Finite presentability of groups acting on locally finite twin buildings

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Abstract

Let G be a group acting strongly transitively on a locally finite twin building. The main examples of such groups are Kac-Moody groups over finite fields. In this case, G has an associated Weyl group W and accompanying Coxeter diagram. G is always finitely generated under these conditions, and it has been conjectured that if G has Coxeter diagram with at least one ∞ label, then G is not finitely presented.

We prove for large classes of diagrams that G does not have the homological finiteness property FP_2 and hence is not finitely presented. In doing so, we provide strong evidence that the conjecture is true by showing that G is not finitely presented if the diagram has exactly one ∞ and the rest is "as spherical as possible". We also show that if G is rank 3 and has at least one ∞ in the diagram, then G is not finitely presented.

The main tool is a theorem of Gandini which states that a group acting on a 2-dimensional contractible space with certain properties is not FP_2. We first introduce non-standard realizations of buildings since the twin building that G naturally acts on is, in general, of too high dimension. Under certain restrictions on the Coxeter diagram, we realize the twin building as a product of two trees on which G acts with the properties desired in order to apply Gandini's theorem.

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Chapter 0

Introduction

Jacques Tits introduced buildings in the 1950s and '60s as a geometric way to study exceptional simple Lie groups. In the late 1980s, Tits and Mark Ronan introduced twin buildings to yield a geometric approach to studying Kac-Moody groups, which are infinite-dimensional generalizations of semisimple Lie groups. Tits described Kac-Moody groups via a group functor $\mathcal{G}$ associating a Kac-Moody group $\mathcal{G}(k)$ to any field k. If k is infinite, then $\mathcal{G}(k)$ is an infinitely generated group; meanwhile, Kac-Moody groups $\mathcal{G}(\mathbb{F}_q)$ over finite fields are finitely generated. It is then a natural question to ask whether they are also finitely presented.

This question can be generalized to the context of groups acting strongly transitively on locally finite twin buildings, with Kac-Moody groups over finite fields being the most well-known examples. These groups come equipped with a Weyl group and an associated Coxeter diagram by which one can classify them. In the case where the Coxeter diagram has all finite labels, the "2-spherical" case, Abramenko and Mühlherr [AM97] showed that $\mathcal{G}(\mathbb{F}_q)$ is finitely presented excepting a few cases when $q = 2$ or 3. On the other hand, it is known that $\mathcal{G}(\mathbb{F}_q)$ is not finitely presented if it is of type $\tilde{A}_1$. In 1980, Stuhler [Stu80] showed that the group $\mathrm{SL}_2(\mathbb{F}_q[t, t^{-1}])$, which is a Kac-Moody group of type $\tilde{A}_1$, is not finitely presented using homological methods. It can be shown for any group of type $\tilde{A}_1$ by looking at the natural action of the group on its associated twin building, which is a twin tree in this case.

It has been conjectured that if the Coxeter diagram for $\mathcal{G}(\mathbb{F}_q)$ has at least one ∞, then $\mathcal{G}(\mathbb{F}_q)$ is not finitely presented. This conjecture, stated in the more general context of groups acting on locally finite twin buildings, is the main subject of this dissertation. We now record the conjecture in the more general context, also supplanting finite presentation with the homological finiteness property FP_2, which is implied by finite presentation.

Conjecture 0.0.1. Let G be a group acting strongly transitively on a thick twin building of type (W, S) with $S = \{s_i | 1 \leq i \leq n\}$, and let $(q_i)_{i=1}^n$ be a set of parameters with $q_i \in \mathbb{N}$, $q_i \geq 2$ for all i such that for any s_i-panel $\mathcal{P}$, the number of chambers containing $\mathcal{P}$ is $q_i + 1$. If $|s_i s_j| = \infty$ for some $i \neq j$, then G is not FP_2.

The result is known for diagrams of type $\tilde{A}_1$. It was expected to be true for a diagram where all labels are ∞, but is not, to our knowledge, recorded anywhere in the literature. However, it was unclear how finite and infinite labels would mix. There were no known results regarding the finite presentability of groups with at least one finite and one infinite label in the diagram.

The main tool employed in attacking this conjecture is the following theorem of Gandini [Gan12], which we state only in the necessary context. The theorem will be stated in full generality in Chapter 7.

Theorem 0.0.1. If a group G acts cellularly on a product of two trees with finite stabilizers of unbounded order, then G is not FP_2 and hence not finitely presented.

By a cellular action, we mean that G permutes the cells and whenever G stabilizes a cell, it fixes it pointwise. Gandini's result follows primarily from two prior results, namely Brown's filtration criterion [Bro87] and a result due to Kropholler [Kro93], which will be stated and discussed in greater detail in Chapter 7. Brown's filtration criterion discusses the condition on stabilizers while Kropholler's result addresses the bounding of finite subgroups. However, Gandini made a key observation that allows for quicker arguments if one wishes to show that G is not FP_2 (or, more generally, not FP_n).

Our basic strategy will be to find an appropriate geometric realization of the twin building that G acts on so that we can apply Gandini's theorem. If the group is of type $\tilde{A}_1$, then we can act on the geometric realization of the twin building itself, as mentioned above, since each half of the building is a tree. If all labels in the Coxeter diagram are ∞, then the theorem can be applied to G acting on the Davis realization [Dav08] of the twin building. However, the dimension of the Davis realization is too high if there is at least one finite label; it is no longer a tree. Therefore we must adapt this approach by making careful choices for the realization of the building. We will discuss this concept further in Chapter 3.

In the first chapter, we will introduce several necessary notions from Coxeter groups and complexes. Coxeter complexes are the building blocks for defining buildings, and their structure will be useful throughout. The basics of Coxeter groups, especially the Coxeter diagram, are important in classifying these groups acting on twin buildings via their Weyl group. In addition, properties of words in Coxeter groups are important in proving that certain complexes are trees in Lemma 7.2.2 and Lemma 7.3.2, which in turn leads to showing that groups with certain Coxeter diagrams are not FP_2. The main result of this chapter is Lemma 1.1.1, where we prove that words in Coxeter groups satisfying certain conditions cannot be reduced to the trivial word. This is the crucial tool in the proof of Lemma 7.2.2, where it implies that circuits cannot exist in the relevant complex.

In the second chapter, we introduce buildings from both the classic simplicial approach as well as the more recent combinatorial or "W-metric" approach. While the groups we study act naturally on twin buildings, we will often look at the actions of groups on just one half of the twin building at a time. Hence it is important to fully understand single buildings first. In particular, we discuss the Weyl distance function and its properties which allows us to make plentiful use of the theory of Coxeter groups in describing the geometry of buildings and groups acting upon them.

The next chapter introduces new non-standard geometric realizations of buildings, which will be the CW-complexes used in applying Gandini's theorem. We do this via the definition of a Z-realization of a building as in Chapter 12 of [AB08], where Z is a topological space used to model a chamber in the building. We provide an introduction to the Davis realization, which, apart from the standard geometric realization of a building as a simplicial complex, is the most well-known example of such realizations and an important motivation for the non-standard realizations we define. Since the dimension of the Davis realization is, in general, too high, the goal is to define a realization which is a tree. However, the difficulty lies in retaining finite cell stabilizers while reducing the dimension of the realization.

The fourth chapter introduces the standard and necessary background of the theory of groups acting on buildings. In particular, we introduce the notion of a BN-pair for a group, which consists of subgroups B and N of G satisfying certain axioms. Common examples of such groups include the classical Lie groups. A group with a BN-pair acts naturally on a building, whose construction we describe in this chapter, and has several interesting properties.

Next we introduce the basic theory of twin buildings and groups acting on them in Chapters 5 and 6. A twin building is a pair of buildings along with a codistance function that takes a pair of chambers, one from each building, and outputs an element of the Weyl group. One can think of it as defining an opposition relation between the two buildings. We state a few quick results on the properties of the codistance function that will be useful in Chapter 6 when examining groups acting on twin buildings. The associated group theory is similar to that in Chapter 4. We introduce the notion of a twin BN-pair, with subgroups B_+ and B_- which correspond to the two halves of the twin building. There is again an associated Weyl group and Birkhoff decomposition similar to the Bruhat decomposition in the case of a single BN-pair.

We then give a brief introduction to groups with an RGD system, where RGD stands for “root group data”, as the main examples of groups with a twin BN-pair. Kac-Moody groups over fields are the most common examples of groups with an RGD system and hence with a twin BN-pair. We define Kac-Moody algebras and Kac-Moody groups briefly as well since they are the motivating examples for the conjecture this thesis addresses. Finally, we close Chapter 6 by studying the cell stabilizers of groups acting on a locally finite twin building. In order to apply Gandini's theorem, we need finite subgroups of unbounded order as well as finite cell stabilizers. Section 6.6 provides an analysis of when both of these conditions hold. The main results are Proposition 6.6.1 and Lemma 6.6.4, where we prove that the cell stabilizers, which are intersections of certain parabolic subgroups in G, are always finite so long as the parabolic subgroups in question are of spherical type and also can have arbitrarily large order so long as the Weyl group W is infinite.

The final chapter addresses the results we have achieved toward proving the conjecture. There are two main results, Theorem 7.2.1 and Theorem 7.3.1. Theorem 7.2.1 is as follows:

Theorem. Suppose that G has Coxeter system (W, S) such that $S = J \sqcup K$, $|K| \geq 2$, such that $J \cup \{s\}$ is spherical for any $s \in K$ and $m(s,t) = \infty$ for any $s \neq t$ in K. Then G is not FP_2 and hence not finitely presented.

In particular, this theorem provides substantial evidence that the conjecture is true. This is pointed out in Corollary 7.2.1.

Corollary. Suppose that G has Weyl group W with generating $S = J \cup \{s, t\}$ such that $m(s,t) = \infty$ and both $J \cup \{s\}$ and $J \cup \{t\}$ are spherical. Then G is not FP_2.

This was expected to be the most difficult case to show that G is not finitely presented since it is the closest to the 2-spherical case where G is known to be finitely presented. Indeed, we assume that there is just one ∞ in the diagram and that the rest of the diagram is "as spherical as possible". It would be shocking if fewer conditions on the sphericity of the diagram would lead to finite presentability of the group, but, with our current methods, it is difficult to show more.

Theorem 7.3.1 includes the case where all labels are ∞ in the diagram and is stated as follows:

Theorem. Suppose that G has Coxeter system (W,S) such that

$$S = \bigsqcup_{i=1}^{n} J_i,\ n \geq 2,$$

where J_i is spherical for $1 \leq i \leq n$ but $m(s,t) = \infty$ whenever $s \in J_i$ and $t \in J_j$ for $i \neq j$. Then G is not FP_2 and hence not finitely presented.

In combination, the two theorems show that all groups acting on locally finite twin buildings with finite stabilizers of unbounded order and non 2-spherical rank 3 Weyl group are not finitely presented. Finally, we state as corollaries that the Borel subgroups B_+ and B_- are not finitely generated in these cases. This follows immediately from the two theorems by applying Gandini's theorem to the action of B_+ and B_- on the realization of the other half of the twin building. The corresponding complex is 1-dimensional, so Gandini's theorem applies to finite generation, or the property FP_1, instead of finite presentation.

Chapter 1

Coxeter systems and complexes

1.1 Coxeter systems

Any description of buildings begins with a description of a *Coxeter group* or *Coxeter system*. These are abstractions of reflection groups named after H.S.M. Coxeter. In the 1930s, Coxeter was studying reflection groups and found that every reflection group admitted a special presentation, which led to the definition of a Coxeter group (see [Cox34]). He also classified all finite Coxeter groups by their Coxeter diagrams; every finite Coxeter group can be viewed as a finite reflection group of some Euclidean space. We now give the basic definitions and some examples.

Definition 1.1.1. A *Coxeter system* is a pair (W, S) consisting of a group W and a generating set S for W such that W admits a presentation

$$W = \langle S | (st)^{m(s,t)} = 1 \text{ for all } s, t \in S \rangle,$$

where $m(s,t) \in \mathbb{N} \cup \{\infty\}$, $m(s,t) = m(t,s)$, and $m(s,t) = 1$ if and only if $s = t$ for all $s, t \in S$. It can be proved that $m(s,t)$ is in fact the order of the element st. Note that the elements of S all have order 2. If $m(s,t) = \infty$, there is no relation between s and t. Such a group W is called a *Coxeter group*. We call $|S|$ the *rank* of the Coxeter group or system. In the case that $|W| < \infty$, then we will say that W, or the system (W, S), is *spherical*. Finally, for any $J \subset S$, define $W_J := \langle J \rangle \leq W$. If $|W_J| < \infty$, then we say that W_J is a *spherical subgroup* of W.

While Coxeter systems where S is infinite are permitted, we will restrict our focus to the case where $|S| < \infty$. It will be useful to give a more visual way of talking about Coxeter systems. One way is via the *Coxeter matrix* $M = (m(s,t))_{s,t \in S}$. If S is a finite generating set of n elements, then this is a $n \times n$ matrix that is symmetric since $m(s,t) = m(t,s)$ and with every diagonal entry equal to 1 since $m(s,s) = 1$ for all $s \in S$.

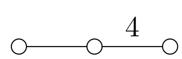
We will more often use the *Coxeter diagram* associated to the Coxeter system (W,S). The diagram has vertex set S and an edge between the vertices s and t if and only if $m(s,t) \geq 3$. If $m(s,t) \geq 4$, then we label the edge connecting $s,t \in S$ with the number $m(s,t)$. Note that if no edge exists between two vertices, then those two generators commute.

Example 1.1.1. Consider the Coxeter system (W,S) with generating set $S = \{s_1, s_2, s_3\}$ and $W = \langle s_1, s_2, s_3 | s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^3 = (s_2s_3)^4 = (s_1s_3)^2 = 1\rangle$.

Then the Coxeter matrix is

$$M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix},$$

and the Coxeter diagram is

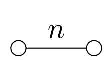


Example 1.1.2. A more recognizable example is the *dihedral group* of order $2n$ which has Coxeter presentation $D_{2n} = \langle s, t | s^2 = t^2 = (st)^n = 1\rangle$.

The Coxeter matrix is

$$M = \begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}$$

and the Coxeter diagram is



We next define a notion of word length and distance in the Coxeter group, both of which will be useful later.

Definition 1.1.2. Let (W, S) be a Coxeter system, and let $w \in W$. A *word* in the generating set S is a sequence $\mathbf{s} = (s_1, \ldots, s_m)$ of elements of S. We will often be less formal and refer to the element $w = s_1 \cdots s_m$ as a word as well, but at times it will be important to distinguish between the word and the element it represents and will use the sequence notation in this case.

Definition 1.1.3. Let (W, S) be a Coxeter system. We define the *word length* of a word $w \in W$ to be $\ell(w) := \min\{m | w = s_1 \cdots s_m \text{ where } s_i \in S \text{ for } 1 \leq i \leq m\}$ for $w \neq 1$ and $\ell(1) = 0$. This length in turn yields a distance function on the Coxeter group, $d : W \times W \to \mathbb{N}_0$ given by $d(v, w) = \ell(v^{-1}w)$.

We say that a word $\mathbf{s} = (s_1, \ldots, s_m)$ is *reduced* if the corresponding element $w = s_1 \cdots s_m$ has length $\ell(w) = m$. In this case, we say that $s_1 \cdots s_m$ is a *reduced decomposition* of w. Note that a reduced decomposition is not at all unique in general.

If W is spherical, then there is a special element that is not present in the case when W is infinite. This is the *longest element* of W.

Proposition 1.1.1. Let (W, S) be a spherical Coxeter system. Then:

(1) There exists a unique element w_0 of maximal length in W.

(2) $\ell(ww_0) = \ell(w_0) - \ell(w)$ for all $w \in W$.

(3) $w_0^2 = 1$, and w_0 normalizes the generating set S.

Now that we have defined a reduced decomposition of an element of W, we can introduce a property of Coxeter groups called the *exchange condition*, which is, in fact, equivalent to W having a presentation as in Definition 1.1.1.

Definition 1.1.4 (The Exchange Condition)**.** Given $w \in W, s \in S$, and any reduced decomposition $w = s_1 \cdots s_m$ of w, either $\ell(sw) = m + 1$ or else there is an index i such that $w = ss_1 \cdots \hat{s}_i \cdots s_m$, where $\hat{s}_i$ indicates that s_i is deleted from the expression.

Definition 1.1.5. An *elementary M-operation* on a word means an operation of one of the following two types:

(I) Delete a subword of the form (s, s).

(II) Given $s, t \in S$ with $s \neq t$ and $m(s,t) < \infty$, replace an alternating subword $(s, t, \ldots)$ of length $m = m(s,t)$ by the alternating word $(t, s, \ldots)$ of the same length.

We say a word is *M-reduced* if it cannot be shortened by any finite sequence of elementary M-operations.

The following theorem is due to Tits [Tit69]. It provides a solution to the word problem for Coxeter groups and will be useful to us in Lemma 1.1.1 when showing what reduced decompositions look like for words with certain restrictions.

Theorem 1.1.1 ([AB08], Theorem 2.33)**.**

(1) A word is reduced if and only if it is M-reduced.

(2) If two words $\mathbf{s}$ and $\mathbf{t}$ are reduced, then they represent the same element of W if and only if $\mathbf{s}$ can be transformed to $\mathbf{t}$ by elementary M-operations of type (II).

A consequence of this theorem is that given any word, any reduced decomposition of that word is obtained through a finite sequence of elementary M-operations.

Lemma 1.1.1. Let $t_1, \ldots, t_m \in S$ such that $m(t_{i-1}, t_i) = \infty$ for $2 \leq i \leq m$. Define $J := S \backslash \{t_1, \ldots, t_m\}$ and $\tilde{W}_i := W_{J \cup \{t_i\}} \backslash W_J$. If $w \in \tilde{W}_1 \cdots \tilde{W}_m$, then any reduced decomposition of w is of the form $\tilde{w}_1 \cdots \tilde{w}_m$ with $\tilde{w}_i \in \tilde{W}_i$. In particular, $\ell(w) \geq m$.

Proof. Suppose $w \in \tilde{W}_1 \cdots \tilde{W}_m$, so we can write $w = w_1 \cdots w_m$ such that $w_i \in \tilde{W}_i$ for each $1 \leq i \leq m$ and such that each w_i is written as a not necessarily reduced word in $J \cup \{t_i\}$. Due to Theorem 1.1.1, any reduced decomposition of w is obtained by a finite sequence of elementary M-operations. Thus it suffices to show that after applying any elementary M-operation, we can still write $w = w_1' \cdots w_m'$ where each $w_i' \in \tilde{W}_i$.

First consider any MI-operation. The first case is when the MI-operation occurs within some w_i. Then the resulting word $w_i' = w_i$ in W and thus still lies in $\tilde{W}_i$. Now consider any MI-operation occurring in $w_{i-1}w_i$ for some $2 \leq i \leq m$. Since $t_{i-1} \neq t_i$, this means that $w_{i-1} = w'_{i-1}s$ and $w_i = sw'_i$ for some $s \neq t_{i-1}, t_i$ and hence $s \in J$ so $w'_{i-1} \in \tilde{W}_{i-1}$, $w'_i \in \tilde{W}_i$. Thus, after applying the MI-operation to delete the (s, s), we obtain the desired decomposition of w.

Now we consider any MII-operation. An MII-operation can occur in some w_i, $w_{i-1}w_i$, or $w_{i-1}w_iw_{i+1}$ since it involves only two letters, and we know that $t_{i-1} \neq t_i \neq t_{i+1}$. We will examine each case in turn.

First suppose that the MII-operation occurs solely in some w_i. Then the resulting word $w'_i = w_i$ in W so $w'_i \in \tilde{W}_i$. Now suppose that the MII-operation occurs in some $w_{i-1}w_i$. Since $m(t_{i-1}, t_i) = \infty$, it cannot involve both letters. Suppose that it involves neither. Then it involves some $s, t \in J$, and we must have $w_{i-1} = v_{i-1}u_{i-1}$ and $w_i = u_iv_i$ where $v_{i-1} \in \tilde{W}_{i-1}, v_i \in \tilde{W}_i$, and u_{i-1}, u_i are alternating words in s and t involved in the MII-operation. After performing the MII-operation, we get an alternating word u in the letters s and t so that $w_{i-1}w_i = v_{i-1}uv_i$. Let $w'_{i-1} = v_{i-1}u$ and $w'_i = v_i$. Then the result is in the desired form.

If the MII-operation involves t_{i-1}, then we must have $w_{i-1} = v_{i-1}u_{i-1}$ and $w_i = sv_i$ with $s \in J, v_{i-1} \in W_{J\cup\{t_{i-1}\}}, v_i \in \tilde{W}_i$, and u_{i-1} an alternating word in s and t_{i-1} ending in t_{i-1}. The MII-operation is on $u_{i-1}s$ and transforms this into a word u of the same length as $u_{i-1}s$ but ending in t_{i-1}. Then $w_{i-1}w_i = v_{i-1}uv_i$. Let $w'_{i-1} = v_{i-1}u$ and let $w'_i = v_i$. Note that $v_{i-1}u = v_{i-1}u_{i-1}s$ since $u = u_{i-1}s$ in W. Since $s \in J$ and $v_{i-1}u_{i-1} = w_{i-1} \in \tilde{W}_{i-1}$, $w'_{i-1} = w_{i-1}s \in \tilde{W}_{i-1}$, so this decomposition is of the desired form. The case where the MII-operation involves t_i is similar to this case.

The final case is when an MII-operation occurs in some $w_{i-1}w_iw_{i+1}$. Since $t_{i-1} \neq t_i \neq t_{i+1}$, $m(t_{i-1}, t_i) = \infty = m(t_i, t_{i+1})$, and the operation involves just two letters, one letter must be t_i and the other some $s \in J$ since there are no relations between t_i and either t_{i-1} or t_{i+1}. In this case, we must have $w_{i-1} = v_{i-1}s, w_i = t_is\cdots st_i$, and $w_{i+1} = sv_{i+1}$, with $v_{i-1} \in \tilde{W}_{i-1}$ and $v_{i+1} \in \tilde{W}_{i+1}$. Then after the MII-operation we are left with $w_{i-1}w_iw_{i+1} = v_{i-1}t_is\cdots st_iv_{i+1}$. Let $w'_{i-1} = v_{i-1}, w'_i = t_is\cdots st_i$, and $w'_{i+1} = v_{i+1}$. Note that $w'_i = sw_is$ and lies in $\tilde{W}_i$ since $s \in J$. Thus this decomposition is in the desired form.

Thus, after either type of MII-operation, we can write $w = w'_1\cdots w'_m$ with $w'_i \in \tilde{W}_i$. Any reduced decomposition of w is obtained from finitely many such operations, so the resulting reduced word is also in this form. □

1.2 Coxeter complexes

1.2.1 Simplicial complexes

Definition 1.2.1. A *simplicial complex* is a nonempty collection Δ of finite subsets of some vertex set V such that:

(1) Every singleton $\{v\} \subset V$ lies in Δ.

(2) If $A \in \Delta$ and $B \subset A$, then $B \in \Delta$.

The elements of Δ are called *simplices*, and the subsets of a simplex A are called the *faces* of A. We call $r := |A|$ the *rank* of A, and $r - 1$ is called the *dimension* of A. A *subcomplex* of Δ is a subset Δ' of Δ which is a simplicial complex in its own right.

Remark 1.2.1. Note that any simplicial complex Δ is also a *poset* under the face relation, i.e. $A \leq B$ if and only if $A \subset B$ as sets. As a poset it has the two following properties:

(a) Any two elements $A, B \in \Delta$ have a greatest lower bound $A \cap B$.

(b) For any $A \in \Delta$, the poset $\Delta_{\leq A}$ of faces of A is isomorphic as a poset to the power set of $\{1, 2, \ldots, r\}$, where $r = \mathrm{rk}(A) \in \mathbb{Z}_{\geq 0}$.

These two properties in fact characterize simplicial complexes. That is, any nonempty poset Δ satisfying (a) and (b) can be identified with the poset of simplices of a simplicial complex (see [AB08], Section A.1.1). We will refer to such posets as simplicial complexes.

Although it is required that any two simplices A and B have a greatest lower bound, it is not necessarily the case that they have an upper bound. If A and B have an upper bound, then we say that A and B are *joinable*. This notion extends to an arbitrary set of simplices in the same manner. This leads to the following definition.

Definition 1.2.2. The *link* of a simplex A in Δ, denoted by $\mathrm{lk}A$ or $\mathrm{lk}_\Delta A$, is the subcomplex of Δ consisting of the simplices B that are disjoint from A and *joinable* to A.

Remark 1.2.2. Given a simplex A, let $\Delta_{\geq A}$ be the set of all simplices in Δ that contain A. Then we have an isomorphism of simplicial complexes between $\mathrm{lk}A$ and $\Delta_{\geq A}$ given by $B \mapsto B \cup A$, where $B \in \mathrm{lk}A$ and $B \cup A$ is the least upper bound of B and A.

Definition 1.2.3. Let $(P, \leq)$ be a partially ordered set with $P \neq \emptyset$. A *flag* in P is a collection of pairwise comparable elements in P, that is, a chain in P. The *flag complex* associated to P, which we will denote $\Delta(P)$, is the simplicial complex with vertex set P and finite flags as simplices.

Example 1.2.1. If P is the poset of nonempty simplices of a simplicial complex Σ partially ordered by inclusion, then the flag complex on P is the *barycentric subdivision* of the simplicial complex Σ.

1.2.2 Chamber complexes

Definition 1.2.4. Let Δ be a finite-dimensional simplicial complex. We say that Δ is a *chamber complex* of rank n (dimension $n-1$) if all maximal simplices have the same rank n (dimension $n-1$) and any two can be connected by a gallery. A *gallery* is a sequence of maximal simplices in which any two consecutive ones are *adjacent* (i.e. are distinct and have a common codimension-1 face). There is a more general notion of *pregallery* in which consecutive maximal simplices may be equal or adjacent.

The maximal simplices of a chamber complex are called *chambers*, and their codimension-1 faces will be called *panels*. A chamber complex is said to be *thin* if each panel is a face of exactly two chambers. Similarly, a chamber complex is said to be *thick* if each panel is a face of at least three chambers. A chamber subcomplex Δ' of a chamber complex Δ is a simplicial subcomplex of Δ that is a chamber complex of the same dimension.

The existence of galleries in a chamber complex lends a natural notion of *gallery distance* between chambers. Given two chambers C and D, we define $d(C, D)$ to be the minimal length of a gallery connecting C and D. This gallery distance can also be extended to arbitrary simplices. That is, given simplices A and B, we define $d(A, B)$ to be the minimal length of galleries of the form $C_0, C_1, \ldots, C_\ell$ with $A \leq C_0$ and $B \leq C_\ell$. Note that the distance function on simplices is not a metric since $d(A, B) = 0$ if and only if there is a chamber with both A and B as faces.

Definition 1.2.5. If Δ and Δ' are chamber complexes of the same dimension, then a simplicial map $\phi : \Delta \to \Delta'$ is called a *chamber map* if it takes chambers to chambers.

Note that a chamber map takes adjacent chambers to chambers that are equal or adjacent and hence takes galleries to pregalleries.

Definition 1.2.6. Let Δ be a chamber complex of rank n with vertex set V, and let I be a set with n elements. We say that Δ is *colorable* if it admits a *type function*, that is, a function $\tau : V \to I$ such that the vertices of every chamber are mapped bijectively onto I. Given a vertex v, we call $\tau(v)$ the *type* of v.

The notion of type can be extended to simplices other than the vertices as well. Given a simplex A of Δ with vertices $v_1, \ldots, v_\ell$, define $\tau(A) := \{\tau(v_1), \ldots, \tau(v_\ell)\}$. Similarly, the *cotype* of A is defined to be $I \backslash \tau(A)$. Thus the type of a chamber is I, and the cotype of a chamber is the empty set. The type of a panel is $I \backslash \{i\}$ for some $i \in I$, and its cotype is $\{i\}$. We call such a panel an *i-panel*.

We now introduce some important terminology for later. Given $i \in I$, two adjacent chambers of Δ will be called i-adjacent if their common panel is an i-panel. Given any subset $J \subset I$, we will say that two chambers are J-equivalent if there exists a gallery $(C_0, \ldots, C_\ell)$ such that any two consecutive chambers C_m and C_{m+1} are j-adjacent for some $j \in J$. This yields the following definition.

Definition 1.2.7. The equivalence classes of chambers under J-equivalence are called *J-residues*.

The following theorem is Proposition A.20 in the appendix of [AB08].

Theorem 1.2.1. Let Δ be a colorable chamber complex where the link of every simplex is a chamber complex and every panel is the face of at least two chambers. Let A be a simplex in Δ, and let $\mathcal{C}_{\geq A}$ be the set of chambers having A as a face. Then:

(1) For every simplex A, the set $\mathcal{C}_{\geq A}$ is a J residue, where J is the cotype of A.

(2) Every residue has the form $\mathcal{C}_{\geq A}$ for some simplex A.

(3) For any simplex A,

$$A = \bigcap_{C \geq A} C.$$

(4) Δ, as a poset, is isomorphic to the set of residues in $\mathcal{C}$, ordered by reverse inclusion.

This theorem is nice because it allows us to identify simplices with residues via the correspondence $A \leftrightarrow \mathcal{C}_{\geq A}$.

1.2.3 Coxeter complexes

We are now ready to introduce Coxeter complexes via the poset definition of a simplicial complex. A Coxeter complex (of type (W, S)) is a simplicial complex on which the Coxeter group W acts in a natural way. First we will introduce some notation. Recall that the subgroup W_J, where $J \subset S$, of W is defined $W_J := \langle J \rangle$. We call W_J a *standard subgroup* of W. Given $w \in W$ and $J \subset S$, we define a *standard (left) coset* of W to be a coset of the form wW_J. Note that $wW_\emptyset = \{w\}$ and $wW_S = W$. Also note that if $J \subset K \subset S$, then $wW_J \subset wW_K$, so W acts on the standard (left) cosets in a way that preserves containment. This motivates the definition of a Coxeter complex:

Definition 1.2.8. Let $\Sigma(W, S)$ be the poset of standard cosets in W, ordered by reverse inclusion. Thus $wW_J \leq vW_K$ in $\Sigma(W, S)$ if and only if $wW_J \supset vW_K$ as subsets of W. In this case we say that wW_J is a *face* of vW_K. We call $\Sigma(W, S)$ the *Coxeter complex* associated to the Coxeter system (W, S). $\Sigma(W, S)$ is called *spherical* if (W, S) is spherical. This terminology comes from the fact that a spherical Coxeter complex is isomorphic to the simplicial complex associated to a finite reflection group ([AB08], Section 1.5), which triangulates a sphere ([AB08], Proposition 1.108).

Theorem 1.2.2 ([AB08], Theorem 3.5)**.** The poset $\Sigma(W, S)$ is a simplicial complex. Moreover, it is a thin chamber complex of rank $|S|$, it is colorable, and the action of W on $\Sigma(W, S)$ is type-preserving. In particular, $\Sigma(W, S)$ has a canonical W-invariant type function with values in S given by $\tau(wW_J) = S \backslash J$. That is, the simplex wW_J has cotype J.

In general, we say that a simplicial complex Σ is a *Coxeter complex* of type (W, S) if Σ is isomorphic to $\Sigma(W, S)$ as a simplicial complex. In this case, Σ inherits the type function from $\Sigma(W, S)$.

Theorem 1.2.3 ([AB08], Proposition 3.32)**.** Let $\Sigma := \Sigma(W, S)$. The image $W \hookrightarrow Aut(\Sigma)$ is the normal subgroup $Aut_0(\Sigma)$ consisting of the type-preserving automorphisms of Σ.

Remark 1.2.3. Note that ordering the standard (left) cosets by reverse inclusion means that the chambers are sets of the form $wW_\emptyset = \{w\}$ and hence we have a bijection between the chambers of $\Sigma(W, S)$ and the elements of W. Futhermore, the action of W on the set of chambers $\mathcal{C}(\Sigma(W, S))$ by left translation is simply transitive. The simplices of the form $w\langle s\rangle = \{w, ws\}$ with $w \in W$ and $s \in S$ are the panels and are contained in exactly the two chambers w and ws. These two chambers are s-adjacent according to our coloring of simplices. The vertices are given by maximal standard left cosets of W.

Example 1.2.2. The symmetric group on n letters S_n has a well-known presentation with the generating set $S = \{(1\ 2), (2\ 3), \ldots (n-1\ n)\}$. Defining $s_i = (i\ i+1)$, S_n has the presentation

$$S_n = \langle s_1, \ldots, s_{n-1} | (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1 \text{ if } |i-j| \neq 1\rangle.$$

This presentation makes (S_n, S) a Coxeter system of type A_{n-1}. The Coxeter complex associated to (S_n, S) is the barycentric subdivision of the boundary of the standard $(n-1)$-simplex.

Here is the Coxeter complex of type A_2, with $W = \langle s_1, s_2 | s_1^2 = s_2^2 = (s_1 s_2)^3 = 1\rangle$ with the chambers and panels labeled by the corresponding standard cosets. Cosets of the form $w\langle s_1\rangle$ are s_1-panels (i.e. of cotype s_1), and cosets of the form $w\langle s_2\rangle$ are s_2-panels.

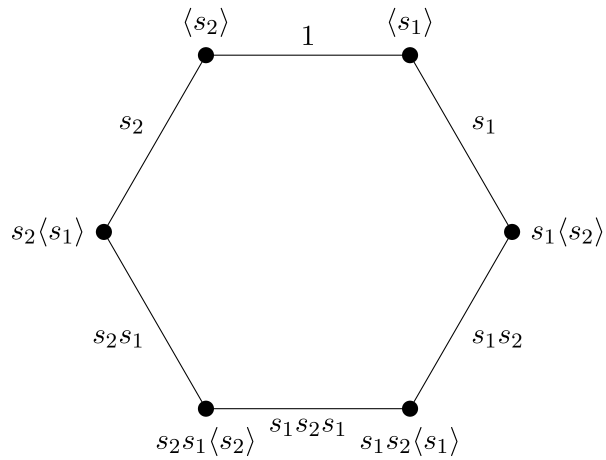
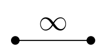


Figure 1.1: Coxeter complex of type A_2

Example 1.2.3. Let $W = D_\infty$ be the infinite dihedral group with presentation

$$\langle s, t \mid s^2 = t^2 = 1 \rangle.$$

That is, the Coxeter diagram is  .

Then the resulting Coxeter complex is a line, where the red vertices are t-panels and blue vertices are s-panels.

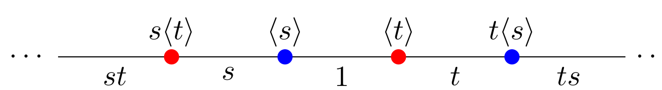
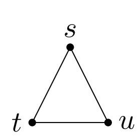


Figure 1.2: Coxeter complex of type $\tilde{A}_1$

Example 1.2.4. Let W be the Coxeter group with presentation

$$\langle s, t, u \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^3 = 1 \rangle.$$

The Coxeter diagram is a triangle:



The group W can be viewed as the group of isometries of the plane generated by the reflections with respect to the sides of an equilateral triangle. The Coxeter complex $\Sigma(W, \{s, t, u\})$ is the plane tiled by equilateral triangles.

Now set $\Sigma := \Sigma(W, S)$. We will distinguish $\{1\}$ from the other chambers and set $C := \{1\}$ and call it the *fundamental chamber*.

Remark 1.2.4. Note that we have a way to construct galleries in Σ since it is a chamber complex. Given two simplices A and B in Σ, a gallery from A to B is of the form $(C_0, C_1, \dots, C_\ell)$ where C_{i-1} and C_i are s_i-adjacent, with $A \in C_0, B \in C_\ell$, and $s_i \in S$. Such a gallery is said to have *type* $(s_1, \dots, s_\ell)$. The *gallery distance* between two simplices is the minimal length of such a gallery. The gallery distance between two chambers $\{v\}$ and $\{w\}$ is $d(v, w) = \ell(v^{-1}w)$.

1.2.4 Opposition

We now introduce the notion of an opposition relation in a spherical Coxeter complex. Introducing it here will make the definition of a twin building in Chapter 5 much more intuitive. Recall from Proposition 1.1.1 that there is a unique longest element w_0 in a spherical Coxeter system (W, S).

Definition 1.2.9. Let (W, S) be a spherical Coxeter system and Σ a Coxeter complex of type (W, S). Then there is a chamber map $\mathrm{op}_\Sigma : \Sigma \to \Sigma$ given by the action of the longest element $w_0 \in W$. Since $w_0^2 = 1$, op_Σ is an involution.
Another way to think about op_Σ is that, given a chamber C in Σ, the gallery distance between C and $\mathrm{op}_\Sigma(C)$ is maximal in Σ, that is $\ell(w_0)$.

If one looks at Figure 1.2 above, the image of a chamber (edge) under op_Σ is the edge opposite it in the hexagon, making sense of the terminology. Also note that op_Σ is given by the action of $s_1 s_2 s_1 = s_2 s_1 s_2$, the longest element of W.

1.2.5 Roots

In this subsection, we will define an important concept in Coxeter complexes and buildings as a whole. This is the notion of a root. The definition of a root is a bit cumbersome, but they are easy enough to picture in the context of a Coxeter complex. We will first define them using the notion of a folding before giving a more hands-on definition in our present context.

Definition 1.2.10. Let Σ be a thin chamber complex. A *folding* of Σ is a chamber map $\phi : \Sigma \to \Sigma$ such that:

(1) $\phi^2 = \phi$

(2) For every chamber $C \in \phi(\Sigma)$, there is exactly one chamber $C' \in \Sigma \backslash \phi(\Sigma)$ such that $\phi(C') = C$.

Suppose that C and C' are adjacent chambers and that $\phi(C') = C$. Then we say that ϕ is *reversible* if and only if there exists a folding ϕ' with $\phi'(C) = C'$.

A *root* α of Σ is the image of Σ under a reversible folding. The subcomplex α' generated by the chambers not in α is also a root, called the *opposite* root of α, often denoted $-\alpha$. It is also the image of the opposite folding ϕ'. The intersection $\partial\alpha := \alpha \cap -\alpha$ will be called the wall bounding the two roots.

Remark 1.2.5. Given a root $\alpha \subset \Sigma$, there is an easy way to describe the chambers contained in α, denoted $\mathcal{C}(\alpha)$. By Lemma 3.43 in [AB08], there always exists a pair of adjacent chambers C, C' such that $C \in \alpha$ and $C' \notin \alpha$. In this case, $\mathcal{C}(\alpha) = \{D \in \mathcal{C}(\Sigma) | d(D, C) < d(D, C')\}$. That is, the chambers in α are all chambers in Σ closer to C than C' in gallery distance.

If $\Sigma = \Sigma(W, S)$ and we identify $\mathcal{C}(\Sigma)$ with W, then define the simple root α_s to be the unique root containing 1 but not s. Then any root $\alpha \subset \Sigma$ can be written $\alpha = w\alpha_s$ for some $w \in W$ and $s \in S$ and is the unique root containing w but not ws.

Example 1.2.5. For an example, we go back to the Coxeter complex of type A_2 where $W = S_3$ as in Example 1.2.2. The half of the hexagon with blue edges in Figure 1.3 below is the simple root α_{s_2}, the unique root containing 1 but not s_2. The other half with red edges is the opposite root $-\alpha_{s_2} = s_2\alpha_{s_2}$. In general a root can be thought of as a half Coxeter complex if the Coxeter complex is spherical.

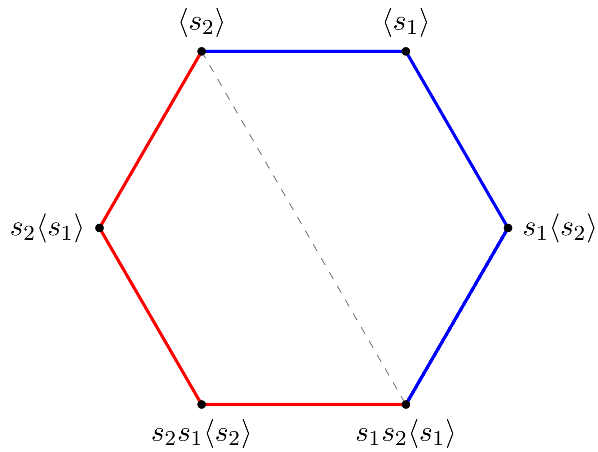


Figure 1.3: The roots α_{s_2} (in blue) and $-\alpha_{s_2}$ (in red)

1.2.6 Weyl Distance Function

In this subsection, we introduce an important tool in the theory of Coxeter complexes and buildings called the *Weyl distance function.* It builds off the notion of gallery distance and tells us not only the distance between two chambers C and D but also the direction from C to D.

Definition 1.2.11. Let $\Sigma := \Sigma(W, S)$ and let $\mathcal{C}(\Sigma)$ be the set of chambers of Σ. Then the *Weyl distance function* is a function $\delta : \mathcal{C}(\Sigma) \times \mathcal{C}(\Sigma) \to W$ defined as follows. Given two chambers C and D in Σ and a gallery of type $(s_1, \ldots, s_m)$ from C to D, define $\delta(C, D) = s_1 \cdots s_m$. We call $\delta(C, D)$ the *Weyl distance* from C to D.
This definition can be extended to all simplices in Σ. Given simplices A and B in Σ, let $\mathcal{C}_A$ and $\mathcal{C}_B$ be the chambers containing A and containing B, respectively. Then define $\delta(A, B) := \delta(C, D)$ where $C \in \mathcal{C}_A$, $D \in \mathcal{C}_B$, and the distance $d(C, D)$ is minimal among such pairs of chambers.

The Weyl distance gives the direction by differentiating between a gallery from C to D and a gallery from D to C. Indeed $\delta(C, D) = \delta(D, C)^{-1}$.

Remark 1.2.6.

(1) Note that this definition allows any choice of gallery between two chambers. It is in fact the case that the definition is independent of the choice of gallery. To see why,

identify $\mathcal{C}(\Sigma)$ with W. Then C and D are identified with two elements w_1 and w_2, respectively, and the gallery from C to D consists of the chambers w_1, $w_1 s_1$, $\ldots$, $w_1 s_1 \cdots s_m = w_2$. Hence $s_1 \cdots s_m = w_1^{-1} w_2$, which is independent of choice of gallery.

(2) This also yields a strong version of the triangle inequality. That is, given three chambers C_1, C_2, C_3, we have $\delta(C_1, C_3) = \delta(C_1, C_2)\delta(C_2, C_3)$. Again identifying $\mathcal{C}(\Sigma)$ with W, identify C_i with w_i. Then $\delta(C_1, C_3) = w_1^{-1} w_3 = w_1^{-1} w_2 w_2^{-1} w_3 = \delta(C_1, C_2)\delta(C_2, C_3)$.

(3) One can also deduce from the definition of Weyl distance that if $w := \delta(C, D)$ for chambers C, D in Σ, then there is a 1-1 correspondence between (minimal) galleries from C to D and (reduced) decompositions of w.

Chapter 2

Buildings

Now that we have discussed Coxeter complexes, we are prepared to define a building. We will discuss buildings from two different points of view in this chapter. The first will be viewing a building as a simplicial complex, or, more specifically, a chamber complex.

2.1 Simplicial buildings

Definition 2.1.1. Let (W, S) be a Coxeter system. A *building* of type (W, S) is a simplicial complex Δ that can be expressed as the union of a collection of subcomplexes $\mathcal{A} := \{\Sigma_i\}_{i \in I}$, called *apartments* satisfying the following axioms:

(B0) Σ_i is isomorphic to the Coxeter complex $\Sigma(W, S)$ for all i.

(B1) For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them.

(B2) If Σ and Σ' are two apartments containing the simplices A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

Any collection $\mathcal{A}$ of apartments of Δ satisfying these three axioms is called a *system of apartments* for Δ. Similar to Coxeter complexes, we say that the building Δ is *spherical* if W is finite.

Remark 2.1.1.

(1) There is an equivalent axiom to (B2) which is helpful (called axiom (B2") in [AB08]): Let Σ and Σ' be two apartments containing a simplex C that is a chamber of Σ Then there is an isomorphism $\Sigma \to \Sigma'$ fixing every simplex in $\Sigma \cap \Sigma'$.

(2) In the literature, a building sometimes requires that every panel is a face of at least three chambers. In this case, the building is called *thick*. With our definition, this is not necessary. In fact, a Coxeter complex is itself a (thin) building with a single apartment.

(3) Δ is a chamber complex since if C and C' are maximal simplices in Δ, then they are also maximal simplices of some apartment Σ by axiom (B1). Since Σ is a chamber complex, C and C' must have the same dimension and are connected by a gallery.

(4) Δ is colorable as a chamber complex. To see why this is true, we can fix an arbitrary chamber C and assign types to its vertices arbitrarily. Since Coxeter complexes are colorable, this coloring extends to a coloring of any apartment containing C. We just need these colorings to match on the intersections. This follows from axiom (B2") as stated in the first remark. That is, if Σ, Σ' are two apartments containing C, then we can define the type function on Σ' by composing the type function of Σ with the isomorphism $\Sigma' \to \Sigma$ which fixes the intersection.

(5) Every building Δ admits a system of apartments. It is a fact that the union of any family of apartment systems is again an apartment system. Hence the union of all apartment systems is an apartment system for Δ. We call this maximal apartment system the *complete* system of apartments. A useful description is given in Remark 4.56 of [AB08]: Let Δ be a building of type (W, S). Then Σ lies in the complete system of apartments if and only if $\Sigma \cong \Sigma(W, S)$.

(6) If Δ is spherical, then Δ admits a unique system of apartments. ([AB08], Theorem 4.70).

(7) Recall that if A is a simplex in the Coxeter complex $\Sigma(W, S)$, then $\text{lk}_\Sigma(A)$ is a Coxeter

complex itself of type (W_J, J), where J is the cotype of A.

The same statement holds true for buildings. That is, if Δ is a building of type (W, S) and A is a simplex of cotype J in Δ, then $\mathrm{lk}_\Delta(A)$ is a building of type (W_J, J). The apartments are the subcomplexes of the form $\mathrm{lk}_\Sigma(A)$, where Σ is an apartment of Δ.

2.1.1 Examples

Example 2.1.1.

(1) As mentioned in Remark 2.1.1, any Coxeter complex is itself a (thin) building.

(2) Suppose Δ is a building of rank 1. Then since the only Coxeter complex of rank 1 is the set of two points (the 0-sphere S^0), each apartment must be S^0, the Coxeter complex of type A_1. Conversely, every rank-1 simplicial complex with at least two vertices is a building by taking every 2-vertex subcomplex to be an apartment.

(3) A tree T with no endpoints (an endpoint is a vertex on only one edge) is a building of type D_∞. The complete system of apartments in this case is the collection of all subcomplexes of T that are lines, which we know are Coxeter complexes of type D_∞ from Example 1.2.3.

Example 2.1.2 (Building associated to a vector space)**.** Let V be a vector space of finite dimension $n \geq 2$ over an arbitrary field k. The *projective space* associated to V consists of all nonzero proper subspaces of V. This is a partially ordered set with $V_1 < V_2$ if and only if $V_1 \subset V_2$. Let $\Delta := \Delta(V)$ be the flag complex on this projective space. The simplices (of dimension $m - 1$) are the flags $V_1 < \cdots < V_m$ of nonzero proper subspaces of V, and the maximal simplices of Δ are flags $V_1 < \cdots V_{n-1}$, where $\dim V_i = i$. It can be shown (see Section 4.3 of [AB08]) that $\Delta(V)$ is a building of dimension $n - 1$, so these maximal flags give the chambers of the building. The apartments are isomorphic to the Coxeter complex of type A_{n-1}; hence this building is spherical.

2.1.2 Retractions

In this subsection, we define and give properties of retractions of a building onto its apartments. Retractions will allow us to extend certain properties of apartments to the entire building containing them.

Definition 2.1.2. Let Δ be a chamber complex and Σ be a chamber subcomplex of Δ. A *retraction* is a chamber map $\rho : \Delta \to \Sigma$ such that ρ is the identity on Σ. In this case, we say that Σ is a *retract* of Δ.

Throughout the rest of this section, assume that Δ is a building.

Proposition 2.1.1. Every apartment is a retract of Δ.

Proof. Let Σ be an apartment of Δ. Fix a chamber C in Σ and consider all apartments containing C. Since any two chambers share some apartment, the collection of all apartments containing C spans the whole building. For any apartment Σ' containing C, there exists a unique isomorphism $\phi_{\Sigma'} : \Sigma' \to \Sigma$ fixing C. Existence follows from axiom (B2), and uniqueness follows from the *standard uniqueness argument* discussed in Remark 3.47 of [AB08]. Essentially, knowing the image of a chamber map on C allows us to figure out what happens as we move away from C along a gallery. For any two such apartments Σ' and Σ'', the isomorphisms $\phi_{\Sigma'}$ and $\phi_{\Sigma''}$ agree on the intersection $\Sigma' \cap \Sigma''$. This follows from the axiom (B2") discussed in Remark 2.1.1 since we can construct $\phi_{\Sigma''}$ by composing $\phi_{\Sigma'}$ with the isomorphism $\Sigma'' \to \Sigma'$ from (B2") that fixes $\Sigma' \cap \Sigma''$ pointwise. Thus the isomorphisms $\phi_{\Sigma'}$ fit together to give a chamber map $\rho : \Delta \to \Sigma$ that is the identity on Σ since $\phi_\Sigma = id_\Sigma$; hence ρ is a retraction. □

The proof of Proposition 2.1.1 actually yields a canonical retraction of Δ onto any apartment Σ. We will use this retraction and its properties again in Chapter 3.

Definition 2.1.3. Given an apartment Σ and chamber C in Σ, there is a *canonical retraction* $\rho = \rho_{\Sigma,C} : \Delta \to \Sigma$ called the retraction onto Σ centered at C. It is the unique chamber map $\Delta \to \Sigma$ that fixes C pointwise and maps every apartment containing C isomorphically onto Σ.

We omit the proof of the following proposition which can be found as Proposition 4.39 in [AB08].

Proposition 2.1.2. The retraction $\rho = \rho_{\Sigma,C}$ has the following properties.

(1) For any face $A \leq C$, $\rho^{-1}(A) = \{A\}$.

(2) ρ preserves distances from C. That is, for any chamber D in Δ, $d(C, D) = d(C, \rho(D))$.

(3) ρ is the unique chamber map $\Delta \to \Sigma$ that fixes C pointwise and preserves distances from C.

The following proposition is an application of retractions.

Proposition 2.1.3. Every apartment Σ is a convex chamber subcomplex of Δ. That is, given chambers C, D in Σ, every minimal gallery in Δ between C and D is contained in Σ. (D can actually be any simplex in Σ, but we only care about chambers in our context.)

Proof. Let $\Gamma : C = C_0, \dots, C_d = D$ be a minimal gallery from C to D. If Γ is not contained in Σ, then there is an index $i \geq 1$ with $C_{i-1} \in \Sigma$ and $C_i \notin \Sigma$. Let C' be the chamber of Σ distinct from C_{i-1} with $C_{i-1} \cap C_i$ as a face, and let $\rho = \rho_{\Sigma,C'}$. Then $\rho(C_i) = C_{i-1}$ since ρ preserves distance from C' by Proposition 2.1.2. Thus, the pregallery $\rho(\Gamma)$ has a repetition, contradicting minimality of Γ. □

This immediately implies the following corollary:

Corollary 2.1.1. If C and D are chambers lying in the same apartment Σ of Δ, then $d_\Delta(C, D) = d_\Sigma(C, D)$, where d denotes gallery distance. Moreover, the diameter of the building is the same as the diameter of any apartment.

2.1.3 Spherical buildings

This section is mostly to develop an intuition for the notion of a twin building in Chapter 5 since twin buildings can be viewed as a generalization of spherical buildings. Recall that a spherical building is a building of type (W, S) where W is finite, or when all its apartments are spherical Coxeter complexes. Since the diameter of a building is the same as that of any apartment (Corollary 2.1.1), a building is spherical if and only if it has finite diameter.

Definition 2.1.4. Two chambers C and D in a spherical building Δ are called *opposite*, denoted C op D, if $d(C, D) = \text{diam } \Delta$.

Lemma 2.1.1 (Lemma 4.69, [AB08])**.** Let C and D be opposite chambers in a spherical building, and let Σ be any apartment containing C and D. Then every chamber of Σ occurs in some minimal gallery from C to D.

The consequence of this lemma and the fact that any apartment Σ is convex is that Σ is the *convex hull* of any pair of opposite chambers C and D, where the convex hull is the smallest convex chamber subcomplex of Δ containing $\{C, D\}$.

2.1.4 Weyl distance

Recall that we defined the Weyl distance in a Coxeter complex $\Sigma := \Sigma(W, S)$ in Definition 1.2.11 as a function $\delta : \mathcal{C}(\Sigma) \times \mathcal{C}(\Sigma) \to W$. Now suppose that Δ is a building of type (W, S). There is a natural extension of this Weyl distance function to the entire building.

Proposition 2.1.4. There is a function $\delta : \mathcal{C}(\Delta) \times \mathcal{C}(\Delta) \to W$ with the following properties:

(1) Given a minimal gallery $\Gamma : C_0, \ldots, C_d$ of type $(s_1, \ldots, s_d)$, $\delta(C_0, C_d)$ is the element $w = s_1 \cdots s_d$ in W.

(2) Let C and D be chambers, and let $w = \delta(C, D)$. Then there is a 1-1 correspondence between minimal galleries from C to D and reduced decompositions of w.

Proof. Given two chambers C, D, choose an apartment Σ containing them. Set $\delta(C, D) = \delta_\Sigma(C, D)$. This is independent of choice of Σ because the isomorphisms between apartments fixing their intersections can be taken to be type-preserving. Since apartments are convex, if we take a minimal gallery from C_0 to C_d as above, then this minimal gallery lies in some apartment, and we appeal to the same properties of the Weyl distance function holding in Coxeter complexes. □

Now we state some important properties of the Weyl distance function that will lead us to an alternate definition of a building. This is Proposition 4.84 in [AB08].

Proposition 2.1.5. Let C and D be chambers in Δ.

(1) $\delta(C, D) = 1$ if and only if $C = D$.

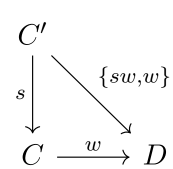
(2) $\delta(D, C) = \delta(C, D)^{-1}$.

(3) If $\delta(C, C') = s \in S$ and $\delta(C, D) = w$, then $\delta(C', D) = sw$ or w. If, in addition, $\ell(sw) = \ell(w) + 1$, then $\delta(C', D) = sw$.

(4) If $\delta(C, D) = w$, then for any $s \in S$, there exists some chamber C' such that $\delta(C', C) = s$ and $\delta(C', D) = sw$. If $\ell(sw) = \ell(w) - 1$, then there is a unique such C'.

Proof. The first two properties are immediate from the definition. Now suppose $\delta(C, D) = w$ and let $(s_1, \ldots, s_d)$ be a reduced decomposition of w. Choose a gallery $C = C_0, C_1, \ldots, C_d = D$ of type $(s_1, \ldots, s_d)$. If $\ell(sw) = \ell(w) + 1$, then for any C' with $\delta(C', C) = s$, the gallery $C', C_0, \ldots, C_d$ has reduced type $(s, s_1, \ldots, s_d)$. It is a fact that a gallery is minimal if and only if it has reduced type ([AB08], Prop 4.41), so this gallery must be minimal and therefore $\delta(C', D) = sw$. If $\ell(sw) = \ell(w) - 1$ instead, then there is some reduced decomposition of w beginning with s, so we can assume our gallery has type $(s_1, \ldots, s_d)$, where $s = s_1$. Then $\delta(C, C_1) = s$ and $\delta(C_1, D) = sw$. If C' is any chamber distinct from C_1 such that $\delta(C', C) = s$, then C', C, C_1 are all s-adjacent, and the gallery $C', C_1, \ldots, C_d$ is reduced of type $(s_1, \ldots, s_d)$. That is, $\delta(C', D) = w$. This proves both (3) and (4), with C_1 being the unique chamber described in (4). □

Note that properties (1) and (2) are mirror the properties of the distance function on a metric space. The first sentence of property (3) is the analogue of the triangle inequality, and we draw a triangle such as



to visualize this property. This diagram may also be useful to visualize the last two axioms of a W-metric building as defined in Definition 2.2.1 in the next section.

2.2 W-Metric Buildings

We have introduced buildings as chamber complexes. However, this simplicial approach will not always be the most useful for us. Another way to look at buildings is a more combinatorial approach that defines a building using the Weyl distance function. This approach no longer relies on apartments in the definition, but the Weyl group is necessary from the beginning instead of arising after choosing a type function. We will call buildings using this new definition *W-metric buildings* as opposed to the *simplicial buildings* we have already defined. The axioms for the W-metric building will look very similar to the properties stated in Proposition 2.1.5, and the name W-metric comes from [AB08] due to the vague similarity between these axioms and the axioms for a metric space.

Definition 2.2.1. A building of type (W, S) is a pair $(\mathcal{C}, \delta)$ consisting of a nonempty set $\mathcal{C}$, whose elements are called *chambers*, and a map $\delta : \mathcal{C} \times \mathcal{C} \to W$ called the *Weyl distance function*, such that for all $C, D \in \mathcal{C}$, the following three axioms hold:

(WD1) $\delta(C, D) = 1$ if and only if $C = D$.

(WD2) If $\delta(C, D) = w$ and $C' \in \mathcal{C}$ satisfies $\delta(C', C) = s \in S$, then $\delta(C', D) = sw$ or w. If, in addition, $\ell(sw) = \ell(w) + 1$, then $\delta(C', D) = sw$.

(WD3) If $\delta(C, D) = w$, then for any $s \in S$, there exists some chamber $C' \in \mathcal{C}$ such that $\delta(C', C) = s$ and $\delta(C', D) = sw$.

Given this definition, we see that Proposition 2.1.5 states that every simplicial building of type (W, S) gives rise to a W-metric building of type (W, S). It turns out that the two concepts are indeed equivalent. We will not prove this here, but the discussion can be found in Section 5.6 of [AB08].

We now introduce some already-used terminology in this new context.

Definition 2.2.2.

(1) Given $s \in S$ we say that two chambers $C, D \in \mathcal{C}$ are *s-adjacent* if $\delta(C, D) = s$. We say that they are s-equivalent, written $C \sim_s D$ if $\delta(C, D) \in \{1, s\}$.

(2) This leads to a new definition of panels. The equivalence classes in $\mathcal{C}$ under s-equivalence are called *s-panels*.

(3) Whereas the Weyl distance was defined using galleries in the simplicial building, we use the Weyl distance to define galleries in this case. We say that two chambers are *adjacent* if they are s-adjacent for some $s \in S$. A *gallery* of length n is a sequence $C_0, \ldots, C_n$ of $n+1$ chambers such that C_{i-1} and C_i are adjacent for each $1 \leq i \leq n$. If there is no gallery of length $< n$ connecting C_0 and C_n, then the *gallery distance* between C_0 and C_n is n, written $d(C_0, C_n) = n$. In this case, the gallery is *minimal*. Lastly, the type of the gallery is $(s_1, \ldots, s_n)$ if $s_i = \delta(C_{i-1}, C_i)$.

(4) This definition can be extended as before to J-residues for any $J \subseteq S$. Given $J \subseteq S$, we say that two chambers $C, D \in \mathcal{C}$ are *J-equivalent*, written $C \sim_J D$ if and only if there is a gallery of type $(s_1, \ldots, s_n)$ connecting C and D with $s_i \in J$ for all $1 \leq i \leq n$. The equivalence classes are called *J-residues*. The J-residue containing a given chamber C will be denoted $R_J(C)$.

Note that panels are residues of type $\{s\}$ for some $s \in S$, chambers themselves are residues of type $\emptyset$, and the entire building $\mathcal{C}$ is a residue of type S.

Remark 2.2.1. Note that (WD3) does not include the statement that the chamber C' is unique if $\ell(sw) = \ell(w) - 1$ as in Proposition 2.1.5. This property actually follows from (WD2). Consider the s-panel containing C and choose a chamber C' as in (WD3). Then C' lies in this panel. Now suppose $C'' \neq C'$ is also in this panel. Then $\delta(C'', C') = s$ and $\delta(C', D) = sw$ with $\ell(s(sw)) = \ell(sw) + 1$, so $\delta(C'', D) = w$ by (WD2). Thus C' is unique.

The notions of thick and thin buildings are adjusted for the new notion of panel. Every panel contains at least two chambers by (WD3). If each panel contains exactly two chambers, then the building is *thin*. If each panel contains at least three chambers, then the building is *thick*.

Example 2.2.1. We give the natural example of a thin building of type (W, S) here, which we call the *standard thin building*. We let $\mathcal{C} = W$ and define $\delta_W : W \times W \to W$ by $\delta_W(w_1, w_2) = w_1^{-1} w_2$. Then (W, δ_W) is a thin building of type (W, S). The chambers are elements of W, and each chamber w is s-adjacent to only ws. Note that for any three chambers C, D, E in W, $\delta_W(C, E) = \delta_W(C, D)\delta_W(D, E)$, which is the same stronger triangle inequality seen in simplicial Coxeter complexes (Remark 1.2.6). The standard thin building plays the role in the W-metric case that $\Sigma(W, S)$ played in the simplicial case; that is, it is the model for apartments.

Both parts of the following lemma can be proved by induction on n. For full details, see Lemma 5.16 in [AB08]. The conclusion is that the Weyl distance function in this context satisfies the same correspondence with galleries as discussed in Proposition 2.1.4 in the simplicial context.

Lemma 2.2.1. Let C and D be chambers and let $w := \delta(C, D)$.

(1) If Γ is a gallery of type $\mathbf{s} = (s_1, \ldots, s_n)$ connecting C and D, then there exists a subword $(s_{i_1}, \ldots, s_{i_m})$ of $\mathbf{s}$ such that $w = s_{i_1} \cdots s_{i_m}$, where $0 \leq m \leq n$ and $1 \leq i_1 < \cdots < i_m \leq n$. If $\mathbf{s}$ is *reduced*, then $w = s_1 \cdots s_n$ and Γ is minimal.

(2) If $w = s_1 \cdots s_n$ with $s_1, \ldots, s_n \in S$, then there exists a gallery Γ of type $\mathbf{s} = (s_1, \ldots, s_n)$ between C and D. If $\mathbf{s}$ is reduced, then this gallery Γ is uniquely determined and minimal.

Chapter 3

Realizations of buildings

In this chapter, we discuss the general method for constructing metric realizations of buildings. This is not a new definition of a building but rather a different way to think about and view them. The idea is to give a metric model for a closed chamber and then to glue copies of this model together to get a model for the building as a metric space.

The most important example in the literature apart from the standard geometric realization of a building is due to Davis [Dav98]; his construction, now called the *Davis realization* provides a CAT(0) realization for every building, even those with spherical or Euclidean Coxeter systems. Davis first introduced the idea for Coxeter complexes in sections 6 and 11 of [Dav83]. His result on buildings in [Dav98] relied upon the significant result by his Ph.D. student Moussong [Mou88] that apartments can always be realized as CAT(0) metric spaces. A recent and more full treatment of this subject by Davis can be found in chapters 5, 7, 12 and 18 in [Dav08]. The following treatment of the subject is a bit more general and can be found in chapter 12 of [AB08].

3.1 The Z-realization of a building

3.1.1 Definition

Throughout this section, let Δ be a building of type (W, S) and $\mathcal{C} = \mathcal{C}(\Delta)$ its set of chambers with Weyl distance function $\delta : \mathcal{C} \times \mathcal{C} \to W$.

Let Z be a set with a family of nonempty subsets Z_s for each $s \in S$. Z is a model for a closed chamber in Δ, and Z_s can be thought of as its s-panel. Indeed, Davis calls this a *panel structure* on Z. However, we note that Z_s need not be codimension 1 inside Z as panels are in a simplicial building. We will construct the Z-realization of the building Δ, denoted by $Z(\Delta)$, by gluing together copies $Z(C)$ of Z for each chamber C of Δ. The gluing should respect the panel structure in Δ. That is, if C and D are s-adjacent chambers, then the copies of Z_s in $Z(C)$ and $Z(D)$ should be identified in $Z(\Delta)$. Therefore we construct $Z(\Delta)$ as follows. Define $S_z := \{s \in S | z \in Z_s\}$ and $W_z := \langle S_z \rangle$. Begin with the product $\mathcal{C} \times Z$, and let $\sim$ be the equivalence relation on $\mathcal{C} \times Z$ defined by $(C, z) \sim (C', z')$ if and only if $z' = z$ and $\delta(C, C') \in W_z$.

Definition 3.1.1. The Z-realization of Δ, denoted by $Z(\Delta)$, is the quotient $(\mathcal{C} \times Z)/\sim$. We denote by $[C, z]$ the equivalence class of (C, z), and we set $Z(C) := \{[C, z] | z \in Z\}$. There is a well-defined function $\tau : Z(\Delta) \to Z$, given by $\tau([C, z]) = z$. We call $\tau(x)$ the *type* of x for $x = [C, z] \in Z(\Delta)$.

Example 3.1.1.

(1) Suppose Z is a geometric closed simplex with vertex set S. For each $s \in S$, let Z_s be the face of Z not containing the vertex s. Then $Z(\Delta)$ is the usual geometric realization of the building Δ, and $Z(C)$ is the closed chamber $\overline{C}$.

(2) Consider the case where Δ consists of just one apartment Σ. Choose a fundamental chamber and identify Σ with the standard Coxeter complex $\Sigma(W, S)$. Then $\mathcal{C} = W$ and $\delta(w, w') = w^{-1}w'$ for $w, w' \in W$. Therefore, to obtain the Z-realization $Z(\Sigma)$, we take the quotient of $W \times Z$ by the equivalence relation $\sim$ defined by $(w, z) \sim (w', z')$ if and only if $z' = z$ and $w' \in w\langle S_z \rangle$.

Note that W acts on $Z(W, S)$ with Z as strict fundamental domain and stabilizers $\langle S_z \rangle$ for z in the fundamental domain $Z(1)$. This is another way to geometrically model a Coxeter system and can in fact be more natural than the Coxeter complex $\Sigma(W, S)$ as seen in the following example.

(3) Let W be the Coxeter group of type $\tilde{A}_1 \times \tilde{A}_1$. That is, W has presentation

$$W = \langle s, t, u, v | s^2 = t^2 = u^2 = v^2 = (su)^2 = (sv)^2 = (tu)^2 = (tv)^2 = 1 \rangle,$$

so s and t commute with u and v. Let Z be a solid square, let Z_s and Z_t be a pair of opposite sides, and let Z_u and Z_v be the other pair of opposite sides as in Figure 3.1. Then $Z(W, S)$ is a tiling of the Euclidean plane by squares. Since W is a product of two copies of D_∞, which we think of as a line, it is natural to end up with a tiling of the plane as a model for W. On the other hand, the standard Coxeter complex $\Sigma(W, S)$ is 3-dimensional in this case.

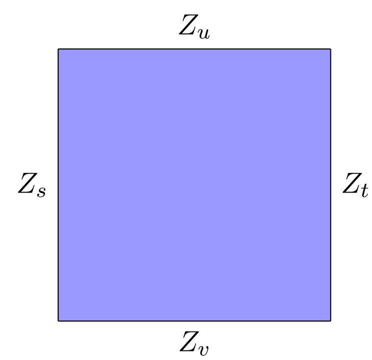


Figure 3.1: $W = D_\infty \times D_\infty$, and Z is a square

(4) Let (W, S) be a Coxeter system, and let $\mathcal{S}$ be the set of spherical subsets of S, i.e. $J \subset S$ such that $\langle J \rangle$ is finite. View $\mathcal{S}$ as a poset ordered by inclusion. Let Z be the geometric realization of the flag complex on $\mathcal{S}$, that is, $Z = |K(\mathcal{S})|$. For each $s \in S$, define $\mathcal{S}_{\geq s}$ to be the subset of $\mathcal{S}$ of all spherical subsets of S containing s. Then let the s-panel Z_s be the geometric realization of the flag complex on $\mathcal{S}_{\geq s}$, i.e., $Z_s = |K(\mathcal{S}_{\geq s})|$. The realization $Z(\Delta)$ is the *Davis realization* of a building, which we will discuss further in subsection 3.1.7.

3.1.2 Cellulation of $Z(\Delta)$

In Chapter 7, we will need to look at the stabilizers of cells in a group acting on a Z-realization of a building. In order to talk about stabilizers of cells, we first need to define a cellulation of $Z(\Delta)$. For our purposes, Z will always be a simplicial complex and hence also a CW-complex where the cells are the closed simplices. Given a cell $\sigma \in Z$, we define the cells $[C, \sigma]$ in $Z(\Delta)$ for all $C \in \mathcal{C}$, where $[C, \sigma] := \bigcup_{z \in \sigma} [C, z]$. We develop an equivalence between cells similar to that between points: $[C, \sigma] = [C', \sigma']$ if and only if $\bigcup_{z \in \sigma} [C, z] = \bigcup_{z' \in \sigma'} [C', z']$ if and only if $\delta(C, C') \in S_z$ for all $z \in \sigma$ and $\sigma = \sigma'$, where the last equivalence follows from equivalence of points from each union. Now we define $S_\sigma := \{s \in S | \sigma \subset Z_s\} = \bigcap_{z \in \sigma} S_z$. Then we can reformulate equivalence between cells by saying $[C, \sigma] = [C', \sigma']$ if and only if $\sigma = \sigma'$ and $\delta(C, C') \in \langle S_\sigma \rangle$. This establishes a CW-complex structure on $Z(\Delta)$.

3.1.3 A metric on $Z(\Delta)$

Assume from now on that Z is a topological space with a metric, denoted by $d(\,\cdot\,,\,\cdot\,)$ or $d_Z(\,\cdot\,,\,\cdot\,)$ if the context is unclear, and that Z_s is a closed subspace for each $s \in S$. For us, Z will usually be the geometric realization of a simplicial (flag) complex and Z_s the geometric realization of a face of Z. We follow the method and terminology of section 12.1.2 in [AB08] and define this metric using *chains* which can be thought of as piecewise geodesic paths on $Z(\Delta)$.

Definition 3.1.2. A *chain* in X is a finite sequence γ of points $x_0, x_1, \ldots, x_m$ $(m \geq 0)$ such that for each $i = 1, \ldots, m$ there is a chamber $C_i \in \mathcal{C}$ such that x_{i-1} and x_i are both in $Z(C_i)$. We say that γ is an m-chain, and we call the subchain x_{i-1}, x_i the ith segment of γ The *length* of γ is

$$\ell(\gamma) := \sum_{i=1}^{m} d_Z(\tau(x_{i-1}), \tau(x_i)).$$

Thinking of a γ as a piecewise geodesic, the length of γ is simply the sum of the lengths of the pieces.

This leads to the definition of our metric on $X = Z(\Delta)$.

Definition 3.1.3. We define a distance function on X by

$$d(x, y) := \inf_\gamma \ell(\gamma)$$

where γ ranges over all chains from x to y.

Remark 3.1.1. In general, we do not require that the chambers $C_1, \ldots, C_m$ in a chain form a gallery in Δ. However, it can always be arranged so that this is the case by repeating points. In fact, given a chain γ from x to y, there is a chain γ' with $\ell(\gamma') \leq \ell(\gamma)$ such that γ' can be represented by an m-chain with an associated minimal gallery (see Proposition 12.25 in [AB08]).

We provide the following proposition without proof. For details, see Proposition 12.10 in [AB08].

Proposition 3.1.1.

(1) The distance function $d : X \times X \to \mathbb{R}$ in Definition 3.1.3 is a metric.

(2) The type function $\tau : X \to Z$ maps $Z(C)$ isometrically onto Z for every chamber $C \in \mathcal{C}$.

3.1.4 Chamber maps

In this subsection, we examine the effect of a chamber map between two Z-realizations. Let Δ and Δ' be buildings of type (W, S) and let $X = Z(\Delta)$ and $X' = Z(\Delta')$ be their Z-realizations, where Z is a metric space with closed nonempty subspaces Z_s. Let $\phi : \Delta \to \Delta'$ be a type-preserving chamber map. Then ϕ induces a map $Z(\phi) : X \to X'$ given by $[C, z] \mapsto [\phi(C), z]$ for all $C \in \mathcal{C}$ and $z \in Z$. We will usually write ϕ instead of $Z(\phi)$ when the meaning is clear from context.

Proposition 3.1.2. Given a type-preserving chamber map $\phi : \Delta \to \Delta'$, the induced map $\phi : X \to X'$ is distance-decreasing. That is,

$$d(\phi(x), \phi(y)) \leq d(x, y)$$

for all $x, y \in X$.

Proof. Note that $\phi : X \to X'$ takes a chain in X to a chain in X' of the same length since $\tau(x) = \tau(\phi(x))$ for all $x \in X$. The result follows immediately. $\square$

The chamber map most relevant to us is the retraction $\rho = \rho_{\Sigma,C}$, where Σ is an apartment of Δ and C is a chamber of Σ. In this case, we can say a bit more than in Proposition 3.1.2. The following is Proposition 12.18 in [AB08].

Proposition 3.1.3. Let $\rho = \rho_{\Sigma,C}$. Then

$$d(\rho(x), \rho(y)) \leq d(x, y)$$

for all $x, y \in X$, with equality if $x \in Z(C)$.

Proof. We know that ρ is distance-decreasing since it is a chamber map. It remains to show that equality holds for $x \in Z(C)$. Choose a chamber D with $y \in Z(D)$, and let Σ' be an apartment containing the chambers C and D. Recall that we defined ρ such that the restriction of ρ to Σ' is a type-preserving isomorphism $\phi : \Sigma' \to \Sigma$. We know that the maps induced by ϕ and ϕ^{-1} are both distance-decreasing, so ϕ induces an isometry $Z(\Sigma') \to Z(\Sigma)$, so $d(\rho(x), \rho(y)) = d(x, y)$. □

3.1.5 Curvature

In this subsection we will briefly discuss the notion of a curvature property of a space called the CAT(κ) property for any real number κ. We will be most interested in the CAT(0) situation, that is, when $\kappa = 0$. A complete treatment of this subject can be found in II.1 of Bridson-Haefliger [BH99].

Definition 3.1.4. A *geodesic* or *geodesic segment* joining points p and q in a metric space X is a path from p to q in X isometric to a closed interval in $\mathbb{R}$ of length $d(p, q)$. We denote by $[p, q]$ the choice of any geodesic segment from p to q.

A *geodesic triangle* in X consists of three points p, q, r in X and geodesic segments $[p, q], [q, r], [r, p]$ joining them.

The model space or *comparison space* M^2_κ is a complete simply connected manifold of constant curvature κ. More specifically, if $\kappa = 0$, this is $\mathbb{R}^2$. If $\kappa > 0$, it is the sphere of radius $1/\sqrt{\kappa}$. If $\kappa < 0$, it is the hyperbolic plane with metric scaled by $1/\sqrt{-\kappa}$. Given a geodesic triangle in X with points p, q, r, there is a *comparison triangle* in M^2_κ with points $\bar{p}, \bar{q}, \bar{r}$

with the same pairwise distances. This comparison triangle is unique up to isometry. If the perimeter of the geodesic triangle in X is less than twice the diameter of M^2_κ, then the comparison triangle always exists. Note that the comparison triangle will always exist if $\kappa = 0$.

Lastly, given a geodesic triangle in X and comparison triangle in M^2_κ, a point $\bar{x} \in [\bar{q}, \bar{r}]$ is a *comparison point* for $x \in [q, r]$ if $d(\bar{q}, \bar{x}) = d(q, x)$. Comparison points on the other two sides of the triangle are defined the same way.

Let D_κ denote the diameter of the comparison space M^2_κ.

Definition 3.1.5. Fix $\kappa \in \mathbb{R}$. Let X be a metric space in which there is a geodesic between all points less than D_κ apart. Furthermore, assume that the perimeter of any geodesic triangle is less than $2D_\kappa$ so that the comparison triangle exists. We say that X is a CAT(κ) *space* if for every geodesic triangle in X, given by say $[p, q], [q, r], [r, p]$, and every point $x \in [q, r]$, the comparison point $\bar{x} \in [\bar{q}, \bar{r}]$ satisfies $d(p, x) \leq d(\bar{p}, \bar{x})$.

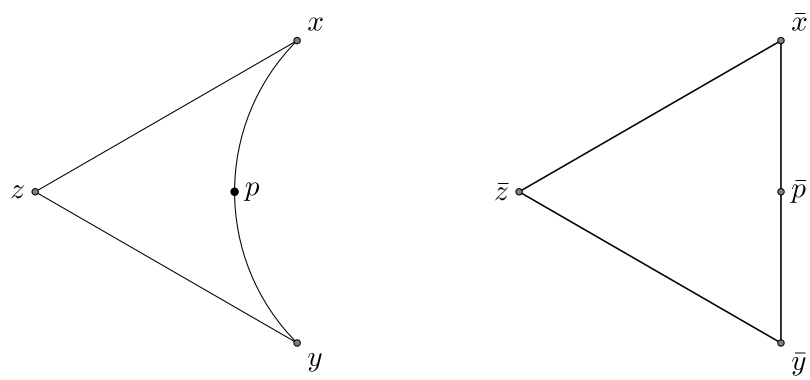


Figure 3.2: A geodesic triangle in X and its comparison triangle in $\mathbb{R}^2$

As mentioned above, we will be most concerned with the CAT(0) property. In this case, the comparison triangle lies in the Euclidean plane $\mathbb{R}^2$. Intuitively, the CAT(0) property relates to nonpositive curvature as one can see in Figure 3.2 below. Indeed, a Riemannian manifold has sectional curvature ≤ 0 if and only if it is locally a CAT(0) space. See the Appendix to Chapter II.1 in [BH99]. CAT(0) spaces have many nice properties and are widely studied in the realm of geometric group theory. In particular, CAT(0) cube complexes have been especially important in the field.

They are *uniquely geodesic*; that is, there is exactly one geodesic $[p, q]$ between two points p and q (see Proposition 11.5, [AB08]). Complete CAT(0) spaces have a nice fixed point property for groups acting on them as shown by Bruhat and Tits [BT72]. The statement is Theorem 11.23 in [AB08] and is followed by a nice discussion. We now state the property most relevant to us. For a proof, see Proposition 11.7 in [AB08].

Proposition 3.1.4. Let X be a CAT(0) space. Then X is contractible.

The following proposition (Proposition 12.29, [AB08]) is the main reason for the discussion of CAT(κ) spaces. Again, we will apply it in the case $\kappa = 0$.

Proposition 3.1.5. Suppose that $Z(W, S)$ is a CAT(κ) space for some real number κ. Then the Z-realization of any building Δ of type (W, S) is a CAT(κ) space.

3.1.6 When is $Z(\Delta)$ a tree?

In this subsection, we will collect some results that will be useful in later chapters when looking at groups acting on $Z(\Delta)$. In particular, it will be important to know some conditions on $Z(\Sigma)$, where Σ is any apartment of Δ, that imply that $Z(\Delta)$ is a *tree*.

First we introduce some key definitions.

Definition 3.1.6. An (unoriented) *graph* $\Gamma = (V, E, I)$ is a triple consisting of a nonempty vertex set V, an edge set E, and a symmetric incidence relation $I \subset V \times E \cup E \times V$ denoted eIv or vIe for $v \in V, e \in E$ such that $|\{v \in V | vIe\}| \in \{1, 2\}$.
The *geometric realization* of a graph is the metric space associated to the graph where we say all edges are length 1.

Remark 3.1.2. If $|\{v \in V | vIe\}| = 1$, this means e is a loop at some vertex v. If $|\{v \in V | vIe\}| = 2$, then e is an edge between two distinct vertices. There is a notion of orientation with graphs, but all of our graphs will be unoriented.

Definition 3.1.7. A *combinatorial tree* is a nonempty connected graph without circuits. Since a combinatorial tree is a graph, we can talk about the geometric realization of a combinatorial tree as a metric space. We will use the term *tree* to mean the geometric realization of a combinatorial tree. Connecting this definition with the discussion of CAT(0) spaces, it is known that a 1-dimensional simplicial complex is a combinatorial tree if and only if it its geometric realization is CAT(0).

A notion we will need later is that of an *endpoint* of a tree. This is a vertex that is contained in only one edge.

Lemma 3.1.1. If $Z(\Sigma)$ is connected for any apartment Σ of Δ, then $Z(\Delta)$ is connected.

Proof. Let x, y be two points in $Z(\Delta)$, say $x \in Z(C), y \in Z(D)$ for chambers C, D of Δ. Then there exists some apartment Σ containing both C and D. Therefore $x, y \in Z(\Sigma)$. Since $Z(\Sigma)$ is connected, there exists some path connecting x and y in $Z(\Sigma)$ and hence also in $Z(\Delta)$. □

Lemma 3.1.2. If $Z(\Sigma)$ is a tree for any apartment Σ of Δ, then $Z(\Delta)$ is a tree.

Proof. Since $Z(\Sigma)$ is a tree, it is a 1-dimensional connected simplicial complex that must be CAT(0) as explained in Definition 3.1.7. By Proposition 3.1.5 and Lemma 3.1.1, $Z(\Delta)$ must be CAT(0) and connected. It also must be 1-dimensional and simplicial since $Z(\Sigma)$ is; hence $Z(\Delta)$ is a tree as well. □

In the cases we will examine, the difficult part in showing that $Z(\Sigma)$ is a tree lies in showing that it contains no circuits. The argument for connectedness will always be the same. Therefore, we will record it here.

Lemma 3.1.3. If Z is connected, then $Z(\Sigma)$ is connected.

Proof. Suppose Z is connected, and let p, q be any two points in $Z(\Sigma)$. Suppose $p \in Z(C)$ and $q \in Z(D)$ for chambers C and D in Σ. Let $\Gamma : C = C_0, C_1, \ldots, C_m = D$ be a gallery from C to D in Σ of type $(t_1, \ldots, t_m)$ with $t_i \in S$ for $1 \leq i \leq m$. This means that $Z(C_{j-1})$ and $Z(C_j)$ are glued along a copy of Z_{t_j}. Then since each $Z(C_j)$ is connected itself (being isometric to Z), we can take a path from p to q through $Z(C), Z(C_1), \ldots, Z(C_{m-1}), Z(D)$. □

There is another approach to showing that $Z(\Delta)$ is a tree when the realization of any apartment is a tree that does not rely on the CAT(0) property. It uses the properties of retractions. We include a couple additional assumptions that will always be satisfied in the cases we study.

Lemma 3.1.4. Let Δ be a building of type (W, S). Suppose that the following three conditions hold:

(i) Z is a tree.

(ii) $Z(W, S)$ is a tree.

(iii) For all $s \in S$, $Z_s = Z$ or Z_s is an endpoint of Z.

Then $Z(\Delta)$ is a tree.

Proof. First, note that $Z(\Sigma)$ is isometric to $Z(W, S)$ for any apartment Σ of Δ. Therefore, (ii) implies that $Z(\Sigma)$ is a tree for any apartment Σ.

In order for $Z(\Delta)$ to be a tree, we must show that it is connected and contains no circuits. Connectedness is now immediate from Lemma 3.1.1.

Suppose that there is a circuit Γ in $Z(\Delta)$. We may assume, without loss of generality, that Γ is a simple closed curve. Given any two chambers C and D of Δ, there exists an apartment Σ containing both. Therefore $Z(C)$ and $Z(D)$ both lie in $Z(\Sigma)$, which is a tree, so there can be no circuit lying in $Z(C) \cup Z(D)$. Hence any circuit must pass through at least 3 distinct Z-chambers. Suppose that Γ begins and ends at $x \in Z(C_0)$ and passes consecutively through $Z(C_0), Z(C_1), \ldots, Z(C_{m-1}), Z(C_m) = Z(C_0)$, where $Z(C_{i-1}) \neq Z(C_i)$ for $1 \leq i \leq m$. We may assume, again without loss of generality, that x is an interior point of $Z(C_0)$, i.e. x is not an endpoint of $Z(C_0)$.

Now choose some apartment Σ_0 containing the chamber C_0, and let $\rho = \rho_{\Sigma_0, C_0}$ be the retraction from Δ to Σ_0 centered at C_0. This induces a map $\tilde{\rho} : Z(\Delta) \to Z(\Sigma_0)$. By Proposition 3.1.3, $\tilde{\rho}$ is distance-decreasing and preserves the distance from $Z(C_0)$. We show that $\tilde{\rho}(\Gamma)$ is a closed path in $Z(\Sigma_0)$, which yields a contradiction.

Let y be the point in $Z(C_0) \cap Z(C_1) \cap \Gamma$ where Γ passes from the interior of $Z(C_0)$ to the interior of $Z(C_1)$, and let z be the point in $Z(C_0) \cap Z(C_{m-1})\Gamma$ where Γ passes from the interior of $Z(C_{m-1})$ to the interior of $Z(C_0)$. These points are uniquely defined by requiring condition (iii). Indeed (iii) implies that distinct Z-chambers in $Z(\Delta)$ intersect only at endpoints. Denote the segments from x to y and z to x by $[x, y]$ and $[z, x]$, respectively. Since Z is a tree, it is uniquely geodesic, so these segments are unique. We know that $y \neq z$ since Γ is has no backtracking. Indeed, if $y = z$, then $[x, y] = [z, x]$ since Z is uniquely geodesic.

Note that $\tilde{\rho}$ fixes the segments $[x, y]$ and $[z, x]$ since these segments lie entirely in $Z(C_0)$. Therefore the image of Γ under $\tilde{\rho}$ is a closed path in $Z(\Sigma)$ passing from x to y and then to z and back to x without intersecting either $[x, y]$ or $[z, y]$ in between. This is due to the fact that Γ has no backtracking and so never intersects these edges itself and $\tilde{\rho}$ is distance-preserving from points in $Z(C_0)$, so it fixes $Z(C_0)$ pointwise and sends points outside $Z(C_0)$ to other points outside $Z(C_0)$. Therefore, a circuit exists in $Z(\Sigma_0)$, contradicting the fact that $Z(\Sigma_0)$ is a tree. Thus, no circuits exist in $Z(\Delta)$.

□

3.1.7 The Davis realization

In this subsection, we discuss briefly the Davis realization of a building as introduced in Example 3.1.1. It is the most important example of a Z-realization of a building currently in the literature, and the construction of the Davis realization strongly influenced our construction of certain Z-realizations in Chapter 7.

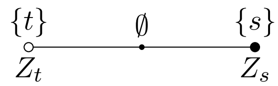
Let us recall the definition.

Definition 3.1.8. Let (W, S) be a Coxeter system, and let $\mathcal{S}$ be the set of spherical subsets of S. Then $Z = |K(\mathcal{S})|$ and $Z_s = |K(\mathcal{S}_{\geq s})|$ are the geometric realizations of the flag complexes on $\mathcal{S}$ and $\mathcal{S}_{\geq s}$, respectively.

Note that the dimension of the Davis realization is equal to the maximum cardinality of a spherical subset of S. In general, $dim(Z(\Delta)) \leq dim(\Delta)$, and the dimensions can be arbitrarily far apart. For example, if $|S| = n$, and $m(s, t) = \infty$ for all $s \neq t$ in S, then $dim(Z(\Delta)) = 1$ and $dim(\Delta) = n - 1$. Now we give a few examples of the Davis realization.

Example 3.1.2.

(1) Let $W = D_\infty$ with generators s and t. Then the set of spherical subsets of S is $\mathcal{S} = \{\emptyset, \{s\}, \{t\}\}$. Therefore Z is the flag complex



where the s-panel Z_s is the vertex corresponding to s, and the t-panel Z_t is the vertex corresponding to t. In this case, $Z(W,S)$, like the Coxeter complex $\Sigma(W,S)$, is a line as seen in Figure 3.3 below. The only difference is that $Z(W,S)$ has more cells because of the point in Z corresponding to $\emptyset$.

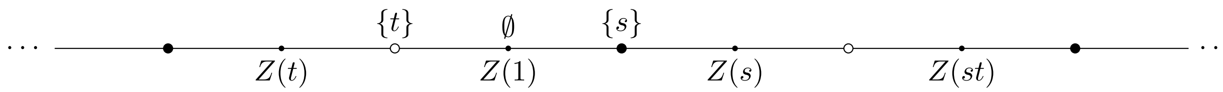
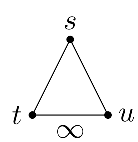


Figure 3.3: Davis realization of an apartment $Z(D_\infty, \{s,t\})$

Note that $Z(s)$ and $Z(st)$ intersect at a copy of Z_t since $\delta(s, st) = t$.

An arbitrary building Δ of type $(D_\infty, \{s,t\})$ is an infinite tree with no endpoints. The same is true for $Z(\Delta)$, albeit with extra cells once again.

Example 3.1.3. Let (W,S) be the Coxeter system with Coxeter diagram



This is an important example moving forward. Here $\mathcal{S} = \{\emptyset, \{s\}, \{t\}, \{u\}, \{s,t\}, \{s,u\}\}$. Therefore Z is the flag complex in Figure 3.4. Note that the bottom edge is $Z_s = |K(\mathcal{S}_{\geq s})|$, the left edge is $Z_t = |K(\mathcal{S}_{\geq t})|$, and the right edge is $Z_u = |K(\mathcal{S}_{\geq u})|$.

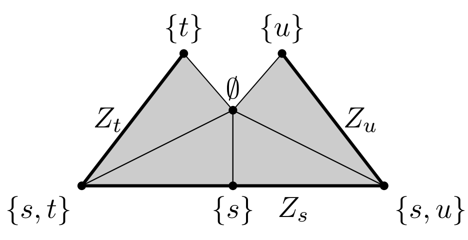


Figure 3.4: $Z = |K(\mathcal{S})|$ with panels Z_s, Z_t, Z_u labeled

Now we show how this extends to $Z(W,S)$ in Figure 3.5. We have labeled a few Z-chambers as well as shown how the copies of Z glue together. Additionally, one can note the "tree-like" skeleton branching through the interior of the hexagons. This can actually be defined as a Z-realization itself by choosing Z to be the s-panel $|K(\mathcal{S}_{\geq\{s\}})|$ in the current example; this skeleton will is the complex to which we can apply Gandini's theorem.

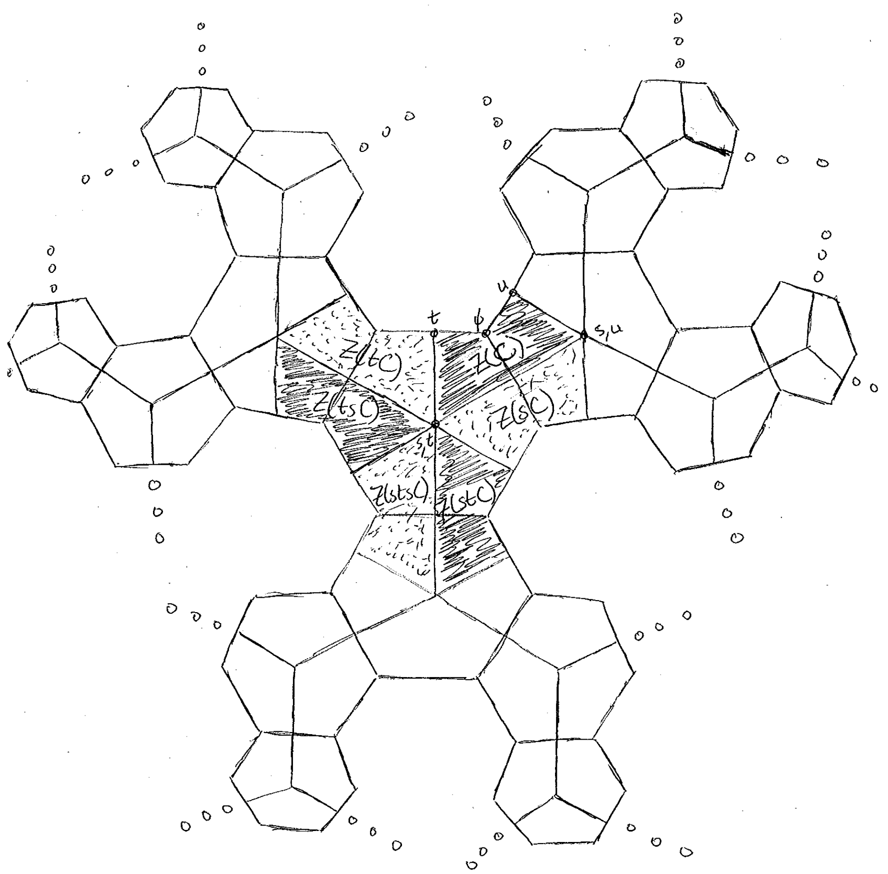


Figure 3.5: $Z(W, S)$

Moussong originally proved that the realization $Z(\Sigma)$ of any apartment Σ is a CAT(0) space. See Theorem 14.1 in [Mou88] for the original proof or Section 12.3 of [Dav08] for a more recent treatment. Gromov [Gro87] proved this in the case that (W, S) is right-angled, i.e. the generators either commute or have no relations between them, by showing that the links of vertices are CAT(1), and Moussong was then able to generalize this result for any Coxeter system.

Davis was then able to prove the following result (Theorem 11.1, [Dav98]) for any building.

Theorem 3.1.1. For any building Δ, its Davis realization $Z(\Delta)$ is a complete CAT(0) space.

This result implies that any building can be realized as a contractible space, even the spherical ones, via the Davis realization. Davis's proof primarily relies upon Moussong's result for apartments and the fact that apartments are retracts of buildings. For a brief discussion of the Davis realization and some applications of it satisfying the CAT(0) property, see Section 12.4 of [AB08].

Chapter 4

Groups with a BN-pair

Thus far, we have mostly discussed the intrinsic properties of buildings, both combinatorial and geometric. However, there is rich group theory that goes along with the theory of buildings. In this chapter, we will discuss a class of groups G for which we can construct an associated building Δ on which G acts by automorphisms.

We will work backwards by first assuming that G already acts "nicely" on a building. This imposes certain conditions on G that will be used as axioms for the class of groups we seek.

4.1 Group actions on buildings

Assume throughout this section that Δ is a simplicial building of type (W, S). First, we must decide what it means for G to act "nicely" on Δ. Of course, we want G to act by simplicial automorphisms. However, this does not preserve enough of the structure of Δ. In particular, the Weyl distance is not preserved. In order to preserve the Weyl distance between chambers, we require the action to be type-preserving. This ensures that galleries are sent to galleries of the same type and hence preserves Weyl distance. Similar to the action of a Coxeter group W on the Coxeter complex $\Sigma(W, S)$, we want the action of G to be transitive on the chambers of Δ. However, this is not quite strong enough for our purposes. We will describe two ways to strengthen chamber transitivity to provide a richer theory. The first arises naturally when one thinks of simplicial buildings and apartment systems.

Definition 4.1.1. Suppose $\mathcal{A}$ is a G-invariant system of apartments for Δ. Then the G-action on Δ is *strongly transitive* if G acts transitively on the set of pairs (Σ, C) consisting of an apartment $\Sigma \in \mathcal{A}$ and a chamber $C \in \Sigma$.

Remark 4.1.1. There are two other equivalent ways to define a strongly transitive action under the same assumptions:

(1) The action is transitive on chambers and the stabilizer of a given chamber C is transitive on the set of apartments in $\mathcal{A}$ containing C.

(2) The action is transitive on $\mathcal{A}$ and the stabilizer of a given apartment $\Sigma \in \mathcal{A}$ is transitive on $\mathcal{C}(\Sigma)$.

There is another notion of transitivity of the action of G on Δ. This is better described in the W-metric situation, so let $(\mathcal{C}, \delta)$ be the corresponding W-metric building where $\mathcal{C} := \mathcal{C}(\Delta)$, δ is the Weyl distance function, and G acts by isometries on $\mathcal{C}$. That is,

$$\delta(gC, gD) = \delta(C, D)$$

for all chambers $C, D \in \mathcal{C}$ and all $g \in G$.

Definition 4.1.2. We say that the action of G on Δ is *Weyl transitive* if for any $w \in W$, the action is transitive on the set $\{(C, D) | \delta(C, D) = w\}$. Equivalently, the action is chamber transitive and the stabilizer of a given chamber C acts transitively on the *w-sphere*

$$\{D \in \mathcal{C} | \delta(C, D) = w\}.$$

One advantage of Weyl transitivity is that it has no reference to an apartment system.

Proposition 6.14 in [AB08] states the relationship between strong transitivity and Weyl transitivity.

Proposition 4.1.1. The following conditions are equivalent:

(1) The G-action on Δ is strongly transitive with respect to some apartment system.

(2) The G-action on Δ is Weyl transitive, and there is an apartment Σ (in the complete system of apartments) such that the stabilizer of Σ acts transitively on $\mathcal{C}(\Sigma)$.

The direction (1) implies (2) follows from the fact that a chamber-transitive G-action is Weyl transitive if and only if $\Delta = \bigcup_{b \in B} b\Sigma$, which is always true if the action is strongly transitive. If (2) holds, then the action is strongly transitive on the set $G\Sigma$, which happens to be an apartment system.

Remark 4.1.2. If Δ is a spherical building, then the notions of strong transitivity and Weyl transitivity are equivalent. However, we will mainly be interested in the non-spherical situation where strong transitivity is the stronger condition.

Assume now that G acts strongly transitively on Δ, and choose an arbitrary pair (Σ, C) as in the definition. We will refer to Σ as the *fundamental apartment* and C as the *fundamental chamber*. We now introduce three important subgroups of G:

$$
\begin{aligned}
B :=& \{g \in G | gC = C\}, \\
N :=& \{g \in G | g\Sigma = \Sigma\}, \\
T :=& \{g \in G | g \text{ fixes } \Sigma \text{ pointwise}\}.
\end{aligned}
$$

By identifying Σ with $\Sigma(W, S)$, we may view W as the group of type-preserving automorphisms of Σ. In this case, let $f : N \to W$ be the homomorphism induced by the action of N on Σ. Note that $T = \ker(f)$, so $T \trianglelefteq N$. Now we show that f is surjective. Since G acts strongly transitively on Δ, the stabilizer of Σ, N acts transitively on $\mathcal{C}(\Sigma)$. Therefore, given $w \in W$, there exists $n \in N$ such that $nC = wC$. Since n, w are both type-preserving, they agree pointwise on C and hence $f(n) = w$ by the standard uniqueness argument. Thus, f is surjective, and $W \cong N/T$. Lastly, $T = B \cap N$ since if $n \in B \cap N$, n fixes C pointwise and thus must act trivially on Σ by the standard uniqueness argument. Figure 4.1 summarizes the notation.

Remark 4.1.3. The reason for the letters B, N, T, W being used here comes from the theory of algebraic groups where B is a Borel subgroup, T is a maximal torus, N is the normalizer of T, and W is the Weyl group.

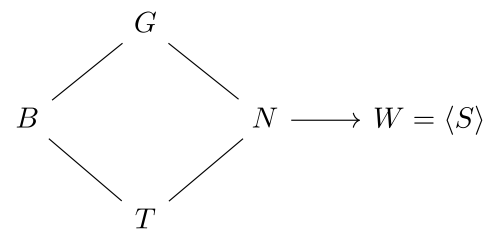


Figure 4.1: BN data

Example 4.1.1. Let P be the *projective plane* over a field k. That is, P consists of "points" and "lines" and an incidence relation satisfying certain axioms (see Definition 4.18, [AB08]). In this case, the points correspond to 1-dimensional subspaces of the vector space k^3, the lines correspond to 2-dimensional subspaces of k^3, and incidence is given by inclusion of subspaces.

Let Δ be the flag complex over P. Then Δ is a spherical rank 2 building with one vertex for each proper nonzero subspace and one edge for each 1-dimensional subspace contained in a 2-dimensional subspace. Since Δ is spherical, it has a unique system of apartments. In this case, there is one apartment for each triple $\{L_1, L_2, L_3\}$ of 1-dimensional subspaces such that $k^3 = L_1 \oplus L_2 \oplus L_3$. That is, $\{L_1, L_2, L_3\} = \{\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle\}$ for some basis $\{e_1, e_2, e_3\}$ of k^3. The corresponding apartment is the subcomplex seen in Figure 4.2.

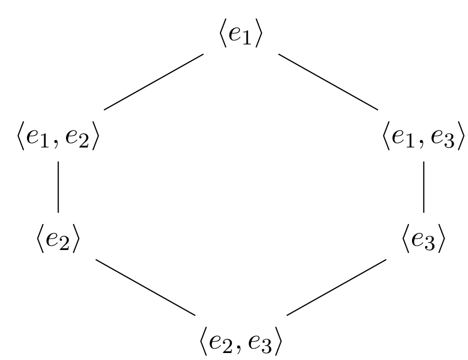


Figure 4.2: An apartment of Δ

Define the fundamental apartment Σ to be associated to the standard basis of k^3, and define the fundamental chamber C in Σ to be the edge $\langle e_1 \rangle - -\langle e_1, e_2 \rangle$. Let $G = \mathrm{GL}_3(k)$

be the group of automorphisms of k^3. Then any $g \in G$ takes subspaces to subspaces and induces a type-preserving automorphism of Δ. This action is strongly transitive.

The group B consists of all automorphisms of k^3 that leave $\langle e_1 \rangle$ and $\langle e_1, e_2 \rangle$ invariant; hence B is the upper triangular group

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

The subgroup N permutes the subspaces $\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle$. Therefore N is the monomial group consisting of matrices with exactly one nonzero element in each row and each column. The subgroup T fixes Σ pointwise so must send $\langle e_i \rangle$ to $\langle e_i \rangle$ for $i = 1, 2, 3$. Therefore T is the group of diagonal matrices. The group $W = N/T$ is isomorphic to the symmetric group on 3 letters, S_3. Equivalently, W can be identified as the set of permutation matrices in G, the matrices with exactly one 1 in each row and column and all other entries 0. Lastly, W can be seen to be generated by the set S consisting of the permutation matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

There are several nice consequences about the structure of G resulting from a strongly transitive action. The following is a combination of Proposition 6.3 and Corollary 6.5 in [AB08].

Proposition 4.1.2. Suppose a group G acts strongly transitively on a building Δ with respect to a system of apartments $\mathcal{A}$.

(1) Then $G = BNB$, where B and N are the subgroups defined above. In particular, $G = \langle B, N \rangle$.

(2) $\Delta = \bigcup_{b \in B} b\Sigma$.

Now we will study some further conditions imposed on the group G by requiring strong transitivity. Let C, Σ be the fundamental chamber and apartment and B, N, T, and W be defined as above. Since G acts transitively on the set of chambers $\mathcal{C} = \mathcal{C}(\Delta)$ and B is the stabilizer in G of C, it follows that there is a bijection between the set of cosets G/B and $\mathcal{C}$ given by $gB \mapsto gC$. Now note that Weyl transitivity, which we have due to strong transitivity, implies that B-orbits in $\mathcal{C}$ are in bijection with the elements of W via the map $B.gC \mapsto \delta(C, gC)$. This follows from the definition of Weyl transitivity stating that the action of G is chamber transitive and that B is transitive on the w-sphere $\{D \in \mathcal{C} | \delta(C, D) = w\}$. Putting these two bijections together, we get a bijection from the set of double cosets $B\backslash G/B$ to W via $BgB \mapsto \delta(C, gC)$.

We can say a bit more when we have strong transitivity. If $\delta(C, gC) = w$, then the double coset BgB from this bijection is simply the double coset $B\tilde{w}B$, where $\tilde{w} \in N$ is some lift of w. Notationally, we will refer to this double coset as BwB from now on. The result is the *Bruhat decomposition* of G.

Theorem 4.1.1. Suppose that G acts strongly transitively on Δ and let B be the stabilizer of a chamber C. Then

$$G = \coprod_{w \in W} BwB.$$

In particular, $G = \langle B, N \rangle$.

Going back to Example 4.1.1, let $G = \mathrm{GL}_3(k)$. Then the Bruhat decomposition becomes

$$\mathrm{GL}_3(k) = \coprod_{w \in W} BwB$$

where B is the group of upper triangular matrices and w ranges over the permutation matrices. This holds for $G = \mathrm{GL}_n(k)$ acting on k^n for all other n as well.

Remark 4.1.4. As a historical remark, this partition of G is named the Bruhat decomposition after F. Bruhat, who discovered this decomposition while studying representation theory of complex Lie groups and verified it for the classical groups in 1954 (Theorem 7.1, [Bru54]).

Now we continue to look at group-theoretic consequences of G acting strongly transitively on Δ. We have already discussed the bijection between G/B and $\mathcal{C}$ given by $gB \mapsto gC$. A natural question to ask is: How is s-adjacency between chambers in $\mathcal{C}$ reflected in the group? Suppose that $h \in G$ such that $\delta(C, hC) = s$. Then $C \cap hC = C \cap sC$ is the panel A in C of cotype s. Since h is type-preserving, we know that $hA = A$. Therefore hC is s-adjacent to C if and only if h stabilizes A. We will denote this stabilizer by P_s. It turns out that $P_s = B \cup BsB$, again writing s to mean a lift of s in N. We now state a more general result (Proposition 6.27, [AB08]) for the stabilizer P_J of the face of C of cotype J.

Proposition 4.1.3. Given $J \subset S$, let A be the face of C of cotype J. Then the stabilizer of A in G is

$$P_J := \bigcup_{w \in W_J} BwB = BW_JB.$$

In particular, this union of double cosets is a subgroup of G.

Proof. Recall that simplices are in 1-1 correspondence with residues (1.2.1). Given $g \in G$, it follows that $gA = A$ if and only if gC and C are in the same J-residue. Indeed, $gA = A$ is a face of both gC and C, and two chambers sharing a face of cotype J can be connected by a gallery of type $(s_1, \ldots, s_n)$ where $s_i \in J$ for $1 \leq i \leq n$. Moreover, gC and C lie in the same J-residue if and only if $\delta(C, gC) \in W_J$ if and only if $g \in BwB$ where $w = \delta(C, gC)$. □

Definition 4.1.3. We call the subgroups of the form $P_J = BW_JB$ *standard parabolic subgroups*, and we call the left cosets gP_J *standard parabolic cosets*. The standard parabolic subgroups are exactly the subgroups of G containing B. In general, we call $Q \leq G$ a *parabolic subgroup* of G if it contains a conjugate of B, or, equivalently, if it is conjugate to a standard parabolic subgroup.

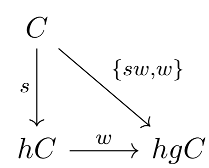
The following results can be proved using the simplicial definitions of buildings as well (see Chapter V, Section 1F, [Bro87]). However, it is more intuitive to use the W-metric approach and Weyl transitivity. Indeed, we can obtain interesting results about products of double cosets from the axioms (WD2) and (WD3) as in Definition 2.2.1. Note first that a product $BgB \cdot BhB$ of double cosets is a set containing the element gh and is closed under left and right multiplication by B. Therefore it is a union of double cosets in $B\backslash G/B$, one of which must be $BghB$. The building structure allows us to say much more than this. The following theorem is Theorem 6.21 in [AB08], with the additional assumption that our action is strongly transitive so that we know the double coset corresponding to $w = \delta(C, gC)$ is exactly BwB.

Theorem 4.1.2. Given $s \in S$ and $w \in W$, we have

$$BswB \subset BsBBwB \subset BswB \cup BwB.$$

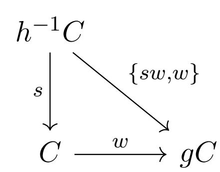
In particular, $BsBBwB$ is either the double coset $BswB$ or the union of two double cosets. In the case that $\ell(sw) = \ell(w) + 1$, $BsBBwB = BswB$.

Proof. Given $h \in BsB$, $g \in BwB$, we want to know which double coset contains the element hg. We know that $\delta(C, hC) = s$ and $\delta(C, gC) = w$; hence $\delta(hC, hgC) = w$ since G acts by isometries. Our goal is then to compute $\delta(C, hgC)$. To do this, we apply axiom (WD2) in the following case:



We conclude that $\delta(C, hgC) = sw$ or w. In addition, if $\ell(sw) = \ell(w) + 1$, then $\delta(C, hgC) = sw$. Translating this back into the language of double cosets, we have $hg \in BswB \cup BwB$, and $hg \in BswB$ if $\ell(sw) = \ell(w) + 1$.

It remains to show that $BswB \subset BsBBwB$. In this case, consider the diagram



obtained through acting on the above diagram by h^{-1}. Then the axiom (WD3) implies that there exists some $h \in BsB$ such that $\delta(h^{-1}C, gC) = sw$. Thus $hg \in BswB$ for this specific choice of h. Therefore $BsBBwB \cap BswB \neq \emptyset$, and hence contains it. $\square$

We continue to assume that we have a strongly transitive action of G on Δ. In the most important applications, including the ones we consider in this thesis, Δ is a *thick* building. This property also has a group-theoretic interpretation.

Proposition 4.1.4. Suppose Δ is thick. For any $s \in S$, $sBs^{-1} \nleq B$.

Proof. Since Δ is thick, there exists a chamber $C' \in \mathcal{C}$ with $C' \neq C, sC$ but $\delta(C, C') = s$. We know that $C' = hC$ for some $h \in BsB$. Since $hC \neq sC$, we know that $h \notin sB$. Therefore $BsB \not\subset sB$. Equivalently, $sB \not\subset Bs$, or $sBs^{-1} \not\subset B$. $\square$

4.1.1 BN-pairs and the associated building

Now that we have collected this group-theoretic data corresponding to a strongly transitive action on a thick building, we compile it into two axioms. This allows us to reconstruct Δ from only the group data.

Definition 4.1.4. We say that a pair of subgroups B and N of a group G is a *BN-pair* if B and N generate G, the intersection $T := B \cap N$ is normal in N, and the quotient $W := N/T$ admits a set of generators S such that the following two conditions hold:

(BN1) For $s \in S$, $w \in W$,

$$sBw \subset BswB \cup BwB.$$

(BN2) For $s \in S$,

$$sBs^{-1} \nleq B.$$

The group W will be called the *Weyl group* associated to the BN-pair. One also says that the quadruple (G, B, N, S) is a *Tits system*.

Remark 4.1.5. For all the properties that we have seen, it turns out that only these two axioms are needed. Indeed, the following statements all follow from (BN1) and (BN2):

(1) S consists of elements of order 2 and (W, S) is a Coxeter system.

(2) $B \cup BsB$ is a subgroup of G for every $s \in S$.

(3) BW_JB is a subgroup of G for all $J \subset S$.

(4) $G = \coprod_{w \in W} BwB$.

(5) $BsBwB = BswB$ if $\ell(sw) = \ell(w) + 1$.

(6) $BsBwB = BwB \cup BswB$ if $\ell(sw) = \ell(w) - 1$.

For details on how the two axioms imply these statements, see Chapter V, Section 2A in [Bro89].

Now we describe the building associated to a BN-pair. We have already described its chambers through the bijection between G/B and $\mathcal{C}$. The key result in reconstructing the building Δ from the group G is the data regarding stabilizers in Proposition 4.1.3. The construction then looks very similar to the construction of a Coxeter complex.
Define $\Delta(G, B)$ to be the poset $\{gP_J | g \in G, J \subset S\}$ of standard parabolic cosets, where $P_J = BW_JB$, ordered by reverse inclusion. The chambers are exactly the elements gB for some $g \in G$, with fundamental chamber corresponding to B. The faces of the fundamental chamber are then the standard parabolic subgroups. Note that the definition of $\Delta(G, B)$ depends only on (G, B) and not on N. However, the subgroup N allows us to define an apartment system of $\Delta(G, B)$ on which G acts strongly transitively. Indeed, define the fundamental apartment to be $\Sigma := \{wP_J | w \in W, J \subset S\}$. Then the chambers of Σ is the set $\mathcal{C}(\Sigma) = \{wB | w \in W\}$, and the set $\mathcal{A} = \{g\Sigma | g \in G\}$ is a system of apartments on which G acts strongly transitively. Now let us state that $\Delta(G, B)$ really is a simplicial complex and a building as we desire.

Lemma 4.1.1. The poset $\Delta(G, B)$ is a simplicial complex.

Proof. To show that $\Delta(G, B)$ is a simplicial complex, we must exhibit a greatest lower bound for any two elements, and we must show that for any element A, the set of faces $\Delta(G, B)_{\leq A}$ is ismorphic to the set of subsets of a finite set.
First, take any two elements gP_J, hP_K in $\Delta(G, B)$, where $g, h \in G$, $J, K \subset S$. A greatest lower bound would be the smallest left coset containing these two since our ordering is by reverse inclusion. The smallest left coset containing these two is gP where $P = \langle P_K, P_J, g^{-1}h \rangle$. Furthermore, we know that $P = P_L$ for some $L \subset S$ since P contains B and hence is a standard parabolic subgroup.

For the second condition, note that we have the poset isomorphism $J \mapsto P_J$ between the subsets of S and the set of standard parabolic subgroups. If $A = P_J$ is a face of the fundamental chamber, then $\Delta(G, B)_{\leq A}$ is the poset of standard parabolic subgroups P_K containing P_J ordered by reverse inclusion. This is isomorphic to the set of subsets of S containing J ordered by reverse inclusion. Lastly, this is isomorphic to the set of subsets of the finite set $S \backslash J$ ordered by inclusion, where this isomorphism is given by taking the complement. That is, for $K \supset J$, $K \mapsto S \backslash K \subset S \backslash J$. □

We will merely state that $\Delta(G, B)$ is a thick building. For full details, see the first theorem in V.3 of [Bro89].

Theorem 4.1.3. The simplicial complex $\Delta(G, B)$ is a thick building of type (W, S) with a type-preserving G-action that is strongly transitive with respect to the set of apartments $\mathcal{A} = G\Sigma$, where $\Sigma = \{wP_J | w \in W, J \subset S\}$.

Remark 4.1.6. We can define $\Delta(G, B)$ as a building in the W-metric sense as well. In this case, $\mathcal{C} = G/B$ and $\delta : G/B \times G/B \to W$ is defined by $\delta(gB, hB) = w$ if and only if $g^{-1}h \in BwB$.

The following theorem (Theorem 6.56, [AB08]) gives the main result connecting a strongly transitive action on a thick building to a BN-pair in G.

Theorem 4.1.4. (1) Given a BN-pair in G, the generating set S is uniquely determined, and (W, S) is a Coxeter system. There is a thick building $\Delta = \Delta(G, B)$ that admits a strongly transitive G-action such that B is the stabilizer of a fundamental chamber and N stabilizes a fundamental apartment and is transitive on its chambers.

(2) Conversely, suppose a group G acts strongly transitively on a thick building Δ with fundamental apartment Σ and a fundamental chamber C. Let B be the stabilizer of C, and let N be a subgroup of G that stabilizes Σ and is transitive on the chambers of Σ. Then (B, N) is a BN-pair in G, and Δ is canonically isomorphic to $\Delta(G, B)$.

Chapter 5

Twin Buildings

Motivated by Tits' study of Kac-Moody groups over fields [Tit87], Ronan and Tits introduced twin buildings in the late 1980s. Twin buildings can be viewed as generalizations of spherical buildings. There is a rich group theory associated to twin buildings corresponding to groups of Kac-Moody type. Just as spherical buildings arose as a way to study groups of Lie type, twin buildings arose to study groups of Kac-Moody type, which can be thought of as infinite-dimensional generalizations of Lie groups. Although these groups are infinite-dimensional, they are of finite rank, so the buildings involved are still finite-dimensional. We will discuss the group theory associated to twin buildings further in Chapter 6. In this chapter we will collect the necessary notions of twin buildings in order to sufficiently develop the group theory later.

We will use the W-metric approach as introduced by Tits in [Tit92] to define twin buildings. The basic idea is that a twin building is a pair of buildings of the same type, which we will denote $(\mathcal{C}_+, \mathcal{C}_-)$, together with an *opposition relation* between chambers of $\mathcal{C}_+$ and $\mathcal{C}_-$. This opposition relation is similar to the opposition relation between chambers in a spherical building, but it should be noted that the two halves of the twin building, $\mathcal{C}_+$ and $\mathcal{C}_-$, need not be spherical and will never be spherical for us.

5.1 Definition and Basic Examples

Fix an arbitrary Coxeter system (W, S).

Definition 5.1.1. A *twin building* of type (W, S) is a triple $(\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ consisting of two buildings $(\mathcal{C}_+, \delta_+)$ and $(\mathcal{C}_-, \delta_-)$ of type (W, S) together with a *codistance* function

$$\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$$

satisfying the following conditions for each $\epsilon \in \{+, -\}$, any $C \in \mathcal{C}_\epsilon$, and any $D \in \mathcal{C}_{-\epsilon}$, where $w := \delta^*(C, D)$:

(Tw1) $\delta^*(C, D) = \delta^*(D, C)^{-1}$.

(Tw2) If $C' \in \mathcal{C}_\epsilon$ satisfies $\delta_\epsilon(C', C) = s$ with $s \in S$ and $\ell(sw) < \ell(w)$, then $\delta^*(C', D) = sw$.

(Tw3) For any $s \in S$, there exists a chamber $C' \in \mathcal{C}_\epsilon$ with $\delta_\epsilon(C', C) = s$ and $\delta^*(C', D) = sw$.

A twin building is called *thick* if both $\mathcal{C}_+$ and $\mathcal{C}_-$ are thick buildings.

Although there are no connecting galleries between a chamber $C \in \mathcal{C}_+$ and a chamber $D \in \mathcal{C}_-$, there is still some notion of "distance" between these chambers called the *numerical codistance*.

Definition 5.1.2. We define the *numerical codistance* between chambers $C \in \mathcal{C}_\epsilon$ and $D \in \mathcal{C}_{-\epsilon}$ for $\epsilon \in \{+, -\}$ by

$$d^*(C, D) := \ell(\delta^*(C, D))$$

We say that C and D are *opposite*, which we denote by C op D, if $d^*(C, D) = 0$ or, equivalently, if $\delta^*(C, D) = 1$.

Remark 5.1.1. The codistance $d^*(C, D)$ for $C \in \mathcal{C}_+$ and $D \in \mathcal{C}_-$ should be thought of as a measure of how far away C and D are from one another in the absence of connecting galleries. Intuitively, small codistance means C and D are far apart, while large codistance means C and D are close together. If C and D have minimal codistance, then they are opposite and can be thought of as being at maximal distance from one another.

In examining the axioms, one sees that (Tw2) resembles the axiom (WD2) from Definition 2.2.1 but has $\ell(sw) < \ell(w)$ instead of $\ell(sw) > \ell(w)$ as in (WD2). The above discussion explains this: decreasing codistance is thought of as increasing distance. (Tw2) also does not state that $\delta^*(C', D) \in \{sw, w\}$ as in (WD2). We will see in Lemma 5.2.1(1) that this follows from the axioms and does not need to be assumed. The condition cannot be dropped from (WD2), however.

Now we discuss two examples of twin buildings. The first is realizing a spherical building as a twin building, showing that twin buildings are indeed a generalization of spherical buildings. Our second example will show that W can be realized as a twin building. Just as W provided a natural example of a thin building and apartment, W provides a natural example of a thin twin building which will be used as a twin apartment. More interesting examples will be introduced in the next chapter in connection with the group theory behind the development of the theory of twin buildings.

Example 5.1.1.

(1) Assume that W is finite with w_0 the longest element. Let (C, δ) be a building of type (W, S), and define $\mathcal{C}_+$ and $\mathcal{C}_-$ to be disjoint copies of $\mathcal{C}$. For any $C \in \mathcal{C}$, denote by C_ϵ the corresponding chamber in $\mathcal{C}_\epsilon$. Then we define $\delta_\epsilon : \mathcal{C}_\epsilon \times \mathcal{C}_\epsilon \to W$ by

$$\delta_+(C_+, D_+) := \delta(C, D) \text{ and } \delta_-(C_-, D_-) := w_0\delta(C, D)w_0.$$

To obtain the twin building $(\mathcal{C}_+, \mathcal{C}_-, \delta^*)$, we define the codistance by

$$\delta^*(C_+, D_-) := \delta(C, D)w_0 \text{ and } \delta^*(D_-, C_+) := w_0\delta(D, C).$$

Defining the codistance this way means that opposition in the twin building coincides with the usual notion of opposition in the spherical building. Indeed, $\delta^*(C_+, D_-) = \delta(C, D)w_0 = 1$ if and only if $\delta(C, D) = w_0$.

(2) Let W be any Coxeter group, not necessarily spherical, and take two disjoint copies W_+ and W_- of W. For $w \in W$, denote by w_+ and w_- the corresponding elements in

W_+ and W_-, respectively. We then define the distance functions $\delta_\pm$ and codistance δ^*_W by setting

$$\delta_+(v_+, w_+) = \delta_-(v_-, w_-) = \delta^*(v_+, w_-) = \delta^*(v_-, w_+) = v^{-1}w$$

for all $v, w \in W$. The twin building (W_+, W_-, δ^*_W) is the *standard thin twin building* of type (W, S).

5.2 Basic Properties of Twin Buildings

Throughout this section, assume that $(\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ is a twin building of type (W, S) and that $\epsilon \in \{+, -\}$.

Lemma 5.2.1. Given $C \in \mathcal{C}_\epsilon$, $D \in \mathcal{C}_{-\epsilon}$, and $s \in S$, let $w := \delta^*(C, D)$. Then we have:

(1) $\delta^*(C', D) \in \{w, sw\}$ for any $C' \in \mathcal{C}_\epsilon$ with $\delta_\epsilon(C', C) = s$.

(2) If $\ell(sw) > \ell(w)$, there exists precisely one chamber $C' \in \mathcal{C}_\epsilon$ satisfying $\delta_\epsilon(C', C) = s$ and $\delta^*(C', D) = sw$.

As mentioned above, note that (1) follows from the axioms in this case whereas the corresponding statement about W-metric buildings is included in axiom (WD2). We also see that (2) is another way of stating that there is uniqueness of the chamber C' in (Tw3) if $\ell(sw) > \ell(w)$. This mirrors what occurs with buildings where it follows from the axioms that the chamber C' is unique in (WD3) if $\ell(sw) < \ell(w)$ (see Remark 2.2.1).

The following lemma will be useful in the next chapter and builds on the previous lemma. The proof is rather short and displays how one can use the axioms to prove basic properties, so we include it here.

Lemma 5.2.2. Given $C, D \in \mathcal{C}_\epsilon$, and $E \in \mathcal{C}_{-\epsilon}$, let $w := \delta^*(D, E)$.

(1) If Γ is a gallery of type $\mathbf{s} = (s_1, \ldots, s_n)$ from C to D in $\mathcal{C}_\epsilon$, then there exists a subword $(s_{i_1}, \ldots, s_{i_m})$ of $\mathbf{s}$ such that $\delta^*(C, E) = s_{i_1} \cdots s_{i_m} w$.

(2) If $v := \delta_\epsilon(C, D)$ satisfies $\ell(vw) = \ell(w) - \ell(v)$, then $\delta^*(C, E) = vw$.

Proof. (1) follows immediately from Lemma 5.2.1(1) by induction on n.

For (2), we can rewrite $\ell(vw) = \ell(w) - \ell(v)$ as $\ell(w) = \ell(v) + \ell(vw) = \ell(v^{-1}) + \ell(vw)$. Thus there exists a reduced decomposition of w, say $w = s_1 \cdots s_n$ such that some initial segment $s_1 \cdots s_k$ is a reduced decomposition of v^{-1}. Now choose a minimal gallery from C to D of type $(s_k, \ldots, s_1)$. Suppose that C_1 is the chamber s_1-adjacent to D in this gallery. Since $\ell(s_1 w) < \ell(w)$, (Tw2) implies that $\delta^*(C_1, E) = s_1 w$. Repeating this application of (Tw2) as we move backwards through the gallery toward C, we obtain $\delta^*(C, E) = s_k \cdots s_1 w = vw$ as desired. □

The following corollary gives an important condition for discerning whether two chambers are opposite in a twin building and states that any apartment will contain a chamber opposite a given chamber in the other half of the twin building. The first statement follows from Lemma 5.2.2(2). The second statement follows from this condition.

Corollary 5.2.1.

(1) If $C, D \in \mathcal{C}_\epsilon$ and $C' \in \mathcal{C}_{-\epsilon}$ satisfy $\delta_\epsilon(D, C) = \delta^*(D, C')$, then C op C'. That is $\delta^*(C, C') = 1$.

(2) Given any chamber $C' \in \mathcal{C}_{-\epsilon}$, any apartment of $\mathcal{C}_\epsilon$ contains at least one chamber opposite C'.

5.3 Twin apartments

Let $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ be a twin building of type (W, S). A twin apartment is a subpair $\Sigma = (\Sigma_+, \Sigma_-)$ of $\mathcal{C}$ satisfying a certain condition. There are a few equivalent ways of characterizing twin apartments, but we will use the opposition relation in the vein of Corollary 5.2.1(2).

Definition 5.3.1. A *twin apartment* of a twin building $\mathcal{C}$ is a pair $\Sigma = (\Sigma_+, \Sigma_-)$ such that Σ_+ is an apartment of $\mathcal{C}_+$, Σ_- is an apartment of $\mathcal{C}_-$, and every chamber in $\Sigma_+ \cup \Sigma_-$ is opposite precisely one other chamber in $\Sigma_+ \cup \Sigma_-$.

We know from Corollary 5.2.1(2) that there is always at least one chamber in $\Sigma_{-\epsilon}$ opposite any given chamber in $\mathcal{C}_\epsilon$, so a twin apartment requires that there is also at most one. In Figure 5.1, we give a schematic representation of a twin apartment through the example of type D_∞.

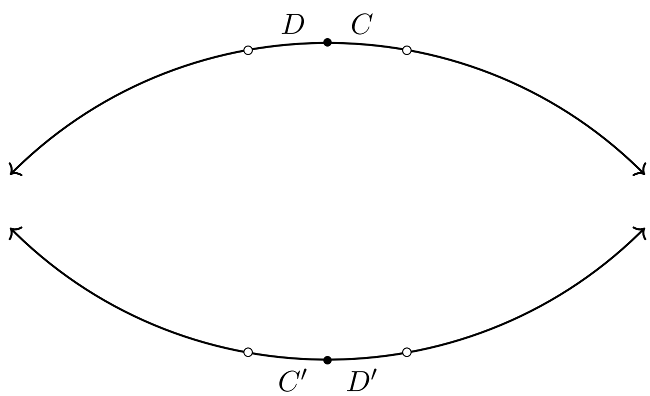


Figure 5.1: A twin apartment of type D_∞ where C op C', D op D'

Definition 5.3.1 also leads to a natural way to define a twin apartment via a generalization of the opposition involution we saw in Coxeter complexes.

Definition 5.3.2. If $\Sigma = (\Sigma_+, \Sigma_-)$ is a twin apartment, then the *opposition involution*, denoted by op_Σ, associates to each chamber $C \in \Sigma_+ \cup \Sigma_-$ the unique chamber $C' = \mathrm{op}_\Sigma(C) \in \Sigma_+ \cup \Sigma_-$ such that C' op C.

Now we gather some useful facts about twin apartments.

Lemma 5.3.1. Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment.

(1) $\mathrm{op}_\Sigma : \Sigma_\epsilon \to \Sigma_{-\epsilon}$ is an isometry. That is, if $C, D \in \mathcal{C}_\epsilon$ such that $\delta_\epsilon(C, D) = w \in W$, then $\delta_{-\epsilon}(\mathrm{op}_\Sigma(C), \mathrm{op}_\Sigma(D)) = w$ as well.

(2) Given $C \in \Sigma_\epsilon$ and $D' \in \Sigma_{-\epsilon}$, set $D := \mathrm{op}_\Sigma(D') \in \Sigma_\epsilon$. Then

$$\delta^*(C, D') = \delta_\epsilon(C, D).$$

(3) Given $C \in \Sigma_\epsilon$ and $w \in W$, there is a unique chamber $D' \in \Sigma_{-\epsilon}$ such that $\delta^*(C, D') = w$.

(4) For any three chambers $C, D, E \in \Sigma_+ \cup \Sigma_-$,

$$\delta(C, E) = \delta(C, D)\delta(D, E)$$

where each δ can be interpreted as δ_+, δ_-, or δ^*, whichever one makes sense in context.

(5) Σ is a thin twin building, isomorphic to the standard thin twin building of type (W, S) as described in Example 5.1.1(2)

To prove the first property, it suffices to show that op_Σ preserves s-adjacency for all $s \in S$. This preservation of s-adjacency is hinted at in Figure 5.1. Properties (3) and (4) will be most useful for us in the next chapter. The third property is a standard property of apartments, which follows from (2). We have seen this statement before in terms of the Weyl distance, and it also holds for the codistance. (4) is the statement that the strong triangle inequality is satisfied in twin apartments, using the codistance or just the Weyl distance in one half of the twin apartment. We have seen this in the case of an ordinary apartment, and property (3) allows us to reduce to that case. The final statement follows from the previous ones and justifies the moniker *standard thin twin building* given to (W_+, W_-, δ^*_W) from Example 5.1.1(2).

We now need one more definition and result before proceeding to the group theory related to twin buildings. We saw in Chapter 2 that an apartment in a spherical building can be defined as the convex hull of any pair of opposite chambers contained in it. This notion extends to twin buildings, although we will not define convexity in the W-metric case.

Definition 5.3.3. Let C and C' be opposite chambers with $C \in \mathcal{C}_\epsilon$. We define

$$\Sigma(C, C') = \{D \in \mathcal{C}_\epsilon | \delta_\epsilon(C, D) = \delta^*(C', D)\}.$$

and then also define

$$\Sigma\{C,C'\} := \begin{cases} (\Sigma(C,C'),\Sigma(C',C)) & \text{if } \epsilon = +, \\ (\Sigma(C',C),\Sigma(C,C')) & \text{if } \epsilon = -. \end{cases}$$

The following proposition is Proposition 5.179(1) in [AB08].

Proposition 5.3.1. If C and C' are opposite chambers, then $\Sigma\{C,C'\}$ is a twin apartment and is in fact the unique twin apartment containing C and C'.

This leads to the definition of certain apartment systems in the buildings $\mathcal{C}_+$ and $\mathcal{C}_-$.

Definition 5.3.4. We define the apartment systems $\mathcal{A}_+$ and $\mathcal{A}_-$ as follows:

$$\mathcal{A}_+ := \{\Sigma(C,C')|C \in \mathcal{C}_+, C' \in \mathcal{C}_-, C \text{ op } C'\},$$
$$\mathcal{A}_- := \{\Sigma(C',C)|C' \in \mathcal{C}_-, C \in \mathcal{C}_+, C \text{ op } C'\}.$$

and the system of twin apartments as

$$\mathcal{A} = \{\Sigma\{C,C'\}|C \in \mathcal{C}_+, C' \in \mathcal{C}_-, C \text{ op } C'\}.$$

Chapter 6

Groups with a twin BN-pair

We saw in Chapter 4 that a strongly transitive action of a group G on a building corresponds to certain group data called a BN-pair. It turns out that there is a corresponding notion of a strongly transitive action on a twin building that results in a"twin BN-pair". The definition of strongly transitive action from a single building does not directly translate to the twin building situation, so we must first decide what we should mean by strongly transitive in this context.

6.1 Group actions on twin buildings

In this section, $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ is a twin building of type (W, S), and G is a group.

Definition 6.1.1. We say that G *acts* on $\mathcal{C}$ if it acts on both $\mathcal{C}_+$ and $\mathcal{C}_-$ and preserves δ_+, δ_-, and δ^*. That is, for all $g \in G$, all $C, C' \in \mathcal{C}_+$, and all $D, D' \in \mathcal{C}_-$, we have

(1) $\delta_+(gC, gC') = \delta_+(C, C')$

(2) $\delta_-(gD, gD') = \delta_-(D, D')$

(3) $\delta^*(gC, gD) = \delta(C, D)$

Now we want to define the analogues to B, N, and T in the case of an action on a twin building. Assume that G acts on $\mathcal{C}$, and fix a fundamental pair of chambers (C_+, C_-) where $C_+ \in \mathcal{C}_+$, $C_- \in \mathcal{C}_-$, and C_+ op C_-. Then we define $B_\pm$ to be the stabilizer of the chamber

$C_\pm$:

$$B_+ := \{g \in G | gC_+ = C_+\},$$
$$B_- := \{g \in G | gC_- = C_-\}.$$

The choice of (C_+, C_-) yields a fundamental twin apartment, namely $\Sigma = (\Sigma_+, \Sigma_-) = \Sigma\{C_+, C_-\}$, the unique twin apartment containing the pair (C_+, C_-) as defined in Definition 5.3.3. This leads to the subgroups

$$N := \{g \in G | g\Sigma = \Sigma\},$$
$$T := \{g \in G | gC = C \text{ for all } C \in \Sigma_+ \cup \Sigma_-\}.$$

Lemma 6.1.1. [Lemma 6.70, [AB08]] The following conditions are equivalent:

(i) For any $w \in W$, G acts transitively on

$$\{(C, D) \in \mathcal{C}_+ \times \mathcal{C}_- | \delta^*(C, D) = w\}.$$

(ii) For $\epsilon = +$ or $-$, G acts transitively on $\mathcal{C}_\epsilon$, and B_ϵ acts transitively on $\{D \in \mathcal{C}_\epsilon | \delta^*(C_\epsilon, D) = w\}$ for each $w \in W$.

(iii) G acts transitively on $\{(C, C') \in \mathcal{C}_+ \times \mathcal{C}_- | C \text{ op } C'\}$.

(iv) G acts transitively on $\mathcal{A}$, and N acts transitively on

$$\{(C, C') \in \Sigma_+ \times \Sigma_- | C \text{ op } C'\}.$$

(v) For $\epsilon = +$ or $-$, G acts transitively on $\mathcal{A}_\epsilon$, and N acts transitively on Σ_ϵ.

Definition 6.1.2. We say that the action of G on $\mathcal{C}$ is *strongly transitive* if it satisfies the five equivalent conditions of Lemma 6.1.1.

Corollary 6.1.1. Suppose that G acts strongly transitively on $\mathcal{C}$.

(1) G acts strongly transitively on the building $\mathcal{C}_\epsilon$ with respect to the apartment system $\mathcal{A}_\epsilon$.

(2) If $\mathcal{C}$ is thick, then (B_+, N) and (B_-, N) are BN-pairs in G with common Weyl group $W \cong N/T$.

6.2 Group-theoretic consequences

We now follow the same path as in Chapter 4 and examine some of the conditions imposed on G by a strongly transitive action on a twin building $\mathcal{C}$. First, the form of strong transitivity in Lemma 6.1.1(ii) implies that B_ϵ-orbits in $\mathcal{C}_{-\epsilon}$ are in 1-1 correspondence with the elements of W, with the orbit of a chamber $D \in \mathcal{C}_{-\epsilon}$ corresponding to $w = \delta^*(C_\epsilon, D)$. However, we know from our work in Chapter 4 that chambers in $\mathcal{C}_{-\epsilon}$ are in 1-1 correspondence with the cosets in $G/B_{-\epsilon}$. Therefore we get a bijection

$$B_\epsilon \backslash G / B_{-\epsilon} \to W,$$

given by

$$B_\epsilon g B_{-\epsilon} \mapsto \delta^*(C_\epsilon, gC_{-\epsilon})$$

for $g \in G$. The inverse is given by

$$w \mapsto B_\epsilon w B_{-\epsilon}$$

for $w \in W$, where we define the double coset $B_\epsilon w B_{-\epsilon} := B_\epsilon n B_{-\epsilon}$ if $n \in N$ is some lift of w. This leads to the analogue of the Bruhat decomposition called the *Birkhoff decomposition.*

Proposition 6.2.1. If a group G acts strongly transitively on a twin building, then

$$G = \coprod_{w \in W} B_\epsilon w B_{-\epsilon}$$

for each $\epsilon \in \{+, -\}$. Given $g \in G$ and $w \in W$, we have

$$g \in B_\epsilon w B_{-\epsilon} \iff \delta^*(\mathcal{C}_\epsilon, g\mathcal{C}_{-\epsilon}) = w.$$

Since the double cosets B_+sB_- and B_+B_- are different, we get the following corollary which will be an axiom for a twin BN-pair:

Corollary 6.2.1. If G acts strongly transitively on a twin building, then

$$B_+s \cap B_- = \emptyset$$

for all $s \in S$.

The following proposition makes use of the properties of δ^*, specifically those in Lemma 5.2.1 and axiom (Tw2). See Proposition 6.77 in [AB08] for the proof utilizing these properties.

Proposition 6.2.2. Assume that G acts strongly transitively on $\mathcal{C}$. For $\epsilon \in \{+,-\}, w \in W, s \in S$, we have:

(1) $B_\epsilon sB_\epsilon wB_{-\epsilon} \subset B_\epsilon\{sw, w\}B_{-\epsilon}$

(2) If $\ell(sw) < \ell(w)$, then $B_\epsilon sB_\epsilon wB_{-\epsilon} = B_\epsilon swB_{-\epsilon}$.

(3) If $\mathcal{C}$ is thick and $\ell(sw) > \ell(w)$, then

$$B_\epsilon sB_\epsilon wB_{-\epsilon} = B_\epsilon\{sw, w\}B_{-\epsilon}.$$

6.3 Twin BN-pairs and the twin building $\mathcal{C}(G, B_+, B_-)$

Now that we have seen that a strongly transitive action of a group G on a twin building imposes certain conditions on G, we reverse the process and provide axioms on a group G from which we can construct a strongly transitive action on a twin building. Since all relevant examples for us will be actions on thick twin buildings, we will only treat the thick case here.

Definition 6.3.1. Let B_+, B_-, and N be subgroups of a group G such that $B_+ \cap N = B_- \cap N =: T$. Assume that $T \trianglelefteq N$, and set $W := N/T$. The triple (B_+, B_-, N) is called a *twin BN-pair* with Weyl group W if W admits a set S of generators such that the following conditions hold for all $w \in W$ and $s \in S$ and each $\epsilon \in \{+,-\}$:

(TBN0) (G, B_ϵ, N, S) is a Tits system as in Definition 4.1.4.

(TBN1) If $\ell(sw) < \ell(w)$, then $B_\epsilon s B_\epsilon w B_{-\epsilon} = B_\epsilon sw B_{-\epsilon}$.

(TBN2) $B_+ s \cap B_- = \emptyset$.

In this situation, we also say that (G, B_+, B_-, N, S) is a *twin Tits system.* The twin BN-pair is said to be *saturated* if $T = B + \cap B_-$.

Now that we have the axioms for a twin BN-pair or twin Tits system, we want to know how to construct a twin building on which a group G with a twin BN-pair acts. Given axiom (TBN0), we know how to construct $\mathcal{C}_\epsilon$ and δ_ϵ already from the discussion in Chapter 4. That is, $\mathcal{C}_\epsilon = G/B_\epsilon$, and $\delta_\epsilon(gB_\epsilon, hB_\epsilon) = w$ if and only if $g^{-1}h \in B_\epsilon w B_\epsilon$ for $g, h \in G$ and $w \in W$. The only thing that remains is to describe the codistance function δ^* to construct the twin building. This utilizes the Birkhoff decomposition:

$$\delta^*(gB_\epsilon, hB_{-\epsilon}) = w \iff g^{-1}h \in B_\epsilon w B_{-\epsilon}$$

for $g, h \in G$ and $w \in W$.

Definition 6.3.2. Given a group G with twin BN-pair (B_+, B_-, N) and associated Coxeter system (W, S), the (thick) twin building of type (W, S) constructed above with $\mathcal{C}_\epsilon = G/B_\epsilon$ will be called the *twin building associated to* (B_+, B_-, N) and denoted by $\mathcal{C}(G, B_+, B_-)$.

Remark 6.3.1. As with the building associated to a BN-pair described in Chapter 4, the definition does not depend on N although we can describe the apartments Σ_ϵ in the fundamental twin apartment using N by $\Sigma_\epsilon = \{n\mathcal{C}_\epsilon | n \in N\}$.

The following theorem gathers the main results of this section relating twin BN-pairs and groups acting strongly transitively on a twin building.

Theorem 6.3.1. [Theorem 6.87, [AB08]]

(1) Let G be a group that acts strongly transitively on a thick twin building $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ of type (W, S). Let $(C_+, C_-) \in \mathcal{C}_+ \times \mathcal{C}_-$ be a pair of opposite chambers, and let $\Sigma = \Sigma\{C_+, C_-\}$ be the associated twin apartment. Then, if we denote by B_+, B_-,

and N the stabilizers of C_+, C_-, and Σ in G, the triple (B_+, B_-, N) is a saturated BN-pair in G with Weyl group W. The twin building $\mathcal{C}(G, B_+, B_-)$ associated to this twin BN-pair is canonically isomorphic to $\mathcal{C}$.

(2) Let (G, B_+, B_-, N, S) be a twin Tits system with Weyl group W. Then $\mathcal{C}(G, B_+, B_-)$ is a thick twin building of type (W, S) on which G acts strongly transitively. If we set $C_+ := B_+$, $C_- = B_-$, and $\Sigma := \Sigma\{C_+, C_-\}$, then we recover B_ϵ as the stabilizer of C_ϵ in G.

We quickly mention one example of a group with a twin BN-pair before describing a large family of examples in the next section.

Example 6.3.1. The group $G := \mathrm{SL}_n(k[t, t^{-1}])$ has a twin BN-pair. For more details, see Section 6.12 of [AB08].

6.4 Groups with an RGD system

In this section, we will introduce a rich family of groups that have twin BN-pairs and therefore act strongly transitively on a twin building. These are groups with an RGD system, where RGD stands for "root group data". A group with an RGD system has even more structure than a group with a twin BN-pair, coming equipped with root groups. These are groups associated to a root system, the set of roots of a Coxeter complex. This additional structure within the group leads to corresponding additional structure in the building upon which it acts, namely the Moufang property, which we will define later in this section.

Before we state the axioms that define an RGD system, we give the setup and gather some relevant definitions. Let (W, S) be a Coxeter system and $\Sigma = \Sigma(W, S)$ the standard Coxeter complex of type (W, S). We denote by Φ the set of roots of Σ and by Φ_+ and Φ_- the sets of positive and negative roots, respectively. That is, letting $1 \in W$ be the fundamental chamber of Σ, we have

$$\Phi_+ = \{\alpha \in \Phi | 1 \in \alpha\}$$

and

$$\Phi_- = \{\alpha \in \Phi | 1 \notin \alpha\}.$$

Definition 6.4.1. Given $\alpha, \beta \in \Phi$, the pair $\{\alpha, \beta\}$ is called *prenilpotent* if $\alpha \cap \beta$ and $(-\alpha) \cap (-\beta)$ each contain at least one chamber. We then define the *closed interval* $[\alpha, \beta]$ to be

$$[\alpha, \beta] := \{\gamma \in \Phi | \gamma \supset \alpha \cap \beta \text{ and } -\gamma \supset (-\alpha) \cap (-\beta)\}$$

and the *open interval* (α, β) to be

$$(\alpha, \beta) := [\alpha, \beta] \setminus \{\alpha, \beta\}.$$

An example of a prenilpotent pair of roots $\{\alpha, \beta\}$ is if the pair is nested, that is, $\alpha \subset \beta$ or $\beta \subset \alpha$.

Now we are ready to define an RGD system. Suppose we are given a triple $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ consisting of a group G, a family of subgroups U_α (the root groups), and a subgroup T. As for notation, we will write U_s to mean U_{α_s} and U_{-s} to mean $U_{-\alpha_s}$ for $s \in S$, and we will write $U^* := U \setminus \{1\}$. for any group U.

Definition 6.4.2. The triple $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is called an *RGD system of type* (W, S) if the following conditions are satisfied:

(RGD0) For all $\alpha \in \Phi$, $U_\alpha \neq \{1\}$.

(RGD1) For all $\alpha \neq \beta$ in Φ such that $\{\alpha, \beta\}$ is prenilpotent,

$$[U_\alpha, U_\beta] \leq U_{(\alpha,\beta)}.$$

(RGD2) For every $s \in S$ there is a function $m : U_s^* \to G$ such that for all $u \in U_s^*$ and $\alpha \in \Phi$,

$$m(u) \in U_{-s} u U_{-s} \text{ and } m(u) U_\alpha m(u)^{-1} = U_{s\alpha}.$$

Moreover $m(u)^{-1} m(v) \in T$ for all $u, v \in U_s^*$.

(RGD3) For all $s \in S$,

$$U_{-s} \not\leq U_+,$$

where $U_\pm := \langle U_\alpha | \alpha \in \Phi \rangle$.

(RGD4) $G = T\langle U_\alpha | \alpha \in \Phi \rangle$.

(RGD5) T normalizes U_α for each $\alpha \in \Phi$, i.e.,

$$T \leq \bigcap_{\alpha \in \Phi} N_G(U_\alpha).$$

Remark 6.4.1. It follows from the axioms that we always have

$$T = \bigcap_{\alpha \in \Phi} N_G(U_\alpha).$$

Tits, in fact, defines T this way and removes axiom (RGD5) and the second assertion of (RGD2) in [Tit92].

There is a Weyl group W associated to an RGD system. Define the group $N := \langle T, \{m(u) | u \in U_s^*, s \in S\} \rangle$, where the $m(u)$ come from the axiom (RGD2). Then we define $W := N/T$. In addition, we define $B_\pm := TU_\pm$. This leads to the desired twin BN-pair:

Theorem 6.4.1 (Theorem 8.80, [AB08])**.** Let $(G, (U_\alpha)_{\alpha \in \Phi}, T)$ be an RGD system with subgroups B_+, B_-, N as defined above and S a generating set for the Weyl group $W = N/T$. Then (G, B_+, B_-, N, S) is a twin Tits system with Weyl group $W = N/T$.

6.4.1 The Moufang property

Let $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ be a thick twin building of type (W, S).

Definition 6.4.3. Fix a twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$, and let $\alpha = (\alpha_+, \alpha_-)$ be a pair of roots in Σ. We call α a *twin root* if $\mathrm{op}_\Sigma(\alpha) = -\alpha := (-\alpha_+, -\alpha_-)$.

Another way to think of this is to take a pair of adjacent chambers $C, D \in \Sigma_+$ and let α_+ be the root containing C but not D. Then define α_- so that $\mathrm{op}_\Sigma(\alpha_-) = -\alpha_+$. That is, α_- contains $D' := \mathrm{op}_\Sigma(D)$ but not $C' := \mathrm{op}_\Sigma(C)$ as in Figure 6.1 below.

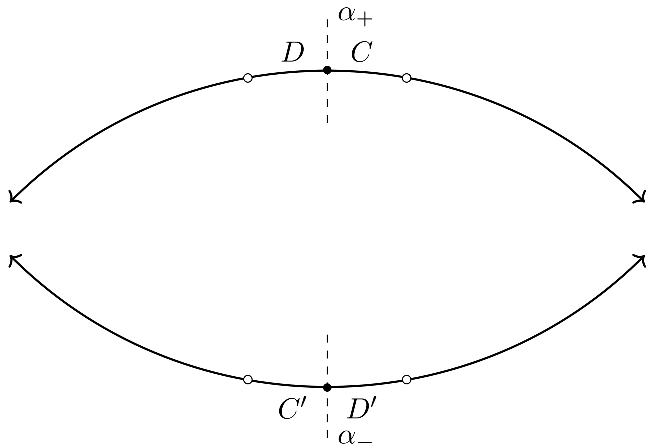


Figure 6.1: A twin root (the two halves right of the dashed lines)

Definition 6.4.4. For any twin root α of $\mathcal{C}$, the *root group* U_α is defined to be the set of automorphisms g of $\mathcal{C}$ such that g fixes α pointwise and g fixes $\mathcal{P}$ pointwise for every interior panel $\mathcal{P}$. By interior panel, we mean that $\mathcal{P} \cap \alpha$ contains more than one chamber.

A panel $\mathcal{P}$ is called a *boundary panel* if $\mathcal{P} \cap \alpha$ contains exactly one chamber. Call this chamber C. Then we define the set $\mathcal{C}(\mathcal{P}, \alpha) := \mathcal{P} \setminus \{C\}$. The root group U_α acts on $\mathcal{C}(\mathcal{P}, \alpha)$. Now we are ready to define the Moufang property.

Definition 6.4.5. We say that $\mathcal{C}$ is a *Moufang* twin building if the action of U_α is transitive on $\mathcal{C}(\mathcal{P}, \alpha)$ for every twin root α of $\mathcal{C}$.

The Moufang property essentially says that the twin building has a great deal of symmetry in that it has plenty of automorphisms. There are many group-theoretic consequences of the Moufang property as well. One way to obtain the RGD axioms is to work backwards from the conditions on the group imposed by acting on a Moufang twin building just as we saw the (twin) BN-pair axioms arise from the conditions on the group imposed by acting strongly transitively on a thick (twin) building. For a detailed discussion of Moufang twin buildings and groups with RGD systems, see Chapter 8 in [AB08]. Lastly, now that we have defined the Moufang property, we can finish off the consequences of a group having an RGD system:

Theorem 6.4.2. [Theorem 8.81, [AB08]] Let $\mathcal{C} = \mathcal{C}(G, B_+, B_-)$ be the twin building associated to the twin Tits system (G, B_+, B_-, N, S) of Theorem 6.4.1. Then $\mathcal{C}$ is a Moufang twin building.

6.5 Kac-Moody groups

The theory of Kac-Moody groups was developed by Robert Moody and Victor Kac. These groups can be viewed as infinite-dimensional versions of semisimple complex Lie groups, and they share many similarities with their finite-dimensional counterparts. Later, Tits defined Kac-Moody groups over arbitrary fields [Tit87] in a manner similar to Chevalley's construction of Chevalley groups [Che55], which are linear algebraic groups defined over any field corresponding to semisimple complex Lie algebras. A few years later, Tits refined the notion of BN-pair in [Tit92] with the concept of root group data and connected these groups with twin buildings. Kac-Moody groups are the most important current examples of groups with RGD systems and were the origin of the conjecture we address in this thesis. In addition, they have many important applications in physics, especially in supergravity theories and string theory (see [BC17] for a survey of these applications as well as a list of more references). In this section, we will give a brief introduction to the theory of Kac-Moody groups and how they fit into our context. This introduction will generally follow the treatment of the subject by Caprace and Rémy in Section 3 of [CR09].

6.5.1 Kac-Moody Lie algebras

We begin with an $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ of rank l with $a_{ij} \in \mathbb{R}$. To obtain a Kac-Moody algebra, we want this to be a generalized Cartan matrix.

Definition 6.5.1. An $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ is called a *generalized Cartan matrix* if $a_{ij} \in \mathbb{Z}$ for $1 \leq i, j \leq n$, $a_{ii} = 2$ for all $1 \leq i \leq n$, $a_{ij} \leq 0$ and $a_{ij} = 0$ if and only if $a_{ji} = 0$ for all $i \neq j \in \{1, \ldots, n\}$.

We then define a *realization* of A to be a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a $\mathbb{C}$-vector space of dimension $2n-l$, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is a linearly independent subset of $\mathfrak{h}^*$, $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$

is a linearly independent subset of $\mathfrak{h}$, and $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ for all $1 \leq i, j \leq n$. Here $\langle \cdot, \cdot \rangle$ is the natural map $\mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ given by evaluating a linear functional on a vector. Such a triple always exists and is unique up to isomorphism.

We then define a Lie algebra $\tilde{\mathfrak{g}}(A)$ generated by $\{e_i, f_i\}_{i=1}^n$ and a basis of $\mathfrak{h}$ subject to the following relations:

$$
\begin{aligned}
&[e_i, f_j] = \delta_{ij}\alpha_i^\vee && 1 \leq i, j \leq n, \\
&[h, h'] = 0 && h, h' \in \mathfrak{h}, \\
&[h, e_i] = \langle \alpha_i, h \rangle e_i && 1 \leq i \leq n, h \in \mathfrak{h}, \\
&[h, f_i] = -\langle \alpha_i, h \rangle f_i && 1 \leq i \leq n, h \in \mathfrak{h}.
\end{aligned}
$$

These generators and relations used to define this Lie algebra ensure the existence of a triangular decomposition and root space decomposition, which are desirable when thinking ahead toward having root group data. We gather this information in the following theorem due to Kac:

Theorem 6.5.1. [Theorem 1.2, [Kac90]] Let $\tilde{\mathfrak{n}}_+$ be the subalgebra generated by $\{e_i\}_{i=1}^n$ and $\tilde{\mathfrak{n}}_-$ be the subalgebra generated by $\{f_i\}_{i=1}^n$. Let $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$, and $Q_\pm = \sum_{i=1}^n \mathbb{Z}_\pm \alpha_i$. Then:

(i) $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$.

(ii) With respect to the adjoint $\mathfrak{h}$-action on $\tilde{\mathfrak{g}}(A)$, we have the decomposition

$$\tilde{\mathfrak{g}}(A) = \left(\bigoplus_{\alpha \in Q_{+\setminus\{0\}}} \tilde{\mathfrak{g}}_\alpha \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in Q_{+\setminus\{0\}}} \tilde{\mathfrak{g}}_{-\alpha} \right),$$

where $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}}(A) | [h, x] = \langle \alpha, h \rangle x \text{ for all } h \in \mathfrak{h}\}$.

(iii) Among all ideals intersecting $\mathfrak{h}$ trivially, there is a unique maximal one, which we call $\mathfrak{r}$.

The Lie algebra we will actually deal with is $\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\mathfrak{r}$, where $\mathfrak{r}$ is the maximal ideal described in (iii) above. The root space decomposition in (ii) induces a root space decomposition

$$\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0 \cong \mathfrak{h}$. We define $\Phi = \{\alpha \in Q \setminus \{0\} | \mathfrak{g}_\alpha \neq 0\}$. The elements of Φ are called *roots*, and Φ itself is called the *root system*. To see how root spaces behave under the bracket, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for any $\alpha, \beta \in \mathfrak{h}^*$.

6.5.2 The Weyl group

There is also a Weyl group corresponding to a generalized Cartan matrix and Kac-Moody algebra. This Weyl group is a Coxeter group which can be associated to a GCM A in a natural way. We define a Coxeter matrix $M = (m_{ij})_{i,j=1}^n$ with $m_{ii} = 1$ and for $i \neq j$, we define m_{ij} in terms of A by

$$\begin{array}{rccccc} m_{ij} = & 2 & 3 & 4 & 6 & \infty \\ \text{if } a_{ij}a_{ji} = & 0 & 1 & 2 & 3 & \geq 4. \end{array}$$

We define a set of generators $S = \{s_1, \ldots, s_n\}$ with the usual relations $(s_i\ s_j)^{m_{ij}} = 1$ for $1 \leq i, j \leq n$. Then the Weyl group $W := \langle S \rangle$ is a Coxeter group.

One can also define linear reflections s_i on $\mathfrak{h}^*$ by

$$s_i(x) = x - \langle x, \alpha_i^\vee \rangle \alpha_i$$

for any $x \in \mathfrak{h}^*$. Then these reflections have the same relations as above (see Section 1.1.4 of [CR09]) and result in the same Weyl group.

Now that we have defined W, we also define the set of *real roots* $\Phi^{re} := \{w\alpha_i | 1 \leq i \leq n, w \in W\}$ to be translates of the simple reflections by elements of the Weyl group.

6.5.3 Kac-Moody groups over $\mathbb{C}$

Now that we have the background of Kac-Moody algebras, we are ready to start constructing a Kac-Moody group, initially just over $\mathbb{C}$, as a subgroup of $Aut(\mathfrak{g}(A))$. We will do this by exponentiating certain operators on $\mathfrak{g}(A)$. In order for exponentiation to be well-defined, the operator must be locally nilpotent. We say that an operator f on a vector space V is *locally nilpotent* if for all $v \in V$, there exists some $m_v \in \mathbb{N}$ depending on v such that $f^{m_v}(v) = 0$.

Lemma 6.5.1. The operators ad e_i and ad f_i are locally nilpotent on $\mathfrak{g}(A)$ for all $1 \leq i \leq n$.

For a proof of this lemma, see Lemma 3.3 in [CR09]. It follows from this lemma that the operators

$$\exp \operatorname{ad} e_i = \sum_{m=0}^{\infty} \frac{1}{m!} (\operatorname{ad} e_i)^m$$

and

$$\exp \operatorname{ad} f_i = \sum_{m=0}^{\infty} \frac{1}{m!} (\operatorname{ad} f_i)^m$$

are well defined elements of $Aut(\mathfrak{g}(A))$.

For $\alpha \in \Phi^{re}$, we define the root group $U_\alpha := \langle \exp \operatorname{ad} x | x \in \mathfrak{g}_\alpha \rangle \leq Aut(\mathfrak{g}(A))$. The subgroup U_α is well-defined since ad x is locally nilpotent by Lemma 6.5.1. We are now ready to define an adjoint Kac-Moody group:

Definition 6.5.2. The *adjoint Kac-Moody group of type A over* $\mathbb{C}$ is $G := \langle U_\alpha | \alpha \in \Phi^{re} \rangle$.

Theorem 6.5.2. [Theorem 3.5, [CR09]] Let G be an adjoint Kac-Moody group as defined above, and define $T := \bigcap_{\alpha \in \Phi^{re}} N_G(U_\alpha)$ in light of Remark 6.4.1. Then the triple $(G, (U_\alpha)_{\alpha \in \Phi^{re}}, T)$ is an RGD system.

6.5.4 Generalization to arbitrary fields

Just as complex semisimple Lie groups can be defined over arbitrary fields as Chevalley groups, Tits showed in [Tit87] that similar constructions can be carried out in the Kac-Moody context. Tits begins with a generalized Cartan matrix A, as well as Π and $\Pi^\vee$

as above. To this data, he associates a group functor $\mathcal{G} : Rgs \to Gps$ from the category of commutative rings to the category of groups. This functor is called a *Tits functor*. Restricting this functor to fields yields the so-called *split Kac-Moody groups* that come naturally equipped with root groups U_α and have an RGD system just as in the adjoint case over the complex numbers.

Example 6.5.1. The groups $\mathrm{SL}_n(k[t,t^{-1}])$ are all Kac-Moody groups of type $\tilde{A}_{n-1}$ over the field k. That is, the Weyl group has Coxeter diagram of this type. For example, the generalized Cartan matrix for $\mathrm{SL}_2(k[t,t^{-1}])$ is

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

However, one should note that these are not the only Kac-Moody groups of this type since the functor also depends on the choice of Π and $\Pi^\vee$.

There are also several other constructions of "non-split" Kac-Moody groups, but we will not discuss these here as this is just a brief introduction to a deep theory.

6.6 Cell stabilizers of a group acting on a twin building

The goal of this section is to determine the conditions necessary to ensure that we can apply Gandini's theorem. That is, we want a group G acting cellularly on a space with finite cell stabilizers such that G contains finite subgroups of unbounded order. In particular, we will impose certain conditions on the thick twin building that will yield the desired properties.

We first give the setup. Let $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ be a thick twin building of type (W, S) where $S = \{s_i | 1 \leq i \leq n\}$ and W is of infinite order. We also have a set of parameters $(q_i)_{1 \leq i \leq n}$ with $q_i \in \mathbb{N}$, $q_i \geq 2$ such that for any s_i-panel $\mathcal{P}$, the number of chambers contained in $\mathcal{P}$ is $|\mathcal{C}(\mathcal{P})| = q_i + 1$ for $1 \leq i \leq n$. We then define $q_{min} := \min q_i$ and $q_{max} := \max q_i$.

Now suppose that G is a group acting strongly transitively on the thick twin building $\mathcal{C}$ and fix a pair of opposite chambers $C_+ \in \mathcal{C}_+$ and $C_- \in \mathcal{C}_-$. Let $\Sigma := \Sigma\{C_+, C_-\}$ to be the fundamental twin apartment defined by this pair of opposite chambers, and set $B_\pm := G_{C_\pm}$ and $N := G_\Sigma$ to be the stabilizers in G of the two fundamental chambers and fundamental twin apartment. By Theorem 6.3.1, (B_+, B_-, N) is a saturated twin BN-pair in G. That is, we know $T := N \cap B_\pm = B_+ \cap B_-$. We additionally require two finiteness assumptions:

1. The parameter q_i is finite for all $1 \leq i \leq n$.

2. The subgroup $T = B_+ \cap B_-$ is finite.

Example 6.6.1.

(1) A special case of this is an RGD system $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ such that the root groups U_α and T are all finite subgroups of G. If $\alpha = \alpha_i$ is a simple root, then we define the parameter $q_i = |U_\alpha|$. Since G has an RGD system, it has a twin BN-pair and thus acts strongly transitively on a corresponding thick twin building. It is a fact that the panels will have the correct orders as well.

(2) The example that spawned the conjecture we address is as follows: Let $\mathcal{G}$ be a Kac-Moody group functor with corresponding Weyl group of type (W, S), and let $G = \mathcal{G}(\mathbb{F}_q)$. Then G has a family of root groups $(U_\alpha)_{\alpha\in\Phi}$ where $|U_\alpha| = q$ since $U_\alpha \cong (\mathbb{F}_q, +)$. Hence all parameters are equal to q. Furthermore, $T \cong (\mathbb{F}_q^*)^k$ for some $k \in \mathbb{N}$, and hence $|T| = (q-1)^k$, so T is finite.

Our focus will not be on the G-action on the twin building but instead on certain Z-realizations of the twin building. We will first look at the general case where we assume only that Z is a simplicial complex, and hence a CW-complex. Given such a Z, we define the complex $X := Z(\mathcal{C}_+) \times Z(\mathcal{C}_-)$. We will focus on the action on just one half of this complex at a time, so let $\Delta \in \{\mathcal{C}_+, \mathcal{C}_-\}$, and let δ be the corresponding Weyl distance function. Recall the cellulation we defined on $Z(\Delta)$ in the subsection 3.1.2: Given a cell $\sigma \in Z$, we define the cells of $Z(\Delta)$ to be $[C, \sigma] := \bigcup_{z\in\sigma}[C, z]$ for all $C \in \Delta$. This establishes a CW-complex structure on $Z(\Delta)$ and hence on the product X.

Now we need to established the G-action on X. We know that G acts strongly transitively on $(\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ and hence acts on Δ. We define the action of G on $Z(\Delta)$ by

$$g.[C, z] = [gC, z]$$

for any $g \in G, C \in \Delta, z \in Z$. This action is well-defined since G preserves δ. This action naturally extends to an action on the cells. We now show that this action is *cellular*, i.e. if an element $g \in G$ stabilizes a cell, then it also fixes the cell pointwise. Let $[C, \sigma]$ be a cell and let $g \in G_{[C,\sigma]}$. We want to show that the cell is fixed pointwise by g. Since $[gC, \sigma] = [C, \sigma]$, we know that $\delta(gC, C) \in \langle S_\sigma \rangle$, where $S_\sigma = \{s \in S | \sigma \subset Z_s\}$. If $z \in \sigma$, then $S_\sigma \subset S_z$, so $\delta(gC, C) \in \langle S_z \rangle$ and hence $[gC, z] = [C, z]$. Thus G acts cellularly on $Z(\Delta)$. This induces a cellular action of G on X by

$$g.([C, z], [C', z']) = ([gC, z], [gC', z'])$$

for any $g \in G$, $C \in \mathcal{C}_+$, $C' \in \mathcal{C}_-$, and $z, z' \in Z$.

Now that we have established the cellular action of G on X, we want to examine the stabilizers in G of points in X and show that, with an appropriate choice of Z, these subgroups will be finite and have no bound on their orders.

A general point in X is of the form $([C, z], [C', z'])$, where $C \in \mathcal{C}_+$, $C' \in \mathcal{C}_-$, and $z, z' \in Z$. Since G acts strongly transitively on the twin building $\mathcal{C}$, there is some $g \in G$ such that $(gC, gC') = (C_+, wC_-)$, where $w = \delta^*(C, C')$ by Lemma 6.1.1(i). Then the stabilizer of the point $([C_+, z], [wC_-, z'])$ is conjugate to the stabilizer of the original point. Since we only wish to show that the stabilizers are finite, it suffices to look only at points of the latter form, which will make computations easier.

Recall that $[C_+, z] = [D, z'']$ if and only $z = z''$ and $\delta(C_+, D) \in \langle S_z \rangle$. Moreover, $G_{C_+} = B_+$, so $G_{[C_+,z]} = B_+ \langle S_z \rangle B_+$. Similarly, since $G_{wC_-} = wG_{C_-}w^{-1} = wB_-w^{-1}$, we have $G_{[wC_-,z']} = wB_-\langle S_{z'}\rangle B_-w^{-1}$. Therefore the stabilizer of the point $([C_+,z],[wC_-,z'])$ is the intersection $B_+\langle S_z\rangle B_+ \cap wB_-\langle S_{z'}\rangle B_-w^{-1}$.

First we will study the case when $S_z = \emptyset$ and $S_{z'} = \emptyset$; that is, when z and z' do not lie in any panel. The stabilizers in this case are the subgroups $B_+ \cap wB_-w^{-1}$. These will provide finite subgroups of G of unbounded order.

The first lemma toward this result gives an upper and lower bound on the number of chambers in the "w-sphere" of a chamber C in one half of the building.

Lemma 6.6.1. Suppose $w \in W$ with reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$ where $\ell = \ell(w)$ and $1 \leq i_1, \ldots, i_\ell \leq n$. Fix a chamber $C \in \Delta$ and define its "w-sphere" in Δ to be $\mathcal{C}_w(C) := \{D \in \Delta | \delta(C,D) = w\}$. Then

$$q_{min}^{\ell(w)} \leq |\mathcal{C}_w(C)| = q_{i_1} \cdots q_{i_\ell} \leq q_{max}^{\ell(w)}.$$

Proof. We proceed by induction. If $\ell = 1$, then $w = s_{i_1}$. By assumption, the s_{i_1}-panel contains q_{i_1} chambers distinct from C, so the statement is easily seen to be true.

Claim: Whenever $s = s_i \in S$ and $\ell(ws) = \ell(w) + 1$, then

$$\mathcal{C}_{ws}(C) = \coprod_{D \in \mathcal{C}_w(C)} \mathcal{C}_s(D).$$

Proof of Claim: If $E \in \mathcal{C}_{ws}(C)$, then $\delta(C,E) = ws$. Therefore, there exists a minimal gallery $C = C_0, C_1, \ldots, C_k = E$ of type $(s_{i_1}, \cdots, s_{i_\ell}, s)$ where $E \in \mathcal{C}_s(C_{k-1})$ and $C_{k-1} \in \mathcal{C}_w(C)$. This proves the $\subset$ inclusion.

On the other hand, if $E \in \mathcal{C}(D)$ for some $D \in \mathcal{C}_w(C)$, then $E \in \mathcal{C}_{ws}(C)$ since $\ell(ws) = \ell(w) + 1$. Hence $\supset$ holds as well.

It remains to show that the union is disjoint. Suppose that $D, D' \in \mathcal{C}_w(C)$ and that $\mathcal{C}_s(D) \cap \mathcal{C}_s(D') \neq \emptyset$. Then there is some chamber s-adjacent to both D and D'; hence D and D' lie in the same s-panel. If $D \neq D'$, then $\delta(D,D') = s$, so $D' \in \mathcal{C}_s(D)$. Since $\ell(ws) = \ell(w) + 1$, $D' \in \mathcal{C}_{ws}(C)$, a contradiction.

Note that, with $s = s_i$, $|C_s(D)| = q_i$ for all $D \in \mathcal{C}_w(C)$. The claim implies $|\mathcal{C}_{ws}(C)| = |\mathcal{C}_w(C)||\mathcal{C}_s(D)|$ for any $D \in \mathcal{C}_w(C)$. Therefore, for $\ell > 1$, we have $|\mathcal{C}_{ws_{i_\ell}}(C)| = q_{i_1} \cdots q_{i_{\ell-1}}$ by the induction hypothesis. We now note that $\ell(w) = \ell(ws_{i_\ell}s_{i_\ell}) = \ell(ws_{i_\ell})+1$, and $|\mathcal{C}_{s_{i_\ell}}(D)| = q_{i_\ell}$ for any D with $\delta(C, D) = ws_{i_\ell}$. The claim then gives $|\mathcal{C}_w(C)| = |\mathcal{C}_{ws_{i_\ell}}(C)||\mathcal{C}_{s_{i_\ell}}(D)| = q_{i_1} \cdots q_{i_\ell}$ for any D with $\delta(C, D) = ws_{i_\ell}$. Clearly $q_{min}^{\ell(w)} \leq q_{i_1} \cdots q_{i_\ell} \leq q_{min}^{\ell(w)}$, so the statement is true. $\square$

Lemma 6.6.2. The group $B_+ \cap wB_-w^{-1} = G_{C_+} \cap G_{wC_-}$ acts transitively on $\mathcal{C}_{w^{-1}}(wC_-)$, the w^{-1}-sphere about wC_- in $\mathcal{C}_-$.

Proof. First we need to show that this group actually acts on $\mathcal{C}_{w^{-1}}(wC_-)$. Clearly wB_-w^{-1} acts on $\mathcal{C}_{w^{-1}}(wC_-)$ since it stabilizes wC_- and acts by isometries on $\mathcal{C}_-$. This action restricts to an action of the subgroup $B_+ \cap wB_-w^{-1}$ as well. Now we must show that this action is transitive. Let $C'_- \in \mathcal{C}_{w^{-1}}(wC_-)$; then $\delta_-(wC_-, C'_-) = w^{-1}$, so $\delta_-(C'_-, wC_-) = w = \delta^*(C_+, wC_-)$. By Corollary 5.2.1(1), we have C'_- op C_+. Since C'_- was arbitrary in $\mathcal{C}_{w^{-1}}(wC_-)$, it follows that $\mathcal{C}_{w^{-1}}(wC_-) \subset C_+^{op} := \{D \in \mathcal{C}_- | \delta^*(C_+, D) = 1\}$.
Since G acts strongly transitively on the twin building, B_+ acts transitively on C_+^{op} by Lemma 6.1.1(ii). Now, given any $C'_- \in \mathcal{C}_{w^{-1}}(wC_-)$, there exists some $b_+ \in B_+$ such that $b_+C_- = C'_-$. We want to show that $b_+ \in wB_-w^{-1}$ as well, which will prove transitivity.
Consider the twin apartment $\Sigma = \Sigma\{C_+, C_-\}$, which also contains wC_-. Then $b_+\Sigma = \Sigma\{C_+, C'_-\}$, which contains the chamber b_+wC_-. Since $\delta^*(C_+, wC_-) = w = \delta_-(C'_-, wC_-)$, $wC_- \in \Sigma\{C_+, C'_-\}$ by definition of this twin apartment. Also note that $\delta^*(C_+, wC_-) = w = \delta^*(C_+, b_+wC_-)$ since b_+ acts as an isometry. Hence $wC_- = b_+wC_-$ by uniqueness of chambers codistance w from C_+ in a twin apartment. Thus $b_+ \in B_+ \cap wB_-w^{-1}$. $\square$

Now we are ready to show that the groups $B_+ \cap wB_-w^{-1}$, for $w \in W$, provide finite subgroups of G of unbounded order.

Proposition 6.6.1. $|T|q_{min}^{\ell(w)} \leq |B_+ \cap wB_-w^{-1}| \leq |T|q_{max}^{\ell(w)}$ for any $w \in W$.

Proof. Here we will make use of the Orbit-Stabilizer Theorem. Consider the action of $B_+ \cap wB_-w^{-1}$ on $\mathcal{C}_{w^{-1}}(wC_-)$ and, in particular, the stabilizer of the chamber C_- in $B_+ \cap$

wB_-w^{-1}. The stabilizer is $B_- \cap B_+ \cap wB_-w^{-1} = T \cap wB_-w^{-1}$, using the fact that $T = B_+ \cap B_-$. Since $T \trianglelefteq N$, $wTw^{-1} = T$. Therefore, since $T \leq B_-$ as well, we have $T \leq wB_-w^{-1}$. Hence the stabilizer of C_- in $B_+ \cap wB_-w^{-1}$ is just T. Since the action of $B_+ \cap wB_-w^{-1}$ is transitive by Lemma 6.6.2, the orbit is all of $\mathcal{C}_{w^{-1}}(wC_-)$. Thus, we obtain $[B_+ \cap wB_-w^{-1} : T] = |\mathcal{C}_{w^{-1}}(wC_-)|$, where T and $\mathcal{C}_{w^{-1}}(wC_-)$ are finite. Hence $|B_+ \cap wB_-w^{-1}| = |T||\mathcal{C}_{w^{-1}}(wC_-)|$, and the result follows from Lemma 6.6.1. □

Corollary 6.6.1. *G has finite subgroups of unbounded order if W is infinite.*

Proof. If W is infinite, then $|T|q_{min}^{\ell(w)}$ goes to infinity as $\ell(w)$ goes to infinity. By Proposition 6.6.1, $|B_+ \cap wB_-w^{-1}| \geq |T|q_{min}^{\ell(w)}$, so the order of $B_+ \cap wB_-w^{-1}$ can be made arbitrarily large. □

Now that we have shown that G has finite subgroups of unbounded order, it remains to show that all cell stabilizers are finite. Recall that, due to conjugacy, these stabilizers are of the form $B_+W_IB_+ \cap wB_-W_JB_-w^{-1}$ where $I, J \subset S$ and $w \in W$. We have already shown that these are finite if $I, J = \emptyset$. We show that these are finite subgroups of G if I, J are spherical subsets of S.

Lemma 6.6.3. *Let $P_J = B_\pm W_J B_\pm$ be a standard parabolic subgroup. Then $[P_J : B_\pm] < \infty$ if and only if $|W_J| < \infty$.*

Proof. Suppose that W_J is finite. We already assume that q_i is finite for $1 \leq i \leq n$. Then the J-residue containing $C_\pm$, $R_J(C_\pm)$ is finite by Lemma 6.6.1 since each w-sphere in $\mathcal{C}_\pm$ is finite, and there are only finitely many to consider due to the assumption that W_J is finite. We know that any chamber in $R_J(C_\pm)$ can be written as $gC_\pm$ with $g \in P_J$. Hence P_J acts transitively on $R_J(C_\pm)$, and the stabilizer of $C_\pm$ is $B_\pm$. The Orbit-Stabilizer Theorem then implies that $[P_J : B_\pm] = |R_J(C_\pm)| < \infty$.
On the other hand, suppose $|W_J|$ is infinite. By the Bruhat decomposition in G, all double cosets $B_\pm wB_\pm$ are distinct for distinct $w \in W_J$. Therefore, there are infinitely many such double cosets and hence infinitely many left cosets in $P_J/B_\pm$. Thus $[P_J : B_\pm] = \infty$. □

Lemma 6.6.4. Let $P_I = B_+ W_I B_+$ and $P_J = B_- W_J B_-$ where W_I and W_J are both spherical. Then

$$[P_I \cap w P_J w^{-1} : B_+ \cap w B_- w^{-1}] < \infty$$

and thus $P_I \cap w P_J w^{-1}$ is a finite group.

Proof. We will utilize the Orbit-Stabilizer Theorem again. Consider the set $P_I/B_+ \times wP_Jw^{-1}/wB_-w^{-1}$. There is a natural action of $P_I \cap wP_Jw^{-1}$ on this product by left multiplication. Now consider the element (B_+, wB_-w^{-1}). The stabilizer of this element is $B_+ \cap wB_-w^{-1}$. Hence

$$[P_I \cap wP_Jw^{-1} : B_+ \cap wB_-w^{-1}] = |Orb(B_+, wB_-w^{-1})| \leq [P_I : B_+][wP_Jw^{-1} : wB_-w^{-1}] = [P_I : B_+][P_J : B_-]$$

where equality would occur only if the action were transitive. Since both $[P_I : B_+]$ and $[P_J : B_-]$ are finite by Lemma 6.6.3, we have

$$[P_I \cap wP_Jw^{-1} : B_+ \cap wB_-w^{-1}] < \infty$$

as desired. The fact that this group is then finite follows from Proposition 6.6.1 which shows that $B_+ \cap wB_-w^{-1}$ is finite. □

In order to utilize Lemma 6.6.4, we must choose Z and Z_s so that the sets S_z for $z \in Z$ are spherical subsets of S. If this is the case, then all cell stabilizers will be finite. Furthermore, G will contain the subgroups $B_+ \cap wB_-w^{-1}$ for $w \in W$ and thus has finite subgroups of unbounded order as we take $\ell(w)$ to ∞. Thus we will have the necessary conditions on G in order to apply Gandini's theorem.

Chapter 7

Results

We assume the same set up as in Section 6.6. That is, G is a group acting strongly transitively on a thick twin building $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ of type (W, S), where W is infinite and $S = \{s_i | 1 \leq i \leq n\}$. We also have a set of parameters $(q_i)_{i=1}^n$ with $q_i \in \mathbb{N}$, $q_i \geq 2$ such that for any s_i-panel $\mathcal{P}$, the number of chambers in $\mathcal{P}$ is $|\mathcal{C}(\mathcal{P})| = q_i + 1$. We assume q_i to be finite for all $1 \leq i \leq n$. Set $q_{min} := \min q_i$ and $q_{max} := \max q_i$.

Fix a pair of opposite chambers $C_+ \operatorname{op} C_-$ with $C_+ \in \mathcal{C}_+$ and $C_- \in \mathcal{C}_-$ and let $\Sigma = \Sigma\{C_+, C_-\}$ be the fundamental twin apartment defined by this pair of opposite chambers. Set $B_\pm = G_{C_+}$ and $N = G_\Sigma$. Then the triple (B_+, B_-, N) is a saturated twin BN-pair in G with Weyl group W. Then the subgroup T, the pointwise fixer of Σ_+ and Σ_-, can be written $T = B_+ \cap B_-$ and is also assumed to be finite.

Finally, we define $X = Z(\mathcal{C}_+) \times Z(\mathcal{C}_-)$. As discussed in Section 6.6, this is a CW-complex on which G acts cellularly.

7.1 Finiteness properties and Gandini's Theorem

In this section, we provide the relevant background results mentioned in the introduction and also define the homological finiteness properties FP_n.

Definition 7.1.1. Let R be a ring with unity. An R-module M is *of type* FP_n if and only if there exists an exact sequence

$$P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

such that P_i is a finitely generated projective R-module for all $i \leq n$. We say that M is of type FP_∞ if it is of type FP_n for all n.

A group G is FP_n, respectively FP_∞, if the trivial G-module $\mathbb{Z}$ is FP_n, respectively FP_∞, as a $\mathbb{Z}[G]$-module.

Remark 7.1.1.

1. Every group is FP_0.

2. A group G is FP_1 if and only if it is finitely generated.

3. If G is finitely presented, then it is FP_2.

4. If G is finite, then G is FP_∞.

Given these definitions, we are now ready to state the implication of Brown's filtration criterion:

Lemma 7.1.1. [Bro87] If a group G acts cellularly on an n-dimensional contractible CW-complex with stabilizers of type FP_∞, then G is of type FP_∞ if and only if it is of type FP_n.

A few years later, Kropholler showed the following result for groups belonging to a large class of groups called LH$\mathfrak{F}$-groups. A discussion of this class of groups can be found in Section 2 of [Kro93]. All of the groups we consider belong to this class.

Lemma 7.1.2. (Theorem B, [Kro93]) If G is an LH$\mathfrak{F}$-group and is of type FP_∞, then there is a bound on the orders of its finite subgroups.

These two results combine to give the following general statement of Gandini's theorem:

Theorem 7.1.1. [Gan12] Let G be a group acting on an n-dimensional contractible CW-complex with finite stabilizers. If G has no bound on the orders of its finite subgroups, then G is not FP_n.

For years, Brown's filtration criterion has been the key tool in proving whether or not a group G is FP_n. Although Gandini's theorem cannot be used to prove positive results for FP_n, it provides a much more efficient tool for proving the negative results since it no longer relies on computing filtrations.

An immediate application of Theorem 7.1.1 is to the Davis realization of a twin building where G is a group acting strongly transitively on a locally finite twin building. In this case, we do not need to impose any conditions on the Coxeter diagram associated to the group G.

Proposition 7.1.1. Suppose that G is a group acting strongly transitively on a locally finite twin building. Let n be the maximal cardinality of a spherical subset of S. Then G is not FP_{2n}, and the Borel subgroups B_+ and B_- are not FP_n.

Proof. We prove the result for G. Let Z and Z_s for $s \in S$ be defined as in Definition 3.1.8, and define $X = Z(\mathcal{C}_+) \times Z(\mathcal{C}_-)$. Recall that the dimension of the Davis realization of one building is equal to the maximal cardinality of a spherical subset of S. Therefore, $\dim Z(\mathcal{C}_+) = \dim Z(\mathcal{C}_-) = n$, and $\dim X = 2n$. We know that X is contractible since it is the product of contractible spaces by Theorem 3.1.1. Proposition 6.6.1 shows that G has finite subgroups of unbounded order, and Lemma 6.6.4 shows that G acts with finite cell stabilizers. Therefore Theorem 7.1.1 implies that G is not FP_{2n}.
The Borel subgroups $B_\pm$ (in fact, the parabolic subgroups of spherical type) act on $Z(\mathcal{C}_\mp)$, which is n-dimensional. The rest of the argument is roughly the same. □

Remark 7.1.2. The Davis realization gives a bound on the finiteness length of the group G. However, this bound is not, in general, sharp. In the next two sections, we will see that one can greatly improve upon this bound in the case that there is at least one ∞ in the diagram by choosing appropriate realizations on which G acts cellularly.

For our context, we want G to act on a product of two trees which is a 2-dimensional, contractible CW-complex. Furthermore, it suffices to require finite stabilizers. If the conditions are satisfied, then the group is not FP_2 and therefore not finitely presented.

7.2 Groups satisfying the condition (A)

In this section we will define a condition (A) that the Weyl group W in G will satisfy so that complex X will be a product of two trees given appropriate choices of Z and Z_s for $s \in S$.

Definition 7.2.1. Suppose that G has Coxeter system (W, S). Then G satisfies (A) if: $S = J \sqcup K$, $|K| \geq 2$, such that $J \cup \{s\}$ is spherical for any $s \in K$ and $m(s,t) = \infty$ for any $s \neq t$ in K.

Let $S = J \cup K$ as in Definition 7.2.1. Recall that we defined $\mathcal{S}$ earlier to be the set of spherical subsets of S. Set $\mathcal{S}' := \mathcal{S}_{\geq J}$ to be the set of spherical subsets of S containing J. We now define $Z = |K(\mathcal{S}')|$ to be the geometric realization of the flag complex on this set $\mathcal{S}'$, and we define the s-panel $Z_s = |K(\mathcal{S}'_{\geq s})|$ to be the geometric realization of the flag complex on the spherical subsets of S containing $J \cup \{s\}$ for all $s \in S$. In this case, the complex Z is easy to describe. The only spherical subsets of S containing J are J and $J \cup \{t\}$ for $t \in K$. If we let $K = \{t_1, \ldots, t_m\}$, then the complex Z is:

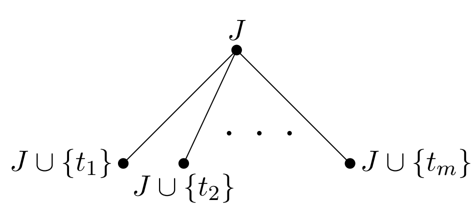


Figure 7.1: The complex $Z = |K(\mathcal{S}')|$ where $K = \{t_1, \ldots, t_m\}$.

The panels are easy to describe as well. For $s \in K$, $Z_s = |K(\mathcal{S}'_{\geq s})|$ is exactly the vertex corresponding to $J \cup \{s\}$ (i.e. all vertices except the top one in Figure 7.1). For $s \in J$, $Z_s = Z$ since $\mathcal{S}'_{\geq s} = \mathcal{S}'$ for any $s \in J$. We now state a few consequences of this panel structure in Z.

Lemma 7.2.1. Let $\Delta \in \{\mathcal{C}_+, \mathcal{C}_-\}$ and δ be the corresponding Weyl distance. Let $C, D \in \Delta$, and recall that $Z(C) = \{[C, z] | z \in Z\}$.

(1) $\delta(C, D) \in W_J$ if and only if $Z(C) = Z(D)$.

(2) Let $s \in K$. Then $\delta(C, D) \in W_{J \cup \{s\}}$ if and only if $[C, z] = [D, z]$ for $z \in Z_s$.

Proof. Since any $z \in Z$ lies in Z_s for all $s \in J$, we have $J \subset S_z$ and hence $W_J \leq \langle S_z \rangle$. Therefore, if $\delta(C, D) \in W_J$ for two chambers $C, D \in \mathcal{C}$, we have $[C, z] = [D, z]$ for all $z \in Z$. That is $Z(C) = Z(D)$. Conversely, if $Z(C) = Z(D)$, then this means in particular that $[C, z] = [D, z]$ for any interior point $z \in Z$, that is, a point that is not in Z_s for any $s \in K$. Then $\delta(C, D) \in W_J$.

The second part is very similar. If $z \in Z_s$ for some $s \in K$, then $S_z = J \cup \{s\}$, so $\delta(C, D) \in W_{J \cup \{s\}}$ if and only if $[C, z] = [D, z]$. □

Remark 7.2.1. In terms of the complex $Z(\Delta)$, Lemma 7.2.1(1) implies that there is exactly one copy of Z for each J-residue in Δ.

Our goal is to show that $Z(\Delta)$ is a tree so that our group acts on a contractible space of the correct dimension. Again by tree, we mean the geometric realization of a combinatorial tree. In order to show that $Z(\Delta)$ is a tree, it suffices to show that $Z(A)$ is a tree for any apartment A of Δ by Lemma 3.1.2.

Lemma 7.2.2. $Z(A)$ is a tree for any apartment A of Δ.

Proof. Since Z is connected, $Z(A)$ is connected by Lemma 3.1.3.

Now we show that there are no circuits. First, it is clear that Z itself is a tree, so a circuit must involve more than one Z-chamber. A natural consequence of (1) from Lemma 7.2.1 is that two Z-chambers that intersect at an interior point of Z must coincide entirely. Therefore distinct Z-chambers may only be glued along a copy of Z_s for $s \in K$. Moreover, they can be glued along exactly one such panel: If $s, t \in K$ with $s \neq t$ and $[C, z] = [D, z]$ for $z \in Z_s$ and $[C, z'] = [D, z']$ $z' \in Z_t$, then $\delta(C, D) \in W_{J \cup \{s\}} \cap W_{J \cup \{t\}} = W_J$, so $Z(C) = Z(D)$.

Now assume that there exists a circuit in $Z(A)$. Fix a point $p = [C_0, z]$ in the circuit; we may assume that p is an interior point of $Z(C_0)$ since the circuit involves at least two distinct Z-chambers. Then we can denote the circuit by a gallery $Z(C_0), Z(C_1), \ldots, Z(C_m) = Z(C_0)$

given by the Z-chambers that the circuit passes through, where $Z(C_{j-1})$ and $Z(C_j)$ are glued along a panel of type $s_{i_j} \in K$ for all $1 \leq j \leq m$. In particular, $Z(C_{j-1}) \neq Z(C_j)$. By fixing an interior point as our start and end point, we have required that the final Z-chamber be the same as the first, and hence we can choose $C_m = C_0$. By definition of our equivalence relation, $\delta(C_{j-1}, C_j) = w_j \in W_{J\cup\{s_{i_j}\}} \backslash W_J$ since $Z(C_{j-1}) \neq Z(C_j)$. To follow the notation of Lemma 1.1.1, set $t_j := s_{i_j}$ and $\tilde{W}_j := W_{J\cup\{s_{i_j}\}} \backslash W_J$. We may assume that $t_{j-1} \neq t_j$ for all $2 \leq j \leq m$ since if $t_{j-1} = t_j$, then the Z-chambers $Z(C_{j-1}), Z(C_j)$, and $Z(C_{j+1})$ all intersect at the panel corresponding to t_j, and therefore the gallery can skip $Z(C_j)$ and move from $Z(C_{j-1})$ directly to $Z(C_{j+1})$.

Since A is an apartment, we know that for any three chambers C, D, E in A, $\delta(C, E) = \delta(C, D)\delta(D, E)$. Therefore, $1 = \delta(C_0, C_0) = w_1 \cdots w_m \in \tilde{W}_1 \cdots \tilde{W}_m$. By Lemma 1.1.1, $\ell(\delta(C_0, C_0)) \geq m$, which is a contradiction. Therefore no circuit can exist, and $Z(A)$ is a tree. □

The following now immediately follows by Lemma 3.1.2.

Lemma 7.2.3. $Z(\Delta)$ is a tree.

Now we are ready to prove the main result of this section.

Theorem 7.2.1. If G is a group satisfying (A), then G is not FP_2 and therefore not finitely presented.

Proof. Let $X = Z(\mathcal{C}_+) \times Z(\mathcal{C}_-)$ with $Z = |K(\mathcal{S}')|$. We have shown that $Z(\mathcal{C}_\pm)$ are both trees in Lemma 7.2.3, and both are clearly CW-complexes. Therefore, the product X is a product of two trees which is contractible as a product of contractible spaces and is a CW-complex, where the cells are products of non-empty cells from the two trees.

Now it remains to show that the G-action on X described above yields finite stabilizers of unbounded order.

Now we examine the cell stabilizers. Let $C_\pm$ be the fundamental chamber in $\mathcal{C}_\pm$. Then $G_{[C_\pm, z]} = B_\pm \langle S_z \rangle B_\pm$. As discussed above, the strong transitivity of the action of G on its twin building implies that every cell stabilizer is conjugate to $G_{([C_+,z],[wC_-,z'])} = B_+\langle S_z \rangle B_+ \cap wB_-\langle S_{z'} \rangle B_- w^{-1}$ for some $z, z' \in Z$, $w \in W$. Since $\langle S_z \rangle$ and $\langle S_{z'} \rangle$ are both finite, this intersection is always finite by Lemma 6.6.4. Thus all cell stabilizers are finite. Moreover, if we let z, z' be interior points of Z, then the stabilizer of the cell $([C_+, z], [wC_-, z'])$ is $B_+W_JB_+ \cap wB_-W_JB_-w^{-1}$ which contains $B_+ \cap wB_-w^{-1}$ as a subgroup. We know that this subgroup grows without bound as $\ell(w) \to \infty$ by Proposition 6.6.1, so we have stabilizers of unbounded order.

Thus Theorem 0.0.1 implies that G is not FP_2. $\square$

We state a special case of this as a quick corollary which yields strong evidence that the conjecture is true:

Corollary 7.2.1. Suppose that G has Weyl group W with generating $S = J \cup \{s, t\}$ such that $m(s, t) = \infty$ and both $J \cup \{s\}$ and $J \cup \{t\}$ are spherical. Then G is not FP_2.

As mentioned in the introduction, this is the case closest to the 2-spherical case where G is known to be FP_2. This was expected to be the most difficult case, and it would be surprising if relaxing the sphericity conditions on the group resulted in a group that is FP_2.

We also record the corresponding result for the action of $B_\pm$ on $\mathcal{C}_\mp$. Note that Lemma 6.1.1(ii) states that $B_\pm$ in fact acts transitively on each w-sphere in $\mathcal{C}_\mp$. The result uses the same arguments as for the whole group, except it focuses only on one half of the building. Hence the realization is a tree instead of a product of two trees. There are still finite subgroups of unbounded order: the examples of such subgroups in G given in Proposition 6.6.1 already lie inside B_+ or B_-. Also, the cell stabilizers are all conjugate in G to $B_\pm \cap B_\mp \langle S_z \rangle B_\mp$, for some $z \in Z$. These stabilizers are finite by Lemma 6.6.4.

Proposition 7.2.1. Suppose that G is a group satisfying (A). Then the subgroups B_+ and B_- are not FP_1 and hence not finitely generated.

Remark 7.2.2. Proposition 7.2.1 in fact holds for any parabolic subgroup of spherical type, i.e. a subgroup of the form $B_\pm W_J B_\pm$ where W_J is finite. This follows from Lemma 6.6.3 since, in general, if H is a finite index subgroup of G, then G is FP_n if and only if H is FP_n.

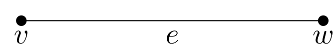
Now we make another remark that allows a nice decomposition of our group so long as there is at least one ∞ in the diagram.

Remark 7.2.3. Note that in proving that $Z(\Delta)$ is a tree, we never made use of the fact that $J \cup \{s\}$ and $J \cup \{t\}$ were spherical. This was only used to establish finite stabilizers when discussing finiteness properties.

The following theorem from Bass-Serre theory becomes useful in light of Remark 7.2.3. Here Serre refers only to the combinatorial context, so we ignore the additional topological structure coming from our context. Before stating the theorem, we need one more definition.

Definition 7.2.2. Let G be a group acting on a graph Γ. A fundamental domain of Γ for the action of G is a subgraph Γ_0 if $\Gamma_0 \cong G\backslash\Gamma$.

Theorem 7.2.2. (Theorem 6, I.4.1 [Ser03]) Suppose G acts on a graph Γ with a segment

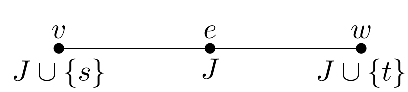
$Z =$ 

as fundamental domain. Then the homomorphism $G_v *_{G_e} G_w \to G$ induced by the inclusions $G_v \to G$ and $G_w \to G$ is an isomorphism if and only if Γ is a combinatorial tree.

We tie Remark 7.2.3 and Theorem 7.2.2 together to obtain the following result:

Proposition 7.2.2. If G has Coxeter system (W, S) such that there exist generators $s, t \in S$ such that $m(s, t) = \infty$, then G acts on a tree with a segment as fundamental domain. Furthermore, if we name the edge e with vertices v and w, then $G = G_v *_{G_e} G_w$ is the amalgamated product of the vertex stabilizers over the edge stabilizer.

Proof. Set $J := S \setminus \{s, t\}$ Then consider the Z-realization of the building Δ of type (W, S) with Z the following edge e:



Define $Z_u = Z$ for $u \in J$, $Z_s = v$, and $Z_t = w$. As noted in Remark 7.2.3, the arguments showing that $Z(\Delta)$ in the more general case will still hold here since it doesn't need J, $J \cup \{s\}$, and $J \cup \{t\}$ to be spherical. Hence $Z(\Delta)$ is a tree. Ignoring the topological structure, it is also a combinatorial tree with the segment Z as fundamental domain for the action of G. By Theorem 7.2.2, we obtain the decomposition $G = G_v *_{G_e} G_w$. $\square$

7.3 Groups satisfying the condition (B)

Definition 7.3.1. Suppose G has Coxeter system (W, S). Then we say G satisfies (B) if $W = \langle S \rangle$ such that

$$S = \coprod_{i=1}^{n} J_i, \ n \geq 2,$$

where all the J_i are spherical subsets of S but $m(s,t) = \infty$ whenever $s \in J_i$ and $t \in J_j$ for $i \neq j$.

Remark 7.3.1. A special case of (B) is when the diagram has all labels ∞. As mentioned in the introduction, it was expected that the group was not finitely presented in this case, but no proof was recorded in the literature. To see how (B) applies, let $S = \{s_1, \ldots, s_n\}$; then set $J_i = \{s_i\}$ for $1 \leq i \leq n$.

Remark 7.3.2. Since $m(s,t) = \infty$ whenever $s \in J_i$ and $t \in J_j$ for $i \neq j$, there are no relations between such generators and hence no relations between words in W_{J_i} and words in W_{J_j}. It follows that $W = W_{J_1} * \cdots * W_{J_n}$, the free product of the spherical subgroups generated by each J_i.

We now repeat the strategy for showing that groups satisfying the condition (A) are not FP_2. That is, we choose an appropriate Z such that $X = Z(\mathcal{C}_+) \times Z(\mathcal{C}_-)$ is a product of two trees and that the action of G on X has the desired properties.

Let $\mathcal{S}$ be the set of spherical subsets of S. Define $\mathcal{S}' := \{\emptyset, J_1, J_2, \ldots, J_n\} \subset \mathcal{S}$ and define $Z = |K(\mathcal{S}')|$ to be the geometric realization of the flag complex of this subset of spherical subsets of S. Define the s-panels to be $Z_s = |K(\mathcal{S}'_{\geq s})|$ for all $s \in S$. Then $Z_s = Z_t$ for all $s, t \in J_i$, $1 \leq i \leq n$, by definition. Z is then the simplicial complex in Figure 7.2, where the panel Z_s for $s \in J_i$ corresponds to the vertex labeled J_i and all other points are interior points of Z which do not lie in any panels.

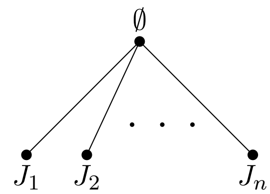


Figure 7.2: $Z = |K(\mathcal{S}')|$

Lemma 7.3.1. Let $\Delta \in \{\mathcal{C}_+, \mathcal{C}_-\}$ and δ be the corresponding Weyl distance. Let $C, D \in \Delta$.

(1) Let $1 \leq i \leq n$ and $z \in Z_s$ for some $s \in J_i$. Then $\delta(C, D) \in W_{J_i}$ if and only if $[C, z] = [D, z]$.

(2) Let z be an interior point of Z. Then $[C, z] = [D, z]$ if and only if $C = D$.

Proof. If $z \in Z_s$ for some $s \in J_i$, then $z \in Z_s$ for all $s \in J_i$. Also, $z \notin Z_t$ for any $t \in J_j$ for $j \neq i$. Hence $S_z = J_i$. Therefore $\delta(C, D) \in W_{J_i}$ if and only if $[C, z] = [D, z]$ by definition of the equivalence relation.

If z is an interior point, then $S_z = \emptyset$ so $\langle S_z \rangle = \{1\}$. Therefore $[C, z] = [D, z]$ if and only if $\delta(C, D) = 1$ if and only if $C = D$. □

Due to the decomposition of W as a free product as noted in Remark 7.3.2, the proof that $Z(\Delta)$ is a tree is much easier. In particular, we do not need to show that the apartments are trees first.

Lemma 7.3.2. $Z(\Delta)$ is a tree.

Proof. Since Z is connected, $Z(A)$ is connected for any apartment A of Δ by Lemma 3.1.3. The fact that any two chambers in Δ share some apartment then implies that $Z(\Delta)$ is connected.

Now suppose that there is a circuit in $Z(\Delta)$. Since Z itself is a tree, the circuit must involve more than one Z-chamber. Fix a point p in the circuit, with $p \in Z(C_0)$ for some chamber C_0. We may assume that p is an interior point of this Z-chamber since the circuit cannot lie in just one Z-chamber. From Lemma 7.3.1, we know that two distinct Z-chambers cannot intersect at an interior point. Therefore, we can only glue Z-chambers along s-panels $s \in S$. Moreover, we can glue distinct Z-chambers along at most one distinct panel (that is, we consider the panel Z_s for all $s \in J_i$ to be just one panel) since if we glue along two panels, we have by 7.3.1(1) that $\delta(C, D) \in W_{J_i} \cap W_{J_j}$ for some $i \neq j$, so $\delta(C, D) = 1$ and thus $C = D$. Suppose our circuit from p back to itself passes through Z-chambers $Z(C_0), Z(C_1), \ldots, Z(C_m) = Z(C_0)$, where we have $Z(C_m) = Z(C_0)$ since p is an interior point of $Z(C_0)$, and $Z(C_{j-1})$ and $Z(C_j)$ are glued together along their s_j-panel for some $s_j \in J_{i_j} \subset S$. We may assume that s_{j-1} and s_j do not lie in the same J_i, that is $J_{i_{j-1}} \neq J_{i_j}$, since $J_{i_{j-1}} = J_{i_j}$ would correspond to having $Z(C_{j-1}), Z(C_j)$, and $Z(C_{j+1})$ all sharing a vertex. In this case, our gallery could move directly from $Z(C_{j-1})$ to $Z(C_{j+1})$, so we could remove $Z(C_j)$. Then we get a corresponding gallery in Δ from C_0 to itself passing successively through $C_1, C_2, \ldots, C_{m-1}$. Since $Z(C_{j-1})$ and $Z(C_j)$ are glued along Z_{s_j}, we have $\delta(C_{j-1}, C_j) = w_j \in W_{J_{i_j}}$. Therefore we have $\delta(C_0, C_0) = w_1 \cdots w_m = 1$. However, given Remark 7.3.2, this is impossible and thus provides a contradiction. Thus there are no circuits in $Z(\Delta)$, so it is a tree. □

Theorem 7.3.1. If G satisfies contiion (B), then G is not FP_2 and therefore not finitely presented.

Proof. Let $X = Z(\mathcal{C}_+) \times Z(\mathcal{C}_-)$. From Lemma 7.3.2, X is a product of two trees and is therefore a 2-dimensional conractible CW-complex. G acts on X as before. By strong transitivity of the action of G on the twin building, all cell stabilizers are conjugate to $B_+\langle S_z\rangle B_+ \cap wB_-\langle S_{z'}\rangle B_- w^{-1}$ for some $w \in W$, $z, z' \in Z$. These groups are always finite since S_z and $S_{z'}$ are always spherical subsets of S. Hence all cell stabilizers are finite. If $z \in Z$ is an interior point, then $G_{([C_+,z],[wC_-,z])} = B_+ \cap wB_-w^{-1}$, which is finite but of unbounded order as $\ell(w) \to \infty$. Therefore Theorem 0.0.1 applies, and G is not FP_2. □

The fact that the Borel subgroups are not finitely generated quickly follows by the same discussion just before Proposition 7.2.1.

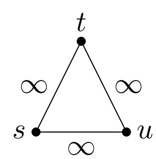
Proposition 7.3.1. Suppose G satisfies condition (B). Then the subgroups B_+ and B_- are not FP_1 and hence not finitely generated.

7.4 Rank 3 cases

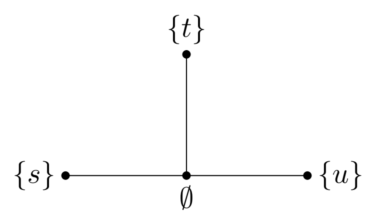
Now that we have proved the conjecture for these two large classes of groups, we can prove that the conjecture is true for all rank 3 cases with at least one ∞ label in the diagram.

Theorem 7.4.1. Suppose that G has rank 3 Weyl group generated by $S = \{s, t, u\}$ with at least one ∞ label in the corresponding Coxeter diagram. Then G is not FP_2 and is therefore not finitely presented.

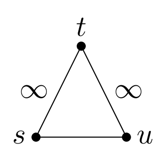
Proof. First, consider the case where all labels are infinite:



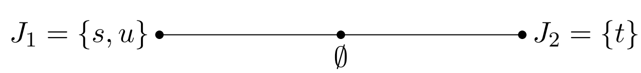
Then we apply Theorem 7.3.1 where $J_1 = \{s\}$, $J_2 = \{t\}$, and $J_3 = \{u\}$ since this decomposition of S satisfies the condition (B). In this case, Z is the complex



Now suppose that the diagram has two infinite labels:



This case also follows from Theorem 7.3.1. Indeed, set $J_1 = \{s, u\}$ and $J_2 = \{t\}$, and this decomposition satisfies (B). With this decomposition, Z is the complex



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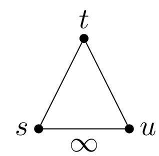
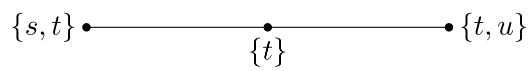
Lastly, suppose there is just one infinite label, say $m(s,u)=\infty$:



Then the result follows from Theorem 7.2.1 since the decomposition of S with $J=\{t\}$ and $K=\{s,u\}$ satisfies (A). In this case, Z is the following complex:



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