Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay

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Abstract

Time delay systems refer to control systems whose change of current state depends on the past values of its state and/or input. Input delay in a control system emerges when the transmission of the control signal from the controller to the actuator takes a certain amount of time. This lagging effect in the input can be caused by long-distance transmission of the control signal or time-consuming computation of a control algorithm carried out by the controller.

A fundamental problem in the control of linear systems with input delay is the problem of stabilization. The importance of such a problem is clear from the observation that feedback laws designed without consideration of the delay mostly fail to stabilize given a significantly large delay. To overcome an arbitrarily large delay, a predictor feedback law predicts the future state of the system as the sum of the zero input solution and the zero state solution of the system. Such a prediction of the future state cancels the effect of the input delay and results in a stable closed-loop system free of delay. However, the term corresponding to the zero state solution in the predictor feedback law is distributed because the zero state solution is a convolution between the state transition matrix and the input term. This causes difficulty in the implementation of the predictor feedback law. A truncated predictor feedback law discards the distributed term to avoid such difficulty. A delay independent truncated predictor feedback law further drops the delay-dependent state transition matrix in the truncated predictor feedback law to overcome uncertain or time-varying delay.

This dissertation presents the stabilization of linear systems with input delay via truncated predictor based feedback designs. The first part of the dissertation is dedicated to the design of truncated predictor based feedback laws with a constant feedback parameter. It is shown through examples that such feedback laws cannot stabilize general, possibly exponentially unstable, linear systems with a sufficiently large delay. Admissible delay bounds with stability guarantee are then established. Comprehensive analysis of stabilization is presented for linear systems with time-varying delay under either state feedback or output feedback.

The second part of the dissertation is dedicated to devising truncated predictor based feedback laws for linear systems with all open loop poles at the origin or in the open left-half plane. A time-varying feedback parameter design for the truncated predictor based feedback laws is motivated by improving the closed-loop performance over constant feedback parameter designs and by enabling the feedback laws to accommodate a completely unknown delay. In particular, time-varying feedback parameters in a delay independent truncated predictor feedback law are proposed to result in a smaller overshoot and a higher convergence rate of the closed-loop system. Such a time-varying parameter design still requires an upper bound of the delay to be known. In the absence of any knowledge of the delay, a control scheme is

proposed that equips the delay independent truncated predictor feedback law with a delay independent update algorithm for the feedback parameter. The implementation of such a control scheme is simple because only the current state, and no knowledge of the delay, is required. Further work would focus on devising novel control schemes that completely adapt to an unknown input delay in general linear systems with possibly exponentially unstable open loop poles.

Notation

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{R}_0^+$	the set of non-negative real numbers
$\mathbb{R}^n$	the set of real vectors of dimension n
$\mathbb{R}^{n\times m}$	the set of real matrices of dimensions $n \times m$
$\mathbb{C}$	the set of complex numbers
j	the imaginary unit $\sqrt{-1}$
$\mathrm{Re}(\cdot)\,(\mathrm{Im}(\cdot))$	the real (imaginary) part of a complex number
$\lvert\cdot\rvert$	the absolute value of a scalar, the Euclidean norm of a vector or the norm of a matrix induced by a vector Euclidean norm
0	a zero scalar, vector or matrix of appropriate dimensions
$I\,(I_n)$	an identity matrix of appropriate dimensions (of dimensions $n \times n$)
$I[a,b]$	the set of integers within the interval $[a,b]$, where $a,b \in \mathbb{R}$ and $a \le b$. Either side of the interval can be open if a or b is replaced by ∞
$\dot{x}(t)$	the first order derivative of $x(t) : \mathbb{R} \to \mathbb{R}^n$ with respect to time t
$C([t_1,t_2],\mathbb{R}^n)$	the set of $\mathbb{R}^n$-valued continuous functions on $t \in [t_1,t_2]$
$D([k_1,k_2],\mathbb{R}^n)$	the set of $\mathbb{R}^n$-valued functions on $k \in I[k_1,k_2]$, where $k,k_1,k_2 \in \mathbb{Z}$
$\lvert\lvert f\rvert\rvert_C$	the continuous norm $\sup_{t\in[t_1,t_2]} \lvert f(t)\rvert$ of $f \in C([t_1,t_2],\mathbb{R}^n)$
$\lvert\lvert f\rvert\rvert_D$	the discrete norm $\max_{k\in I[k_1,k_2]} \lvert f(k)\rvert$ of $f \in D([k_1,k_2],\mathbb{R}^n)$
$L_2([t_1,t_2],\mathbb{R}^n)$	the set of $\mathbb{R}^n$-valued square integrable functions on $t \in [t_1,t_2]$
$AC([t_1,t_2],\mathbb{R}^n)$	the set of $\mathbb{R}^n$-valued absolutely continuous functions f on $t \in [t_1,t_2]$ with $\dot{f} \in L_2([t_1,t_2],\mathbb{R}^n)$
$\lvert\lvert f\rvert\rvert_{AC}$	the absolutely continuous norm $\sup_{t\in[t_1,t_2]} \lvert f(t)\rvert^2 + \int_{t_1}^{t_2} \lvert\dot{f}(t)\rvert^2 \mathrm{d}s$ of $f \in AC([t_1,t_2],\mathbb{R}^n)$
$C^k([t_1,t_2],\mathbb{R}^n)$	the set of $\mathbb{R}^n$-valued functions having continuous kth order time derivative on $t \in [t_1,t_2]$

$PC([t_1,t_2],\mathbb{R}^n)$	the set of $\mathbb{R}^n$-valued piecewise continuous functions on $t\in[t_1,t_2]$
MF	the set of multivariate functions $f(x,t):[0,1]\times\mathbb{R}\to\mathbb{R}^n$
$f_x(x,t)$	$\frac{\partial}{\partial x}f(x,t)$ for $f(x,t)\in MF$
$f_{xt}(x,t)$	$\frac{\partial^2}{\partial t\partial x}f(x,t)$ for $f(x,t)\in MF$
$\vert\vert f(t)\vert\vert$	$\sqrt{\int_0^1 \vert f(x,t)\vert^2\mathrm{d}x}$ for $f(x,t)\in MF$
L^1	$\left\{x(t)\ \middle\vert\ x(t):[0,\infty)\to\mathbb{R}^n \text{ and } \int_0^\infty \vert x(t)\vert\mathrm{d}t<\infty\right\}$
x_t	the restriction of $x(s):\mathbb{R}\to\mathbb{R}^n$ to $s\in[t-\tau,t]$, for some $\tau\in\mathbb{R}_0^+$
$\dot{x}_t$	the restriction of $\dot{x}(s):\mathbb{R}\to\mathbb{R}^n$ to $s\in[t-\tau,t]$, for some $\tau\in\mathbb{R}_0^+$
$x_t(\theta)$	$x(t+\theta)$, where $x(s):\mathbb{R}\to\mathbb{R}^n$ and $\theta\in[-\tau,0]$, for some $\tau\in\mathbb{R}_0^+$
x_k	the restriction of $x(p):\mathbb{Z}\to\mathbb{R}^n$ to $p\in I[k-r,k]$, for some $r\in\mathbb{N}$
$x_k(l)$	$x(k+l)$, where $x(p):\mathbb{Z}\to\mathbb{R}^n$ and $l\in I[-r,0]$, for some $r\in\mathbb{N}$
$\mathrm{tr}(\cdot)$	the trace of a square matrix
$\det(\cdot)$	the determinant of a square matrix
$\lambda(\cdot)$	the set of eigenvalues of a square matrix
$\lambda_{\min}(\cdot)\,(\lambda_{\max}(\cdot))$	the minimum (maximum) eigenvalue of a real symmetric matrix
$v^{\mathrm{T}}(A^{\mathrm{T}})$	the transpose of a vector v (a matrix A)
$A>B\,(A\geq B)$	$A-B$ is positive definite (semidefinite), where A and B are real symmetric matrices
$A<B\,(A\leq B)$	$A-B$ is negative definite (semidefinite), where A and B are real symmetric matrices

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1. Introduction

1.1. Introduction to Time Delay Systems

The phenomenon of time delay is a commonplace in almost every scientific discipline. Time delay refers to the amount of time it takes for the matter, energy or information in a dynamic system to transfer from one place to another or to make their full impact on the system after their emergence. Such a lagging effect causes the change of current state of the system to rely on past values of its state and/or input. For instance, the economic model in [150] reveals that economic growth relies on population growth and technological advancement. In particular, the population growth does not take effect on the economic growth until it transitions to the labor growth, which potentially takes a couple of decades. Similarly, the technological advancement does not boost the economy until the productivity of the workforce is improved through technology innovations. Other examples of time delay in the study of biology, physics, mathematics and engineering are many, and we will mention a few in the following subsection as examples.

1.1.1. Examples of Time Delay Systems:

1.1.1.1. A Predator-prey Model: In biology studies, the Lotka–Volterra equations are differential equations that describe predator-prey interactions in a natural ecosystem. A typical set of the Lotka–Volterra equations takes the following form (see [116]),

$$\dot{x}_1(t) = \alpha x_1(t)\Big(1 - \frac{x_1(t)}{K}\Big) - \beta x_1(t)x_2(t), \tag{1.1}$$

$$\dot{x}_2(t) = -\gamma x_2(t) + \omega x_1(t-\tau)x_2(t-\tau), \tag{1.2}$$

where $x_1(t)$ and $x_2(t)$ are the populations of the prey and the predator, respectively, and α, K, β, γ and ω are positive constants determined by the characteristics of the prey and the predator and the natural ecosystem in which they inhabit. The first term on the right-hand side of (1.1) suggests that even in the absence of predation, the population of the prey cannot exceed K, which represents the carrying capacity of the ecosystem for the prey population. The second term on the right-hand side of the equation describes the negative effect of the predation on the population of the prey. Basically, the predation causes the population of the prey to decrease, and the rate of this decrease is proportional to the number of interactions between the prey and the predator, which can be characterized by the product of the population of the prey and that of the predator.

The right-hand side of (1.2) also contains two terms. The first term suggests that, in the absence of the prey, the population of the predator decreases exponentially toward zero. The second term indicates that the positive effect of the predation on the population of the predator does not occur instantly. The positive constant τ represents the time it takes for the predation to show its impact on the growth of the predator population. Compared to the predator-prey model without consideration of the delay τ (see [50]), the differential equations (1.1) and (1.2) are more accurate in describing the actual population dynamics of the two species.

1.1.1.2. The Distribution of Primes: In the study of the distribution of primes, mathematicians formulated the asymptotic behavior of the prime counting function $\pi(x)$ with respect to the prime x in the Prime Number Theorem [1]. The prime countering function $\pi(x)$ is the number of prime numbers that are not greater than the prime number x. It turns out that such a theorem can be proved from the perspective of time delay systems ([114], [129]). Reference [114] defined a smooth curve $y(x)$ that best fits the actual variation of $\pi(x)$. By using a probability argument for the distribution of primes, [114] obtained

$$2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x}\left(x^{\frac{1}{2}}\right)=0, \tag{1.3}$$

where the first and second order derivatives with respect to x are defined as $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, respectively. Reference [129] defined two functions of x, v and w, as

$$2^v=\ln x$$

and

$$w(v)=\frac{\mathrm{d}y}{\mathrm{d}x}\log x-1,$$

which, together with (1.3), imply that

$$\frac{\mathrm{d}x}{\mathrm{d}v}=\alpha x\log x, \tag{1.4}$$

$$\frac{\frac{\mathrm{d}w}{\mathrm{d}v}}{1+w(v)}=\alpha\left(1-2^{v-1}\frac{\mathrm{d}y}{\mathrm{d}x}\left(x^{\frac{1}{2}}\right)\right), \tag{1.5}$$

$$1+w(v-1)=2^{v-1}\frac{\mathrm{d}y}{\mathrm{d}x}\left(x^{\frac{1}{2}}\right), \tag{1.6}$$

[1]The Prime Number Theorem: $\lim_{x\to\infty}\frac{\pi(x)}{\frac{x}{\log(x)}}=1$.

where $\alpha = \ln 2$. Then, the following time delay system is obtained from (1.5) and (1.6),

$$\frac{\mathrm{d}w}{\mathrm{d}v} = -\alpha w(v-1)(1+w(v)). \tag{1.7}$$

According to [129], the solution to (1.7) satisfies that $w(v) \to 0$ as $v \to \infty$. Therefore, $\frac{\mathrm{d}y}{\mathrm{d}x}$ approaches $1/\log x$ as x goes to infinity, which coincides with the statement of the Prime Number Theorem. This example shows that the study of time delay systems facilitates the development of number theory.

1.1.1.3. A Traffic Flow Model: The efficiency of a transportation system relies on the smooth flow of traffic. A simple mathematical model for traffic flow can be established by considering n number of automobiles that move along a straight road (see [34]). Denote the position of the ith automobile at time t as $x_i(t)$ and let the $(i+1)$th automobile move in front of the ith one, $i \in I[1,n]$. The following sequence of equations describe the movement of the automobiles in terms of their positions,

$$\ddot{x}_i(t) = k(\dot{x}_i(t-\tau_i) - \dot{x}_{i+1}(t-\tau_i)), \;\; i \in I[1,n]. \tag{1.8}$$

Basically, the ith equation implies that the acceleration of the ith automobile at time t is proportional to the relative velocity between the ith and $(i+1)$th automobiles at a past time instant $t-\tau_i$, with the coefficient of proportionality $k < 0$. The constant $\tau_i \in \mathbb{R}_0^+$ represents the time it takes for the driver of the ith automobile to sense the traffic condition, determine a necessary action to accelerate or decelerate the automobile, and regulate the gas paddle or the brake based on his/her decision. These time delays are caused by the lagged behaviors of the drivers, which cannot be neglected, especially in modeling fast moving traffic.

1.1.2. Delay Differential Equations: The study of delay phenomena scatters among different scientific disciplines. A systematic study of time delay and its effects within a unified framework relies on modeling time delay systems by delay differential equations (also known as functional differential equations, differential-difference equations, equations with dead time, after-effects or deviating arguments). A typical delay differential equation is given by

$$\dot{x}(t) = f(t, x_t, \dot{x}_t), \tag{1.9}$$

where $f : \mathbb{R} \times C^1([-\tau,0],\mathbb{R}^n) \times C([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$ is a general nonlinear functional, x_t and $\dot{x}_t$ are respectively the restrictions of the state $x(s)$ and its derivative $\dot{x}(s)$ to $s \in [t-\tau,t]$, and $\tau \in \mathbb{R}_0^+$ is the amount of the delay. The equation indicates that the change of the current state $x(t)$ depends on the values of the state over the time interval $[t-\tau, t]$, rather than solely on the current value of the state $x(t)$.

Equation (1.9) belongs to the group of delay differential equations of neutral type whose right-hand side depends on the derivative of x_t. In contrast to the neutral type is the retarded type of delay differential equations whose right-hand side does not depend on the derivative of x_t. A retarded equation takes the form of

$$\dot{x}(t) = f(t, x_t), \tag{1.10}$$

where $f : \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a general nonlinear functional . The focus of this dissertation is on the study of delay differential equations of retarded type. We refer to the rich literature on delay differential equations of neutral type (see [47], [33], [34], [60], [63], and the references therein).

We note that the expression for a general delay free ordinary differential equation is given by

$$\dot{x}(t) = f(t, x(t)), \tag{1.11}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a general nonlinear function. On the other hand, f in (1.9) or (1.10) has two arguments, the time t and a function $x(s)$ restricted to $s \in [t-\tau, t]$. It is thus natural that delay differential equations are also referred to as functional differential equations.

The way x_t appears in f in (1.9) or (1.10) determines whether the delay is distributed or discrete (pointwise). As the names suggest, if f depends on x_t over a time interval $[t_1, t_2]$, where $t - \tau \le t_1 \le t_2 \le t$, then the delay is distributed. For example, the following scalar equation with distributed delay describes the circumnutation of a plant to geotropic movements [53],

$$\dot{\alpha}(t) = -k \int_1^\infty f(n) \sin \alpha(t - nt_0) \mathrm{d}n, \tag{1.12}$$

where $\alpha(t)$ denotes the angle of the plant with the plumb line, k is some constant, $f(n)$ is an exponentially decaying function and t_0 denotes the geotropic reaction time of the plant. If, on the other hand, f depends only on the value of x_t at distinct time instants, then the delay is discrete or pointwise. Examples of delay differential equations with discrete delay are (1.2), (1.7) and (1.8).

1.1.3. The Initial Condition, the Cauchy Problem and the Step Method: The initial condition of a delay differential equation is defined differently in comparison with that of an ordinary differential equation without delay. We take equation (1.7) for illustration of its initial condition. We first rewrite the equation as

$$\dot{x}(t) = -\alpha x(t-1)(1 + x(t)). \tag{1.13}$$

Let $x(t)$ start its evolution from $t = 0$. The value of $x(t-1)$ is required for computing $\dot{x}(t)$, $t \in [0, 1]$. As time elapses beyond $t = 1$, no information of $x(t)$ on $t \leq 0$ is required. In order to depict the complete evolution of $x(t)$ on $t \in [0, \infty)$, it is necessary for (1.13) to take

$$x_0 = \phi(\theta), \ \ \theta \in [-1, 0], \tag{1.14}$$

as its initial condition, where ϕ is typically assumed to be a piecewise continuous function. Similarly, the initial condition of (1.9) or (1.10) can be defined by

$$x_0 = \psi(t_0 + \theta), \ \ \theta \in [-\tau, 0], \tag{1.15}$$

where the state of the equation starts its evolution from $t_0 \in \mathbb{R}$ and ψ is a piecewise continuous function. On the other hand, the initial condition of an ordinary differential equation without delay is defined by $x_0 = x(t_0)$ if the state of the equation starts its evolution from t_0. The distributed feature of the initial condition of a delay differential equation implies that the equation is infinite-dimensional. In contrast to delay differential equations, ordinary differential equations without delay are finite-dimensional .

Closely related to the initial condition of a delay differential equation is the Cauchy problem for the equation. The Cauchy problem, also referred to as the initial value problem, is to determine whether the equation admits a unique solution given an initial condition. For both the neutral type equation (1.9) and the retarded type equation (1.10), comprehensive analysis was established in [47] that addresses the problem. Sufficient conditions on the general, possibly nonlinear, functional f that guarantee the existence and uniqueness of the solution were established in [47], [40], [45], [60] and [34]. Most of these sufficient conditions do not involve the explicit solution. When f allows a delay differential equation to be explicitly solved, we can address the Cauchy problem by directly solving the equation.

Consider equation (1.13) for example. Given the initial condition $\phi(\theta) = c \in \mathbb{R} \setminus \{0\}$, $\theta \in [-1, 0]$, $x(t)$ satisfies the ordinary differential equation without delay on $t \in [0, 1]$,

$$\dot{x}(t) = -\alpha c(1 + x(t)), \tag{1.16}$$

and thus,

$$x(t) = (1+c)\mathrm{e}^{-\alpha ct} - 1, \;\; t \in [0,1]. \tag{1.17}$$

Forwarding the time interval of interest to $t \in [1,2]$, we see that $x(t)$ satisfies the ordinary differential equation without delay,

$$\dot{x}(t) = -\alpha\Big((1+c)\mathrm{e}^{-\alpha c(t-1)} - 1\Big)(1+x(t)), \tag{1.18}$$

which is obtained by using the solution of $x(t)$ on $t \in [0,1]$ in equation (1.13). From (1.18), we obtain

$$x(t) = (1+c)\mathrm{e}^{\alpha(t-1-c)+\frac{1+c}{c}(\mathrm{e}^{-\alpha c(t-1)}-1)} - 1, \;\; t \in [1,2]. \tag{1.19}$$

Following the same solution procedure for each of the time interval $[k, k+1]$, $k \in \mathbb{N}$, in a successive manner, we obtain the explicit solution of $x(t)$ on $t \in [0,\infty)$, which obviously implies the existence and uniqueness of the solution. Such a method to obtain the explicit solution to a delay differential equation is referred to as the step method, which was originally proposed in [12]. The step method is applicable to both equations of neutral type and retarded type with initial conditions that are possibly time-varying.

The existence and uniqueness of the solution to a delay differential equation are only byproducts of the use of the step method. In most circumstances, the step method provides more intricate properties of the solution. For example, the solution of a retarded type equation becomes smoother as time progresses, while it is not the case for a neutral type equation. Such conclusions were reached in [12] by employing the step method. Moreover, the solution to a neutral type equation obtained by the use of the step method facilitates the stability analysis of the time delay system described by the equation (see [63]).

1.2. Stability of Time Delay Systems

1.2.1. Stability Definitions: Major interest in the study of a time delay system is in the evolution of the state of the system as time tends to infinity. It is often desirable to find an equilibrium point in the state space and that the state stays within a small neighborhood of the point after some finite time, if such a point exists. Otherwise, a diverging state implies that the system is unstable. We here present stability definitions for system (1.10). It is assumed that given the initial condition $x_{t_0} = \psi(t_0+\theta) \equiv 0$, $\theta \in [-\tau, 0]$, $x(t) \equiv 0$, $t \geq t_0$, is a trivial solution of the system. We made this assumption without loss of generality because the stability of a non-trivial solution $\bar{x}(t)$ of the system is equivalent to that of the trivial solution $\tilde{x}(t) \equiv 0$ of the following system

$$\dot{\tilde{x}}(t) = f(t, (\tilde{x} + \bar{x})_t) - f(t, \bar{x}_t), \tag{1.20}$$

which is the delay differential equation governing the evolution of $\tilde{x}(t) = x(t) - \bar{x}(t)$ (see [45]).

Definition 1. *The trivial solution $x(t) \equiv 0$ of system (1.10) is stable if for every $\epsilon > 0$ and every $t_0 \geq 0$, there exists $\delta(\epsilon, t_0) > 0$ such that*

$$||x_{t_0}||_C \leq \delta \ \textit{implies} \ |x(t)| \leq \epsilon, \ \ t \geq t_0.$$

Definition 2. *The trivial solution $x(t) \equiv 0$ of system (1.10) is uniformly stable if δ in Definition 1 is independent of t_0.*

Definition 3. *The trivial solution $x(t) \equiv 0$ of system (1.10) is attractive if there exists $\delta(t_0) > 0$ such that*

$$||x_{t_0}||_C \leq \delta \ \textit{implies} \ \lim_{t \to \infty} x(t) = 0.$$

Definition 4. *The trivial solution $x(t) \equiv 0$ of system (1.10) is asymptotically stable if it is both stable and attractive.*

Definition 5. *The trivial solution $x(t) \equiv 0$ of system (1.10) is uniformly asymptotically stable if it is uniformly stable and there exists $\delta > 0$, independent of t_0, such that, for every $\epsilon > 0$, there exists $T = T(\delta, \epsilon)$ such that*

$$||x_{t_0}||_C \leq \delta \ \textit{implies} \ |x(t)| \leq \epsilon, \ \ t \geq t_0 + T.$$

Definition 6. *The trivial solution $x(t) \equiv 0$ of system (1.10) is globally uniformly asymptotically stable if δ in Definition 5 can be any positive number.*

In the case of a linear system, we often refer to stability (asymptotic stability) of its trivial solution as stability (asymptotic stability) of the system.

1.2.2. Lyapunov Stability Theorems: A straightforward approach to analyzing the stability of a time delay system is to obtain the analytic solution of the system by employing the step method. However, the step method ceases to work when the analytic solution is not available. Even when an analytic solution is available, it is in many instances challenging to express the solution as a simple function of $t \in \mathbb{R}$, as seen in the example for the illustration of the step method in Subsection 1.1.3. Therefore, stability analysis that does not involve the analytic solution is preferable. Similar to Lyapunov function based stability analysis for systems without delay, Lyapunov functional based stability analysis for time delay systems does not require an analytic solution. By picking a positive definite Lyapunov functional $V(t, x_t)$ and taking its time derivative along the system trajectory, we conclude that the system is stable if this time derivative is negative definite. The following theorem on the stability of system (1.10) lays foundation for such Lyapunov functional based stability analysis.

Theorem 1.1. ***(The Krasovskii Stability Theorem)*** *Suppose f in (1.10) maps bounded sets in $\mathbb{R} \times C([-\tau, 0], \mathbb{R}^n)$ to bounded sets in $\mathbb{R}^n$, $u, v, w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous and nondecreasing functions, $u(s), v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. The trivial solution of system (1.10) is uniformly stable if there exists a continuously differentiable functional $V(t, \phi) : \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}_0^+$ such that*

$$u(|\phi(0)|) \leq V(t, \phi) \leq v(||\phi||_C), \tag{1.21}$$

and the time derivative of $V(t, \phi)$ along the trajectory of the system satisfies

$$\dot{V}(t, \phi) \leq -w(|\phi(0)|). \tag{1.22}$$

If $w(s) > 0$ for $s > 0$, then, the trivial solution is uniformly asymptotically stable . If in addition $\lim_{s \to \infty} u(s) = \infty$, then, the trivial solution is globally uniformly asymptotically stable.

It was pointed out in [34] that the inclusion of the time derivative of x_t as an additional argument of the Lyapunov functional in Theorem 1.1 might make the stability conditions in the theorem easier to satisfy. The following theorem extends Theorem 1.1 by allowing such an additional argument in the Lyapunov functionals .

Theorem 1.2. ***(An extension of the Krasovskii Stability Theorem)*** *Suppose f in (1.10) maps bounded sets in $\mathbb{R} \times C([-\tau, 0], \mathbb{R}^n)$ to bounded sets in $\mathbb{R}^n$, $u, v, w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous and nondecreasing functions, $u(s), v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. The trivial solution of system (1.10) is uniformly stable if there exists a continuously differentiable functional $V(t, \phi, \dot{\phi}) : \mathbb{R} \times AC([-\tau, 0], \mathbb{R}^n) \times$*

$L_2([-\tau,0],\mathbb{R}^n) \to \mathbb{R}_0^+$ *such that*

$$u(|\phi(0)|) \le V(t,\phi,\dot{\phi}) \le v(||\phi||_{AC}), \tag{1.23}$$

and the time derivative of $V(t,\phi)$ *along the trajectory of the system satisfies*

$$\dot{V}(t,\phi,\dot{\phi}) \le -w(|\phi(0)|). \tag{1.24}$$

If $w(s) > 0$ *for* $s > 0$*, then, the trivial solution is uniformly asymptotically stable. If in addition* $\lim_{s\to\infty} u(s) = \infty$*, then, the trivial solution is globally uniformly asymptotically stable.*

In the application of the Lyapunov-Krasovskii Theorem or its extension, the choice of a delicate Lyapunov functional $V(t,x_t)$ that satisfies all the stability conditions is the key. Typically, the time derivative of a Lyapunov functional along the trajectory of a time delay system depends on x_t in one form or another. This makes the majorization of the time derivative by a negative definite term that depends solely on the current state $x(t)$ of the system difficult (see (1.22) or (1.24)). To overcome such difficulty, the following stability theorem takes an approach differently from Theorem 1.1 or 1.2. By picking a positive definite Lyapunov function $V(t,x(t))$, we can conclude the stability of the system if $\dot{V}(t,x(t))$ along the system trajectory is negative definite under a condition on the evolution of $V(s,x(s))$ over the time interval $s \in [t-\tau,t]$. This condition facilitates the majorization of the time derivative of the Lyapunov function by a negative definite term that depends solely on the current state $x(t)$.

Theorem 1.3. ***(The Razumikhin Stability Theorem)*** *Suppose* f *in (1.10) maps bounded sets in* $\mathbb{R} \times C([-\tau,0],\mathbb{R}^n)$ *to bounded sets in* $\mathbb{R}^n$*,* $u,v,w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *are continuous and nondecreasing functions,* $u(s)$ *and* $v(s)$ *are positive for* $s > 0$*,* $u(0) = v(0) = 0$*, and* v *is strictly increasing. The trivial solution of system (1.10) is uniformly stable if there exists a continuously differentiable function* $V(t,x(t)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_0^+$ *such that*

$$u(|x|) \le V(t,x) \le v(|x|), \tag{1.25}$$

and the time derivative of $V(t,x(t))$ *along the trajectory of the system satisfies*

$$\dot{V}(t,x(t)) \le -w(|x(t)|) \ \textit{whenever} \ V(t+\theta,x(t+\theta)) \le V(t,x(t)), \ \ \theta \in [-\tau,0]. \tag{1.26}$$

If in addition, $w(s) > 0$ *for* $s > 0$*, and there exists a continuous and nondecreasing function* $p(s) > s$

for $s > 0$ such that condition (1.26) is strengthened to

$$\dot{V}(t, x(t)) \leq -w(|x(t)|) \text{ whenever } V(t+\theta, x(t+\theta)) \leq p(V(t, x(t))), \;\; \theta \in [-\tau, 0], \tag{1.27}$$

then the trivial solution is uniformly asymptotically stable. If in addition $\lim_{s\to\infty} u(s) = \infty$, then the trivial solution is globally uniformly asymptotically stable.

1.3. Control Systems with Time Delay

1.3.1. Input and State Delays: In a control system, time delay can take place in the input and/or state of the system. One form of input delay is induced in the implementation of a digital controller in continuous-time control systems. A digital controller typically consists of a computer that generates an input signal according to a control algorithm, an A/D converter before the computer and a D/A converter after the computer. Given a complex control algorithm that demands heavy computation, the computer generates the input signal $u(t-\tau)$ in a non-negligible time $\tau \in \mathbb{R}^+$. Moreover, transformation between digital and analog signals carried out by the A/D and D/A converters also adds to the delay. Another form of input delay is induced by long-distance transmission of the input signal between controllers and controlled plants. This form of input delay typically appears in control of large networks where the controller and the controlled plant are located far apart. In general, all the time consumption related to the generation, processing and transmission of the input signal can be modeled as input delay.

State delay is also typical in control systems. One form of state delay appears in open loop systems. Examples of this form of state delay were given in Subsection 1.1.1, where the time delay systems are autonomous because no external input is applied to the systems. Another form of state delay appears in closed-loop systems when their open loop systems are subject to state feedback and input delay simultaneously. In this case, the controller utilizes the current state $x(t)$ to generate the current input signal $u(t)$. However, the actual input signal injected to the open loop systems lags behind the current input signal by the amount of the input delay. Therefore, the actual input appears as a function of past state in the closed-loop system.

The mathematical description for a control system with input and state delays is given by

$$\begin{cases} \dot{x}(t) \;=\; f(t, x_t, u_t), \\ y(t) \;=\; w(t, x(t)), \end{cases} \tag{1.28}$$

where $f : \mathbb{R} \times C([-\sigma, 0], \mathbb{R}^n) \times C([-\tau, 0], \mathbb{R}^m) \to \mathbb{R}^n$ is a general nonlinear functional, $w : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^q$ is a general nonlinear function, and $\tau, \sigma \in \mathbb{R}_0^+$ are the amount of the input delay and the state delay, respectively. A typical state feedback law takes the form of

$$u(t) = z(t, x_t, u_t), \tag{1.29}$$

where $z : \mathbb{R} \times C([-\tilde{\sigma}, 0], \mathbb{R}^n) \times C([-\tilde{\tau}, 0], \mathbb{R}^m) \to \mathbb{R}^m$, for some $\tilde{\sigma}, \tilde{\tau} \in \mathbb{R}_0^+$, is an appropriate functional and might contain linear operations such as addition, multiplication by a constant and integration, as well as nonlinear operations.

On the other hand, the output of a control system typically consists of physical quantities that can be directly measured. Therefore, output feedback laws that employ only the measurement of the output are more practical than state feedback laws. A common output feedback law design for time delay systems first seeks to utilize the measurement of u_t and $y(t)$ to construct an observer whose state $\hat{x}(t)$ approaches $x(t)$ asymptotically. Then, based on the observed state $\hat{x}(t)$, a feedback law is constructed whose structure replicates that of a state feedback law. If the state feedback law (1.29) achieved a certain control objective, the output feedback law

$$\begin{cases} \dot{\hat{x}}(t) = g(t, \hat{x}_t, u_t, y(t)), \\ u(t) = z(t, \hat{x}_t, u_t), \end{cases} \tag{1.30}$$

potentially achieves the same control objective, where $g : \mathbb{R} \times C([-\tilde{\sigma}, 0], \mathbb{R}^n) \times C([-\tilde{\tau}, 0], \mathbb{R}^m) \times \mathbb{R}^q \to \mathbb{R}^n$, for some $\tilde{\sigma} \in \mathbb{R}_0^+$, is a proper functional that guarantees

$$\lim_{t \to \infty} (x(t) - \hat{x}(t)) = 0.$$

While most control systems behave in a nonlinear manner when the state of the system is far away from the equilibrium point, near the equilibrium point, their dynamics can be efficiently approximated by a linear system. Such an approximation can be obtained by Jacobian linearization of the control systems at the equilibrium point. The mathematical description for a linear system with input and state delays is as follows,

$$\begin{cases} \dot{x}(t) = \int_{-\sigma}^{0} A(t, s) x(t+s) \mathrm{d}s + \int_{-\tau}^{0} B(t, s) u(t+s) \mathrm{d}s, \\ y(t) = C(t) x(t), \end{cases} \tag{1.31}$$

where $\tau, \sigma \in \mathbb{R}_0^+$ represent the input delay and the state delay, respectively, $A(t, s) : \mathbb{R} \times [-\sigma, 0] \to \mathbb{R}^{n \times n}, B(t, s) : \mathbb{R} \times [-\tau, 0] \to \mathbb{R}^{n \times m}$ are the dynamics matrix and control matrix of the system, respectively, and are of bounded variation with respect to s, and $C(t) : \mathbb{R} \to \mathbb{R}^{q\times n}$ is the sensor matrix of the system. System (1.31) is time-varying and the delay is of distributed type. The time-invariant version of system (1.31) is given by

$$\begin{cases} \dot{x}(t) = \int_{-\sigma}^{0} A(s)x(t+s)\mathrm{d}s + \int_{-\tau}^{0} B(s)u(t+s)\mathrm{d}s, \\ y(t) = Cx(t), \end{cases} \tag{1.32}$$

where $A(s) : [-\sigma, 0] \to \mathbb{R}^{n\times n}, B(s) : [-\tau, 0] \to \mathbb{R}^{n\times m}$ are the dynamics matrix and the control matrix of the system, respectively, and are of bounded variation, and $C \in \mathbb{R}^{q\times n}$ is the sensor matrix of the system. Furthermore, the discrete delay version of system (1.32) is given by

$$\begin{cases} \dot{x}(t) = \sum_{i=0}^{l} A_i x(t-\sigma_i) + \sum_{j=0}^{k} B_j u(t-\tau_j), \\ y(t) = Cx(t), \end{cases} \tag{1.33}$$

where $A_{\text{I}} \in \mathbb{R}^{n\times n}$, $i \in I[0, l]$, and $B_j \in \mathbb{R}^{n\times m}$, $j \in I[0, k]$, are the dynamics matrices and control matrices of the system, respectively, and $\tau_j \in \mathbb{R}_0^+$, $j \in I[0, k]$, and $\sigma_{\text{I}} \in \mathbb{R}_0^+$, $i \in I[0, l]$, are the lengths of the input and state delays, respectively. The attention of this dissertation is restricted to linear time-invariant systems with a single input delay,

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t-\tau), \\ y(t) = Cx(t), \end{cases} \tag{1.34}$$

and its stabilization via either state or output feedback. Occasionally in this dissertation, the constant delay τ in system (1.34) is allowed to be time-varying but bounded, namely, $\tau(t) : \mathbb{R} \to [0, D]$, where $D \in \mathbb{R}_0^+$ is an upper bound on the delay.

1.3.2. An Overview of Stabilization of Time Delay Systems: The study of control systems with time delay spans a wide range of problems, including stabilization, trajectory tracking, output regulation, disturbance rejection, performance improvement, robust control and adaptation to accommodate unknown system parameters.

Central to these problems is the problem of stabilization. The stabilization of a time delay system can be accomplished through two different paths. The first path extensively involves Lyapunov funtional based stability analysis. We pick a positive definite Lyapunov functional and compute its time derivative along the trajectory of the system. By designing feedback that renders the time derivative negative definite, we achieve closed-loop stability (see [55]). It turns out that such analysis is effective for systems with either input or state delay (see [45], [34], [33], [76], [97], [98], [99], [100], [101], [141] and references therein). Furthermore, the Lyapunov functional based stability analysis has shown its universal applicability to the study of various time delay systems, including time-varying, nonlinear and stochastic systems with delays allowed to be time-varying or even unknown (see [11], [13], [55], [14], [15], [47], [66], [67], [108] and [115], for a small sample of the literature).

The second path to the stabilization of a time delay system is more straightforward. Given a linear system with a single constant input delay, a feedback law completely cancels the effect of the delay by multiplying a feedback gain matrix with the state of the system at the future time ahead of the current time by the amount of the delay. Thanks to the linearility of the system, the future state can be explicitly predicted as the sum of the zero input solution and the zero state solution of the system. Such a feedback law is referred to as the predictor feedback law [79] due to the prediction of the future state. The spectrum of the closed-loop system under the predictor feedback is finite and can be arbitrarily assigned in the complex plane by an appropriate choice of the feedback gain matrix. Therefore, such a predictor feedback design is also referred to as finite spectrum assignment [77]. An alternative design that also leads to the predictor feedback law involves a key step of transforming the original open loop system with delay into an open loop system free of delay, and is referred to as the model reduction technique [7].

The predictor feedback laws for linear systems with input delay were generalized to stabilize linear systems with input and state delays in [61], [62] and [64], where the state of the systems at a future time is predicted by the use of the variation-of-constants formula (see [12] for the formula). Such feedback laws designed by the use of the variation-of-constants formula were further generalized to stabilize neutral type systems with input delay in [63]. Besides these generalizations, the combinated use of the finite spectrum assignment or the model reduction technique with control techniques for nonlinear systems such as cross-term forwarding, backstepping and/or recursive methods induced various prediction methods for the stabilization of linear and nonlinear systems with input and state delays (see [56], [57], [58], [67], [9] and [11]). Predictor based feedback laws were also developed for linear systems with input and state delays [149], [137].

In all these predictor based feedback designs, a distributed delay term appears in the resulting predictor based feedback laws. Such a distributed nature of the feedback laws originates from predicting the state of the system at a future time by the use of the variation-of-constants formula . It was pointed out that these distributed delay terms would cause difficulty in their implmenetation (see [113] and [91]).

Recently, a sequential predictors approach to the stabilization of linear systems with input delay was proposed that manage to observe the state of the system at the future time ahead of the current time by the amount of the delay without using the variation-of-constants formula (see [13], [85], [86] and [17]). The sequential predictors are a sequence of dynamic predictors that observe the future state of the system in a progressive manner. Specifically, the state of the first predictor observes that of the system at the future time ahead of the current time by a small amount. Repeating such an observation manner, the state of the next predictor observes that of the previous predictor at the future time ahead of the current time by the same small amount. Given sufficiently large number of sequential predictors, the state of the final predictor would observe the state of the system at the future time ahead of the current time by the amount of the delay. The method of sequential prediction has an advantage over other prediction methods because the sequential predictors only contain discrete delay terms that can be readily implemented.

In the remainder of the dissertation, we will focus our attention on predictor based feedback designs for linear systems with input delay. The development of our feedback laws is inspired by predictor feedback design methods, while the stability analysis of the resulting closed-loop system is carried out by employing Lyapunov functional based methods. The scope of the dissertation is not limited to basic stabilization. It expands to more challenging problems such as performance improvement and adaptation to accommodate unknown system parameters. The proposed feedback laws for stabilization will be modified to address each of these challenging problems effectively. In the following section, we review some predictor feedback laws and their roles in the stabilization of linear systems with input and state delays.

1.4. Predictor Feedback

For a linear time-invariant system with a single input delay,

$$\dot{x}(t) = Ax(t) + Bu(t - \tau), \tag{1.35}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$, $\tau \in \mathbb{R}_0^+$ and (A, B) is controllable, the feedback law

$$u(t) = Fx(t + \tau), \tag{1.36}$$

where F is a feedback gain matrix such that $A + BF$ is Hurwitz, completely cancels the effect of the input delay. Under such a feedback law, the closed-loop system

$$\dot{x}(t) = (A + BF)x(t) \tag{1.37}$$

is free of delay and is asymptotically stable. Also, the spectrum of the closed-loop system is finite and can be arbitrarily assigned on the complex plane by an appropriate choice of F. We note that feedback law (1.36) cannot be directly implemented because it requires the future state of the system $x(t+\tau)$ to be known. However, since system (1.35) is linear, the explicit solution of the future state can be obtained as the sum of the zero input solution and the zero state solution of the system,

$$\begin{aligned} x(t+\tau) &= \mathrm{e}^{A\tau}x(t) + \int_t^{t+\tau} \mathrm{e}^{A(t+\tau-s)}Bu(s-\tau)\mathrm{d}s \\ &= \mathrm{e}^{A\tau}x(t) + \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}Bu(s)\mathrm{d}s. \end{aligned} \tag{1.38}$$

Then, the feedback law (1.36) is expressed as

$$u(t) = F\mathrm{e}^{A\tau}x(t) + F\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}Bu(s)\mathrm{d}s, \tag{1.39}$$

which is referred to as the predictor feedback law due to the prediction of the future state [79].

The predictor feedback law consists of two terms. The first term is a static feedback term that is readily implementable. The second term is a distributed delay term that requires the input to be integrated over the past time interval $[t-\tau, t]$. A straightforward method to implement the distributed delay term is to approximate the term by a finite sum. By using the backward rectangular rule to approximate the distributed delay term, we arrive at the following feedback law,

$$u(t) = F\mathrm{e}^{A\tau}x(t) + F\frac{1}{N}\sum_{i=0}^{N-1} \mathrm{e}^{A\tau(1-\frac{i}{N})}Bu\left(t-\tau\Big(1-\frac{i}{N}\Big)\right), \tag{1.40}$$

where $N \in \mathbb{N}\backslash\{0\}$ is the number of the integration steps used by the backward rectangular approximation. The feedback law (1.40) can be readily implemented, and intuitively, would achieve stabilization if N is sufficiently large. However in [113], it was shown through an example, where the system is given by

$$\dot{x}(t) = x(t) + u(t-1), \tag{1.41}$$

and the feedback gain is given by $F = -2$, that the feedback law (1.40) for system (1.41) fails to stabilize no matter how large N is. The use of other numerical integration methods such as the composite trapezoidal rule and the Simpson rule to approximate the distributed delay term by a finite sum may encounter similar difficulty in achieving stabilization.

The cause of the instability of the closed-loop system consisting of system (1.35) and the feedback law (1.40) was examined in [28] and [91]. The characteristic equation of the closed-loop system is given by

$$\det\left(\begin{bmatrix} sI - A & -B\mathrm{e}^{-s\tau} \\ -F\mathrm{e}^{A\tau} & I - \frac{F}{N}\sum_{i=0}^{N-1}\mathrm{e}^{-\tau(1-\frac{i}{N})(sI-A)}B \end{bmatrix}\right) = 0, \tag{1.42}$$

which is the characteristic equation of a delay differential equation of neutral type. According to [28] and [91], it is the neutral feature of this characteristic equation that leads to instability.

1.5. Truncated Predictor Feedback and Delay Independent Truncated Predictor Feedback

The implemetation issue with the predictor feedback law when it comes to the approximation of its distributed delay term by finite sums stimulated the design of truncated predictor feedback. By discarding the distributed delay term of the predictor feedback law, the truncated predictor feedback law (see [72]) takes the form of

$$u(t) = F\mathrm{e}^{A\tau}x(t), \tag{1.43}$$

leading to a static linear state feedback law. The implementation of the truncated predictor feedback law requires the eaxct knowledge of the delay as appeared in the exponential factor $\mathrm{e}^{A\tau}$. To compensate an unknown delay or a time-varying delay, the following delay independent truncated predictor feedback law (see [72])

$$u(t) = Fx(t) \tag{1.44}$$

further discards the delay-involved exponential factor in (1.43). The remarkable advantage of the feedback laws (1.43) and (1.44) over other distributed feedback laws in the literature including the predictor feedback law (1.39), is their easy implementation.

Under an eigenstructural assignment based low gain feedback design ([70]) of the feedback gain matrix $F = F(\gamma)$, where γ is the feedback parameter, [72] showed that given any τ, (1.43) stabilizes system (1.35) with all its open loop poles in the closed left-half plane by tuning γ to be small enough, and (1.44) stabilizes system (1.35) with all all its open loop poles at the origin or in the open left-half plane by tuning γ to be small enough. An alternative parameterized low gain feedback design, the Lyapunov equation based approach (see [147]) constructs

$$F(\gamma) = -B^{\mathrm{T}}P(\gamma),$$

where $P(\gamma)$ is the unique positive definite solution to the parametric algebraic Riccati equation

$$A^{\mathrm{T}}P(\gamma) + P(\gamma)A - P(\gamma)BB^{\mathrm{T}}P(\gamma) = -\gamma P(\gamma), \;\; \gamma > -2\min\{\mathrm{Re}(\lambda) : \lambda \in \lambda(A)\}. \tag{1.45}$$

The inequality of γ in (1.45) guarantees the existence and uniqueness of the solution $P(\gamma)$. The name of Lyapunov based feedback design is attributed to the fact that $P(\gamma) = W^{-1}(\gamma)$, where $W(\gamma)$ is the unique positive definite solution to the Lyapunov equation

$$\left(A + \frac{\gamma}{2}I\right)^{\mathrm{T}} W(\gamma) + W(\gamma)\left(A + \frac{\gamma}{2}I\right) = BB^{\mathrm{T}}.$$

Under the Lyapunov equation based design, same results on the stabilization of system (1.35) with those of [72] were established in [147] and [144].

1.6. Contributions

In the first part of this dissertation, we develop truncated predictor based feedback laws with a constant feedback parameter for the stabilization of linear systems with input delay. Examples are established to show that these feedback laws fail to stabilize general, possibly exponentially unstable, linear systems with a sufficiently large delay. Admissible delay bounds that guarantee closed-loop stability are established. To comprehensively address the stabilizing effects of the truncated predictor based feedback laws, we study the stabilization of linear systems with either a constant delay or time-varying delay. Both state feedback and output feedback are considered.

In the second part of this dissertation, time-varying feedback parameters replace the constant feedback parameter in the truncated predictor based feedback laws. Such a replacement is motivated by solving the challenging control problem of designing feedback laws in the absence of any knowledge of the delay. For linear systems with all open loop poles at the origin or in the open left-half plane, a family of time-varying feedback parameters in the delay independent truncated predictor feedback law is proposed to improve the closed-loop performance with a smaller overshoot and a higher convergence rate. An upper bound of the delay is still required in the design of these time-varying parameters. To completely eliminate the requirement of any knowledge of the delay, a control scheme is proposed that equips the delay independent truncated predictor feedback law with a delay independent update algorithm for the feedback parameter. Such a control scheme allows ease of implementation in the sense that only the current state, and no knowledge of the delay, is required.

The line of work in this dissertation has been focused on continuous-time linear systems with input delay. To facilitate the digital implementation of our novel feedback designs, we developed counterparts results of all the abovementioned work in the discrete-time setting. The followings are a list of my publications during the preparation for this dissertation, and the list reflects my work in both the continuous-time setting and the discrete-time setting.

Books

- Y. Wei and Z. Lin, “Truncated Predictor Based Feedback Designs for Linear Systems with Input Delay,” *Springer*, to be published in 2020.

Book Chapters

- S. A. A. Rizvi, Y. Wei and Z. Lin, “ Reinforcement Learning for Optimal Adaptive Control of Time Delay Systems,” Chapter in Handbook on Reinforcement Learning and Control, D. H. Cansever, F. L. Lewis and Y. Wan, *Springer*, to be published in 2020.

Journals

- Y. Wei and Z. Lin, “Regulation of discrete-time linear systems in the absence of any knowledge of the input delay,” *IEEE Transactions on Automatic Control*, under revision.
- Y. Wei and Z. Lin, “A delay independent output feedback law for discrete-time linear systems with bounded input delay,” *Automatica*, under revision.
- Y. Xie, Y. Wei and Z. Lin, “Stabilization of linear systems with time-varying input delay by event-triggered delay independent truncated predictor feedback,” *International Journal of Robust and Nonlinear Control*, under revision.
- Y. Wei and Z. Lin, “Vision-based tracking by a quadrotor on ROS,” *Unmanned Systems*, Vol. 07, No. 04, pp. 233–244, 2019.
- Y. Wei and Z. Lin, “Regulation of a class of linear input delayed systems without delay knowledge,” *SIAM Journal on Control and Optimization*, Vol. 57, No. 2, pp. 999-1022, 2019.
- S. Su, Y. Wei and Z. Lin, “Stabilization of discrete-time linear systems with an unknown time-varying delay by switched low gain feedback,” *IEEE Transactions on Automatic Control*, Vol. 64, No. 5, pp. 2069-2076, 2019.
- Y. Wei and Z. Lin, “ Stabilization of discrete-time linear systems by delay independent truncated predictor feedback,” *Control Theory and Technology*, Vol. 17, No. 1, pp. 112-118, 2019.

- Y. Wei and Z. Lin, "Time-Varying low gain feedback for linear systems with unknown input delay," *Systems & Control Letters*, Vol. 123, pp. 98-107, 2019.
- Y. Wei and Z. Lin, "A delay-independent output feedback for linear systems with time-varying input delay," *International Journal of Robust and Nonlinear Control*, Vol. 28, No. 8, pp. 2950-2960, 2018.
- Y. Wei and Z. Lin, "Adaptation in truncated predictor feedback to overcome uncertainty in the delay," *International Journal of Robust and Nonlinear Control*, Vol. 28, No. 8, pp. 3127-3139, 2018.
- Y. Wei and Z. Lin, "Stability criteria of linear systems with multiple input delays under truncated predictor feedback," *Systems & Control Letters*, Vol. 111, pp. 9-17, 2018.
- Y. Wei and Z. Lin, "Maximum delay bounds of linear systems under delay independent truncated predictor feedback," *Automatica*, Vol. 83, pp. 65-72, 2017.
- Y. Wei and Z. Lin, "Stabilization of exponentially unstable discrete-time linear systems by truncated predictor feedback," *Systems & Control Letters*, Vol. 97, pp. 27-35, 2016.

Conferences

- Y. Wei and Z. Lin, "A delay independent output feedback law for discrete-time linear systems with bounded input delay," *Proc. the 2020 American Control Conference*, accepted.
- S. A. A. Rizvi, Y. Wei and Z. Lin, "Model-free optimal stabilization of unknown time delay systems using adaptive dynamic programming," *Proc. the 58th IEEE Conference on Decision and Control*, 6536-6541, Nice, France, 2019.
- Y. Xie, Y. Wei and Z. Lin, "Stabilization of linear systems with input delay by event-triggered delay independent truncated predictor feedback," *Proc. the 2019 American Control Conference*, pp. 3708-3713, Philadelphia, U.S.A., July 2019.
- Y. Wei and Z. Lin, "Regulation of a class of linear input delayed systems without delay knowledge," *Proc. the 57th IEEE Conference on Decision and Control*, pp. 6240-6245, Miami Beach, U.S.A., December 2018.
- Y. Wei and Z. Lin, "A delay independent output feedback for linear systems with time-varying input delay," *Proc. the 2018 American Control Conference*, pp. 4087-4092, Milwaukee, U.S.A., June 2018.

- Y. Wei and Z. Lin, “Vision-based tracking by a quadrotor on ROS,” *The 20th IFAC World Congress*, pp. 11447-11452, Toulouse, France, July, 2017.
- Y. Wei and Z. Lin, “Delay independent truncated predictor feedback for stabilization of linear systems with multiple time-varying input delays,” *Proc. the 2017 American Control Conference*, pp. 5732-5737, Seattle, U.S.A., May 2017.
- Y. Wei and Z. Lin, “Stabilization of exponentially unstable linear systems with multiple input delays by truncated predictor feedback,” *Proc. the 35th Chinese Control Conference*, pp. 1621-1626, Chengdu, China, July 2016.
- Y. Wei and Z. Lin, “On the delay bounds of discrete-time linear systems under delay independent truncated predictor feedback,” *Proc. the 2016 American Control Conference*, pp. 89-94, Boston, U.S.A., July 2016.
- Y. Wei and Z. Lin, “On the delay bounds of linear systems under delay independent truncated predictor feedback: The state feedback case,” *Proc. the 54th IEEE Conference on Decision and Control*, pp. 4642-4647, Osaka, Japan, December 2015.

1.7. Dissertation Outline

In Section 2, we focus on the design of delay independent truncated predictor based feedback laws. In the absence of the exact knowledge of the delay, the truncated predictor feedback law is no longer implementable due to its delay-dependent term. The removal of the delay-dependent term from the truncated predictor feedback law results in the delay independent truncated predictor feedback law. It is shown through an example that the delay independent feedback cannot compensate an arbitrarily large delay in a linear system with purely imaginary open loop poles. Admissible delay bounds with stability guarantee are then established for general, possibly exponentially unstable, linear systems. However, for a system with all its open loop poles at the origin or in the open left-half plane, the delay independent truncated predictor feedback law compensates an arbitrarily large delay as long as the low gain feedback design technique is applied to parameterize the feedback gain matrix.

In Section 3, we develop a delay independent output feedback law for linear systems with input delay. Existing truncated predictor based output feedback laws contain the exact value of the delay in the dynamics of their state observers. We propose a delay independent truncated predictor output feedback law that do not contain any information of the delay in neither the dynamics of the state observer nor the input expression. Admissible upper bounds of the delay are established to guarantee the stabilizability of a general linear system that is possible exponentially unstable. In the special case where a linear system has all its open loop poles at the origin or in the open left-half plane, the stabilization can be achieved for an arbitrarily large delay. This implies that we achieve the same level of stabilization with that under the existing truncated predictor based output feedback laws, while our output feedback design does not require the exact knowledge of the delay in its implementation.

The low gain nature of the feedback parameter of the delay independent truncated predictor feedback law leads to a large overshoot and a low convergence rate of the closed-loop system. In Section 4, a time-varying feedback parameter design of the delay independent truncated predictor feedback law is proposed to improve the closed-loop performance under the constant feedback parameter design. We proactively let the values of the time-varying feedback parameter be relatively large. After the closed-loop system enters a small neighborhood of zero, we decrease the parameter drastically to a sufficiently small amount. The evolution of the feedback parameter at the starting phase of system evolution improves closed-loop performance in terms of reducing overshoot and increasing convergence rate. Also, the small amount of the feedback parameter as time goes to infinity guarantees closed-loop stability.

The time-varying feedback parameter design in Section 4 requires an upper bound of the delay to be known for the stabilization. We manage to not require any knowledge of the delay in feedback designs in Section 5. In the absence of any knowledge of the delay, a control scheme is proposed that equips the delay independent truncated predictor feedback law with an update algorithm, which is also delay independent, for the feedback parameter. This control scheme allows ease of implementation because only current state, and no knowledge of the delay, is required.

2. Delay Independent Truncated Predictor Feedback for Linear Systems with Input Delay

The truncated predictor feedback design simplifies the predictor feedback design by discarding the distributed delay term. The implementation of the remaining static feedback term of the predictor feedback law requires the exact knowledge of the input delay. The truncated predictor state feedback law for continuous-time linear systems with a constant delay is such an example. The exact value of the delay appears in the exponential factor $\mathrm{e}^{A\tau}$ of the truncated predictor feedback law. Moreover, the determination of the value of the feedback parameter of the truncated predictor feedback law requires, although not the exact value of the delay, an upper bound of the delay to be known. This requirement of the information of the delay, which explicitly or implicitly appears in the truncated predictor feedback law, suggests that the truncated predictor feedback law is delay-dependent. The delay-dependency of the truncated predictor feedback law also manifests itself in the observer based feedback design. To construct an observer whose state asymptotically approaches the state of the open loop system, the dynamics of the observer generally contains the delayed input. This implies that the exact value of the delay is also required in the construction of the observer used in the observer based truncated predictor feedback law.

The truncated predictor feedback design compensates a constant delay by requiring the exact value of the delay. Such compensation becomes trickier when the delay is time-varying. The compensation of a bounded time-varying delay via a truncated predictor feedback design relies on the prediction of the future state of the system at the future time instant $\phi^{-1}(t)$, where $\phi(t)$ represents the past time instant at which the input is injected into the system. Clearly, the implementation of a truncated predictor feedback law in the face of a time-varying delay fails if $\phi(t)$ does not admit an inverse. Consider a standard form of

$$\phi(t) = t - d(t),$$

where $d(t)$ is the time-varying delay. The inverse of $\phi(t)$ does not exist whenever

$$d(t) = t - t_1,$$

on some time interval $t \in [t_1, t_2]$. It is noteworthy that such a time-varying delay is physically meaningful. The requirement on the existence of $\phi^{-1}(t)$ restricts the application of truncated predictor feedback in compensating time-varying delays. Therefore, the exact knowledge of time-varying delays is only necessary, but not necessarily sufficient, for the success in the implementation of a truncated predictor feedback law.

Compared to feedback laws whose realization relies heavily on the knowledge of the delay, those that require less such knowledge are preferable. The craving for feedback laws that are delay independent, at least partially if not completely, is driven by the lack of the information of the delay in practice. In practice, the delay is hardly precisely known. Oftentimes, only its upper bound and/or lower bound is known. In the worst case, no knowledge of the delay can be assumed. Basically, the overall objective of the rest of this dissertation is to relax the assumption on the availability of the knowledge of the delay for control design.

In this section, we introduce delay independent truncated predictor feedback laws that discard the exponential factor of the truncated predictor feedback laws. The remaining feedback gain of the delay independent truncated predictor feedback laws is parameterized in a feedback parameter, whose value is determined based on the knowledge of an upper bound of the delay. The delay independent truncated predictor feedback laws are less delay dependent than the truncated predictor feedback laws because their implementation no longer requires the exact knowledge of the delay to be known. Besides the basic stabilization problem, we further consider the problem of improving the performance of a closed-loop system under a delay independent truncated predictor feedback law. Such a problem originates from the low gain nature of the delay independent truncated predictor feedback law in the stabilization of a linear system with all its open loop poles at the origin or in the open left-half plane. It has been observed that an excessively small value of the feedback parameter results in a large overshoot and a slow convergence rate of the closed-loop system. The poor closed-loop performance inspires the design of a time-varying feedback parameter whose value is chosen large at the beginning phase of the system evolution and is decreased as needed to facilitate the proof of stability. The large value of the parameter at the starting phase of the system evolution reduces the overshoot and increases the convergence rate. Benefits of the time-varying parameter design in the closed-loop performance are demonstrated in the convergence rate analysis and the numerical studies.

In a predictor feedback law for a linear system with input delay, the future state is predicted by the solution of the linear system. The zero input solution contains the transition matrix. The zero state solution gives rise to the distributed nature of the feedback law. In [72], it is established that, when the system is not exponentially unstable, low gain feedback can be designed such that the predictor feedback law, with the distributed delay term truncated, still achieves stabilization in the presence of an arbitrarily large delay. Furthermore, in the absence of purely imaginary poles, the transition matrix in the truncated predictor feedback can be safely dropped, resulting in a delay independent truncated predictor feedback law , which is simply a delay independent linear state feedback. In this section, we first construct an example to show that, in the presence of purely imaginary poles, the linear delay independent truncated predictor feedback in general cannot stabilize the system for an arbitrarily large delay. By using the extended Krasovskii Stability Theorem (Theorem 1.2), we derive a bound on the delay under which a delay independent truncated predictor feedback law achieves stabilization for a general system that may be exponentially unstable.

We consider the asymptotic stabilization problem for the following linear system with time-varying delay in the input,

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(\phi(t)), \\ x(\theta) &= \phi(\theta),\ \theta \in [-D, 0], \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are state and input, respectively. The time-varying delay function $\phi(t)$: $\mathbb{R}^+ \to \mathbb{R}$ is assumed to take the standard form of

$$\phi(t) = t - d(t), \tag{2.2}$$

where $d(t) : \mathbb{R}^+ \to \mathbb{R}^+$ denotes time-varying delay which is bounded by a finite positive constant D, i.e.,

$$0 \leq d(t) \leq D,\ \ t \geq 0. \tag{2.3}$$

Only the information on the bound D, but not the delay $d(t)$ itself, will be required in our feedback design and stability analysis. We also assume that the pair (A, B) is stabilizable.

In [72], it is shown that, when the open loop system is not exponentially unstable, a parametrized feedback gain matrix $F(\gamma)$ can be designed by the low gain feedback design technique [70] such that the finite-dimensional truncated predictor feedback law

$$u(t) = F(\gamma)e^{Ad}x(t) \tag{2.4}$$

would still asymptotically stabilize system (2.1) for an arbitrarily large constant delay d as long as the low gain parameter γ is tuned small enough. In the absence of purely imaginary poles, the transition matrix in the truncated predictor feedback law (2.4) can be dropped and the feedback law further simplifies to a delay independent truncated predictor state feedback law (also referred to as the delay independent state feedback TPF law),

$$u(t) = F(\gamma)x(t). \tag{2.5}$$

Such a feedback law, parameterized in the low gain parameter γ, is referred to as the delay independent truncated predictor feedback law. The truncated predictor feedback design originally proposed in [72] uses the eigenstructure assignment based low gain feedback design method. The design was simplified in [144], where a parametric Lyapunov equation based low gain feedback design was adopted.

In this section, we will examine the properties of the delay independent truncated predictor feedback for general systems, which may have purely imaginary (nonzero poles on the imaginary axis) or exponentially unstable poles. In particular, we will first construct an example to show that, in the presence of purely imaginary poles, the delay independent state feedback TPF law in general does not have the ability to stabilize the system for an arbitrarily large delay. We then derive, by applying the extended Krasovskii Stability Theorem (Theorem 1.2), a bound on the delay under which a delay independent truncated predictor feedback law achieves stabilization for the system. The expression of this bound indicates that, when all the closed right-half plane poles are at the origin, stabilization of the system would be achieved for an arbitrarily large delay as long as the low gain parameter is chosen to be sufficiently small. This observation coincides with the results in both [72] and [144]. Moreover, it will be shown that, for a given delay with an arbitrarily larg upper bound, the upper bound of the low gain parameter deived in this section that guarantees stability is less conservative than the one given in [144].

2.1. Preliminaries

It is shown in [72] that a linear system with all its open loop poles at the origin or in the open left-half plane can be stabilized for an arbitrarily large delay by a delay independent truncated predictor feedback law,

$$u(t) = F(\gamma)x(t), \ \gamma > 0. \tag{2.6}$$

The construction of $F(\gamma)$ was given in [147] by utilizing the Lyapunov equation based low gain design technique [143]. That is, for a controllable pair (A, B), the parametrized feedback gain matrix $F(\gamma)$ in

(2.6) is constructed as,

$$F(\gamma) = -B^{\mathrm{T}} P(\gamma), \tag{2.7}$$

where the positive definite matrix $P(\gamma)$ is the solution to the parametric algebraic Riccati equation

$$A^{\mathrm{T}} P(\gamma) + P(\gamma) A - P(\gamma) B B^{\mathrm{T}} P(\gamma) = -\gamma P(\gamma) \tag{2.8}$$

with

$$\gamma > -2\min\{\mathrm{Re}(\lambda(A))\}. \tag{2.9}$$

In the case where all eigenvalues of A are at the origin or in the open left-half plane, the delay is allowed to be arbitrarily large but bounded, and the value of the parameter γ is required to approach zero as the bound on the delay increases to infinity. As a result, the parametric algebraic Riccati equation (2.8) is a low gain feedback design and the feedback parameter is referred to as the low gain parameter.

In this section, we will first show that, when system (2.1) is not exponentially unstable but has purely imaginary poles, the delay independent truncated predictor feedback law (2.6) is in general not able to achieve asymptotic stabilization for a large enough delay. We will then derive bound on the delay under which the delay independent truncated predictor feedback law would achieve stabilization for a general system that may be exponentially unstable.

To achieve our objectives, we need some technical preliminaries. We first recall some properties of the solution $P(\gamma)$ of the algebraic Riccati equation (2.8) from [147].

Lemma 2.1. *Assume that all eigenvalues of A are on the closed right-half plane. The unique positive definite solution $P(\gamma)$ to (2.8) satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} P(\gamma) > 0, \tag{2.10}$$

$$\mathrm{tr}\left(B^{\mathrm{T}} P(\gamma) B\right) = 2\mathrm{tr}(A) + n\gamma, \tag{2.11}$$

$$P(\gamma) B B^{\mathrm{T}} P(\gamma) \leq (2\mathrm{tr}(A) + n\gamma) P(\gamma), \tag{2.12}$$

$$\mathrm{e}^{A^{\mathrm{T}} t} P(\gamma) \mathrm{e}^{At} \leq \mathrm{e}^{\omega\gamma t} P(\gamma), \tag{2.13}$$

where $\gamma > 0$, $t \geq 0$ and $\omega \geq 2\frac{\mathrm{tr}(A)}{\gamma} + n - 1$. Moreover, the eigenvalues of A and those of $A + BF(\gamma)$

are symmetric with respect to $\mathrm{Re}\{s\} = -\frac{\gamma}{2}$ *in the complex plane, i.e.,*

$$\lambda(A) + \lambda(A + BF(\gamma)) = -\gamma, \tag{2.14}$$

and $P(\gamma)$ *is a rational matrix in* γ*. When all eigenvalues of* A *are on the imaginary axis,*

$$\lim_{\gamma \to 0^+} P(\gamma) = 0. \tag{2.15}$$

We now recall the following lemma.

Lemma 2.2. *([43]) For any positive semi-definite matrix* $Q \geq 0$*, two scalars* γ_2 *and* γ_1 *with* $\gamma_2 \geq \gamma_1$*, and a vector valued function* $\omega : [\gamma_1, \gamma_2] \to \mathbb{R}^n$ *such that the integrals in the following are well-defined, then*

$$\left(\int_{\gamma_1}^{\gamma_2} \omega^{\mathrm{T}}(\beta)\,\mathrm{d}\beta\right) Q \left(\int_{\gamma_1}^{\gamma_2} \omega(\beta)\,\mathrm{d}\beta\right) \leq (\gamma_2 - \gamma_1) \int_{\gamma_1}^{\gamma_2} \omega^{\mathrm{T}}(\beta)\, Q \omega(\beta)\,\mathrm{d}\beta. \tag{2.16}$$

We next establish the following simple fact and its two corollaries.

Lemma 2.3. *If all eigenvalues of* $A \in \mathbb{R}^{n\times n}$ *are on the closed right-half plane, then*

$$\mathrm{tr}^2(A) \geq \mathrm{tr}(A^2). \tag{2.17}$$

Moreover, the equality sign holds if and only if all the eigenvalues of A *are real and at most one of them is positive.*

Proof. Let the eigenvalues of A be $\lambda_1, \lambda_2, \cdots, \lambda_p$, $\alpha_1 \pm j\beta_1, \alpha_2 \pm j\beta_2, \cdots, \alpha_q \pm j\beta_q$, where

$$p + 2q = n \tag{2.18}$$

and

$$\begin{cases} \lambda_i \geq 0,\ i = 1, 2, \cdots, p, \\ \alpha_i \geq 0,\ i = 1, 2, \cdots, q, \\ \beta_i > 0,\ i = 1, 2, \cdots, q. \end{cases} \tag{2.19}$$

It then follows that

$$\mathrm{tr}(A) = \sum_{i=1}^{p} \lambda_i + 2\sum_{i=1}^{q} \alpha_i \tag{2.20}$$

and

$$\mathrm{tr}(A^2) = \sum_{i=1}^{p} \lambda_i^2 + 2\sum_{i=1}^{q} \Big(\alpha_i^2 - \beta_i^2\Big). \tag{2.21}$$

Then, the difference between $\mathrm{tr}^2(A)$ and $\mathrm{tr}(A^2)$ can be expressed as

$$\begin{aligned}
\mathrm{tr}^2(A) - \mathrm{tr}(A^2) \;&=\; \Big(\sum_{i=1}^{p} \lambda_i\Big)^2 + 4\Big(\sum_{i=1}^{q} \alpha_i\Big)^2 + 4\Big(\sum_{i=1}^{p} \lambda_i\Big)\Big(\sum_{i=1}^{q} \alpha_i\Big) \\
&\quad -\sum_{i=1}^{p} \lambda_i^2 - 2\sum_{i=1}^{q} \Big(\alpha_i^2 - \beta_i^2\Big) \\
&=\; 2\sum_{1\le i<j\le p} \lambda_i\lambda_j + 4\sum_{i=1}^{p} \lambda_i \sum_{i=1}^{q} \alpha_i + 2\sum_{i=1}^{q} \alpha_i^2 \\
&\quad +8\sum_{1\le i<j\le q} \alpha_i\alpha_j + 2\sum_{i=1}^{q} \beta_i^2 \\
&\ge\; 0,
\end{aligned} \tag{2.22}$$

from which we can readily conclude by invoking (2.19) that, if A has at least one pair of imaginary eigenvalues, then

$$\mathrm{tr}^2(A) > \mathrm{tr}(A^2). \tag{2.23}$$

Therefore, a necessary condition for the equality sign to hold is that all the eigenvalues of A are real. Under this condition, (2.22) can be simplified as

$$\mathrm{tr}^2(A) - \mathrm{tr}(A^2) = \Big(\textstyle\sum_{i=1}^{p} \lambda_i\Big)^2 - \textstyle\sum_{i=1}^{p} \lambda_i^2, \tag{2.24}$$

where $p \ge 1$. Now, we consider two separate cases, $p = 1$ and $p \ge 2$. If $p = 1$, $\mathrm{tr}^2(A) - \mathrm{tr}(A^2) = 0$ for any $\lambda_1 \ge 0$. If $p \ge 2$,

$$\mathrm{tr}^2(A) - \mathrm{tr}(A^2) = 2\textstyle\sum_{1\le i<j\le p} \lambda_i\lambda_j, \tag{2.25}$$

from which it readily follows that the necessary and sufficient condition for the equality sign to hold is that at most one of the eigenvalues of A is positive. Combining the two separate cases, we can conclude that the equality sign in (2.17) holds if and only if all eigenvalues of A are real and at most one of them is positive. $\square$

Corollary 1. *If all eigenvalues of $A \in \mathbb{R}^{n\times n}$ are on the closed right-half plane, then the following inequality holds for any real $\gamma > 0$,*

$$\left(\frac{\gamma}{2\Big(2\mathrm{tr}(A)+n\gamma\Big)\left(\Big(2\mathrm{tr}(A)+n\gamma\Big)\Big(\mathrm{tr}(A)+\frac{n+3}{2}\gamma\Big)-\gamma\mathrm{tr}(A)-\mathrm{tr}(A^2)\right)}\right)^{\frac{1}{2}}$$
$$< \frac{1}{\sqrt{2}\Big(2\mathrm{tr}(A)+n\gamma\Big)}. \tag{2.26}$$

Proof. First we denote the left-hand side and the right-hand side of inequality (2.26) as $D_1(\gamma)$ and $D_2(\gamma)$, respectively. The functions $D_1(\gamma)$ and $D_2(\gamma)$ are well defined because of the assumption that all eigenvalues of A are in the closed right-half plane, Lemma 2.3 and $\gamma > 0$. We notice that $D_1(\gamma)$ can be written as follows,

$$D_1(\gamma) = \left(2\Big(2\mathrm{tr}(A)+n\gamma\Big)^2\left(\frac{n+3}{2}+\frac{\mathrm{tr}(A)}{\gamma}-\frac{\mathrm{tr}(A)+\frac{\mathrm{tr}(A^2)}{\gamma}}{2\mathrm{tr}(A)+n\gamma}\right)\right)^{-\frac{1}{2}}.$$

In order to show that inequality (2.26) holds, it suffices to show that

$$\frac{n+3}{2}+\frac{\mathrm{tr}(A)}{\gamma}-\frac{\mathrm{tr}(A)+\frac{\mathrm{tr}(A^2)}{\gamma}}{2\mathrm{tr}(A)+n\gamma} > 1,$$

which is equivalent to

$$2n\gamma\mathrm{tr}(A)+\frac{n(n+1)}{2}\gamma^2+2\mathrm{tr}^2(A)-\mathrm{tr}(A^2) > 0,$$

which can be easily verified by using the facts that $\mathrm{tr}(A) \geq 0$ and $\gamma > 0$, along with Lemma 2.3. $\square$

Corollary 2. *If $A \in \mathbb{R}^{n\times n}$ is exponentially unstable with all eigenvalues on the closed right-half plane, then $D_1(\gamma)$ as defined in the proof of Corollary 1, where $\gamma > 0$, has a unique maximal value $D_1(\gamma^\star)$ achieved at $\gamma = \gamma^\star$, where*

$$\gamma^\star = \frac{n\Big(2\mathrm{tr}^2(A)-\mathrm{tr}(A^2)+(n+1)\mathrm{tr}(A)\Big)+\sqrt{\Delta}}{n(n+3)\mathrm{tr}(A)}, \tag{2.27}$$

and

$$\Delta = n^2\Big(2\mathrm{tr}^2(A) - \mathrm{tr}(A^2) + (n+1)\mathrm{tr}(A)\Big)^2$$
$$+2n(n+3)\mathrm{tr}^2(A)\Big(2\mathrm{tr}^2(A) - \mathrm{tr}(A^2)\Big). \tag{2.28}$$

Proof. We determine the monotonicity of $D_1(\gamma)$, which is equivalent to the monotonicity of $D_1^2(\gamma)$. The derivative of $D_1^2(\gamma)$ with respect to γ is derived as follows,

$$\frac{\mathrm{d}D_2^2(\gamma)}{\mathrm{d}\gamma} = \frac{h(\gamma) - \gamma\dfrac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma}}{h^2(\gamma)},$$

where

$$h(\gamma) = 2\big(2\mathrm{tr}(A) + n\gamma\big)\bigg(\Big(2\mathrm{tr}(A) + n\gamma\Big)\Big(\mathrm{tr}(A) + \frac{n+3}{2}\gamma\Big) - \gamma\mathrm{tr}(A) - \mathrm{tr}(A^2)\bigg). \tag{2.29}$$

Then, it suffices to determine the sign of $h(\gamma) - \gamma\dfrac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma}$, whose expression can be obtained as,

$$h(\gamma) - \gamma\frac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma} \;=\; -2n(n+3)\mathrm{tr}(A)\gamma^2 + 2n\Big(4\mathrm{tr}^2(A) - 2\mathrm{tr}(A^2) + (2n+2)\mathrm{tr}(A)\Big)\gamma$$
$$+4\mathrm{tr}(A)(2\mathrm{tr}^2(A) - \mathrm{tr}(A^2)). \tag{2.30}$$

It can be observed that the right-hand side of (2.30) is a quadratic function of γ. By Lemma 2.3, it has a unique positive root $\gamma^\star$ as long as A is exponentially unstable with all its eigenvalues in the closed right-half plane, where $\gamma^\star$ is defined as in (2.27). Moreover, for each $\gamma \in (0, \gamma^\star)$,

$$h(\gamma) - \gamma\frac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma} > 0, \tag{2.31}$$

and for each $\gamma \in [\gamma^\star, \infty)$,

$$h(\gamma) - \gamma\frac{\mathrm{d}h(\gamma)}{\mathrm{d}\gamma} \leq 0. \tag{2.32}$$

Therefore, the continuity of $D_1(\gamma)$ with respect to γ implies that $D_1(\gamma)$ reaches its unique maximal value $D_1(\gamma^\star)$ at $\gamma^\star$. □

2.2. Stability Analysis

We first provide an example to show that for a system that is not exponentially unstable but has purely imaginary poles, delay independent truncated predictor feedback in general is not able to achieve stabilization for a sufficiently large delay.

2.2.0.1. **Example 5.1:** Consider system (2.1) with a constant delay

$$d(t) = \tau$$

and

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.33}$$

The system is controllable with $\lambda(A) = \{\pm j\}$. Consider the delay independent state feedback TPF law (2.6) with

$$F(\gamma) = -\begin{bmatrix} \gamma^2 & 2\gamma \end{bmatrix}, \tag{2.34}$$

where $\gamma > 0$ by the inequality in (2.9). The characteristic equation of the closed-loop system is given by

$$\det\left(sI - A + BB^{\mathrm{T}}P(\gamma)\mathrm{e}^{-\tau s}\right) = s^2 + 2\gamma s e^{-\tau s} + \gamma^2 e^{-\tau s} + 1 = 0.$$

We first define two real sequences as

$$\begin{cases} \tau_{i,1} &= \dfrac{1}{\omega_{\mathrm{R}}} \cos^{-1}\left(\dfrac{\omega_{\mathrm{R}}^2 - 1}{4\omega_{\mathrm{R}}^2 + \gamma^2}\right) + i\dfrac{2\pi}{\omega_{\mathrm{R}}}, \;\; i \in \mathbb{N}, \\ \tau_{i,2} &= \dfrac{1}{\omega_{\mathrm{L}}} \cos^{-1}\left(\dfrac{\omega_{\mathrm{L}}^2 - 1}{4\omega_{\mathrm{L}}^2 + \gamma^2}\right) + i\dfrac{2\pi}{\omega_{\mathrm{L}}}, \;\; i \in \mathbb{N}, \end{cases} \tag{2.35}$$

where $\tau_{i,1}$ is defined on $\gamma \in (0, +\infty)$, $\tau_{i,2}$ is defined on $\gamma \in (0, 1)$,

$$\omega_{\mathrm{R}} = \left(1 + 2\gamma^2 + \gamma\sqrt{5\gamma^2 + 4}\right)^{\frac{1}{2}} \tag{2.36}$$

and

$$\omega_{\mathrm{L}} = \left(1 + 2\gamma^2 - \gamma\sqrt{5\gamma^2 + 4}\right)^{\frac{1}{2}}. \tag{2.37}$$

We consider two cases with respect to the value of γ.

(I) Fix a $\gamma \geq 1$. Then, according to [25], the closed-loop system is unstable if

$$\tau > \tau_{0,1}.$$

Note from (2.35) that

$$\tau_{0,1} < \frac{\pi}{2\sqrt{6}},$$

because

$$\omega_{\mathrm{R}} > \sqrt{6}$$

and

$$\cos^{-1}\left(\frac{\omega_{\mathrm{R}}^2 - 1}{4\omega_{\mathrm{R}}^2 + \gamma^2}\right) \leq \frac{\pi}{2}.$$

This implies that the closed-loop system is unstable if

$$\tau > \frac{\pi}{2\sqrt{6}}.$$

(II) Fix a $\gamma \in (0, 1)$. According to [25], because the closed-loop system in the absence of delay is stable and

$$\omega_{\mathrm{R}} > \omega_{\mathrm{L}} > 0,$$

there exists $k \in \mathbb{N} \setminus \{0\}$ such that the sequences in (2.35) satisfy

$$0 < \tau_{0,1} < \tau_{0,2} < \tau_{1,1} < \tau_{1,2} < ... < \tau_{k-1,1} < \tau_{k-1,2} < \tau_{k,1} < \tau_{k+1,1} < \tau_{k,2} < ...,$$

and the closed-loop system is unstable if

$$\tau \in \cup_{i \in I[0,k-1]}[\tau_{i,1}, \tau_{i,2}] \cup [\tau_{k,1}, \infty).$$

Note from (2.35) that

$$\tau_{i,1} < \frac{\pi}{2} + 2\pi i$$

and

$$\tau_{i,2} > \frac{\pi}{2} + 2\pi i, \;\; i \in \mathbb{N},$$

because

$$\omega_{\mathrm{R}} > 1,$$

$$\cos^{-1}\left(\frac{\omega_{\mathrm{R}}^2 - 1}{4\omega_{\mathrm{R}}^2 + \gamma^2}\right) \leq \frac{\pi}{2},$$

$$\omega_{\mathrm{L}} < 1,$$

and

$$\cos^{-1}\left(\frac{\omega_{\mathrm{L}}^2 - 1}{4\omega_{\mathrm{L}}^2 + \gamma^2}\right) \geq \frac{\pi}{2}.$$

Therefore, if

$$\tau = \frac{\pi}{2} + 2\pi l, \;\; l \in \mathbb{N},$$

the closed-loop system is unstable.

Combining the two cases, we see that, for any of the delays,

$$\tau = \frac{\pi}{2} + 2\pi l, \;\; l \in \mathbb{N},$$

the closed-loop system is unstable for each $\gamma > 0$. This implies that the delay independent truncated predictor feedback law (2.6) fails to stabilize the system with

$$\tau = \frac{\pi}{2} + 2\pi l, \;\; l \in \mathbb{N}.$$

□

Unlike the class of linear systems with open loop poles at the origin, linear systems with purely imaginary open loop poles cannot be stabilized by tuning the feedback parameter of the delay independent state feedback law if the delay is too large. However, the stabilization of a general linear system can be achieved by tuning the feedback parameter as long as the delay value is small enough. We now present a theorem on this fact. Before presenting the theorem, we make the following assumption on system (2.1).

For a general system (2.1), the eigenvalues of A can be anywhere on the complex plane. Without loss of generality, let the pair (A, B) be given in the following form,

$$A = \begin{bmatrix} A_{\mathrm{L}} & 0 \\ 0 & A_{\mathrm{R}} \end{bmatrix}, \; B = \begin{bmatrix} B_{\mathrm{L}} \\ B_{\mathrm{R}} \end{bmatrix},$$

where all eigenvalues of A_{L} are in the open left-half plane and all eigenvalues of A_{R} are in the closed right-half plane. Correspondingly, system (2.1) can be written as

$$\begin{cases} \dot{x}_{\mathrm{L}}(t) = A_{\mathrm{L}} x_{\mathrm{L}}(t) + B_{\mathrm{L}} u(\phi(t)), \\ \dot{x}_{\mathrm{R}}(t) = A_{\mathrm{R}} x_{\mathrm{R}}(t) + B_{\mathrm{R}} u(\phi(t)), \end{cases}$$

where

$$x(t) = \begin{pmatrix} x_{\mathrm{L}}^{\mathrm{T}}(t) & x_{\mathrm{R}}^{\mathrm{T}}(t) \end{pmatrix}^{\mathrm{T}}.$$

It is clear that any linear state feedback law that stabilizes x_{R} subsystem would stabilize the entire system. Thus, we will assume, without loss of generality, that all eigenvalues of A are on the closed right-half plane.

Theorem 2.1. *Consider system (2.1). Let (A, B) be controllable and all eigenvalues of A be on the closed right-half plane. If, for each $\gamma > 0$,*

$$D < \left(\frac{\gamma}{2\Big(2\mathrm{tr}(A) + n\gamma\Big)\left(\Big(2\mathrm{tr}(A) + n\gamma\Big)\left(\mathrm{tr}(A) + \dfrac{n+3}{2}\gamma\right) - \gamma \mathrm{tr}(A) - \mathrm{tr}(A^2)\right)} \right)^{\frac{1}{2}}, \tag{2.38}$$

then the delay independent state feedback TPF law (2.6) asymptotically stabilizes the system.

Proof. Under the assumption that all eigenvalues of A are on the closed right-half plane, we let $\gamma > 0$ satisfy (2.9). The closed-loop system consisting of system (2.1) and the delay independent truncated predictor feedback law (2.6) is expressed as

$$\dot{x}(t) = Ax(t) + B(-B^{\mathrm{T}}P)x(\phi(t)))$$

$$= (A - BB^{\mathrm{T}}P)x(t) + BB^{\mathrm{T}}P(x(t) - x(\phi(t)))$$

$$= A_{\mathrm{c}}x(t) + BB^{\mathrm{T}}P\lambda(t), \tag{2.39}$$

where

$$\begin{aligned} A_{\mathrm{c}} =& A + BF(\gamma) \\ =& A - BB^{\mathrm{T}}P(\gamma) \end{aligned} \tag{2.40}$$

is defined in Lemma 2.1 and

$$\lambda(t) = x(t) - x(\phi(t)).$$

Consider a Lyapunov functional

$$V(x_t) = V_1(x) + V_2(x_t), \tag{2.41}$$

where

$$V_1(x) = x^{\mathrm{T}}(t)Px(t), \tag{2.42}$$

$$V_2(x_t) = \epsilon \int_{-D}^{0} \int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s\mathrm{d}\theta,$$

and ϵ is some real positive constant whose value is to be determined later. Then the derivative of $V(x_t)$ along the trajectory of the closed-loop system is

$$\dot{V}(x_t) = \dot{V}_1(x) + \dot{V}_2(x_t).$$

On the one hand, $\dot{V}_1(x)$ can be evaluated as

$$\begin{aligned} \dot{V}_1(x) &= \dot{x}^{\mathrm{T}}(t)Px(t) + x^{\mathrm{T}}(t)P\dot{x}(t) \\ &= x^{\mathrm{T}}(t)(A_{\mathrm{c}}^{\mathrm{T}}P + PA_{\mathrm{c}})x(t) + 2\lambda^{\mathrm{T}}(t)PBB^{\mathrm{T}}Px(t) \\ &= x^{\mathrm{T}}(t)\left(-\gamma P - PBB^{\mathrm{T}}P\right)x(t) + 2\lambda^{\mathrm{T}}(t)PBB^{\mathrm{T}}Px(t) \\ &\leq -\gamma x^{\mathrm{T}}(t)Px(t) + \lambda^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\lambda(t) \end{aligned}$$

$$\leq -\gamma x^{\mathrm{T}}(t)Px(t) + \Big(2\mathrm{tr}(A) + n\gamma\Big)\lambda^{\mathrm{T}}(t)P\lambda(t), \tag{2.43}$$

where we have used Young's Inequality and the properties of $P(\gamma)$ as established in Lemma 2.1. Considering that

$$\lambda(t) = \int_{\phi(t)}^{t} \dot{x}(s)\mathrm{d}s,$$

we can continue (2.43) by using Lemma 2.2 as follows,

$$\begin{aligned}
\dot{V}_1(x) &\leq -\gamma x^{\mathrm{T}}(t)Px(t) + \Big(2\mathrm{tr}(A) + n\gamma\Big)\left(\int_{\phi(t)}^{t} \dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}} P \left(\int_{\phi(t)}^{t} \dot{x}(s)\mathrm{d}s\right) \\
&\leq -\gamma x^{\mathrm{T}}(t)Px(t) + \Big(2\mathrm{tr}(A) + n\gamma\Big)d(t)\int_{t-d(t)}^{t} \dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s \\
&\leq -\gamma x^{\mathrm{T}}(t)Px(t) + \Big(2\mathrm{tr}(A) + n\gamma\Big)D\int_{t-D}^{t} \dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s. \qquad (2.44)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\dot{V}_2(x_t) &= \epsilon\int_{-D}^{0}\Big(\dot{x}^{\mathrm{T}}(t)P\dot{x}(t) - \dot{x}^{\mathrm{T}}(t+\theta)P\dot{x}(t+\theta)\Big)\mathrm{d}\theta \\
&= \epsilon\left(D\dot{x}^{\mathrm{T}}(t)P\dot{x}(t) - \int_{t-D}^{t}\dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s\right). \qquad (2.45)
\end{aligned}$$

Combining (2.44) and (2.45), we have

$$\begin{aligned}
\dot{V}(x_t) \leq\ & -\gamma x^{\mathrm{T}}(t)Px(t) + \Big(2\mathrm{tr}(A) + n\gamma\Big)D\int_{t-D}^{t} \dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s \\
& +\epsilon D\dot{x}^{\mathrm{T}}(t)P\dot{x}(t) - \epsilon\int_{t-D}^{t}\dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s. \qquad (2.46)
\end{aligned}$$

By (2.39), Young's Inequality, Lemma 2.1 and Lemma 2.2, $\dot{x}^{\mathrm{T}}(t)P\dot{x}(t)$ can be evaluated as

$$\begin{aligned}
\dot{x}^{\mathrm{T}}(t)P\dot{x}(t) &= \Big(A_c x(t) + BB^{\mathrm{T}}P\lambda(t)\Big)^{\mathrm{T}} P\Big(A_c x(t) + BB^{\mathrm{T}}P\lambda(t)\Big) \\
&\leq 2x^{\mathrm{T}}(t)A_c^{\mathrm{T}}PA_c x(t) + 2\lambda^{\mathrm{T}}(t)PBB^{\mathrm{T}}PBB^{\mathrm{T}}P\lambda(t) \\
&\leq 2\varpi x^{\mathrm{T}}(t)Px(t) + 2\Big(2\mathrm{tr}(A) + n\gamma\Big)^2\lambda^{\mathrm{T}}(t)P\lambda(t) \\
&\leq 2\varpi x^{\mathrm{T}}(t)Px(t) + 2\Big(2\mathrm{tr}(A) + n\gamma\Big)^2\left(\int_{t-d(t)}^{t}\dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}} P\left(\int_{t-d(t)}^{t}\dot{x}(s)\mathrm{d}s\right) \\
&\leq 2\varpi x^{\mathrm{T}}(t)Px(t) + 2\Big(2\mathrm{tr}(A) + n\gamma\Big)^2 D\int_{t-D}^{t}\dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s, \qquad (2.47)
\end{aligned}$$

where ϖ is as defined in Lemma 2.1.

Substitution of (2.47) into (2.46) results in

$$\dot{V}(x_t) \leq (-\gamma + 2\epsilon D\varpi)x^{\mathrm{T}}(t)Px(t) + \left(\left(2\mathrm{tr}(A) + n\gamma\right)D + 2\epsilon D^2\left(2\mathrm{tr}(A) + n\gamma\right)^2 - \epsilon\right)$$
$$\times \int_{t-D}^{t} \dot{x}^{\mathrm{T}}(s)P\dot{x}(s)\mathrm{d}s. \tag{2.48}$$

Let

$$\epsilon = \frac{\left(2\mathrm{tr}(A) + n\gamma\right)D}{1 - 2D^2\left(2\mathrm{tr}(A) + n\gamma\right)^2} \tag{2.49}$$

be such that the second term in (2.48) is zero. Notice that ϵ is positive if and only if

$$D < \frac{1}{\sqrt{2}\left(2\mathrm{tr}(A) + n\gamma\right)}. \tag{2.50}$$

Hence, under condition (2.50),

$$\dot{V}(x_t) \leq (-\gamma + 2\epsilon D\varpi)x^{\mathrm{T}}(t)Px(t).$$

Furthermore, if

$$-\gamma + 2\epsilon D\varpi < 0, \tag{2.51}$$

then there exists some positive constant $\rho(\gamma)$, depending on γ, such that

$$\begin{aligned}\dot{V}(x_t) &\leq -\rho(\gamma)x^{\mathrm{T}}(t)Px(t)\\ &\leq -\rho(\gamma)\lambda_{\min}(P)\|x(t)\|_2^2 < 0,\ x(t) \neq 0,\end{aligned}$$

where $\lambda_{\min}(P)$ denotes the minimal eigenvalue of P and $\|x(t)\|_2$ is the Euclidean norm of $x(t)$. Recalling the structure of $V(x_t)$, we have

$$\begin{aligned}\lambda_{\min}(P)\|x(t)\|_2^2 &\leq V(x_t)\\ &\leq \lambda_{\max}(P)\max_{s\in[-D,0]}\|x_t(s)\|_2^2 + \epsilon D\lambda_{\max}(P)\int_{-D}^{0}\|\dot{x}_t(s)\|_2^2\mathrm{d}s,\end{aligned}$$

where $\lambda_{\max}(P)$ is the maximum eigenvalue of P. By Theorem 1.1, stabilization of system (2.1) is achieved under the delay independent truncated predictor feedback law (2.6) as long as (2.50) and (2.51) hold.

Substitution of the expressions of ϵ and ϖ into (2.51) gives

$$\gamma > 2D^2 \frac{\Big(2\text{tr}(A) + n\gamma\Big)}{1 - 2D^2\Big(2\text{tr}(A) + n\gamma\Big)^2} \left(\frac{1}{2}\Big(n\gamma + 2\text{tr}(A)\Big)\Big((n+1)\gamma + 2\text{tr}(A)\Big) - \gamma\text{tr}(A) - \text{tr}(A^2) \right),$$

which is equivalent to (2.38). In view of (2.50), if the delay bound D satisfies

$$D < \min\{D_1, D_2\},$$

where D_1 and D_2 are as defined in the proof of Corollary 1, then the the closed-loop system is asymptotically stable. Noting that $0 < D_1 < D_2$ for any $\gamma > 0$ by Corollary 1, we have

$$\min\{D_1, D_2\} = D_1.$$

This completes the proof. □

Remark 2.1. *To examine the conservativeness of Theorem 2.1, we now compare (2.38) with the existing results in the literature. Considering the case where all eigenvalues of the open loop system are at the origin, we have*

$$\text{tr}(A) = \text{tr}(A^2) = 0.$$

Inequality (2.38) then simplifies to

$$D < \frac{1}{n\sqrt{n + 3\gamma}}, \ \ \gamma > 0,$$

which is equivalent to

$$\gamma < \frac{1}{n\sqrt{n + 3D}}, \ \ D > 0. \tag{2.52}$$

This implies that for any time-varying delay that is bounded by an arbitrarily large D, the system would be stabilized under the delay independent truncated predictor feedback law as long as γ is chosen sufficiently small. Moreover, the upper bound on γ is given by (2.52). Recall the result of Theorem 2 *from [144], which establishes an upper bound of γ as*

$$\gamma < \frac{1}{3\sqrt{3}n\sqrt{n}D}, \quad D > 0. \tag{2.53}$$

Comparing (2.52) with (2.53), we can easily observe that, for any system whose eigenvalues are all at the origin, our result on the upper bound of γ is less conservative than that of [144]. Note that the Razumikhin Stability Theorem was adopted to obtain (2.53). We consider the Krasovskii stability analysis an advantage over the Razumikhin stability analysis. □

Remark 2.2. *We consider a system whose open loop poles are all on the imaginary axis and there exist at least one pair of nonzero poles. In this case, $\mathrm{tr}(A) = 0$ and $\mathrm{tr}(A^2) < 0$. Consequently, (2.38) reduces to*

$$D < \left(\frac{1}{n\Big(n(n+3)\gamma^2 - 2\mathrm{tr}(A^2)\Big)}\right)^{\frac{1}{2}}, \quad \gamma > 0. \tag{2.54}$$

Moreover, the right-hand side of (2.54) is strictly decreasing with respect to γ. Hence, any

$$D < \left(\frac{1}{-2n\mathrm{tr}(A^2)}\right)^{\frac{1}{2}} \tag{2.55}$$

is a valid bound for some $\gamma > 0$.

In addition, recall that the system in Example 5.1 is unstable under any feedback parameter if

$$\tau = \frac{\pi}{2}$$

On the other hand, by (2.55), the delay bound of the system can be arbitrarily close to $\frac{1}{2\sqrt{2}} < \frac{\pi}{2}$, which requires γ to approach zero. This implies that Theorem 2.1 does not conflict with Example 5.1. □

Remark 2.3. *We now consider exponentially unstable systems whose open loop poles are all on the closed right-half plane. By Corollary 2, we observe that the delay bound given in (2.38) has the unique maximal value $D_1(\gamma^\star)$ at $\gamma = \gamma^\star$, where $\gamma^\star$ is the unique positive solution to (2.27), $D_1(\gamma)$ is as defined in Corollary 1 and $D_1(\gamma^\star)$ is independent of γ. Furthermore, we note that*

$$\begin{aligned}\lim_{\gamma\to 0^+} D_1(\gamma) =& \lim_{\gamma\to +\infty} D_1(\gamma) \\ =& 0.\end{aligned} \tag{2.56}$$

Therefore, any delay bound that satisfies $0 < D \leq D_1(\gamma^\star)$ *is valid for some* $\gamma > 0$. □

Remark 2.4. *We note that only the information of the delay bound, not even the information on the derivative of the delay, is involved in (2.38), which guarantees the stabilization of system (2.1) under the delay independent truncated predictor feedback law (2.6). Therefore, Theorem 2.1 can be applied to systems whose input delays are fast-varying, which will be shown in the simulation to be presented in Section 2.3. It is worth mentioning here that, for the practical application of Theorem 2.1, the delay bound* D *of the time-varying delay* $d(t)$ *needs to be known in order to determine an appropriate value of* γ *for the stability of the closed-loop system. However, no knowledge of the delay itself is needed in the construction of the delay independent state feedback TPF law (2.6). This illustrates the robustness of the feedback law (2.6) with respect to uncertainty in the delay.* □

2.3. Numerical Examples

In this section, we provide simulation results to demonstrate the theoretical conclusion of Theorem 2.1. We consider three differnet cases of system (2.1): a system whose open loop poles are all at the origin, a system whose open loop poles are all on the imaginary axis with at least one pair of nonzero poles and an exponentially unstable system whose open loop poles are all on the closed right-half plane.

2.3.0.1. **Example 5.2 (A system with all poles at the origin):** Consider system (2.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly, the system is controllable with all its open loop poles at the origin. For a linear system with all its open loop poles at the origin, the delay independent state feedback TPF law stabilizes the system in the presence of any bounded input delay as long as the value of the low gain feedback parameter in the feedback law is chosen sufficiently small. Consider a fast-varying time delay

$$d(t) = 0.5\left(1 + \sin^2(100t)\right), \tag{2.57}$$

with an upper bound $D = 1$. We pick a feedback parameter $\gamma = 0.3$. Let the initial condition of the system be

$$x(\theta) = \begin{bmatrix} 1 & 0 & -1 & 2 \end{bmatrix}^{\mathrm{T}}, \ \theta \in [-D, 0]. \tag{2.58}$$

The state response and control evolution under the delay independent state feedback TPF law are shown in Fig. 2.1. To examine the low gain nature of the feedback law (2.6), we pick a smaller value for γ as 0.1 and observe its effects on the closed-loop performance of the system. Fig. 2.2 illustrates the state response and input evolution of the closed-loop system with the same initial condition as given by (2.58). A comparison of the two figures clearly shows that, although the stabilization of the system in the presence of a large input delay requires the feedback parameter to be sufficiently small, an excessively small feedback parameter typically leads to poor closed-loop performance in the form of a large overshoot and a small convergence rate. Thus, we need to avoid choosing an unnecessarily small value of the low gain parameter. □

2.3.0.2. **Example 5.3 (A system with all poles on the imaginary axis):** Consider system (2.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly, (A, B) is controllable with $\lambda(A) = \{\pm j, \pm j\}$. Also, $\mathrm{tr}(A) = 0$ and $\mathrm{tr}(A^2) = -4$. Let $\gamma = 0.5$ be the feedback parameter, and the corresponding delay bound in (2.54) is given as $D < 0.1291$. For the simulation purpose, let $D = 0.12$. In order to show that the delay independent truncated predictor feedback law (2.6) is applicable to systems with a fast-varying input delay, we choose the time-varying delay as follows,

$$d(t) = 0.06\Big(1 + \sin^2(100t)\Big),$$

from which it can be easily verified that $d(t) \in [0.06, 0.12]$ and $|\dot{d}(t)| \leq 6$, $t \geq 0$. Simulation is run for 20 seconds with the initial condition

$$x^{\mathrm{T}}(\theta) = \begin{bmatrix} -2 & 1 & 0 & 2 \end{bmatrix}, \ \theta \in [-0.12, 0].$$

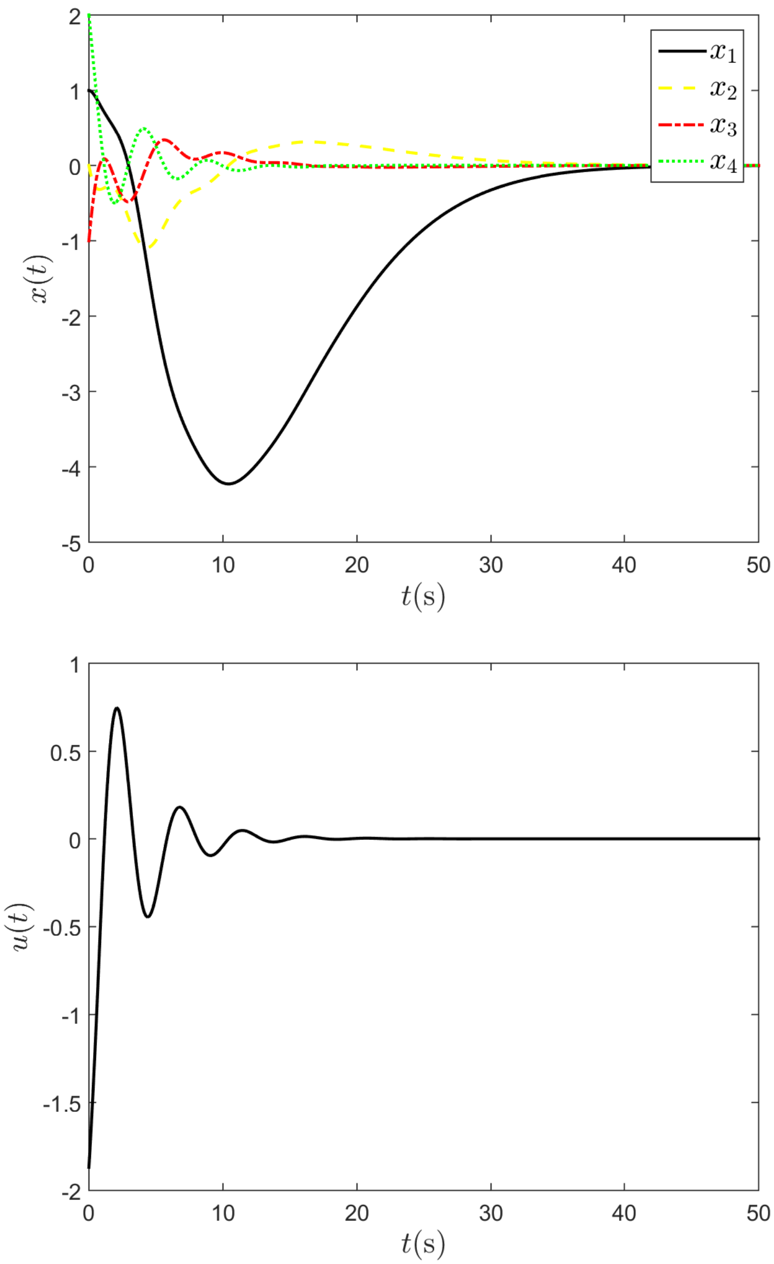


Fig. 2.1: State response and control input under the delay independent state feedback TPF law (2.6): $\gamma = 0.3$.

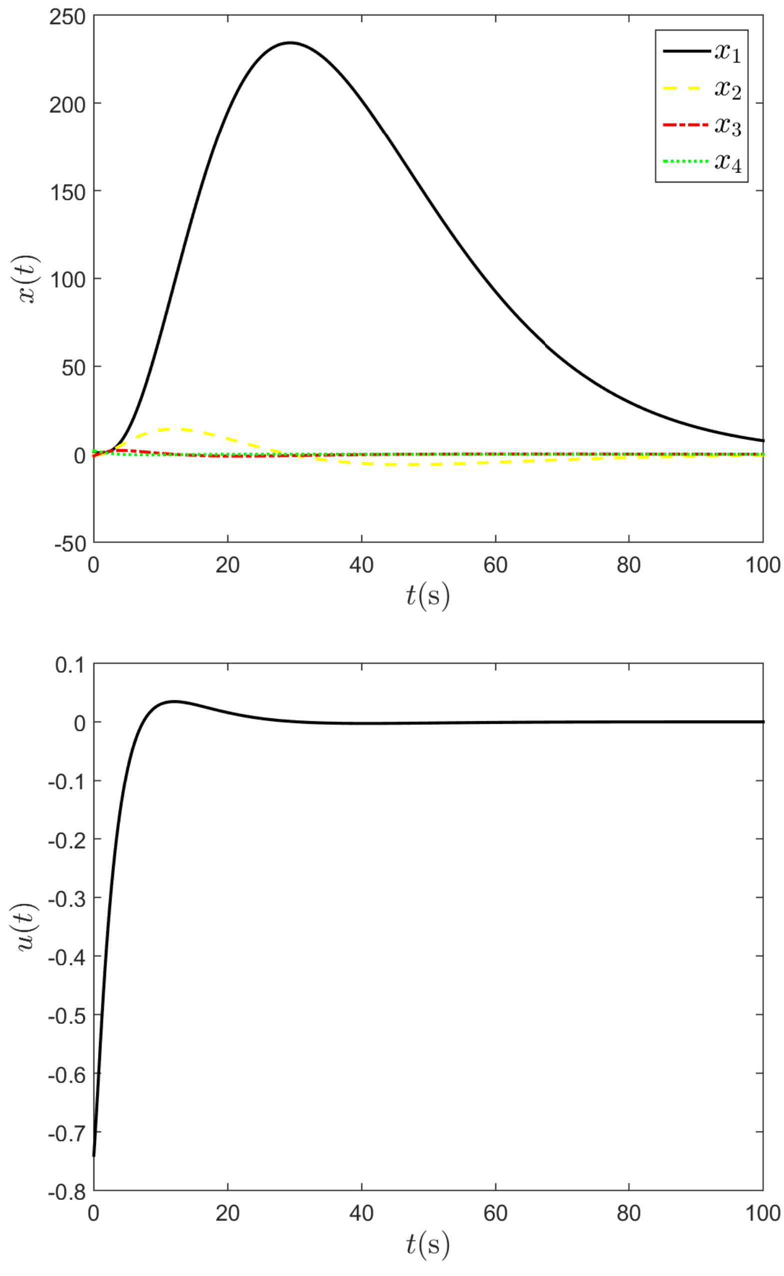


Fig. 2.2: State response and control input under the delay independent state feedback TPF law (2.6): $\gamma = 0.1$.

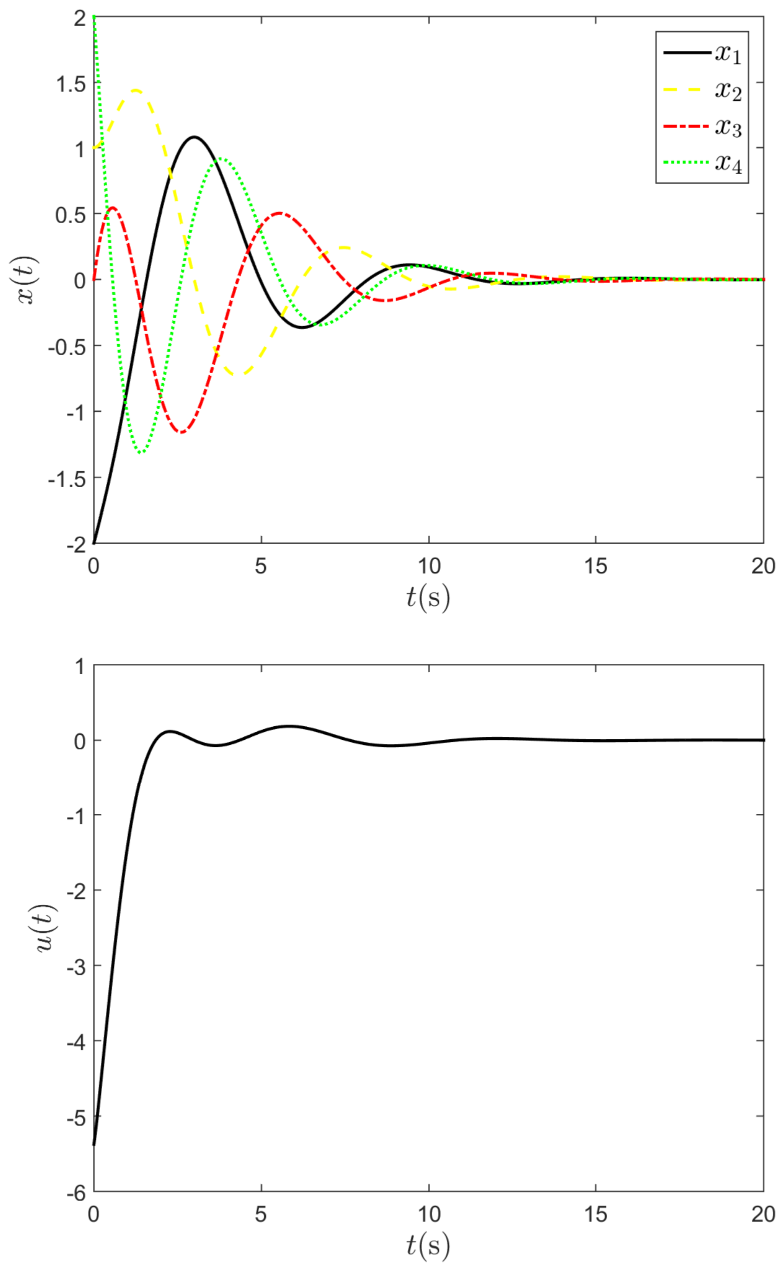


Fig. 2.3: State response and control input under the delay independent state feedback TPF law (2.6): $\gamma = 0.5$.

Shown in Fig. 2.3 are the state response $x(t)$ and the control input $u(t)$, for $\gamma = 0.5$. The stability of the closed-loop system is clear. To examine the conservativeness of delay bound in Theorem 2.1, we determine through simulation the tight bound on the constant delay for $\gamma = 0.5$ to be $D = 0.50$. $\square$

2.3.0.3. **Example 5.4 (An exponentially unstable system):** Consider system (2.1) with a controllable pair

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

It can be verified that $\lambda(A) = \{\pm j, \pm j, 1\}$. Also, $\mathrm{tr}(A) = 1$ and $\mathrm{tr}(A^2) = -3$. Let $\gamma = 0.1$. The corresponding delay bound in (2.38) is given as $D < 0.1768$. For the simulation purpose, we choose $D = 0.17$, and a fast-varying delay

$$d(t) = 0.085\Big(1 + \sin^2(100t)\Big).$$

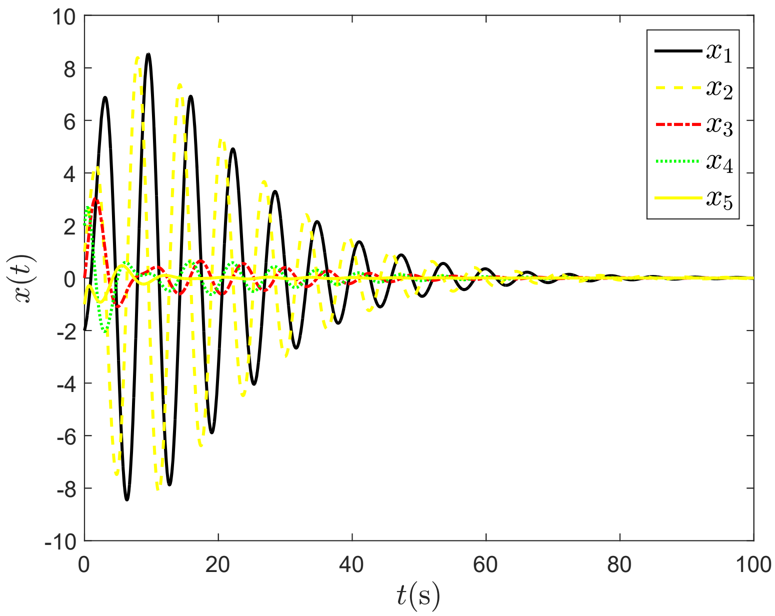
Clearly, $d(t) \in [0.085, 0.17]$ and $|\dot{d}(t)| \leq 8.5, t \geq 0$. With the initial condition given by

$$x^{\mathrm{T}}(\theta) = \begin{bmatrix} -2 & 1 & 0 & 2 & -1 \end{bmatrix}, \quad \theta \in [-0.17, 0],$$

simulation is run for 100 seconds. Shown in Fig. 2.4 are the state response and the control input, which show that the system is asymptotically stable. As in Example 1, we determine through simulation the tight bound on the constant delay for $\gamma = 0.1$ to be $D = 0.45$. □

2.4. Summary

We considered stabilization of a linear system with time-varying delay in the input by using a delay independent linear state feedback law that is inspired by the classical predictor feedback. It is known that such a feedback law is capable of stabilizing the time delay system for an arbitrary large delay if the open loop system is not exponentially unstable and has no purely imaginary poles. In this section, we first constructed an example to show that the feedback law is in general not able to stabilize the system for an arbitrarily large delay if the system has purely imaginary poles. We then derived a bound on the delay under which the delay independent truncated predictor feedback law achieves stabilization for a general system that may be exponentially unstable.



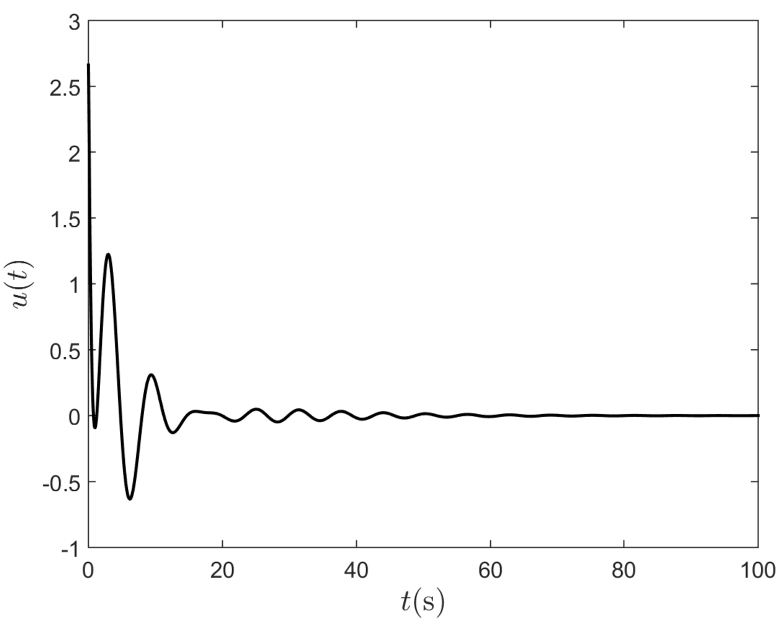


Fig. 2.4: State response and control input under the delay independent state feedback TPF law (2.6): $\gamma = 0.1$.

3. A Delay Independent Output Feedback Law for Linear Systsems with Input Delay

A delay-dependent or delay-independent truncated predictor state feedback law stabilizes a general linear system in the presence of a certain amount of input delay. Results in Section 2 provides estimates of the maximum delay bound under which the closed-loop stability can be achieved. In the face of time-varying or unknown delay, delay independent feedback laws are preferable over delay-dependent feedback laws as the former provide robustness to uncertainties in the delay. We present in this section a delay-independent observer based output feedback law that stabilizes the system. Our design is of the delay independence nature. We establish an estimate of the maximum allowable delay bound through a Razumikhin-type stability analysis. This delay bound result reveals the capability of the proposed output feedback law in handling an arbitrarily large input delay in linear systems with all open loop poles at the origin or in the open left-half plane. Compared with that of the delay-dependent output feedback laws in the literature, this same level of stabilization result is not sacrificed by the absence of the prior knowledge of the delay.

More specifically, we consider stabilization of a general linear system with time-varying input delay by delay independent output feedback. A state observer is constructed, without resorting to any knowledge of the delay, not even its upper bound. The design of such an observer based output feedback law, seemingly straightforward at the first glance, turns out to be a meticulous one because of the feed of the current input signal rather than the delayed version into the dynamic of the state observer to guarantee the delay independence of the observer dynamics. An estimated maximum delay bound for the stability of a general linear system under the influence of the proposed observer-based output feedback law is derived. The expression of the delay bound indicates that, for a class of systems whose open loop poles are at the origin or in the open left-half plane, the proposed delay independent output feedback law would handle an arbitrarily large delay, just as the delay dependent truncated predictor output feedback laws constructed in [72].

3.1. Feedback Design

We consider a linear system with a time-varying delay in the input,

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(\phi(t)), \ \ t \geq 0, \\ y(t) = Cx(t), \\ x(0) = x_0, \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^q$ and $x_0 \in \mathbb{R}^n$ are the state, the input, the output and the initial state, respectively. The time-varying delay function $\phi(t) : \mathbb{R}^+ \to \mathbb{R}$ is assumed to be expressed by

$$\phi(t) = t - d(t),$$

where the time-varying delay $d(t)$ is continuous and satisfies

$$0 \leq d(t) \leq D, \;\; t \geq 0, \tag{3.2}$$

and $D \in \mathbb{R}^+$ represents an upper bound of the delay. We also assume that the pair (A, B) is stabilizable and the pair (A, C) is detectable.

Without loss of generality, it is assumed that the pair (A, B) are in the form of

$$A = \begin{bmatrix} A_{\mathrm{L}} & 0 \\ 0 & A_{\mathrm{R}} \end{bmatrix}, \;\; B = \begin{bmatrix} B_{\mathrm{L}} \\ B_{\mathrm{R}} \end{bmatrix},$$

where all eigenvalues of $A_{\mathrm{L}} \in \mathbb{R}^{n_{\mathrm{L}} \times n_{\mathrm{L}}}$ are on the open left-half plane and all eigenvalues of $A_{\mathrm{R}} \in \mathbb{R}^{n_{\mathrm{R}} \times n_{\mathrm{R}}}$ are on the closed right-half plane. The block diagonal form of matrix A indicates that the stabilizability of the pair (A, B) implies the controllability of the pair $(A_{\mathrm{R}}, B_{\mathrm{R}})$.

It was established in [72] that an output feedback law built upon a delay dependent state observer,

$$\begin{cases} \dot{\hat{x}}(t) \;=\; A\hat{x}(t) + Bu(\phi(t)) - L(y(t) - C\hat{x}(t)), \\ u(t) \;=\; F(\gamma)e^{Ad(t)}\hat{x}(t), \end{cases} \tag{3.3}$$

stabilizes the system with all open loop poles on the closed left-half plane for an arbitrarily large delay as long as the feedback gain matrix $F(\gamma)$ is designed by the use of the eigenstructure assignment based low gain feedback technique [70] and the value of the low gain parameter γ is sufficiently small. Moreover, a slightly simpler observer-based output feedback law,

$$\begin{cases} \dot{\hat{x}}(t) \;=\; A\hat{x}(t) + Bu(\phi(t)) - L(y(t) - C\hat{x}(t)), \\ u(t) \;=\; F(\gamma)\hat{x}(t), \end{cases} \tag{3.4}$$

is shown to stabilize a system with all open loop poles at the origin or in the open left-half plane for an arbitrarily large delay if $F(\gamma)$ is constructed by using the low gain technique (see [70]). However, in (3.4), the complete knowledge of the time-varying delay, $d(t)$, is still required in the observer. Thus, to achieve the objective of a delay independent observer-based output feedback law, we propose the following observer-based delay independent truncated predictor output feedback law, also referred to as the delay independent output feedback TPF law,

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L(y(t) - C\hat{x}(t)), \;\; t \geq 0, \\ u(t) = F(\gamma)\hat{x}(t), \\ \hat{x}(\theta) = \psi(\theta), \;\; \theta \in [-D, 0], \end{cases} \tag{3.5}$$

where $\hat{x}$ is the state of the observer, $\psi(\theta)$ is the initial condition of $\hat{x}(t)$ and is assumed to be piecewise continuous on $\theta \in [-D, 0]$, $L \in \mathbb{R}^{n\times p}$ is such that all eigenvalues of $A + LC$ have a negative real part,

$$F(\gamma) = -B^{\mathrm{T}} P(\gamma),$$

and

$$P(\gamma) = \begin{bmatrix} 0 & 0 \\ 0 & P_{\mathrm{R}}(\gamma) \end{bmatrix}, \tag{3.6}$$

with $P_{\mathrm{R}}(\gamma) \in \mathbb{R}^{n_{\mathrm{R}}\times n_{\mathrm{R}}}$ being the unique positive definite solution to the parametric algebraic Riccati equation

$$A_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}(\gamma) + P_{\mathrm{R}}(\gamma) A_{\mathrm{R}} - P_{\mathrm{R}}(\gamma) B_{\mathrm{R}} B_{\mathrm{R}}^{\mathrm{T}} P_{\mathrm{R}}(\gamma) = -\gamma P_{\mathrm{R}}(\gamma), \tag{3.7}$$

for $\gamma > -2\min\{\mathrm{Re}(\lambda(A_{\mathrm{R}}))\}$. Note that the controllability of the pair $(A_{\mathrm{R}}, B_{\mathrm{R}})$ guarantees the existence and uniqueness of such P_{R} [144], and the detectability of the pair (A, C) ensures the existence of such L.

By defining an error signal as

$$e(t) = x(t) - \hat{x}(t), \tag{3.8}$$

we obtain the derivative of $e(t)$ from (3.1) and (3.5),

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\ &= Ae(t) + Bu(\phi(t)) - Bu(t) + LCe(t) \\ &= (A + LC)e(t) - BB^{\mathrm{T}} P(\hat{x}(\phi(t)) - \hat{x}(t)) \\ &= (A + LC)e(t) + BB^{\mathrm{T}} P(\alpha(t) - \beta(t)), \end{aligned}$$

where

$$\alpha(t) = x(t) - x(\phi(t)) \tag{3.9}$$

and

$$\beta(t) = e(t) - e(\phi(t)). \tag{3.10}$$

Then, the closed-loop system consisting of system (3.1) and the output feedback law (3.5) can be written as

$$\begin{cases} \dot{x}(t) \; = \; Ax(t) - BB^{\mathrm{T}}P(x(\phi(t)) - e(\phi(t))), \\ \dot{e}(t) \; = \; (A + LC)e(t) + BB^{\mathrm{T}}P(\alpha(t) - \beta(t)). \end{cases} \tag{3.11}$$

Decompose the state $x(t)$ as

$$x(t) = \begin{bmatrix} x_{\mathrm{L}}(t) \\ x_{\mathrm{R}}(t) \end{bmatrix},$$

we rewrite the closed-loop system as

$$\begin{cases} \dot{x}_{\mathrm{L}}(t) \; = \; A_{\mathrm{L}}x_{\mathrm{L}}(t) - B_{\mathrm{L}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(x_{\mathrm{R}}(\phi(t)) - e_{\mathrm{R}}(\phi(t))), \\ \dot{x}_{\mathrm{R}}(t) \; = \; A_{\mathrm{c}}x_{\mathrm{R}}(t) + B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) + e_{\mathrm{R}}(\phi(t))), \\ \dot{e}(t) \; = \; (A + LC)e(t) + BB_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t)), \end{cases} \tag{3.12}$$

where

$$A_{\mathrm{c}} = A_{\mathrm{R}} - B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}} \tag{3.13}$$

and signals $e_{\mathrm{L}}(t) \in \mathbb{R}^{n_{\mathrm{L}}}$, $e_{\mathrm{R}}(t) \in \mathbb{R}^{n_{\mathrm{R}}}$, $\alpha_{\mathrm{L}}(t) \in \mathbb{R}^{n_{\mathrm{L}}}$, $\alpha_{\mathrm{R}}(t) \in \mathbb{R}^{n_{\mathrm{R}}}$, $\beta_{\mathrm{L}}(t) \in \mathbb{R}^{n_{\mathrm{L}}}$ and $\beta_{\mathrm{R}}(t) \in \mathbb{R}^{n_{\mathrm{R}}}$ correspond to the following partitions

$$e(t) = \begin{bmatrix} e_{\mathrm{L}}(t) \\ e_{\mathrm{R}}(t) \end{bmatrix}, \; \alpha(t) = \begin{bmatrix} \alpha_{\mathrm{L}}(t) \\ \alpha_{\mathrm{R}}(t) \end{bmatrix}, \; \beta(t) = \begin{bmatrix} \beta_{\mathrm{L}}(t) \\ \beta_{\mathrm{R}}(t) \end{bmatrix},$$

respectively.

Clearly, the asymptotic stability of

$$\begin{cases} \dot{x}_{\mathrm{R}}(t) \; = \; A_{\mathrm{c}}x_{\mathrm{R}}(t) + B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) + e_{\mathrm{R}}(\phi(t))), \\ \dot{e}(t) \; = \; (A + LC)e(t) + BB_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t)) \end{cases} \tag{3.14}$$

guarantees that of system (3.12) because A_{L} is Hurwitz. Hence, without loss of generality, we only need to consider the asymptotic stability of system (3.14) for simplicity.

It is no longer without loss of generality to make the assumption on system (3.1) that all the eigenvalues of A are on the closed right-half plane, as is the case for stabilization analysis of truncated predictor based state feedback laws (see [147] or [144]).

3.2. Stability Analysis

Theorem 3.1. *For each $\gamma > 0$, there exists $D^* > 0$ such that, for each $D < D^*$, system (3.14) is asymptotically stable.*

Proof. We adopt the following Lyapunov function for the closed-loop system (3.14),

$$V(x_{\mathrm{R}}(t), e(t)) = x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}(\gamma)x_{\mathrm{R}}(t) + e^{\mathrm{T}}(t)Re(t),$$

where $P_{\mathrm{R}}(\gamma)$ is the unique positive definite solution to (3.7) with

$$\gamma > -2\min\{\mathrm{Re}(\lambda(A_{\mathrm{R}}))\}, \tag{3.15}$$

R is the unique positive definite solution to the Lyapunov equation

$$(A + LC)^{\mathrm{T}}R + R(A + LC) = -\xi I, \tag{3.16}$$

and ξ is some positive constant whose value is to be determined later. By using Young's Inequality and Lemma 2.1, and defining

$$\sigma = \mathrm{tr}(B^{\mathrm{T}}R^2B), \tag{3.17}$$

the derivative of $V(x_{\mathrm{R}}(t), e(t))$ along the trajectory of system (3.14) can be evaluated as follows,

$$\begin{aligned}
&\dot{V}(x_{\mathrm{R}}(t), e(t)) \\
=&\big(A_{\mathrm{c}}x_{\mathrm{R}}(t) + B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) + e_{\mathrm{R}}(\phi(t)))\big)^{\mathrm{T}}P_{\mathrm{R}}x_{\mathrm{R}}(t) + x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\big(A_{\mathrm{c}}x_{\mathrm{R}}(t) + B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}} \\
&\times (\alpha_{\mathrm{R}}(t) + e_{\mathrm{R}}(\phi(t)))\big) + \big((A + LC)e(t) + BB_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t))\big)^{\mathrm{T}}Re(t) \\
&+ e^{\mathrm{T}}(t)R\big((A + LC)e(t) + BB_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t) - \beta_{\mathrm{R}}(t))\big) \\
=&x_{\mathrm{R}}^{\mathrm{T}}(t)(A_{\mathrm{c}}^{\mathrm{T}}P_{\mathrm{R}} + P_{\mathrm{R}}A_{\mathrm{c}})x_{\mathrm{R}}(t) + 2\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}x_{\mathrm{R}}(t) + 2e_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}x_{\mathrm{R}}(t)
\end{aligned}$$

$$
\begin{aligned}
&+ e^{\mathrm{T}}(t)\big((A+LC)^{\mathrm{T}}R + R(A+LC)\big)e(t) + 2(\alpha_{\mathrm{R}}(t)-\beta_{\mathrm{R}}(t))^{\mathrm{T}}P_{\mathrm{R}}B_{\mathrm{R}}B^{\mathrm{T}}Re(t)\\
\leq& - x_{\mathrm{R}}^{\mathrm{T}}(t)(\gamma P_{\mathrm{R}} + P_{\mathrm{R}}B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}})x_{\mathrm{R}}(t) + 2\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)\\
&+ 2e_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}e_{\mathrm{R}}(\phi(t)) + x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}x_{\mathrm{R}}(t) - \xi e^{\mathrm{T}}(t)e(t) + \frac{1}{2}e^{\mathrm{T}}(t)e(t)\\
&+ 2(\alpha_{\mathrm{R}}(t)-\beta_{\mathrm{R}}(t))^{\mathrm{T}}P_{\mathrm{R}}B_{\mathrm{R}}B^{\mathrm{T}}R^2BB_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t)-\beta_{\mathrm{R}}(t))\\
\leq& - \gamma x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t) + 2(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t) + 4(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)e_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}e_{\mathrm{R}}(t)\\
&+ 4(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)\beta_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\beta_{\mathrm{R}}(t) - \left(\xi - \frac{1}{2}\right)e^{\mathrm{T}}(t)e(t) + 4(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)\\
&\times \sigma\big(\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t) + \beta_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\beta_{\mathrm{R}}(t)\big)\\
=& - \gamma x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t) + 2(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)(1+2\sigma)\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t) + 4(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)\\
&\times (1+\sigma)\beta^{\mathrm{T}}(t)P\beta(t) + e^{\mathrm{T}}(t)\left(-\left(\xi - \frac{1}{2}\right)I + 4(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)P\right)e(t).
\end{aligned}
\tag{3.18}
$$

For each $\gamma > -2\min\{\mathrm{Re}(\lambda(A_{\mathrm{R}}))\}$, there exists a sufficiently large ξ such that

$$
\gamma I + 4(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)P(\gamma) \leq \left(\xi - \frac{1}{2}\right)I, \tag{3.19}
$$

$$
P(\gamma) \leq R, \tag{3.20}
$$

$$
1 < \lambda_{\max}(R), \tag{3.21}
$$

$$
4(A+LC)^{\mathrm{T}}P(\gamma)(A+LC) \leq 3R(\varpi_{\mathrm{R}}(\gamma) + 2(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)^2), \tag{3.22}
$$

where $\varpi_{\mathrm{R}}(\gamma)$ is given by

$$
\varpi_{\mathrm{R}}(\gamma) = \frac{1}{2}\Big(n_{\mathrm{R}}\gamma + 2\mathrm{tr}(A_{\mathrm{R}})\Big)\Big((n_{\mathrm{R}}+1)\gamma + 2\mathrm{tr}(A_{\mathrm{R}})\Big) - \gamma\mathrm{tr}(A_{\mathrm{R}}) - \mathrm{tr}(A_{\mathrm{R}}^2).
$$

Fix this ξ. Based on inequality (3.19), which is equivalent to

$$
-\left(\xi - \frac{1}{2}\right)I + 4(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)P(\gamma) \leq -\gamma I, \tag{3.23}
$$

we estimate $\dot{V}$ as

$$
\begin{aligned}
\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq& -\gamma x_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}x_{\mathrm{R}} - \gamma e^{\mathrm{T}}e + 2(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)(1+2\sigma)\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)\\
&+4(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)(1+\sigma)\beta^{\mathrm{T}}(t)P\beta(t).
\end{aligned}
\tag{3.24}
$$

It follows from the definition of $\alpha_{\mathrm{R}}(t)$ and (3.14) that

$$\alpha_{\mathrm{R}}(t)=\int_{\phi(t)}^{t}\Big(A_{\mathrm{c}}x_{\mathrm{R}}(t)+B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)+B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}e_{\mathrm{R}}(\phi(t))\Big)\mathrm{d}t.$$

By using Lemma 2.1, Lemma 2.2 and Young's Inequality, we evaluate

$$\begin{aligned}
&\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)\\
\le\ &(t-\phi(t))\int_{\phi(t)}^{t}\Big(A_{\mathrm{c}}x_{\mathrm{R}}(t)+B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)+B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}e_{\mathrm{R}}(\phi(t))\Big)^{\mathrm{T}}\\
&\times P_{\mathrm{R}}\Big(A_{\mathrm{c}}x_{\mathrm{R}}(t)+B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)+B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}e_{\mathrm{R}}(\phi(t))\Big)\mathrm{d}t\\
\le\ &3D\int_{\phi(t)}^{t}\Big(x_{\mathrm{R}}^{\mathrm{T}}(t)A_{\mathrm{c}}^{\mathrm{T}}P_{\mathrm{R}}A_{\mathrm{c}}x_{\mathrm{R}}(t)+\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)\\
&+e_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}B_{\mathrm{R}}B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}e_{\mathrm{R}}(\phi(t))\Big)\mathrm{d}t\\
\le\ &3D\int_{\phi(t)}^{t}\Big(\varpi_{\mathrm{R}}(\gamma)x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)+2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)\\
&+2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2x_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}x_{\mathrm{R}}(\phi(t))+(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2e_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}e_{\mathrm{R}}(\phi(t))\Big)\mathrm{d}t\\
\le\ &3D\int_{\phi(t)}^{t}\Big(\Big(\varpi_{\mathrm{R}}(\gamma)+2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\Big)x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)\\
&+2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\Big(x_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}x_{\mathrm{R}}(\phi(t))+e^{\mathrm{T}}(\phi(t))Pe(\phi(t))\Big)\Big)\mathrm{d}t.
\end{aligned}\tag{3.25}$$

Similarly, by the definition of $\beta(t)$ and (3.14), we have

$$\beta(t)=\int_{\phi(t)}^{t}\Big((A+LC)e(t)+BB_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t)-\beta_{\mathrm{R}}(t))\Big)\mathrm{d}t.$$

Then, by using Lemma 2.1, Lemma 2.2, Young's Inequality and the fact that

$$B^{\mathrm{T}}PB=B_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}B_{\mathrm{R}},$$

we evaluate

$$\begin{aligned}
\beta^{\mathrm{T}}(t)P\beta(t)\ \le\ &(t-\phi(t))\int_{\phi(t)}^{t}\Big((A+LC)e(t)+BB_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t)-\beta_{\mathrm{R}}(t))\Big)^{\mathrm{T}}\\
&\times P\Big((A+LC)e(t)+BB_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t)-\beta_{\mathrm{R}}(t))\Big)\mathrm{d}t\\
\le\ &2D\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+(\alpha_{\mathrm{R}}(t)-\beta_{\mathrm{R}}(t))^{\mathrm{T}}
\end{aligned}$$

$$
\begin{aligned}
&\times P_{\mathrm{R}}B_{\mathrm{R}}B^{\mathrm{T}}PBB_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}(\alpha_{\mathrm{R}}(t)-\beta_{\mathrm{R}}(t))\Big)\mathrm{d}t\\
\le\;& 2D\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\\
&\times\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)+2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\beta_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\beta_{\mathrm{R}}(t)\Big)\mathrm{d}t\\
\le\;& 2D\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+4(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\\
&\times\Big(x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)+x_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}x_{\mathrm{R}}(\phi(t))+e_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}e_{\mathrm{R}}(t)\\
&+e_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}e_{\mathrm{R}}(\phi(t))\Big)\Big)\mathrm{d}t\\
=\;& 2D\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+4(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\\
&\times\Big(x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)+x_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}x_{\mathrm{R}}(\phi(t))+e^{\mathrm{T}}(t)Pe(t)\\
&+e^{\mathrm{T}}(\phi(t))Pe(\phi(t))\Big)\Big)\mathrm{d}t. \qquad (3.26)
\end{aligned}
$$

We employ (3.20) to simplify (3.25) and (3.26) as

$$
\begin{aligned}
\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)\;\le\;& 3D\int_{\phi(t)}^{t}\Big(\Big(\varpi_{\mathrm{R}}(\gamma)+2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\Big)x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)\\
&+2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2V(x_{\mathrm{R}}(\phi(t)),e(\phi(t))\Big)\mathrm{d}t,
\end{aligned}
$$

and

$$
\begin{aligned}
\beta^{\mathrm{T}}(t)P\beta(t)\;\le\;& 2D\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+4(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\\
&\times\Big(V(x_{\mathrm{R}}(t),e(t))+V(x_{\mathrm{R}}(\phi(t)),e(\phi(t)))\Big)\Big)\mathrm{d}t,
\end{aligned}
$$

respectively.

It then follows from (3.24) that

$$
\begin{aligned}
&\dot{V}(x_{\mathrm{R}}(t),e(t))\\
\le\;& -\gamma(x_{\mathrm{R}}^{\mathrm{T}}P_{\mathrm{R}}x_{\mathrm{R}}+e^{\mathrm{T}}e)+6D(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)(1+2\sigma)\\
&\times\int_{\phi(t)}^{t}\Big(\Big(\varpi_{\mathrm{R}}(\gamma)+2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\Big)x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)\\
&+2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2V(x_{\mathrm{R}}(\phi(t)),e(\phi(t))\Big)\mathrm{d}t+8D(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)
\end{aligned}
$$

$$
\begin{aligned}
&\times(1+\sigma)\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+4(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\\
&\times\Big(V(x_{\mathrm{R}}(t),e(t))+V(x_{\mathrm{R}}(\phi(t)),e(\phi(t)))\Big)\Big)\mathrm{d}t. \qquad (3.27)
\end{aligned}
$$

We simplify (3.27) by using (3.20) and (3.22) as follows,

$$
\begin{aligned}
&\dot{V}(x_{\mathrm{R}}(t),e(t))\\
\leq\ & -\frac{\gamma}{\lambda_{\max}(R)}V(x_{\mathrm{R}}(t),e(t))+6D(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)(1+2\sigma)\\
&\times\int_{\phi(t)}^{t}\Big(\Big(\varpi_{\mathrm{R}}(\gamma)+2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\Big)V(x_{\mathrm{R}}(t),e(t))\\
&+2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2V(x_{\mathrm{R}}(\phi(t)),e(\phi(t))\Big)\mathrm{d}t\\
&+32D(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^3(1+\sigma)\int_{\phi(t)}^{t}\Big(V(x_{\mathrm{R}}(t),e(t))+V(x_{\mathrm{R}}(\phi(t)),e(\phi(t)))\Big)\mathrm{d}t.
\end{aligned}
$$

When $V(x_{\mathrm{R}}(t+\theta),e(t+\theta))<\eta V(x_{\mathrm{R}}(t),e(t))$, $\theta\in[-2D,0]$, for a constant $\eta>1$, we have

$$
\begin{aligned}
&\dot{V}(x_{\mathrm{R}}(t),e(t))\\
\leq\ & -\frac{\gamma}{\lambda_{\max}(R)}V(x_{\mathrm{R}}(t),e(t))+6D^2\eta(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)(1+2\sigma)\\
&\times\Big(\varpi_{\mathrm{R}}(\gamma)+4(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\Big)V(x_{\mathrm{R}}(t),e(t))\\
&+64D^2\eta(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^3(1+\sigma)V(x_{\mathrm{R}}(t),e(t))\\
=\ & -V(x_{\mathrm{R}}(t),e(t))\Big(\frac{\gamma}{\lambda_{\max}(R)}-6D^2\eta(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)(1+2\sigma)\\
&\times\Big(\varpi_{\mathrm{R}}(\gamma)+4(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\Big)-64D^2\eta(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^3(1+\sigma)\Big).
\end{aligned}
$$

It then follows that if

$$
\begin{aligned}
\frac{\gamma}{\lambda_{\max}(R)}\ >\ & 6D^2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)(1+2\sigma)\Big(\varpi_{\mathrm{R}}(\gamma)+4(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^2\Big)\\
&+64D^2(2\mathrm{tr}(A_{\mathrm{R}})+n_{\mathrm{R}}\gamma)^3(1+\sigma), \qquad (3.28)
\end{aligned}
$$

then, $\dot{V}(x_{\mathrm{R}}(t),e(t))<-\rho(\gamma)V(x_{\mathrm{R}}(t),e(t))$ for some positive constant $\rho(\gamma)$, which implies that system (3.14) is asymptotically stable by the Rzumikhin Stability Theorem (Theorem 1.3). By

$$
\gamma>-2\mathrm{min}\{\mathrm{Re}(\lambda(A_{\mathrm{R}}))\},
$$

the definitions of σ and $\varpi_{\mathrm{R}}(\gamma)$, the right-hand side of (3.28) is non-negative. Thus, $\gamma > 0$, which naturally satisfies $\gamma > -2\min\{\mathrm{Re}(\lambda(A_{\mathrm{R}}))\}$, is necessary for (3.28) to hold. We further note that the right-hand side of (3.28) is a strictly increasing function of D, and it goes to zero as D goes to zero and goes to infinity as D goes to infinity. On the other hand, the left-hand side of (3.28) is a positive constant independent of D. Therefore, for each $D < D^*$, where D^* is the unique positive solution to the following nonlinear equation,

$$\begin{aligned}\frac{\gamma}{\lambda_{\max}(R)} =& 6D^2(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)(1+2\sigma)\Big(\varpi_{\mathrm{R}}(\gamma) + 4(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)^2\Big)\\ &+ 64D^2(2\mathrm{tr}(A_{\mathrm{R}}) + n_{\mathrm{R}}\gamma)^3(1+\sigma), \end{aligned} \tag{3.29}$$

system (3.14) is asymptotically stable. This completes the proof. □

Theorem 3.2. *Assume that all the eigenvalues of A_{R} are at the origin. For an arbitrarily large delay bound D, there exists γ^*¿0 such that, for each $\gamma \in (0, \gamma^*)$, system (3.14) is asymptotically stable.*

Proof. We adopt the same Lyapunov function (3.15) and the estimate (3.18) of its time derivative along the trajectory of system (3.14) for our stability analysis. In view of the assumption that all the eigenvalues of A_{R} are at the origin, we rewrite (3.18) as

$$\begin{aligned}\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq& -\gamma x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t) + 2n_{\mathrm{R}}\gamma(1+2\sigma)\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)\\ &+ 4n_{\mathrm{R}}\gamma(1+\sigma)\beta^{\mathrm{T}}(t)P\beta(t) + e^{\mathrm{T}}(t)\left(-\left(\beta - \frac{1}{2}\right)I + 4n_{\mathrm{R}}\gamma P\right)e(t). \end{aligned} \tag{3.30}$$

By the low gain feedback design,

$$\lim_{t\to\infty} P(\gamma) = 0,$$

and hence there exist $\nu > 0$ and $\gamma_1 > 0$ such that

$$P(\gamma) \leq \gamma\nu I, \;\; \gamma \leq \gamma_1. \tag{3.31}$$

Pick $\xi > \max\{\frac{1}{2}, \nu\}$. Then, there exists $\gamma_2 > 0$ such that, for each $\gamma \in (0, \gamma_2)$,

$$-\left(\xi - \frac{1}{2}\right)I + 4n_{\mathrm{R}}\gamma P(\gamma) \leq -\gamma R, \tag{3.32}$$

which implies that

$$\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq - \gamma V(x_{\mathrm{R}}(t), e(t)) + 2n_{\mathrm{R}}\gamma(1+2\sigma)\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)$$
$$+ 4n_{\mathrm{R}}\gamma(1+\sigma)\beta^{\mathrm{T}}(t)P\beta(t). \tag{3.33}$$

To further estimate the time derivative (3.33) of V, we follow (3.25) and (3.26) to derive

$$\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t) \leq 3D\int_{\phi(t)}^{t}\Big(\Big(\varpi_{\mathrm{R}}(\gamma)+2(n_{\mathrm{R}}\gamma)^2\Big)x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)$$
$$+2(n_{\mathrm{R}}\gamma)^2\Big(x_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}x_{\mathrm{R}}(\phi(t))+e^{\mathrm{T}}(\phi(t))Pe(\phi(t))\Big)\Big)\mathrm{d}t$$

and

$$\beta^{\mathrm{T}}(t)P\beta(t) \leq 2D\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+4(n_{\mathrm{R}}\gamma)^2$$
$$\times\Big(x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)+x_{\mathrm{R}}^{\mathrm{T}}(\phi(t))P_{\mathrm{R}}x_{\mathrm{R}}(\phi(t))+e^{\mathrm{T}}(t)Pe(t)$$
$$+e^{\mathrm{T}}(\phi(t))Pe(\phi(t))\Big)\Big)\mathrm{d}t,$$

respectively.

There exists $\gamma_3 > 0$ such that, for each $\gamma \in (0, \gamma_3)$,

$$P(\gamma) \leq R, \tag{3.34}$$

by which $\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t)$ and $\beta^{\mathrm{T}}(t)P\beta(t)$ can be respectively majorized further as

$$\alpha_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}\alpha_{\mathrm{R}}(t) \leq 3D\int_{\phi(t)}^{t}\Big(\Big(\varpi_{\mathrm{R}}(\gamma)+2(n_{\mathrm{R}}\gamma)^2\Big)x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t)$$
$$+2(n_{\mathrm{R}}\gamma)^2V(x_{\mathrm{R}}(\phi(t)), e(\phi(t))\Big)\mathrm{d}t,$$

and

$$\beta^{\mathrm{T}}(t)P\beta(t) \leq 2D\int_{\phi(t)}^{t}\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t)+4(n_{\mathrm{R}}\gamma)^2$$
$$\times\Big(V(x_{\mathrm{R}}(t), e(t))+V(x_{\mathrm{R}}(\phi(t)), e(\phi(t)))\Big)\Big)\mathrm{d}t.$$

Inequality (3.33) can then be continued as

$$
\begin{aligned}
\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq & -\gamma V(x_{\mathrm{R}}(t), e(t)) + 6Dn_{\mathrm{R}}\gamma(1+2\sigma)\int_{\phi(t)}^{t}\bigg(\Big(\varpi_{\mathrm{R}}(\gamma) \\
& + 2(n_{\mathrm{R}}\gamma)^2\Big)x_{\mathrm{R}}^{\mathrm{T}}(t)P_{\mathrm{R}}x_{\mathrm{R}}(t) + 2(n_{\mathrm{R}}\gamma)^2 V(x_{\mathrm{R}}(\phi(t)), e(\phi(t)))\bigg)\mathrm{d}t \\
& + 8Dn_{\mathrm{R}}\gamma(1+\sigma)\int_{\phi(t)}^{t}\bigg(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t) \\
& + 4(n_{\mathrm{R}}\gamma)^2\Big(V(x_{\mathrm{R}}(t), e(t)) + V(x_{\mathrm{R}}(\phi(t)), e(\phi(t)))\Big)\bigg)\mathrm{d}t. \qquad (3.35)
\end{aligned}
$$

When $V(x_{\mathrm{R}}(t+\theta), e(t+\theta)) < \eta V(x_{\mathrm{R}}(t), e(t))$, $\theta \in [-2D, 0]$, for a constant $\eta > 1$, we obtain

$$
\begin{aligned}
\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq & -\gamma V(x_{\mathrm{R}}(t), e(t)) + 6D^2 n_{\mathrm{R}}\gamma(1+2\sigma)(\varpi_{\mathrm{R}}(\gamma) \\
& + 4(n_{\mathrm{R}}\gamma)^2)V(x_{\mathrm{R}}(t), e(t)) + 8D^2 n_{\mathrm{R}}\gamma(1+\sigma) \\
& \times\Big(e^{\mathrm{T}}(t)(A+LC)^{\mathrm{T}}P(A+LC)e(t) + 8(n_{\mathrm{R}}\gamma)^2 V(x_{\mathrm{R}}(t), e(t))\Big), \qquad (3.36)
\end{aligned}
$$

which, by (3.31), can be continued as,

$$
\begin{aligned}
\dot{V}(x_{\mathrm{R}}(t), e(t)) \leq & -\gamma V(x_{\mathrm{R}}(t), e(t)) + 6D^2 n_{\mathrm{R}}\gamma(1+2\sigma)(\varpi_{\mathrm{R}}(\gamma) + 4(n_{\mathrm{R}}\gamma)^2)V(x_{\mathrm{R}}(t), e(t)) \\
& + 8D^2 n_{\mathrm{R}}\gamma^2(1+\sigma)\left(\frac{\nu}{\lambda_{\max}(R)} + 8(n_{\mathrm{R}}\gamma)^2\right)V(x_{\mathrm{R}}(t), e(t)) \\
= & -\gamma\Bigg(1 - 6D^2 n_{\mathrm{R}}(1+2\sigma)(\varpi_{\mathrm{R}}(\gamma) + 4(n_{\mathrm{R}}\gamma)^2) - 8D^2 n_{\mathrm{R}}\gamma(1+\sigma) \\
& \times\left(\frac{\nu}{\lambda_{\max}(R)} + 8(n_{\mathrm{R}}\gamma)^2\right)\Bigg)V(x_{\mathrm{R}}(t), e(t)). \qquad (3.37)
\end{aligned}
$$

Then, there exists $\gamma_4 > 0$ such that, for each $\gamma \in (0, \gamma_4)$,

$$
6D^2 n_{\mathrm{R}}(1+2\sigma)(\varpi_{\mathrm{R}}(\gamma) + 4(n_{\mathrm{R}}\gamma)^2) + 8D^2 n_{\mathrm{R}}\gamma(1+\sigma)\left(\frac{\nu}{\lambda_{\max}(R)} + 8(n_{\mathrm{R}}\gamma)^2\right) < 1.
$$

Taking $\gamma^* = \min\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ completes the proof. □

3.3. Numerical Examples

We consider two examples. The first example illustrates the ability of the delay independent output feedback law (3.5) in stabilizing a linear system with all open loop poles at the origin and in the presence of a large fast-varying input delay. The second example demonstrates the stabilization of an exponentially unstable system with a fast-varying input delay under the delay independent observer based feedback law (3.5).

3.3.0.1. **Example 5.5 (A system with all open loop poles at the origin):** Consider system (3.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tag{3.38}$$

and initial conditions

$$x(0) = \begin{bmatrix} 1 & 0 & -1 & 2 \end{bmatrix}^{\mathrm{T}}, \ \hat{x}(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \ \theta \in [-D, 0]. \tag{3.39}$$

To study the non-sensitivity of the controller in handling a fast-varying delay, we consider a delay function,

$$d(t) = 1 + \sin(10t). \tag{3.40}$$

This delay function has an upper bound of 2 and its time derivative reaches a peak value of 10.

For linear systems whose open loop poles are all at the origin, we note from Theorem 3.2 that, in the face of an arbitrarily large bounded delay, the delay independent observer-based feedback law asymptotically stabilizes the system as long as γ is chosen small enough. With the pole placement of $\lambda(A + LC) = \{-1, -2, -3, -4\}$, the selection of a small $\gamma = 0.3$ completes the design elements of the feedback law (3.5) for simulation of the closed-loop system. Figs. 3.1 and 3.2 show the evolution of the system states, the error signal and the control input.

3.3.0.2. **Example 5.6 (An exponentially unstable system):** Consider system (3.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \tag{3.41}$$

and initial conditions

$$x(0) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^{\mathrm{T}}, \ \hat{x}(\theta) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \ \theta \in [-D, 0]. \tag{3.42}$$

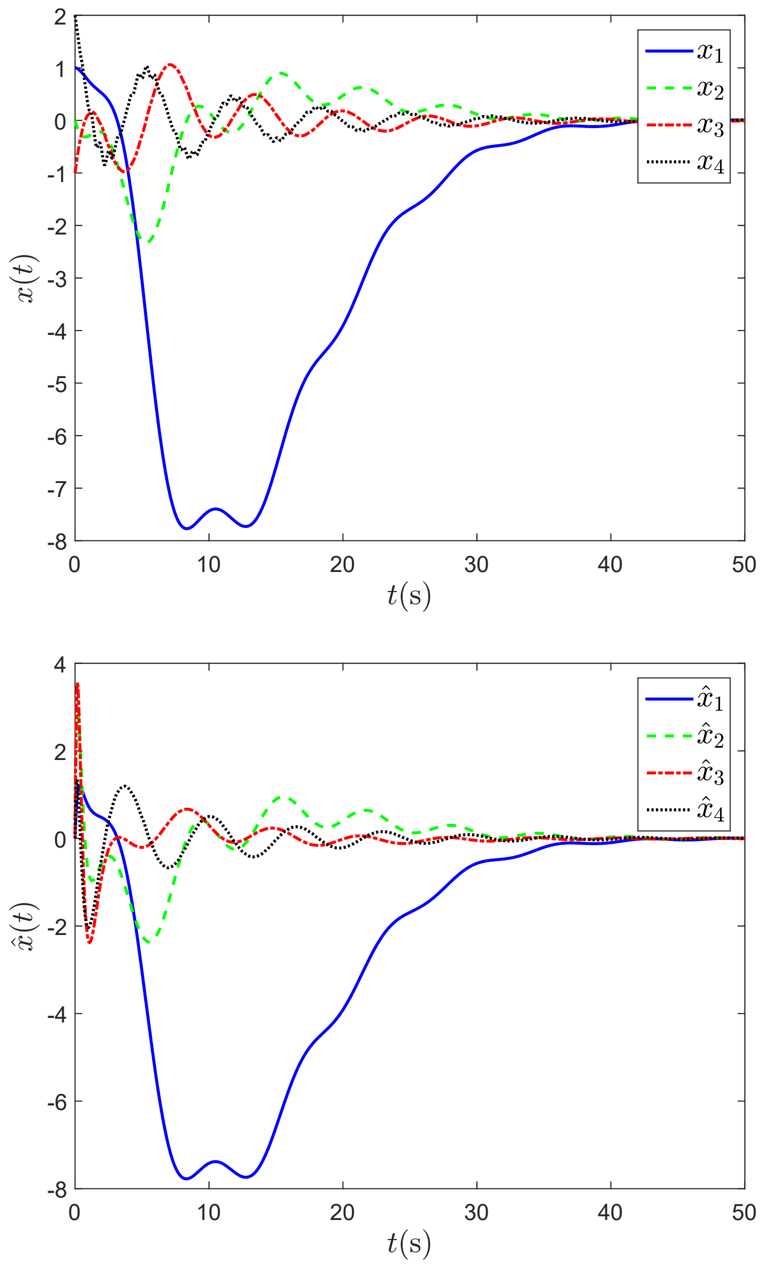


Fig. 3.1: The evolution of x and $\hat{x}$ in system (3.38) under the delay independent output feedback TPF law (3.5).

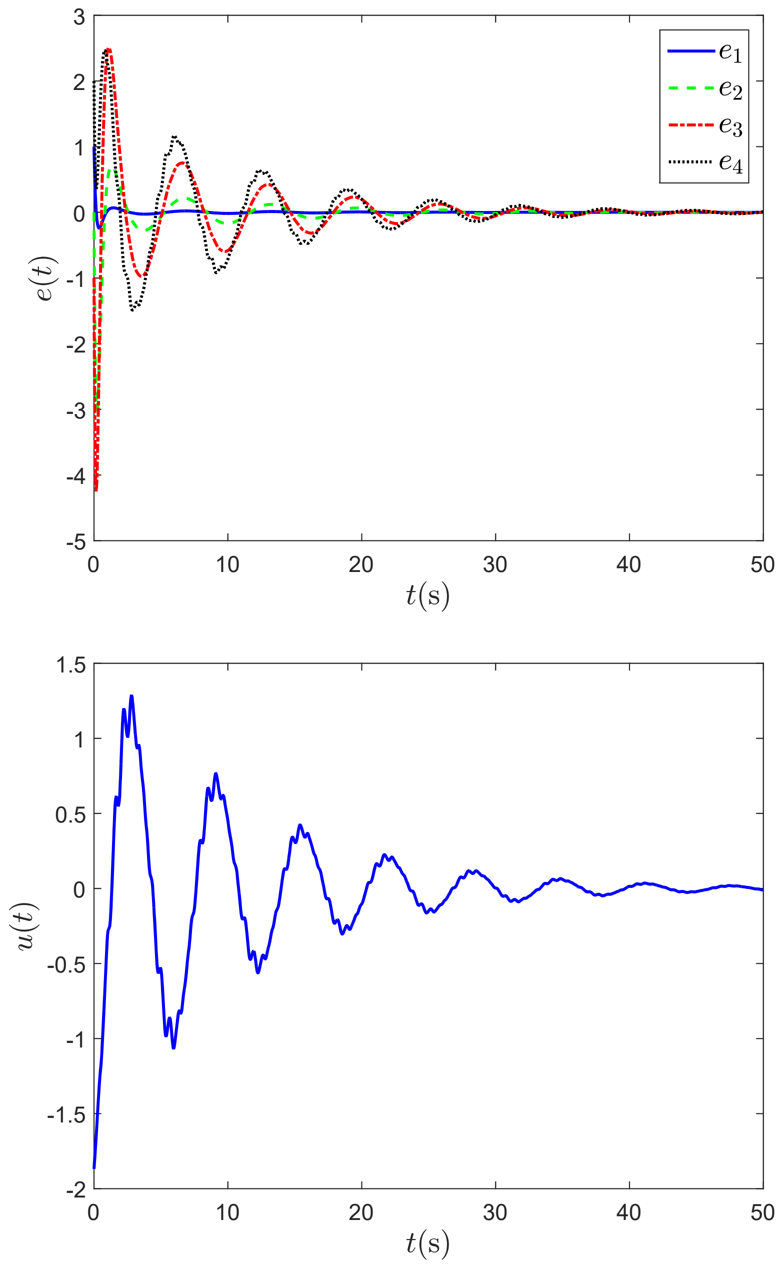


Fig. 3.2: The evolution of e and u in system (3.38) under the delay independent output feedback TPF law (3.5).

The time-varying delay is specified to be

$$d(t) = 0.5(1 + \sin(10t)). \tag{3.43}$$

Placing the eigenvalues of $A+LC$ at $\lambda = \{-1, -2, -3\}$ and picking a small low gain parameter $\gamma = 0.1$, we run the simulation and obtain the closed-loop evolution of system (3.41) under (3.5), which, as shown in Figs. 3.3 and 3.4, verifies the stabilizing effect of controller (3.5).

3.4. Summary

To overcome the lack of the exact knowledge of the delay, we constructed truncated predictor based feedback laws that do not involve the delay in their expression. We formulated the stability conditions in terms of the robustness of the feedback laws to a certain amount of input delay. Specifically, for each value of the feedback parameter, a general linear system that is possibly exponentially unstable is stabilized as long as the delay is below a certain bound. The proposed feedback laws are not completely delay independent because the design of the feedback parameter requires an upper bound of the delay to be known. Examination of the stability conditions shows that as long as the delay is below some upper bound, there exists some intervals of the feedback parameter such that any feedback parameter within such intervals would stabilize the system.

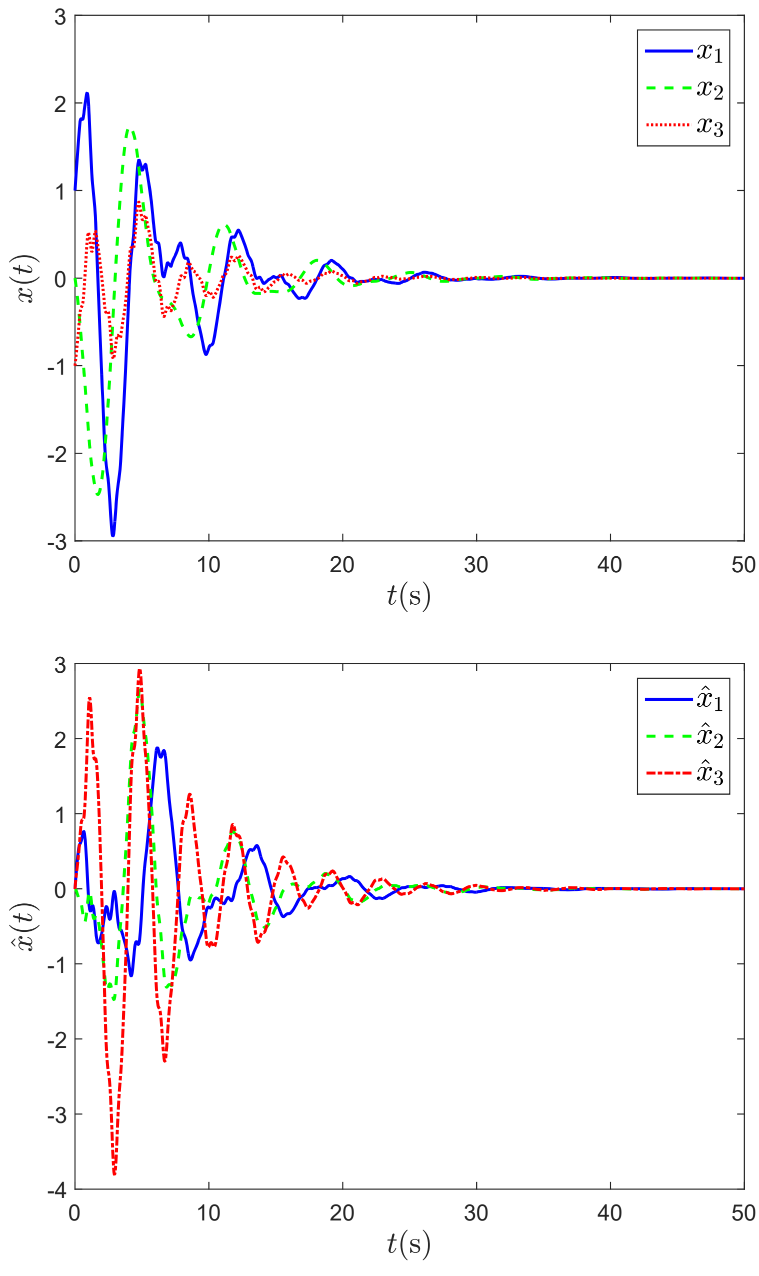


Fig. 3.3: The evolution of x and $\hat{x}$ in system (3.41) under the delay independent output feedback TPF law (3.5).

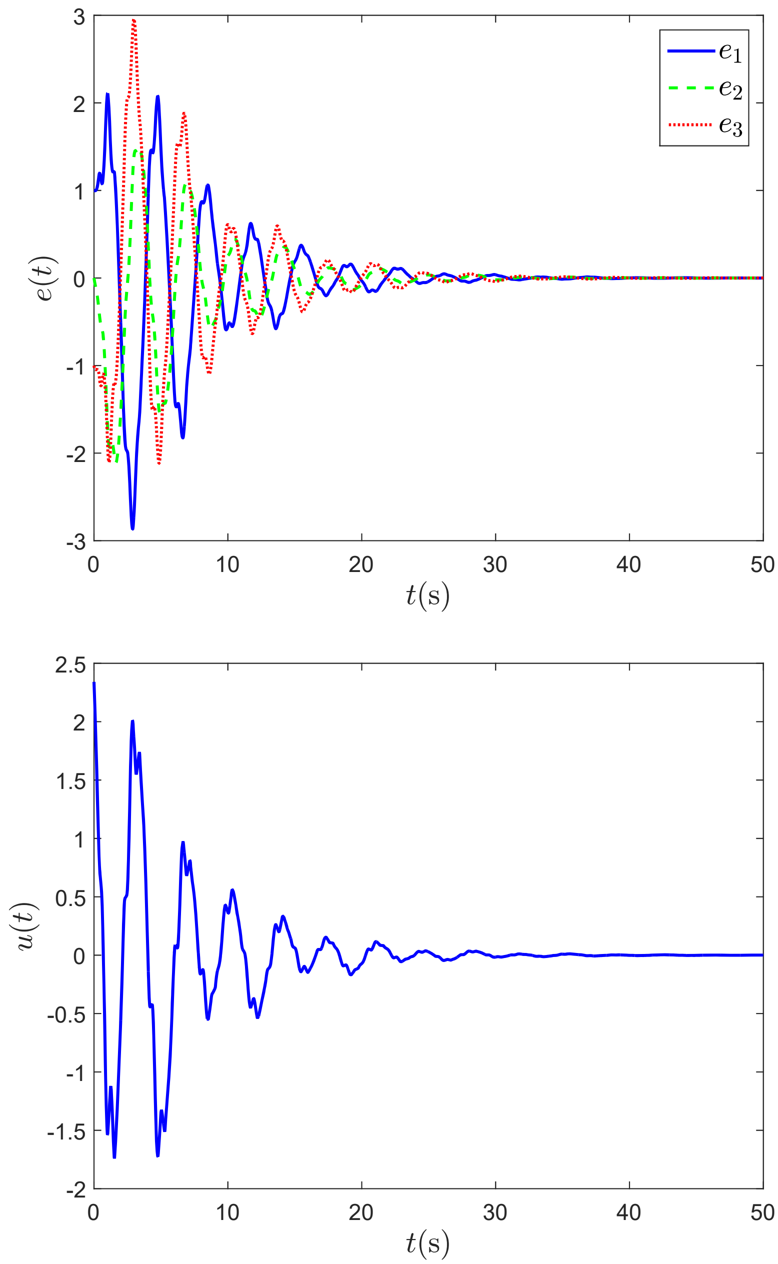


Fig. 3.4: The evolution of e and u in system (3.41) under the delay independent output feedback TPF law (3.5).

4. Improvement on Closed-loop Performance

Example 5.2 in Section 2 numerically verifies that a small value of the feedback parameter increases the ability of the delay independent truncated predictor feedback law to stabilize a linear system with all its open loop poles at the origin or in the open left-half plane. However, too small a feedback parameter degrades the closed-loop performance in terms of the overshoot and the convergence rate. Generally speaking, poor closed-loop performance with a large overshoot and a slow convergence rate is the consequence of the application of low gain feedback designs. We will examine the side effects of small values of the feedback parameter on the closed-loop performance in the stabilization of a linear system with all its open loop poles at the origin or in the open left-half plane. Furthermore, we will propose an approach to improving the closed-loop performance under a delay independent truncated predictor feedback law. Specifically, the traditional low gain feedback design with a constant feedback parameter is generalized to a time-varying parameter design. For an unknown delay with a known upper bound, a delay independent truncated predictor feedback law with a time-varying feedback parameter, constructed by using the parametric Lyapunov equation based approach, globally regulates the system to zero as long as the time-varying low gain parameter has a continuous second derivative and approaches a sufficiently small constant with its derivative approaching zero as time goes to infinity. Improvement on the closed-loop performance over the traditional constant parameter design is demonstrated through a convergence rate analysis and then observed in simulation.

The value of the low gain parameter embedded in the feedback gain matrix of the delay independent truncated prfedictor feedback law is determined by an upper bound of the delay (see [72] and [144]). The upper bound of the feedback parameter derived based on a Lyapunov analysis is conservative due to numerous estimations made throughout the derivation. Inspired by the spirit of the traditional low gain feedback design [70], we propose the concept of time-varying low gain feedback, where the low gain parameter is time-dependent. To improve the closed-loop performance, we design the parameter that starts from a relatively large value, and approaches a sufficiently small positive constant dependent on the known bound on the delay as time goes to infinity. Intuitively, such a time-varying parameter reduces the overshoot and increases the convergence rate in the early stage of the state evolution. As the closed-loop system reaches a state around zero, the value of the parameter is decreased to a sufficiently small constant that would not affect the closed-loop performance while guaranteeing stability.

A fundamental difficulty in stability analysis that involves a feedback law with a time-varying feedback parameter lies in the Lyapunov analysis. Typically, Lyapunov analysis requires $\dot{V}(x(t)) \leq -w(|x(t)|)$, where $V(x(t))$ is a Lyapunov function, $w(\cdot)$ is a positive scalar function and $|x(t)|$ is the Euclidean norm of the current state of system. In the case of time-varying feedback laws, taking the time derivative of a typical Lyapunov function $V(x(t), t) = x^{\mathrm{T}}(t)P(t)x(t)$, where $P(t)$ is a time-varying positive definite matrix, results in terms involving different time instants such as $x^{\mathrm{T}}(s)P(\theta)x(s)$ with $s \neq \theta$. Majorization of the time derivative by $-w(|x(t)|)$ poses challenges. However, the partial differential equation (PDE) representation of the closed-loop system, which has been extensively explored in [67], provides us with more freedom to construct a Lyapunov functional that facilitates such majorization. Indeed, this PDE representation has been exploited in [14] and [15] to carry out design and stability analysis of adaptive predictor feedback laws. Furthermore, as seen in these references, the Lyapunov analysis is direct, without resorting to the Krasovskii or Razumikhin Stability Theorem. Recent advancements in designing predictor-based feedback laws have been made by employing the PDE method and various backstepping transformations for linear time-invariant systems with distinct input delays [112] and nonlinear systems with multiple input delays [11] or even distributed input delays [10]. In this section, we will also adopt the PDE representation of the closed-loop system and direct stability analysis to obtain design specifications for the feedback parameter that guarantee closed-loop stability.

4.1. Time-varying Low Gain Feedback Design

We consider the regulation of a linear system with delayed input,

$$\dot{X}(t) = AX(t) + BU(t - \tau), \quad t \geq 0, \tag{4.1}$$

where $X \in \mathbb{R}^n$ and $U \in \mathbb{R}^m$ are the state and the input, respectively, and $\tau \in \mathbb{R}^+$ is an unknown constant delay with a known upper bound $\overline{\tau} \geq \tau$. In this section, we adopt capital letters X and U to denote the state and the input of the system because their lower cases are reserved for other purposes. The initial condition is given by

$$X(\theta) = \phi(\theta), \tag{4.2}$$

where $\phi(\theta) \in C[-\tau, 0]$. It is assumed that (A, B) is controllable with all eigenvalues of A at the origin.

Remark 4.1. *Alternatively, we can define the initial condition of the delayed system (4.1) as*

$$U(\theta) = \psi(\theta) \in C[-\tau, 0] \tag{4.3}$$

and

$$X(0) = X_0 \in \mathbb{R}^n. \tag{4.4}$$

Because we are considering the closed-loop system under a state feedback law, it is more convenient to define the initial condition solely in terms of the state $X(t)$. □

It was pointed out in Theorem 1 of [144] that the delay independent truncated predictor state feedback law (also referred to as the delay independent state feedback TPF law)

$$\begin{aligned} U(t) =& F(\gamma)X(t) \\ =& -B^{\mathrm{T}}P(\gamma)X(t), \end{aligned}$$

with a constant feedback parameter γ, asymptotically stabilizes system (4.1) for an arbitrarily large delay τ if

$$\gamma \in \left(0, \frac{1}{3\sqrt{3}n\sqrt{n}\overline{\tau}}\right], \tag{4.5}$$

where $P(\gamma)$ is the unique positive definite solution to the parametric algebraic Riccati equation,

$$A^{\mathrm{T}}P(\gamma) + P(\gamma)A - P(\gamma)BB^{\mathrm{T}}P(\gamma) = -\gamma P(\gamma), \;\; \gamma > 0. \tag{4.6}$$

Notice that the feedback gain matrix $F(\gamma)$ in the delay independent state feedback TPF law is constant when γ is fixed. Also, the theoretical bound of γ given by (4.5) is considerably smaller compared with the bound observed from simulation.

To overcome this conservativeness in determining the value of γ, we design a time-varying low gain feedback law whose feedback parameter is time-dependent,

$$\begin{aligned} U(t) =& F(\gamma(t))X(t) \\ =& -B^{\mathrm{T}}P(\gamma(t))X(t), \;\; t \geq -\tau. \end{aligned} \tag{4.7}$$

Considering the inverse proportionality relationship between the upper bound of γ and $\overline{\tau}$ as shown in

(4.5), we design the time-varying feedback parameter as

$$\gamma(t) = \frac{h}{\hat{\tau}(t)}. \tag{4.8}$$

Here h is a positive constant and $\hat{\tau}(t)$ satisfies the following conditions,

$$\hat{\tau}(t) \in C^2[-\tau, \infty), \ \ \hat{\tau}(t) > 0, \ \ \lim_{t\to\infty} \hat{\tau}(t) = \overline{\tau}, \ \ \lim_{t\to\infty} \dot{\hat{\tau}}(t) = 0. \tag{4.9}$$

4.2. PDE-ODE Cascade Representation

As pointed out in [67], the delayed input $U(t-\tau)$ in system (4.1) can be considered as the boundary value of

$$u(x,t) = U(t + \tau(x-1)), \ \ x \in [0,1] \tag{4.10}$$

at $x = 0$, where $u(x,t)$ is the solution of a transport PDE

$$\tau u_t(x,t) = u_x(x,t), \tag{4.11}$$

with the boundary condition

$$u(1,t) = U(t). \tag{4.12}$$

Thus, system (4.1) is equivalent to the cascade of an ordinary differential equation (ODE) with a transport PDE,

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{4.13}$$

$$\tau u_t(x,t) = u_x(x,t), \tag{4.14}$$

$$u(1,t) = U(t). \tag{4.15}$$

Following the idea in analyzing an adaptive predictor feedback law for linear systems with unknown input delay [15], we define a signal associated with $\hat{\tau}(t)$,

$$\hat{u}(x,t) = U(t + \hat{\tau}(t)(x-1)), \tag{4.16}$$

which can be verified to satisfy the PDE,

$$\hat{\tau}(t)\hat{u}_t(x,t) =(1+\dot{\hat{\tau}}(t)(x-1))\hat{u}_x(x,t), \tag{4.17}$$

$$\hat{u}(1,t) =U(t). \tag{4.18}$$

In view of the expression for the time-varying low gain feedback law (4.7), we have the difference between $\hat{u}(x,t)$ and $U(t)$ as

$$\begin{aligned}\hat{w}(x,t) &= \hat{u}(x,t) - U(t)\\ &= \hat{u}(x,t) + B^{\mathrm{T}}P(\gamma(t))X(t).\end{aligned} \tag{4.19}$$

To measure the distance between $\hat{\tau}(t)$ and the actual delay τ, we define

$$\tilde{\tau}(t) = \tau - \hat{\tau}(t), \tag{4.20}$$

and correspondingly,

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t), \tag{4.21}$$

from which, along with (4.7), (4.13), (4.17), (4.18) and (4.19), we obtain the PDE for $\hat{w}(x,t)$ as

$$\begin{aligned}\hat{\tau}(t)\hat{w}_t(x,t) = {}& \hat{w}_x(x,t)(1+\dot{\hat{\tau}}(t)(x-1)) + \hat{\tau}(t)B^{\mathrm{T}}\frac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t)\\ &+ \hat{\tau}(t)B^{\mathrm{T}}P(\gamma(t))\Big(\big(A - BB^{\mathrm{T}}P(\gamma(t))\big)X(t)\\ &+ B\tilde{u}(0,t) + B\hat{w}(0,t)\Big),\end{aligned} \tag{4.22}$$

$$\hat{w}(1,t) = 0. \tag{4.23}$$

By (4.13), (4.19) and (4.21), the closed-loop system under the time-varying low gain feedback law (4.7) takes the form,

$$\dot{X}(t) = \big(A - BB^{\mathrm{T}}P(\gamma(t))\big)X(t) + B\tilde{u}(0,t) + B\hat{w}(0,t). \tag{4.24}$$

By (4.14), (4.17), (4.18), (4.19), (4.20) and (4.21), the PDE for $\tilde{u}(x,t)$ is obtained as

$$\tau\tilde{u}_t(x,t) =\tilde{u}_x(x,t) - \frac{\tilde{\tau} + \tau\dot{\hat{\tau}}(t)(x-1)}{\hat{\tau}(t)}\hat{w}_x(x,t), \tag{4.25}$$

$$\tilde{u}(1,t) =0. \tag{4.26}$$

The constructed Lyapunov functional used in the stability analysis to be carried out in the next subsection also contains $\hat{w}_x(x,t)$ besides $\tilde{u}(x,t)$ and $\hat{w}(x,t)$. Thus, in view of (4.17), (4.19), (4.22) and (4.23), we derive the governing PDE for $\hat{w}_x(x,t)$ as

$$\hat{\tau}(t)\hat{w}_{xt}(x,t) = \hat{w}_{xx}(x,t)(1+\dot{\hat{\tau}}(t)(x-1)) + \dot{\hat{\tau}}(t)\hat{w}_x(x,t), \tag{4.27}$$

$$\begin{aligned}\hat{w}_x(1,t) =& -\hat{\tau}(t)B^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\dot{\gamma}(t)X(t) - \hat{\tau}(t)B^{\mathrm{T}}P(\gamma(t))\Big((A-BB^{\mathrm{T}}P(\gamma(t)))X(t)\\ &+ B\tilde{u}(0,t) + B\hat{w}(0,t)\Big).\end{aligned} \tag{4.28}$$

The following lemma establishes estimates of cross terms between $\tilde{u}(x,t)$, $\hat{w}(x,t)$, $\hat{w}_x(x,t)$, $\hat{w}_{xx}(x,t)$ and $X(t)$.

Lemma 4.1. *The following properties hold for system (4.1),*

$$\begin{aligned}&-\int_0^1 (1+x)(\tilde{\tau}+\tau\dot{\hat{\tau}}(t)(x-1))\tilde{u}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x\\ \le&\Big(|\tilde{\tau}| + \frac{1}{2}\tau|\dot{\hat{\tau}}(t)|\Big)\Big(\epsilon||\tilde{u}(t)||^2 + \frac{1}{\epsilon}||\hat{w}_x(t)||^2\Big),\end{aligned} \tag{4.29}$$

for any positive constant ϵ,

$$\begin{aligned}&\int_0^1 (1+x)(1+\dot{\hat{\tau}}(t)(x-1))\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x\\ \le&\frac{1}{2}(|\dot{\hat{\tau}}(t)|-1)|\hat{w}(0,t)|^2 + \Big(|\dot{\hat{\tau}}(t)| - \frac{1}{2}\Big)||\hat{w}(t)||^2,\end{aligned} \tag{4.30}$$

$$\begin{aligned}&\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}(t)B^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\dot{\gamma}(t)X(t)\mathrm{d}x\\ \le&h^{\frac{1}{2}}||\hat{w}(t)||^2 + \Big(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\Big)^2 h^{\frac{3}{2}}X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t),\end{aligned} \tag{4.31}$$

$$\begin{aligned}&\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}(t)B^{\mathrm{T}}P(\gamma(t))\Big(\big(A-BB^{\mathrm{T}}P(\gamma(t))\big)X(t) + B\tilde{u}(0,t) + B\hat{w}(0,t)\Big)\mathrm{d}x\\ \le&\hat{\tau}(t)n\gamma(t)||\hat{w}(t)||^2 + \frac{3}{2}\hat{\tau}(t)n(n+1)\gamma^2(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t)\\ &+ 3\hat{\tau}(t)n\gamma(t)\left(|\tilde{u}(0,t)|^2 + |\hat{w}(0,t)|^2\right),\end{aligned} \tag{4.32}$$

$$\begin{aligned}&\int_0^1 (1+x)(1+\dot{\hat{\tau}}(t)(x-1))\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\mathrm{d}x\\ \leq &|\hat{w}_x(1,t)|^2+\frac{1}{2}(|\dot{\hat{\tau}}(t)|-1)|\hat{w}_x(0,t)|^2+\Big(|\dot{\hat{\tau}}(t)|-\frac{1}{2}\Big)||\hat{w}_x(t)||^2,\end{aligned}\tag{4.33}$$

and,

$$\begin{aligned}|\hat{w}_x(1,t)|^2\leq &2\hat{\tau}^2(t)\dot{\gamma}^2(t)X^{\mathrm{T}}(t)\frac{\partial P}{\partial \gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial \gamma}X(t)+6\hat{\tau}^2(t)n^2\gamma^2(t)\\ &\times\Big(\frac{n+1}{2}\gamma(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t)+|\tilde{u}(0,t)|^2+|\hat{w}(0,t)|^2\Big).\end{aligned}\tag{4.34}$$

Proof. (4.29): By using Young's Inequality, we obtain

$$\begin{aligned}&-\int_0^1 (1+x)(\tilde{\tau}+\tau\dot{\hat{\tau}}(t)(x-1))\tilde{u}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x\\ \leq&\Big(|\tilde{\tau}|+\frac{1}{2}\tau|\dot{\hat{\tau}}(t)|\Big)\int_0^1\Big(\epsilon|\tilde{u}(x,t)|^2+\frac{1}{\epsilon}|\hat{w}_x(x,t)|^2\Big)\mathrm{d}x\\ =&\Big(|\tilde{\tau}|+\frac{1}{2}\tau|\dot{\hat{\tau}}(t)|\Big)\Big(\epsilon||\tilde{u}(t)||^2+\frac{1}{\epsilon}||\hat{w}_x(t)||^2\Big),\end{aligned}$$

where ϵ is any positive constant.

(4.30): Noting that $\hat{w}(1,t)=0$ and using integration by parts, we have

$$\begin{aligned}&\int_0^1(1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x\\ =&(1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{w}(x,t)|_0^1-\int_0^1\hat{w}^{\mathrm{T}}(x,t)\hat{w}(x,t)\mathrm{d}x-\int_0^1(1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x,\end{aligned}$$

which implies that

$$\int_0^1(1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x=-\frac{1}{2}(|\hat{w}(0,t)|^2+||\hat{w}(t)||^2).\tag{4.35}$$

On the other hand,

$$\int_0^1(x^2-1)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x=\frac{1}{2}|\hat{w}(0,t)|^2-\int_0^1 x|\hat{w}(x,t)|^2\mathrm{d}x$$

implies that

$$\dot{\hat{\tau}}(t)\int_0^1(x^2-1)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x\leq|\dot{\hat{\tau}}(t)|(\frac{1}{2}|\hat{w}(0,t)|^2+||\hat{w}(t)||^2).\tag{4.36}$$

Thus, (4.30) readily follows from adding (4.35) and (4.36).

(4.31): By (4.8) and Young's Inequality, we have

$$
\begin{aligned}
&\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}(t)B^{\mathrm{T}}\frac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t)\mathrm{d}x\\
=&-\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}hB^{\mathrm{T}}\frac{\partial P}{\partial \gamma}X(t)\mathrm{d}x\\
\leq& 2\int_0^1 \left| h^{\frac{1}{4}}\hat{w}^{\mathrm{T}}(x,t)\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}h^{\frac{3}{4}}B^{\mathrm{T}}\frac{\partial P}{\partial \gamma}X(t)\right|\mathrm{d}x\\
\leq& h^{\frac{1}{2}}||\hat{w}(t)||^2+\Big(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\Big)^2 h^{\frac{3}{2}}X^{\mathrm{T}}(t)\frac{\partial P}{\partial \gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial \gamma}X(t).
\end{aligned}
$$

(4.32): By Lemma 2.1 and Young's Inequality, we derive

$$
\begin{aligned}
&\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}(t)B^{\mathrm{T}}P(\gamma(t))\Big(\big(A-BB^{\mathrm{T}}P(\gamma(t))\big)X(t)+B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)\mathrm{d}x\\
=&\hat{\tau}(t)\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)B^{\mathrm{T}}P^{\frac{1}{2}}(\gamma(t))P^{\frac{1}{2}}(\gamma(t))\Big((A-BB^{\mathrm{T}}P(\gamma(t)))X(t)\\
&+B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)\mathrm{d}x\\
\leq&\hat{\tau}(t)\int_0^1 \hat{w}^{\mathrm{T}}(x,t)B^{\mathrm{T}}P(\gamma(t))B\hat{w}(x,t)\mathrm{d}x+\hat{\tau}(t)\int_0^1\Big((A-BB^{\mathrm{T}}P(\gamma(t)))X(t)\\
&+B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)^{\mathrm{T}}P(\gamma(t))\Big((A-BB^{\mathrm{T}}P(\gamma(t)))X(t)+B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)\mathrm{d}x\\
\leq&\hat{\tau}(t)n\gamma(t)||\hat{w}(t)||^2+3\hat{\tau}(t)\int_0^1\Big(X^{\mathrm{T}}(t)(A-BB^{\mathrm{T}}P(\gamma(t)))^{\mathrm{T}}P(\gamma(t))(A\\
&-BB^{\mathrm{T}}P(\gamma(t)))X(t)+\tilde{u}^{\mathrm{T}}(0,t)B^{\mathrm{T}}P(\gamma(t))B\tilde{u}(0,t)+\hat{w}^{\mathrm{T}}(0,t)B^{\mathrm{T}}P(\gamma(t))B\hat{w}(0,t)\Big)\mathrm{d}x\\
\leq&\hat{\tau}(t)n\gamma(t)||\hat{w}(t)||^2+\frac{3}{2}\hat{\tau}(t)n(n+1)\gamma^2(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t)\\
&+3\hat{\tau}(t)n\gamma(t)\left(|\tilde{u}(0,t)|^2+|\hat{w}(0,t)|^2\right).
\end{aligned}
$$

(4.33): It can be verified that

$$
\int_0^1 (1+x)\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\mathrm{d}x=|\hat{w}_x(1,t)|^2-\frac{1}{2}|\hat{w}_x(0,t)|^2-\frac{1}{2}||\hat{w}_x(t)||^2.
$$

On the other hand,

$$
\int_0^1 \left(x^2-1\right)\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\mathrm{d}x=\frac{1}{2}|\hat{w}_x(0,t)|^2-\int_0^1 x|\hat{w}_x(x,t)|^2\mathrm{d}x
$$

implies that

$$\dot{\hat{\tau}}(t)\int_0^1 \left(x^2-1\right)\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\mathrm{d}x \leq |\dot{\hat{\tau}}(t)|\left(\frac{1}{2}|\hat{w}_x(0,t)|^2+||\hat{w}_x(t)||^2\right).$$

Thus, (4.33) holds.

(4.34): We evaluate $\hat{w}_x(1,t)$ by using (4.28), Young's Inequality and Lemma 2.1 as follows,

$$\begin{aligned}
|\hat{w}_x(1,t)|^2 \leq & 2\hat{\tau}^2(t)\dot{\gamma}^2(t)X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t)+2\hat{\tau}^2(t)\Big((A-BB^{\mathrm{T}}P(\gamma(t)))X(t)\\
&+B\tilde{u}(0,t)+B\hat{w}(0,t))\Big)^{\mathrm{T}}P(\gamma(t))BB^{\mathrm{T}}P(\gamma(t))\Big((A-BB^{\mathrm{T}}P(\gamma(t)))X(t)\\
&+B\tilde{u}(0,t)+B\hat{w}(0,t))\Big)\\
\leq & 2\hat{\tau}^2(t)\dot{\gamma}^2(t)X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t)+2n\gamma(t)\hat{\tau}^2(t)\Big((A-BB^{\mathrm{T}}P(\gamma(t)))X(t)\\
&+B\tilde{u}(0,t)+B\hat{w}(0,t))\Big)^{\mathrm{T}}P(\gamma(t))\Big((A-BB^{\mathrm{T}}P(\gamma(t)))X(t)\\
&+B\tilde{u}(0,t)+B\hat{w}(0,t))\Big)\\
\leq & 2\hat{\tau}^2(t)\dot{\gamma}^2(t)X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t)+6n\gamma(t)\hat{\tau}^2(t)\Big(X^{\mathrm{T}}(t)(A-BB^{\mathrm{T}}P(\gamma(t)))^{\mathrm{T}}\\
&\times P(\gamma(t))(A-BB^{\mathrm{T}}P(\gamma(t)))X(t)+\tilde{u}^{\mathrm{T}}(0,t)B^{\mathrm{T}}P(\gamma(t))B\tilde{u}(0,t)\\
&+\hat{w}^{\mathrm{T}}(0,t)B^{\mathrm{T}}P(\gamma(t))B\hat{w}(0,t)\Big)\\
\leq & 2\hat{\tau}^2(t)\dot{\gamma}^2(t)X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t)\\
&+6\hat{\tau}^2(t)n^2\gamma^2(t)\Big(\frac{n+1}{2}\gamma(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t)+|\tilde{u}(0,t)|^2+|\hat{w}(0,t)|^2\Big).
\end{aligned}$$

□

4.3. Direct Stability Analysis

Based on the fact that $\hat{\tau}(t)$ has a continuous second derivative, we can establish the second order differentiability of the feedback gain matrix $F(\gamma(t))$, the state $X(t)$ and the input $U(t)$. The differentiability of these signals is extensively involved in the construction of the Lyapunov functional and stability analysis to be carried out next.

Lemma 4.2. *The time-varying feedback gain matrix $F(\gamma(t))$ in (4.7) is bounded and has a continuous second derivative on $t \in [-\tau, \infty)$.*

Proof. By the time-varying low gain feedback design (4.9), $\hat{\tau}(t) \in C[-\tau, \infty)$ is positive and has a finite limit $\overline{\tau}$, and hence $\hat{\tau}(t)$ is bounded on $t \in [-\tau, \infty)$. Suppose that

$$\inf_{t \geq -\tau} \hat{\tau}(t) = \tau_{\min}, \tag{4.37}$$

and

$$\sup_{t \geq -\tau} \hat{\tau}(t) = \tau_{\max}, \tag{4.38}$$

where $0 < \tau_{\min} \leq \tau_{\max}$. By the inverse proportionality relationship between $\gamma(t)$ and $\hat{\tau}(t)$, $\gamma(t)$ is also bounded, with

$$\inf_{t \geq -\tau} \gamma(t) = h/\tau_{\max} \tag{4.39}$$

and

$$\sup_{t \geq -\tau} \gamma(t) = h/\tau_{\min}. \tag{4.40}$$

The increasing monotonicity of $P(\gamma)$ with respect to γ by Lemma 2.1 implies that $P(\gamma(t))$ is bounded with

$$P(h/\tau_{\max}) \leq P(\gamma(t)) \leq P(h/\tau_{\min}). \tag{4.41}$$

Therefore, the boundedness of $F(\gamma(t))$ follows readily from the construction of

$$F(\gamma(t)) = -B^{\mathrm{T}} P(\gamma(t)).$$

On the other hand, as noted in [143], $P(\gamma)$ is a rational matrix function of γ, which implies that $P(\gamma)$ is infinitely differentiable with respect to the feedback parameter. The first and second derivatives of $F(\gamma(t))$ with respect to t are given as follows,

$$\dot{F}(\gamma(t)) = B^{\mathrm{T}} \frac{\partial P}{\partial \gamma} \frac{h}{\hat{\tau}^2(t)} \dot{\hat{\tau}}(t), \tag{4.42}$$

$$\ddot{F}(\gamma(t)) = B^{\mathrm{T}} \frac{h}{\hat{\tau}^2(t)} \Big(-\frac{\partial^2 P}{\partial \gamma^2} \frac{h}{\hat{\tau}^2(t)} \dot{\hat{\tau}}^2(t) - 2 \frac{\partial P}{\partial \gamma} \frac{\dot{\hat{\tau}}^2(t)}{\hat{\tau}(t)} + \frac{\partial P}{\partial \gamma} \ddot{\hat{\tau}}(t) \Big), \tag{4.43}$$

from which it follows that $F(\gamma(t))$ has a continuous second derivative with respect to t since $\hat{\tau}(t) \in C^2[-\tau, \infty)$. $\square$

Lemma 4.3. *With the initial condition $X(\theta) = \psi(\theta) \in C[-\tau, 0]$, system (4.1) under the time-varying low gain feedback law (4.7) has a unique solution $X(t) \in C[-\tau, \infty)$. Moreover, $U(t) \in C[-\tau, \infty)$ and $X(t), U(t) \in C^1(0, \infty) \cap C^2(\tau, \infty)$, where the notation $C^1(0, \infty) \cap C^2(\tau, \infty)$ denotes the set of functions defined on $(0, \infty)$ that has continuous first order derivative on $(0, \infty)$ and has continuous second order derivative on (τ, ∞).*

Proof. The proof follows the general idea of the proof of Theorem 3.1 in [12]. Under the time-varying low gain feedback law (4.7), system (4.1) becomes

$$\dot{X}(t) = AX(t) + BF(\gamma(t - \tau))X(t - \tau), \;\; t \geq 0. \tag{4.44}$$

Considering the continuity of both $F(\gamma(\theta))$ and $\psi(\theta)$ on $\theta \in [-\tau, 0]$, there exists a unique solution

$$X(t) = \mathrm{e}^{At}\psi(0) + \int_0^t \mathrm{e}^{A(t-s)} BF(\gamma(s - \tau))\psi(s - \tau)\mathrm{d}s,$$

on $t \in [0, \tau]$, which implies that $X(t) \in C[-\tau, \tau]$. Furthermore, the solution $X(t)$ on $t \in [\tau, 2\tau]$ is obtained as follows,

$$X(t) = \mathrm{e}^{A(t-\tau)}X(\tau) + \int_\tau^t \mathrm{e}^{A(t-s)} BF(\gamma(s - \tau))X(s - \tau)\mathrm{d}s, \;\; t \in [\tau, 2\tau],$$

which implies that $X(t) \in C[-\tau, 2\tau]$ due to the continuity of $X(t)$ and $F(\gamma(t))$ on $[0, \tau]$. Similarly, the existence and uniqueness of $X(t)$ can be established along the time axis toward positive infinity. The continuity and uniqueness of $X(t)$ on $t \in [-\tau, \infty)$ follow readily.

From the right-hand side of (4.44), we obtain $\dot{X}(t) \in C(0, \infty)$ in view of the fact that $X(t - \tau)$ is continuous on $t \in (0, \infty)$. Taking the time derivative of both sides of (4.44) yields

$$\begin{aligned}\ddot{X}(t) =& A^2X(t) + ABF(\gamma(t - \tau))X(t - \tau) + B\dot{F}(\gamma(t - \tau))X(t - \tau)\\ &+ BF(\gamma(t - \tau))AX(t - \tau) + BF(\gamma(t - \tau))BF(\gamma(t - 2\tau))X(t - 2\tau),\end{aligned}$$

which implies that $X(t) \in C^2(\tau, \infty)$ since $X(t) \in C[-\tau, \infty)$ and $F(\gamma(t))$ is continuously differentiable on $t \in [-\tau, \infty)$, as established in Lemma 4.2.

Considering the time-varying low gain feedback law

$$U(t) = F(\gamma(t))X(t), \tag{4.45}$$

we know that $U(t) \in C[-\tau, \infty)$ because both $X(t)$ and $F(\gamma(t))$ are continuous on $t \in [-\tau, \infty)$. Also, the first and second derivatives of $U(t)$ take the following form,

$$\dot{U}(t) = \dot{F}(\gamma(t))X(t) + F(\gamma(t))\dot{X}(t), \tag{4.46}$$

$$\ddot{U}(t) = \ddot{F}(\gamma(t))X(t) + 2\dot{F}(\gamma(t))\dot{X}(t) + F(\gamma(t))\ddot{X}(t), \tag{4.47}$$

which, in view of $X(t) \in C[-\tau, \infty) \cap C^1(0, \infty) \cap C^2(\tau, \infty)$ and the continuity of the second derivative of $F(\gamma(t))$, imply that $U(t) \in C^1(0, \infty) \cap C^2(\tau, \infty)$. □

With the time-varying low gain feedback law and the PDE representation of the closed-loop system in hand, we can now establish global regulation of the system by a direct Lyapunov stability analysis.

Theorem 4.1. *There exists a sufficiently small positive constant $h^\star$ such that, for each $h \in (0, h^\star]$, the time-varying low gain feedback law (4.7) globally regulates $X(t)$ and $U(t)$ of system (4.1) to zero, that is,*

$$\lim_{t\to\infty} X(t) = 0, \quad \lim_{t\to\infty} U(t) = 0,$$

for any given $X(\theta) \in C[-\tau, 0]$ and $U(\theta) \in C[-\tau, 0]$.

Proof. Inspired by the Lyapunov functional in the stability analysis of linear systems under an adaptive predictor feedback law [15], where a term involving $\tilde{\tau}^2(t)$ is introduced to bound $\tilde{\tau}(t)$, we consider

$$\begin{aligned} V(x_t, \gamma(t)) =& X^{\mathrm{T}}(t)P(\gamma(t))X(t) + b_1\tau \int_0^1 (1+x)|\tilde{u}(x,t)|^2 \mathrm{d}x \\ &+ b_2\hat{\tau}(t) \int_0^1 (1+x)\Big(|\hat{w}(x,t)|^2 + |\hat{w}_x(x,t)|^2\Big)\mathrm{d}x, \end{aligned} \tag{4.48}$$

by discarding the term associated with $\tilde{\tau}^2(t)$ because $\hat{\tau}(t)$ in our case is not an estimate of τ but only provides information of an upper bound of delay to the time-varying low gain feedback law in order to achieve regulation. Here, b_1 and b_2 are two positive constants whose values are to be determined later. Taking the time derivative of V along the closed-loop trajectory gives,

$$\begin{aligned} \dot{V}(X_t, \gamma(t)) \;=&\; 2\dot{X}^{\mathrm{T}}(t)P(\gamma(t))X(t) + X^{\mathrm{T}}(t)\dot{P}(\gamma(t))X(t) \\ &+2b_1 \int_0^1 (1+x)\tilde{u}^{\mathrm{T}}(x,t)\Big(\tilde{u}_x(x,t) \end{aligned}$$

$$
\begin{aligned}
&-(\tilde{\tau}+\tau\dot{\hat{\tau}}(t)(x-1))\frac{\hat{w}_x(x,t)}{\hat{\tau}(t)}\Big)\mathrm{d}x+2b_2\int_0^1(1+x)\hat{w}^{\mathrm{T}}(x,t)\\
&\times\Big(\hat{w}_x(x,t)(1+\dot{\hat{\tau}}(t)(x-1))+\hat{\tau}(t)B^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\dot{\gamma}(t)X(t)+\hat{\tau}(t)B^{\mathrm{T}}P(\gamma(t))\\
&\times\Big(\big(A-B^{\mathrm{T}}P(\gamma(t))\big)X(t)+B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)\Big)\mathrm{d}x\\
&+2b_2\int_0^1(1+x)\hat{w}_x^{\mathrm{T}}(x,t)\Big(\hat{w}_{xx}(x,t)(1+\dot{\hat{\tau}}(t)(x-1))+\dot{\hat{\tau}}(t)\hat{w}_x(x,t)\Big)\mathrm{d}x\\
&+b_2\dot{\hat{\tau}}(t)\int_0^1(1+x)(|\hat{w}(x,t)|^2+|\hat{w}_x(x,t)|^2)\mathrm{d}x,
\end{aligned}
\tag{4.49}
$$

where (4.17), (4.22), (4.25) and (4.27) are used.

To determine a time domain where $\dot{V}(X_t,\gamma(t))$ is well defined, we first observe from the right-hand side of (4.49) that the highest order derivative is $\hat{w}_{xx}(x,t)$, which can be computed as,

$$
\begin{aligned}
\hat{w}_{xx}(x,t)=&\hat{u}_{xx}(x,t)\\
=&\frac{\partial^2U}{\partial s^2}\hat{\tau}^2(t)\Big|_{s=t+\hat{\tau}(t)(x-1)}
\end{aligned}
\tag{4.50}
$$

by virtue of (4.19) and (4.16). Recall from Lemma 4.3 that $U(t)$ has a continuous second derivative on $t\in(\tau,\infty)$. Then, $\hat{w}_{xx}(x,t)$ is well defined if

$$
t>\tau+\tau_{\max},
\tag{4.51}
$$

where $\tau_{\max}$ is defined in the proof of Lemma 4.2 as the supremum of $\hat{\tau}(t)$ on $t\in[-\tau,\infty)$. This guarantees that

$$
s=t+\hat{\tau}(t)(x-1)>\tau
\tag{4.52}
$$

for any $x\in[0,1]$. It then follows from $X(t)\in C^1(0,\infty)$, the continuous differentiability of $P(\gamma(t))$ with respect to t, which is equivalent to that of $F(\gamma(t))$ as indicated in Lemma 4.2, and $\hat{\tau}(t),\gamma(t)\in C^1[-\tau,\infty)$, which is implied by the time-varying low gain feedback design (4.8) and (4.9), that $\dot{V}\in C[\tau+\tau_{\max}+1,\infty)$ is well defined. Let

$$
t_s=\tau+\tau_{\max}+1
\tag{4.53}
$$

denote the starting point of the consideration of $\dot{V}(X_t,\gamma(t))$ as a function of t.

With the help of the parametric algebraic Riccati equation (4.6) and the closed-loop representation

(4.24), we obtain from (4.49) that

$$\begin{aligned}
\dot{V}(X_t,\gamma(t)) \;=\; & X^{\mathrm{T}}(t)\Big(-\gamma(t)P(\gamma(t))-P(\gamma(t))BB^{\mathrm{T}}P(\gamma(t))\Big)X(t)\\
&+2X^{\mathrm{T}}(t)P(\gamma(t))B\tilde{u}(0,t)+2X^{\mathrm{T}}(t)P(\gamma(t))B\hat{w}(0,t)\\
&+X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}\dot{\gamma}(t)X(t)\\
&+2b_1\int_0^1(1+x)\tilde{u}^{\mathrm{T}}(x,t)\tilde{u}_x(x,t)\mathrm{d}x\\
&-\frac{2b_1}{\hat{\tau}(t)}\int_0^1(1+x)\left(\tilde{\tau}+\tau\dot{\hat{\tau}}(t)(x-1)\right)\tilde{u}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x\\
&+2b_2\int_0^1(1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)(1+\dot{\hat{\tau}}(t)(x-1))\mathrm{d}x\\
&+2b_2\int_0^1(1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}(t)B^{\mathrm{T}}P(\gamma(t))\\
&\times\Big((A-BB^{\mathrm{T}}P(\gamma(t)))X(t)+B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)\mathrm{d}x\\
&+2b_2\int_0^1(1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}(t)B^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\dot{\gamma}(t)X(t)\mathrm{d}x\\
&+2b_2\int_0^1(1+x)\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)(1+\dot{\hat{\tau}}(t)(x-1))\mathrm{d}x\\
&+2b_2\int_0^1(1+x)|\hat{w}_x(x,t)|^2\dot{\hat{\tau}}(t)\mathrm{d}x\\
&+b_2\dot{\hat{\tau}}(t)\int_0^1(1+x)(|\hat{w}(x,t)|^2+|\hat{w}_x(x,t)|^2)\mathrm{d}x.
\end{aligned}$$

Next, we use

$$\int_0^1(1+x)\tilde{u}^{\mathrm{T}}(x,t)\tilde{u}_x(x,t)\mathrm{d}x=-\frac{1}{2}(|\tilde{u}(0,t)|^2+||\tilde{u}(t)||^2),$$

Young's Inequality and the properties (4.29)-(4.33) in Lemma 4.1 to estimate $\dot{V}(X_t,\gamma(t))$ as,

$$\begin{aligned}
\dot{V}(X_t,\gamma(t)) \;\le\; & -\gamma(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t)+2|\tilde{u}(0,t)|^2\\
&+2|\hat{w}(0,t)|^2+X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}\dot{\gamma}(t)X(t)\\
&-b_1\left(|\tilde{u}(0,t)|^2+||\tilde{u}(t)||^2\right)+2b_1\frac{|\tilde{\tau}|+\frac{1}{2}\tau|\dot{\hat{\tau}}(t)|}{\hat{\tau}(t)}\Big(\epsilon||\tilde{u}(t)||^2\\
&+\frac{1}{\epsilon}||\hat{w}_x(t)||^2\Big)+2b_2\Big(\frac{1}{2}(|\dot{\hat{\tau}}(t)|-1)|\hat{w}(0,t)|^2\\
&+\Big(|\dot{\hat{\tau}}(t)|-\frac{1}{2}\Big)||\hat{w}(t)||^2\Big)\\
&+2b_2\Big(\hat{\tau}(t)n\gamma(t)||\hat{w}(t)||^2+\frac{3}{2}\hat{\tau}(t)n(n+1)\gamma^2(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t)
\end{aligned}$$

$$
\begin{aligned}
&+3\hat{\tau}(t)n\gamma(t)(|\tilde{u}(0,t)|^2+|\hat{w}(0,t)|^2)\Big)\\
&+2b_2\Big(h^{\frac{1}{2}}||\hat{w}(t)||^2+\Big(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\Big)^2 h^{\frac{3}{2}}X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t)\Big)\\
&+2b_2\Big(|\hat{w}_x(1,t)|^2+\frac{1}{2}(|\dot{\hat{\tau}}(t)|-1)|\hat{w}_x(0,t)|^2+\Big(|\dot{\hat{\tau}}(t)|-\frac{1}{2}\Big)||\hat{w}_x(t)||^2\Big)\\
&+4b_2|\dot{\hat{\tau}}(t)||\hat{w}_x(t)||^2+2b_2|\dot{\hat{\tau}}(t)|\left(||\hat{w}(t)||^2+||\hat{w}_x(t)||^2\right).
\end{aligned}\tag{4.54}
$$

Substitution of $|\hat{w}_x(1,t)|^2$ in (4.54) by its estimate (4.34) and arranging the terms in (4.54) give

$$
\begin{aligned}
\dot{V}(X_t,\gamma(t))\le{}&X^{\mathrm{T}}(t)P(\gamma(t))X(t)\Big(-\gamma(t)+6b_2\hat{\tau}^2(t)n^2(n+1)\gamma^3(t)\\
&+3b_2\hat{\tau}(t)n(n+1)\gamma^2(t)\Big)+|\tilde{u}(0,t)|^2\Big(2-b_1+6b_2\hat{\tau}(t)n\gamma(t)\\
&+12b_2\hat{\tau}^2(t)n^2\gamma^2(t)\Big)+|\hat{w}(0,t)|^2\Big(2+b_2\left(|\dot{\hat{\tau}}(t)|-1\right)+6b_2\hat{\tau}(t)n\gamma(t)\\
&+12b_2\hat{\tau}^2(t)n^2\gamma^2(t)\Big)+||\tilde{u}(t)||^2\Big(-b_1+2b_1\epsilon\frac{|\tilde{\tau}|+\frac{1}{2}\tau|\dot{\hat{\tau}}(t)|}{\hat{\tau}(t)}\Big)\\
&+X^{\mathrm{T}}(t)\Big(\frac{\partial P}{\partial\gamma}\dot{\gamma}(t)+2b_2h^{\frac{3}{2}}\Big(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\Big)^2\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\\
&+4b_2\hat{\tau}^2(t)\dot{\gamma}^2(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\Big)X(t)\\
&+||\hat{w}_x(t)||^2\Big(\frac{2b_1}{\epsilon}\frac{|\tilde{\tau}|+\frac{1}{2}\tau|\dot{\hat{\tau}}(t)|}{\hat{\tau}(t)}-b_2+8b_2|\dot{\hat{\tau}}(t)|\Big)\\
&+||\hat{w}(t)||^2\left(-b_2+2b_2\hat{\tau}(t)n\gamma(t)+2b_2h^{\frac{1}{2}}+4b_2|\dot{\hat{\tau}}(t)|\right)\\
&+|\hat{w}_x(0,t)|^2b_2(|\dot{\hat{\tau}}(t)|-1).
\end{aligned}\tag{4.55}
$$

Recall that

$$
\gamma(t)=\frac{h}{\hat{\tau}(t)}.\tag{4.56}
$$

We have

$$
h=\gamma(t)\hat{\tau}(t).
$$

Substitution of $\gamma(t)\hat{\tau}(t)$ in (4.55) by h simplifies the estimate of $\dot{V}(X_t,\gamma(t))$ as follows,

$$
\begin{aligned}
\dot{V}(X_t,\gamma(t))\ \le\ &\gamma(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t)\Big(-1+6b_2n^2(n+1)h^2\\
&+3b_2n(n+1)h\Big)+|\tilde{u}(0,t)|^2\Big(2-b_1+6b_2nh+12b_2n^2h^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+|\hat{w}(0,t)|^2\Big(b_2(|\dot{\hat{\tau}}(t)|-1)+2+6b_2nh+12b_2n^2h^2\Big)\\
&+||\tilde{u}(t)||^2 b_1\Big(-1+2\epsilon\frac{|\tilde{\tau}|+\frac{1}{2}\tau|\dot{\hat{\tau}}(t)|}{\hat{\tau}(t)}\Big)\\
&+X^{\mathrm{T}}(t)\Big(-\frac{h}{\hat{\tau}^2(t)}\dot{\hat{\tau}}(t)\frac{\partial P}{\partial\gamma}+2b_2\Big(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\Big)^2\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}h^{\frac{3}{2}}\\
&+4b_2\Big(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\Big)^2\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}h^2\Big)X(t)\\
&+||\hat{w}_x(t)||^2\Big(\frac{2b_1}{\epsilon}\frac{|\tilde{\tau}|+\frac{1}{2}\tau|\dot{\hat{\tau}}(t)|}{\hat{\tau}(t)}+8b_2|\dot{\hat{\tau}}(t)|-b_2\Big)\\
&+||\hat{w}(t)||^2 b_2\Big(-1+2nh+2h^{\frac{1}{2}}+4|\dot{\hat{\tau}}(t)|\Big)\\
&+|\hat{w}_x(0,t)|^2 b_2(|\dot{\hat{\tau}}(t)|-1). \qquad (4.57)
\end{aligned}
$$

To group terms that involve $\frac{\partial P}{\partial\gamma}$ in (4.57), we consider

$$
\begin{aligned}
\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma} &= \Big(\frac{\partial P}{\partial\gamma}\Big)^{\frac{1}{2}}\Big(\frac{\partial P}{\partial\gamma}\Big)^{\frac{1}{2}}BB^{\mathrm{T}}\Big(\frac{\partial P}{\partial\gamma}\Big)^{\frac{1}{2}}\Big(\frac{\partial P}{\partial\gamma}\Big)^{\frac{1}{2}}\\
&\le \mathrm{tr}\Big(B^{\mathrm{T}}\frac{\partial P}{\partial\gamma}B\Big)\frac{\partial P}{\partial\gamma}\\
&= \frac{\partial}{\partial\gamma}\left(\mathrm{tr}\left(B^{\mathrm{T}}P(\gamma)B\right)\right)\frac{\partial P}{\partial\gamma}\\
&= n\frac{\partial P}{\partial\gamma},
\end{aligned}
$$

where Lemma 2.1 is used. It then follows that

$$
\begin{aligned}
&-\frac{h}{\hat{\tau}^2(t)}\dot{\hat{\tau}}(t)\frac{\partial P}{\partial\gamma}+2b_2\Big(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\Big)^2\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\Big(h^{\frac{3}{2}}+2h^2\Big)\\
\le\ & h\frac{\partial P}{\partial\gamma}\frac{|\dot{\hat{\tau}}(t)|}{\hat{\tau}^2(t)}\left(1+2b_2|\dot{\hat{\tau}}(t)|h^{\frac{1}{2}}n+4b_2|\dot{\hat{\tau}}(t)|hn\right)\\
\le\ & \gamma(t)P(\gamma(t))\frac{\tau_{\max}\lambda_{\max}\Big(\max_{\gamma\in[h/\tau_{\max},h/\tau_{\min}]}\{\frac{\partial P}{\partial\gamma}\}\Big)}{\lambda_{\min}(P(\frac{h}{\tau_{\max}}))}\frac{|\dot{\hat{\tau}}(t)|}{\tau_{\min}^2}\\
&\times\left(1+2b_2|\dot{\hat{\tau}}(t)|h^{\frac{1}{2}}n+4b_2|\dot{\hat{\tau}}(t)|hn\right), \qquad (4.58)
\end{aligned}
$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote respectively the minimum and the maximum eigenvalue of a real symmetric matrix, $\tau_{\min}$ and $\tau_{\max}$ are respectively the infimum and the supremum of $\hat{\tau}(t)$ over $[-\tau,\infty)$, and we have used the boundedness of $\hat{\tau}(t)$, $\gamma(t)$ and $P(\gamma(t))$ as shown in the proof of Lemma 4.2. By denoting

$$\sigma = \frac{\tau_{\max}\lambda_{\max}\Big(\max_{\gamma\in[h/\tau_{\max},h/\tau_{\min}]}\{\frac{\partial P}{\partial\gamma}\}\Big)}{\tau_{\min}^2\lambda_{\min}(P(\frac{h}{\tau_{\max}}))},$$

and recalling from the design of $\hat{\tau}(t)$ that

$$\lim_{t\to\infty}\hat{\tau}(t) = \overline{\tau} \tag{4.59}$$

and

$$\lim_{t\to\infty}\dot{\hat{\tau}}(t) = 0, \tag{4.60}$$

we see that there exists a sufficiently large time constant $t_0 \geq t_s$ such that, for each $t \geq t_0$,

$$|\dot{\hat{\tau}}(t)| \leq \min\left\{\frac{1}{2b_2h^{\frac{1}{2}}n + 4b_2hn}, \frac{h}{\sigma}\right\}.$$

Thus, (4.58) can be continued as follows,

$$-\frac{h}{\hat{\tau}^2(t)}\dot{\hat{\tau}}(t)\frac{\partial P}{\partial\gamma} + 2b_2\Big(\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}\Big)^2\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\Big(h^{\frac{3}{2}} + 2h^2\Big) \leq 2h\gamma(t)P(\gamma(t)).$$

The estimate of $\dot{V}(X_t, \gamma(t))$ then takes the form,

$$\begin{aligned}
\dot{V}(X_t,\gamma(t)) \leq\ & \gamma(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t)\Big(-1+2h+6b_2n^2(n+1)h^2+3b_2n(n+1)h\Big)\\
&+|\tilde{u}(0,t)|^2\Big(2-b_1+6b_2nh+12b_2n^2h^2\Big)\\
&+|\hat{w}(0,t)|^2\Big(b_2(|\dot{\hat{\tau}}(t)|-1)+2+6b_2nh+12b_2n^2h^2\Big)\\
&+||\tilde{u}(t)||^2b_1\Big(-1+2\epsilon\frac{|\tilde{\tau}|+\frac{1}{2}\tau|\dot{\hat{\tau}}(t)|}{\hat{\tau}(t)}\Big)\\
&+||\hat{w}_x(t)||^2\Big(\frac{2b_1}{\epsilon}\frac{|\tilde{\tau}|+\frac{1}{2}\tau|\dot{\hat{\tau}}(t)|}{\hat{\tau}(t)}+8b_2|\dot{\hat{\tau}}(t)|-b_2\Big)\\
&+||\hat{w}(t)||^2b_2\Big(-1+2nh+2h^{\frac{1}{2}}+4|\dot{\hat{\tau}}(t)|\Big)\\
&+|\hat{w}_x(0,t)|^2b_2\left(|\dot{\hat{\tau}}(t)|-1\right).
\end{aligned} \tag{4.61}$$

We next simplify those terms in (4.61) that contain $|\dot{\hat{\tau}}(t)|$. In view of

$$\lim_{t\to\infty}\hat{\tau}(t) = \overline{\tau}$$

and

$$\lim_{t\to\infty} \dot{\hat{\tau}}(t) = 0,$$

both of which are required by the time-varying parameter design, as given in (4.9), there exists a sufficiently large positive constant $t_1 \geq t_0$ such that, for each $t \geq t_1$,

$$\begin{aligned}
0 \leq & \frac{\tau}{\hat{\tau}(t)} \leq 2 \left(\text{or } \frac{|\tilde{\tau}|}{\hat{\tau}(t)} \leq 1\right), \\
|\dot{\hat{\tau}}(t)| \leq & \frac{\tau_{\min}}{\tau},
\end{aligned}$$

which together guarantee that

$$\begin{aligned}
\frac{|\tilde{\tau}| + \frac{1}{2}\tau|\dot{\hat{\tau}}(t)|}{\hat{\tau}(t)} \leq & 1 + \frac{\tau_{\min}}{2\hat{\tau}(t)} \\
\leq & \frac{3}{2}.
\end{aligned} \tag{4.62}$$

Let $\epsilon = \frac{1}{4}$. Then,

$$-1 + 2\epsilon\frac{|\tilde{\tau}| + \frac{1}{2}\tau|\dot{\hat{\tau}}(t)|}{\hat{\tau}(t)} < 0.$$

Then, the term associated with $||\tilde{u}(t)||^2$ in (4.61) becomes negative. Also, there exists a sufficiently large positive constant $t_2 \geq t_1$ such that,

$$|\dot{\hat{\tau}}(t)| \leq \frac{1}{16}, \;\; t \geq t_2. \tag{4.63}$$

Thus, with the help of (4.62), we obtain

$$\frac{2b_1}{\epsilon}\frac{|\tilde{\tau}| + \frac{1}{2}\tau|\dot{\hat{\tau}}(t)|}{\hat{\tau}(t)} + 8b_2|\dot{\hat{\tau}}(t)| - b_2 \leq 12b_1 - \frac{b_2}{2}. \tag{4.64}$$

To make the right-hand side of (4.64) non-positive, we let b_1 and b_2 satisfy

$$24b_1 \leq b_2. \tag{4.65}$$

It then follows that the term corresponding to $||\hat{w}_x(t)||^2$ in (4.61) is non-positive.

Choose $b_2 > 4$. Then, there exists a sufficiently large constant $t_3 \geq t_2$ such that, for each $t \geq t_3$,

$$|\dot{\hat{\tau}}(t)| \leq \frac{1}{2} - \frac{2}{b_2}, \tag{4.66}$$

which is equivalent to

$$1-|\dot{\hat{\tau}}(t)|-\frac{2}{b_2}\geq\frac{1}{2}. \tag{4.67}$$

Then,

$$b_2\left(|\dot{\hat{\tau}}(t)|-1\right)+2+6b_2nh+12b_2n^2h^2<0, \tag{4.68}$$

which is implied by

$$6nh+12n^2h^2<\frac{1}{2}, \tag{4.69}$$

leads to the negativeness of the coefficient associated with $|\hat{w}(0,t)|^2$ in (4.61). Note that the definition of t_3 naturally gives rise to the negativeness of the coefficient associated with $|\hat{w}_x(0,t)|^2$ in (4.61), and

$$-1+2nh+2h^{\frac{1}{2}}+4|\dot{\hat{\tau}}(t)|<0 \tag{4.70}$$

if

$$2nh+2h^{\frac{1}{2}}<\frac{1}{2}. \tag{4.71}$$

In view of (4.61), we see that there exists a sufficiently large constant t_3 such that, for each $t\geq t_3$, the coefficients associated with $||\tilde{u}(t)||^2$, $||\hat{w}_x(t)||^2$, $||\hat{w}(t)||^2$, $|\hat{w}(0,t)|^2$ and $|\hat{w}_x(0,t)|^2$ in (4.61) are all non-positive as long as

$$\begin{cases} b_2>4, \\ 24b_1\leq b_2, \\ nh(1+2nh)<\frac{1}{12}, \\ nh+h^{\frac{1}{2}}<\frac{1}{4}. \end{cases}$$

It is then clear that $\dot{V}(X_t, \gamma(t)) < 0$ on $t \geq t_3$ if

$$\begin{cases} 2h + 6b_2n^2(n+1)h^2 + 3b_2n(n+1)h < 1, \\ 6b_2nh(1+2nh) < b_1 - 2, \\ nh(1+2nh) < \frac{1}{12}, \\ nh + h^{\frac{1}{2}} < \frac{1}{4}, \\ 24b_1 \leq b_2, \\ b_1 > 2, \\ b_2 > 4 \end{cases} \tag{4.72}$$

holds. We first choose b_1 and b_2 according to the last row of (4.72). Then, substitution of b_1 and b_2 in the rest of (4.72) shows that there exists a sufficiently small $h^\star$ such that, for each $h \in (0, h^\star]$,

$$\dot{V}(X_t, \gamma(t)) < 0, \;\; t \geq t_3. \tag{4.73}$$

It is worth mentioning here that $h^\star$ is independent of any information of the delay, including $\overline{\tau}$, but only depends on the dimension of the system n.

To complete the proof, it remains to establish global regulation of both $X(t)$ and $U(t)$. We first demonstrate the square integrability of $X(t)$ over $t \geq 0$. By denoting

$$1 - 2h - 6b_2n^2(n+1)h^2 - 3b_2n(n+1)h = \eta,$$

which is a positive constant as long as h is chosen small enough, we have from (4.61) that

$$\dot{V}(X_t, \gamma(t)) \leq -\eta\gamma(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t), \;\; t \geq t_3.$$

Then, in view of the boundedness of $\gamma(t)$ and $P(\gamma(t))$, as established in the proof of Lemma 4.2, we have,

$$\eta\lambda_{\min}\Big(P\left(\frac{h}{\tau_{\max}}\right)\Big)\frac{h}{\tau_{\max}}|X(t)|^2 \leq -\dot{V}, \;\; t \geq t_3,$$

which leads to

$$\int_{t_3}^{\infty} |X(t)|^2 \mathrm{d}t \leq \frac{\tau_{\max}(V(X_{t_3}, \gamma(t_3)) - V(X_\infty, \gamma(\infty))}{\eta\lambda_{\min}\Big(P\left(\frac{h}{\tau_{\max}}\right)\Big)h}$$

$$\leq \frac{\tau_{\max} V(X_{t_3}, \gamma(t_3))}{\eta \lambda_{\min}\Big(P\left(\frac{h}{\tau_{\max}}\right)\Big) h}.$$

Recalling that V is continuously differentiable on $t \in [t_s, \infty)$, we see that V is bounded for $t \in [t_s, t_3]$, where $t_3 \geq t_s$ according to the definition of t_3. Then,

$$\int_{t_3}^{\infty} |X(t)|^2 \mathrm{d}t < \infty,$$

which, together with the fact that $X(t) \in C[0, t_3]$, as indicated in Lemma 4.2, imply that

$$\begin{aligned} \int_0^{\infty} |X(t)|^2 \mathrm{d}t &= \int_0^{t_3} |X(t)|^2 \mathrm{d}t + \int_{t_3}^{\infty} |X(t)|^2 \mathrm{d}t \\ &\leq t_3 \max_{t \in [0, t_3]} \{|X(t)|^2\} + \int_{t_3}^{\infty} |X(t)|^2 \mathrm{d}t \\ &< \infty, \end{aligned}$$

that is, $X(t)$ is square integrable on $t \in [0, \infty)$.

On the other hand, from the definition of $V(X_t, \gamma(t))$ in (4.48) and the boundedness of $V(X_t, \gamma(t))$ for $t \in [t_s, \infty)$, which can be seen by noting the boundedness of $V(X_t, \gamma(t))$ on $t \in [t_s, t_3]$ and $\dot{V}(X_t, \gamma(t)) < 0$ on $t \geq t_3$, we establish the boundedness of $X(t)$ on $t \in [t_s, \infty)$. Then, $X(t) < \infty$ for $t \in [-\tau, \infty)$ readily follows from the continuity of $X(t)$ on $t \in [-\tau, t_s]$. Thus, (4.44) implies the boundedness of $\dot{X}(t)$ on $t \geq 0$ since $F(\gamma(t))$ is bounded, as indicated in Lemma 4.2. With the square integrability of $X(t)$ and the boundedness of $\dot{X}(t)$, global regulation of $X(t)$, which further implies that

$$\lim_{t \to \infty} U(t) = 0 \tag{4.74}$$

by (4.7) and the boundedness of $F(\gamma(t))$, follows from the Barbalat's lemma. □

Remark 4.2. *The requirement for $\hat{\tau}(t)$ to have a continuous second derivative comes from the requirement for the well-definedness of the partial derivatives in the PDEs (4.17), (4.22), (4.25) and (4.27), where $\hat{w}_{xt}$ and $\hat{w}_{xx}$ are the highest order derivatives. From the definition of $\hat{w}(x,t)$ in (4.19) and the time-varying low gain feedback law (4.7) it follows that both the second order partial derivatives contain the same term $\ddot{\hat{\tau}}(s)|_{s=t+\hat{\tau}(x-1)}$. Take $\hat{w}_{xx}(x,t)$ for example,*

$$\hat{w}_{xx}(x,t) = \Big(B^{\mathrm{T}} \frac{h}{\hat{\tau}^2(s)} \Big(- \frac{\partial^2 P}{\partial \gamma^2(s)} \frac{h}{\hat{\tau}^2(s)} \dot{\hat{\tau}}^2(s) - 2 \frac{\partial P}{\partial \gamma(s)} \frac{\dot{\hat{\tau}}^2(s)}{\hat{\tau}(s)} + \frac{\partial P}{\partial \gamma(s)} \ddot{\hat{\tau}}(s) \Big) X(s)$$

$$+2B^{\mathrm{T}}\frac{\partial P}{\partial\gamma(s)}\frac{h}{\hat{\tau}^2(s)}\dot{\hat{\tau}}(s)\dot{X}(s)-B^{\mathrm{T}}P(\gamma(s))\ddot{X}(s)\bigg)\hat{\tau}^2(t)\bigg|_{s=t+\hat{\tau}(t)(x-1)},$$

where we have used (4.42), (4.43), (4.47), (4.50) and the fact that

$$F(\gamma(t))=-B^{\mathrm{T}}P(\gamma(t)). \tag{4.75}$$

Thus, the existence of $\ddot{\hat{\tau}}(t)$ is necessary for that of $\hat{w}_{xx}(x,t)$. Without loss of generality, we assume that $\hat{\tau}(t)\in C^2[-\tau,\infty)$ as in the design of the time-varying low gain feedback laws in Section 4.1. This requirement on $\hat{\tau}(t)$, which is necessary in our design, facilitates all our derivations in Sections 4.2 and 4.3, and in general, cannot be relaxed unless a new Lyapunov analysis on the closed-loop system is established involving at most the first derivative of $\hat{\tau}(t)$. □

Theorem 4.1 reveals a group of time-varying low gain feedback laws which achieve global regulation of the closed-loop system. By the structure of the time-varying feedback parameter in (4.8) and Theorem 4.1, h and $\hat{\tau}(t)$ need to be designed first in order to construct $\gamma(t)$. Thus, for the sake of simplicity, we consider directly designing $\gamma(t)$ without involving h and $\hat{\tau}(t)$.

Corollary 3. *There exists a sufficiently small positive constant $\gamma^\star$, which is inversely proportional to the delay bound $\overline{\tau}$, such that the time-varying low gain feedback law (4.7) globally regulates $X(t)$ and $U(t)$ of system (4.1) as long as*

$$\gamma(t)\in C^2[-\tau,\infty),\ \ \gamma(t)>0,\ \ \lim_{t\to\infty}\gamma(t)\in(0,\gamma^\star],\ \ \lim_{t\to\infty}\dot{\gamma}(t)=0. \tag{4.76}$$

Proof. Given a $\gamma(t)$ satisfying (4.76), we write $\gamma(t)$ in the form of (4.8), where h and $\hat{\tau}(t)$ are selected as

$$h=\overline{\tau}\lim_{t\to\infty}\gamma(t)>0 \tag{4.77}$$

and

$$\hat{\tau}(t)=\frac{h}{\gamma(t)}. \tag{4.78}$$

It can be readily verified that

$$\lim_{t\to\infty}\hat{\tau}(t)=\overline{\tau}. \tag{4.79}$$

Also, $\gamma(t) \in C^2[-\tau, \infty)$, $\gamma(t) > 0$ and $\lim_{t\to\infty} \dot{\gamma}(t) = 0$ imply that $\hat{\tau}(t) \in C^2[-\tau, \infty)$, $\hat{\tau}(t) > 0$ and $\lim_{t\to\infty} \dot{\hat{\tau}}(t) = 0$, respectively, because

$$\begin{aligned}\dot{\hat{\tau}}(t) &= \frac{-h\dot{\gamma}(t)}{\gamma^2(t)},\\ \ddot{\hat{\tau}}(t) &= \frac{h}{\gamma^2(t)}\left(\frac{2\dot{\gamma}^2(t)}{\gamma(t)} - \ddot{\gamma}(t)\right).\end{aligned}$$

Note from the selection of

$$h = \overline{\tau} \lim_{t\to\infty} \gamma(t) \tag{4.80}$$

that $\lim_{t\to\infty} \gamma(t) \in (0, \gamma^\star]$ is equivalent to $h \in (0, \overline{\tau}\gamma^\star]$. Theorem 4.1 concludes that there exists a sufficiently small positive constant $h^\star$, which is independent of any information of the delay, such that, for any $h \in (0, h^\star]$, the time-varying low gain feedback law

$$U(t) = -B^{\mathrm{T}} P(\gamma(t))X(t) \tag{4.81}$$

globally regulates the system. Thus, there exists a sufficiently small positive constant

$$\gamma^\star = \frac{h^\star}{\overline{\tau}}, \tag{4.82}$$

which is inversely proportional to $\overline{\tau}$, such that the regulation of the closed-loop system is guaranteed if $\lim_{t\to\infty} \gamma(t) \in (0, \gamma^\star]$. □

Remark 4.3. *The traditional constant low gain feedback law is a special case of the time-varying low gain feedback law. In particular, Corollary 4.1 concludes that there exists a sufficiently small positive constant $\gamma^\star$, which is inversely proportional to the upper bound of delay $\overline{\tau}$ such that, for each $\gamma \in (0, \gamma^\star]$, the low gain feedback law*

$$U(t) = -B^{\mathrm{T}} P(\gamma)X(t) \tag{4.83}$$

globally regulates system (4.1). This observation is consistent with the observation in Remark 2.1. □

Remark 4.4. *Dealing with the conservativeness incurred in the Lyapunov stability analysis on the upper bound of feedback parameter γ under the constant parameter low gain feedback design, as given by (4.5), the time-varying parameter design proposed in this section takes a proactive selection of γ at a relatively* *large value, which corresponds to a fast convergence rate, during the starting phase of the system evolution. After the system reaches a state near the equilibrium point zero, reducing γ to a sufficiently small constant, as required by Corollary 3, would not affect the closed-loop transient performance, while guaranteeing stability. Therefore, such a time-varying parameter design would manifest its merits in the closed-loop performance compared with that of the constant parameter design.* □

Remark 4.5. *Remark 4.4 provides a guideline for the construction of a time-varying $\gamma(t)$ that outperforms a constant γ in terms of the closed-loop performance, as demonstrated by simulation. A theoretical proof of such an improvement is challenging due to the time-varying design of $\gamma(t)$ and remains to be carried out.* □

4.4. Convergence Rate Analysis

To illustrate the merits of the time-varying low gain feedback design in comparison with the constant parameter design in terms of the closed-loop performance, we compare the convergence rates of the closed-loop system under a constant parameter feedback with different values of the feedback parameter within the range where exponential stability of the closed-loop system is ensured.

Theorem 4.2. *The delay independent state feedback TPF law with a constant parameter,*

$$U(t) = -B^{\mathrm{T}}P(\gamma)X(t), \tag{4.84}$$

for a sufficiently small $\gamma > 0$, exponentially stabilizes system (4.1) with

$$|X(t)|^2 \leq \lambda_{\min}^{-1}(P(\gamma))\mathrm{e}^{-\frac{\beta}{\zeta}t}V(X_0, 0), \;\; t \geq 0, \tag{4.85}$$

where

$$\begin{aligned}
\beta &= \min\left\{\gamma\Big(1 - 120n^2(n+1)\overline{\tau}^2\gamma^2 - 60n(n+1)\overline{\tau}\gamma\Big), 20\Big(1 - 2n\overline{\tau}\gamma - 2\overline{\tau}^{\frac{1}{2}}\gamma^{\frac{1}{2}}\Big), 1\right\}, \\
\zeta &= \max\{1, 40\overline{\tau}\},
\end{aligned}$$

and $V(X_t, t)$ is as given in (4.48) with $\gamma(t) = \gamma$ and $\hat{\tau}(t) = \overline{\tau}$.

Proof. The proof follows an analysis similar to that in the proof of Theorem 4.1, except the distinction between a constant feedback parameter and a time-varying parameter. With a Lyapunov functional

$V(X_t, \gamma(t))$ given by (4.48), its estimated time derivative along the trajectory of the closed-loop system (4.44), (4.55), for the special case of the constant parameter design takes the form of

$$
\begin{aligned}
&\dot{V}(X_t, \gamma(t)) \\
\le\ & \gamma X^{\mathrm{T}} P X \Big(-1 + 6b_2 n^2 (n+1) \overline{\tau}^2 \gamma^2 + 3b_2 n(n+1) \overline{\tau} \gamma \Big) \\
&+ |\tilde{u}(0,t)|^2 \Big(2 - b_1 + 6b_2 n \overline{\tau} \gamma + 12 b_2 n^2 \overline{\tau}^2 \gamma^2 \Big) \\
&+ |\hat{w}(0,t)|^2 \Big(-b_2 + 2 + 6b_2 n \overline{\tau} \gamma + 12 b_2 n^2 \overline{\tau}^2 \gamma^2 \Big) \\
&+ ||\tilde{u}(t)||^2 b_1 \Big(-1 + 2\epsilon \frac{|\tilde{\tau}|}{\hat{\tau}(t)} \Big) + ||\hat{w}_x(t)||^2 \Big(\frac{2b_1}{\epsilon} \frac{|\tilde{\tau}|}{\hat{\tau}(t)} - b_2 \Big) \\
&+ ||\hat{w}(t)||^2 b_2 \Big(-1 + 2n \overline{\tau} \gamma + 2 \overline{\tau}^{\frac{1}{2}} \gamma^{\frac{1}{2}} \Big) - |\hat{w}_x(0,t)|^2 b_2,
\end{aligned}
\tag{4.86}
$$

where we have replaced h with $\gamma \overline{\tau}$ and all the terms involving $\dot{\gamma}$ or $\dot{\hat{\tau}}(t)$ disappear because of the constant parameter design. Noting that

$$
\frac{|\tilde{\tau}|}{\hat{\tau}(t)} = \frac{\overline{\tau} - \tau}{\overline{\tau}} < 1
$$

and taking $\epsilon = \frac{1}{3}$ in (4.86), we arrive at a further estimate of $\dot{V}(X_t, \gamma(t))$,

$$
\begin{aligned}
\dot{V}(X_t, \gamma(t)) \le\ & \gamma X^{\mathrm{T}} P X \Big(-1 + 6b_2 n^2 (n+1) \overline{\tau}^2 \gamma^2 + 3b_2 n(n+1) \overline{\tau} \gamma \Big) \\
&+ |\tilde{u}(0,t)|^2 \Big(2 - b_1 + 6b_2 n \overline{\tau} \gamma + 12 b_2 n^2 \overline{\tau}^2 \gamma^2 \Big) \\
&+ |\hat{w}(0,t)|^2 \Big(-b_2 + 2 + 6b_2 n \overline{\tau} \gamma + 12 b_2 n^2 \overline{\tau}^2 \gamma^2 \Big) \\
&- \frac{1}{3} ||\tilde{u}(t)||^2 b_1 + ||\hat{w}_x(t)||^2 (6b_1 - b_2) \\
&+ ||\hat{w}(t)||^2 b_2 \Big(-1 + 2n \overline{\tau} \gamma + 2 \overline{\tau}^{\frac{1}{2}} \gamma^{\frac{1}{2}} \Big) - |\hat{w}_x(0,t)|^2 b_2.
\end{aligned}
\tag{4.87}
$$

With the selection of $b_1 > 2$ and $b_2 > 6b_1$, the terms involving $||\tilde{u}(0,t)||^2$, $|\hat{w}(0,t)|^2$ or $||\hat{w}_x(t)||^2$ in (4.87) are negative if

$$
6b_2 n \overline{\tau} \gamma + 12 b_2 n^2 \overline{\tau}^2 \gamma^2 < b_1 - 2.
$$

For illustration, we choose $b_1 = 3$ and $b_2 = 20$. Then, a sufficiently small γ satisfying

$$
\max \Big\{ 120 n^2 (n+1) \overline{\tau}^2 \gamma^2 + 60 n(n+1) \overline{\tau} \gamma, 2n \overline{\tau} \gamma + 2 \overline{\tau}^{\frac{1}{2}} \gamma^{\frac{1}{2}} \Big\} < 1
$$

results in

$$\begin{aligned}\dot{V}(X_t,\gamma(t)) \leq & -\beta\Big(X^{\mathrm{T}}PX+||\tilde{u}(t)||^2+||\hat{w}(t)||^2+||\hat{w}_x(t)||^2\Big)\\ \leq & -\frac{\beta}{\zeta}V(X_t,\gamma(t)),\end{aligned}$$

where we have used

$$\begin{aligned}V(X_t,\gamma(t)) \ \leq & \ \max\{1,2b_1\tau,2b_2\hat{\tau}(t)\}\Big(X^{\mathrm{T}}PX+||\tilde{u}(t)||^2+||\hat{w}(t)||^2+||\hat{w}_x(t)||^2\Big)\\ \leq & \ \zeta\Big(X^{\mathrm{T}}PX+||\tilde{u}(t)||^2+||\hat{w}(t)||^2+||\hat{w}_x(t)||^2\Big).\end{aligned}$$

Consequently, an estimate of $V(X_t,\gamma(t))$ readily follows from the comparison lemma,

$$V(X_t,\gamma(t)) \leq \mathrm{e}^{-\frac{\beta}{\zeta}t}V(X_0,0),\ \ t\geq 0,$$

which, by the definition of $V(X_t,\gamma(t))$ in (4.48), further implies the exponential convergence of $X(t)$, as expressed in (4.85). □

Remark 4.6. *Theorem 4.2 establishes a guaranteed convergence rate of $\frac{\beta}{2\zeta}$ for the state of the closed-loop system under the constant parameter design of the delay independent truncated predictor feedback law. Examination of the guaranteed convergence rate indicates that γ only appears in the expression of β and*

$$\begin{aligned}\lim_{\gamma\to 0}\beta = & \lim_{\gamma\to\overline{\gamma}}\beta\\ = & \ 0,\end{aligned}$$

where

$$\overline{\gamma}=\min\{\gamma_1,\gamma_2\} \tag{4.88}$$

with γ_1 and γ_2 being the unique positive solutions to the nonlinear equations,

$$\begin{aligned}120n^2(n+1)\overline{\tau}^2\gamma^2+60n(n+1)\overline{\tau}\gamma=1,\\ 2n\overline{\tau}\gamma+2\overline{\tau}^{\frac{1}{2}}\gamma^{\frac{1}{2}}=1,\end{aligned}$$

respectively. Thus, there exists a maximum β on the interval $\gamma\in(0,\overline{\gamma})$, corresponding to the fastest *convergence rate.* □

4.5. A Numerical Example

To compare the closed-loop performance of system (4.1) under the traditional constant parameter low gain feedback and the time-varying parameter low gain feedback, we consider system (4.1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \ \tau = \overline{\tau} = 1, \ \ \phi(\theta) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \ \theta \in [-1, 0],$$

as an example. The values of $\gamma(t)$ is designed to decrease slowly from 0.3. On the other hand, $\lim_{t\to\infty} \gamma(t)$ is computed as 7.6×10^{-4} by employing (4.72), where $b_1 = 3$ and $b_2 = 72$, Corollary 3 and the fact that $\overline{\tau} = 1$.

Simulation shows that, regardless of how conservative the bound on $\lim_{t\to\infty} \gamma(t)$ is, it does not prevent us from designing a time-varying low gain feedback law with better closed-loop performance than a traditional constant parameter design. This can be demonstrated by selecting

$$\gamma(t) = -0.0953 \arctan(t - 20) + 0.1504, \ \ t \geq -1, \tag{4.89}$$

which satisfies all the design requirements in (4.76). Evolution of the time-varying low gain parameter, the system state and the control input under the time-varying parameter design is illustrated in Figs. 4.1 and 4.2. The system evolution under the constant parameter design with $\gamma = 0.068$, which is the theoretical upper bound given by (4.5), is also given for comparison. Obviously, this choice of $\gamma(t)$ achieves better closed-loop performance than the choice of a constant γ.

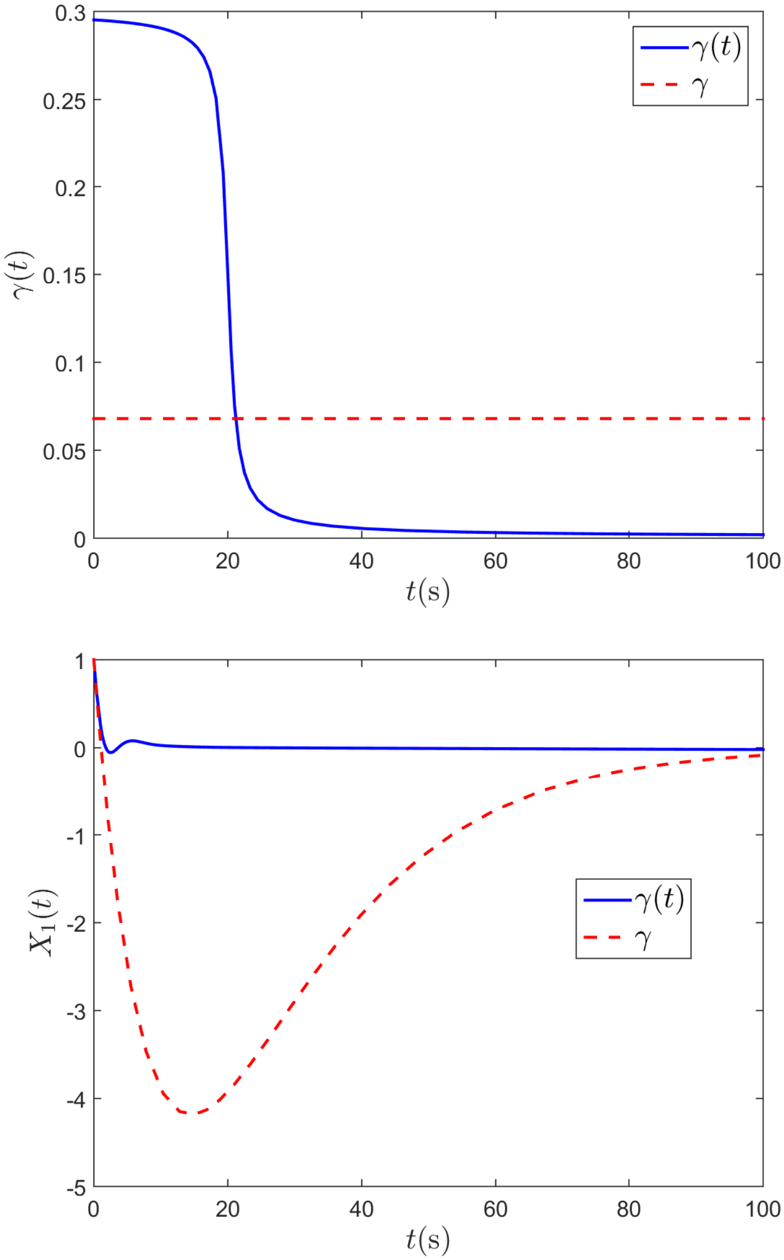


Fig. 4.1: $\gamma(t)$ given by (4.89) and $\gamma = 0.068$, and the corresponding evolution of $X_1(t)$.

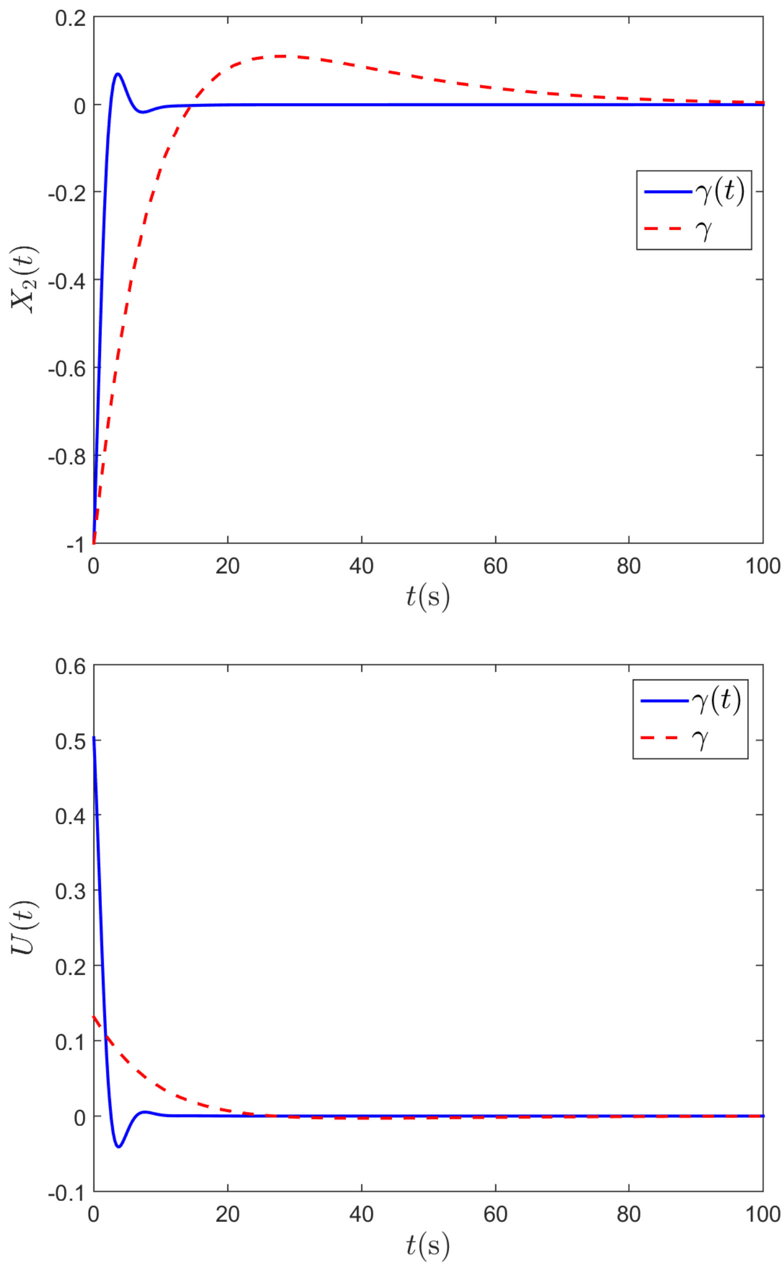


Fig. 4.2: The evolution of $X_2(t)$ and $U(t)$ corresponding to $\gamma(t)$ given by (4.89) and $\gamma = 0.068$.

4.6. Summary

Global regulation of a linear system whose open loop poles are at the origin or in the open left-half plane with an arbitrarily large unknown input delay was studied. With the knowledge of an upper bound on delay, a low gain feedback design with a time-varying low gain parameter was proposed to replace the traditional design with a constant low gain parameter. Both convergence rate analysis and numerical simulation demonstrate improved closed-loop performance under the time-varying parameter design over the traditional constant parameter design.

5. Regulation of Linear Systems with Input Delay in the Absence of any Delay Knowledge

In this section, we propose a control scheme that, in the absence of any knowledge of the delay, regulates to zero the state and the control input of a linear input delayed system whose open loop poles are at the origin or in the open left-half plane. Two main features of our control scheme are its non-distributed nature in the sense that only the current state is used in the feedback and its delay independence in the sense that no knowledge of the delay, neither its exact value nor its upper bound, is required. The main ingredients of our control scheme and the regulation proof include the design of a delay independent truncated predictor feedback law with a time-varying feedback parameter, Lyapunov function based adaptation of the time-varying parameter, a mechanism for switching between two update laws of the time-varying parameter, and the partial differential equation based analysis of the closed-loop system.

Both the truncated predictor feedback law and the delay independent truncated predictor feedback law require knowledge of the input delay. The delay appears in the exponential factor of the truncated predictor feedback law. An upper bound of the delay is necessary for determining the value of a low gain feedback parameter of the delay independent truncated predictor feedback laws. In Section 2, a delay independent truncated predictor feedback law was proposed for a general linear system and an admissible delay bound for the closed-loop stability was establihsed. A delay independent output feedback law was proposed in Section 3 for a general linear system, and an admissible delay bound that assures closed-loop stability was given. Also, an upper bound of the delay is required to be known for the stability guarantee. To study the potential of the delay independent truncated predictor feedback law to achieving faster convergence of the closed-loop system, Section 4 presented a time-varying feedback parameter that guarantees closed-loop stability of linear systems with all open loop poles at the origin or in the open left-half plane. An upper bound of the delay is still required for the design of the time-varying feedback parameter therein. For systems with open loop poles in the closed left-half plane, Reference [121] developed adaptation of the truncated predictor feedback law to accommodate unknown delay, but the delay is required to be in a sufficiently small range whose upper and lower bounds are known. A control scheme that employs the full actuator states $U(t+\theta)$, $\theta \in [-\tau, 0]$, regulates a general linear system with input delay, but requires the upper bound of the delay to be known [66]. Typically, a delay-adaptive control scheme for a general linear system utilizes a lower bound and an upper bound of the delay (see [14] and [15]). Regardless of different types of systems and various control schemes, regulation of linear systems by using a feedback law independent of any knowledge of the delay still remains an open problem.

The delay independent truncated predictor feedback law as a non-distributed controller allows easy implementation and requires relatively less knowledge of the delay. To achieve regulation, in the absence of any knowledge of the delay, of linear systems with all open loop poles at the origin or in the open left-half plane, in this section, we consider the delay independent feedback law as the nominal controller for the design of a regulation scheme. The Lyapunov equation based low gain feedback design results in a feedback gain matrix with a single feedback parameter. An update algorithm for the feedback parameter consists of two update laws with a switching mechanism. This update algorithm guarantees the regulation of the state and the control input of the system to zero and the regulation
indexregulation!continuous-time of the feedback parameter to a positive constant. The regulation proof is carried out by the use of the partial differential equation (PDE) based system representation and a Lyapunov analysis of the closed-loop system. The development and the applications of the PDE approach to the representation and analysis of delayed systems can be found in great detail in [67].

5.1. A feedback law with a time-varying parameter

We consider the design of a delay compensation scheme that regulates to zero the state and the control input of the following linear system with input delay,

$$\begin{cases} \dot{X}(t) \,=\, AX(t) + BU(t-\tau), \;\; t \geq 0, \\ X(\theta) \,=\, \psi(\theta), \;\; \theta \in [-\tau, 0], \end{cases} \tag{5.1}$$

where $X \in \mathbb{R}^n$ and $U \in \mathbb{R}^m$ are the state vector and the input vector, respectively, $\tau \geq 0$ is an arbitrarily large unknown delay, and the initial condition

$$\psi(\theta) \in PC[-\tau, 0].$$

It is assumed that all eigenvalues of A are at the origin and (A, B) is controllable.

In Section 2, the delay independent truncated predictor feedback law with the Lyapunov equation based low gain feedback design takes the form of

$$U(t) = -B^{\mathrm{T}} P(\gamma) X(t), \;\; t \geq -\tau, \tag{5.2}$$

where γ is the low gain feedback parameter, $-B^{\mathrm{T}}P(\gamma)$ is the feedback gain matrix and $P(\gamma)$ is the unique positive definite solution to the parametric algebraic Riccati equation,

$$A^{\mathrm{T}}P(\gamma) + P(\gamma)A - P(\gamma)BB^{\mathrm{T}}P(\gamma) = -\gamma P(\gamma), \;\; \gamma > 0. \tag{5.3}$$

With the knowledge of an upper bound of the delay, an upper bound of γ that ensures the asymptotic stability of the closed-loop system was established.

In the absence of any knowledge of the delay value, an upper bound of the low gain parameter with stability guarantee cannot be determined. Inspired by the time-varying low gain feedback design in Section 4, we adopt

$$U(t) = -B^{\mathrm{T}}P(\gamma(t))X(t), \;\; t \geq -\tau, \tag{5.4}$$

with a time-varying feedback parameter

$$\gamma(t) > 0.$$

The initial condition of $\gamma(t)$ is given as

$$\gamma(\theta) = \phi(\theta), \;\; \theta \in [-\tau, 0].$$

It is assumed that

$$\phi(\theta) \in PC[-\tau, 0],$$

and satisfies

$$\phi(\theta) > 0, \;\; \theta \in [-\tau, 0].$$

For notational brevity, $\gamma(0)$ is denoted as γ_0 in the rest of the section.

Remark 5.1. *We have defined the initial condition of the closed-loop system consisting of (5.1) and (5.4) as $\psi(\theta)$ and $\phi(\theta)$ on $\theta \in [-\tau, 0]$. The initial condition for $U(t)$ is determined by the initial conditions $\psi(\theta)$ and $\phi(\theta)$. If we consider $t = 0$ as the starting time instant of the feedback law (5.4), then, the initial condition of the closed-loop system can be defined by $X(0)$, γ_0 and*

$$U(\theta) = \upsilon(\theta) \in PC[-\tau, 0).$$

$\square$

The bottleneck to achieving the regulation of closed-loop signals including $X(t)$, $\gamma(t)$ and $U(t)$ under (5.4), without resorting to any knowledge of the delay, is the design of an appropriate delay independent update algorithm for $\gamma(t)$. Such an algorithm and its main features are presented in the next section.

5.2. An update algorithm for the feedback parameter

An update algorithm for $\gamma(t)$, which constructs a continuous $\gamma(t)$ on $t \geq 0$, consists of two update laws and a mechanism that governs the switching between them. Update law I takes the form of

$$\dot{\gamma}(t) = -\alpha \frac{V^p(t)}{V^p(t) + \beta} \gamma^q(t), \ \ t \geq 0, \tag{5.5}$$

where

$$\alpha, \beta > 0, p \geq 1, q \geq 2,$$

and

$$V(X(t), t) = X^{\mathrm{T}}(t) P(\gamma(t)) X(t)$$

is a Lyapunov function commonly used for stability analysis of systems with or without delays. For notational convenience, we will denote

$$V(t) = V(X(t), t)$$

in the rest of the section.

Update law II takes the following form,

$$\dot{\gamma}(t) = -\zeta \gamma^r(t), \ \ t \geq 0, \tag{5.6}$$

where

$$\zeta > 0, \ \ r \geq 2.$$

The switching between the two update laws is based on an event-triggered switching condition according to the value of the Lyapunov function $V(t)$. The update law is switched from I to II whenever

$$V(t) \geq \epsilon,$$

where $\epsilon > 0$ is a threshold. Denote the time instant of a switch from update law I to update law II as T_{12}. We consider isolated time instants

$$T_{12} + \delta_i, \;\; i \in \mathbb{N},$$

such that

$$\gamma(T_{12} + \delta_i) = \frac{\gamma(T_{12})}{\xi^i}, \tag{5.7}$$

where

$$\xi > 1$$

and δ_i is computed by the use of (5.6) and (5.7) as

$$\delta_i = \frac{\gamma^{1-r}(T_{12})}{(r-1)\zeta} \left(\xi^{(r-1)i} - 1 \right). \tag{5.8}$$

Then, the time instant of the next switch from update law II back to update law I is

$$T_{21} = T_{12} + \min_{i \in \mathbb{N}} \{ \delta_i : V(T_{12} + \delta_i) < \epsilon \}. \tag{5.9}$$

Note from (5.8) that δ_i is strictly increasing with respect to i. Thus, (5.9) indicates that after a switch from update law I to update law II, we check the value of V at the isolated time instants

$$T_{12} + \delta_i, \;\; i \in \mathbb{N},$$

and switch from update law II back to update law I at the first time instant

$$T_{12} + \delta_i$$

at which

$$V(T_{12} + \delta_i) < \epsilon.$$

Two main features of the update algorithm for $\gamma(t)$ are its non-distributed nature and delay independence. In particular, the update algorithm utilizes exclusively the current state as the feedback, rather than resorting to the past values of any closed-loop signal. On the other hand, no knowledge of the delay is required in the update algorithm. As a result, the integrated control scheme consisting of the feedback law (5.4) and the update algorithm is also non-distributed and delay independent.

Remark 5.2. *According to the switching mechanism, the value of $V(0)$ uniquely determines which update law is to implement at $t = 0$. If*

$$V(0) < \epsilon,$$

then the closed-loop system starts to evolve under update law I. Otherwise, it starts to evolve under update law II. By the definition of $V(t)$, the selection of the update law for $\gamma(t)$ at $t = 0$ solely depends on $\psi(0)$ and $\phi(0)$. □

Remark 5.3. *Both T_{12} and T_{21} can be infinite, provided that the switching condition from one update law to another is never satisfied. In particular,*

$$T_{l12} = \infty$$

if

$$V(t) < \epsilon, \;\; t \geq T_{l21},$$

where T_{l12} and T_{l21} are the time instants of the last time of switch from update law I to update law II and from II to I, respectively. Similarly,

$$T_{l21} = \infty$$

if

$$V(T_{l12} + \delta_i) \geq \epsilon, \;\; i \in \mathbb{N},$$

where δ_i is defined in (5.8) with T_{12} replaced by T_{l12}. □

Remark 5.4. *On every time interval $[T_{12}, T_{21}]$, where T_{12} is the time instant of a switch from update law I to update law II and T_{21} is the time instant of the next switch from update law II back to update* *law I, $\gamma(t)$ evolves according to update law II. Equation (5.7) shows that the values of $\gamma(t)$ at the time instants of the sequence*

$$\{T_{12} + \delta_i\}_{i=0}^{i=J},$$

where

$$J = \min\{j \in \mathbb{N} : V(T_{12} + \delta_j) < \epsilon\},$$

form a decreasing geometric sequence of a ratio

$$\xi^{-1} < 1.$$

From this perspective, update law II decreases the value of $\gamma(t)$ geometrically.

Another feature of update law II is that the switching condition from update law II to update law I is only required to be examined at the isolated time instants of the sequence

$$\{T_{12} + \delta_i\}_{i=0}^{i=J},$$

while for the rest of the time on the interval $[T_{12}, T_{21}]$, there is no feedback from other closed-loop signals to $\gamma(t)$. □

Remark 5.5. *The frequency of switching between the two update laws for $\gamma(t)$ can be partially described by an examination of (5.9). Note that*

$$T_{21} - T_{12} \geq \delta_1 = \frac{\gamma^{1-r}(T_{12})}{(r-1)\zeta}\left(\xi^{r-1} - 1\right) \geq \frac{\gamma_0^{1-r}}{(r-1)\zeta}\left(\xi^{r-1} - 1\right),$$

because of the non-increasing monotonicity of $\gamma(t)$ on $t \in [0, \infty)$ and $r \geq 2$. Therefore, every time update law I switches to update law II, the time it takes to switch from update law II back to update law I is bounded from below by a positive constant. This implies that on any bounded time interval, the number of switches between the two update laws must be finite. Thus, the time-varying function $\dot{\gamma}(t)$ is piecewise continuous on every bounded time interval. □

In the rest of the section, we aim to prove the following two theorems on the well-definedness and regulation to zero of the closed-loop signals under the proposed control scheme.

Theorem 5.1. *Under the update algorithm for $\gamma(t)$ and for any given initial conditions $\psi(\theta), \phi(\theta) \in PC[-\tau, 0]$, there exist unique solutions*

$$X(t), \gamma(t), U(t) \in C[0, \infty).$$

Moreover,

$$\gamma(t) \in (0, \gamma_0], \;\; t \in [0, \infty).$$

Theorem 5.2. *The feedback law (5.4) with $\gamma(t)$ updated by the update algorithm achieves*

$$\lim_{t\to\infty} X(t) = 0, \;\; \lim_{t\to\infty} U(t) = 0.$$

Moreover,

$$\lim_{t\to\infty} \gamma(t)$$

exists and is positive. In particular, on $t \in [0, \infty)$, the number of switches between the two update laws for $\gamma(t)$ is finite, and the last switch happens from update law II to update law I.

The proof of Theorem 5.1 is given in Section 5.3. Preliminary results for the proof of Theorem 5.2 are presented in Sections 5.4 and 5.5, and the proof of Theorem 5.2 is given at the end of Section 5.5.

5.3. Proof of the properties of the closed-loop signals

Proof of Theorem 5.1: The proof is inspired by the continuation progress employed in the existence and the uniqueness proof of the solution of a functional differential equation (see Theorem 3.1 in [12]).

In view of the facts that

$$\psi(\theta), \phi(\theta) \in PC[-\tau, 0],$$

the open loop system (5.1) and the feedback law (5.4), there exists a unique solution $X(t)$ on $t \in [0, \tau]$ expressed as

$$X(t) = \mathrm{e}^{At} X(0) - \int_0^t BB^{\mathrm{T}} P(\phi(s-\tau))\psi(s-\tau)\mathrm{d}s,$$

which leads to

$$X(t) \in C[0, \tau].$$

With the solution $X(t)$, $t \in [0, \tau]$, the evolution of $\gamma(t)$ on $t \in [0, \tau]$ follows the update algorithm presented in Section 5.2. Consider an auxiliary signal

$$s(t) = \gamma^{-1}(t)$$

on $t \in [0, \tau]$. Then, $s(t)$ either satisfies

$$\dot{s}(t) = \alpha \frac{V^p(t)}{V^p(t) + \beta} s^{2-q}(t), \tag{5.10}$$

or

$$\dot{s}(t) = \zeta s^{2-r}(t),$$

corresponding to update laws I and update law II, respectively.

The existence of the signal $s(t)$ on $t \in [0, \tau]$ is shown by using the technique of proof by contradiction. Suppose that the solution $s(t)$ only exists on $t \in [0, t_{\mathrm{f}})$, where $0 < t_{\mathrm{f}} \leq \tau$. Note from Remark 5.5 that the number of switches between the two update laws on $t \in [0, t_{\mathrm{f}})$ is finite. Denote the time instant of the last time of switch as t_{s}. On $t \in [t_{\mathrm{s}}, t_{\mathrm{f}})$, either update law I or update law II is implemented. Suppose that it is the first case. According to (5.10),

$$\dot{s}(t) \leq \alpha s^{2-q}(t), \;\; t \in [t_{\mathrm{s}}, t_{\mathrm{f}}),$$

which leads to

$$s(t) \leq \big(\alpha(q-1)(t - t_{\mathrm{s}}) + s^{q-1}(t_{\mathrm{s}})\big)^{\frac{1}{q-1}} < \infty, \;\; t \in [t_{\mathrm{s}}, t_{\mathrm{f}}).$$

This contradicts with

$$\lim_{t \to t_{\mathrm{f}}^-} s(t) = \infty.$$

Applying a similar argument to the second case, where update law II is implemented on $t \in [t_{\mathrm{s}}, t_{\mathrm{f}})$, also results in a contradiction. Therefore, $s(t)$, and thus $\gamma(t)$, exist on $t \in [0, \tau]$. Note that

$$\dot{s}(t) \geq 0.$$

Thus,

$$s(t) \geq \gamma_0^{-1} > 0,$$

and

$$\gamma(t) > 0, \ \ t \in [0, \tau].$$

The uniqueness of the solution $\gamma(t)$ on $t \in [0, \tau]$ again follows from proof by contradiction. Suppose that the time instant of the first time the solution $\gamma(t)$ becomes non-unique is t_d, where $t_d \in [0, \tau)$. Again, recall from Remark 5.5 that the number of switches between the two update laws is finite on any bounded time interval. Therefore, there exists a sufficiently small

$$\varepsilon > 0$$

such that either update law I or update law II, is implemented on $t \in [t_d, t_d + \varepsilon]$. Consider the first case where update law I is implemented. The right hand side of (5.5) is continuous with respect to both t and γ at the point $(t_d, \gamma(t_d))$ because

$$X(t) \in C[0, \tau]$$

and $P(\gamma)$ is infinitely differentiable with respect to γ (see Lemma 2.1). Furthermore,

$$\frac{\partial}{\partial \gamma}\left(-\alpha \frac{V^p(t)}{V^p(t) + \beta}\gamma^q\right) = -\alpha \frac{V^{p-1}(t)\gamma^{q-1}}{V^p(t) + \beta}\left(\frac{p\gamma\beta X^{\mathrm{T}}(t)\frac{\partial P}{\partial \gamma}X(t)}{V^p(t) + \beta} + qV(t)\right),$$

and is continuous with respect to t and γ at $(t_d, \gamma(t_d))$. By the existence and uniqueness theorem on the solution of an ordinary differential equation, there exists an

$$\eta \in (0, \varepsilon]$$

such that the solution $\gamma(t)$ on $[t_d, t_d + \eta]$ is unique. This contradicts the fact that $\gamma(t)$ starts to become non-unique from t_d. Applying a similar argument to the second case where update law II is implemented also leads to a contradiction. Therefore, $\gamma(t)$ has a unique solution on $t \in [0, \tau]$.

With

$$X(t), \gamma(t) \in C[0, \tau],$$

the existence and the uniqueness of $X(t)$ and $\gamma(t)$ on $t \in [\tau, 2\tau]$ can be obtained similarly. Repeatedly applying the analysis along the time axis leads to the conclusion that there exists unique solutions

$$X(t), \gamma(t) \in C[0, \infty).$$

In view of the feedback law (5.4), the existence, the uniqueness and the continuity of $U(t)$ then directly follow from those of $X(t)$ and $\gamma(t)$. On the other hand, by the use of the continuation progress, the positiveness of $\gamma(t)$ on $t \in [0, \infty)$ follows readily from the fact that

$$\gamma(t) > 0$$

on $t \in [0, \tau]$. By the non-increasing monotonicity of $\gamma(t)$, we get

$$\gamma(t) \in (0, \gamma_0], \;\; t \in [0, \infty).$$

□

Remark 5.6. *The positiveness of $\gamma(t)$ on $t \in [0, \infty)$ as a conclusion of Theorem 5.1 shows that the update algorithm for $\gamma(t)$ successfully avoids the singularity at*

$$\gamma = 0.$$

According to Section 2, such a singularity destroys the existence of a positive definite solution to (5.3). □

Remark 5.7. *Theorem 5.1 also holds when $\gamma(t)$ is updated according to either update law I or update law II on $t \in [0, \infty)$. The proof for the case of a single update law is simpler, without the consideration of the other update law and the switching mechanism.* □

In the rest of this section, we establish a lower bound of $\gamma(t)$ on $t \in [0, \infty)$ as a motivation for the more refined non-zero lower bound to be established in the proof of Theorem 5.2 in Section 5.5.

Proposition 1. *$\gamma(t) \geq 1/\omega(t)$ under the update algorithm for $\gamma(t)$, where*

$$\begin{aligned}\dot{\omega}(t) &= \max\left\{\alpha\omega^{2-q}(t), \zeta\omega^{2-r}(t)\right\} \\ &\triangleq f(\omega),\ \ t \geq 0,\end{aligned} \tag{5.11}$$

and

$$\omega(0) = \gamma_0^{-1}.$$

Proof. Consider an auxiliary signal

$$s(t) = 1/\gamma(t),\ \ t \in [0, \infty).$$

The upper Dini derivative of $s(t)$ satisfies

$$\begin{aligned}D^+ s(t) &= \limsup_{a \to 0^+} \frac{s(t+a) - s(t)}{a} \\ &\leq \max\left\{\alpha\frac{V^p(t)}{V^p(t)+\beta}s^{2-q}(t), \zeta s^{2-r}(t)\right\} \\ &\leq \max\left\{\alpha s^{2-q}(t), \zeta s^{2-r}(t)\right\},\ \ t \geq 0.\end{aligned} \tag{5.12}$$

Regardless of the values of α, q, ζ, r and γ_0 given in Section 5.2, $\omega(t)$ has a unique solution on $t \in [0, \infty)$. In fact, the explicit solution of $\omega(t)$ can be obtained. Take the case where

$$q > r$$

and

$$\gamma_0 > (\zeta/\alpha)^{1/(q-r)}$$

for example. We compute

$$\begin{aligned}\omega(t) &= \begin{cases} \left((q-1)\alpha t + \gamma_0^{1-q}\right)^{\frac{1}{q-1}},\ \ t \in [0, t_\omega], \\ \left((r-1)\zeta(t - t_\omega) + \left(\frac{\alpha}{\zeta}\right)^{\frac{r-1}{q-r}}\right)^{\frac{1}{r-1}},\ \ t \in [t_\omega, \infty), \end{cases} \\ t_\omega &= \frac{\left(\frac{\alpha}{\zeta}\right)^{\frac{q-1}{q-r}} - \gamma_0^{1-q}}{\alpha(q-1)}.\end{aligned}$$

The solution of $\omega(t)$ for other cases regarding the values of α, q, ζ, r can be obtained similarly. For brevity, we omit the solution of $\omega(t)$ for other cases.

By its definition in (5.11), $f(\omega)$ is locally Lipschitz in $w \in \mathbb{R}^+ \subset \mathbb{R}$. On the other hand, the interval $[0, \infty)$ is the maximum time interval of existence of both $\omega(t)$ and $s(t)$. By the continuity and the positiveness of $s(t)$ on $t \in [0, \infty)$,

$$s(0) = w(0)$$

and (5.12), it follows from the comparison lemma (Lemma 3.4 in [59]) that

$$s(t) \leq \omega(t), \;\; t \in [0, \infty).$$

□

5.4. The PDE description of the closed-loop system

The modeling of an input delayed system as a cascade of an ordinary differential equation (ODE) with a PDE brings a wealth of tools in the PDE analysis to the control systems analysis, advancing control techniques for state regulation, trajectory tracking and compensation for unknown delays (see [4], [11], [14], [15] and [67]). While the PDE analysis in those earlier works applies to general linear or certain nonlinear delayed systems under predictor-based feedback laws, Section 4 introduces the PDE method for the analysis of system (5.1) under the feedback law (5.4). We recall from 4 the PDE-based modeling of such a closed-loop system.

We first define a series of functions and some auxiliary signals,

$$u(x,t) = U(t + \tau(x-1)), \tag{5.13}$$

$$\hat{u}(x,t) = U(t + \hat{\tau}(t)(x-1)), \tag{5.14}$$

$$\hat{\tau}(t) = \frac{h}{\gamma(t)}, \tag{5.15}$$

$$\hat{w}(x,t) = \hat{u}(x,t) - U(t), \tag{5.16}$$

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t), \tag{5.17}$$

$$\tilde{\tau}(t) = \tau - \hat{\tau}(t), \tag{5.18}$$

where

$$x \in [0,1], \;\; t \geq 0,$$

and h is some positive constant. Note from Theorem 5.1 that

$$\gamma(t) > 0, \;\; t \in [0,\infty).$$

Thus, $\hat{\tau}(t)$ is well-defined on the interval $[0,\infty)$. With the actuator state $u(x,t)$, the open loop system (5.1) is represented as a cascade of an ODE with a PDE,

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{5.19}$$

$$\tau u_t(x,t) = u_x(x,t), \tag{5.20}$$

$$u(1,t) = U(t). \tag{5.21}$$

Then, we arrive at the following closed-loop system representation,

$$\dot{X}(t) = \big(A - BB^{\mathrm{T}}P(\gamma(t))\big)X(t) + B\tilde{u}(0,t) + B\hat{w}(0,t). \tag{5.22}$$

Next, we recall from Section 4 the governing PDEs for $\hat{u}(x,t)$, $\hat{w}(x,t)$, $\tilde{u}(x,t)$ and $\hat{w}_x(x,t)$ as

$$\begin{cases} \hat{\tau}\hat{u}_t(x,t) \;=\; \big(1 + \dot{\hat{\tau}}(x-1)\big)\hat{u}_x(x,t), \\ \hat{u}(1,t) \;=\; U(t), \end{cases} \tag{5.23}$$

$$\begin{cases} \hat{\tau}\hat{w}_t(x,t) \;=\; \hat{w}_x(x,t)(1 + \dot{\hat{\tau}}(x-1)) + \hat{\tau}B^{\mathrm{T}}\dfrac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t) + \hat{\tau}B^{\mathrm{T}}P(\gamma(t)) \\ \qquad\qquad \times\Big(\big(A - BB^{\mathrm{T}}P(\gamma(t))\big)X(t) + B\tilde{u}(0,t) + B\hat{w}(0,t)\Big), \\ \hat{w}(1,t) \;=\; 0, \end{cases} \tag{5.24}$$

$$\begin{cases} \tau\tilde{u}_t(x,t) \;=\; \tilde{u}_x(x,t) - \dfrac{\tilde{\tau} + \tau\dot{\hat{\tau}}(x-1)}{\hat{\tau}}\hat{w}_x(x,t), \\ \tilde{u}(1,t) \;=\; 0, \end{cases} \tag{5.25}$$

and

$$\begin{cases} \hat{\tau}\hat{w}_{xt}(x,t) \;=\; \hat{w}_{xx}(x,t)(1 + \dot{\hat{\tau}}(x-1)) + \dot{\hat{\tau}}\hat{w}_x(x,t), \\ \hat{w}_x(1,t) \;=\; -\hat{\tau}B^{\mathrm{T}}\dfrac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t) - \hat{\tau}B^{\mathrm{T}}P(\gamma(t)) \\ \qquad\qquad \times\Big((A - BB^{\mathrm{T}}P(\gamma(t)))X(t) + B\tilde{u}(0,t) + B\hat{w}(0,t)\Big), \end{cases} \tag{5.26}$$

respectively.

We finally recall a lemma from Section 4 on some properties of the functions (5.13)-(5.18).

Lemma 5.1. *Consider the closed-loop system consisting of (5.1) and (5.4). The following properties hold,*

$$
\begin{aligned}
&-\int_0^1 (1+x)(\tilde{\tau}+\tau\dot{\hat{\tau}}(x-1))\tilde{u}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x \\
\leq&\Big(|\tilde{\tau}|+\frac{1}{2}\tau|\dot{\hat{\tau}}|\Big)\Big(\rho||\tilde{u}(t)||^2+\frac{1}{\rho}||\hat{w}_x(t)||^2\Big),
\end{aligned}
$$

where $\rho > 0$ is any constant,

$$
\int_0^1 (1+x)(1+\dot{\hat{\tau}}(x-1))\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x \leq \frac{1}{2}(|\dot{\hat{\tau}}|-1)|\hat{w}(0,t)|^2+\Big(|\dot{\hat{\tau}}|-\frac{1}{2}\Big)||\hat{w}(t)||^2,
$$

$$
\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}B^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\dot{\gamma}(t)X(t)\mathrm{d}x \leq h^{\frac{1}{2}}||\hat{w}(t)||^2+\Big(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\Big)^2 h^{\frac{3}{2}}X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t),
$$

$$
\begin{aligned}
&\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}B^{\mathrm{T}}P(\gamma(t))\Big(\big(A-BB^{\mathrm{T}}P(\gamma(t))\big)X(t)+B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)\mathrm{d}x \\
\leq\ & \hat{\tau}n\gamma(t)||\hat{w}(t)||^2+\frac{3}{2}\hat{\tau}n(n+1)\gamma^2(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t) \\
&+3\hat{\tau}n\gamma(t)(|\tilde{u}(0,t)|^2+|\hat{w}(0,t)|^2),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 (1+x)(1+\dot{\hat{\tau}}(x-1))\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\mathrm{d}x \\
\leq\ & |\hat{w}_x(1,t)|^2+\frac{1}{2}(|\dot{\hat{\tau}}|-1)|\hat{w}_x(0,t)|^2+\Big(|\dot{\hat{\tau}}|-\frac{1}{2}\Big)||\hat{w}_x(t)||^2,
\end{aligned}
$$

$$
\begin{aligned}
|\hat{w}_x(1,t)|^2 \leq\ & 2\hat{\tau}^2\dot{\gamma}^2(t)X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(t)+6\hat{\tau}^2n^2\gamma^2(t) \\
&\times\Big(\frac{n+1}{2}\gamma(t)X^{\mathrm{T}}(t)P(\gamma(t))X(t)+|\tilde{u}(0,t)|^2+|\hat{w}(0,t)|^2\Big).
\end{aligned}
$$

In our regulation analysis later in this section, we adopt the following Lyapunov functional

$$\begin{aligned}\mathcal{V}(X_t,\gamma(t)) \;=\; & V(t)+b_1\tau\int_0^1(1+x)|\tilde{u}(x,t)|^2\mathrm{d}x+b_2\hat{\tau}(t)\int_0^1(1+x)|\hat{w}(x,t)|^2\mathrm{d}x\\ & +b_2\hat{\tau}(t)\int_0^1(1+x)|\hat{w}_x(x,t)|^2\mathrm{d}x, \end{aligned} \tag{5.27}$$

where

$$b_1, b_2 > 0$$

and their values are to be determined.

In the following lemma, we establish the continuity of $\mathcal{V}(t)$ with respect to time under the update algorithm for $\gamma(t)$.

Lemma 5.2. *Under the update algorithm for $\gamma(t)$ and with the initial conditions*

$$\psi(\theta),\phi(\theta)\in PC[-\tau,0],$$

there exists

$$t_c>0$$

such that

$$\mathcal{V}(t)\in C[t_c,\infty)$$

if

$$h\le\frac{1}{2\max\left\{\alpha\gamma_0^{q-2},\zeta\gamma_0^{r-2}\right\}}. \tag{5.28}$$

Proof. We study the continuity of $\mathcal{V}(X_t,\gamma(t))$ term by term. The first term in $\mathcal{V}(X_t,\gamma(t))$ is continuous on $t\in[0,\infty)$ because of the continuity of $X(t)$ and $\gamma(t)$ on the same interval. By a change of variables,

$$\begin{aligned}&\tau\int_0^1(1+x)|\tilde{u}(x,t)|^2\mathrm{d}x\\ =&\int_{t-\tau}^{t}\left(2+\frac{s-t}{\tau}\right)|U(s)|^2\mathrm{d}s+\frac{\tau}{\hat{\tau}(t)}\int_{t-\hat{\tau}(t)}^{t}\left(2+\frac{s-t}{\hat{\tau}(t)}\right)|U(s)|^2\mathrm{d}s\end{aligned}$$

$$-\frac{2\tau}{\hat{\tau}(t)}\int_{t-\hat{\tau}(t)}^{t}\left(2+\frac{s-t}{\hat{\tau}(t)}\right)U^{\mathrm{T}}(s)U\left(\frac{\tau}{\hat{\tau}(t)}s+t\left(1-\frac{\tau}{\hat{\tau}(t)}\right)\right)\mathrm{d}s,$$

when $\tau \neq 0$. The continuity of $U(t)$ on $t \in [0,\infty)$ implies that the second term of $\mathcal{V}(X_t,\gamma(t))$ is continuous with respect to t if

$$t \geq \max\left\{\tau, \hat{\tau}(t)\right\}.$$

The continuity of the second term of $\mathcal{V}(X_t,\gamma(t))$ is obvious when

$$\tau = 0.$$

Noting that

$$\begin{aligned}\hat{\tau}(t)\int_0^1(1+x)|\hat{w}(x,t)|^2\mathrm{d}x \;=\; & \int_{t-\hat{\tau}(t)}^{t}\left(2+\frac{s-t}{\hat{\tau}(t)}\right)|U(s)|^2\mathrm{d}s+\frac{3}{2}\hat{\tau}(t)|U(t)|^2\\ & -2\int_{t-\hat{\tau}(t)}^{t}\left(2+\frac{s-t}{\hat{\tau}(t)}\right)U(s)\mathrm{d}sU(t),\end{aligned}$$

we deduce that the third term of $\mathcal{V}(X_t,\gamma(t))$ is continuous if

$$\begin{aligned}t \geq & \max\left\{\hat{\tau}(t),0\right\}\\ = & \hat{\tau}(t).\end{aligned}$$

The last term of $\mathcal{V}(X_t,\gamma(t))$ needs careful examination because it involves $\dot{\gamma}(t)$, which is not necessarily continuous on $t \in [0,\infty)$ because of possible switching between the two update laws for $\gamma(t)$. It follows from a change of variables that

$$\begin{aligned}&\int_0^1(1+x)|\hat{w}_x(x,t)|^2\mathrm{d}x\\ =&\int_{t-\hat{\tau}(t)}^{t}(2\hat{\tau}(t)+s-t)\Bigg(\dot{\gamma}^2(s)\left|B^{\mathrm{T}}\frac{\partial P}{\partial\gamma}X(s)\right|^2+|B^{\mathrm{T}}P(\gamma(s))AX(s)|^2\\ &+|B^{\mathrm{T}}P(\gamma(s))BB^{\mathrm{T}}P(\gamma(s-\tau))X(s-\tau)|^2+3\dot{\gamma}(s)X^{\mathrm{T}}(s)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}P(\gamma(s))AX(s)\\ &-3\dot{\gamma}(s)X^{\mathrm{T}}(s)\frac{\partial P}{\partial\gamma}BB^{\mathrm{T}}P(\gamma(s))BB^{\mathrm{T}}P(\gamma(s-\tau))X(s-\tau)\\ &-3X^{\mathrm{T}}(s)A^{\mathrm{T}}P(\gamma(s))BB^{\mathrm{T}}P(\gamma(s))BB^{\mathrm{T}}P(\gamma(s-\tau))X(s-\tau)\Bigg)\mathrm{d}s. \qquad (5.29)\end{aligned}$$

On the time interval $[t - \hat{\tau}(t), t]$, where

$$t \geq \hat{\tau}(t),$$

$\dot{\gamma}(t)$ is piecewise continuous according to Remark 5.5. Also,

$$\begin{aligned} 0 \geq & \dot{\gamma}(t) \\ & \geq \min\left\{-\alpha\gamma^q(t), -\zeta\gamma^r(t)\right\} \\ & \geq \min\left\{-\alpha\gamma_0^q, -\zeta\gamma_0^r\right\} \end{aligned} \tag{5.30}$$

implies that $\dot{\gamma}(t)$ is bounded. In view of the continuity and the boundedness of $X(t)$ and $\gamma(t)$ on $[t - \hat{\tau}(t), t]$, the integrand of the right hand side of (5.29) is Riemann integrable on the same interval as long as

$$t \geq \tau + \hat{\tau}(t).$$

Combining the continuity analysis for all the terms of $\mathcal{V}(X_t, \gamma(t))$, we conclude that, if

$$t \geq \tau + \hat{\tau}(t),$$

$\mathcal{V}(X_t, \gamma(t))$ is continuous. Under the update algorithm for $\gamma(t)$, the term

$$\frac{\dot{\gamma}(t)}{\gamma^2(t)}$$

is bounded on $t \in [0, \infty)$ because

$$\begin{aligned} 0 \geq & \frac{\dot{\gamma}(t)}{\gamma^2(t)} \\ & \geq \min\left\{-\alpha\gamma^{q-2}(t), -\zeta\gamma^{r-2}(t)\right\} \\ & \geq \min\left\{-\alpha\gamma_0^{q-2}, -\zeta\gamma_0^{r-2}\right\}. \end{aligned}$$

By the use of the comparison lemma, we compute

$$\gamma(t) \geq \frac{1}{\frac{1}{\gamma_0} + \max\left\{\alpha\gamma_0^{q-2}, \zeta\gamma_0^{r-2}\right\}t}, \quad t \geq 0. \tag{5.31}$$

If h satisfies (5.28), we have

$$h < \frac{1}{\max\left\{\alpha\gamma_0^{q-2}, \zeta\gamma_0^{r-2}\right\}}.$$

Therefore, there exists

$$t_{\rm c} > 0$$

such that, for each $t \geq t_{\rm c}$,

$$t \geq \tau + h(\gamma_0^{-1} + \max\left\{\alpha\gamma_0^{q-2}, \zeta\gamma_0^{r-2}\right\} t), \tag{5.32}$$

which implies that

$$t \geq \tau + \hat{\tau}(t)$$

according to (5.31). Then, the continuity of $\mathcal{V}(X_t, \gamma(t))$ follows.

$\square$

Remark 5.8. *If $\gamma(t)$ is updated by either update law I or update law II on $t \in [0, \infty)$ alone, Lemma 5.2 also holds with the denominator of the right-hand side of (5.28) becoming*

$$2\alpha\gamma_0^{q-2}$$

or

$$2\zeta\gamma_0^{r-2},$$

respectively. The proof for the case of a single update law resembles the proof of Lemma 5.2, with a difference in the estimate of the lower bound of $\dot{\gamma}(t)$ in (5.30). $\square$

We compute the time derivative of $\mathcal{V}(X_t, \gamma(t))$ along the trajectory of the closed-loop system between switchings for $\gamma(t)$ as follows,

$$\begin{aligned}
\dot{\mathcal{V}}(X_t, \gamma(t)) \;=\; & X^{\rm T}(t)\Big(-\gamma(t)P(\gamma(t)) - P(\gamma(t))BB^{\rm T}P(\gamma(t))\Big)X(t) \\
& +2X^{\rm T}(t)P(\gamma(t))B\tilde{u}(0,t) + 2X^{\rm T}(t)P(\gamma(t))B\hat{w}(0,t) \\
& +X^{\rm T}(t)\frac{\partial P}{\partial \gamma}\dot{\gamma}(t)X(t) + 2b_1\int_0^1 (1+x)\tilde{u}^{\rm T}(x,t)\tilde{u}_x(x,t)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&-\frac{2b_1}{\hat{\tau}}\int_0^1 (1+x)\left(\tilde{\tau}+\tau\dot{\hat{\tau}}(x-1)\right)\tilde{u}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)\mathrm{d}x \\
&+2b_2\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{w}_x(x,t)(1+\dot{\hat{\tau}}(x-1))\mathrm{d}x \\
&+2b_2\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}B^{\mathrm{T}}P(\gamma(t))\Big((A-BB^{\mathrm{T}}P(\gamma(t)))X(t) \\
&+B\tilde{u}(0,t)+B\hat{w}(0,t)\Big)\mathrm{d}x+2b_2\int_0^1 (1+x)\hat{w}^{\mathrm{T}}(x,t)\hat{\tau}B^{\mathrm{T}}\frac{\partial P}{\partial\gamma}\dot{\gamma}(t)X(t)\mathrm{d}x \\
&+2b_2\int_0^1 (1+x)\hat{w}_x^{\mathrm{T}}(x,t)\hat{w}_{xx}(x,t)\left(1+\dot{\hat{\tau}}(x-1)\right)\mathrm{d}x \\
&+2b_2\int_0^1 (1+x)|\hat{w}_x(x,t)|^2\dot{\hat{\tau}}\mathrm{d}x+b_2\dot{\hat{\tau}}\int_0^1 (1+x)(|\hat{w}(x,t)|^2 \\
&+|\hat{w}_x(x,t)|^2)\mathrm{d}x, \qquad (5.33)
\end{aligned}
$$

where we have used the closed-loop system representation (5.22), the Riccati equation (5.3), and the governing PDEs (5.23)-(5.26). It then follows that

$$
\begin{aligned}
&\dot{\mathcal{V}}(X_t,\gamma(t)) \\
\le\ & \gamma(t)V(t)\left(-1+6b_2n^2(n+1)h^2+3b_2n(n+1)h\right)+|\tilde{u}(0,t)|^2\Big(2-b_1 \\
&+6b_2nh+12b_2n^2h^2\Big)+|\hat{w}(0,t)|^2\left(b_2(\dot{\hat{\tau}}-1)+2+6b_2nh+12b_2n^2h^2\right) \\
&+||\tilde{u}(t)||^2b_1\left(-1+2\rho\frac{|\tilde{\tau}|+\frac{1}{2}\tau\dot{\hat{\tau}}}{\hat{\tau}}\right)+X^{\mathrm{T}}(t)\frac{\partial P}{\partial\gamma}X(t)\Bigg(-\frac{h}{\hat{\tau}^2}\dot{\hat{\tau}}+2b_2n\Big(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\Big)^2h^{\frac{3}{2}} \\
&+4b_2n\Big(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\Big)^2h^2\Bigg)+||\hat{w}_x(t)||^2\Big(\frac{2b_1}{\rho}\frac{|\tilde{\tau}|+\frac{1}{2}\tau\dot{\hat{\tau}}}{\hat{\tau}}+8b_2\dot{\hat{\tau}}-b_2\Big) \\
&+||\hat{w}(t)||^2b_2\Big(-1+2nh+2h^{\frac{1}{2}}+4\dot{\hat{\tau}}\Big)+|\hat{w}_x(0,t)|^2b_2(\dot{\hat{\tau}}-1), \qquad (5.34)
\end{aligned}
$$

where we have employed

$$\int_0^1 (1+x)\tilde{u}^{\mathrm{T}}(x,t)\tilde{u}_x(x,t)\mathrm{d}x=-\frac{1}{2}(|\tilde{u}(0,t)|^2+||\tilde{u}(t)||^2)$$

and

$$\frac{\partial}{\partial\gamma}P(\gamma)BB^{\mathrm{T}}\frac{\partial}{\partial\gamma}P(\gamma)\le n\frac{\partial}{\partial\gamma}P(\gamma),$$

from Section 4, Young's Inequality, Lemma 5.1, and the facts that

$$h=\gamma(t)\hat{\tau}(t)$$

and $\gamma(t)$ is non-increasing.

5.5. Regulation under the update algorithm

The regulation analysis of the closed-loop signals under the update algorithm for $\gamma(t)$ is not straightforward due to the intrinsic mechanism of switching between the two update laws. We first establish two propositions on the regulation effects of update law I on $\gamma(t)$.

Proposition 2. *There exists $\gamma_0^\star > 0$ such that, for each $\gamma_0 \in (0, \gamma_0^\star]$, the feedback law (5.4) with $\gamma(t)$ updated by update law I as given in (5.5) achieves*

$$\lim_{t\to\infty} X(t) = 0, \ \ \lim_{t\to\infty} U(t) = 0. \tag{5.35}$$

Also,

$$\lim_{t\to\infty} \gamma(t)$$

exists and is positive.

Proof. Let

$$h = \gamma_0$$

in the definition of $\hat{\tau}(t)$ and take $\mathcal{V}(X_t, \gamma(t))$ given by (5.27) as the Lyapunov functional. Choose

$$b_1 = 3 > 0, \ \ b_2 = 50\left(\tau + 1 + \frac{1}{2}\tau\alpha\gamma_0^{q-1}\right)^2 > 0,$$

in (5.27), and

$$\rho = \frac{1}{4(\tau + 1 + \frac{1}{2}\tau\alpha\gamma_0^{q-1})} > 0$$

in (5.34). Define a positive constant

$$\gamma_0^* = (2\alpha)^{1/(1-q)}.$$

In view of update law I,

$$0 \leq \dot{\hat{\tau}}(t) = \alpha\gamma_0 \frac{V^p(t)}{V^p(t) + \beta}\gamma^{q-2}(t)$$

$$\leq \alpha\gamma_0^{q-1}, \tag{5.36}$$

which implies that the inequality

$$2n\alpha b_2(\gamma_0^{q-\frac{1}{2}} + 2\gamma_0^q) \leq 1$$

suffices for

$$-\frac{\gamma_0}{\hat{\tau}^2}\dot{\hat{\tau}} + 2b_2\left(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\right)^2 n\gamma_0^{\frac{3}{2}} + 4b_2\left(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\right)^2 n\gamma_0^2 \leq 0$$

to hold. Replacing $\dot{\hat{\tau}}$ by its upper bound $\alpha\gamma_0^{q-1}$ on the right-hand side of (5.34), we see that the non-positiveness of $\dot{\mathcal{V}}(X_t, \gamma(t))$ is guaranteed by the following inequalities,

$$\begin{cases} -1 + 6b_2n^2(n+1)\gamma_0^2 + 3b_2n(n+1)\gamma_0 < 0, \\ 2 - b_1 + 6b_2n\gamma_0 + 12b_2n^2\gamma_0^2 < 0, \\ 2n\alpha b_2\left(\gamma_0^{q-\frac{1}{2}} + 2\gamma_0^q\right) \leq 1, \\ -1 + 2n\gamma_0 + 2\gamma_0^{\frac{1}{2}} + 4\alpha\gamma_0^{q-1} < 0, \\ \alpha\gamma_0^{q-1} < 1, \\ 2 + b_2\left(\alpha\gamma_0^{q-1} - 1\right) + 6b_2n\gamma_0 + 12b_2n^2\gamma_0^2 < 0, \\ -1 + 2\rho\dfrac{|\tilde{\tau}| + \frac{1}{2}\tau\alpha\gamma_0^{q-1}}{\hat{\tau}} < 0, \\ \dfrac{2b_1}{\rho}\dfrac{|\tilde{\tau}| + \frac{1}{2}\tau\alpha\gamma_0^{q-1}}{\hat{\tau}} + 8b_2\alpha\gamma_0^{q-1} - b_2 < 0. \end{cases} \tag{5.37}$$

Note that the last two inequalities in (5.37) still contain the time-varying term $\hat{\tau}(t)$. By the non-increasing monotonicity of $\gamma(t)$, we obtain

$$\hat{\tau}(t) \geq 1,$$

which implies that

$$\left(|\tilde{\tau}| + \frac{1}{2}\tau\alpha\gamma_0^{q-1}\right)/\hat{\tau} \leq \tau + 1 + \frac{1}{2}\tau\alpha\gamma_0^{q-1}.$$

Thus, the value of ρ makes the last but one inequality in (5.37) hold. On the other hand, the last inequality of (5.37) holds if

$$8\left(\tau + 1 + \frac{1}{2}\tau\alpha\gamma_0^{q-1}\right)^2 b_1 < b_2\left(1 - 8\alpha\gamma_0^{q-1}\right). \tag{5.38}$$

In view of (5.37), (5.38), and the values of b_1, b_2 and ρ, we have that, if

$$\begin{cases} b_2\alpha\gamma_0^{q-1} + 6b_2n\gamma_0 + 12b_2n^2\gamma_0^2 < b_2 - 2, \\ \max\Big\{3b_2n(n+1)\gamma_0 + 6b_2n^2(n+1)\gamma_0^2,\ \ 6b_2n\gamma_0 + 12b_2n^2\gamma_0^2, \\ 2n\alpha b_2\left(\gamma_0^{q-\frac{1}{2}} + 2\gamma_0^q\right),\ \ 2n\gamma_0 + 2\gamma_0^{\frac{1}{2}} + 4\alpha\gamma_0^{q-1},\ \ 16\alpha\gamma_0^{q-1}\Big\} < 1, \end{cases}$$

then, (5.37) holds. Notice that b_2 approaches

$$50(\tau+1)^2$$

as

$$\gamma_0 \to 0^+.$$

This shows that there exists a sufficiently small $\gamma_0^\star \le \gamma_0^*$ such that, for each $\gamma_0 \in (0, \gamma_0^\star]$,

$$\begin{aligned} \dot{\mathcal{V}}(X_t, \gamma(t)) \le& -\mu\gamma(t)V(t) \\ \le& 0, \end{aligned} \tag{5.39}$$

where

$$\begin{aligned} \mu =& 1 - 3b_2n(n+1)\gamma_0 - 6b_2n^2(n+1)\gamma_0^2 \\ >& 0 \end{aligned}$$

and we have used (5.34).

Fix a $\gamma_0 \in (0, \gamma_0^\star]$. It follows from

$$h = \gamma_0,\ \ \gamma_0 \le \gamma_0^*$$

and Remark 5.8 that there exists $t_1 > 0$ such that

$$\mathcal{V}(X_t, \gamma(t)) \in C[t_1, \infty).$$

In fact, Remark 5.7 and the proof of Lemma 5.2 imply that

$$\dot{\mathcal{V}}(X_t, \gamma(t)) \in C[t_1, \infty).$$

Then, by the use of the mean value theorem and (5.39), we have

$$\mathcal{V}(X_t, \gamma(t)) \leq \mathcal{V}(X_{t_1}, \gamma(t_1)), \ \ t \in [t_1, \infty).$$

Furthermore, (5.39) implies that

$$\begin{aligned} \int_{t_1}^{t} \mu\gamma(s)V(s)\mathrm{d}s \leq & -\int_{t_1}^{t} \dot{\mathcal{V}}(X_s, \gamma(s))\mathrm{d}s \\ =& \mathcal{V}(X_{t_1}, \gamma(t_1)) - \mathcal{V}(X_t, \gamma(t)) \\ \leq& \mathcal{V}(X_{t_1}, \gamma(t_1)), \ \ t \geq t_1, \end{aligned} \tag{5.40}$$

where we have used the fundamental theorem of calculus based on the fact that

$$\dot{\mathcal{V}}(X_t, \gamma(t)) \in C[t_1, \infty).$$

By update law I, we obtain

$$\frac{\mathrm{d}\gamma(t)}{\gamma^{q-1}(t)} = -\alpha \frac{V^{p-1}(t)}{V^p(t)+\beta} V(t)\gamma(t)\mathrm{d}t. \tag{5.41}$$

When $q = 2$, integrating both sides of (5.41) from t_1 to t gives

$$\begin{aligned} \gamma(t) =& \gamma(t_1)\exp\left(-\alpha\int_{t_1}^{t}\gamma(s)V(s)\frac{V^{p-1}(s}{V^p(s)+\beta}\mathrm{d}s\right), \\ \geq& \gamma(t_1)\exp\left(-\frac{\alpha}{\beta}\int_{t_1}^{t}\gamma(s)V(s)\mathrm{d}s\mathcal{V}^{p-1}(X_{t_1}, \gamma(t_1))\right) \\ \geq& \gamma(t_1)\exp\left(-\frac{\alpha}{\beta\mu}\mathcal{V}^{p}(X_{t_1}, \gamma(t_1))\right) \\ >& 0, \ \ t \geq t_1, \end{aligned} \tag{5.42}$$

where

$$p \geq 1, \ \ V(t) \leq \mathcal{V}(X_t, \gamma(t)),$$

and (5.39) and (5.40) are used. When $q > 2$, following a similar procedure leads to

$$\begin{aligned} \gamma(t) \geq & \left(\gamma^{2-q}(t_1) + \frac{\alpha}{\mu\beta}(q-2)\mathcal{V}^p(X_{t_1}, \gamma(t_1))\right)^{\frac{1}{2-q}} \\ >& 0, \ \ t \geq t_1. \end{aligned} \tag{5.43}$$

The lower bounds on $\gamma(t)$ in (5.42) and (5.43) and the non-increasing monotonicity of $\gamma(t)$ imply that

$$\lim_{t\to\infty}\gamma(t)$$

exists and is positive.

It remains to prove the regulation to zero of $X(t)$ and $U(t)$. By (5.40), the positive lower bounds of $\gamma(t)$ in (5.42) and (5.43), and the boundedness of $X(t)$ on $t\in[0,t_1]$, we get the square integrability of $X(t)$ on $[0,\infty)$. Moreover, the positive lower bounds of $\gamma(t)$ in (5.42) and (5.43), the continuity of $X(t)$, and

$$V(t)\le\mathcal{V}(X_{t_1},\gamma(t_1))$$

on $t\ge t_1$ imply the boundedness of $X(t)$ on $[0,\infty)$, which leads to the boundedness of $\dot{X}(t)$ in view of (5.1) and (5.4). Then,

$$\lim_{t\to\infty}X(t)=0$$

follows from the Barbalat's lemma (Lemma 2.14 in [109]), and

$$\begin{aligned}\lim_{t\to\infty}U(t)=&-B^{\mathrm{T}}P\Big(\lim_{t\to\infty}\gamma(t)\Big)\lim_{t\to\infty}X(t)\\ =&0.\end{aligned}$$

This completes the proof. □

Proposition 2 reveals the regulation effect of update law I when γ_0 is small. To illustrate the global behavior of the closed-loop system under update law I with respect to γ_0, we present a regulation result concerning an arbitrarily large γ_0 in the following proposition.

Proposition 3. *Under the feedback law (5.4) with $\gamma(t)$ updated by update law I as given in (5.5), for any given $\gamma_0>0$,*

$$\liminf_{t\to\infty}|X(t)|=0,\ \ \liminf_{t\to\infty}|U(t)|=0, \tag{5.44}$$

and

$$\lim_{t\to\infty}\gamma(t) \tag{5.45}$$

exists and is positive. Moreover, if $X(t)$ is bounded on $t \in [0, \infty)$, then,

$$\lim_{t\to\infty} X(t) = 0, \;\; \lim_{t\to\infty} U(t) = 0. \tag{5.46}$$

Proof. We consider the following two cases.

a) There exists $t_0 > 0$ such that

$$\gamma(t_0) \leq \gamma_0^\star,$$

where $\gamma_0^\star$ is as described in Proposition 2. Reset the starting point of system evolution at $t = t_0$, and define the initial conditions as $X(\theta)$ and $\gamma(\theta)$, $\theta \in [t_0 - \tau, t_0]$. Then, the regulation of $X(t)$ and $U(t)$ to zero as time goes to infinity, and the facts that

$$\lim_{t\to\infty} \gamma(t)$$

exists and is positive, are straightforward from Proposition 2.

b) There does not exist $t_0 > 0$ such that

$$\gamma(t_0) \leq \gamma_0^\star.$$

This implies

$$\gamma(t) > \gamma_0^\star, \;\; t \geq 0,$$

With the non-increasing monotonicity of $\gamma(t)$, we have that

$$\lim_{t\to\infty} \gamma(t)$$

exists and is positive. Denote this limit as $\underline{\gamma}$, which satisfies

$$\underline{\gamma} \geq \gamma_0^\star.$$

We claim that

$$\liminf_{t\to\infty} |X(t)| = 0.$$

Suppose the opposite. There exist $\nu > 0$ and $t_1 > 0$ such that

$$|X(t)| \geq \nu, \;\; t \geq t_1.$$

Then,

$$\begin{aligned} V(t) \geq & \lambda_{\min}(P(\gamma_0^\star))\nu^2 \\ \triangleq & M \end{aligned}$$

on $t \in [t_1, \infty)$. By update law I,

$$\dot{\gamma}(t) \leq \frac{-\alpha}{1 + \frac{\beta}{M^p}}\gamma^q(t),$$

from which we have

$$\gamma(t) \leq \left(\gamma^{1-q}(t_1) + \frac{\alpha(q-1)}{1 + \frac{\beta}{M^p}}(t - t_1)\right)^{\frac{1}{1-q}} \to 0 \;\text{ as }\; t \to \infty.$$

This contradicts the fact that

$$\gamma(t) > \gamma_0^\star > 0$$

on $t \in [0, \infty)$, and the claim follows.

It follows from

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma) > 0$$

(see Lemma 2.1),

$$|U(t)| \leq |B||P(\gamma_0)||X(t)|$$

and

$$\liminf_{t \to \infty} |X(t)| = 0$$

that

$$\liminf_{t \to \infty} |U(t)| = 0.$$

The remainder of the proof is to show the regulation to zero of $X(t)$ and $U(t)$ under the assumption that $X(t)$ is bounded on $t \in [0, \infty)$.

Consider the factor

$$\frac{V^p(t)}{V^p(t) + \beta}$$

on the right-hand side of (5.5). We compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{V^p(t)}{V^p(t)+\beta}\right) = {} & \frac{p\beta V^{p-1}(t)}{(V^p(t)+\beta)^2}\Bigg(2X^{\mathrm{T}}(t)P(\gamma(t))\Big(AX(t) \\ & -BB^{\mathrm{T}}P(\gamma(t-\tau))X(t-\tau)\Big) - X^{\mathrm{T}}(t)\frac{\partial P}{\partial \gamma}X(t)\alpha \\ & \times \frac{V^p(t)}{V^p(t)+\beta}\gamma^q(t)\Bigg),\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{V^p(t)}{V^p(t)+\beta}\right)\right| \leq {} & \frac{p}{\beta}\lambda_{\max}^{p-1}(P(\gamma_0))|X(t)|^{2(p-1)}\Bigg(2|A||P(\gamma_0)||X(t)|^2 \\ & +2|B|^2|P(\gamma_0)|^2|X(t)||X(t-\tau)| \\ & +\alpha\gamma_0^q \max_{\gamma \in [\underline{\gamma}, \gamma_0]}\left\{\frac{\partial P}{\partial \gamma}\right\}|X(t)|^2\Bigg) \\ & < \infty.\end{aligned}$$

In view of its boundedness, we deduce that the factor

$$\frac{V^p(t)}{V^p(t) + \beta}$$

is uniformly continuous. On the other hand, (5.5) implies that

$$\begin{aligned}\int_0^t \frac{V^p(s)}{V^p(s)+\beta}\mathrm{d}s = & \frac{-1}{\alpha}\int_0^t \frac{\mathrm{d}\gamma(s)}{\gamma^q(s)} \\ = & \frac{\gamma^{1-q}(t) - \gamma_0^{1-q}}{\alpha(q-1)} \\ \leq & \frac{\underline{\gamma}^{1-q}}{\alpha(q-1)},\end{aligned}$$

from which it follows that

$$\frac{V^p(t)}{V^p(t) + \beta} \in L^1.$$

Therefore, by the Barbalat's lemma (Lemma 2.16 in [109]),

$$\lim_{t\to\infty} \frac{V^p(t)}{V^p(t)+\beta} = 0,$$

and thus

$$\lim_{t\to\infty} V(t) = 0.$$

By the fact that

$$\gamma(t) > \gamma_0^\star$$

on $t \in [0,\infty)$ and the use of the squeeze theorem of limit, we obtain (5.46). $\square$

Remark 5.9. *Proposition 3 shows that, given any γ_0, update law I regulates $\gamma(t)$ to a positive constant, but the regulation of $X(t)$ and $U(t)$ to zero requires the boundedness of $X(t)$. Under update law I alone, we are unable to exclude the case where $X(t)$ is not bounded while still concurring with (5.44) and (5.45) exists. To guarantee that $X(t)$ is bounded, we introduce update law II and the switching mechanism to ensure that after some finite time, update law I remains in effect with a bounded $X(t)$.* $\square$

Before proceeding to the proof of Theorem 5.2, we establish a result on the regulation of the Lyapunov function $V(t)$ to zero by the update algorithm that is assumed to achieve the regulation of $\gamma(t)$ to zero. This regulation result is the key to designing the event-triggered switching mechanism between the two update laws for $\gamma(t)$ in the sense that an update law switches only when the value of $V(t)$ crosses a certain threshold.

Proposition 4. *If the closed-loop system consisting of (5.1) and (5.4) with $\gamma(t)$ updated by the update algorithm achieved*

$$\lim_{t\to\infty} \gamma(t) = 0,$$

then,

$$\lim_{t\to\infty} V(t) = 0.$$

Proof. Consider the Lyapunov functional $\mathcal{V}(X_t, \gamma(t))$ in (5.27). Pick

$$b_1 = 3, \;\; b_2 = 100$$

of $\mathcal{V}(X_t, \gamma(t))$ in (5.27),

$$\rho = 1/8$$

in (5.34), and a small $h > 0$ such that

$$\begin{aligned} \max\Big\{ \; & 6b_2 n^2(n+1)h^2 + 3b_2 n(n+1)h, \;\; 6b_2 nh + 12b_2 n^2 h^2, \\ & b_2 Dh + 6b_2 nh + 12 b_2 n^2 h^2, \;\; 8b_2 Dh, \;\; 2nh + 2h^{\frac{1}{2}} + 4Dh, \\ & 2b_2 nDh^{\frac{3}{2}}(1 + 2h^{\frac{1}{2}}) \Big\} < 1, \end{aligned} \tag{5.47}$$

and

$$\begin{aligned} & \big(1 - 6b_2 n^2(n+1)h^2 - 3b_2 n(n+1)h\big)\, 2b_2 h \le \min\left\{ \frac{b_1}{2}, b_2 - 32b_1 - 8b_2 Dh, \right. \\ & \left. 1 - 2nh, -2h^{\frac{1}{2}} - 4Dh \right\}, \end{aligned} \tag{5.48}$$

where

$$D = \max\left\{ \alpha\gamma_0^{q-2}, \zeta\gamma_0^{r-2} \right\},$$

which, according to the update algorithm in Section 5.2, is an upper bound of

$$\left| \frac{\dot{\gamma}(t)}{\gamma^2(t)} \right|.$$

Note from Lemma 5.2 that such a selection of h implies that there exists $t_1 > 0$ such that

$$\mathcal{V}(X_t, \gamma(t)) \in C[t_1, \infty).$$

In view of the boundedness of

$$\frac{\dot{\gamma}(t)}{\gamma^2(t)},$$

and the continuity of $\gamma(t)$ on $t \in [0, \infty)$, as given in Theorem 5.1, we compute by using the comparison lemma,

$$\gamma(t) \geq \frac{1}{\dfrac{1}{\gamma_0} + Dt}, \ \ t \geq 0. \tag{5.49}$$

The definition of $\hat{\tau}(t)$ and the boundedness of

$$\frac{\dot{\gamma}(t)}{\gamma^2(t)}$$

imply that

$$\begin{aligned} 0 \leq & \dot{\hat{\tau}}(t) \\ = & -\frac{h}{\gamma^2(t)}\dot{\gamma}(t) \\ \leq & Dh. \end{aligned} \tag{5.50}$$

Therefore, the inequality

$$2b_2nDh^{\frac{3}{2}}\left(1 + 2h^{\frac{1}{2}}\right) \leq 1,$$

which is implied by (5.47), leads to

$$-\frac{h}{\hat{\tau}^2}\dot{\hat{\tau}} + 2b_2n\left(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\right)^2 h^{\frac{3}{2}} + 4b_2n\left(\frac{\dot{\hat{\tau}}}{\hat{\tau}}\right)^2 h^2 \leq 0.$$

The boundedness of

$$\frac{\dot{\gamma}(t)}{\gamma^2(t)}$$

implies that

$$\left|\frac{\dot{\gamma}(t)}{\gamma(t)}\right| \leq D\gamma(t).$$

By

$$\lim_{t\to\infty} \gamma(t) = 0$$

and the use of the squeeze theorem of limit, we obtain

$$\lim_{t\to\infty} \frac{\dot{\gamma}(t)}{\gamma(t)} = 0,$$

and thus

$$\lim_{t\to\infty}\frac{\dot{\hat{\tau}}(t)}{\hat{\tau}(t)}=0.$$

Then,

$$\frac{|\tilde{\tau}|+\frac{1}{2}\tau\dot{\hat{\tau}}}{\hat{\tau}}\leq\frac{\tau}{h}\gamma(t)+1+\frac{1}{2}\tau\frac{\dot{\hat{\tau}}}{\hat{\tau}}\to 1 \text{ as } t\to\infty,$$

indicating that there exists $t_2\geq t_1$ such that, for each $t\geq t_2$,

$$\frac{|\tilde{\tau}|+\frac{1}{2}\tau\dot{\hat{\tau}}}{\hat{\tau}}\leq 2.$$

In view of (5.34) and (5.50), we deduce that the non-positiveness of $\dot{\mathcal{V}}(X_t,\gamma(t))$ on $t\geq t_2$ is guaranteed by (5.47). Therefore,

$$\dot{\mathcal{V}}(X_t,\gamma(t))\leq-\kappa\gamma(t)V(t)-\sigma(||\tilde{u}(t)||^2+||\hat{w}(t)||^2+||\hat{w}_x(t)||^2),\ \ t\geq t_2,$$

where

$$\begin{aligned}\kappa=&1-6b_2n^2(n+1)h^2-3b_2n(n+1)h\\ >&0\end{aligned}$$

and

$$\begin{aligned}\sigma=&\min\{\frac{b_1}{2},b_2-32b_1-8b_2Dh,1-2nh-2h^{\frac{1}{2}}-4Dh\}\\ >&0.\end{aligned}$$

The definition of $\mathcal{V}(X_t,\gamma(t))$ leads to

$$V(t)\geq\mathcal{V}(X_t,\gamma(t))-W(t)(||\tilde{u}(t)||^2+||\hat{w}(t)||^2+||\hat{w}_x(t)||^2),$$

where

$$W(t)=\max\Big\{2b_1\tau,2b_2\hat{\tau}(t)\Big\},$$

and thus

$$\dot{\mathcal{V}}(X_t,\gamma(t))\leq-\kappa\gamma(t)\mathcal{V}(X_t,\gamma(t))+(\kappa W(t)\gamma(t)-\sigma)(||\tilde{u}(t)||^2+||\hat{w}(t)||^2+||\hat{w}_x(t)||^2).$$

Note from (5.48) and

$$\lim_{t\to\infty} \gamma(t) = 0$$

that there exists $t_3 \geq t_2$ such that, for $t \geq t_3$,

$$\kappa W(t)\gamma(t) - \sigma \leq 0.$$

This implies that

$$\dot{\mathcal{V}}(X_t, \gamma(t)) \leq -\kappa\gamma(t)\mathcal{V}(X_t, \gamma(t)), \ \ t \geq t_3.$$

By the use of the definition and the continuity of $\mathcal{V}(X_t, \gamma(t))$, and (5.49), we obtain by using the comparison lemma that, for each $t \in [t_3, \infty)$,

$$\begin{aligned} V(t) \leq &\mathcal{V}(X_t, \gamma(t)) \\ \leq &\mathcal{V}(X_{t_3}, \gamma(t_3)) \exp\left(-\kappa \int_{t_3}^{t} \gamma(s)\mathrm{d}s\right) \\ \leq &\mathcal{V}(X_{t_3}, \gamma(t_3)) \left(\frac{1 + D\gamma_0 t}{1 + D\gamma_0 t_3}\right)^{-\frac{\kappa}{D}}. \end{aligned}$$

Proposition 4 then follows from the squeeze theorem of limit. □

We now present the proof of Theorem 5.2.

Proof of Theorem 5.2: We claim that the number of switches between update law I and update law II is finite on $t \in [0, \infty)$. Suppose the number of switches from update law I to update law II is infinite, then, there are an infinite number of disjoint time intervals on which update law II is in effect. By Remarks 5.4 and 5.5, and the fact that

$$\dot{\gamma}(t) \leq 0, \ \ t \in [0, \infty),$$

we get

$$\lim_{t\to\infty} \gamma(t) = 0.$$

Therefore,

$$\lim_{t\to\infty} V(t) = 0$$

follows from Proposition 4. On the other hand, an infinite number of switches from update law I to update law II and Remark 5.5 imply that there exists a sequence $\{t_k\}_{k=0}^{\infty}$ such that

$$\lim_{k\to\infty} t_k = \infty$$

and

$$V(t_k) = \epsilon, \;\; k \in \mathbb{N}.$$

This contradicts with

$$\lim_{t\to\infty} V(t) = 0.$$

Therefore, the claim follows.

We next claim that the last switch between the two update laws happens from update law II to update law I. Suppose the opposite. Denote the time instant of the last switch as t_l. Then, $\gamma(t)$ evolves according to update law II on $t \geq t_l$ and the closed-loop system under update law II satisfies

$$V(t_l + \delta_i) \geq \epsilon, \;\; i \in \mathbb{N},$$

where δ_i is as defined in (5.8) with T_{12} replaced by t_l. By (5.6), we obtain

$$\lim_{t\to\infty} \gamma(t) = 0.$$

It then follows from Proposition 4 that

$$\lim_{t\to\infty} V(t) = 0,$$

which again contradicts the fact that

$$V(t_l + \delta_i) \geq \epsilon, \;\; i \in \mathbb{N}.$$

We conclude that the last switch happens from update law II to update law I. Note that a natural consequence of this conclusion is that

$$V(t) < \epsilon, \;\; t \geq t_I,$$

according to the switching condition from update law I to update law II, where t_I is the time instant of the last switch and is also the time instant of the system evolution from which update law I remains in effect all the time. Recall from Proposition 3 that $\gamma(t)$ is bounded below by a positive constant. Therefore, $X(t)$ is bounded on $t \geq t_I$. The regulation of $X(t)$ and $U(t)$ to zero then follows directly from Proposition 3. □

Remark 5.10. *The assumption on system (5.1) that all its open loop poles are at the origin can be relaxed to that all its open loop poles are at the origin or in the open left-half plane. Without loss of generality, we assume that the pair (A, B) has the following stability structural decomposition,*

$$A = \begin{bmatrix} A_{\mathrm{L}} & 0 \\ 0 & A_o \end{bmatrix}, \quad B = \begin{bmatrix} B_{\mathrm{L}} \\ B_o \end{bmatrix},$$

where $A_{\mathrm{L}} \in \mathbb{R}^{n_{\mathrm{L}} \times n_{\mathrm{L}}}$ is Hurwitz, all eigenvalues of $A_o \in \mathbb{R}^{n_o \times n_o}$ are at the origin, and $n_{\mathrm{L}} + n_o = n$. Accordingly, we decompose system (5.1) into the following two subsystems,

$$\begin{cases} \dot{X}_{\mathrm{L}}(t) &= A_{\mathrm{L}} X_{\mathrm{L}}(t) + B_{\mathrm{L}} U(t-\tau), \\ \dot{X}_o(t) &= A_o X_o(t) + B_o U(t-\tau), \end{cases}$$

where

$$X(t) = \begin{bmatrix} X_{\mathrm{L}}(t) & X_o(t) \end{bmatrix}^{\mathrm{T}}$$

is the corresponding decomposition of the state vector $X(t)$, $U(t)$ is constructed for the X_o subsystem by following the design of our proposed control scheme. By Theorem 5.2, the regulation of the X_o subsystem is achieved under the constructed $U(t)$. It is then clear that the regulation of the whole system is achieved.
□

5.6. Numerical examples

Consider a linear system (5.1) with

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The unknown input delay is

$$\tau = 1$$

and the initial condition of the state is given by

$$\psi(\theta) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}, \;\; \theta \in [-\tau, 0].$$

Note that (A, B) is controllable with all eigenvalues of A located at the origin. The parameters of our control scheme are chosen as

$$\alpha = 1, \;\; \beta = 1, \;\; p = 1, \;\; q = 2, \;\; \zeta = 1, \;\; r = 2, \;\; \epsilon = 1, \;\; \xi = 1.1,$$

and the initial condition of $\gamma(t)$ is set to be

$$\phi(\theta) = 0.3, \;\; \theta \in [-\tau, 0].$$

The evolutions of the closed-loop signals including $X(t)$, $U(t)$, $V(t)$, $\gamma(t)$, $\dot{\gamma}(t)$ and $\dot{\gamma}(t)/\gamma^2(t)$ are shown in Figs. 5.1-5.3.

Ample information on the mechanism of the update algorithm for $\gamma(t)$ can be observed from Figs. 5.1-5.3. With

$$\begin{aligned} V(0) &= 2.7 \\ &> \epsilon \\ &= 1, \end{aligned}$$

update law II is implemented at the beginning phase of the system evolution. A switch from update law II to update law I does not happen at

$$\begin{aligned} t &= \delta_1 \\ &= 0.3 \end{aligned}$$

because

$$V(\delta_1) > 1.$$

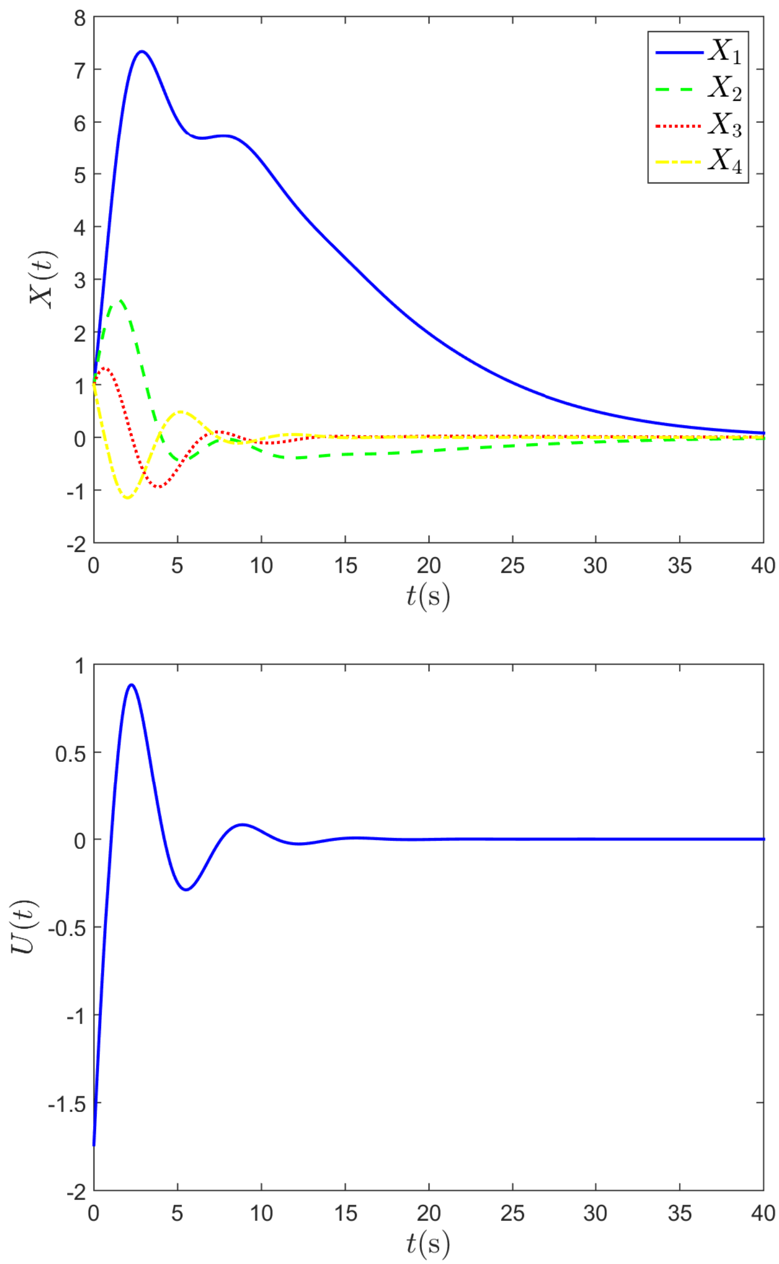


Fig. 5.1: Evolutions of the state and the input of the closed-loop system.

However, a switch from update law II to update law I happens at

$$t = \delta_2 = 0.7$$

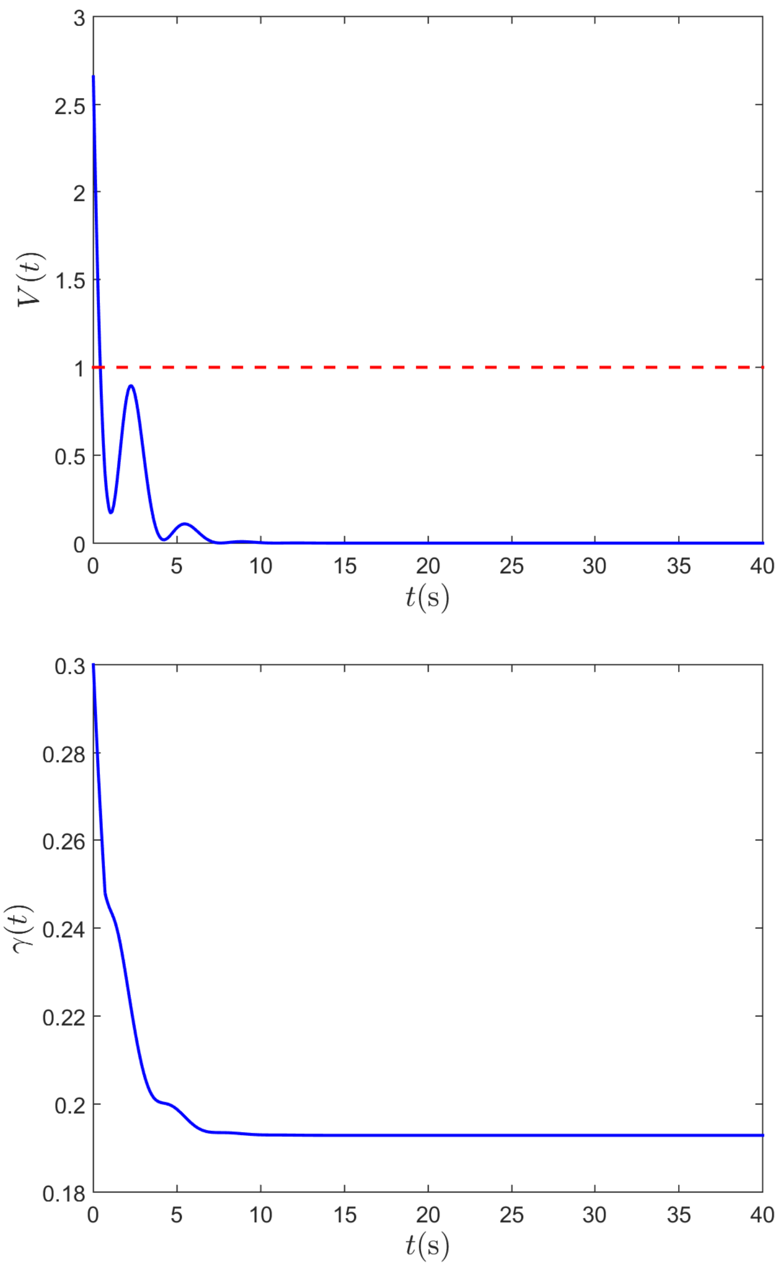


Fig. 5.2: Evolutions of the closed-loop signals $V(t)$ and $\gamma(t)$.

because

$$V(\delta_2) < 1.$$

Note from the plot of $\dot{\gamma}(t)/\gamma^2(t)$ that the first switch, which in fact is the only switch between update

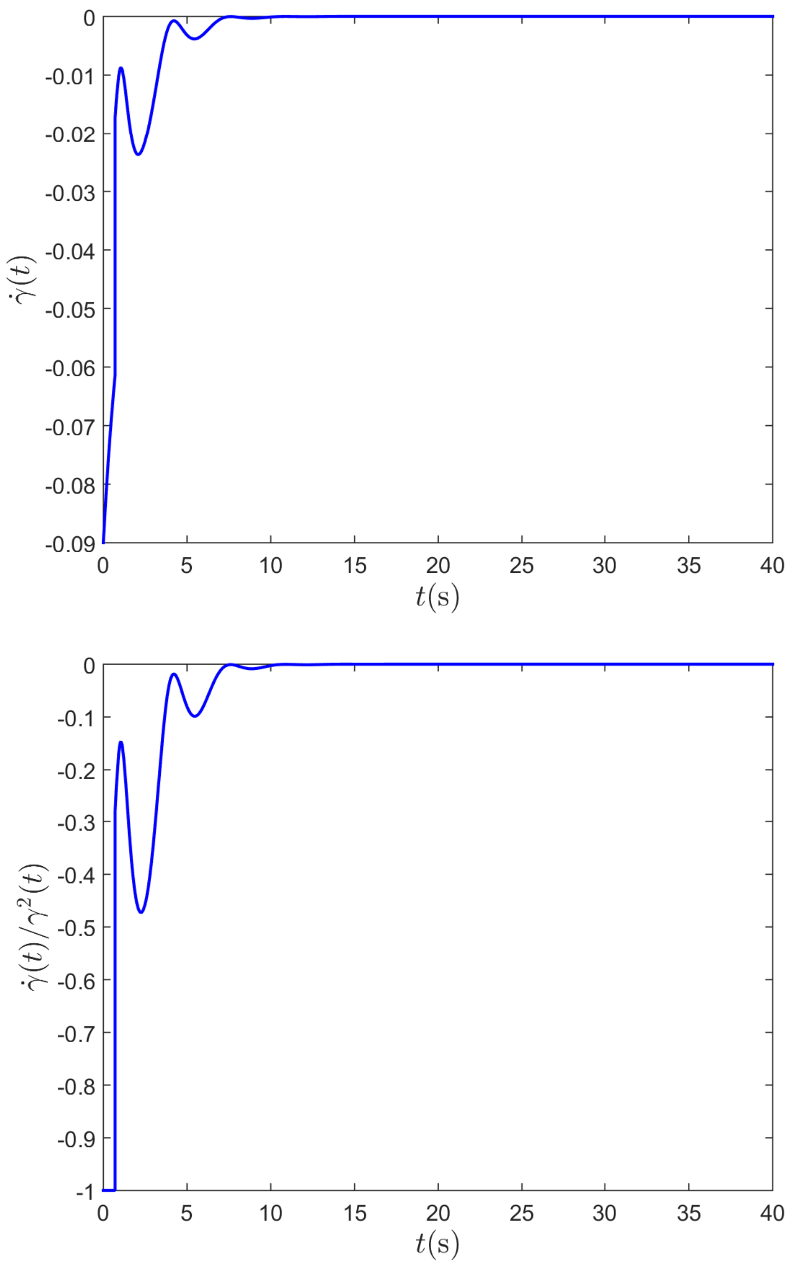


Fig. 5.3: Evolutions of the closed-loop signals $\dot{\gamma}(t)$ and $\dot{\gamma}(t)/\gamma^2(t)$.

laws I and II, happens at

$$t = \delta_2$$

by the discontinuity of $\dot{\gamma}(t)/\gamma^2(t)$ at

$$t = \delta_2.$$

After

$$t = \delta_2,$$

update law I remains in effect all the time because

$$V(t) < 1, \;\; t \geq \delta_2.$$

It is interesting to mention here that a switch from update law II to update law I does not happen at $t = 0.4$ when $V(t)$ crosses the threshold $\epsilon = 1$, as marked in the plot of $V(t)$. This is because the switching condition from update law II to update law I is checked only at the isolated time instants

$$T_{12} + \delta_i, \;\; i \in \mathbb{N},$$

where in this simulation,

$$T_{12} = 0.$$

To study the regulation performance of the control scheme, we carry out more simulation runs. In this simulation, we pick a larger

$$\gamma_0 = 1.$$

The rest of system parameters are the same as those in the previous simulation, except that

$$\epsilon = 50.$$

Figs. 5.4-5.6 illustrate the evolutions of the closed-loop signals with such a choice of γ_0. Note from the evolution of $X(t)$ that the overshoot and the convergence time of $X(t)$ are larger than those in the previous simulation. This is caused by the excessively large value of γ_0, which tends to destabilize the system at the starting phase of system evolution. The feedback law (5.4) starts to stabilize the system only after $\gamma(t)$ decreases to a small value. Also, we observe from the evolution of $\dot{\gamma}(t)/\gamma^2(t)$ that $\gamma(t)$

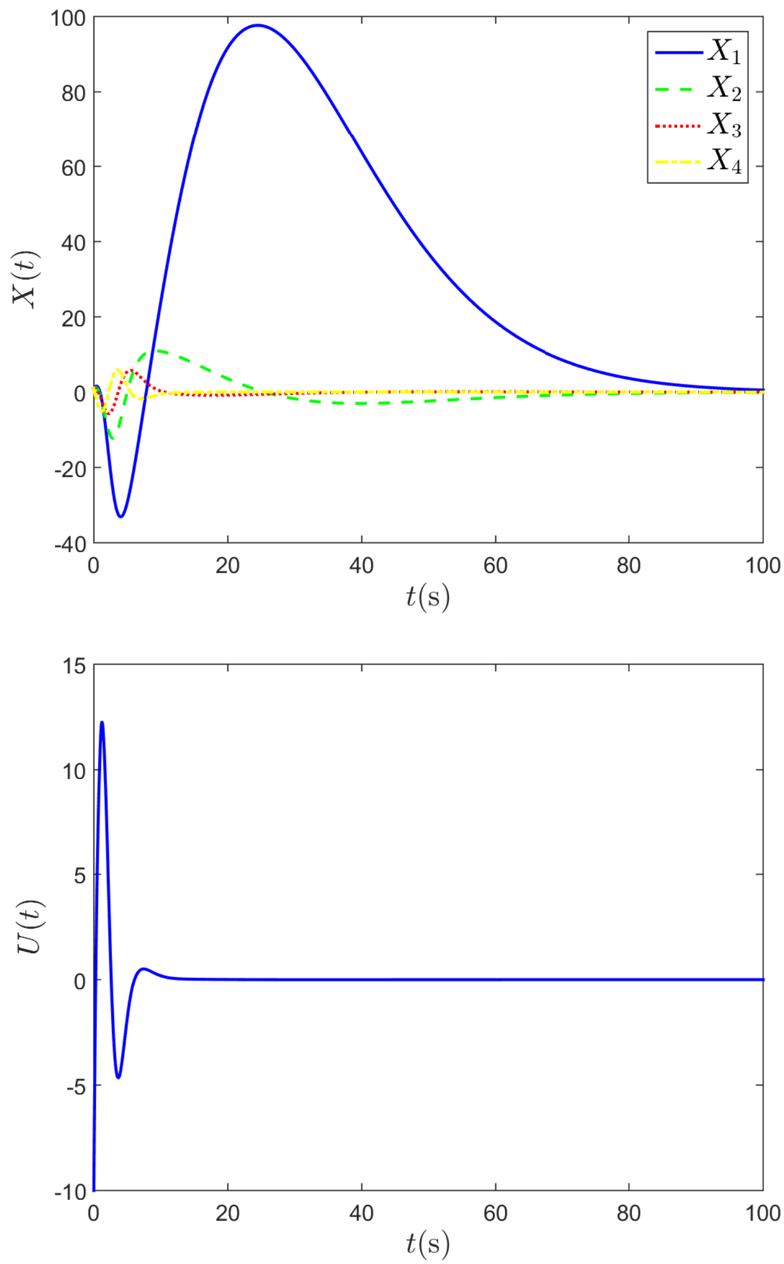


Fig. 5.4: Evolutions of the state and the input of the closed-loop system.

is updated by update law II only on the time interval $t \in [0.8135, 2.0154]$.

The regulation performance of the control scheme can also be examined under larger values of τ. Consider a closed-loop system whose system parameters are the same as those in the first simulation except that $\tau = 5$ and $\epsilon = 10$. Picking a larger ϵ is due to the expectation of a large overshoot of the closed-loop system in the presence of a large τ. The closed-loop evolution is presented in Figs.

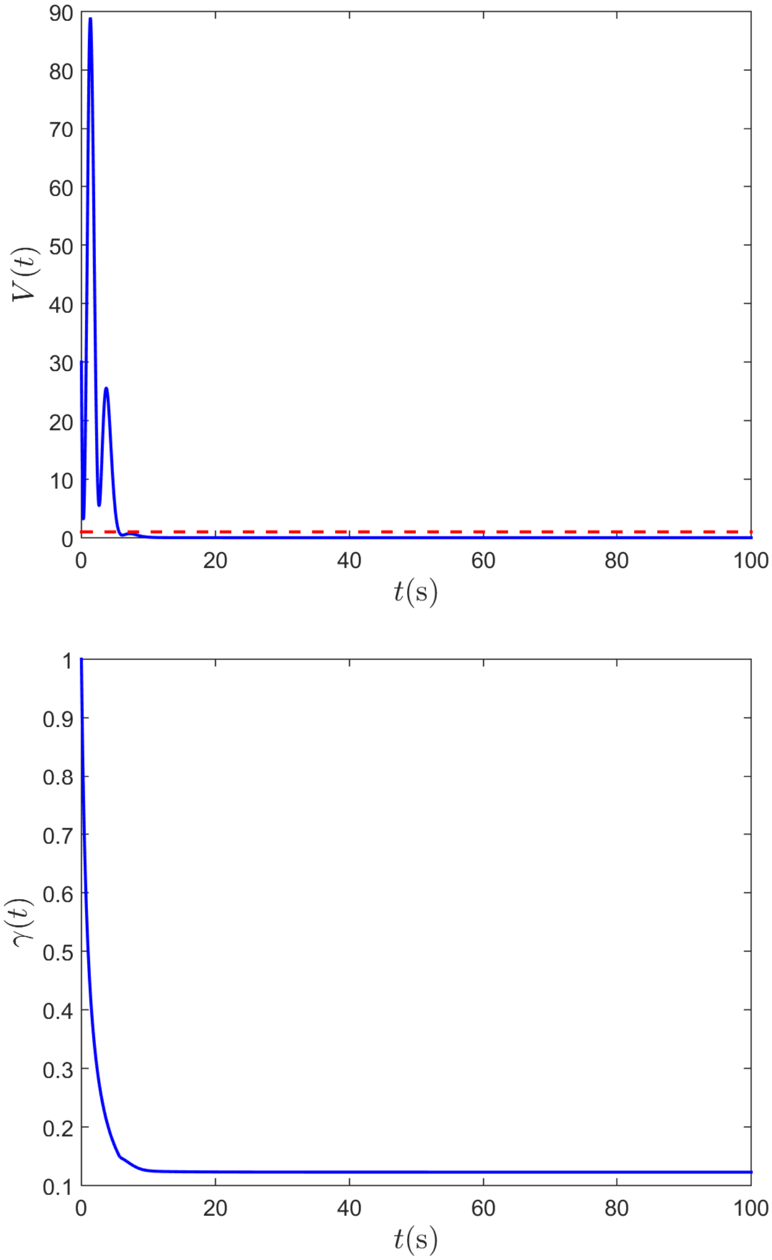


Fig. 5.5: Evolutions of the closed-loop signals $V(t)$ and $\gamma(t)$.

5.7-5.9. Consider an even larger $\tau = 10$ and an $\epsilon = 10$. The simulation results are presented in Figs. 5.10-5.12. Note that, as the value of τ increases, the regulation effects of the control scheme becomes weaker, resulting in a larger overshoot and a slower convergence rate of the closed-loop system. This can be easily explained by the low gain nature of the feedback law (5.4). Smaller values of the feedback parameter increase the ability to achieve regulation at the cost of slower convergence rate of the closed-

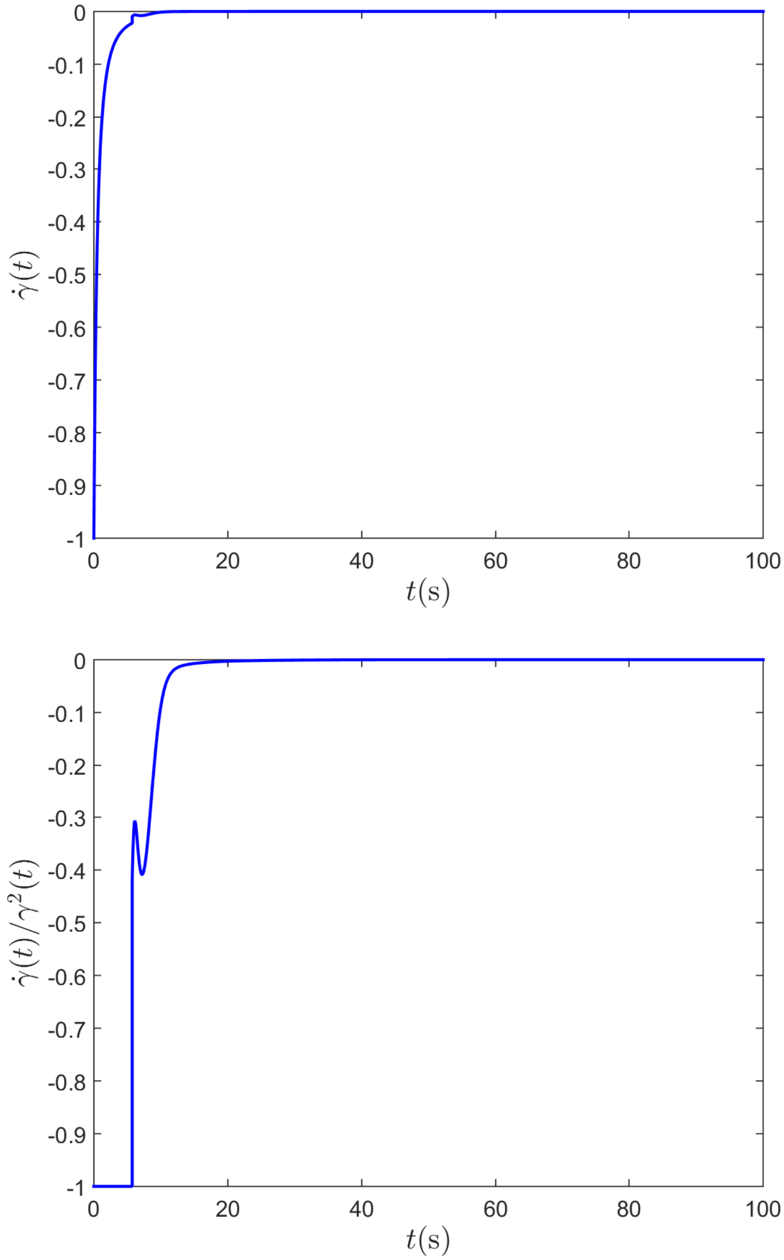


Fig. 5.6: Evolutions of the closed-loop signals $\dot{\gamma}(t)$ and $\dot{\gamma}(t)/\gamma^2(t)$.

loop system. We also note that the computation of our control scheme is heavier in comparison with the delay independent truncated predictor feedback law (5.2) with a constant feedback parameter due to the adaptation of the time-varying feedback parameter. In particular, the real-time solution of the algebraic Riccati equation (5.3) contributes significantly to the computational burden.

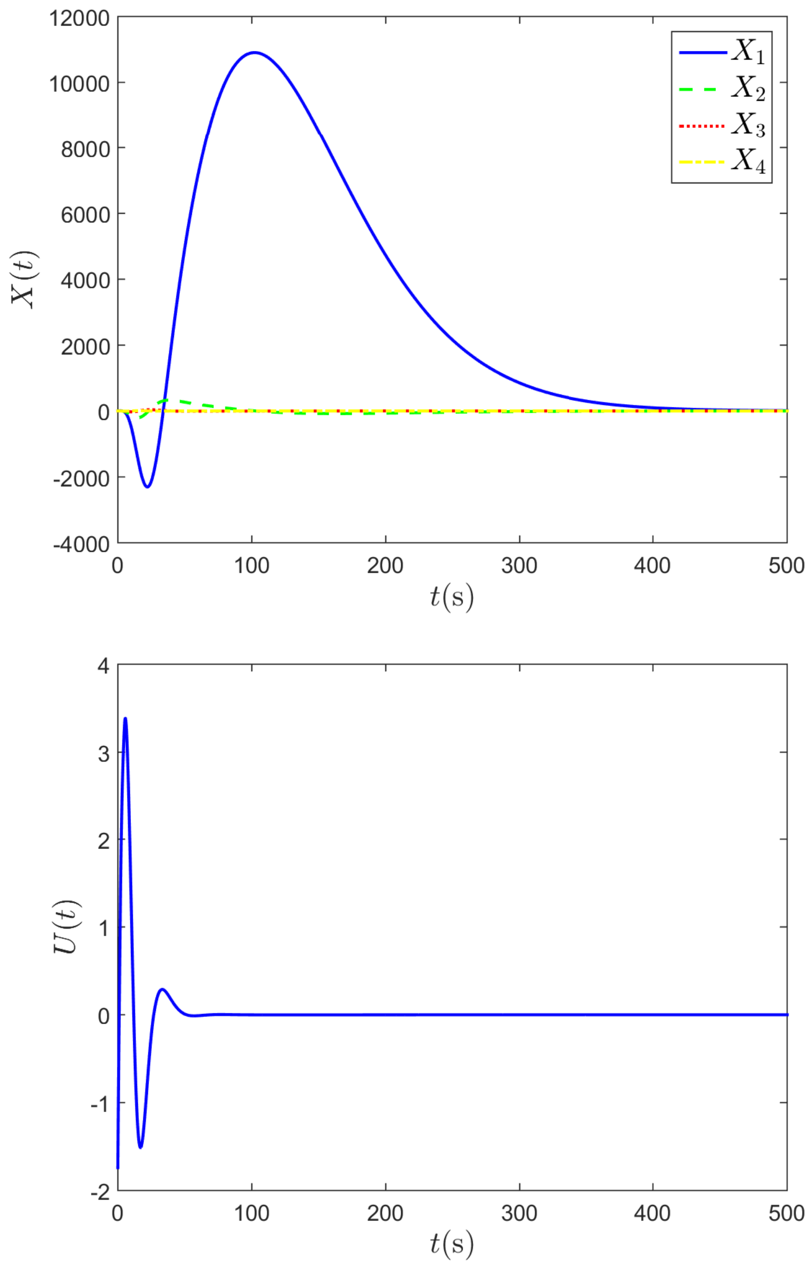


Fig. 5.7: Evolutions of the state and the input of the closed-loop system.

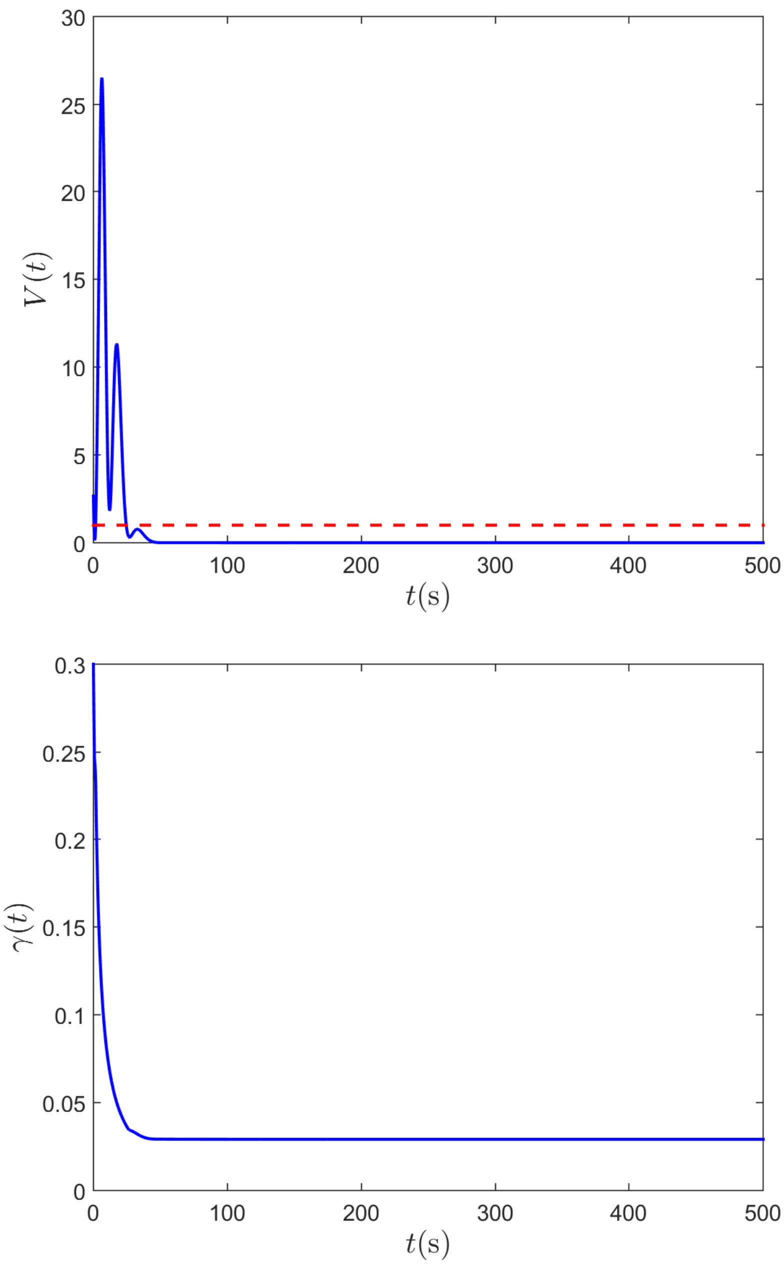


Fig. 5.8: Evolutions of the closed-loop signals $V(t)$ and $\gamma(t)$.

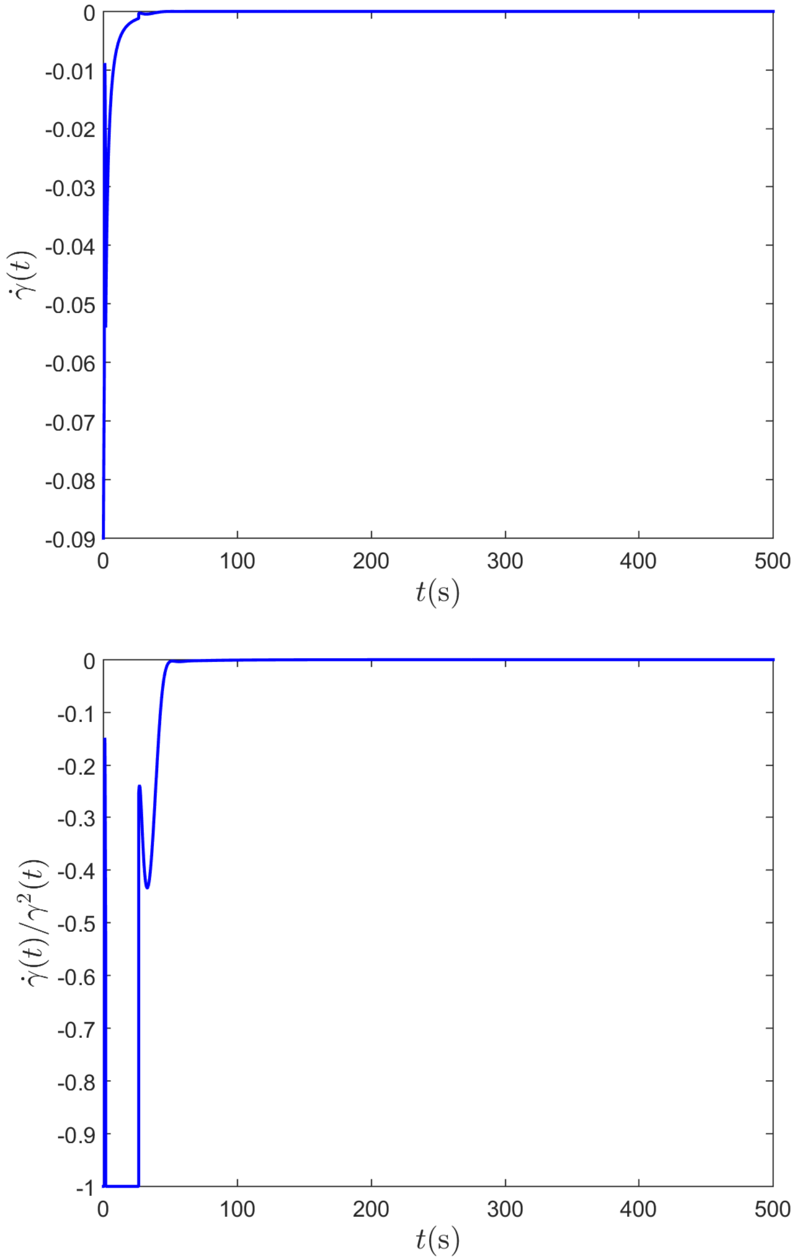


Fig. 5.9: Evolutions of the closed-loop signals $\dot{\gamma}(t)$ and $\dot{\gamma}(t)/\gamma^2(t)$.

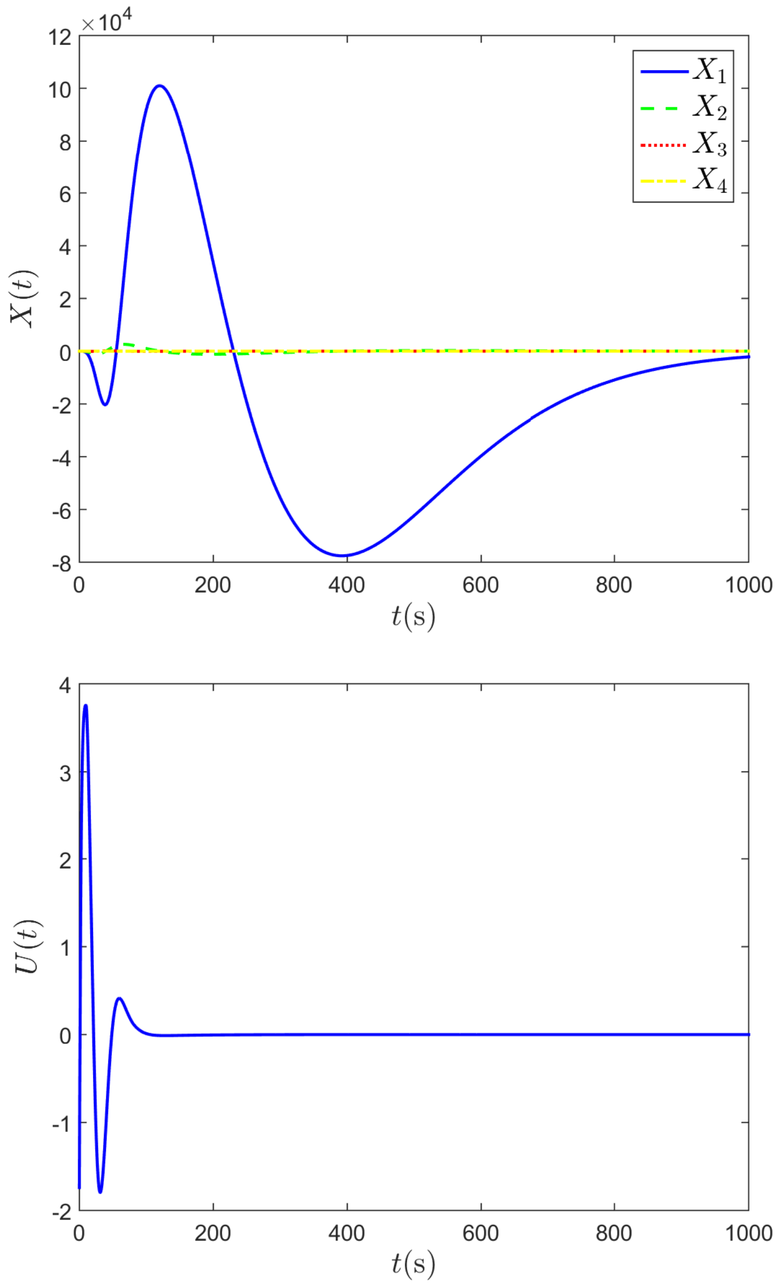


Fig. 5.10: Evolutions of the state and the input of the closed-loop system.

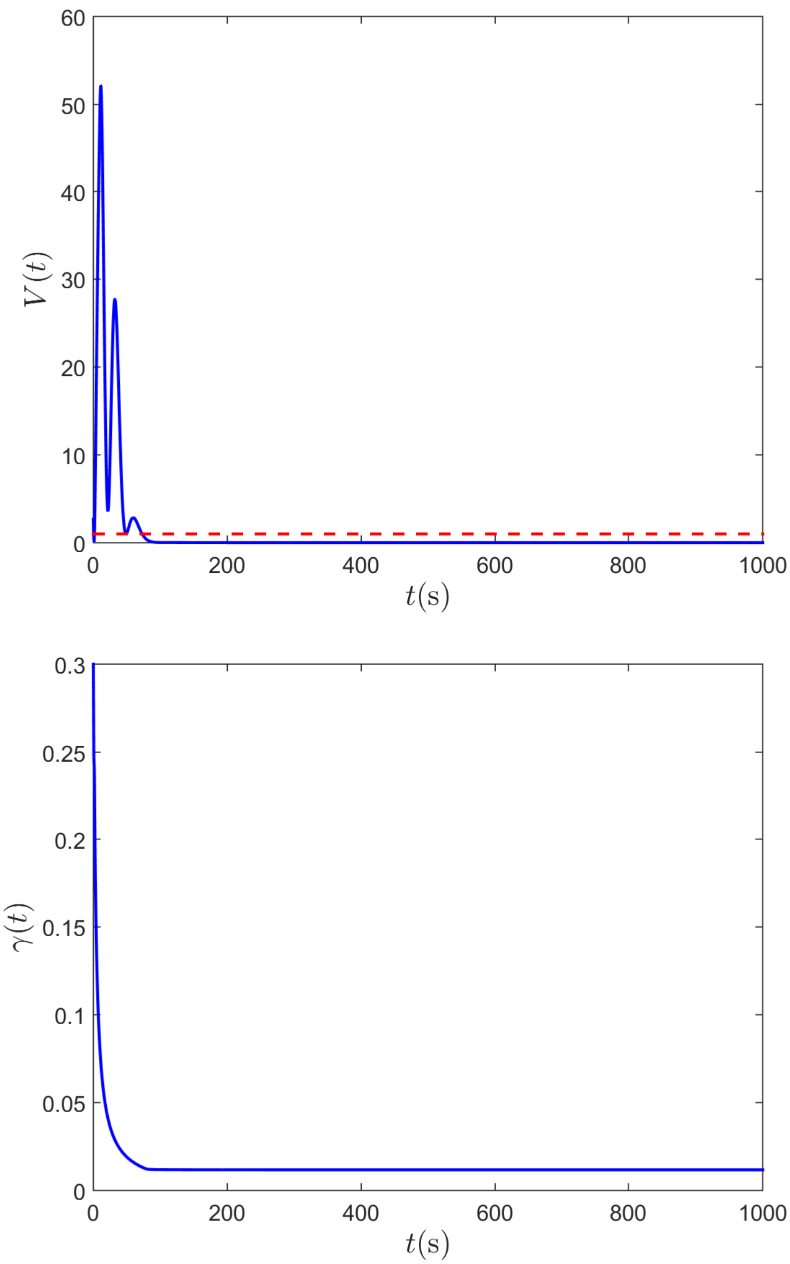


Fig. 5.11: Evolutions of the closed-loop signals $V(t)$ and $\gamma(t)$.

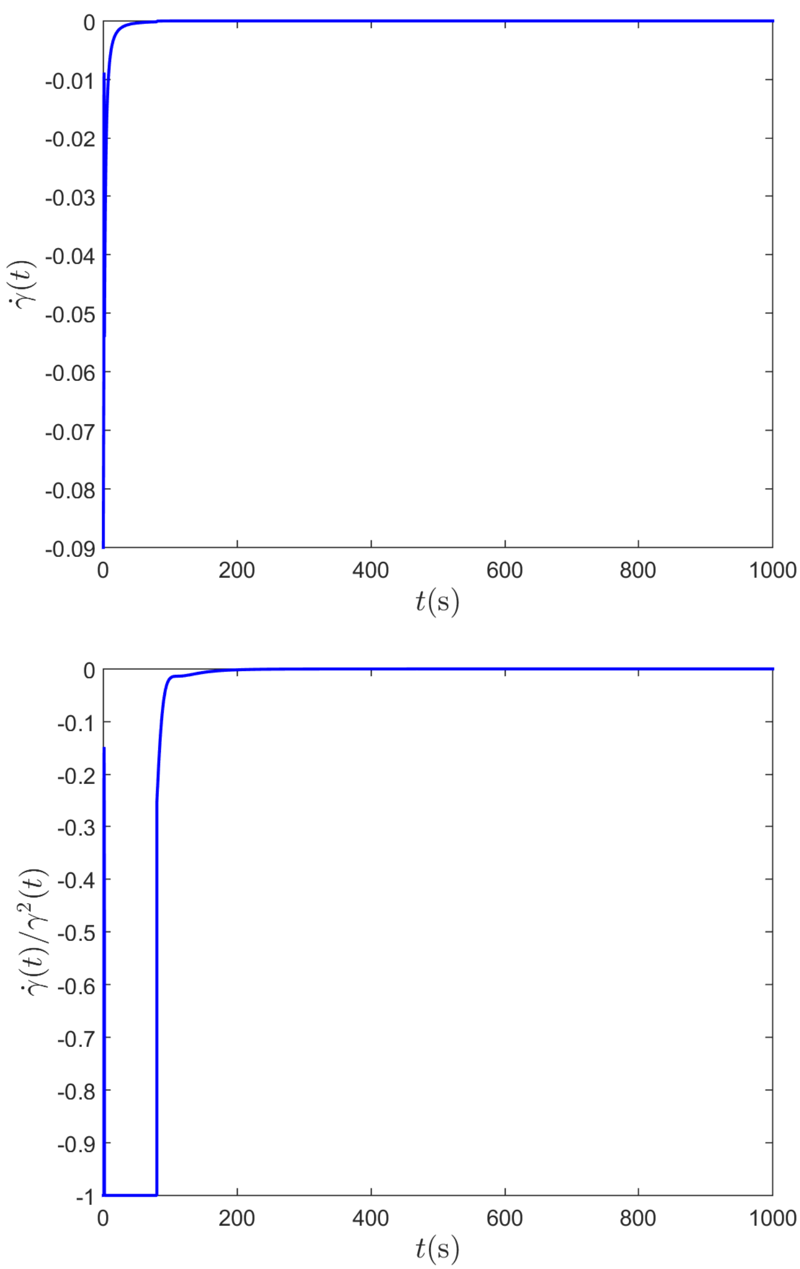


Fig. 5.12: Evolution of the closed-loop signals $\dot{\gamma}(t)$ and $\dot{\gamma}(t)/\gamma^2(t)$.

Remark 5.11. *As seen in Section 5.2, the update algorithm for $\gamma(t)$ provides a parameter space in which we can choose the values of $(\alpha, \beta, p, q, \zeta, r, \gamma_0)$ freely to achieve the regulation of the system. Given an open loop system, the analysis on the closed-loop performance with respect to the choice of the parameters in the space remains to be carried out.* $\square$

5.7. Summary

For an input delayed linear system with open loop poles at the origin or in the open left-half plane, the regulation of its state and control input is achieved without any knowledge of the delay. This is made possible by the adaptation of the delay independent truncated predictor feedback law with a time-varying feedback parameter. An update algorithm for the feedback parameter is proposed to compensate an arbitrarily large unknown delay. The use of only the current state as the feedback contributes to the non-distributed nature of the control scheme. A limitation of the results in this section is the restriction of the open loop systems to those with poles at the origin or in the open left-half plane. Generalization of the method for the regulation of more general systems entails further investigation.

6. Conclusions and Future Work

This dissertation covers novel feedback designs for the control of linear systems with input delay. Our feedback methods have their origin in the truncated predictor feedback, and are capable of solving control problems not limited to stabilization, but also including improving closed-loop performance and accommodating unknown delay. The key of achieving these is a novel design of the feedback parameter in our truncated predictor based feedback laws. The value of the parameter is devised to vary with respect to time, no matter such a variation is an off-line design or an online update.

The regulation of a general linear system without any delay knowledge is a challenging problem that has puzzled the community of control theory to date. Basically, in order to cancel the effect of an arbitrarily large delay in a control system, one has to know the amount of the delay for determining the right amount of control effort applied to the system. This is exactly how the predictor feedback ensures closed-loop stability. Thus, the knowledge of the delay is critical to the design of any control law that compensates delay. In the absence of any delay knowledge, an intuitive thought would be estimating the amount of the delay according to closed-loop performance, and then resorting to existing control methods to generate the right amount of control effort. Following this line of thought, we plan to modify the truncated predictor feedback law through equipping it with an update law for estimating the amount of the delay that appears in its exponential factor. The success of devising such an adaptive control scheme would give hope to solving the regulation problem for linear systems with all open loop poles in the closed left-half plane. A bigger picture where the problem can be solved for general linear systems with possibly exponentially unstable open loop poles remains unclear at this time, and we expect any achievement in the literature succeeding this dissertation.

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