta_t = (1-\theta)\left[\frac{1-\theta\pi_{H_t}^{\epsilon-1}}{1-\theta}\right] + \theta\pi_{H,t}^{\epsilon(1+\varphi)}\Delta_{t-1}$$

Price dispersion is the source of welfare losses from inflation variability.

Appendix 2.E Optimal policy setup

The optimal Ramsey problem chooses optimal paths $\{Y_t, \pi_{H,t}, \tau_t, b_t, \Delta_t, J_t, K_t, Q_t^M, R_t\}$ to maximize the welfare of households given by

$$\text{Max } E_0 \sum_{t=0}^{\infty} \beta^t U(\vec{Y}_t, \Delta_t, \xi_t)$$

subject to

$$\left(\frac{1-\theta\pi_{H,t}^{\epsilon-1}}{1-\theta}\right)^{\frac{1+\epsilon\varphi}{1-\epsilon}} = \frac{\epsilon}{\epsilon-1}\frac{K_t}{J_t} \tag{2.69}$$

$$K_t = \mu_t^W\left(\frac{Y_t}{A_t}\right)^{\varphi+1} + \beta\theta E_t K_{t+1}\pi_{H,t+1}^{\epsilon(1+\varphi)} \tag{2.70}$$

$$J_t = (1-\tau_t)C_t^{-\sigma}Y_t + \beta\theta E_t J_{t+1}\pi_{H,t+1}^{\epsilon-1} \tag{2.71}$$

$$\Delta_t = (1-\theta)\left(\frac{1-\theta\pi_{H,t}^{\epsilon-1}}{1-\theta}\right)^{\frac{\epsilon(1+\varphi)}{\epsilon-1}} + \theta\pi_{H,t}^{\epsilon(1+\varphi)}\Delta_{t-1} \tag{2.72}$$

$$1 = (1-\alpha)p_{H,t}^{1-\eta} + \alpha\mathcal{Q}_t^{1-\eta} \tag{2.73}$$

$$(C_t^*)^{-\sigma} = C_t^{-\sigma}\mathcal{Q}_t \tag{2.74}$$

$$(1+\tilde{\rho}Q_t^M)\frac{\tilde{B}_{t-1}^M}{P_{t-1}} = \tilde{Q}_t^M\frac{\tilde{B}_t^M}{P_t} + p_{H,t}(\tau_t Y_t - Z_t - G_t) \tag{2.75}$$

$$\beta E_t\left\{\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\frac{P_t}{P_{t+1}}\right\} = Q_t^{LC,S} \tag{2.76}$$

$$\beta E_t\left\{\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\frac{P_t}{P_{t+1}}\left(1+\rho Q_{t+1}^{LC,M}\right)\right\} = Q_t^{LC,M} \tag{2.77}$$

$$\beta E_t\left\{\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\right\} = q_t^S \tag{2.78}$$

$$\beta E_t\left\{\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\left(1+\rho q_{t+1}^M\right)\right\} = q_t^M \tag{2.79}$$

$$\beta E_t\left\{\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\frac{P_t}{P_{t+1}}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right\} = Q_t^{FC,S} \tag{2.80}$$

$$\beta E_t\left\{\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\frac{P_t}{P_{t+1}}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\left(1+\rho Q_{t+1}^{FC,M}\right)\right\} = Q_t^{FC,M} \tag{2.81}$$

$$\beta E_t\left\{\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\frac{P_t}{P_{t+1}}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right\} = \frac{1}{R_t^*} \tag{2.82}$$

$$\beta E_t\left\{\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\frac{P_t}{P_{t+1}}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\left(1+\rho^* Q_{t+1}^{*M}\right)\right\} = Q_t^{*M} \tag{2.83}$$

where (2.69), (2.70) and (2.71) define the short-run aggregate-supply, (2.72) represents to the law of motion of the price dispersion, (2.73) gives the relationship between relative prices, (2.74) represents the risk sharing condition, (2.75) corresponds to the government budget constraint (making use of definitions introduced in Appendix

2.C), (2.76)-(2.80) are the Euler equations for short-term bonds denominated in local currency, inflation-indexed units and foreign currency, respectively, (2.77)-(2.81) are the Euler equation for long-term domestic bonds denominated in local currency, inflation-indexed units and foreign currency, respectively, (2.83) is the Euler equation for short-term international bonds and (2.83) is the Euler equation for long-term international bonds.

Household's transversality condition:

$$\lim_{j\to\infty} E_t\left[Q_{t,t+1+j}\frac{D_{t-1+j}}{P_{t+j}}\right]=0 \tag{2.84}$$

If we rule out the case in which assets and liabilities shoot out to +/- infinity, in which case one country will be indefinitely borrowing and the other one will be indefinitely saving, we obtain the usual limiting condition over the government bonds:

$$\lim_{j\to\infty}\left\{Q^{LC,M}_{t+j}\frac{B^{LC,M}_{t+j}}{P_{t+j}}+q^{M}_{t+j}b^{M}_{t+j}+Q^{FC,M}_{t+j}\frac{\mathcal{E}_{t+j}B^{FC,M}_{t+j}}{P_{t+j}}+Q^{LC,S}_{t+j}\frac{B^{LC,S}_{t+j}}{P_{t+j}}+q^{S}_{t+j}b^{S}_{t+j}+Q^{FC,S}_{t+j}\frac{\mathcal{E}_{t+j}B^{FC,S}_{t+j}}{P_{t+j}}\right\}=0 \tag{2.85}$$

Appendix 2.F The symmetric deterministic steady state

Here I define a steady state with zero net domestic inflation, $\bar{\pi}_H=1$. I normalize $\bar{P}_H=\bar{P}_F$, and use the equilibrium conditions to obtain:

Price index: $\bar{P}=\left[\vartheta(\bar{P}_H)^{1-\eta}+(1-\vartheta)(\bar{P}_F)^{1-\eta}\right]^{\frac{1}{1-\eta}}=\bar{P}_H$, and $\bar{p}_H=\frac{\bar{P}_H}{\bar{P}}=1$, so $\bar{\pi}_H=\bar{\pi}=1$.

For the rest of the world: $\bar{P}^*=\bar{P}^*_F$.

Relative prices: $1=\left[(1-\alpha)\left(\frac{\bar{P}_H}{\bar{P}}\right)^{1-\eta}+\alpha\bar{\mathcal{Q}}^{1-\eta}\right]=1-\alpha+\alpha\bar{\mathcal{Q}}^{1-\eta}$, with $\alpha>0$ (open economy), $\bar{\mathcal{Q}}=1$.

In the rest of the world, $\bar{\mathcal{Q}}=1$, then $\bar{P}=\bar{P}^*\bar{\mathcal{E}}$.

Risk sharing condition, assuming symmetry: $\bar{C}^{-\sigma} = \bar{C^*}^{-\sigma}\bar{\mathcal{Q}}$, with $\bar{\mathcal{Q}} = 1$, $\bar{C} = \bar{C^*}$.

Aggregate demand (2.61):

$$\begin{aligned}\bar{Y} &= \left(\frac{\bar{P_H}}{\bar{P}}\right)^{-\eta}\left[(1-\alpha)\bar{C} + \alpha\left(\frac{1}{\bar{\mathcal{Q}}}\right)^{-\eta}\bar{C^*}\right] + \bar{G} \\ &= (1-\alpha)\bar{C} + \alpha\bar{C^*} + \bar{G} \\ &= \bar{C} + \bar{G}\end{aligned}$$

Short-run aggregate-supply (2.47-2.49):

$$\left(\frac{1-\theta\bar{\pi}_H^{\epsilon-1}}{1-\theta}\right)^{\frac{1+\epsilon\varphi}{1-\epsilon}} = \frac{\epsilon}{\epsilon-1}\frac{\bar{K}}{\bar{J}} \Rightarrow \frac{\epsilon-1}{\epsilon} = \frac{\bar{K}}{\bar{J}} \tag{2.86}$$

$$\bar{K} = \bar{\mu}^W\left(\frac{\bar{Y}}{\bar{A}}\right)^{\varphi+1} + \beta\theta\bar{K}\bar{\pi}_H^{\epsilon(1+\varphi)} \Rightarrow \bar{K} = \frac{\bar{\mu}^W}{1-\beta\theta}\left(\frac{\bar{Y}}{\bar{A}}\right)^{\varphi+1} \tag{2.87}$$

$$\bar{J} = (1-\bar{\tau})\bar{C}^{-\sigma}\bar{Y} + \beta\theta\bar{J}\bar{\pi}_H^{\epsilon-1} \Rightarrow \bar{J} = \frac{1}{1-\beta\theta}(1-\bar{\tau})\bar{C}^{-\sigma}\bar{Y} \tag{2.88}$$

Then,

$$\frac{\bar{\mu^W}\left(\frac{\bar{Y}}{\bar{A}}\right)^{\varphi+1}}{(1-\bar{\tau})\bar{C}^{-\sigma}\bar{Y}} = \frac{\epsilon-1}{\epsilon} \Rightarrow \frac{\epsilon}{(\epsilon-1)(1-\bar{\tau})}\frac{\frac{\bar{\mu}^W}{\bar{A}}\left(\frac{\bar{Y}}{\bar{A}}\right)^{\varphi}}{\bar{C}^{-\sigma}} = 1$$

And using the aggregate demand equation ($\bar{C} = \bar{Y} - \bar{G}$):

$$\frac{\epsilon}{\epsilon-1}\frac{\bar{\mu}^W}{\bar{A}}\left(\frac{\bar{Y}}{\bar{A}}\right)^{\varphi} = (1-\bar{\tau})(\bar{Y}-\bar{G})^{-\sigma}$$

Similarly for the rest of the world, using the aggregate demand equation (2.62): $\bar{Y^*} = \left(\frac{\bar{P_F}}{\bar{P}}\right)^{-\eta}\bar{C^*} + \bar{G^*} = \bar{C^*} + \bar{G^*}$.

Law of motion of the price dispersion across firms:

$$\bar{\Delta} = (1-\theta)\left[\frac{1-\theta\bar{\pi}_H^{\epsilon-1}}{1-\theta}\right] + \theta\bar{\pi}_H^{\epsilon(1+\varphi)}\bar{\Delta} \quad\Rightarrow\quad \bar{\Delta} = (1-\theta) + \theta\bar{\Delta} \quad\Rightarrow\quad \bar{\Delta} = 1$$

Steady state price of short-term debt: $\bar{Q}^{LC,S} = \bar{q}^S = \bar{Q}^{FC,S} = \beta$.

The steady-state price of long-term debt is given by:

$$\bar{Q}^{LC,M} = \bar{q}^M = \bar{Q}^{FC,M} = \frac{\beta}{1-\beta\rho}$$

which is increasing in average maturity, ρ. The intuition is straightforward: long-term debt yields more coupon payments, therefore demands a higher price.

The steady state government budget constraint implies

$$\bar{\tilde{Q}}^M \left(\frac{\bar{\tilde{B}}}{P}\right) + \bar{p}_H(\bar{\tau}\bar{Y} - \bar{Z} - \bar{G}) = (1+\rho\bar{\tilde{Q}}^M)\left(\frac{\bar{\tilde{B}}}{P}\right) \Rightarrow \bar{\tilde{Q}}^M \left(\frac{\bar{\tilde{B}}}{P}\right) = \frac{\beta}{1-\beta}(\bar{\tau}\bar{Y} - \bar{Z} - \bar{G}) = \frac{\bar{S}\beta}{1-\beta} \tag{2.89}$$

To sum up, in the symmetric deterministic steady state with zero net domestic inflation: $\bar{\pi}_H = 1$, $\bar{\pi} = 1$, $\bar{p}_H = 1$, $\bar{\mathcal{Q}} = 1$, $\bar{\Delta} = 1$, $\bar{C} = \bar{C}^*$, $\bar{Y} = \bar{C} + \bar{G}$, $\bar{Y}^* = \bar{C}^* + \bar{G}^*$. And assume $\bar{\pi}^* = 1$.

Appendix 2.G A second-order approximation to utility

The life-time utility of the households in the small open economy is defined by

$$U_0 \equiv E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_{jt}) = E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \int_0^1 \frac{N_t(h)^{1+\varphi}}{1+\varphi} dh\right) \tag{2.90}$$

First we note that

$$\int_0^1 N_t(h) dh = \frac{Y_t}{A_t} \Delta_t^{\frac{1}{1+\varphi}}$$

where $\Delta_t = \int_0^1 \left(\frac{P_{H,t}(h)}{P_{H,t}}\right)^{-\epsilon(1+\varphi)} dh$ denotes the measure of price dispersion across firms and satisfies the recursive relation

$$\Delta_t = (1-\theta)\left[\frac{1-\theta\pi_{H_t}^{\epsilon-1}}{1-\theta}\right] + \theta\pi_{H,t}^{\epsilon(1+\varphi)}\Delta_{t-1}$$

Second, combining the aggregate demand, the price index and the risk sharing condition we obtain:

$$Y_t = \left[\frac{1-\alpha\left(\frac{C_t}{C_t^*}\right)^{\sigma(1-\eta)}}{1-\alpha}\right]^{\frac{-\eta}{1-\eta}}\left[(1-\alpha)C_t + \alpha\left(\frac{C_t}{C_t^*}\right)^{\sigma\eta}C_t^*\right] + G_t \Rightarrow C_t = f(Y_t; C_t^*, G_t) \tag{2.91}$$

Then,

$$U_0 \equiv E_0\sum_{t=0}^{\infty}\beta^t\left[u(Y_t; G_t, C_t^*) - v(Y_t; A_t)\Delta_t\right] \tag{2.92}$$

We use a second-order Taylor expansion for a variable X_t:

$$\frac{X_t}{\bar{X}} = e^{\frac{X_t}{\bar{X}}} = 1 + \hat{X}_t + \frac{1}{2}\hat{X}_t^2$$

and

$$\tilde{X}_t = X_t - \bar{X} = \bar{X}\left(\hat{X}_t + \frac{1}{2}\hat{X}_t^2\right)$$

The first term in (2.92) can be approximated using a second-order Taylor expansion around the steady state defined in the previous section as

$$
\begin{aligned}
u(Y_t; G_t, C_t^*) &= \bar{u} + \bar{u}_Y \tilde{Y}_t + \bar{u}_G \tilde{G}_t + \bar{u}_{C^*} \tilde{C}_t^* + \frac{1}{2}\bar{u}_{YY}\tilde{Y}_t^2 + \frac{1}{2}\bar{u}_{GG}\tilde{G}_t^2 + \frac{1}{2}\bar{u}_{C^*C^*}(\tilde{C}^*)_t^2 + \bar{u}_{YG}\tilde{Y}_t\tilde{G}_t \\
&+ \bar{u}_{YC^*}\tilde{Y}_t\tilde{C^*}_t + \bar{u}_{GC^*}\tilde{G}_t\tilde{C}_t^* + \mathcal{O}(||\xi||^3) \\
&= \bar{u} + \bar{u}_Y \bar{Y}\left(\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2\right) + \frac{1}{2}\bar{u}_{YY}\bar{Y}^2\hat{Y}_t^2 + \bar{u}_{YG}\bar{Y}\bar{G}\hat{Y}_t\hat{G}_t + \bar{u}_{YC^*}\bar{Y}\bar{C}\hat{Y}_t\hat{C}_t^* + \text{t.i.p.} \\
&+ \mathcal{O}(||\xi||^3) \\
&= \bar{u} + \bar{u}_Y \bar{Y}\left[\hat{Y}_t + \frac{1}{2}\left(1 + \frac{\bar{u}_{YY}\bar{Y}}{\bar{u}_Y}\right)\hat{Y}_t^2 + \frac{\bar{u}_{YG}}{\bar{u}_Y}\bar{G}\hat{Y}_t\hat{G}_t + \frac{\bar{u}_{YC^*}}{\bar{u}_Y}\bar{C}\hat{Y}_t\hat{C}_t^*\right] + \text{t.i.p.} \\
&+ \mathcal{O}(||\xi||^3) \\
&= \bar{u} + \bar{u}_Y \bar{Y}\left\{\hat{Y}_t + \frac{1}{2}\left\{1 + \left[-\frac{\sigma}{s_C} - \bar{Y}\left(\frac{d\bar{C}}{dY}\right)\partial C\left(\frac{d\bar{Y}}{dC}\right)\right]\left(\frac{d\bar{C}}{dY}\right)\right\}\hat{Y}_t^2\right. \\
&+ \left[\frac{\sigma s_G}{s_C} + \bar{G}\left(\frac{d\bar{C}}{dY}\right)\partial C\left(\frac{d\bar{Y}}{dC}\right)\right]\left(\frac{d\bar{C}}{dY}\right)\hat{Y}_t\hat{G}_t \\
&+ \left.\left[-\sigma - \left(\frac{d\bar{C}}{dY}\right)\bar{C}\partial C\left(\frac{d\bar{Y}}{dC}\right)\right]\left[1 - \left(\frac{d\bar{C}}{dY}\right)\right]\hat{Y}_t\hat{C}_t^*\right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3)
\end{aligned}
\tag{2.93}
$$

where $\left(\frac{d\bar{Y}}{dC}\right) = 1 - \alpha\left[1 - \sigma\eta\left(1 + \frac{1}{1-\alpha}\right)\right] > 1$ with $0 < \alpha < 1$, and "t.i.p." represents the terms that are independent of policy.

The second term in (2.92) can be approximated by:

$$
\begin{aligned}
v(Y_t;A_t)\Delta_t &= \bar{v}\bar{\Delta} + \bar{v}_Y\bar{\Delta}\tilde{Y}_t + \bar{v}_A\bar{\Delta}\tilde{A}_t + \bar{v}\tilde{\Delta}_t + \frac{1}{2}\bar{v}_{YY}\bar{\Delta}\tilde{Y}_t^2 + \frac{1}{2}\bar{v}_{AA}\bar{\Delta}\tilde{A}_t^2 + \bar{v}_{YA}\bar{\Delta}\tilde{Y}\tilde{A} + \bar{v}_Y\bar{\Delta}\tilde{Y}_t\tilde{\Delta}_t \\
&+ \bar{v}_A\bar{\Delta}\tilde{A}_t\tilde{\Delta}_t + \mathcal{O}(||\xi||^3) \\
&= \bar{v} + \bar{v}_Y\bar{Y}\left(\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2\right) + \bar{v}\hat{\Delta}_t + \frac{1}{2}\bar{v}_{YY}\bar{Y}^2\hat{Y}_t^2 + \bar{v}_{YA}\bar{Y}\bar{A}\hat{Y}_t\hat{A}_t + \bar{v}_Y\bar{Y}\hat{Y}_t\hat{\Delta}_t + \bar{v}_A\bar{A}\hat{A}_t\hat{\Delta}_t \\
&+ \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= \bar{v} + \bar{v}_Y\bar{Y}\left[\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2 + \frac{\bar{v}\hat{\Delta}_t}{\bar{v}_Y\bar{Y}} + \frac{1}{2}\frac{\bar{v}_{YY}\bar{Y}\hat{Y}_t^2}{\bar{v}_Y} + \frac{\bar{v}_{YA}\bar{A}\hat{Y}_t\hat{A}_t}{\bar{v}_Y} + \hat{Y}_t\hat{\Delta}_t + \frac{\bar{v}_A\bar{A}\hat{A}_t\hat{\Delta}_t}{\bar{v}_Y\bar{Y}}\right] \\
&+ \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= \bar{v} + \bar{v}_Y\bar{Y}\left[\hat{Y}_t + \frac{1}{2}(1+\varphi)\hat{Y}_t^2 + \frac{\hat{\Delta}_t}{1+\varphi} - (\varphi+1)\hat{Y}_t\hat{A}_t + \hat{Y}_t\hat{\Delta}_t - \hat{A}_t\hat{\Delta}_t\right] + \text{t.i.p.} \\
&+ \mathcal{O}(||\xi||^3)
\end{aligned}
$$

From Benigno and Woodford (2004) we know that a second order approximation to the law of motion of price dispersion yields

$$
\hat{\Delta}_t = \theta\hat{\Delta}_{t-1} + \frac{\theta\epsilon}{1-\theta}(1+\varphi)(1+\epsilon\varphi)\frac{\hat{\pi}_{H,t}^2}{2} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \tag{2.94}
$$

which implies that $\hat{\Delta}_t = \mathcal{O}(\pi_{H,t}^2)$. This in turn allows us to approximate $v(Y_t;A_t)\Delta_t$ as

$$
v(Y_t;A_t)\Delta_t = \bar{v} + \bar{v}_Y\bar{Y}\left[\hat{Y}_t + \frac{1}{2}(1+\varphi)\hat{Y}_t^2 + \frac{\hat{\Delta}_t}{1+\varphi} - (\varphi+1)\hat{Y}_t\hat{A}_t\right] + \text{t.i.p.} + \mathcal{O}(||\xi||^3)
$$

Note that in steady state:

$$
\begin{aligned}
\bar{v}_Y &= (1-\Phi)\bar{u}_C \\
\bar{v}_Y &= (1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\bar{u}_Y = (1-\Phi)\left[\frac{\alpha\sigma\eta + (1-\alpha)^2 + \alpha\sigma\eta(1-\alpha)}{1-\alpha}\right]\bar{u}_Y
\end{aligned}
$$

where

$$\Phi \equiv 1 - \left(\frac{\epsilon - 1}{\epsilon}\right)\left(\frac{1-\bar{\tau}}{\bar{\mu}^w}\right) < 1$$

$-U_n/U_c = (1-\Phi)MPN$, so Φ, which measures the inefficiency of steady-state output $\bar{Y}$, depends on the steady state tax rate, $\bar{\tau}$, and the elasticity of substitution between differentiated goods, ϵ.

Then, we can rewrite

$$v(Y_t; A_t)\Delta_t = \bar{v} + (1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\bar{u}_Y\bar{Y}\left[\hat{Y}_t + \frac{1}{2}(1+\varphi)\hat{Y}_t^2 + \frac{\hat{\Delta}_t}{1+\varphi} - (\varphi+1)\hat{Y}_t\hat{A}_t\right] + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \tag{2.95}$$

Combining (2.93) and (2.95) we approximate the life-time utility (2.92) as

$$\begin{aligned} U_0 - \bar{U}_0 = \bar{u}_Y\bar{Y}E_0\sum_{t=0}^{\infty}\beta^t\Bigg\{&\left[1-(1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\right]\hat{Y}_t \\ &+\frac{1}{2}\left[1-\frac{\sigma}{s_C}\left(\frac{d\bar{C}}{dY}\right)-\bar{Y}\left(\frac{d\bar{C}}{dY}\right)^2\partial C\left(\frac{d\bar{Y}}{dC}\right)-(1+\varphi)(1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\right]\hat{Y}_t^2 \\ &-\frac{(1-\Phi)}{1+\varphi}\left(\frac{d\bar{Y}}{dC}\right)\hat{\Delta}_t + \left\{\left[\frac{\sigma s_G}{s_C}+\bar{G}\left(\frac{d\bar{C}}{dY}\right)\partial C\left(\frac{d\bar{Y}}{dC}\right)\right]\left(\frac{d\bar{C}}{dY}\right)\hat{G}_t\right. \\ &\left.+\left[-\sigma-\left(\frac{d\bar{C}}{dY}\right)\bar{C}\partial C\left(\frac{d\bar{Y}}{dC}\right)\right]\left[1-\left(\frac{d\bar{C}}{dY}\right)\right]\hat{C}_t^* + (\varphi+1)(1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\hat{A}_t\right\}\hat{Y}_t\Bigg\} \\ &+\text{t.i.p.} + \mathcal{O}(||\xi||^3) \end{aligned}$$

From Benigno and Woodford (2004) we observe

$$E_0\sum_{t=0}^{\infty}\beta^t\hat{\Delta}_t = \frac{\theta\epsilon}{(1-\theta)(1-\beta\theta)}(1+\varphi)(1+\epsilon\varphi)\sum_{t=0}^{\infty}\beta^t\beta^t\frac{\hat{\pi}_{H,t}^2}{2}$$

Therefore, the second-order approximation to the life-time utility can be further expressed as

$$U_0 - \bar{U}_0 = \bar{u}_Y \bar{Y} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ A_{Y\alpha} \hat{Y}_t + \frac{1}{2} A_{YY\alpha} \hat{Y}_t^2 - \frac{A_{\pi_H \alpha}}{2} \hat{\pi}_{H,t}^2 + \left(A_{G\alpha} \hat{G}_t + A_{C^*} \hat{C}_t^* + A_{A\alpha} \hat{A}_t \right) \hat{Y}_t \right\}$$
$$+ \text{t.i.p.} + \mathcal{O}(||\xi||^3)$$

where

$$A_{Y\alpha} = 1 - (1-\Phi)\left(\frac{d\bar{Y}}{d\bar{C}}\right) = 1 - \left(\frac{d\bar{Y}}{d\bar{C}}\right) + \Phi\left(\frac{d\bar{Y}}{d\bar{C}}\right) = \alpha\left[1 - \sigma\eta\left(1 + \frac{1}{1-\alpha}\right)\right]$$
$$+ \Phi\left\{1 - \alpha\left[1 - \sigma\eta\left(1 + \frac{1}{1-\alpha}\right)\right]\right\}$$

$$A_{YY\alpha} = 1 - \frac{\sigma}{s_C}\left(\frac{d\bar{C}}{dY}\right) - (1-\Phi)(1+\varphi)\left(\frac{d\bar{Y}}{dC}\right) - \left(\frac{d\bar{C}}{dY}\right)^2 \bar{Y}\partial C\left(\frac{d\bar{Y}}{dC}\right)$$
$$= 1 - \frac{\sigma}{s_C}\left(\frac{d\bar{C}}{dY}\right) - \frac{1}{s_C}\left(\frac{d\bar{C}}{dY}\right)^2 \frac{\alpha\eta\sigma}{1-\alpha}\left(\frac{\sigma}{1-\alpha} + \alpha\eta\sigma - \alpha\right) - (1-\Phi)(1+\varphi)\left(\frac{d\bar{Y}}{dC}\right)$$
$$= 1 - \frac{\sigma}{s_C}\left(\frac{d\bar{C}}{dY}\right)\left[1 + \frac{\alpha\sigma\eta}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)\right] - \frac{1}{s_C}\left(\frac{d\bar{C}}{dY}\right)^2 \frac{\alpha^2\eta\sigma}{1-\alpha}(\eta\sigma - 1)$$
$$- (1-\Phi)(1+\varphi)\left(\frac{d\bar{Y}}{dC}\right)$$

$$A_{\pi_H \alpha} = (1-\Phi)\frac{\epsilon\theta(1+\epsilon\varphi)}{(1-\theta)(1-\beta\theta)}\left(\frac{d\bar{Y}}{dC}\right)$$

$$A_{G\alpha} = \sigma\frac{s_G}{s_C}\left(\frac{d\bar{C}}{dY}\right) + \left(\frac{d\bar{C}}{dY}\right)^2 \bar{G}\partial C\left(\frac{d\bar{Y}}{dC}\right) = \sigma\frac{s_G}{s_C}\left(\frac{d\bar{C}}{dY}\right)$$
$$+ \sigma\frac{s_G}{s_C}\left(\frac{d\bar{C}}{dY}\right)^2\left(\frac{\alpha}{1-\alpha}\right)\eta\left(\frac{\sigma}{1-\alpha} - \alpha + \alpha\sigma\eta\right)$$
$$= \sigma\frac{s_G}{s_C}\left(\frac{d\bar{C}}{dY}\right)\left[1 + \frac{\alpha\sigma\eta}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)\right] + \frac{s_G}{s_C}(\sigma\eta - 1)\frac{\alpha^2\eta\sigma}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)^2$$

$$A_{C^*} = \left[-\sigma - \left(\frac{d\bar{C}}{dY}\right)\bar{C}\partial C\left(\frac{d\bar{Y}}{dC}\right)\right]\left[1 - \left(\frac{d\bar{C}}{dY}\right)\right]$$
$$= \left[-\sigma - \frac{\alpha\eta\sigma}{1-\alpha}\left(\frac{\sigma}{1-\alpha} + \alpha\eta\sigma - \alpha\right)\left(\frac{d\bar{C}}{dY}\right)\right]\left[1 - \left(\frac{d\bar{C}}{dY}\right)\right]$$
$$= \left\{-\sigma\left[1 + \frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)\right] - \frac{\alpha^2\eta\sigma}{1-\alpha}(\eta\sigma - 1)\left(\frac{d\bar{C}}{dY}\right)\right\}\left[1 - \left(\frac{d\bar{C}}{dY}\right)\right]$$

$$A_{A\alpha} = (1-\Phi)(1+\varphi)\left(\frac{d\bar{Y}}{dC}\right)$$

Note that when $\sigma\eta = 1$:

$$A_{Y\alpha} = \frac{\Phi - \alpha}{1-\alpha} = \frac{A_Y - \alpha}{1-\alpha}$$

$$A_{YY\alpha} = 1 - \frac{\sigma}{s_C} - \frac{(1+\varphi)(1-\Phi)}{1-\alpha} = \frac{A_{YY} - \alpha\left(1 - \frac{\sigma}{s_C}\right)}{1-\alpha}$$

$$A_{\pi_H\alpha} = \frac{1-\Phi}{1-\alpha}\frac{\epsilon\theta(1+\epsilon\varphi)}{(1-\theta)(1-\beta\theta)} = \frac{A_{\pi_H}}{1-\alpha}$$

$$A_{G\alpha} = \frac{\sigma s_G}{s_C} = A_G$$

$$A_{C^*} = -\sigma\frac{\alpha}{1-\alpha}$$

$$A_{A\alpha} = \frac{(1-\Phi)(1+\varphi)}{1-\alpha} = \frac{A_A}{1-\alpha}$$

And when $\alpha = 0$, we recover the results obtained by Leeper and Zhou (2021) for a closed economy.

Appendix 2.H Second-order approximation to the intertemporal government solvency condition

Recall the government budget constraint

$$Q_t^{LC,M}\frac{B_t^{LC,M}}{P_t} + q_t^M b_t^M + Q_t^{FC,M}\frac{\mathcal{E}_t B_t^{FC,M}}{P_t} + Q_t^{LC,S}\frac{B_t^{LC,S}}{P_t} + q_t^S b_t^S + Q_t^{FC,S}\frac{\mathcal{E}_t B_t^{FC,S}}{P_t} + p_{H,t}S_t$$
$$= (1 + Q_{t-1}^{LC,M})\frac{B_{t-1}^{LC,M}}{P_t} + q_{t-1}^M b_{t-1}^M + Q_{t-1}^{FC,M}\frac{\mathcal{E}_t B_{t-1}^{FC,M}}{P_t} + \frac{B_{t-1}^{LC,S}}{P_t} + b_{t-1}^S + \frac{\mathcal{E}_t B_{t-1}^{FC,S}}{P_t} \quad (2.96)$$

and the no-arbitrage conditions

$$\begin{aligned} Q_t^{LC,M} &= \beta E_t \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \left(1 + \rho Q_{t+1}^{LC,M}\right) \\ q_t^M &= \beta E_t \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \left(1 + \rho q_{t+1}^M\right) \\ Q_t^{FC,M} &= \beta E_t \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \left(1 + \rho Q_{t+1}^{FC,M}\right) \end{aligned} \tag{2.97}$$

Define

$$W_t = C_t^{-\sigma} \left[\left(1 + \rho Q_t^{LC,M}\right) \frac{B_{t-1}^{LC,M}}{P_{t-1}} + \left(1 + \rho q_t^M\right) b_{t-1}^M + \left(1 + \rho Q_t^{FC,M}\right) \frac{\mathcal{E}_{t-1} B_{t-1}^{FC,M}}{P_{t-1}} \right] \tag{2.98}$$

By applying (2.97) and (2.98), (2.96) can be rewritten as

$$W_t = C_t^{-\sigma} p_{H,t} S_t + \beta E_t W_{t+1} \tag{2.99}$$

and

$$W_t = E_t \sum_{k=0}^{\infty} C_{t+1}^{-\sigma} p_{H,t+k} S_{t+k} \tag{2.100}$$

A second-order approximation to $C_t^{-\sigma} p_{H,t} S_t$ yields

$$
\begin{aligned}
C_t^{-\sigma} p_{H,t} S_t = \bar{C}^{-\sigma}\bar{S} + \bar{C}^{-\sigma}\bar{S}\Bigg\{ & \left[-\frac{\sigma}{s_C} + \bar{Y}\left(\frac{\partial \bar{p}_H}{\partial C}\right) + \frac{\bar{\tau}\bar{Y}}{\bar{S}}\left(\frac{d\bar{Y}}{dC}\right)\right]\left(\frac{d\bar{C}}{dY}\right)\left(\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2\right) \\
& + \left[\frac{\sigma s_G}{s_C} - \bar{G}\left(\frac{\partial \bar{p}_H}{\partial C}\right) - \frac{s_G}{s_D}\left(\frac{d\bar{Y}}{dC}\right)\right]\left(\frac{d\bar{C}}{dY}\right)\hat{G}_t + \left[-\sigma\left(\frac{d\bar{C}}{dC^*}\right) + \bar{C}\left(\frac{\partial \bar{p}_H}{\partial C^*}\right)\right]\hat{C}_t^* \\
& + \frac{\bar{\tau}\bar{Y}}{\bar{S}}\left(\hat{\tau} + \frac{1}{2}\hat{\tau}^2\right) - \frac{s_Z}{s_D}\hat{Z}_t \\
& + \frac{1}{2}\Bigg[\frac{\sigma(\sigma+1)}{s_C^2}\left(\frac{d\bar{C}}{dY}\right) - \frac{\sigma}{s_C}\bar{Y}\partial C\left(\frac{d\bar{C}}{dY}\right) - 2\frac{\sigma}{s_C}\bar{Y}\left(\frac{\partial \bar{p}_H}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right) - 2\frac{\sigma}{s_C}\frac{\bar{Y}\bar{\tau}}{\bar{S}} + \\
& + \bar{Y}^2\left(\frac{\partial^2 \bar{p}_H}{\partial C^2}\right)\left(\frac{d\bar{C}}{dY}\right) + \bar{Y}^2\left(\frac{\partial \bar{p}_H}{\partial C}\right)\partial C\left(\frac{d\bar{C}}{dY}\right) + 2\bar{Y}\left(\frac{\partial \bar{p}_H}{\partial C}\right)\frac{\bar{Y}\bar{\tau}}{\bar{S}}\Bigg]\left(\frac{d\bar{C}}{dY}\right)\hat{Y}_t^2 \\
& + \Bigg[-\frac{\sigma(\sigma+1)s_G}{s_C^2}\left(\frac{d\bar{C}}{dY}\right) + \frac{\sigma}{s_C}\bar{G}\partial C\left(\frac{d\bar{C}}{dY}\right) + 2\frac{\sigma}{s_C}\bar{G}\left(\frac{d\bar{C}}{dY}\right)\left(\frac{\partial \bar{p}_H}{\partial C}\right) + \frac{\sigma}{s_G}\frac{s_G}{s_D} \\
& - \bar{G}\bar{Y}\left(\frac{\partial^2 \bar{p}_H}{\partial C^2}\right)\left(\frac{d\bar{C}}{dY}\right) - \bar{G}\bar{Y}\left(\frac{\partial \bar{p}_H}{\partial C}\right)\partial C\left(\frac{d\bar{C}}{dY}\right) - \frac{1}{s_D}\bar{G}\left(\frac{\partial \bar{p}_H}{\partial C}\right) + \sigma\frac{s_G}{s_C}\frac{\bar{\tau}\bar{Y}}{\bar{S}} \\
& - \bar{G}\left(\frac{\partial \bar{p}_H}{\partial C}\right)\frac{\bar{\tau}\bar{Y}}{\bar{S}}\Bigg]\left(\frac{d\bar{C}}{dY}\right)\hat{G}_t\hat{Y}_t \\
& + \Bigg[\frac{\sigma(\sigma+1)}{s_C}\left(\frac{d\bar{C}}{dC^*}\right) - \sigma\bar{Y}\partial C^*\left(\frac{dC}{dY}\right)\left(\frac{d\bar{Y}}{dC}\right) - \sigma\bar{Y}\left(\frac{\partial \bar{p}_H}{\partial C^*}\right) - \sigma\bar{Y}\left(\frac{\partial \bar{p}_H}{\partial C}\right)\left(\frac{d\bar{C}}{dC^*}\right) \\
& + \bar{Y}\bar{C}\left(\frac{\partial^2 \bar{p}_H}{\partial C \partial C^*}\right) + \bar{Y}\bar{C}\left(\frac{\partial \bar{p}_H}{\partial C}\right)\partial C^*\left(\frac{dC}{dY}\right)\left(\frac{d\bar{Y}}{dC}\right) \\
& - \sigma\frac{\bar{\tau}\bar{Y}}{\bar{S}}\left(\frac{d\bar{C}}{dC^*}\right)\left(\frac{d\bar{Y}}{dC}\right) + \frac{\bar{\tau}\bar{Y}}{\bar{S}}\bar{C}\left(\frac{\partial \bar{p}_H}{\partial C^*}\right)\left(\frac{d\bar{Y}}{dC}\right)\Bigg]\left(\frac{d\bar{C}}{dY}\right)\hat{Y}_t\hat{C}_t^* \\
& + \left[-\frac{\sigma}{s_C}\frac{\bar{\tau}\bar{Y}}{\bar{S}} + \frac{\bar{\tau}\bar{Y}^2}{\bar{S}}\left(\frac{\partial \bar{p}_H}{\partial C}\right) + \frac{\bar{\tau}\bar{Y}}{\bar{S}}\left(\frac{d\bar{Y}}{dC}\right)\right]\left(\frac{d\bar{C}}{dY}\right)\hat{\tau}_t\hat{Y}_t \\
& - \left[-\frac{\sigma}{s_C}\frac{s_Z}{s_D} + \frac{s_Z}{s_D}\bar{Y}\left(\frac{\partial \bar{p}_H}{\partial C}\right)\right]\left(\frac{d\bar{C}}{dY}\right)\hat{Z}_t\hat{Y}_t + \left[\sigma\frac{s_G}{s_C}\frac{\bar{\tau}\bar{Y}}{\bar{S}} - \frac{\bar{\tau}\bar{Y}}{\bar{S}}\bar{G}\left(\frac{\partial \bar{p}_H}{\partial C}\right)\right]\left(\frac{d\bar{C}}{dY}\right)\hat{\tau}_t\hat{G}_t \\
& + \left[-\sigma\left(\frac{d\bar{C}}{dC^*}\right) + \bar{C}\left(\frac{\partial \bar{p}_H}{\partial C^*}\right)\right]\frac{\bar{\tau}\bar{Y}}{\bar{S}}\hat{\tau}_t\hat{C}_t^*\Bigg\} + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3)
\end{aligned}
$$

Here "s.o.t.i.p." refers to second-order (or higher) terms independent of policy; the first-order terms have been kept as these will matter for the log-linear aggregate-supply relation that appears as a constraint in our policy problem.

Therefore,

$$\frac{W_0 - \bar{W}}{\bar{W}} = (1-\beta)E_0 \sum_{t=0}^{\infty} \beta^t \Big(B_Y \hat{Y}_t + B_\tau \hat{\tau}_t + B_{\tau Y} \hat{\tau}_t \hat{Y}_t + \frac{1}{2} B_{YY} \hat{Y}_t^2 + \frac{1}{2} B_{\tau\tau} \hat{\tau}_t^2 + B_G \hat{G}_t$$
$$+ B_{C^*} \hat{C}_t^* + B_Z \hat{Z}_t + B_{GY} \hat{G}_t \hat{Y}_t + B_{C^*Y} \hat{C}_t^* \hat{Y}_t + B_{ZY} \hat{Z}_t \hat{Y}_t + B_{G\tau} \hat{G}_t \hat{\tau}_t + B_{C^*\tau} \hat{C}_t^* \hat{\tau}_t \Big)$$
$$+ \text{t.i.p.} + \mathcal{O}(||\xi||^3) \tag{2.101}$$

where

$$
\begin{aligned}
B_{Y\alpha} &= -\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\frac{\sigma}{s_C} + \frac{\bar{\tau}\bar{Y}}{\bar{S}} \\
B_{\tau\alpha} &= B_{\tau\tau} = \frac{\bar{\tau}\bar{Y}}{\bar{S}} \quad \text{Note that this term does not depend on } \eta \text{ or } \alpha. \\
B_{\tau Y\alpha} &= \frac{\bar{\tau}\bar{Y}}{\bar{S}}\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\frac{\sigma}{s_C}\right] \\
B_{YY\alpha} &= \frac{\bar{\tau}\bar{Y}}{\bar{S}}\left[1 - \frac{2\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right] - \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right) \\
&+ \frac{\sigma^2}{s_C^2}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)^2\left[1 + \frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)\right] + \frac{\sigma}{s_C^2}\left[1 + \frac{\alpha}{1-\alpha}\left(\frac{\sigma\eta}{1-\alpha} + 1\right)\right]\left(\frac{d\bar{C}}{dY}\right)^2 \\
&+ \frac{\sigma}{s_C^2}\frac{\alpha^2}{(1-\alpha)^2}\sigma\eta(\eta\sigma - 1)\left(\frac{d\bar{C}}{dY}\right)^3 \\
B_{G\alpha} &= \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\sigma\frac{s_G}{s_C} - \frac{s_G}{s_D} \\
B_{C^*} &= -\sigma\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]
\end{aligned}
$$

$$
B_{Z\alpha} = -\frac{s_Z}{s_D}
$$

$$
\begin{aligned}
B_{GY\alpha} &= -\sigma\frac{s_G}{s_C}\left\{\frac{\sigma+1}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)^2 - \frac{1}{s_C}\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^2\left[1 + \sigma\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\right. \\
&\left. - \frac{\bar{\tau}\bar{Y}}{\bar{S}}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right) - \frac{1}{s_D}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right) + \frac{\sigma\eta}{s_C}(\sigma\eta - 1)\frac{\alpha^2}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^3\right\} \\
B_{C^*Y} &= \left\{\sigma\frac{\bar{\tau}\bar{Y}}{\bar{S}}\left[\frac{1}{1-\alpha} - \left(\frac{d\bar{Y}}{dC}\right)\right] + \frac{\sigma}{s_C}\frac{\alpha^2\sigma\eta}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)\left[1 - \left(\frac{d\bar{C}}{dY}\right)\right](\sigma\eta - 1)\right. \\
&+ \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right] \\
&\left. + \frac{\sigma^2}{s_C}\frac{1}{1-\alpha}\left[1 - \left(\frac{d\bar{C}}{dY}\right)\right]\left[1 + \frac{\alpha\sigma\eta}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)\right]\right\}\left(\frac{d\bar{C}}{dY}\right) \\
B_{ZY\alpha} &= \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\sigma\frac{s_Z}{s_C}\frac{1}{s_D} \\
B_{G\tau\alpha} &= \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\sigma\frac{s_G}{s_C}\frac{\bar{\tau}\bar{Y}}{\bar{S}} \\
B_{C^*\tau} &= -\sigma\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\frac{\bar{\tau}\bar{Y}}{\bar{S}}
\end{aligned}
$$

and $s_G = \frac{\bar{G}}{\bar{Y}}$ is the steady state government expenditure to GDP ratio, $s_Z = \frac{\bar{Z}}{\bar{Y}}$ is the steady state government transfer payment to GDP ratio, $s_D = \frac{\bar{S}}{\bar{Y}}$ is the steady state surplus to GDP ratio.

Note that when $\sigma\eta = 1$ $\left(\frac{d\bar{C}}{dY} = 1-\alpha\right)$:

$$B_{Y\alpha} = B_Y = -\frac{\sigma}{s_C} + \frac{\bar{\tau}\bar{Y}}{\bar{S}}, \qquad B_{\tau\alpha} = B_{\tau\tau\alpha} = B_\tau = B_{\tau\tau} = \frac{\bar{\tau}\bar{Y}}{\bar{S}}$$

$$B_{YY\alpha} = B_{YY} = \frac{\bar{\tau}\bar{Y}}{\bar{S}}\left(1 - 2\frac{\sigma}{s_C}\right) - \frac{\sigma}{s_C} + \frac{\sigma^2}{s_C^2} + \frac{\sigma}{s_C^2}, \qquad B_{G\alpha} = B_G = \sigma\frac{s_G}{s_C} - \frac{s_G}{s_D}$$

$$B_{GY\alpha} = B_{GY} = -\sigma\frac{s_G}{s_C}\left[\frac{\sigma+1}{s_C} - \frac{\bar{\tau}\bar{Y}}{\bar{S}} - \frac{1}{s_D}\right], \qquad B_{C^*Y} = \frac{\alpha}{1-\alpha}\frac{\sigma^2}{s_C}$$

$$B_{G\tau\alpha} = B_{G\tau} = \sigma\frac{s_G}{s_C}\frac{\bar{\tau}\bar{Y}}{\bar{S}}, \qquad B_{C^*\tau} = 0$$

$$B_{\tau Y\alpha} = B_{\tau Y} = \frac{\bar{\tau}\bar{Y}}{\bar{S}}\left(1 - \frac{\sigma}{s_C}\right), \qquad B_{C^*} = 0$$

$$B_{ZY\alpha} = B_{ZY} = \sigma\frac{s_Z}{s_C}\frac{1}{s_D}$$

And when $\alpha = 0$ $\left(\frac{d\bar{C}}{dY} = 1\right)$, we recover the results obtained by Leeper and Zhou (2021) for a closed economy.

Appendix 2.I Second-order approximation to the aggregate supply

The aggregate supply relation is defined by the equations

$$J_t\left(\frac{1-\theta\pi_{H,t}^{\epsilon-1}}{1-\theta}\right)^{\frac{1+\epsilon\varphi}{1-\epsilon}} = \frac{\epsilon}{\epsilon-1}K_t \tag{2.102}$$

$$K_t = \mu_t^w\left(\frac{Y_t}{A_t}\right)^{\varphi+1} + \beta\theta E_t K_{t+1}\pi_{H,t+1}^{\epsilon(1+\varphi)} \tag{2.103}$$

$$J_t = (1-\tau_t)p_{H,t}C_t^{-\sigma}Y_t + \beta\theta E_t J_{t+1}\pi_{H,t+1}^{\epsilon-1} \tag{2.104}$$

A second-order approximation to (2.102) can be written as

$$\frac{\epsilon}{\epsilon-1}\tilde{K}_t - \tilde{J} = \bar{J}\frac{\theta}{1-\theta}(1+\epsilon\varphi)\left\{\tilde{\pi}_{H,t} + \frac{1}{2}\left[\frac{\theta}{1-\theta}\epsilon(\varphi+1) + (\epsilon-2)\right]\tilde{\pi}_{H,t}^2\right\} + \frac{\theta}{1-\theta}(1+\epsilon\varphi)\tilde{J}_t\tilde{\pi}_{H,t} + \mathcal{O}(||\xi||^3) \tag{2.105}$$

A second-order approximation to (2.103) can be written as

$$\tilde{K}_t = \beta\theta E_t\tilde{K}_{t+1} + \beta\theta\epsilon(1+\varphi)\bar{K}\left\{E_t\tilde{\pi}_{H,t+1} + \frac{1}{2}[\epsilon(1+\varphi)-1]E_t\tilde{\pi}_{H,t+1}^2\right\} + \beta\theta\epsilon(1+\varphi)E_t\tilde{K}_{t+1}\tilde{\pi}_{H,t+1} + \bar{\mu}^W(1+\varphi)\bar{Y}^{\varphi}\tilde{Y}_t - \bar{\mu}^W(1+\varphi)^2\bar{Y}^{\varphi}\tilde{Y}_t\tilde{A}_t + (1+\varphi)\bar{Y}^{\varphi}\tilde{\mu}_t^W\tilde{Y}_t + \frac{1}{2}\bar{\mu}^W\varphi(1+\varphi)\bar{Y}^{\varphi-1}\tilde{Y}_t^2 - \bar{\mu}^W(1+\varphi)\bar{Y}^{\varphi+1}\tilde{A}_t + \bar{Y}^{\varphi+1}\tilde{\mu}_t^W + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \tag{2.106}$$

A second-order approximation to (2.104) can be written as

$$\begin{aligned}
\tilde{J}_t &= \beta\theta E_t \tilde{J}_{t+1} + \beta\theta(\epsilon-1)\bar{J}\left[E_t\tilde{\pi}_{H,t+1} + \frac{1}{2}(\epsilon-2)E_t\tilde{\pi}^2_{H,t+1}\right] + \beta\theta(\epsilon-1)E_t\tilde{J}_{t+1}\tilde{\pi}_{H,t+1} \\
&+ (1-\bar{\tau})\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dY}\right)\bar{Y} + \bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial \bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right) + \bar{C}^{-\sigma}\right]\tilde{Y}_t + \\
&(1-\bar{\tau})\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dG}\right)\bar{Y} + \bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial \bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dG}\right)\right]\tilde{G}_t + \\
&(1-\bar{\tau})\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dC^*}\right)\bar{Y} + \bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial \bar{p}_{H,t}}{\partial C^*}\right)\right]\tilde{C}^*_t - \bar{C}^{-\sigma}\bar{Y}\tilde{\tau}_t \\
&+ \frac{1}{2}(1-\bar{\tau})\left[2\left(\frac{\partial \bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma} - 2\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dY}\right) - 2\sigma\left(\frac{\partial \bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)^2\bar{C}^{-\sigma-1}\bar{Y} + \right. \\
&\left(\frac{\partial \bar{p}_{H,t}}{\partial C}\right)\partial C\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}\bar{Y} + \left(\frac{\partial^2 \bar{p}_{H,t}}{\partial C^2}\right)\left(\frac{d\bar{C}}{dY}\right)^2\bar{C}^{-\sigma}\bar{Y} - \sigma\bar{C}^{-\sigma-1}\partial C\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{Y} + \\
&\left.\sigma(\sigma+1)\bar{C}^{-\sigma-2}\left(\frac{d\bar{C}}{dY}\right)^2\bar{Y}\right]\tilde{Y}^2_t - \left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dY}\right)\bar{Y} + \bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial \bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right) + \bar{C}^{-\sigma}\right]\tilde{Y}_t\tilde{\tau}_t - \\
&\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dG}\right)\bar{Y} + \bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial \bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dG}\right)\right]\tilde{G}_t\tilde{\tau}_t - \left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dC^*}\right)\bar{Y} + \bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial \bar{p}_{H,t}}{\partial C^*}\right)\right]\tilde{C}^*_t\tilde{\tau}_t \\
&+ (1-\bar{\tau})\left[\left(\frac{\partial^2 \bar{p}_{H,t}}{\partial C^2}\right)\left(\frac{d\bar{C}}{dG}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}\bar{Y} + \left(\frac{\partial \bar{p}_{H,t}}{\partial C}\right)\partial C\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dG}\right)\bar{C}^{-\sigma}\bar{Y} - \right. \\
&2\sigma\left(\frac{\partial \bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dG}\right)\bar{C}^{-\sigma-1}\bar{Y} + \sigma(\sigma+1)\bar{C}^{-\sigma-2}\bar{Y}\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dG}\right) - \sigma\bar{C}^{-\sigma-1}\partial C\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dG}\right)\bar{Y} + \\
&\left.\left(\frac{\partial \bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dG}\right)\bar{C}^{-\sigma} - \sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dG}\right)\right]\tilde{Y}_t\tilde{G}_t + \\
&(1-\bar{\tau})\left[\left(\frac{\partial^2 \bar{p}_{H,t}}{\partial C\partial C^*}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}\bar{Y} + \left(\frac{\partial \bar{p}_{H,t}}{\partial C}\right)\partial C^*\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}\bar{Y} - \sigma\left(\frac{\partial \bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dC^*}\right)\bar{Y} - \right. \\
&\sigma\left(\frac{\partial \bar{p}_{H,t}}{\partial C^*}\right)\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dY}\right)\bar{Y} + \sigma(\sigma+1)\bar{C}^{-\sigma-2}\left(\frac{d\bar{C}}{dC^*}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{Y} - \sigma\bar{C}^{-\sigma-1}\partial C^*\left(\frac{d\bar{C}}{dY}\right)\bar{Y} + \\
&\left.\left(\frac{\partial \bar{p}_{H,t}}{\partial C^*}\right)\bar{C}^{-\sigma} - \sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dC^*}\right)\right]\tilde{Y}_t\tilde{C}^*_t + \text{t.i.p.} + \mathcal{O}(||\xi||^3)
\end{aligned} \tag{2.107}$$

Therefore, $\frac{\epsilon}{\epsilon-1}$(2.106)-(2.107) can be expressed as

$$
\begin{aligned}
&\frac{\epsilon}{\epsilon-1}\tilde{K}_t - \tilde{J} = \beta\theta E_t\left(\frac{\epsilon}{\epsilon-1}\tilde{K}_{t+1} - \tilde{J}_{t+1}\right) \\
&+ \frac{\epsilon}{\epsilon-1}\beta\theta\epsilon(1+\varphi)\bar{K}\left\{E_t\tilde{\pi}_{H,t+1} + \frac{1}{2}[\epsilon(1+\varphi)-1]E_t\tilde{\pi}^2_{H,t+1}\right\} - \beta\theta(\epsilon-1)\bar{J}\left[E_t\tilde{\pi}_{t+1} + \frac{1}{2}(\epsilon-2)E_t\tilde{\pi}^2_{H,t+1}\right] \\
&+ \frac{\epsilon}{\epsilon-1}\beta\theta\epsilon(1+\varphi)E_t\tilde{K}_{t+1}\tilde{\pi}_{H,t+1} - \beta\theta(\epsilon-1)E_t\tilde{J}_{t+1}\tilde{\pi}_{H,t+1} \\
&+ \frac{\epsilon}{\epsilon-1}\left[\bar{\mu}^W(1+\varphi)\bar{Y}^{\varphi}\tilde{Y}_t - \bar{\mu}^W(1+\varphi)^2\bar{Y}^{\varphi}\tilde{Y}_t\tilde{A}_t + (1+\varphi)\bar{Y}^{\varphi}\tilde{\mu}^W_t\tilde{Y}_t + \frac{1}{2}\bar{\mu}^W\varphi(1+\varphi)\bar{Y}^{\varphi-1}\tilde{Y}^2_t\right. \\
&\left.-\bar{\mu}^W(1+\varphi)\bar{Y}^{\varphi+1}\tilde{A}_t + \bar{Y}^{\varphi+1}\tilde{\mu}^W_t\right] \\
&- \left\{(1-\bar{\tau})\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dY}\right)\bar{Y} + \bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right) + \bar{C}^{-\sigma}\right]\tilde{Y}_t\right. \\
&+(1-\bar{\tau})\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dG}\right)\bar{Y} + \bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dG}\right)\right]\tilde{G}_t \\
&+(1-\bar{\tau})\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dC^*}\right)\bar{Y} + \bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial\bar{p}_{H,t}}{\partial C^*}\right)\right]\tilde{C}^*_t - \bar{C}^{-\sigma}\bar{Y}\tilde{\tau}_t \\
&+\frac{1}{2}(1-\bar{\tau})\left[2\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma} - 2\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dY}\right) - 2\sigma\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)^2\bar{C}^{-\sigma-1}\bar{Y}\right. \\
&+\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\partial C\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}\bar{Y} + \left(\frac{\partial^2\bar{p}_{H,t}}{\partial C^2}\right)\left(\frac{d\bar{C}}{dY}\right)^2\bar{C}^{-\sigma}\bar{Y} - \sigma\bar{C}^{-\sigma-1}\partial C\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{Y} \\
&\left.+\sigma(\sigma+1)\bar{C}^{-\sigma-2}\left(\frac{d\bar{C}}{dY}\right)^2\bar{Y}\right]\tilde{Y}^2_t - \left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dY}\right)\bar{Y} + \bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right) + \bar{C}^{-\sigma}\right]\tilde{Y}_t\tilde{\tau}_t \\
&-\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dG}\right)\bar{Y} + \bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dG}\right)\right]\tilde{G}_t\tilde{\tau}_t - \left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dC^*}\right)\bar{Y} + \bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial\bar{p}_{H,t}}{\partial C^*}\right)\right]\tilde{C}^*_t\tilde{\tau}_t \\
&+(1-\bar{\tau})\left[\left(\frac{\partial^2\bar{p}_{H,t}}{\partial C^2}\right)\left(\frac{d\bar{C}}{dG}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}\bar{Y} + \left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\partial C\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dG}\right)\bar{C}^{-\sigma}\bar{Y}\right. \\
&-2\sigma\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dG}\right)\bar{C}^{-\sigma-1}\bar{Y} + \sigma(\sigma+1)\bar{C}^{-\sigma-2}\bar{Y}\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dG}\right) - \sigma\bar{C}^{-\sigma-1}\partial C\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dG}\right)\bar{Y} \\
&\left.+\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dG}\right)\bar{C}^{-\sigma} - \sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dG}\right)\right]\tilde{Y}_t\tilde{G}_t \\
&+(1-\bar{\tau})\left[\left(\frac{\partial^2\bar{p}_{H,t}}{\partial C\partial C^*}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}\bar{Y} + \left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\partial C^*\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}\bar{Y} - \sigma\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dC^*}\right)\bar{Y}\right. \\
&-\sigma\left(\frac{\partial\bar{p}_{H,t}}{\partial C^*}\right)\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dY}\right)\bar{Y} + \sigma(\sigma+1)\bar{C}^{-\sigma-2}\left(\frac{d\bar{C}}{dC^*}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{Y} - \sigma\bar{C}^{-\sigma-1}\partial C^*\left(\frac{d\bar{C}}{dY}\right)\bar{Y} \\
&\left.\left.+\left(\frac{\partial\bar{p}_{H,t}}{\partial C^*}\right)\bar{C}^{-\sigma} - \sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dC^*}\right)\right]\tilde{Y}_t\tilde{C}^*_t\right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3)
\end{aligned}
\tag{2.108}
$$

Then we plug (2.105) into (2.108) and obtain

$$
\bar{J}\frac{\theta}{1-\theta}(1+\epsilon\varphi)\left\{\tilde{\pi}_{H,t} + \frac{1}{2}\left[\frac{\theta}{1-\theta}\epsilon(\varphi+1) + (\epsilon-2)\right]\tilde{\pi}^2_{H,t}\right\} + \frac{\theta}{1-\theta}(1+\epsilon\varphi)\tilde{J}_t\tilde{\pi}_{H,t}
$$

$$
\begin{aligned}
&= \beta\theta\bar{J}\frac{\theta}{1-\theta}(1+\epsilon\varphi)\left\{E_t\tilde{\pi}_{H,t+1}+\frac{1}{2}\left[\frac{\theta}{1-\theta}\epsilon(\varphi+1)+(\epsilon-2)\right]E_t\tilde{\pi}^2_{H,t+1}\right\}+\beta\theta\frac{\theta}{1-\theta}(1+\epsilon\varphi)E_t\tilde{J}_{t+1}\tilde{\pi}_{H,t+1}\\
&+\frac{\epsilon}{\epsilon-1}\beta\theta\epsilon(1+\varphi)\bar{K}\left\{E_t\tilde{\pi}_{H,t+1}+\frac{1}{2}[\epsilon(1+\varphi)-1]E_t\tilde{\pi}^2_{H,t+1}\right\}-\beta\theta(\epsilon-1)\bar{J}\left[E_t\tilde{\pi}_{H,t+1}+\frac{1}{2}(\epsilon-2)E_t\tilde{\pi}^2_{H,t+1}\right]\\
&+\beta\theta(1+\varphi)\epsilon\left[E_t\tilde{J}_{t+1}\tilde{\pi}_{H,t+1}+\bar{J}\frac{\theta}{1-\theta}(1+\epsilon\varphi)E_t\tilde{\pi}^2_{H,t+1}\right]-\beta\theta(\epsilon-1)E_t\tilde{J}_{t+1}\tilde{\pi}_{H,t+1}\\
&+\frac{\epsilon}{\epsilon-1}\left[\bar{\mu}^W(1+\varphi)\bar{Y}^{\varphi}\tilde{Y}_t-\bar{\mu}^W(1+\varphi)^2\bar{Y}^{\varphi}\tilde{Y}_t\tilde{A}_t+(1+\varphi)\bar{Y}^{\varphi}\tilde{\mu}^W_t\tilde{Y}_t+\frac{1}{2}\bar{\mu}^W\varphi(1+\varphi)\bar{Y}^{\varphi-1}\tilde{Y}^2_t\right.\\
&\left.-\bar{\mu}^W(1+\varphi)\bar{Y}^{\varphi+1}\tilde{A}_t+\bar{Y}^{\varphi+1}\tilde{\mu}^W_t\right]\\
&-\left\{(1-\bar{\tau})\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dY}\right)\bar{Y}+\bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)+\bar{C}^{-\sigma}\right]\tilde{Y}_t\right.\\
&+(1-\bar{\tau})\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dG}\right)\bar{Y}+\bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dG}\right)\right]\tilde{G}_t\\
&+(1-\bar{\tau})\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dC^*}\right)\bar{Y}+\bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial\bar{p}_{H,t}}{\partial C^*}\right)\right]\tilde{C}^*_t-\bar{C}^{-\sigma}\bar{Y}\tilde{\tau}_t\\
&+\frac{1}{2}(1-\bar{\tau})\left[2\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}-2\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dY}\right)-2\sigma\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)^2\bar{C}^{-\sigma-1}\bar{Y}\right.\\
&+\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\partial C\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}\bar{Y}+\left(\frac{\partial^2\bar{p}_{H,t}}{\partial C^2}\right)\left(\frac{d\bar{C}}{dY}\right)^2\bar{C}^{-\sigma}\bar{Y}-\sigma\bar{C}^{-\sigma-1}\partial C\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{Y}\\
&\left.+\sigma(\sigma+1)\bar{C}^{-\sigma-2}\left(\frac{d\bar{C}}{dY}\right)^2\bar{Y}\right]\tilde{Y}^2_t-\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dY}\right)\bar{Y}+\bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)+\bar{C}^{-\sigma}\right]\tilde{Y}_t\tilde{\tau}_t-\\
&\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dG}\right)\bar{Y}+\bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dG}\right)\right]\tilde{G}_t\tilde{\tau}_t-\left[-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dC^*}\right)\bar{Y}+\bar{C}^{-\sigma}\bar{Y}\left(\frac{\partial\bar{p}_{H,t}}{\partial C^*}\right)\right]\tilde{C}^*_t\tilde{\tau}_t\\
&+(1-\bar{\tau})\left[\left(\frac{\partial^2\bar{p}_{H,t}}{\partial C^2}\right)\left(\frac{d\bar{C}}{dG}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}\bar{Y}+\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\partial C\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dG}\right)\bar{C}^{-\sigma}\bar{Y}\right.\\
&-2\sigma\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dG}\right)\bar{C}^{-\sigma-1}\bar{Y}+\sigma(\sigma+1)\bar{C}^{-\sigma-2}\bar{Y}\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dG}\right)-\sigma\bar{C}^{-\sigma-1}\partial C\left(\frac{d\bar{C}}{dY}\right)\left(\frac{d\bar{C}}{dG}\right)\bar{Y}\\
&\left.+\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dG}\right)\bar{C}^{-\sigma}-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dG}\right)\right]\tilde{Y}_t\tilde{G}_t\\
&+(1-\bar{\tau})\left[\left(\frac{\partial^2\bar{p}_{H,t}}{\partial C\partial C^*}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}\bar{Y}+\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\partial C^*\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma}\bar{Y}-\sigma\left(\frac{\partial\bar{p}_{H,t}}{\partial C}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dC^*}\right)\bar{Y}\right.\\
&-\sigma\left(\frac{\partial\bar{p}_{H,t}}{\partial C^*}\right)\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dY}\right)\bar{Y}+\sigma(\sigma+1)\bar{C}^{-\sigma-2}\left(\frac{d\bar{C}}{dC^*}\right)\left(\frac{d\bar{C}}{dY}\right)\bar{Y}-\sigma\bar{C}^{-\sigma-1}\partial C^*\left(\frac{d\bar{C}}{dY}\right)\bar{Y}\\
&\left.\left.+\left(\frac{\partial\bar{p}_{H,t}}{\partial C^*}\right)\bar{C}^{-\sigma}-\sigma\bar{C}^{-\sigma-1}\left(\frac{d\bar{C}}{dC^*}\right)\right]\tilde{Y}_t\tilde{C}^*_t\right\}+\text{t.i.p.}+\mathcal{O}(||\xi||^3)
\end{aligned}
\tag{2.109}
$$

Note that in steady state we have the relations: $\frac{\bar{K}}{\bar{J}}=\frac{\epsilon-1}{\epsilon}$, $(1-\beta\theta)\bar{K}=\frac{\bar{\mu}^W}{\bar{A}}\bar{Y}^{\varphi+1}$ and $(1-\beta\theta)\bar{J}=(1-\bar{\tau})\bar{C}^{-\sigma}\bar{Y}$. Set $\bar{A}=1$.

$$
\bar{J}\frac{\theta}{1-\theta}(1+\epsilon\varphi)\left\{\hat{\pi}_{H,t}+\frac{1}{2}\left[\frac{\theta}{1-\theta}\epsilon(\varphi+1)+(\epsilon-2)\right]\hat{\pi}^2_{H,t}\right\}+\frac{\theta}{1-\theta}(1+\epsilon\varphi)\bar{J}\hat{J}_t\hat{\pi}_{H,t}
$$

$$
\begin{aligned}
&= \beta\theta \bar{J} \frac{\theta}{1-\theta}(1+\epsilon\varphi)\left\{E_t \hat{\pi}_{H,t+1} + \frac{1}{2}\left[\frac{\theta}{1-\theta}\epsilon(\varphi+1) + (\epsilon-2)\right] E_t \hat{\pi}^2_{H,t+1}\right\} + \beta\theta \frac{\theta}{1-\theta}(1+\epsilon\varphi)\bar{J} E_t \hat{J}_{t+1}\hat{\pi}_{H,t+1} \\
&+ \beta\theta\epsilon(1+\varphi)\bar{J}\left\{E_t\hat{\pi}_{H,t+1} + \frac{1}{2}[\epsilon(1+\varphi)-1]E_t\hat{\pi}^2_{H,t+1}\right\} - \beta\theta(\epsilon-1)\bar{J}\left[E_t\hat{\pi}_{H,t+1} + \frac{1}{2}(\epsilon-2)E_t\hat{\pi}^2_{H,t+1}\right] \\
&+ \beta\theta\epsilon(1+\varphi)\left[\bar{J}E_t\hat{J}_{t+1}\hat{\pi}_{H,t+1} + \bar{J}\frac{\theta}{1-\theta}(1+\epsilon\varphi)E_t\hat{\pi}^2_{H,t+1}\right] - \beta\theta(\epsilon-1)\bar{J}E_t\hat{J}_{t+1}\hat{\pi}_{H,t+1} \\
&+ \frac{\epsilon}{\epsilon-1}\frac{\bar{\mu}^W}{\bar{A}}\bar{Y}^{\varphi+1}\left[(1+\varphi)\bar{A}\left(\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2\right) - (1+\varphi)^2\bar{A}^2\hat{Y}_t\hat{A}_t + (1+\varphi)\bar{A}\hat{\mu}^W_t\hat{Y}_t + \frac{1}{2}\bar{A}\varphi(1+\varphi)\hat{Y}_t^2 - (1+\varphi)\bar{A}^2\hat{A}_t + \bar{A}\hat{\mu}^W_t\right] \\
&- (1-\bar{\tau})\bar{C}^{-\sigma}\bar{Y}\left\{\left[1 - \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\left(\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2\right) - \frac{\sigma s_G}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dG}\right)\hat{G}_t - \sigma\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\hat{C}^*_t - \right. \\
&\frac{\bar{\tau}}{1-\bar{\tau}}\left(\hat{\tau}_t + \frac{1}{2}\hat{\tau}_t^2\right) + \frac{1}{2}\left\{-\frac{2\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right) + \frac{\sigma(\sigma+1)}{s_C^2}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)^2 + \right. \\
&\left.\frac{\sigma}{s_C^2}\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^2\left[1 + \left(\frac{d\bar{C}}{dY}\right)\left(\frac{\sigma}{1-\alpha} + \alpha\eta\sigma - \alpha\right)\right]\right\}\hat{Y}_t^2 - \frac{\bar{\tau}}{1-\bar{\tau}}\left[1 - \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\hat{Y}_t\hat{\tau}_t - \\
&\left(\frac{\bar{\tau}}{1-\bar{\tau}}\right)\frac{\sigma s_G}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\hat{G}_t\hat{\tau}_t + \left(\frac{\bar{\tau}}{1-\bar{\tau}}\right)\sigma\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\hat{C}^*_t\hat{\tau}_t + \\
&\left\{\frac{\sigma}{s_C}s_G\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right) - \frac{\sigma(\sigma+1)}{s_C^2}s_G\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)^2 - \frac{\sigma}{s_C^2}s_G\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^2\left[1 + \left(\frac{d\bar{C}}{dY}\right)\left(\frac{\sigma}{1-\alpha} + \alpha\eta\sigma - \alpha\right)\right]\right\}\hat{Y}_t\hat{G}_t + \\
&\left\{-\sigma\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right] + \frac{\sigma(\sigma+1)}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\left[1 - \left(\frac{d\bar{C}}{dY}\right)\right] - \frac{\sigma}{s_C}\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^2 + \right. \\
&\left.\left.\frac{\sigma}{s_C}\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^2\left[1 - \left(\frac{d\bar{C}}{dY}\right)\right]\left(\frac{\sigma}{1-\alpha} + \alpha\eta\sigma - \alpha\right)\right\}\hat{Y}_t\hat{C}^*_t\right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= \beta\bar{J}\frac{\theta}{1-\theta}(1+\epsilon\varphi)E_t\hat{\pi}_{H,t+1} + \beta\bar{J}\frac{\theta}{1-\theta}(1+\epsilon\varphi)E_t\hat{J}_{t+1}\hat{\pi}_{H,t+1} + \beta\bar{J}\frac{\theta}{1-\theta}(1+\epsilon\varphi)\frac{1}{2}\left[\frac{1}{1-\theta}\epsilon(1-\varphi) + (\epsilon-2)\right]E_t\hat{\pi}^2_{H,t+1} + \\
&(1-\beta\theta)\bar{J}\left\{\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\hat{Y}_t + \frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t + \left[1 - \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\frac{\bar{\tau}}{1-\bar{\tau}}\hat{\tau}_t\hat{Y}_t + \right. \\
&\frac{1}{2}\left\{(1+\varphi)\bar{A} - 1 + \frac{3\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right) + \varphi(1+\varphi)\bar{A} - \frac{\sigma(1+\sigma)}{s_C^2}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)^2 - \right. \\
&\left.\frac{\sigma}{s_C^2}\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^2\left[1 + \left(\frac{d\bar{C}}{dY}\right)\left(\frac{\sigma}{1-\alpha} + \alpha\eta\sigma - \alpha\right)\right]\right\}\hat{Y}_t^2 + \frac{1}{2}\left(\frac{\bar{\tau}}{1-\bar{\tau}}\right)\hat{\tau}_t^2 - \\
&(1+\varphi)\bar{A}^2\hat{A}_t + \bar{A}\hat{\mu}_t - \frac{\sigma s_G}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\hat{G}_t + \sigma\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\hat{C}^*_t - \bar{A}^2(1+\varphi)^2\hat{A}_t\hat{Y}_t + (1+\varphi)\bar{A}\hat{\mu}^W_t\hat{Y}_t - \\
&\left\{\frac{\sigma}{s_C}s_G\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right) - \frac{\sigma(\sigma+1)}{s_C^2}s_G\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)^2 - \frac{\sigma}{s_C^2}s_G\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^2\left[1 + \left(\frac{d\bar{C}}{dY}\right)\left(\frac{\sigma}{1-\alpha} + \alpha\eta\sigma - \alpha\right)\right]\right\}\hat{Y}_t\hat{G}_t - \\
&\left\{-\sigma\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right] + \frac{\sigma(\sigma+1)}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\left[1 - \left(\frac{d\bar{C}}{dY}\right)\right] - \frac{\sigma}{s_C}\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^2 + \right. \\
&\left.\frac{\sigma}{s_C}\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^2\left[1 - \left(\frac{d\bar{C}}{dY}\right)\right]\left(\frac{\sigma}{1-\alpha} + \alpha\eta\sigma - \alpha\right)\right\}\hat{Y}_t\hat{C}^*_t + \\
&\left.\left(\frac{\bar{\tau}}{1-\bar{\tau}}\right)\frac{\sigma s_G}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\hat{G}_t\hat{\tau}_t - \left(\frac{\bar{\tau}}{1-\bar{\tau}}\right)\sigma\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\hat{C}^*_t\hat{\tau}_t\right\} + \frac{1}{2}\epsilon(1+\varphi)\hat{\pi}_{H,t} + \text{t.i.p.} + \mathcal{O}(||\xi||^3)
\end{aligned}
\tag{2.110}
$$

Define $V_t = \hat{\pi}_{H,t} + \frac{1}{2}\left[\frac{\epsilon(\varphi+1)}{1-\theta} + (\epsilon-1)\right]\hat{\pi}^2_{H,t} + \hat{J}_t\hat{\pi}_{H,t}$, and substitute into (2.110), we obtain a recursive relation

$$V_t = \kappa_\alpha \left\{ \hat{Y}_t + C_\tau \hat{\tau}_t + C_{\tau Y}\hat{\tau}_t \hat{Y}_t + \frac{1}{2}C_{YY}\hat{Y}_t^2 + \frac{1}{2}C_{\tau\tau}\hat{\tau}_t^2 + \frac{1}{2}C_\pi \hat{\pi}_{H,t}^2 + C_A \hat{A}_t + C_{\mu^W}\hat{\mu}_t^W + C_G \hat{G}_t \right.$$
$$\left. + C_{C^*}\hat{C}_t^* + C_{AY}\hat{A}_t\hat{Y}_t + C_{\mu^W Y}\hat{\mu}_t^W \hat{Y}_t + C_{GY}\hat{G}_t\hat{Y}_t + C_{C^*Y}\hat{C}_t^*\hat{Y}_t + C_{G\tau}\hat{G}_t\hat{\tau}_t + C_{C^*\tau}\hat{C}_t^*\hat{\tau}_t \right\}$$
$$+ \beta E_t V_{t+1} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \tag{2.111}$$

where

$$C_{\tau\alpha} = C_{\tau\tau\alpha} = \psi_\alpha$$

$$C_{\tau Y\alpha} = \left[1 - \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\psi_\alpha$$

$$C_{YY\alpha} = \left\{(1+\varphi)\bar{A} - 1 + \frac{3\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right) + \varphi(1+\varphi)\bar{A} - \frac{\sigma}{s_C^2}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)^2\left(1+\frac{\alpha\eta\sigma}{1-\alpha}\right) - \right.$$
$$\left. \frac{\sigma^2}{s_C^2}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)^2\left[1+\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)\right] - \frac{\sigma}{s_C^2}\frac{\alpha^2\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^2(\sigma\eta - 1)\right\}\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]^{-1}$$

$$C_{\pi\alpha} = \frac{\epsilon(1+\varphi)}{\kappa_\alpha}$$

$$C_{A\alpha} = -(1+\varphi)\bar{A}^2\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]^{-1}$$

$$C_{\mu^W\alpha} = \bar{A}\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]^{-1}$$

$$C_{G\alpha} = -\frac{\sigma s_G}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]^{-1}$$

$$C_{C^*} = \sigma\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]^{-1}$$

$$C_{AY\alpha} = -(1+\varphi)^2\bar{A}^2\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]^{-1}$$

$$C_{\mu^W Y\alpha} = (1+\varphi)\bar{A}\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]^{-1}$$

$$C_{GY\alpha} = -\left\{\frac{\sigma}{s_C}s_G\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right) - \frac{\sigma}{s_C^2}s_G\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)^2\left(1+\frac{\alpha\eta\sigma}{1-\alpha}\right) - \frac{\sigma^2}{s_C^2}s_G\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)^2\left[1+\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)\right] - \right.$$
$$\left. \frac{\sigma}{s_C^2}s_G\frac{\alpha^2\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^2(\sigma\eta - 1)\right\}\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]^{-1}$$

$$C_{C^*Y} = -\left\{-\sigma\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right] + \frac{\sigma^2}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\left[1 - \left(\frac{d\bar{C}}{dY}\right)\right]\left[1+\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)\right] + \right.$$
$$\frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\left[1 - \left(1+\frac{\alpha\eta\sigma}{1-\alpha}\right)\left(\frac{d\bar{C}}{dY}\right)\right] +$$
$$\left. \frac{\sigma}{s_C}\frac{\alpha^2\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)^2\left[1 - \left(\frac{d\bar{C}}{dY}\right)\right](\sigma\eta - 1)\right\}\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]^{-1}$$

$$C_{G\tau\alpha} = \sigma\frac{s_G}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\psi_\alpha$$

$$C_{C^*\tau} = -\sigma\left[1 - \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]\psi_\alpha$$

and

$$\kappa_\alpha = \left(\frac{1-\theta}{\theta}\right)\left(\frac{1}{1+\epsilon\varphi}\right)(1-\beta\theta)\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]$$

$$w_\tau = \frac{\bar{\tau}}{1-\bar{\tau}}$$

$$\psi_\alpha = w_\tau\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]^{-1}$$

Note that when $\sigma\eta = 1$ $\left(\frac{d\bar{C}}{dY} = 1-\alpha\right)$, $\bar{A} = 1$:

$$\kappa_\alpha = \kappa = \left(\frac{1-\theta}{\theta}\right)\left(\frac{1}{1+\epsilon\varphi}\right)(1-\beta\theta)\left(\varphi + \frac{\sigma}{s_C}\right)$$

$$C_{\tau\alpha} = C_{\tau\tau\alpha} = C_\tau = C_{\tau\tau} = \psi$$

$$C_{YY\alpha} = C_{YY} = \left[\varphi^2 + 2\varphi + \frac{3\sigma}{s_C} - \frac{\sigma(1+\sigma)}{s_C^2}\right]\left(\varphi + \frac{\sigma}{s_C}\right)^{-1}$$

$$C_{A\alpha} = C_A = -(1+\varphi)\left(\varphi + \frac{\sigma}{s_C}\right)^{-1}$$

$$C_{G\alpha} = C_G = -\frac{\sigma s_G}{s_C}\left(\varphi + \frac{\sigma}{s_C}\right)^{-1}$$

$$C_{AY\alpha} = C_{AY} = -(1+\varphi)^2\left(\varphi + \frac{\sigma}{s_C}\right)^{-1}$$

$$C_{GY\alpha} = C_{GY} = \left[\frac{\sigma(1+\sigma)s_G}{s_C^2} - \frac{\sigma s_G}{s_C}\right]\left(\varphi + \frac{\sigma}{s_C}\right)^{-1}$$

$$C_{G\tau\alpha} = C_{G\tau} = \sigma\frac{s_G}{s_C}\psi$$

$$\psi_\alpha = \psi = \frac{w_\tau}{\varphi + \sigma s_C^{-1}}$$

$$C_{\tau Y\alpha} = C_{\tau Y} = \left(1 - \frac{\sigma}{s_C}\right)\psi$$

$$C_{\pi\alpha} = C_\pi = \frac{\epsilon(1+\varphi)}{\kappa}$$

$$C_{\mu^W\alpha} = C_{\mu^W} = \left(\varphi + \frac{\sigma}{s_C}\right)^{-1}$$

$$C_{C^*} = 0$$

$$C_{\mu^W Y\alpha} = C_{\mu^W Y} = (1+\varphi)\left(\varphi + \frac{\sigma}{s_C}\right)^{-1}$$

$$C_{C^*Y} = -\frac{\alpha}{1-\alpha}\frac{\sigma^2}{s_C}\left(\varphi + \frac{\sigma}{s_C}\right)^{-1}$$

$$C_{C^*\tau} = 0$$

And when $\alpha = 0$ $\left(\frac{d\bar{C}}{dY} = 1\right)$ and $\bar{A} = 1$, we recover the results obtained by Leeper and Zhou (2021) for a closed economy.

Integrating (2.111) forward from $t = 0$, we have

$$V_0 = E_0 \sum_{t=0}^{\infty} \beta^t \kappa \Big(\hat{Y}_t + C_{\tau\alpha}\hat{\tau}_t + C_{\tau Y\alpha}\hat{\tau}_t\hat{Y}_t + \frac{1}{2}C_{YY\alpha}\hat{Y}_t^2 + \frac{1}{2}C_{\tau\tau\alpha}\hat{\tau}_t^2 + \frac{1}{2}C_{\pi\alpha}\hat{\pi}_t^2 + C_{A\alpha}\hat{A}_t + C_{\mu^W\alpha}\hat{\mu}_t^W$$
$$+ C_{G\alpha}\hat{G}_t + C_{C^*}\hat{C}_t^* C_{AY\alpha}\hat{A}_t\hat{Y}_t + C_{\mu^W Y\alpha}\hat{\mu}_t^W\hat{Y}_t + C_{GY\alpha}\hat{G}_t\hat{Y}_t + C_{C^*Y}\hat{C}_t^*\hat{Y}_t + C_{G\tau\alpha}\hat{G}_t\hat{\tau}_t + C_{C^*\tau}\hat{C}_t^*\hat{\tau}_t \Big)$$
$$+ \text{t.i.p.} + \mathcal{O}(||\xi||^3) \tag{2.112}$$

Appendix 2.J Quadratic approximation to the objective function

Now we use a linear combination of (2.101) and (2.112) to eliminate the linear terms ($\hat{Y}_t$ and $\hat{\tau}_t$) in the second order approximation to the welfare measure. The coefficients μ_B, μ_C must satisfy:

$$\mu_{B\alpha}B_{Y\alpha} + \mu_{C\alpha}C_{Y\alpha} = -A_{Y\alpha} = -\left[1 - (1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\right]$$
$$\mu_{B\alpha}B_{\tau\alpha} + \mu_{C\alpha}C_{\tau\alpha} = 0$$

The solution is:

$$\mu_{B\alpha} = \left[1 - (1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\right]\frac{w_\tau}{\Gamma_\alpha}$$
$$\mu_{C\alpha} = -\left[1 - (1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\right]\frac{(1+w_G)\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]}{\Gamma_\alpha}$$

where $w_G = \frac{\bar{G}+\bar{Z}}{\bar{S}}$ is the steady-state government output outlays to surplus ratio, and satisfies $1 + w_G = \frac{\bar{\tau}\bar{Y}}{\bar{S}}$. $\Gamma_\alpha = (1+w_G)\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right) - w_\tau\right] + \frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\frac{\sigma}{s_C}w_\tau$.

Therefore, we can finally express the objective function in the linear quadratic form of:

$$\frac{U_0 - \bar{U}_0}{\bar{u_Y}\bar{Y}} + \mu_{B\alpha}\left(\frac{1}{1-\beta}\right)\frac{W_0 - \bar{W}}{\bar{W}} + \mu_{C\alpha}\frac{V_0}{\kappa} = -E_0\sum_{t=0}^{\infty}\beta^t\frac{1}{2}\left\{q_{x\alpha}\left(\hat{Y}_t - \hat{Y}_t^e\right)^2 + q_{\pi\alpha}\hat{\pi}_t^2\right\}$$
$$+ \text{t.i.p.} + \mathcal{O}(||\xi||^3)$$

with

$$\begin{aligned}
q_{x\alpha} &= -A_{YY\alpha} - \mu_{B\alpha}B_{YY\alpha} - \mu_{C\alpha}C_{YY\alpha} \\
&= -\left[1 - \frac{\sigma}{s_C}\left(\frac{d\bar{C}}{dY}\right) - \frac{1}{s_C}\left(\frac{d\bar{C}}{dY}\right)^2\frac{\alpha\eta\sigma}{1-\alpha}\left(\frac{\sigma}{1-\alpha} + \alpha\eta\sigma - \alpha\right) - (1-\Phi)(1+\varphi)\frac{1}{1-\alpha}\left(\frac{d\bar{Y}}{dC}\right)\right] \\
&\quad - \left[1 - (1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\right]\frac{1}{\Gamma}\left\{w_\tau B_{YY\alpha} - (1+w_G)C_{YY\alpha}\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\left(\frac{d\bar{C}}{dY}\right)\right]\right\} \\
q_{\pi\alpha} &= -A_{\pi_H\alpha} - \mu_{C\alpha}C_{\pi_H\alpha} \\
&= \frac{\left[1 - (1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\right](1+w_G)\epsilon(1+\varphi)\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]}{\Gamma_\alpha\kappa_\alpha} \\
&\quad + \frac{(1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\epsilon\left[(1+\varphi)\bar{A} - 1 + \frac{\sigma}{s_C}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\right]}{\kappa_\alpha}
\end{aligned}$$

and $\hat{Y}_t^e$ denotes the efficient level of output, which is exogenous and depends on the exogenous shocks $\hat{A}_t$, $\hat{\mu}_t^W$, $\hat{G}_t$, $\hat{Z}_t$ and $\hat{C}_t^*$:

$$\begin{aligned}
\hat{Y}_t^e &= q_{x\alpha}^{-1}\Big[\left(A_{A\alpha} + \mu_{C\alpha}C_{YA\alpha}\right)\hat{A}_t + \left(A_{G\alpha} + \mu_{B\alpha}B_{YG\alpha} + \mu_{C\alpha}C_{YG\alpha}\right)\hat{G}_t \\
&\quad + \left(A_{C^*} + \mu_{B\alpha}B_{YC^*} + \mu_{C\alpha}C_{YC^*}\right)\hat{C}_t^* + \mu_{B\alpha}B_{YZ\alpha}\hat{Z}_t + \mu_{C\alpha}C_{\mu_W Y\alpha}\hat{\mu_W}_t\Big] \\
&= q_{A\alpha}\hat{A}_t + q_{G\alpha}\hat{G}_t + q_{C^*}\hat{C}_t^* + q_{Z\alpha}\hat{Z}_t + q_{\mu_W\alpha}\hat{\mu_W}_t
\end{aligned}$$

where

$$q_{A\alpha} = q_{x\alpha}^{-1} \left\{ (1-\Phi)(1+\varphi)\left(\frac{d\bar{Y}}{dC}\right) + \left[1-(1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\right] \frac{(1+w_G)}{\Gamma_\alpha}(1+\varphi)^2\bar{A}^2 \right\}$$

$$q_{G\alpha} = q_{x\alpha}^{-1} \left\{ \sigma\frac{s_G}{s_C}\left(\frac{d\bar{C}}{dY}\right)\left[1+\frac{\alpha\sigma\eta}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)\right] + \frac{s_G}{s_C}(\sigma\eta-1)\frac{\alpha^2\eta\sigma}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)^2 \right.$$

$$\left. +\mu_{B\alpha}B_{YG\alpha} + \mu_{C\alpha}C_{YG\alpha} \right\}$$

$$q_{C^*} = q_{x\alpha}^{-1} \left\{ \left\{ -\sigma\left[1+\frac{\alpha\eta\sigma}{(1-\alpha)^2}\left(\frac{d\bar{C}}{dY}\right)\right] - \frac{\alpha^2\eta\sigma}{1-\alpha}(\eta\sigma-1)\left(\frac{d\bar{C}}{dY}\right) \right\} \left[1-\left(\frac{d\bar{C}}{dY}\right)\right] \right.$$

$$\left. +\mu_{B\alpha}B_{YC^*} + \mu_{C\alpha}C_{YC^*} \right\}$$

$$q_{Z\alpha} = q_{x\alpha}^{-1} \left\{ \left[1-(1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\right] \frac{w_\tau}{\Gamma_\alpha}\frac{1}{1-\alpha}\left(\frac{d\bar{C}}{dY}\right)\sigma\frac{s_Z}{s_C}\frac{1}{s_D} \right\}$$

$$q_{\mu_W\alpha} = -q_{x\alpha}^{-1} \left\{ (1+\varphi)\bar{A}\left[1-(1-\Phi)\left(\frac{d\bar{Y}}{dC}\right)\right] \frac{1+w_G}{\Gamma_\alpha} \right\}$$

Note that when $\sigma\eta = 1$ $\left(\frac{d\bar{C}}{dY} = 1-\alpha\right)$ and $\bar{A} = 1$:

$$\kappa_\alpha = \kappa = \left(\frac{1-\theta}{\theta}\right)\left(\frac{1}{1+\epsilon\varphi}\right)(1-\beta\theta)\left(\varphi+\frac{\sigma}{s_C}\right)$$

$$\Gamma_\alpha = \Gamma = (1+w_G)\left(\varphi+\frac{\sigma}{s_C}-w_\tau\right)+\frac{\sigma}{s_C}w_\tau$$

$$\mu_{B\alpha} = \left(1-\frac{1-\Phi}{1-\alpha}\right)\frac{\mu_B}{\Phi}$$

$$\mu_{C\alpha} = \left(1-\frac{1-\Phi}{1-\alpha}\right)\frac{\mu_C}{\Phi}$$

$$
\begin{aligned}
q_{x\alpha} = &- \left[1 - \frac{\sigma}{s_C} - \left(\frac{1-\Phi}{1-\alpha}\right)(1+\varphi)\right] \\
&- \left(1 - \frac{1-\Phi}{1-\alpha}\right)\frac{1}{\Gamma}\left\{w_\tau\left[(1+w_G)\left(1 - 2\frac{\sigma}{s_C}\right) - \frac{\sigma}{s_C} + \frac{\sigma^2}{s_C^2} + \frac{\sigma}{s_C^2}\right] - \right. \\
&\left.(1+w_G)\left[\varphi^2 + 2\varphi + \frac{3\sigma}{s_C} - \frac{\sigma(1+\sigma)}{s_C^2}\right]\right\} \\
q_{\pi\alpha} = &\frac{\left(1 - \frac{1-\Phi}{1-\alpha}\right)(1+w_G)\epsilon(1+\varphi)\left(\varphi + \frac{\sigma}{s_C}\right)}{\Gamma\kappa} + \frac{\left(\frac{1-\Phi}{1-\alpha}\right)\epsilon\left(\varphi + \frac{\sigma}{s_C}\right)}{\kappa} \\
q_{A\alpha} = &q_{x\alpha}^{-1}\left[\frac{(1-\Phi)(1+\varphi)}{(1-\alpha)} + \left(1 - \frac{1-\Phi}{1-\alpha}\right)\frac{(1+w_G)}{\Gamma}(1+\varphi)^2\right] \\
q_{G\alpha} = &q_{x\alpha}^{-1}\left\{\frac{\sigma s_G}{s_C} - \left(1 - \frac{1-\Phi}{1-\alpha}\right)\frac{1}{\Gamma}\left\{w_\tau \sigma \frac{s_G}{s_C}\left(\frac{\sigma+1}{s_C} - \frac{\bar{\tau}\bar{Y}}{\bar{S}} - \frac{1}{s_D}\right)\right.\right. \\
&\left.\left. + (1+w_G)\left[\frac{\sigma(1+\sigma)s_G}{s_C^2} - \frac{\sigma s_G}{s_C}\right]\right.\right. \\
q_{C^*} = &q_{x\alpha}^{-1}\frac{\alpha}{1-\alpha}\sigma\left[-1 + \left(1 - \frac{1-\Phi}{1-\alpha}\right)\frac{1}{\Gamma}\frac{\sigma}{s_C}(1 + w_G + w_\tau)\right] \\
q_{Z\alpha} = &q_{x\alpha}^{-1}\left(1 - \frac{1-\Phi}{1-\alpha}\right)\frac{w_\tau}{\Gamma}\frac{\sigma}{s_C}\frac{s_Z}{s_D} \\
q_{\mu^W\alpha} = &-q_{x\alpha}^{-1}\left(1 - \frac{1-\Phi}{1-\alpha}\right)\frac{(1+w_G)(1+\varphi)}{\Gamma}
\end{aligned}
$$

For the loss function to be convex, i.e. $q_{x\alpha} > 0$ and $q_{\pi\alpha} > 0$ we need:

$$
\begin{aligned}
q_{x\alpha} > 0 \Rightarrow &- \left[1 - \frac{\sigma}{s_C} - \left(\frac{1-\Phi}{1-\alpha}\right)(1+\varphi)\right] > \\
&\left(1 - \frac{1-\Phi}{1-\alpha}\right)\frac{1}{\Gamma}\left\{w_\tau\left[(1+w_G)\left(1 - 2\frac{\sigma}{s_C}\right) - \frac{\sigma}{s_C} + \frac{\sigma^2}{s_C^2} + \frac{\sigma}{s_C^2}\right] - \right. \\
&\left.(1+w_G)\left[\varphi^2 + 2\varphi + \frac{3\sigma}{s_C} - \frac{\sigma(1+\sigma)}{s_C^2}\right]\right\} \\
q_{\pi\alpha} > 0 \Rightarrow &\alpha < \frac{\Gamma(1-\Phi)}{(1+w_G)(1+\varphi)} + \Phi
\end{aligned}
$$

When $\alpha = 0$ $\left(\frac{d\bar{C}}{dY} = 1\right)$ and $\bar{A} = 1$, we recover the results obtained by Leeper and Zhou (2021) for a closed economy. Notice that when $\sigma\eta = 1$ the differences with the closed economy are (with $0 < \alpha < 1$):

$$
\begin{array}{lcl}
A_{Y\alpha} = 1 - \dfrac{1-\Phi}{1-\alpha} & \text{vs} & A_Y = \Phi \\
A_{YY\alpha} = 1 - \dfrac{\sigma}{s_C} - \dfrac{(1+\varphi)(1-\Phi)}{1-\alpha} & < & A_{YY} = 1 - \dfrac{\sigma}{s_C} - (1+\varphi)(1-\Phi) \\
A_{\pi_H\alpha} = \dfrac{1-\Phi}{1-\alpha}\dfrac{\epsilon\theta(1+\epsilon\varphi)}{(1-\theta)(1-\beta\theta)} & > & A_\pi = \dfrac{(1-\Phi)\epsilon\theta(1+\epsilon\varphi)}{(1-\theta)(1-\beta\theta)} \\
A_{A\alpha} = \dfrac{(1+\varphi)(1-\Phi)}{1-\alpha} & > & A_A = (1+\varphi)(1-\Phi)
\end{array}
$$

$$
A_{C^*} = -\frac{\alpha}{1-\alpha}\sigma, \quad B_{C^*Y} = \frac{\alpha}{1-\alpha}\frac{\sigma^2}{s_C} \quad \text{and} \quad C_{C^*Y} = -\frac{\alpha}{1-\alpha}\frac{\sigma^2}{s_C}
$$

The effect of α on inflation and output gap weights in the welfare loss function, and on the coefficients of the efficient output are summarized in Figure 2.J.1.

Appendix 2.K Log-Linear CPI inflation, Real Exchange Rate and Net Exports

Before we introduce the linear constraints of the problem, it is useful to specify the log-linearized CPI inflation and the real exchange rate. By log-linearizing equations (2.4) and (2.74) we obtain

$$
\hat{\pi}_t = \hat{\pi}_{H,t} + \frac{\alpha}{1-\alpha}\Delta\hat{\mathcal{Q}}_t \tag{2.113}
$$

$$
\hat{\mathcal{Q}}_t = \sigma\left(\hat{C}_t - \hat{C}_t^*\right) \tag{2.114}
$$

and using the risk sharing condition (2.74), the relation between relative prices (2.73), and the market clearing in the goods market (2.65) we can re-express the log-linearized

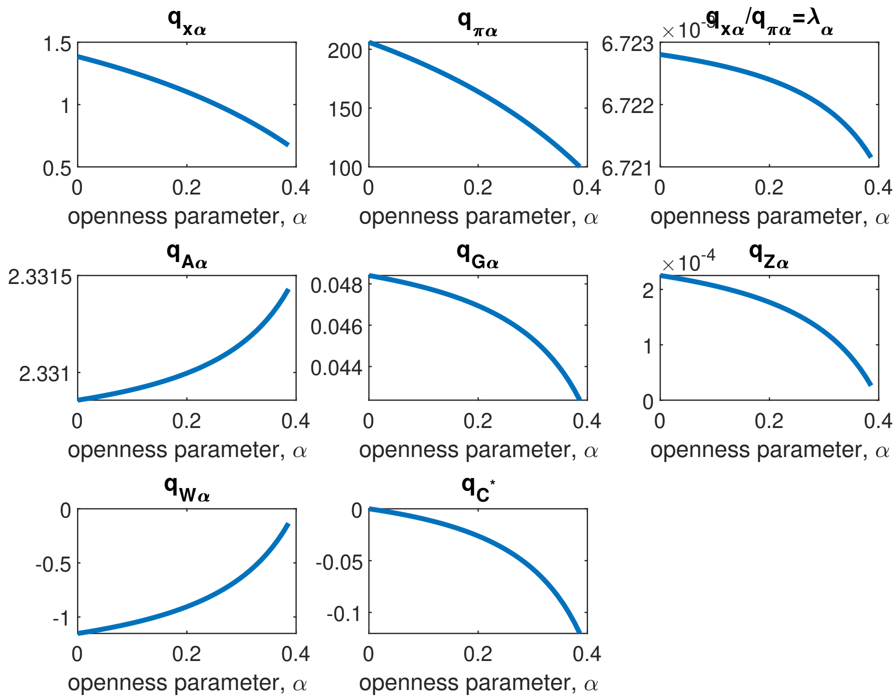

Figure 2.J.1: Effect of the small open economy's openness parameter, α, on inflation and output gap weights in the welfare loss function, $q_{\pi\alpha}$ and $q_{x\alpha}$, and on the coefficients of the efficient output, $q_{A\alpha}, q_{G\alpha}, q_{Z\alpha}, q_{W\alpha}, q_{C^*}$

CPI inflation, when $\sigma\eta = 1$, as

$$\hat{\pi}_t = \hat{\pi}_{H,t} + \alpha\frac{\sigma}{s_C}\left(\Delta\hat{Y}_t - s_G\Delta\hat{G}_t - s_C\Delta\hat{C}_t^*\right) \tag{2.115}$$

$$\hat{\mathcal{Q}}_t = \frac{\sigma}{s_C}(1-\alpha)\left(\hat{Y}_t - s_G\hat{G}_t - s_C\hat{C}_t^*\right) = \frac{\sigma}{s_C}(1-\alpha)\left(\hat{x}_t - h_t\right) \tag{2.116}$$

where $h_t = \hat{Y}_t^e - s_G\hat{G}_t - s_C\hat{C}_t^* = q_{A\alpha}\hat{A}_t + (q_{G\alpha} - s_G)\hat{G}_t + q_{Z\alpha}\hat{Z}_t + q_{W\alpha}\hat{\mu}_t^W + (q_{C^*} - s_C)\hat{C}_t^*$.

Note that when $\alpha = 0$, $\hat{\pi}_t = \hat{\pi}_{H,t}$.

Let $NX_t \equiv \frac{1}{\bar{Y}}\left(Y_t - G_t - \frac{P_t}{P_{H,t}}C_t\right)$ denote net exports in terms of domestic output, expressed as a fraction of steady state output $\bar{Y}$. A first order approximation around the symmetric steady state yields $\hat{NX}_t = \hat{Y}_t - s_G\hat{G}_t - s_C\hat{C}_t - \frac{\alpha}{1-\alpha}s_C\hat{\mathcal{Q}}_t$, which combined with the aggregate domestic output ($\hat{Y}_t = s_G\hat{G}_t + s_C\hat{C}_t + \frac{\alpha\omega}{\sigma}s_C\hat{\mathcal{T}}_t$) implies

$$\hat{NX}_t = \alpha s_C \left(\frac{\omega}{\sigma} - 1\right)\hat{\mathcal{T}}_t$$

where $\omega \equiv \sigma\eta + (1-\alpha)(\sigma\eta - 1)$. In the special case of $\sigma = \eta = 1$ we have $\hat{NX}_t = 0$ for all t, though the latter property will also hold for any configuration of those parameters satisfying $\sigma(\eta - 1) + (1-\alpha)(\sigma\eta - 1) = 0$. More generally, the sign of the relationship between the terms of trade and net exports is ambiguous, depending on the relative size of σ and η.

Appendix 2.L Linear Constraint 1: Aggregate supply

A first-order Taylor series expansion of (2.49) around the zero-inflation steady state yields the log-linear aggregate-supply relation (first order terms from the second-order approximation conducted in Appendix 2.I)

$$\hat{\pi}_{H,t} = \beta E_t \hat{\pi}_{H,t+1} + \kappa_\alpha(\hat{x}_t + \psi_\alpha \hat{\tau}_t) + u_{\alpha,t} \tag{2.117}$$

where, when $\sigma\eta = 1$

$$\kappa_\alpha = \kappa = \frac{(1-\theta)(1-\beta\theta)}{\theta}\frac{1}{1+\epsilon\varphi}\left(\varphi + \frac{\sigma}{s_C}\right)$$
$$\psi_\alpha = \psi = \frac{w_\tau}{\varphi + \sigma s_C^{-1}}$$

and $u_{\alpha,t}$ is a composite cost-push shock that depends on the four domestic exogenous disturbances and the four foreign exogenous disturbances

$$u_{\alpha,t} = \underbrace{\kappa\left(q_{A\alpha} - \frac{1+\varphi}{\varphi+\sigma s_C^{-1}}\right)}_{u_{A\alpha}}\hat{A}_t + \underbrace{\kappa\left(q_{G\alpha} - \frac{s_G}{s_C}\frac{\sigma}{\varphi+\sigma s_C^{-1}}\right)}_{u_{G\alpha}}\hat{G}_t + \underbrace{\kappa q_{Z\alpha}}_{u_{Z\alpha}}\hat{Z}_t + \underbrace{\kappa\left(q_{W\alpha} + \frac{1}{\varphi+\sigma s_C^{-1}}\right)}_{u_{W\alpha}}\hat{\mu}_t^W$$
$$+ \underbrace{\frac{\kappa q_{C^*}}{s_C^*}\left(q_{A^*} + \frac{c_{x^*}}{q_{x^*}c_{L^*}}F_{A^*}^*\right)}_{u_{A^*}}\hat{A}_t^* + \underbrace{\frac{\kappa q_{C^*}}{s_C^*}\left(q_{G^*} - s_G^* + \frac{c_{x^*}}{q_{x^*}c_{L^*}}F_{G^*}^*\right)}_{u_{G^*}}\hat{G}_t^* + \underbrace{\frac{\kappa q_{C^*}}{s_C^*}\left(q_{Z^*} + \frac{c_{x^*}}{q_{x^*}c_{L^*}}F_{Z^*}^*\right)}_{u_{Z^*}}\hat{Z}_t^*$$
$$+ \underbrace{\frac{\kappa q_{C^*}}{s_C^*}\left(q_{W^*} + \frac{c_{x^*}}{q_{x^*}c_{L^*}}F_{W^*}^*\right)}_{u_{W^*}}\hat{\mu}_t^{W*} \qquad (2.118)$$

with

$$F_i^* = \frac{1}{1-\beta^*\rho_i}\left[f_i^* + (1-\beta^*)\frac{\bar{\tau}^*}{s_D^*}\frac{1}{\kappa^*\psi^*}u_i^* + \beta^*\frac{\sigma^*}{s_C^*}\upsilon_i^*\right] - \frac{\beta^*\rho^*}{1-\beta^*\rho^*\rho_i}\frac{\sigma^*}{s_C^*}\upsilon_i^*, \quad i = A^*, G^*, Z^*, W^*$$
$$f_t^* = -(1-\beta^*)\frac{\bar{\tau}^*}{s_D^*}q_{A^*}\hat{A}_t^* + (1-\beta^*)\left(\frac{s_G^*}{s_D^*} - \frac{\bar{\tau}^*}{s_D^*}q_{G^*}\right)\hat{G}_t^* + (1-\beta^*)\left(\frac{s_Z^*}{s_D^*} - \frac{\bar{\tau}^*}{s_D^*}q_{Z^*}\right)\hat{Z}_t^*$$
$$- (1-\beta^*)\frac{\bar{\tau}^*}{s_D^*}q_{W^*}\hat{\mu}_t^{W*}$$
$$c_{x^*} = \left(\frac{1}{\psi^*} - 1\right)(1-\beta^*)\frac{\bar{\tau}^*}{s_D^*} - \frac{\sigma^*}{s_C^*}\beta^*(1-\rho^*)$$
$$c_{L^*} = -\frac{b_\pi^*}{q_{\pi^*}}\left(\frac{1-\beta^*}{\kappa^*\psi^*}\frac{\bar{\tau}^*}{s_D^*} - 1\right) - \frac{\beta^*\rho^*}{1-\beta^*\rho^*}\frac{1}{q_{\pi^*}} + \frac{\beta^*\rho^*(1-\rho^*)}{1-\beta(\rho^*)^2}\frac{1}{q_{\pi^*}}$$
$$+ \frac{1}{q_{x^*}}\left(\frac{m_{x^*}}{1-\beta^*\rho^*} - \frac{n_{x^*}}{1-\beta^*}\right)\left(\frac{1}{\psi^*} - 1\right)(1-\beta^*)\frac{\bar{\tau}^*}{s_D^*}$$
$$- \frac{1}{q_{x^*}}\frac{\sigma^*}{s_C^*}\left(\frac{m_{x^*}}{1-\beta^*(\rho^*)^2} - \frac{n_{x^*}}{1-\beta^*\rho^*}\right)\beta^*(1-\rho^*)$$
$$+ \frac{1}{q_{x^*}}\frac{\sigma^*}{s_C^*}\left(\frac{\rho^* m_{x^*}}{1-\beta^*(\rho^*)^2} - \frac{n_{x^*}}{1-\beta^*\rho^*}\right)\beta^*\rho^*(1-\rho^*)$$

$$m_{x^*} = (1-\beta^*\rho^*)\frac{\sigma^*}{s_C^*} \qquad n_{x^*} = (1-\beta^*)\left(\frac{b_\tau^*}{\psi^*} - b_x^*\right)$$
$$b_\tau^* = \frac{\bar{\tau}^*}{s_D^*} \qquad b_x^* = \frac{\bar{\tau}^*}{s_D^*} - \frac{\sigma^*}{s_C^*}$$

The exogenous disturbances generate cost-push effects through $u_{\alpha,t}$ because with a distorted steady state, they generate a time-varying gap between the flexible-price equilibrium level of output and the efficient level of output. If the steady state were not distorted, only variations in wage markups would have cost-push effects. This is why wage markups are regarded as "pure" cost-push disturbances.

If $\hat{\tau}_t$ were exogenous, $\kappa\psi\hat{\tau}_t + u_{\alpha,t}$ prevents complete stabilization of inflation and the welfare-relevant output gap. Iterating forward on (2.117) yields

$$\hat{\pi}_{H,t} = E_t \sum_{k=0}^{\infty} \beta^k \kappa \hat{x}_{t+k} + U_{\alpha,t}$$

where $U_{\alpha,t} \equiv E_t \sum_{k=0}^{\infty} (\kappa\psi\hat{\tau}_{t+k} + u_{\alpha,t+k})$ determines the degree to which stabilization of inflation and output gap is not possible. This is the only source of trade off between stabilization of inflation and output gap in conventional new Keynesian optimal monetary policy analyses (Galí, 1991).

When $\hat{\tau}_t$ is chosen optimally along with monetary policy, then $\hat{\tau}_t$ can be set to fully absorb cost-push shocks, making simultaneous stabilization of inflation and the output gap possible. Benigno and Woodford (2004) rewrite (2.117) as

$$\hat{\pi}_{H,t} = \beta E_t \hat{\pi}_{H,t+1} + \kappa \hat{x}_t + \kappa\psi(\hat{\tau}_t - \hat{\tau}^e_{\alpha,t}) \qquad (2.119)$$

where $\hat{\tau}^e_{\alpha,t} \equiv -\frac{u_{\alpha,t}}{\kappa\psi}$ is the tax rate that offsets the cost-push shock. Expression (2.119) describes the trade-off relation between inflation and output that fiscal policy faces because tax rates can help stabilize output and inflation by offsetting variations in cost-push distortions. Notice that it does not depend on the currency composition of the government's debt.

Finally, notice that these expressions are analogous to the closed economy version of the model, with the addition of foreign shocks. Indeed, the first four terms of the composite cost-push shock, which correspond to domestic shocks, are identical to the closed economy version, with the welfare loss function weights and coefficients of the efficient output being affected by the degree of openness. The remaining four terms, which correspond to foreign shocks, are only present when we consider a small open economy ($q_{C^*} = 0$ when $\alpha = 0$).

Appendix 2.M Linear Constraint 2: Aggregate demand

A second constraint arises from the household's Euler equation for short-term bonds (equation 2.41). Using the log-linear approximations to the resource constraint (equation 2.67), the aggregate price level (equation 2.113), and the real exchange rate (equation 2.114), and the definition of the output gap, we obtain

$$\hat{x}_t = E_t \hat{x}_{t+1} - \frac{s_C}{\sigma_{\alpha\gamma}} \left(\hat{\tilde{R}}_t - \gamma_{LC} E_t \hat{\pi}_{H,t+1} \right) + \upsilon_{\alpha,t}$$

where

$$\begin{aligned} \hat{\tilde{R}}_t &= \gamma_{LC,t} \hat{R}_t + \gamma_{CPI,t} \hat{r}_t + \gamma_{FC,t} \hat{R}_t^* \\ \sigma_{\alpha\gamma} &= \sigma \left[1 - \gamma_{FC} + \gamma_{LC} \left(\frac{\alpha}{1-\alpha} \right) \right] \left(\frac{d\bar{C}}{dY} \right) \\ \text{and when } \sigma\eta = 1,\ \sigma_{\alpha\gamma} &= \sigma \left[1 - \alpha + \alpha \gamma_{LC} - (1-\alpha) \gamma_{FC} \right] \end{aligned}$$

and the composite aggregate demand shock, $\upsilon_{\alpha,t}$, is

$$\upsilon_{\alpha,t} = \underbrace{q_{A\alpha}\left(\rho_A - 1\right)}_{\upsilon_{A\alpha}} \hat{A}_t + \underbrace{q_{\mu_W \alpha}\left(\rho_{\mu_W} - 1\right)}_{\upsilon_{\mu_W \alpha}} \hat{\mu}_t^W + \underbrace{\left(q_{G\alpha} - s_G\right)\left(\rho_G - 1\right)}_{\upsilon_{G\alpha}} \hat{G}_t + \underbrace{q_{Z\alpha}\left(\rho_Z - 1\right)}_{\upsilon_{Z\alpha}} \hat{Z}_t$$
$$+ \underbrace{\left\{-\frac{s_C}{\sigma_{\alpha\gamma}}\gamma_{FC}\rho^* \frac{F^*_{A^*}}{c_{L^*}}\frac{1}{q_{\pi^*}} + \left[q_{C^*} + \frac{\alpha s_C}{1-\alpha} - \frac{s_C}{\sigma_{\alpha\gamma}}\left(\frac{\alpha\gamma_{LC}}{1-\alpha} - \gamma_{FC}\right)\right]\frac{\left(\rho_{A^*} - 1\right)}{s^*_C}\left(q_{A^*} + \frac{c_{x^*}F^*_{A^*}}{q_{x^*F^*_{A^*}}c_{L^*}}\right)\right\}}_{\upsilon_{A^*}} \hat{A}^*_t$$
$$+ \underbrace{\left\{-\frac{s_C}{\sigma_{\alpha\gamma}}\gamma_{FC}\rho^* \frac{F^*_{G^*}}{c_{L^*}}\frac{1}{q_{\pi^*}} + \left[q_{C^*} + \frac{\alpha s_C}{1-\alpha} - \frac{s_C}{\sigma_{\alpha\gamma}}\left(\frac{\alpha\gamma_{LC}}{1-\alpha} - \gamma_{FC}\right)\right]\frac{\left(\rho_{G^*} - 1\right)}{s^*_C}\left(q_{G^*} - s^*_G + \frac{c_{x^*}F^*_{G^*}}{q_{x^*}c_{L^*}}\right)\right\}}_{\upsilon_{G^*}} \hat{G}^*_t$$
$$+ \underbrace{\left\{-\frac{s_C}{\sigma_{\alpha\gamma}}\gamma_{FC}\rho^* \frac{F^*_{Z^*}}{c_{L^*}}\frac{1}{q_{\pi^*}} + \left[q_{C^*} + \frac{\alpha s_C}{1-\alpha} - \frac{s_C}{\sigma_{\alpha\gamma}}\left(\frac{\alpha\gamma_{LC}}{1-\alpha} - \gamma_{FC}\right)\right]\frac{\left(\rho_{Z^*} - 1\right)}{s^*_C}\left(q_{Z^*} + \frac{c_{x^*}F^*_{Z^*}}{q_{x^*F^*_{Z^*}}c_{L^*}}\right)\right\}}_{\upsilon_{Z^*}} \hat{Z}^*_t$$
$$+ \underbrace{\left\{-\frac{s_C}{\sigma_{\alpha\gamma}}\gamma_{FC}\rho^* \frac{F^*_{W^*}}{c_{L^*}}\frac{1}{q_{\pi^*}} + \left[q_{C^*} + \frac{\alpha s_C}{1-\alpha} - \frac{s_C}{\sigma_{\alpha\gamma}}\left(\frac{\alpha\gamma_{LC}}{1-\alpha} - \gamma_{FC}\right)\right]\frac{\left(\rho_{W^*} - 1\right)}{s^*_C}\left(q_{W^*} + \frac{c_{x^*}F^*_{W^*}}{q_{x^*F^*_{W^*}}c_{L^*}}\right)\right\}}_{\upsilon_{W^*}} \hat{\mu}^{W*}_t$$

the parameters that refer to the rest of the world are defined in Appendix 2.L.

Note that the aggregate demand depends on the openness of the economy and the currency composition of the government's debt, and that when $\alpha = 0$ ($q^*_{C^*} = 0$) we recover the results obtained by Leeper and Zhou (2021) for a closed economy.

Rewrite the Euler equation as

$$\hat{x}_t = E_t\hat{x}_{t+1} + \gamma_{LC}\frac{s_C}{\sigma_{\alpha\gamma}}E_t\hat{\pi}_{H,t+1} - \frac{s_C}{\sigma_{\alpha\gamma}}\left(\hat{\tilde{R}}_t - \hat{\tilde{R}}^e_{\alpha,t}\right)$$

where $\hat{\tilde{R}}^e_{\alpha,t} \equiv \frac{\sigma_{\alpha\gamma}}{s_C}\upsilon_{\alpha,t}$ is the setting of the short-term interest rate that exactly offsets the composite demand-side shock.

The sensitivity of the expected output gap with respect to interest rate, $\frac{s_C}{\sigma_{\alpha\gamma}}$, depends on the openness of the economy and the currency composition of the government's debt. When the small open economy government only issues bonds in local currency, $\gamma_{LC} = 1$, an increase in openness raises that sensitivity (through the stronger effects of the induced terms of trade changes on demand). In the special case that $\sigma\eta = 1$, $\sigma_{\alpha\gamma} = \sigma$, so, regardless of the level of openness, the sensitivity of the expected output gap with respect to interest rate is the same as in the closed economy. When the government does not issue inflation-indexed debt but issues some amount in local and foreign currency, $\gamma_{CPI} = 0$ and $\gamma_{LC} \neq 0$ and $\gamma_{FC} \neq 0$, regardless of the level of openness, the sensitivity of the expected output gap with respect to interest rate is higher than in the closed economy, $\sigma_{\alpha\gamma} < \sigma$. Finally, when the small open economy government only issues bonds in foreign currency, $\gamma_{FC} = 1$, it takes the interest rate as given.

Appendix 2.N Linear Constraint 3: Government solvency condition

Define:

$$\frac{\hat{\tilde{B}}_t^M}{P_t} = \gamma_{LC}\left(\frac{\widehat{B_t^{M,LC}}}{P_t}\right) + \gamma_{CPI}\hat{b}_t^M + \gamma_{FC}\left(\frac{\widehat{\mathcal{E}_t B_t^{M,FC}}}{P_t}\right)$$
$$\hat{\tilde{Q}}_t^M = \gamma_{LC}\hat{Q}_t^{LC,M} + \gamma_{CPI}\hat{q}_t + \gamma_{FC}\hat{Q}_t^{M,FC}$$

with

$$\tilde{B}^M = B^M + Pb^M + \mathcal{E}B^{FC,M}$$
$$\gamma_{LC} = \frac{B^M}{\tilde{B}^M}, \qquad \gamma_{CPI} = \frac{Pb^M}{\tilde{B}^M}, \qquad \gamma_{FC} = \frac{\mathcal{E}B^{FC,M}}{\tilde{B}^M}$$

If the government issues local currency, inflation-indexed and foreign currency bonds with average maturity indexed by ρ, fiscal solvency implies the additional constraint

$$\left(\frac{\hat{\tilde{B}}^M_{t-1}}{P_{t-1}}\right) + \tilde{f}_t = \beta\left(\frac{\hat{\tilde{B}}^M_t}{P_t}\right) + (1-\beta)\frac{\bar{\tau}}{s_D}(\hat{x}_t + \hat{\tau}_t) + \gamma_{LC}\hat{\pi}_{H,t} + \beta(1-\rho)\hat{\tilde{Q}}^M_t$$
$$- \alpha\frac{\sigma}{s_C}(1-\beta)\hat{x}_t + \frac{\sigma}{s_C}\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\Delta\hat{x}_t$$

where $s_D \equiv \bar{S}/\bar{Y}$ is the steady-state surplus-to-output ratio and $\tilde{f}_t$ is the composite fiscal shock that reflects all eight exogenous disturbances, four domestic and four foreign, to the government's flow constraint

$$\tilde{f}_t = f_t + \left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\Delta f_{H,t} + \gamma_{FC}\hat{\pi}^*_t$$

$$f_t = \underbrace{-(1-\beta)\left(\frac{\bar{\tau}}{s_D} - \alpha\frac{\sigma}{s_C}\right)q_A}_{f_A}\hat{A}_t \underbrace{-(1-\beta)\left(\frac{\bar{\tau}}{s_D} - \alpha\frac{\sigma}{s_C}\right)q_{\mu W}}_{f_{\mu W}}\hat{\mu}^W_t$$
$$+ \underbrace{(1-\beta)\left[\frac{s_G - \bar{\tau}q_G}{s_D} - \alpha\frac{\sigma}{s_C}(s_G - q_G)\right]}_{f_G}\hat{G}_t + \underbrace{(1-\beta)\left(\frac{s_Z - \bar{\tau}q_Z}{s_D} + \frac{\sigma}{s_C}\alpha q_Z\right)}_{f_Z}\hat{Z}_t$$
$$- \underbrace{(1-\beta)\left[\frac{\bar{\tau}q_{C^*}}{s_D} + \alpha\frac{\sigma}{s_C}(s_C - q_{C^*})\right]\frac{1}{s^*_C}\left(\frac{c_{x^*}}{q_{x^*}c_{L^*}}F^*_{A^*} + q_{A^*}\right)}_{f_{A^*}}\hat{A}^*_t$$
$$- \underbrace{(1-\beta)\left[\frac{\bar{\tau}q_{C^*}}{s_D} + \alpha\frac{\sigma}{s_C}(s_C - q_{C^*})\right]\frac{1}{s^*_C}\left(\frac{c_{x^*}}{q_{x^*}c_{L^*}}F^*_{G^*} + q_{G^*} - s^*_G\right)}_{f_{G^*}}\hat{G}^*_t$$
$$- \underbrace{(1-\beta)\left[\frac{\bar{\tau}q_{C^*}}{s_D} + \alpha\frac{\sigma}{s_C}(s_C - q_{C^*})\right]\frac{1}{s^*_C}\left(\frac{c_{x^*}}{q_{x^*}c_{L^*}}F^*_{Z^*} + q_{Z^*}\right)}_{f_{Z^*}}\hat{Z}^*_t$$
$$- \underbrace{(1-\beta)\left[\frac{\bar{\tau}q_{C^*}}{s_D} + \alpha\frac{\sigma}{s_C}(s_C - q_{C^*})\right]\frac{1}{s^*_C}\left(\frac{c_{x^*}}{q_{x^*}c_{L^*}}F^*_{W^*} + q_{W^*}\right)}_{f_{W^*}}\hat{\mu}^{W*}_t$$

$$\Delta f_{H,t} = \underbrace{-\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\frac{\sigma}{s_C} q_A\left(\rho_A - 1\right)}_{\Delta f_{H,A}} \hat{A}_t + \underbrace{\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\frac{\sigma}{s_C}(s_G - q_G)\left(\rho_G - 1\right)}_{\Delta f_{H,G}} \hat{G}_t$$
$$\underbrace{-\frac{\sigma}{s_C}\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right] q_Z(\rho_Z - 1)}_{\Delta f_{H,Z}} \hat{Z}_t \underbrace{-\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\frac{\sigma}{s_C} q_{\mu_W}(\rho_W - 1)}_{\Delta f_{H,\mu^W}} \hat{\mu_W}_t$$
$$+\underbrace{\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\frac{\sigma}{s_C}(s_C - q_{C^*})\frac{1}{s_C^*}\left(\frac{c_{x^*}}{q_{x^*}c_{L^*}} F^*_{A^*} + q_{A^*}\right)\left(\rho_{A^*} - 1\right)}_{\Delta f_{H,A^*}} \hat{A}^*_t$$
$$+\underbrace{\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\frac{\sigma}{s_C}(s_C - q_{C^*})\frac{1}{s_C^*}\left(\frac{c_{x^*}}{q_{x^*}c_{L^*}} F^*_{G^*} + q_{G^*} - s^*_G\right)\left(\rho_{G^*} - 1\right)}_{\Delta f_{H,G^*}} \hat{G}^*_t$$
$$+\underbrace{\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\frac{\sigma}{s_C}(s_C - q_{C^*})\frac{1}{s_C^*}\left(\frac{c_{x^*}}{q_{x^*}c_{L^*}} F^*_{Z^*} + q_{Z^*}\right)\left(\rho_{Z^*} - 1\right)}_{\Delta f_{H,Z^*}} \hat{Z}^*_t$$
$$+\underbrace{\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\frac{\sigma}{s_C}(s_C - q_{C^*})\frac{1}{s_C^*}\left(\frac{c_{x^*}}{q_{x^*}c_{L^*}} F^*_{W^*} + q_{W^*}\right)\left(\rho_{W^*} - 1\right)}_{\Delta f_{H,W^*}} \hat{\mu}^{W*}_t$$

$$\hat{\pi}^*_t = -\frac{b^*_\pi}{q_{\pi^*}}\frac{F^*_{A^*}}{c_{L^*}}\hat{A}^*_t - \frac{b^*_\pi}{q_{\pi^*}}\frac{F^*_{G^*}}{c_{L^*}}\hat{G}^*_t - \frac{b^*_\pi}{q_{\pi^*}}\frac{F^*_{Z^*}}{c_{L^*}}\hat{Z}^*_t - \frac{b^*_\pi}{q_{\pi^*}}\frac{F^*_{W^*}}{c_{L^*}}\hat{\mu}^{W*}_t$$

with $b^*_\pi = 1 + (1-\beta^*)\frac{b^*_\tau}{\kappa^*\psi^*}$. The remaining parameters that refer to the rest of the world are defined in Appendix 2.L.

Naturally, the government solvency condition depends on the currency composition of the government's debt. Also, note that when $\alpha = 0$ ($q_{C^*} = 0$) we recover the results obtained by Leeper and Zhou (2021) for a closed economy.

Appendix 2.O Government's intertemporal equilibrium condition

Solving the government budget identity forward when the government issues some fraction of local currency bonds ($\gamma_{LC} \neq 0$), and imposing transversality and the term structure relation yields an intertemporal version of the solvency condition

$$\left(\frac{\hat{\tilde{B}}^{M}_{t-1}}{P_{t-1}}\right) + \tilde{F}_t = \gamma_{LC}\hat{\pi}_{H,t} - \frac{\sigma}{s_C}\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\hat{x}_{t-1} + \frac{\sigma_{\alpha\gamma}}{s_C}\hat{x}_t$$
$$+ (1-\beta)E_t\sum_{k=0}^{\infty}\beta^k\left[b_\tau\left(\hat{\tau}_{t+k} - \hat{\tau}^e_{t+k}\right) + b_x\hat{x}_{t+k}\right] + E_t\sum_{k=0}^{\infty}(\beta\rho)^{k+1}\left(\hat{\tilde{R}}_{t+k} - \hat{\tilde{R}}^e_{t+k}\right)$$

where $b_\tau = \frac{\bar{\tau}}{s_D}$, $b_x = \frac{\bar{\tau}}{s_D} - \frac{\sigma}{s_C}$ and

$$\tilde{F}_t = E_t\sum_{k=0}^{\infty}\beta^k\tilde{f}_{t+k} - (1-\beta)\frac{\bar{\tau}}{s_d}E_t\sum_{k=0}^{\infty}\beta^k\hat{\tau}^e_{t+k} + E_t\sum_{k=0}^{\infty}\left[\beta^{k+1} - (\beta\rho)^{k+1}\right]\hat{\tilde{R}}^e_{t+k}$$

When the government does not issue bonds in local currency, $\gamma_{LC} = 0$, and issues some fraction of inflation-indexed bonds, $\gamma_{CPI} > 0$ ($\gamma_{FC} \neq 1$), the intertemporal version of the solvency condition is

$$\left(\frac{\hat{\tilde{B}}^{M}_{t-1}}{P_{t-1}}\right) + \tilde{F}_t = -\gamma_{FC}\frac{\sigma}{s_C}(1-\alpha)\Delta\hat{x}_t + (1-\beta)E_t\sum_{k=0}^{\infty}\beta^k\left[b_\tau\hat{\tau}_{t+k} + \left(b_\tau - \alpha\frac{\sigma}{s_C}\right)\hat{x}_{t+k}\right]$$
$$+ E_t\sum_{k=0}^{\infty}\left[(\beta\rho)^{k+1} - \beta^{k+1}\left(1 + \frac{\gamma_{FC}}{1-\gamma_{FC}}\right)\right]\left(\hat{\tilde{R}}_{t+k} - \hat{\tilde{R}}^e_{t+k}\right)$$

And when the government only issues bonds in foreign currency, $\gamma_{FC} = 1$,

$$\left(\frac{B^{M,\hat{FC}}_{t-1}\mathcal{E}_{t-1}}{P_{t-1}}\right) + \tilde{F}_t = -\frac{\sigma}{s_C}(1-\alpha)\Delta\hat{x}_t + (1-\beta)E_t\sum_{k=0}^{\infty}\beta^k\left[b_\tau\hat{\tau}_{t+k} + \left(b_\tau - \alpha\frac{\sigma}{s_C}\right)\hat{x}_{t+k}\right]$$

with $\hat{\tilde{R}}^e_t = \hat{R}^*_t$.

Appendix 2.P Optimal Policy Analytics: Flexible prices

Consider the optimal equilibrium when prices are completely flexible, which eliminates the trade off between inflation and the output gap. The setup connects to V.V Chari, Christiano, and P. J. Kehoe (1996) and Varadarajan Chari and P. Kehoe (1999), except that they consider only real debt, while I also consider debt in local and foreign currency. Flexible prices emerge when $\theta = 0$, which implies $\kappa = \infty$ and $q_{\pi\alpha} = 0$. Costless inflation converts the loss function from (2.8) to

$$\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t q_x \hat{x}_t^2 \tag{2.120}$$

I examine two cases, one when the government issues some fraction of bonds denominated in local currency, and one when all the bonds are indexed to CPI and/or denominated in foreign currency. In the first case, unexpected inflation variations (which have no welfare cost under flexible prices) can be used to relax the intertemporal government solvency condition, but this is no longer possible in the second case. This makes the optimal solution to the problem intrinsically different.

- **Case 1: Some fraction of bonds in local currency** ($\gamma_{LC} \neq 0$)

When the government issues some bonds in local currency, $\gamma_{LC} \neq 0$, the optimal policy problem minimizes loss function (2.120) subject to the constraints

$$\hat{x}_t + \psi(\hat{\tau}_t - \hat{\tau}_t^e) = 0$$

$$\hat{x}_t + \frac{s_C}{\sigma_\alpha}\left(\hat{\tilde{R}}_t - \hat{\tilde{R}}_t^e - \gamma_{LC} E_t \hat{\pi}_{H,t+1}\right) - E_t \hat{x}_{t+1} = 0$$

$$\left(\frac{\hat{\tilde{B}}_{t-1}^M}{P_{t-1}}\right) + \tilde{F}_t = \gamma_{LC}\hat{\pi}_{H,t} - \frac{\sigma}{s_C}\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\hat{x}_{t-1} + \frac{\sigma_\alpha}{s_C}\hat{x}_t$$
$$+ (1-\beta)E_t\sum_{k=0}^{\infty}\beta^k\left[b_\tau\left(\hat{\tau}_{t+k} - \hat{\tau}_{t+k}^e\right) + b_x\hat{x}_{t+k}\right] + E_t\sum_{k=0}^{\infty}(\beta\rho)^{k+1}\left(\hat{\tilde{R}}_{t+k} - \hat{\tilde{R}}_{t+k}^e\right)$$

The optimal solution sets $\hat{x}_t = 0$ at all times, which can be achieved if fiscal policy follows $\hat{\tau}_t = \hat{\tau}_t^e$ and monetary policy sets $\hat{\tilde{R}}_t - \gamma_{LC} E_t \hat{\pi}_{H,t+1} = \hat{\tilde{R}}_t^e$. In this optimal policy assignment, fiscal policy stabilizes the output gap, and monetary policy stabilizes expected domestic inflation and the maturity structure of debt determines the timing of inflation. Equilibrium domestic inflation satisfies

$$\left(\frac{\hat{\tilde{B}}_{t-1}^M}{P_{t-1}}\right) + \tilde{F}_t = \gamma_{LC}\hat{\pi}_{H,t} + \gamma_{LC} E_t\sum_{k=1}^{\infty}(\beta\rho)^k\hat{\pi}_{H,t+k} \tag{2.121}$$

so increases in fiscal stress, $\left(\frac{\hat{\tilde{B}}_{t-1}^M}{P_{t-1}}\right) + \tilde{F}_t$, raise expected domestic inflation. When $\rho > 0$, (2.121) implies that long-term debt allows the government to trade off domestic inflation today for domestic inflation in the future. The longer the average maturity, the farther into the future inflation can be postponed. This conclusion resembles Cochrane (2001)'s optimal inflation-smoothing result.

When $\rho = 0$ and all bonds are one-period, (2.121) collapses to

$$\left(\frac{\hat{\tilde{B}}_{t-1}^M}{P_{t-1}}\right) + \tilde{F}_t = \gamma_{LC}\hat{\pi}_{H,t}$$

and, as Benigno and Woodford (2007) emphasize, "optimal policy will involve highly volatile and extreme sensitivity of inflation to fiscal shocks."

Then, in this case, unexpected inflation variations occur as needed in order to prevent taxes from ever having to be varied in order to respond to variations in fiscal stress, as in the analyses of Bohn (1990) and Varadarajan Chari and P. Kehoe (1999). This allows a model with only riskless nominal government debt to achieve the same state-contingent allocation of resources as the government would choose to bring about if it were to issue state-contingent debt, as in the model of Lucas and Stokey (1983).

Because taxes do not have to adjust in response to variations in fiscal stress, as in the tax-smoothing model of Barro (1979), it is possible to "smooth" them across states and over time. Note that it is really the "tax gap", $\hat{\tau}_t - \hat{\tau}_t^e$, that should be smoothed, rather than the tax rate itself.

- **Case 2: No bonds in local currency** ($\gamma_{LC} = 0$)

When the government does not issue bonds in local currency, the optimal policy problem minimizes loss function (2.120) subject to the constraints

$$\hat{x}_t + \psi(\hat{\tau}_t - \hat{\tau}_t^e) = 0 \tag{2.122}$$

$$\hat{x}_t + \frac{s_C}{\sigma_\alpha}\left(\hat{\tilde{R}}_t - \hat{\tilde{R}}_t^e\right) - E_t\hat{x}_{t+1} = 0 \tag{2.123}$$

$$\left(\frac{\hat{\tilde{B}}_{t-1}^M}{P_{t-1}}\right) + \tilde{f}_t = \beta\left(\frac{\hat{\tilde{B}}_t^M}{P_t}\right) + (1-\beta)\frac{\bar{\tau}}{s_D}(\hat{x}_t + \hat{\tau}_t) + \beta(1-\rho)\hat{\tilde{Q}}_t^M - \alpha\frac{\sigma}{s_C}(1-\beta)\hat{x}_t - \frac{\sigma}{s_C}(1-\alpha)\gamma_{FC}\Delta\hat{x}_t \tag{2.124}$$

$$\hat{\tilde{Q}}_t^M = \beta\rho E_t\hat{\tilde{Q}}_{t+1}^M - \hat{\tilde{R}}_t \tag{2.125}$$

The solution to this problem is now less trivial, as complete stabilization of the output gap is not generally possible. The optimal state-contingent evolution of output and taxes can be determined using a Lagrangian method.

In this case, variations in fiscal stress will require changes in the tax rate, as in the analysis of Barro (1979). With only inflation-indexed bonds and foreign currency bonds, the real value of private claims on the government at the beginning of period t, $\left(\frac{\hat{\tilde{B}}^M_{t-1}}{P_{t-1}}\right)$, is a predetermined variable. This means that unexpected inflation variations are no longer able to relax the intertemporal solvency condition of the government. In fact, we can observe from the system of equations above, (2.122)-(2.125), that the path of inflation is now completely irrelevant to welfare.

First-order conditions with respect to $\hat{x}_t$, $\hat{\tau}_t$, $\hat{r}_t$, $\left(\frac{\hat{\tilde{B}}^M_{t-1}}{P_{t-1}}\right)$ and $\hat{\tilde{Q}}^M_t$ yield optimality conditions

$$q_x\hat{x}_t = \left[\left(\frac{1}{\psi}-1\right)b_\tau + \alpha\frac{\sigma}{s_C} + (1-\alpha)\gamma_{FC}\frac{\sigma}{s_C}\right](1-\beta)L^b_t - \frac{\sigma_\alpha}{s_C}L^q_t + \frac{1}{\beta}\frac{\sigma_\alpha}{s_C}L^q_{t-1} \tag{2.126}$$

$$\beta(1-\rho)L^b_t = L^q_t - \rho L^q_{t-1} \tag{2.127}$$

$$E_tL^b_{t+1} = L^b_t \tag{2.128}$$

where L^b_t and L^q_t are Lagrange multipliers corresponding to (2.124) and (2.125).

The optimal evolution of output gap and tax rate are given by

$$q_x\hat{x}_t = (1-\beta)\left[\left(\frac{1}{\psi}-1\right)b_\tau + \alpha\frac{\sigma}{s_C} + (1-\alpha)\gamma_{FC}\frac{\sigma}{s_C}\right]L^b_t - \frac{\sigma_\alpha}{s_C}\beta\left(\frac{1-\rho}{1-\rho\mathbb{L}}\right)(1-\beta^{-1}\mathbb{L})L^b_t \tag{2.129}$$

$$\hat{\tau}_t = \hat{\tau}^e_t - \frac{1}{\psi}\hat{x}_t \tag{2.130}$$

The evolution of inflation remains indeterminate.

The optimality condition for debt that requires L^b_t to be a martingale may be written as $(1-\mathbb{B})E_{t-1}L^b_t = 0$, where $\mathbb{B}$ is the backshift operator, defined as $\mathbb{B}^{-j}E_t\xi_t \equiv E_t\xi_{t+j}$.

Taking expectations of (2.28) and (2.29) and applying the backshift operator, we obtain general expressions for the k-step-ahead expectations of inflation and output gap

$$E_t\hat{\pi}_{H,t+k} = \rho^k\hat{\pi}_{H,t} + \gamma_{LC}\rho^k\alpha_\pi(L_t^b - L_{t-1}^b) \tag{2.131}$$

$$E_t\hat{x}_{t+k} = \rho^k\hat{x}_t + (1-\rho^k)\alpha_x L_t^b \tag{2.132}$$

where $\alpha_\pi = \frac{1-\beta}{\kappa\psi q_{\pi\alpha}} b_\tau$ and $\alpha_x = \frac{1-\beta}{q_x}\left(\frac{b_\tau}{\psi} - b_x\right)$.

Without local currency bonds, inflation is not affected by a pure fiscal shock under the optimal policy, but instead the output gap and the tax rates are. Optimal policy "smooths" L_t^b, the value (in units of marginal utility) of additional government revenue in period t, so that it follows a random walk. As highlighted by Benigno and Woodford (2004), this is the proper generalization of the Barro tax-smoothing result, though it only implies smoothing of tax rates in fairly special cases. A similar result is found when prices are sticky, even when government debt is denominated in local currency, as I show below.

Note that when the government issues only foreign currency bonds, $\gamma_{FC} = 0$, the optimal evolution of output gap is given by

$$\hat{x}_t = \frac{1}{q_x}(1-\beta)\left(\frac{b_\tau}{\psi} - b_x\right)L_t^b \tag{2.133}$$

Condition (2.133) makes the output gap proportional to L_t^b, so the gap inherits the martingale property of L_t^b to perfectly smooth the gap: $E_t\hat{x}_{t+k} = \hat{x}_t$.

Flexible prices neglect the welfare costs of domestic inflation. When prices are sticky and domestic inflation volatility is costly, the optimal allocation balances variations in domestic inflation against variations in the output gap.

Appendix 2.Q Sticky Prices: Derivation of the optimality conditions

Under sticky prices and when the government issues some fraction of its bonds in local currency, policy seeks paths for $\left\{\hat{\pi}_{H,t}, \hat{x}_t, \hat{\tau}_t, \hat{\tilde{R}}_t, \hat{\tilde{B}}_t^M, \hat{\tilde{Q}}_t^M\right\}$ that minimize (2.8) subject to (2.9)-(2.12). The Lagrangian for this problem when the government issues some bonds in local currency, $\gamma_{LC} \neq 0$, can be written as

$$
\begin{aligned}
\mathcal{L}_0 = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \frac{1}{2}\left(q_{\pi\alpha}\hat{\pi}_{H,t}^2 + q_x \hat{x}_t^2\right) - L_t^{\pi}\left[\hat{\pi}_{H,t} - \beta E_t \hat{\pi}_{t+1}^H - \kappa \hat{x}_t - \kappa\psi\left(\hat{\tau}_t - \hat{\tau}_t^e\right)\right] \\
& - L_t^x \left[\hat{x}_t - E_t \hat{x}_{t+1} - \gamma_{LC}\frac{s_C}{\sigma_\alpha} E_t \hat{\pi}_{H,t+1} + \frac{s_C}{\sigma_\alpha}\left(\hat{\tilde{R}}_t - \hat{\tilde{R}}_t^e\right)\right] \\
& - L_t^b \Bigg\{ \left(\frac{\hat{\tilde{B}}_{t-1}^M}{P_{t-1}}\right) + \tilde{f}_t - \gamma_{LC}\hat{\pi}_{H,t} - \beta(1-\rho)\hat{\tilde{Q}}_t^M - \beta\left(\frac{\hat{\tilde{B}}_t^M}{P_t}\right) - (1-\beta)\frac{\bar{\tau}}{s_D}(\hat{x}_t + \hat{\tau}_t) \\
& + \alpha\frac{\sigma}{s_C}(1-\beta)\hat{x}_t - \frac{\sigma}{s_C}\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\Delta\hat{x}_t \Bigg\} \\
& - L_t^q \left[\hat{\tilde{Q}}_t^M - \beta\rho E_t \hat{\tilde{Q}}_{t+1}^M + \hat{\tilde{R}}_t\right] \Bigg\}
\end{aligned}
$$

First-order conditions with respect to $\hat{\pi}_{H,t}$, $\hat{x}_t$, $\hat{\tau}_t$, $\hat{\tilde{R}}_t$, $\hat{\tilde{B}}_t^M$ and $\hat{\tilde{Q}}_t^M$ yield optimality conditions

$$\hat{\pi}_{H,t}\rfloor\;\; \beta^t\left(q_{\pi\alpha}\hat{\pi}_{H,t} - L^\pi_t + \gamma_{LC}L^b_t\right) + \beta^{t-1}\left(\beta L^\pi_{t-1} + \gamma_{LC}L^x_{t-1}\frac{s_C}{\sigma_\alpha}\right) = 0$$

$$\Rightarrow q_{\pi\alpha}\hat{\pi}_{H,t} = L^\pi_t - L^\pi_{t-1} - \gamma_{LC}L^b_t - \gamma_{LC}\frac{s_C}{\sigma_\alpha}\frac{1}{\beta}L^x_{t-1}$$

$$\hat{x}_t\rfloor\;\; -\beta^{t+1}E_tL^b_{t+1}\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\frac{\sigma}{s_C} + \beta^t\left\{q_x\hat{x}_t + \kappa L^\pi_t - L^x_t + L^b_t(1-\beta)\frac{\bar{\tau}}{s_D}\right.$$

$$\left. -\alpha L^b_t\frac{\sigma}{s_C}(1-\beta) + L^b_t\frac{\sigma}{s_C}\left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\right\} + \beta^{t-1}L^x_{t-1} = 0$$

$$\Rightarrow q_x\hat{x}_t = \left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\frac{\sigma}{s_C}\left(\beta E_tL^b_{t+1} - L^b_t\right) - \kappa L^\pi_t + L^x_t - L^b_t(1-\beta)\left(\frac{\bar{\tau}}{s_D} - \alpha\frac{\sigma}{s_C}\right)$$

$$-\beta^{-1}L^x_{t-1}$$

$$\hat{\tau}_t\rfloor\;\; \kappa\psi L^\pi_t + (1-\beta)\frac{\bar{\tau}}{s_D}L^b_t = 0 \Rightarrow L^\pi_t = -\frac{1-\beta}{\kappa\psi}\frac{\bar{\tau}}{s_D}L^b_t$$

$$\hat{\tilde{R}}_t\rfloor\;\; -L^x_t\frac{s_C}{\sigma_\alpha} - L^q_t = 0 \Rightarrow L^x_t = -\frac{\sigma_\alpha}{s_C}L^q_t$$

$$\hat{\tilde{Q}}^M_t\rfloor\;\; \beta^t\left[\beta(1-\rho)L^b_t - L^q_t\right] + \beta^t\rho L^q_{t-1} = 0 \Rightarrow \beta(1-\rho)L^b_t - L^q_t + \rho L^q_{t-1} = 0$$

$$\left(\frac{\hat{\tilde{B}}^M_t}{P_t}\right)\rfloor\;\; -\beta^{t+1}E_tL^b_{t+1} + \beta^{t+1}L^b_t = 0 \Rightarrow E_tL^b_{t+1} - L^b_t = 0$$

Then

$$q_{\pi\alpha}\hat{\pi}_{H,t} = -\frac{1-\beta}{\kappa\psi}b_\tau(L^b_t - L^b_{t-1}) - \gamma_{LC}L^b_t + \gamma_{LC}\frac{1}{\beta}L^q_{t-1} \tag{2.134}$$

$$q_x\hat{x}_t = \left\{\left(\frac{1}{\psi} - 1\right)b_\tau + \alpha\frac{\sigma}{s_C} - \left[\alpha\gamma_{LC} - (1-\alpha)\gamma_{FC}\right]\frac{\sigma}{s_C}\right\}(1-\beta)L^b_t - \frac{\sigma_\alpha}{s_C}L^q_t + \frac{1}{\beta}\frac{\sigma_\alpha}{s_C}L^q_{t-1} \tag{2.135}$$

$$\beta(1-\rho)L^b_t = L^q_t - \rho L^q_{t-1} \tag{2.136}$$

$$E_tL^b_{t+1} = L^b_t \tag{2.137}$$

where L^b_t and L^q_t are Lagrange multipliers corresponding to (2.11) and (2.12).

Chapter 3

Optimal Time-Consistent Monetary Policy and Sovereign Debt Currency Denomination

3.1 Introduction

Historically, external sovereign debt in emerging economies has largely been denominated in foreign currency, a phenomenon known in the international finance literature as the "original sin" (Eichengreen and Hausmann 1999). However, over the past two decades, the share of external sovereign debt denominated in foreign currency has evidenced a reduction (Arslanalp and Tsuda 2014). In this chapter I address two questions. Has the debt denominated in foreign currency been replaced by debt denominated in local currency or indexed? What are the policy implications of these types of bonds and how does the government choose the currency denomination of its debt?

To answer the first question, I construct a dataset on foreign holdings of government debt issued internationally and domestically for five Latin American countries: Brazil, Colombia, Mexico, Peru and Uruguay. I follow Arslanalp and Tsuda (2014) and extend their work by distinguishing between debt denominated in local currency and indexed units. I find that in Brazil, Colombia, Mexico and Peru, external debt in foreign currency was mainly substituted by debt in local currency. In Uruguay, on the contrary, debt in foreign currency was largely substituted by inflation-indexed debt.

To answer the second question, I use a small open economy model in which inflation is costly and exchange rate is determined endogenously. Each period, a government that cannot default and lacks commitment regarding its monetary policy and bond issuance, optimally chooses inflation and the issuance of debt denominated in local currency, foreign currency and inflation-indexed units. I follow Ottonello and Perez (2019)'s analytical framework and extend it by introducing inflation-indexed debt to the government's portfolio choice.

Firstly, I analyze the trade-offs of each policy instrument by looking at the first order conditions of the problem. On one hand, local currency bonds and inflation-indexed bonds are useful tools in terms of hedging. As exchange rate (nominal and real) and tradable endowment are negatively correlated, the payoffs of these bonds decline in real terms when tradable endowment falls, helping to smooth tradable consumption. On the other hand, when unable to commit, the government has an ex-post incentive to generate surprise inflation and real exchange rate depreciation to dilute the real value of its local currency (both) and inflation-indexed bonds (only real exchange rate depreciation). Foreign investors anticipate these incentives, lowering the price for local currency and inflation-indexed debt. The government takes this into account, and chooses a lower amount of debt in local currency and inflation-indexed units than it will choose under commitment.

Secondly, I calibrate the model to the average Latin American country in the sample, Peru and Uruguay, and solve the model numerically. I find that large external debt-to-GDP ratios, long debt duration, and low inflation costs encourage more borrowing in inflation-indexed bonds and can explain the larger share of inflation-indexed debt in Uruguay compared to other Latin American countries.

In addition to Ottonello and Perez (2019), some recent papers examine the currency composition of sovereign debt, considering local and foreign currency bonds. S. Liu, Ma, and Shen (2021) embed the debt currency denomination in a sudden stop model, while Engel and Park (2022) analyze an optimal contract problem with outright default. These papers emphasize the hedging benefits and the inflationary bias of local currency bonds, but none of them incorporate inflation-indexed bonds as a policy tool. The innovation of this paper is to study empirically and theoretically the role of inflation-indexed bonds as an additional policy instrument for the government.

The chapter proceeds as follows. The answer to the first (empirical) research question is provided in Section 3.2. To answer to the second (theoretical) research question, I first describe the model in Section 3.3, introduce the optimal policy problem in Section 3.4, and present the quantitative analysis in Section 3.5. Section 4 concludes.

3.2 Data

This section presents the main facts about the currency composition of external sovereign debt for five Latin American countries: Brazil, Colombia, Mexico, Peru and Uruguay. The countries were selected based on the size of the economy -they represent 70% of Latin America's gross domestic output- and data availability. The objective is to analyze how large debt positions in local currency and in inflation-indexed units are relative to foreign currency and how these relative positions have evolved over the last decades and across countries.

I construct a dataset on foreign holdings of government debt issued internationally

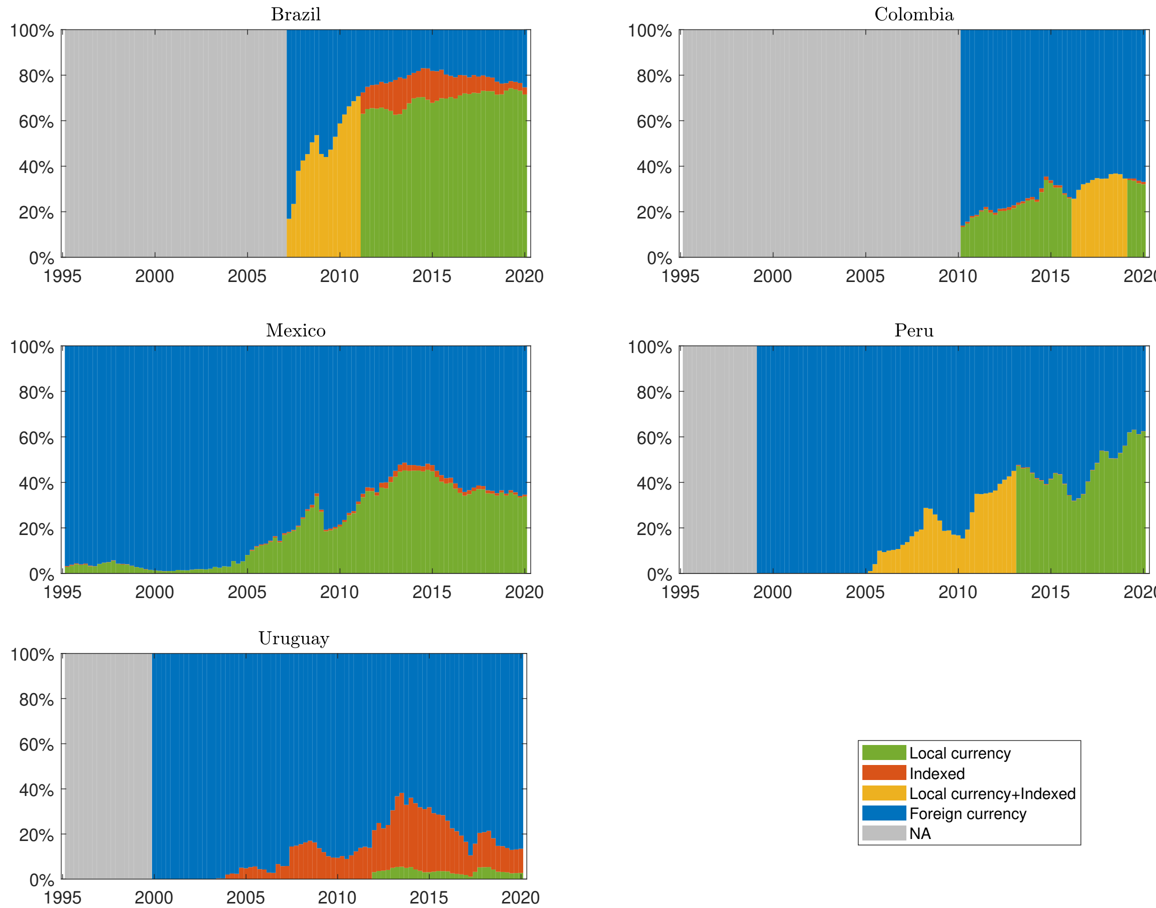

Figure 1: Currency composition of foreign holdings of sovereign debt.

and domestically. For this purpose, I follow Arslanalp and Tsuda (2014) and extend their work by distinguishing between debt denominated in local currency and inflation-indexed units, which will be shown to have different characteristics and policy implications. In the case of Brazil, the data corresponds to Federal Public debt obtained from the monthly public debt reports, elaborated by the National Treasury. For Colombia, the data corresponds to Central Government debt obtained from the Ministry of Finance. In the case of Mexico, the data corresponds to Federal Public debt, obtained from the Ministry of Finance, for the internationally issued debt, and from the Central Bank of Mexico, for the domestically issued debt held by nonresidents. For Peru, the data corresponds to Non-Monetary Public Sector debt. The data that corresponds to internationally issued debt was obtained from the Central Bank of Peru, and the data for domestically issued debt from the Ministry of Finance. In the case of Uruguay, the data corresponds to Non-Monetary Public Sector debt, and it was obtained from the Central Bank of Uruguay. (See Appendix 3.A for more details) The frequency and time-period of the data varies considerably across countries. For instance, the estimates for Mexico are constructed on a monthly basis starting in 1991, whereas the estimates for Colombia are quarterly and start in 2010.

Figure 1 depicts the dynamics of the currency composition of government debt held by foreigners in the five Latin American countries under study. Debt composition is tilted towards foreign currency, a well documented and studied characteristic of emerging economies. More than half of the foreign holdings of government debt are denominated in foreign currency, except for Brazil after 2010 and Peru in recent years. Moreover, in the last two decades there has been a reduction in the share of foreign currency debt, another fact more recently highlighted in the literature (see for example Ottonello and Perez 2019, S. Liu, Ma, and Shen 2021, and Engel and Park 2022). An innovation of this paper is to study the evolution of indexed debt. For the countries under study, this type of debt is indexed to the evolution of aggregate inflation, is issued domestically and its share is small compared to local currency debt. The exception is Uruguay, which most of its non-foreign currency denominated debt corresponds to inflation-indexed debt and some of it is issued internationally. Brazil also registered a significant share of inflation-indexed debt over a period of time. It reached 16% in 2013, but it fell to 3% by the end of the sample.

In the following sections, I build a model and derive analytical and numerical results to explain the differences across countries documented in this section.

3.3 Model

This section introduces a model to study optimal monetary policy and debt currency denomination. It closely follows Ottonello and Perez (2019), augmented to include inflation-indexed debt. A small open economy is populated by a continuum of identical risk-averse households that live forever, consume tradable and non-tradable goods, receive a stochastic endowment and lump-sum transfers net of taxes from the government, but cannot borrow or save. The government, who chooses inflation and debt issuance, is benevolent but lacks commitment and cannot default. The government can issue debt denominated in foreign currency, local currency, and indexed to inflation, which is priced by risk-neutral foreign investors that also have access to a risk-less bond denominated in foreign currency.

3.3.1 Households

In the small open economy, the preferences of the representative household are defined over an infinite stream of consumption and inflation:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_t) - l(\pi_t) \right) \tag{3.1}$$

where $\beta \in (0, 1)$ is the subjective discount factor, c_t denotes consumption, and $\pi_t \equiv P_t/P_{t-1}$ is the gross inflation rate, with P_t the aggregate price level. $u : \mathbb{R}_+ \to \mathbb{R}$ is a differentiable, increasing and concave function, and $l : \mathbb{R}_+ \to \mathbb{R}$ is a differentiable and convex function. The disutility from inflation captures the distortionary costs associated with this variable. In addition, the consumption good is a composite of tradable goods, $c_{T,t}$, and non-tradable goods, $c_{N,t}$

$$c_t = C(c_{T,t}, c_{N,t}) \tag{3.2}$$

where $C : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a differentiable function, increasing in both arguments, concave, and homogeneous of degree one.

The representative household receives a stream of endowments of tradable goods,

$y_{T,t}$, and non-tradable goods, $y_{N,t}$, and lump-sum transfers (net of taxes) from the government, T_t. Each period, the household faces the following budget constraint

$$P_{T,t}c_{T,t} + P_{N,t}c_{N,t} = P_{T,t}y_{T,t} + P_{N,t}y_{N,t} + T_t \tag{3.3}$$

where $P_{T,t}$ and $P_{N,t}$ are the prices of tradable and non-tradable goods, respectively, measured in local currency.

The representative household chooses state-contingent consumption plans, $\{c_t, c_{T,t}, c_{N,t}\}_{t=0}^{\infty}$, that maximize the lifetime utility (3.1), subject to the aggregation technology (3.2), the sequence of period budget constraints (3.3), and the given sequences of prices, $\{P_{T,t}, P_{N,t}\}_{t=0}^{\infty}$, endowments and government lump-sum transfers, $\{y_{T,t}, y_{N,t}, T_t\}_{t=0}^{\infty}$. The optimal choice between tradable and non-tradable goods is obtained by taking the ratio of the first order conditions (FOCs)

$$\frac{C_{c_{T,t}}}{C_{c_{N,t}}} = \frac{P_{T,t}}{P_{N,t}} \tag{3.4}$$

where $f_{x_i} = \partial f(x_1, ..., x_i, ..., x_n)/\partial x_i$.

3.3.2 Government

The government chooses inflation and external debt. There are three types of bonds available to the government: denominated in foreign currency, b_t^*, local currency, B_t, and indexed to inflation, b_t^π. Each bond is sold at price Q_t^*, Q_t, and Q_t^π, respectively. Following Woodford (2001), I assume that the government bonds are actually a portfolio of many bonds that pay a declining coupon of δ^i local currency, inflation-indexed or foreign currency units, $j+1$ periods after they were issued, where $0 < \delta \leq \beta^{-1}$.[1] A measure of the duration of the bond is given by $(1-\beta\delta)^{-1}$, which

[1]By issuing one bond in local currency (inflation-indexed or foreign currency) in period t, the government promises to repay one unit of local currency (inflation-indexed or foreign currency) in period $t+1$, δ in period $t+2$, δ^2 in period $t+3$, and so on, and in exchange receives Q_t (Q_t^π or Q_t^*)

is used to calibrate δ to capture the observed maturity structure of government debt. Note that when $\delta = 0$, the bonds are reduced to one period bonds.

The government budget constraint expressed in local currency is then given by

$$e_t Q_t^* b_t^* + P_t \left(Q_t b_t + Q_t^\pi b_t^\pi \right) = e_t (1 + \delta Q_t^*) b_{t-1}^* + P_t \left[(1 + \delta Q_t) \frac{b_{t-1}}{\pi_t} + (1 + \delta Q_t^\pi) b_{t-1}^\pi \right] + T_t \tag{3.5}$$

where $Q_t b_t = Q_t B_t / P_t$ is real value of local currency bonds, and e_t is the nominal exchange rate, i.e. the price of foreign currency in terms of local currency.

3.3.3 Rest of the World

The rest of the world is populated by a continuum of identical risk-neutral households that have access to the bonds issued by the small open economy's government and a risk-free bond denominated in foreign currency that pays gross international rate R. Bond prices are then given by

$$Q_t^* = \frac{1}{R - \delta} \tag{3.6}$$

$$Q_t = \frac{1}{R} E_t \left[\frac{e_t}{e_{t+1}} (1 + \delta Q_{t+1}) \right] \tag{3.7}$$

$$Q_t^\pi = \frac{1}{R} E_t \left[\frac{e_t}{e_{t+1}} \frac{P_{t+1}}{P_t} (1 + \delta Q_{t+1}^\pi) \right] = \frac{1}{R} E_t \left[\frac{r_t}{r_{t+1}} (1 + \delta Q_{t+1}^\pi) \right] \tag{3.8}$$

where $r_t = e_t / P_t$ is the real exchange rate (RER).

3.3.4 Equilibrium

In equilibrium, the market for non-tradable goods clears

$$c_{N,t} = y_{N,t} \tag{3.9}$$

units of local currency (inflation-indexed or foreign currency) in period t.

Assuming the law of one price holds for tradable goods ($P_{T,t} = e_t P^*_{T,t}$) and normalizing the international price of tradable goods to one, we get that $P_{T,t} = e_t$. Given the assumed preferences (3.1) and the aggregation technology (3.2), the aggregate price level of the small open economy is given by $P_t = e_t \left(C_{c_T} \left(\frac{c_{T,1}}{y_{N,t}}, 1 \right) \right)^{-1}$.[2] For later use, we define the inverse of the equilibrium nominal exchange rate, n, as

$$e_t^{-1} = n\left(P_t, \frac{c_{T,t}}{y_{N,t}}\right), \qquad n\left(P_t, \frac{c_{T,t}}{y_{N,t}}\right) = \frac{1}{P_t}\left(C_{c_T}\left(\frac{c_{T,t}}{y_{N,t}}, 1\right)\right)^{-1} \tag{3.10}$$

where $n\left(P_t, \frac{c_{T,t}}{y_{N,t}}\right)$ is a differentiable function, decreasing in its first argument and increasing in its second argument. Note that

$$\left(\frac{e_t}{P_{t-1}}\right)^{-1} = n\left(\pi_t, \frac{c_{T,t}}{y_{N,t}}\right), \qquad r_t^{-1} = n\left(1, \frac{c_{T,t}}{y_{N,t}}\right)$$

Aggregating in the small open economy households and government budget constraints, (3.3) and (3.5), imposing the market clearing condition in the non-tradable sector (3.9), and using the definition of the inverse of the equilibrium nominal rate (3.10), we obtain the resource constraint of the small open economy, expressed in units of tradable consumption

$$\begin{aligned} c_{T,t} = y_{T,t} - b^*_{t-1} - n\left(\pi_t, \frac{c_{t,T}}{y_{N,t}}\right) b_{t-1} - n\left(1, \frac{c_{t,T}}{y_{N,t}}\right) b^\pi_{t-1} \\ + Q^*_t\left(b^*_t - \delta b^*_{t-1}\right) + n\left(1, \frac{c_{t,T}}{y_{N,t}}\right)\left[Q_t\left(b_t - \delta\frac{b_{t-1}}{\pi_t}\right) + Q^\pi_t\left(b^\pi_t - \delta b^\pi_{t-1}\right)\right] \end{aligned} \tag{3.11}$$

Given initial levels of government debt b^*_{-1}, b_{-1}, and b^π_{-1}, a state-contingent sequence of endowments $\{y_{T,t}, y_{N,t}\}_{t=0}^\infty$, and government policies $\{b^*_t, b_t, b^\pi_t\}_{t=0}^\infty$, a competitive equilibrium is a state-contingent sequence of private sector allocations $\{c_{T,t}, c_{N,t}\}_{t=0}^\infty$ and prices $\{Q^*_t, Q_t, Q^\pi_t, e_t, \pi_t\}_{t=0}^\infty$, such that: private allocations solve the household's problem given equilibrium prices, transfers (net of taxes) satisfy the government

[2] See Uribe and Schmitt-Grohé (2017).

budget constraint (3.5), debt prices satisfy (3.6), (3.7) and (3.8), and market for non-tradable goods clears (3.9).

3.4 Optimal Policy

This section outlines the policy problem of the government, which has to choose a set of government policies $\{\pi_t, b_t^*, b_t, b_t^\pi\}_{t=0}^\infty$ in order to maximize the utility of the representative household (3.1), subject to the constraints implied by the competitive equilibrium defined in the previous section. Before introducing the time-consistent problem, which is the focus of this analysis, I briefly lay out the problem under commitment to use as a benchmark. We focus on the notion of a Markov perfect equilibrium in which policies depend on payoff-relevant states. In other words, the solution to the system of equations that defines the policy problem will be a set of time-invariant Markov-perfect equilibrium policy rules mapping the vector of states to the optimal decisions for each variable and for all moment in time. Regarding the sequence of endowments, I assume that y_N is constant over time while y_T follows a Markov process with transition probability $g_y(y_T, y_T')$.

3.4.1 Commitment and Discretion

Commitment

When the government can commit to future policies, the policy problem is given by choosing $\{c_{T,t}, \pi_t, b_t^*, b_t, b_t^\pi\}$ in order to maximize household's lifetime utility (3.1), subject to the resource constraint (3.11) and bond prices (3.6, 3.7, 3.8). By committing to an entire path of policy instruments, the government is able to influence expectations in order to improve the policy trade-offs they face. The Lagrangian and the resultant set of FOCs are derived in Appendix A.

The commitment equilibrium is determined by the system given by the FOCs

(3.32) in Appendix 3.B.1, the constraints (3.11, 3.6, 3.7, 3.8) and the exogenous process for the endowment of tradable and nontradable goods.

Discretion

When the government lacks commitment, it seeks to maximize the value function:

$$V(s_t) = \max_{c_{T,t},\pi_t,b_t^*,b_t,b_t^\pi} u\left(C\left(c_{T,t}, y_N\right)\right) - l(\pi_t) + \beta E_t\left[V(s_{t+1})\right] \tag{3.12}$$

subject to the resource constraint (3.11) and bond prices (3.6, 3.7, 3.8). Even though the government chooses the same variables than under commitment, it cannot make time-inconsistent promises about their future behavior to have a beneficial impact on current policy trade-offs through expectations. Instead, foreign lenders anticipate the incentives faced by the government each period and form expectations accordingly. However, the government can still influence those expectations by affecting the states the next period's government inherits. To capture this, future expectations are replaced by the following state-dependent auxiliary functions:

$$\mathcal{X}(s_{t+1}) = n\left(\pi_{t+1}, \frac{c_{T,t+1}}{y_N}\right)(1 + \delta Q_{t+1}) \tag{3.13}$$

$$\mathcal{Z}(s_{t+1}) = n\left(1, \frac{c_{T,t+1}}{y_N}\right)(1 + \delta Q^\pi_{t+1}) \tag{3.14}$$

in the local currency and indexed bond pricing equations, respectively. These auxiliary functions reflect the fact that, in equilibrium, we can map exogenous variables to the steady state and expectations are formed rationally based on that mapping. The current government takes this into account when setting its policy. The price of local currency and inflation-indexed debt are determined by the following recursive equations

$$Q(s_t) = \frac{1}{n\left(1, \frac{c_{T,t}}{y_N}\right)} \frac{1}{R} E_t[\mathcal{X}(s_{t+1})] \tag{3.15}$$

$$Q^{\pi}(s_t) = \frac{1}{n\left(1, \frac{c_{T,t}}{y_N}\right)} \frac{1}{R} E_t[\mathcal{Z}(s_{t+1})] \tag{3.16}$$

The discretionary equilibrium is determined by the system given by the FOCs (3.34) in Appendix 3.B.1; the resource constraint (3.11); auxiliary equations (3.13) and (3.14); bond prices (3.15) and (3.16); and the exogenous process for the endowment of tradable goods. Using the notion of Markov perfect equilibrium, the solution to this system is a set of time-invariant policy rules $x_t = H(s_t)$ mapping the vector of states $s_t = \{b^*_{t-1}, b_{t-1}, b^{\pi}_{t-1}, y_{T,t}\}$ to the optimal decisions for $x_t = \{c_{T,t}, \pi_t, b^*_t, b_t, b^{\pi}_t, Q^*_t, Q_t, Q^{\pi}_t\}$ for all $t \geq 0$.

3.4.2 Policy Trade-Offs

The policy trade-offs faced by the government can be characterized by analyzing the FOCs.

Dilution through real exchange rate

First, consider the FOC for tradable consumption

$$
\begin{aligned}
&u'(c_t)C_{c_T,t} - \lambda_t \underbrace{\left[1 + n_c\left(\pi_t, \frac{c_{T,t}}{y_N}\right) b_{t-1} + n_c\left(1, \frac{c_{T,t}}{y_N}\right) b^{\pi}_{t-1}\right]}_{\text{Dilution through RER}} \\
&\underbrace{-\lambda_t \left\{1 - n_c\left(1, \frac{c_{T,t}}{y_N}\right)\left[Q_t\left(b_t - \delta\frac{b_{t-1}}{\pi_t}\right) + Q^{\pi}_t\left(b^{\pi}_t - \delta b^{\pi}_{t-1}\right)\right]\right\}}_{\text{Only under commitment}} \\
&\underbrace{+\mu_t \frac{1}{R}\frac{n_c\left(\pi_t, \frac{c_{T,t}}{y_N}\right)}{n\left(1, \frac{c_{T,t}}{y_N}\right)} Q_t + \mu^{\pi}_t \frac{1}{R}\frac{n_c\left(1, \frac{c_{T,t}}{y_N}\right)}{n\left(1, \frac{c_{T,t}}{y_N}\right)} Q^{\pi}_t}_{\text{Only under commitment}} \\
&\underbrace{-\frac{1}{\beta}\frac{1}{R}\frac{1}{n\left(1, \frac{c_{T,t-1}}{y_N}\right)}\left[\mu_{t-1} n_c\left(\pi_t, \frac{c_{T,t}}{y_N}\right)(1 + \delta Q_t) + \mu^{\pi}_{t-1} n_c\left(1, \frac{c_{T,t}}{y_N}\right)(1 + \delta Q^{\pi}_t)\right]}_{\text{Only under commitment}} = 0
\end{aligned} \tag{3.17}
$$

Lines two to four in expression (3.17) are only present under commitment and reflect the fact that the government can directly affect expectations and therefore bond prices. The first line is present both under commitment and discretion. Under discretion and with only foreign currency bonds, the marginal benefit of raising one additional unit of good with debt is $u'(c_t)C_{c_T,t}$. But in the presence of local currency or inflation-indexed bonds, increasing one additional unit of good with debt increases tradable consumption by less than one, i.e. the denominator in (3.17) is greater than one. This is because higher tradable consumption appreciates the real exchange rate ($n_c(1, c_{T,t}/y_N) > 0$), which increases the value of local currency or indexed debt repayment and decreases tradable consumption. This creates an incentive for the government to depreciate the real exchange rate ex-post to dilute the real value of debt repayment. To do it, tradable consumption has to be postponed, result that can be attained by issuing less debt at the cost of distorting intertemporal consumption decisions.

Dilution through inflation

Second, consider the FOC for inflation, which highlights the nature of the inflationary bias contained in the model,

$$-l'(\pi_t) = \lambda_t \underbrace{n_\pi\left(\pi_t, \frac{c_{T,t}}{y_N}\right)(1+\delta Q_t)b_{t-1}}_{\text{Dilution through inflation}} + \underbrace{\mu_{t-1}\frac{1}{\beta R}n_\pi\left(\pi_t, \frac{c_{T,t}}{y_N}\right)(1+\delta Q_t)r_{t-1}}_{\text{Only under commitment}} \quad (3.18)$$

The left-hand-side measures the cost of raising inflation. The terms on the right-hand-side of (3.18) measure the benefits of raising inflation, and are only present when there is a non-zero stock of debt in local currency. The first term captures the fact that higher inflation dilutes the value of outstanding debt in local currency measured in units of tradable consumption and enables a higher consumption by saving resources. This benefit is higher, the higher the stock and the longer the maturity of debt in local currency. The second term, only present under commitment, captures the government's promise not to use inflation to reduce bond prices. Then, under discretion, ex-post the government has an incentive to increase inflation.

Hedging benefits and costs derived from the lack of commitment

Third, consider the FOC for each type of bond:

$$
\begin{aligned}
[b_t^*]: \quad & \underbrace{\beta E_t\left[\lambda_{t+1}\right]\left(1+\frac{\delta}{R-\delta}\right)-\lambda_t\frac{1}{R-\delta}}_{\text{Consumption smoothing}} \\
& \underbrace{-\lambda_t\left\{\frac{1}{R}E_t[\mathcal{X}_{b^*}(s_{t+1})]\left(b_t-\delta\frac{b_{t-1}}{\pi}\right)+\frac{1}{R}E_t[\mathcal{Z}_{b^*}(s_{t+1})]\left(b_t^\pi-\delta b_{t-1}^\pi\right)\right\}}_{\text{Only under discretion}}=0
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
[b_t]: \quad & \underbrace{\beta E_t\left[\lambda_{t+1}\frac{1}{r_{t+1}}\frac{1}{\pi_{t+1}}\left(1+\delta Q_{t+1}\right)\right]-\lambda_t\frac{1}{r_t}Q_t}_{\text{Consumption smoothing}} \\
& \underbrace{-\lambda_t\frac{1}{R}\left\{E[\mathcal{X}_b(s_{t+1})]\left(b_t-\delta\frac{b_{t-1}}{\pi_t}\right)+E_t[\mathcal{Z}_b(s_{t+1})]\left(b_t^\pi-\delta b_{t-1}^\pi\right)\right\}}_{\text{Only under discretion}}=0
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
[b_t^\pi]: \quad & \underbrace{\beta E_t\left[\lambda_{t+1}\frac{1}{r_{t+1}}\left(1+\delta Q_{t+1}^\pi\right)\right]-\lambda_t\frac{1}{r_t}Q_t^\pi}_{\text{Consumption smoothing}} \\
& \underbrace{-\lambda_t\frac{1}{R}\left\{E_t[\mathcal{X}_{b^\pi}(s_{t+1})]\left(b_t-\delta\frac{b_{t-1}}{\pi_t}\right)+E_t[\mathcal{Z}_{b^\pi}(s_{t+1})]\left(b_t^\pi-\delta b_{t-1}^\pi\right)\right\}}_{\text{Only under discretion}}=0
\end{aligned}
\tag{3.21}
$$

Equations (3.19), (3.20) and (3.21) describe the government's optimal debt policy regarding foreign currency, local currency and inflation-indexed bonds, respectively.

- **Hedging benefits**

The first line in each of these equations gives the optimal trade-off between current and future distortions associated with the need to satisfy the government's intertemporal budget constraint - consumption smoothing (hand-to-mouth households). The government trades-off the short-run costs of reducing the stock of debt against the discounted value of the long-term benefits of lower debt. These terms are present under both commitment and discretion.

To study this effect in isolation, consider a case where inflation is costless ($l(\pi_t) = 0$), the real exchange rate is independent from consumption ($n_c\left(1, \frac{c_{T,t}}{y_N}\right) = 0$) and bonds are only one-period ($\delta = 0$). Then,

$$u'(c_t)C_{c_T,t} = \beta R E_t\left[u'(c_{t+1})C_{c_T,t+1}\right] \tag{3.22}$$

$$COV\left[u'(c_t)C_{c_T,t}, (e_t/P_{t-1})^{-1}\right] = 0 \quad \text{with local currency bonds} \tag{3.23}$$

$$COV\left[u'(c_t)C_{c_T,t}, r_t^{-1}\right] = 0 \quad \text{with inflation-indexed bonds} \tag{3.24}$$

Equation (3.22) defines the optimal path for tradable consumption. Equations (3.23) and (3.24) arise from the possibility of issuing bonds in local currency and inflation-indexed, respectively. According to these equations, the marginal utility of tradable consumption is isolated from fluctuations in the nominal exchange rate, in the case of being able to issue local currency bonds, and in the real exchange rate, in the case of inflation-indexed bonds. When nominal exchange rate is perfectly negatively (positively) correlated with tradable income, the government can obtain perfect tradable consumption smoothing by issuing positive (negative) debt in local currency. [3] Similarly, when real exchange rate is perfectly negatively (positively) correlated with tradable income, the government can obtain perfect tradable consumption smoothing by issuing positive (negative) inflation-indexed debt. A well-known stylized fact in emerging economies,

[3] See Ottonello and Perez (2019) and Korinek (2009).

documented for our sample of Latin American countries in Table 3, is that exchange rates are negatively correlated with aggregate tradable income. Then, local currency and inflation-indexed bonds can be a useful hedging against endowment risk.

- **Discipline effects**

The second line in equations (3.19), (3.20) and (3.21) captures wedges that are introduced when the government lacks commitment. Under discretion, these expressions are generalized Euler equations, which include partial derivatives of policy functions with respect to debt due to time-consistency requirements. These partial derivatives capture the effect of higher debt on bond prices. In general, the form of these auxiliary functions, $\mathcal{X}(s_t)$ and $\mathcal{Z}(s_t)$, is unknown, which is why we need to use numerical methods to solve the policy problem.

If the government cannot keep its promises, foreign lenders anticipate that higher debt in local currency increases the government's desire to introduce inflation and real exchange rate depreciation surprises, and demand a higher return on local currency bonds. Similarly, foreign lenders anticipate that higher inflation-indexed debt increases government's temptation to generate real exchange rate depreciation surprises, and demand a higher return on inflation-indexed bonds.

I focus on one-period bonds ($\delta = 0$) and assume that the endowment of tradables is constant to analytically explore how the government internalizes the lower prices on local currency and indexed bonds. The generalized Euler equations can be expressed in this case as (see derivation in Appendix 3.B.2):

$$[b_t^*] : u_t' C_{c_T,t} = \beta R u_{t+1}' C_{c_T,t+1} \underbrace{\left(1 + n_{c,t} b_{t-1} + m_{c,t} b_{t-1}^{\pi}\right)}_{\text{Dilution through RER}} \frac{1}{\underbrace{1 + \underbrace{b_t n_{\pi,t+1} \pi_{\boldsymbol{b}^*,t+1}}_{\substack{\text{due to dilution}\\ \text{through inflation}}} + \underbrace{\left(n_{c,t+1} b_t + m_{c,t+1} b_t^{\pi}\right) \frac{1}{R} \frac{\partial \left(n_{t+2} b_{t+1} + m_{t+2} b_{t+1}^{\pi} + b_{t+1}^*\right)}{\partial \boldsymbol{b}_t^*}}_{\substack{\text{due to dilution}\\ \text{through RER}}}}_{\text{Discipline effect}}} \tag{3.25}$$

$$[b_t] : u_t' C_{c_T,t} = \beta R u_{t+1}' C_{c_T,t+1} \underbrace{\left(1 + n_{c,t} b_{t-1} + m_{c,t} b_{t-1}^{\pi}\right)}_{\text{Dilution through RER}} \frac{1}{\underbrace{1 + \underbrace{\frac{b_t n_{\pi,t+1} \pi_{\boldsymbol{b},t+1}}{n_{t+1}}}_{\substack{\text{due to dilution}\\ \text{through inflation}}} + \underbrace{\frac{\left(n_{c,t+1} b_t + m_{c,t+1} b_t^{\pi}\right) \frac{1}{R} \frac{\partial\left(n_{t+2} b_{t+1} + v_{t+2} b_{t+1}^{\pi} + b_{t+1}^*\right)}{\partial \boldsymbol{b}_t}}{n_{t+1}}}_{\substack{\text{due to dilution}\\ \text{through RER}}}}_{\text{Discipline effect}}} \tag{3.26}$$

$$[b_t^{\pi}] : u_t' C_{c_T,t} = \beta R u_{t+1}' C_{c_T,t+1} \underbrace{\left(1 + n_{c,t} b_{t-1} + m_{c,t} b_{t-1}^{\pi}\right)}_{\text{Dilution through RER}} \frac{1}{\underbrace{1 + \underbrace{\frac{b_t n_{\pi,t+1} \pi_{\boldsymbol{b}^{\pi},t+1}}{m_{t+1}}}_{\substack{\text{due to dilution}\\ \text{through inflation}}} + \underbrace{\frac{\left(n_{c,t+1} b_t + m_{c,t+1} b_t^{\pi}\right) \frac{1}{R} \frac{\partial\left(n_{t+2} b_{t+1} + v_{t+2} b_{t+1}^{\pi} + b_{t+1}^*\right)}{\partial \boldsymbol{b}_t^{\pi}}}{m_{t+1}}}_{\substack{\text{due to dilution}\\ \text{through RER}}}}_{\text{Discipline effect}}} \tag{3.27}$$

where $n_t = (\pi_t, c_{T,t})$, $m_t = (1, c_{T,t})$, $n_{\pi,t} = n_{\pi}(\pi_t, c_{T,t})$, $n_{c,t} = n_c(\pi_t, c_{T,t})$ and $m_{c,t} =$

$m_c(1, c_{T,t})$. The term that captures the dilution through real exchange rate is common to (3.25), (3.26), and (3.27). This concept was explained above when analyzing the FOC for tradable consumption, and refers to the fact that when rising one additional unit of good with debt, consumption of tradables increases by $1/\left(1 + n_{c,t}b_{t-1} + m_{c,t}b^{\pi}_{t-1}\right)$. The term that captures the discipline effect reflects the higher returns demanded by the foreign lenders as a consequence of the ex-post incentives for the government to dilute the real value of local currency and inflation-indexed debt. When the government issues bonds in local currency, it is tempted to inflate away its real value by generating surprise inflation. The term that captures the discipline effect related to the dilution through inflation depends on the reduction in debt repayment in local currency, $n_{\pi,t+1}b_t$ (common for all types of bonds), and on the sensitivity of the optimal inflation to the type of bond considered, $\pi_{b^*,t+1}$, $\pi_{b,t+1}$, $\pi_{b^{\pi},t+1}$. Moreover, when the government issues bonds in local currency or indexed to inflation, it is tempted to dilute their real value by depreciating the real exchange rate. The term that captures the discipline effect related to the dilution through real exchange rate depends on the reduction in debt repayment in local currency and inflation-indexed, $n_{c,t+1}b_t + m_{c,t+1}b^{\pi}_t$ (common for all types of bonds), and on the sensitivity of future debt choices to the current choice of bond considered, $\frac{\partial\left(n_{t+2}b_{t+1}+v_{t+2}b^{\pi}_{t+1}+b^*_{t+1}\right)}{\partial \boldsymbol{b}^*_t}$, $\frac{\partial\left(n_{t+2}b_{t+1}+v_{t+2}b^{\pi}_{t+1}+b^*_{t+1}\right)}{\partial \boldsymbol{b}_t}$, $\frac{\partial\left(n_{t+2}b_{t+1}+v_{t+2}b^{\pi}_{t+1}+b^*_{t+1}\right)}{\partial \boldsymbol{b}^{\pi}_t}$.

To summarize, from analyzing the FOCs for consumption and inflation we conclude that, under discretion, there is a real exchange rate depreciation and an inflation bias. With local currency and indexed debt, the government has an incentive, ex-post, to depreciate the real exchange rate to dilute the real value of these type of bonds. Similarly, with local currency debt, the government has an incentive to increase inflation to reduce the real value of this type of debt. Looking at the FOCs for each type of bond, we first observe that local currency and inflation-indexed bonds are useful instruments to smooth tradable consumption. As nominal and real exchange rate are negatively correlated with tradable endowment, the real value of their payoffs go up in "bad times", providing hedging against tradable endowment risk. Finally, we note that foreign investors anticipate the ex-post incentives of the government to generate surprise inflation and real exchange rate depreciation, lowering the price for local currency and inflation-indexed debt. The government takes this into account, and chooses a lower amount of debt in local currency and indexed units than it will choose under commitment.

3.5 Quantitative Analysis

In this section I calibrate the model to match the average Latin American country in the sample and two particular countries: Peru and Uruguay. I use the quantified model to offer potential explanations for the differences between countries.

3.5.1 Calibration

Before moving to the calibration of the model, we need to make some assumptions. One period corresponds to one year. The period utility function takes the following form

$$u(c_{T,t}, c_{N,t}) - l(\pi_t) = \frac{(c_{T,t}^{\alpha} c_{N,t}^{1-\alpha})^{1-\sigma}}{1-\sigma} - \frac{\psi_\pi}{2}(\pi_t - \bar{\pi})^2 \tag{3.28}$$

where α is the share of tradables in aggregate consumption, σ corresponds to the coefficient of risk aversion, ψ_π is the parameter that governs the disutility of excess inflation, and $\bar{\pi}$ is some benchmark inflation rate.

The process for the tradable endowment is assumed to follow an AR(1) process in logs

$$\log\, y_{T,t} = \rho_{y_T} \log\, y_{T,t-1} + \epsilon_t, \tag{3.29}$$

where $\epsilon_t \sim N\left(0, \sigma_{y_T}^2\right)$.

Additionally, I introduce a quadratic cost for the issuance of bonds, which is meant to capture the distortions not discussed in this paper. The introduction of foreign currency bond issuance costs is extensively used in small open economies, see Schmitt-Grohé and Uribe (2003). S. Liu, Ma, and Shen (2021) incorporate local currency bond issuance costs. I am extending this approach to inflation-indexed bonds. Then, the resource constraint becomes:

$$\begin{aligned} c_{T,t} = {} & y_{T,t} - b_{t-1}^* - \frac{c_{T,t}^{1-\alpha}}{\pi_t}\frac{b_{t-1}}{\alpha} - c_{T,t}^{1-\alpha}\frac{b_{t-1}^{\pi}}{\alpha} \\ & + \frac{1}{R-\delta}(b_t^* - \delta b_{t-1}^*) + \frac{1}{R}E\left[\mathcal{X}(s_{t+1})\right]\frac{1}{\alpha}\left(b_t - \delta\frac{b_{t-1}}{\pi_t}\right) + \frac{1}{R}E\left[\mathcal{Z}(s_{t+1})\right]\frac{1}{\alpha}\left(b_t^{\pi} - \delta b_{t-1}^{\pi}\right) \\ & - \frac{\psi_b}{2}\left(\frac{1}{R-\delta}b_t^* - \bar{B}^*\right)^2 - \frac{\psi_b}{2}\left(\frac{1}{R}E\left[\mathcal{X}(s_{t+1})\right]\frac{b_t}{\alpha} - \bar{B}\right)^2 - \frac{\psi_b}{2}\left(\frac{1}{R}E\left[\mathcal{Z}(s_{t+1})\right]\frac{b_t^{\pi}}{\alpha} - \bar{B}^{\pi}\right)^2 \end{aligned} \tag{3.30}$$

where $s_t = \left(\frac{b_{t-1}^*}{\alpha}, \frac{b_{t-1}}{\alpha}, \frac{b_{t-1}^{\pi}}{\alpha}, y_{T,t}\right)$. The FOCs of the calibrated model are described in Appendix 3.B.3.

The calibrated parameter values are summarized in Table 1. The risk aversion coefficient, σ, the share of tradables in the consumption aggregator, α, the international risk-free interest rate, R, and the cost of inflation, ψ_π are taken from Ottonello and Perez (2019). These parameters are used to calibrate the model for the average Latin American country as well as for Peru and Uruguay. Following their work, the risk aversion coefficient is set to 5, which is within the upper values considered in the macroeconomic literature and in the lower values considered in the finance literature. The share of tradables in the consumption aggregator is set to 0.5, which is similar

Table 1: Calibration for the average Latin American country

Description	Parameter	Value	Source/Target
From literature:			
Risk aversion	σ	5	Ottonello and Perez (2019)
Tradable share in utility	α	0.5	
Cost of inflation	ψ_π	7.08	
Risk free interest rate	R	1.04	
Nontradable endowment	y_N	1	Normalization
Calibrated to fit targets:			
Discount factor	β	0.960	Average external debt = 17%
Benchmark inflation	$\bar{\pi}$	1.043	Average inflation = 1.049
Decay rate of bonds	δ	0.89	Average debt duration = 7.0
Autocorrelation of y_T	ρ_{y_T}	0.53	Estimation, data tradable output
Standard deviation of y_T	σ_{y_T}	0.03	Estimation, data tradable output
Cost of issuing bonds	ψ_b	0.065	Same across types of bonds
Target of FC bonds	$\bar{B}^*$	0.22	Average share of FC bonds = 67%
Target of LC bonds	$\bar{B}$	0.10	Average share of LC bonds = 27%
Target of indexed bonds	$\bar{B}^\pi$	0.01	Average share indexed bonds = 6%

Table 2: Calibration for Peru and Uruguay

		Value		
Description	Parameter	Peru	Uruguay	Target
Calibrated to fit targets:				
Discount factor	β	0.961	0.953	Average external debt = 14%, 31%
Target of inflation	$\bar{\pi}$	1.023	1.063	Average inflation = 1.028, 1.076
Decay rate of bonds	δ	0.90	0.95	Average debt duration = 7.5, 10.6
Autocorrelation of y_T	ρ_{y_T}	0.40	0.71	Estimation, data tradable output
Standard deviation of y_T	σ_{y_T}	0.02	0.03	Estimation, data tradable output

to the value used by S. Liu, Ma, and Shen (2021) (0.4). The international risk-free interest rate, R, is set to 1.04, a standard value in the literature. The cost of inflation is set to 7.08. The level of nontradable endowment, y_N, is normalized to one.

The remaining parameters are calibrated to match moments in the data, considering the period 2005 (or latest available) to 2019. The calibrated parameters for Peru and Uruguay are displayed in Table 2. The discount factor, β, is calibrated to target an average stock of external sovereign debt of 17% of GDP for the average Latin American country in our sample. For Peru and Uruguay the targets are 14% and 31%. The calibrated values are 0.960, 0.961 and 0.953, respectively. The benchmark rate of inflation, $\bar{\pi}$, is chosen to match the average inflation rate, which for the average country in the sample is 4.9%, for Peru is 2.8% and for Uruguay is 7.6%. The calibrated values are 1.043, 1.023 and 1.063, respectively. The decay rate of bonds, δ, is chosen to match the duration of bonds, which is 7 years for the sample average, 7.5 for Peru and 10.6 for Uruguay.[4] The calibrated values are 0.89, 0.90, and 0.95, respectively. The process for tradable endowment is estimated with annual data on tradable GDP (agricultural and manufacturing sectors) for the period 1990-2019 for the panel countries analyzed in section 3.2. The data of tradable GDP was detrended using a log-linear trend. The parameters that govern the bond issuance cost, $\psi_b, \bar{B}^*, \bar{B}, \bar{B}^\pi$, are calibrated in the following way. ψ_b is set to 0.065, chosen to be relatively small but at the same time accommodate the cases of Peru and Uruguay.[5] $\bar{B}^*, \bar{B}, \bar{B}^\pi$ are chosen to match average foreign currency, local currency and inflation-indexed shares in the average Latin American country, which are %67, 23% and 6%, respectively. The calibrated values are 0.22, 0.10 and 0.1, respectively.

For this model, the equilibrium policy functions cannot be computed and the model's steady state is endogenously determined as part of the model solution, so it is a priori unknown. Therefore, we cannot solve the model using a perturbation around the steady state. Following the solution method used by Bacchetta, Wincoop, and Young (2022), I use the Taylor projection method developed in Levintal (2018) to solve this problem. This involves approximating the solution locally at various nodes of the state space, and then combining these local solutions to form the global solution. (See brief description of the solution method in Appendix 3.C)

3.5.2 Numerical Results

Table 3 summarizes the data and model moments, calibrated for the average

[4] Total debt duration obtained from the ministry of finance of each country, and IMF's Article IV for Brazil.

[5] When adjusting β and δ to match the average debt-to-GDP ratio and duration of each country, we need to remember that $0 < \delta < \beta^{-1}$.

Latin American country (LAC), Peru and Uruguay. To compute the model's moments I simulate the exogenous stochastic process for tradable endowment for 100,000 periods, and discard the first 10,000 observations. Average level of total sovereign external debt to GDP and average inflation rate match the data counterpart, since there are targets of the calibration. Similarly, the currency composition of the external sovereign debt matches the data for the average Latin American country. However, when analyzing Peru and Uruguay I keep the parameters that govern the bond issuance cost, ψ_b, $\bar{B}^*$, $\bar{B}$ and $\bar{B}^\pi$, as calibrated for the average country in the sample. In the case of Peru, a smaller sovereign external debt-to-GDP, lower inflation rate and slightly higher bond duration relative to the average Latin American country lead to a smaller share of inflation-indexed debt and a larger share of foreign currency. On the contrary, in the case of Uruguay, a larger sovereign external debt-to-GDP, higher inflation rate and debt duration relative to the average Latin American country result in a larger share of inflation-indexed debt and a smaller share of foreign currency debt. The model predicts a 0 percent average share of inflation-indexed debt in the case of Peru, as observed in the data. For Uruguay, it predicts a 19 percent average share of inflation-indexed debt, close to the 16 percent observed in the data. Nevertheless, the model overestimates the observed share of foreign currency debt (and underestimates the share of local currency) in the case of Peru, and underestimates it in the case of Uruguay (and overestimates the share of local currency). The model correctly predicts a strong and negative correlation between exchange rate, nominal and real, and GDP.

- **Role of Average Debt, Duration and Inflation Costs**

To analyze these results, I consider the effect of each parameter individually. Figure 2 plots the outcomes for key parameters, considering the model calibrated to the average Latin American country as the baseline. An increase in the discount factor, β, all other things being equal, reduces the average external sovereign debt-to-GDP

Table 3: Numerical Results

	LAC		Peru		Uruguay	
Moment	Data	Model	Data	Model	Data	Model
Average Level						
Debt to GDP	17%	17%	14%	14%	31%	31%
Share of debt in FC	67%	67%	66%	74%	82%	53%
Share of debt in LC	27%	27%	34%	26%	2%	28%
Share of indexed debt	6%	6%	0%	0%	16%	19%
Inflation	4.9%	4.9%	2.8%	2.8%	7.6%	7.6%
Correlation with GDP						
Inflation	-0.05	0.27	0.50	0.36	-0.51	-0.01
Nominal exchange rate	-0.52	-0.75	-0.65	-0.68	-0.83	-0.89
Real exchange rate	-0.52	-0.80	-0.67	-0.74	-0.90	-0.92

ratio. As analyzed in subsection 3.4.2, lower local currency and indexed debt levels, result in smaller hedging benefits, given the negative correlation between exchange rate, nominal and real, and tradable endowment. But at the same time, this reduces the government incentives to dilute the value of the debt ex-post through inflation and real exchange rate depreciation. This makes the extra return on local currency and inflation-indexed bonds required by foreign lenders smaller, and their prices higher. Additionally, as debt decreases so does the cost of issuing bonds introduced in subsection 3.5.1. As a result of balancing these benefits and costs, the share of inflation-indexed debt decreases when β increases, the share of foreign currency debt increases, while the share of local currency debt remains stable.

An increase in δ, all other things being equal, translates into an increase in debt duration.[6] As discussed in subsection 3.4.2, a larger duration increases the hedging benefits provided by local currency and inflation-indexed bonds, when there is a negative correlation between exchange rate, nominal and real, and tradable endowment. At the same time, a higher duration increases the benefits from diluting debt through surprise inflation. Therefore, the required return on bonds in local currency increases, and their prices decrease. As a result of balancing these benefits

[6] As mentioned in the model section (3.3), duration is given by $(1-\beta\rho)^{-1}$.

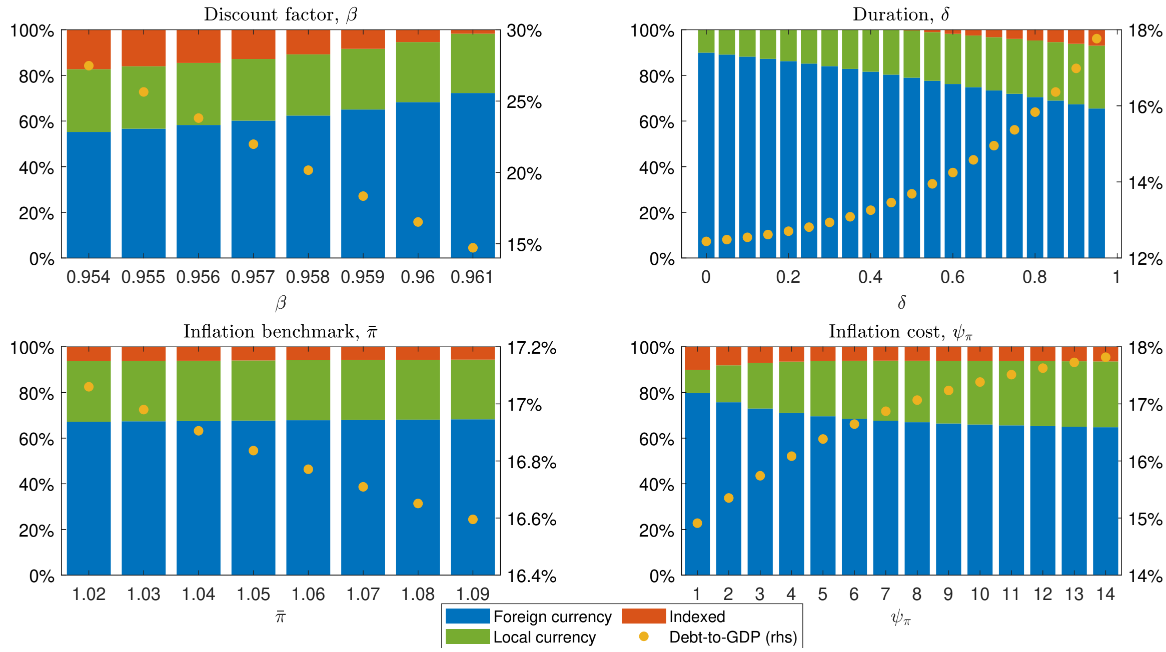

Figure 2: Currency composition (left-hand side axis) and average external debt-to-GDP (right-hand-side axis) for different parameter values. Except for the parameter subject to change, the calibration corresponds to the average Latin American country.

and costs, when δ rises, the share of local currency increases and the share of foreign currency debt decreases. For $\delta > 0.5$, i.e. duration larger than 2 years, as δ increases not only the share of local currency increases but also the share of inflation-indexed debt. Moreover, when duration increases, so does the average debt-to-GDP. Same logic applies as for a reduction in β.

Increases in inflation costs are governed by $\bar{\pi}$ and ψ_π, and affect the inflationary bias and the price of bonds, as explained in section 3.4.2. All other things being equal, the larger the inflation costs, the smaller the incentives for the government to generate ex-post inflation, and higher the price of bonds denominated in local currency (given the smaller extra return demanded by foreign lenders). Quantitatively, a rise in the inflation benchmark, $\bar{\pi}$, does not alter currency composition and it slightly reduces average debt-to-GDP ratio. An increase in ψ_π reduces the share of foreign currency and inflation-indexed debt, boosting the share of local currency bonds. This

Table 4: Numerical results changing one parameter at a time with LAC as the baseline

	LAC	Debt to GDP		Duration		Average inflation		Inflation cost	
Moment	Baseline	14%	31%	7.5y	10.6y	2.8%	7.6%	1.8	28.3
Average Level									
Debt to GDP	17%	14%	31%	17%	17%	17%	17%	17%	17%
Share of debt in FC	68%	73%	53%	67%	67%	67%	68%	72%	66%
Share of debt in LC	26%	26%	28%	27%	27%	27%	26%	15%	30%
Share of indexed debt	6%	1%	19%	6%	6%	6%	6%	12%	5%
Inflation	4.9%	4.9%	4.9%	4.9%	4.9%	2.8%	7.6%	4.9%	4.9%
Correlation with GDP									
Inflation	0.27	0.30	0.15	0.28	0.30	0.29	0.26	0.34	0.18
Nominal exchange rate	-0.75	-0.74	-0.77	-0.75	-0.76	-0.75	-0.79	-0.75	-0.76
Real exchange rate	-0.80	-0.79	-0.81	-0.80	-0.81	-0.80	-0.74	-0.80	-0.78

effect gets smaller as ψ_π increases. Additionally, the larger ψ_π, the larger the average debt-to-GDP ratio. Same logic applies as for a reduction in β.

Table 4 presents the numerical results changing one parameter at a time using the calibrated values for Peru and Uruguay. The first column refers to the results for the average Latin American country. Columns 2 to 3 show the effect of adjusting the calibration to match the average debt-to-GDP, duration and inflation rate of Peru and Uruguay. Even though each parameter has different effects on the currency composition, as noted in figure 1, in the case of Peru and Uruguay, the difference in the share of inflation-indexed debt is explained by the debt-to-GDP ratio. Column 4 introduces changes in the inflation cost parameter ψ_π, considering costs 2.5 times higher and smaller than the baseline. All other things being equal, the higher the inflation costs, the smaller the share of foreign currency and inflation-indexed debt. As inflation gets more costly, the incentives for the government to dilute the value of outstanding local currency debt get reduced. This lowers the extra return demanded by foreign lenders, and increases the price of local currency bonds. Therefore, the government can benefit from the hedging properties of local currency bonds, without paying a much higher return on them.

3.6 Conclusion

This chapter examines empirically and theoretically the role of inflation-indexed sovereign external bonds as an additional policy tool for the government. I build a dataset on foreign holdings of government debt distinguishing between local currency, foreign currency and indexed bonds. In Latin American countries, the reduction in the external debt denominated in foreign currency experienced in the last two decades lead to an increase in the share of local currency debt. Inflation-indexed bonds represented a small fraction, with the exception of Uruguay.

I develop a model of optimal time-consistent monetary policy and choice of debt denominated in foreign currency, local currency and inflation-indexed units. The choice of debt currency denomination balances hedging benefits and costs derived from discipline effects, due to the ex-post incentive to dilute the real value of local currency and inflation-indexed bonds. I calibrate the model for the average Latin American country in the sample, Peru and Uruguay. According to the model, a larger debt-to-GDP ratio, longer debt duration and lower inflation costs encourage more borrowing in inflation-indexed bonds and can explain the larger share of inflation-indexed debt in Uruguay compared to other Latin American countries.

I leave for future research studying the optimal decision of debt duration and distortionary taxes. Over the last decades, duration has increased in Latin American countries, and exploring its brake-down by type of bonds can introduce interesting considerations for optimal policy. Distortionary taxes introduce an extra cost to altering intertemporal consumption decisions, and will affect hedging benefits and costs derived from dilution incentives. These two considerations are analyzed by Leeper, Leith, and D. Liu (2021) in a close economy framework, and it would be interesting to extend the analysis to an open economy.

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Appendix

Appendix 3.A Description of the Data

- **Brazil**. Foreign holdings of sovereign debt correspond to Federal Public Debt issued in domestic markets and held by nonresidents and debt issued in international markets, which is assumed to be held by nonresidents. The information was obtained from the annex of the monthly public debt report, elaborated by the National Treasury. For debt issued in international markets, data is available since January 2004. For debt issued in domestic markets, data is available since December 1999. However, its decomposition by bond-holder is available since January 2007. To obtain the information by bond-holder and currency denomination I created a dataset based on the information included in the monthly public debt reports, available since January 2011.

- **Colombia**. Foreign holdings of sovereign debt correspond to Central Government External Debt issued in domestic markets and held by nonresidents and debt issued in international markets, which are assumed to be held by nonresidents. For debt issued in domestic markets, the information was compiled based on Domestic Debt Profile Treasury Securities Reports, elaborated by the Ministry of Finance. I could gather this data since January 2010, but there is a discontinuity from January 2016 to December 2018. For debt issued in international markets, the information was obtained from the historical Central Government debt data, elaborated by the Ministry of Finance, and it is available since June 2001.

- **Mexico**. Foreign holdings of sovereign debt correspond to Public Sector Debt issued in domestic markets and held by nonresidents and debt issued in international markets, which is assumed to be held by nonresidents. The data about debt issued in international markets was obtained from the Federal Public Sector debt statistics elaborated by the Ministry of Finance since January 1990. The information about debt issued in domestic markets held by nonresidents was obtained from the Central Bank of Mexico, since January 1991.

- **Peru**. Foreign holdings of sovereign debt correspond to Non-Monetary Public Sector. Information about debt issued in international markets was obtained from the Public Debt Statistics elaborated by the Central Bank of Peru available since the first quarter of 1999. For debt issued in domestic markets, I constructed a dataset based on the Sovereign Bond Holding Reports elaborated by the General Directorate of the Public Treasury, Ministry of Economy and Finance. Information by currency was available since the first quarter of 2013.

- **Uruguay**. Foreign holdings of sovereign debt correspond to Non-Monetary Public Sector. Debt held by nonresidents was obtained from the Central Bank of Uruguay, available since the last quarter of 1999. To distinguish between local currency and inflation-indexed debt I assume that external debt (60% of total debt) is split as total debt. Before 2012 there was no debt denominated in local currency, not issued in domestic nor international markets.

In all possible cases I verify that the data is similar to Arslanalp and Tsuda (2014).

Appendix 3.B Optimal Policy Derivations

3.B.1 Optimal Policy under Commitment and Discretion

Commitment

When the government can commit to future policies, the Lagrangian for the policy problem is given by

$$\begin{aligned}
\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \{ & u\left(C\left(c_{T,t}, y_N\right)\right) - l(\pi_t) \\
& + \lambda_t \left[y_{T,t} - c_{T,t} - n\left(\pi_t, \frac{c_{T,t}}{y_N}\right)(1+\delta Q_t) b_{t-1} - n\left(1, \frac{c_{T,t}}{y_N}\right)(1+\delta Q_t^{\pi}) b_{t-1}^{\pi} - \frac{R}{R-\delta} b_{t-1}^{*} \right. \\
& \left. + n\left(1, \frac{c_{T,t}}{y_N}\right) Q_t b_t + n\left(1, \frac{c_{T,t}}{y_N}\right) Q_t^{\pi} b_t^{\pi} + \frac{1}{R-\delta} b_t^{*} \right] \\
& + \mu_t \left\{ Q_t - \frac{1}{R} \frac{1}{n\left(1, \frac{c_{T,t}}{y_N}\right)} E_t \left[n\left(\pi_{t+1}, \frac{c_{T,t+1}}{y_N}\right)(1+\delta Q_{t+1}) \right] \right\} \\
& + \mu_t^{\pi} \left\{ Q_t^{\pi} - \frac{1}{R} \frac{1}{n\left(1, \frac{c_{T,t}}{y_N}\right)} E_t \left[n\left(1, \frac{c_{T,t+1}}{y_N}\right)(1+\delta Q_{t+1}^{\pi}) \right] \right\} \}
\end{aligned} \tag{3.31}$$

By committing to an entire path of policy instruments, the government is able to influence expectations in order to improve the policy trade-offs they face.

The resultant set of FOCs are given by

$$
\begin{aligned}
[c_{T,t}]: \quad & u'(c_t)C_{c_T,t} \\
& -\lambda_t\left[1+n_c\left(\pi_t,\frac{c_{T,t}}{y_N}\right)(1+\delta Q_t)b_{t-1}-n_c\left(1,\frac{c_{T,t}}{y_N}\right)(1+\delta Q_t^{\pi})b_{t-1}^{\pi}\right. \\
& \left.-n_c\left(1,\frac{c_{T,t}}{y_N}\right)(Q_t b_t+Q_t^{\pi}b_t^{\pi})\right] \\
& +\mu_t\frac{1}{R}\frac{n_c\left(1,\frac{c_{T,t}}{y_N}\right)}{n\left(1,\frac{c_{T,t}}{y_N}\right)^2}E_t\left[n\left(\pi_{t+1},\frac{c_{T,t+1}}{y_N}\right)(1+\delta Q_{t+1})\right] \\
& +\mu_t^{\pi}\frac{1}{R}\frac{n_c\left(1,\frac{c_{T,t}}{y_N}\right)}{n\left(1,\frac{c_{T,t}}{y_N}\right)^2}E_t\left[n\left(1,\frac{c_{T,t+1}}{y_N}\right)(1+\delta Q_{t+1}^{\pi})\right] \\
& -\beta^{-1}\mu_{t-1}\frac{1}{R}\frac{1}{n\left(1,\frac{c_{T,t-1}}{y_N}\right)}n_c\left(\pi_t,\frac{c_{T,t}}{y_N}\right)(1+\delta Q_t) \\
& -\beta^{-1}\mu_{t-1}^{\pi}\frac{1}{R}\frac{1}{n\left(1,\frac{c_{T,t-1}}{y_N}\right)}n_c\left(1,\frac{c_{T,t}}{y_N}\right)(1+\delta Q_t^{\pi})=0 \\
[\pi_t]: \quad & -l'(\pi_t)-\lambda_t n_{\pi}\left(\pi_t,\frac{c_{T,t}}{y_N}\right)(1+\delta Q_t)b_{t-1} \\
& -\frac{1}{\beta}\mu_{t-1}\frac{1}{R}\frac{1}{n\left(1,\frac{c_{T,t-1}}{y_N}\right)}n_{\pi}\left(\pi_t,\frac{c_{T,t}}{y_N}\right)(1+\delta Q_t)=0 \\
[b_t]: \quad & \lambda_t n\left(1,\frac{c_{T,t}}{y_N}\right)Q_t-\beta E_t\left[\lambda_{t+1}n\left(\pi_{t+1},\frac{c_{T,t+1}}{y_N}\right)(1+\delta Q_{t+1})\right]=0 \\
[b_t^{\pi}]: \quad & \lambda_t n\left(1,\frac{c_{T,t}}{y_N}\right)Q_t^{\pi}-\beta E_t\left[\lambda_{t+1}n\left(1,\frac{c_{T,t+1}}{y_N}\right)(1+\delta Q_{t+1}^{\pi})\right]=0 \\
[b_t^{*}]: \quad & \lambda_t\frac{1}{R-\delta}-\beta E_t\left[\lambda_{t+1}\right]\frac{R}{R-\delta}=0\Rightarrow\lambda_t-\beta E_t\left[\lambda_{t+1}\right]R=0 \\
[Q_t]: \quad & -\lambda_t\left[n\left(\pi_t,\frac{c_{T,t}}{y_N}\right)\delta b_{t-1}+n\left(1,\frac{c_{T,t}}{y_N}\right)b_t\right]+\mu_t \\
& -\beta^{-1}\mu_{t-1}\frac{1}{R}\frac{1}{n\left(1,\frac{c_{T,t-1}}{y_N}\right)}n\left(\pi_t,\frac{c_{T,t}}{y_N}\right)\delta=0 \\
[Q_t^{\pi}]: \quad & -\lambda_t\left[n\left(1,\frac{c_{T,t}}{y_N}\right)\delta b_{t-1}^{\pi}+n\left(1,\frac{c_{T,t}}{y_N}\right)b_t^{\pi}\right]+\mu_t^{\pi} \\
& -\beta^{-1}\mu_{t-1}^{\pi}\frac{1}{R}\frac{1}{n\left(1,\frac{c_{T,t-1}}{y_N}\right)}n\left(1,\frac{c_{T,t}}{y_N}\right)\delta=0
\end{aligned}
\tag{3.32}
$$

Discretion

The Lagrangian for the policy problem under discretion is given by

$$\begin{aligned}\mathcal{L} = {} & u(C(c_{T,t}, y_N)) - l(\pi_t) + \beta E_t[V(s_{t+1})] \\ & + \lambda_t \Bigg\{ y_{T,t} - c_{T,t} - b^*_{t-1} - n\left(\pi_t, \frac{c_{T,t}}{y_N}\right) b_{t-1} - n\left(1, \frac{c_{T,t}}{y_N}\right) b^{\pi}_{t-1} + \frac{1}{R-\delta}\left(b^*_t - \delta b^*_{t-1}\right) \\ & + \frac{1}{R} E_t\left[\mathcal{X}(s_{t+1})\left(b_t - \delta \frac{b_{t-1}}{\pi_t}\right) + \mathcal{Z}(s_{t+1})\left(b^{\pi}_t - \delta b^{\pi}_{t-1}\right)\right]\Bigg\} \end{aligned} \tag{3.33}$$

where the model equilibrium also involves the bond prices definitions given by (3.15) and (3.16).

The FOCs for the policy problem are

$$\begin{aligned}
[c_{T,t}]: \quad & u'(c_t) C_{c_T,t} - \lambda_t \left[1 + n_c\left(\pi_t, \frac{c_{T,t}}{y_N}\right) b_{t-1} + n_c\left(1, \frac{c_{T,t}}{y_N}\right) b^{\pi}_{t-1}\right] = 0 \\
[\pi_t]: \quad & - l'(\pi_t) - \lambda_t \left\{ n_\pi \left(\pi_t, \frac{c_{T,t}}{y_N}\right) - \delta \frac{1}{R} E_t[\mathcal{X}(s_{t+1})] \frac{1}{\pi_t^2} \right\} b_{t-1} = 0 \\
[b^*_t]: \quad & \beta E_t\left[\lambda_{t+1}\right]\left(1 + \frac{\delta}{R-\delta}\right) \\
& - \lambda_t \left\{ \frac{1}{R-\delta} + \frac{1}{R} E_t[\mathcal{X}_{b^*}(s_{t+1})]\left(b_t - \delta \frac{b_{t-1}}{\pi}\right) + \frac{1}{R} E_t[\mathcal{Z}_{b^*}(s_{t+1})]\left(b^{\pi}_t - \delta b^{\pi}_{t-1}\right) \right\} = 0 \\
[b_t]: \quad & \beta E_t\left[\lambda_{t+1} \mathcal{X}(s_{t+1})\right] \\
& - \lambda_t \frac{1}{R} \left\{ E[\mathcal{X}_b(s_{t+1})]\left(b_t - \delta \frac{b_{t-1}}{\pi_t}\right) + E_t[\mathcal{Z}_b(s_{t+1})]\left(b^{\pi}_t - \delta b^{\pi}_{t-1}\right) + E_t[\mathcal{X}(s_{t+1})] \right\} = 0 \\
[b^{\pi}_t]: \quad & \beta E_t\left[\lambda_{t+1} \mathcal{Z}(s_{t+1})\right] \\
& - \lambda_t \frac{1}{R} \left\{ E_t[\mathcal{X}_{b^\pi}(s_{t+1})]\left(b_t - \delta \frac{b_{t-1}}{\pi_t}\right) + E_t[\mathcal{Z}_{b^\pi}(s_{t+1})]\left(b^{\pi}_t - \delta b^{\pi}_{t-1}\right) + E_t[\mathcal{Z}(s_{t+1})] \right\} = 0
\end{aligned} \tag{3.34}$$

Note that from the FOC with respect to consumption we obtain:

$$\lambda_t = \frac{u'(c_t) C_{c_T,t}}{1 + n_c(\pi_t, c_{T,t}) b_{t-1} + n_c(1, c_{T,t}) b^{\pi}_{t-1}} \tag{3.35}$$

3.B.2 Generalized Euler Equations

The trade-offs associated to the government choices arise from the analysis of the fist order conditions. Define $\mathcal{C}(b_t^*, b_t, b_t^\pi), \mathcal{P}(b_t^*, b_t, b_t^\pi), \mathcal{B}^*(b_t^*, b_t, b_t^\pi), \mathcal{B}(b_t^*, b_t, b_t^\pi), \mathcal{B}^\pi(b_t^*, b_t, b_t^\pi)$ the expected tradable consumption, inflation, and debt policies in foreign currency, local currency, and inflation-indexed units, respectively. In equilibrium, these expectations are consistent with optimal policies. Without loss of generality we set $y_N = 1$, and assume y_T is constant. The recursive government problem can be expressed as

$$V(b_{t-1}^*, b_{t-1}, b_{t-1}^\pi) = \max_{c_{T,t}, \pi_t, b_t^*, b_t, b_t^\pi} u\left(C\left(c_{T,t}, 1\right)\right) - l(\pi_t) + \beta E_t\left[V(b_t^*, b_t, b_t^\pi)\right] \tag{3.36}$$

subject to the household's budget constraint (3.3), the government's budget constraint (3.5) after using bond prices (3.6-3.8) and the inverse of the nominal exchange rate function (3.10):

$$\begin{aligned} &-y_{T,t} - c_{T,t} - b_{t-1}^* - n\left(\pi_t, c_{T,t}\right) b_{t-1} - n\left(1, c_{T,t}\right) b_{t-1}^\pi + \frac{1}{R} b_t^* \\ &\quad + \frac{1}{R} E_t\left[n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) b_t + n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) b_t^\pi\right] = 0 \end{aligned} \tag{3.37}$$

The FOCs of this problem are given by

$$[c_t]: u'(c_t)C_{c_T,t} = \lambda_t + \lambda_t n_c(\pi_t, c_{T,t})b_{t-1} + \lambda_t n_c(1, c_{T,t})b^{\pi}_{t-1} \tag{3.38}$$

$$[\pi_t]: -l'(\pi_t) = \lambda_t n_{\pi}(\pi_t, c_{T,t})b_{t-1} \tag{3.39}$$

$$[b_t]: \beta V_b(b_t, b^{\pi}_t, b^*_t) = -\lambda_t \frac{1}{R}\left\{\frac{\partial[n\left(\mathcal{P}(b_t, b^{\pi}_t, b^*_t), \mathcal{C}(b_t, b^{\pi}_t, b^*_t)\right) b_t]}{\partial b_t} + \frac{\partial n\left(1, \mathcal{C}(b_t, b^{\pi}_t, b^*_t)\right)}{\partial b_t} b^{\pi}_t\right\} \tag{3.40}$$

$$[b^{\pi}_t]: \beta V_{b^{\pi}}(b_t, b^{\pi}_t, b^*_t) = -\lambda_t \frac{1}{R}\left\{\frac{\partial n\left(\mathcal{P}(b_t, b^{\pi}_t, b^*_t), \mathcal{C}(b_t, b^{\pi}_t, b^*_t)\right)}{\partial b^{\pi}_t} b_t + \frac{\partial[n\left(1, \mathcal{C}(b_t, b^{\pi}_t, b^*_t)\right) b^{\pi}_t]}{\partial b^{\pi}_t}\right\} \tag{3.41}$$

$$[b^*_t]: \beta V_{b^*}(b_t, b^{\pi}_t, b^*_t) = -\lambda_t \frac{1}{R}\left[1 + \frac{\partial n\left(\mathcal{P}(b_t, b^{\pi}_t, b^*_t), \mathcal{C}(b_t, b^{\pi}_t, b^*_t)\right)}{\partial b^*_t} b_t + \frac{\partial n\left(1, \mathcal{C}(b_t, b^{\pi}_t, b^*_t)\right)}{\partial b^*_t} b^{\pi}_t\right] \tag{3.42}$$

and the envelope conditions are

$$[b_{t-1}]: \beta V_b(b_{t-1}, b^{\pi}_{t-1}, b^*_{t-1}) = -\lambda_t n(\pi_t, c_{T,t}) \tag{3.43}$$

$$[b^{\pi}_{t-1}]: \beta V_{b^{\pi}}(b_{t-1}, b^{\pi}_{t-1}, b^*_{t-1}) = -\lambda_t n(1, c_{T,t}) \tag{3.44}$$

$$[b^*_{t-1}]: \beta V_{b^*}(b_t, b^{\pi}_{t-1}, b^*_{t-1}) = -\lambda_t \tag{3.45}$$

Then

$$\lambda_t = \frac{u'(c_t)C_{c_T,t}}{1 + n_c(\pi_t, c_{T,t})b_{t-1} + n_c(1, c_{T,t})b^{\pi}_{t-1}} \tag{3.46}$$

Combining (3.46) with the envelope conditions we obtain three Euler equations:

$$\lambda_t \left\{ \frac{\partial [n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) b_t]}{\partial b_t} + \frac{\partial n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)}{\partial b_t} b_t^\pi \right\} = \beta R \lambda_{t+1} n(\pi_{t+1}, c_{T,t+1}) \tag{3.47}$$

$$\lambda_t \left\{ \frac{\partial n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)}{\partial b_t^\pi} b_t + \frac{\partial [n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) b_t^\pi]}{\partial b_t^\pi} \right\} = \beta R \lambda_{t+1} n(1, c_{T,t+1}) \tag{3.48}$$

$$\lambda_t \left[1 + \frac{\partial n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)}{\partial b_t^*} b_t + \frac{\partial n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)}{\partial b_t^*} b_t^\pi \right] = \beta R \lambda_{t+1} \tag{3.49}$$

The price sensitivity of debt can be calculated. The resource constraint of this economy at $t+1$ is

$$\begin{aligned}
&\mathcal{C}(b_t, b_t^\pi, b_t^*) - y_{T,t+1} - n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) b_t - n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) b_t^\pi - b_t^* + \frac{1}{R}\mathcal{B}^*(b_t, b_t^\pi, b_t^*) \\
&+ \frac{1}{R} E_t \left[n\left(\mathcal{P}(\mathcal{B}(b_t, b_t^\pi, b_t^*), \mathcal{B}^\pi(b_t, b_t^\pi, b_t^*), \mathcal{B}^*(b_t, b_t^\pi, b_t^*)\right), \right. \\
&\left. \mathcal{C}(\mathcal{B}(b_t, b_t^\pi, b_t^*), \mathcal{B}^\pi(b_t, b_t^\pi, b_t^*), \mathcal{B}^*(b_t, b_t^\pi, b_t^*))\right) \mathcal{B}(b_t, b_t^\pi, b_t^*) \right] \\
&+ \frac{1}{R} E_t \left[n\left(1, \mathcal{C}(\mathcal{B}(b_t, b_t^\pi, b_t^*), \mathcal{B}^\pi(b_t, b_t^\pi, b_t^*), \mathcal{B}^*(b_t, b_t^\pi, b_t^*))\right) \mathcal{B}^\pi(b_t, b_t^\pi, b_t^*) \right] = 0
\end{aligned}$$

To simplify notation define

$$\begin{aligned}
&n(\pi_{t+2}, c_{T,t+2}) b_{t+1} + n(1, c_{T,t+2}) b_{t+1}^\pi + b_{t+1}^* \equiv \\
&n\left(\mathcal{P}(\mathcal{B}(b_t, b_t^\pi, b_t^*), \mathcal{B}^\pi(b_t, b_t^\pi, b_t^*), \mathcal{B}^*(b_t, b_t^\pi, b_t^*)), \mathcal{C}(\mathcal{B}(b_t, b_t^\pi, b_t^*), \mathcal{B}^\pi(b_t, b_t^\pi, b_t^*), \mathcal{B}^*(b_t, b_t^\pi, b_t^*))\right) \mathcal{B}(b_t, b_t^\pi, b_t^*) \\
&+ n\left(1, \mathcal{C}(\mathcal{B}(b_t, b_t^\pi, b_t^*), \mathcal{B}^\pi(b_t, b_t^\pi, b_t^*), \mathcal{B}^*(b_t, b_t^\pi, b_t^*))\right) \mathcal{B}^\pi(b_t, b_t^\pi, b_t^*) + \mathcal{B}^*(b_t, b_t^\pi, b_t^*)
\end{aligned} \tag{3.50}$$

Differentiating the resource constraint at $t+1$ with respect to b_t, b_t^π and b_t^*:

$$\begin{aligned}
\frac{\partial \mathcal{C}(b_t, b_t^\pi, b_t^*)}{\partial b_t} = &- \frac{\partial [n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) b_t]}{\partial b_t} - \frac{\partial n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)}{\partial b_t} b_t^\pi \\
&+ \frac{1}{R} \frac{\partial [n(\pi_{t+2}, c_{T,t+2}) b_{t+1} + n(1, c_{T,t+2}) b_{t+1}^\pi + b_{t+1}^*]}{\partial b_t}
\end{aligned} \tag{3.51}$$

$$\frac{\partial \mathcal{C}(b_t, b_t^\pi, b_t^*)}{\partial b_t^\pi} = -\frac{\partial n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)}{\partial b_t^\pi} b_t - \frac{\partial [n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) b_t^\pi]}{\partial b_t^\pi} + \frac{1}{R}\frac{\partial [n(\pi_{t+2}, c_{T,t+2}) b_{t+1} + n(1, c_{T,t+2}) b_{t+1}^\pi + b_{t+1}^*]}{\partial b_t^\pi} \tag{3.52}$$

$$\frac{\partial \mathcal{C}(b_t, b_t^\pi, b_t^*)}{\partial b_t^*} = -\frac{\partial n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)}{\partial b_t^*} b_t - \frac{\partial n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)}{\partial b_t^*} b_t^\pi - 1 + \frac{1}{R}\frac{\partial [n(\pi_{t+2}, c_{T,t+2}) b_{t+1} + n(1, c_{T,t+2}) b_{t+1}^\pi + b_{t+1}^*]}{\partial b_t^*} \tag{3.53}$$

Applying the chain rule to the first two terms of the previous equations

$$\begin{aligned}\frac{\partial [n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) b_t]}{\partial b_t} &= n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) \\ &+ b_t n_\pi\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) \mathcal{P}_b(b_t, b_t^\pi, b_t^*) \\ &+ b_t n_c\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) \mathcal{C}_b(b_t, b_t^\pi, b_t^*)\end{aligned} \tag{3.54}$$

$$\frac{\partial n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)}{\partial b_t} = n_c\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) \mathcal{C}_b(b_t, b_t^\pi, b_t^*) \tag{3.55}$$

$$\begin{aligned}\frac{\partial n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)}{\partial b_t^\pi} &= n_\pi\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) \mathcal{P}_{b^\pi}(b_t, b_t^\pi, b_t^*) \\ &+ n_c\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) \mathcal{C}_{b^\pi}(b_t, b_t^\pi, b_t^*)\end{aligned} \tag{3.56}$$

$$\frac{\partial [n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) b_t^\pi]}{\partial b_t^\pi} = n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) + b_t^\pi n_c\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) \mathcal{C}_{b^\pi}(b_t, b_t^\pi, b_t^*) \tag{3.57}$$

$$\frac{\partial n\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)}{\partial b_t^*} = n_{\pi}\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\mathcal{P}_{b^*}(b_t, b_t^{\pi}, b_t^*) + n_c\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\mathcal{C}_{b^*}(b_t, b_t^{\pi}, b_t^*) \tag{3.58}$$

$$\frac{\partial n\left(1, \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)}{\partial b_t^*} = n_c\left(1, \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\mathcal{C}_{b^*}(b_t, b_t^{\pi}, b_t^*) \tag{3.59}$$

To obtain the Euler equation for debt in local currency, we first combine equations (3.51), (3.54) and (3.55)

$$\begin{aligned}
&\frac{\partial[n\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)b_t]}{\partial b_t} + \frac{\partial n\left(1, \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)}{\partial b_t}b_t^{\pi} = \\
&n\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right) + b_t n_{\pi}\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\mathcal{P}_b(b_t, b_t^{\pi}, b_t^*) \\
&+ b_t n_c\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\mathcal{C}_b(b_t, b_t^{\pi}, b_t^*) + b_t^{\pi} n_c\left(1, \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\mathcal{C}_b(b_t, b_t^{\pi}, b_t^*) = \\
&n\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right) + b_t n_{\pi}\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\mathcal{P}_b(b_t, b_t^{\pi}, b_t^*) \\
&+ \left[b_t n_c\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right) + b_t^{\pi} n_c\left(1, \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\right]\mathcal{C}_b(b_t, b_t^{\pi}, b_t^*) = \\
&n\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right) + b_t n_{\pi}\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\mathcal{P}_b(b_t, b_t^{\pi}, b_t^*) \\
&- \left[b_t n_c\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right) + b_t^{\pi} n_c\left(1, \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\right]\frac{\partial[n\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)b_t]}{\partial b_t} \\
&- \left[b_t n_c\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right) + b_t^{\pi} n_c\left(1, \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\right]\frac{\partial m\left(\mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)}{\partial b_t}b_t^{\pi} \\
&+ b_t n_c\left(\mathcal{P}(b_t, b_t^{\pi}, b_t^*), \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\frac{1}{R}\frac{\partial[n(\pi_{t+2}, c_{T,t+2})b_{t+1} + n(1, c_{T,t+2})b_{t+1}^{\pi} + b_{t+1}^*]}{\partial b_t} \\
&+ b_t^{\pi} n_c\left(1, \mathcal{C}(b_t, b_t^{\pi}, b_t^*)\right)\frac{1}{R}\frac{\partial[n(\pi_{t+2}, c_{T,t+2})b_{t+1} + n(1, c_{T,t+2})b_{t+1}^{\pi} + b_{t+1}^*]}{\partial b_t}
\end{aligned} \tag{3.60}$$

Rearranging terms we get

$$\frac{\partial[n\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)b_t]}{\partial b_t}+\frac{\partial n\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)}{\partial b_t}b_t^\pi =$$
$$\frac{1}{1+b_t n_c\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)+b_t^\pi n_c\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)}\left\{n\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\right.$$
$$+b_t n_\pi\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\mathcal{P}_b(b_t,b_t^\pi,b_t^*)$$
$$+b_t n_c\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\frac{1}{R}\frac{\partial[n(\pi_{t+2},c_{T,t+2})b_{t+1}+n(1,c_{T,t+2})b_{t+1}^\pi+b_{t+1}^*]}{\partial b_t}$$
$$\left.+b_t^\pi n_c\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\frac{1}{R}\frac{\partial[n(\pi_{t+2},c_{T,t+2})b_{t+1}+n(1,c_{T,t+2})b_{t+1}^\pi+b_{t+1}^*]}{\partial b_t}\right\} \tag{3.61}$$

Finally, we substitute this equation into (3.47), to obtain (using (3.46)) the modified Euler equation for debt in local currency

$$\frac{u'(c_t)C_{c_T,t}}{1+n_c(\pi_t,c_{T,t})b_{t-1}+n_c(1,c_{T,t})b_{t-1}^\pi}=\beta R\frac{u'(c_{t+1})C_{c_T,t+1}}{1+n_c(\pi_{t+1},c_{T,t+1})b_t+n_c(1,c_{T,t+1})b_t^\pi}$$
$$n(\pi_{t+1},c_{T,t+1})\left\{\frac{\partial[n\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)b_t]}{\partial b_t}\right.$$
$$\left.+\frac{\partial n\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)}{\partial b_t}b_t^\pi\right\}^{-1}$$
$$=\beta R u'(c_{t+1})C_{c_T,t+1}$$
$$\frac{1}{1+\frac{b_t n_\pi(t+1)\pi_b(t+1)+[b_t n_c(t+1)+b_t^\pi m_c(t+1)]\frac{1}{R}\frac{\partial\left[n(t+2)b_{t+1}+m(t+2)b_{t+1}^\pi+b_{t+1}^*\right]}{\partial b_t}}{n(t+1)}} \tag{3.62}$$

where $m(t)=n(1,c_{T,t})$.

So

$$u'_t C_{c_T,t} = \beta R u'_{t+1} C_{c_T,t+1} \underbrace{\left(1 + n_{c_T,t} b_{t-1} + m_{c,t} b^{\pi}_{t-1}\right)}_{\text{Dilution through RER}} \underbrace{\frac{1}{1 + \frac{b_t n_{\pi,t+1}\pi_{b,t+1} + (b_t n_{c_T,t+1} + b^{\pi}_t m_{c,t+1}) \frac{1}{R} \frac{\partial\left(n_{t+2} b_{t+1} + m_{t+2} b^{\pi}_{t+1} + b^{*}_{t+1}\right)}{\partial b_t}}{n_{t+1}}}}_{\text{Discipline effect}} \tag{3.63}$$

where, to simplify notation, $f(x_t, y_t) = f_t$ and $\partial f(x_t, y_t)/\partial x_t = f_{x,t}$.

To obtain the Euler equation for inflation-indexed debt, we first combine equations

(3.52), (3.56) and (3.57)

$$
\begin{aligned}
&\frac{\partial n\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)}{\partial b_t^\pi}b_t+\frac{\partial[n\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)b_t^\pi]}{\partial b_t^\pi}=\\
&b_t n_\pi\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\mathcal{P}_{b^\pi}(b_t,b_t^\pi,b_t^*)\\
&+b_t n_c\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\mathcal{C}_{b^\pi}(b_t,b_t^\pi,b_t^*)\\
&+n\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)+b_t^\pi n_c\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\mathcal{C}_{b^\pi}(b_t,b_t^\pi,b_t^*)=\\
&b_t n_\pi\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\mathcal{P}_{b^\pi}(b_t,b_t^\pi,b_t^*)+n\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\\
&+\left[b_t n_c\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)+b_t^\pi n_c\left(\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\right]\mathcal{C}_{b^\pi}(b_t,b_t^\pi,b_t^*)=\\
&b_t n_\pi\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\mathcal{P}_{b^\pi}(b_t,b_t^\pi,b_t^*)+n\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\\
&-\left[b_t n_c\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)+b_t^\pi n_c\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\right]\frac{\partial n\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)}{\partial b_t^\pi}b_t\\
&-\left[b_t n_c\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)+b_t^\pi n_c\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\right]\frac{\partial[n\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)b_t^\pi]}{\partial b_t^\pi}\\
&+\left[b_t n_c\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\right]\frac{1}{R}\frac{\partial[n(\pi_{t+2},c_{T,t+2})b_{t+1}+n(1,c_{T,t+2})b_{t+1}^\pi+b_{t+1}^*]}{\partial b_t^\pi}\\
&+\left[b_t^\pi n_c\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\right]\frac{1}{R}\frac{\partial[n(\pi_{t+2},c_{T,t+2})b_{t+1}+n(1,c_{T,t+2})b_{t+1}^\pi+b_{t+1}^*]}{\partial b_t^\pi}=\\
&b_t n_\pi\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\mathcal{P}_{b^\pi}(b_t,b_t^\pi,b_t^*)+n\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\\
&-\left[b_t n_c\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\right]\left\{\frac{\partial n\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)}{\partial b_t^\pi}b_t+\frac{\partial[n\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)b_t^\pi]}{\partial b_t^\pi}\right\}\\
&-\left[b_t^\pi n_c\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\right]\left\{\frac{\partial n\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)}{\partial b_t^\pi}b_t+\frac{\partial[n\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)b_t^\pi]}{\partial b_t^\pi}\right\}\\
&+\left[b_t n_c\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\right]\frac{1}{R}\frac{\partial[n(\pi_{t+2},c_{T,t+2})b_{t+1}+n(1,c_{T,t+2})b_{t+1}^\pi+b_{t+1}^*]}{\partial b_t^\pi}\\
&+\left[b_t^\pi n_c\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\right]\frac{1}{R}\frac{\partial[n(\pi_{t+2},c_{T,t+2})b_{t+1}+n(1,c_{T,t+2})b_{t+1}^\pi+b_{t+1}^*]}{\partial b_t^\pi}
\end{aligned}
\tag{3.64}
$$

Rearranging terms we get

$$\begin{aligned}
&\frac{\partial n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)}{\partial b_t^\pi} b_t + \frac{\partial [n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) b_t^\pi]}{\partial b_t^\pi} \\
&= \frac{1}{1 + b_t n_c\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) + b_t^\pi n_c\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)} \\
&\{b_t n_\pi\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) \mathcal{P}_{b^\pi}(b_t, b_t^\pi, b_t^*) + n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) \\
&+ [b_t n_c\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)] \frac{1}{R} \frac{\partial [n(\pi_{t+2}, c_{T,t+2}) b_{t+1} + n(1, c_{T,t+2}) b_{t+1}^\pi + b_{t+1}^*]}{\partial b_t^\pi} \\
&+ [b_t^\pi n_c\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)] \frac{1}{R} \frac{\partial [n(\pi_{t+2}, c_{T,t+2}) b_{t+1} + n(1, c_{T,t+2}) b_{t+1}^\pi + b_{t+1}^*]}{\partial b_t^\pi} \Bigg\}
\end{aligned} \tag{3.65}$$

Finally, we substitute this equation into (3.48), to obtain (using (3.46)) the modified Euler equation for inflation-indexed debt

$$\begin{aligned}
\frac{u'(c_t) C_{c_T,t}}{1 + n_c(\pi_t, c_{T,t}) b_{t-1} + n_c(1, c_{T,t}) b_{t-1}^\pi} &= \beta R \frac{u'(c_{t+1}) C_{c_T,t+1}}{1 + n_c(\pi_{t+1}, c_{T,t+1}) b_t + n_c(1, c_{T,t+1}) b_t^\pi} \\
&\quad n(1, c_{T,t+1}) \left\{ \frac{\partial n\left(\mathcal{P}(b_t, b_t^\pi, b_t^*), \mathcal{C}(b_t, b_t^\pi, b_t^*)\right)}{\partial b_t^\pi} b_t \right. \\
&\quad \left. + \frac{\partial [n\left(1, \mathcal{C}(b_t, b_t^\pi, b_t^*)\right) b_t^\pi]}{\partial b_t^\pi} \right\}^{-1} \\
&= \beta R u'(c_{t+1}) C_{c_T,t+1} \\
&\quad \frac{1}{1 + \frac{b_t n_\pi(t+1) \pi_{b^\pi}(t+1) + [b_t n_c(t+1) + b_t^\pi m_c(t+1)] \frac{1}{R} \frac{\partial \left[n(t+2) b_{t+1} + m(t+2) b_{t+1}^\pi + b_{t+1}^*\right]}{\partial b_t^\pi}}{m(t+1)}}
\end{aligned} \tag{3.66}$$

So

$$u_t' C_{c_T,t} = \beta R u_{t+1}' C_{c_T,t+1} \underbrace{\left(1 + n_{c_T,t} b_{t-1} + m_{c,t} b_{t-1}^\pi\right)}_{\text{Dilution through RER}} \frac{1}{\underbrace{1 + \frac{b_t n_{\pi,t+1} \pi_{b^\pi,t+1} + (b_t n_{c_T,t+1} + b_t^\pi m_{c,t+1}) \frac{1}{R} \frac{\partial \left(n_{t+2} b_{t+1} + m_{t+2} b_{t+1}^\pi + b_{t+1}^*\right)}{\partial b_t^\pi}}{m_{t+1}}}_{\text{Discipline effect}}} \tag{3.67}$$

Finally, to obtain the Euler equation for debt in foreign currency, we first combine equations (3.53), (3.58) and (3.59)

$$\begin{aligned}
&1+\frac{\partial n\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)}{\partial b_t^*}b_t+\frac{\partial n\left(1,\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)}{\partial b_t^*}b_t^{\pi}=1\\
&+b_t n_{\pi}\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\mathcal{P}_{b^*}(b_t,b_t^{\pi},b_t^*)\\
&+b_t n_{c}\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\mathcal{C}_{b^*}(b_t,b_t^{\pi},b_t^*)\\
&+b_t^{\pi} n_{c}\left(1,\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\mathcal{C}_{b^*}(b_t,b_t^{\pi},b_t^*)\\
&=1+b_t n_{\pi}\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\mathcal{P}_{b^*}(b_t,b_t^{\pi},b_t^*)\\
&+\left[b_t n_{c}\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)+b_t^{\pi} n_{c}\left(1,\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\right]\mathcal{C}_{b^*}(b_t,b_t^{\pi},b_t^*)\\
&=1+b_t n_{\pi}\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\mathcal{P}_{b^*}(b_t,b_t^{\pi},b_t^*)\\
&-\left[b_t n_{c}\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)+b_t^{\pi} n_{c}\left(1,\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\right]\frac{\partial n\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)}{\partial b_t^*}b_t\\
&-\left[b_t n_{c}\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)+b_t^{\pi} n_{c}\left(1,\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\right]\frac{\partial n\left(1,\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)}{\partial b_t^*}b^{\pi}\\
&-\left[b_t n_{c}\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)+b_t^{\pi} n_{c}\left(1,\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\right]\\
&+\left[b_t n_{c}\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\right]\frac{1}{R}\frac{\partial[n(\pi_{t+2},c_{T,t+2})b_{t+1}+n(1,c_{T,t+2})b_{t+1}^{\pi}+b_{t+1}^*]}{\partial b_t^*}\\
&+\left[b_t^{\pi} n_{c}\left(1,\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\right]\frac{1}{R}\frac{\partial[n(\pi_{t+2},c_{T,t+2})b_{t+1}+n(1,c_{T,t+2})b_{t+1}^{\pi}+b_{t+1}^*]}{\partial b_t^*}\\
&=1+b_t n_{\pi}\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\mathcal{P}_{b^*}(b_t,b_t^{\pi},b_t^*)\\
&-\left[b_t n_{c}\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\right]\left[1+\frac{\partial n\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)}{\partial b_t^*}b_t+\frac{\partial n\left(1,\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)}{\partial b_t^*}b_t^{\pi}\right]\\
&-\left[b_t^{\pi} n_{c}\left(1,\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\right]\left[1+\frac{\partial n\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)}{\partial b_t^*}b_t+\frac{\partial n\left(1,\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)}{\partial b_t^*}b_t^{\pi}\right]\\
&+\left[b_t n_{c}\left(\mathcal{P}(b_t,b_t^{\pi},b_t^*),\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\right]\frac{1}{R}\frac{\partial[n(\pi_{t+2},c_{T,t+2})b_{t+1}+n(1,c_{T,t+2})b_{t+1}^{\pi}+b_{t+1}^*]}{\partial b_t^*}\\
&+\left[b_t^{\pi} n_{c}\left(1,\mathcal{C}(b_t,b_t^{\pi},b_t^*)\right)\right]\frac{1}{R}\frac{\partial[n(\pi_{t+2},c_{T,t+2})b_{t+1}+n(1,c_{T,t+2})b_{t+1}^{\pi}+b_{t+1}^*]}{\partial b_t^*}
\end{aligned} \tag{3.68}$$

Rearranging terms we get

$$
\begin{aligned}
&1+\frac{\partial n\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)}{\partial b_t^*}b_t+\frac{\partial n\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)}{\partial b_t^*}b_t^\pi \\
&=\frac{1}{1+b_t n_c\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)+b_t^\pi n_c\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)} \\
&\{1+b_t n_\pi\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\mathcal{P}_{b^*}(b_t,b_t^\pi,b_t^*) \\
&+\left[b_t n_c\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\right]\frac{1}{R}\frac{\partial[n(\pi_{t+2},c_{T,t+2})b_{t+1}+n(1,c_{T,t+2})b_{t+1}^\pi+b_{t+1}^*]}{\partial b_t^*} \\
&+\left[b_t^\pi n_c\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)\right]\frac{1}{R}\frac{\partial[n(\pi_{t+2},c_{T,t+2})b_{t+1}+n(1,c_{T,t+2})b_{t+1}^\pi+b_{t+1}^*]}{\partial b_t^*}\Bigg\}
\end{aligned}
\tag{3.69}
$$

Finally, we substitute this equation into (3.49), to obtain (using (3.46)) the modified Euler equation for debt in foreign currency

$$
\begin{aligned}
&\frac{u'(c_t)C_{c_T,t}}{1+n_c(\pi_t,c_{T,t})b_{t-1}+n_c(1,c_{T,t})b_{t-1}^\pi} \\
&=\beta R\frac{u'(c_{t+1})C_{c_T,t+1}}{1+n_c(\pi_{t+1},c_{T,t+1})b_t+n_c(1,c_{T,t+1})b_t^\pi} \\
&\left\{1+\frac{\partial n\left(\mathcal{P}(b_t,b_t^\pi,b_t^*),\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)}{\partial b_t^*}b_t+\frac{\partial n\left(1,\mathcal{C}(b_t,b_t^\pi,b_t^*)\right)}{\partial b_t^*}b_t^\pi\right\}^{-1} \\
&=\beta R u'(c_{t+1})C_{c_T,t+1} \\
&\frac{1}{1+b_t n_\pi(t+1)\pi_{b^*}(t+1)+[b_t n_c(t+1)+b_t^\pi m_c(t+1)]\frac{1}{R}\frac{\partial\left[n(t+2)b_{t+1}+m(t+2)b_{t+1}^\pi+b_{t+1}^*\right]}{\partial b_t^*}}
\end{aligned}
\tag{3.70}
$$

So

$$
\begin{aligned}
u_t'C_{c_T,t}=\beta R u_{t+1}'C_{c_T,t+1}&\underbrace{\left(1+n_{c_T,t}b_{t-1}+m_{c,t}b_{t-1}^\pi\right)}_{\text{Dilution through RER}} \\
&\underbrace{\frac{1}{1+b_t n_{\pi,t+1}\pi_{b^*,t+1}+\left(b_t n_{c_T,t+1}+b_t^\pi m_{c,t+1}\right)\frac{1}{R}\frac{\partial\left(n_{t+2}b_{t+1}+m_{t+2}b_{t+1}^\pi+b_{t+1}^*\right)}{\partial b_t^*}}}_{\text{Discipline effect}}
\end{aligned}
\tag{3.71}
$$

3.B.3 Calibrated model

Let $\mathbf{s}_t = \left(b^*_{t-1}, \hat{b}_{t-1}, \hat{b}^{\pi}_{t-1}, y_{T,t}\right)$ denote the vector of state variables, where $\hat{b}_t = b_t/\alpha$ and $\hat{b}^{\pi}_t = b^{\pi}_t/\alpha$. The Lagrangian for the policy problem in the calibrated section is given by

$$
\begin{aligned}
\mathcal{L} &= u(C(c_{T,t}, y_N)) - l(\pi_t) + \beta E_t[V(s_{t+1})] \\
&+ \lambda_t \left\{ y_{T,t} - c_{T,t} - b^*_{t-1} - \frac{c_{T,t}^{1-\alpha}}{\pi}\hat{b}_{t-1} - c_{T,t}^{1-\alpha}\hat{b}^{\pi}_{t-1} \right. \\
&+ \frac{1}{R-\delta}(b^*_t - \delta b^*_{t-1}) + \frac{1}{R}E\left[\mathcal{X}(\mathbf{s}_{t+1})\right]\left(\hat{b}_t - \delta\frac{\hat{b}_{t-1}}{\pi_t}\right) + \frac{1}{R}E\left[\mathcal{Z}(\mathbf{s}_{t+1})\right]\left(\hat{b}^{\pi}_t - \delta\hat{b}^{\pi}_{t-1}\right) \\
&\left. - \frac{\psi_b}{2}\left(\frac{1}{R-\delta}b^*_t - \bar{B}^*\right)^2 - \frac{\psi_b}{2}\left(\frac{1}{R}E\left[\mathcal{X}(\mathbf{s}_{t+1})\right]\hat{b}_t - \bar{B}\right)^2 - \frac{\psi_b}{2}\left(\frac{1}{R}E\left[\mathcal{Z}(\mathbf{s}_{t+1})\right]\hat{b}^{\pi}_t - \bar{B}^{\pi}\right)^2 \right\}
\end{aligned} \tag{3.72}
$$

where the model equilibrium also involves the bond prices definitions given by

$$Q(s_t) = \frac{1}{c_{T,t}^{1-\alpha}}\frac{1}{R}E_t[\mathcal{X}(s_{t+1})] \tag{3.73}$$

$$Q^{\pi}(s_t) = \frac{1}{c_{T,t}^{1-\alpha}}\frac{1}{R}E_t[\mathcal{Z}(s_{t+1})] \tag{3.74}$$

and the process for tradable endowment (3.29).

The first order conditions of the problem are

$$c_T : \alpha c_T^{\alpha(1-\sigma)-1} = \lambda \left[1 + (1-\alpha) c_T^{-\alpha} \left(\frac{\hat{b}}{\pi} + \hat{b}^{\pi}\right)\right] \tag{3.75}$$

$$\pi : \psi(\pi - \bar{\pi}) = \lambda \left\{ \frac{c_T^{1-\alpha}}{\pi^2} \hat{b} + \frac{1}{R} E\left[\mathcal{X}(\mathbf{s}')\right] \frac{\delta \hat{b}}{\pi^2} \right\} \tag{3.76}$$

$$\begin{aligned} b^{*\prime} : \beta E\left[V_{b^*}(\mathbf{s}')\right] = -\lambda \Bigg\{ & \frac{1}{R-\delta} + \frac{1}{R} E[\mathcal{X}_{b^*}(\mathbf{s}')] \left(\hat{b}' - \delta \frac{\hat{b}}{\pi}\right) + \frac{1}{R} E[\mathcal{Z}_{b^*}(\mathbf{s}')] \left(\hat{b}^{\pi\prime} - \delta \hat{b}^{\pi}\right) \\ & - \psi_b \left(\frac{1}{R-\delta} b^{*\prime} - \bar{B}^*\right) \frac{1}{R-\delta} - \psi_b \left(\frac{1}{R} E\left[\mathcal{X}(\mathbf{s}')\right] \hat{b}' - \bar{B}\right) \frac{1}{R} E\left[\mathcal{X}_{b^*}(\mathbf{s}')\right] \hat{b}' \\ & - \psi_b \left(\frac{1}{R} E\left[\mathcal{Z}(\mathbf{s}')\right] \hat{b}^{\pi\prime} - \bar{B}^{\pi}\right) \frac{1}{R} E\left[\mathcal{Z}_{b^*}(\mathbf{s}')\right] \hat{b}^{\pi\prime} \Bigg\} \end{aligned} \tag{3.77}$$

$$\begin{aligned} \hat{b}' : \beta E\left[V_b(\mathbf{s}')\right] = -\lambda \frac{1}{R} \Bigg\{ & E[\mathcal{X}_b(\mathbf{s}')] \left(\hat{b}' - \delta \frac{\hat{b}}{\pi}\right) + E[\mathcal{Z}_b(\mathbf{s}')] \left(\hat{b}^{\pi\prime} - \delta \hat{b}^{\pi}\right) + E[\mathcal{X}(\mathbf{s}')] \\ & - \psi_b \left(\frac{1}{R} E\left[\mathcal{X}(\mathbf{s}')\right) \hat{b}' - \bar{B}\right) \left(\frac{1}{R} E\left[\mathcal{X}_b(\mathbf{s}')\right] \hat{b}' + \frac{1}{R} E\left[\mathcal{X}(\mathbf{s}')\right]\right) \\ & - \psi_b \left(\frac{1}{R} E\left[\mathcal{Z}(\mathbf{s}')\right] \hat{b}^{\pi\prime} - \bar{B}^{\pi}\right) \frac{1}{R} E\left[\mathcal{Z}_b(\mathbf{s}')\right] \hat{b}^{\pi\prime} \Bigg\} \end{aligned} \tag{3.78}$$

$$\begin{aligned} \hat{b}^{\pi\prime} : \beta E\left[V_{b^{\pi}}(\mathbf{s}')\right] = -\lambda \frac{1}{R} \Bigg\{ & E[\mathcal{X}_{b^{\pi}}(\mathbf{s}')] \left(\hat{b}' - \delta \frac{\hat{b}}{\pi}\right) + E[\mathcal{Z}_{b^{\pi}}(\mathbf{s}')] \left(\hat{b}^{\pi\prime} - \delta \hat{b}^{\pi}\right) + E[\mathcal{Z}(\mathbf{s}')] \\ & - \psi_b \left(\frac{1}{R} E\left[\mathcal{X}(\mathbf{s}')\right] \hat{b}' - \bar{B}\right) \frac{1}{R} E\left[\mathcal{X}_{b^{\pi}}(\mathbf{s}')\right] \hat{b}' \\ & - \psi_b \left(\frac{1}{R} E\left[\mathcal{Z}(\mathbf{s}')\right] \hat{b}^{\pi\prime} - \bar{B}^{\pi}\right) \left(\frac{1}{R} E\left[\mathcal{Z}_{b^{\pi}}(\mathbf{s}')\right] \hat{b}^{\pi\prime} + \frac{1}{R} E\left[\mathcal{Z}(\mathbf{s}')\right]\right) \Bigg\} \end{aligned} \tag{3.79}$$

Appendix 3.C Solution method

This section describes the solution method used to solve the model described in section 3.3.

There are 4 state variables; three endogenous, b^*, $\hat{b} = \frac{1}{\alpha} b$ and $\hat{b}^{\pi} = \frac{1}{\alpha} b^{\pi}$; and one exogenous, y_T (assumption: $y_N = 1$). Let $\mathbf{s} = \left(b^*, \hat{b}, \hat{b}^{\pi}, y_T\right)$ denote the vector of state variables. There are 7 control variables: $c_T, \pi, b^{*\prime}, \hat{b}', \hat{b}^{\pi\prime}, Q, Q^{\pi}$. Correspondingly, there are 7 functional equations, defined by the set of FOCs, (3.76)-(3.79), the resource constraint (3.30), and the bond prices, (3.73) and (3.74), with

$$\lambda = \frac{\alpha c_T^{\alpha(1-\sigma)-1}}{1 + (1-\alpha) c_T^{-\alpha} \left(\frac{\hat{b}}{\pi} + \hat{b}^{\pi} \right)}$$

$$\mathcal{X}(\mathbf{s}') \equiv \frac{{c'_T}^{1-\alpha}}{\pi'} \left(1 + \delta Q(\mathbf{s}')\right)$$

$$\mathcal{Z}(\mathbf{s}') \equiv {c'_T}^{1-\alpha} \left(1 + \delta Q^{\pi}(\mathbf{s}')\right)$$

$$E[\mathcal{X}_{b^*}(\mathbf{s}')] = (1-\alpha) \frac{{c'_T}^{-\alpha}}{\pi'} \left(1 + \delta Q(\mathbf{s}')\right) \frac{\partial c'}{\partial b^{*\prime}} - \frac{{c'_T}^{1-\alpha}}{(\pi')^2} \left(1 + \delta Q(\mathbf{s}')\right) \frac{\partial \pi'}{\partial b^{*\prime}} + \delta \frac{{c'_T}^{1-\alpha}}{\pi'} \frac{\partial Q(s')}{\partial b^{*\prime}}$$

$$E[\mathcal{X}_{b}(\mathbf{s}')] = (1-\alpha) \frac{{c'_T}^{-\alpha}}{\pi'} \left(1 + \delta Q(\mathbf{s}')\right) \frac{\partial c'}{\partial b'} - \frac{{c'_T}^{1-\alpha}}{(\pi')^2} \left(1 + \delta Q(\mathbf{s}')\right) \frac{\partial \pi'}{\partial b'} + \delta \frac{{c'_T}^{1-\alpha}}{\pi'} \frac{\partial Q(s')}{\partial b'}$$

$$E[\mathcal{X}_{b^{\pi}}(\mathbf{s}')] = (1-\alpha) \frac{{c'_T}^{-\alpha}}{\pi'} \left(1 + \delta Q(\mathbf{s}')\right) \frac{\partial c'}{\partial b^{\pi\prime}} - \frac{{c'_T}^{1-\alpha}}{(\pi')^2} \left(1 + \delta Q(\mathbf{s}')\right) \frac{\partial \pi'}{\partial b^{\pi\prime}} + \delta \frac{{c'_T}^{1-\alpha}}{\pi'} \frac{\partial Q(s')}{\partial b^{\pi\prime}}$$

$$E[\mathcal{Z}_{b^*}(\mathbf{s}')] = (1-\alpha) {c'_T}^{1-\alpha} \left(1 + \delta Q^{\pi}(\mathbf{s}')\right) \frac{\partial c'}{\partial b^{*\prime}} + \delta {c'_T}^{1-\alpha} \frac{\partial Q^{\pi}(s')}{\partial b^{*\prime}}$$

$$E[\mathcal{Z}_{b}(\mathbf{s}')] = (1-\alpha) {c'_T}^{1-\alpha} \left(1 + \delta Q^{\pi}(\mathbf{s}')\right) \frac{\partial c'}{\partial b'} + \delta {c'_T}^{1-\alpha} \frac{\partial Q^{\pi}(s')}{\partial b'}$$

$$E[\mathcal{Z}_{b^{\pi}}(\mathbf{s}')] = (1-\alpha) {c'_T}^{1-\alpha} \left(1 + \delta Q^{\pi}(\mathbf{s}')\right) \frac{\partial c'}{\partial b^{\pi\prime}} + \delta {c'_T}^{1-\alpha} \frac{\partial Q^{\pi}(s')}{\partial b^{\pi\prime}}$$

I approximate a linear policy function for each control variable and, following the Taylor projection method developed in Levintal (2018), I set to zero all the functional equations and their partial derivatives with respect to each state variables. Thus, I have a system of 35 equations (7 functional equations and 28 partial derivatives) and 35 unknowns (constant plus 4 coefficients corresponding to each state variable, for each control variable). I first obtain a solution at one point of the grid using Ottonello and Perez (2019) as a benchmark. Then, I simulate the exogenous stochastic process for tradable endowment for 100,000 periods.

Appendix 3.D Analysis of additional parameters

In addition to the parameters analyzed in subsection 3.5.2, I consider the effect of changing the risk aversion parameter, σ, and the share of tradables in aggregate consumption, α. Figure 3.D.1 plots these outcomes, considering the model calibrated to the average Latin American country as the baseline.

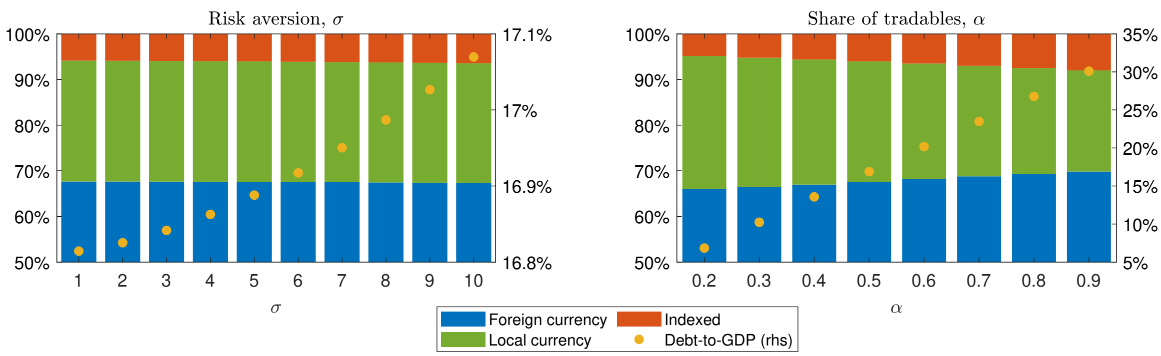

Figure 3.D.1: Currency composition (left-hand side axis) and average external debt-to-GDP (right-hand-side axis) for different parameter values. Except for the parameter subject to change, the calibration corresponds to the average Latin American country.

On the left panel of Figure 3.D.1 we observe that both currency composition and average external debt-to-GDP remain relatively stable for different levels of risk aversion, σ. This affects the marginal utility of consumption, therefore affects the hedging benefits (consumption smoothing), as well as the inflation and real exchange rate devaluation incentives. In this numerical results we observe that these benefits and costs get almost offset.

On the right panel of Figure 3.D.1 we observe that as α increases, there is a slight increase in the share of foreign currency and inflation-indexed debt, to the detriment of the share of local currency debt. Additionally, as α increases, so does the average external debt-to-GDP ratio. The share of tradables in aggregate consumption, α, affects the marginal utility of consumption as well as the real exchange rate, and therefore hedging benefits and costs derived from inflation and real exchange rate depreciation bias.

Chapter 4

Conclusion

This document examines the links between optimal monetary and fiscal policy and sovereign debt structure, in light of the changes in currency composition and maturity structure verified by emerging economies in the last twenty years. First, I study how different debt structures, in terms of currency composition and maturity, affect the optimal monetary and fiscal policy response of the government. Second, I endogenize the currency composition decision, and use this model to explain the differences observed across Latin American countries.

I leave for future research studying the optimal decision of debt duration and distortionary taxes. Optimal debt duration is study by Leeper, Leith, and Liu (2021) in a closed economy framework. Distortionary taxes introduce an extra cost to altering intertemporal consumption decisions, and will affect hedging benefits and costs derived from dilution incentives. Additionally, I would like to incorporate domestic debt to the analysis. Recently, emerging economies have registered an increase in the share of debt held by locals. This type of bonds creates different incentives for the government, since it cares about the welfare of these bondholders. The government can interfere in its regulation, as long as these bonds are issued under domestic jurisdiction, which could bring financial repression concerns.