Quantum Mechanical Operator Algebras and The Logic of Quantum Measurement

Morrison Turnansky

Scottsdale, Arizona

B.S. Mathematics, College of William and Mary, 2016

M.S. Mathematics, University of Virginia, 2019

A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia

May 2023

Comittee Members:

Christian Gromoll

Advisor: Benjamin Hayes

Robert Jones

David Sherman

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April 27, 2023

Abstract

In quantum mechanics, the connection between the operator algebraic realization and the logical models of measurement of state observables has long been an open question. In the approach that is presented here, we introduce a new mathematical structure called a cubic lattice. We claim that the cubic lattice may be faithfully realized as a subset of the self-adjoint space of a von Neumann algebra. Furthermore, we obtain a unitary representation of the symmetry group of the cubic lattice. In so doing, we re-derive the classic quantum gates and gain a description of how they govern a system of qubits of arbitrary cardinality. Evidently, this setup gives rise to a new empirical logical model of the quantum measurement problem. We note that all previous attempts to construct an empirical logical model for quantum mechanical measurement have failed. We conclude with a new generalization of the cubic lattice relating to higher spin systems, which leads us to new operator algebraic structures derived from its symmetry group.

Acknowledgements

This thesis would not have been possible without the assistance both directly and indirectly of many. My sincerest gratitude to the following. My parents and sister for their infinite love and support that is far beyond what words can express. Dr. Joseph Oliveira for sharing his PhD work, which became the starting point of this thesis, reinvigorating my passion for mathematics, and suggesting literature for inspiration. My advisor, Dr. Benjamin Hayes, for the patience throughout many discussions, guidance, kindness, and technical expertise. Dr. Charles R. Johnson, my undergraduate mathematics advisor, for taking a chance on a young freshman asking to try math research and stoking my love of mathematics. The friends I met while at UVA for helping to make Charlotesville feel like home. The gymnastics community, especially the College of William and Mary, for giving me a sense of belonging, many of my best memories, and shaping me during the most formative years of my life. To all of you, I am forever in your debt.

Contents

Outline

The intention of this thesis is to explore the mathematics of qubits from an axiomatic perspective.

Summary

1. Section 1 introduces the cubic lattice and outlines some necessary definitions from lattice theory.

2. Section 2 lifts the lattice theoretic properties of the cubic lattice into a Banach space structure, and we introduce the notion of its dual space. The novel results are concentrated in sections 3,4,5, and 6.

3. Section 3 demonstrates how one can move from a purely geometric version of the cubic lattice to an operator theory based version contained in a Hilbert lattice.

4. Section 4 discusses the various operations one can perform on a cubic lattice embedded in a Hilbert lattice and the outline for a potential logic. In particular, we discuss a unitary operator that agrees in some sense with the $(\cdot)^{\perp}$ operation of the Hilbert lattice.

5. Section 5 deduces the standard set of universal quantum gates from the cubic lattice in the formal context of operator algebras. By doing so, we extend these notions beyond the traditional finite dimensional restrictions associated with such a set of standard quantum gates to vector spaces of any cardinal dimension. We also lift the relations of the cubic lattice to quantum relations.

6. Section 6 generalizes the above results for the cubic lattice to the multi-cubic lattice.

This work may be of potential interest to a wide audience. Please review sections 1 and 2 per your background. Of course, I would hope that everyone would find all of the sections of interest, however for the respective communities, I would recommend the following. For those interested in operator algebras, sections 3, 5, and 6 are of interest. For those readers interested in logic and quantum computations, I would refer to sections 3,4, and 5.

Main Results

We create the first Hilbert Lattice realization of a cubic lattice.

Main Result 1 (Theorem 3.1.13)**.** *Let H be a Hilbert space constructed as a tensor product of* 2 *dimensional spaces over an index set I. For the given Hilbert lattice HL of H, there exists a cubic lattice CL such that $CL \subseteq HL$, and the atoms of CL are projections onto subspaces H forming an orthonormal basis of H.*

We consider the minimal von Neumann algebra containing CL as well.

Main Result 2 (Theorem 5.2.8)**.** *The atoms of $W^*(\{s_i\}_{i\in I})$ are the atoms of CL.*

We proceed to describe the algebra in our embedding of the cubic lattice.

Main Result 3 (Theorem 5.2.11)**.** $B(H) = W^*(\{Us_iU^*\}_{i\in I}, \{s_i\}_{i\in I})$.

As a consequence, we generalize the Pauli matrices to infinite systems of qubits in our choice of matrix units when considered as a representation of $M_2(B)$ as opposed to $M_2(\mathbb{C})$, where $B \cong I_2 \otimes B(H_{I-i})$ for an indexing set I.

$$U_{\Delta_i} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ s_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \text{and } is_iU_{\Delta_i} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

We introduce a new structure, the critical multi-cubic lattice, and use it to generalize the operator algebraic structure of the cubic lattice. One can view the critical multi-cubic lattice as the a generalization of the reflection symmetries of a cube. In this sense, $Aut(\mathbb{Z}_{2k+1})$ enforces this symmetry.

Main Result 4 (Theorem 6.2.19)**.** *Let M be an $|I|$-critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$. Then $Per_{Aut(\mathbb{Z}_{2k+1})}(C_M) \cong C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1})) \wr S_I$. Let $Aut(\mathbb{Z}_{2k+1})$ be generated by $\{\sigma_i\}_{i=1}^k$, then $C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1})) = \cap_{i=1}^k C_{S_{2k}}(\sigma_i)$, and $C_{S_{2k}}(\sigma_i) \cong \Pi_{j=1}^{l_i}(\mathbb{Z}_{j_i} \wr S_{N_{j_i}})$.*

We highlight that this result reduces to the case of the cubic lattice when $2k + 1 = 3$, and the group reduces to an infinite version of the Coxeter group B_n. We also get a similar structure to the cubic lattice when $2k + 1$ is prime.

Main Result 5 (Theorem 6.4.10)**.** $B(H) = W^*(\{U_H p_c U_H^*\}_{c\in C}, \{p_c\}_{c\in C})$ *if and only if $2k + 1$ is prime.*

1 Background and Definitions

1.1 Background

Quantum mechanics has its roots in Planck's theory of black body radiation and the realization by Planck that the quantized energy of the radiator could be derived by the equation $E = h\nu$. In the following decades the nature of quantum mechanics began to unveil itself with the progenitors of the field such as Boltzmann, Planck, and Einstein. As experimental data accumulated, the theory began to standardize into the two interpretations of Schrödinger and Heisenberg. The key physical characteristics of a quantum mechanical system in either interpretation are the Hamiltonian describing the energy state of the system, the states of the system such as position or momentum, and the observables representing the action of measuring the system. Finally in the 1930's, there was a unification of ideas beginning with the study of operator theory under the guidance of Schrödinger, Pauli, Heisenberg, de Broglie, Dirac, von Neumann, and Bohr and his school. To this end, the von Neumann axioms, sometimes referred to as the Dirac-von Neumann axioms are given as follows:

1) The states of a quantum system are unit vectors in a complex Hilbert space, H.

2) The observables are self adjoint operators in H.

3) The probability that an observable, A, is in state ϕ is given by the inner product $\langle \phi, A\phi \rangle$.

There are some exceptions to these axioms in particular statement 1. In full generality, the observables of a quantum system are self adjoint operators represented by their projection valued measures on their respective spectra where the states are linear functionals. To be concrete, if A is an observable, and ω is a state of the system, then the expectation value of an observable, a random variable, A in state ω is given by $\omega(A) :=$

$$\int_{\sigma(A)} \lambda d\omega(e_\lambda),$$

where $\lambda \to \omega(e_\lambda)$ is a cumulative distribution function, whose law is determined by the inner product $\langle \omega, \omega \rangle$ acting on e_λ such that $||\omega|||_2 = 1$. By Borel functional calculus, we can obtain new observables $f(A)$ by applying a Borel function f to the projection valued measure of A. For our application, these Borel functions will often be the indicator functions $\omega(\chi_E(A))$, which is the probability that the observable A will have a value in the set E when the system is in state ω [2]. As indicator functions are projections when viewed in a von Neumann algebra, we see that that the study of projection operators will be central to our work.

The notion of a Boolean logic and its associated Boolean algebra for computation has become ubiquitous as opposed to the inherent non Boolean nature of quantum mechanics. There has been and currently is great interest in discovering the subtleties of the non-standard logic of quantum computation in order to close the gap between the theory of quantum mechanics, computation, and the measurements of entangled states using quantum mechanical systems. Perhaps most famously posed by Deutsch [6] and Feynman is the ideal of true quantum computation systems derived from true quantized simulations. In this vein, there is the traditional notion of quantum logic, which relies upon the notion of orthogonality. However, as we will see there is much more to this story, which is currently mathematically incomplete.

Definition 1.1.1 ([10]). *An associative ortho-algebra (AOA) is system* $(L, +, 0, 1)$ *where* L *is a set* $0, 1 \in L$*, and* $+$ *is a binary operation with domain* $D(+)$ *satisfying the following four conditions:*

i) If $(p, q) \in D(+)$*, then* $(q, p) \in D(+)$ *and* $p + q = q + p$*.*

ii) If (p, q)*,* $(p + q, r) \in D(+)$*, then* $(q, r), (p, q + r) \in D(+)$ *and* $(p + q) + r = p + (q + r)$*.*

iii) For each $p \in L$ *there exists a unique* $q \in L$ *such that* $(p, q) \in D(+)$ *and* $p + q = 1$*.*

iv) If $(p, p) \in D(+)$*, then* $p = 0$*.*

Example 1.1.2. *Let* H *be a Hilbert space,* $L = L(H)$*,* 0 *the zero operator, and* $1 = I$*. Then the orthogonal projection operators on the space,* $(L, +, 0, 1)$ *form an AOA.*

In physics, the notion of reflection symmetries is paramount as they reveal isometric invariants with respect to a system of parameters. For a given space, the reflection symmetries under consideration here can be represented as a subgroup of the unitary operators. However, isometry groups associated with ortho-algebras and their symmetry groups have not been fully explored. This fact will be discussed and demonstrated to be in some sense contradictory. A goal here will be to develop a model in which this apparent contradiction is eliminated thereby establishing a consistent set of structures for our applications.

1.2 Lattice Definitions

The primary idea is to combine two distinct notions of quantum logic by considering their corresponding lattices. We begin with a discussion of lattices.

Definition 1.2.1 ([9]). *A lattice,* L*, is a partially ordered set in which every pair of elements* $a, b \in L$ *has a greatest lower bound represented by* $a \wedge b$ *and a least upper bound represented by* $a \vee b$*.*

Definition 1.2.2. *A lattice* L *is distributive if the following two properties hold for all* $x, y, z \in L$

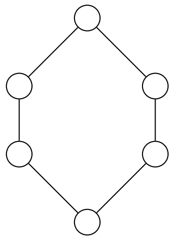
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{1}$$

$$x \vee (y \wedge z) = (x \wedge y) \vee (x \vee z) \tag{2}$$

Definition 1.2.3. *The complement of an element $x \in L$ is an element $x^\perp$ such that $x \wedge x^\perp = 0$ and $x \vee x^\perp = 1$.*

Definition 1.2.4. *A complemented lattice is a lattice in which all elements of the lattice have a complement. Note that in general the complement need not be unique.*

Example 1.2.5. *Assume the ordering is induced vertically in the hexagonal lattice, so that the top vertex has the highest order while the bottom vertex has the lowest order. Then there is a unique orthocomplement, which can be see as a reflection about the horizontal axis, and there is an order preserving complement seen as a reflection about the vertical axis.*



It is worth adressing that in our context, the notion of complement is far more general than the standard Boolean notion. For example, De Morgan's laws need not hold.

Definition 1.2.6. *A orthocomplemented lattice is a complete lattice with an order reversing complement.*

Definition 1.2.7. *An orthomodular lattice is an orthocomplemnted lattice such that $x \leq y$ implies $x \vee (x^\perp \wedge y) = x \vee y$.*

Orthomodular lattices and distributive lattices are two ideas one takes for granted in standard logic. It is worth noting that distributivity forces an already orthomodular lattice to have a unique orthocomplement and is therefore Boolean [17].

Definition 1.2.8. *A lattice L is Boolean if L is distributive lattice, with a largest element 1 and a least element 0, and has a unique orthocomplement.*

As one may realize, a lattice having both of these properties is far too rigid to allow for quantum logic, as quantum logic is much more general than standard Boolean logic. We need to consider a structure that allows for a sufficiently more general logic.

Definition 1.2.9. *An order ideal of a poset P is subset $I \subseteq P$ such that if $x \in I$, $y \in P$, and $y \leq x$, then $y \in I$. A principle order ideal of a poset P is denoted $(x) = \{y \in P : y \leq x\}$. A lattice ideal of a lattice L is $I \subseteq L$ such that $a, b \in I$ implies $a \wedge b \in I$, and if $b \in L$, $a \in I$, and $b \leq a$,then $b \in I$.*

Definition 1.2.10. *An order filter of a poset P is a subset $J \subseteq P$ such that if $x \in J$, $y \in P$, $y \geq x$, then $y \in J$. A principle order filter of a poset P is denoted $[x] = \{y \in P : y \geq x\}$. A lattice filter of lattice L is $J \subseteq L$ such that $a, b \in J$ implies $a \vee b \in I$ and if $b \in L$, $a \in J$, and $b \geq a$, then $b \in J$.*

Definition 1.2.11. *Let L be a lattice with a minimal element, 0. An element $x \in L$, $x \neq 0$ is an atom if for all $y \in L$ such that $y \leq x$, $y = x$ or $y = 0$. Let L be a lattice with maximal element, 1. An element $y \in L$, $y \neq 1$ is a coatom if for all $x \in L$ such that $y \leq x$, then $x = y$ or $x = 1$.*

Definition 1.2.12. *An atomistic lattice is a lattice L where every element $x \in L$ may be written as the join of atoms. A coatomistic lattice is a lattice L where every element $x \in L$ may be written as the meet of coatoms.*

Definition 1.2.13. *A lattice L is atomic if $0 \neq x \in L$, then there exists an atom $p \in L$ such that $p \leq x$. A lattice L is coatomic if $1 \neq y \in L$, then there exists a coatom $q \in L$ such that $q \geq x$.*

1.3 Common Approaches

The standard approach for approach for describing the spin states of n qubits is to consider a tensor product of the form $\otimes_{i=1}^{n}\mathbb{C}^2$ creating of vector space dimension 2^n. In this setting each pure state is represented by an orthonormal basis vector

Example 1.3.1. *Using this system we have the classical notation and the Dirac Bra-ket notation. For a quantum state in which the kth quibit has an up spin we let the first coordinate be a 1 and 0 if the state is down. Thus, the 2 quibit state where both quibits have an up spin is reprented by $11\rangle$ or $[1111]^t$ written without the normallization constant of $\frac{1}{2}$. The remaining three pure states are the following:*

$$01\rangle = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, 10\rangle = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \text{ and } 00\rangle = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

The observables in this setting act on the vector space and as such are self adjoint matrices of $M_{2^n}(\mathbb{C})$.

Definition 1.3.2. *Let H be a Hilbert Space. We define the lattice of closed linear subspaces of H to be the Hilbert Lattice henceforth referred to as HL. In this context, $u \vee v = span\{u, v\}$, and $u \wedge v = span\{u\} \cap span\{v\}$. For finite dimensional subspaces, the lattice rank coincides with the standard rank.*

In some literature the Hilbert lattice is referred to as a standard lattice. The term is used because this is the standard construction of lattice of projection operators of a Hilbert space, we refer the reader to [21] for an in depth discussion. We will call the lattice HL.

The major issue with the above approach is that the geometry of the state space is not preserved because the dimensionality is too large. There are many unitary transformations that violate physical meaning, so we need a more restrictive symmetry group. The above construction has the following corresponding lattice structure.

From the above realization, we next introduce the isometries of an n-dimensional cubic face lattice and its dual in the category of posets, the octahedral lattice. This approach was developed by Dr. N. Metropolis and Dr. G. C. Rota [15]. The axiomatic structure was made correct by Dr. J S. Oliveira, and its generalization to arbitrary cardinals corresponding to an infinite dimensional cubic Banach lattice, was first considered by Dr. H. R. Fisher and then further developed by by Dr. J. S. Oliveira [7]. We introduce the necessary structure:

The cubic lattice can be thought of as a lattice of the faces of an n-cube.

Definition 1.3.3. *Let I^n denote the n-dimensional cube $[-1,1]^n$ embedded in $\mathbb{R}^n$.*

1) The set I^n is a convex subset of $\mathbb{R}^n$ whose extreme points are the points x with $x_i = \pm 1$ for $1 \leq i \leq n$.

2) The n-dimensional cube I^n is given by the closed convex hull of these finitely many extreme points, as such it is a polytope in $\mathbb{R}^n$. One can also view this as an application of the well known Krein Milman Theorem.

3) The cubic polytope may be obtained as the intersection of hyperplanes,

a) $H_i = \{x \in \mathbb{R}^n | x_i = 1\}$,

b) $L_i = \{x \in \mathbb{R}^n | x_i = -1\}$.

4) We next consider the corresponding half spaces,

a) $H_i^- = \{x \in \mathbb{R}^n | x_i \leq 1\}$,

b) $L_i^+ = \{x \in \mathbb{R}^n | x_i \geq -1\}$.

5) Then $I^n = (\cap_{i=1}^n H_i^-) \cap (\cap_{i=1}^n L_i^+)$.

Now then, let $\{\epsilon^i\}_{i=1}^n$ be the standard orthonormal basis of dual vectors.

Definition 1.3.4. *We describe a face $F \subseteq I^n$ as follows: Let A^+ and A^- be subsets of $\{1, 2, \ldots, n\}$ such that $x \in F$ if and only if $x \in I^n$ and $\epsilon^i(x) = 1$ for $i \in A^+$ and $\epsilon^i(x) = -1$ for $i \in A^-$.*

Definition 1.3.5. *The pair (A^+, A^-) uniquely determines the face F if $A^+ \cap A^- = \emptyset$.*

Definition 1.3.6. *If F, G are faces of I^n such that $F, G \neq \emptyset$, with $F = (A^+, A^-)$ and $G = (B^+, B^-)$, then $G \subseteq F$ if and only if $A^+ \subseteq B^+$ and $A^- \subseteq B^-$.*

Definition 1.3.7. *Let $\mathscr{F}(I^n)$ be the set of all faces of I^n ordered by the above notion, so that $\mathscr{F}(I^n)$ forms a complete lattice, where $\vee$ is the union of faces, and $\wedge$ is the intersection of faces.*

We now tie together the geometric notion of the faces of the n-cube to the lattice of signed sets.

Definition 1.3.8. *Let $S = \{1, 2, \ldots, n\}$ a signed set on S is a pair $x = (A^+, A^-)$ of subsets of S such that $A^+ \cap A^- = \emptyset$. The collection of signed sets is denoted by $L^+(S)$ is a poset with order relation $\leq$ defined by reverse inclusion $x = (A^+, A^-) \leq y = (B^+, B^-)$ if and only if $B^+ \subseteq A^+$ and $B^- \subseteq A^-$.*

Definition 1.3.9. *With the addition of a 0 element, $L^+(S)$ becomes a lattice denoted by $L(S)$ where for $x, y \in L(S)$, $x \vee y = (A^+ \cap B^+, A^- \cap B^-) \in L(S)$ and $x \wedge y = (A^+ \cup B^+, A^- \cup B^-) \in L(S)$ if $B^+ \cap A^- = \emptyset = B^- \cap A^+$ or $x \wedge y = 0 \in L(S)$ otherwise.*

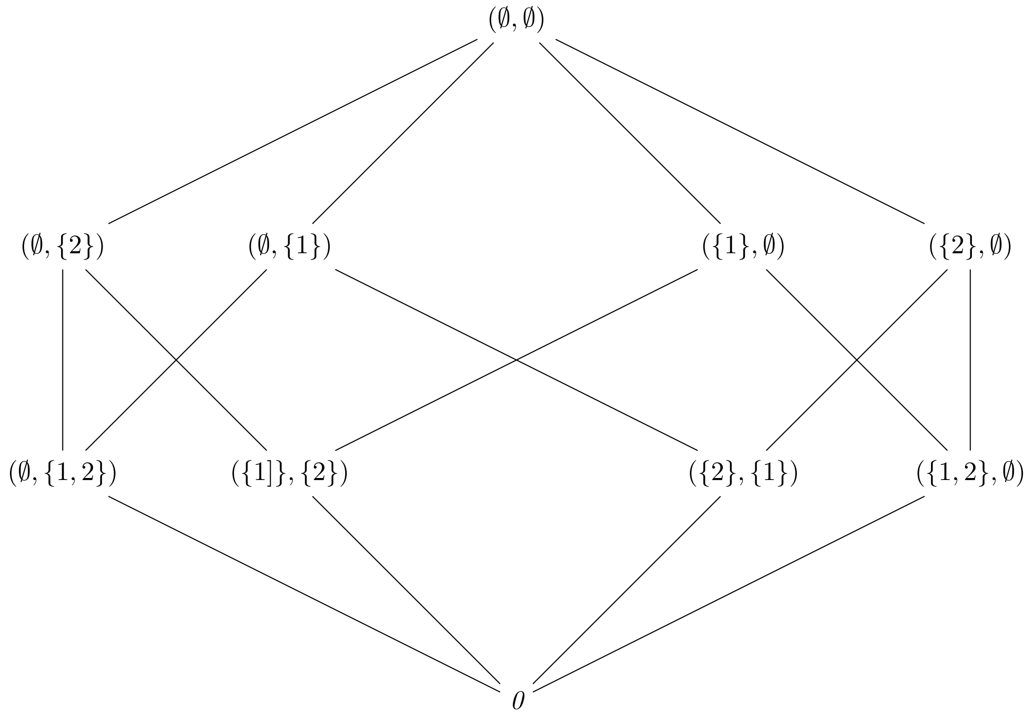
Definition 1.3.10. *Every lattice $L(S)$, in addition to the operations $\vee, \wedge$ also admits a partially defined operation $\Delta : L(S) \times L(S) \to L(S)$ defined by $\Delta(x, 0) = 0$, and if $0 < x = (A^+, A^-)$, $0 < y = (B^+, B^-)$, $y \leq x$, then $\Delta(x, y) = (A^+ \cup (B^- - A^-), A^- \cup (B^+ - A^+))$.*

Now we are ready to provide a generalization of the lattice of faces of the n-cube to higher ordered cardinals by defining the additional axioms of an infinite dimensional cubic lattice.

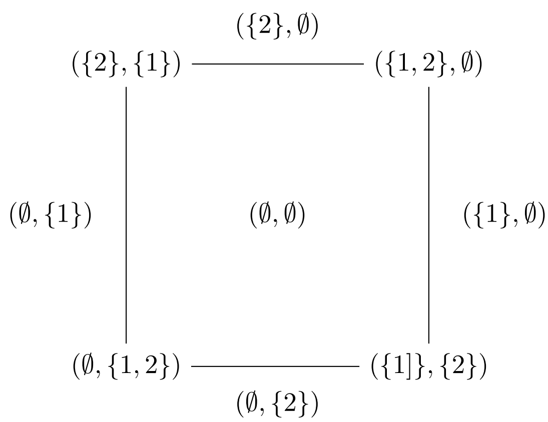
Definition 1.3.11 ([16])**.** *A cubic lattice, C, is a lattice with 0 and 1 satisfies the following axioms:*

1. *For $x \in L$, there is an order-preserving map $\Delta_x : (x) \to (x)$, (x) denotes the principal ideal generated by (x).*
2. *If $0 < a, b < x$, then $a \vee \Delta_x(b) < x$ if and only if $a \wedge b = 0$.*
3. *L is complete.*
4. *L is atomistic.*
5. *L is coatomistic.*

Example 1.3.12.



The face lattice of the 2-cube



Signed Sets of the 2-cube

We now have geometric realizations with their corresponding lattice realizations. New results will be derived by the merging of these realizations and deducing additional properties.

In this setting, we have an n dimensional vector with notation similar to the n length Bra-ket. An up spin for a quantum particle in the kth dimension is represented as a 1, a down spin is represented as a 0, but in this case we also have indeterminate measurable states represented by an X. The geometry of the observables of this system is an n-cube. The geometry of the state space is preserved allowing for reflection symmetries to act in a natural way. However, it is shown in [16] that orthogonality is not well defined for these structures. Repairs are in order here as they will be demonstrated in the following sections of this document.

Example 1.3.13.

$$\begin{tikzcd}
10|\rangle \arrow[r, dash, "1X|\rangle"] \arrow[d, dash, "X0|\rangle"'] \arrow[dr, phantom, "XX|\rangle"] & 11|\rangle \arrow[d, dash, "X1|\rangle"] \\
00|\rangle \arrow[r, dash, "0X|\rangle"'] & 01|\rangle
\end{tikzcd}$$

Signed Sets of the 2-cube using Bra-Ket Notation

Lastly, we have introduced the poset dual of the cubic lattice, the octahedral lattice.

Definition 1.3.14. *The octahedral lattice is the lattice derived by reversing the order of the poset corresponding to the cubic lattice.*

It is worth noting a distinction between the cubic lattice and the cube. The cubic lattice is a purely set theoretic object while the cube is a faithful geometric representation of the lattice as constructed via an intersection of hyperplanes as discussed above. The notion of reflection symmetries of a lattice is in a sense vacuous and only really has meaning when considering the cube as a Euclidean geometric object. Although Δ is originally defined on the cubic lattice, we give a geometric interpretation as a reflection of the cubic lattice about a plane in space.

We have now given a very terse description of cubic algebras. The focus of this thesis is to demonstrate a faithful realization of the cubic algebra as an operator algebra. A large amount of technology must be developed as we do not yet even have a linear space of operators with which to begin. Therefore we will begin with a substantial amount of work in defining the proper notion of addition and multiplication for a cubic algebra. We demonstrate that the logic induced by the lattice of signed sets and its reflection symmetries have a locally Boolean structure and its relation to $\perp$, as in a Hilbert space, and how it relates to this non standard Boolean structure. We the generalize this structure to critical multi-cubic lattices, which will be defined in section 6.

2 Analytic Properties of the Cubic Lattice

The ultimate goal is to show that the cubic lattice is the correct structure for quantum circuits with a well defined automorphism group, quantum logic, and Feynman walks. The definition of a cubic lattice is purely axiomatic, and in a sense algebraic, in order to talk about the above properties, we first need to discuss the analytic properties of the cubic lattice. In order to talk about the analytic properties of the cubic lattice, we will begin with a discussion of its dual in the category of posets.

2.1 Dual Space of Cubic Lattice

Definition 2.1.1. *We define the octahedral lattice as the dual in the category of posets of the cubic lattice. Equivalently, one obtains the octahedral lattice by reversing the direction of the partial order of the cubic lattice.*

For conciseness, we often times refer to the cubic lattice as CL, and the octahedral lattice as OL. While the construction of octahedral lattice is concise given a cubic lattice, we want to describe the properties of the octahedral lattice in terms of Δ. Furthermore, we differentiate the cubic and/or octahedral lattice from the geometric object of the cube, C, and the octahedron, O.

Definition 2.1.2. *We denote $\wedge_{(\cdot)}$ and $\vee_{(\cdot)}$ to be the operations in a lattice $(\cdot)$. In particular, $\wedge_{CL}$, $\vee_{CL}$, and $\wedge_{OL}$, $\vee_{OL}$ are lattice operations of the cubic lattice and octahedral lattice respectively.*

Definition 2.1.3. *Let L_1, L_2 be lattices. A lattice homomorphism $h : L_1 \to L_2$ is a map such that for all $a, b \in L_1$, $h(a \wedge_{L_1} b) = h(a) \wedge_{L_2} h(b)$, and $h(a \vee_{L_1} b) = h(a) \vee_{L_2} h(b)$, whenever these operations meet and join are defined between a and b. A lattice anti-homomorphism is a map such that for all $a, b \in L_1$, $h(a \wedge_{L_1} b) = h(a) \vee_{L_2} h(b)$, and $h(a \vee_{L_1} b) = h(a) \wedge_{L_2} h(b)$, again when the operations of meet and join are defined for $a, b \in L_1$.*

Definition 2.1.4. *A lattice isomorphism is a bijective lattice homomorphism and a lattice anti-isomorphism is a bijective lattice anti-homomorphism.*

Proposition 2.1.5. *Let $\phi : CL \to OL$ be the natural poset dual map. Then ϕ is a lattice anti-isomorphism.*

We have now shown that the natural dual poset map, ϕ, is in fact a lattice anti-isomorphism. In the following sections, we create a topology defined on the cubic lattice. We then show that ϕ can in some notion be extended in a reasonable way and functions as the dual map in this constructed topology. Before continuing in this theme, we discuss consequences of the lattice anti-isomorphism.

Definition 2.1.6. *Let $\Delta^C(b,a)$ denote the Δ with base a acting on b where $a \leq b$ the order on the cubic lattice. Then we define $\Delta^O(b,a) := \phi(\Delta^C(\phi^{-1}(b), \phi^{-1}(a)))$.*

With this definition, many of the dual statements for axioms of a cubic lattice are easily shown for Δ^O on the octahedron.

Lemma 2.1.7. *The following conditions hold for $a \leq_O b$ equivalently $b \leq_C a$:*

1. $\Delta^O(a,b) \wedge_O b = a$
2. $\Delta^O(a,b) \vee_O b = \emptyset$
3. $\Delta^O(a, \Delta^O(a,b)) = b$ *(involution)*

Proof. The results follow from the following computations.

1.

$$\begin{aligned}
\Delta^O(a,b) \wedge_O b &= a \\
\phi(\Delta^C(\phi^{-1}(a), \phi^{-1}(b))) \wedge_O b &= a \\
\phi(\Delta^C(\phi^{-1}(a), \phi^{-1}(b)) \vee_C \phi^{-1}(b)) &= a \\
\phi(\Delta^C(a,b) \vee_C b) &= a \\
\phi(a) &= a
\end{aligned}$$

2.

$$\begin{aligned}
\Delta^O(a,b) \vee_O b &= \emptyset \\
\phi(\Delta^C(\phi^{-1}(a), \phi^{-1}(b)) \vee_O b &= \emptyset \\
\phi(\Delta^C(\phi^{-1}(a), \phi^{-1}(b) \wedge_C \phi^{-1}(b)) &= \emptyset \\
\phi(\Delta^C(a,b) \wedge_C b) &= \emptyset \\
\phi(\emptyset) &= \emptyset
\end{aligned}$$

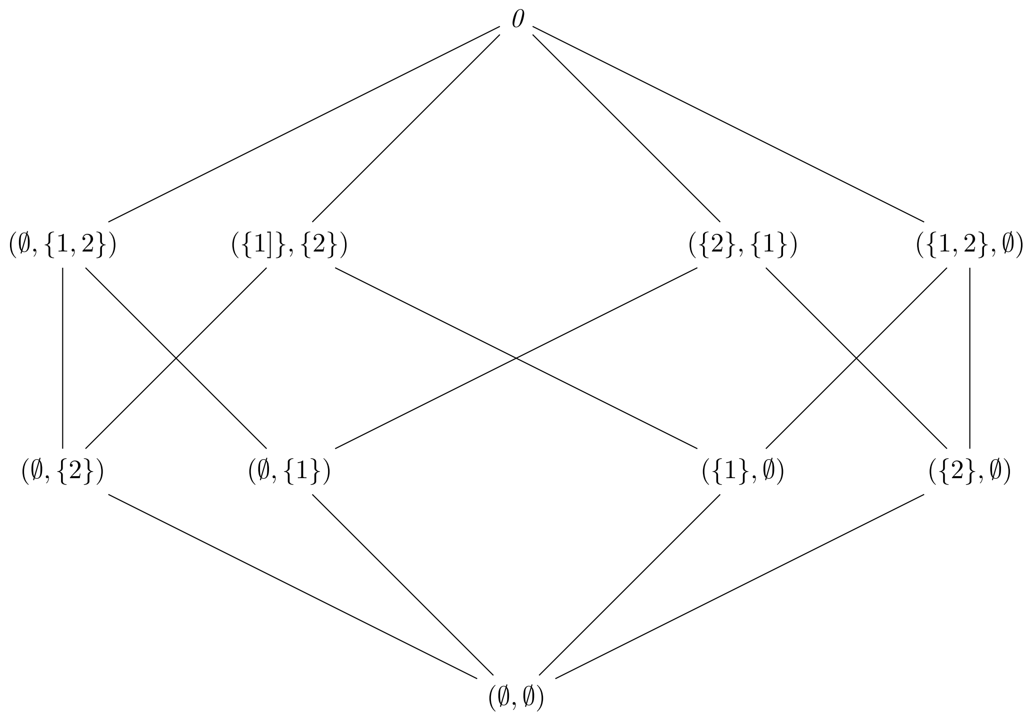
3.

$$
\begin{aligned}
\Delta^O(a, \Delta^O(a,b)) &= b \\
\Delta^O(a, \phi(\Delta^C(\phi^{-1}(a), \phi^{-1}(b)))) &= b \\
\phi(\Delta^C(\phi^{-1}(a), \phi^{-1} \circ \phi(\Delta^C(\phi^{-1}(a), \phi^{-1}(b)))) &= b \\
\phi(\Delta^C(a, \Delta^C(a,b))) &= b \\
\phi(b) &= b
\end{aligned}
$$

□

Therefore for a given set of reflection symmetries of the cube, we can consider the dual notion of reflections of the octahedron. In a very similar sense we get a geometric representation of the octahedron.

Example 2.1.8.



The face lattice of the 2-cube

We now have both a lattice realization and a geometric realization for both the cube and its dual the octahedron. We have also shown that the group of reflection symmetries is the same for both geometric realizations. We are now in a position to discuss a third realization of the cube and octahedron as Banach algebras. At each step we grow closer to a physical notion, which will be discussed at the end of the section. As will be seen in section 3, each instance of the cube and its dual add mathematical structure necessary for the final results.

2.2 Operator Structure of Cubic Lattices

In [16], it is mentioned that the cubic lattice is a base norm space and its dual is an order unit space. We first present this argument in more detail from a geometric perspective. Next we add a more analytic structure to the cubic lattice that in some sense generalizes its signed set structure and also its dual the octahedral lattice. This gives us the above results as an immediate corollary. Also it is noteworthy that we do not use any cardinality arguments assuming compact operators over a separable Hilbert space to obtain this result, which allow us to potentially use the full result of the characterization of signed sets which did not have a cardinality limitations. The algebraic and geometric perspective discussed here will be combined with the analytic perspective to derive the desired results. Lastly, we discuss some of the physical consequences of these new mathematical structures.

We first state the definitions following from [2].

Definition 2.2.1. *An ordered vector space is a vector space V ordered by a proper cone V^+ which generates V, i.e. $V = V^+ - V^+$. A subset $K \subseteq V^+$ is a base if $K = V^+ \cap H$, where H is a hyperplane, $0 \notin H$, and for all $v \in V^+$, $v = \lambda k$, where $\lambda \in \mathbb{R}_+$ and $k \in K$.*

Definition 2.2.2. *An ordered vector space, V, is a base norm space with distinguished base $B = co(K \cup -K)$ if K is a the base of V^+ and B is radially compact i.e. that $\{\lambda : \lambda\rho \in B\}$ is a compact subset of $\mathbb{R}$ for every $0 \neq \rho \in B$. The norm on V is defined by the Minkowski functional relative to B.*

Definition 2.2.3. *A positive element e of an ordered vector space A is to be an order unit if for all $a \in A$, there exists $\lambda \in \mathbb{R}_+$, such that $-\lambda e \leq a \leq \lambda e$. The order unit is called Archimedean if for all $a \in A$ and for all $n \in \mathbb{N}$, $na \leq e$ implies $a \leq 0$. If A has norm $||a|| = inf\{\lambda > 0 : -\lambda e \leq a \leq \lambda e\}$, and e is a Archimedean order unit, then A is an order unit space with distinguished order unit e henceforth denoted by 1.*

Definition 2.2.4. *The Banach space $l^\infty(S)$ is defined as the norm closure of the real linear span of the extreme points of the cube with the supremum norm [7].*

We note this fact this implies that the boundary of the unit cube has norm 1.

Definition 2.2.5. *In [7], they define $l^1(S)$ to be the Banach space of the closure of the real linear span of the extreme points of the octahedron equipped with the l^1 norm.*

Again we note that the boundary of the octehedron has norm 1. However, while required for these investigations, this realization does not lend itself easily to the operator theory based quantum mechanics.

Thus, we adapt the spirit of the arguments of [16] to apply to operators, as opposed to a purely geometric object in Euclidean space.

Theorem 2.2.6. *Let H be a separable Hilbert space. The space $l^\infty(S)$ is an ordered unit space with distinguished order unit of the identity, 1, and its dual the Octahedron $l^1(S)$ is the base norm space with distinguished base of the upper orthant denoted by K.*

Proof. The proof follows from the more general case concerning C^* algebras as a result of Proposition 2.2, 2.3, and 2.13 in [2]. □

As an immediate result of this geometric case, one can observe that the extreme points of $l^\infty(S)$, $l^1(S)$ are atoms in their respective lattice of subspaces of Euclidean space. In particular, the lattice definitions from the original axioms lift to a more standard euclidean geometry in a Banach space. At this point, for a n dimensional cubic lattice, we have an n dimensional euclidean cube.

A key property of quantum logic is the notion of orthomodularity. In particular, the perpendicular operation, $^\perp$, needs to be well defined. However, for the cubic lattice, there is no $^\perp$ operation. Although as a Banach space, the euclidean cube has a notion of perpendicularity, the induced operations of the cubic lattice fail to be orthomodular. One can see in [7] that orthomodularity of the lattice of a cubic system of two dimensional subspaces fails. Consider a vector in the space distinct from the basis vectors and take the complement of it's join with any other given basis vector. The issue is that the dimensionality of the space is too small given the size of the lattice.

We refer to the Hilbert lattice. It is worth noting that the above arguments had the cubic lattice embedded in cardinality I space while the following embeddings are in 2^I space using the constructions mentioned in the introduction.

We now have a higher dimensional space, in fact exponentially so. Each vertex in the original euclidean space is now represented by a vector in the new space. As an example of this coordinatization given an ordering on the vector space, suppose a vector representing a spin up in every direction will be represented by the one's vector. For each position of the n vector there are 2 choices leading to 2^n possible vectors, one for each vertex of an n dimensional cube. Therefore a major goal is to have a space with a reasonably defined notion of $\perp$ while maintaining the notion of a cubic lattice.

In order to to so, we now provide a more traditional operator theory perspective. The key observation is that $l^\infty(S)$ can be viewed as the self-adjoint parts of a more general von Neumann algebras. We have only shown that $l^\infty(S)$ is a Banach space and in particular an order unit space, but in the following section, we show how $l^\infty(S)$ can be viewed as subset of the standard construction defined above. This will give us a von Neumann algebra realization, a substantially richer structure than we had originally stated.

We next present some intuition to the standard construction and its relation to the physicality of quantum systems. Consider $B(H)$, the bounded compact linear operators on a complex Hilbert space H. To be concrete assume $H = \mathbb{C}^\infty$, as it will in general be built from tensor products of $\mathbb{C}^2$, this is a reasonable assumption of the most general case for later purposes. Alternatively, $B(\mathbb{C}^n) \cong M_n(\mathbb{C})$ for finite dimensional vector spaces. Furthermore, given the norm of a normal operator x defined by $||x|| = \sup_{\lambda \in \sigma(x)} |\lambda|$ the L^∞ norm or l^∞ norm for $\{\lambda \in \sigma(x)\}$, conincides with our previous norm of $l^\infty(H)$.

Definition 2.2.7. *Let $\mathscr{M}$ be a von Neumann algebra with unit denoted by* 1. *The state space K of a von Neumann algebra is the set of continuous linear functionals of $\rho \in \mathscr{M}^*$ such that $\rho(1) = 1$ and $\rho(a) \geq 0$ for all $a \in \mathscr{M}^+$.*

Definition 2.2.8. *Let K be the state space of von Neumann algebra. The normal state space K_* is the set of $\rho \in K$ such that ρ is weak* continuous.*

Proposition 2.2.9. *The self adjoint part M_*^+ of the predual of a von Neumann algebra $\mathscr{M}$ is a base norm space whose distinguished base is the normal state space K_* of $\mathscr{M}$. [2]*

Therefore, we can consider the normal state space as subset of $\mathscr{M}_*^+$ or its canonical embedding in $\mathscr{M}^{*+}$.

Now we see that both $l^\infty(S)$ and the self adjoint part of an order unit space share the same type of Banach space. However, it still remains to show that there exists a von Neumann algebra over a specific Hilbert space such that $l^\infty(S)$ is truly contained its self adjoint space and the atoms of the Hilbert lattice agree with the atoms of CL. Let H be the Hilbert space over which L(S) is defined.

Definition 2.2.10. *A convex subset F of a convex set C is a face if for any $x, y \in C$, if there exists $0 < \lambda < 1$, $\lambda x + (1-\lambda)y \in$, then $x, y \in F$.*

Definition 2.2.11. *A face is of finite rank if it contains finitely many linearly independent elements.*

Definition 2.2.12. *For each positive operator $a \in B(H)$, we define the trace of a, $tr(a) = \sum_\gamma \langle a\xi_\gamma, \xi_\gamma \rangle$ where $\{\xi_\gamma\}$ is any orthonormal basis. For arbitrary $a \in B(H)$, we define $||a||_1 = tr(|a|)$. [2]*

Since, the unit ball in the dual is weak * compact, we did not need to make an additional compactness assumption on the states. If one required separability, one can equivalently say that trace class operators in a separable Hilbert space are compact, which is sometimes included in the definition of trace class, and in general is easily derivable statement.

Furthermore, for our application of a von Neumann algebra, which will be defined as the bounded linear operators over a specifically constructed Hilbert space, the linear combination of extreme points in some sense tells the full story as we will have a type-I JBW algebra, see [2] for definition.

Proposition 2.2.13. *If A is an atomic JBW algebra, then its normal state space is the norm closed convex hull of its extreme points. [1]*

We now present some physical terminology from the derived structures.

2.3 Relevant Physics Definitions

Definition 2.3.1. *Let $A \in B(H)$ be a compact self adjoint bounded linear operator. Then A is an observable of the system. We refer to the set of compact self adjoint operators as the observables. Note that in some contexts the assumptions of compactness or even boundedness are waived.*

Definition 2.3.2. *Let ϕ be a linear functional on the space B(H) with $\phi(1) = 1$. ϕ is called a state. Let $h \in H$. Then h defines a dual map on $B(H)$, by $\omega_h \in B(H)^*$, $\omega_h := \langle Ah, h\rangle$, and ω_h is a state and in particular a vector state.*

Definition 2.3.3. *Let H be formed by tensor products of Hilbert spaces $\{H_\alpha\}_{\alpha \in I}$, I a typically finite indexing set. The product states are the subsets projection operators associated with the simple tensors in H. In some literature, product states are called separable states. Mathematically speaking they are the simple tensors in the respective basis.*

Definition 2.3.4. *A state is a pure state if and only if it is a vector state. One can also consider the pure states as one dimensional rays of the unit ball.*

Definition 2.3.5. *A state is mixed if it is not pure.*

Example 2.3.6. *Let $H = \mathbb{C}^2 \otimes \mathbb{C}^2$ and $\{e_i\}_{i=1}^2$, $\{f_i\}_{i=1}^2$ be the respective standard basis of each element of the product. Then $e_1 \otimes f_1 = [1, 0, 0, 0]^t$ and $e_2 \otimes f_2 = [0, 0, 0, 1]^t$ are both pure states and in particular product states, but*

$$\frac{1}{2}\left[(e_1 \otimes f_1|\rangle\langle|e_1 \otimes f_1) + (e_2 \otimes f_2|\rangle\langle|e_2 \otimes f_2)\right] = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} = \rho.$$

One can observe that ρ has trace 1, but ρ can not be written as projection operator onto a singe vector *as otherwise it would have eigenvalues 1 or 0 as opposed to* $\frac{1}{2}$*. For clarity observe that,*

$$\frac{1}{2}[(e_1 \otimes f_1 + e_2 \otimes f_2)|\rangle\langle|(e_1 \otimes f_1 + e_2 \otimes f_2)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which can be seen to the projection operator on the vector state $\frac{1}{2}[(e_1 \otimes f_1 + e_2 \otimes f_2)|\rangle\langle|(e_1 \otimes f_1 + e_2 \otimes f_2)]$*, which is a pure state but not a product state.*

One also now observes that the density operators associated with pure states are not equal to the full dual space of $B(H)$ as we only have rank one operators. We define the mixed states as the remaining density operators in the space, i.e. linear transformations with rank strictly greater than 1 noting that not all mixed states have a physical realization.

We have discussed how the closure of vertices of the geometric cube and octahedron form Banach spaces with the same metric as self adjoint parts of a von Neumann algebra and its predual. However, the atoms of the lattice of Hilbert space can not in general be viewed as the atoms of a cubic lattice. Consider a Hilbert space on 3 dimensions, which is obviously not a cubic lattice as a finite cubic lattice must have a 2^n number of atoms.

3 Embeddings of the Cubic Lattice and Octehedral Lattice

In this section, we build the necessary embedding to demonstrate that cubic lattices have a realization as a von Neumann algebra. In addition, we discuss the algebraic structure and compare it to the poset structure of the lattice. Lastly, we compare the dual spaces with respect to both spaces categories. We show that there is, in a reasonable sense, a direct relationship between the dual of the poset and the dual of the analytic structure.

3.1 Cubic Lattice as a subset of a Hilbert Lattice

We adapt the following definitions and proposition from [19] to our notation.

Proposition 3.1.1. *The Hilbert lattice is an atomic, (completely) atomistic, complete, orthomodular lattice. [19]*

For the following theorem, we will be constructing a Hilbert space from an infinite tensor product. We do so in an established but non-standard way. We outline the necessary definitions for expository purposes and use the results from [22]. Unless otherwise stated, when we refer to a Hilbert space formed by infinite tensor products, we mean the following construction, not the standard construction.

For the following, I is an index set of not necessarily countable cardinality, H_α is a finite dimensional Hilbert space for all $\alpha \in I$, and the norm on $f_\alpha \in H_\alpha$ is the norm of the Hilbert space.

Definition 3.1.2. *[22] $\Pi_{\alpha \in I} z_\alpha$, $z_\alpha \in \mathbb{C}$, $\alpha \in I$, is convergent, and a is its respective value if there exists for every $\delta > 0$, a finite set $I_0 = I_0(\delta) \subseteq I$, such that for every finite set $J = \{\alpha_1, \ldots, \alpha_n\}$ (mutually distinct α_i) with $I_0 \subset J \subset I$*

$$|z_{\alpha_1} \cdot \cdots \cdot z_{\alpha_n} - a| \leq \delta.$$

Definition 3.1.3. *$\Pi_{\alpha \in I} z_\alpha$ is quasi-convergent if and only if $\Pi_{\alpha \in I} |z_\alpha|$ is convergent. It value is*

1. *the value of $\Pi_{\alpha \in I} z_\alpha$ if it is convergent*
2. *0, if it is not convergent.*

Definition 3.1.4. *A sequence f_α, $\alpha \in I$, is a C-sequence if and only if $f_\alpha \in H_\alpha$ for all $\alpha \in I$, and $\Pi_{\alpha \in I} ||f_\alpha||$ converges.*

Lemma 3.1.5. *If f_α, $\alpha \in I$, and g_α, $\alpha \in I$ are two C-sequences then $\Pi_\alpha \langle f_\alpha, g_\alpha \rangle$ is quasi-convergent. [22]*

Definition 3.1.6. *Let $\Phi(f_\alpha; \alpha \in I)$ be the set of functionals on the product $\Pi_{\alpha \in I} H_\alpha$ which is conjugate linear in each $f_\alpha \in I$ separately over C-sequences.*

Definition 3.1.7. *The set of all such* Φ *for any C-sequence will be denoted by* $\Pi \odot_{\alpha \in I} H_\alpha$. *We note that* $\Pi \odot_{\alpha \in I} H_\alpha$ *is a linear space, but it is not an inner product space.*

Definition 3.1.8. *Given a C-sequence* f_α^0, $\alpha \in I$, *we form the functional* $\Phi(f_\alpha; \alpha) = \Pi_{\alpha \in I}(f_\alpha^0, f_\alpha)$ *where* f_α, $\alpha \in I$ *runs over all C-sequences. Denote such a functional by* $\Pi \otimes_{\alpha \in I} f_\alpha^0$.

Definition 3.1.9. *Consider the set of all finite linear aggregates of the above elements:*

$$\Phi = \sum_{v=1}^{p} \Pi \otimes_{\alpha \in I} f_{\alpha,v}^0$$

where $p = 0, 1, \ldots, p$ *and* $f_{\alpha,v}^0$, $\alpha \in I$ *is a C-sequence for each* $v = 1, 2, \ldots, p$. *Denote the set of these* Φ *by* $\Pi' \otimes_{\alpha \in I} H_\alpha$.

Definition 3.1.10. *For* $\Phi = \sum_{v=1}^{p} \Pi \otimes_{\alpha \in I} f_{\alpha,v}^0$, $\Psi = \sum_{\mu=1}^{q} \Pi \otimes_{\alpha \in I} g_{\alpha,\mu}^0 \in \Pi' \otimes_{\alpha \in I} H_\alpha$ *we define the inner product by:*

$$\langle \Phi, \Psi \rangle = \sum_{v=1}^{p} \sum_{\mu=1}^{q} \Pi_{\alpha \in I} \langle f_{\alpha,v}^0, g_{\alpha,\mu}^0 \rangle.$$

Lemma 3.1.11. *Let* $\Phi, \Psi \in \Pi' \otimes_{\alpha \in I} H_\alpha$. *The value of* $\langle \Phi, \Psi \rangle$ *is independent of the choice of their respective decompositions. [22]*

Proof. It suffices to prove that $\langle \Phi, \Psi \rangle$ is unchanged, if only for $\Phi's$ decomposition is changed, or if only $\Psi's$ is unchanged. As $\langle \Phi, \Psi \rangle = \overline{\langle \Psi, \Phi \rangle}$, (for the same decompositions), we need to show only the first case only. Instead of comparing two decompositions of Φ, we might compare their formal difference with 0. In other words: We must prove $\langle \Phi, \Psi \rangle = 0$ for $\Phi = 0$. That is $\Phi(f_\alpha; \alpha \in I) = 0$ for every possible decomposition of this Φ.

Now in this case:

$$\begin{aligned}
\langle \Phi, \Psi \rangle &= \sum_{\mu=1}^{q} (\sum_{v=1}^{p} \Pi_{\alpha \in I} \langle f_{\alpha,v}^0, g_{\alpha,\mu}^0 \rangle) \\
&= \sum_{\mu=1}^{q} (\sum_{v=1}^{p} (\Pi \otimes_{\alpha \in I} f_{\alpha,v}^0)(g_{\alpha,\mu}^0; \alpha \in I)) \\
&= \sum_{\mu=1}^{q} \Phi(g_{\alpha,\mu}^0; \alpha \in I) = 0.
\end{aligned}$$

□

Definition 3.1.12. *Consider the functions* $\Phi \in \Pi \otimes_{\alpha \in I} H_\alpha$ *for which a sequence* $\Phi_1, \Phi_2, \cdots \in \Pi' \otimes_{\alpha \in I} H_\alpha$ *exists such that*

1. $\Phi(f_\alpha; \alpha \in I) = \lim_{r \to \infty} \Phi_r(f_\alpha; \alpha \in I)$ *for all C-sequences* f_α, $\alpha \in I$,

2. $\lim_{r,s\to\infty} ||\Phi_r - \Phi_s|| = 0$

The set they form is the complete direct product of H_α, $\alpha \in I$ to be denoted by $\Pi \otimes_{\alpha\in I} H_\alpha$. Note that $\Pi' \otimes_{\alpha\in I} H_\alpha \subseteq \Pi \otimes_{\alpha\in I} H_\alpha \subseteq \Pi \odot_{\alpha\in I} H_\alpha$.

For our application, the convergence criteria of Definition 3.1.2 is acceptable. We will only be concerned with forming the tensors of elementary basis elements of the respective H_α, so all of our elements are functionals derived from C-sequences as in Definition 3.1.8. We can then consider their span in the natural way.

Theorem 3.1.13. *Let H be a Hilbert space constructed as a tensor product of 2 dimensional spaces over an index set I. For the given Hilbert lattice HL of H, there exists a cubic lattice CL such that $CL \subseteq HL$, and the atoms of CL are projections onto subspaces H forming an orthonormal basis of H.*

Proof. We begin with the standard construction of a basis over a tensor product of index I. Let e_i^+, e_i^- represent the 2 basis vectors for $i \in I$.

We now have that each elementary tensor is C-sequence as each element $||e_i|| = 1$, so we have a linear functional of the form in Definition 3.1.8 in H, and it can be represented by its respective projection operator. As these are projections onto 1 dimensional subspaces, they are atoms in HL, and in the cone $B(H)^+$.

For each atomic elementary tensor described above, we use the notation, $v = \{A^+, A^-\}$, where $A^+ = \{i \in I : v_i = e_i^+\}$ and $A^- = \{i \in I : v_i = e_i^-\}$. By the construction of v, we have that $A^+ \cap A^- = \emptyset$, and $A^+ \cup A^- = I$. Now we observe that the all such v form the atoms of a signed set over the indexing set I.

We define $CL = L(S_I)$, the lattice of signed sets generated by the closure of the above atoms under the operations of meet and join from the definition of cubic lattices. Recall by Definition 1.3.10, that $\Delta : L(S) \times L(S) \to L(S)$ can be defined on any signed set.

As we have a description of the atoms of the cubic lattice in $\mathscr{M}^+$, we need to show that the atoms are closed under $\vee$. Consider $a, b \in CL \cap HL$, where $a = \{A^+, A^-\}$ and $b = \{B^+, B^-\}$. Then $a \vee_{CL} b = \{A^+ \cap B^+, A^- \cap B^-\}$. We now have that $a \vee_{CL} b$ is the projection P_V onto the subspace $V = \otimes_{i\in I} V_i$ where $V_i = e_i^+$ for $i \in A^+ \cap B^+$, $V_i = e_i^-$ for $i \in A^- \cap B^-$, and $V_i = span\{e_i^+, e_i^-\}$ otherwise, so that $a \vee_{CL} b \in HL$ and $P_V \in \mathscr{M}^+$. Therefore $CL \subseteq HL$. In addition as any element of CL is a join of its atoms by atomisticity, and $0 \in HL$ trivially the result follows.

The atoms of CL form an orthonormal system in H. For any distinct atoms a, $b \in CL$, we have that there exists $i \in I$ such that $a_i \neq b_i$, so $\langle a_i, b_i\rangle_{H_\alpha} = 0$ which implies that $\langle a, b\rangle_H = 0$. Furthermore, these vectors span $\Pi' \otimes_{\alpha\in I} H_\alpha$, and therefore are dense in H. $\square$

Remark 3.1.14. *The Hilbert lattice is not a cubic lattice. Suppose not, then there exists a signed set realization of HL, $L(S)$ [16]. Let $r(\cdot)$ denote the rank of a subspace. Consider the join of two linearly independent atoms $a = \{A^+, A^-\}, b = \{B^+, B^-\}$ such that $|\{A^+ - B^+\}| > 1$, so $r(a \vee_C b) = |\{(A^+ \cap B^+) \cup (A^- \cap B^-)\}| > 2$. Then $2 = r(a \vee_H b) < r((A^+ \cap B^+) \cup (A^- \cap B^-)) = r(a \vee_C b)$.*

Definition 3.1.15. *Let $V = \otimes_{i \in I} V_i$ for some index set I over vector spaces $\{V_i\}_{i \in I}$. A generalized simple tensor of V is a subspace of V of the form $\otimes_{i \in I} U_i$, where U_i is a subspace of V_i.*

Corollary 3.1.16. *The set of projection operators in $\mathscr{M}^+$ that are in CL are exactly the operators represented by generalized simple tensors in the orthonormal basis.*

Definition 3.1.17. *We say that $CL \subseteq \mathscr{M}$, where $\mathscr{M} = B(H)$ and H is constructed as in Theorem 3.1.13 to mean the set of orthogonal projections onto their respective closed subspaces of CL are in $B(H)$. We will use the notation: $a \in CL$, and $p_a \in B(H)$.*

3.2 Ideals of Cubic Lattices and Ideals of von Neumann Algebras

Now that we have a Hilbert space constructed in Theorem 3.1.13, we can consider the von Neumann algebra $\mathscr{M} = B(H)$. As our underlying Hilbert space is non-separable in full generality, we include a standard fact to be applied later in the paper without reference.

Proposition 3.2.1. *If X is a Banach space, then its dual X' in the weak* topology is a Hausdorff locally convex topological vector space.*

Since von Neumann algebras, their duals, and preduals are Banach spaces, we will freely use that a von Neumann algebra and its dual space are locally convex Hausdorff spaces with their respective weak* topology.

We are now ready to move forward with the geometric structure of the cubic lattice and the octehedral lattice. In [16], there is a notion of an ideal in a lattice, and in [2] there is a different notion of an ideal as the kernel of a state. We discuss the relations of these structures.

Definition 3.2.2. *In a lattice, L, the principle lattice ideal generated by $x \in L$ denoted $(x) \subseteq L$ is the set of all $y \in L$ such that $y \leq x$. Specifically for $x \in C$, the principle ideal generated by a face x is the set of all sub-faces contained in x. We use the notation $(x)_L \leq L$.*

Although not immediately relevant, it is convenient to describe the dual notion of an ideal to be used in the future.

Definition 3.2.3. *In a lattice, L, the principle lattice filter generated by $x \in L$ denoted $[x] \subseteq L$ is the set of all $y \in L$ such that $y \geq x$. Specifically for $x \in C$, the principle filter generated by a face x is the set of all faces that contain x as a sub-face.*

Definition 3.2.4. *Let C be a cone. A face F generated by $b \in C$ is the set of all all elements $a \in C$ such that $a \leq b$.*

The following two definitions are derived from [2].

Definition 3.2.5. *Let M be an ordered vector space. A point $p \in M$ is an extreme point if for all $u, v \in M$ such that $0 \leq \lambda \leq 1$ and $\lambda u + (1-\lambda)v = p$ implies $p = u$ or $p = v$.*

Definition 3.2.6. *Let C be a convex set. An point $x \in C$ is an exposed point if there exists a continuous linear functional, l such that $l(x) = \alpha$ and $l(y) < \alpha$ for all $x \neq y \in C$.*

Definition 3.2.7. *If a is a positive element in a von Neumann algebra $\mathscr{M}$, then the face generated by a in the positive cone $\mathscr{M}^+$, denoted by $face(a)$, consists of all $b \in \mathscr{M}$ such that $0 \leq b \leq \lambda a$ for some $\lambda \in \mathbb{R}^+$.*

One can see that the above definition is a specific instance of the general definition of faces in a cone. To distinguish from our lattice definitions, we include the following:

Definition 3.2.8. *An ideal in a von Neumann algebra $\mathscr{M}$ is the standard notion of an ideal when considering M as a ring.*

We first restrict the argument to extreme points in $\mathscr{M}$.

Definition 3.2.9. *For $p \in \mathscr{M}^+$, denote $face(p) = span(p)_{\geq 0} = span(p) \cap \mathscr{M}^+$.*

Lemma 3.2.10. *If $p \in \mathscr{M}^+$ is an exposed point of the unit ball of $\mathscr{M}$ with the order unit topology, then p is a rank one orthogonal projection operator.*

Proof. We use that the extreme points of of the unit ball intersected with $\mathscr{M}^+$ are projection operators, see Proposition 2.23 [2]. As exposed points are extreme points, we have that the exposed points are also projections. The full result follows since being an exposed point forces p to be an atom i.e. a projection onto a closed one dimensional subspace and the result follows. $\square$

We now have a relationship between extreme points, projection operators, and faces. Just as von Neumann algebras are generated by their projections, cubic lattices are generated by the convex hull of their vertices when considered as a geometric object. We proceed to formalize this relationship.

We are now in a position to discuss the relation between lattice ideals of CL and ideals in $\mathscr{M}$. As a direct application of [2]:

Proposition 3.2.11 (Alfsen Theorem 3.13)**.** *There is a natural 1-1 correspondence between the σ-weakly closed left ideals $\mathscr{J}$ in a von Neumann algebra $\mathscr{M}$ and the σ-weakly closed faces $\mathscr{D}$ of the positive cone $\mathscr{M}^+$ given by $\mathscr{D} = \mathscr{J}^+$, and $\mathscr{J} = \mathscr{M}\mathscr{D}$.*

Lemma 3.2.12. *Let $(a)_{CL}$ be a principal ideal in CL and $(a)_R \leq \mathscr{M}^+$ be a face in the cone $\mathscr{M}^+$, then for all $b \in CL$, $b \in (a)_{CL}$ if and only if $b \in (a)_R$.*

Proof. We have shown that ordering of cubic lattice coincides with the ordering of the Hilbert lattice. Therefore for all $b \in CL$, $b \leq_{CL}$ if and only $b \leq_R a$, equivalently $b \in (a)_{CL}$ if and only if $b \in (a)_{HL}$. $\square$

Theorem 3.2.13. *Let $(a)_{CL}$ be a principal ideal in CL and $(a)_R \leq \mathscr{M}$ be an ideal in the ring $\mathscr{M}$, then for all $b \in CL$, $b \in (a)_{CL}$ if and only if $b \in (a)_R$.*

Proof. We have already shown the correspondence between ideals in the cubic lattice and faces of the cone, as direct application of Proposition 3.2.11, we conclude the result. $\square$

We now have a direct correspondence between ideals in a cubic lattice, which can be viewed geometrically as a face of the cube and its respective sub-faces, and the ideals of a von Neumann algebra that are in bijective correspondence with the faces of a cone defined purely algebraically.

3.3 The Lattice Dual as an Algebra Anti Isomorphism.

In order to expand our discussion of CL and HL as sets, we would benefit from compactness. Therefore, we consider the pre dual space $\mathscr{M}_*$ and dual $\mathscr{M}^*$ of $\mathscr{M}$.

Definition 3.3.1 (Definition 3.24 [2])**.** *Let $\sigma, \omega \in \mathscr{M}_*^+$, where $\mathscr{M}$ is a von Nuemann algebra. We say that σ is absolutely continuous with respect to ω, written as $\sigma << \omega$, if $\sigma(q) = 0$ for all projections $q \in M$ such that $\omega(q) = 0$.*

Theorem 3.3.2 (Theorem 3.27 [2])**.** *If $\mathscr{M}$ is a von Neumann algebra and $\omega \in M_*^+$, then the norm closure of the face generated by $\omega \in M_*^+$ consists of all $\sigma \in M_*^+$ such that $\sigma << \omega$.*

Proposition 3.3.3. *For a base norm space X with generating hyperplane K, there is an order isomorphism from the non-zero faces of X to the faces of K.*

This is a standard fact, where the morphism is defined by a face F in X induces a face $F \cap K$ in K. One can also see this as a map from $0 \neq x \in X$ to $x/||x||$ assuming X is a normed space and observing the induced facial structure.

Proposition 3.3.4. *If F is a face in $\mathscr{M}_*^+$, then there is an order isomorphism to faces in the normal state space K_*.*

Proof. A direct result of Proposition 2.2.9 and Proposition 3.3.3. $\square$

We use a direct application of [2] with slight abbreviation to avoid introducing notation that we will not use. For the full statement see references.

Proposition 3.3.5 ([2] Theorem 3.35). *Let $\mathscr{M}$ be a von Neumann algebra with normal state space K_*, and denote $\mathscr{F}$ the set of all norm closed faces of K_*, by $\mathscr{P}$ the set of all projections in $\mathscr{M}$, and by $\mathscr{J}$ the set of all σ-weakly closed left ideals in $\mathscr{M}$, each equipped with the natural ordering. Then there is an order preserving bijection $\Phi : p \to F$ from $\mathscr{P}$ to $\mathscr{F}$, and an order reversing bijection $\Psi : p \to J$ from $\mathscr{P}$ to $\mathscr{J}$, and hence also an order reversing bijection $\Theta = \Psi \circ \Phi^{-1}$ from $\mathscr{F}$ to $\mathscr{J}$. The maps Φ, Ψ, and Θ and the final inverse are explicitly given by the equations*

(i) $F = \{\sigma \in K_* | \sigma(p) = 1\}$,

(ii) $J = \{a \in \mathscr{M} | ap = 0\}$

(iii) $J = \{a \in \mathscr{M} | \sigma(a^*a) = 0 \text{ all } \sigma \in F\}$, $F = \{\sigma \in K_* | \sigma(a^*a) = 0 \text{ all } a \in J\}$

Definition 3.3.6. *In the higher dimensional embedding, we lose the $+1$, -1 directionality to gain orthogonality. Therefore each $i \in S+$ and $j \in S^-$ corresponds to a mutually linearly independent linear functional for a total of $2|S|$ linear functionals. As an example let $j \in A^+$, and $f_i \in \{e_i^+, e_i^-\}$, and $p_{\otimes_{i\in S} f_i}$ for all $i \in S$ be the projection onto $\otimes_{i\in S} f_i$, then*

$$\epsilon_j(p_{\otimes_{i\in S} f_i}) = \begin{cases} 1 & f_j = e_j^+ \\ 0 & f_j = e_j^- \end{cases}$$

and extend linearly.

Definition 3.3.7. *Define $\phi : CL \to \mathscr{F}$ by $\phi((A^+, A^-))$ as the norm closed convex hull of the linear functionals $\{\epsilon_i : i \in A^+\} \cup \{\epsilon_j : j \in A^-\}$.*

As we will show the above ϕ will be the the analytic equivalent of our ϕ defined as a lattice anti isomorphism, and it will agree on the corresponding lattices, so the reuse of notation is intentional.

Definition 3.3.8. *We define a unitary operator denoted U_Δ by linearly extending its action on the basis of H, and letting U_Δ act by inner automorphism on orthogonal projections of subspaces of HL.*

Lemma 3.3.9. *Let $CL \subseteq HL$ as in Theorem 3.1.13 with corresponding projections in $B(H)$. Then the restriction of the anti-isomorphism $\Theta^{-1} : \mathscr{J} \to \mathscr{F}$ of Proposition 3.3.5 to CL is equal to $\phi \circ U_\Delta : CL \to \mathscr{F}$.*

Proof. Let J be the left ideal generated by a projection operator, $p_{(A^+,A^-)}$, onto a subspace of $(A^+, A^-) \in CL \subseteq HL$. In addition, $U_\Delta p_{(A^+,A^-)} U_\Delta = p_{\Delta(A^+,A^-)}$. For simplicity, we will assign $(B^+, B^-) = \Delta(A^+, A^-) = (A^-, A^+)$.

We claim that the face $\phi(U_\Delta p U_\Delta)$ in the normal state space, $F_{U_\Delta(p)} = \overline{\{\sigma \in K_* : \sigma << \phi(U_\Delta p U_\Delta)\}}^{||\cdot||}$ is equal to $\Theta^{-1}(J)$. Firstly we have for any state $\omega \in \phi(U_\Delta p U_\Delta)$, $\omega(p) = 0$ as $supp(\omega)$ is orthogonal to p. Therefore, $F_{U_\Delta(p)} \subseteq \Theta^{-1}(p)$.

Suppose $F_{U_\Delta(p)} \subset \Theta^{-1}(p)$, and there exists a state $\gamma \in \Theta^{-1}(J)$ such that γ is not absolutely continuous with respect to $F_{U_{\Delta(p)}}$. In particular, γ is not absolutely continuous with respect to a subset of $F_{U_{\Delta(p)}}$, namely the extreme points of $F_{U_{\Delta(p)}}$, consisting of $\{\epsilon_i : i \in B^+\} \cup \{\epsilon_j : j \in B^-\}$, so there exists some projection $a \in \mathscr{M}$ such that $0 \neq a \subseteq (\cap_{i\in B^+} Ker(\epsilon_i)) \cap (\cap_{j \in B^-} Ker(\epsilon_j))$ and $\gamma(a) \neq 0$. By construction, any projection in $(\cap_{i\in B^+} Ker(\epsilon_i)) \cap (\cap_{j \in B^-} Ker(\epsilon_j))$ is less than or equal to p, so $\gamma(p) \neq 0$, which is a contradiction. □

Theorem 3.3.10. *For the normal state space K_* of $B(H)$ where H is constructed as in Theorem 3.1.13, there exists an OL such that the coatoms of OL are contained in the coatoms of K_*.*

Proof. By Theorem 3.1.13, the atoms of the cubic lattice form an orthonormal basis of H, and the map $\phi : CL \to OL$ as defined Lemma 3.3.9 is an order reversing map. As ϕ is the restriction of the map in M_*^+ whose facial structure is equivalent to K_*, we have our result. □

Example 3.3.11. *The above results do not hold for the coatoms of CL. Referring back to Example 1.3.12, we see that coatoms are rank 2 projection operators onto a given half space while the coatoms of the respective Hilbert lattice must be rank 3 operators.*

4 The Logic of a Hilbert Lattice

We are now in a place to relate the importance of Δ to a general quantum mechanical system. Let $\mathscr{G}$ be a mechanical system potentially classical or quantum, which will be defined below.

For expository purposes, we reduce to a classical mechanical system, and then obtain a quantum mechanical system as a generalization.

A classical mechanical system, $\mathscr{G}(S, O)$ is a set of physical states or physically realizable possibilities for objects in the system, S, and a set of observables or physical qualities of interest to the observer of the system, O.

Definition 4.0.1. *The observables of a classical system are real valued Borel measurable functions $f : S \to \mathbb{R}$, and thus an experimentally verifiable question is if $f^{-1}(E) = F$ for some Borel set $F \subseteq S$.*

With implication acting as a partial ordering of the experimentally verifiable statements, and the action of negation, we obtain a logic $L(\mathscr{G}(S, O))$. The logic of the state space is a set of subsets of S and, as defined in [21], is then the set of experimentally verifiable questions one can ask of the physical system. To be precise:

Definition 4.0.2 ([21])**.** *Let L be a complete lattice with orthocomplementation, $^{\perp}$, then L is a logic if*

1. *for any countably infinite sequence a_1, a_2, $\ldots$ of elements of L, $\wedge_n a_n, \vee_n a_n \in L$.*
2. *if $a_1, a_2 \in L$ and $a_1 < a_2$, there exists $b \in L$ such that $b < a_1^{\perp}$ and $b \vee a_1 = a_2$.*

In the classical case and our application, The orthocomplement creates an orthomodular lattice, but this is not the case in full generality.

In the classical case, the notion of and, $\wedge$, is always defined and commutative, and thus we obtain a Boolean or σ-finite Boolean algebra when considering the lattice of the logic. However, in a quantum mechanical setting, one does not have perfect knowledge of the state space nor their respective observables, due to the uncertainty principle, which can be seen as a consequence of non-commuting operators. Therefore the issue comes down to the fact that $\wedge$ is not always experimentally verifiable. As a consequence, we lose distributivity and retain some form of ortho-algebraic conditions. Thus, a logic of a quantum mechanical system can be defined as a non distributive mechanical system. In other words, it is the set of experimentally verifiable questions of a system with a natural partial ordering defined by implication and some notion of ortho-complementation.

Definition 4.0.3. *Let H be a Hilbert Space. We define the lattice of closed linear subspaces of H to be the Hilbert Lattice henceforth referred to as HL. In this context, $u \vee v = span\{u, v\}$, and $u \wedge v = span\{u\} \cap span\{v\}$. For finite dimensional subspaces, the lattice rank coincides with the standard rank.*

One can directly observe that the Hilbert lattice HL satisfies all the criteria of a logic. Therefore, our choice of embedding space in Theorem 3.1.13, has a logical realization. In order to tie things together: the observables are self adjoint linear operators in the von Neumann algebra $B(H)$, and the states S are considered as vector states in $B(H)^*$.

Definition 4.0.4. *Let A be a Hermitian operator and $0 \neq \lambda \in \mathbb{R}$, then $A(\{\lambda\}) = span\{h \in H : Ah = \lambda h\}$, so that λ is in the point spectrum of A if $0 < A(\{\lambda\}) \in L(H)$. [21]*

For compact Hermitian operators we have that the point spectrum gives the full picture, and this is sufficient to describe our general case up to a tolerable approximation error.

4.1 Merging the implication structures of a Cubic lattice, a Hilbert Lattice, and a von Neumann Algebra

We now have an order embedding of the poset of ideals of the cubic lattice to the poset of faces of a von Neumann algebra. In particular, all of these maps can be shown to be well defined for addition and multiplication in the von Neumann algebra. Both structures have a fairly diverse set of operations. This section explores how the cubic lattice operations $\wedge_C$, $\vee_C$, and Δ, the orthomodular Hilbert lattice operations, $\wedge_H$, $\vee_H$, and $^\perp$, and the von Neumann algebra operations $+$, $\cdot$, and Jordan algebra multiplication,$\circ$, relate.

The first question is can we lift lattice structure of the lattice of signed sets L(S) to the algebraic structure of the $SA(\mathscr{M})_+ = (\mathscr{M}, +, \circ)$, where $\mathscr{M}$ is a von Neumann algebra. The answer is partially yes, and utilizes a standard result from logic and computing.

We first use a standard result of von Neumann algebras.

Proposition 4.1.1. *The self adjoint operators of von Neumann algebra form an order unit algebra where multiplication is the Jordan product induced by the multiplication of the von Neumann algebra.*

However, this order unit algebra is not unique to the given von Neumann algebra. There are many possible associative products defining many distinct von Neumann algebras that are consistent with a single underlying Jordan product. For more information, see a discussion of global orientations and Hilbert balls [2].

We now demonstrate the value of choosing an orthonormal basis. Of course, orthogonal projection operators need not commute even in the finite dimensional case.

Lemma 4.1.2. *Let H be a Hilbert space with an orthonormal basis $\mathscr{B}$. If F, $G \subseteq \mathscr{B}$, then $P_F P_G = P_{F \cap G} = P_G P_F$.*

Proof. We show the equality for a given element $h \in H$. By definition we have $P_F(h) = \sum_{f\in F}\langle f, h\rangle f$, $P_G(h) = \sum_{g\in G}\langle g,h\rangle g$, and $P_{F\cap G} = \sum_{f\in F\cap G}\langle f,h\rangle f$. Now,

$$\begin{aligned}
P_G P_F(h) &= \sum_{g\in G}\langle g, P_F h\rangle g \\
&= \sum_{g\in G}\langle g, \sum_{f\in F}\langle f,h\rangle f\rangle g \\
&= \sum_{g\in G\cap F}\langle g, \sum_{f\in F}\langle f,h\rangle f\rangle g + \sum_{g\in G-F}\langle g, \sum_{f\in F}\langle f,h\rangle f\rangle g \\
&= \sum_{g\in G\cap F}\langle g, \sum_{f\in F}\langle f,h\rangle f\rangle g + \sum_{g\in G-F}\sum_{f\in F}\overline{\langle f,h\rangle}\langle g,f\rangle g \\
&= \sum_{g\in G\cap F}\langle g, \sum_{f\in F}\langle f,h\rangle f\rangle g + \sum_{g\in G-F}\sum_{f\in F}\overline{\langle f,h\rangle}0 \\
&= \sum_{g\in G\cap F}\langle g, \sum_{f\in F}\langle f,h\rangle f\rangle g + 0 \\
&= \sum_{g\in G\cap F}\langle g, P_F h\rangle g \\
&= \sum_{g\in G\cap F}\langle P_F g, h\rangle g \\
&= \sum_{g\in G\cap F}\langle g, h\rangle g \\
&= P_{F\cap G}(h)
\end{aligned}$$

By symmetry, we have $P_F P_G(h) = P_{F\cap G}(h) = P_G P_F(h)$, and as h was arbitrary, we conclude the result.

□

Definition 4.1.3. *Let P be the set of projections in $\mathscr{M}^+$ onto a fixed orthonormal basis $\mathscr{B}$. The binary operation XOR denoted by $\oplus : P \times P \to P$ is defined by $A \oplus B = B^\perp A + A^\perp B$.*

We have already discussed that CL can be embedded in $\mathscr{M}$ as a set of projection operators, in particular idempotent operators. Now we have a binary operation from projections onto subsets of a fixed orthonormal basis, A, B defined by $A \vee_H B = A \oplus B \oplus AB$ where $\oplus$ is the symmetric addition (XOR) defined above, and $A \wedge B = A \circ B$ is the Jordan product.

Lemma 4.1.4. *Let $A, B \in B(H)$ be projection operators onto different subsets of the same orthonormal basis of H. Then $A \vee_H B = A + B - A \circ B$, where $+, -$ are the standard addition and subtraction induced by the von Neumann algebra.*

Proof. By Lemma 4.1.2, $AB = A \wedge_H B = B \wedge_H A = BA$ by commutativity, and now we can also freely use associativity. Since we are dealing with projection operators, we also have that all operators are idempotent. Now the result follows by a standard computation.

$$
\begin{aligned}
A \vee_H B &= A \oplus B \oplus AB \\
&= (A \oplus B) \oplus AB \\
&= (A(1-B) + B(1-A)) \oplus AB \\
&= (A + B - 2AB) \oplus AB \\
&= (1 - (A + B - 2AB))(AB) + (1 - AB)(A + B - 2AB) \\
&= AB - AB - AB + 2AB + A + B - 2AB - AB - AB + 2AB \\
&= A + B - AB
\end{aligned}
$$

□

Now we have everything to finish the final major result of the subsection:

Lemma 4.1.5. *For a proper principal lattice filter of the cubic lattice, $F \subseteq CL \subseteq HL$, $\wedge_H : F \times F \to HL = \wedge_C : F \times F \to F$. Equivalently the join of and ideal OL agrees with meet of the Hilbert lattice.*

Proof. Let $a, b \in F$. By definition we can write a, b as the joins of atoms that are members of the orthonormal basis constructed in Theorem 3.1.13, so we can write $a \wedge_H b$ in the same orthonormal basis as well. Therefore, we have the same relevant set of atoms for both HL and CL and reduce to this case implicitly for the remainder of the proof.

If α is an atom of $CL \subseteq HL$ such that $\alpha \leq a$ and $\alpha \leq b$ then $\alpha \leq a \wedge_C b$ and $\alpha \leq a \wedge_H b$. In addition, these are the only atoms in the commutative Boolean sub-lattice of HL that are less than or equal to $a \wedge_C b$ or $a \wedge_H b$. By atomisticity of the cubic lattice and the Boolean sub-lattice of the Hilbert lattice, $a \wedge_C b = \vee_C \{\alpha : \alpha \leq a \text{ and } \alpha \leq b\}$, $a \wedge_H b = \vee_H \{\alpha : \alpha \leq a \text{ and } \alpha \leq b\}$.

As the ordering of CL is inherited from HL, $\alpha \vee_H \beta = inf\{c \in HL : c \geq \alpha,\, c \geq \beta\} \leq inf\{c \in CL : c \geq \alpha,\, c \geq \beta\}$. Therefore, $a \wedge_H b \leq a \wedge_C b$. Now by reversing the above argument, $\alpha \wedge_H \beta = sup\{c \in HL : c \leq \alpha,\, c \leq \beta\} \geq sup\{c \in CL : c \geq \alpha,\, c \geq \beta\}$, and $a \wedge_H b \geq a \wedge_C b$. □

Remark 4.1.6. *The subtle and key part of this proof is that one does not lose atoms when finding lattice meets in the cubic lattice i.e. all atoms that are contained in both subfaces remain. However, when we consider joins of two subfaces, one gains atoms that were in neither subface. One can also not substitute the argument with co-atomisiticity as the set of co-atoms of HL and CL are not the same. In the finite dimensional case, one can view that the intersection of two subfaces contains all common lower dimensional subfaces, but the next largest subface containing both may include additional vertices.*

Theorem 4.1.7. *For a principal lattice filter of the cubic lattice, $F \subseteq CL \subseteq HL$, $\wedge_C : F \times F \to F$ is equal to the Jordan product and associative product of $B(H)$.*

Proof. The theorem follows as a result of Lemma 4.1.2 and Lemma 4.1.5. □

In many standard arguments, XOR is taken to be addition modulus 2, so we present the explicit computations for demonstrating that distributivity of the ring is equivalent to distributivity in the Boolean algebra.

Lemma 4.1.8. *Let $\mathscr{B}$ be an orthonormal basis of a Hilbert space H and A, B, C be projections onto subsets of $\mathscr{B}$.*

(1) $\wedge_H$ distributivity and multiplicative distributivity are equivalent.

(2) $\vee_H$ distributivity and XOR distributivity are equivalent.

Proof. We first demonstrate the $\wedge_H$ distributivity and multiplicative distributivity are equivalent using Lemma 4.1.4:

$$\begin{aligned} A \wedge_H (B \vee_H C) &= A(B + C - BC) \\ &= AB + AC - ABC \\ &= (A \wedge_H B) \vee_H (A \wedge_H C) \end{aligned}$$

In order to prove (2), we first use a simplification that $A \oplus B = (A \vee_H B) - (A \wedge_H B)$:

$$\begin{aligned} A \oplus B &= AB^\perp + BA^\perp \\ &= A(1 - B) + B(1 - A) \\ &= A - AB + B - AB \\ &= (A + B - AB) - AB \\ &= (A \vee_H B) - (A \wedge_H B). \end{aligned}$$

Then,

$$\begin{aligned} A \oplus (BC) &= (A \vee_H (B \wedge_H C)) - (A \wedge_H (B \wedge_H C)) \\ &= ((A \wedge_H B) \vee_H (A \wedge_H C)) - (A \wedge_H B \wedge_H C) \\ &= (A \wedge_H B) \oplus (A \wedge_H C) \\ &= (AB) \oplus (AC). \end{aligned}$$

□

Theorem 4.1.9. *The lattice of projections onto subsets of a fixed orthonormal basis, $L(\vee_H, \wedge_H)$, form a Boolean algebra.*

Proof. By 4.1.2, we have shown L when considered as ring $L(\oplus, \cdot)$ is firstly closed under $\cdot$, and secondly commutative. A ring of commutative idempotents is Boolean, so L is a Boolean ring. Then L is distributive as a ring, so we have L is distributive as a lattice by Lemma 4.1.8. As L is an orthomodular sublattice of HL, we have that L is a Boolean algebra by [14]. □

Theorem 4.1.7 demonstrates that the cubic lattice and the Hilbert lattice share a common meet described by the product of a von Neumann algebra. However the join structure of the lattices is very different. One can view this as a different question asked of the system. The join of two observables in a Hilbert lattice asks the minimum subspace where all events are observable. The join of two observables in a cubic lattice asks for the minimum subspace where both observables are entangled.

Theorem 4.1.9 shows a Boolean algebra containing the cubic lattice when embedded as in Theorem 3.1.13. However, the Boolean complement is order reversing and inherently irreconcilable with the unitary propagation of a quantum system. In addition, the Boolean structure is in a sense too rigid and well defined to fully encode the entanglement of a quantum system as stipulated by requirements in [17]:

1. Completeness
2. Atomicity
3. Superposition principle: (atom c is a superposition of atoms a and b if $c \neq a$, $c \neq b$, and $c \leq a \vee b$.)
   (a) Given two different atoms a and b, there is at least one other atom c that is a superposition of a and b.
   (b) If the atom c is a superposition of the distinct atoms a and b, then a is a superposition of b and c.
4. Unitary Operators: Given any two orthogonal atoms a and b in HL, there is a unitary operator U such that $U(a) = b$.
5. Infinite Orthogonality: HL contains a countably infinite sequence of orthogonal elements.

One can see that the Hilbert lattice satisfies these criteria while the Boolean lattice defined above fails the the superposition principle. The following subsection discusses a notion of order preserving complementation of cubic lattice and its Boolean subalgebras.

4.2 A Unitary Negation

Of course one's interest in physical systems involves more than their states and their experimentally verifiable questions. One also wants to know how a state changes over time based on its current state up to one's ability to observe the state experimentally. In the classical setting, if one observes the states of the system at a time, t_0, and we have perfect knowledge of some dynamic law $U(t)$ of the system, then we can predict the states of the system at time $t_0 + t$. However, in a quantum mechanical system, we will not have such predictive power, but we still have a dynamic law $U(t)$ represented by a unitary operator as discussed in the axiomitization of quantum mechanics stated in section 1.

A major issue that immediately occurs is that the notion of orthocomplementation of the logic is order reversing, while unitary operators are order preserving. Therefore, these two structures, which are necessary for the understanding of a quantum mechanical system, are, at some level, irreconcilable. We have discussed in the previous subsection that our embedding space is consistent with the modern literature. Now we will show that Δ when represented as a unitary operator may help bridge the gap to a unitary negation in some sense.

As we have discussed the basic operations $\wedge$ and $\vee$ in the respective lattices, we want to discuss the relation between the remaining operations: $^\perp$ and Δ. We first need to introduce some additional definitions. For additional context, see [2].

Definition 4.2.1. *A symmetry in a von Neumann algebra is a Hermitian involution. An e-symmetry in a von Neumann algebra is a Hermitian element s such that $s^2 = e$, where e is an orthogonal projection. When the specific projection e is not relevant, we will just say partial symmetry [2].*

Definition 4.2.2. *A projection in a von Neumann algebra is halvable if it is the sum of two Murray-von Neumann equivalent projections.*

Definition 4.2.3. *A partial symmetry s in a von Neumann algebra $\mathscr{M}$ will be called balanced if it has a canonical decomposition $s = p - q$, and $p \sim q$ where $\sim$ denotes Murray-von Neumann equivalence. [2]*

Proposition 4.2.4 (Proposition 7.7 [2])**.** *If e is a projection in a von Neumann algebra $\mathscr{M}$, then the following are equivalent:*

i) e is halvable

ii) there exists a Jordan orthogonal pair of e-symmetries

iii) there exists a balanced e-symmetry.

Theorem 4.2.5. *There exists a unitary representation of $Aut(L(S))$ on $B(H)$ where H is constructed in Theorem 3.1.13.*

Proof. As the Hilbert lattice is a atomic and atomistic lattice, and the atoms CL form an orthonormal basis of H, an automorphism on the atoms of CL defines a permutation of the atoms of HL. Therefore, lattice automorphism of CL embed into the permutation group in $B(H)$, in particular they are unitary. $\square$

Note that in the original representation of automorphisms of CL are reflections of the half plane or permutations of a basis of size $|S|$, specifically the representations are embedded in the orthogonal group. One can view this difference as choice in matrix representation.

Proposition 4.2.6. *An involution U in a von Neumann algebra is Hermitian if and only if U is unitary.*

Proof. Let $U \in B(H)$ involution. Suppose U is unitary, then

$$\begin{aligned} U^2 &= I \\ U^*U^2 &= U^* \\ (U^*U)U &= U^* \\ U &= U^* \end{aligned}$$

If U is Hermitian, then for any $h \in H$

$$\begin{aligned} \langle U^2 h, h\rangle &= \langle h, h\rangle \\ \langle Uh, U^*h\rangle &= \langle h, h\rangle \\ \langle Uh, Uh\rangle &= \langle h, h\rangle \\ ||Uh|| &= ||h|| \end{aligned}$$

Therefore, U is an isometry, and as $U^2 = I$ is surjective, so U must be as well. $\square$

Corollary 4.2.7. *The unitary representation of Δ, U_Δ on $B(H)$ is a symmetry.*

Proof. Δ is an involution by definition, so the result follows from the above arguments. $\square$

As a symmetry, we have the a canonical decomposition of $U_\Delta = \frac{1+U_\Delta}{2} - \frac{1-U_\Delta}{2}$, where $\frac{1+U_\Delta}{2}$ and $\frac{1-U_\Delta}{2}$ are orthogonal projection operators.

Proposition 4.2.8. *The symmetry U_Δ is a balanced symmetry of $B(H)$.*

Proof. We need only consider coatoms of $c, \Delta(c) \in CL$ with associated projections p_c, $p_\Delta(c)$, and let U_Δ be the unitary representation of Δ. Then we have the action of the unitary symmetry $U_\Delta p_c U_\Delta = p_{\Delta(c)}$.

Now let $s = p_c - p_{\Delta(c)}$. Then,

$$\begin{aligned} U_\Delta s U_\Delta &= U_\Delta(p_c - p_{\Delta(c)})U_\Delta \\ &= p_{\Delta(c)} - p_c \\ &= -s \end{aligned}$$

Therefore,

$$\begin{aligned} U_\Delta s &= U_\Delta s U_\Delta^2 \\ &= (U_\Delta s U_\Delta)U_\Delta \\ &= -sU_\Delta. \end{aligned}$$

The result follows by 4.2.4. Explicitly the symmetry similarity transformation, s, takes $\frac{1+U_\Delta}{2}$ to $\frac{1-U_\Delta}{2}$, and unitary similarity implies Murray-von Neumann equivalence. $\square$

Of course the choice of coatom was arbitrary in the above proof, so we have the following slightly more general result.

Corollary 4.2.9. *For any given coatom $c \in CL$, the symmetry $s = p_c - p_{\Delta(c)}$ anticommutes with U_Δ.*

Theorem 4.2.10. *The action of $^\perp$ on HL on the coatoms of CL is a symmetry and coincides element-wise with the unitary symmetry associated with Δ.*

Proof. The result follows as for all $c \in CL$, $p_c^\perp = 1 - p_c = p_{\Delta(c)} = U_\Delta p_c U_\Delta$. $\square$

Corollary 4.2.11. *Let c, $\Delta(c) \in CL$ be coatoms, then the lattice operations of CL and HL coincide.*

Proof. Consider $c, \Delta(c) \in CL$ and the associated projection operators p_c and $p_{\Delta(c)}$, which project on a generalized simple tensor u of the form $u_i = V_i$ for all $i \in I - \{j\}$ and for exactly one $j \in I$, $u_j = v$, $v \in V_j$. By lemma 4.1.5, we need only show that $p_c \vee_H p_{\Delta(c)} = p_c \vee_C p_{\Delta(c)}$. Further, we reduce to showing $p_c \vee_H p_{\Delta(c)} \geq p_c \vee_C p_{\Delta(c)}$ by the argument in the proof of Lemma 4.1.5.

Now by Theorem 4.2.10, $p_c^\perp = p_{\Delta c}$, so $p_c \vee_H p_{\Delta c} = p_c + p_{\Delta c} = 1 \geq p_{\Delta c} \vee_C p_c$, where orthogonality is implicitly used in the first set of equalities. $\square$

Example 4.2.12. *Let $H = \mathbb{C}^2$, and U_Δ, s be constructed as above for the standard basis.*

$$U_\Delta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$iU_\Delta s = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

We recognize these matrices as the famous Pauli matrices!

Corollary 4.2.13. *The operators of Corollary 4.2.9 give an explicit construction for the generalization of Pauli matrices to a cubit system up to arbitrary cardinality, ω. Furthermore, as a computational device, we can use our original lower dimensional matrix representation to perform these matrix computations on a $O(\omega)$ vector space as opposed to the exponentially larger $O(2^\omega)$ vector space used in the standard construction.*

We now have an order preserving lattice isomorphism of HL, U_Δ, and an order reversing lattice anti-isomorphism, $^\perp$, that agree on certain subset specifically the coatoms of CL. Furthermore, we have that for pairs of coatoms which half the identity, we have that the canonical symmetry associated with the pair anti-commutes with the action U_Δ, and therefore the action of $^\perp$ when we consider the action element wise. However, U_Δ is distributive on the difference of the projections while $^\perp$ is not except on a principle filter. In essence, we now have a linear distributive operation that agrees element-wise with $^\perp$ on the coatoms of CL.

As we move forward, this relationship will become more relevant. The lattice automorphisms of CL, permutations of coatoms of CL that commute with Δ, and the commutant of Δ are all intimately related in a way that will be discussed. The above section shows that in some sense this relationship holds with $^\perp$ on the Hilbert lattice as well and the reader should keep this in mind.

5 The Necessity of the Cubic Lattice

Throughout this document, we have created a sufficient structure to characterize the algebraic relations of an n-qubit system when considered in an analytic space. However, we now raise the question, what other structures suffice? Is there perhaps an entire set of such objects and what is the underlying characterizing feature? We now demonstrate that the symmetries required for an n-qubit system in fact require a cubic lattice structure. Furthermore, we show that these algebraic relations are in fact measurable in the sense of [24]. We also show which von Neumann algebras contain a cubic lattice of a given cardinality up to $*$-isomorphism.

We will think of the commutant of U_Δ is in some sense generated by the automorphism group of the lattice of signed sets. We will discuss this more in the following section.

5.1 The Symmetry Group of the Cubic Lattice and Quantum Relations

In the finite case, the automorphism (symmetry) group of the cubic lattice is the Coxeter group B_n otherwise known as the hyperoctahedral group O_n.

Definition 5.1.1. *Let $Per(C)$ be the group of permutations of coatoms of CL, $Per_\Delta(C)$ be the centralizer of Δ in $Per(C)$, and $L(S)$ the lattice of signed sets over S.*

Theorem 5.1.2. *For a cubic lattice of cardinality $\aleph$, $L(S)$, $Aut(L(S)) \cong Per_\Delta(C) \cong \mathbb{Z}_2 \wr S_\aleph$, where $\wr$ denotes the unrestricted wreath product. [7]*

For explicit details, we refer to [7]. As a brief sketch, for each cubic lattice over a signed set S, we have an order 2 involution, which is a reflection about the respective axis. We also have the permutation action of each index of S, leading to a a semi direct product on the n-fold direct product S_2 with S_n. Then [7] extends this to a general infinite case. However, their choice of embedding space is a Banach space of dimension equal to the indexing set S, as opposed to our exponentially larger Hilbert embedding. We now generalize these arguments to von Neumann algebras over the Hilbert space constructed in Theorem 3.1.13.

Proposition 5.1.3. *The C^* algebra generated by U_Δ is a von Neumann algebra.*

Proof. Since U_Δ is a self adjoint unitary operator, we have that $\sigma(U_\Delta) \subseteq \{-1, 1\}$. In particular the spectrum of U_Δ is a finite set. Therefore, we have a *-isomorphism: $C^*(U_\Delta) \cong C(\sigma(U_\Delta)) = C(K)$, where K is a finite set. Now we apply that $C(K)$ is a reflexive, as K is finite. Then $C^*(U_\Delta)$ is a finite dimensional C^* algebra, so it is a von Neumann algebra. $\square$

Lemma 5.1.4. *Let ′ denote the commutant. Then $W^*(U_\Delta) = Z(W^*(U_\Delta)')$.*

Proof. As $W^*(U_\Delta)$ is an abelian unital W^* algebra, $W^*(U_\Delta) \subseteq Z(W^*(U_\Delta)')$. Since $W^*(U_\Delta)$ is also a von Neumann algebra, we have that $Z(W^*(U_\Delta)') \subseteq W^*(U_\Delta)'' = W^*(U_\Delta)$, where the last equality follows by the double commutant theorem. □

We now have a large amount of insight into the structure $W^*(U_\Delta)$. There are three views to consider, firstly as a finite dimensional abelian von Neumann algebra $W^*(U_\Delta)$ is isomorphic to an $l^\infty(\{1, 2, \ldots, n\})$ for $n \in \mathbb{N}$. On the other hand, we know that $W^*(U_\Delta)$ as a unital commutative Banach algebra, so $W^*(U_\Delta)$ is also isomorphic to $C(K)$, and lastly that $W^*(U_\Delta)$ is isomorphic to $p(U_\Delta)$, which as Δ is an involution, is a two dimensional $\mathbb{C}$ vector space. Of course, this all ultimately follows from the general principle that continuous maps over a finite space are a vacuous concept and devolves to a map from a finite set to the complex numbers. We describe this in general in the following statement.

Proposition 5.1.5. *Let $A \in B(H)$ be a normal operator such that $A^n = I$ for some $n \in \mathbb{N}$, then $C^*(A)$ is a von Neumann algebra and equal to the center of its commutant.*

Before further discussing $W^*(U_\Delta)'$, we introduce some theory about the automorphism groups of the Hilbert lattice and the cubic lattice. To begin, the embedding of $L(S)$ in $H(L)$ as constructed in Theorem 3.1.13 is minimal in a fairly strict sense.

Theorem 5.1.6. *Let $f : L(S) \to HL$ where the atoms of $L(S)$ are contained in the atoms of HL, and f is an injective order morphism. Then there exists an unique injective order morphism $\psi : \tilde{H}(L) \to HL$, where $\tilde{H}(L)$ is the embedding, j of Theorem 3.1.13 such that $\psi \circ j = f$*

Proof. First we show existence of such an f. We only need to use that the Hilbert lattice is atomic and complete by Proposition 3.1.1. Therefore, if we have two Hilbert lattices, we have an injective order morphism if we have an injective mapping of orthonormal bases to the Hilbert spaces of the respective Hilbert lattices. Then we have $\psi = f \circ j^{-1}$. Now we see that uniqueness follows as j is a bijection between the orthonormal basis of $\tilde{H}(L)$ and the atoms of $L(S)$, so if there exists another map ρ satisfying our criteria, then $\rho \circ j = f$ implies $\rho = f \circ j^{-1} = \psi$. □

Definition 5.1.7. *A conjugate linear operator is a linear operator except for scalar multiplication is treated as conjugate scalar multiplication.*

Definition 5.1.8. *Let H be a Hilbert space and consider $\Phi : B(H) \to B(H)$. Φ is said to be implemented by a (conjugate) unitary if there is a (conjugate) unitary map $U : H \to H$ such that $\Phi a = UaU^*$ for all $a \in B(H)$. [2]*

Lemma 5.1.9. *Let $g \in Aut(H(L))$, then there exists a unitary or conjugate linear unitary operator $U_g : H \to H$ such that g is implemented by U_g.*

Proof. If $g \in Aut(H(L))$, then g is a unital order automorphism, so by [2, Proposition 4.19], g is a Jordan automorphism. If g is a Jordan automorphism, then g is either a $*$-isomorphism or $*$-anti-isomorphism by [2, Proposition 5.69]. If g is a $*$-isomorphism then g is implemented by a unitary, and if g is a $*$-anti isomorphism then g is implemented by a conjugate unitary by [2, Theorem 4.27]. $\square$

Definition 5.1.10. *Let $A \in \mathscr{M}$ be invertible, then $Ad_A : \mathscr{M} \to \mathscr{M}$ is defined by $Ad_A(\cdot) = A(\cdot)A^{-1}$. Equivalently, one can view Ad_A as the inner automorphism induced by A on $\mathscr{M}$. [2]*

Definition 5.1.11. *We say that the action of two unitary operators commute in a von Neumann algebra $\mathscr{M}$ if their action by inner automorphism commutes.*

Theorem 5.1.12. *Let $g \in Aut(H(L))$, then $Ad_g \in Aut(L(S))$ if and only if the action of g commutes with the action of U_Δ on $W^*(L(S))$ where H is constructed in the manner of Theorem 3.1.13.*

Proof. By Lemma 5.1.9, we know that g can be implemented by a unitary or conjugate unitary operator, U. Without loss of generality, we assume that U is a unitary operator as this affects the associative multiplication consistent with the Jordan algebra of the Hilbert lattice, but it does not affect the action as an order automorphism.

Assume that the action U commutes with action of U_Δ on $L(S)$. It is sufficient to show that $Ad_g \in Per_\Delta(C)$. Now c, $\Delta(c)$ be coatoms in $L(S)$. Then

$$\Delta(g(p_c)) = U_\Delta U_g p_c U_g^* U_\Delta = U_g U_\Delta p_c U_\Delta U_g^*. = g(\Delta(p_c))$$

We have that g maps to coatoms to coatoms in some isomorphic lattice to our original $L(S)$ in particular an order isomorphism, so $Ad_g \in Aut(L(S))$.

Now for the converse. If the inner automorphisms do not commute on $L(S)$, then there exists $c \in C$ such that $\Delta(g(p_c)) \neq g(\Delta(p_c))$. As $g \in Aut(L(S))$, $g(c)^\perp = \Delta(g(c))$ by Theorem 4.2.10. Therefore $g(\Delta(p_c)) \neq g(p_c)^\perp$, but by linearity $g(\Delta(p_c)) + g(p_c) = g(\Delta(p_c) + p_c) = g(I) = I$, which leads to a contradiction. $\square$

One can observe that there are many more unitary transformations, and therefore, automorphims of the Hilbert lattice, then there are automorphisms of the cubic lattice. Now that we know that $Aut(H(L))$ are unitary or conjugate unitary transformation, we deduce exactly which unitary operators are automorphisms of the cubic lattice.

We see that we have a choice of equivalence class when we represent $Aut(L(S))$ by its action on $L(S)$, as there are automorphisms acting as the identity on $L(S)$ that to do not act as the identity on HL. Namely in the abelian von Neumann algebra, $W^*(\{p_c\}_{c \in C})$, the symmetries associated $L(S)$ are such an example.

Due to this ambiquity, we choose to define a group representation of Aut(L(S)) up to group isomorphism acting on H as opposed to inner automorphisms acting on HL. From another perspective, we have for $U \in Aut(L(S))$ the action $Uh\rangle\langle Uh_1 \mapsto h_2\rangle\langle h_2$ on the lattice of orthogonal projections, HL, and we are instead considering the action $Uh_1\rangle \mapsto h_2\rangle$ for any $h_1, h_2 \in H$. One can see that any action in $Aut(L(S))$ on HL can be induced by the action on H and vice versa, but we have removed the ambiguity of the representation. We make this more formal below.

Lemma 5.1.13. *There exists a unitary representation $\rho : Aut(L(S)) \to B(H)$ such that $U_g \in W^*(U_\Delta)'$ for all $g \in Aut(L(S))$.*

Proof. When considered as an automorphism group acting on the orthonormal basis constructed in Theorem 3.1.13, we have a group representation of $Aut(L(S))$ contained in the permutation group over H, so we conclude that the group representation is a unitary representation.

Now we apply $Aut(L(S)) \cong Per_\Delta(C)$, the permutations of the coatoms that commute with Δ to see that $\Delta \subseteq Z(Aut(L(S))$. As commutativity of the group implies commutativity of its representation, we conclude the result. □

Proposition 5.1.14. *Every complete Boolean algebra, $\mathscr{B}$, corresponds to a unique Stonean completion $\mathscr{A}$ whose set of projections is equal to $\mathscr{B}$. [11]*

Proposition 5.1.15. *If $\mathscr{A} \leq B(H)$ is an atomic abelian von Neumann algebra whose lattice of projections form an atomic complete Boolean algebra, which is maximal in the Hilbert lattice of H, then $\mathscr{A}$ is a maximal abelian algebra.*

Proof. From Proposition 5.1.14, we know that Boolean lattice of projections correspond to abelian subalgebras of a von Neumann algebra. Let A be the atoms of $\mathscr{A}$, and $p \in \mathscr{A}' - \mathscr{A}$ be a projection. Furthermore, we can assume that p is orthogonal to every $a \in A$, otherwise let $p = p - (\vee_{a \in A} p \wedge a)$. Then p commutes the atoms $a \in \mathscr{A}$, so $a \geq a \wedge p \geq= ap = 0$ as a is an atom. The Hilbert lattice is atomic by Proposition 3.1.1, so let $b \leq p$ be an atom and by the above $a \wedge b \leq a \wedge p = 0$.Therefore, the lattice containing $\mathscr{B}$ and b is an atomic complete Boolean lattice strictly containing $\mathscr{B}$, contradicting maximality. Therefore $\mathscr{A}$ and $\mathscr{A}'$ contain the same projections and must be equal. The result follows as abelian von Neumann algebra equal to its commutant is maximal. □

Lemma 5.1.16. *Let $CL \subseteq HL$ as in Theorem 3.1.13 and $U \in W^*(\Delta)'$ be unitary. There exists a unitary $V \in \rho(Aut(L(S)))$ such that $Ad_U = Ad_V : CL \to CL$ and $U = VS$ for $S \in W^*(\{p_c\}_{c \in C}) \cap W^*(U_\Delta)'$.*

Proof. If $U \in W^*(\Delta)'$, then $Ad_U \in Aut(L(S))$ by Theorem 5.1.12. Now let $V = \rho(Ad_U) \subseteq W^*(U_\Delta)'$. Then $Ad_{V^*} = Ad_V^{-1}$, so $Ad_{UV^*}|_{CL} = Ad_I|_{CL}$. As the action of inner automorphism stabilizes CL, $UV^* \in W^*(\{p_c\}_{c \in C})'$ and $W^*(\{p_c\}_{c \in C})' = W^*(\{p_c\}_{c \in C})$ by Proposition 5.1.15.

Therefore, there exists $S \in W^*(\{p_c\}_{c\in C})$ such that $U = VS$. Furthermore, $S = UV^*$, so $S \in W^*(U_\Delta)'$ as well. $\square$

The above representation when considered as an action of inner automorphism on $B(H)$ can be seen to be identical to our previous notion where we fix $S = I$.

Theorem 5.1.17. *$W^*(U_\Delta)' = W^*(\rho(Aut(L(S))), W^*(\{p_c\}, U_\Delta)')$.*

Proof. We use the above lemmas, to show both von Neumann algebras have the same set of unitaries for an appropriate representation of $Aut(L(S))$. In particular any unitary in $W^*(U_\Delta)'$ is a product of the two algebras $W^*(\rho(Aut(L(S))))$ and $W^*(\{p_c\}, U_\Delta)'$. Now we use that von Neumann algebras are generated by their unitaries, see Proposition I.4.9 in [20], so the result follows. $\square$

Corollary 5.1.18. *$W^*(U_\Delta) = Z(W^*(\rho(Aut(L(S)))))$.*

Proof. Follows immediately from the result $Z(Aut(L(S)) = \{1, \Delta\}$ in [7], the definition of ρ, and the spectral theorem. $\square$

We have extended the purely group theoretic ideas of [7] to the more general von Neumann algebra setting. Now that we have a legitimate and well understood von Neumann algebra, $W^*(\Delta)$, and some insight into its commutator, we can finally discuss the quantum relations that Δ induces. We demonstrate that the relations specified by the cubic lattice are natural and measurable in the sense of [24]. We first define a standard relation:

Definition 5.1.19. *Let X be a set, then a binary relation is a set of ordered pairs $(a, b) \in X \times X$, where $a, b \in X$. In some literature, the notation aRb is used to describe a set with a relation R, denoted (X, R).*

The obvious issue with the classic notion of a relation is that when one considers a non-atomic measure, these finite relations become vacuous. In [24], they generalize this notion to a measurable relation.

Definition 5.1.20. *A measure space (X, μ) is finitely decomposable if X can be partitioned into a (possibly uncountable) family of finite measure subspaces X_λ such that a set $S \subseteq X$ is measurable if and only if its intersection with each X_λ is measurable, in which case $\mu(S) = \sum_\lambda \mu(S \cap X_\lambda)$.*

As pointed out in [24], a measure space (X, μ) is finitely decomposable exactly when $L^\infty(X, \mu)$ is an abelian von Neumann algebra. A full explanation can be seen in [4].

Definition 5.1.21. *Let (X, μ) be a finitely decomposable measure space. A measurable relation on X is a family R of ordered pairs of nonzero projections in $L^\infty(X, \mu)$ such that $(\vee p_\lambda, \vee q_\kappa) \in R$ if and only if some $(p_\lambda, q_\kappa) \in R$ for any pair of families of nonzero projections $\{p_\lambda\}$ and $\{q_\kappa\}$.*

Equivalently, we can impose the two conditions

$p_1 \leq p_2$, $q_1 \leq q_2$, $(p_1, q_1) \in R \Rightarrow (p_2, q_2) \in R$

and

$(\vee p_\lambda, \vee q_\kappa) \in R \Rightarrow$ *some* $(p_\lambda, q_\kappa) \in R$. *[24]*

Of course we are dealing with a more general not necessarily abelian structure. In [24], they define a quantum relation on a von Neumann algebra.

Definition 5.1.22. *A quantum relation on a von Neumann algebra $\mathscr{M} \subseteq B(H)$ is a W^*-bimodule over its commutant $\mathscr{M}'$,i.e., it is weak* closed subspace of $V \subseteq B(H)$ satisfying $\mathscr{M}' V \mathscr{M}' \subseteq V$.*

Now we argue from the reverse perspective. If we a priori argued that a quantum logic must respect the symmetry group of a possibly infinite dimensional cube, the infinite hyperoctahedral group, $Aut(L(S))$, we could consider the von Neumann algebra generated by $Aut(L(S))$.

Proposition 5.1.23. *Let $\mathscr{B}$ be the basis of atoms on $L(S)$ constructed in Theorem 3.1.13, then $W^*(U_\Delta)$ acts transitively in $\mathscr{B}$.*

Proof. If $u, v \in \mathscr{B}$, then we consider the composition of $\mathbb{Z}_2$ actions on each disagreeing index, which is contained in $W^*(U_\Delta)$ by construction. $\square$

Geometrically in the finite dimensional case, we can observe that the Coxeter group, B_n, is the group of rigid motions of the cube and must be able to permute any 2 vertices.

Corollary 5.1.24. *The quantum relations associated with $W^*(U_\Delta)$ are weak* closed subspaces V satisfying $W^*(U_\Delta) V W^*(U_\Delta) \subseteq V$.*

We now have that the operator systems discussed above, i.e. the ideals of CL are have well defined quantum relations. Furthermore, by presupposing the lattice, we have re-derived both Δ, and the invariant subspaces of the cube.

From an experimental setting, this invariant subspace is a natural requirement, as one can rotate the axis for detection of a spin $\frac{1}{2}$ particle, but we still need cubic symmetry as the experiment takes place in Euclidean space. Therefore, our notion Δ can be viewed as a necessary condition of relations in the experiment. In addition, these symmetries can be verified by the single relation Δ as opposed to the (infinite) hyperoctahedral group.

Furthermore, our original lattice based definitions of the cube are now seen to be measurable in a much more general sense.

As the principle ideals of a cubic lattice are again cubic lattices, we can further infer that the principle ideals form von Neumann subalgebras and therefore operator systems.

Definition 5.1.25. *An operator system is unital $*$-closed subspace contained in a unital C^* algebra.*

To be precise we present the following theorem.

Theorem 5.1.26. *Let $p_a \in CL \subseteq HL$ as in Theorem 3.1.13, and $\mathscr{M} = B(H)$. Then $p_a\mathscr{M}p_a$ forms a von Neumann sub algebra, and $[a]_C$ forms a Boolean lattice.*

Proof. The fact that $p_a\mathscr{M}p_a \leq \mathscr{M}$ is a standard result. We refer to [16] for the construction of the complement ${}^c(\cdot) = a \vee_C \Delta\Delta(\cdot, a)$ making $[a]_C$ a Boolean algebra. □

Therefore, an element in a cubic lattice can be seen as a dividing line between a Boolean algebra and a von Neumann algebra. This echoes back to our notion that the projection operators of a cubic lattice detect the minimum entangled state containing the respective atoms and above that level of detection, elements have become disentangled and therefore Boolean.

Corollary 5.1.27. *Let (p) be a principle ideal in CL. Then $C^*(\Delta_p)$ is a von Neumann algebra pDp.*

Proof. This follows as the principle ideals of cubic lattice are themselves cubic lattices. □

5.2 Operator Algebras containing a cubic lattice

We can see that the above results can can be generalized in a straightforward manner.

Definition 5.2.1. *Let C be the co-atoms of CL. Then for each $c \in C$, we get a symmetry in the canonical form of $p_c - p_{\Delta c}$. We denote the set by $\{s_i\}_{i\in I}$.*

Importantly, ith coordinate in the tensor product is equal to the matrix $s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Lemma 5.2.2. *With our previous choice of representation, $\rho : Aut(L(S)) \to B(H)$, the mutual commutant of U_Δ, s_i is equal to $W^*(W^*(\rho(\mathbb{Z}_2 \wr S_{I-i})), W^*(\{p_c\}_{c\in C}) \cap W^*(U_\Delta)')$, again by $\wr$ we mean the unrestricted wreath product.*

Proof. By Lemma 5.1.16, we already have an explicit definition of the unitaries that commute with U_Δ, so we only need to consider the subset that also commute s_i.

We consider the elements of $Per_\Delta(C)$, that fix p_{c_i}, $p_{\Delta c_i}$. They are the permutations fixing the ith coordinate that commute with U_Δ, so we have that it is again the infinite hyperoctahedral group but on one less coordinate or $\mathbb{Z}_2 \wr S_J$, where $|J| = |I| - 1$ or elements of $W^*(\{p_c\}_{c\in C}) \cap W^*(U_\Delta)'$.

Now the result follows by taking the WOT closure of the algebra generated by its unitary operators, which fully defines the von Neumann algebra again by Proposition I.4.9 [20]. □

Theorem 5.2.3. *Let H be constructed in the manner of Theorem 3.1.13, then $B(H) \cong M_2(B)$, where $B \cong I_2 \otimes B(H_{I-i})$.*

Proof. Let U_{Δ_i} be the tensor product whose ith index is equal to U_Δ's ith index and I_2 elsewhere. We claim the following form matrix units for $B(H)$.

$$\begin{aligned}
e_{11} &= \frac{I+s_i}{2} \\
e_{12} &= \frac{(I+s_i)U_{\Delta_i}}{2} \\
e_{21} &= \frac{U_{\Delta_i}(I+s_i)}{2} \\
e_{22} &= \frac{I-s_i}{2}
\end{aligned}$$

We can directly compute that $e_{11}+e_{22} = I$, $e_{12} = e_{21}^*$, and $e_{ij}e_{kl} = \delta_{jk}e_{il}$. Therefore, $B(H) \cong M_2(B)$, where B commutes with all of the matrix units, see Lemma 4.27 of [1].

Now we show $N = W^*(\{e_{ij}\}_{i,j\in\{1,2\}}) = W^*(U_{\Delta_i}, s_i)$.

Firstly, $U_\Delta \in N$,

$$\begin{aligned}
U_{\Delta_i} &= \frac{U_{\Delta_i} + s_iU_{\Delta_i} - s_iU_{\Delta_i} + U_{\Delta_i}}{2} \\
&= \frac{U_{\Delta_i} + s_iU_{\Delta_i} + U_{\Delta_i}s_i + U_{\Delta_i}}{2} \\
&= \frac{(I+s_i)U_{\Delta_i}}{2} + \frac{U_{\Delta_i}(I+s_i)}{2} \\
&= e_{12} + e_{21}
\end{aligned}$$

Secondly, $s \in N$

$$\begin{aligned}
s_i &= \frac{2s_i - I + I}{2} \\
&= \frac{I+s_i}{2} - \frac{I-s_i}{2} \\
&= e_{11} - e_{22}
\end{aligned}$$

Therefore $W^*(U_\Delta, s_i) \subseteq N$. For the reverse containment, the generators of N are in the algebra generated by U_Δ, s_i, so they are in the WOT closure of the algebra.

Now we apply that $M_2(\mathbb{C})_i \otimes I_{I-i} = W^*(U_{\Delta_i}, s_i)$, so that $N' = I_2 \otimes B(H_{I-i})$, where $H_{I-i} = \otimes_{j\in(I-i)}\mathbb{C}^2$ in the manner of Theorem 3.1.13. □

Example 5.2.4. *We see that in our choice of matrix units, we again obtain that*

$$U_{\Delta_i} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ s_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \textit{and } is_iU_{\Delta_i} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

This is considered as a representation of $M_2(B)$ as opposed to $M_2(\mathbb{C})$. Of course if we reduce to the single qubit case, we have that the $B \cong \mathbb{C}$ and only one choice of index for s_i, so our result is consistent.

We relate the above construction to a more familiar general object.

Definition 5.2.5. *[2] A Cartesian triple is a set of operators r, s, t in a von Neumann algebra such that*

1. $r \circ s = s \circ t = t \circ r = 0$.
2. $Ad_r Ad_s Ad_t = I$.

Corollary 5.2.6. *For any $s_i \in S$, the set U_Δ, s_i, and $iU_\Delta s_i$ form a Cartesian triple in B(H).*

Proof. Given our representation, the result follows from standard facts about Pauli matrices. □

We can consider another von Neumann subalgebra of B(H). Namely, $W^*(\{s_i\}_{i\in I})$.

Lemma 5.2.7. *Given our representation of Δ, the coatoms of CL, C, are exactly a generating set of projections of $W^*(\{s_i\}_{i\in I})$.*

Proof. We have that $C = \left\{\dfrac{1 \pm s_i}{2}\right\}_{i\in I}$ generates $\{s_i\}_{i\in I}$ and vice versa. Therefore, $W^*(C)$ generates the unitaries of $W^*(\{s_i\}_{i\in I})$, and therefore generates all of $W^*(\{s_i\}_{i\in I})$. □

Theorem 5.2.8. *The atoms of $W^*(\{s_i\}_{i\in I})$ are the atoms of CL.*

Proof. We have shown that the coatoms of CL are in $W^*(\{s_i\}_{i\in I})$, by Lemma 5.2.7. By coatomicity of CL, and Lemma 4.1.5, we have that the atoms of CL are contained in $W^*(\{s_i\}_{i\in I})$.

Now for the reverse direction, we consider the complete lattice of projections L generated by the canonical projections of $\{s_i\}_{i\in I}$. Here we mean complete in the sense of lattice theory not necessarily complete with respect to the norm and generated in the sense closure of meet and joins. As the canonical projections of $\{s_i\}_{i\in I}$ are exactly the coatoms of CL, we have that the atoms of L are exactly the set of atoms of CL by Lemma 4.1.5, and in addition, L is a complete lattice generated by an orthonormal basis and therefore Boolean.

In our specific application of Proposition 5.1.14, the atoms of L form a maximal set of mutually orthogonal projections, and the subalgebra of bounded operators of $\mathscr{A}$, $C^*(L)$, is abelian, so we have that $C^*(\{s_i\}_{i\in I}) = C^*(L)$ is a von Neumann algebra [11, Remark 10.8] whose atoms are the atoms of the cubic lattice. □

Therefore, we now have a minimal von Neumann algebra containing the $CL = L(S)$ for a given $|S|$. Furthermore, we have shown that $W^*(\{s_i\}_{i\in I}) \leq B(H)$, where H is minimal as in Theorem 5.1.6.

Example 5.2.9. *When reducing the one qubit case, we see that $W^*(U_\Delta, s)$ contain the Pauli matrices, which are a W^* algebra over $\mathbb{C}$ generating all of $M_2(\mathbb{C})$, which is a well known result, as required. Furthermore, we have a unitary matrix $T \in M_2(\mathbb{C})$,*

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

which is a unitary similarity sending s to U_Δ. We recognize this as the normalized Hadamard matrix.

Definition 5.2.10. *We define $U = \otimes_{i \in I} T$.*

Theorem 5.2.11. $B(H) = W^*(\{U s_i U^*\}_{i\in I}, \{s_i\}_{i\in I})$.

Proof. We only need to show that $W^*(\{U s_i U^*\}_{i\in I}, \{s_i\}_{i\in I})' = W^*(\{U s_i U^*\}_{i\in I})' \cap W^*(\{s_i\}_{i\in I})' = Z(B(H)) = \mathbb{C}I$.

Suppose that V is a unitary operator commuting with U_Δ, then by Lemma 5.1.16, $Ad_V \in Aut(L(S))$, when considering its action by inner automorphism on $L(S)$. As V commutes with each co-atom of $L(S)$, V acts trivially on the co-atoms of $L(S)$, so by coatomisticity of L(S), V acts trivially on $L(S)$. Then $V \in W^*(\{s_i\}_{i\in I})$. By symmetry, $V \in W^*(\{U s_i U^*\}_{i\in I})$.

Consider canonical projections p_i of $U s_i U^*$ and $q_i \in s_i$ for some fixed index $i \in I$. Then $p_i \wedge q_i = \lim_{n\to\infty}(p_i q_i p_i)^n = \lim_{n\to\infty}(\frac{1}{2}p_i)^n = 0$. By construction, any atom $a \in \{U s_i U^*\}$, a is bounded by a canonical projection of $U s_i U^*$, so we assume without loss of generality that $a \leq p$, and by symmetry we assume $b \leq q$. Then $a \wedge b \leq p \wedge q = 0$. Therefore the atomistic Boolean lattice of projections associated with $\{U s_i U^*\}_{i\in I}$ and $\{s_i\}_{i\in I}$ have distinct sets of atoms. By atomisticity, $W^*(\{s_i\}_{i\in I})$ and $W^*(\{U s_i U^*\}_{i\in I})$ are abelian von Neumann algebras whose only common projections are 0 and I, so their intersection is $\mathbb{C}I$ by Proposition 5.1.14. □

As we have demonstrated now, that not only is the Hilbert Lattice we originally embedded CL into minimal, the von Neumann algebra as a whole is generated by two copies of CL with orthogonal atoms.

Corollary 5.2.12. *Let $L(C)$ be the meet semi lattice generated by the C the coatoms of the cubic lattice adjoin 1. Then the meets and joins of $L(S)$ are exactly $CL \subseteq B(H)$.*

We can see that our generation of $B(H)$ is therefore a generalization of the single qubit case to arbitrary cardinals.

Now we have shown that $B(H)$ is generated by Δ, CL directly. We also see that $B(H)$ is a minimal structure containing both, and as such is a necessary structure if one considers an operator algebraic structure of the cubic lattice under the conditions [17] detailed at the conclusion of section 4.1.

5.3 Phase Rotations

So far we have re-derived the Pauli and Hadamard gates, referred to as the X, Z, and H gates in the literature, and their respective role in the underlying von Neumann algebra. As shown this von Neumann algebra is over a Hilbert space constructed in the standard manner and generalized to arbitrary cardinals. The question now becomes what types of observables can we obtain as functions of our already constructed observables? We will show that continuous functional calculus can be used to construct universal quantum gates in the sense of the Solovay Kitaev theorem, [12].

Definition 5.3.1. *Let U_Δ, s be represented in $M_2(\mathbb{C})$, then*

$$R_x(\theta) = e^{iU_\Delta \frac{\theta}{2}} = \begin{bmatrix} cos\left(\frac{\theta}{2}\right) & -isin\left(\frac{\theta}{2}\right) \\ -isin\left(\frac{\theta}{2}\right) & cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$R_y(\theta) = e^{iU_\Delta s \frac{\theta}{2}} = \begin{bmatrix} cos\left(\frac{\theta}{2}\right) & -sin(\frac{\theta}{2}) \\ sin\left(\frac{\theta}{2}\right) & cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$R_z(\theta) = e^{is\frac{\theta}{2}} = \begin{bmatrix} e^{\frac{-i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{bmatrix}$$

We will discuss group theoretic properties that can be shown directly from a computation in the case of $M_2(\mathbb{C})$, but we highlight a more general technique to extend these results.

Proposition 5.3.2. *Let A be a normal operator in a C^* algebra, $\mathscr{A}$. Then for any $f \in C(\sigma(A))$, and unitary $U \in \mathscr{A}$, $Uf(A)U^* = f(UAU^*)$.*

Proof. Let $\rho(1) = I$, $\rho(z) = A$ be the standard continuous functional calculus on A. Let $\gamma = U\rho U^*$, and let $\tau(1) = I$, $\tau(z) = UAU^*$. As a transformation by unitary similarity does not change the spectrum of A, our mappings γ, and τ both have the same domain $C(\sigma(A))$. We have $\gamma(1) = U\rho(1)U^* = UIU^* = I = \tau(1)$, and $\gamma(z) = U\rho(z)U^* = UAU^* = \tau(z)$, and the result follows for any continuous function by the uniqueness of the continuous functional calculus. □

For the interested reader, we note that the above result can be generalized without much additional effort; however, we are only concerned with the continuous functions for the remainder of the section. In summary, the embedding of the continuous functions over the spectrum of an operator into its C^* algebra is independent of its concrete representation. However, if we were to extend the result to the bounded Borel measurable functions on $\sigma(A)$, $B_b(\sigma(A))$, the underlying Hilbert space becomes relevant. If $W^*(A) \leq \mathscr{A} \leq B(H)$, where H is separable, then the map $B_b(\sigma(A)) \to W^*(A)$ is a $*$-isomorphism, and the associated measure spaces are σ-finite. If H is non-separable, then the map is just a $*$-homomorphism whose measure space need only be finitely decomposable. Recall that our constructed Hilbert space in 3.1.13 is either finite dimensional or non-separable, but never separable and infinite, so this distinction is relevant. As we are only concerned with the continuous functions, we leave the technical details to a discussion in Section 9 of [5].

Lemma 5.3.3. *Let U be a unitary operator and A be a normal operator in a C^* algebra such that $UA = -AU$. Then for any $t \in \mathbb{C}$, $Ue^{tA}U^* = e^{-tA}$.*

Proof. We apply Proposition 5.3.2 to see that $Ue^{tA}U^* = e^{tUAU^*} = e^{-tA}$. □

Now we can use the above lemmas to immediately deduce that the action of unitary similarity of any member of a Cartesian triple acts as inversion of rotation of any other member of the same Cartesian triple. Explicitly, $e^x e^{-x} = 1$ when considered as standard continuous functions over $\mathbb{C}$, and we have an algebra homomorphism for the respective operator valued functions. Furthermore, the action of unitary similarity of any normal element on its own exponent function is trivial.

Theorem 5.3.4. *Let $G = \langle U_\Delta, e^{2\pi\theta i s_i}\rangle$. Then $G \cong D_{2n}$ for some $n \in \mathbb{N}$, if θ is a rational or D_∞ if θ is an irrational.*

Proof. We recognize from the above discussion that U_Δ embeds to the automorphism group generated by $e^{2\pi\theta i s_i}$ as inversion, so we take the semidirect product. With the presentation $\langle U_\Delta, e^{2\pi\theta i s_i} : U_\Delta e^{2\pi\theta i s_i} U_\Delta = e^{-2\pi\theta i s_i}\rangle$, and we see that the isomorphism type of the group follows by the order of $e^{2\pi\theta i s_i}$, which is finite if θ is a rational and infinite otherwise, so the result follows. □

Corollary 5.3.5. *Let $G = \langle U_\Delta, e^{i\sum_{i\in I} 2\pi\theta_i s_i}\rangle$. Then $G \cong D_{2n}$ for some $n \in \mathbb{N}$ if $\theta = 1$ for all but finitely $i \in I$ and $\theta_i \in \mathbb{Q}$ for finitely many i, or $G \cong D_\infty$ otherwise.*

Proof. We need only apply the previous theorem to each s_i and use that continuous functions of commuting operators commute by functional calculus. If there are only finitely many rational θ not equal to one, then we can consider the lcm of their respective orders to obtain a finite n satisfying the claim. □

Corollary 5.3.6. *Let $\mathscr{A} = C^*(D_{2n})$, $3 \leq n \in \mathbb{N}$ in the representation $\pi : G \to B(H)$, where H is of Theorem 3.1.13, and $\mathscr{B}$ be the reduced C^* algebra of D_{2n} with left regular representation $\lambda : G \to B(l^2(G))$. Then $\mathscr{A}$ is a nontrivial quotient of $\mathscr{B}$.*

Proof. We start with $n \geq 4$ and assume that $\theta_i = 1$ for all but exactly one $k \in I$. Without loss of generality, we assume $k = 1$. As D_{2n} is a group extension of discrete groups, D_{2n} is amenable, and we have that the reduced C^* algebra and the universal C^* algebra are isomorphic, so we only need to show that $||\pi(a)|| < ||\lambda(a)||$ for some $a \in \mathscr{A}$. Let us consider the group ring $\mathbb{C}[G]$ and restrict to elements over the cyclic subgroup $\mathbb{Z}_n \cong \langle r \rangle$ in each representation.

Then $\lambda(\sum_{j=0}^{n-1} c_j r^j) \neq 0$ for any choice of $c_j \in \mathbb{C}$ as the r_j are linearly independent. However, $\pi(r) = R \otimes (\otimes_{i \in \{I-1\}} I_2)$, for an appropriate rotation matrix $R \in M_2(\mathbb{C})$, and as a vector space, $C^*(R)$ has dimension at most 3 because $U_\Delta \notin C^*(R)$ as U_Δ does not commute with R and $C^*(R)$ is an abelian algebra. Therefore, $\pi(\sum_{j=0}^{n-1} c_j r^j) = 0$ for some choice of $c_j \in \mathbb{C}$.

Now let $n = 3$, we can directly compute that $\pi(a) = I + R + R^2 = 0$, so $\pi(a) = 0 < ||\lambda(a)||$, again using linear independence.

We have shown the result for a single coordinate of the tensor product, and if we extend to the multi-coordinate tensor case, we have for some element $a \in \mathscr{A}$, $\pi(a_i) < \lambda(a_i)$, so the same must be true for the product of the norms across the indexing set. $\square$

Remark 5.3.7. *We want to highlight that this behavior is quite different when considering the relation of anti-commutativity of the product of s_i. U_Δ and $\Pi_{j \in J}\ s_i$ anti commute exactly when J is odd, and the relationship is non-obvious when J is infinite. This is because -1 factors through the tensor product, and we get a term $(-1)^n$ as a leading coefficient. However, as described above this does not occur when we consider exponentiation of the respective product.*

We conclude having demonstrated that many of the "classical" quantum gates are a direct consequence of our construction of the cubic lattice as an orthomodular lattice of orthogonal projections. Due to our construction we have another natural choice of representation in a more geometric view as a cube in dimension $|I|$ as opposed to the larger $2^{|I|}$, which may have interesting application on its own. In addition, we have shown a number of group theoretic properties of their respective algebras, and that the remaining gates can be naturally constructed as functions of the already constructed gates both in a direct sense via the exponential map, and in a more general sense as an observable constructed of a continuous or Borel function over the spectrum of a Cartesian triple using the spectral mapping theorems. From a physical perspective, we have given a mathematically formal description of the lattice of observables for a system of spin -$\frac{1}{2}$ of arbitrary cardinal. Furthermore, the gates I, De, $\sqrt{s}$, U, or more standardly I, X, $\sqrt{Z}$, H combined with classical circuits generate a set of universal quantum gates. In the following section we look to generalize the previous results to more general spin systems.

6 A generalization to Multi-Cubic Lattices

6.1 Introduction to the Multi-Cubic Lattice

We have demonstrated that a cubic lattice has a realization within a von Neumann algebra over a specific Hilbert space, namely the tensor products of $\mathbb{C}^2$. We now have the question, are there other similar lattice embeddings into a Hilbert space? The case of the cubic lattice was seen to be similar to the spin $\frac{1}{2}$ system of qubits, so it is natural to consider higher order systems. In order to do so, we must first generalize the notion of a cubic lattice. We are fortunate that such a structure already exists, namely the multi-cubic lattice. It is worth noting that to our knowledge, no attempt of any analytic structure has been attempted on the multi-cubic lattice, much less something as strong as an embedding into a von Neumann algebra.

Definition 6.1.1. *Let $\Omega = \{1, 2, \dots, n\}$. For each $i \in \Omega$, let $M_i = \{-n_i, \dots, 0, \dots, n_i\}$ be a finite $\mathbb{Z}$-module of odd size. We define $\mathscr{V} = \Pi_{i=1}^n M_i$. Elements of $\mathscr{V}$ will be denoted by $\vec{v}$ with the i^{th} component $(\vec{v})_i$. Note also that M_i being of odd size is equivalent to the property $2m = 0$ implies $m = 0$. [3]*

We have now described the set $\mathscr{V}$ over which our lattice will be defined. We begin with defining the poset on $\mathscr{V}$.

Definition 6.1.2. *For $A \subseteq \Omega$ define $X_A = \langle e_i | i \in A \rangle$, the submodule generated by the standard basis indexed by the set A. We now have a poset $P = (\{\vec{a} + X_A | \vec{a} \in \mathscr{V}, A \subseteq \Omega\}, \subseteq)$, henceforth P will be known as a multi-cube. [3]*

To be clear, when we say the ordering is determined by $\subseteq$, we mean as the poset of the submodule, $\vec{a} + X_A \leq \vec{b} + X_B$ if and only if $A \subseteq B$ and $\vec{b} - \vec{a} \in X_B$.

We now make P into a join lattice and meet semi lattice. We define the partial meet as follows:

Definition 6.1.3. *Let P be a multi-cube and define $(\vec{a} + X_A) \wedge (\vec{b} + X_B) = \vec{c} + X_{A \cap B}$ if there exists $c \in (\vec{a} + X_A) \cap (\vec{b} + X_B)$ and undefined otherwise. [3]*

Definition 6.1.4. *Now let us define $\delta_{\vec{b} \neq \vec{a}} = \{i \in \Omega | b_i \neq a_i\}$, this is analogous to a support function as $\delta_{\vec{a} \neq \vec{0}} = \{i \in \Omega | a_i \neq 0\}$, precisely the non-zero coordinates of $\vec{a}$. [3]*

Definition 6.1.5. *We can now turn P into a join lattice by defining join as $(\vec{a} + X_A) \vee (\vec{b} + X_B) = \vec{a} + X_{A \cup B \cup \delta_{\vec{a} \neq \vec{b}}}$. For a given indexing set, I, and odd $k \in \mathbb{N}$, we say that M is a I-multi-cubic lattice over $\mathbb{Z}_k$ when M is the lattice constructed from the multi-cube P for $\mathscr{V} = \Pi_{i \in I} \mathbb{Z}_k$. [3]*

Hence M when considered as a lattice is only complete under the join operation but not meet. The operation of Δ was critical in the definition of the cubic lattice, and we have gone on to define the generalization without a notion of Δ. Therefore, this is the next natural discussion. We first present the notion of Δ applied to a multi-cube as it was originally defined. We then redefine it for purposes that will become clear later.

We first need to define a notion of a critical element.

Definition 6.1.6. *Let M be a multi-cubic lattice. A vector $\vec{x} \in \vec{a} + X_A$ is critical if $x_i \neq 0$ implies that $e_i \notin X_A$. We denote the unique critical element of $x \in M$ as $\Gamma(a)$. [3]*

We leave the proofs of uniqueness and existence of the critical element to [3]. We need to show the full decomposition of the an element into its critical element and minimal submodule.

Definition 6.1.7. *Let M be a multi-cubic lattice and $a \in M$. Then define $\sigma(a)$ to be the subset of Ω $i \notin \sigma(a)$ if and only if $a_i \neq 0$.*

Now we are ready to define Δ.

Definition 6.1.8. *Let M be a multi-cubic lattice and $a, b \in M$ and $a \leq b$. Then $\Delta : M \times M \to M$ is defined by $\Delta(b, a) = 2\Gamma(b) - \Gamma(a) + X_{\sigma(a)}$.*

To see that this Δ defined over multi-cubic lattice functions in a similar way to the Δ defined as reflection symmetries over signed sets, we cite the following proposition.

Proposition 6.1.9. *Let M be a multi-cubic lattice, $a, b \in M$. Then the following hold. [3]*

1. $\Delta(a, a) = a$
2. *if* $a \leq b$*, then* $\Delta(b, a) \leq b$
3. *if* $a \leq b$*, then* $\Gamma(\Delta(b, a)) = 2\Gamma(b) - \Gamma(a)$
4. *if* $a \leq b$ *then* $\Delta(b, \Delta(b, a)) = a$
5. *if* $a \leq b \leq c$ *then* $\Delta(c, a) \leq \Delta(c, b)$
6. *if* $a \leq b$*, then* $\Delta(b, a) = b$ *if and only if* $\Gamma(a) = \Gamma(b)$
7. *if* $a < b$*, then* $\Delta(b, a) = b$ *or* $\Delta(b, a) \vee b = \emptyset$

We can see that Δ is order preserving (2,4,5), and it acts as the identity on the diagonal elements (1) of the product. In addition, it retains a notion of a local complement (6,7).

By the definition of Δ, one can directly see that it utilizes the $\mathbb{Z}$ module structure of the MCL. However, we can abstract Δ by only focusing on its action and define it in the context of a multivalued signed set of the MCL. This will allow us to generalize the ideas proven for the cubic lattice.

6.2 Introduction to the Critical Multi-Cubic Lattice and its Automorphisms

In order to have the multi-cubic lattice be a true generalization of the cubic lattice, we will need to make some slight modifications. We create generalized the notion of a signed set from a three valued set $\{-1, X, 1\}$ to an $2n+1$ valued set $\{-n, -n-1, \ldots, -1, X, 1, 2, \ldots n-1, n\}$. With the mild change of definition, the 0 of the original multi-cubic lattice is now referred to as X. The importance of this will become clear as we move from a Cartesian product of $\mathbb{Z}$ modules to a tensor product of $\mathbb{Z}$ modules as the algebraic structure in terms of scalar multiplication, addition, and multiplication will be of course different, but more importantly consistent with an embedding into a von Neumann algebra of similar structure to the previous sections.

With the above discussion, we remove the ambiguity of non-critical elements.

Definition 6.2.1. *We define a critical multi-cubic, M, lattice as a multi-cubic lattice, $\overline{M}$, modulo the equivalence relation where for any $m, n \in \overline{M}$, we have $m \sim n$ if and only if $\Gamma(m) - \Gamma(n) \in X_{\sigma(m)}$ and $X_{\sigma(m)} = X_{\sigma(n)}$. From this point forward, we also allow a critical multi-cubic lattice to be a potentially infinite direct product of $\mathbb{Z}$ algebras. If M is a critical multi-cubic lattice indexed by a set I over a ring $\mathbb{Z}_{2k+1}$, we say that M is an $|I|$ critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$.*

For a critical multi-cubic lattice, M, if we let $n = \Gamma(m) + X_{\sigma(m)}$, we see that with this new notion of equality, all elements are critical. In addition, we have the property that for $m, n \in M$, $m \neq n$, then $\Gamma(m) - \Gamma(n) \neq 0$.

One can view a critical multi-cubic lattice as a mixture between a direct product of finite $\mathbb{Z}$ algebras, and the lattice of submodules defined by the closure of $\{\{\mathbb{Z}e_i\}_{i\in I}, 0\}$ under meets and joins. By reducing to the equivalence class we lose addition of the product ring, preserve to some degree both ring multiplication and $\mathbb{Z}$ scalar multiplication, and gain a structure of a non-zero valued outcomes with an additional indeterminate value seen in the cubic lattice.

Before going forward we show that our preserved operations are well defined. The essential concept is that two elements in a multi-cubic lattice map to the same equivalence class in a critical multi-cubic lattice if their value of the indices which are neither 0 nor X are equal.

Proposition 6.2.2. *Let $\overline{M}$ be a multi-cubic lattice over $\mathbb{Z}_{2k+1}$. Then the operations of scalar multiplication by units in the base ring and ring multiplication by units are independent of equivalence class representative in the corresponding critical multi-cubic lattice M.*

Proof. Let $P : \overline{M} \to M$ be the natural projection map, and $m, n \in \overline{M}$ such that $P(m) = P(n)$. Since $P(m) = P(n)$, $m_i = n_i$ for all $m_i, n_i \neq 0$. For all $m_i = 0$, $P_i(m_i) = X$, and similarly for all $n_j = 0$, $P_j(n_j) = X$. As we assumed that all non-zero entries of m, n are identical, we now have that $P(m)$ and

$P(n)$ have identical nonzero entries and all other values as X.

For nonzero divisor scalar multiplication, $c \in Aut(\mathbb{Z}_{2k+1})$. We have $c \cdot m = c \cdot \Pi_{i \in I - \sigma(m)} m_i + X_{\sigma(m)} = \Pi_{i \in I - \sigma(m)} c m_i + X_{\sigma(m)}$, $m_i \in \mathbb{Z}_{2k+1}$, and $c \cdot n = c \cdot \Pi_{i \in I - \sigma(n)} n_i + X_\sigma(n) = \Pi_{i \in I - \sigma(n)} c n_i + X_\sigma(n)$, $n_i \in \mathbb{Z}_{2k+1}$. Then $cm_i = cn_i$ for all $m_i, n_i \neq 0$, so $cP(m) = P(cm) = P(cn) = cP(n)$ as nonzero divisor scalar multiplication does not change which indices had 0 values.

Similarly we show multiplication of units is well defined. Let $\mu \in \overline{M}$, and further all nonzero induces of m and μ are units in $\mathbb{Z}_{2k+1}$. As $m_i = n_i$ for all nonzero entries, we have $m_i \mu_i + X_{\sigma(m) \cup \sigma(\mu)} = n_i \mu_i + X_{\sigma(n) \cup \sigma(\mu)}$. Then $P(m\mu) = P(n\mu)$ as they agree on all nonzero entries. □

As in the case of the cubic lattice, the atoms and coatoms will play a significant role in our results.

Example 6.2.3. *Let M be a 2 critical multi-cubic lattice over $\mathbb{Z}_5$. The atoms of M are of the form $\{a, b\}$ where $a, b \in \{-2, -1, 1, 2\}$. This generalizes the two dimensional cubic lattice case, where atoms are of the form (A^-, A^+), $|A^- \cup A^+| = |S| = 2$. We assign each index in S to a either the set A^- or the set A^+. We can equivalently assign each index in S to either value -1 or the value 1, and see that this is the reduction to the 2 critical multi-cubic lattice over $\mathbb{Z}_3$. In general, an $|I|$ critical multi-cubic lattice has $(2k)^{|I|}$ atoms of the form $\Pi_{i \in I} a_i$, $a_i \in \mathbb{Z}_{2k+1} - \{0\}$.*

Example 6.2.4. *Let M be a 2 critical multi-cubic lattice over $\mathbb{Z}_5$. The coatoms of M are of the form $\{a, b\}$ where exactly one $a, b \in \{-2, -1, 1, 2\}$ and all others are assigned to X. This generalizes the two dimensional cubic lattice case, where atoms are of the form (A^-, A^+), $|A^- \cup A^+| = |1|$. We assign exactly one index in S to a either the set A^- or the set A^+. We can equivalently assign exactly one index in S to either value -1 or the value 1, and see that this is the reduction to the 2 critical multi-cubic lattice over $\mathbb{Z}_3$. In general, an $|I|$ critical multi-cubic lattice has $2k|I|$ coatoms of the form $\Pi_{i \in I} V$, where $V = X$ for all but exactly one $i \in I$ and $V \in \mathbb{Z}_{2k+1} - \{0\}$ otherwise. One may find it helpful to think of X as the submodule $\mathbb{Z}_{2k+1}$ on each index.*

The fact that the atoms and coatoms in the 2 critical multi-cubic lattice of $\mathbb{Z}_3$ are the same as the 2 dimensional cubic lattice is not a coincidence. We show that the case of the cubic lattice is just a specific case of the critical multi-cubic lattice with an automorphism.

Theorem 6.2.5. *Let M be a $|S|$-critical multi-cubic lattice over $\mathbb{Z}_3$. Then M is a cubic lattice.*

Proof. We need only show that $M \cong L(S)$ as lattices. For each $m \in M$, we have that $m = \Pi_{i \in I - \sigma(m)} a_i + X_{\sigma(m)}$, $a_i \in \mathbb{Z}_3^* = \{-1, 1\}$, is mapped to an element $a = (A^+, A^-) \in L(S)$ defined by $i \in A^+$ if $a_i = 1$ and $i \in A^-$ if $a_i = -1$, so that $A^+ \cap A^- = \emptyset$. We denote this mapping by $f : M \to L(S)$, and one can see that f is a bijection.

It remains to show that we have an order homomorphism. If $m \leq n$, $f(m) = (A^+, A^-)$, and $f(n) = (B^+, B^-)$, then $m - n \in X_{\sigma(n)}$, which implies that $B^+ \subseteq A^+$ and $B^- \subseteq A^-$. Therefore, $f(n) = (B^+, B^-) \subseteq (A^+, A^-) = f(m)$. As order homomorphism induce lattice homomorphism we have that f is also a lattice homomorphism. □

We are now in a position to readdress the definition of Δ in the context of a critical multi-cubic lattice. We first define Δ purely in terms of its action on the multi-cubic lattice.

Definition 6.2.6. *Let M be a multi-cubic lattice and $a, b \in M$, $a \leq b$. Then we define $\Delta : M \times M \to M$ by*

$$\Delta(b, a)_i = \begin{cases} -a_i & a_i \in X_{\sigma(b)-\sigma(a)} \\ a_i & \textit{otherwise} \end{cases}$$

for all $i \in I$.

We have that this new definition of Δ induces the same action on the multi-cubic lattice M, so the results of Proposition 6.1.9 hold. As a quick justification, we note that both actions flip the sign of a_i if and only if $\Gamma(a)_i$ is nonzero and $\Gamma(a)_i \in X_{\sigma(b)}$. Another phrasing is that if a and b are critical with $a \leq b$, then the coordinate, i, will change sign exactly when $i \in \{B - A\}$. However, this action defined on product $\mathbb{Z}$ modules will be defined as just another module homomorphism when $b = 1$.

Definition 6.2.7. *Let M be a critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$. For all $n \in M$, we define $\Delta_M(n, \cdot)$ as multiplication by $2k$ on the respective ideal (n).*

Lemma 6.2.8. *Let M be a critical multi-cubic lattice, $n \in M$. Then $\Delta_M(n, \cdot)$ is always exists, is well defined, and has properties of Proposition 6.1.9.*

Proof. For any odd $k \geq 3$, Δ_M is just scalar multiplication by $2k$ which exists by construction and is well defined by Proposition 6.2.2. Let $m \in M$, such that $m \leq n$. As multiplication by $2k \bmod 2k+1$ is equivalent to multiplication by $-1 \bmod 2k$, and the action of $\Delta_M(n, m)$ on the multi-cubic lattice only applies to the nonzero indices of m that are not equal to n, we see that our notion of $\Delta_M(n, \cdot)$ on the corresponding critical multi-cubic lattice must also have the same properties as the properties of Proposition 6.1.9 hold when restricted the critical elements of a multi-cubic lattice. □

The fact that such a map factors through the projection should not be surprising as the original definition operated on on the equivalence class representatives.

Theorem 6.2.9. *Let M be a critical multi-cubic lattice such that $M \cong L(S)$ for an indexing set S for the lattice isomorphism f of Theorem 6.2.5. The action of $\Delta_M(n, m)$ on M is equal to the action of $\Delta(f(n), f(m))$ on $L(S)$.*

Proof. As the cubic lattice atomistic, we need only show that the claim holds for the atoms of the respective ideal. By the proof of Lemma 6.2.8, we have a well defined $\Delta_M(n, m)$ that is equivalent to multiplication by -1 on the ideal (n).

When considering the action of the pushforward on $L(S)$, $f \circ \Delta_M(n, m)$, we swap all nonzero indices of $m \in \sigma(n)$ that are not equal to 1 to -1 or vice versa. If we let $(B^-, B^+) = f(n)$, and $(A^-, A^+) = f(m)$, then the pushforward is equal to $(B^- \cup \{A^+ - B^+\}, B^+ \cup \{A^- - B^-\}) = \Delta(f(m), f(n))$. $\square$

We have shown many similarities between cubic lattices and multi-cubic lattices, we now highlight that they are in fact a weaker object as general critical multi-cubic lattices do not meet the axiomatic definition of cubic lattices.

Proposition 6.2.10. *Let M be a critical multi-cubic lattices over $\mathbb{Z}_{2k+1}$, $k \geq 2$. Then M does not meet axiom 2 of Definition 1.3.11.*

Proof. Let ϕ be any order automorphism of M. Then for any coatoms $a, b \in M$ such that $a \wedge b = 0$ implies $a, b \in \{a_j e_i : a_j \in \mathbb{Z}_{2k+1}\}$ for some fixed $i \in I$. Since ϕ is an order automorphism, $\phi(b)$ is a coatom as well, and $a \vee \phi(b) < 1$ implies $\phi(b) = a$ because the join of any two distinct coatoms is equal to 1. Then for a fixed a, $\phi(b) = a$ for all $b \in \mathbb{Z}_{2k+1} - \{a, 0\}$, so ϕ is not injective. $\square$

In summation, the failure of a general multi-cubic lattice comes down to the fact that $|\mathbb{Z}_{2k+1} - \{a, 0\}| > 1$ for $k \geq 2$.

Proposition 6.2.11. *Let M be a critical multi-cubic lattice. Then M is a coatomistic.*

Proof. As M is a critical multi-cubic lattice, for all $m \in M$, $m_i \in \mathbb{Z}_{2k+1} - \{0\}$ or $m_i = \mathbb{Z}_{2k+1}$. Therefore, for any $m \in M$, we have that $m = \Pi_{i \in I - \sigma(m)} a_i + X_{\sigma(m)}$ where $a_i \in \mathbb{Z}_{2k+1} - \{0\}$, so $m = \wedge_{i \in I} c_i(a)$ where $c_i(a)$ denotes the coatom in the ith index equal to a. $\square$

Therefore, any homomorphism of a critical multi-cubic lattice, which is also an order homomorphism of a critical multi-cubic lattice must map coatoms to coatoms in addition to preserving the scalar module multiplicative. With this new definition of a critical multi-cubic lattice, we will show that in addition to Δ, there exists a whole family of central automorphisms. We formalize the above in the following definition.

Definition 6.2.12. *Let $k \in \mathbb{Z}$, I_M, I_N be indexing sets, and M be a critical $|I|$ multi-cubic lattice over $\mathbb{Z}_{2k+1}$ with coatoms C_M, and N be a I_N-multi-cube over $\mathbb{Z}_{2k+1}$ with coatoms C_N. A map $\phi : M \to N$ is a critical multi-cubic homomorphism if ϕ is a multiplicative group homomorphism over the units of M considered as a ring, and a scalar homomorphism over the units of the base ring such that $\phi(c) \subseteq C_N$ or $\phi(c) = 0$ for all $c \in C_M$. In particular if $N = M$ and ϕ is a $\mathbb{Z}$ module automorphism, then $\phi \in Per(C_M)$.*

Both conditions are necessary, as coatomistic lattices, any lattice homomorphism must map coatoms to coatoms or 0, and any homomorphism preserving the modified $\mathbb{Z}$ algebra structure must be a $\mathbb{Z}$ scalar and multiplication homomorphism. Sufficiency follows as we are guaranteed to keep the $\mathbb{Z}$ linear structure and as a function of coatoms between coatomistic lattices, we have an order homomorphism, which is necessarily a lattice homomorphism as well. We have shown that this notion is a generalization of Δ of the cubic lattice.

Definition 6.2.13. *Let $k \in \mathbb{N}$ and M be a critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$. We define $Aut(M)$ as the automorphism group of M.*

We show by example that the homomorphism conditions are not equivalent.

Example 6.2.14. *Let M be a 1-critical multi-cubic lattice over $\mathbb{Z}_3$ i.e. a cubic lattice. Now we demonstrate that there are $\mathbb{Z}$ module automorphisms that are not order automorphisms.*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

$A[0,1]^t = [1,1]^t \notin C_M$, so A is a $\mathbb{Z}$ linear transformation that is not even a permutation of coatoms and therefore not an order homomorphism.

We will show that there are order automorphisms that are not $\mathbb{Z}$ module automorphisms. Let N be a 2-critical multi-cubic lattice over $\mathbb{Z}_3$. The atoms of N are the standard basis. Let P_{ij} denote the projection onto the sum of standard basis vectors e_i, e_j, so that $C_N = \{P_{12}, P_{13}, P_{13}, P_{34}\}$, and with this ordering let $\sigma = (1234) \in S_4 \cong Per(C_N)$. We claim that σ is not representable as an invertible linear transformation. Recall that permutations act on the lattice of projections by inner automorphism. Then $\sigma(P_{12})\sigma^{-1} = P_{13}$, and $\sigma(P_{34})\sigma^{-1} = P_{12}$, but $\sigma(P_{12} + P_{34})\sigma^{-1} = \sigma I \sigma^{-1} = I \neq P_{13} + P_{12} = \sigma P_{12}\sigma^{-1} + \sigma P_{34}\sigma^{-1}$.

With this view in mind we have a new and equivalent notion of the poset of a critical multi-cubic lattice.

Definition 6.2.15. *Let Π_i be the coordinate-wise projection onto the ith index and Π_J be the projection onto the $J \subseteq I$ coordinates. Implicitly, if $m = \Gamma(m) + X_{\sigma(m)}$, we define Π_J as Π_J^m, $J^m \subseteq I - \sigma(m)$ where $J^m = J - \sigma(m)$.*

Proposition 6.2.16. *Let M be a critical multi-cubic lattice and m, $n \in M$ such that $m \leq n$. Then there exists Π_J, $J \subseteq I - \sigma(n)$ such that $\Pi_J(m) = n$.*

Proof. Recall that $m \leq n$, then $m - n \in X_{\sigma(n)}$, so let $J = \sigma(n) - \sigma(m)$. □

Because we fix a generating set of the $\mathbb{Z}$ algebra, we can view critical multi-cubic homomorphisms as an automorphism composed with a projection as defined above. With the importance of the critical multi-cubic automorphisms now highlighted, we proceed to classify them as a generalization of [7].

Definition 6.2.17. *Let $Per_{Aut(\mathbb{Z}_{2k+1})}(C_M)$ be the permutations of coatoms of M that commute with the action of $Aut(\mathbb{Z}_{2k+1})$ defined by $c \mapsto ac$ for $c \in C_M$ and $a \in Aut(\mathbb{Z}_{2k+1})$.*

Lemma 6.2.18. *Let M be an $|I|$-critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$. Then $Aut(M) \cong Per_{Aut(\mathbb{Z}_{2k+1})}(C_M)$.*

Proof. By definition, we have an injective group homomorphism $i : Aut(M) \to Per(C_M)$ where i is inclusion map. We want to show the range of i is strictly contained in $Per_{Aut(\mathbb{Z}_{2k+1})}(C_M)$. Any $\phi \in Aut(M)$ is a $\mathbb{Z}$ module homomorphism of $\mathbb{Z}^I_{2k+1}$, so we have that for any $a \in Aut(\mathbb{Z}_{2k+1})$, a induces an automorphism on $\mathbb{Z}_{2k+1}$ by multiplication. Now by the property of module homomorphisms over commutative rings for any $m \in M$, $\phi(a) \circ \phi(m) = a\phi(m) = \phi(am) = \phi(ma) = \phi(m) \circ \phi(a)$. Thus, $Aut(M) \leq Per_{Aut(\mathbb{Z}_{2k+1})}(C_M)$.

For the reverse direction, we show that any $\psi \in Per_{Aut(\mathbb{Z}_{2k+1})}(C_M)$ defines a critical multi-cubic automorphism. Firstly, it is a permutation of coatoms of M by definition, and thus bijective. Secondly, for the linearity condition, let ψ_a be multiplication by $a \in Aut(\mathbb{Z}_{2k+1})$. Then $a\psi(m) = \psi_a \circ \psi(m) = \psi \circ \psi_a(m) = \psi(am)$. □

Theorem 6.2.19. *Let M be an $|I|$-critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$. Then $Per_{Aut(\mathbb{Z}_{2k+1})}(C_M) \cong C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1})) \wr S_I$. Let $Aut(\mathbb{Z}_{2k+1})$ be generated by $\{\sigma_i\}_{i=1}^k$, then $C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1})) = \cap_{i=1}^k C_{S_{2k}}(\sigma_i)$, and $C_{S_{2k}}(\sigma_i) \cong \Pi_{j=1}^{l_i}(\mathbb{Z}_{j_i} \wr S_{N_{j_i}})$.*

Proof. Any $\phi \in Per_{Aut(\mathbb{Z}_{2k+1})}(C_M)$ is determined by the image of its coatoms. Note that for any two distinct coatoms their meet is empty if and only if they are of the form, ae_i and be_i for distinct $a, b \in \mathbb{Z}_{2k+1} - \{0\}$ and some $i \in S$. By Lemma 6.2.18, ϕ defines a critical multi-cubic automorphism, so we have that for any $i \in S$ and distinct a, $b \in \mathbb{Z}_{2k+1} - \{0\}$, $\emptyset = \phi(ae_i \wedge be_i) = \phi(ae_i) \wedge \phi(be_i)$. Hence, for all $\alpha \in \mathbb{Z}_{2k+1} - \{0\}$, $\phi(\alpha e_i) = \beta_\alpha e_j$ for some $\beta_\alpha \in \mathbb{Z}_{2k+1} - \{0\}$ and a fixed $j \in S$.

We now need only consider ϕ for each individual index. $Aut(\mathbb{Z}_{2k+1})$ acts on the coatoms of fixed index, $C_i = \{a \in \mathbb{Z}_{2k+1} - \{0\}\}$ by left multiplication, so by relabeling, we consider the action on $\{a : 1 \leq a \leq 2k\}$. Let $Per_{Aut(\mathbb{Z}_{2k+1})}(C_i)$ denote the permutations of C_i commuting with the action of $Aut(\mathbb{Z}_{2k+1})$. We observe that $Per_{Aut(\mathbb{Z}_{2k+1})}(C_i)$ is isomorphic to the centralizer of $Aut(\mathbb{Z}_{2k+1})$ in S_{2k}. Thus, $\phi \in C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1})) \wr S_I$.

For a given $\sigma \in S_m$, with a standard cycle decomposition consisting of N_j cycles of length j, we use that $C_{S_m}(\sigma) \cong \Pi_{j=1}^{l}(\mathbb{Z}_j \wr S_{N_j})$. As the centralizer of subgroup is the intersection of the centralizer of its generators, and we conclude our result. □

It is known that the hyperoctehedral group can be represented as the signed permutation group. We have used this terminology to inspire our generalization.

Definition 6.2.20. *Let $M_I(\cdot)$ be the $(\cdot)$ valued matrices indexed by I.*

Definition 6.2.21. *Let the $Aut(\mathbb{Z}_{2k+1})$-value permutation group of degree I be a generalization of the elements of the permutation group in $M_I(\mathbb{Z}_{2k+1})$ where each nonzero element can take any value of $Aut(\mathbb{Z}_{2k+1})$.*

Proposition 6.2.22. *Let M be an $|I|$ critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$ for $k \in \mathbb{N}$. Then there exists a group representation $\rho : Aut(\mathbb{Z}_{2k+1}) \wr S_I \to M_I(\mathbb{Z}_{2k+1})$ as the $Aut(\mathbb{Z}_{2k+1})$-value permutation group of degree I. Furthermore, when considering as the base ring $Aut(\mathbb{Z}_{2k+1}) \subseteq Z(\rho(Aut(\mathbb{Z}_{2k+1}) \wr S_I))$.*

Proof. We have that the standard basis vectors $\{e_i\}_{i\in I}$ form a generating set of $\mathbb{Z}_{2k+1}^I$ as a $\mathbb{Z}$ module. Now $Aut(\mathbb{Z}_{2k+1}) \wr S_I$ can be represented as the $Aut(\mathbb{Z}_{2k+1})$-value permutation group by considering the action $e_i \mapsto a_i e_\sigma(i)$ for a given $(\times_{i\in I} a_i, \sigma) \in Aut(\mathbb{Z}_{2k+1}) \wr S_I$, which we define as ρ. As a commutative base ring, $\mathbb{Z}_{2k+1}^* \subseteq Z(M_I(\mathbb{Z}_{2k+1}))$, which are in the $Aut(\mathbb{Z}_{2k+1})$-value permutation group by definition and in $Z(Aut(\mathbb{Z}_{2k+1}) \wr S_I)$ because ρ is a multiplicative group representation. □

Lemma 6.2.23. *Let M be an $|I|$ critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$, and ρ be defined by Proposition 6.2.22. Then $\rho(Aut(\mathbb{Z}_{2k+1}) \wr S_I) \leq Aut(M)$.*

Proof. We claim that $\rho(Aut(\mathbb{Z}_{2k+1}) \wr S_I) \leq Aut(M)$. We have a bijection, $f : \{ae_i : a \in \mathbb{Z}_{2k+1} - \{0\}, i \in I\} \to C$, where e_i denotes the standard basis vector, and C is the coatoms of the form specified in Example 6.2.4, and f is defined by $ae_i \mapsto \Pi_{j\in I} V_j$, $V_i = a$, and $V_j = X$ for all $j \neq i$. Then we have that $\rho(Aut(\mathbb{Z}_{2k+1}) \wr S_I)$ defines a permutation of the coatoms by its action on $\{ae_i : a \in \mathbb{Z}_{2k+1} - \{0\}, i \in I\}$. We have already shown in Proposition 6.2.22 that $\rho(Aut(\mathbb{Z}_{2k+1}) \wr S_I)$ commutes with the action of $\mathbb{Z}$ scalar multiplication of units on $\{ae_i : a \in \mathbb{Z}_{2k+1} - \{0\}, i \in I\}$, so we conclude our result. □

Theorem 6.2.24. *Let M be an $|I|$ critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$. Then $Aut(\mathbb{Z}_{2k+1}) \wr S_I \leq Aut(M) \cong Per_{Aut(\mathbb{Z}_{2k+1})}(C_M)$.*

Proof. As a result of Lemma 6.2.23 and Lemma 6.2.18, we have $Aut(\mathbb{Z}_{2k+1}) \wr S_I$ is isomorphic to a subgroup of $Per_{Aut(\mathbb{Z}_{2k+1})}(C_M)$. □

We have now rederived the group theoretic results of [7] and shown it in much more generality. We highlight the cyclic case below as the operator algebraic structure will be most similar to the results of Section 5, and identical if $k = 1$.

Corollary 6.2.25. *If $2k + 1$ is prime, $\mathbb{Z}_{2k} \wr S_I \cong Aut(M) \cong Per_{Aut(\mathbb{Z}_{2k+1})}(C_M)$.*

Proof. By Theorem 6.2.19, $Per_{Aut(\mathbb{Z}_{2k+1})}(C_M) \cong C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1})) \wr S_I$. If $2k+1$ is an odd prime, then $Aut(\mathbb{Z}_{2k+1}) \cong \mathbb{Z}_{2k}$, and $C_{S_{2k}}(\mathbb{Z}_{2k}) \cong (e \times e) \times (\mathbb{Z}_{2k} \times e) \cong \mathbb{Z}_{2k}$, where e denotes the group consisting of only the identity. □

6.3 Logic of the Critical Multi-Cubic Lattice

In a manner similar to the previous section, we now consider the analytic structure of a critical multi-cubic lattice. Firstly, we proceed to generalize Theorem 3.1.13 to our more general object.

Lemma 6.3.1. *A critical multi-cubic lattice is atomistic.*

Proof. Let M be an $|I|$ critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$, and $\Pi_{i \in I-\sigma(m)} a_i + X_{\sigma(m)} = m \in M$, $a_i \in \mathbb{Z}_{2k+1} - \{0\}$. Then $m = \vee_{j \in \sigma(m)} m_j$, where $m_{j_i} = m_i$ for all $i \in I - \sigma(m)$, $m_{j_j} = -1$, and $m_{j_i} = 1$ for all $i \in \sigma(m) - \{j\}$. We have that each m_j is an atom of M, so the result follows. □

Theorem 6.3.2. *Let H be Hilbert space constructed using our previous notion of the infinite tensor product for vector spaces of dimension $2k$, $k \in \mathbb{N}$ over an index set I. For the Hilbert lattice HL, there exists a critical multi-cubic lattice M such that $M \subseteq HL$, and the atoms of M are projections onto subspaces H forming an orthonormal basis of H.*

Proof. The proof is similar to Theorem 3.1.13.

We see that each simple tensor $\otimes_{i \in I} a_i$, $a_i \in \mathbb{Z}_{2k+1} - \{0\}$, is a C-sequence. We again have a linear functional of the form Definition 3.1.8, and we represent it by a respective projection operator. We use the notation $\Pi_{i \in I} a_i$ to define an element m forming the atoms of M an $|I|$ critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$.

Let $\vee_M$ denote join in M and $i : M \to HL$. We need only show that for all $a, b \in M$, $i(a \vee_M b) \in HL$. Let $a = \Pi_{i \in I} a_i$ and $b = \Pi_{i \in I} b_i$, then $a \vee_M b = \Pi_{i \in J} a_i + X_{I-J}$, where $J = \{i \in I : a_i = b_i\}$, and $i(a \vee_M b)$ be the projection $V = \otimes_{i \in I} V_i$, $V_i = a_i$ for $i \in J$ and $V_i = \mathbb{C}^{2k}$, $i \in J$. By atomisticity of Lemma 6.3.1, i is an order homomorphism and therefore a lattice homomorphism.

The set of simple tensors are all rank 1 and therefore atoms in $B(H)^+$. The proof that this set of projections form an orthonormal system with norm dense span is exactly the same as Theorem 3.1.13. □

To relate to our previous notation, we can see that for all $a \in M$, a can be considered as an element of $p \in HL$ by the coordinatization $p_i = 1_{\mathbb{C}^{2k}}$ for $i \in \sigma(a)$ and $p_i = p_{a_i}$ otherwise.

Remark 6.3.3. *We want to highlight that the lattice homomorphism defined in Theorem 6.3.2 is defined on the lattice M considered as a subset of HL and not a a lattice homomorphism from M to the lattice HL.*

We have seen that we can embed a critical multi-cubic lattice to a Hilbert lattice in much the same way the cubic lattice to the Hilbert lattice. From an analytic perspective these objects have been shown to share many of the same qualities. However this is where the similarities stop for the most part. By the arguments of the previous section, we have removed the cardinality constraints of [3], and defined and abstracted multi-cubic auotomorphisms.

One of interesting characteristics of the cubic lattice is that we had an order preserving complementation, Δ. In fact, the action of Δ on the coatoms of cubic lattice embedded in the Hilbert lattice exactly matched the action of negation on the same coatoms. However, we lose this in the general case.

Proposition 6.3.4. *The action of* $^\perp$ *on a critical multi-cubic lattice over* $\mathbb{Z}_{2k+1}$*,* $k > 2$*,* M*, embedded in Hilbert lattice,* HL*, in the manner of Theorem 6.3.2 does not define a permutation on* C*, the coatoms of* C *of* M*.*

Proof. We can show that $^\perp$ does not map coatoms to coatoms. Fix a a coatom $ae_i = c \in C$, then $c^\perp = \vee_{b\in\mathbb{Z}_{2k+1}-\{a,0\}} be_i$, which is not even in M. □

6.4 Operator Algebras of a Critical Multi-Cubic Lattice

We can see that the results in section 5 can can be generalized in a straight forward manner.

Definition 6.4.1. *For each coatom of* CL*, we consider the projection operator onto the space denoted* p_{c_i}*.*

Proposition 6.4.2. *Let* M *be* $|I|$*-multi-cubic lattice over* $\mathbb{Z}_{2k+1}$*. Then there exists a unitary representation* $\rho : Aut(M) \to B(H)$ *where* H *is constructed in Theorem* 6.3.2.

Proof. For any $\phi \in Aut(M)$, we associate $\phi \in Per_{Aut(\mathbb{Z}_{2k+1})}C$. We use coatomicity of the critical multi-cubic lattice to define an action on the atoms which form an orthonormal basis of H and extend the map linearly. As $\rho(\phi)$ sends an orthonormal basis to an orthonormal basis, we have that it must be unitary. □

Proposition 6.4.3. *Let* $M \subseteq HL$ *as in Theorem 6.3.2 and* $U \in W^*(Aut(\mathbb{Z}_{2k+1})'$ *be unitary. There exists a unitary* $V \in \rho(Aut(M))$ *such that* $Ad_U = Ad_V : M \to M$ *and* $U = VS$ *for* $S \in W^*(\{p_c\}_{c\in C})' \cap W^*(\rho(Aut(\mathbb{Z}_{2k+1})))'$*.*

Proof. If $U \in W^*(\rho(Aut(\mathbb{Z}_{2k+1}))'$, then $Ad_U \in \rho(Aut(M))$ by Lemma 6.2.18. Now let $V = \rho(Ad_U) \subseteq W^*(U_\Delta)'$. Then $Ad_{V^*} = Ad_V^{-1}$, so $Ad_{UV^*}|_M = Ad_I|_M$. As the action of inner automorphism stabilizes M, $UV^* \in W^*(\{p_c\}_{c\in C})'$, so there exists $S \in W^*(\{p_c\}_{c\in C})'$ such that $U = VS$. Furthermore, $S = UV^*$, and we conclude that $S \in W^*(\rho(Aut(\mathbb{Z}_{2k+1})))'$ as well. □

Proposition 6.4.4. $W^*(Aut(\mathbb{Z}_{2k+1}))' = W^*(\rho(Per_{Aut(\mathbb{Z}_{2k+1})}C), W^*(\{p_c\}_{c\in C}) \cap W^*(Aut(\mathbb{Z}_{2k+1}))')$.

Proof. This proof is a direct application Proposition 6.4.3. $\square$

Corollary 6.4.5. *Let M be a critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$. Then $Z(\rho(Aut(M))) = \rho(Aut(\mathbb{Z}_{2k+1}))$.*

Proof. We only show one containment in Proposition 6.2.22. The reverse containment follows by Proposition 6.4.4 and that $\rho(Aut(M)) \cap W^*(\{p_c\}_{c\in C}) = I$. $\square$

We now reconstruct the relevant matrix unit structure of $B(H)$ in terms of multi-cubic lattice automorphisms.

Lemma 6.4.6. *Let M be an $|I|$-critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$, C_α be the coatoms of M for a fixed index $\alpha \in I$, and H be constructed in the manner of Theorem 6.3.2, then $B(H) \cong M_{2k}(B)$, where B is the mutual commutant of a set of matrix units.*

Proof. Let $(\cdot)$ denote the respective element in the permutation group contained in $M_{2k}(\mathbb{Z}_2)$ represented in the standard basis. Now we claim the following matrix units form matrix units of $B(H)$. For $i \in C_\alpha$:

$$e_{ii} = p_c$$
$$e_{ij} = e_{ii}(ij)$$
$$e_{ji} = (ij)e_{ii}$$

We can directly compute that $\sum_{i\in C_\alpha} e_{ii} = I$, $e_{ij} = e_{ji}^*$ as permutation group is subgroup of the unitary group, and $e_{ij}e_{kl} = \delta_{jk}e_{il}$. Therefore, $B(H) \cong M_{2k}(B)$, where B is the commutant of the matrix units, see Lemma 4.27 of [1]. $\square$

Lemma 6.4.7. *With the conditions of Lemma 6.4.6, $W^*(\rho(C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1}))), \{p_{C_i}\}_{l=1}^{2k}) \subseteq W^*(\{e_{ij}\}_{i,j=1}^{2k})$.*

Proof. We claim $W^*(\rho(C_{S_{2k}}Aut(\mathbb{Z}_{2k+1})), \{p_{C_i}\}_{l=1}^{2k}) \subseteq W^*(\{e_{ij}\}_{i,j=1}^{2k})$.

The projections onto the appropriate coatoms are the diagonal elements by construction. In place of Δ enforcing the conditions to be a cubic lattice, we have that the module conditions of the critical multi-cubic lattice must be preserved.

We identify $\phi \in C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1})))$, where ρ is defined in Proposition 6.4.2, with $\sigma_\phi \in S_{2k} \subseteq M_{2k}(\mathbb{Z}_2)$ represented by the permutation group. Thus, any element of $\rho(C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1})))$ is a linear combination of matrix units. $\square$

Note that our previous results of section 5 only fully generalize to a particular subset of critical multi-cubic lattices.

Lemma 6.4.8. *With the conditions of Lemma 6.4.6, $B = W^*(\rho(C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1}))), \{p_{C_i}\}_{l=1}^{2k})'$ if and only if $2k+1$ is prime.*

Proof. It is sufficient to consider when $W^*(\{e_{ij}\}_{i,j=1}^{2k}) \subseteq W^*(\rho(Aut(\mathbb{Z}_{2k+1})), \{p_{C_i}\}_{l=1}^{2k})$, which is equivalent to the case when the action of $\rho(C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1})))$ on $\{e_{ii}\}_{i=1}^{2k}$ generates $\{e_{ij}\}_{i,j=1}^{2k}$. This occurs exactly when $C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1}))$ acts transitively on $\mathbb{Z}_{2k+1} - \{0\}$ with our relabeling of Theorem 6.2.19.

Let $e \neq \sigma \in Aut(\mathbb{Z}_{2k+1})$. Consider $C_{S_{2k}}(\sigma) \cong \Pi_{j=1}^{l}(\mathbb{Z}_j \wr S_{N_j})$. The action of $\Pi_{j=1}^{l}(\mathbb{Z}_j \wr S_{N_j})$ on σ must preserve the cycle type of σ. Let $1 \leq i < j \leq 2k$, then the action of $\Pi_{j=1}^{l}(\mathbb{Z}_j \wr S_{N_j})$ can map i to j only if i and j are both in cycles of the same length. Thus, if $\Pi_{j=1}^{l}\mathbb{Z}_j \wr S_{N_j}$ acts transitively, σ must have a cycle decomposition of m cycles of length $\frac{2k}{m}$ where m divides $2k$. On the other hand, these cycles are the orbits of the action $\langle\sigma\rangle$ on $\mathbb{Z}_{2k+1}$ disregarding the orbit of $e \in \mathbb{Z}_{2k+1}$. For $a \in \mathbb{Z}_{2k+1}$, if $|a| = 2k+1$, then $|orb(a)| = \phi(a)$, where ϕ denotes the Euler totient function. If $|a| < 2k+1$, then $|orb(a)| = \phi(2k+1)/\phi(d)$ for some $1 < d \,|\, (2k+1)$. As $2k+1$ is odd, $d > 2$ and $\phi(d) > 1$. Therefore, $\phi(2k+1)/\phi(d)) \neq \phi(2k+1)$, so $2k+1$ must be prime.

Conversely, if $2k+1$ is prime, recall that $C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1}) = C_{S_{2k}}(\sigma) \cong \mathbb{Z}_{2k}$ whose action as a $2k$ cycle in S_{2k} is transitive on $\{a : 1 \leq a \leq 2k\}$. □

We generalize the Hadamard matrix for a critical multi-cubic lattice. Let M be an $|1|$ critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$ where $2k+1$ is prime. Let $\mathbb{Z}_{2k} \cong C_{2k}(\rho(Aut(\mathbb{Z}_{2k+1})))$ be generated by the unitary X_{2k}. As X_{2k} is unitary, there exists a unitary U_{2k} such that $U_{2k}X_{2k}U_{2k}^* = D$, where D is the diagonal matrix of the roots of $2k$ roots of unity in counterclockwise order. In the case where $2k+1 = 3$, $X_{2k} = X$, and $U_{2k} = H$.

Definition 6.4.9. *Let M be an $|I|$ critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$ where $2k+1$ is prime. Define $U_H = \otimes_{i\in I} U_{2k}$.*

Theorem 6.4.10. *$B(H) = W^*(\{U_H p_c U_H^*\}_{c\in C}, \{p_c\}_{c\in C})$ if and only if $2k+1$ is prime.*

Proof. We need only show that the mutual commutant of $\{U_H p_c U_H^*\}_{c\in C}, \{p_c\}_{c\in C})\}$ is $\mathbb{C}I$. By Lemma 6.4.8, we know the commutant of $W^*(\rho(C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1}))), \{p_{c_{ij}}\}_{j=1}^{2k})$ must fix the coatoms for each index $i \in I$ if and only if $2k+1$ is prime. As they must also commute with $\rho(Aut(\mathbb{Z}_{2k+1}))$, any unitary in $W^*(\{p_c\}_{c\in C}, \rho(Aut(\mathbb{Z}_{2k+1})))'$ must be in $Per_{\rho(Aut(\mathbb{Z}_{2k+1}))}C$ and fix all coatoms, so must be in $\{p_c\}_{c\in C})$ and by symmetry be in $\{U_H p_c U_H^*\}_{c\in C}$.

Consider a coatom in $p_i \in \{Up_cU^*\}_{c\in C}$ and $q_i \in \{p_c\}_{c\in C}$ for some fixed index $i \in I$. Then $p_i \wedge q_i = \lim_{n\to\infty}(p_i q_i p_i)^n = \lim_{n\to\infty}(\frac{1}{2k}p_i)^n = 0$. By construction, any atom $a \in W^*(\{Up_cU^*\}_{c\in C})$, a is bounded by a co-atom for all $i \in I$, so we assume without loss of generality that $a \leq p_i$, and by symmetry we assume $b \leq q_i$. Then $a \wedge b \leq p \wedge q = 0$. Therefore the atomistic Boolean lattice of projections associated with $\{Us_iU^*\}_{i\in I}$ and $\{s_i\}_{i\in I}$ have distinct sets of atoms. By atomisticity, $W^*(\{p_c\}_{c\in C})$ and $W^*(\{U_H p_c U_H^*\}_{c\in C})$ are abelian von Nuemann algebras whose only common projections are 0 and I, so their intersection is $\mathbb{C}I$ by Proposition 5.1.14. $\square$

Corollary 6.4.11. *Let M be an $|I|$-critical multi-cubic lattice over $\mathbb{Z}_{2k+1}$. The action of $Aut(M)$ acts transitively on the atoms if and only if $2k+1$ is prime.*

Proof. As $Aut(M) \cong C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1})) \wr S_I$, we need only show that $C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1}))$ acts transitively on the coatoms on fixed index $i \in I$. As the action is just standard modular multiplication, we apply $C_{S_{2k}}(Aut(\mathbb{Z}_{2k+1}))$ acts transitively on $\mathbb{Z}_{2k+1} - \{0\}$ if and only if $2k+1$ is prime. By coatomisticity of the multi-cubic lattice, we conclude the result. $\square$

We have shown a logic of observables of a quantum system and re-derived a universal set of quantum gates in the sense of the Solovay–Kitaev theorem. These observations were derived from a purely axiomatic view of our original cubic lattice. In addition, we have shown how to adapt these classic ideas to an infinite dimensional setting. We then proceeded to generalize these ideas to the critical multi-cubic lattice. However, in order to do so we have lost the Boolean-adjacent properties of the cubic lattice namely axiom 2, and therefore a nice characterization of the logic that followed. An interesting future pursuit would be a logic that is more than 3 valued for the critical multi-cubic lattice. Another pursuit is to discover an intermediate structure between the generality of the critical multi-cubic lattice and the Boolean-like logic of the cubic lattice. From an applications perspective, physical interpretations of a set of universal quantum gates represented in an exponentially smaller space would also be a novel pursuit in the field of quantum computation.

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