Local and Quasi-Local $\mathfrak{sl}_2$ Link Homology

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Abstract

We study algebraic structures in the $\mathfrak{sl}_2$ link homology theories as defined by Khovanov [Kho00] and Bar-Natan [BN05] and extended by Cooper-Krushkal [CK12]. New duality results are applied to the projector P_n which governs the local behavior of these link invariants. As an application, we construct an action of the polynomial ring $\mathbb{Z}[u_1, \ldots, u_n]$ on P_n and prove that the colored $\mathfrak{sl}_2$ link homology is finitely generated over a tensor product of such rings. Replacing P_n by the associated Koszul complex $\mathbb{Z} \otimes_{\mathbb{Z}[u_1,\ldots,u_n]} P_n$ gives a categorification of the $\mathfrak{sl}_2$ Reshetikhin-Turaev invariant (up to normalization) via bounded chain complexes. The invariant is quasi-local, i.e. extends to tangles and respects gluing up to taking many direct sum copies. We conjecture that our link invariant is functorial under link cobordisms (up to sign), and as evidence for this we show that the intermediate chain complexes $Q_n = \mathbb{Z} \otimes_{\mathbb{Z}[u_n]} P_n$ are Frobenius algebra objects in an appropriate monoidal category. Combining these results allows us to simplify the endomorphism algebra $\mathrm{End}^\bullet(P_n)$, lending credence to recent conjectures of Gorsky, Oblomkov, Rasmussen, and Shende [GOR12, GORS12, OS12, ORS12].

Chapter 1

Introduction

One of the main reasons for growing interest in homological link invariants over the last decade is that it is expected that they should have applications to 4-dimensional topology. An essential ingredient in such applications is functoriality of the link homology theory under surface cobordisms in $S^3 \times [0,1]$; indeed, precisely this sort of functoriality of Khovanov homology allowed Rasmussen [Ras10] to prove Milnor's conjecture on the 4-ball genus of torus knots, a result which was previously only known using gauge theory. Unfortunately, most categorifications of Reshetikhin-Turaev link invariants associate infinite total rank homology groups to even the simplest knots, and so cannot be functorial under link cobordisms for formal reasons. Roughly speaking, the problem occurs when the categorified invariants are defined locally (i.e. by defining them first on tangles), in which case infinite chain complexes are necessary to categorify the denominators which appear in the Reshetikhin-Turaev tangle invariant.

This thesis is motivated by the desire to understand, and perhaps control, the infinity of such categorifications of the Reshetikhin-Turaev invariants. To this end, on one hand we show that colored $\mathfrak{sl}_2$ link homology theory of Cooper and Krushkal [CK12] is finitely generated over a certain canonically defined action of a polynomial algebra. One the other hand, we construct a new categorification of a (normalized version of) the $\mathfrak{sl}_2$-Reshetikhin-Turaev invariant via bounded chain complexes. The new invariant provides a categorical analogue of clearing denominators. It extends to tangles, but respects gluing of tangles only up to taking many direct sum copies.

The relationship between the two invariants is essentially Koszul duality between modules over polynomial and exterior algebras. The construction reveals many new symmetries in Cooper-Krushkal colored $\mathfrak{sl}_2$ link homology, allowing us to address fundamental questions about the structure of this invariant. The methods used to address the $\mathfrak{sl}_2$ case are expected to generalize to $\mathfrak{sl}_n$ (colored by the symmetric powers of the standard representation, i.e. the one-row partitions), as well as to other homology theories for which such a local presentation is available (see [FSS12, Roz10, Ros12]).

1.1 Compromising locality

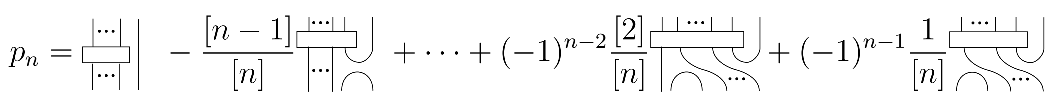
One of the major themes of this thesis is the need to compromise the notion of locality for the categorified Reshetikhin-Turaev invariants, which we now explain. Firstly, an invariant of links $L \in S^3$ is said to be *local* if it extends to a functor from the category of tangles, i.e. links with boundary. The most important example in this thesis is the $\mathfrak{sl}_2$ Reshetikhin-Turaev link invariant—also called the colored Jones polynomial—defined in §3.2, which is actually a $\mathbb{Z}[q, q^{-1}]$ valued invariant of framed, oriented links $L \subset S^3$ whose components are labelled by non-negative integers, called the colors. This invariant is local in the appropriate sense, i.e. it extends to a functor $P : \mathscr{C} \to \mathrm{TL}$, where

- $\mathscr{C}$ is the category of colored, framed, oriented tangles in $\mathbb{R}^2 \times [0, 1]$. An object of $\mathscr{C}$ is a possibly empty sequence of $(n_1, \ldots, n_r)$ of positive integers, and a morphism is a colored, framed, oriented tangle $(T, \partial T) \subset (\mathbb{R}^2 \times [0, 1], \mathbb{R}^2 \times \{1, 0\})$.

- TL is the *Temperley-Lieb category*. Objects of TL are non-negative integers, and the set TL_n^m of morphisms $n \mapsto m$ is defined to be the $\mathbb{C}(q)$-vector space generated by properly embedded 1-submanifolds of the rectangle $[0, 1]^2$ with boundary equal to a standard set of m points on the "top" $[0,1]\times\{1\}$ of the rectangle and n points on the "bottom" $[0,1]\times\{0\}$. Here $\mathbb{C}(q)$ is the field of rational functions in an indeterminate q. We regard the generators modulo planar isotopy and the relation $D\sqcup U=(q+q^{-1})D$, where U is a circle disjoint from the rest of the diagram. Composition of morphisms is given composition of planar tangles. Note that TL_n^m has a $\mathbb{C}(q)$ basis given by tangles without any circle components. We will call such a basis element a *Temperley-Lieb diagram.*

- the functor $P:\mathscr{C}\to\mathrm{TL}$ acts on objects as $(n_1,\dots,n_r)\mapsto n_1+\cdots+n_r$. In particular the image of a colored, framed, oriented tangle $T:(n_1,\dots,n_r)\to(m_1,\dots,m_s)$ is an element $P(T)\in\mathrm{TL}_{n_1+\cdots n_r}^{m_1+\cdots+m_s}$.

Now, let 1_n denote the trivial n-colored arc, regarded as a morphism $n\to n$ in $\mathscr{C}$. Clearly $1_n\circ 1_n=1_n$. By locality (functoriality) we have $P(1_n)\circ P(1_n)=P(1_n)$. That is to say, $p_n:=P(1_n)$ is an idempotent element of $\mathrm{TL}_n:=\mathrm{TL}_n^n$, called the *Jones-Wenzl projector.* The p_n can be defined recursively by (1) $p_1=1$ is the multiplicative identity of TL_1, and (2):



(1.1)

where $[k]=\frac{q^k-q^{-k}}{q-q^{-1}}$ is the quantum integer and a white box denotes p_{n-1}. It follows that the tangle invariant $P(T)\in\mathrm{TL}_n^m$ has rational coefficients with respect to the standard basis of TL_n^m given by Temperley-Lieb diagrams.

Denominators typically complicate the program of categorification; for example, to category the recursion (1.1) it is necessary for Cooper-Krushkal to expand the denominators in power series in positive powers of q, resulting in a link homology theory which associates infinite rank homology groups to even the simplest knots (e.g. the unknots colored by $n\geq 2$). It is well known that such infiniteness obstructs functoriality of the Cooper-Krushkal invariant under link cobordisms. The question now arises of how to deal with the denominators. As we see it, there are two ways one may remove the denominators in the $\mathfrak{sl}_2$ Reshetikhin-Turaev invariant:

1. Restrict to tangles without boundary, i.e. links. In this case $P(L) \in \mathbb{Z}[q, q^{-1}] \subset \mathbb{C}(q) \cong \mathrm{TL}_0$ is actually a polynomial, and there are no denominators to worry about.

2. Clear the denominators. It is not hard to see from the recursion (1.1) that $\prod_{1\leq k\leq n}(1-q^{2k})p_n \in \mathrm{TL}_n$ is a polynomial linear combination of Temperley-Lieb diagrams, hence a similar normalization of the $\mathfrak{sl}_2$ Reshetikhin-Turaev invariant will be polynomial as well.

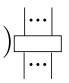
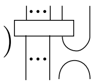
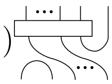
This suggests two potential approaches for obtaining a categorification of the $\mathfrak{sl}_2$ Reshetikhin-Turaev invariant with the required finiteness properties: (1) ignore tangles altogether and attempt to categorify the colored Jones polynomial for links via bounded chain complexes, and (2) find a categorical analogue of clearing denominators. The approach (1) was carried out by Khovanov in [Kho05]. However, it is not known whether the resulting colored $\mathfrak{sl}_2$ link homology can be made (projectively) functorial under link cobordisms, although this idea has been pursued in [Weh07]. Rather, in this thesis we propose to follow the approach of (2).

1.2 Clearing denominators for the categorified Jones-Wenzl idempotent

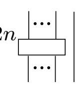
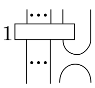
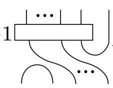
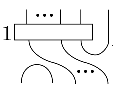
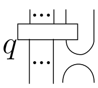
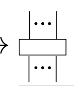
The colored $\mathfrak{sl}_2$ link homology considered in this thesis is that due to Cooper-Krushkal [CK12], and takes place in Bar-Natan's setting [BN05] for Khovanov homology [Kho00]. In §3.3 we recall Bar-Natan's tangle categories $\mathcal{TL}_n^m$ and the bilinear functor $\odot$: $\mathcal{TL}_k^m \times \mathcal{TL}_n^k \to \mathcal{TL}_n^m$ given by composition of tangles. In §3.5-3.6 we recall the Cooper-Krushkal categorification of the Jones-Wenzl projectors $p_n \in \mathrm{TL}_n$, which are semi-infinite chain complexes over $\mathcal{TL}_n$.

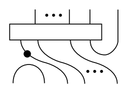
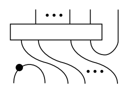
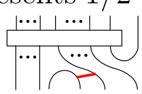
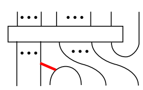
Recall the Temperley-Lieb algebra TL_n ($n \in \mathbb{Z}_{\geq 0}$) defined in the previous section, as well as the recursion (1.1) which the Jones-Wenzl projectors $p_n \in \mathrm{TL}_n$ satisfy.

Multiplying both sides of (1.1) by $(1-q^{2n})$ gives

$(1-q^{2n})p_n = (1-q^{2n})$  $-\,(q-q^{2n-1})$  $+\cdots+(-1)^{n-1}(q^{n-1}-q^{n+1})$ 

Now, interpreting each term as a chain complex over Bar-Natan's category $\mathcal{TL}_n = \mathcal{TL}_n^n$, q as the grading shift functor and the minus sign as an object lying in odd homological degree, one is naturally led to the following sequence $E_\bullet$ of chain complexes and chain maps which is an analogue of the right-hand side above:

q^{2n}  $\to q^{2n-1}$  $\to\cdots\to q^{n+1}$  $\to q^{n-1}$  $\to\cdots\to q$  $\to$  (1.2)

Here, the white box denotes a Cooper-Krushkal projector P_{n-1}, and the maps between adjacent terms are (1)  $-$  between the two middle terms, where the dot represents 1/2 times the cobordism given by a punctured torus, and (2) saddle cobordisms  or  between all other terms. The composition of successive maps $d_{i+1,i}$ is nonzero, but satisfies

$$d_{i+2,i+1}\circ d_{i+1,i}\simeq 0 \tag{1.3}$$

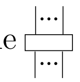
as a chain map $E_i\to E_{i+2}$. That is, $E_\bullet$ represents a chain complex over the homotopy category of chain complexes, in which the underlined term is regarded as the degree zero chain group. In order to define an honest chain complex from (1.2) it is necessary to introduce higher length components $d_{ij}\in \mathrm{Hom}^{1-i+j}(E_j,E_i)$ $(i\geq j)$ of the differential such that $d_{ii} := (-1)^i d_{E_i}$, $d_{i+1,i}$ are the chain maps already defined, and

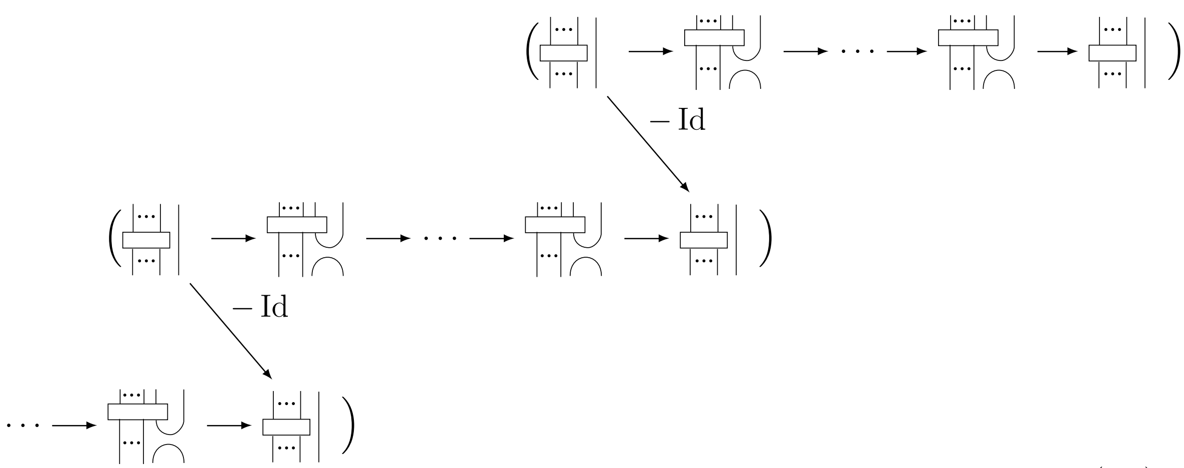
$$\Big(\sum_{i\geq j} d_{ij}\Big)^2 = 0$$

The existence of the "length-two" component $d_{i+2,i}$ is implied by (1.3) and, in general, $d_{i+k,i}$ exists only if the higher Massey product of the length-one components $d_{j+1,j}$ vanishes in $\mathrm{Ext}^{2-k}(E_i,E_{i+k})$ (chain maps $t^{2-k}E_i\to E_{i+k}$ modulo homotopy). We call a chain complex obtained in this way a *convolution* of the homotopy chain complex

$E_\bullet$, following standard terminology for the flattening of a chain complex over a triangulated category [Kap88]. We call any convolution of (1.2) a *symmetric projector.* In §7.1 we prove:

Theorem 1.4. *For each integer $n \geq 2$ there exists a unique convolution $Q_n \in \mathrm{Kom}(n)$ of the sequence (1.2) up to homotopy equivalence.*

There is a chain map $\partial_n : t^{2n-1}q^{-2n}Q_n \to Q_n$ given by (minus) the projection-followed-by inclusion of the  summand, and P_n is homotopy equivalent to $\mathbb{Z}[u_n] \otimes Q_n$ with differential $1 \otimes d_{Q_n} + u_n \otimes \partial_n$ where u_n is a formal indeterminate of bidegree $(2-2n, 2n)$. In other words, P_n is homotopy equivalent to the following chain complex:



(1.5)

in which each row is Q_n and we are omitting all of the grading shifts because of space limitations. We thus obtain an attractive expression of P_n as a periodic chain complex built from Q_n. This reduces the complexity in computing the higher differentials appearing in the Cooper-Krushkal recursion to the apparently much simpler problem of computing the higher differentials required to obtain an actual chain complex from (1.2).

We can now sketch the construction of our quasi-local $\mathfrak{sl}_2$ link homology. The above expression of P_n as a periodic chain complex can be used to construct an action of $\mathbb{Z}[u_1, \ldots, u_n]$ on P_n, where u_k is an indeterminate of homological degree

$2-2n$ and q-degree $2n$. The generators of $\mathbb{Z}[u_1,\ldots,u_n]$ correspond to chain maps $U_k^{(n)}: t^{2-2k}q^{2k}P_n \to P_n$ for $1 \leq k \leq n$, where t and q denote the upward grading shift functors in homological, respectively q-degree. For any sequence of integers $\mathbf{i} = (i_1,\ldots,i_r)$ with $1 \leq i_1,\ldots,i_r \leq n$, define a chain complex:

$$P_n(\mathbf{i}) := \mathrm{Cone}(U_{i_1}^{(n)}) \odot \cdots \odot \mathrm{Cone}(U_{i_r}^{(n)})$$

We use the convention that if $\mathbf{i} = \varnothing$ is the empty sequence, then $P_n(\mathbf{i}) = P_n$. Since P_n is idempotent up to homotopy it is reasonable to refer to $P_n(\mathbf{i})$ as the *Koszul complex* associated to the sequence $u_{i_1},\ldots,u_{i_r}$. Note that $P_n(\mathbf{i}) \odot P_n(\mathbf{j}) \simeq P_n(\mathbf{i}\cdot\mathbf{j})$, where $\mathbf{i}\cdot\mathbf{j}$ denotes the concatenation of sequences (this equivalence is an isomorphism unless $\mathbf{i}$ or $\mathbf{j}$ is empty). We summarize some of the basic properties of the $P_n(\mathbf{i})$ in the following theorem, which is proven in Chapter §7.

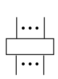
Theorem 1.6. *We have*

1. *$P(2,3,\ldots,n)$ is homotopy equivalent to a bounded chain complex.*

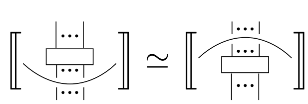
2. *The ordering of indices is irrelevant up to homotopy: $P_n(i,j) \simeq P_n(j,i)$ for all i,j.*

3. *Quasi-idempotency: $P_n(i,i) \simeq (1+t^{1-2i}q^{2i})P_n(i)$.*

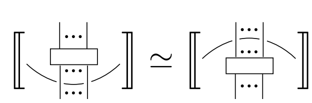
4. *Each $P_n(\mathbf{i})$ has the symmetries of the rectangle. That is, if $g: \mathcal{TL}_n \to \mathcal{TL}_n$ is a covariant functor given by reflection of diagrams across the vertical or horizontal axis, then $g(P_n(\mathbf{i})) \simeq P_n(\mathbf{i})$. In particular,*

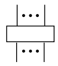
   where  $= P_n(\mathbf{i})$.

5. *For each sequence $\mathbf{i}$, the complex $P_n(\mathbf{i})$ can be slid past strands up to homotopy equivalence:*



and



where  $= P_n(\mathbf{i})$ *and* $[\![$  $]\!]$ *denotes the chain complex associated to the crossing in Bar-Natan's [BN05] extension of Khovanov homology to tangles.*

Thus, replacing the projector $P_n = P_n(\varnothing)$ with the complex $P_n(2, 3, \ldots, n)$ in the definition of the Cooper-Krushkal $\mathfrak{sl}_2$-link invariant (and leaving $P_1 = 1_1$ unchanged) gives a categorification of the $\mathfrak{sl}_2$ Reshetikhin-Turaev invariant (up to normalization) via bounded chain complexes. The new invariant extends to tangles, but only respects gluing up to taking many direct sum copies.

1.3 Outline of the thesis

In Chapter 2 we set up some basic framework on convolutions, deformation retracts, and differential graded categories which will be used throughout.

In Chapter 3 we recall the definition of Bar-Natan's categories $\mathcal{TL}_n^m$ and the Cooper-Krushkal construction of colored $\mathfrak{sl}_2$ link homology.

In Chapter 4 we use a notion of duality in Bar-Natan's categories to give a graphical description of Hom complexes between chain complexes over $\mathcal{TL}_n^m$. We use the Cooper-Krushkal axioms to develop a calclus for simpifying Hom complexes between planar compositions of Cooper-Krushkal projectors, and apply the calculus to several examples.

In Chapter 5 we apply the aforemention calculus to the study of the Ext algebra of P_m, which describes the local behavior of the Cooper-Krushkal invariant. We show that this algebra is graded commutative, and study its action on planar compositions of P_n's for various n.

In Chapter 6 we relate the existence of a polynomial action on P_n to the existence of certain symmetric chain complexes Q_n, which we call symmetric projectors. Assuming the existence of the Q_n we establish several structural properties of P_n culminating in an attractive expression for P_n as a certain periodic chain complex built out of the product $C'_n \simeq (Q_1 \sqcup 1_{n-1}) \odot \cdots \odot Q_n$. We then use this periodic expression to simplify the chain complex $\mathrm{End}^{\bullet,\bullet}(P_n)$ as a dg $\mathbb{Z}[u_1,\ldots,u_n]$-module and give a partial result toward a conjecture in [GOR12] on the structure of the $\mathfrak{sl}_2$ homology on the n-colored unknot.

In Chapter 7 we use results from Chapter 6 to inductively construct Q_n; we then study properties of Q_n, showing that the chain complex is quasi-idempotent, and is a Frobenius algebra object in the monoidal category of (the homotopy category of) chain complexes fixed by $P_{n-1} \sqcup 1_1$. We then assemble the Q_n to construct the quasi-local $\mathfrak{sl}_2$ link homology mentioned in the title. We conclude with some remarks on functoriality of this invariant.

Chapter 2

Homological algebra preliminaries

In this chapter we introduce some basic algebraic notions such as convolutions and deformation retracts. Theorems 2.10 and 2.15 provide the some of the main technical tools in this thesis. We include their proofs here, because we do not know of elsewhere in the literature where one can find these results as stated. The reader may wish to read the main definitions 2.2, 2.3, and 2.12, and then refer to this section only as needed.

Call a category $\mathscr{A}$ $\mathbb{Z}$-*linear* if the morphism spaces are abelian groups and composition is bilinear and *additive* if, in addition, $\mathscr{A}$ is closed under finite direct sums (equivalently direct products). For a $\mathbb{Z}$-linear category, let $\mathrm{Kom}(\mathscr{A})$ denote the category of potentially unbounded chain complexes over $\mathscr{A}$ with differentials of degree $+1$, with morphisms given by degree zero chain maps.

Definition 2.1. For chain complexes $(A^\bullet, d_A)$, $(B^\bullet, d_B)$ over any $\mathbb{Z}$-linear category $\mathscr{A}$ define the *hom complex* to be the chain complex $\mathrm{Hom}^\bullet_{\mathscr{A}}(A,B)$ of homogeneous multi-maps $A \to B$ with differential given by the super-commutator. The homological degree k piece is

$$\mathrm{Hom}^k_{\mathscr{A}}(A,B) = \prod_{i\in\mathbb{Z}}(\mathrm{Hom}(A^i, B^{i+k}))$$

and the differential sends $f \mapsto [d,f] := d_B \circ f - (-1)^{|f|} f \circ d_A$. Here and throughout we use $|f| = \deg_h(f) \in \mathbb{Z}$ to denote the homological degree of a homogeneous map.

In this thesis, most of the additive categories are graded, and come equipped with a grading shift functor $q : \mathscr{A} \to \mathscr{A}$. Even though all of the arrows $A \to B$ will be assumed to be homogenous of q-degree zero, it is convenient to consider maps of arbitrary degree. This is accomplished with the following:

Definition 2.2. Let $\mathrm{Hom}^{\bullet,\bullet}_{\mathscr{A}}(A,B) := \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}^{\bullet}_{\mathscr{A}}(q^jA,B)$ denote the chain complex of graded abelian groups generated by bihomogeneous multimaps of arbitrary bidegree and differential given by the super-commutator $[d,f] = d_B \circ f - (-1)^{|f|} f \circ d_A$. By an element of this hom complex we will always mean a bihomogeneous element, and we let $\deg(f) = (\deg_h(f), \deg_q(f))$ denote the bidegree. Let $\mathrm{Ext}^{i,j}(A,B)$ denote the (i,j)-th homology group of $\mathrm{Hom}^{\bullet,\bullet}(A,B)$, which is simply the group of chain maps $t^iq^jA \to B$ modulo chain homotopy.

2.1 Convolutions

Suppose we have chain complexes E_i over some additive category and chain maps $\alpha_i : E_i \to E_{i+1}$. If $\alpha_{i+1}\circ\alpha_i = 0$, then the sequence $\cdots \xrightarrow{\alpha_{i-1}} E_i \xrightarrow{\alpha_i} E_{i+1} \xrightarrow{\alpha_{i+1}} \cdots$ is called a *bicomplex* or *double complex*, and we can try flatten to get an honest chain complex, called the total complex. As a graded object, the total complex is equal to $\bigoplus_i t^iE_i$, and the differential is a represented by a $\mathbb{Z}\times\mathbb{Z}$ matrix with diagonal entries equal to $(-1)^i d_{E_i}$, subdiagonal entries equal to α_i, and all other entries equal to zero. The total complex is undefined if the infinite direct sum $\bigoplus_i t^iE_i$ fails to exist, but there is no problem if, for example, $E_i = 0$ for $i > 0$ and each E_i is supported in non-positive homological degrees.

If, on the other hand $\alpha_{i+1}\circ\alpha_i$ is *homotopic* to zero, rather than zero on the nose, then the notion of total complex is replaced by convolution:

Definition 2.3. Let E_i be chain complexes over an additive category and $\alpha_i : E_i \to E_{i+1}$ chain maps such that $\alpha_{i+1}\circ\alpha_i \simeq 0$ for all $i \in \mathbb{Z}$. Any such sequence will be called a *homotopy chain complex*, and will be denoted as

$$E_\bullet = \cdots \xrightarrow{\alpha_{i-1}} E_i \xrightarrow{\alpha_i} E_{i+1} \xrightarrow{\alpha_{i+1}} \cdots \tag{2.4}$$

A convolution of a homotopy chain complex $E_\bullet$ is any chain complex which, as a graded object equals $\bigoplus_{i\in\mathbb{Z}} t^i E_i$ and whose differential d satisfies the following conditions: if $d_{ij} \in \mathrm{Hom}^{1-i+j}(E_j, E_i)$ is the corresponding component of d, then

- $d_{ii} = (-1)^i d_{E_i}$.
- $d_{i+1,i} = \alpha_i$.
- $d_{ij} = 0$ for $i < j$.

We will denote a convolution (2.4) by $M = \mathrm{Tot}(E_\bullet)$, or with a parenthesized notation in which we write all of the degree shifts explicitly:

$$M = (\cdots \xrightarrow{\alpha_{i-1}} t^i E_i \xrightarrow{\alpha_i} t^{i+1} E_{i+1} \xrightarrow{\alpha_{i+1}} \cdots)$$

We have a dual notion, in which $\bigoplus$ is replaced by $\prod$ in the above definition. We will refer to convolutions using $\bigoplus$ (respectively $\prod$) as being of type I (respectively type II).

We warn the reader that convolutions are typically far from unique, so the notation $M = \mathrm{Tot}(E_\bullet)$ is abusive. Rather, one should speak of the set of convolutions of a given homotopy chain complex $E_\bullet$. Note that $\alpha_{i+1} \circ \alpha_i \simeq 0$ is simply the vanishing of the Massey product$\langle [\alpha_{i+1}], [\alpha] \rangle$ in the homology of the differential graded algebra $\mathrm{End}^\bullet_{\mathbb{Z}}(\bigoplus_i E_i)$ (see [May69]). In general a convolution of (2.4) exists if and only if all higher Massey products $\langle [\alpha_{i+r}], [\alpha_{i+r-1}], \ldots, [\alpha_i] \rangle$ vanish (for a small illustration see Lemma 2.9). The notion of convolution is standard in the theory of triangulated categories, where the term is used more generally to mean a flattening $\mathrm{Tot}(E_\bullet) \in \mathscr{T}$ of a chain complex $E_\bullet \in \mathrm{Kom}(\mathscr{T})$ over a triangulated category; see [Kap88]. Since the triangulated categories here are homotopy categories, we typically a regard convolution as simply an iterated mapping cone, or a chain complex $(M^\bullet, d_M)$ with an additional grading $M = M_\bullet^\bullet$ with respect to which d_M is filtered, $d(M_j^i) \subset \bigoplus_{k \geq j} M_k^{i+1}$ (see [BK90]). In case M is a convolution of complexes of graded modules over a graded ring, then there are actually three gradings in total: the homological degree, q-degree, and convolution degree. The differential is $(\deg_h, \deg_q)$ bihomogenous of degree $(1,0)$ and $\deg_{conv}$ non-increasing. Throughout this section we will suppress the q-degree.

Example 2.5. If $f : A \to B$ is a chain map then the mapping cone $\mathrm{Cone}(f)$ is defined to be the chain complex $\mathrm{Cone}(f)^k = A^{k+1} \oplus B^k$ with differential given by

$$d_{\mathrm{Cone}(f)} = \begin{bmatrix} (-1)^{k+1} d_A & 0 \\ f & d_B \end{bmatrix}.$$

This is precisely the two term convolution $\mathrm{Cone}(f) = (t^{-1}A \xrightarrow{f} B)$.

Note that a convolution $\mathrm{Tot}(E_\bullet)$ is naturally equipped with a filtration. We give a special name to the "convolution degree" of a map $\mathrm{Tot}(E_\bullet) \to \mathrm{Tot}(F_\bullet)$:

Definition 2.6. Suppose $M = \mathrm{Tot}(E_\bullet)$ and $N = \mathrm{Tot}(F_\bullet)$ are convolutions. Say that an element $f \in \mathrm{Hom}^\bullet_{\mathscr{A}}(M, N)$ of M has *length* k if the component

$$f_{ij} \in \mathrm{Hom}^\bullet_{\mathscr{A}}(E_j, F_i)$$

vanishes unless $i - j = k$.

We can write any element $f \in \mathrm{Hom}^\bullet_{\mathscr{A}}(M, N)$ in terms of its length k components, $f = \sum_{k \in \mathbb{Z}} f_k$, where $f_k := (f_{i+k,i})_i \in \prod_i \mathrm{Hom}^\bullet(E_i, F_{i+k})$ is regarded as an element of $\mathrm{Hom}^\bullet_{\mathscr{A}}(M, N)$ of length k. Let us say that f is a map of convolutions if $f_k = 0$ for $k < 0$. Suppose $F_\bullet$ is bounded from above, i.e. $F_i = 0$ for $i \gg 0$, and $f_k \in \mathrm{Hom}^\bullet(M, N)$ are any elements of length k, each of some fixed homological degree r. Then any infinite sum $f_0 + f_1 + \cdots$ is finite on restriction to each E_j, hence is a well-defined element of $\mathrm{Hom}^\bullet_{\mathscr{A}}(M, N)$ by the universal property of direct sums.

Moreover, length is additive under composition of morphisms, so that if $f = f_0 + f_1 + \cdots$, $g = g_0 + g_1 + \cdots$, and $f \circ g = (f \circ g)_0 + (f \circ g)_1 + \cdots$ are written in terms of length k components, then $(f \circ g)_k = \sum_{i+j=k} f_i \circ g_j$. We have proven:

Lemma 2.7. *Let M and N be convolutions which are bounded above, fix $r \in \mathbb{Z}$, and suppose we have elements $f_k \in \operatorname{Hom}^r(M, N)$ of length k, for each $k \in \mathbb{Z}_{\geq 0}$. Then the series $f = f_0 + f_1 + \cdots$ is a well defined element of $\operatorname{Hom}^r(M, N)$. In particular, if $\alpha \in \operatorname{End}^0(M)$ has length $k > 0$, then $\operatorname{Id}_M - \alpha$ and $\operatorname{Id} + \alpha + \alpha^2 + \cdots$ are mutual inverses.* □

If $E_\bullet$ is a homotopy chain complex as in (2.4), then the differential on a convolution $M = \operatorname{Tot}(E_\bullet)$ can be written in terms of its length k components as

$$d_M = \Delta_0 + \Delta_1 + \cdots$$

where $\Delta_k \in \operatorname{End}^1(M)$ has length k. In particular $\Delta_0|_{E_i} = (-1)^i d_{E_i}$ and $\Delta_1|_{E_i} = \alpha_i$. Consider the equation

$$(\Delta_0 + \Delta_1 + \Delta_2 \cdots)^2 = 0. \tag{2.8}$$

Taking the length $k \geq 0$ components, we see that (2.8) holds if and only if

- $\Delta_0^2 = 0$, which automatically holds since Δ_0 is the sum of differentials on the E_i, shifted in homological degree.
- $[\Delta_0, \Delta_1] = 0$, which is automatically satisfied since we assume that each α_i is chain map.
- For $k \geq 1$, $[\Delta_0, \Delta_k] = -\sum_{i,j} \Delta_i \circ \Delta_j$ where the sum is over $1 \leq i, j \leq k-1$ such that $i + j = k + 1$ (k fixed).

We have proven the following, which can be used in an inductive construction of convolutions, as in Theorem 7.1:

Lemma 2.9. *Let A, B, C, D be chain complexes over an additive category, and suppose we have elements $\alpha \in \operatorname{Hom}^1(A, B)$, $\beta \in \operatorname{Hom}^1(B, C)$, and $\gamma \in \operatorname{Hom}^1(C, D)$.*

1. *A convolution* $(A \xrightarrow{\alpha} B)$ *exists if and only if* α *is a cycle.*

2. *A convolution* $(A \xrightarrow{\alpha} B \xrightarrow{\beta} C)$ *exists if and only if* α, β *are cycles and* $\beta \circ \alpha \in \mathrm{Hom}^2(A, C)$ *is a boundary. In particular, such a convolution exists if* α *and* β *are cycles and* $\mathrm{Ext}^2(A, C) \cong 0$.

3. *A convolution* $(A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D)$ *exists if and only if* α, β, γ *are cycles,* $\beta \circ \alpha = [d, \rho]$ *and* $\gamma \circ \beta = [d, \sigma]$ *are boundaries, and* $\gamma \circ \rho + \sigma \circ \alpha \in \mathrm{Hom}^2(A, D)$ *is a boundary. In particular such a convolution exists if the sub-convolutions* $(A \to B \to C)$ *and* $(B \to C \to D)$ *exist and* $\mathrm{Ext}^2(A, D) \cong 0$. □

The following proposition says that we can perturb the length $k \geq 1$ component of the differential of a convolution up to homotopy, at the expense of introducing higher length components. In particular, if $E_\bullet$ and $F_\bullet$ represent the same object of $\mathrm{Kom}(\mathrm{Kom}(\mathscr{A})_{/h})$, then any convolution of $E_\bullet$ is isomorphic to a convolution of $F_\bullet$ and vice versa.

Theorem 2.10. *Suppose we are given* $E_i \in \mathrm{Kom}(\mathscr{A})$, $E_i = 0$ *for* $i \gg 0$, *and cycles* $\alpha_i \in \mathrm{Hom}^1(E_i, E_{i+1})$ *such that* $\alpha_{i+1} \circ \alpha_i \simeq 0$ *for all* i. *Suppose* $M = (\cdots \xrightarrow{\alpha_{i-1}} E_i \xrightarrow{\alpha_i} E_{i+1} \xrightarrow{\alpha_{i+1}} \cdots)$ *is a convolution of the corresponding homotopy chain complex. Fix an integer* $k \geq 1$ *and assume there are elements* $\phi_i \in \mathrm{Hom}^1(E_i, E_{i+k})$ *such that*

$$d_{i+k,i} - \phi_i \simeq 0$$

for all $i \in \mathbb{Z}$, *where* $d_{i+k,i} \in \mathrm{Hom}^1(E_i, E_{i+k})$ *is the component of* d_M. *Then up to isomorphism of convolutions, each* $d_{i+k,i}$ *can be replaced by* ϕ_k *at the expense of affecting only the length* $> k$ *components of* d_M.

Proof. Fix an integer $k \geq 1$; we will perturb the length k component of d_M. Write the differential on M as $\Delta = \Delta_0 + \Delta_1 + \Delta_2 + \cdots$ in terms of its length l components, so that in particular $\Delta_0|_{E_i} = d_{E_i}$ and $\Delta_1|_{E_i} = \alpha_i$. Now, fix an element $H \in \mathrm{End}^0(M)$ of length k. By Lemma 2.7, the infinite sum $\mathrm{Id}_M + H + H^2 + \cdots$ is a well defined element of $\mathrm{End}^0(C)$, and is a two sided inverse for $(\mathrm{Id}_M - H)$. Conjugating the differential Δ by $(\mathrm{Id}_M - H)$ gives

$$\Delta' := (\mathrm{Id}_M - H) \circ (\Delta_0 + \Delta_1 + \Delta_2 + \cdots) \circ (\mathrm{Id}_M + H + H^2 + \cdots) \;=\; d_0 + \Delta'_1 + \Delta'_2 + \cdots \tag{2.11}$$

Recall that length is additive under function composition and H has length k, so $\Delta'_l = \Delta_l$ for $0 \leq l < k$ and $\Delta'_k = \Delta_k - \Delta_0 H + H\Delta_0$. This is to say, a perturbation of the length k part of Δ up to homotopy is realized by the isomorphism $(\mathrm{Id}_M - H) : (M, \Delta) \xrightarrow{\cong} (M, \Delta')$ of convolutions, where the length l components of Δ and Δ' agree for $0 \leq l < k$. □

2.2 Deformation retracts

Here we recall the standard notion of (strong) deformation retracts, which are a particular nice class of chain homotopy equivalences which interact nicely with convolutions.

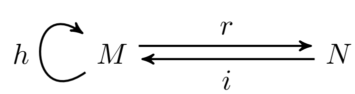


Figure 2.1: The data (r, i, h) of a deformation retract $M \to N$

Definition 2.12. Let $\mathscr{A}$ be a $\mathbb{Z}$-linear category, and M, N chain complexes over $\mathscr{A}$. A chain map $r : M \to N$ is called a *deformation retract* if there exist a chain map $i : N \to M$ and a homotopy $h \in \mathrm{End}^{-1}_{\mathscr{A}}(M)$ such that

- $h \circ i = r \circ h = 0$.
- $r \circ i = \mathrm{Id}_N$.
- $\mathrm{Id}_M - i \circ r = d_M \circ h + h \circ d_M$.

In this case we say (r, i, h) give the data of the deformation retract.

Lemma 2.13. *Suppose $A, B \in \mathrm{Kom}(\mathscr{A})$ and (r, i, h) give the data of a strong deformation retract $A \to B$. Then:*

1. $\mathrm{Id}_A = i \circ r + d_A \circ h + h \circ d_A$ *is a decomposition of* Id_A *into mutually orthogonal idempotents.*

2. *h may be assumed to satisfy $h^2 = 0$.*

Proof. The proof of (1) is straightforward. For part (2), put $h' = hdh$. Then

- $(h')^2 = (hdh)(hdh) = h(dh)(hd)h = 0$ since hd and dh are orthogonal, and
- $dh' + h'd = d(hdh) + (hdh)d = (dh)^2 + (hd)^2 = dh + hd = \mathrm{Id}_A - ir$ since hd and dh are idempotent.

Thus, h' has the desired properties. □

Proposition 2.14 (Gaussian elimination)**.** *Suppose we have graded objects $A = (A^k)_{k\in\mathbb{Z}}$ and $B = (B^k)_{k\in\mathbb{Z}}$ over an additive category $\mathscr{A}$, and suppose $C = A \oplus B$ is a chain complex with differential $\left[\begin{smallmatrix} {}_Ad_A & {}_Ad_B \\ {}_Bd_A & {}_Bd_B \end{smallmatrix}\right]$). Suppose also that ${}_Bd_B^2 = 0$. If $(B, {}_Bd_B)$ is a contractible chain complex, then there is a deformation retract $C \to A'$ where $A' = A$ with differential $d'_A = {}_Ad_A - {}_Ad_B \circ h \circ {}_Bd_A$, where h is a nulhomotopy for B which satisfies $h^2 = 0$.*

Proof. By hypotheses there is some nulhomotopy $h : B \to B$, and by Lemma 2.13 we may assume $h^2 = 0$. The relevant maps are defined in the following diagram:

$$H = \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \circlearrowright \begin{matrix} A \\ \oplus \\ B \end{matrix} \underset{i = \begin{bmatrix} \mathrm{Id} \\ -h \circ {}_Bd_A \end{bmatrix}}{\overset{r = \begin{bmatrix} \mathrm{Id} & -{}_Ad_B \circ h \end{bmatrix}}{\rightleftarrows}} A$$

It is straightforward to check that (1) r and i are chain maps, (2) $r \circ i = \mathrm{Id}_A$, (3) $\mathrm{Id}_{A\oplus B} = i \circ r + \left[\begin{smallmatrix} {}_A d_A & {}_A d_B \\ {}_B d_A & {}_B d_B \end{smallmatrix}\right] H + H \left[\begin{smallmatrix} {}_A d_A & {}_A d_B \\ {}_B d_A & {}_B d_B \end{smallmatrix}\right]$, (3) $r \circ H = 0$, (4) $H \circ i = 0$. That is, (r, i, H) give the data of a strong deformation retract. □

We want to see how convolutions interact with deformation retracts. Suppose we are given the data (π, σ, h) of a deformation retract $M \to N$. Suppose further that each of π, σ, and h are maps of convolutions. That is, writing everything in terms of its length k components (Definition 2.6) gives

- $\pi = \pi_0 + \pi_1 + \cdots$
- $\sigma = \sigma_0 + \sigma_1 + \cdots$
- $h = h_0 + h_1 + \cdots$
- $d_M = d_0 + d_1 + \cdots$

Taking the length zero parts of the equation $d_N = \pi \circ d_M \circ \sigma$ gives that the length zero part of d_N is precisely $\pi_0 \circ d_0 \circ \sigma_0$. Taking the length zero parts of the equations $d_N \circ \pi - \pi \circ d_M = 0$ and $d_M \circ \sigma - \sigma \circ d_N = 0$ shows that π_0 and σ_0 are chain maps $(M, d_0) \leftrightarrow (N, \pi_0 \circ d_0 \circ \sigma_0)$, and taking the length zero components of

(i) $\mathrm{Id}_N = \pi \circ \sigma$,

(ii) $\mathrm{Id}_M - \sigma \circ \pi = d_M \circ h + h \circ d_M$,

(iii) $\pi \circ h = 0$, and

(iv) $h \circ \sigma = 0$,

we see that (π_0, σ_0, h_0) give the data of a deformation retract $(M, d_0) \to (N, \pi_0 \circ d_0 \circ \sigma_0)$. Under a mild assumption on M, the converse also holds:

Theorem 2.15. *Suppose we have chain complexes E_i, F_i for $i \in \mathbb{Z}$ and the data (π_0, σ_0, h_0) of deformation retract $\bigoplus_k E_k \to \bigoplus_k F_k$. Suppose $M = (\cdots \xrightarrow{\alpha_{i-1}} E_i \xrightarrow{\alpha_i}$*

$E_{i+1} \xrightarrow{\alpha_{i+1}} \cdots)$ *is a convolution of the* E_i*. If* $E_i = 0$ *for* $i \gg 0$ *then there is a convolution* $N = (\cdots \xrightarrow{\beta_{i-1}} F_i \xrightarrow{\beta_i} F_{i+1} \xrightarrow{\beta_{i+1}} \cdots)$ *onto which* M *deformation retracts, where* $\beta_i = \pi_0|_{E_{i+1}} \circ \alpha_i \circ \sigma_0|_{F_i}$*. For each integer* $k \geq 0$*, the length* k *components of the data* (π, σ, h) *of the retract* $M \to N$ *are polynomials in the* π_0, σ_0, h_0*, and the components* d_l *of* d_M*. These polynomials are universal in the sense that they do not depend on any of the initial data.*

Before proving, we note that there are various extensions of this theorem which will be of use to us:

1. One could allow generalized convolutions, in which the terms are indexed by any partially ordered set X, instead of $\mathbb{Z}_{\leq 0}$. The boundedness condition should be replaced by $|\{y \in X \mid y > x\}| < \infty$ for all $x \in X$.

2. One could replace the condition $|\{y \in X \mid y > x\}| < \infty$ for all $x \in X$ with the condition $|\{y \in X \mid y < x\}| < \infty$ for all $x \in X$ while simultaneously replacing $\bigoplus$ with $\prod$ in the definition of convolution.

3. Morally speaking if π_0, σ_0, h_0, and d_M preserve some additional structure on M, then the same is true of π, σ, h and d_N, since the components of the latter are certain polynomials evaluated on the components of the former.

Proof. Let M, E_k, F_k be as in the hypotheses, and put $N = \bigoplus_k F_k$. We can write the differential d_M in terms of its length k components as $d_M = d_0 + d_1 + \cdots$. By hypothesis we have length zero maps $\pi_0 \in \mathrm{Hom}^0(M, N)$, $\sigma_0 \in \mathrm{Hom}^0(N, M)$, and $h_0 \in \mathrm{Hom}^{-1}(M, M)$ such that

(i) $\mathrm{Id}_N = \pi_0 \circ \sigma_0$.

(ii) $\mathrm{Id}_M - \sigma_0 \circ \pi_0 = d_0 \circ h_0 + h_0 \circ d_0$

(iii) $\pi_0 \circ h_0 = 0$.

(iv) $h_0 \circ \sigma_0 = 0$.

Put $e := \sigma_0 \circ \pi_0$, and consider following statement, where $k \in \mathbb{N} \cup \{\infty\}$:

Hyp(k)**.** *There exist elements* $\alpha_l \in \mathrm{End}^0(M)$ *of length* l*, for* $1 \le l < k$ *such that (i) each* α_l *is a polynomial in* $h_0, e, d_0, d_1, d_2, \ldots$*, and (ii) in terms of length* l *components*

$$\Delta := (\mathrm{Id}_M + \alpha_k) \circ \cdots \circ (\mathrm{Id}_M + \alpha_1) \circ d_M \circ (\mathrm{Id}_M + \alpha_1)^{-1} \circ \cdots \circ (\mathrm{Id}_M + \alpha_k)^{-1} \tag{2.16}$$

satisfies $\Delta_0 = d_0$ *and* $\Delta_l = e \circ \Delta_l \circ e$ *for all* $1 \le l < k$*.*

Let us assume that **Hyp**(∞) holds. Then define Φ to be the infinite composition $\Phi := \cdots \circ (\mathrm{Id}_M + \alpha_2) \circ (\mathrm{Id}_M + \alpha_1)$, which is a well defined series $\Phi = \mathrm{Id}_M + \Phi_1 + \Phi_2 + \cdots$. By Lemma 2.7 Φ and Φ^{-1} are well defined elements of $\mathrm{End}^0(M)$. Put

$$\pi := \pi_0 \circ \Phi, \qquad \sigma := \Phi^{-1} \circ \sigma_0, \qquad h = \Phi^{-1} \circ h_0 \circ \Phi, \qquad d_N = \pi_0 \circ \Phi \circ d_M \circ \Phi^{-1} \circ \sigma_0$$

An elementary calculation shows that (π, σ, h) give the data of a deformation retract $(M, d_M) \to (N, d_N)$. For example, by statement (ii) of **Hyp**(∞) the conjugated differential $\Delta := \Phi \circ d_M \circ \Phi^{-1}$ satisfies $\Delta = d_0 + \Delta_1 + \Delta_2 + \cdots$ with $\Delta_k = e \circ \Delta_k \circ e$ for all $k \ge 1$. Since $h_0 \circ e = 0 = e \circ h_0$, we have $h_0 \circ \Delta_k = 0 = \Delta_k \circ h_0$ for all $k \ge 1$, and it follows that

$$\begin{aligned} d_M \circ h + h \circ d_M &= d_M \circ (\Phi^{-1} \circ h_0 \circ \Phi) + (\Phi^{-1} \circ h_0 \circ \Phi) \circ d_M \\ &= \Phi^{-1} \circ (\Delta \circ h_0 + h_0 \circ \Delta) \circ \Phi \\ &= \Phi^{-1} \circ (d_0 \circ h_0 + h_0 \circ d_0) \circ \Phi \\ &= \Phi^{-1} \circ (\mathrm{Id}_M - \sigma_0 \circ \pi_0) \circ \Phi \\ &= \mathrm{Id}_M - \sigma \circ \pi \end{aligned}$$

The other relations are immediate. Finally, the length k components of π, σ, h, and d_N are polynomials in π_0, σ_0, h_0 and the d_k since the same is true of $\mathrm{Id}_M + \alpha_k$ and $(\mathrm{Id}_M + \alpha_k)^{-1}$. It remains to show that **Hyp**(∞) holds. This is taken care of by the following lemma. □

Lemma 2.17. *The statement* ***Hyp***(∞) *holds.*

Proof. We will construct by induction on $k \geq 1$ a stable family of elements $\{\alpha_1, \ldots, \alpha_{k-1}\}$ for which **Hyp**(k) holds. The base case $k = 1$ is vacuous. Assume by induction that $\{\alpha_1, \ldots, \alpha_{k-1}\}$ satisfy **Hyp**(k), and define $\Delta := d_0 + \Delta_1 + \Delta_2 + \ldots$ to be the differential d_M conjugated by $\prod_{k>l\geq 1}(\mathrm{Id}_M + \alpha_l)$ as in equation (2.16). By the induction hypothesis, the α_l are polynomial in the $h_0, e, d_0, d_1, \ldots$, hence so are the Δ_l.

Taking the length k part of the equation $\Delta^2 = 0$ gives

$$d_0 \circ \Delta_k + \Delta_k \circ d_0 = -\sum_{i,j} \Delta_i \circ \Delta_j \tag{2.18}$$

where the sum on the right-hand side is over $1 \leq i, j < k$ such that $i + j = k$. Now, by the induction hypothesis we have $\Delta_l = e\Delta_l e$ for $1 \leq l < k$. Since $h_0 e = eh_0 = 0$, composing (2.18) on the left (resp. right) with h_0 gives

$$h_0 \circ [d_0, \Delta_k] = 0 \qquad \text{(resp. } [d_0, \Delta_k] \circ h_0 = 0\text{).} \tag{2.19}$$

Define

$$\alpha_k := h_0 \circ \Delta_k - e \circ \Delta_k \circ h_0$$

Since Δ_k is polynomial in $h_0, e, d_0, d_1, \ldots$, the same is true of α_k. Compute

$$\begin{aligned} [d_0, \alpha_k] &= [d_0, h_0] \circ \Delta_k - h_0 \circ [d_0, \Delta_k] - e \circ [d_0, \Delta_k] \circ h_0 + e \circ \Delta_k \circ [d_0, h_0] \\ &= (\mathrm{Id}_M - e) \circ \Delta_k + e \circ \Delta_k \circ (\mathrm{Id}_M - e) \\ &= \Delta_k - e\Delta_k e \end{aligned}$$

Here we have used that the super-commutator $[d_0, -]$ satisfies the graded Leibniz rule with respect to function composition, together with (2.19) and the facts that $[d_0, h_0] = \mathrm{Id}_M - e$ and $[d_0, e] = 0$. Therefore

$$\Delta' := (\mathrm{Id}_M + \alpha_k) \circ \Delta \circ (\mathrm{Id}_M - \alpha_k + \alpha_k^2 - \cdots) = d_0' + \Delta_1' + \Delta_2' + \cdots$$

with $\Delta_l' = \Delta_l$ for $1 \leq l < k$ and $\Delta_k' = \Delta_k + \alpha_k \circ d_0 - d_0 \circ \alpha_k = e \circ \Delta_k \circ e$. This shows that $\alpha_1, \ldots, \alpha_k$, satisfy the conditions of **Hyp**$(k + 1)$. This completes the inductive step and completes the proof. $\square$

2.3 Differential graded categories

A *differential graded category* [Kel06], or simply dg category, is an additive category in which (i) morphism spaces are chain complexes of abelian groups and (ii) composition of morphisms satisfies a graded Leibniz rule $d_{\mathcal{A}}(f \circ g) = d_{\mathcal{A}}(f) \circ g + (-1)^{|f|} f \circ d_{\mathcal{A}}(g)$ where $|f|$ denotes the degree of f. By an isomorphism $A \cong B$ between objects of a dg category we will always mean an invertible degree zero cycle $\phi \in \mathrm{Hom}^0_{\mathcal{A}}(A, B)$.

If $\mathcal{A}, \mathcal{B}$ are dg categories, a functor $F : \mathcal{A} \to \mathcal{B}$ is said to be *differential graded* if F induces a chain map on morphism spaces. The natural transformations $\alpha : F \to G$ between dg functors $F, G : \mathcal{A} \to \mathcal{B}$ can be assembled into a chain complex, which makes the collection of dg functors $\mathcal{A} \to \mathcal{B}$ into a dg category. The details follow. Suppose $F, G : \mathcal{A} \to \mathcal{B}$ are dg functors. The degree k elements of $\mathrm{Nat}^\bullet(F, G)$ are sequences $\alpha = (\alpha_A)$ of elements $\alpha_A \in \mathrm{Hom}^k_{\mathcal{B}}(F(A), G(A))$ for each $A \in \mathrm{Obj}(\mathcal{A})$ such that for any $f \in \mathrm{Hom}^l_{\mathcal{A}}(A, A')$ we have $\alpha_{A'} \circ F(f) - (-1)^{kl} G(f) \circ \alpha_A = 0$. Clearly $\mathrm{Nat}^k(F, G)$ is an abelian group. Further, there is a differential $\mathrm{Nat}^k(F, G) \to \mathrm{Nat}^{k+1}(F, G)$ given by $d(\alpha)_A := d_{\mathcal{B}}(\alpha_A)$. One needs only to check dg naturality of $d(\alpha)$, which we leave to the reader.

Example 2.20. An additive category $\mathscr{A}$ will be considered a dg category, where the morphisms have the trivial grading and zero differential. The category $\mathrm{Kom}_{dg}(\mathscr{A})$ of chain complexes with morphisms given by $\mathrm{Hom}^\bullet$ complexes are differential graded, and the obvious inclusion $\mathscr{A} \to \mathrm{Kom}(\mathscr{A})$ is a dg functor. We also have full dg subcategories $\mathrm{Kom}^{\pm,b}_{dg}(\mathscr{A})$ of $\mathrm{Kom}_{dg}(\mathscr{A})$ consisting of chain complexes with various conditions on the gradings.

2.4 Lifting multilinear functors

Suppose $F : \mathscr{A} \to \mathscr{B}$ is a linear functor. Extend F to a dg functor $F : \mathrm{Kom}_{dg}(\mathscr{A}) \to \mathrm{Kom}_{dg}(\mathscr{B})$ as follows: for each $A \in \mathrm{Kom}_{dg}(\mathscr{A})$ put $F(A)^k = F(A^k)$ and if in terms of components we have $d_A = (d_k)$, then put $(d_{F(A)}) = (F(d_k))$. It is easy to check that this defines a dg functor. We can also lift multilinear functors.

If $\mathcal{A}$ and $\mathcal{B}$ are dg categories, let $\mathcal{A} \otimes \mathcal{B}$ denote the dg category with objects the pairs (A, B) with $A \in \mathcal{A}$, $B \in \mathcal{B}$ and morphism complexes

$$\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{B}}\Big((A,B),(A',B')\Big) := \mathrm{Hom}_{\mathcal{A}}(A,A') \otimes \mathrm{Hom}_{\mathcal{B}}(B,B')$$

with composition $(f \otimes g) \circ (f' \otimes g') = (-1)^{|f'||g|}(f \circ f') \otimes (g \circ g')$. If $\mathcal{A}_1, \ldots, \mathcal{A}_r, \mathcal{B}$ are dg categories, a *differential graded multilinear functor* $\mathcal{A}_1 \times \cdots \times \mathcal{A}_r \to \mathcal{B}$ is defined to be a dg functor $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_r \to \mathcal{B}$. Regarding an additive category as a dg category where the morphism spaces have trivial grading and differential gives the corresponding notion of tensor product of additive categories. The following definition gives the procedure for lifting bilinear functors to dg bilinear functors on the corresponding categories of chain complexes, in the spirit of the usual tensor product of chain complexes of, say, abelian groups.

Definition 2.21. Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be additive categories and $\odot : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ a bilinear functor. Define the *dg lift of F* to the dg bilinear functor $\odot : \mathrm{Kom}^-_{dg}(\mathscr{A}) \otimes \mathrm{Kom}^-_{dg}(\mathscr{B}) \to \mathrm{Kom}^-_{dg}(C)$ as follows. On objects, put

$$(A \odot B)^k = \bigoplus_{i+j=k} A^i \odot B^j, \qquad d_{A\odot B} := d_A \odot \mathrm{Id}_B + \mathrm{Id}_A \odot d_B \tag{2.22}$$

The direct sum in (2.22) is finite since A and B are assumed to be bounded from above. In the definition of the differential above we have already used the action of $\odot$ on morphisms, which is defined by

$$(f \odot g)|_{A^i \odot B^j} = (-1)^{i|g|} f|_{A^i} \odot g|_{B^j}.$$

for all $f \in \mathrm{Hom}^\bullet_{\mathscr{A}}(A, A')$ and all $g \in \mathrm{Hom}^\bullet_{\mathscr{B}}(B, B')$.

Note that the dg lift of the ordinary tensor product $\otimes_{\mathbb{Z}}$ of abelian groups is exactly the usual tensor product of chain complexes of abelian groups, and the action of $f \otimes g$ on $A \otimes B$ is given by the Koszul sign rule:

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b)$$

Proposition 2.23. *The lift of a bilinear functor is a dg bilinear functor, and isomorphic bilinear functors have dg isomorphic lifts.*

Proof. Left to the reader. □

Remark 2.24. We can define similarly the dg lift of a multilinear functor $F : \mathscr{A}_1 \times \cdots \times \mathscr{A}_r \to \mathscr{B}$, which will be a dg multilinear functor $\mathrm{Kom}^-_{dg}(\mathscr{A}_1) \times \cdots \times \mathrm{Kom}^-_{dg}(\mathscr{A}_r) \to \mathrm{Kom}^-_{dg}(\mathscr{B})$. We also have dg lifts on categories of chain complexes bounded from below defined by precisely the same formulas, with the symbol Kom^-_{dg} replaced by Kom^+_{dg} everywhere. If $\mathscr{B}$ contains countable direct sums or products then the boundedness conditions can be removed, obtaining a dg multilinear functor $\mathrm{Kom}_{dg}(\mathscr{A}_1) \times \cdots \times \mathrm{Kom}_{dg}(\mathscr{A}_r) \to \mathrm{Kom}_{dg}(\mathscr{B})$ (if $\mathcal{B}$ contains countable products then we replace $\bigoplus$ with $\prod$ in (2.22)). If $\mathcal{B}$ contains countable direct sums *and* direct products (for example if $\mathscr{B}$ is the category of abelian groups), then there are two distinct dg lifts of F, and it will be necessary to distinguish between them in our notation when confusion can arise.

2.5 Lifting contravariant functors

Suppose $\mathcal{A}$ is a dg category, and let $\mathcal{A}^{op}$ denote the *opposite dg category*, which is the dg category with the same objects as $\mathcal{A}$ and morphisms

$$\mathrm{Hom}_{\mathcal{A}^{op}}(A, B) := \mathrm{Hom}_{\mathcal{A}}(B, A)$$

and composition $f \circ^{op} g := (-1)^{|f||g|} g \circ f$. A *differential graded contravariant functor* $\mathcal{A} \to \mathcal{B}$ is defined to be a dg functor $\mathcal{A}^{op} \to \mathcal{B}$. Contravariant functors between additive categories can be lifted to dg contravariant functor between categories of chain complexes.

Definition 2.25. Let $(\ )^\vee : \mathscr{A} \to \mathscr{B}$ be a linear contravariant functor. Define the *dg lift* of F to be the dg contravariant functor $(\ )^\vee : \mathrm{Kom}_{dg}(\mathscr{A}) \to \mathrm{Kom}_{dg}(\mathscr{B})$ which acts on objects as

$$(A^\vee)^k := (A^{-k})^\vee \qquad\qquad d_{A^\vee} = -d_A^\vee$$

In the definition of the differential we have already used the action of $(\ )^\vee$ which is defined as follows. For $f \in \mathrm{Hom}^k_{\mathscr{A}}(A, B)$, define $f^\vee \in \mathrm{Hom}^k_{\mathscr{B}}(B^\vee, A^\vee)$ by commutativity of the following square

$$\begin{array}{ccc} (B^\vee)^i & \longrightarrow & (A^\vee)^{i+k} \\ \downarrow & & \downarrow \\ (B^{-i})^\vee & \xrightarrow{(-1)^{ik} f^\vee_{-i-k}} & (A^{-i-k})^\vee. \end{array}$$

Here, the vertical arrows are identities, and the top-most arrow is the component of $f^\vee$ which we are trying to define. In other words, $(f^\vee)_i = (f_{-i-k})^\vee$

The proof of the following is straightforward.

Proposition 2.26. *The dg lift of a linear contravariant functor is a dg contravariant functor.*

Proof. Left to the reader. □

Remark 2.27. The dg lift of a linear contravariant functor restricts to contravariant dg functors

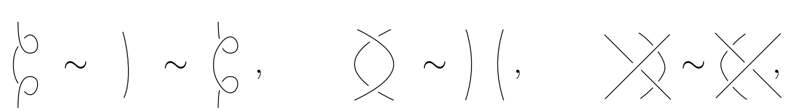
$$\mathrm{Kom}^{\pm}_{dg}(\mathscr{A}) \to \mathrm{Kom}^{\mp}_{dg}(\mathscr{B})$$

Chapter 3

Basics of colored $\mathfrak{sl}_2$ link homology

3.1 Locality of link invariants

Let $D^2 \subset \mathbb{R}^2$ be the standard disk of radius one centered at the origin, and put $B_n := \frac{1}{n}\{1-n, 3-n, \ldots, n-1\}$, regarded as a subset of $(\mathbb{R} \times 0) \subset \mathbb{R}^2$. B_n is simply a standard, ordered set of n points in the interior of D^2 which is invariant under the reflections across the the x- and y-axes. An (m, n)-tangle is a properly embedded 1-submanifold $(T, \partial T) \subset (D^2 \times [-1, 1], B_m \times \{1\} \cup B_n \times \{-1\})$. If a tangle T is generic with respect to the orthogonal projection $(x, y, z) \mapsto (x, z)$, then we call the image $D = \pi(T) \in [-1, 1]^2$, together with over-crossing and undercrossing information near the double points, an (m, n)-tangle diagram which represents T. The (m, n) tangles (respectively tangle diagrams) form a category with objects given by the non-negative integers and morphisms $n \to m$ given by (m, n)-tangles (respectively tangle diagrams) and composition given by gluing followed by reparametrization. A framed tangle is a tangle together with a choice of trivialization of the normal bundle (considered up to homotopy) which is restricts to a standard framing at the boundary, and there is a natural bijection between the set of framed (m, n)-tangles modulo framed isotopy and the set of (m, n)-tangle diagrams modulo the following local relations which we call the *framed Reidemeister moves*:



An invariant of (framed) links is said to be *local* if it extends to a functor from the category of (framed) tangles or, equivalently, from the category of tangle diagrams modulo the (framed) Reidemeister moves.

Similarly we can define what it means to be a local invariant of (framed) oriented links whose components are labelled by elements from some fixed label set S.

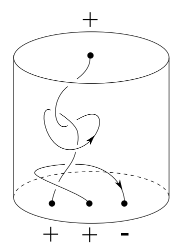


Figure 3.1: An oriented $(1,3)$-tangle, which could also be regarded as a morphism $(+,+,-) \to (+)$ in the category of oriented tangles.

3.2 The $\mathfrak{sl}_2$ link polynomial

In the 1980's and 1990's a beautiful connection between low-dimensional topology, representation theory, and quantum field theory produced a vast family of invariants of links in 3-manifolds, known collectively as the Witten-Reshetikhin-Turaev (WRT) invariants. Specializing to the case where the ambient manifold is S^3, one obtains what are called the Reshetikhin-Turaev link invariants, which associate to a complex semisimple Lie algebra $\mathfrak{g}$ and a framed, oriented link $L \subset S^3$ a Laurent polynomial $P(L) = P_{\mathfrak{g}}(L) \in \mathbb{Z}[q, q^{-1}]$. One allows the components of L to be labelled by finite dimensional representations of $\mathfrak{g}$, called the colors. The polynomial is multilinear with respect to direct sum of colors, hence without loss of generality one may assume that the colors are all irreducible representations of $\mathfrak{g}$.

The case $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$ is the only case which interests us in this thesis. The irreducible representations of $\mathfrak{sl}_2$ are the symmetric powers $\mathrm{Sym}^n(\mathbb{C}^2)$ of the standard representation, and so the $\mathfrak{sl}_2$ link polynomial is naturally an invariant $P_{\mathfrak{sl}_2}$ of framed, oriented links whose components are labelled by natural numbers. This invariant is called the *colored Jones polynomial.* When all the colors are 1, the link invariant is the famous Jones polynomial.

The $\mathfrak{g}$ Reshetikhin-Turaev invariant is local, in the sense that it extends to a functor from a category of framed, oriented, colored, tangles. We define the target category in case $\mathfrak{g} = \mathfrak{sl}_2$ next.

Let TL_n^m be the $\mathbb{C}(q)$-vector space generated by properly embedded 1-submanifolds of the rectangle $[-1,1]^2$ with boundary equal to a standard set of m points $B_m \times \{1\}$ on the "top" of the rectangle and n points $B_n \times \{-1\}$ on the "bottom" of the rectangle, where $B_m = \frac{1}{m}\{1-m, 3-m, \ldots, m-1\} \subset [-1,1]$. Here $\mathbb{C}(q)$ is the field of rational functions in an indeterminate q. We regard the generators modulo planar isotopy and the relation $D \sqcup U = (q+q^{-1})D$, where U is a circle disjoint from the rest of the diagram. By a *diagram* or a Temperley-Lieb diagram, we will simply mean the image of a 1-manifold with no circle components inside TL_n^m.

We have a pairing $\mathrm{TL}_k^m \otimes \mathrm{TL}_n^k \to \mathrm{TL}_n^m$ given by vertical stacking, which we denote by $a \cdot b$, or simply ab. The pairing makes the TL_n^m into a $\mathbb{C}(q)$-linear category TL with object given by non-negative integers and morphisms $n \to m$ given by elements of TL_n^m. In particular, composition makes the vector space $\mathrm{TL}_n := \mathrm{TL}_n^n$ into an algebra, called the *Temperley-Lieb algebra* on n strands. For a diagram $a \in \mathrm{TL}_n^m$, define the *through degree* $\tau(a)$ to be the minimal k such that $a = b \cdot c$ with $b \in \mathrm{TL}_k^m$, $c \in \mathrm{TL}_n^k$. For a linear combination $b = \sum_a f_a a$ of diagrams, let $\tau(b) := \max\{\tau(a) \mid f_a \neq 0\}$. The elements $p_n \in \mathrm{TL}_n$ characterized in the following definition are precisely the $\mathfrak{sl}_2$

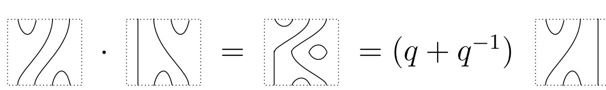


Figure 3.2: Multiplication in TL_4. Each of the diagrams above has through degree 2.

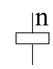
Reshetikhin-Turaev invariants of the trivial n-colored arcs.

Definition 3.1. The Jones-Wenzl projector $p_n \in \mathrm{TL}_n$ is the (unique, by the following theorem) element satisfying

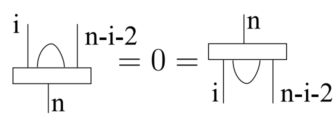
(JW1) $p_n = 1_n + a$ with $\tau(a) < n$.

(JW2) $a \cdot p_n = p_n \cdot b = 0$ whenever $\tau(a), \tau(b) < n$.

We refer to axiom (JW2) by saying that p_n *kills turnbacks*. Indeed, using the graphical notation in which we denote a parallel strands by 

and $p_n :=$ 

, this axiom becomes equivalent to

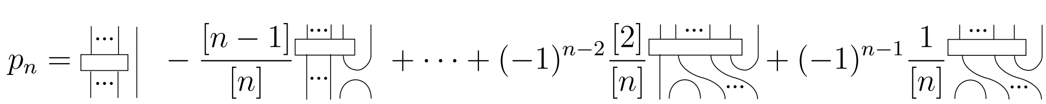


for $0 \leq i \leq n-2$. Similarly, if $f \in \mathrm{TL}_n$ is such that $a \cdot f = 0$ (respectivley $f \cdot a$) whenever $\tau(a) < n$, then we say f kills turnbacks from above (respectively below). The following is classical [Wen87].

Theorem 3.2. *The Jones-Wenzl projectors $p_n \in \mathrm{TL}_n$ exist and are unique. Further, axiom (JW2) may be relaxed in the following sense: if $f \in \mathrm{TL}_n$ is any element which satisfies axiom (JW1) and kills turnbacks from below, then $f = p_n$.*

Proof. For uniqueness, suppose we had two elements $f, f' \in \mathrm{TL}_n$ satisfying (1) and (2). Then $\tau(f - 1_n) < n$ implies that $(f - 1_n)f' = 0$. Similarly, $f'(f - 1_n) = 0$. Hence $f = ff' = f'$. This establishes uniqueness. Actually, the previous argument only uses axiom (JW1) and the fact that f' and f kill turnbacks from below, respectively above, i.e. $f \cdot a = 0 = a \cdot f'$ for $\tau(a) < n$. Given existence of a Jones-Wenzl projector p_n, this will establish the last statement of the theorem.

For existence, put $p_1 = 1_1 \in \mathrm{TL}_1$ and assume by induction that $p_{n-1} \in \mathrm{TL}_{n-1}$ is a Jones-Wenzl projector. Define an element $p_n \in \mathrm{TL}_n$ by



(3.3)

where $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$ is the quantum integer and a white box denotes p_{n-1}. Clearly $\tau(p_n - 1_n) < n$. We leave it to the reader to show that $p_n \cdot a = 0$ for $\tau(a) < n$. This will be proven on the level of categorifications in Proposition 6.10. Let $(\ )^\vee : \mathrm{TL}_n \to \mathrm{TL}_n$ denote the $\mathbb{C}(q)$-linear algebra anti-automorphism which reflects diagrams across the x-axis. Then $a \cdot p_n^\vee = (p_n \cdot a^\vee)^\vee = 0$ whenever $\tau(a) < n$, hence $p_n^\vee \cdot p_n \in \mathrm{TL}_n$ satisfies axioms (JW1) and (JW2). This proves existence, and completes the proof. □

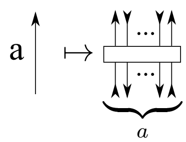
Note that the last statement of the theorem implies that p_n defined in (3.3) is already a Jones-Wenzl projector; we did not need to double p_n as in the proof.

Remark 3.4. Even though it is not strictly for the definition of the $\mathfrak{sl}_2$ Reshetikhin-Turaev invariant to do so, let us relate the definitions thus far to representation theory. The contents of this remark could easily fill a much larger volume, so we will keep ourselves brief. We have an isomorphism $\mathrm{TL}_n^m \cong \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_1^{\otimes n}, V_1^{\otimes m})$ where $U_q(\mathfrak{sl}_2)$ is a q-deformed version of the universal enveloping algebra of $\mathfrak{sl}_2$ and V_1 is a q-deformed version of the standard representation $\mathbb{C}^2$ of $\mathfrak{sl}_2$ [Jim85, Kas95]. The representation theory of $U_q(\mathfrak{sl}_2)$ mimics that of $\mathfrak{sl}_2$ in many ways; in particular there is a *Clebsch-Gordan* type rule $V_n \otimes V_1 \cong V_{n+1} \oplus V_{n-1}$, where V_k is a q-version of the symmetric power $V_k = \mathrm{Sym}_q^k(V_1)$. Iterating this, one can prove inductively that

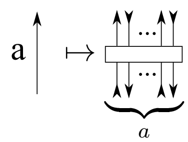
$$V_1^{\otimes n} = V_n \oplus \underbrace{\bigoplus_{0 \le k < n} m_{k,n} V_k}_{W} =: V_n \oplus W$$

for some multiplicities $m_{k,n}$. The Jones-Wenzl projector corresponds exactly to the projection-followed-by-inclusion of the unique V_n summand. Now, the W summand can be thought of as being spanned by the images of all the maps $V_1^{\otimes n} \to V_1^{\otimes n}$ which factor through $V^{\otimes k}$ with $k < n$. Hence the projection onto the V_n summand annihilates all such maps. This is the representation theoretic analogue of axiom (JW2). Axiom (JW1) is simply a normalization which ensures that $p_n^2 = p_n$; this property, called *idempotency* ensures that p_n corresponds to a projection operator.

We illustrate the idea of the $\mathfrak{sl}_2$ Reshetikhin-Turaev invariant in figure 3.2. More precisely, suppose D is a colored, framed, oriented tangle diagram. Mark some points $z_1, \ldots, z_k \subset D$ away from the crossings and away from the boundary, such that there is at least one marked point on each component of the underlying tangle. Replace an a-colored component with a parallel copies of itself, except near each marked point, where we insert a white box:



if a is odd,



if a is even

Interpret the white boxes as Jones-Wenzl projectors and the crossings as the elements


(3.5)

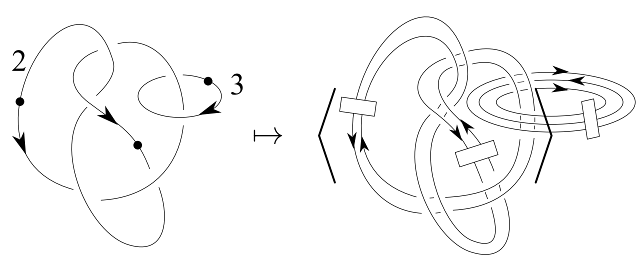
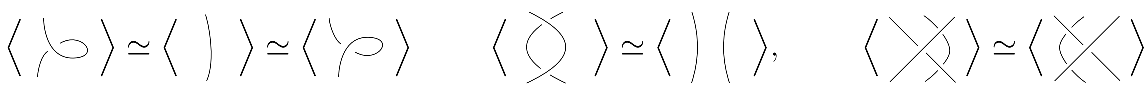
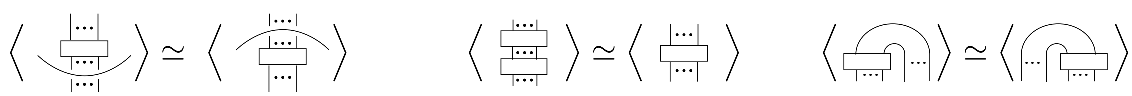


Figure 3.3: An illustration of the $\mathfrak{sl}_2$ Reshetikhin-Turaev link invariant. Starting with a suitably decorated oriented link diagram, one obtains a cabled diagram as above. The parallel strands should have alternating orientations. Each crossing and white box corresponds to a certain linear combination of planar diagrams (see (3.3) and (3.5)); the whole picture is interpreted as an element of $\mathrm{TL}_0^0 \cong \mathbb{C}(q)$ in the obvious multilinear way. The illustration is similar for tangle diagrams, in which case the result is interpreted as an element of the appropriate TL_n^m.

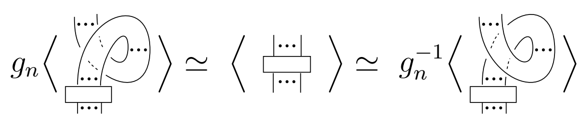
Because of the topological nature of the algebras TL_a, it is clear how to glue these elements together in the plane to obtain an element of TL_n^m. It is a straightforward exercise to show that the following relations hold, hence this procedure gives a well-defined invariant of colored, framed oriented tangles:



where each strand is arbitrarily oriented, and



The dependence on framing is given by



where the white box denotes p_n and the strands are alternately oriented. Here, $g_n = q^{(n^2+2n)/2}$ for n even and $g_n = q^{(n^2+2n-3)/2}$ for n odd.

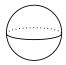
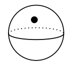
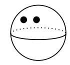
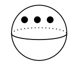
3.3 The tangle categories

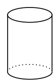
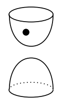
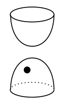
In [BN05] Bar-Natan interprets the Temperley-Lieb diagrams as objects of a category in which the morphisms ensure that the Temperley-Lieb relations lift to isomorphisms.

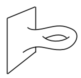
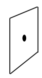
Definition 3.6. For each integer, fix a standard set $K_n \subset \partial D^2$ of $2n$ points, and define a category Cob_n as follows:

- The objects of Cob_n: symbols $q^j T$, where $T \subset D^2$ is a properly embedded 1-submanifold with boundary $K_n \subset \partial D^2$ and $j \in \mathbb{Z}$.

- A morphism $f : q^i T \to q^j T'$ is a formal $\mathbb{Z}$-linear combination of cobordisms $T \to T'$ in $D^2 \times [0,1]$, decorated with dots, regarded modulo (1) isotopy of the underlying surfaces (rel boundary), (2) dots are allowed to move freely about the components of the cobordism, and (3) the following local relations:

1.  $= 0$,  $= 1$,  $= 0$, and  $= 0$

2.  $=$  $+$ 

3.  $= 2$  .

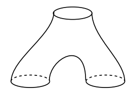
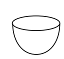
Here, a cobordism $S : q^i T \to q^j T'$ is a properly embedded surface $S \in D^2 \times I$ with boundary $\partial S = (T \times \{0\}) \cup (T' \times \{1\}) \cup (B_n \times [0,1])$. The degree of $S : q^i T \to q^j T'$ is defined by

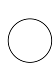
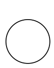
$$\deg_q(S) = n + j - i - \chi(S) + 2(\# \text{ of dots})$$

where $\chi(S)$ is the Euler characteristic of the surface S, and we allow only homogeneous morphisms of $\deg_q$ zero.

Composition of morphisms in Cob_n is induced by gluing of cobordisms, extended bilinearly to arbitrary morphisms. Since Euler characteristic is additive under gluing, the composition of degree zero morphisms is again degree zero, so that Cob_n is well-defined.

Let us motivate our definition of degree of a cobordism $S : T \to T'$. Suppose for the moment that T and T' are tangles without boundary, i.e. embeddings of some number of circles in the interior of D^2. On one hand, the TQFT philosophy suggests that the unknot should be an associative algebra object in Cob_0 with multiplication given by a pair of pants μ and unit given by the bowl η:

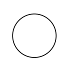
$\mu :=$  $\eta :=$ 

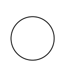
These surfaces have Euler characteristic -1, respectively $+1$. On the other hand, we want an isomorphism  $\cong (q\varnothing) \oplus (q^{-1}\varnothing)$ in Cob_0, in order to categorify the corresponding relation in the Temperley-Lieb algebra. This forces the unit map $\varnothing \to$  to have degree ± 1, hence the multiplication map to have to degree ∓ 1. Thus

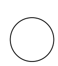
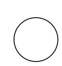
$\deg_q(S) = \pm\chi(S)$ for the elementary cobodisms above. Since both degree and Euler characteristic are additive under composition of morphisms, and every cobordism is a composition of pairs of pants, bowls, and their reflections, we see that the degree of a cobordism should be defined to be $\deg_q(S) = \pm\chi(S)$ for every cobordism S between tangles without boundary. We prefer the choice $\deg(S) = -\chi$.

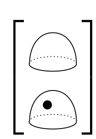
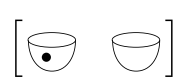
For a tangle with boundary, the identity cobordism $\mathrm{Id}_T : T \to T$ is homeomorphic to the union of cylinders $S^1 \times [0,1]$, one for each circle component of T, and rectangles $[0,1]^2$, one for each arc component of T. If T as $2n$ boundary points, then there must be n arc components of T, hence $\chi(\mathrm{Id}_T) = n$. Thus $\deg_q(S) = n - \chi(S)$ when S is an identity cobordism. It is natural to require that degree add under all possible ways gluings of cobordisms, which forces $\deg_q(S) = n - \chi(S)$ for each cobordism $S : T \to T'$ between tangles with $2n$ boundary points.

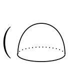
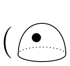
Finally, we regard the functor $a \mapsto qa$ as the upward shift in degree. So applying q to the target (resp. source) of a morphism should increase (resp. decrease) the degree by 1.

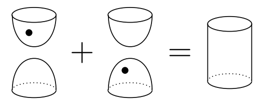
Now that we have motivated the definition of degree, let us motivate the relations. As mentioned already, we expect an isomorphism  $\cong q\varnothing \oplus q^{-1}\varnothing$. For degree reasons, any cobordism $S : \varnothing \to \varnothing$ is zero unless $\chi(S) = 0$. This forces all but one of the closed surface relations in Definition 3.6 (recall that a dot is 1/2 a handle). For the remaining relations, we take the following as justification:

Proposition 3.7. *We have*  $\cong q\varnothing \oplus q^{-1}\varnothing$ *in* Cob_0.

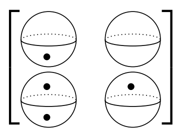
Proof. Define maps $\phi :$  $\to q\varnothing \oplus q^{-1}\varnothing$ and $\psi : q\varnothing \oplus q^{-1}\varnothing \to$  by

$\phi :=$  $\psi :=$ 

First check that $\chi($$) = 1$, $\chi($$) = -1$, etcetera. Hence after degree shifts the components of ϕ and ψ have $\deg_q$ zero. Check:

$\psi \circ \phi =$ 

by the "neck-cutting" relation in Definition 3.6. Because of the closed surface relations in definition 3.6, any closed surface can be regarded as a multiple of the 2-dimensional empty cobordism, regarded as an endomorphism of the empty 1-manifold in Cob_0. Thus we can identify $\mathrm{End}_{\mathrm{Cob}_0}(\varnothing) = \mathbb{Z}$. Under this identification we have:

$\phi \circ \psi =$  $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So ϕ and ψ are inverses. □

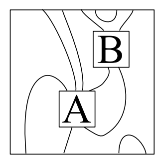
It will be convenient to define categories $\mathrm{Cob}_n^m \cong \mathrm{Cob}_{(n+m)/2}$ in which the diagrams are regarded as embedded in the rectangle $[0,1] \times [0,1]$ with a standard set $C_m \times \{1\}$ (resp. $C_n \times \{0\}$) of boundary points on the "top" (resp. "bottom") of the rectangle. We will use the notation Cob_n and Cob_n^n interchangeably. The categories Cob_n^m do not contain all direct sums, so we formally add them via a standard construction, obtaining a category whose objects should be thought of as column vectors (formal direct sums) of objects of Cob_n and whose morphisms are matrices of morphisms (with appropriate source and target) in Cob_n.

Definition 3.8. Let $\mathcal{TL}_n^m$ denote the category with objects the symbols $\bigoplus_{k=1}^r a_k$, where $a_1, \ldots, a_r \in \mathrm{Cob}_n^m$ are objects. Morphisms in $\mathcal{TL}_n^m$ from $\bigoplus_{k=1}^r a_k \to \bigoplus_{l=1}^s b_l$ are matrices $f = (f_{ij})$ where $f_{ij} \in \mathrm{Hom}_{\mathrm{Cob}_n^m}(a_j, b_i)$. Composition is given by matrix multiplication $(f \circ g)_{ij} = \sum_k f_{ik} \circ g_{kj}$. Let $\mathrm{Kom}(m,n)$ denote the category of chain complexes over $\mathcal{TL}_n^m$, and similarly $\mathrm{Kom}^+(m,n)$, $\mathrm{Kom}^-(m,n)$ and $\mathrm{Kom}^b(m,n)$ the full subcategories of chain complexes bounded from below, respectively bounded from above, respectively bounded.

It is easy to see that $\mathcal{TL}_n^m$ contains all finite direct sums, hence is an additive category with zero object given by the empty sum.

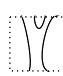
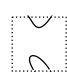
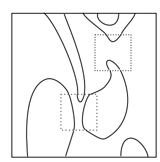
3.4 Interpreting the pictures

Because of the topological nature of the categories $\mathcal{TL}_n^m$ objects of can be glued together in the plane in precisely the same way as elements of TL_n^m (extending by multilinearity to direct sums of diagrams). But more is true, we can also glue morphisms (linear combinations of surfaces in $[0,1]^3$) in the same multi-linear way. Collections of categories with this sort of algebraic structure are called canopolies in [BN05] and canopolises in [MN08]. We illustrate the idea with an example, and refer to [BN05] for the details. Fix $A \in \mathrm{Kom}^-(4,2)$, $B \in \mathrm{Kom}^-(2,2)$, and consider the picture

$T(A,B) =$ 

We will describe in steps how to interpret $T(A,B)$ as an object of $\mathrm{Kom}^-(6,4)$ in a functorial way.

1. If $A = a$ and $B = b$ are objects of the appropriate Cob_n^m, then we may define $T(a,b)$ to be the object given by gluing diagrams together, for example

   $T\Big(q^3$  $, q$  $\Big) = q^4$ 

   Similarly, if $f : a \to a'$, $g : b \to b'$ are morphisms of q-degree zero, then we can define a morphism $T(f,g) : T(a,b) \to T(a',b')$ by gluing cobordisms f, g, together with the identity cobordism away from a, b.

2. We can extend $T(\ ,\ )$ linearly in each argument, obtaining a bilinear functor $T(\ ,\ ) : \mathcal{TL}_2^4 \times \mathcal{TL}_2^2 \to \mathcal{TL}_4^6$.

3. Now, if A and B are chain complexes, then we have a bicomplex $T(A,B)^{ij} = T(A^i, B^j)$ with a pair of anticommuting differentials given by $\delta_{ij} = T(d|_{A^i}, \mathrm{Id}\,|_{B^j})$

and $\delta'_{ij} = (-1)^j T(\mathrm{Id}|_{A^i}, d|_{B^j})$. We define $T(A, B)$ to be the total complex:

$$T(A,B)^k = \bigoplus_{i+j=k} T(A^i, B^i) \qquad d_{T(A,B)} = \delta + \delta'$$

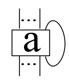
If $f : A \to A'$ and $g : B \to B'$ are degree zero chain maps, then we have $T(f, g) : T(A, B) \to T(A', B')$ defined in terms of components as $T(f, g)_{i,j} = T(f_i, g_j)$, which will clearly be a chain map. By introducing signs in the appropriate places, we can actually extend T to a dg bilinear functor (see Definition 2.21).

In general, if $T(A_1, \ldots, A_r)$ is a similar looking picture with r inputs, then we can regard $A_1, \ldots, A_r \mapsto T(A_1, \ldots, A_r)$ as a dg multilinear functor via an analagous procedure. Notice that the definition of $T(\ , \ldots, \ )$ requires us to order the inputs, but the resulting functor is invariant under reordering of the inputs in T while simultaneously permuting the arguments. I.e. if $\pi \in S_r$ is a permutation, then $\pi T(A_{\pi^{-1}(1)}, \ldots, A_{\pi^{-1}(r)}) \cong T(A_1, \ldots, A_r)$ naturally, where πT is the diagram obtained from T by reordering the boxes: the i-th box in T is the $\pi(i)$-th box in πT. The independence up to natural isomorphism on the ordering of the inputs is a standard fact about multicomplexes obtained from commutative lattices in this way, and is familiar to anyone who has proven that the complex which computes Khovanov homology is independent from the ordering of the crossings in a knot projection up to isomorphism [Kho00].

We will eventually want to compose unbounded chain complexes together in the plane, but for this will will need to formally adjoin countable direct products or direct sums to $\mathcal{TL}_{n_0}^{m_0}$. We postpone this until section 4.1.

We give special notation to certain planar compositions:

Definition 3.9. Let $\odot : \mathcal{TL}_k^m \times \mathcal{TL}_n^k \to \mathcal{TL}_n$ be the bilinear functor induced by vertical stacking followed by reparametrization of rectangles, so that $a \odot b$ is 'a on top of b.' Let $\sqcup : \mathcal{TL}_n \times \mathcal{TL}_m \to \mathcal{TL}_{n+m}$ be the bilinear functor induced by horizontal juxtaposition. Let $T : \mathcal{TL}_n \to \mathcal{TL}_{n-1}$ be the partial trace functor:

$T(a) =$ 

Denote similarly the extensions of these functors to the appropriate categories $\mathrm{Kom}^{\pm}(m,n) := \mathrm{Kom}^{\pm}(\mathcal{TL}_n^m)$ of semi-infinite chain complexes.

It is clear that $\odot$ and $\sqcup$ are associative up to natural isomorphism. The following is well-known:

Proposition 3.10. *For each integer $n \geq 0$, $(\mathcal{TL}_n, \odot, 1_n)$ is a monoidal category [ML98], where 1_n denotes the diagram consisting of n vertical strands. Similarly, the categories $\mathrm{Kom}^{\pm}(\mathcal{TL}_n)$ are monoidal with monoid $\odot$ and unit 1_n, regarded as a chain complex concentrated in degree 0.* □

Notation. It will be useful to have notation for the *graded* abelian groups generated by maps which are homogeneous of arbitrary degree. Our notation will have to support our eventual passage to the differential graded categories of chain complexes over $\mathcal{TL}_n^m$. So we put

$$\mathrm{Hom}^{0,\bullet}_{\mathcal{TL}_n^m}(a,b) := \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{TL}_n^m}(q^k a, b)$$

Generally, if $A, B \in \mathrm{Kom}(\mathcal{TL}_n^m)$ are chain complexes, we let

$$\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_n^m}(A,B) := \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^{\bullet}_{\mathcal{TL}_n^m}(q^k A, B)$$

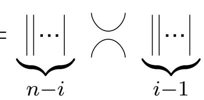
denote the chain complex of graded abelian groups. Note that a degree zero morphism $q^k a \to b$ could also be regarded as a degree k morphism $a \to b$, so the above direct sums really do give the graded morphism spaces. Additivity of Euler-characteristic under gluing ensures that the degrees add under composition of morphisms. By an element of a (bi)graded abelian group we will always mean a (bi)homogeneous element. In the case of an element $f \in \mathrm{Hom}^{\bullet,\bullet}(A,B)$ we let $\deg(f) = (\deg_h(f), \deg_q(f))$ denote the bidegree. Sometimes we write $|f| = \deg_h(f)$. Note that our notations $\mathrm{Hom}^{0,\bullet}$ and $\mathrm{Hom}^{\bullet,\bullet}$ are compatible once we regard objects of $\mathcal{TL}_n^m$ as chain complexes concentrated in homological degree zero.

3.5 Turnback killing and Cooper-Krushkal projectors

In [CK12] Cooper and Krushkal define a chain complex P_n over $\mathcal{TL}_n$ which categorifies the Jones-Wenzl projector $p_n \in \mathrm{TL}_n$.

Definition 3.11. Define the *Temperley-Lieb generators* $e_i \in \mathcal{TL}_n$ by

$$e_i =$$



Say $M \in \mathrm{Kom}(n)$ kills turnbacks from below (resp. above) if $M \odot e_i \simeq 0$ (resp. $e_i \odot M \simeq 0$) for $1 \leq i \leq n-1$. Say M kills turnbacks if M kills turnbacks from above and below.

We now define the Cooper-Krushkal projectors. Our definition differs from the original definition of Cooper and Krushkal (definition 3.1 in [CK12]) in that the projectors considered here are supported in non-positive homological degrees, rather than non-negative. Also, we omit the condition on quantum grading here; this is only necessary to have a well-defined notion of Euler characteristic, which we discuss in §3.7.

Definition 3.12. Call a chain complex $C \in \mathrm{Kom}(n)$ a Cooper-Krushkal projector if

(CK0) C is supported in non-positive homological degrees.

(CK1) The degree zero chain group is $C^0 = 1_n$, the monoidal identity. Moreover, 1_n does not appear as a summand of any other chain group.

(CK2) C kills turnbacks.

The following appears as Theorem 3.2, corollary 3.4, and corollary 3.5 in [CK12].

Theorem 3.13 (Cooper-Krushkal)**.** *Cooper-Krushkal projectors exist and are unique up to homotopy equivalence. Such a complex* $P_n \in \mathrm{Kom}(n)$ *is also idempotent up to homotopy:* $P_n \odot P_n \simeq P_n$. □

Note that by Proposition 4.26, axiom (CK2) can be relaxed while still retaining uniqueness of Cooper-Krushkal projectors up to homotopy equivalence.

The turnback killing property (definition 3.11) plays a central role in the $\mathfrak{sl}_2$ quantum invariants. Note that if $b \in \mathcal{TL}_n$ is a direct sum of non-identity diagrams up to shifts and $A \in \mathrm{Kom}(n)$ kills turnbacks, then $A \odot b \simeq 0$. The axiom (CK1) says that a Cooper-Krushkal projector $P_n \in \mathrm{Kom}(n)$ can be written $P_n = \mathrm{Cone}(N \to 1_n)$ for some $N \in \mathrm{Kom}^{\leq 0}(n)$ whose chain groups N^k are direct sums of non-identity diagrams up to shifts. Hence if $A \in \mathrm{Kom}(n)$ kills turnbacks we have $A \odot N^k \simeq 0$. Theorem 2.15 now implies that $A \odot N \simeq 0$. Hence an application of Gaussian elimination (proposition 2.14) establishes the following:

Proposition 3.14. *Suppose* $A \in \mathrm{Kom}(n)$ *kills turnbacks and* $P_n \in \mathrm{Kom}(n)$ *is a Cooper-Krushkal projector. Then there is a deformation retract* $A \odot P_n \xrightarrow{\simeq} A$ *with section given by the composition*

$$A = A \odot 1_n \xrightarrow{\mathrm{Id}_A \odot \iota} A \odot P_n$$

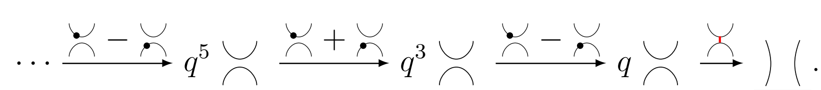
where $\iota : 1_n \to P_n$ *is the inclusion of the degree zero chain group. A similar result holds for* $P_n \odot A$. □

Specializing the above to the case $A = P_n$ we recover idempotency: $P_n \odot P_n \simeq P_n$.

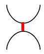
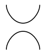
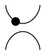
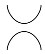
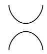
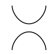
3.6 The Cooper-Krushkal recursion

The Cooper-Krushkal axioms force P_1 to be equal to the monoidal identity 1_1. For a Cooper-Krushkal projector $P_2 \in \mathrm{Kom}(2)$ we may choose the following minimal representative:

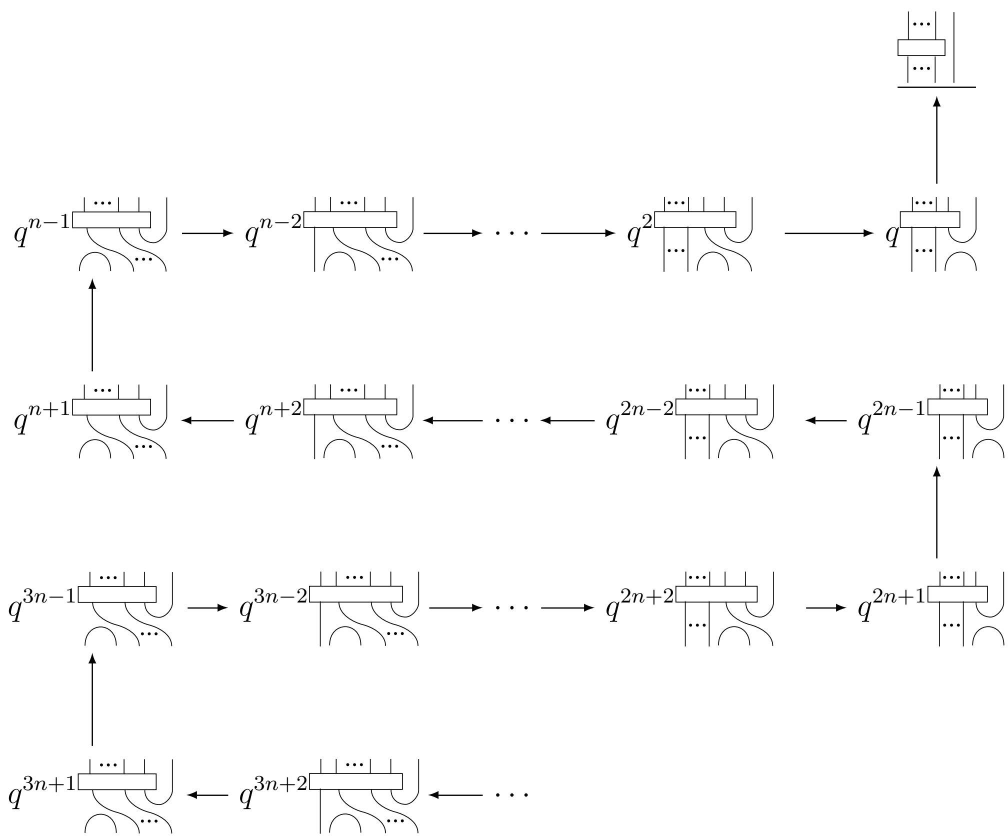
$P_2 \quad :=$



(3.15)

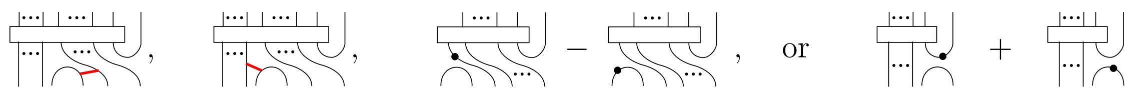
The underlined term indicates the degree zero chain group, and we use the convention that  : q  $\to$  denotes the map corresponding to the saddle cobordism, and e.g.  : q^2  $\to$  denotes an identity cobordism with a dot on one of the sheets. It is a fun and useful exercise to show that P_2 kills turnbacks (hint: there aren't that many choices for self-maps of $P_2 \odot$  of homological degree -1 and q-degree zero; write down the obvious one and check that it defines a nulhomotopy $d \circ h + h \circ d = \mathrm{Id}_{P_2}$). In general, Cooper-Krushkal construct the projector P_n as a convolution (Definition 2.3) built out of copies of P_{n-1}.

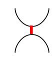
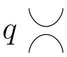
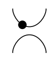
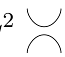
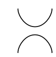
Definition 3.16. Let $P_{n-1} \in \mathrm{Kom}(n-1)$ be a Cooper-Krushkal projector, and define the *Frenkel-Khovanov sequence* (relative to P_{n-1}) to be the following semi-infinite sequence of chain complexes and chain maps



(3.17)

where the white box denotes P_{n-1} and the maps are

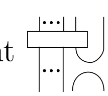


between adjacent terms. We remind the reader that  : q  $\to$ denotes the map corresponding to the saddle cobordism, and  : q^2  $\to$  denotes an identity cobordism with a dot on one of the sheets. Since the gluing of diagrams together in the plane is functorial, it is clear how to interpret the indicated pictures as chain maps.

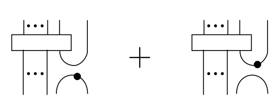
This sequence $E_\bullet = (\cdots \xrightarrow{\alpha_{-2}} E_{-1} \xrightarrow{\alpha_{-1}} E_0)$ is not a bicomplex since the composition $\alpha_{i+1} \circ \alpha_i$ of successive maps is nulhomotopic rather than zero on the nose. That is, $(E_\bullet, \alpha)$ defines an object of $\mathrm{Kom}(\mathrm{Kom}(n)_{/h})$. In [CK12] it is proven that:

Theorem 3.18 (The Cooper-Krushkal recursion)**.** *If $P_{n-1} \in \mathrm{Kom}(n-1)$ is a Cooper-Krushkal projector, then there exists a convolution (Definition 2.3) $P_n \in \mathrm{Kom}(n)$ of the Frenkel-Khovanov sequence relative to P_{n-1}, and any such convolution is a Cooper-Krushkal projector.* □

We will give an independent proof of the existence of such a convolution in Proposition 6.13.

Remark 3.19. In [CK12, §7.4] the map between adjacent  terms is defined to be

rather than our



But by dot-hopping (Lemma 6.42) the two maps are homotopic and, by Theorem 2.10, any convolution of one sequence is isomorphic to a convolution of the other. Also, in [CK12, §7.16] a convolution of the Frenkel-Khovanov sequence is doubled in order to obtain a chain complex which kills turnbacks from above as well as from below. However by Proposition 4.26 this step is unnecessary.

3.7 A well-defined Euler characteristic, and categorification

Put $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$ and let $\mathrm{TL}'_n \subset \mathrm{TL}_n$ denote the $\mathcal{A}$-subalgebra generated by the Temperley-Lieb diagrams. It is well known that $\mathcal{TL}_n$ is isomorphic to the category $H^n - \mathbf{pgmod}$ of finitely generated, graded, projective modules over Khovanov's ring H^n [Kho02] and that the Grothendieck group satisfies

$$K_0(H^n - \mathbf{pgmod}) \cong \mathrm{TL}'_n$$

as $\mathcal{A}$-algebras, where the algebra structure on the left-hand side is inherited from the monoidal product $\odot$ on $\mathcal{TL}_n$. Indeed if S denotes a set of representatives of isotopy classes of Temperley-Lieb diagrams $a \in \mathrm{TL}_n$ without circle components, then there is a family $\{P(a) \mid a \in S\}$ of pairwise non-isomorphic graded, indecomposable, projective H^n-modules such that any indecomposable projective is isomorphic to some $P(a)$ up to a shift. The isomorphism of categories becomes the easily checked fact that $\mathrm{Hom}_{\mathcal{TL}_n}(q^k a, b) \cong \mathrm{Hom}_{H^n}(q^k P(a), P(b))$ for all $k \in \mathbb{Z}$ and all $a, b \in S$, and that these isomorphisms are compatible with composition of morphisms. This is done in Proposition 4.10. In terms of H^n-modules, all of the chain complexes of interest in this paper can be assumed to lie in a certain full subcategory of $\mathrm{Kom}(H^n - \mathbf{pgmod})$ whose bigradings are supported in a certain angle shaped region of $\mathbb{Z} \times \mathbb{Z}$:

Definition 3.20. Let $\mathrm{Kom}^{\angle}(n)$ denote the full subcategory of $\mathrm{Kom}(H^n - \mathbf{pgmod})$ consisting of chain complexes $\bigoplus_{i,j\in\mathbb{Z}} M^{ij}$ such that i is homological degree, j is the q-degree, and

1. $M^{i,j} = 0$ for $i \gg 0$.
2. the integers $m_i := \min\{j \mid M^{i,j} \neq 0\}$ form a sequence which tends to ∞ as $i \to -\infty$.

The following should not be surprising:

Proposition 3.21. *We have a well defined Euler characteristic*

$$\chi(M^{\bullet,\bullet}) = \sum_{i\in\mathbb{Z}}(-1)^i[M^{i,\bullet}] \in \mathbb{Z}[[q]][q^{-1}] \otimes_{\mathcal{A}} \mathrm{TL}'_n$$

for every $M \in \mathrm{Kom}^{\angle}(n)$*, where* $[M^{i,\bullet}]$ *denotes the image in the Grothendieck group* $K_0(H^n - \mathbf{pgmod}) \cong \mathrm{TL}'_n$*. Further,* $M \simeq N$ *implies* $\chi(M) = \chi(N)$*.*

Proof. Each chain group $M^{i,\bullet}$ of $M \in \mathrm{Kom}^{\angle}(n)$ can be written as a finite direct sum

$$M^{i,\bullet} \cong q^{s_i} \bigoplus_k q^{r_{ik}} P(a_k)$$

with $r_{ij} \geq 0$ and $\lim_{i\to-\infty} s_i = \infty$. It follows that $\chi(M)$ is a well-defined Laurent series with values in TL'_n.

Now, let us argue that homotopy equivalent chain complexes have the same value of χ. Recall that we have fixed a set S of isotopy representatives of tangles $(T, \partial T) \subset (D^2, \partial D^2)$, regarded as objects of $\mathcal{TL}_n$. It is not hard to see [BN05] that the graded morphism space $\bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}(q^k P(a), P(b))$ is supported in non-negative degrees for all $a, b \in S$, and that the degree zero part satisfies

$$\mathrm{Hom}_{H^n}(P(a), P(b)) \cong \mathrm{Hom}_{\mathcal{TL}_n}(a, b) = \begin{cases} \mathbb{Z}\cdot \mathrm{Id}_{P(a)} & \text{for } b = a \\ 0 & \text{otherwise} \end{cases}$$

for $a, b \in S$. So the chain groups of $M \in \mathrm{Kom}^{\angle}(n)$ split as a direct sum of their $q^k P(a)$ isotypic components, and we can put each component $q^k P(a)^{\oplus i} \to q^k P(a)^{\oplus j}$ of the differential simultaneously into Smith normal form (a non-square diagonal matrix with integer entries). By iteratively splitting off and contracting summands of the form

$$0 \to P(a) \xrightarrow{\pm\mathrm{Id}} P(a) \to 0$$

we obtain a chain complex which we denote M_{red}. Note that $\chi(M_{red}) = \chi(M)$. From our remarks regarding gradings, it now easily follows that if $f \simeq \mathrm{Id}_{M_{red}}$ then we must actually have $f = \mathrm{Id}_{M_{red}}$, hence an equivalence of reduced chain complexes is an isomorphism. It follows that $\chi(M) = \chi(M_{red}) = \chi(N_{red}) = \chi(N)$ whenever $M \simeq N$. This completes the proof. □

Definition 3.22. Suppose $m \in \mathrm{TL}_n$ is a linear combination of Temperley-Lieb diagrams with coefficients of the form $f(q)/g(q)$ with $f(q), g(q) \in \mathbb{Z}[q, q^{-1}]$. By expanding into Laurent power-series, we obtain an element $m' \in \mathbb{Z}[[q]][q^{-1}] \otimes_{\mathcal{A}} \mathrm{TL}'_n$, and we say $M \in \mathrm{Kom}^{\angle}(n)$ *categorifies* m if $\chi(M) = m'$.

We will leave it to the reader to check that all of the chain complexes in this paper satisfy the conditions on bigradings which define $\mathrm{Kom}^{\angle}(n)$. For example, note the increasing sequence of q-degree shifts appearing in the Cooper-Krushkal recursion in §3.6, and that P_n categorifies $p_n \in \mathrm{TL}_n$ in the above sense.

3.8 The local colored $\mathfrak{sl}_2$ link homology

We recall the categorification of the colored Jones-polynomial due to Cooper-Krushkal [CK12]. We call their homology theory *local colored $\mathfrak{sl}_2$-link homology*, and will sometimes omit the adjectives "colored" and "local." The construction agrees with Bar-Natan's extension of Khovanov homology to tangles when all of the colors are 1; in order to obtain Khovanov's cube complex associated to an (1-colored) oriented link diagram D, one applies the functor $\mathrm{Hom}^{\bullet,\bullet}(\varnothing, -)$ to the chain complex $C(D)$ constructed here. Here, the word local refers to the fact that the Cooper-Krushkal invariant extends to tangles in a way which respects gluing, and is defined more precisely in the introduction. Because of the topological nature of the categories $\mathcal{TL}_n$ gluing diagrams together in the plane corresponds to a multilinear functor as explained in Definition 3.9. Hence the cabling procedure illustrated in figure 3.2 can be interpreted now as a chain complex over $\mathcal{TL}_n$; we need only define the object associated to a crossing and then compose them together in the plane according to a link diagram. Of course, one then needs to check that assignment depends on the choice of diagram only up an isomorphism.

Definition 3.23. Define chain the following 2-term chain complexes over $\mathcal{TL}_2$:

(3.24)

For each pair of integers $n, m \geq 1$, define chain complexes $X_{\pm}(n,m) \in \mathcal{TL}_{n+m}$ by

interpreted as the corresponding planar composition of complexes (3.24). We have shown the orientations only in the case where m and n are odd; if either m or n is even, orient the parallel strands in one of the two possible alternating ways; the resulting chain complex will not depend on the choice.

Definition 3.25. Fix as auxilliary data a family of Cooper-Krushkal projectors $P_m \in \mathrm{Kom}^-(m)$ for each integer $m \geq 1$. Let D be an oriented, colored tangle diagram with crossings $w_1, \ldots, w_r$ and some marked points $z_1, \ldots, z_s \subset D$ away from the crossings and away from the boundary. Assume that there is at least one marked point on each component of the underlying tangle. Define $C(D) = C(D, \{P_m\})$ to be the chain complex obtained from D by replacing an n-colored component with n parallel copies of itself, except near each marked point, where we insert a white box:

Interpret the white boxes as the chain complexes P_m and the crossings as the chain chain complexes $X_{\pm}(1,1)$ from Definition 3.23. Taking planar composition defines $C(D)$ up to canonical isomorphism (given by reordering the set of crossings and marked points on D).

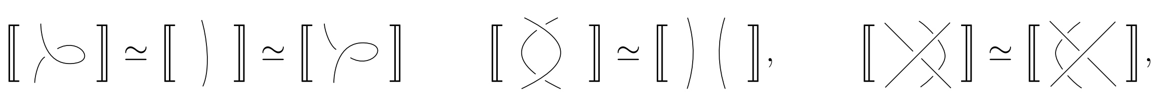
Alternatively, let F_D denote the planar composition functor with $r + s$ inputs obtained from D by removing small disks near each crossing and each marked point, and replacing an arc labelled by n with n-parallel copies of itself. In order to be precise, let $B_1, \ldots, B_{r+s} \subset D^2$ be a disjoint family of small disks in the interior of D^2 such that $B_i \cap D$ (respectively B_{r+i}) is a small neighborhood of w_i for $1 \leq i \leq r$ (respectively of z_i for $1 \leq i \leq s$). Let F_D be the planar diagram given by D minus the interior of $B_1 \cup \cdots \cup B_{r+s}$. For $1 \leq i \leq r$ let $X_i = X_\pm(n, m)$ denote the chain complex obtained by replacing the arcs incident to crossing w_i by parallel copies of themselves. Suppose the marked point z_i lies on a component colored by n_i. Then we have

$$C(D, \{P_m\}) = F_D(X_1, \ldots, X_r, P_{n_1}, \ldots, P_{n_s})$$

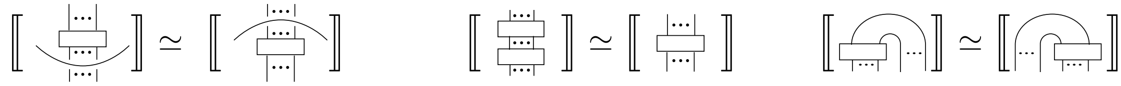
If the boundary points of D lie on components which are colored with $n_1, \ldots, n_k$, then $C(D) \in \mathrm{Kom}^-(n_1 + \cdots + n_k)$.

Results of Bar-Natan [BN05] imply that, away from the marked points, the chain complex $C(D, \{P_m\})$ defines a tangle invariant. From [CK12] we know that the Cooper-Krushkal projectors $P_m \in \mathrm{Kom}(m)$ satisfy certain isotopy relations which imply that $C(D; \{P_n\})$ is an invariant of the underlying colored, framed, oriented tangle up to homotopy equivalence.

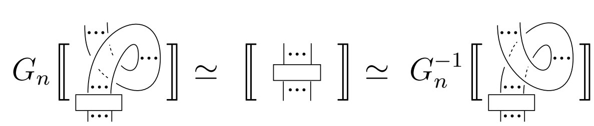
Theorem 3.26. *The chain complexes*  *and*  *satisfy the following local relations*



where each diagram is arbitrarily oriented, and interpreted as a chain complex over the appropriate category $\mathcal{TL}_n$. Moreover, all but the last can be chosen to be deformation retracts. Further:



The dependence on framing is given by



(3.27)

where $G_n = t^{-n^2/2}q^{(n^2+2n)/2}$ *for* n *even and* $G_n = t^{(1-n^2)/2}q^{(n^2+2n-3)/2}$ *for* n *odd. As a corollary, if* D *and* D' *are diagrams which represent isotopic (colored, framed, oriented) tangles, then* $C(D;\{P_n\})$ *and* $C(D';\{P_n\})$ *are chain homotopy equivalent.*

□

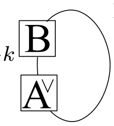
Definition 3.28. Suppose L is a colored, framed, oriented, link. Define the *local colored* $\mathfrak{sl}_2$ *link homology of* L to be the homology of the chain complex $\mathrm{Hom}^{\bullet,\bullet}(\varnothing, C(D))$ where D is any diagram for L.

Chapter 4

$\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_n}$ is a planar pairing

This chapter is dedicated to the proof and consequences of the fact that morphism complexes in Kom(m, n) can be computed via a graphical pairing:

$$\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_n^m}(A, B) \cong q^{(m+n)/2}\, \mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_0^\Pi}\Big(\varnothing, q^{-k}$$



$$\Big)^{\Pi}$$

naturally. Here, $(\ )^\vee$ is a certain contravariant functor which reflects diagrams and reverses all degree shifts. The pairing is deduced from the usual notion of duality in categories (in this case embedded cobordisms with corners [BD95]), together with formal properties of differential graded lifts. In applications, B and A will be semi-infinite chain complexes, say bounded from above in homological degree. So in order to define the planar composition $B \odot A^\vee$ it will be necessary to embed the categories $\mathcal{TL}_m^n$ into a category $(\mathcal{TL}_m^n)^\Pi$ which contains countable direct products.

4.1 Formally adjoining direct sums and products

Definition 4.1. Let $(\mathcal{TL}_n^m)^\oplus$ and $(\mathcal{TL}_n^m)^\Pi$ be the closures of $\mathcal{TL}_n^m$ under countable direct sums, respectively countable direct products. That is to say,

1. Let $(\mathcal{TL}_n^m)^\oplus$ be the category with objects the symbols $\bigoplus_{i\geq 1} a_i$ with $a_i \in \mathcal{TL}_n^m$, $i \in \{1, 2, \ldots\}$ and morphisms $\bigoplus_{i\geq 1} a_i \to \bigoplus_{j\geq 1} b_j$ the matrices

$$(f_{ij}) \in \prod_{i,j\geq 1} \mathrm{Hom}_{\mathcal{TL}_n^m}(a_j, b_i)$$

with finite columns, i.e. for fixed j, $f_{ij} = 0$ for all but finitely many i.

2. Let $(\mathcal{TL}_n^m)^\Pi$ be the category with objects the symbols $\prod_{i\geq 1} a_i$ with $a_i \in \mathcal{TL}_n^m$, $i \in \{1, 2, \dots\}$, and morphisms $\prod_{i\geq 1} a_i \to \prod_{j\geq 1} b_j$ given by matrices
$$(f_{ij})_\in \prod_{i,j\geq 1} \mathrm{Hom}_{\mathcal{TL}_n^m}(a_j, b_i)$$
with finite rows, i.e. for fixed i, $f_{ij} = 0$ for all but finitely many j.

In any case composition of morphisms is given by matrix multiplication: $(f \circ g)_{ij} = \sum_k f_{ik} \circ g_{kj}$ which is always a finite sum, in light of the finiteness conditions on morphisms.

Let $\mathrm{Kom}(m,n)^\Pi := \mathrm{Kom}(\mathcal{TL}_n^{m\Pi})$ and $\mathrm{Kom}(m,n)^\oplus := \mathrm{Kom}(\mathcal{TL}_n^{m\oplus})$ denote the categories of potentially unbounded chain complexes with morphisms given by degree $(0,0)$ chain maps.

The following is clear:

Proposition 4.2. *The categories* $\mathrm{Kom}(m,n)^\Pi$ *and* $\mathrm{Kom}(m,n)^\oplus$ *naturally contain* $\mathrm{Kom}(m,n)$ *as a full subcategory. The planar composition functors extend naturally to multilinear functors on the categories* $\mathrm{Kom}(m,n)^\Pi$*, respectively* $\mathrm{Kom}(m,n)^\oplus$*, and these restrict to the usual planar composition functors on* $\mathrm{Kom}^\pm(n)$*, up to isomorphism.* □

Definition 4.3. If a planar composition of potentially unbounded chain complexes is to be evaluated in $\mathrm{Kom}(m,n)^\Pi$ or $\mathrm{Kom}(m,n)^\oplus$, then we will place the symbol Π, respectively $\oplus$ somewhere in our picture. We denote the extension of $\odot$ to $\mathrm{Kom}(m,n)^\Pi$ by $\odot^\Pi$, and to $\mathrm{Kom}(m,n)^\oplus$ by $\odot$.

4.2 Duality in $\mathcal{TL}_n^m$

In category theory one has a well-defined notion of duality, which can be motivated from the point of view of vector spaces. Let V, W be finite dimensional vector spaces over a field k, and put $V^* = \mathrm{Hom}_k(V, k)$. Let $\otimes = \otimes_k$. We have maps $\eta_V : k \to V \otimes V^*$ and $\varepsilon_V : V^* \otimes V \to k$ defined as follows: fix a basis $\{v_1, \ldots, v_r\} \subset V$ and let $\{v_1^*, \ldots, v_r^*\} \in V^*$ be the dual basis: $v_j^*(v_i) = \delta_{ij}$, the Kronecker delta.

1. $\varepsilon_V(\phi \otimes v) := \phi(v)$ for all $\phi \in V^*$ and all $v \in V$.

2. $\eta_V(1) := \sum_{i=1}^r v_i \otimes v_i^*$, which is independent from the choice of basis.

It is straightforward to check that

$$(\mathrm{Id}_V \otimes \varepsilon_V) \circ (\eta_V \otimes \mathrm{Id}_V) = \mathrm{Id}_V \qquad \text{and} \qquad (\varepsilon_A \odot \mathrm{Id}_{V^*}) \circ (\mathrm{Id}_{V^*} \odot \eta_V) = \mathrm{Id}_{V^*} \quad (4.4)$$

These "unit" and "counit" maps can be used to define a familiar natural isomorphism $\phi_{V,W} : \mathrm{Hom}_k(V, W) \cong \mathrm{Hom}_k(k, W \otimes V^*)$ given by:

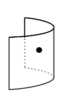
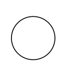
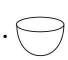
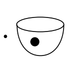
$$\phi_{V,W}(f)(1) = (f \otimes \mathrm{Id}_{V^*}) \circ \eta_V(1) = \sum_{i=1}^r f(v_i) \otimes v_i^*$$

for all $f \in \mathrm{Hom}_k(V, W)$. On the other hand, given such an isomorphism we could define η_V to be the image of Id_V in $\mathrm{End}_k(V) \cong \mathrm{Hom}(k, V \otimes V^*)$, and we could let $\varepsilon_V = (\eta_{V^*})^*$. Naturality of the isomorphism is equivalent to the properties (4.4).

Precisely the same sort of duality exists in the cobordism categories $\mathcal{TL}_n$.

Definition 4.5. Let $(\ )^\vee : \mathcal{TL}_m^n \to \mathcal{TL}_n^m$ be the contravariant functor which reverses the q-degree shifts, reflects all diagrams about the x-axis, and acts on morphisms (linear combinations of decorated surfaces in $D^2 \times I$) by the transformation $(x, y, z) \mapsto (x, -y, 1 - z)$. We call $(\ )^\vee$ the *duality functor*.

As a motivating example note that

$\mathrm{End}^{0,\bullet}_{\mathcal{TL}_2^0}($  $) \cong \mathbb{Z}\cdot\mathrm{Id} \oplus \mathbb{Z}\cdot$  and $\mathrm{Hom}^{0,\bullet}_{\mathcal{TL}_0}(\varnothing,$  $) \cong \mathbb{Z}\cdot$  $\oplus\ \mathbb{Z}\cdot$ 

We will produce an isomorphism between these two groups, but since as graded abelian groups the former is $\mathbb{Z} \oplus q^2\mathbb{Z}$ and the latter is $q^{-1}\mathbb{Z} \oplus q\mathbb{Z}$, we must first shift the gradings. This example is useful to keep in mind.

Definition 4.6. Fix integers $m, n \geq 0$ and put $\delta = (n-m)/2$. Let $a \in \mathcal{TL}_n^m$ be a planar tangle. Define $\eta_a \in \mathrm{Hom}_{\mathcal{TL}_m}(1_m, q^\delta a \odot a^\vee)$ for each $a \in \mathcal{TL}_n^m$ as follows. If a is a planar tangle, define η_a to be the result of bending the identity cobordism upward; i.e. η_a is the result of applying the operation shown in figure 4.1 to the identity cobordism $\mathrm{Id}_a \in D^2 \times [0,1]$. Extend by linearity to every object of $\mathcal{TL}_n^m$. That is to say, put $\eta_{qa} = q(\eta_a)$ and

$$\eta_{a \oplus b} = (\eta_a, \eta_b) \in \mathrm{Hom}(1_m, q^\delta a) \oplus \mathrm{Hom}(1_m, q^\delta b) \cong \mathrm{Hom}(1_m, q^\delta (a \oplus b))$$

Finally, put $\varepsilon_a := (\eta_{a^\vee})^\vee \in \mathrm{Hom}_{\mathcal{TL}_n}(q^\delta a^\vee \odot a, 1_n)$ for each $a \in \mathcal{TL}_n^m$.

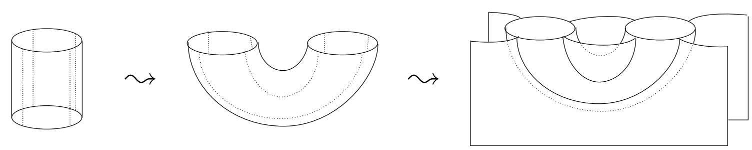
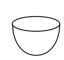


Figure 4.1: Bending a cylinder upward. This operation sends a cobordism $T \in \mathrm{Hom}_{\mathcal{TL}_n^m}(a, b)$ to a cobordism $T' \in \mathrm{Hom}_{\mathcal{TL}_m}(1_m, b \odot a^\vee)$ and is an isomorphism by Proposition 4.8. The picture drawn here corresponds to $m = n = 2$. Recall that we read cobordisms from bottom to top.

For example if $a \in \mathcal{TL}_{2n}^0$ is a diagram without circle components, then

- the diagram $a \odot a^\vee \in \mathcal{TL}_0$ consists of n disjoint circles, and $\eta_a : \varnothing \to q^n a \odot a^\vee$ is the cobordism given by n disjoint disks  which cap off each of these components.

- the diagram $a^\vee \odot a \in \mathcal{TL}_{2n}$ is the disjoint union of a and its reflection, and $\varepsilon_a : a^\vee \odot a \to 1$ is the iterated saddle cobordism given by attaching 1-handles on matching pairs of components (n in total).

The next proposition says that the usual, categorical notion of duality is satisfied.

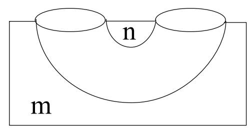
Lemma 4.7. *For each $a \in \mathcal{TL}_n^m$ the maps $\eta_a \in \mathrm{Hom}_{\mathcal{TL}_m}(1_m, q^\delta a \odot a^\vee)$ and $\varepsilon_a \in \mathrm{Hom}_{\mathcal{TL}_n}(q^\delta a^\vee \odot a, 1_n)$ of Definition 4.6 satisfy:*

1. $(\mathrm{Id}_a \odot \varepsilon_a) \circ (\eta_a \odot \mathrm{Id}_a) = \mathrm{Id}_a$ *and* $(\varepsilon_a \odot \mathrm{Id}_{a^\vee}) \circ (\mathrm{Id}_{a^\vee} \odot \eta_a) = \mathrm{Id}_{a^\vee}$

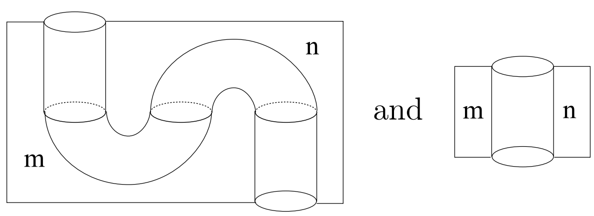
2. $(f \odot \mathrm{Id}_{a^\vee}) \circ \eta_a = (\mathrm{Id}_a \odot f^\vee) \circ \eta_b$

3. $\varepsilon_b \circ (\mathrm{Id}_{b^\vee} \odot f) = \varepsilon_a \circ (f^\vee \odot \mathrm{Id}_a)$

Proof. For a tangle $a \in \mathcal{TL}_n^m$, let us denote η_a as

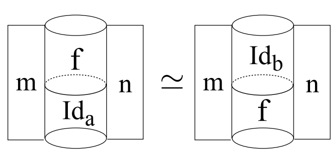
$\eta_a =$ 

where the sheet labelled n denotes n parallel copies, and similarly for m. It is understood that the identity cobordism of a is piped through the cylinder. The bilinear functor $\odot$ corresponds to gluing, and composition of morphisms corresponds to vertical stacking, so that $f \circ g$ is f on-top-of g. Note that

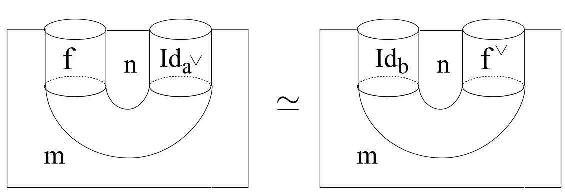


are isotopic embeddings. Piping the identity cobordism Id_a through the tubes gives $(\mathrm{Id}_a \odot \varepsilon_a) \circ (\eta_a \odot \mathrm{Id}_a) = \mathrm{Id}_a$, which is the first part of (1). The second part is similar. This proves (1) in the case where a is a planar tangle. Since every object is a formal direct sum of shifts of such objects, (1) follows by linearity.

Now, let $f : q^k a \to q^l b$ be a dotted cobordism, and note that we have an isotopy



Bending upward gives an isotopy



which implies (2) in this case. Since the dotted cobordisms generate the morphism spaces linearly, (2) follows by linearity.

Finally, (3) follows from (2) by an application of the duality functor $(\ )^\vee$.

$$\begin{aligned}((f \odot \mathrm{Id}_{a^\vee}) \circ \eta_a)^\vee &= ((\mathrm{Id}_a \odot f^\vee) \circ \eta_b)^\vee \\ \eta_a^\vee \circ (\mathrm{Id}_a \odot f^\vee) &= \eta_b^\vee \circ (f \odot \mathrm{Id}_{a^\vee}) \\ \varepsilon_{a^\vee} \circ (\mathrm{Id}_a \odot f^\vee) &= \varepsilon_{b^\vee} \circ (f \odot \mathrm{Id}_{a^\vee})\end{aligned}$$

Replacing a and b by by $b^\vee$, respectively $a^\vee$ gives (3). □

As motivated in the introduction of this subsection, we can compute the space of morphisms $a \to b$ in terms of η_a and ε_a via a standard procedure:

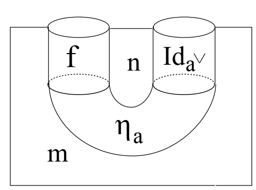
Proposition 4.8. *Fix integers m, n, and put $\delta = (n-m)/2$. The map $\phi_{a,b}(f) := (f \odot \mathrm{Id}_{a^\vee}) \circ \eta_a = (\mathrm{Id}_b \odot f^\vee) \circ \eta_b$ defines an isomorphism*

$$\mathrm{Hom}_{\mathcal{TL}_n^m}(a,b) \cong \mathrm{Hom}_{\mathcal{TL}_m}(1_m, q^\delta b \odot a^\vee)$$

Dually the mapping $f \mapsto \varepsilon_a \circ (f^\vee \odot \mathrm{Id}_a) = \varepsilon_b \circ (\mathrm{Id}_{b^\vee} \odot f)$ defines an isomorphism

$$\mathrm{Hom}_{\mathcal{TL}_n^m}(a,b) \cong \mathrm{Hom}_{\mathcal{TL}_n}(q^\delta b^\vee \odot a, 1_n)$$

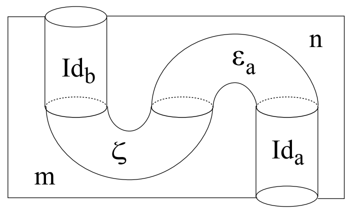
Proof. Put $\delta = (n-m)/2$, and define a degree zero map $\phi = \phi_{a,b} : \mathrm{Hom}_{\mathcal{TL}_n^m}(a,b) \to \mathrm{Hom}_{\mathcal{TL}_m}(1_m, q^\delta b \odot a^\vee)$ as in the hypotheses by $\phi(f) := (f \odot \mathrm{Id}_{a^\vee}) \circ \eta_a$, and define its proposed inverse $\psi = \psi_{a,b} : \mathrm{Hom}_{\mathcal{TL}_m}(1_m, q^\delta b \odot a^\vee) \to \mathrm{Hom}_{\mathcal{TL}_n^m}(a,b)$ by $\psi(\zeta) := (\mathrm{Id}_b \odot \varepsilon_a) \circ (\zeta \odot \mathrm{Id}_a)$. The fact that ϕ and ψ are inverses follows formally from Lemma 4.7. Graphically we may write

$\phi_{a,b}(f) \ =$ 

for all $f \in \mathrm{Hom}_{\mathcal{TL}_n^m}(a,b)$

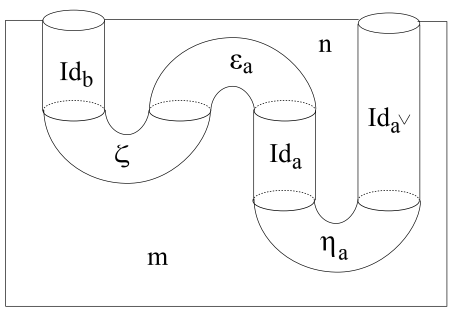
and

$\psi_{a,b}(\zeta)$ =



for all $\zeta \in \mathrm{Hom}(1_m, q^\delta b \odot a^\vee)$

Thus $\phi_{a,b} \circ \psi_{a,b}(\zeta)$ equals



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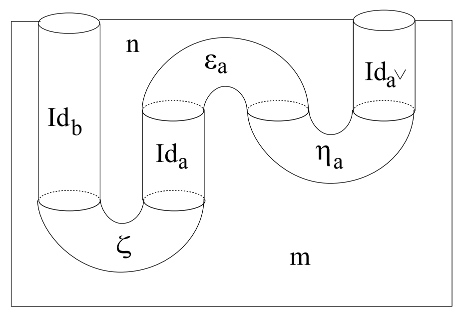
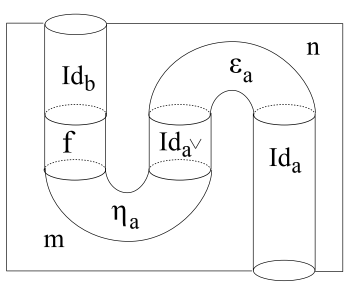


which equals ζ by straightening out the s-bend on the right: $(\varepsilon_a \odot \mathrm{Id}_{a^\vee}) \circ (\mathrm{Id}_a \odot \eta_a) = \mathrm{Id}_{a^\vee}$ by part (1) of Lemma 4.7. Similarly, $\psi_{a,b} \circ \phi_{a,b}(f)$ is



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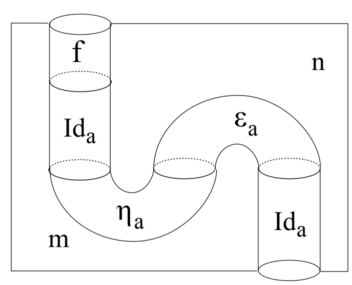


which equals f, again by straightening out the s-bend on the right. This completes the proof of the first statement.

The second statement follows by an application of $(\ )^\vee$. Indeed:

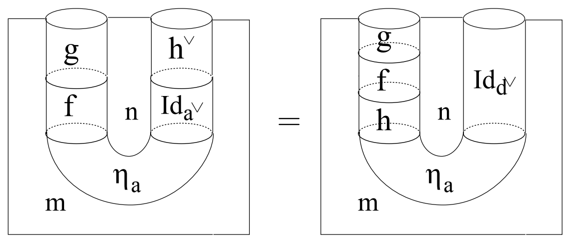
$$\mathrm{Hom}_{\mathcal{TL}_n^m}(a, b) \cong \mathrm{Hom}_{\mathcal{TL}_m^n}(b^\vee, a^\vee) \cong \mathrm{Hom}_{\mathcal{TL}_n}(1_n, q^{-\delta} a^\vee \otimes b) \cong \mathrm{Hom}_{\mathcal{TL}_n}(q^\delta b^\vee \otimes a, 1_n)$$

The first and third isomorphisms are given by $f \mapsto f^\vee$, and the second comes from part 1. This composition sends

$$f \mapsto \phi_{b^\vee, a^\vee}(f^\vee)^\vee = ((f^\vee \odot \mathrm{Id}_b) \circ \eta_{b^\vee})^\vee = \eta_{b^\vee}^\vee \circ (\mathrm{Id}_{b^\vee} \odot f) = \varepsilon_b \circ (\mathrm{Id}_{b^\vee} \odot f)$$

This proves the dual statement.

For naturality of these isomorphisms, suppose we have $h \in \mathrm{Hom}_{\mathcal{TL}_n^m}(c, a)$ and $g \in \mathrm{Hom}_{\mathcal{TL}_n^m}(b, d)$. We need to check that $(g \odot h^\vee) \circ \phi_{a,b}(f) = \phi_{c,d}(g \circ f \circ h)$ for all $f \in \mathrm{Hom}_{\mathcal{TL}_n^m}(a, b)$. This is a straightforward computation:



by sliding $h^\vee$ down and through the identity cylinder η_a using part (2) of Lemma 4.7. This completes the proof. □

This proposition will primarily be used as a method for computing hom spaces between (chain complexes of) objects of $\mathcal{TL}_n^m$, but we also obtain a proof that Bar-Natan's cobordism categories are isomorphic to the categories of finitely generated projective modules over Khovanov's rings H^n [Kho02].

Definition 4.9. Define a functor $\mathcal{TL}_0 \to \mathbb{Z}$-**gmod** into the category of finitel generated graded abelian groups by $|a\rangle = \mathrm{Hom}^{0,\bullet}_{\mathcal{TL}_n}(\varnothing, a)$. On morphisms $|f\rangle : |a\rangle \to |b\rangle$ is simply post-composition with $f \in \mathrm{Hom}_{\mathcal{TL}_0}(a, b)$. For each integer $n \geq 0$, define a ring $H^n := \bigoplus_{a,b} q^n |b \odot a^\vee\rangle$ where the sum is over a set (finite) of fixed representatives of isotopy classes of tangles $a, b \in \mathcal{TL}_{2n}^0$ without circle components. The multiplication on H^n is induced from the saddle maps:

$$\zeta \cdot \zeta' = (\mathrm{Id}_c \odot \varepsilon_b \odot \mathrm{Id}_{a^\vee}) \circ (\zeta \odot \zeta')$$

for $\zeta \in |c \odot b^\vee\rangle$, $\zeta \in |b \odot a^\vee\rangle$.

The ring H^n is precisely Khovanov's ring from [Kho02], non-negatively graded with unit lying in degree zero. The following is well-known, but its proof seems not to be written down anywhere.

Proposition 4.10. *The category $\mathcal{TL}_n$ is isomorphic to the category H^n-* **pgmod** *of finitely generated, graded, projective (left) H^n modules.*

Proof. Put ${}_b(H^n)_a = |b \odot a^\vee\rangle$. It is clear that $b \odot a^\vee$ is a closed diagram with some number, say k, of circles, and that ${}_b(H^n)_a$ is isomorphic to the tensor product of k copies of $|\bigcirc\rangle = \mathbb{Z}[x]/(x^2)$. The element $1_a \in {}_a(H^n)_a$ corresponding to $1 \otimes \cdots \otimes 1 \in (\mathbb{Z}[x]/(x^2))^{\otimes n}$ is an idempotent in H^n, and so each $P(a) := H^n 1_a$ is a graded projective H^n module. In [Kho02] Khovanov shows that every finitely generated graded projective module is a direct sum of the $P(a)$ shifted in grading. So we have a correspondence on objects $\mathcal{TL}_n \to H^n -$ **pgmod** given by

$$\bigoplus_{k=1}^{r} q^{j_k} T_k \to \bigoplus_{k=1}^{r} q^{j_k} P(a_k)$$

where $T_k \subset D^2$ is a 1-submanifold without circle components, regarded as an object of $\mathcal{TL}_n$, and $a_k \in$ is the representative of the isotopy class of T_k, the set of which was implicitly fixed in Definition 4.9. Every object of $H^n -$ **pgmod** is isomorphic to an object in the image of this mapping, so we need only check that the morphism spaces and composition maps agree. It is a formal property of modules of the form $H^n \cdot r$, $r \in H^n$ that the space of morphisms $P(a) \to P(b)$ is simply $1_b(H^n)1_a = {}_b(H^n)_a$, and the composition is induced from the multiplication in the ring H^n. Consider the following diagram:

$$\begin{array}{ccccc} \mathrm{Hom}(b,c)\otimes\mathrm{Hom}(a,b) & \xrightarrow{\mathrm{Id}\otimes\phi_{a,b}} & \mathrm{Hom}(b,c)\otimes|b\odot a^\vee\rangle & \xleftarrow{\phi_{b,c}^{-1}\otimes\mathrm{Id}} & |c\odot b^\vee\rangle\otimes|b\odot a^\vee\rangle \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}(a,c) & \xrightarrow{\phi_{a,c}} & |c\odot a^\vee\rangle & \xrightarrow{=} & |c\odot a^\vee\rangle \end{array}$$

where the left-most vertical map is given by composition of morphisms, the middle vertical map is $f \otimes \zeta \mapsto (f \odot \mathrm{Id}) \circ \zeta$ and the right-most vertical map is given by multiplication in H^n, i.e. $\zeta \otimes \zeta' \mapsto (\mathrm{Id} \odot \varepsilon_b \odot \mathrm{Id}) \circ (\zeta \odot \zeta')$. The first square commutes by naturality of $\phi_{a,b}$. Note that from the proof of Proposition 4.8 we have $\phi_{b,c}^{-1}(\zeta') =$

$(\mathrm{Id}_c \odot \varepsilon_b) \circ (\zeta' \odot \mathrm{Id}_b)$. Hence

$$(\phi_{b,c}^{-1}(\zeta') \odot \mathrm{Id}_{a^\vee}) \circ \zeta = (\mathrm{Id}_c \odot \varepsilon_b \odot \mathrm{Id}_{a^\vee}) \circ (\zeta' \odot \zeta)$$

so the second square commutes as well. This shows that multiplication in H^n and composition in $\bigoplus_{a,b} \mathrm{Hom}(a,b)$ are intertwined by the isomorphism $\phi_{a,b}$ from Proposition 4.8 and completes the proof. $\square$

4.3 Differential graded duality

In this section we extend the isomorphism of Proposition 4.8 to categories of unbounded chain complexes over $\mathcal{TL}_n$. We start by recalling the differential graded lift of the contravariant functor $(\ )^\vee : \mathcal{TL}_n^m \to \mathcal{TL}_m^n$ from Definition 4.5.

Definition 4.11. By abuse, denote by $(\ )^\vee$ the functor $\mathcal{TL}_n^{m\oplus} \to \mathcal{TL}_m^{n\,\Pi}$ defined on objects by $(\bigoplus_{i=1}^\infty a_i)^\vee = \prod_{i=1}^\infty a_i^\vee$. On morphisms (infinite matrices of morphisms in $\mathcal{TL}_m^n$), $(\ )^\vee$ acts as $(f_{ij})^\vee = (f_{ji}^\vee)$. Denote also by $(\ )^\vee$ the inverse functor $\mathcal{TL}_n^{m\Pi} \to \mathcal{TL}_m^{n\,\oplus}$ obtained by swapping the roles of $\oplus$ and Π. Finally, let $(\ )^\vee$ denote (once again by abuse of notation) the mutually inverse functors $\mathrm{Kom}_{dg}(m,n)^\oplus \leftrightarrow \mathrm{Kom}(n,m)^\Pi$ which are given by the differential graded lifts (Definition 2.25). Specifically, $(A^\vee)^k = (A^{-k})^\vee$ with differential $d_{A^\vee} = -d_A^\vee$, where on morphisms we have

$$(f^\vee)_{(B^{-k})^\vee} = (-1)^{k|f|}(f|_{A^{-k-|f|}})^\vee$$

and $|f|$ denotes homological degree.

In particular, if $A \in \mathrm{Kom}(m,n)$, we have a chain complex $A^\vee \in \mathrm{Kom}(n,m)$, but these are no longer dual in the category theoretic sense (this should be compared with the similar situation for infinite dimensional vector spaces). Nonetheless still have unit and counit maps, which we define next.

Definition 4.12. Fix integers $m, n \geq 0$ and put $\delta = (n-m)/2$. Let $A \in \mathrm{Kom}(m,n)$ be arbitrary. For each $i \in \mathbb{Z}$ we have

$$\eta_{A^i} : 1_m \to q^\delta A^i \odot (A^i)^\vee$$

given in Definition 4.6. Define η_A to be the direct product of these maps:

$$\eta_A : 1_m \to q^\delta \prod_{i\in\mathbb{Z}} A^i \odot (A^i)^\vee$$

The right hand side is nothing other than the degree zero chain group of $q^\delta A \odot^\Pi A^\vee$, hence we regard η_A as an element $\eta_A \in \mathrm{Hom}^{0,0}_{\mathcal{TL}^\Pi_m}(1_m, q^\delta A \odot^\Pi A^\vee)$. Similarly, define $\varepsilon_A \in \mathrm{Hom}^{0,0}_{\mathcal{TL}^\oplus_n}(q^\delta A^\vee \odot A, 1_n)$ to be the element $\varepsilon_A = (\eta_{A^\vee})^\vee$.

It will turn out that η_A and ε_A are chain maps, but this needs to be proven. We do not have a dg version of statement (1) in Lemma 4.7, but we do still have versions of (2) and (3), which is all we will need for our dg duality theorem.

Lemma 4.13. *For each $f \in \mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}^m_n}(A,B)$ we have*

(2) $(f \odot^\Pi \mathrm{Id}_{A^\vee}) \circ \eta_A = (\mathrm{Id}_B \odot^\Pi f^\vee) \circ \eta_B$

(3) $\varepsilon_A \circ (f^\vee \odot \mathrm{Id}_A) = \varepsilon_B \circ (\mathrm{Id}_{B^\vee} \odot f)$

Proof. By the formulas which define the dg extension of $\odot$ and $(\ )^\vee$ (definitions 2.21 and 2.25, we have

$$\begin{aligned}
(\mathrm{Id}_B \odot^\Pi f^\vee) \circ \eta_B &\overset{(1)}{=} (\mathrm{Id}_B \odot^\Pi f^\vee) \circ \prod_{i\in\mathbb{Z}} \eta_{B^i} \\
&\overset{(2)}{=} \prod_{i\in\mathbb{Z}} (\mathrm{Id}_B \odot^\Pi f^\vee)|_{B^i \odot (B^i)^\vee} \circ \eta_{B^i} \\
&\overset{(3)}{=} \prod_{i\in\mathbb{Z}} (-1)^{ik} (\mathrm{Id}_{B^i} \odot (f^\vee|_{(B^i)^\vee})) \circ \eta_{B^i} \\
&\overset{(4)}{=} \prod_{i\in\mathbb{Z}} (-1)^{ik+ik} (\mathrm{Id}_{B^i} \odot f|^\vee_{A^{i-k}}) \circ \eta_{B^i} \\
&\overset{(5)}{=} \prod_{i\in\mathbb{Z}} (f|_{A^{i-k}} \odot \mathrm{Id}) \circ \eta_{A^{i-k}} \\
&\overset{(6)}{=} (f \odot^\Pi \mathrm{Id}_{A^\vee}) \circ \eta_A
\end{aligned}$$

In the third and fourth identities we used the definition of the dg lifts of $\odot$ (using direct product) and $(\ )^\vee$, respectively. In the fifth we used the property $(\alpha \odot \mathrm{Id}) \circ \eta_a = (\mathrm{Id} \odot \alpha^\vee) \circ \eta_b$ from Lemma 4.7 This proves (1). (2) follows by an application of $(\ )^\vee$. □

Corollary 4.14. *The maps η_A and ε_A of Definition 4.12 are chain maps.*

Proof. Recall that (1) the differential on the $\mathrm{Hom}^\bullet$ complexes is given by super-commutator with the differential, (2) the differential on $A \odot^\Pi A^\vee$ is $d_A \odot^\Pi \mathrm{Id} - \mathrm{Id}_A \odot^\Pi d_A^\vee$, (3) the differential on 1_m is zero. An entirely formal calculation now gives

$$\begin{aligned}
[d, \eta_A] &= d_{A \odot^\Pi A^\vee} \circ \eta_A \\
&= (d_A \odot^\Pi \mathrm{Id}_{A^\vee}) \circ \eta_A - (\mathrm{Id}_A \odot^\Pi d_A^\vee) \circ \eta_A \\
&= (d_A \odot^\Pi \mathrm{Id}_{A^\vee}) \circ \eta_A - (d_A \odot^\Pi \mathrm{Id}_{A^\vee}) \circ \eta_A \\
&= 0
\end{aligned}$$

In the third identity we used Lemma 4.13. So η_A is a chain map. Applying $(\ )^\vee$ shows that ε_A is a chain map as well. □

Theorem 4.15. *Fix integers m, n, and put $\delta = (n-m)/2$. The chain map $\phi_{A,B}(f) := (f \odot^\Pi \mathrm{Id}_{A^\vee}) \circ \eta_A = (\mathrm{Id}_B \odot^\Pi f^\vee) \circ \eta_B$ defines an isomorphism*

$$\mathrm{Hom}^\bullet_{\mathcal{TL}_n^m}(A, B) \cong \mathrm{Hom}^\bullet_{\mathcal{TL}_m^\Pi}(1_m, q^\delta B \odot^\Pi A^\vee)$$

Dually the mapping $f \mapsto \varepsilon_a \circ (f^\vee \odot \mathrm{Id}_a) = \varepsilon_b \circ (\mathrm{Id}_{b^\vee} \odot f)$ defines an isomorphism

$$\mathrm{Hom}^\bullet_{\mathcal{TL}_n^m}(A, B) \cong \mathrm{Hom}^\bullet_{\mathcal{TL}_n^\oplus}(q^\delta B^\vee \odot A, 1_n)$$

By replacing A by $q^k A$ and taking the direct sum over $k \in \mathbb{Z}$ we obtain the analogous isomorphisms for $\mathrm{Hom}^{\bullet,\bullet}$ complexes.

Define $L_f(\alpha) := f \circ \alpha$ and $R_f(\alpha) = (-1)^{|f||a|} \alpha \circ f$ whenever these compositions make sense in $\mathrm{Kom}(m, n)$. The isomorphisms of Theorem 4.15 are natural in A and B in the following sense: $\phi_{A,C} \circ L_f = L_{f \odot \mathrm{Id}} \circ \phi_{A,B}$ for all $f \in \mathrm{Hom}^{\bullet,\bullet}(B, C)$ and $\phi_{C,B} \circ R_f = L_{\mathrm{Id} \odot f^\vee} \circ \phi_{A,B}$ for all $f \in \mathrm{Hom}^{\bullet,\bullet}(C, A)$.

Proof of Theorem 4.15. This follows formally from Proposition 4.8 and properties of dg functors. Note that $f \mapsto f \odot^\Pi \mathrm{Id}_{A^\vee}$ is a chain map $\mathrm{Hom}^\bullet(A,B) \to \mathrm{Hom}^\bullet(A \odot^\Pi A^\vee, B \odot^\Pi A^\vee)$. Further, precomposition with the chain map $\eta_A : q^{-\delta} 1_m \to A \odot^\Pi A^\vee$ gives a chain map $\mathrm{Hom}^\bullet(A \odot^\Pi A^\vee, B \odot^\Pi A^\vee) \to \mathrm{Hom}^\bullet(q^{-\delta} 1_m, B \odot^\Pi A^\vee)$. Composing these maps gives the chain map $\phi_{A,B}$ defined in the hypotheses. To see that this is an isomorphism, it suffices to show that it induces an isomorphism of the underlying bigraded objects (i.e. forgetting the differentials).

Let$f_{ij} \in \mathrm{Hom}^{\bullet,\bullet}(A^j, B^i)$ denote the components of $f \in \mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_n^m}(A,B)$. Assume that f has q-degree zero (not especially important here) and homological degree k, so in particular $f_{ij} = 0$ unless $i - j = k$. By definition, $\phi_{A,B}(f) = (f \odot^\Pi \mathrm{Id}_{A^\vee}) \circ \eta_A$ is the product over i of the following compositions of maps:

$$1_m \xrightarrow{\eta_{A^i}} A^i \odot (A^i)^\vee \xrightarrow{f_{i+k,i} \odot \mathrm{Id}_{(A^i)^\vee}} B^{i+k} \odot (A^i)^\vee$$

In other words, in terms of components, $\phi_{A,B}$ induces a map

$$\prod_{i-j=k} \mathrm{Hom}_{\mathcal{TL}_n^m}(A^j, B^i) \to \prod_{i-j=k} \mathrm{Hom}_{\mathcal{TL}_m}(1_m, q^\delta B^i \odot (A^j)^\vee)$$

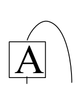
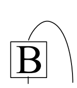
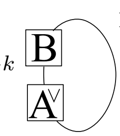
which is nothing other than the component-wise application of the isomorphism ϕ_{A^j,B^i} from Proposition 4.8. This shows that $\phi_{A,B}$ is an isomorphism. The dual statement follows from an application of $(\ )^\vee$. Naturality follows as in the proof of Proposition 4.8. $\square$

Corollary 4.16. *We have an isomorphism*

$$\theta_{A,B} : \mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_m^n}(A,B) \cong q^{(m+n)/2} |\mathrm{Tr}(B \odot^\Pi A^\vee)\rangle$$

which is natural in $A, B \in \mathrm{Kom}(m,n)$*, where* $\mathrm{Tr} : \mathrm{Kom}(m) \to \mathrm{Kom}(0)$ *is the Markov trace and* $|C\rangle := \mathrm{Hom}^{\bullet,\bullet}(\varnothing, C)$.

Proof. Compute

$\mathrm{Hom}^\bullet(q^k A, B) \cong \mathrm{Hom}^\bullet\Big(q^k$  ,  $\Big) \cong q^{(n+m)/2}\,\mathrm{Hom}^\bullet\Big(\varnothing, q^{(m+n)/2-k}$  $^{\Pi}\Big),$

The first isomorphism comes from an isomorphism of categories $\mathcal{TL}_m^n \cong \mathcal{TL}_{n+m}^0$ obtained by bending the the top-most strands to the right and down. The second isomorphism is the $m = 0$ special case of Theorem 4.15 (note that $1_0 = \varnothing$). Taking the direct sum over $k \in \mathbb{Z}$ of these isomorphisms gives the result. $\square$

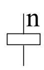
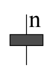
As an immediate corollary we have:

Corollary 4.17. *For $M \in \mathrm{Kom}(n-1)$, $N \in \mathrm{Kom}(n)$, we have natural isomorphisms*

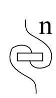
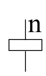
1. $\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_n}(M \sqcup 1, N) \cong q\,\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_{n-1}}(M, T(N))$
2. $\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_n}(N, M \sqcup 1) \cong q\,\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_{n-1}}(T(N), M)$

where $T(N)$ denotes the partial trace functor from Definition 3.9. $\square$

4.4 Graphical calculus

Theorem 4.15 says that we can compute the chain complex of morphisms between planar compositions of projectors in terms of a related planar compositions of $P_n =$  and $P_n^\vee =$ . We give some rules which can be used to simplify many such compositions, as well as examples illustrating the danger of mistreating them. We will frequently perform planar isotopies on our diagrams, and we will need to check that the corresponding chain complex is well defined. Propositions 4.21 and 4.26 will be used throughout this thesis.

Proposition 4.18. *Let M and N be planar compositions of P_n's for various n. If the underlying diagrams (a union of arcs and rectangles in the disk) for M and N are isotopic rel boundary, then M and N are homotopy equivalent.*

Proof. It is clear that  is a universal projector (Definition 3.12, hence is homotopy equivalent to . This fact together with isotopy invariance in the underlying categories $\mathcal{TL}_n^m$, gives the result. $\square$

Definition 4.19. Let $c \in \mathrm{Cob}_n^m \subset \mathcal{TL}_n^m$ be any object. Define the *through-degree* of c, denoted $\tau(c)$, to be the minimal k such that $c = a \odot b$, with $a \in \mathcal{TL}_k^m$, $b \in \mathcal{TL}_n^k$. Define the through-degree of a chain complex $C \in \mathrm{Kom}(m,n)^\Pi$ or $\mathrm{Kom}(m,n)^\oplus$ to be $\tau(C) = \max\{\tau(c)\}$, where c ranges over all direct summands (or factors) of all chain groups of C.

It is clear that $\tau(A^\vee) = \tau(A)$ and $\tau(A \odot B), \tau(A \odot^\Pi B) \leq \min\{\tau(A), \tau(B)\}$. All of the results in the remainder of this section are consequences of

1. The projectors P_n kill turnbacks.

2. P_n can be written as a mapping cone $P_n = \mathrm{Cone}(N \xrightarrow{f} 1_n)$ with $\tau(N) < n$, which follows from axiom (CK1) of Definition 3.12.

3. The following lemma:

Lemma 4.20 (Turnback killing lemma)**.** *Suppose $Q \in \mathrm{Kom}(n,m)$ kills turnbacks from above and $N \in \mathrm{Kom}(k,n)$ has through degree $\tau(N) < n$. If Q is bounded from below or N is bounded from above, then $N \odot Q \simeq 0$. The same is true if we allow $N \in \mathrm{Kom}(k,n)^\oplus$.*

Proof. The chain complex $N \odot Q \in \mathrm{Kom}(k,m)^\oplus$ is the total complex (using $\bigoplus$) of a bicomplex

$$\cdots \to N^{i-1} \odot Q \to N^i \odot Q \to N^{i+1} \odot Q \to \cdots$$

By assumption each N^i is a direct sum $\bigoplus_x a_x$ where a_x is a diagram with $\tau(a_x) < n$. Up to isomorphism, each a_x is a product of Temperley-Lieb generators e_i. Since Q kills turnbacks, we have $a_x \odot Q \simeq 0$ for each x, hence $N^i \odot Q \simeq 0$ for all i.

Assume first that N is bounded from above, i.e. $N^i = 0$ for $i \gg 0$. Then Theorem 2.15 applies, and so we can contract each of these terms, obtaining $N \odot Q \simeq 0$ as desired. If N is not bounded from below, write $N = \mathrm{Cone}(M \to N')$ where N' is supported in non-negative homological degrees and M is supported in non-positive homological degrees. By what has been said, $M \odot Q \simeq 0$ and so we have

$$N \odot Q \cong \mathrm{Cone}(M \odot Q \to N' \odot Q) \simeq N' \odot Q.$$

This latter chain complex is the total complex (using $\bigoplus$) of the bicomplex

$$N^0 \odot Q \to N^1 \odot Q \to \cdots$$

which as a bigraded object is the direct sum $\bigoplus_{i\geq 0} t^i(N^i \odot Q)$. If Q is bounded from below, then this direct sum is finite in each degree, hence equivalent to a direct product in the graded category. We can once again use Theorem 2.15 to contract each term, obtaining $N' \odot Q \simeq 0$. This completes the proof. □

We will refer to the application of either of following two propositions as *projector absorbing*; we will repeatedly use these results throughout the rest of this thesis.

Proposition 4.21. *If $Q \in \mathrm{Kom}(n,m)^{\oplus}$ kills turnbacks from above, then $P_n \odot Q \simeq Q$. In fact, the composition*

$$Q \quad = \quad 1_n \odot Q \xrightarrow{\iota \odot \mathrm{Id}_Q} P_n \odot Q$$

is the section of a deformation retract, where $\iota : 1_n \to P_n$ is the inclusion of the degree zero chain group. Dually, if $Q \in \mathrm{Kom}(m,n)^{\Pi}$ kills turnbacks from above, then $\iota^{\vee} \odot \mathrm{Id}_Q : P_n^{\vee} \odot^{\Pi} Q \to Q$ is a deformation retract. We have similar facts if Q kills turnbacks from below.

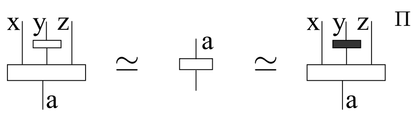
Proof. By the axioms for universal projectors, we may write $P_n = \mathrm{Cone}(N \to 1_n)$, where $N \in \mathrm{Kom}^{-}(\mathcal{TL}_n)$ has through degree $\tau(N) < n$. Then $N \odot Q \simeq 0$ by Lemma 4.20, and so Gaussian elimination (proposition 2.14) gives a deformation retract

$$P_n \odot Q \cong \mathrm{Cone}(N \odot Q \to 1_n \odot Q) \to 1_n \odot Q \tag{4.22}$$

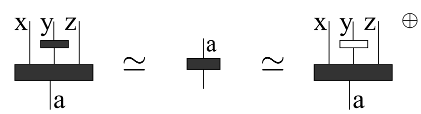
This deformation retract has section given by the obvious inclusion of the $1_n \odot Q$ summand on the left hand side of (4.22), which is precisely the map $\iota \odot \mathrm{Id}_Q$. Symmetrically different versions of this argument, and an application of $(\ )^\vee$, establish the other statements of the proposition. □

The next three results constitute the most important relations in our graphical calculus.

Proposition 4.23 (Absorption rule)**.** *Fix non-negative integers x, y, z, and put $a := x + y + z$. Then we have*



and

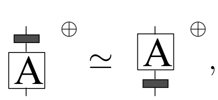


and vertical reflections of these.

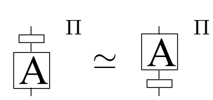
In fact by Proposition 4.21 we can find nice representative for each of these equivalences, each of which can be chosen to be a deformation retract. For example if $\iota_n : 1_y \to P_y$ is the inclusion of the degree zero chain group then $(\mathrm{Id}_{1_x} \sqcup \iota_y \sqcup \mathrm{Id}_{1_z}) \odot \mathrm{Id}_{P_a} : P_a \to (1_x \sqcup P_y \sqcup 1_z) \odot P_a$ is the section of a deformation retract, and $(\mathrm{Id}_{1_x} \sqcup \iota^\vee \sqcup \mathrm{Id}_{1_z}) \odot \mathrm{Id}_{P_a} : (1_x \sqcup P_y^\vee \sqcup 1_z) \odot P_a \to P_a$ is a deformation retract. This takes care of the equivalences on the left above; there are similar descriptions of the the equivalences on the right.

Proof. Regard P_a as an object of $\mathrm{Kom}(y, x + a + z)$ by bending the x top left-most strands to the left and down and the z top right-most strands to the right and down. Call the resulting chain complex Q. Then Q kills turnbacks from above. By Proposition 4.21 we have $P_y \odot Q \simeq Q$, which is the first equivalence in the statement above. The remaining equivalences are proven similarly. □

Proposition 4.24 (Commuting rule)**.** *Let $A \in \mathrm{Kom}(n)^\oplus$ (respectively $A \in Kom(n)^\Pi$) be arbitrary. We have*



respectively



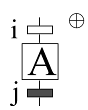
Proof. Fix $A \in \text{Kom}(n)^{\oplus}$, and let $e_i \in \mathcal{TL}_n$ be a Temperley-Lieb generator. Then $\tau(e_i \odot A) < n$, and so $e_i \odot (A \odot P_n^{\vee}) \cong (e_i \odot A) \odot P_n^{\vee} \simeq 0$ by Lemma 4.20 (which applies since $P_n^{\vee}$ is bounded from below). This is to say, $A \odot P_n^{\vee}$ kills turnbacks from above. $A \odot P_n^{\vee}$ clearly kills turnbacks from below since $P_n^{\vee}$ does. Similarly, $P_n^{\vee} \odot A$ kills turnbacks from above and below. Two applications of Proposition 4.21 now give

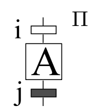
$$A \odot P_n^{\vee} \simeq P_n^{\vee} \odot (A \odot P_n^{\vee}) \cong (P_n^{\vee} \odot A) \odot P_n^{\vee} \simeq P_n^{\vee} \odot A.$$

This proves the first statement. The second follows by an application of $(\ )^{\vee}$. □

The proof of the next proposition uses a result from later in the thesis, but there is no circularity.

Proposition 4.25 (Orthogonality rule)**.** *Suppose $i \neq j$ and let $A \in \text{Kom}(i,j)^{\oplus}$ (respectively $A \in \text{Kom}(i,j)^{\Pi}$) be a planar composition of projectors and dual projectors. We have*



$\simeq 0$ *respectively*


$\simeq 0.$

We remark that if $i = j$ then the absorption and commuting rules tell us how to simplify either of these expressions (simply erase the white box in the $\oplus$ case and erase the black box in the Π case).

Proof. Suppose $j < i$, and let $A \in \text{Kom}(i,j)^{\Pi}$ be arbitrary. Note that $\tau(A \odot P_j^{\vee}) \leq i < j$. Since P_i is bounded from above, a dual version of Lemma 4.20 applies, and we have $P_i \odot^{\Pi} (A \odot^{\Pi} P_j^{\vee}) \simeq 0$. Note that this holds without the assumption that A is a planar composition of projectors and dual projectors.

If, on the other hand, $j > i$ then we need to do more work. By Observation 6.29, P_n is homotopy equivalent to the inverse limit of bounded chain complexes:

$$P_n \simeq \lim_{\infty \leftarrow k} E_k^{(n)}$$

Assume A is a planar composition of projectors and dual projectors. Planar composition T^{Π} commutes with inverse limits, so by "expanding" the projectors, we see that

$P_i \odot A$ is the inverse limit

$$P_i \odot A \simeq \lim_{\infty \leftarrow k} E_k$$

where each E_k is bounded from below, and $\tau(E_k) \leq i < j$. A dual version of Lemma 4.20 implies that $E_k \odot^{\Pi} P_j^{\vee} \simeq 0$ for each k. Therefore

$$(P_i \odot^{\Pi} A) \odot^{\Pi} P_j^{\vee} \simeq \lim_{\infty \leftarrow k} E_k \odot^{\Pi} P_j \simeq 0$$

This completes the proof. □

We have concluded the set-up of our calculus, and in the following sections we will illustrate their power with some interesting examples; we will also illustrate the necessity of the conditions on gradings in the various hypotheses with some counterexamples. But first, we conclude this section with a simple observation:

Proposition 4.26. *If $Q \in \mathrm{Kom}^{\pm}(n)$ kills turnbacks from below, then Q kills turnbacks from above as well, and vice versa.*

Proof. Assume $Q \in \mathrm{Kom}^{-}(n)$ kills turnbacks from below. Since Q is bounded from above, we have $Q \odot P_n = Q \odot^{\Pi} P_n$ and so Proposition 4.24 implies that $Q \odot P_n \simeq P_n \odot Q$. Since Q kills turnbacks from below, Proposition 4.21 says also that $Q \odot P_n \simeq Q$. In other words

$$Q \simeq Q \odot P_n \simeq P_n \odot Q$$

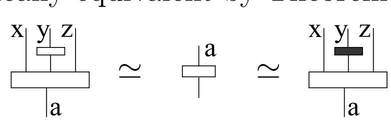
This latter chain complex kills turnbacks from below since Q does, and from above since P_n does. This completes the proof in this case.

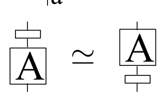
If on the other hand $Q \in \mathrm{Kom}^{+}(n)$ kills turnbacks from below, it is necessary to replace P_n by $P_n^{\vee}$ in the previous discussion, obtaining $Q \simeq Q \odot P_n^{\vee} \simeq P_n^{\vee} \odot Q$, which kills turnbacks. The remaining cases follow by symmetry. □

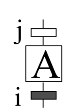
Remark 4.27. This result allows us to simplify the expression for the universal projector constructed in [CK12]. The result of the Cooper-Krushkal [CK12] construction (up to the penultimate step) is a chain complex which is a convolution of the Frenkel-Khovanov sequence (3.17). Cooper and Krushkal prove that such a chain complex P'_n satisfies axiom (CK1) from Definition 3.12 and kills turnbacks from below, hence $P_n := s_x(P'_n) \odot P'_n$ is a universal projector, where $s_x : \mathrm{Kom}(n) \to \mathrm{Kom}(n)$ is the functor given by vertical reflection. Once we know that a universal projector exists, Proposition 4.26 implies that we already had one at the previous step: P'_n is bounded above in homological degree and kills turnbacks from below, so it kills turnbacks from above.

4.5 Some computations

In this section we will implicitly be working with categories $\mathcal{TL}_n^{m\Pi}$, and so we will omit the symbol Π from our planar compositions. Our work up to this point says that we can compute the chain complex $\mathrm{Hom}^{\bullet}_{\mathcal{TL}_n^m}(M, N)$ of morphisms between planar compositions of P_n in the following way:

1. Reflect M and replace all the white boxes with black boxes to obtain $M^\vee$
2. Glue up all of the loose ends of N with the corresponding loose ends of $M^\vee$
3. Simplify using the following relations:
   (a) Diagrams which are isotopic rel boundary give homotopy equivalent chain complexes (canonically equivalent by Theorem 5.3).
   (b) Absorption rules: 

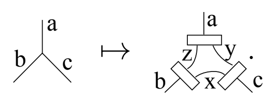
(proposition 4.23).
   (c) Commuting rule: 

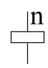
for every chain complex A over $\mathcal{TL}_n$ (proposition 4.24).
   (d) Orthogonality rule: if $i \neq j$ then 

$\simeq 0$ for every chain complex $A \in \mathrm{Kom}(j, i)^\Pi$ which is a planar composition of projectors and dual projectors (proposition 4.25).

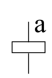
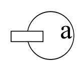
4. Take $\mathrm{Hom}^\bullet(\varnothing, -)$ of the result.

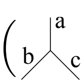
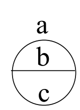
The planar composition of complexes over $\mathcal{TL}_n^{m\Pi}$ is spherical, in the sense that it doesn't matter how we connect up the loose ends in (2) above, as long as the strands are paired correctly. By invariance under planar isotopy, we also know that the line of reflection is irrelevant. Throughout the rest of this section, let a labelled trivalent graph denote a planar composition of projectors via the following rule

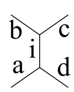
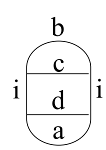


We are using the convention that a strand labelled by a non-negative integer n denotes n parallel copies of itself and  denotes P_n. In order to form the above planar composition it is necessary and sufficent that $a+b+c \in 2\mathbb{Z}$ and that any sum of two elements from $\{a, b, c\}$ is no smaller than the third.

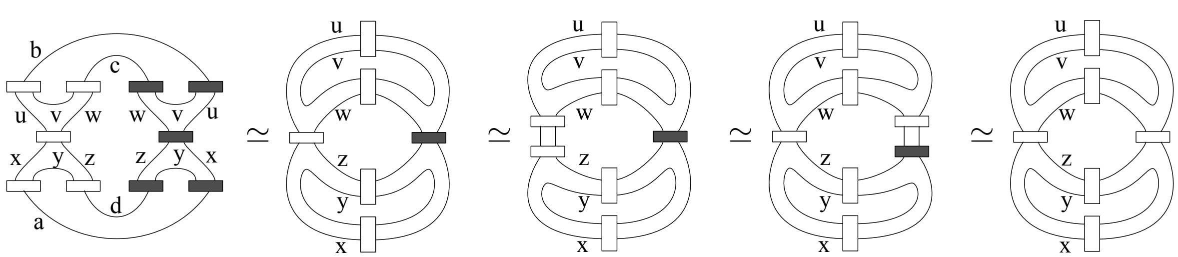
Proposition 4.28. *Put* $|A\rangle := \mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_0}(\varnothing, A)$ *for all* $A \in \mathrm{Kom}(0)$. *We have*

1. $\mathrm{End}^\bullet\Big($  $\Big) \simeq q^a|$  $\rangle$.

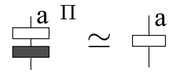
2. $\mathrm{End}^\bullet\Big($  $\Big) \simeq q^{(a+b+c)/2}|$  $\rangle$.

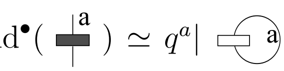
3. $\mathrm{End}^\bullet\Big($  $\Big) \simeq q^{(a+b+c+d)/2}|$  $\rangle$.

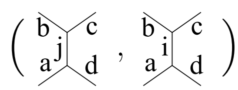
Proof. Let us prove (3) only. The other parts are special cases of this one. Observe that in the category $\mathcal{TL}_0^\Pi$ we have

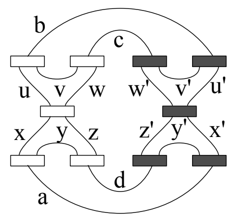


,

In the first equivalence we repeatedly used that  (proposition 4.23), and in the third we the commuting rule (proposition 4.24). An application of $|\ \ \rangle$, together with the isomorphism from Corollary 4.16, now gives the result. □

Note that the duality functor automatically gives an isomorphism $\mathrm{End}^\bullet_{\mathcal{TL}_n}(A) \cong \mathrm{End}^\bullet_{\mathcal{TL}_n}(A^\vee)$ for any $A \in \mathrm{Kom}(\mathcal{TL}_n)$. For example . This illustrates a preference for chain complexes which are bounded above, and is our reason for choosing our conventions the way we have: we want the colored unknots give the endomorphism rings of the colored arcs.

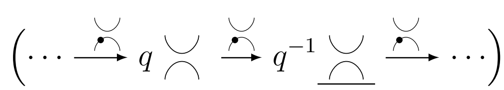
Proposition 4.29. *If $j \neq i$ then* $\mathrm{Hom}^\bullet$  $\simeq 0$.

Proof. Proposition 4.25 says that  $\simeq 0$ if $j := x' + z' < x + z =: i$.

Duality (theorem 4.15) now gives the result. □

The following (counter) example is useful to keep in mind:

Example 4.30. Let $A \in \mathrm{Kom}(\mathcal{TL}_2)$ be the chain complex

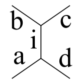
$A :=$ 

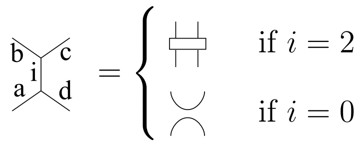
It is not hard to show that this chain complex kills turnbacks from below but not from above. Therefore

$$A \odot P_2 \simeq A \not\simeq P_2 \odot A,$$

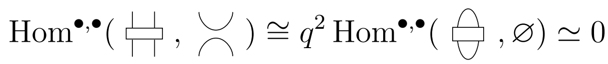
since the latter chain complex kills turnbacks from above as well as below, and the former does not. This gives a counter-example to the statements "$P_n \odot A \simeq A \odot P_n$ for $A \in \mathrm{Kom}(\mathcal{TL}_n)$" and "$P_n \odot A \simeq 0$ for $\tau(A) < n$." In particular, the hypotheses of the commuting and semi-orthogonality rules (propositions 4.24 and 4.25, respectively) are necessary.

4.6 Higher order computatons

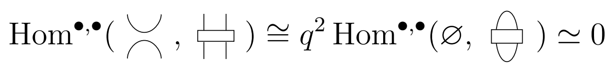
Propositions 4.28 and 4.29 are straightforward applications of the graphical calculus, but many important Hom complexes are more interesting. For example, the orthogonality in Proposition 4.29 is destroyed if the arguments are instead replaced by *convolutions* built out of these networks. Consider the case $a = b = c = d = 1$. The only possibilities for the networks  in this case are

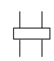
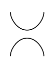


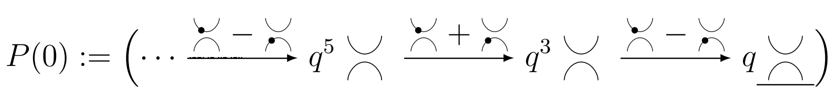
In agreement with Proposition 4.29 we have



and



On the other hand there do exist nontrivial chain maps from  to the following chain complex built out of copies of  :



Indeed, $P(0)$ is the "tail" of $P(2) := P_2$ (see (3.15), and the projection map $\pi : tP(2) \to P(0)$ cannot possibly be nulhomotopic, for otherwise $1_2 \simeq \mathrm{Cone}(\pi)$ would split as a direct sum $P(2) \oplus P(0)$, which is absurd. In fact, the complexes $P(0)$ and $P(2)$ are special cases of the higher order projectors constructed by the author and Benjamin Cooper in [CH12]. The hom complexes between the higher order projectors are highly nontrivial in general, and are important in understanding the categories $\mathcal{TL}_n$. We illustrate some of the techniques for studying them in the case $n = 2$ below:

Proposition 4.31. *Write* $\mathrm{End} = \mathrm{End}^{\bullet,\bullet}$, $\mathrm{Hom} = \mathrm{Hom}^{\bullet,\bullet}$, *and* $1 = 1_2$. *The complexes of morphisms between* $P(0)$ *and* $P(2)$ *satisfy*

1. $\mathrm{End}(P(2)) \simeq \mathrm{Hom}(1, P(2))$ *is supported in non-positive homological degrees.*

2. $\mathrm{End}(P(0)) \simeq \mathrm{Hom}(1, P(0)^\vee)$ *is supported in non-negative homological degrees.*

3. $\mathrm{Hom}(P(2), P(0)) \simeq \mathrm{Hom}(1, P(0)_\infty)$, *where* $P(0)_\infty$ *is the bi-infinite chain complex (4.37).*

4. $\mathrm{Hom}(P(0), P(2))$ *is contractible.*

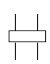
In fact, we can show that the chain complexes appearing in statements (1), (2), and (3) of Proposition 4.31 are finitely generated modules over rings $\mathbb{Z}[u_2], \mathbb{Z}[u_2^{-1}], \mathbb{Z}[u_2, u_2^{-1}]$, respectively, where u_2 is an indeterminate of bidegree $(-2, 4)$ (see Theorem 6.37). It is interesting to note that each of these hom complexes can be computed in terms of a limit of the torus braids on 2-strands.

Proof. Observe that $P(0)$ is simply the tail of the projector $P(2)$, i.e. $P(2) \simeq \mathrm{Cone}(P(0) \to 1)$. It follows that $1 = 1_2$ can be expressed as a convolution

$$1_2 \simeq (P(2) \to P(0)) \tag{4.32}$$

Taking duals gives

$$1_2 \simeq (P(0)^\vee \to P(2)^\vee) \tag{4.33}$$

Since $P(2) =$  kills turnbacks and $P(0)$ has through degree zero, Lemma 4.20 implies that

$$P(2) \odot^\Pi P(0)^\vee \simeq 0. \tag{4.34}$$

Hence by duality (theorem 4.15), we have statement (4) of the proposition.

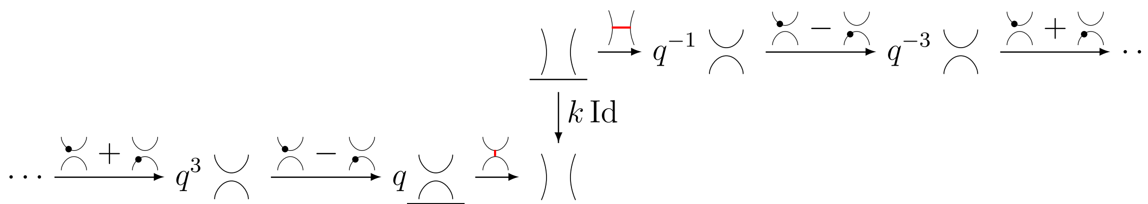
Applying $P(2) \odot^\Pi (-)$ to (4.33) and contracting the term (4.34) gives $P(2) \simeq P(2) \odot^\Pi P(2)^\vee$, which we already knew by projector absorbing. Duality now gives statement (1).

Applying $(-) \odot^\Pi P(0)^\vee$ to (4.32) and contracting the term (4.34) gives $P(0)^\vee \simeq P(0) \odot^\Pi P(0)^\vee$ which implies statement (2).

Finally, applying $(-) \odot^{\Pi} P(2)^{\vee}$ to $P(0) \simeq (1_2 \to tP(2))$ gives

$$P(0) \odot^{\Pi} P(2)^{\vee} \simeq (P(2)^{\vee} \xrightarrow{\alpha} tP(2))$$

which is a chain complex of the form:

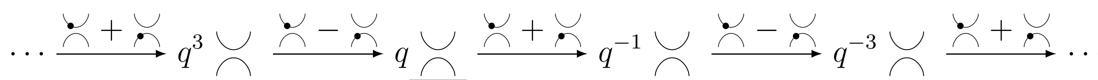


(4.35)

for some $k \in \mathbb{Z}$. In Proposition 4.36 it is proven that $k = \pm 1$, from which (3) follows. □

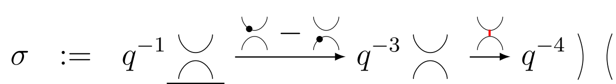
The following was used in the proof of the above.

Proposition 4.36. *Let $P(2)$ and $P(0)$ be as before. Then $P(0) \odot^{\Pi} P(2)^{\vee}$ is homotopy equivalent to*



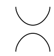
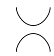
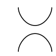
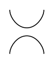
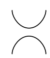
(4.37)

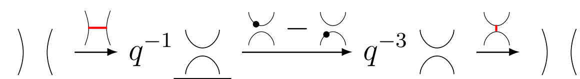
Proof. From the proof of Proposition 4.31 we know that $P(0) \odot^{\Pi} P(2)^{\vee}$ is homotopy equivalent to the chain complex (4.35) for some $k \in \mathbb{Z}$. Our strategy is to argue that $P(0) \odot^{\Pi} P(2)^{\vee}$ is equivalent to $t^2 q^{-4}$ times itself; this periodicity will force $k = \pm 1$ and the result will follow. Now, consider the following chain complex, which happens to be homotopy equivalent to the full right-handed twist on 2-strands:



Since $P(2)^{\vee}$ kills turnbacks, we have $\sigma \odot P(2)^{\vee} \simeq t^2 q^{-4} P(2)^{\vee}$ which is a special case of (3.27). It follows that

$$P(0) \odot^{\Pi} \sigma \odot^{\Pi} P(2)^{\vee} \simeq t^2 q^{-4} P(0) \odot^{\Pi} P(2)^{\vee} \qquad (4.38)$$

On the other hand, we claim that $P(0) \odot \sigma \simeq P(0)$. That this should hold is clear on the level of chain groups: (1) the chain groups of $P(0)$ are all shifts of , (2) σ is the full twist on 2 strands, and (3) the chain complexes built out of  are invariant under isotopy of tangles, up to homotopy equivalence, which implies $\sigma \odot$  $\simeq$ . But in order to take into account the differential requires an idea. The idea is that not only does σ preserve , but *there exists a map* $\to \sigma$ which becomes an equivalence after applying $(-) \odot$ . Indeed consider the following (co)augmentation of σ:

$\hat{\sigma} =$ 

We leave it to the reader to show that $\hat{\sigma} = \mathrm{Cone}(1 \to \sigma)$ kills turnbacks (this is actually a special case of Proposition 6.10). Since $\tau(P(0)) = 0$, the turnback killing lemma (Lemma 4.20) gives

$$0 \simeq P(0) \odot \hat{\sigma} \cong \mathrm{Cone}(P(0) \odot 1 \to P(0) \odot \sigma)$$

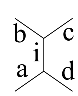
It is a simple fact from homological algebra that $\mathrm{Cone}(f) \simeq 0$ implies f is a homotopy equivalence. Thus $P(0) \odot \sigma \simeq P(0)$, and so

$$P(0) \odot^{\Pi} \sigma \odot^{\Pi} P(2)^{\vee} \simeq P(0) \odot^{\Pi} P(2)^{\vee} \tag{4.39}$$

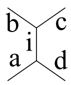
Combining (4.38) and (4.39), we conclude that $P(0) \odot^{\Pi} P(2)$ is periodic, i.e.

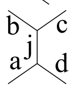
$$P(0) \odot^{\Pi} P(2)^{\vee} \simeq t^{-2} q^{4} P(0) \odot^{\Pi} P(2)^{\vee}.$$

The proposition follows. $\square$

Even though orthogonality of the networks 

is destroyed upon taking convolutions, we still retain semi-orthogonality, as in the computation that $\mathrm{Hom}(P(0), P(2)) \simeq 0$ in Proposition 4.31.

Proposition 4.40 (Strong semi-orthogonality). *Let $A \in \mathrm{Kom}(b + c, a + d)$ be a convolution $A = (\cdots \to t^{-1}E_{-1} \to E_0)$ where each E_k is a finite direct sum of* *copies of* 

, up to shifts in q-degree. Suppose B is a similar such convolution of complexes 

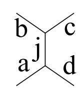
. If $i < j$ then $\mathrm{Hom}^{\bullet,\bullet}(A, B) \simeq 0$.

Proof. Since $\mathrm{Hom}^{\bullet,\bullet}(-, B)$ is a (contravariant) dg functor, $\mathrm{Hom}^{\bullet,\bullet}(A, B)$ is the total complex (using $\prod$) of a bicomplex of the form

$$\mathrm{Hom}^{\bullet,\bullet}(A^0, B) \to \mathrm{Hom}^{\bullet,\bullet}(A^{-1}, B) \to \mathrm{Hom}^{\bullet,\bullet}(A^{-2}, B) \to \cdots$$

where A^k denotes the k-th chain group of A. In order to show that this total complex is contractible, it suffices by Theorem 2.15 to show that each $\mathrm{Hom}^{\bullet,\bullet}(A^{-k}, B) \simeq 0$. By hypothesis

$$B = (\cdots \to t^{-2}F_{-2} \to t^{-1}F_{-1} \to F_0)$$

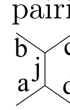
is a convolution, where each F_l is a finite direct sum of q-degree shifts of 

. As a bigraded object $B \cong \bigoplus_{l \leq 0} t^l F_l$. Since each F_i is supported in non-positive homological degrees and A^{-k} is a chain complex supported in a single homological degree, for any homogeneous map $f \in \mathrm{Hom}^{\bullet,\bullet}(A^{-k}, B)$ only finitely many components ${}_lf \in \mathrm{Hom}^{\bullet,\bullet}(A^{-k}, t^l F_l)$ $(l \leq 0)$ can be nonzero. This implies that

$$\mathrm{Hom}^{\bullet,\bullet}(A^{-k}, B) \cong \bigoplus_{l \leq 0} t^l \,\mathrm{Hom}^{\bullet,\bullet}(A^{-k}, F_l)$$

as bigraded abelian groups. Taking into account the differentials gives a convolution:

$$\mathrm{Hom}^{\bullet,\bullet}(A^{-k}, B) \cong \Big(\cdots \to t^{-1}\,\mathrm{Hom}^{\bullet,\bullet}(A^{-k}, F_{-1}) \to \mathrm{Hom}^{\bullet,\bullet}(A^{-k}, F_0)\Big)^{\oplus}$$

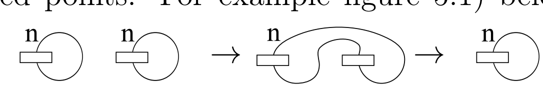
By Theorem 2.15 to show this is contractible, it suffices to show each term is contractible. Note that $\mathrm{Hom}(A^{-k}, F_l)$ can be computed in terms of the planar pairing $F_l \odot (A^{-k})^\vee$ as in Corollary 4.16. But since each F_l is a sum of shifts of 

, $\tau(A^{-k}) \leq i < j$ (recall the hypotheses on A), $F_l \odot (A^{-k})^\vee \simeq 0$ since P_j kills turnbacks (see Lemma 4.20). This shows that $\mathrm{Hom}(A^{-k}, F_l) \simeq 0$ for all l and completes the proof. □

Chapter 5

Sheet algebra and colored unknots

We are interested in functoriality properties of the colored $\mathfrak{sl}_2$ link homology under link cobordisms. A cobordism $\Sigma : L_0 \to L_1$ between oriented links in S^3 is an oriented surface in $S^3 \times I$ with boundary $\partial\Sigma = L_1 \times \{1\} \sqcup (-L_0 \times \{0\})$, regarded modulo isotopy rel boundary (with appropriate adjustments for colored, framed links), where $-L_1$ denotes L_1 with the opposite orientation. Functoriality is the property that there should be some family of cobordisms $\Sigma : L_1 \to L_2$ which induce well-defined maps $H_{\mathfrak{sl}_2}(\Sigma) : H_{\mathfrak{sl}_2}(L_1) \to H_{\mathfrak{sl}_2}(L_2)$, up to sign, and such that composition of cobordisms corresponds to composition of maps.

Now, recall that in order to define the colored $\mathfrak{sl}_2$ link homology of a link L it was necessary to mark some number of points on a diagram D for L. The marked points indicate where to place a Cooper-Krushkal projector in a certain cabling of D. Since projectors absorb one another and can be slid under strands up to homotopy equivalence, the precise location and number of marked points is irrelevant, as long as there is at least one on each component of L. Nonetheless, the markings are necessary in order to construct the associated chain complex. So in order to define the map $H_{\mathfrak{sl}_2}(\Sigma) : H_{\mathfrak{sl}_2}(L_1) \to H_{\mathfrak{sl}_2}(L_2)$ associated to a (colored, framed, oriented) cobordism $\Sigma : L_1 \to L_2$ we would first need to mark Σ, i.e. fix a certain kind of embedded graph $\Gamma \subset \Sigma$ which describes the behavior (merging, sliding under strands, etc.) of the marked points. For example figure 5.1) below is a graphical representation of the map  where the first map is n-parallel saddle cobordisms, and the second merges the two projectors. In this next section we

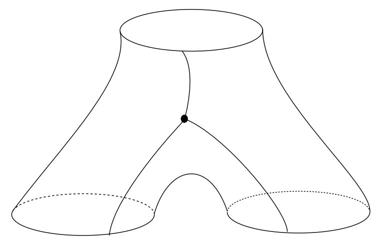
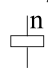
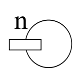
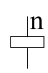


Figure 5.1: Some marked surfaces. Away from the embedded graphs, a surface is to be interpreted as n parallel copies of the corresponding morphism in Bar-Natan's cobordism category. The trivalent vertex is the merging of two projectors, and a univalent vertex is the inclusion of the degree zero chain group of a projector.

examine the skein theory of these graphs, i.e. the local relations which these graphs satisfy. We call this type of skein theory *sheet algebra* since the graphs can be regarded as embedded in the identity cobordism on a single, colored arc, which we will draw as a vertical sheet. Propositions 5.1 and 5.5 are the basis for our future work on studying the chain complexes $\mathrm{End}^{\bullet,\bullet}(P_n)$. We also prove that P_n is an associative algebra object in the homotopy category $\mathrm{Kom}(n)_{/h}$, and the $\mathrm{Ext}^{\bullet,\bullet}(P_n, P_n)$ is a graded commutative algebra whose action on $P_n =$ 

can be recovered faithfully from an action of the n-colored unknot 

on 

via saddle cobordisms.

5.1 Endomorphisms of P_n

The main piece of machinery behind all of our work on sheet algebra is the following proposition:

Proposition 5.1. *Suppose* $P \in \mathrm{Kom}(n)$ *is a Cooper-Krushkal projector, and let* $\iota : 1_n \to P$ *denote the inclusion of the degree zero chain group. If* $Q \in \mathrm{Kom}(n)$ *kills turnbacks then* $(\ ) \circ \iota : \mathrm{Hom}^{\bullet,\bullet}(P, Q) \to \mathrm{Hom}^{\bullet,\bullet}(1_n, Q)$ *is a deformation retract.*

Proof. By Lemma 4.20 we know that $\iota \odot \mathrm{Id}_{Q^\vee} : Q^\vee = 1_n \odot Q^\vee \to P \odot Q^\vee$ is the section of a deformation retact. Applying the dg functor $(\ )^\vee : \mathrm{Kom}_{dg}(n)^\oplus \to \mathrm{Kom}_{dg}(n)^\Pi$, we see that $\mathrm{Id}_Q \odot \iota^\vee : Q \odot^\Pi P^\vee \to Q$ is a deformation retract. Now, from the naturality of the isomorphism in Theorem 4.15 we have a commutative square

$$\begin{array}{ccc} \mathrm{Hom}^{\bullet,\bullet}(P,Q) & \xrightarrow{\cong} & \mathrm{Hom}^{\bullet,\bullet}(1_n, Q \odot^\Pi P^\vee) \\ {\scriptstyle (\ )\circ\iota}\downarrow & & \downarrow{\scriptstyle (\mathrm{Id}_Q \odot (\iota)^\vee)\circ(\ )} \\ \mathrm{Hom}^{\bullet,\bullet}(1_n, Q) & \xleftarrow{\cong} & \mathrm{Hom}^{\bullet,\bullet}(1_n, Q \odot 1_n^\vee), \end{array}$$

Since the right most arrow is a deformation retract, so is the left-most. The completes the proof. □

Suppose $P, Q \in \mathrm{Kom}(\mathcal{TL}_n)$ are Cooper-Krushkal projectors, and let $\iota_P : 1_n \to P$, $\iota_Q : 1_n \to Q$ denote the inclusions of the degree zero chain groups (chain maps since P and Q are supported in non-positive homological degrees). Proposition 5.1 above says that

$$(\ ) \circ \iota_P : \mathrm{Hom}^{\bullet,\bullet}(P,Q) \to \mathrm{Hom}^{\bullet,\bullet}(1_n, Q)$$

is a deformation retract. The bidegree $(0,0)$ chain group of the right-hand side is

$$\mathrm{Hom}_{\mathcal{TL}_n}(1_n, Q^0) = \mathrm{End}_{\mathcal{TL}_n}(1_n) = \mathbb{Z},$$

and so any two bidegree $(0,0)$ chain maps $P \to Q$ (in particular any two homotopy equivalences) are homotopic up to a scalar. We can fix the scalar:

Definition 5.2. Let P and Q be universal projectors and $\iota_P \to P$, $\iota_Q : 1_n \to Q$ be the inclusions of the degree zero chain groups. Call a map $\psi : P \simeq Q$ a *canonical equivalence* if $\psi \circ \iota_P = \iota_Q$.

The following proposition says that canonical equivalences exist, are unique up to homotopy, and are in fact homotopy equivalences as the name suggests. This theorem gives a refinement of the uniqueness statement in [CK12].

Theorem 5.3. *Let $P, Q \in \mathrm{Kom}(\mathcal{TL}_n)$ be Cooper-Krushkal projectors and ι_P, ι_Q the inclusions of the degree zero chain groups as before. There exists a map $\psi : P \to Q$ uniquely characterized up to homotopy by $\psi \circ \iota_P = \iota_Q$. Any such map is a homotopy equivalence, and any other homotopy equivalence is homotopic to $\pm\psi$. In particular, any two Cooper-Krushkal projectors are canonically homotopy equivalent.*

Proof. Proposition 5.1 gives a deformation retract

$$(\ ) \circ \iota_P : \mathrm{Hom}^{\bullet,\bullet}(P, Q) \to \mathrm{Hom}^{\bullet,\bullet}(1_n, Q).$$

Let σ be a section for this deformation retract, and put $\psi := \sigma(\iota_Q)$. Then $\psi \circ \iota_P = \sigma(\iota_Q) \circ \iota_P = \iota_Q$ since σ is a section for the retract $(\ ) \circ \iota_P$. This proves existence of ψ. For uniqueness, suppose we have some other $\psi' : P \to Q$ such that $\psi' \circ \iota_P = \iota_Q$. Then $(\psi - \psi') \circ \iota_P = 0$. Since $(\ ) \circ \iota_P$ is a homotopy equivalence, we must have $\psi \simeq \psi'$. This proves uniqueness.

To see that this ψ is a homotopy equivalence, let $\phi : Q \to P$ be a chain map such that $\phi \circ \iota_Q = \iota_P$, which exists by the above. Then $(\phi \circ \psi) \circ \iota_P = \phi \circ \iota_Q = \iota_P$, which implies $\phi \circ \psi \simeq \mathrm{Id}_P$ by the uniqueness statement above. Similarly, $\psi \circ \phi \simeq \mathrm{Id}_Q$, so ψ and ϕ are homotopy inverses.

Let $\psi' : P \to Q$ be any homotopy equivalence. For degree reasons, $\psi' \circ \iota_P = k\iota_Q$ for some $k \in \mathbb{Z}$. So $(\psi' - k\psi) \circ \iota_P) = 0$. Since $(\ ) \circ \iota_P : \mathrm{Hom}^{\bullet,\bullet}(P, Q) \to \mathrm{Hom}^{\bullet,\bullet}(1_n, Q)$ is a homotopy equivalence, it follows that $\psi' \simeq k\psi$.

By Corollary 6.45, we know that $H^{0,0}(\mathrm{End}^{\bullet,\bullet}(P_n)) \cong \mathbb{Z}$, from which it follows that the coefficient $k \in \mathbb{Z}$ above must be ± 1. That is, any two equivalences $P \to Q$ of Cooper-Krushkal projectors are homotopic up to a sign. This completes the proof. $\square$

Note that even though the proof of the above used a result from later in the thesis, there is no circularity.

5.2 Canonical representations of $\mathrm{End}^{\bullet,\bullet}(P_n)$

For simplicity of notation, let us write $\mathrm{End} = \mathrm{End}^{\bullet,\bullet}$ and $\mathrm{Hom} = \mathrm{Hom}^{\bullet,\bullet}$. Let $P \in \mathrm{Kom}(n)$ be a universal projector and suppose we have some chain complex $Q \in \mathrm{Kom}(n)$ and a homotopy equivalence $\phi : P \to Q$. Conjugating by ϕ gives a homotopy equivalence $\mathrm{End}(P) \to \mathrm{End}(Q)$ which we call a *canonical representation*. Any other homotopy equivalence $\phi' : P \to Q$ is homotopic to $\pm\phi$, hence canonical representations are unique up to homotopy. It is clear that the composition of canonical representations is canonical, and that a canonical representation $\mathrm{End}(P) \to \mathrm{End}(P)$ is homotopic to the identity map.

As an application, suppose we have Cooper-Krushkal projectors $P_m \in \mathrm{Kom}(m)$ and $P_n \in \mathrm{Kom}(n)$. Then

$$(P_m \sqcup 1_{n-m}) \odot P_n \simeq P_n \simeq P_n \odot (P_m \sqcup 1_{n-m})$$

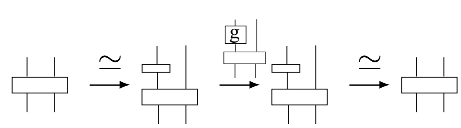
and we have canonical maps $\rho^n_m, \bar{\rho}^n_m : \mathrm{End}(P_m) \to \mathrm{End}(P_n)$ given by the action of $\mathrm{End}^{\bullet,\bullet}(P_m)$ on the appropriate factor, followed by a canonical representation. We intend to show that these maps coincide and are canonical in an appropriate sense.

Definition 5.4. For integers $1 \leq m \leq n$ define maps $\rho^n_m, \bar{\rho}^n_m : \mathrm{End}(P_m) \to \mathrm{End}(P_n)$ as follows.

- Let ρ^n_m denote the composition $\mathrm{End}(P_m) \to \mathrm{End}((P_m \sqcup 1_{n-m}) \odot P_n) \to \mathrm{End}(P_n)$ where the first map is $g \mapsto (g \sqcup 1_{n-m}) \odot \mathrm{Id}_{P_n}$ and the second is a canonical representation.

- Let $\bar{\rho}^n_m$ denote the composition $\mathrm{End}(P_m) \to \mathrm{End}(P_n \odot (P_m \sqcup 1_{n-m})) \to \mathrm{End}(P_n)$ where the first map is $g \mapsto \mathrm{Id}_{P_n} \odot (g \sqcup 1_{n-m})$ and the second is the canonical representation.

Graphically $\rho^n_m(g)$ is the composition



The following will be used, for example, in Theorem 7.11 to show that certain mapping cones commute up to homotopy: $(\mathrm{Cone}(g) \sqcup 1_{n-m}) \odot \mathrm{Cone}(f) \simeq \mathrm{Cone}(f) \odot (\mathrm{Cone}(g) \sqcup 1_{n-m})$ for particular $g \in \mathrm{End}^{\bullet,\bullet}(P_m)$ and $f \in \mathrm{End}^{\bullet,\bullet}(P_n)$.

Proposition 5.5. *We have*

1. $\rho^n_n \simeq \bar{\rho}^n_n \simeq \mathrm{Id}_{\mathrm{End}(P_n)}$.
2. $\rho^n_m \simeq \bar{\rho}^n_m$ *for all* $1 \leq m \leq n$.
3. $\rho^n_k \circ \rho^k_m \simeq \rho^n_m$ *for all* $1 \leq m \leq k \leq n$.

Proof. Fix n, and let $P_n \in \mathrm{Kom}(n)$ be a Cooper-Krushkal projector. Let $\Psi : \mathrm{End}(P_n \odot P_n) \to \mathrm{End}(P_n)$ be a canonical representation, and define $L, R : \mathrm{End}(P_n) \to \mathrm{End}(P_n \odot P_n)$ by $L(f) = f \odot \mathrm{Id}_{P_n}$ and $R(f) = \mathrm{Id}_{P_n} \odot f$ for all $f \in \mathrm{End}(P_n)$, so that $\rho^n_n = \Psi \circ L$ and $\bar{\rho}^n_n = \Psi \circ R$.

By Proposition 4.21 there is a deformation retract $\pi : P_n \odot P_n \to P_n$ with section $\sigma := \mathrm{Id}_{P_n} \odot \iota$, where $\iota : 1_n \to P_n$ is the inclusion of the degree zero chain group. Then Ψ is homotopic to the map $F \mapsto \pi \circ F \circ (\mathrm{Id} \odot \iota)$, and so ρ^n_n is homotopic to the map

$$\rho^n_n = \Psi \circ L \simeq \Big(f \mapsto \pi \circ (f \odot \mathrm{Id}_{P_n}) \circ (\mathrm{Id}_{P_n} \odot \iota) = \pi \circ (\mathrm{Id}_{P_n} \odot \iota) \circ f = f\Big) = \mathrm{Id}_{\mathrm{End}(P_n)}$$

A similar argument shows that $\bar{\rho}^n_n = \Psi \circ R \simeq \mathrm{Id}_{\mathrm{End}(P_n)}$. This proves (1). Moreover, since Ψ is an isomorphism, $\Psi \circ L \simeq \Psi \circ R$ implies that $L \simeq R$. That is, the two obvious actions of $\mathrm{End}(P_n)$ on $P_n \odot P_n$ coincide up to homotopy.

Suppose now we have $1 \leq m \leq n$ and a Cooper-Krushkal projector $P_m \in \mathrm{Kom}(m)$. Retain notation as before, so that Ψ, L, and R are defined. For brevity put $I = P_m \sqcup 1_{n-m}$. By projector absorbing (proposition 4.23) we have an equivalence $P_n \odot I \simeq P_n$. Composing on the right with P_n gives an equivalence $\pi_1 : (P_n \odot I) \odot P_n \to P_n \odot P_n$. Similarly we have an equivalence $\pi_2 : P_n \odot (I \odot P_n) \simeq P_n \odot P_n$ obtained by applied Proposition 4.23 to the parenthesized term. Forming the associated canonical representations gives a diagram which commutes up to homotopy:

$$\begin{array}{ccccc}
\mathrm{End}(P_n) & \xrightarrow{\alpha} & \mathrm{End}(P_n \odot I \odot P_n) & \xrightarrow{\Phi_1} & \mathrm{End}(P_n \odot P_n) \\
 & & \Phi_2 \big\downarrow & & \Psi \big\downarrow \\
 & & \mathrm{End}(P_n \odot P_n) & \xrightarrow{\Psi} & \mathrm{End}(P_n)
\end{array}$$

where $\alpha(g) = \mathrm{Id}_{P_n} \odot (g \sqcup 1_{n-m}) \odot \mathrm{Id}_{P_m}$. By inspection we have $\Phi_1 \circ \alpha = L \circ \bar{\rho}^n_m$ and $\Phi_2 \circ \alpha = R \circ \rho^n_m$. Now, by commutativity we have $\Psi \circ L \circ \bar{\rho}^n_m \simeq \Psi \circ R \circ \rho^n_m$. Since $\Psi \circ L \simeq \Psi \circ R \simeq \mathrm{Id}$, this implies (2).

For (3), suppose we have $1 \leq m \leq k \leq n$ and a Cooper-Krushkal projector $P_k \in \mathrm{Kom}(k)$. Retain notation as before, and put $J := P_k \sqcup 1_{n-k}$. Consider the equivalence $(P_m \sqcup 1_{k-m}) \odot P_k \simeq P_k$ implied by Proposition 4.23. Applying the functor $(-) \sqcup 1_{n-k}$ gives an equivalence $I \odot J \simeq J$, and then applying the functor $(-) \odot P_n$ gives an equivalence $I \odot J \odot P_n \simeq J \odot P_n$. The associated canonical equivalence $\Theta : \mathrm{End}(I \odot J \odot P_n) \to \mathrm{End}(J \odot P_n)$ satisfies $\Theta((g \sqcup 1_{n-m}) \odot \mathrm{Id}_J \odot \mathrm{Id}_{P_n}) = (\rho^k_m(g) \sqcup 1_{n-k}) \odot \mathrm{Id}_{P_n}$. Now, consider the following diagram which commutes up to homotopy:

$$\begin{array}{ccccc}
\mathrm{End}(P_m) & \xrightarrow{\beta} & \mathrm{End}(I \odot P_n) & \longrightarrow & \mathrm{End}(P_n) \\
 & & \big\downarrow & & \big\uparrow \\
 & & \mathrm{End}(I \odot J \odot P_n) & \xrightarrow{\Theta} & \mathrm{End}(J \odot P_n)
\end{array}$$

where $\beta(g) = (g \sqcup 1_{n-m}) \odot \mathrm{Id}_{P_n}$ and all of the other maps are canonical representations. By definition the composition along the top row is precisely ρ^n_m. The composition corresponding to the other path is precisely $\rho^n_k \circ \rho^k_m$. Since the diagram commutes up to homotopy, this proves (3). □

As a corollary we obtain:

Corollary 5.6. *The algebra* $\mathrm{Ext}^{\bullet,\bullet}(P_n, P_n)$ *is graded commutative.*

Proof. Recall our notation $\mathrm{Ext}(P_n) = \mathrm{Ext}^{\bullet,\bullet}(P_n, P_n)$. By Proposition 5.5 we have an isomorphism of bigraded algebras $\mathrm{Ext}(P_n) \cong \mathrm{Ext}(P_n \odot P_n)$ given by $L([f]) \mapsto$

$[f \odot \mathrm{Id}] = [\mathrm{Id} \odot f]$. Compute

$$L([f][g]) = L([f]) \circ L([g]) = [f \odot \mathrm{Id}][\mathrm{Id} \odot g] = (-1)^{|f||g|}[\mathrm{Id} \odot g][f \odot \mathrm{Id}] = (-1)^{|f||g|}L([g][f])$$

In the third equality we used the general property of differential graded bilinear functors. Namely $(f \odot \mathrm{Id}) \circ (\mathrm{Id} \odot g) = (f \odot g) = (-1)^{|f||g|}(\mathrm{Id} \odot g) \circ (f \odot \mathrm{Id})$. Applying the inverse isomorphism gives the result. □

Remark 5.7. It is true $f \otimes g \mapsto f \circ g$ and $f \otimes g \mapsto (-1)^{|f||g|} g \circ f$ are homotopic chain maps $\mathrm{End}^{\bullet,\bullet}(P_n) \otimes \mathrm{End}^{\bullet,\bullet}(P_n) \to \mathrm{End}^{\bullet,\bullet}(P_n)$, but we have chosen to prove the result on the level of homology because the proof simplifies greatly.

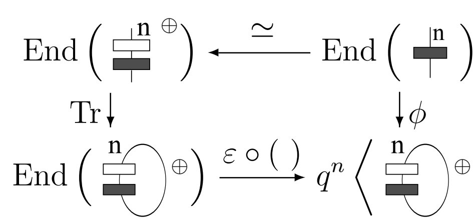
We have another source of actions of $\mathrm{End}(P_n)$ which is important:

Proposition 5.8. *The map* $\xi : \mathrm{End}(P_n) \to \mathrm{End}(P_n^\vee)$ *induced by* $P_n \odot P_n^\vee \simeq P_n^\vee$ *is homotopic to the map given by* $f \mapsto f^\vee$.

Proof. Recall that we set $\mathrm{Hom} = \mathrm{Hom}^{\bullet,\bullet}$ and $\mathrm{End} = \mathrm{End}^{\bullet,\bullet}$. Put $P := P_n$. Let $\langle \ |$ denote the functor $C \mapsto \mathrm{Hom}_{\mathcal{TL}_0}(C, \varnothing)$, and let $\phi : \mathrm{End}(P^\vee) \cong q^n \langle \mathrm{Tr}(P \otimes P^\vee)|$ be the isomorphism from Corollary 4.16, and put $\varepsilon = \phi(\mathrm{Id}_{P^\vee})$. From the explicit expression for ϕ, we have

$$\phi(f^\vee) = \varepsilon \circ \mathrm{Tr}(\mathrm{Id}_P \odot f^\vee) = \varepsilon \circ \mathrm{Tr}(f \odot \mathrm{Id}_{P^\vee}) \tag{5.9}$$

for all $f^\vee \in \mathrm{End}(P_n^\vee)$. Consider the following diagram:



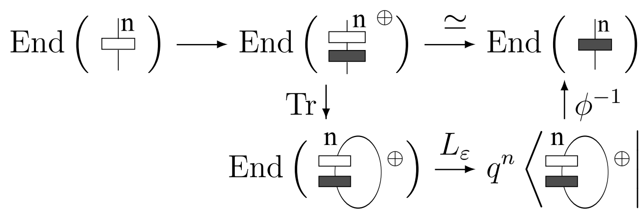
(5.10)

where the top arrow is $f^\vee \mapsto \mathrm{Id}_P \otimes f^\vee$. The square commutes by (5.9).

Now, by Proposition 4.21 the map $\iota \odot \mathrm{Id}_{P^\vee} : P_n^\vee \to P_n \odot P_n^\vee$ is a the section of a deformation retract π. Conjugating with these equivalences gives us an equivalence $\Psi : \mathrm{End}(P_n \odot P_n^\vee) \to \mathrm{End}(P_n)$ which satisfies

$$\Psi(\mathrm{Id}_P \odot f^\vee) = \pi(\mathrm{Id}_P \odot f^\vee) \circ (\iota \odot \mathrm{Id}_{P^\vee}) = \pi \circ (\iota \odot \mathrm{Id}_{P^\vee}) \circ f^\vee = f^\vee$$

That is to say, the map $f^\vee \mapsto \mathrm{Id}_P \odot f^\vee$ is a right inverse for the homotopy equivalence Ψ. It follows that the top-most arrow of (5.10) is a homotopy equivalence. Inverting this arrow and the right-most arrow of (5.10) yields a diagram which commutes up to homotopy

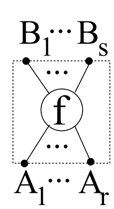


where we have introduced an addition horizontal arrow on the left which sends $g \mapsto g \otimes \mathrm{Id}_{P^\vee}$. The composition along the top row is precisely ξ in the statement of this proposition. One checks that the composition in the other direction sends g to ϕ^{-1} of the map $\varepsilon \circ \mathrm{Tr}(g \otimes \mathrm{Id}_{P^\vee}) = \phi(g^\vee)$ (see equation (5.9)). In other words, ξ is homotopic to $g \mapsto g^\vee$. This completes the proof. $\square$

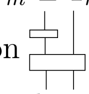
5.3 P_n is a unital algebra

In this section our pictures coincide with the usual graphical notation for morphisms in a monoidal category, which we describe next.

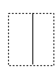
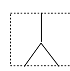
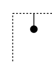
Let $(\mathscr{A}, \otimes, 1)$ be a monoidal category [ML98]. Denote objects of $\mathscr{A}$ by labeled dots on a line segment such that $A \otimes B$ is drawn as "the dot labelled by A sitting to the left of the dot labeled by B." Dots labeled by the monoidal identity 1 will be omitted from the diagrams since they do not affect the corresponding object up to *canonical* isomorphism by MacLane's coherence theorem (see chapter VII of [ML98]). We will draw a morphism $f : A_1 \otimes \cdots \otimes A_r \to B_1 \otimes \cdots \otimes B_s$ generically



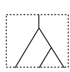
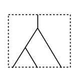
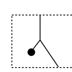
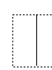
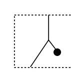
where the disk labeled with f could itself be a more interesting diagram. Composition of morphisms corresponds to vertical stacking, so that $f \circ g$ is "f on top of g," and $f \otimes g$ corresponds to horizontal juxtaposition. Identity maps will be denoted by vertical line segments, and the identity map $\mathrm{Id}_1 : 1 \to 1$ of the monoidal identity will be denoted by the empty diagram.

Remark 5.11. Note that the pictures here are of a different nature than the ones considered earlier. For example, the compositon of projectors $(P_m \sqcup 1_{n-m}) \odot P_n$ would in earlier sections have been represented by vertical composition  . Here we reserve the vertical direction (in the coordinates of the page) for *morphisms* between such pictures. So here we would denote the composition $(P_m \sqcup 1_{n-m}) \odot P_n$ by two labelled dots along a horizontal line segment.

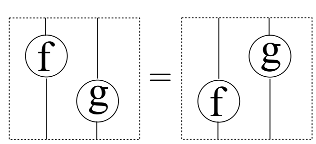
To illustrate the diagrammatics, suppose (A, μ, η) is a unital associative algebra object in $\mathscr{A}$. That is, A is an object of $\mathscr{A}$ and $\mu : A \otimes A \to A$, $\eta : 1 \to A$ satisfy $\mu \circ (\mu \otimes \mathrm{Id}_A) = \mu \circ (\mathrm{Id}_A \otimes \mu)$, $\mu \circ (\mathrm{Id}_A \otimes \eta) = \mu \circ (\eta \otimes \mathrm{Id}_A) = \mathrm{Id}_A)$. We denote Id_A, μ, and η as in:

$\mathrm{Id}_A =$  $\mu =$  , $\eta =$  ,

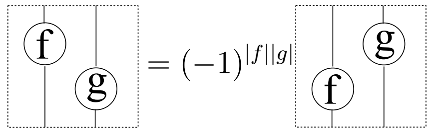
where the label A is understood. The relations for μ and η become:

 =  , and  =  = 

Remark 5.12. Any bilinear functor $F : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ satisfies $F(f, \mathrm{Id}) \circ F(\mathrm{Id}, g) = F(f, g) = F(\mathrm{Id}, g) \circ F(f, \mathrm{Id})$ by definition. If F is the monoid in a monoidal category, then this is represented diagrammatically as



The monoidal categories we will be interested in are categories of chain complexes, for example $(\mathrm{Kom}(n), \odot, 1_n)$. If x is some shift functor, then we have $(xA) \odot B \cong x(A \odot B) \cong A \odot (xB)$ naturally. In this interest of not cluttering our notation, from this point on we will omit the grading shifts from the diagrammatic representation of morphisms. In short, our morphisms will be allowed to have nonzero degrees, and we have



The category $(\mathrm{Kom}^-(n), \odot, 1_n)$ is monoidal since it is the category of semi-infinite chain complexes on a monoidal category. In this subsection, all of the diagrams will represent morphisms in this category, and all of the objects will be compositions of a fixed Cooper-Krushkal projector $P_n \in \mathrm{Kom}(n)$. Note that $P_n \odot \cdots \odot P_n$ is a Cooper-Krushkal projector, and the inclusion of the degree zero chain group is precisely $\iota \odot \cdots \odot \iota$. Then Proposition 5.1 applied to this case gives the following:

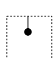
Lemma 5.13. *Precomposition with $\iota^{\odot k}$ gives a deformation retract*

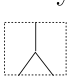
$$\mathrm{Hom}^{\bullet,\bullet}(P_n^{\odot k}, P_n) \to \mathrm{Hom}^{\bullet,\bullet}(1_n, P_n).$$

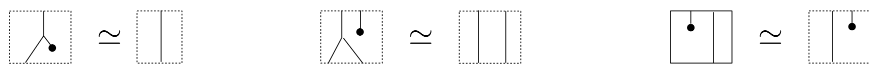
In particular if $f \circ \iota^{\odot k} \simeq g \circ \iota^{\odot k}$, then $f \simeq g$. □

From this lemma one can derive a number of diagrammatic identities involving the maps μ and ι. In particular we have the following, which should be compared to Theorem 7.40.

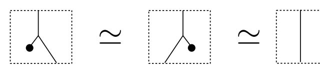
Proposition 5.14. *There is a chain map $\mu : P_n \odot P_n \to P_n$ uniquely characterized by the fact that ι is a right or left unit for μ. This makes (P_n, μ, ι) into an associative algebra object in the homotopy category of* $\mathrm{Kom}(n)$*.*

Proof. Let $\iota =$  $: 1_n \to P_n$ denote the inclusion of the degree zero chain group. For uniqueness, suppose we had two maps $\mu, \mu' : P_n \odot P_n \to P_n$ such that $\mu \circ (\mathrm{Id}_{P_n} \odot \iota) \simeq \mathrm{Id} \simeq \mu \circ (\mathrm{Id}_{P_n} \odot \iota)$. Then $(\mu - \mu') \circ (\iota \odot \iota) = \iota - \iota = 0$, and so lemma 5.13 implies $\mu \simeq \mu'$.

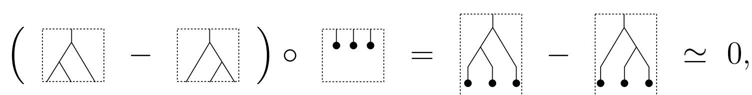
By Proposition 4.21, $\mathrm{Id} \odot \iota$ is a homotopy equivalence, and so we can define $\mu =$  $: P_n \odot P_n \to P_n$ to be a homotopy inverse. We have the following graphical relations:

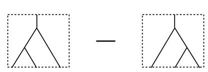
 (5.15)

The first two are restatements that μ and $\mathrm{Id}\odot\iota$ are homotopy inverses. The last follows since each $\iota\odot\mathrm{Id}$ and $\mathrm{Id}\odot\iota$ are canonical equivalences, and hence homotopic by Theorem 5.3. This implies

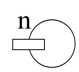
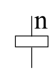


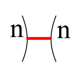
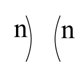
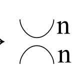
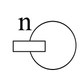
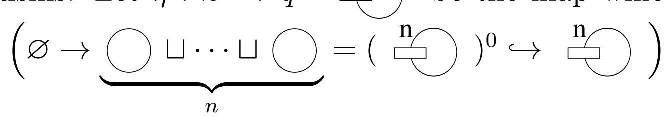
hence ι is a two-sided unit for μ. We need only see that μ is associative. But this follows from lemma 5.13: since ι is a two-sided unit for μ we have

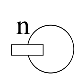
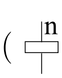


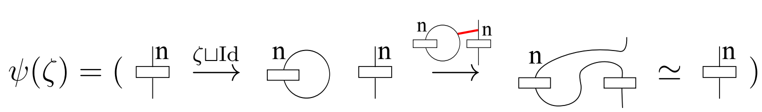
hence  $\simeq 0$. □

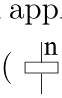
5.4 The unknots as algebras

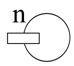
We give an explicit description of the action of $q^n|$  $\rangle$ on . This section is reminiscent of §4.2.

Definition 5.16. Let $s =$  denote the map q^n  $\rightarrow$  consisting of n parallel saddle cobordisms. Let $\eta : \varnothing \rightarrow q^n$  be the map which is n parallel "cap" cobordisms $\eta =$ .

The saddle induces a chain map $\psi : q^n \mid$  $\rangle \rightarrow \mathrm{End}($  $)$ given by



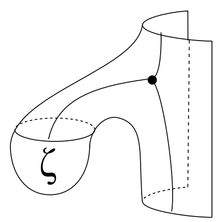
where the final map is given by sliding the projectors so they are adjacent and applying a standard equivalence which merges them. We also have a map $\phi : \mathrm{End}($  $) \rightarrow$

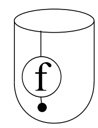
$q^n|$ 

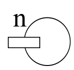
$\rangle$ defined by $\phi(f) = \mathrm{Tr}(f) \circ \eta$, where Tr is the Markov trace. For the following proposition it is useful to recall the definitions and results of §4.2. In particular, the diagrams in that section still give a useful way for visualizing the maps η, s, ϕ, and ψ.

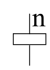
Proposition 5.17. *The maps ϕ and ψ from the preceding discussion are homotopy inverses, and $\psi(\eta) \simeq \mathrm{Id}_{P_n}$.*

Proof. We have the following illustrations of ϕ and ψ:

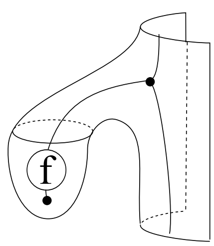
$\psi(\zeta) =$ 

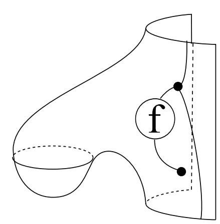
$\phi(f) =$ 

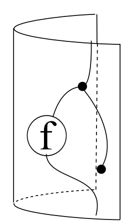
for all $\zeta \in \mathrm{Hom}^{\bullet,\bullet}(\varnothing,$ 

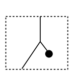
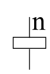
$)$ and all $f \in \mathrm{End}^{\bullet,\bullet}($ 

$)$. If f is a cycle then we have

$\psi(\phi(f)) =$ 

$\overset{(1)}{=}$ 

$\overset{(2)}{\simeq}$ 

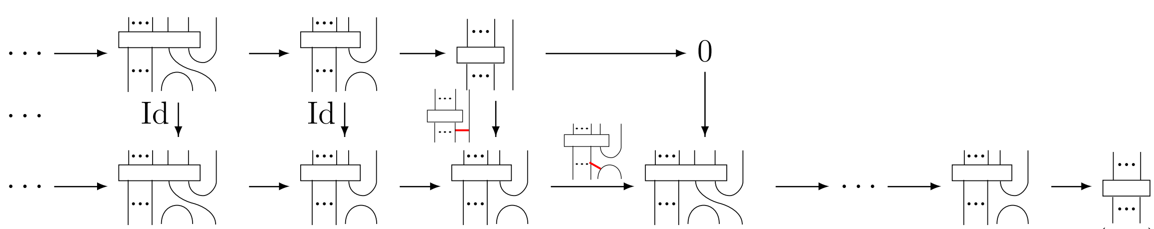
where (1) holds since distant maps commute, and (2) holds by the right-most relation in (5.15) as well as isotopy invariance of morphisms in Bar-Natan's categories (used in cancelling the cup and saddle cobordisms). This latter map is homotopic to f since  $\simeq \mathrm{Id}$. This shows that $\psi \circ \phi(f) \simeq f$ whenever $f \in \mathrm{End}^{\bullet,\bullet}($ 

$)$ is a cycle. A slight modification of the argument establishes more generally that $\psi \circ \phi \simeq \mathrm{Id}_{\mathrm{End}(P)}$. We leave the arguments that $\phi \circ \psi \simeq \mathrm{Id}$ and $\psi(\eta) \simeq \mathrm{Id}_{P_n}$ to the reader. □

Chapter 6

A polynomial action on $\mathfrak{sl}_2$-link homology

In this section we construct an action of the polynomial ring $\mathbb{Z}[u_1, \ldots, u_n]$ on the Cooper-Krushkal projector $P_n \in \mathrm{Kom}(n)$. The motivation for such an action comes from the apparent periodicity in the Cooper-Krushkal recursion which the P_n satisfy, described in §3.5. Consider the following diagram, in which each row is the Frenkel-Khovanov sequence $E_\bullet$ and we have omitted all degree shifts:



(6.1)

The right-most nontrivial square commutes up to homotopy, and every other square commutes on the nose. That is to say, (6.1) defines a map of homotopy complexes $q^{2n}E[2-2n]_\bullet \to E_\bullet$, where [1] denotes the upward grading shift, $E[1]_k = E_{k-1}$. It is one of the goals of this paper to realize this homotopy chain map as an honest chain map. That is to say, we wish to add some more maps pointing to the right and (non-strictly) down such that (1) the rows become the projector P_n and (2) the non-horizontal components define a chain map $U_n : t^{2-2n}q^{2n}P_n \to P_n$. Constructing U_n directly is quite difficult because of the higher differentials required to make the Frenkel-Khovanov sequence a chain complex. Nonetheless, if U_n were to exist, then after contracting a large contractible summand, the mapping cone $\mathrm{Cone}(U_n)$ would be homotopy equivalent to a much simpler chain complex, in fact a convolution of the homotopy chain complex (6.7). We construct U_n indirectly by first constructing a chain complex $Q_n \in \mathrm{Kom}(n)$ which is a convolution of (6.7), hence behaves as if it were the mapping cone $\mathrm{Cone}(U_n)$. We call Q_n a *symmetric projector*, and our strategy for constructing it is as follows:

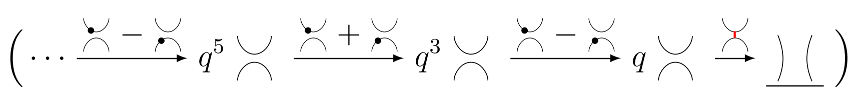
1. assume the existence of $Q_2, \ldots, Q_n$,
2. deduce that P_n is homotopy equivalent to a chain complex of the form $\mathbb{Z}[u_n] \otimes Q_n$ with differential $1 \otimes d_{Q_n} + u_n \otimes \partial_n$.
3. use this description to compute some groups $\mathrm{Ext}^{i,j}(P_n, P_n)$ in §6.5,
4. use the Ext group computation to establish existence and uniqueness of Q_{n+1} in §7.1.

Thus, the next three sections of this paper constitute one very large inductive step in a proof of all of the results contained therein. We first consider the case $n = 2$.

6.1 The polynomial action in the case $n = 2$

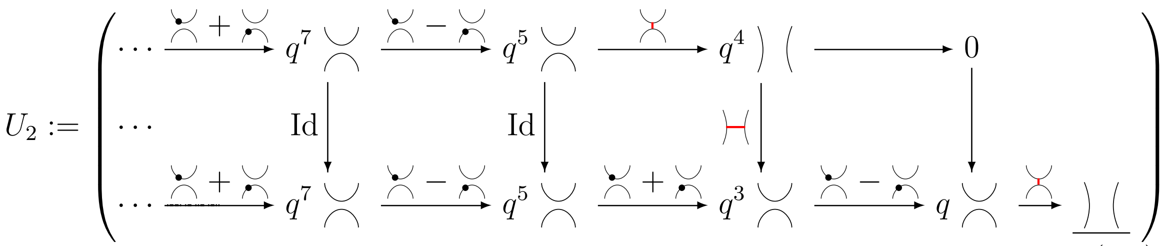
Note that the Cooper-Krushkal axioms force $P_1 = 1_1$, the identity strand and the Cooper-Krushkal recursion in case $n = 2$ produces the expression

$P_2 :=$



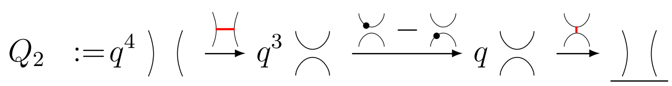
(6.2)

The homotopy chain map (6.1) induces the following honest chain map $U_2 : t^{-2}q^4P_2 \to P_2$:



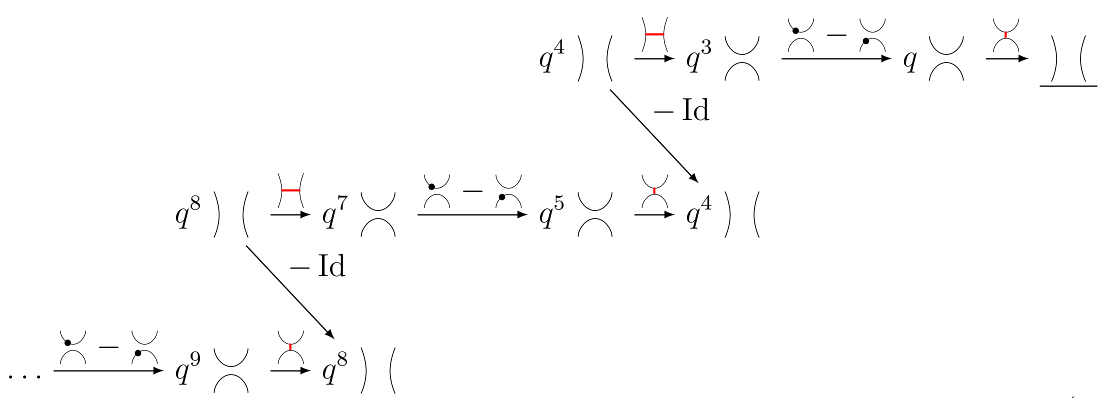
(6.3)

By contracting the identity maps (Gaussian elimination, proposition 2.14) we see that Cone(U_2) deformation retracts onto the much simpler chain complex



(6.4)

We can now recover P_2 as a periodic chain complex built out of copies of Q_2 via a construction which should recall Koszul duality relating modules over the polynomial algebra $\mathbb{Z}[u_2]$ and the exterior algebra $\Lambda[\partial_2]$. Let $\partial_2 : t^3q^{-4}Q_2 \to Q_2$ denote the chain map which is (minus) projection-followed-by-inclusion of the $)\,($ chain group. Then $(\partial_2)^2 = 0$ and we can form a periodic bicomplex, the total complex of which is

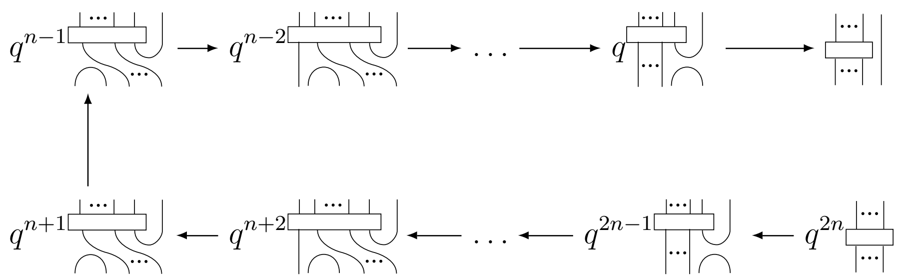


(6.5)

Contracting the contractible summands (Gaussian elimination, Proposition 2.14) we see that in fact P_2' deformation retracts onto P_2. It is remarkable that this description of P_2 generalizes to all of the projectors P_n.

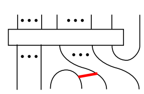
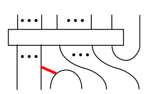
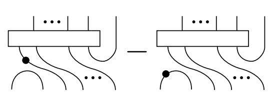
6.2 The symmetric Frenkel-Khovanov sequence

Definition 6.6. Let $P_{n-1} \in \mathrm{Kom}(n-1)$ denote a fixed Cooper-Krushkal projector, and define the *symmetric Frenkel-Khovanov sequence* (relative to P_{n-1}) to be the following sequence of chain complexes in $\mathrm{Kom}(n)$ and chain maps:



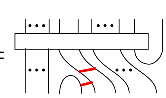
(6.7)

where the white box denotes P_{n-1}. The maps in this sequence are given by

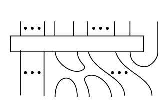
-  between two terms in the top row.
-  between two terms in the bottom row.
-  between the two terms in the left column.

Proposition 6.8. *The symmetric Frenkel-Khovanov sequence (6.7) is a homotopy chain complex.*

Proof. We need to check that the composition of consecutive maps is nulhomotopic. Let us write the sequence (6.7) as $E_{1-2n} \xrightarrow{\alpha_{1-2n}} \cdots \xrightarrow{\alpha_{-2}} E_{-1} \xrightarrow{\alpha_{-1}} E_0$. The proof splits up into cases. If $1-n \leq i < -1$, then generically $\alpha_{i+1} \circ \alpha_i$ is the composition of saddle cobordisms,

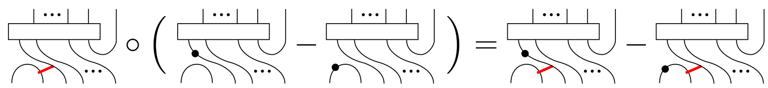
$\alpha_{i+1} \circ \alpha_i =$ 

By isotopy invariance of morphisms the saddle maps can be performed in any order, and so $\alpha_{i+1} \circ \alpha_i$ factors through the chain complex



which is contractible since P_{n-1} kills turnbacks. Hence $\alpha_{i+1} \circ \alpha_i \simeq 0$ in this case.

The case $i = -n$ is taken care of by the observation

$\alpha_{1-n} \circ \alpha_{-n} =$  $= 0$

by sliding dots.

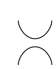
For the remaining cases, $i = -1-n$ is similar to the case $i = -n$, and $1 - 2n \leq i < -1 - n$ is similar to $1 - n \leq i < -1$. This completes the proof. □

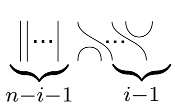
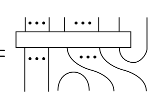
Definition 6.9. Call a chain complex $Q_n \in \mathrm{Kom}(n)$ a symmetric projector if either (1) $n = 1$ and $Q_1 := \mathrm{Cone}(b)$ where $b : q^2 1_1 \to 1_1$ is a dotted identity cobordism or (2) $n \geq 2$ and Q_n is convolution of the symmetric Frenkel-Khovanov sequence (6.7).

6.3 The relationship between P_n and Q_n

The existence and uniqueness of symmetric projectors is postponed until Chapter 7. In meantime, we will assume that $Q_1, \ldots, Q_n$ exist, and will deduce an inter-relationship with P_n. In particular we will be able to compute some groups $\mathrm{Ext}^{i,j}(P_n, P_n)$ of chain maps $t^i q^j P_n \to P_n$ modulo chain homotopy. The vanishing of $\mathrm{Ext}^{1-2n, 2+2n}(P_n, P_n)$ is then used in an inductive construction of Q_{n+1} in Chapter 7.

Proposition 6.10. *If $Q_n \in \mathrm{Kom}(n)$ is a symmetric projector, then Q_n kills turnbacks.*

Proof. Let $P_{n-1} \in \mathrm{Kom}(n-1)$ be a Cooper-Krushkal projector. For each $1 \leq i \leq n-1$, let $e_i = 1_{n-i-1} \sqcup e \sqcup 1_{i-1}$ denote the Temperley-Lieb generator, where $e =$ . Define chain complexes $F(i) = (P_{n-1} \sqcup 1) \odot e_1 \odot \cdots \odot e_i$ for $1 \leq i < n-1$ and $F(0) = (P_{n-1} \sqcup 1)$, and note that

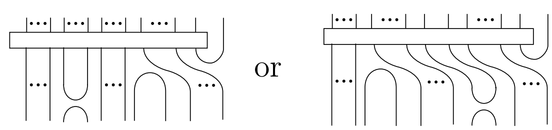
$e_1 \odot \cdots \odot e_i =$  $\qquad F(i) =$  (6.11)

The symmetric Frenkel-Khovanov sequence can be written

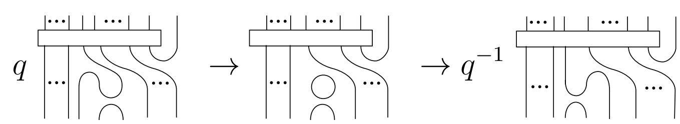
$$E_\bullet \quad = \quad \begin{array}{ccccccc} F(n-1) & \longrightarrow & F(n-2) & \longrightarrow \cdots \longrightarrow & F(1) & \longrightarrow & F(0) \\ \uparrow & & & & & & \\ F(n-1) & \longleftarrow & F(n-2) & \longleftarrow \cdots \longleftarrow & F(1) & \longleftarrow & F(0) \end{array}$$

where the maps are given by saddle cobordisms $F(i) \to F(i \pm 1)$, and a difference of dotted identity maps $F(n-1) \to F(n-1)$. Here we are omitting the degree shifts, and we had to fold up the sequence $F(0) \to \cdots \to F(n-1) \to F(n-1) \to \cdots \to F(0)$ because of space limitations. Assume that $Q_n = \mathrm{Tot}(E_\bullet)$ is a symmetric projector. Applying $(-) \odot e_k$ to $E_\bullet$ gives a sequence which can be split up into subsequences of the form

1. 1-term sequences $F(j) \odot e_k$ where $j \notin \{k-1, k, k+1\}$, which are contractible chain complexes of the form

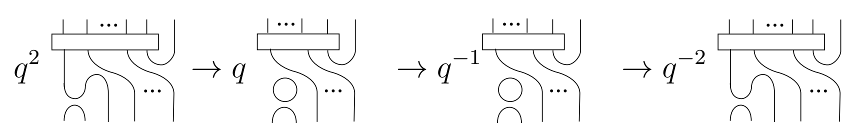


2. if $k \neq n-1$, two 3-term subquences $F(k \pm 1) \odot e_k \to F(k) \odot e_k \to F(k \mp 1) \odot e_k$, which up to a shift are of the form



   or the reverse, where the maps merge or split off a disjoint unknotted circle.

3. if $k = n-1$, a 4-term subsequence $F(n-2) \odot e_{n-1} \to F(n-1) \odot e_{n-1} \to F(n-1) \odot e_{n-1} \to F(n-2) \odot e_{n-1}$, which up to a shift is



   where the first and last maps merge or split off a disjoint unknotted circle, and the middle map is a difference of dotted identity maps.

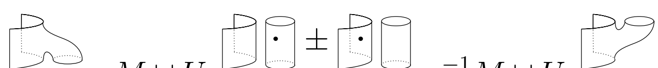
By Lemma 6.12 following this proposition, any convolutions of the sequences of type (2) or (3) are contractible. Clearly the chain complexes in (1) are contractible since P_{n-1} kills turnbacks. So $Q_n \odot e_k$ can be reassociated into a convolution of contractible chain complexes, hence is contractible by Theorem 2.15. So Q_n kills turnbacks from below. By Proposition 4.26, Q_n kills turnbacks from above as well. $\square$

The following was used in the proof of the above proposition, and will be used again in the proof of Theorem 7.1.

Lemma 6.12. *Let $M \in \mathrm{Kom}(\mathcal{TL}_n)$ be arbitrary, and let $E_\bullet$ be a 3- or 4-term homotopy chain complex of the form*

$$E_\bullet \;=\; qM \longrightarrow M \sqcup U \longrightarrow q^{-1}M,$$

or



$$E_\bullet \;=\; q^2M \longrightarrow qM \sqcup U \longrightarrow q^{-1}M \sqcup U \longrightarrow q^{-2}M.$$

Here, $U \in \mathcal{TL}_0$ is an unknot and the maps above are as shown in some disk near a fixed point p on a free strand of M, and identities elsewhere. Then any convolution of $E_\bullet$ is contractible.

Proof. This is proven in [CK12], where such a chain complex is called a triple, respectively quadruple. Alternatively, one can use Lemma 6.39 to iteratively split off contractible summands. $\square$

The fact that triples are contractible is related to invariance of Khovanov homology under the Reidemester II move.

Proposition 6.13 (Obtaining P_n from Q_n)**.** *If Q_m is a symmetric projectors then there is a periodic bicomplex built out of Q_n, the total complex of which deformation retracts onto a Cooper-Krushkal projector $P_n \in \mathrm{Kom}(n)$:*

$$P_n \;\simeq\; \mathrm{Tot}(Q_n \xrightarrow{\partial_n} t^{2-2n}q^{2n}Q_n \xrightarrow{\partial_n} t^{4-4n}q^{4n}Q_n \xrightarrow{\partial_n} \cdots) \tag{6.14}$$

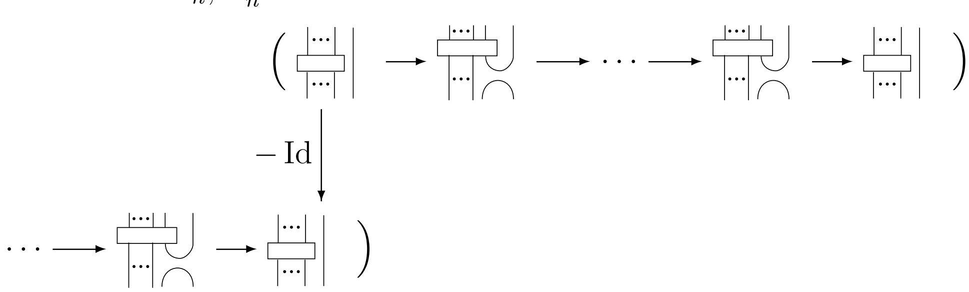
for some chain map $\partial_n : Q_n \to t^{1-2n}q^{2n}Q_n$ *satisfying* $\partial_n^2 = 0$*. The result holds for* $n = 1$ *if we enlarge* $\mathrm{Kom}(n)$ *to allow for chain complexes whose chain groups are not necessarily finite direct sums of diagrams.*

Because the grading conventions may be confusing, let us disambiguate. The claim is that there is a chain complex P_n' which, as a bigraded object, is $P_n' = \bigoplus_{k\geq 0}(t^{2-2n}q^{2n})^k Q_n$, and whose differential is represented by a $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ matrix with diagonal entries all equal to d_{Q_n} (no sign since the shift in homological degree is even), sub-diagonal entries all equal to ∂_n, and all other entries equal to zero. The component $\partial_n : Q_n \to t^{2-2n}q^{2n}Q_n$ of the differential must have homological degree 1, q-degree 0, which forces ∂_n to be a bidegree $(0,0)$ map $Q_n \to t^{1-2n}q^{2n}Q_n$. Writing $d^2 = 0$ in terms of components, we see that $d^2 = 0$ if and only if ∂_n is a chain map and $\partial_n^2 = 0$.

Proof. Assume $n \geq 2$, and let $E_\bullet = E_{1-2n} \to \cdots \to E_0$ be the symmetric Frenkel-Khovanov sequence (Definition 6.6), so that $E_{1-2n} = q^{2n}P_{n-1}\sqcup 1_1$ and $E_0 = P_{n-1}\sqcup 1_1$. If $Q_n = \mathrm{Tot}(E_\bullet)$ is a symmetric projector then we have chain maps $\eta_n : P_{n-1}\sqcup 1 \to Q_n$ and $\varepsilon_n : Q_n \to t^{1-2n}q^{2n}P_{n-1}\sqcup 1$ given by the inclusion of E_0, respectively the projection onto $t^{1-2n}E_{1-2n}$. Put $\partial_n := -\eta_n \circ \varepsilon_n$. Note that $\varepsilon_n \circ \eta_n = 0$, so that $\partial_n^2 = 0$. Thus we can form the bicomplex as in the statement, and we can define P_n' to be the total complex:

$$P_n' := \mathrm{Tot}(Q_n \xrightarrow{\partial_n} t^{2-2n}q^{2n}Q_n \xrightarrow{\partial_n} t^{4-4n}q^{4n}Q_n \xrightarrow{\partial_n} \cdots)$$

By definition of ∂_n, P_n' can also be written as:



(6.15)

Contracting the vertical maps in (6.15) using Gaussian elimination (proposition 2.14) gives a deformation retract of P_n' onto a chain complex P_n such that the degree zero chain group is $(P_n)^0 = 1_n$, and the object 1_n does not appear as a direct summand of any other chain group. That is, P_n satisfies axiom (CK1) for Cooper-Krushkal projectors (Definition 3.12).

We want to see that P'_n kills turnbacks from below. Since $Q_n \in \mathrm{Kom}(n)$ is supported in non-positive homological degrees and $t^{2-2n}q^{2n}$ involves a negative shift in homological degree (recall that we assume $n \geq 2$), the infinite direct sum $\bigoplus_{k\geq 0}(t^{2-2n}q^{2n})^k Q_n$ is finite in each degree, hence exists in $\mathrm{Kom}(n)$ and is isomorphic to the infinite product $\prod_{k\geq 0}(t^{2-2n}q^{2n})^k Q_n$. The same is true if Q_n is replaced by $Q_n \odot e$ or $e \odot Q_n$ for some Temperley-Lieb generator e. Theorem 2.15 and the remarks following apply, and so the contractions $Q_n \odot e \simeq 0$ and $e \odot Q_n$ implied by Proposition 6.10 can be applied simultaneously to each term of $P'_n \odot e$, respectively $e \odot P'_n$. This shows that $P_n \simeq P'_n$ kills turnbacks, so P_n is a Cooper-Krushkal projector. This completes the proof in case $n \geq 2$.

In case $n = 1$, precisely the same argument works, where $Q_1 = \mathrm{Cone}(b)$, $b : q^2 1_1 \to 1_1$, and $\partial_1 : Q_1 \to t^{-1}q^2 Q_1$ is the (minus) projection followed by inclusion of the $t^{-1}q^2 1_1$ summand. It is necessary to only embed $\mathrm{Kom}(1)$ in a category which contains $\bigoplus_{k\geq 0} q^{2k} Q_1$, for example the category $\mathrm{Kom}(1)^{\oplus}$ of Definition 4.1. It happens that this infinite direct sum is equivalent to an infinite direct product, as was the case for $n \geq 2$, but this is actually not needed here since the condition that P_1 kill turnbacks is vacuously true. □

We conclude this section with a simple yet important observation:

Proposition 6.16 (Obtaining Q_n from P_n)**.** *If $Q_n \in \mathrm{Kom}(n)$ is a symmetric projector, then there is a Cooper-Krushkal projector $P_n \in \mathrm{Kom}(n)$ and a chain map $U_n : t^{2-2n}q^{2n}P_n \to P_n$ such that $\mathrm{Cone}(U_n) \simeq Q_n$.*

Proof. Fix an integer $n \geq 2$, let Q_n be symmetric projector, and let $P'_n = (Q_n \to x_n Q_n \to \dots)$ be as in (6.15), where we have let $x_n := t^{2-2n}q^{2n}$. Let $U'_n : t^{2-2n}q^{2n}P'_n \to$

P_n' denote the *periodicity map*

$$
\begin{array}{ccccccccccccc}
x_n P_n' & = & \Big(x_n Q_n & \xrightarrow{\partial_n} & x_n^2 Q_n & \xrightarrow{\partial_n} & x_n^3 Q_n & \xrightarrow{\partial_n} & x_n^4 Q_n & \xrightarrow{\partial_n} & \dots \Big) \\
{\scriptstyle U_n'}\big\downarrow & & & \searrow^{\mathrm{Id}} & & \searrow^{\mathrm{Id}} & & \searrow^{\mathrm{Id}} & & \searrow^{\mathrm{Id}} & \\
P_n' & = & \Big(Q_n & \xrightarrow{\partial_n} & x_n Q_n & \xrightarrow{\partial_n} & x_n^2 Q_n & \xrightarrow{\partial_n} & x_n^3 Q_n & \xrightarrow{\partial_n} & \dots \Big)
\end{array}
\tag{6.17}
$$

Clearly $\mathrm{Cone}(U_n') \simeq Q_n$. Conjugating with a deformation retract $P_n' \overset{\simeq}{\rightarrow} P_n$ onto a Cooper-Krushkal projector one obtains the result. □

6.4 Periodic chain complexes

For fixed n, we can tensor the expressions $P_m \simeq \mathrm{Tot}(Q_m \xrightarrow{\partial_m} t^{2-2m}q^{2m}Q_m \xrightarrow{\partial_n} \cdots)$ for $2 \le m \le n$, obtaining an expression for $P_n \simeq (P_2 \sqcup 1_{n-2}) \odot \cdots \odot P_n$ in terms of a multiperiodic multicomplex built out of $(Q_2 \sqcup 1_{n-2}) \odot \cdots \odot Q_n$. The periodicity can be built in as an action of the polynomial ring $\mathbb{Z}[u_2, \dots, u_n]$. We will introduce this language next. In this section and the next, we will identify $\mathcal{TL}_n$ with the category of finitely generated graded projective modules over Khovanov's ring H^n, as justified by Proposition 4.10. This is simply a matter of convenience, so that for a chain complex $A \in \mathrm{Kom}(n)$ we can tensor A with an abelian group, and we can talk about elements of A, etcetera.

Let $x_1, \dots, x_k$ be indeterminates of bidegree $\deg(x_i) = (a_i, b_i)$, and assume that the homological degrees a_i are even. We regard the polynomial algebra $R = \mathbb{Z}[x_1, \dots, x_k]$ as a differential bigraded algebra with zero differential. Put $a = (a_1, \dots, a_k)$ and $b = (b_1, \dots, b_k)$. Let $G^v = t^{a\cdot v}q^{b\cdot v}$ denote the corresponding grading shift functor, for each $v \in \mathbb{Z}^k$, where $\cdot$ denotes the usual dot product. For, $E \in \mathrm{Kom}(n)$, assume that the direct sum

$$M := \bigoplus_{v \ge 0} G^v(E)$$

exists in $\mathrm{Kom}(n)$ and is isomorphic to the direct product

$$\bigoplus_{v\geq 0} G^v(E) \cong \prod_{v\geq 0} G^v(E) \tag{6.18}$$

where each is indexed by $v \in (\mathbb{Z}_{\geq 0})^k$.

Definition 6.19. Let $R = \mathbb{Z}[x_1, \ldots, x_r]$ be a differential bigraded algebra with zero differential and bigrading $\deg(x_i) = (a_i, b_i)$, and assume the homological degrees a_i are even. If $E \in \mathrm{Kom}(n)$ is a chain complex such that (6.18) holds, denote by $R \vec{\otimes} E$ any chain complex $(\prod_{v\geq 0} G^v(E), \Delta)$ such that the components $\Delta_{vw} \in \mathrm{Hom}^{1,0}(G^w(E), G^v(E))$ of the differential satisfy:

1. $\Delta_{00} = d_E$

2. $\Delta_{vw} = G^w(\Delta_{v-w,0})$.

3. $\Delta^2 = 0$.

Note that $R \vec{\otimes} E$ is a periodic chain complex built out of E; the periodicity of the differential is precisely the fact that the component Δ_{vw} depends only the difference $v - w$. More precisely, put $M := \prod_{v\geq 0} G^v(E)$. Since $G^v \circ G^w \cong G^{v+w}$ we can identify $G^v(M)$ with the obvious subcomplex of M. The inclusions $X^v : G^v(M) \hookrightarrow M$ commute with one-another and generate an an action of R on M, i.e. a map of differential bigraded algebras $R \to \mathrm{End}^{\bullet,\bullet}(M)$. Note that the differential bigraded R-module M is isomorphic to $R \otimes E$ with differential $d(r \otimes e) = r \otimes d_E(e)$, bigrading $\deg(r \otimes e) = \deg(r) + \deg(e)$, and obvious action of R on the left. The identification is via $x^v \otimes E \cong G^v(E)$, where $x^v = x_1^{i_1} \cdots x_k^{i_k}$ for all $v = (i_1, \ldots, i_k) \in (\mathbb{Z}_{\geq 0})^k$.

Now, suppose $\Delta \in \mathrm{End}^{1,0}(M)$ is a differential such that $(M, \Delta) = R \vec{\otimes} E$. Such a Δ induces an element $d \in \mathrm{End}^{1,0}(R \otimes E)$ via the isomorphism $M \cong R \otimes E$, and d satisfies the analogous properties

1. $d(1 \otimes e) \in 1 \otimes d_E(e) + I \otimes E$, where $I \leq R$ is the ideal generated by the $x_1, \ldots, x_k$.

2. $d(r \otimes e) = rd(1 \otimes e)$.

3. $d^2 = 0$.

So $R \vec{\otimes} E$ is obtained from $R \otimes E$ by twisting the differential via higher degree R-equivariant terms.

Our main reason for introducing the notation $R \vec{\otimes} E$ is to describe and simplify the periodic chain complexes built out of symmetric projectors Q_m:

Proposition 6.20. *Let u_m be a formal indeterminate of bidegree $(2-2m, 2m)$. Then there is a chain complex $\mathbb{Z}[u_m] \vec{\otimes} Q_m$ which deformation retracts onto a Cooper-Krushkal projector $P_m \in \mathrm{Kom}(n)$.*

Proof. By Proposition 6.13 there is a bicomplex

$$Q_m \xrightarrow{\partial_m} t^{2-2m}q^{2m}Q_m \xrightarrow{\partial_m} t^{4-4m}q^{4m}Q_m \xrightarrow{\partial_m} \cdots$$

the total complex of which deformation retracts onto a Cooper-Krushkal projector. This total complex T is isomorphic as a graded object to $\mathbb{Z}[u_m] \otimes Q_m$ where $u_m^k \otimes Q_m$ is identified with the copy of Q_m appearing with the shift by $(t^{2-2m}q^{2m})^k$. In terms of the isomorphism of bigraded objects $T \cong \mathbb{Z}[u_m] \otimes Q_m$, the differential satisfies

$$d(u_m^k \otimes z) = u_m^k \otimes d_{Q_n}(z) + u_m^{k+1} \otimes \partial_m(z)$$

It is clear that d_T commutes with the natural $\mathbb{Z}[u_m]$-action (this is just the observation that T is periodic), and that $T \cong \mathbb{Z}[u_m] \vec{\otimes} Q_m$. This proves the proposition. □

We now set up some elementary theory for manipulating and simplifying periodic the chain complexes $R \vec{\otimes} E$.

Theorem 6.21. *Let $R = \mathbb{Z}[x_1, \ldots, x_r]$ be as in Definition 6.19, and suppose we have a deformation retract $E \overset{\simeq}{\rightarrow} F$ of chain complexes $E, F \in \mathrm{Kom}(n)$. The any chain complex $R \vec{\otimes} E$ deformation retracts onto some $R \vec{\otimes} F$, and the data of the deformation retract can be chosen to commute with the R-action.*

Proof. We want to describe $M = R \vec{\otimes} E$ as a convolution over the indexing set $S = (\mathbb{Z}_{\geq 0})^r$, then use Theorem 2.15 and the comments following. We use the partial order on $\mathbb{Z}^r$ given by $v \geq w$ if the coordinates of $v - w$ are non-negative. Since S has a unique minimal element rather than a unique maximal element, in order to use the result of Theorem 2.15 we need to be working with convolutions using $\prod$ instead of $\bigoplus$. But this holds since, from Definition 6.19 we always assume that (6.18) holds.

Let us use a multi-index notation. I.e. let $x^v = x_1^{i_1} \cdots x_r^{i_r}$ for each $v = (i_1, \ldots, i_r) \in (\mathbb{Z}_{\geq 0})^r$, and for any $f \in \mathrm{End}^{\bullet}(R \otimes E)$ let f_{vw} denote the component $f_{vw} \in \mathrm{Hom}^{\bullet}(x^w \otimes E, x^v \otimes E)$. Let $|v| = v_1 + \cdots + v_r$ for each $v \in (\mathbb{Z}_{\geq 0})^r$, and say an element $f \in \mathrm{End}^{\bullet,\bullet}(R \otimes M)$ has length $k \in \mathbb{Z}$ if $f_{vw} = 0$ unless $|v - w| = k$. We can write d_M in terms of its length k components as $d_M = \sum_k d_k$, where $d_k \in \mathrm{End}^1(R \otimes E)$ is an element of length k. By the conditions we place on the differential of $M = R \vec{\otimes} E$ the d_k satisfy

1. $d_k = 0$ for $k < 0$.

2. $d_0 = \mathrm{Id}_R \otimes d_E$, so that the chain complex (M, d_0) is isomorphic to $\prod_{v \geq 0} t^{a \cdot v} q^{b \cdot v} E$.

3. Each d_k commutes with the R-action.

Suppose (π', σ', h') give the data of a deformation retract $E \to F$, and put $\pi_0 := \mathrm{Id}_R \otimes \pi'$, $\sigma_0 := \mathrm{Id}_R \otimes \sigma'$, and $h_0 := \mathrm{Id}_R \otimes h'$. Then (π_0, σ_0, h_0) give the data of a deformation retract $R \otimes E \to R \otimes F$, and each commutes with the R-action. We are now in a situation where we can use Theorem 2.15, obtaining a deformation retract (π, σ, h) $M \to N$, where the length zero part of the differential d_N satisfies $(N, (d_N)_0) = R \otimes F$. To see that $N = R \vec{\otimes} F$ and that the maps π, σ, h commute with the R-action, Theorem 2.15 says that we can assume that the components of π, σ, h, and d_N are polynomial in the π_0, σ_0, h_0 and the d_k. The former commute with the R-action since the latter do. This completes the proof. $\square$

Proposition 6.22 (Transitivity of $\vec{\otimes}$)**.** *Suppose we have a chain complex* $(R\otimes S)\vec{\otimes}E \in \mathrm{Kom}(n)$*. Then there is a factorization* $(R \otimes S) \vec{\otimes} E \cong R \vec{\otimes} (S \vec{\otimes} E)$*. Further, if there is a deformation retract* $E \to F$ *then the following diagram commutes*

$$\begin{array}{ccc} (R \otimes S) \vec{\otimes} E & \longrightarrow & (R \otimes S) \vec{\otimes} F \\ \cong \big\downarrow & & \cong \big\downarrow \\ R \vec{\otimes} (S \vec{\otimes} E) & \longrightarrow & R \vec{\otimes} (S \vec{\otimes} F) \end{array}$$

where the horizontal arrows are the deformation retracts implied by Theorem 6.21.

Proof. Put $R := \mathbb{Z}[x_1, \ldots, x_k]$ and $S := \mathbb{Z}[x_{k+1}, \ldots, x_{k+l}]$. Suppose $\deg(x_i) = (a_i, b_i)$ with a_i even, and let G^{v+w} be the corresponding shift functor. Throughout, we will use letters v, v' to denote elements of $\mathbb{Z}^k \times 0 \subset \mathbb{Z}^k \times \mathbb{Z}^l$ and letters w, w' to denote elements of $0 \times \mathbb{Z}^l \subset \mathbb{Z}^k \times \mathbb{Z}^l$.

By hypothesis we have a chain complex $M = (R \otimes S) \vec{\otimes} E$:

$$M = \prod_{v,w\geq 0} G^{v+w}(E)$$

Write d_M in terms of components as

$$d_{(v,w),(v',w')} \in \mathrm{Hom}^{1,0}(G^{v'+w'}(E), G^{v+w}(E)).$$

Put $\partial_{w,w'} := d_{(0,w),(0,w')}$. Then $\partial_{0,0} = d_E$ and $\partial_{w,w'}$ depends only on the difference $w - w'$. This is to say, we have a chain complex $N = (\prod_{w\geq 0} G^w(E), \sum_{w,w'\geq 0} \partial_{w,w'}) \cong S \vec{\otimes} E$. To see that $M \cong R \vec{\otimes} N$, define $\Delta_{v,v'} \in \mathrm{Hom}^{1,0}(G^{v'}(N), G^v(N))$ to be

$$\Delta_{v,v'} = \sum_{w,w'\geq 0} d_{(v,w),(v',w')}.$$

It is clear that $d_M = \sum_{v,v'\geq 0} \Delta_{v,v'}$ and that

1. $\Delta_{0,0} = \sum_{w,w'\geq 0} d)_{(0,w'),(0,w)} = d_N$
2. $G^{v'}(\Delta_{v-v',0}) = \sum_{w,w'\geq 0} G^{v'}(d_{(v-v',w),(0,w')}) = \sum_{w,w'\geq 0} d_{(v,w),(v',w')}) = \Delta_{v,v'}$.

This is to say $M \cong R \vec{\otimes} N$. The statement about deformation retracts follows from naturality of the deformation retracts implied by Theorem 2.15. This completes the proof. $\square$

The following is straightforward: linear functors send periodic chain complexes to periodic chain complexes.

Proposition 6.23. *Suppose $T : \mathcal{TL}_m \to \mathcal{TL}_n$ is a linear functor, and let T also denote the extension to categories of chain complexes. Then $T(R \vec{\otimes} E) \cong R \vec{\otimes} T(E)$.*

Proof. Put $R := \mathbb{Z}[x_1, \ldots, x_k]$. Suppose $\deg(x_i) = (a_i, b_i)$ with a_i even, and let G^v be the corresponding shift functor. Let us recall the action of a T on chain complexes. For a chain complex $(A^\bullet, d_A)$ over $\mathcal{TL}_m$ we have $T(A)^i = T(A^i)$ and $d_{T(A)} = T(d_A)$ where the action on morphisms is $T(f)|_{T(A^i)} = T(f|_{A^i})$ for all $f \in \mathrm{Hom}^{\bullet;\bullet}_{\mathcal{TL}_m}(A, B)$. Because T treats all of the chain groups equally, T commutes with the grading shift functors. Hence, applying T to a chain complex

$$M = R \vec{\otimes} E = \Big(\prod_{v \geq 0} G^v(E) \ , \ \sum_{v \geq w} d_{vw} \Big)$$

yields

$$T(M) \cong \Big(\prod_{v \geq 0} G^v(T(E)) \ , \ \sum_{v \geq w} T(d_{vw}) \Big)$$

It is clear that this latter chain complex satisfies the conditions of Definition 6.19, i.e. $T(M) \cong R \vec{\otimes} T(E)$. $\square$

More generally, if $T(-,-)$ is a bilinear functor then $T(R \vec{\otimes} E, S \vec{\otimes} F) \cong (R \otimes S) \vec{\otimes} T(E, F)$, by combining the previous two propositions.

Now, fix an integer $n \geq 2$ and assume that we have symmetric projectors $Q_m \in \mathrm{Kom}(m)$ for each $1 \leq m \leq n$. By Proposition 6.23, tensoring together the chain complexes $\mathbb{Z}[u_m] \vec{\otimes} (Q_m \sqcup 1_{n-m})$ for $1 \leq m \leq n$ gives a chain complex $\mathbb{Z}[u_1, \ldots, u_m] \vec{\otimes} M_n$, where $M_n = (Q_1 \sqcup 1_{n-1}) \odot \cdots \odot Q_n$. By Proposition 6.20 each $\mathbb{Z}[u_m] \vec{\otimes} Q_m$ is homotopy equivalent to a Cooper-Krushkal projector P_m, so gluing them together gives a chain complex homotopy equivalent to $(P_1 \sqcup 1_{n-2}) \odot \cdots \odot P_n$. By projector absorbing (proposition 4.23), this latter chain complex is homotopy equivalent to P_n. This gives an expression for P_n in terms of a multiperiodic chain complex:

$$P_n \simeq \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} M_n \in \mathrm{Kom}(n)^{\oplus}$$

We want to simplify this chain complex, but category $\mathrm{Kom}(n)^{\oplus}$ is too big for many purposes.

Definition 6.24. Recall the notation $\mathrm{Kom}(n)^{\oplus} = \mathrm{Kom}(\mathcal{TL}_n^{\oplus})$. Let $\mathrm{Kom}'(n) \subset \mathrm{Kom}(n)^{\oplus}$ denote the full category of chain complexes whose chain groups can be written

$$M^i \cong \bigoplus_{j=1}^{\infty} q^{j+k_i} a_{ij}$$

where (1) each $a_{ij} \in \mathcal{TL}_n$ is a finite direct sum of tangles without circle components and without grading shifts, zero for $i \gg 0$, and (2) $k_i \in \mathbb{Z}$ is some sequence such that $\lim_{i \to -\infty} k_i + i = \infty$.

$$\begin{array}{cccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\ddots & \bullet & \bullet & \bullet & 0 & \cdots \\
\cdots & 0 & \bullet & \bullet & 0 & \cdots \\
\cdots & 0 & 0 & \bullet & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Figure 6.1: The bigradings on chain complexes in $\mathrm{Kom}(m, n)'$ are supported in a region of the $\mathbb{Z} \times \mathbb{Z}$ lattice as shown above, up to translations. Moreover, the sum along the skew diagonals $\{(x, y) \mid x + y = k\}$ is finite.

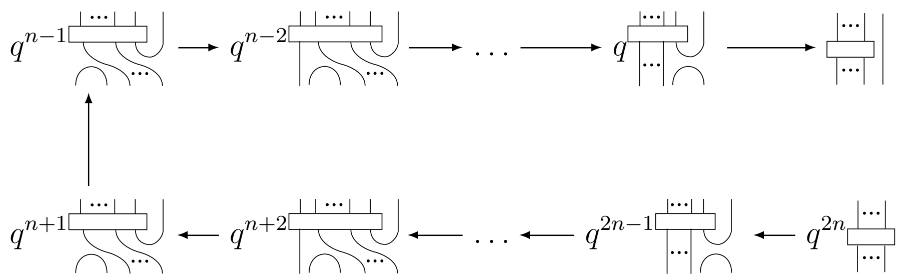
Remark 6.25. The isomorphism of categories in Proposition 4.10 allows us to regard $N \in \mathrm{Kom}(n)^{\oplus}$ as a chain complex of (not necessarily finitely generated) graded projective H^n modules. The category $\mathrm{Kom}(n)'$ consists precisely of those chain complexes whose image is a chain complex $M^\bullet \in \mathrm{Kom}(H^n\text{-}\mathbf{pgmod})$ such that (1) each chain group M^k is a graded projective H^n-module $M^i = \bigoplus_{j\in\mathbb{Z}} M^{ij}$ with each M^{ij} finitely generated as an abelian group, (2) there exist $k, l \in \mathbb{Z}$ such that $M^{ij} = 0$ for $i > k$ or $i + j < l$, and (3) for each $k \in \mathbb{Z}$ the sum $\bigoplus_{i+j=k} M^{ij}$ is finite.

Deformation retracts preserve our conditions on gradings, so the result of Theorem 6.28 shows that there exist Cooper-Krushkal projectors in the intersection $\mathrm{Kom}'(n)\cap \mathrm{Kom}(n)$, provided that the symmetric projectors $Q_1, \ldots, Q_n$ exist. If, then, Q_{n+1} were shown to exist, then we may as well assume that Q_{n+1} is a convolution of the symmetric Frenkel-Khovanov sequence relative to $P_n \in \mathrm{Kom}(n) \cap \mathrm{Kom}(n)'$, so Q_{n+1} would also lie in this intersection. Thus all of the chain complexes of interest can be assumed to lie in $\mathrm{Kom}(n) \cap \mathrm{Kom}(n)'$. Note also that the (completed) Grothendieck group of $\mathrm{Kom}(n)'$ is non-trivial, in contrast with that of $\mathrm{Kom}(n)$. The following facts are clear.

1. $\mathrm{Kom}(n)'$ is closed under planar composition.
2. $\mathrm{Kom}(n)'$ contains $\mathrm{Kom}^b(\mathcal{TL}_n)$.
3. Suppose $x_1, \ldots, x_r$ are indeterminates of bidegree $\deg(x_i) = (a_i, b_i)$, and assume $a_i \in \{-2, -4, \ldots\}$ and $a_i + b_i > 0$. If $E \in \mathrm{Kom}(n)'$ then any chain complex $\mathbb{Z}[x_1, \ldots, x_r] \otimes E \in \mathrm{Kom}(n)^\oplus$ is a well defined chain complex in $\mathrm{Kom}'(n)$ and equation (6.18) is still valid.

Recall that n is fixed and that we assume symmetric projectors $Q_m \in \mathrm{Kom}(m)$ exist for $1 \le m \le n$.

Definition 6.26. For any chain complex $Z \in \mathrm{Kom}(n-1)$ which kills turnbacks, let $E_\bullet(Z)$ denote the symmetric Frenkel-Khovanov sequence relative to Z, which is the homotopy chain complex obtained from the usual symmetric Frenkel-Khovanov sequence (Definition 6.6) by replacing P_{n-1} with Z everywhere:



where each white box denotes Z. The maps $\alpha_k : E_k(Z) \to E_{k+1}(Z)$ between adjacent terms are saddle maps or a difference of dotted identities, and the fact that $E_\bullet(Z)$ is a homotopy chain complex follows from the same argument as in the proof of Proposition 6.8.

Lemma 6.27. *Suppose* $Q_n = \mathrm{Tot}(E_\bullet(P_{n-1})) \in \mathrm{Kom}(n)$ *is a symmetric projector, and let* $Z \in \mathrm{Kom}(n-1)$ *be any chain complex which kills turnbacks. Then* $(Z \sqcup 1_1) \odot Q_n$ *deformation retracts onto some* $\mathrm{Tot}(E_\bullet(Z))$.

Proof. Note that that $E_k(W) = (W \sqcup 1) \odot a_k$ for some diagram $a_k \in \mathcal{TL}_n$ and the maps between adjacent terms act only on the a_k factors. Hence if $Q_n = \mathrm{Tot}(E_\bullet(P_{n-1})$ is a symmetric projector, then $(Z \sqcup 1_1) \odot Q_n = \mathrm{Tot}(E_\bullet(Z \odot P_{n-1}))$. By projector absorbing we have a deformation retract $Z \odot P_{n-1} \to Z$; applying this to each term of $E_\bullet(Z \odot P_{n-1})$ gives the result. □

Gluing together the chain complexes $P_m \simeq \mathbb{Z}[u_m] \vec{\otimes} Q_m$ from Proposition 6.20 gives an expression for $P_n \simeq (P_1 \sqcup 1_{n-1}) \odot \cdots \odot P_n$ as a periodic chain complex $\mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} (Q_1 \sqcup 1_{n-1}) \odot \otimes \odot Q_m$. It turns out that $(Q_1 \sqcup 1_{n-1}) \odot \cdots \odot Q_n$ deformation retracts onto a bounded chain complex, and we obtain the following attractive description of P_n. The next theorem constructs a standard family of chain complexes to which we will refer frequently in the sequel.

Theorem 6.28. *Fix an integer* $n \geq 1$*, and assume that symmetric projectors exist in* $\mathrm{Kom}(m)$ *for* $1 \leq m \leq n$*. Then there exist chain complexes* $C_m, Q'_m, P'_m \in \mathrm{Kom}(m)'$

and $Q_m, P_m \in \mathrm{Kom}(m) \cap \mathrm{Kom}(m)'$ *such that* Q_m *is a symmetric projector,* P_m *is a Cooper-Krushkal projector, and*

(1) $C_m = \mathrm{Tot}(E_\bullet(C_{m-1}))$ *is a bounded chain complex which kills turnbacks and is homotopy equivalent to* $(Q_1 \sqcup 1_{n-1}) \odot \cdots \odot Q_n$

(2) $Q'_m = \mathbb{Z}[u_1, \ldots, u_{m-1}] \vec{\otimes} C_m$ *is homotopy equivalent to* Q_m*,*

(3) $P'_m = \mathbb{Z}[u_n] \vec{\otimes} Q'_n = \mathbb{Z}[u_1, \ldots, u_m] \vec{\otimes} C_m$ *deformation retracts onto* P_m*,*

for each $1 \leq m \leq n$*.*

Proof. Put $P'_1 := \mathbb{Z}[u_1] \vec{\otimes} Q_1$ and $C_1 := Q'_1 := Q_1$. For $m \geq 2$, assume by induction that C_m, Q'_m, and P'_n are defined and satisfy (1),(2),(3) above. If a symmetric projector Q_{m+1} exists, by Lemma 6.27 we may as well assume that $Q_{m+1} = \mathrm{Tot}(E_\bullet(P_m))$. Since C_m kills turnbacks, lemma 6.27 gives us a deformation retract

$$(C_m \sqcup 1_1) \odot Q_m \xrightarrow{\simeq} \mathrm{Tot}(E_\bullet(C_m)).$$

Define $C_{m+1} := \mathrm{Tot}(E_\bullet(C_m))$ to be this last chain complex; C_{m+1} kills turnbacks from below since Q_m does and from above by Proposition 4.26. Clearly C_{m+1} is bounded and is homotopy equivalent to $(Q_1 \sqcup 1_{n-1}) \odot \cdots \odot (Q_{n-1} \sqcup 1) \odot Q_n$ by the induction hypothesis. So (1) holds.

Now define $Q'_{m+1} = \mathbb{Z}[u_1, \ldots, u_m] \vec{\otimes} C_{m+1}$ to be the result of applying the deformation retract $(C_m \sqcup 1_1) \odot Q_m \to C_{m+1}$ to each term of

$$(P'_m \sqcup 1) \odot Q_{m+1} = \mathbb{Z}[u_1, \ldots, u_m] \vec{\otimes} (C_m \sqcup 1) \odot Q_{m+1}$$

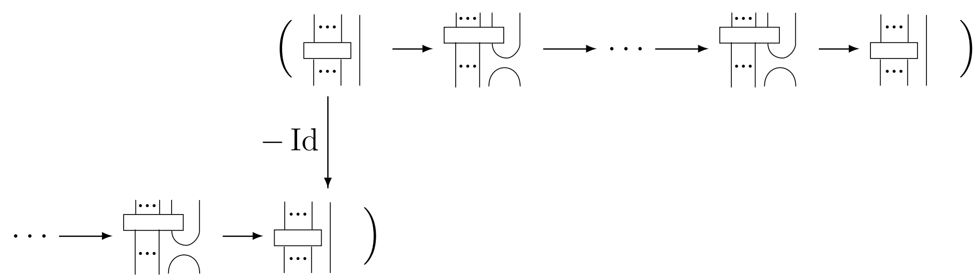
With this definition, (2) holds. Recall the chain complex $\mathbb{Z}[u_{m+1}] \vec{\otimes} Q_{m+1}$ from Proposition 6.20, and define $P'_{m+1} = \mathbb{Z}[u_{m+1}] \vec{\otimes} Q'_{m+1}$ to be the result of applying the deformation retract $(C_m \sqcup 1_1) \odot Q_m \to C_{m+1}$ to each term of

$$\begin{aligned}(P'_m \sqcup 1) \odot (\mathbb{Z}[u_{m+1}] \vec{\otimes} Q_{m+1}) &\overset{(*)}{\cong} \mathbb{Z}[u_{m+1}] \vec{\otimes} ((P'_m \sqcup 1_1) \odot Q_{m+1}) \\ &\cong \mathbb{Z}[u_{m+1}] \vec{\otimes} \Big(\mathbb{Z}[u_1, \ldots, u_m] \vec{\otimes} (C_m \sqcup 1) \odot Q_{m+1}\Big)\end{aligned}$$

where $(*)$ holds by Proposition 6.22. The result will be a chain complex

$$\mathbb{Z}[u_{m+1}] \vec{\otimes} Q'_{m+1} \cong \mathbb{Z}[u_1, \ldots, u_{m+1}] \vec{\otimes} C_{m+1}.$$

As in the proof of Proposition 6.13 the chain complex appearing on the left-hand side of the isomorphism $(*)$ above can be written as



where the white box denotes $P'_m \odot P_m$ and each row is $(P'_m \sqcup 1_1) \odot Q'_{m+1}$. Applying the retract $P'_m \odot P_m \to P'_m$ to each term yields P'_{m+1}, which has precisely the same form as the above, except where the white boxes now denote P'_m and each row is Q'_{m+1}. By the inductive hypothesis there is a deformation retract $P'_m \to P_m$; applying this to each white box and then contracting the vertical maps gives a deformation retract of P'_{m+1} onto a Cooper-Krushkal projector P_{m+1}. This completes the inductive step and completes the proof. □

Observation 6.29. Since P_n is supported in non-positive homological degrees, it is the direct limit of its truncations $P_n = \mathrm{colim}_{k\to\infty}((P_n)^{-k} \to (P_n)^{-k+1} \to \cdots \to (P_n)^0)$, where $(P_n)^l$ denotes the l-th chain group. By contrast, Theorem 6.28 allows us to express P_n as an *inverse* limit of bounded chain complexes which kill turnbacks. Indeed, put $R = \mathbb{Z}[u_1, \ldots, u_n]$, and let $I \leq R$ be the ideal generated by the u_i, and $P'_n := R \vec{\otimes} C_n$. Then each $E_k := P'_n / I^k P'_n$ is bounded (this chain complex corresponds to the direct sum of the $f \otimes C_n$, where $f = f(u_1, \ldots, u_n)$ runs over the monomials of total degree $\leq k$). Then each E_k kills turnbacks, and $P_n \simeq \lim_{\infty \leftarrow k} E_k$. The fact is often useful. For one application, see Proposition 4.25.

Remark 6.30. It is interesting to note that we have given a positive answer the following question: does there exist a bounded chain complex $C \in \mathrm{Kom}(n)$ which kills turnbacks? This question can be stated without mention of categorified Jones-Wenzl projectors, but constructing such a C from scratch (i.e. without the aid of the P_n) seems to be quite difficult.

6.5 Application I: sheet algebra and the GOR conjecture

Throughout this section, assume that symmetric projectors $Q_1, \ldots, Q_n$ exist. Our goal here will be to use the periodic presentation of the Cooper-Krushkal projectors from the previous section to study $\mathrm{End}^{\bullet,\bullet}(P_n)$ and give a partial result toward a conjecture of Gorsky-Oblomkov-Rasmussen [GOR12] on $\mathfrak{sl}_2$-link homology of the colored unknots, refined to include the $\mathbb{Z}[u_1, \ldots, u_n]$-action. Their conjecture was motivated by a relationship between $\mathfrak{sl}_2$-link homology and Homfly homology, and a (conjectural) relationship between Homfly homology of an algebraic link and the Hilbert scheme of points on its defining singular curve. Here is our version:

Conjecture 6.31. *Let $P'_n = \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} C_n$ denote the chain complex from Theorem 6.28. Then there is a deformation retract of $\mathbb{Z}[u_1, \ldots, u_n]$-modules $\mathrm{End}^{\bullet,\bullet}(P'_n) \to W_n$ where $W_n = \mathbb{Z}[u_1, \ldots, u_n] \otimes \Lambda[w_1, \ldots, w_n]$ with bigrading $\deg(u_m) = (2-2m, 2m)$, $\deg(w_m) = (1-2m, 2+2m)$, and differential given by*

$$d(u_m) = 0 \qquad\qquad d(w_m) = \sum_{i+j=m+1} u_i u_j$$

for all $1 \leq m \leq n$, together with the graded Leibniz rule. In particular W_n computes the $\mathfrak{sl}_2$-link homology of the n-colored unknot and the limiting Khovanov homology of the (n, m)-torus links.

Note that such a deformation retract would endow W_n with the structure of an A_∞-algebra [Val12], but we should not expect the induced multiplication μ_2 to coincide with the obvious multiplication on W_n. For example we can check in the $n = 2$ case that $\mu_2(\xi_2, \xi_2) = u_2^3$. Rather, the retract should preserve the $\mathbb{Z}[u_1, \ldots, u_n]$-action. We will prove a weakened version of this conjecture in Theorem 6.37.

Let $P'_n = \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} C'_n$ be as in Theorem 6.28, and let P_n be a Cooper-Krushkal projector onto which P'_n deformation retracts. Then we have the following sequence of deformation retracts and isomorphisms

$$\mathrm{End}^{\bullet,\bullet}(P'_n) \to \mathrm{Hom}^{\bullet,\bullet}(P_n, P'_n) \to \mathrm{Hom}^{\bullet,\bullet}(1_n, P'_n) = \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} \mathrm{Hom}^{\bullet,\bullet}(1_n, C_n) \tag{6.32}$$

The first is the functor $\mathrm{Hom}^{\bullet,\bullet}(-, P'_n)$ applied to the deformation retract $P'_n \to P_n$ the second is by proposition 5.1, and the last is an isomorphism implied by linearity of $\mathrm{Hom}^{\bullet,\bullet}$ and Proposition 6.19. Each is a deformation retract or isomorphism of $\mathbb{Z}[u_1, \ldots, u_n]$-modules. Thus, as a first step toward understanding $\mathrm{End}^{\bullet,\bullet}(P'_n)$, we might try to simplify $\mathrm{Hom}^{\bullet,\bullet}(1_n, C_n)$:

Proposition 6.33. *There is a deformation retract of* $\mathrm{Hom}^{\bullet,\bullet}(1_n, C_n)$ *onto the exerior algebra* $\Lambda[w_1, \ldots, w_n]$ *with bigrading* $\deg(w_k) = (1 - 2k, 2 + 2k)$ *and zero differential.*

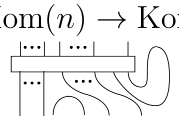
Proof. Induction on $n \geq 1$. In case $n = 1$, we have $C'_1 = \mathrm{Cone}(b)$, where $b : q^2 1_1 \to 1_1$ is the dotted identity cobordism. Note that $\mathrm{End}^{\bullet,\bullet}(1_1) = \mathbb{Z}[b]/b^2$, hence

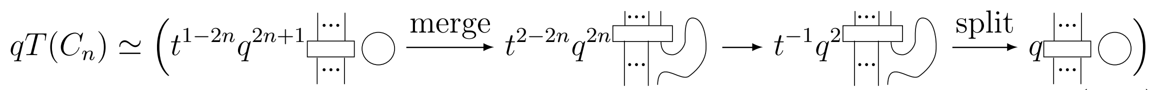
$$\mathrm{Hom}^{\bullet,\bullet}(1_1, C'_1) = \left(t^{-1}q^2\mathbb{Z}[b]/b^2 \xrightarrow{b} \mathbb{Z}[b]/b^2\right) \simeq t^{-1}q^4\mathbb{Z} \oplus \mathbb{Z}$$

The homotopy equivalence is a Gaussian elimination, hence a deformation retract. This gives the result in case $n = 1$.

Assume by induction that we have a deformation retract $\mathrm{Hom}^{\bullet,\bullet}(1_{n-1}, C_{n-1}) \to \Lambda[w_1, \ldots, w_{n-1}]$ as in the statement. By construction of C_n from Theorem 6.28, we have $C_n = \mathrm{Tot}(E_\bullet(C_{n-1}))$ where $E_\bullet(-)$ denotes the symmetric Frenkel-Khovanov sequence (see Definition 6.26). By Corollary 4.17 we have

$$\mathrm{Hom}^{\bullet,\bullet}(1_n, C_n) \cong \mathrm{Hom}^{\bullet,\bullet}(1_{n-1}, qT(C_n))$$

where $T : \mathrm{Kom}(n) \to \mathrm{Kom}(n-1)$ is the partial trace functor. Contracting complexes of the form  , we can see that the partial trace $qT(C_n)$ deformation retracts onto a convolution of the form



(6.34)

where the white boxes here denotes C_{n-1}. Lemma 6.39 implies that $qT(C_n)$ deformation retracts onto a $(t^{1-2n}q^{2+2n}C_{n-1} \to C_{n-1})$. Applying $\mathrm{Hom}^{\bullet,\bullet}(1_{n-1}, -)$ and using the inductive hypothesis, we see that $\mathrm{Hom}^{\bullet,\bullet}(1_n, C_n)$ deformation retracts onto a chain complex which as a bigraded abelian group is $t^{1-2n}q^{2+2n}\Lambda[w_1, \ldots, w_{n-1}] \oplus \Lambda[w_1, \ldots, w_{n-1}] \cong \Lambda[w_1, \ldots, w_n]$.

Degree considerations force the differential on $\Lambda[w_1, \ldots, w_n]$ to vanish. Perhaps the quickest way to see this is to collapse the bigrading to the single grading $\deg_s = \deg_h + \deg_q$ so that $\deg_s(w_k) = 3$. Since the degrees add under multiplication in the exterior algebra, we have $\deg_s(v) \in 3\mathbb{Z}$ for all homogeneous elements v. The differential is $\deg_s$-homogeneous of degree 1, so the only possibility is $d(v) = 0$ for all homogeneous elements $v \in \Lambda[w_1, \ldots, w_n]$. Since the homogeneous elements span linearly, the result follows. □

As an immediate corollary we obtain the following first step toward conjecture 6.31. Even though the result of Theorem 6.37 (appearing in a moment) is much stronger, the proof of the following is more immediate, and serves at least to take care of the $n = 1$ case of Theorem 6.37.

Corollary 6.35. *Assume that symmetric projectors* $Q_1, \ldots, Q_n$ *exist and let* $P'_n = \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} C_n$ *be the periodic chain complex from Theorem 6.28. There is a deformation retract of* $\mathbb{Z}[u_1, \ldots, u_n]$*-modules* $\mathrm{End}^{\bullet,\bullet}(P'_n) \to V_n$*, where*

$$V_n = \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} \Lambda[w_1, \ldots, w_n]$$

is some periodic complex with bigrading $\deg(u_k) = (2-2k, 2k)$*,* $\deg(w_k) = (1-2k, 2+2k)$ *and differential satisfying*

$$d(u_k) = 0 \qquad d(w_k) = \sum_{i+j=k+1} a_{ij} u_i u_j$$

for some integers $a_{ij} \in \mathbb{Z}$.

Proof. From (6.32) we have a deformation retract

$$\mathrm{End}^{\bullet,\bullet}(P_n') \to \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} \mathrm{Hom}^{\bullet,\bullet}(1_n, C_n).$$

We can use Theorem 6.21 to apply the retract $\mathrm{Hom}^{\bullet,\bullet}(1_n, C_n)$ from Proposition 6.33 to each term, obtaining a deformation retract $\mathrm{End}^{\bullet,\bullet}(P_n') \to \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} \Lambda[w_1, \ldots, w_n]$. Note that the bidegrees are as stated. For the statement about the differentials, it is useful to collapse the bigrading to $\deg_s = \deg_h + \deg_q$, so that $\deg_s(u_k) = 2$ and $\deg_s(w_k) = 3$ for all $1 \leq k \leq n$. The differential is $(\deg_h, \deg_q)$ bihomogeneous of degree $(1, 0)$, hence $\deg_s$ homonogeneous of degree $+1$. It follows that $d(1) = 0$ and $d(w_k) = \sum_{i,j} a_{ij}^k u_i u_j$ is quadratic in the the u_i. By equating q-degrees of each side of this equation (remember that the differential is still *bihomogeneous*) we see $a_{ij}^k = 0$ unless $2i + 2j = 2 + 2k$, i.e. $i + j = k + 1$. This shows that the differential is as in the statement. □

Before attempting to improve this result, let us consider the $n = 1$ case of Corollary 6.35, which gives us a deformation retract of $\mathbb{Z}[u_1]$-modules

$$\mathrm{End}^{\bullet,\bullet}(P_1') \xrightarrow{\simeq} \mathbb{Z}[u_1] \vec{\otimes} \Lambda[w_1]$$

Now, the dotted identity map is a chain map of bidegree $(0, 2)$, hence it must be homotopic to a multiple of the periodicity map u_1 for degree reasons. Let us examine this in more detail. By definition, P_1' (see Theorem 6.28) is the chain complex

$$P_1' := \mathbb{Z}[u_1] \vec{\otimes} Q_1 = \begin{array}{ccccccc} & & & & t^{-1}q^2 1 & \xrightarrow{b} & 1 \\ & & & & {\scriptstyle -\mathrm{Id}} \downarrow & & \\ & & t^{-1}q^4 1 & \xrightarrow{b} & q^2 1 & & \\ & & {\scriptstyle -\mathrm{Id}} \downarrow & & & & \\ \cdots & \xrightarrow{b} & q^4 1 & & & & \end{array}$$

where $b : q^2 1_1 \to 1_1$ is the dotted identity cobordism. Now consider the element $h \in \mathrm{End}^{-1,2}(P_1')$ given in the following diagram:

$$t^{-1}q^2 1 \xleftarrow{\mathrm{Id}} 1$$

$$t^{-1}q^4 \cdot 1 \xleftarrow{\mathrm{Id}} q^2 \cdot 1$$

$$\cdots \xleftarrow{\mathrm{Id}} q^4 1$$

Since h is periodic, h commutes with u_1. By direct observation, we have $[d, h] = b - u_1$, where $b : q^2 P_1' \to P_1'$ is the dotted identity map. We have proven that

Lemma 6.36. *$b - u_1$ is a nulhomotopic element of $\mathrm{End}^{0,2}(\mathbb{Z}[u_1] \vec{\otimes} Q_1)$, and the nulhomotopy can be chosen to commute with the $\mathbb{Z}[u_1]$-action.* □

We now state our main result on the chain complexes $\mathrm{End}^{\bullet,\bullet}(P_n)$, which is stronger than Corollary 6.35 but not as strong as the statement of conjecture 6.31. We remark that this result will be used in an inductive construction of the symmetric projectors Q_m in Chapter 7; we can retroactively omit the assumption that the Q_m exist. Let $\mathrm{End} = \mathrm{End}^{\bullet,\bullet}$ in the following, for aesthetic reasons.

Theorem 6.37. *Assume that symmetric projectors $Q_m \in \mathrm{Kom}(m)$ exist for $1 \leq m \leq n$, and let $P_n' = \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} C_n$ be the periodic chain complex constructed in Theorem 6.28. There are integers $a_{ij} \in \mathbb{Z}$, independent from n, satisfying $a_{i1} = a_{1j} = 1$, and such that*

1. *There is a deformation retract $\mathrm{End}(P_n') \to V_n$ where*

$$V_n := \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} \Lambda[w_1, \ldots, w_n]$$

with bigrading $\deg(u_k) = (2-2k, 2k)$, $\deg(w_k) = (1-2k, 2+2k)$, and differential

$$d(u_k) = 0 \qquad d(w_m) = \sum_{\substack{i+j=m+1 \\ 1 \leq i,j \leq m}} a_{ij} u_i u_j$$

for all $1 \leq k \leq n$.

2. *the map* $V_n \to V_n$ *given by* $v \mapsto d(w_n v) + w_n d(v)$ *is homotopic to left-multiplication by* $d(w_n)$.

Consequently, V_n *computes the Cooper-Krushkal homology of the* n*-colored unknot and the limiting Khovanov homology of the torus knots* $T_{n,\infty}$*, as bigraded* $\mathbb{Z}[u_1, \ldots, u_n]$*-modules.*

Note that the bigradings on V_n are inherited from the algebra structure, but we do not assume that the differential respects the Leibniz rule with respect to the obvious multiplication on V_n. This is related to the fact that the isomorphism in homology does not preserve the algebra structure, and so the differential of a general word $w_{i_1} \cdots w_{i_r}$ is as yet unspecified, even the differential respects the Leibniz rule with respect to *some* bilinear operator on V_n. In fact, the existence of a deformation retract $\mathrm{End}^{\bullet,\bullet}(P'_n) \to V_n$ endows V_n with the structure of a unital A_∞-algebra [Val12].

Nonetheless, statement (2) of Theorem 6.37 implies that V_n is isomorphic to a dg $\mathbb{Z}[u_1, \ldots, u_n]$-module $V'_n = \mathbb{Z}[u_1, \ldots, u_n] \otimes \Lambda[w_1, \ldots, w_n]$ with differential satisfying $d(w_n v) = d(w_n)v - w_n d(v)$, which is a special case of the Leibniz rule for the *obvious* multiplication on V'_n. The issue is that even though both V_n and V'_n are dg $\mathbb{Z}[u_1, \ldots, u_n]$-modules, the isomorphism $V_n \cong V'_n$ may destroy this action. We want to be careful not to disturb the polynomial action, so that we can perform every simplification in a category of dg modules. This is the reason for our failure to characterize more specifically the differential on V_n.

Proof of Theorem 6.37. Induction on $n \geq 1$. In case $n = 1$ Corollary 6.35 gives a deformation retract $\mathrm{End}^{\bullet,\bullet}(P'_1) \to \mathbb{Z}[u_1] \vec{\otimes} \Lambda[w_1]$. The only possibility for the differential on this chain complex is $d(w_1) = a_{11} u_1^2$ for some $a_{11} \in \mathbb{Z}$. In order to have the correct homology groups (we know that $\mathrm{End}^{\bullet,\bullet}(1_1) = \mathbb{Z}[b]/(b^2)$) we must have $a_{11} = \pm 1$, and the sign is irrelevant up to replacing $w_1 \leftrightarrow -w_1$. This proves statement (1) in this case. For statement (2) compute

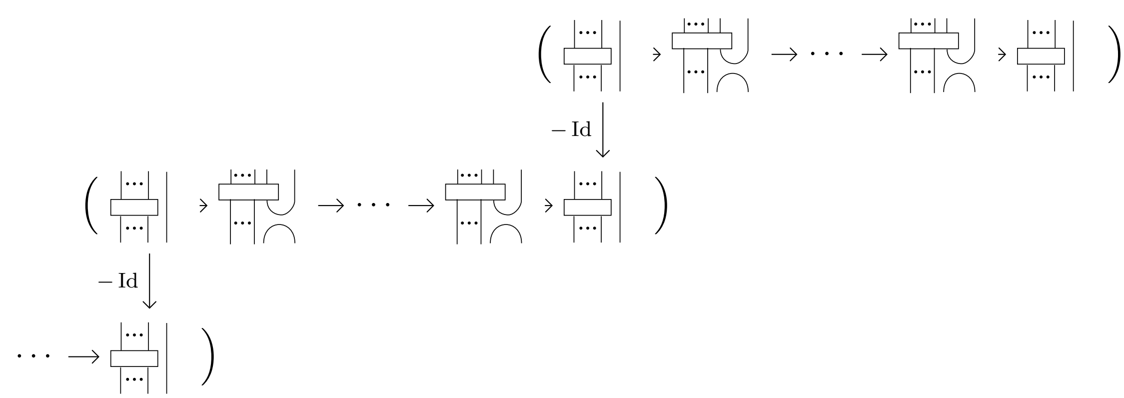
$$d(w_1 u_1^k) = u_1^k d(w_1) = u_1^{k+2}$$

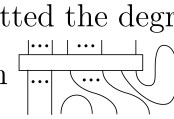
by $\mathbb{Z}[u_1]$-equivariance. Hence the quantity $d(w_1 v) + w_1 d(v)$ is equal to $u_1^2 v$ for all $v \in V_1$. This proves (2).

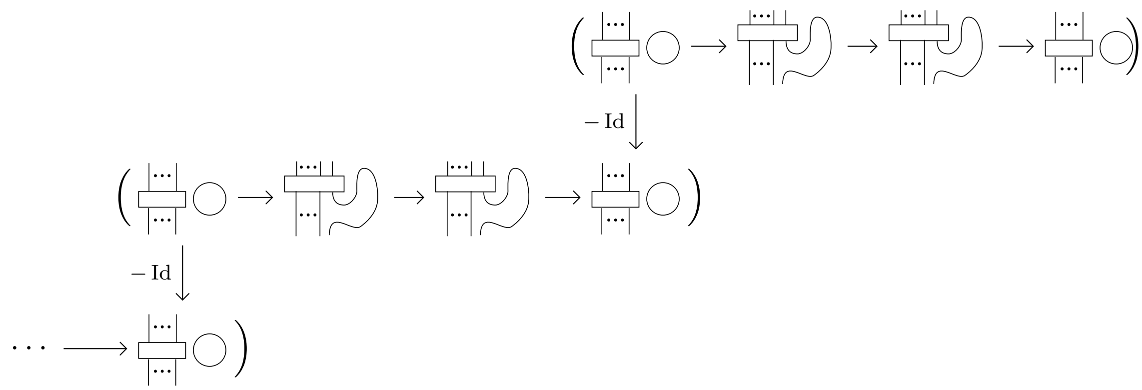
Suppose that symmetric projectors $Q_m \in \mathrm{Kom}(m)$ exist for $1 \leq m \leq n$, and assume by induction that we have a deformation retract $\mathrm{End}^{\bullet,\bullet}(P'_{n-1}) \to V_{n-1}$ as in the statement. Let $P'_n = \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} C_n \in \mathrm{Kom}(n)'$ be the periodic chain complex from Theorem 6.28, which deformation retracts onto a Cooper-Krushkal projector $P_n \in \mathrm{Kom}(n)$. We have a sequence of deformation retracts

$$\mathrm{End}^{\bullet,\bullet}(P'_n) \to \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} \mathrm{Hom}^{\bullet,\bullet}(1_n, C_n) \cong \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} \mathrm{Hom}^{\bullet,\bullet}(1_{n-1}, qT(C_n))$$

where T is the partial trace functor. The first is the deformation retract of $\mathbb{Z}[u_1, \ldots, u_n]$-modules from equation (6.32) and the second is many simultaneous (and $\mathbb{Z}[u_1, \ldots, u_n]$ equivariant) applications of Corollary 4.17. We will essentially repeat the argument in the proof of Proposition 6.33 in the context of the periodic chain complex P'_n. By construction, P'_n is a chain complex of the form

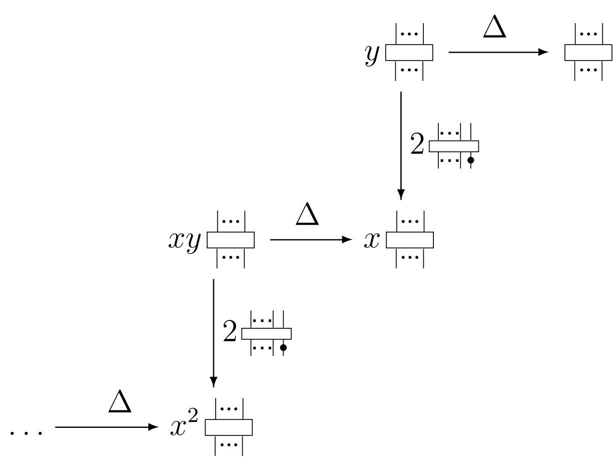


where each row is Q'_n, each white box is P'_{n-1}, and we have omitted the degree shifts because of space limitations. Contracting complexes of the form , we can see that the partial trace $qT(P_n')$ deformation retracts onto a convolution of the form

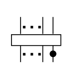


We can simplify each row by delooping and canceling. These deformation retracts can be applied simultaneously to each row via Theorem 2.15, and the local computation (Lemma 6.39) implies that the resulting chain complex will be of the form (remembering what the degree shifts are; see the proof of Proposition 6.33)

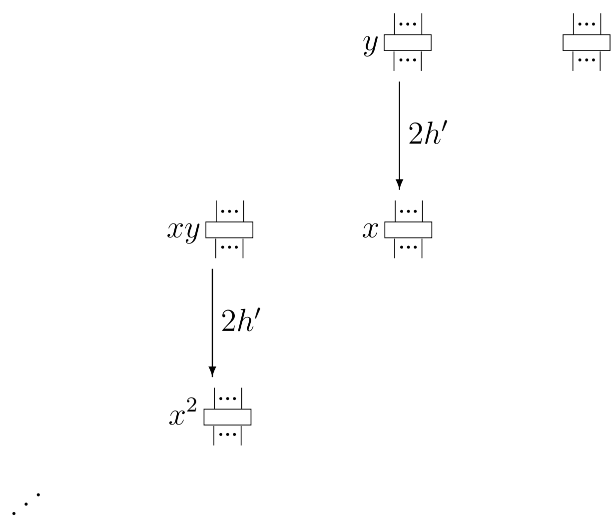
$Z \quad :=$



where $x = t^{2-2n}q^{2n}$ and $y = t^{1-2n}q^{2n+2}$. Here $\Delta \in \mathrm{End}^{\bullet,\bullet}(P_{n-1})$ is some cycle which after degree shifts Δ must have bidegree $(1,0)$; this means $\Delta \in \mathrm{End}^{\bullet,\bullet}(P_{n-1})$ has bidegree $(2-2n, 2+2n)$. Now, each of the above simplifications was nothing other than the application of a corresponding simplification (see the proof of Proposition 6.33) of $qT(C_n)$ to each term of $qT(P_n') \cong \mathbb{Z}[u_1,\ldots,u_n]\vec{\otimes}qT(C_n)$ using Theorem 6.21. Thus we actually have a deformation retract of $\mathbb{Z}[u_1,\ldots,u_n]$-modules $qT(P_n') \to Z$.

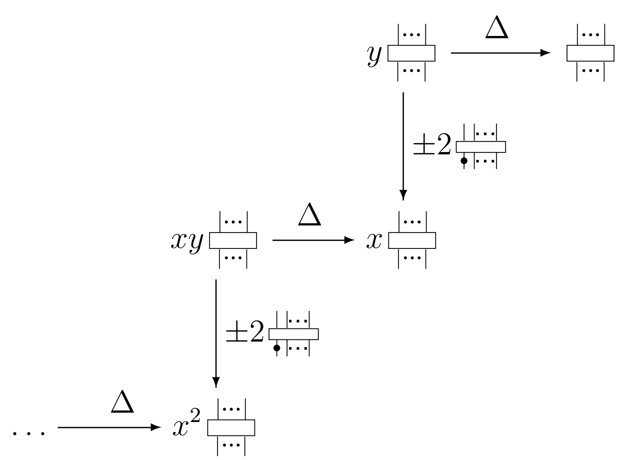
Now we want to relate the dotted identity  to the action of u_1. But by the inductive definition of the $P_1',\ldots,P_n'$, the action of u_1 is inherited from a factorization

$P'_n = (P'_1 \sqcup 1_{n-1}) \odot F$ for some F. So informally speaking, we need to move the dot from the southeast corner of to the northwest corner, and then use Lemma 6.36. The reader may now want to take a look at Lemma 6.42 following this proof. By that lemma we have an element $h \in \mathrm{End}^{-1,2}(P'_{n-1})$ which commutes with the $\mathbb{Z}[u_1, \ldots, u_{n-1}]$-action, and such that $[d, h'] =$ $+ (-1)^n$. Consider now $h \in \mathrm{End}^{-1,0}(Z)$ given by

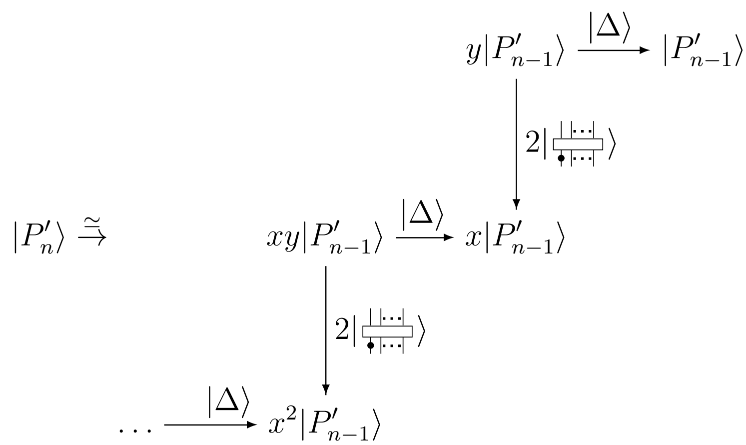


Then h commutes with the $\mathbb{Z}[u_1, \ldots, u_n]$-action. Conjugating the differential d_Z by $(1 + h)$ gives Z' with differential $d'_Z = d_Z + h \circ d_Z - d_Z \circ h$:

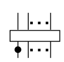
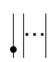
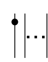
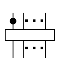
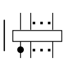
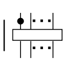
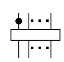
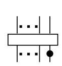
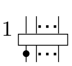
$Z' =$



where $\pm = (-1)^{n-1}$. After conjugating the differential by a $\mathbb{Z}[u_1, \ldots, u_n]$-equivariant invertible map (which acts on the upper-left corners as $\pm\,\mathrm{Id}$ and on the lower-right corners as Id) and replacing Δ by $\pm\Delta$ we may remove the $\pm$ sign in the above. Now, put $|A\rangle := \mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_m}(1_m, A)$ for each $A \in \mathrm{Kom}(n)$, and for each $f \in \mathrm{Hom}^{\bullet,\bullet}_{\mathcal{TL}_m}(A, B)$, let $|f\rangle \in \mathrm{Hom}^{\bullet,\bullet}(|A\rangle, |B\rangle)$ denote the map given by post-composition with f. Then $|\ \ \rangle$ is a functor. Applying this functor to what has been said so far, we have a deformation retract of left $\mathbb{Z}[u_1, \ldots, u_n]$-modules



(6.38)

Note that post-composition with  gives the same map $q^2|P_{n-1}\rangle \to |P_{n-1}\rangle$ as precomposition with  = , which in turn is the same as post-composition with . That is to say, |⟩ = |⟩. Now,  $\simeq U_1 \in \mathrm{End}^{0,2}(P'_{n-1})$ by Lemma 6.36, and in fact the homotopy can be chosen to commute the the $\mathbb{Z}[u_1, \ldots, u_{n-1}]$ action. So we can replace the vertical maps in the right-hand side of (6.38) by $2|U_1\rangle$ up to a $\mathbb{Z}[u_1, \ldots, u_n]$-equivariant isomorphism (see the argument earlier in the proof by which we replaced the maps  by $(-1)^{n-1}$).

Finally we can apply the deformation retract $(\pi, \sigma, h) : |P'_{n-1}\rangle \to V_{n-1}$ (which exists by the induction hypothesis) to each term of the result obtaining a chain complex

$$V_n \quad := \quad \begin{array}{ccccc} & & & & yV_{n-1} \xrightarrow{\delta} V_{n-1} \\ & & & & \downarrow 2u_1 \\ & & xyV_{n-1} & \xrightarrow{\delta} & xV_{n-1} \\ & & \downarrow 2u_1 & & \\ \cdots & \xrightarrow{\delta} & x^2V_{n-1} & & \end{array}$$

onto which $|P'_n\rangle$ deformation retracts as a dg $\mathbb{Z}[u_1, \ldots, u_n]$-module. As a bigraded $\mathbb{Z}[u_1, \ldots, u_n]$-module, the above is obviously isomorphic to $\mathbb{Z}[u_n] \otimes \Lambda[w_n] \otimes V_{n-1}$, where the $u_n^k \otimes 1 \otimes V_{n-1}$ summand is identified with the copy of V_{n-1} appearing with shift x^k, and the the $u_n^k \otimes w_n \otimes V_{n-1}$ summand is identified with the copy of V_{n-1} appearing with the shift $x^k y$. Denote a simple tensor $u_n^k \otimes w_n^\varepsilon \otimes v$ simply by juxtaposition. The differential on V_n induced by the deformation retract satisfies (and is characterized by)

- the obvious inclusion $V_{n-1} \hookrightarrow V_n$ is a chain map
- $d(w_n v) = 2u_1 u_n v + \delta(v) - w_n d(v)$ for each $v \in V_{n-1}$, where $\delta(v) = \pi(\Delta \circ (\sigma(v)))$, and
- $d(u_n v) = u_n d(v)$ for all $v \in V_n$.

Now, for degree reasons, $\delta(1)$ must be a quadratic polynomial $g(u_2, \ldots, u_{n-1}) \in \mathbb{Z}[u_2, \ldots, u_{n-1}]$ and $d(w_n) = 2u_1 u_n + g$ is as in the statement. This proves (1).

For (2), let $f : V_n \to V_n$ denote the map $f(v) = d(w_n v) + w_n d(v)$. A simple computation shows that

$$f(w_n v) = 2u_1 u_n w_n v + w_n \delta(v)$$

and

$$f(v) = 2u_1 u_n v + \delta(v)$$

for all $v \in V_{n-1}$. Hence to prove part (2) it suffices to show that $v \mapsto \delta(v)$ is homotopic to left-multiplication by $\delta(1) = g(u_2, \ldots, u_{n-1}) = d(w_n) - 2u_1 u_n$. By definition, $\delta(v) = \pi(\Delta \circ \sigma(v))$. Putting $v = 1$ and applying σ to both sides gives

$$\Delta \circ \sigma(1) \simeq \sigma(g(u_2, \ldots, u_{n-1})) = g(U_2, \ldots, U_{n-1}) \circ \sigma(1)$$

by $\mathbb{Z}[u_1, \ldots, u_n]$-equivariance, where $U_m \in \mathrm{End}(P'_{n-1})$ denote the periodicity maps on P'_{n-1}. Now, for degree reasons $\mathrm{Ext}^{0,0}(P'_{n-1}) \cong \mathbb{Z}$ is generated by Id (see Corollary 6.45). So $\sigma(1) \simeq \pm \mathrm{Id}_{P'_n}$, hence $\Delta \simeq g(U_2, \ldots, U_{n-1})$. It follows that δ is homotopic to the map

$$v \mapsto \pi(g(U_2, \ldots, U_{n-1}) \circ \sigma(v)) = g(u_2, \ldots, u_{n-1})\pi(\sigma(v)) = g(u_2, \ldots, u_{n-1})v$$

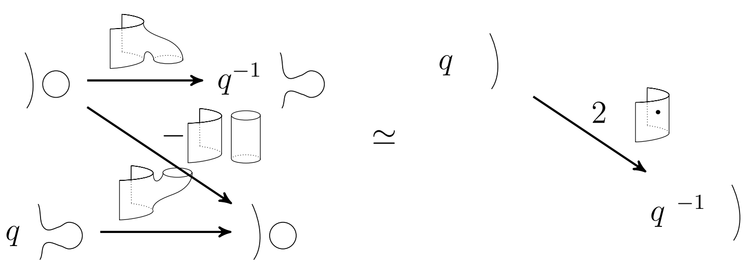
again by $\mathbb{Z}[u_1, \ldots, u_{n-1}]$-equivariance, and the fact that $\pi \circ \sigma = \mathrm{Id}_{V_n}$. That is to say, δ is homotopic to left multiplication by $\delta(1)$. This completes the proof. □

The following two lemmas were used in the argument above:

Lemma 6.39. *The chain complexes*

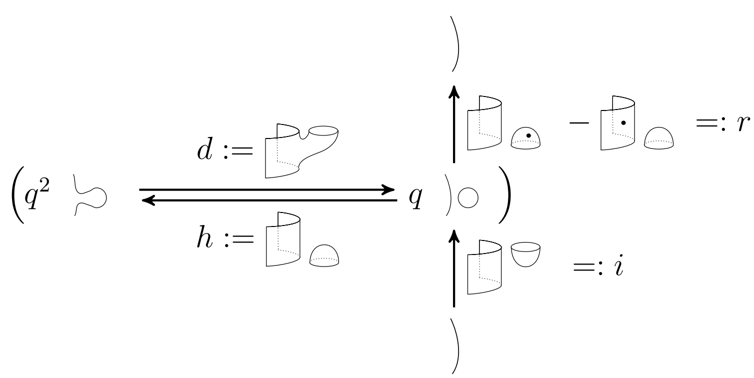
(6.40)

deformation retract onto *, and applying these equivalences row-by-row to the following chain complex gives a deformation retract:*

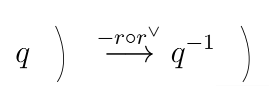


(6.41)

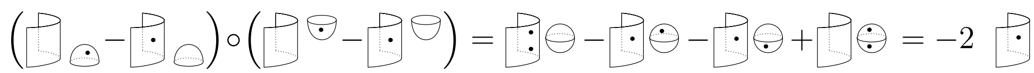
Proof. We leave it to the reader to check that the following diagram defines the data (r, i, h) of a deformation retract $C_1 \to 1$ as in the statement:



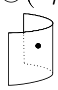
Since $C_2 = C_1^\vee$, $(i^\vee, r^\vee, h^\vee)$ are the data of a deformation retract $C_2 \to 1$. This proves the first statement. For the second, applying the aforementioned retracts to the rows of left-hand side of (6.41) gives a chain complex



(see, for example Theorem 2.15 in the very special case of a two-term convolution). By definition of r we have

$r \circ r^\vee =$ 

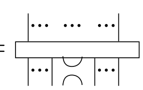
This completes the proof. $\square$

Lemma 6.42 (Dot-hopping)**.** *Let $Z \in \mathrm{Kom}(n)$ be any chain complex, and let $b_i = \mathrm{Id}_Z \odot (1_{n-i} \sqcup b \sqcup 1_{i-1})$, $c_i = (1_{n-i} \sqcup b \sqcup 1_{i-1}) \odot \mathrm{Id}_Z$ be the dotted identity maps, where $b =$*  *$: q^2 1_1 \to 1_1$ is the dotted identity cobordism.*

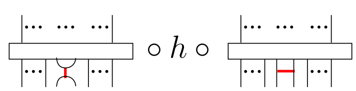
1. *If Z kills turnbacks then $c_i \simeq -c_{i+1}$ and $b_i \simeq -b_{i+1}$.*

2. *If $Z = \mathbb{Z}[x_1, \dots, x_r] \vec{\otimes} C$, where C kills turnbacks, then the homotopies $c_{i+1} + c_i \simeq 0$ and $b_{i+1} + b_i \simeq 0$ cn be chosen to commute with the $\mathbb{Z}[x_1, \dots, x_r]$-action.*

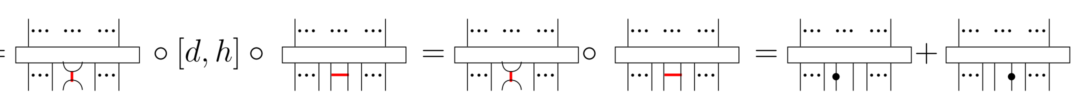
Proof. Let us handle only the case of the dotted identity maps b_i. The argument for the c_i is similar. For (1) let $h \in \mathrm{End}^{-1,0}(Z \odot e_i)$ denote the homotopy which contracts

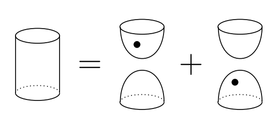
$Z \odot e_i =$ 

Here we let the white box denote Z. I.e. $[d, h] = \mathrm{Id}_{Z \odot e_i}$. Now let $h' \in \mathrm{End}^{-1,2}(Z)$ denote the composition

$h' =$ 

Then we have

$[d, h'] =$ 

The last equality follows from the neck-cutting relation  in $\mathcal{TL}_n$. That is to say, $[d, h'] = b_{i+1} + b_i$. This proves (1).

For (2), note that if C kills turnbacks, then so does any chain complex $Z = \mathbb{Z}[x_1, \dots, x_r] \vec{\otimes} C$, and furthermore each nulhomotopy which contracts $Z \odot e_i$ can be chosen to commute with the $\mathbb{Z}[x_1, \ldots, x_r]$ action by Theorem 6.21. That is to say, the map h' in the proof of part (1) commutes with the $\mathbb{Z}[x_1, \ldots, x_r]$ action. This completes the proof. □

We have an important corollary to the above computation. But first, recall the canonical representations $\rho_m^n : \mathrm{End}^{\bullet,\bullet}(P_m, P_m) \to \mathrm{End}^{\bullet,\bullet}(P_n, P_n)$ from Definition 5.4.

Proposition 6.43. *Suppose symmetric projectors $Q_1, \ldots, Q_n$ exist. Then for each $1 \leq m \leq n$ the following square commutes up to homotopy:*

$$\begin{array}{ccc} \mathrm{End}(P_m) & \xrightarrow{\rho_m^n} & \mathrm{End}(P_n) \\ \simeq\downarrow & & \simeq\downarrow \\ V_m & \xrightarrow{i} & V_n \end{array}$$

where ρ_m^n is the canonical representation (Definition 5.4), i is the obvious inclusion, and the vertical maps are the deformation retracts from Theorem 6.37

Proof. Since $\rho_k^n \circ \rho_m^k \simeq \rho_m^n$, it suffices to prove the proposition for $m = n-1$. Run through the proof of Theorem 6.37 with $P_n' = \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} C_n$ replaced by $(P_{n-1}' \sqcup 1_1) \odot P_n')$. Let us omit the primes everywhere; it will be understood throughout the rest of the proof that P_m is the periodic chain complex. The partial trace satisfies $T((P_{n-1} \sqcup 1_1) \odot A) \cong P_{n-1} \odot T(A)$ for all $A \in \mathrm{Kom}(n)$, hence the argument in the proof of Theorem 6.37 produces a deformation retract π from $\mathrm{End}((P_{n-1} \sqcup 1_1) \odot P_n)$ onto the chain complex

$$\begin{array}{ccccc} & & y|P_{n-1}^{\odot 2}\rangle & \xrightarrow{|\mathrm{Id} \odot \Delta\rangle} & |P_{n-1}^{\odot 2}\rangle \\ & & \downarrow 2|\mathrm{Id} \odot U_1\rangle & & \\ xy|P_{n-1}^{\odot 2}\rangle & \xrightarrow{|\mathrm{Id} \odot \Delta\rangle} & x|P_{n-1}^{\odot 2}\rangle & & \\ \downarrow 2|\mathrm{Id} \odot U_1\rangle & & & & \\ \cdots \xrightarrow{|\mathrm{Id} \odot \Delta\rangle} x^2|P_{n-1}^{\odot 2}\rangle & & & & \end{array}$$

As a bigraded abelian group, there is an obvious isomorphism of the above chain complex with $A \otimes |P_{n-1}^{\odot 2}\rangle$, where $A := \mathbb{Z}[u_n] \otimes \Lambda[w_n]$. Hence, we let $A \vec{\otimes} |P_{n-1}^{\odot 2}\rangle$

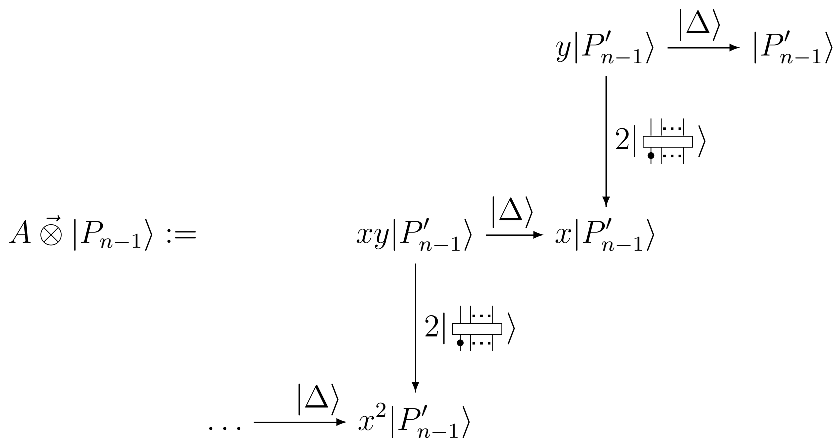
denote this chain complex. Now, the deformation retract $\pi : \mathrm{End}((P_{n-1} \sqcup 1_1) \odot P_n) \to A \vec{\otimes} |P_{n-1}^{\odot 2}\rangle$ commutes with the diagonal action of $\mathrm{End}(P_{n-1})$ on the first factor everywhere, and one can check that $\pi(\mathrm{Id}_{P_n}) = 1 \otimes \iota$, where $\iota : 1_{n-1} \to P_{n-1}^{\odot 2}$ is the inclusion of the degree zero chain group. This gives commutativity of the bottom triangle of the following diagram:

$$\begin{array}{ccc}
\mathrm{End}(P_{n-1}) & \xrightarrow{\rho^n_{n-1}} & \mathrm{End}(P_n) \\
\Big\downarrow L & \searrow^{\nu} & \Big\uparrow \simeq \\
A \vec{\otimes} |P_{n-1}^{\odot 2}\rangle & \xleftarrow{\pi} & (\mathrm{End}((P_{n-1} \sqcup 1_1) \odot P_n)
\end{array}$$

where $L(f) = 1 \otimes |f \odot \mathrm{Id}\rangle(\iota)$, $\nu(f) = (f \sqcup 1_1) \odot \mathrm{Id}_{P_n}$, and π is the aforementioned deformation retract. The top triangle commutes by definition of ρ^n_{n-1}. The map L is homotopic to $R(f) = |\mathrm{Id} \odot f\rangle(\iota)$ by Proposition 5.5. We have established commutativity of the first square of the following diagram up to homotopy:

$$\begin{array}{ccccccc}
\mathrm{End}(P_{n-1}) & \xrightarrow{\alpha_1} & |P_{n-1}^{\odot 2}\rangle & \xrightarrow{\beta_1} & |P_{n-1}\rangle & \xrightarrow{\gamma_1} & V_{n-1} \\
\Big\downarrow \rho^n_{n-1} & & \Big\downarrow 1 \otimes (-) & & \Big\downarrow 1 \otimes (-) & & \Big\downarrow i \\
\mathrm{End}(P_n) & \xrightarrow{\alpha_2} & A \vec{\otimes} |P_{n-2}^{\odot 2}\rangle & \xrightarrow{\beta_2} & A \vec{\otimes} |P_{n-1}\rangle & \xrightarrow{\gamma_2} & V_n
\end{array} \tag{6.44}$$

Let us describe the maps: $\alpha_1(f) = |\mathrm{Id} \odot f\rangle(\iota)$, α_2 is an equivalence $\mathrm{End}(P_n) \simeq \mathrm{End}((P_{n-1} \sqcup 1) \odot P_n)$ followed by π, β_i are induced from the deformation retract $P_{n-1}^{\odot 2} \to P_{n-1}$, and γ_i are the retracts from Theorem 6.37 and its proof. The first square commutes up to homotopy by what has been said. The second square commutes by naturality of the deformation retracts constructed via Theorem 2.15, and the third square commutes by the construction in Theorem 6.37. Moreover, the compositions along each row are the retracts from Theorem 6.37. To see these last two statements, recall that Proposition 4.21 says that $\iota \odot \mathrm{Id}_{P_{n-1}} : P_{n-1} \to P_{n-1}^{\odot 2}$ is the section of a deformation retract. Since $\iota \odot \mathrm{Id}$ commutes with all of the components of the differential of $A \vec{\otimes} |P_{n-1}^{\odot 2}\rangle$, applying this retract to each term yields exactly the chain complex



from (6.38). Commutativity of (6.44) gives the proposition. $\square$

Corollary 6.45. *Assume that symmetric projectors* $Q_m \in \mathrm{Kom}(m)$ *exist for* $2 \leq m \leq n$. *Then the group* $\mathrm{Ext}^{i,j}(P_n, P_n)$ *of chain maps* $t^i q^j P_n \to P_n$ *modulo chain homotopy satisfies*

1. $\mathrm{Ext}^{k-i,i}(P_n, P_n) = 0$ *for all* i, *if* $k < 0$.
2. $\mathrm{Ext}^{0-i,i}(P_n, P_n) = \mathbb{Z}$ *for* $i = 0$ *and zero otherwise.*
3. $\mathrm{Ext}^{1-i,i}(P_n, P_n) = 0$ *for all* i.
4. $\mathrm{Ext}^{2-i,i}(P_n, P_n) = \mathbb{Z}$ *for* $i = 2, 4, \dots, 2n$ *and zero otherwise.*
5. $\mathrm{Ext}^{3-i,i}(P_n, P_n) = 0$ *for all* i.

For generators of the groups $\mathrm{Ext}^{2-2k,2k}(P_m, P_m)$ *we may take a family of classes* $[U_k^{(m)}]$ *induced from the* $\mathbb{Z}[u_1, \dots, u_n]$*-action on* $\mathbb{Z}[u_1, \dots, u_n] \vec{\otimes} C_n$; *these can be assumed to satisfy* $\rho_m^n(U_k^{(m)}) \simeq U_k^{(n)}$, *where* $\rho_m^n : \mathrm{End}^{\bullet,\bullet}(P_m) \to \mathrm{End}^{\bullet,\bullet}(P_n)$ *is a canonical representation.*

Proof. Let us forget the differential for a moment and regard $V_n = \mathbb{Z}[u_1, \dots, u_n] \otimes \Lambda[w_1, \dots, w_n]$ simply as a bigraded algebra. The bigrading is induced by $\deg(u_m) =$

$(2-2m, 2m)$ and $\deg(w_m) = (1-2m, 2+2m)$. It is useful to collapse the bigrading to the single grading $\deg_s = \deg_h + \deg_q$. Then the generators of V_n satisfy $\deg_s(u_m) = 2$, $\deg_s(w_m) = 3$. Theorem 6.37 says that there is some differential on V_n such that in particular (1) $d(w_m) \neq 0$ and $d(u_m) = 0$ for all $1 \leq m \leq n$, (2) the homology of V_n is isomorphic to the homology of $\mathrm{End}^{\bullet,\bullet}(P_n)$ as bigraded $\mathbb{Z}[u_1, \ldots, u_n]$ modules. We remind the reader that we don't make any claim that the Leibniz rule is satisfied; we only know that the differential is $\mathbb{Z}[u_1, \ldots, u_n]$-equivariant, where the $\mathbb{Z}[u_1, \ldots, u_n]$ action on the homology of $\mathrm{End}^{\bullet,\bullet}(P_n)$ is induced by an equivalence

$$\mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} C_n \simeq P_n.$$

Clearly there are no elements $x \in V_n$ of degree $\deg_s(x) < 0$ or $\deg_s(x) = 1$, and the only bihomogeneous elements with $\deg_s(x) = 3$ are the multiples of w_m. But none of the w_m are cycles. This proves statements (1), (3), and (5) of the corollary.

The only elements $x \in V_n$ with $\deg_s(x) = 0$ are multiplies of the identity. If any of these were a boundary $d(h) = a \cdot 1$, then $\deg_s(h) = -1$ forces h to be zero. (2) follows. Similarly, the only bihomogeneous elements $x \in V_n$ with $\deg_s(x) = 2$ are multiples of some u_m. If any multiple of u_m were a boundary, say $d(h) = au_m$, then $\deg_s(h) = 1$ forces $h = 0$. This shows that the u_m generate homology groups isomorphic to $\mathbb{Z}$, which is (4).

Recall the chain complexes C'_m, Q'_m, P'_m given in Theorem 6.28, $1 \leq m \leq n$. Let $U_k^{(m)} : t^{2-2k}q^{2k}P'_m \to P'_m$ be the chain map induced by the action of u_k on $P'_m = \mathbb{Z}[u_1, \ldots, u_m] \vec{\otimes} C'_m$, and let $\sigma_m : V_m \to \mathrm{End}^{\bullet,\bullet}(P'_m)$ denote the sections of the deformation retracts given in Theorem 6.37. It is clear from the $n = 1$ case of Theorem 6.37 that $\sigma_1(1) \simeq \mathrm{Id}_{P'_1}$, hence

$$\sigma_m(1) \simeq \rho_1^m(\sigma(1)) \simeq \rho_1^m(\mathrm{Id}_{P'_1}) \simeq \mathrm{Id}_{P'_m}$$

Hence by $\mathbb{Z}[u_1, \ldots, u_m]$-equivariance we have

$$\sigma_m(u_k) = U_k^{(m)} \circ \sigma_m(1) \simeq U_k^{(m)}$$

hence the $U_k^{(m)}$ represent generators of the corresponding Ext groups. The statement regarding generators now follows: Proposition 6.43 implies that$\sigma_n|_{V_m} \simeq \rho_m^n \circ \sigma_m$, hence

$$U_k^{(n)} \simeq \sigma_n(u_k) \simeq \rho_m^n(\sigma_m(u_k)) \simeq \rho_m^n(U_k^{(m)})$$

for all $1 \leq k \leq m \leq n$. This completes the proof. □

The following inequality may be of independent interest. In this thesis we will use it show that $\eta^{\odot k} : \mathrm{Hom}(Q_n^{\odot k}, Q_n) \to \mathrm{Hom}(1, Q_n)$ induces an isomorphism of degree $(0,0)$ homology groups (proposition 7.33).

Lemma 6.46. *Any nonzero element $v \in V_n$ satisfies $\deg_q \leq n \deg_s(v)$, where $\deg_s = \deg_q + \deg_h$. In particular the same is true of $\mathrm{Ext}^{\bullet,\bullet}(P_n, P_n)$*

Proof. By linearity it suffices to check the inequality on the generators u_i and w_i for $1 \leq i \leq n$. This is straightforward □

We also have the following, which shows in particular that the $\mathfrak{sl}_2$-link homology of the n-colored unknot has infinite total rank for $n \geq 2$.

Corollary 6.47. *If $n \geq 2$, then there is an inclusion $\mathbb{Z}[u_n] \hookrightarrow \mathrm{Ext}^{\bullet,\bullet}(P_n)$ sending $u_n \mapsto [U_n]$. Further, the images of u_n^k generate the corresponding groups $\mathrm{Ext}^{k(2-2n,2kn)} = \mathbb{Z}$ for all $k \geq 0$.*

Proof. We show that $u_n^k \in V_n$ generate the corresponding homology group. This will be a degree argument, and is fairly technical. Any element of V_n can be written uniquely as $z = u_n^l w_n^\varepsilon v$ where $v \in V_{n-1}$, $l \in \{0, 1, 2, \ldots, \}$, and $\varepsilon \in \{0, 1\}$. Assume that $z \in V_n$ is a nonzero element with $\deg(z) = k(2 - 2n, 2n)$. By equating $\deg_s = \deg_q + \deg_h$ and $\deg_q$ components, the following equations hold:

1. $2(k - l) = 3\varepsilon + \deg_s(v)$

2. $2n(k - l) = \varepsilon(2 + 2n) + \deg_q(v)$.

Clearly $l \leq k$ is necessary, since the right-hand side of (1) is non-negative. Suppose $k - l =: a \geq 0$, and compute

$$\begin{aligned} \delta &:= \deg_q(v) - (n-1)\deg_s(v) \\ &= 2na - 2 - 2n + (1-n)(2a-3) \\ &= 2a - 5\varepsilon + n \end{aligned}$$

If $n \geq 3$ and $a > 0$, then $\delta > 0$ and Lemma 6.46 gives a contradiction.

If $n = 2$ and $a > 0$, then $\delta > 0$ unless $\varepsilon = 1$. In this case, $\deg_h(v) = \deg_s(v) - \deg_q(v) = 3 - 2a$. But $v \in \mathbb{Z}[u_1] \otimes \Lambda[w_1]$ must have homological degree 0 or -1. The only possibility is $a = 2$, which by (1) forces $\deg_s(v) = 1$, hence $v = 0$.

In any case, the only possibility for nonzero z is when $a = 0$, i.e. $z = cu_n^k \in V_n$ for some $c \in \mathbb{Z}$. Suppose one of these were a boundary, say $cu_n^k = d(h)$ for some $h \in V_n$, and write $h = u_n^l w_n^\varepsilon v$ for some $l \geq 0$, $\varepsilon \in \{0, 1\}$, and $v \in V_{n-1}$. Then we must have $\deg(h) = (-1, 0) + k(2 - 2n, 2n)$. Equating the $\deg_s$ and $\deg_q$ components gives

(1)′ $2(k - l) - 1 = 3\varepsilon + \deg_s(v)$

(2)′ $2n(k - l) = \varepsilon(2 + 2n) + \deg_q(v)$.

Clearly $l < k$ is necessary. So suppose $k - l =: a \geq 1$, and compute

$$\begin{aligned} \delta' &:= \deg_q(v) - (n-1)\deg_s(v) \\ &= 2na - \varepsilon(2 + 2n) + (n-1)(1 + 3\varepsilon - 2a) \\ &= 2a + (n-5)\varepsilon + n - 1 \end{aligned}$$

is positive unless $n = 2$, $a = 1$, $\varepsilon = 1$. In this case (1)′ gives $\deg_s(v) = -2$, hence $v = 0$. This observation together with Lemma 6.46 implies that $h = 0$, which is a contradiction in either case. Hence there are no non-trivial boundaries of this degree. This completes the proof. □

6.6 Application II: finite generation of $\mathfrak{sl}_2$-link homology

Let D be an oriented tangle diagram whose components are labelled by non-negative integers, and which is marked with some points $z_1, \ldots, z_r$ away from the crossings and away from the boundary, at least one on each component. In Definition 3.25 we constructed a chain complex

$$C(D; \{P_n\}) = F_D(P_{n_1}, \ldots, P_{n_r}, X_1, \ldots, X_s)$$

Here F_D is the planar diagram obtained by deleting small disks around the crossings and marked points of D and replacing an n-colored arc by n parallel copies of itself, P_n are Cooper-Krushkal projectors, and the X_i are bounded chain complexes associated to the crossings of D. Planar composition respects homotopy equivalence of its arguments, so we may as well replace each P_{n_i} by the periodic chain complex $\mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} C_n$ from Theorem 6.28. By proposition 6.23 and 6.22, we have

$$C(D; \{P_n\}) \simeq \Big(\mathbb{Z}[u_1, \ldots, u_{n_1}] \otimes \cdots \otimes \mathbb{Z}[u_1, \ldots, u_{n_k}]\Big) \vec{\otimes} C(D; \{C_n\}) \tag{6.48}$$

where $C(D; \{C_n\})$ is obtained by replacing P_n everywhere by C_n in Definition 3.25.

We immediately obtain the following finite generation result for Cooper-Krushkal homology:

Theorem 6.49. *Let L be an oriented, framed link in S^3 whose components are labelled by non-negative integers $n_1, \ldots, n_r$, and let $R(L) = \mathbb{Z}[u_1, \ldots, u_{n_1}] \otimes \cdots \otimes \mathbb{Z}[u_1, \ldots, u_{n_r}]$. The Cooper-Krushkal homology of L is naturally finitely generated R-module (bi-graded).*

Proof. Fix an oriented diagram D for L, given the blackboard framing. Mark D with points $z_1, \ldots, z_r$ exactly one on each component of L. Then (6.48) says that

$$C(D; \{P_n\}) \simeq R(L) \vec{\otimes} C(D; \{C_n\})$$

where $C(D;\{C_n\})$ is a planar composition of bounded chain complexes, hence bounded. The $\mathfrak{sl}_2$-link homology of L is defined to be the homology of

$$\mathrm{Hom}^{\bullet,\bullet}(\varnothing, C(D;\{P_n\}) \simeq R(L) \vec{\otimes} \mathrm{Hom}^{\bullet,\bullet}(\varnothing, C(D;\{C_n\}))$$

It is clear that $R(L) \vec{\otimes} \mathrm{Hom}^{\bullet,\bullet}(\varnothing, C(D;\{C_n\}))$ is finitely generated as an $R(L)$-module since the second tensor factor is a bounded chain complex of finitely generated abelian groups. Since $R(L)$ is a polynomial ring it is Noetherian, and so submodules of finitely generated modules are finitely generated. Hence the cycles of $R(L) \vec{\otimes} \mathrm{Hom}^{\bullet,\bullet}(\varnothing, C(D;\{C_n\}))$ form a finitely generated $R(L)$-module, and the same is true of homology. This completes the proof. □

Recall that the generators u_1 act by dotted identity maps, hence $[u_1]^2 = 0$. Thus we can omit these generators, obtaining that the Cooper-Krushkal homology is finitely generated over the corresponding tensor product of algebras $\mathbb{Z}[u_2, \ldots, u_n]$. In fact, we expect that roughly half the number of generators are unnecessary for finite generation. Indeed, let $V_n = \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} \Lambda[w_1, \ldots, w_n]$ be differential bigraded algebra from conjecture 6.31. Observe that $d(w_1) = u_1^2$, $d(w_2) = 2u_1u_2$, $d(w_3) = u_2^2 + 2u_1u_3$, and in general

$$d(w_{2k-1}) \in u_k^2 + 2 \sum_{1 \le j < k} u_j u_{2k+1-k} \tag{6.50}$$

Definition 6.51. For each integer $n \ge 1$, put $R_n := \mathbb{Z}[u_{n-r+1}, \ldots, u_n]$ where $n = 2r+1$ or $n = 2r$.

For example,

$R_1 = \mathbb{Z}$,

$R_2 = \mathbb{Z}[u_2]$, $R_3 = \mathbb{Z}[u_3]$,

$R_4 = \mathbb{Z}[u_3, u_4]$, $R_5 = \mathbb{Z}[u_4, u_5]$.

Proposition 6.52. *Let $W_n = \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} \Lambda[w_1, \ldots, w_n]$ be as in conjecture 6.31. Then the homology of W_n is finitely generated as an R_n-module.*

Proof. Define two functions on monomials:

- the weighted degree $|u_1^{i_1} \cdots u_n^{i_n}| = \sum_{k=1}^n k i_k$ and
- the total degree $\deg_u(u_1^{i_1} \cdots u_n^{i_n}) = \sum_{k=1}^n i_k$.

Let $f = u_1^{i_1} \cdots u_n^{i_n}$ with total degree l. Suppose $i_k \geq 2$ for some $1 \leq k \leq n-r$ where we are writing $n = 2r$ if n is even, and $n = 2r+1$ if n is odd. The inequality $2r+1 \geq n$ implies $1 \leq 2k-1 \leq 2n-2r-1 \leq n$, hence w_{2k-1} is an element of W_n. Equation (6.50) implies that f is homotopic to a linear combination of monomials with strictly higher weighted degree, and total degree l. Since there are finitely many monomials of total degree l, iterating this procedure must terminate after finitely many stages. This realizes f as homotopic to a linear combination of monomials in which the variables $u_1, \ldots, u_{n-r}$ appear at most once each. Since f was arbitrary, this shows that the image of $\mathbb{Z}[u_1, \ldots, u_n]$ in homology is $H(W_n)$ is finitely generated over $R_n := \mathbb{Z}[u_{n-r+1}, \ldots, u_n]$. Since $H(W_n)$ is finitely generated over $\mathbb{Z}[u_1, \ldots, u_n]$, it follows that $H(W_n)$ is finitely generated over R_n. This completes the proof. □

We expect that the dg algebra W_n/R_nW_n will play an important role in the colored $\mathfrak{sl}_2$ link homology, and we conclude with:

Conjecture 6.53. *The $\mathfrak{sl}_2$ link homology of a framed oriented link L whose components are colored $n_1, \ldots, n_k$ is a finitely generated bigraded module over $R_{n_1} \otimes \cdots \otimes R_{n_k}$.*

Chapter 7

Quasi-local $\mathfrak{sl}_2$-link homology theory

In this section we construct the symmetric projectors used throughout the previous section. They will be assembled to form a new categorification of the colored Jones polynomial which is related to the Cooper-Krushkal categorification in the following way. Let D be an oriented tangle diagram whose components are labelled by non-negative integers, and mark D with some number of points $z_1, \ldots, z_r$ away from the crossings and away from the boundary. Let $\{P_n\}$ be a family of Cooper-Krushkal projectors and $\{C_n\}$ the family of bounded chain complexes constructed in Theorem 6.28. As in equation (6.48) we have

$$C(D; \{P_n\}) \cong R \vec{\otimes} C(D; \{C_n\})$$

In this section we show that if D_1 and D_2 are two marked diagrams representing isotopic (colored, framed, oriented) tangles, then $C(D_1; \{C_n\}) \simeq C(D_2; \{C_n\})$, hence this chain complex defines a tangle invariant, at least when the numbers of marked points on each component are the same. The number of marked points on a single component is no longer irrelevant since the chain complexes C_n are not idempotent, in contrast with the P_n. Rather the C_n are quasi-idempotent, as is shown in Theorem 7.11. That is,

$$C_n \odot C_n \simeq \prod_{1 \leq k \leq n} (1 + t^{1-2k} q^{2k}) C_n$$

So the number of marked points is irrelevant up to taking many direct sum copies. In order to motivate why one expects this, recall from Theorem 6.28 that $C_n \simeq$

$(Q_1 \sqcup 1_{n-1}) \odot \cdots \odot Q_n$, where the Q_m are symmetric projectors. In Theorem 7.2 we show that $Q_m \simeq \mathrm{Cone}(U_m)$ where $U_m \in \mathrm{Ext}^{2-2m,2m}(P_m, P_m) \cong \mathbb{Z}$ is a generator. In particular, Q_m categorifies a multiple of the Jones-Wenzl projector

$$\chi(Q_m) = (1 - q^{2n})\chi(P_m)$$

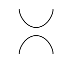
where $\chi : \mathrm{Kom}(m)' \to \mathbb{C}((q)) \otimes \mathbb{C}(q)\,\mathrm{TL}_n$ denotes the graded Euler characteristic (which is defined because of the conditions we place on gradings in $\mathrm{Kom}(m)'$ in Definition 6.24). Since $\chi(P_m) = p_m$ is idempotent, any multiple of it is quasi-idempotent, and we have an interesting categorical realization of this fact: in Theorem 7.11 we prove that $Q_m \odot Q_m \simeq Q_m \oplus t^{1-2m}q^{2m}Q_m$. In the same theorem, we show that the symmetric projectors commute with one another, which shows that C_n is quasi-idempotent.

We expect that the link invariant $C(D; \{C_n\})$ is functorial under link cobordisms up to sign and homotopy. As evidence for this, in Theorem 7.40 we prove that the symmetric projectors Q_n are Frobenius algebra objects in the categories of chain complexes preserved by $P_{n-1} \sqcup 1_1$ up to homotopy.

7.1 Existence of Q_n

We begin by constructing the symmetric projectors $Q_n \in \mathrm{Kom}(n)$, whose existence has so far been assumed. We do this by induction on $n \geq 1$. Note that by convention we have $Q_1 = \mathrm{Cone}(b)$, and the only possibility for Q_2 is the chain complex already defined in section 6.1. Assuming that $Q_1, \ldots, Q_{n-1}$ exist, we will then construct a convolution of the truncation $0 \to \cdots \to 0 \to E_{-k} \to \cdots \to E_0$ of the symmetric Frenkel-Khovanov sequence by induction on $1 \leq k \leq 2n-1$. The inductive step will use lemma 2.9

Theorem 7.1 (Existence theorem)**.** *For each integer $n \geq 1$ there exists a symmetric projector $Q_n \in \mathrm{Kom}(n)$.*

Proof. Induction on $n \geq 1$. In the base case Q_1 is already done. Assume by induction that $Q_m \in \mathrm{Kom}(m)$ exists for each $1 \leq m \leq n-1$. For each $1 \leq i \leq n-1$, let $e_i = 1_{n-i-1} \sqcup e \sqcup 1_{i-1}$ denote the Temperley-Lieb generator, where $e =$ . Let $P_{n-1} \in \mathrm{Kom}(n-1)$ be a Cooper-Krushkal projector. As in the proof of Proposition 6.10, define chain complexes $F(i) = (P_{n-1} \sqcup 1) \odot e_1 \odot \cdots \odot e_i$ for $1 \leq i < n-1$ and $F(0) = (P_{n-1} \sqcup 1)$, and note that the symmetric Frenkel-Khovanov sequence can be written

$$E_\bullet \quad = \quad \begin{array}{ccccccccc} F(n-1) & \longrightarrow & F(n-2) & \longrightarrow & \cdots & \longrightarrow & F(1) & \longrightarrow & F(0) \\ \uparrow & & & & & & & & \\ F(n-1) & \longleftarrow & F(n-2) & \longleftarrow & \cdots & \longleftarrow & F(1) & \longleftarrow & F(0) \end{array}$$

where the maps are given by saddle cobordisms $F(i) \to F(i \pm 1)$ and a difference of dotted identity maps $F(n-1) \to F(n-1)$. Here we are omitting the degree shifts, and we had to fold up the sequence $F(0) \to \cdots \to F(n-1) \to F(n-1) \to \cdots \to F(0)$ because of space limitations.

We will show that there exists a convolution $M_k = \mathrm{Tot}(E_{-k} \to \cdots \to E_0)$ of the corresponding truncation by induction on $1 \leq k \leq 2n-1$. The chain complex M_{1-2n} will be the desired symmetric projector.

For the base case $k = 1$ there is no choice: $M_1 = \mathrm{Cone}(E_{-1} \stackrel{\alpha_{-1}}{\to} E_0)$. Assume by induction that $M_k = \mathrm{Tot}(\cdots \to 0 \to E_{-k} \to \cdots \to E_0)$ exists. The proof that M_{k+1} exists splits into four cases. The first three are very similar, and follow from corollary 4.17, the fact that P_{n-1} kills turnbacks, and lemma 2.9. The fourth case is unique, and is the only case where we use the inductive assumption that $Q_1, \ldots, Q_{n-1}$ exist. In the interest of preventing clutter, we will omit all explicit shifts in bidegree until case 4, where the shifts are essential.

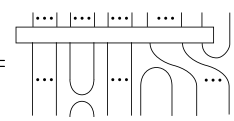
Case 1. If $1 \leq k \leq n-1$, then we seek a convolution of

$$\underbrace{F(k+1)}_{A} \to \underbrace{F(k)}_{B} \to \underbrace{F(k-1) \to \cdots \to F(0)}_{C}$$

By the inductive hypothesis, we assume that there exists a convolution M_k of all terms except for the left-most. Reassociating as indicated by the braces, we write $M_k = (B \xrightarrow{\beta} C)$, where $B = F(k)$, $C = (F(k-1) \to \cdots \to F(0))$, and β is the corresponding component of the differential on M_k. By lemma 2.9, it suffices to show that $\mathrm{Hom}^\bullet(A, C) \simeq 0$. Note that A can be factored as $A = A' \odot e_{k+1}$, and by corollary 4.17 we have

$$\mathrm{Hom}^{\bullet,\bullet}(A, C) = \mathrm{Hom}^{\bullet,\bullet}(A' \odot e_{k+1}, C) \cong \mathrm{Hom}^{\bullet,\bullet}(A', C \odot e_{k+1})$$

The terms of the sequence $(F(k-1) \to \cdots \to F(0)) \odot e_{k+1}$ are each of the form $F(j) \odot e_{k+1}$ for $j \leq k-1$, which is a contractible chain complex

$$F(j) \odot e_{n-k-1} =$$ 

since P_{n-1} kills turnbacks. Hence $C \odot e_{k+1}$ is a convolution of contractible complexes, which is contractible by Theorem 2.15. The implies $\mathrm{Hom}^{\bullet,\bullet}(A, C) \simeq 0$ and completes the inductive step in this case.

Case 2. If $k = n-1$, then we seek a convolution of

$$\underbrace{F(n-1)}_{A} \to \underbrace{F(n-1)}_{B} \to \underbrace{F(n-2)}_{C} \to \underbrace{F(n-3) \to \cdots \to F(0)}_{D}$$

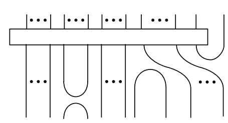
By the inductive hypothesis, we can assume that there is a convolution M_{n-1} of all terms excluding the left-most. Let us reassociate as indicated by the braces above, and write $M_{n-1} = (B \xrightarrow{\beta} C \xrightarrow{\gamma} D)$. Additionally, a convolution $(A \xrightarrow{\alpha} B \xrightarrow{\beta} C)$ of the first three terms exists since $\beta \circ \alpha \simeq 0$ (recall that $E_\bullet$ is a homotopy chain complex). So by lemma 2.9 it suffices to show that $\mathrm{Hom}^{\bullet,\bullet}(A, D) \simeq 0$. But A factors as $A = A' \odot e_{n-1}$, and $\mathrm{Hom}^{\bullet,\bullet}(A' \odot e_{n-1}, D) \simeq 0$ follows by the same argument as in case 1.

Case 3. If $n \leq k < 2n-2$, then we can write $k = 2n-r-2$ for a unique $1 \leq r < n-1$. We seek a convolution of

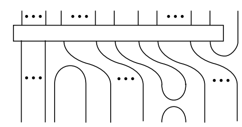
$$\underbrace{F(r)}_{A} \to \underbrace{F(r+1)}_{B} \to \underbrace{F(r+2) \to \cdots \to F(n-1) \to F(n-1) \to \cdots \to F(0)}_{C}$$

By the inductive hypothesis, we assume that there exists a convolution $M_k = (B \to C)$ of all terms except for the left-most. Once again, it suffices by lemma 2.9 to show that $\mathrm{Hom}^{\bullet,\bullet}(A, C) \simeq 0$. Note that $A = A' \odot e_r$ for some A'. As in case 1, we have $\mathrm{Hom}^{\bullet,\bullet}(A, C) \cong \mathrm{Hom}^{\bullet,\bullet}(A', C \odot e_r)$. Applying the functor $(-) \odot e_r$ to $F(r+1) \to \cdots \to F(n-1) \to F(n-1) \to \cdots \to F(0)$ gives a sequence which can be reassociated into subsequences of the form

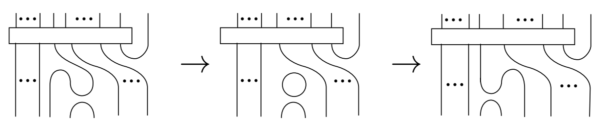
1. 1-term sequences $F(j) \odot e_r$ where $j \notin \{r-1, r, r+1\}$, which are contractible chain complexes of the form



or



2. a 3-term subquence $F(r+1) \odot e_r \to F(r) \odot e_r \to F(r-1) \odot e_r$ of the form

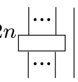
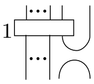
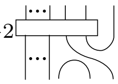
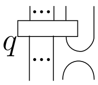
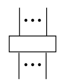


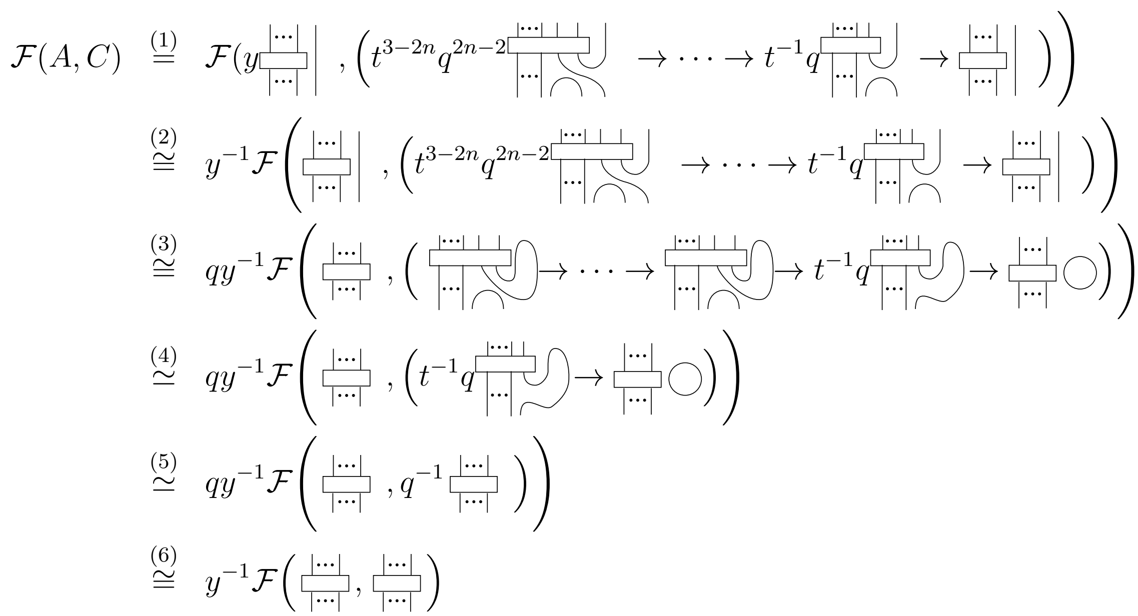
where the maps merge or split off a disjoint unknotted circle.

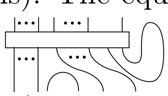
Each of the complexes (1) is contractible since P_{n-1} kills turnbacks, and a convolution of (2) is contractible by lemma 6.12. So $C \odot e_r$ can be reassociated into a convolution of contractible chain complexes, hence is contractible by Theorem 2.15. This completes the inductive step in this case. We remark that here $\mathrm{Hom}^{\bullet,\bullet}(A, (C \to D)) \simeq \mathrm{Hom}^{\bullet,\bullet}(A, C) \not\simeq 0$. This is why it was necessary to "peel-off" an additional term from $E_\bullet$.

Case 4. If $k = 2n-2$ then we seek a convolution of the form

$$\underbrace{F(0)}_{A} \to \underbrace{F(1)}_{B} \to \underbrace{F(2) \to \cdots \to F(n-1) \to F(n-1) \to \cdots \to F(0)}_{C}$$

By the inductive hypothesis, we assume that there is a convolution $M_{2n-2} = (B \to C)$ of all terms except for the left-most. In contrast to the previous cases, we no longer have $\mathrm{Hom}^{\bullet,\bullet}(A, C) \simeq 0$. Nonetheless, it turns out that the corresponding homology group $\mathrm{Ext}^{2,0}(A, C)$ vanishes. Indeed, taking the grading shifts into account, we have $A := t^{1-2n}q^{2n}$ , $B := t^{2-2n}q^{2n-1}$ , and $C := \Big(t^{3-2n}q^{2n-2}$  $\to \cdots \to t^{-1}q$  $\to$  $\Big)$. Abbreviate $\mathcal{F}(-,-) := \mathrm{Hom}^{\bullet,\bullet}(-,-)$ and $y = t^{1-2n}q^{2n}$, and compute:



Let us explain. First, note that $\mathrm{Hom}^{\bullet,\bullet}$ invariant under homotopy equivalence of its arguments. (1) is by definition of A and C. (2) and (6) hold by the easily proven fact that if x and y are shifts in bidegree, then $\mathrm{Hom}^{\bullet,\bullet}(xM, zN) \cong x^{-1}z\,\mathrm{Hom}(M, N)$. (3) holds by corollary 4.17 (we have omitted some of the grading shifts because of space limitations). The equivalence (4) holds by contracting the contractible complexes of the form  and using Theorem 2.15. Finally, (5) holds by delooping and cancelling (Lemma 6.39).

By the above computation, we have an isomorphism of homology groups

$$\mathrm{Ext}^{2,0}(A, C) \cong \mathrm{Ext}^{3-2n,2n}(P_{n-1}, P_{n-1}).$$

This group is zero by Corollary 6.45, which applies to P_{n-1} since we assume that $Q_1, \ldots, Q_{n-1}$ exist. This shows that a convolution $Q_n = M_{2n-1} = (A \to B \to C)$ exists, and completes the proof. $\square$

Now that we know that the symmetric projectors Q_n exist, all of the results and constructions of section 6 are valid.

7.2 Uniqueness and symmetries of Q_n

Theorem 7.2 (Uniqueness theorem). *Let $P_n, Q_n \in \mathrm{Kom}(n)$ be any Cooper-Krushkal projector, respectively any symmetric projector. If $U_n : t^{2-2n}q^{2n}P_n \to P_n$ represents a generator of the corresponding* Ext *group then* $\mathrm{Cone}(U_n) \simeq Q_n$. *In particular, symmetric projectors are unique up to homotopy equivalence.*

Proof. Let Q_n and P_n be as in the hypotheses. Then by Proposition 6.16 there is *some* Cooper-Krushkal projector P'_n and a chain map $U'_n : t^{2-2n}q^{2n}P'_n \to P'_n$ such that $\mathrm{Cone}(U'_n) \simeq Q_n$, and Corollary 6.45 says that the class $[U'_n]$ is a generator of the corresponding $\mathrm{Ext}^{\bullet,\bullet}$ group. In order to relate this to our fixed P_n, compute

$$\begin{aligned} Q_n &\simeq Q_n \odot P_n \\ &\simeq \mathrm{Cone}(U'_n) \odot P_n \\ &\simeq (t^{1-2n}q^{2n}P'_n \odot P_n \xrightarrow{U'_n \odot \mathrm{Id}_{P_n}} P'_n \odot P_n) \\ &\simeq (t^{1-2n}q^{2n}P_n \xrightarrow{U_n} P_n) \\ &= \mathrm{Cone}(U_n) \end{aligned}$$

In the first equivalece we used the fact that Q_n kills turnbacks together with Proposition 4.21; in the last equivalence we applied the deformation retract (π, σ, h) : $P'_n \odot P_n \to P_n$ implied by Proposition 4.21, and we let $U_n = \pi \circ U'_n \circ \sigma$. Conjugation by the homotopy equivalence $P'_n \simeq P_n$ induces an isomorphism on $\mathrm{Ext}^{\bullet,\bullet}$ groups. Hence $[U_n]$ is a generator since $[U'_n]$ is. Moreover, since the corresponding $\mathrm{Ext}^{\bullet,\bullet}$ group is isomorphic to $\mathbb{Z}$, any two generators are homotopic up to sign, hence have homotopy equivalent mapping cones. Since P_n was fixed but arbitrary, this implies the proposition. □

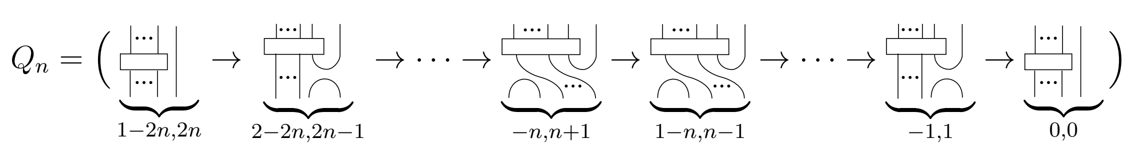
Corollary 7.3. *Let g be a symmetry of the rectangle, regarded as a covariant functor* $\mathrm{Kom}(n) \to \mathrm{Kom}(n)$*. If $P_n \in \mathrm{Kom}(n)$ is a symmetric projector, then $g(Q_n) \simeq Q_n$.*

Proof. One can see that $g^{-1}(P_n)$ is a universal projector, so by Theorem 7.2 we have a map $U_n : t^{2-2n}q^{2n}g^{-1}(P_n) \to g^{-1}(P_n)$ such that $\mathrm{Cone}(U_n) \simeq Q_n$, and such that U_n generates the corresponding $\mathrm{Ext}^{\bullet,\bullet}$ group. Applying g to this, we see that $\mathrm{Cone}(g(U_n)) \simeq g(Q_n)$. But $g(U_n) : t^{2-2n}q^{2n}P_n \to P_n$ is a generator of the corresponding $\mathrm{Ext}^{\bullet,\bullet}$ group since the invertible functor g induces isomorphism $\mathrm{End}^{\bullet,\bullet}(P_n) \to \mathrm{End}^{\bullet,\bullet}(g(P_n))$. Again by Theorem 7.2 we have $g(Q_n) \simeq \mathrm{Cone}(g(U_n)) \simeq Q_n$. This completes the proof. □

For $n \geq 4$, Q_n is not homotopy equivalent to a bounded chain complex, and so $Q_n^\vee \not\simeq Q_n$, even up to a shift in grading. Nonetheless, Q_n is self-dual with respect to a certain duality functor D relative to P_{n-1}. This duality functor will be studied in more detail in a later section, so for now we only define D and show that $D(Q_n) \simeq t^{2n-1}q^{-2n}Q_n$. Throughout the rest of this section, fix an integer $n \geq 1$.

Definition 7.4. Let $D : \mathrm{Kom}(n) \to \mathrm{Kom}(n)^\Pi$ denote the contravariant functor defined by $D(A) = (P_{n-1} \sqcup 1_1) \odot^\Pi A^\vee \odot^\Pi (P_{n-1} \sqcup 1_1)$.

In future sections, particularly in the proof of Proposition 7.33, we will want to see that the equivalence $D(Q_n) \simeq t^{2n-1}q^{-2n}Q_n$ intertwines a pair of maps ε and η. Recall that a symmetric projector $Q_n \in \mathrm{Kom}(n)$ is a convolution of the form:



(7.5)

where the decorations indicate the overall shifts in bidegree and the white box denotes $P_{n-1} \in \mathrm{Kom}(n-1)$.

Definition 7.6. Recall n is fixed. Put $I := P_{n-1} \sqcup 1_1$, let $y = t^{1-2n}q^{2n}$ be the shift functor, and let $Q_n \in \mathrm{Kom}(n)$ be a symmetric projector (relative to P_{n-1}).

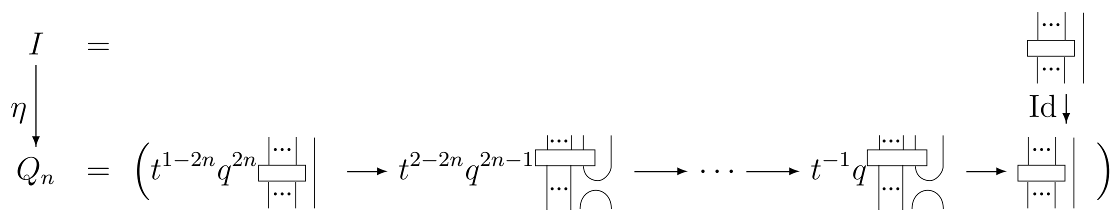
Define $\eta = \eta_n : I \to Q_n$ to be the inclusion of the right-most term of (7.5), and let $\varepsilon = \varepsilon_n : y^{-1}Q_n \to I$ be given by projection onto the left-most term of (7.5).

In Proposition 7.28 we will show that D behaves as though it were the duality functor on the monoidal category $(\mathcal{A}_{/h}, \odot, I)$ of chain complexes preserved by I. That is, $\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(I, B \odot^{\Pi} D(A))$ for chain complexes A, B with $A \simeq I \odot A \odot I$ and $B \simeq I \odot B \odot I$. We will use the following to simplify such Hom complexes involving Q_n and I (most importantly in Proposition 7.33):

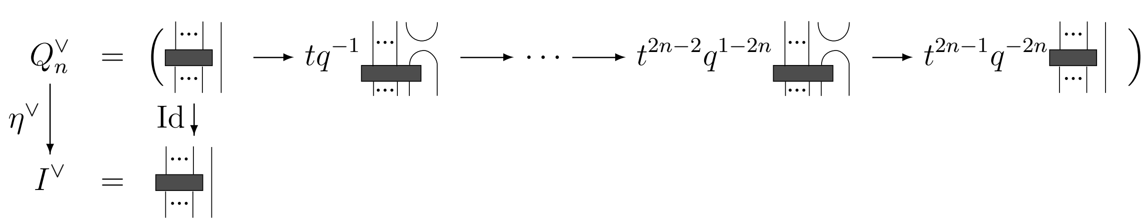
Theorem 7.7. *In the notation of Definition 7.6 we have $D(I) \simeq I$ and $D(Q_n) \simeq y^{-1}Q_n$. Moreover, the following square commutes:*

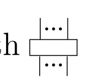
$$\begin{array}{ccc} D(Q_n) & \xrightarrow{\simeq} & y^{-1}Q_n \\ {\scriptstyle D(\eta)}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ D(I_n) & \xrightarrow{\simeq} & I \end{array} \tag{7.8}$$

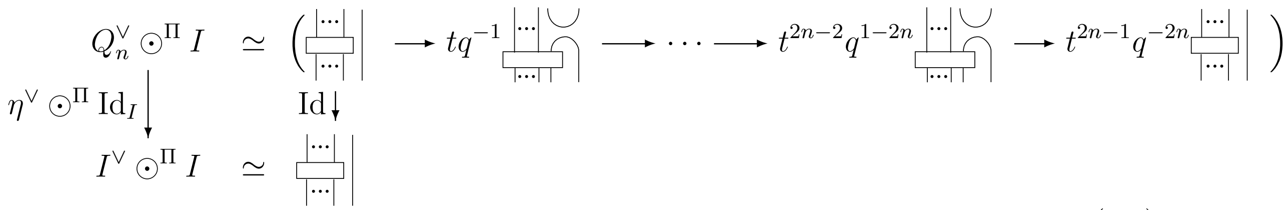
Proof. Note that $\eta : I \to Q_n$ is the inclusion:



where the white box denotes P_{n-1}. Taking $(\ )^{\vee}$ gives a chain map



where the black box denotes $P_{n-1}^{\vee}$. Now, compose with  from below using direct product, and merge the white boxes with black boxes. That is to say, Proposition 4.23 gives an equivalence $P_{n-1}^{\vee} \odot^{\Pi} P_{n-1} \xrightarrow{\simeq} P_{n-1}$; this equivalence can be assumed to be a deformation retract (π, σ, h), hence it can be applied to each term of the diagram above. The vertical map becomes the conjugate $\pi \circ \mathrm{Id} \circ \sigma = \mathrm{Id}$, and we obtain a diagram



(7.9)

The top row is a chain complex which, on one hand is homotopy equivalent to $Q_n^\vee \odot^\Pi I$, and on the other hand is of the form $y^{-1}\,\mathrm{Tot}(s_x E_\bullet)$, where $s_x E_\bullet$ is obtained from the symmetric Frenkel-Khovanov sequence by reflecting across the x-axis. Corollary 7.3 implies that $Q_n^\vee \odot^\Pi I \simeq y^{-1} Q_n$. Moreover, the map $\eta^\vee \odot^\Pi \mathrm{Id}_I$ shown in (7.9) clearly corresponds to ε under this equivalence. This is to say, we have a square

$$\begin{array}{ccc} Q_n^\vee \odot^\Pi I & \xrightarrow{\simeq} & y^{-1}Q \\ {\scriptstyle \eta^\vee \odot^\Pi \mathrm{Id}_I}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ I^\vee \odot^\Pi I & \xrightarrow{\simeq} & I \end{array} \tag{7.10}$$

which commutes up to homotopy. Since I and Q_n are bounded from above $\odot^\Pi$ coincides with $\odot$, applying the functor $I \odot^\Pi (-)$ to the above gives the first square in the following diagram, which commutes up to homotopy:

$$\begin{array}{ccccc} D(Q_n) & \xrightarrow{\simeq} & y^{-1} I \odot Q_n & \xleftarrow{\iota \odot \mathrm{Id}_{Q_n}} & y^{-1}Q_n \\ {\scriptstyle D(\eta)}\downarrow & & {\scriptstyle \mathrm{Id}_I \odot \varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ D(I_n) & \xrightarrow{\simeq} & I \odot I & \xleftarrow{\iota \odot \mathrm{Id}_I} & I \end{array}$$

where $\iota : 1_n \to I$ is the inclusion of the degree zero chain group. The square on the right commutes by inspection. Now, by Proposition 4.21 the compositions

$$Q_n \cong 1_n \odot Q_n \xrightarrow{\iota \odot \mathrm{Id}_{Q_n}} I \odot Q_n$$

and

$$I \cong 1_n \odot I \xrightarrow{\iota \odot \mathrm{Id}_I} I \odot I$$

are homotopy equivalences; inverting them establishes that $D(Q_n) \simeq y^{-1} Q_n$, and that the square (7.8) commutes up to homotopy. This completes the proof. □

7.3 Quasi-idempotency and commuting properties

In general, if $e \in A$ is an idempotent of a $\mathbb{C}(q)$-algebra A, then any multiple $f = ae$, $a \in \mathbb{C}(q)$, is quasi-idempotent. That is, $f^2 = af$. Since $Q_n = \mathrm{Cone}(U_n) = (t^{1-2n}q^{2n}P_n \to P_n)$, we say that Q_n categorifies a multiple of the Jones-Wenzl projector. For example, assuming $\mathcal{TL}_n$ is embedded in an abelian category, we have a short exact sequence

$$0 \to P_n \to \mathrm{Cone}(U_n) \to t^{1-2n}q^{2n}P_n \to 0$$

hence $[\mathrm{Cone}(U_n)] = (1 - q^{2n})[P_n]$ in the Grothendieck group. It is natural to ask whether Q_n is quasi-idempotent up to homotopy, and it is a pleasant surprise that it actually is: $Q_n^{\odot 2} \simeq Q_n \oplus t^{1-2n}q^{2n}Q_n$. This justifies the word "projector" in the term symmetric projector.

Theorem 7.11. *We have*

1. *Symmetric projectors commute with one another:* $(Q_m \sqcup 1_{n-m}) \odot Q_n \simeq Q_n \odot (Q_m \sqcup 1_{n-m})$.

2. *Any symmetric projector* $Q_n \in \mathrm{Kom}(n)$ *is quasi-idempotent in the homotopy category:* $Q_n^{\odot 2} \simeq Q_n \oplus t^{1-2n}q^{2n}Q_n$.

Proof. Let $U_k^{(n)} \in \mathrm{End}^{2-2k,2k}(P_n)$ be a family of generators as in Corollary 6.45, so that

$$\rho_m^n(U_k^{(m)}) \simeq U_k^{(n)} \simeq \bar{\rho}_m^n(U_k^{(m)}) \tag{7.12}$$

for all $1 \le k \le m \le n$, where $\rho_m^n : \mathrm{End}^{\bullet,\bullet}(P_m) \to \mathrm{End}^{\bullet,\bullet}(P_n)$ is a canonical representation (Definition 5.4). By replacing P_n by $(P_n)^{\odot n}$ and $U_k^{(n)}$ by $\mathrm{Id}^{\odot k-1} \odot U_k^{(n)} \odot \mathrm{Id}^{\odot n-k}$ if necessary, we may assume that the $U_k^{(n)}$ commute for all $1 \le k \le n$. We will often abuse notation and omit the superscripts, $U_k = U_k^{(k)}$ whenever there is no possibility of confusion. By Theorem 7.2 we have $Q_m \simeq \mathrm{Cone}(U_m^{(m)})$ and $Q_n \simeq \mathrm{Cone}(U_n^{(n)})$. For each integer $k \geq 1$, let $y_k = t^{1-2k}q^{2k}$ denote the grading shift functor. Compute:

$$(Q_m \sqcup 1_{n-m}) \odot Q_n \cong \left(\begin{array}{ccc} y_m y_n (P_m \sqcup 1_{n-m}) \odot P_n & \xrightarrow{(U_m \sqcup \mathrm{Id}) \odot \mathrm{Id}} & y_n (P_m \sqcup 1_{n-m}) \odot P_n \\ {\scriptstyle -\mathrm{Id} \odot U_n} \downarrow & & {\scriptstyle \mathrm{Id} \odot U_n} \downarrow \\ y^m (P_m \sqcup 1_{n-m}) \odot P_n & \xrightarrow{(U_m \sqcup \mathrm{Id}) \odot \mathrm{Id}} & (P_m \sqcup 1_{n-m}) \odot P_n \end{array} \right)$$

$$\simeq \left(\begin{array}{ccc} y_m y_n P_n & \xrightarrow{\rho^n_m(U_m)} & y_n P_n \\ {\scriptstyle -U_n} \downarrow & \searrow^{z'} & {\scriptstyle U_n} \downarrow \\ y^m P_n & \xrightarrow{\rho^n_m(U_m)} & P_n \end{array} \right)$$

In the last step we applied a deformation retract $\pi : (P_m \sqcup 1_{n-m}) \odot P_n \to P_n$ which by Proposition 4.21 can be assumed to have section given by $\sigma = (\iota \sqcup 1_{n-m}) \odot \mathrm{Id}_{P_n}$. In particular the vertical maps become $\pi \circ (\mathrm{Id} \odot U_n) \circ \sigma = \pi \circ \sigma \circ (\mathrm{Id}_{1_n} \odot U_n) = U_n$ as indicated, and the horizontal maps become $\rho^n_m(U^{(m)}_m) \simeq U^{(n)}_m$, where ρ^n_m is a canonical representation. Theorem 2.10 together with (7.12) says we can replace the horizontal maps by $U^{(n)}_m$ at the expense of affecting the higher length components of the differential. Thus:

$$(Q_m \sqcup 1_{n-m}) \odot Q_n \cong \left(\begin{array}{ccc} y_m y_n P_n & \xrightarrow{U_m} & y_n P_n \\ {\scriptstyle -U_n} \downarrow & \searrow^{z} & {\scriptstyle U_n} \downarrow \\ y^m P_n & \xrightarrow{U_m} & P_n \end{array} \right)$$

Since U_m and U_n commute, taking components of the equation $d^2 = 0$ gives that z is a cycle. For degree reasons we must have $z \in \mathrm{End}^{3-2n-2m, 2m+2n}(P_n)$, hence z is a boundary by Corollary 6.45. Theorem 2.10 again says that up to isomorphism we can replace z by zero at the expense of introducing higher length components of the differential, of which there can be none. This is to say

$$(Q_m \sqcup 1_{n-m}) \odot Q_n \simeq \left(\begin{array}{ccc} y_m y_n P_n & \xrightarrow{U_m} & y_n P_n \\ {\scriptstyle -U_n} \downarrow & & {\scriptstyle U_n} \downarrow \\ y^m P_n & \xrightarrow{U_m} & P_n \end{array} \right)$$

An entirely symmetric argument, made possible by the symmetry in (7.12), shows that $Q_n \odot (Q_m \sqcup 1_{n-m})$ is homotopy to precisely the same chain complex. This gives (1).

In case $m = n$, the above gives

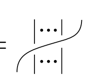
$$Q_n \odot Q_n \simeq \left(\begin{array}{ccc} y_n^2 P_n & \xrightarrow{U_n} & y_n P_n \\ {\scriptstyle -U_n}\downarrow & & \downarrow{\scriptstyle U_n} \\ y^n P_n & \xrightarrow{U_n} & P_n \end{array} \right)$$

After performing an elementary similarity transform to the matrix $\begin{bmatrix} d & 0 & 0 & 0 \\ U_n & -d & 0 & 0 \\ -U_n & 0 & -d & 0 \\ 0 & U_n & U_n & d \end{bmatrix}$ (namely add the second row to the third while subtracting the third column from the second) we can replace the vertical maps with zeroes up to isomorphism. The result will be a chain complex which is isomorphic to $Q_n \oplus t^{1-2n} q^{2n} Q_n$. This proves (2). $\square$

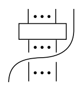
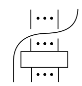
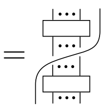
7.4 Quasi-local categorifications of the colored Jones polynomial

We define the quasi-local $\mathfrak{sl}_2$ link homology by analogy with the local theory from Definition 3.25, replacing P_n by a chain complex $C_n \simeq (Q_1 \sqcup 1_{n-1}) \odot \cdots \odot Q_n$. By Theorem 6.28, we have $P_n \simeq \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} C_n$, and the two invariants are closely related. We will focus on showing that the Q_n can be slid under strands up to homotopy equivalence, from which it will follow that the quasi-local theory defines an invariant of colored framed oriented tangles. Since $Q_n \simeq \mathrm{Cone}(U_n)$ for a generator $[U_n] \in \mathrm{Ext}^{2-2n,2n}(P_n, P_n)$, a natural starting place will be an understanding of why the Cooper-Krushkal projector P_n can be slid under strands.

Fix $n \geq 1$, and define a chain complex $X \in \mathrm{Kom}(n+1)$ by

$X :=$ 

We suppress the orientations from our notation, but we assume that the strands on the bottom have alternating orientations, and the strand on top is oriented arbitrarily. Let $F(\ ,\ )$ denote the dg functor $\mathrm{Kom}_{dg}(n) \otimes \mathrm{Kom}_{dg}(n) \to \mathrm{Kom}_{dg}(n)^{\Pi}$ defined by $F(A, B) := (A \sqcup 1_1) \odot X \odot (1_1 \sqcup B)$. We are interested in the chain complexes

$F(P_n, 1_n) =$  $\qquad F(1_n, P_n) =$  , $\qquad F(P_n, P_n) =$ 

and related chain complexes. The statement that that P_n can be slid under strands up to homotopy equivalence is simply the statement that $F(P_n, 1_n) \simeq F(1_n, P_n)$.

By axiom (CK1) of Definition 3.12, P_n can be written as $P_n = \mathrm{Cone}(N \to 1_n)$ where $N \in \mathrm{Kom}^{\leq 0}(n)$ has through-degree $\tau(N) < n$. Suppose $E \in \mathrm{Kom}(n)$ is any chain complex which kills turnbacks. By invariance of Bar-Natan's tangle invariant under the Reidemester II move, we can slide Temperley-Lieb diagrams under strands. In particular if $N^k \in \mathcal{TL}_n$ denotes the k-th chain group of N, then $F(E, N^k) \simeq F(E \odot N^k, 1_n) \simeq 0$ since E kills turnbacks. Now, by bilinearity, $F(E, N)$ is the total complex of a bicomplex of the form

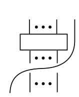
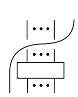
$$\cdots \to F(E, N^{-1}) \to F(E, N^0)$$

so if E kills turnbacks then $F(E, N) \simeq 0$ by Theorem 2.15. Again by bilinearity, we have $F(E, P_n) \cong \mathrm{Cone}(F(E, N) \to F(E, 1_n)) \simeq F(E, 1_n)$. A similar computation shows that $F(P_n, E) \simeq F(1_n, E)$. We have proven:

Lemma 7.13. *If $E \in \mathrm{Kom}(n)$ is any chain complex which kills turnbacks, then*

$$F(E, P_n) \simeq F(E, 1_n) \qquad \text{and} \qquad F(P_n, E) \simeq F(1_n, E) \tag{7.14}$$

In fact each of these equivalences can be chosen to be a deformation retract with sections given by $F(\mathrm{Id}_E, \iota)$, respectively $F(\iota, \mathrm{Id}_E)$, where $\iota : 1_n \to P_n$ is the inclusion of the degree zero chain group. □

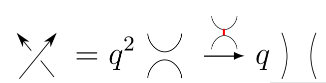
Taking $E = P_n$ gives  $\simeq$  . We are ready to prove that:

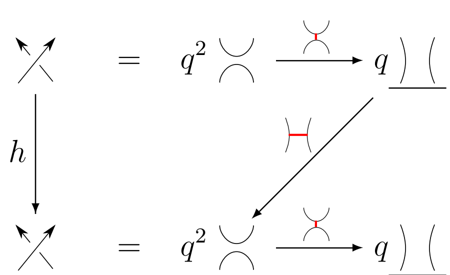
Proposition 7.15. *Symmetric projectors can be slid under strands, up to homotopy equivalence.*

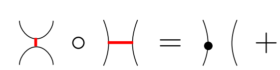
Proof. Let Q_n be a symmetric projector, and retain the notation of the preceding discussion. We need to prove that $F(Q_n, 1_n) \simeq F(1_n, Q_n)$. By Theorem 7.2 we know that $Q_n \simeq \mathrm{Cone}(U_n)$, where $U_n : t^{2-2n}q^{2n}P_n \to P_n$ represents a generator of the corresponding Ext group, and by Lemma 7.13 we know that $F(P_n, 1_n) \simeq F(1_n, P_n)$. We need only see how this equivalence interacts with U_n. We will prove that $F(U_n, \mathrm{Id}_{P_n}) \simeq \pm F(\mathrm{Id}_{P_m}, U_n)$ as elements of $\mathrm{End}^{2-2n,2n}(F(P_n, P_n))$. Indeed if this were the case then we would have

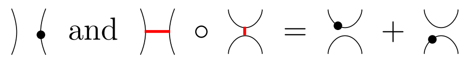
$$\begin{aligned} F(\mathrm{Cone}(U_n), 1_n) &\overset{(1)}{\simeq} F(\mathrm{Cone}(U_n), P_n) \\ &\overset{(2)}{\cong} \mathrm{Cone}(F(U_n, \mathrm{Id}_{P_n})) \\ &\overset{(3)}{\simeq} \mathrm{Cone}(F(\mathrm{Id}_{P_n}, U_n)) \\ &\overset{(4)}{\cong} F(P_n, \mathrm{Cone}(U_n)) \\ &\overset{(5)}{\simeq} F(1_n, \mathrm{Cone}(U_n)) \end{aligned}$$

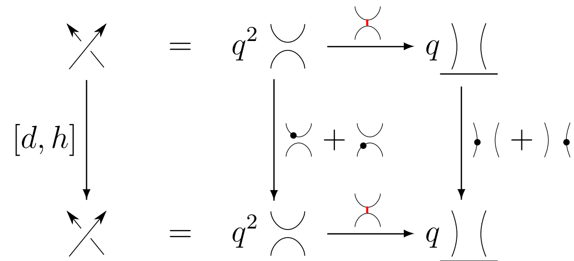
The first and fifth steps are by Lemma 7.13, the second and fourth steps follow by bilinearity of F, and the third is by homotopy invariance of the mapping cone. Since $Q_n \simeq \mathrm{Cone}(U_n)$, the proposition will follow. It remains to show that $F(U_n, \mathrm{Id}_{P_n}) \simeq \pm F(\mathrm{Id}_{P_n}, U_n)$.

Let us handle the case $n = 1$ by itself. In this case $X =$ .
Consider the element $h \in \mathrm{End}^{-1,2}($  $)$ defined by the following diagram:

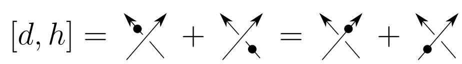


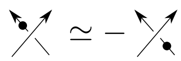
Now, by the neck-cutting relation in the categories $\mathcal{TL}_n$ we have 

 It follows that $[d,h] \in \mathrm{End}^{0,2}$ is the chain map

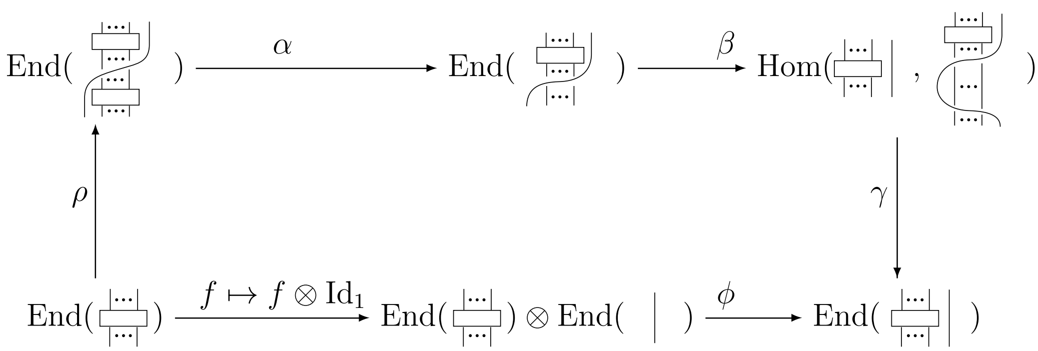


This is to say,



In particular, , i.e. $F(U_1, \mathrm{Id}) \simeq -F(\mathrm{Id}, U_1)$. This shows that Q_1 can be slid under strands.

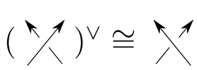
We claim that there is a commutative diagram

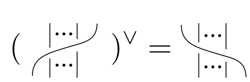


(7.16)

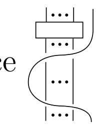
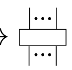
where $\rho(f) = F(f, \mathrm{Id}_{P_n})$ and $\alpha, \beta, \gamma, \phi$ are homotopy equivalences. Indeed,

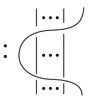
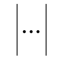
- By Lemma 7.13, $F(\mathrm{Id}_{P_n}, \iota) : F(P_n, 1_n) \to F(P_n, P_n)$ is the section of a deformation retract π, where $\iota : 1_n \to P_n$ is the inclusion of the degree zero chain group. Define α to be conjugation by these homotopy inverses.

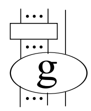
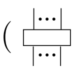
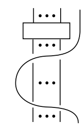
- Upon unpacking the definitions, one sees that the duality functor $(\ )^\vee$ satisfies . Since $(\ )^\vee$ reflects planar compositions, it follows that



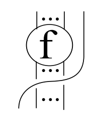
  We can now define β to be the isomorphism $\mathrm{End}((P_n \sqcup 1) \odot X) \cong \mathrm{Hom}(P_n \sqcup 1, (P_n \sqcup 1) \odot X \odot X^\vee)$ from Theorem 4.15.

- Define γ to be post-composition with a homotopy equivalence  $\rightarrow$  .

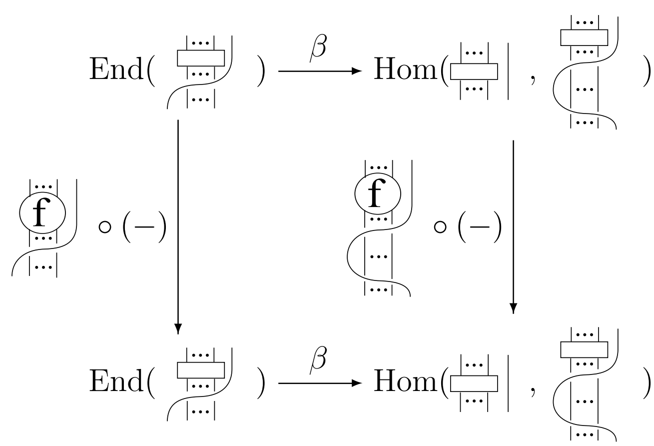
  More specifically, fix a homotopy equivalence g :  $\rightarrow$  , and put

  $\gamma(h) :=$  $\circ\, h$ for all $h \in \mathrm{Hom}($  ,  $)$

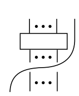
Clearly α, β, γ are homotopy equivalences. Before defining ϕ, let us consider the composition $\gamma \circ \beta \circ \alpha \circ \rho$. By definition of α we have

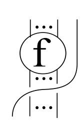
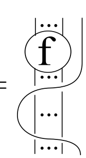
$\alpha(\rho(f)) = \pi \circ F(f, \mathrm{Id}_{P_n}) \circ F(\mathrm{Id}_{P_n}, \iota) = \pi \circ F(\mathrm{Id}, \iota) \circ F(f, \mathrm{Id}_{1_n}) = F(f, \mathrm{Id}_{1_n}) =$ 

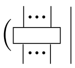
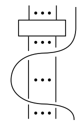
Now, by naturality of the isomorphism of Theorem 4.15, for each $f \in \mathrm{End}(P_n)$ we have a commutative square

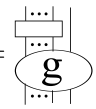
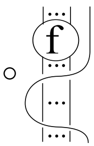
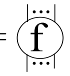
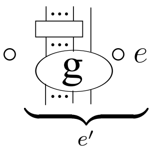


(7.17)

Evaluating on Id $\in \mathrm{End}($  $)$ gives

$\beta \circ \alpha \circ \rho(f) = \beta($  $) =$  $\circ\, e$

where $e = \beta(\mathrm{Id}) \in \mathrm{Hom}($ , $)$. Applying γ gives

$\gamma \circ \beta \circ \alpha \circ \rho(f) =$  $\circ$  $\circ\, e =$  $\circ$  $\circ\, e$

since distant maps super-commute and g has homological degree zero. Put $e' := ((\mathrm{Id}_{P_n} \sqcup \mathrm{Id}_1) \odot g) \circ e \in \mathrm{End}(P_n \sqcup 1_1)$, so that $\gamma \circ \beta \circ \alpha \circ \rho(f) = (f \sqcup \mathrm{Id}_1) \circ e'$. This suggests:

- Define $\phi(f \otimes a) := (f \sqcup a) \circ e'$ for all $f \in \mathrm{End}(P_n)$ and all $a \in \mathrm{End}(1_1)$.

Now, e' is the image of $\mathrm{Id}_{F(P_n, 1_n)}$ under the equivalence $\gamma \circ \beta : \mathrm{End}(F(P_n, 1_n)) \to \mathrm{End}(P_n \sqcup 1_n)$. By Corollary 6.45 the degree $(0, 0)$ homology groups are isomorphic to $\mathbb{Z}$, and must be generated by the respective identity maps. The isomorphism in homology induced by $\gamma \circ \beta$ sends generators to generators, so we must have $e' = \gamma(\beta(\mathrm{Id})) \simeq \pm \mathrm{Id}$. It follows that ϕ is homotopic to the map $f \otimes a \mapsto \pm f \sqcup a$, which is an isomorphism $\mathrm{End}(P_n) \otimes \mathrm{End}(1_1) \to \mathrm{End}(P_n \sqcup 1_1)$. This shows that ϕ is a homotopy equivalence. Finally, the diagram (7.16) commutes by construction. C

So we have a homotopy equivalence $\Psi := \alpha^{-1} \circ \beta^{-1} \circ \gamma^{-1} \circ \phi : \mathrm{End}(P_n) \otimes \mathrm{End}(1) \to \mathrm{End}(F(P_n, P_n))$, which by commutativity of (7.16) satisfies

$$\Psi(f \otimes 1) = F(f, \mathrm{Id}_{P_n})$$

By Corollary 6.45, for $n \geq 2$ the degree $(2 - 2n, 2n)$ homology group is

$$\begin{aligned}
\mathrm{Ext}^{2-2n,2n}(F(P_n, P_n)) &\cong H^{2-2n,2n}(\mathrm{End}(P_n) \otimes \mathrm{End}(1)) \\
&\cong \mathrm{Ext}^{2-2n,2n}(P_n, P_n) \oplus \mathrm{Ext}^{2-2n,2n-2}(P_n, P_n) \\
&\cong \mathbb{Z} \oplus 0
\end{aligned}$$

generated by $[U_n \otimes \mathrm{Id}_1]$. It follows that $F(U_n, \mathrm{Id}_{P_n})$ is a generator of the corresponding Ext group, being the image of $U_n \otimes \mathrm{Id}_1$ under a homotopy equivalence. An entirely symmetric argument shows that $F(\mathrm{Id}_{P_n}, U_n)$ is also a generator of the same group. Hence $F(U_n, \mathrm{Id}_{P_n}) \simeq \pm F(\mathrm{Id}_{P_n}, U_n)$. This completes the proof. $\square$

We now can construct our bounded, quasi-local $\mathfrak{sl}_2$ link homology theory. Actually, we can define a family of quasi-local theories, corresponding to our flexibility in choosing which chain complexes should replace the Cooper-Krushkal projectors in Definition 3.25:

Definition 7.18. Fix a family of generators $U_k^{(n)} \in \mathrm{Ext}^{2-2k,2k}(P_n) \cong \mathbb{Z}$ for each $1 \leq k \leq n$. For any sequence of integers $1 \leq m_1, \ldots, m_r \leq n$ define the *Koszul complex*

$$P_n(m_1, \ldots, m_r) := \mathrm{Cone}(U_{m_1}^{(n)}) \odot \cdots \odot \mathrm{Cone}(U_{m_r}^{(n)})$$

If $r = 0$, so (m_i) is the empty sequence, then put $P_n(\varnothing) := P_n$, a Cooper-Krushkal projector.

The following is clear:

Proposition 7.19. *We have*

1. $P_n(m) \simeq (Q_m \sqcup 1_{n-m}) \odot P_n$,
2. *If at least one of the integers m_i equals n, then* $P_n(m_1, \ldots, m_r) \simeq (Q_{m_1} \sqcup 1_{n-m_1}) \odot \cdots \odot (Q_{m_r} \sqcup 1_{n-m_r})$. $\square$

The following follows from Theorem 7.11.

Proposition 7.20. *The $P_n(m_1, \ldots, m_r)$ are quasi-idempotent up to homotopy:*

$$P_n(m_1, \ldots, m_r) \odot P_n(m_1, \ldots, m_r) \simeq \prod_{1 \leq k \leq r} (1 + t^{1-2m_k} q^{2m_k}) P_n(m_1, \ldots, m_r)$$

$\square$

Definition 7.21. Fix a family $\mathcal{A} = \{A_n\}$ of chain complexes such that each $A_n \in \mathrm{Kom}(n)$ is one of the complexes $P_n(m_1, \ldots, m_r)$ from Definition 7.18 (P_n is allowed).

Let D be an oriented, colored tangle diagram which is marked with a number of points away from the boundary and crossings, with exactly one on each component of the underlying tangle. Let $C(D;\mathcal{A}) \in \mathrm{Kom}(m,k)$ denote the chain complex obtained by replacing P_n by A_n in Definition 3.25.

Theorem 7.22. *Fix a family of chain complexes $\mathcal{A} = \{A_n\}$ as in Definition 7.21. Let D_1 and D_2 be marked, oriented, colored tangle diagrams so that $C(D_i;\mathcal{A}$ are defined. If D_1 and D_2 represent isotopic framed oriented, colored tangles, then $C(D_1;\mathcal{A}) \simeq C(D_1;\mathcal{A})$. This tangle invariant is*

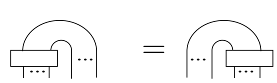
1. *quasi-local, i.e. if D_1 and D_2 are suitably decorated diagrams which are composable, then*

$$C(D_1;\{A_n\}) \odot C(D_2;\mathcal{A}) \simeq f(q,t)C(D_1D_2;\mathcal{A})$$

   where $f(q,t) \in \mathbb{Z}[q,t^{-1}]$ is some polynomial which depends only on the common boundary of D_1 and D_2.

2. *homotopic to a bounded chain complex if $A_n = P_n(2,3,\ldots,n)$ or $A_n = P_n(1,2,\ldots,n)$ for all n.*

Proof. This follows immediately from the invariance of Bar-Natan's tangle invariant under the Reidemeister moves, and the fact (proposition 7.15) that the symmetric projectors can be slid under crossings. Quasi-locality follows from Proposition 7.20. It may concern the reader that we haven't yet proved an analogue of the relation

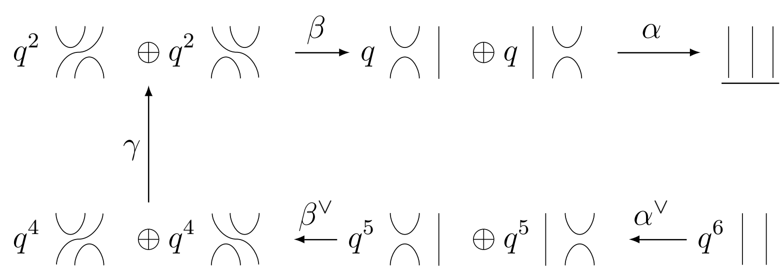


Such a relation does hold in this context, but is not needed for invariance; using the orientation on the diagram one resolves the ambiguity of whether one should glue in the complex A_n or its rotation. $\square$

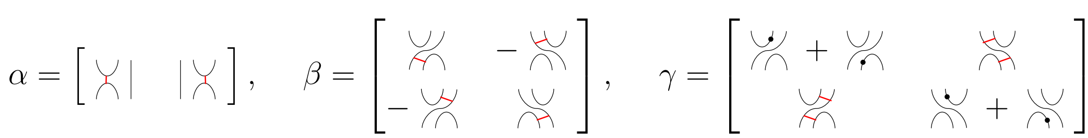
Note that if D is a suitably decorated diagram representing a colored, framed, oriented link L then the homology of $\mathrm{Hom}^{\bullet,\bullet}(\varnothing, C(D;\mathcal{A}))$ categorifies a normalized version of the colored Jones-polynomial. One of our motivations for introducing the quasi-local categorifications was to have a categorification of a normalized $\mathfrak{sl}_2$ Reshetikhin-Turaev invariant via *bounded* chain complexes, and it is natural to ask for a choice $\{A_n\}$ of complexes such that corresponding invariant is bounded up to homotopy equivalence, and the A_n are minimal in an appropriate sense. At one extreme, we have the case $\{A_n\} = \{P_n\}$ which gives the usual local, unnormalized categorification of the colored Jones-polynomial which we know is not bounded. At the other extreme we have the chain complexes $\{A_n\} = \{P_n(1, 2, \ldots, n)\} \simeq \{C_n\}$ which we know is bounded but is not minimal:

Proposition 7.23. *Put* $A_1 = 1_1$, $A_2 = Q_2$, $A_3 = Q_3$, *and* $A_n = P_n(3, 4, \ldots, n)$ *for* $n > 3$. *Then each* $A_n \in \mathrm{Kom}(n)$ *is homotopy equivalent to a bounded chain complex.*

Proof. Clearly A_1 and A_2 are bounded. For A_3, consider the following chain complex $Q \in \mathrm{Kom}^b(3)$:



where



Compare this to the expression for P_3 in [CK12]. We will leave it as an exercise to show that this is a chain complex and kills turnbacks (by Proposition 4.26 it suffices to show that it kills turnbacks from below). Since Q kills turnbacks, we have $Q \odot P_3 \simeq Q$ by Proposition 4.21. On the other hand, expanding Q into its chain groups and contracting terms with turnbacks gives

$$Q \odot P_3 \simeq (t^{-5} q^6 P_3 \xrightarrow{f} P_3) = \mathrm{Cone}(f)$$

for some $f : t^{-4}q^6 P_3 \to P_3$. The corresponding Ext group is isomorphic to $\mathbb{Z}$, generated by U_3, by Corollary 6.45; it follows that $f \simeq kU_3$ for some $k \in \mathbb{Z}$. It is more or less clear that the only possibility is $k = \pm 1$. One way to see this is to apply $\mathrm{Hom}^{\bullet,\bullet}(P_3, -)$ to the short exact sequence associated to the mapping cone $\mathrm{Cone}(kU_3)$, obtaining the short exact sequence

$$0 \to \mathrm{End}^{\bullet,\bullet}(P_3) \to \mathrm{Hom}^{\bullet,\bullet}(P_3, \mathrm{Cone}(kU_3)) \to t^{-5}q^6 \,\mathrm{End}^{\bullet,\bullet}(P_3) \to 0$$

The middle term is homotopy equivalent to $\mathrm{Hom}^{\bullet,\bullet}(1_3, \mathrm{Cone}(kU_3))$ by Proposition 5.1, which in turn is homotopy equivalent to $\mathrm{Hom}^{\bullet,\bullet}(1_3, Q)$, a bounded chain complex. This implies that the connecting differential ∂ in the associated long exact sequence is an isomorphism for all but finitely many i, j. It is not hard to see that $\partial = k[U_3] \circ (-) : \mathrm{Ext}^{i,j}(P_3, P_3) \to \mathrm{Ext}^{i-4,j+6}(P_3, P_3)$. Since infinitely many Ext groups are isomorphic to $\mathbb{Z}$ (see Corollary 6.47), the only possibility is $k = \pm 1$, i.e. $Q \simeq \mathrm{Cone}(U_3) \simeq Q_3$. This shows that $A_3 = Q_3$ is bounded. Now, for $n \geq 3$, the proof that A_n deformation retracts onto a bounded chain complex proceeds by induction on $n \geq 3$ as in the proof of Theorem 6.28. □

Motivated by this proposition and Proposition 6.52, we

Conjecture 7.24. *Let $\{Q_n\}$ be a family of symmetric projectors. If $n = 2r$ or $n = 2r + 1$ then $(Q_{n-r+1} \sqcup 1_{r-1}) \odot \cdots \odot (Q_{n-1} \sqcup 1_1) \odot Q_n$ deformation retracts onto a bounded chain complex $B_n \in \mathrm{Kom}^b(n)$. Further:*

1. *B_n is a Frobenius algebra object in the homotopy category $\mathrm{Kom}^b(n)_{/h}$.*
2. *The chain complex $C(D; \{B_n\})$ constructed in Definition 7.21 defines a functorial link invariant up to sign and homotopy.*

7.5 A special monoidal category

Throughout this section fix an integer $n \geq 1$. In Chapter 5 we were able to describe the algebra of merging and splitting copies of P_n in terms of a unital algebra structure on P_n. There were two aspects to our approach, namely (1) an explicit equivalence $P_n \odot P_n \simeq P_n$, and (2) the fact that precomposition with the unit map $\iota^{\odot k} : 1_n \to P_n^{\odot k}$ gave a homotopy equivalence $\mathrm{Hom}(P_n^{\odot k}, P_n)$. Our strategy for studying sheet algebra involving symmetric projectors is similar, and will involve

1. find a reasonable explicit description of the equivalence $Q_n \odot Q_n \simeq Q_n \oplus t^{1-2n}q^{2n}Q_n$
2. show that iterated precomposition with the "unit map" $\eta : P_{n-1} \sqcup 1 \to Q_n$ gives an isomorphism of groups $\mathrm{Ext}^{0,0}(Q_n^{\odot k}, Q_n) \cong \mathrm{Ext}^{0,0}(P_{n-1} \sqcup 1, Q_n) \cong \mathbb{Z}$.

In this section we introduce a certain monoidal category $\mathcal{A}$ which provides the right framework in which to study the quasi-local $\mathfrak{sl}_2$ link homology theory from Definition 7.21; ultimately we will show that Q_n is a Frobenius algebra object in $\mathcal{A}$. Actually we are already familiar with the counit and unit maps. Recall the notation from 7.6: put $I := P_{n-1} \sqcup 1_1$, $y := t^{1-2n}q^{2n}$. Note that a symmetric projector Q_n can be written as a convolution $Q_n = (yI \to N \to I)$ for some N, let $\eta : I \to Q_n$, $\varepsilon : y^{-1}Q_n \to I$ be the obvious chain maps of degree $(0,0)$.

It will turn out that η is the two sided unit with respect to the map $\mu : Q_n \odot Q_n \to Q_n$ which is projection onto the first summand of $Q_n \odot Q_n \simeq Q_n \oplus yQ_n$, and that a similar statement holds for ε. But we are getting ahead of ourselves. The fact that the unit and counit have source, respectively target, equal to the chain complex I suggests that the role of the monoidal identity here is played by $I := P_{n-1} \sqcup 1_1$ rather than 1_n.

Definition 7.25. Let $\mathcal{A} \subset \mathrm{Kom}^{-}(n)$ be the full subcategory consisting of complexes B such that $I \odot B \simeq B \simeq I \odot B$. Let $\mathcal{A}_{/h}$ denote the homotopy category, i.e. the category with the same objects as $\mathcal{A}$ but with morphism spaces given by degree zero chain maps mod the nulhomotopic ones.

Theorem 7.26. *The homotopy category* $(\mathcal{A}_{/h}, \odot, I)$ *is a monoidal category.*

Proof. Note that $(\mathrm{Kom}^{-}(n), \odot, 1_n)$ is monoidal, since it is the category of chain complexes on a monoidal category. Now, clearly $\mathcal{A}$ is closed under $\odot$. By definition of a monoidal category we must define

- a natural transformation $\mu_{A,B,C} : (A \odot B) \odot C \simeq A \odot (B \odot C)$ which is a homotopy equivalence for all $A, B, C \in \mathcal{A}$, and
- natural transformations $\phi_A : A \odot I_m \simeq A$, respectively $\psi_A : I \odot A \simeq A$

such that the relevant coherence conditions hold. The first coherence condition is the pentagon identity coming from the associahedron A_2, and the second is

$$\begin{array}{ccc} (A \odot I) \odot B & \xrightarrow{\mu} & A \odot (I \odot B) \\ & \searrow^{\psi_A \odot \mathrm{Id}_B} & \downarrow \mathrm{Id}_A \odot \phi_B \\ & & A \odot B \end{array} \tag{7.27}$$

We define the maps $\mu_{A,B,C}$ to be precisely the same as those coming from the monoidal structure on $Kom^{-}(n)$. With this definition the pentagon coherence condition holds on the nose, not just up to homotopy. Let $\iota : 1_n \to I = P_{n-1} \sqcup 1_1$ denote the inclusion of the degree zero chain group. If $A \in \mathcal{A}$, then the composition

$$A \xrightarrow{\cong} 1_n \odot A \xrightarrow{\iota \odot \mathrm{Id}_A} I \odot A$$

is a homotopy equivalence by Proposition 4.21, which applies since by definition of $\mathcal{A}$ we have $I \odot A \simeq A$. Let ϕ_A denote any homotopy inverse of this map. Similarly the composition $A \cong A \odot 1_n \to A \odot I$ is a homotopy equivalence, and we let $\psi_A : A \odot I_m \to A$ denote a homotopy inverse. To see that ϕ_A and ψ_A are natural transformations consider the following diagrams

$$\begin{array}{ccccc} A & \xrightarrow{\cong} & 1_n \odot A & \xrightarrow{\iota \odot \mathrm{Id}_A} & I \odot A \\ f \downarrow & & \mathrm{Id}_{1_n} \odot f \downarrow & & \mathrm{Id}_I \odot f \downarrow \\ B & \xrightarrow{\cong} & 1_n \odot B & \xrightarrow{\iota \odot \mathrm{Id}_B} & I \odot B \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{\cong} & A \odot 1_n & \xrightarrow{\mathrm{Id}_A \odot \iota} & A \odot I \\ f \downarrow & & f \odot \mathrm{Id}_{1_n} \downarrow & & f \odot \mathrm{Id}_I \downarrow \\ B & \xrightarrow{\cong} & B \odot 1_n & \xrightarrow{\mathrm{Id}_B \odot \iota} & B \odot I \end{array}$$

The left square of each diagram commutes by naturality of the isomorphisms $A \cong A \odot 1_n$ and $A \cong 1_n \odot A$ (recall that $(\mathrm{Kom}^-(n), \odot, 1_n)$ is monoidal). The right square of each diagram commutes by inspection; for example

$$(f \odot \mathrm{Id}_I) \circ (\mathrm{Id}_A \odot \iota) = (f \odot \iota) = (\mathrm{Id}_B \odot \iota) \circ (f \odot \mathrm{Id}_{1_n})$$

since $\odot$ is a dg bilinear functor and homological degree of ι is zero. The compositions along each row are homotopy equivalences, and the inverses are ϕ_A, ψ_A, etc., by definition. Inverting each row gives squares wich commute up to homotopy, which implies naturality of ϕ and ψ in the homotopy category.

It remains to check that the coherence condition (7.27) is satisfied. Let $A, B \in \mathcal{A}$ be arbitrary and consider the following diagram

$$\begin{array}{ccccc} A \odot B & \xrightarrow{\cong} & (A \odot 1_n) \odot B & \xrightarrow{((\mathrm{Id}_A \odot \iota) \odot \mathrm{Id}_B}} & (A \odot I) \odot B \\ = \downarrow & & \mu_{A,1_n,B} \downarrow & & \mu_{A,I,B} \downarrow \\ A \odot B & \xrightarrow{\cong} & A \odot (1_n \odot B) & \xrightarrow{(\mathrm{Id}_A \odot (\iota \odot \mathrm{Id}_B)}} & A \odot (I \odot B) \end{array}$$

The square on the left commutes by the triangle coherence condition (7.27) in the monoidal category $(\mathrm{Kom}^n(n), \odot, 1_n)$. The square on the right commutes by naturality of μ. The compositions along the rows are homotopy equivalences with inverses $\psi_A \odot \mathrm{Id}_B$, respectively $\mathrm{Id}_A \odot \phi_B$; inverting them gives a diagram which commutes up to homotopy, and gives the coherence relation (7.27) in the homotopy category. This completes the proof. □

The category $\mathcal{A}_{/h}$ has duals in a weak sense. Recall the functor $D : \mathrm{Kom}(n) \to \mathrm{Kom}(n)^\Pi$ defined by $D(A) = I \odot^\Pi A^\vee \odot^\Pi I$.

Proposition 7.28. *For $A, B \in \mathcal{A}$ we have a homotopy equivalence*

$$\theta_{A,B} : \mathrm{Hom}^{\bullet,\bullet}(A, B) \simeq \mathrm{Hom}^{\bullet,\bullet}(I, B \odot^{\Pi} D(A))$$

and this equivalence is natural in the following sense: for each pair of chain maps $f : X \to A$ and $g : B \to Y$ the square

$$\begin{array}{ccc} \mathrm{Hom}^{\bullet,\bullet}(A, B) & \xrightarrow{\theta_{A,B}} & \mathrm{Hom}^{\bullet,\bullet}(I, B \odot^{\Pi} D(A)) \\ {\scriptstyle g \circ (\ ) \circ f} \big\downarrow & & \big\downarrow {\scriptstyle (g \odot D(f)) \circ (\ )} \\ \mathrm{Hom}^{\bullet,\bullet}(X, Y) & \xrightarrow{\theta_{X,Y}} & \mathrm{Hom}^{\bullet,\bullet}(I, Y \odot^{\Pi} D(X)) \end{array}$$

commutes up to homotopy. We also have a natural equivalence

$$\mathrm{Hom}^{\bullet,\bullet}(A, B) \simeq \mathrm{Hom}^{\bullet,\bullet}(I, D(A) \odot^{\Pi} B)$$

Proof. Compute:

$$\begin{aligned} \mathrm{Hom}^{\bullet,\bullet}(A, B) &\overset{(1)}{\simeq} \mathrm{Hom}^{\bullet,\bullet}(A, I \odot^{\Pi} B \odot^{\Pi} I) \\ &\overset{(2)}{\cong} \mathrm{Hom}^{\bullet,\bullet}(I^{\vee} \odot A \odot I^{\vee}, B) \\ &\overset{(3)}{\simeq} \mathrm{Hom}^{\bullet,\bullet}(I \odot (I^{\vee} \odot A \odot I^{\vee}), B) \\ &\overset{(4)}{\cong} \mathrm{Hom}^{\bullet,\bullet}(I, B \odot^{\Pi} (I \odot^{\Pi} A^{\vee} \odot^{\Pi} I)) \\ &\overset{(5)}{=} \mathrm{Hom}^{\bullet,\bullet}(I, B \odot^{\Pi} D(A)) \end{aligned}$$

Let us explain the steps. By definition of $\mathcal{A}$, $B \in \mathcal{A}$ implies $B \simeq I \odot^{\Pi} B \odot^{\Pi} I$. In fact, since I is the monoidal identity of $\mathcal{A}_{/h}$ this equivalence is natural in the homotopy category. This gives (1). Theorem 4.15 gives isomorphisms

$$\mathrm{Hom}^{\bullet,\bullet}(X \odot Z^{\vee}, Y) \cong \mathrm{Hom}^{\bullet,\bullet}(X, Y \odot^{\Pi} Z) \cong \mathrm{Hom}^{\bullet,\bullet}(Y^{\vee} \odot X, Z)$$

which are natural in $X, Y, Z \in \mathrm{Kom}(n)$. This gives (2). Now, by projector absorbing (proposition 4.23) we have a homotopy equivalence $\pi : I \odot I^{\vee} \simeq I^{\vee}$. The equivalence (3) is given by precomposition with $\pi \odot \mathrm{Id}_A \odot \mathrm{Id}_{I^{\vee}} : I \odot I^{\vee} \odot A \odot I^{\vee}$, which is clearly natural in A and B. The isomorphism (4) is the natural isomorphism of Theorem

4.15, together with the observation that $(X \odot Y)^\vee \cong Y^\vee \odot^\Pi X^\vee$ naturally. Finally (5) holds by definition of $D(A)$.

Since each equivalence is natural in A and B, up to homotopy, it follows that the same is true of their composition $\phi_{A,B}$. This completes the proof. □

7.6 Hom complexes between symmetric projectors

Throughout this section, write Hom = $\mathrm{Hom}^{\bullet,\bullet}$ and End = $\mathrm{End}^{\bullet,\bullet}$. Fix an integer $n \geq 1$ and put $I := P_{n-1} \sqcup 1_1$, $y := t^{1-2n}q^{2n}$.

Recall Definition 6.19: suppose $R = \mathbb{Z}[x_1, \ldots, x_k]$ is a differential bigraded algebra with zero differential and bigrading $\deg(x_i) = (a_i, b_i) \in (2\mathbb{Z}) \times \mathbb{Z}$, and suppose $E \in \mathrm{Kom}(n)$ is such that $R \otimes E$ exists in $\mathrm{Kom}(n)$ and is isomorphic to a direct product (see equation (6.18)) of the complexes $x_1^{i_1} \cdots x_k^{i_k} \otimes E$. Then $R \vec{\otimes} E$ denotes any chain complex $(R \otimes E, d)$ whose differential (1) commutes with the obvious left R-action on $R \otimes E$ and (2) agrees with $\mathrm{Id}_R \otimes d_E$ up to higher degree terms.

If $M := \mathbb{Z}[x] \vec{\otimes} E$ is a chain complex over an abelian category, then we can recover E as the quotient $E \cong M/xM$. In general we can recover E as the homotopy quotient, or mapping cone: if $X : t^{a_i}q^{b_i}M \to M$ is the map given by left multiplication by x, then $\mathrm{Cone}(X)$ deformation retracts onto E. Applying this to the case of $P_n \simeq \mathbb{Z}[u_n] \vec{\otimes} Q_n$ gives the following:

Lemma 7.29. *Let* $V_n = \mathbb{Z}[u_1, \ldots, u_n] \vec{\otimes} \Lambda[w_1, \ldots, w_n]$ *be the chain complex from Theorem 6.37. We have* $\mathrm{Hom}(I, Q_n) \simeq V_n/(u_n V_n)$.

Proof. Theorem 4.15 gives an isomorphism $\mathrm{Hom}(I, Q_n) \cong \mathrm{Hom}(1_n, Q_n \odot^\Pi I^\vee)$. Since Q_n kills turnbacks, projector absorbing (proposition 4.21) gives $Q_n \odot^\Pi I^\vee \simeq Q_n$. Hence $\mathrm{Hom}(I, Q_n) \simeq \mathrm{Hom}(1_n, Q_n)$.

Let $Q_n' = \mathbb{Z}[u_1, \ldots, u_{n-1}] \vec{\otimes} C_n$ and $P_n' = \mathbb{Z}[u_n] \vec{\otimes} Q_n'$ be the periodic chain complexes from Theorem 6.28. Theorem 6.37 says that there is a deformation retract of differential bigraded $\mathbb{Z}[u_1,\ldots,u_n]$-modules $\mathrm{Hom}(1_n,P'_n)\to V_n$. Let $U_n : t^{2-2n}q^{2n}P'_n \to P'_n$ denote the chain map induced by the action of u_n, and note that $Q_n \simeq Q'_n \simeq \mathrm{Cone}(U_n)$. Thus

$$\mathrm{Hom}(I,Q_n)\simeq \mathrm{Hom}(1_n,\mathrm{Cone}(U_n))\cong \Big(t^{1-2n}q^{2n}\,\mathrm{Hom}(1_n,P'_n)\xrightarrow{L_{U_n}}\mathrm{Hom}(1_n,P_n)\Big)$$

where $L_{U_n} : t^{2-2n}q^{2n}\,\mathrm{Hom}(1_n,P'_n)\to \mathrm{Hom}(1_n,P'_n)$ is given by post-composition with U_n. Let us apply the deformation retract $(\pi,\sigma,h) : \mathrm{Hom}(1_n,P'_n)\to V_n$ to each term of the above, obtaining

$$\mathrm{Hom}(1_n,Q_n)\simeq \Big(t^{1-2n}q^{2n}V_n \xrightarrow{f} V_n\Big)$$

where $f(v)=\pi(U_n\circ\sigma(v)) = u_n\pi(\sigma(v)) = u_n v$ by $\mathbb{Z}[u_n]$-equivariance of π. That is to say,

$$\mathrm{Hom}(1_n,Q_n)\simeq \mathrm{Cone}(u_n)\cong V_n/u_nV_n$$

as desired. $\square$

Proposition 7.30. *Let $Q_n\in \mathrm{Kom}(n)$ be a symmetric projector. The group $\mathrm{Ext}^{i,j}(Q_n,Q_n)$ of chain maps $t^iq^jQ_n\to Q_n$ mod homotopy satisfies*

1. $\mathrm{Ext}^{k-i,i}(Q_n,Q_n)\cong 0$ *for all i, if $k<-1$.*
2. $\mathrm{Ext}^{k(1-2n,2n)}(Q_n,Q_n)\cong \mathbb{Z}$ *for $k\in\{0,-1\}$ and is zero otherwise.*
3. $\mathrm{Ext}^{0-i,i}(Q_n,Q_n)\cong \mathbb{Z}$ *for $i=0$ and is zero otherwise.*

Proof. Recall that Q_n is self dual and quasi-idempotent. I.e.

$$D(Q_n)\simeq y^{-1}Q_n \qquad \text{and} \qquad Q_n\odot Q_n\simeq (1+y)Q_n \tag{7.31}$$

We use the convention that for any Laurent polynomial $f(t,q)\in\mathbb{Z}[q,q^{-1},t,t^{-1}]$ and any chain complex $A\in\mathrm{Kom}(n)$, $f(t,q)A$ denotes the direct sum of copies of A, shifted in bidegree according f. Taken together with Proposition 7.28 we have an effective method for simplifying the hom-complexes between symmetric projectors:

$$\begin{aligned} \mathrm{End}(Q_n) &\simeq \mathrm{Hom}(I, Q_n \odot D(Q_n)) \\ &\simeq \mathrm{Hom}(I, Q_n \odot (y^{-1}Q_n)) \\ &\simeq \mathrm{Hom}(I, Q_n \oplus y^{-1}Q) \\ &= (1 + y^{-1})\,\mathrm{Hom}(I, Q_n) \end{aligned}$$

Since $y^{-1} = t^{2n-1}q^{-2n}$ and $\mathrm{Hom}(I, Q_n) \simeq V_n/u_nV_n$ (Lemma 7.29), we have

$$\mathrm{Ext}^{i,j}(Q_n, Q_n) \cong H^{i,j}(V_n/u_nV_n) \oplus H^{i+1-2n,j+2n}(V_n/u_nV_n)$$

The proposition follows from elementary reasoning involving bidegrees, as in the proof of Corollary 6.45. □

Using this lemma we can give a more explicit form for the inverse equivalences $Q_n \odot Q_n \simeq Q_n \oplus yQ_n$. This, in turn, will be used to give an explicit isomorphism $\mathrm{Ext}^{0,0}(Q_n^{\odot k}, Q_n) \cong \mathbb{Z}$. Both of these will be used heavily in our graphical calculus.

Lemma 7.32. *Let $Q_n \in \mathrm{Kom}(n)$ be a symmetric projector, and put $I = P_{n-1} \sqcup 1$, $y := t^{1-2n}q^{2n}$. Let $\varepsilon : y^{-1}Q_n \to I$ and $\eta : I \to Q_n$ be as in Definition 7.6. There exist maps $\mu : Q_n \odot Q_n \to Q_n$ and $\Delta : Q \to Q_n \odot Q_n$ such that the compositions*

$$Q_n \odot Q_n \xrightarrow{\begin{bmatrix} \mu \\ \mathrm{Id} \odot (y\varepsilon) \end{bmatrix}} Q_n \oplus (Q_n \odot yI) \simeq Q_n \oplus yQ_n$$

and

$$Q_n \oplus yQ_n \simeq (Q_n \odot I) \oplus yQ_n \xrightarrow{\begin{bmatrix} \mathrm{Id} \odot \eta & \Delta \end{bmatrix}} Q_n \odot Q_n$$

are inverse equivalences.

Proof. Let us write the symmetric Frenkel-Khovanov sequence as $E_\bullet = E_{1-2k} \to \cdots \to E_0$, where

- $E_0 = I$
- $E_{1-2n} = q^{2n} I$
- for $1-2n < k < 0$ the through degree of E_k is $\tau(E_k) < n$.

Then $Q_n \odot Q_n$ is a convolution of the form

$$\begin{aligned} Q_n \odot Q_n &\overset{(1)}{=} \left(t^{1-2n} Q_n \odot E_{1-2n} \to t^{2-2n} Q_n \odot E_{2-2n} \to \cdots \to t^{-1} Q_n \odot E_{-1} \to Q_n \odot E_0\right) \\ &\overset{(2)}{\simeq} (t^{1-2n} q^{2n} Q_n \odot I \to 0 \to \cdots \to 0 \to Q_n \odot I) \\ &\overset{(3)}{\simeq} (t^{1-2n} q^{2n} Q_n \xrightarrow{z} Q_n) \\ &\overset{(4)}{\cong} (t^{1-2n} q^{2n} Q_n \xrightarrow{0} Q_n) \end{aligned}$$

The first equivalence is obtained by contracting each term $Q_n \odot E_k$, $2-2n \leq k \leq -1$, which is contractible since Q_n kills turnbacks and $\tau(E_k) < n$ for $1-2n < k < 0$ (see the turnback killing Lemma 4.20). In the second equivalence Q_n absorbs $I = P_{n-1} \sqcup 1$, again since Q_n kills turnbacks (see Proposition 4.21). Now the equation $d^2 = 0$ in the third line implies that $z \in \mathrm{End}^{2-2n,2n}(Q_n)$ is a cycle; a calculation shows that the corresponding homology group vanishes, hence z must be a boundary. We can therefore replace z by zero up to higher length arrows, of which there can be none. This gives the last step.

As a bigraded object the right hand side of (1) is a direct sum $(Q_n \odot I) \oplus (Q_n \odot E_{2-2n}) \oplus \cdots \oplus (Q_n \odot E_{-1}) \oplus (Q_n \odot I)$ (omitting degree shifts), and in terms of this decomposition the homotopy equivalence between the right-hand side of (1) and the right-hand side of (4) is represented by a $2 \times (1-2n)$ matrix. Since each of the above simplifications preserves the convolution (i.e. "left-to-right") filtration, the top row of this matrix is simply $[\alpha, 0, \ldots, 0]$ where $\alpha : t^{1-2n} q^{2n} Q_n \odot I \to t^{1-2n} q^{2n} Q_n$ is is the standard homotopy equivalence. This is to say, the equivalence $Q_n \odot Q_n \to t^{1-2n} q^{2n} Q_n \oplus Q_n$ followed by projection onto the first summand factors as $\alpha \circ (\mathrm{Id}_{Q_n} \odot (y\varepsilon))$. Thus the equivalence $Q_n \odot Q_n \to t^{1-2n} q^{2n} Q_n \oplus Q_n$ is precisely as in the hypotheses. An entirely symmetric argument establishes the corresponding property for the inverse map. □

The following plays an essential role in our proof that Q_n is a Frobenius algebra object.

Proposition 7.33. *Pre-composition with $\eta^{\odot k}$ gives an isomorphism*

$$\mathrm{Ext}^{0,0}(Q^{\odot k}, Q) \to \mathrm{Ext}^{0,0}(I, Q) \cong \mathbb{Z}$$

Dually, post-composition with $\varepsilon^{\odot k}$ gives an isomorphism

$$\mathrm{Ext}^{0,0}(y^{-1}Q, (y^{-1}Q)^{\odot k}) \to \mathrm{Ext}^{0,0}(y^{-1}Q, I) \cong \mathbb{Z}$$

Proof. Recall that for simplicity, we denote $\mathrm{Hom}^{\bullet,\bullet}$ simply by Hom, etc. We prove part (1) of the lemma first in the case $k = 1$. I.e. we prove that pre-composition with $\eta : I \to Q_n$ gives an isomorphism in homology

$$(-) \circ [\eta] : \mathrm{Ext}^{0,0}(Q_n, Q_n) \to \mathrm{Ext}^{0,0}(I, Q_n) \cong \mathbb{Z}$$

Consider the diagram

$$\begin{array}{ccccc}
Q_n \odot^{\Pi} D(Q_n) & \xrightarrow{\simeq} & Q_n \odot (y^{-1}Q_n) & \xrightarrow{\simeq} & Q_n \oplus y^{-1}Q_n \\
\Big\downarrow {\scriptstyle \mathrm{Id}\odot D(\eta)} & & \Big\downarrow {\scriptstyle \mathrm{Id}\odot\varepsilon} & & \Big\downarrow {\scriptstyle \pi_1} \\
Q_n \odot^{\Pi} D(I) & \xrightarrow{\simeq} & Q_n \odot I & \xrightarrow{\simeq} & Q_n
\end{array}$$

The horizontal arrows on the left are given by self-duality of Q_n and I (theorem 7.7), the top right horizontal arrow is quasi-idempotency (theorem 7.11), the bottom right arrow is projector absorbing (proposition 4.21), and the right-most vertical map is projection onto the first summand. The first square commutes up to homotopy by Theorem 7.7, and the second by Lemma 7.32. Applying $\mathcal{F} := \mathrm{Hom}(I, -)$ to this diagram gives a diagram which commutes up to homotopy:

$$\begin{array}{ccccccc}
\mathrm{End}(Q_n) & \xrightarrow{\simeq} & \mathcal{F}(Q_n \odot^{\Pi} D(Q_n)) & \xrightarrow{\simeq} & \mathcal{F}(Q_n \odot (y^{-1}Q_n)) & \xrightarrow{\simeq} & \mathcal{F}(Q_n \oplus y^{-1}Q_n) \\
\Big\downarrow {\scriptstyle \mathcal{F}(\eta)} & & \Big\downarrow {\scriptstyle \mathcal{F}(\mathrm{Id}\odot D(\eta))} & & \Big\downarrow {\scriptstyle \mathcal{F}(\mathrm{Id}\odot\varepsilon)} & & \Big\downarrow {\scriptstyle \mathcal{F}(\pi_1)} \\
\mathrm{Hom}(I, Q_n) & \xrightarrow{\simeq} & \mathcal{F}(Q_n \odot^{\Pi} D(I)) & \xrightarrow{\simeq} & \mathcal{F}(Q_n \odot I) & \xrightarrow{\simeq} & \mathcal{F}(Q_n)
\end{array} \tag{7.34}$$

The horizontal arrows on the left are from Proposition 7.28, and the first square commutes by naturality of the equivalence in that proposition. Consider the composition along the top row:

$$\mathrm{End}(Q_n) \simeq \mathrm{Hom}(I, Q_n \oplus y^{-1}Q_n) \cong \mathrm{Hom}(I, Q_n) \oplus y^{-1}\,\mathrm{Hom}(I, Q_n) \tag{7.35}$$

Since $y = t^{1-2n}q^{2n}$, taking degree $(0,0)$ homology groups gives

$$\mathrm{Ext}^{0,0}(Q_n, Q_n) \cong \mathrm{Ext}^{0,0}(I, Q_n) \oplus \mathrm{Ext}^{1-2n,2n}(I, Q_n) \cong \mathbb{Z} \oplus 0$$

This latter isomorphism is by Proposition 7.30. Therefore, the map from the top-left corner of (7.34) to the bottom-right corner is an isomorphism of degree $(0,0)$ homology groups. Since the maps along the bottom row are homotopy equivalences, this shows that $(-)\circ\eta$ induces an isomorphism on degree $(0,0)$ homology groups. This proves part (1) of the lemma in case $k = 1$.

If $k > 1$, we have $Q_n^{\odot k} \simeq (1+y)^{k-1}Q_n$. By equivariance of $\mathrm{Hom}^{\bullet,\bullet}$ with respect to the shift functors, we have $\mathrm{Hom}(yQ_n, Q_n) \cong y^{-1}\,\mathrm{Hom}(Q_n, Q_n)$. Hence

$$\mathrm{Hom}(Q_n^{\odot k}, Q_n) \simeq \mathrm{Hom}((1+y)^{k-1}Q_n, Q_n) \cong (1+y^{-1})^{k-1}\,\mathrm{End}(Q_n, Q_n)$$

Since $y^{-1} = t^{2n-1}q^{-2n}$, taking degree $(0,0)$ homology groups gives

$$\mathrm{Ext}^{0,0}(Q_n^{\odot k}, Q_n) \cong \bigoplus_{i=0}^{k-1} \binom{k-1}{i} \mathrm{Ext}^{i(1-2n,2n)}(Q_n, Q_n) \tag{7.36}$$

By Proposition 7.30, the only nonzero summand one the right-hand side above is the one corresponding to $i = 0$. That is to say, $\mathrm{Ext}^{0,0}(Q_n^{\odot k}, Q_n) \cong \mathrm{Ext}^{0,0}(Q_n, Q_n)$ is contributed to by the unique unshifted Q_n summand of $Q_n^{\odot k}$. By iterating Lemma 7.32 we see that $\mathrm{Id}_{Q_n} \odot \eta^{\odot(k-1)} : Q_n \simeq Q_n \odot I^{\odot(k-1)} \to Q_n^{\odot k}$ is the inclusion of this summand, hence precomposition with this map gives an isomorphism of degree $(0,0)$ homology groups

$$(-)\circ[\mathrm{Id}_{Q_n} \odot \eta^{\odot(k-1)}] : \mathrm{Ext}^{0,0}(Q_n^{\odot k}, Q_n) \xrightarrow{\cong} \mathrm{Ext}^{0,0}(Q_n, Q_n)$$

Following this map with $(-)\circ[\eta] : \mathrm{Ext}^{0,0}(Q_n, Q_n) \to \mathrm{Ext}^{0,0}(I, Q_n)$ gives, by the $k = 1$ one case of part (1) of the lemma (already proven), an isomorphism of homology groups

$$(-)\circ[\eta^{\odot k}] : \mathrm{Ext}^{0,0}(Q_n^{\odot k}, Q_n) \xrightarrow{\cong} \mathrm{Ext}^{0,0}(I, Q_n) \cong \mathbb{Z}$$

This completes the proof of part (1).

The proof of part (2) is entirely similar. Put $\bar{Q}_n := y^{-1}Q_n$, so that $\varepsilon : \bar{Q}_n \to I$ is a degree $(0,0)$ chain map. Self-duality and quasi-idempotency of Q_n imply that $D(\bar{Q}_n) \simeq y\bar{Q}_n$ and $\bar{Q}_n^{\odot 2} \simeq \bar{Q}_n \oplus y^{-1}\bar{Q}_n$. Compute

$$\mathrm{Hom}(\bar{Q}_n, \bar{Q}_n^{\odot k}) \simeq \mathrm{Hom}(\bar{Q}_n, (1+y^{-1})^{k-1}\bar{Q}_n) \cong (1+y^{-1})^{k-1}\,\mathrm{End}(\bar{Q}_n)$$

By an analogue of (7.36) and the comments following, we see that the degree $(0,0)$ homology group of $\mathrm{Hom}(\bar{Q}_n, \bar{Q}_n^{\odot k})$ is contributed to only by the unique unshifted $\bar{Q}_n$ summand of $\bar{Q}_n^{\odot k} \simeq (1+y^{-1})^{k-1}\bar{Q}_n$. By iterating Lemma 7.32 we see that the projection onto this summand can be assumed to be the composition

$$\bar{Q}_n^{\odot k} \xrightarrow{\mathrm{Id}\odot\varepsilon^{\odot(k-1)}} \bar{Q}_n \odot I^{\odot(k-1)} \simeq \bar{Q}_n.$$

Hence post-composition with $\mathrm{Id}\odot\varepsilon^{\odot(k-1)}$ induces an isomorphism in homology

$$[\mathrm{Id}\odot\varepsilon^{\odot(k-1)}]\circ(-) : \mathrm{Ext}^{0,0}(\bar{Q}_n, \bar{Q}_n^{\odot k}) \xrightarrow{\cong} \mathrm{Ext}^{0,0}(\bar{Q}_n, \bar{Q}_n)$$

Therefore part (2) of the lemma will follow from the $k = 1$ case. In this case, we have the following diagram, where $\mathcal{F} = \mathrm{Hom}(I, -)$ as before:

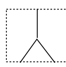
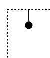
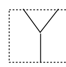
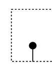
$$\begin{array}{ccccccc}
\mathrm{End}(\bar{Q}_n) & \xrightarrow{\simeq} & \mathcal{F}(D(\bar{Q}_n)\odot^{\Pi}\bar{Q}_n) & \xrightarrow{\simeq} & \mathcal{F}(y\bar{Q}_n\odot\bar{Q}_n) & \xrightarrow{\simeq} & \mathcal{F}(y\bar{Q}_n\oplus\bar{Q}_n) \\
\downarrow{\scriptstyle \mathcal{F}(\varepsilon)} & & \downarrow{\scriptstyle \mathcal{F}(\mathrm{Id}\odot\varepsilon)} & & \downarrow{\scriptstyle \mathcal{F}(\mathrm{Id}\odot\varepsilon)} & & \downarrow{\scriptstyle \mathcal{F}(\pi_1')} \\
\mathrm{Hom}(\bar{Q}_n, I) & \xrightarrow{\simeq} & \mathcal{F}(D(\bar{Q}_n)\odot^{\Pi} I) & \xrightarrow{\simeq} & \mathcal{F}(y\bar{Q}_n\odot I) & \xrightarrow{\simeq} & \mathcal{F}(y\bar{Q}_n)
\end{array} \tag{7.37}$$

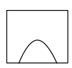
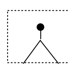
which commutes up to homotopy. Our argument from part (1) implies that $\mathcal{F}(\pi_1')$ is an isomorphism of degree $(0,0)$ homology groups, hence so is $\varepsilon\circ(-)$. This proves part (2) of the lemma. □

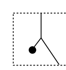
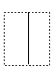
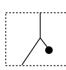
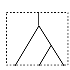
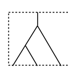
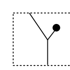
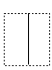
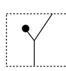
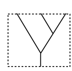
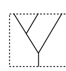
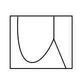
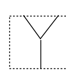
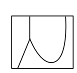
7.7 Sheet algebra for symmetric projectors: Q_n is a Frobenius algebra

Let us recall the definition of a Frobenius algebra object of a monoidal category. Here we allow the counit and comultiplication to have nonzero degree so, strictly speaking we have to work over a (differential) graded monoidal category.

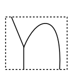
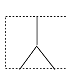
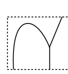
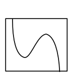
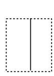
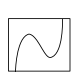
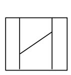
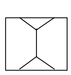
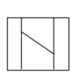
Definition 7.38. Let $(\mathcal{A}, \odot, I)$ be a graded monoidal category. Suppose we have an object $A \in \mathcal{A}$ together with morphisms $\mu : A \odot A \to A$, $\eta : I \to A$, $\Delta : A \to A \odot A$, and $\varepsilon : A \to I$. Graphically we denote these morphisms as

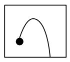
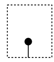
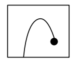
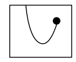
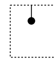
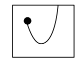
$\mu =$  , $\eta =$  , $\Delta =$  , $\varepsilon =$ 

Assume that μ and η have degree zero; Δ and ε may have some nonzero degree v, respectively $-v$. Put  $:=$  . Say that $(A, \mu, \eta, \Delta, \varepsilon)$ is a *Frobenius algebra object* over $\mathcal{A}$ if

1. (A, μ, η) is an associative algebra object with two-sided unit η, i.e.  $=$  $=$  and  $=$  .
2. (A, Δ, ε) is a graded coalgebra with graded two-sided counit ε, i.e.  $=$  $= (-1)^v$  and  $= (-1)^v$  .
3. Δ is dual to μ, in the sense that  $=$  $=$  .

Remark 7.39. Suppose $(A, \mu, \eta, \Delta, \varepsilon)$ is a Frobenius algebra object. One also has the following relations, which are easy to check.

-  $=$  $= (-1)^v$  .
-  $=$  $= (-1)^v$  .
-  $=$  $= (-1)^v$  .

-  =  =  and  =  $= (-1)^v$  .

In particular, graphs which are isotopic rel boundary correspond to homotopic maps, up to a sign.

We are ready to prove our main result on the quasi-local sheet algebra.

Theorem 7.40. *There exist cycles $\mu \in \mathrm{Hom}^{0,0}(Q_n \odot Q_n, Q_n)$ and $\Delta \in \mathrm{Hom}^{1-2n,2n}(Q_n, Q_n \odot Q_n)$ uniquely characterized up to homotopy by the property that η is a right unit for μ, respectively ε is a right counit for Δ (up to homotopy). These maps make $(Q_n, \mu, \eta, \Delta, \varepsilon)$ into a Frobenius algebra object in the homotopy category of chain complexes preserved by $I = P_{n-1} \sqcup 1$.*

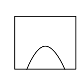
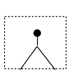
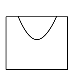
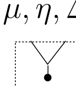
Proof. Let us establish uniqueness first: suppose μ and μ' are degree $(0,0)$ chain maps $Q_n \odot Q_n \to Q_n$ such that $\mu \circ (\mathrm{Id} \odot \eta) \simeq \mu' \circ (\mathrm{Id} \odot \eta) \simeq \mathrm{Id}_{Q_n}$. Then $(\mu' - \mu) \circ (\eta \odot \eta) \simeq \eta - \eta \simeq 0$, and Proposition 7.33 implies that $\mu' - \mu \simeq 0$, hence $\mu \simeq \mu'$. A similar argument establishes uniqueness of Δ up to homotopy.

For existence, let μ and Δ be as in Lemma 7.32. That is to say, μ is the composition

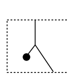
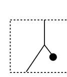
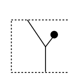
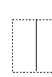
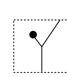
$$\mu : Q_n \odot Q_n \simeq Q_n \oplus yQ_n \to Q_n$$

and Δ is the composition

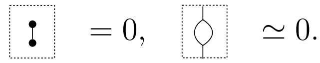
$$\Delta : yQ_n \to Q_n \oplus yQ_n \simeq Q_n \odot Q_n$$

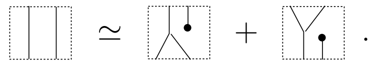
Now, let $\mathcal{A} = (\mathrm{Kom}^-(n), \odot, I)$ be as in Definition 7.25. Throughout the rest of the proof adopt the graphical notation for morphisms in $\mathcal{A}_{/h}$, which is monoidal by Theorem 7.26. Denote $\mu, \eta, \Delta, \varepsilon$ graphically as in Definition 7.38, and put  :=  and  :=  . Note that that Δ and ε have odd homological degree.

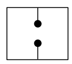
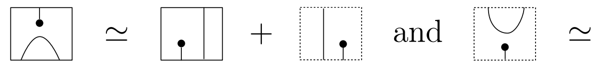
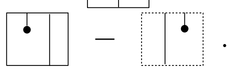
The proof will amount to establishing the following diagrammatic relations. Relations (1), (2), and (6) are equivalent to Q_n being a Frobenius algebra as stated.

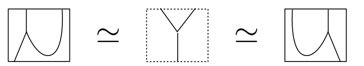
1. (co)unit relations:  $\simeq$  $\simeq$  and  $\simeq$  $\simeq -$  .

2. (co)associativity relations: 

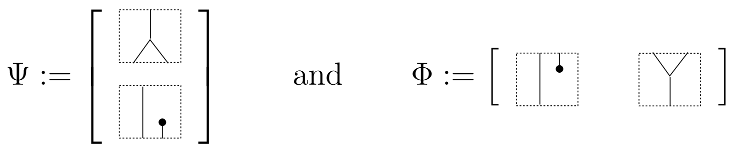
3. orthogonality relations: 

4. decomposition of identity: 

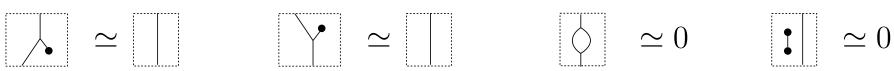
5. $\partial :=$  is a (co)derivation: 


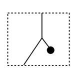
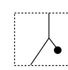
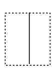
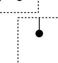
6. Isotopy relations: 

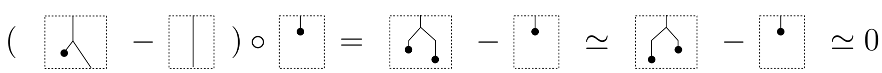
Define

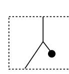
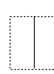
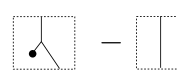
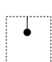


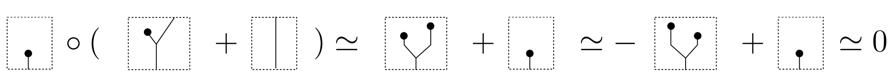
By Lemma 7.32 Ψ and Φ are homotopy inverses $Q_n \odot Q_n \simeq Q_n \oplus t^{1-2n}q^{2n}Q_n$. Expanding $\Psi \circ \Phi \simeq \mathrm{Id}$ into components gives us the relations

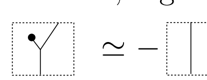
 (7.41)

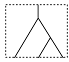
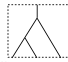
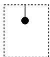
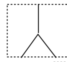
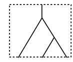
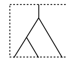
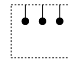
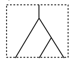
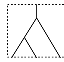
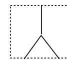
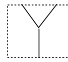
Now, note that  is a degree $(0, 0)$ chain map $Q_n \to Q_n$. To see that  $\simeq$  we need only see that they are homotopic upon pre-composing with $\eta =$ , by Proposition 7.33. Compute:

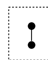
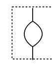


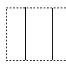
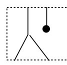
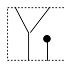
Here we have used that distant maps of even homological degree commute, together with the first relation in (7.41), i.e.  $\simeq$ . This implies  $\simeq 0$, hence  is a two-sided unit. Similarly, observe that

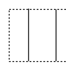
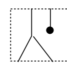
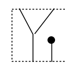
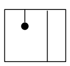


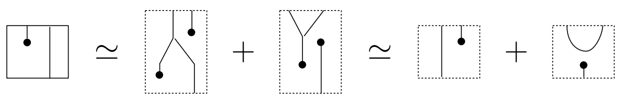
In the second equivalence we used that odd degree maps anti-commute, together with the second relation in (7.41). Proposition 7.33 implies that . This proves (1).

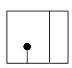
For (2), note that  −  is a degree $(0,0)$ chain map $Q_n^{\odot 3} \to Q_n$. Since  is two-sided unit for , we have ( − ) $\circ$  $\simeq 0$, hence Proposition 7.33 implies that  −  $\simeq 0$. That is,  is associative. A similar argument establishes that  is graded coassociative. This proves (2).

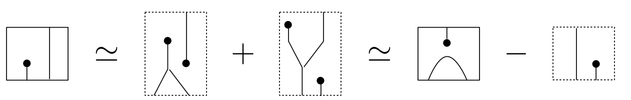
For relation (3), note that Q_n can be written as a convolution $Q_n = (yI \to N \to I)$. η is the inclusion of the I summand and ε is projection onto the yI summand. Their composition is obviously zero, which is  $= 0$. The relation  $\simeq 0$ was already established in 7.41. This proves (3).

The relation  $\simeq$  +  is just a restatement of $\mathrm{Id}_{Q_n \odot Q_n} \simeq \Phi \circ \Psi$. This establishes (4).

Compose the decomposition of identity  $\simeq$  +  from below with  to obtain

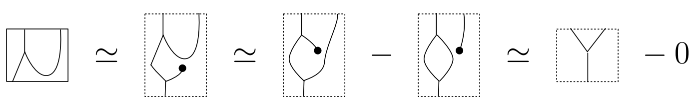


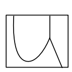
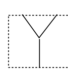
and from above with  to obtain



Rearranging gives (5).

Relation (6) follows from the calculation



In the first equivalence we used the counit relation (1), in the second we used (5), and in the third we used the unit relation (1) and the bubble relation (3). A similar argument shows that  $\simeq$ . This proves (6), and completes the proof. □

7.8 Toward functoriality

In this section we do not include any proofs, but only indicate a potential direction for future work. Recall the notation $P_n(m_1, \ldots, m_r) = (Q_{m_1} \sqcup 1_{n-m_1}) \odot \cdots \odot (Q_{m_r} \sqcup$

1_{n-m_r}) from Definition 7.18. Recall the chain complex $C(D; \{A_n\}$ associated to a suitably marked link diagram, constructed in Definition 7.21. Each A_n is of the form $P_n(m_1, \dots, m_r)$ for some sequence $1 \leq m_1, \dots, m_r \leq n$, and the markings tell us where to place copies of A_n on a certain cabling of D (see figure 3.2). Now, a cobordism between links may be described as a sequence (or *movie*) of marked diagrams. The object represented by such a movie can be regarded as a link cobordism which is marked with a certain graph which describes the histories of the marked points (merges, splits, births, deaths, sliding through crossings, each inherited from a corresponding map involving symmetric projectors). As described in the introduction of section 5, away from the markings the surface corresponds to some parallel copies of a morphism in Bar-Natan's categories. To prove invariance of the corresponding map, it will be necessary to study the local relations, i.e. sheet algebra satisfied by these graphs. Isotopy relations among such graphs implies that certain products $P_n(m_1, \dots, m_r)$ of the Q_m are Frobenius algebra objects in an appropriate monoidal category. Let us focus on establishing this Frobenius algebra property. Fix throughout the rest of this section an integer $n \geq 1$.

Definition 7.42. For each $1 \leq m \leq n$, put $I_m := P_m \sqcup 1_{n-m}$, and let $\mathcal{A}_m \subset \mathrm{Kom}^{\leq 0}(\mathcal{TL}_n)$ be the full subcategory consisting of complexes X which are preserved by I_m, i.e. $X \simeq I_m \odot X \odot I_m$.

In other words, $\mathcal{A}_m$ is precisely the category of complexes on which $I_m = P_m \sqcup 1_{n-m}$ acts as a unit. If $1 \leq k \leq m$, then Proposition 4.23 implies that $I_k \odot I_m \simeq I_m \simeq I_m \odot I_k$, hence $\mathcal{A}_k \supset \mathcal{A}_m$.

An argument similar to that of Proposition 7.25 proves that:

Proposition 7.43. *The homotopy categories of* $(\mathcal{A}_m, \odot, I_m)$ *are monoidal.* □

Suppose $(\mathcal{A}, \odot, 1_{\mathcal{A}})$ and $(\mathcal{B}, \odot, 1_{\mathcal{B}})$ are monoidal categories, and $\mathcal{A} \subset \mathcal{B}$. If $A \in \mathcal{A}$ is a Frobenius algebra in $\mathcal{B}$, then A is automatically a Frobenius algebra in $\mathcal{A}$. For example the counit $A \to 1_{\mathcal{A}}$ can be defined to be the composition $A \cong 1_{\mathcal{A}} \odot A \to$

$1_{\mathcal{A}} \odot 1_{\mathcal{B}} \cong 1_{\mathcal{A}}$. The first isomorphism exists because $A \in \mathcal{A}$, and the last isomorphism exists since $1_{\mathcal{B}}$ is the monoidal identity inside the *larger* category $\mathcal{B}$, hence fixes $1_{\mathcal{A}}$. So given a Frobenius algebra object $A \in \mathcal{A}$, it is natural to ask if there is a larger category $\mathcal{B} \supset \mathcal{A}$ in which A is a Frobenius algebra. For example in Theorem 7.40 we proved:

Theorem 7.44. *$Q_m \sqcup 1_{n-m} \in \mathcal{A}_m$ is a Frobenius algebra object in the larger category $\mathcal{A}_{m-1}$, or more precisely the homotopy category of $(\mathcal{A}_{m-1}, \odot, I_{m-1})$.* □

This suggests a method for proving the following:

Conjecture 7.45. *For each $1 \leq m \leq n$, $P_n(m, m+1, \ldots, n)$ is a Frobenius algebra object in the homotopy category of $(\mathcal{A}_{m-1}, \odot, I_{m-1})$.*

For example, one can construct the counit $\varepsilon : P_n(m, m+1, \ldots, n) \to I_{m-1}$ inductively as follows:

- if $m = 1$, then $P_n(n) \simeq Q_n$ is the usual symmetric projector, and the counit $Q_n \to I_{n-1} = P_{n-1} \sqcup 1$ has been constructed already (see Theorem 7.40).

- Put $A := Q_m \sqcup 1_{n-m}$ and $B := P_n(m+1, m+2, \ldots, n)$ so that $P_n(m, m+1, \ldots, n) \simeq A \odot B$. Assume by induction that we have a counit map $\varepsilon_1 : B \to I_m$. By projector absorbing we have $A \odot I_m \simeq P_n(m)$. Also, the usual counit for symmetric projectors gives us $\varepsilon_2 : A \to I_{m-1}$. Then we can define $\varepsilon : P_n(m, m+1, \ldots, n) \to I_{m-1}$ to be the composition

$$A \odot B \xrightarrow{\mathrm{Id} \odot \varepsilon_1} A \odot I_m \simeq \quad A \xrightarrow{\varepsilon_2} I_{m-1}$$

The following is an abstraction of this idea:

Proposition 7.46. *Suppose $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$ are inclusions of monoidal categories (same monoid, but possibly different monoidal identities). Let $A \in \mathcal{A}$ be a Frobenius algebra in $\mathcal{B}$ and $B \subset \mathcal{B}$ be a Frobenius algebra in $\mathcal{C}$. If we have an isomorphism $\tau : B \odot A \to$*

$A \odot B$ which is compatible with the structure maps, then $A \odot B$ is a Frobenius algebra in $\mathcal{C}$. □

So to prove our conjecture 7.45, we need to show that the Frobenius algebra structure maps for the various $Q_m \sqcup 1_{n-m}$ are compatible up to homotopy with the equivalences which commute the symmetric projectors past one another. Commutativity of the symmetric projectors is the statement $P_n(k,m) \simeq P_n(m,k)$ for all $1 \leq k, m \leq n$, and compatibility with the Frobenius structure maps amounts to commutativity diagrams such as

$$\begin{array}{ccccc} P_n(m,m,k) & \longrightarrow & P_n(m,k,m) & \longrightarrow & P_n(k,m,m) \\ \downarrow & & & & \downarrow \\ P_n(m,k) & & \longrightarrow & & P_n(k,m) \end{array}$$

where the horizontal maps commute factors and the vertical maps are given by the product $Q_m \odot Q_m \to Q_m$. It will be profitable to compute the complex of morphisms between the $P_n(m_1, \ldots, m_r)$, the first step toward which is the following generalization of Lemma 7.29:

Lemma 7.47. *Let $1 \leq m_1 < \cdots < m_r = n$, and let $V_n = \mathbb{Z}[u_1, \ldots, u_n]\vec{\otimes}\Lambda[w_1, \ldots, w_n]$ be the chain complex from Theorem 6.37. Then*

$$\mathrm{Hom}^{\bullet,\bullet}(1_n, P_n(m_1, \ldots, m_r)) \simeq V_n/(u_{m_1}V_n + \cdots + u_{m_r}V_n)$$

□

The next step will be to develop a graphical calculus similar to that in the proof of Theorem 7.40, in which we allow strands labelled by different complexes Q_m ($1 \leq m \leq n$), and which is rich enough to prove that the $P_n(m, m+1, \ldots, n)$ are Frobenius.

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