Partial Trace Ideals, The Conductor and Berger's Conjecture

Sarasij Maitra
Kolkata, India

M.Math, Indian Statistical Institute Kolkata, 2016
B.Math. Indian Statistical Institute Bangalore, 2014

A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia
May, 2022

Contents

Acknowledgements

I will try my best to express my sincerest gratitude as concisely as possible. If you are reading this and even though I haven't listed your name explicitly, please know that I am deeply grateful to you.

Let me start by thanking my advisor, Craig Huneke. I doubt I would ever have such an impact on our field as you did, but I would be very proud of myself even if I can become an advisor like you. Thank you Sir for teaching me so much. I repeat what I have already said, "You are the smartest person I have ever met and it has been so much fun working with you!" I have had tremendous support from Vivek Mukundan; at a time when I was still mustering courage to ask questions to Craig, you provided me a safe and encouraging space to come up with questions and had the patience of dealing with them. I hope to be as supportive and encouraging to my students and peers as I embark on my postdoctoral journey. I am very grateful to Thomas Polstra, Hailong Dao and Eloísa Grifo for encouraging me on this Commutative Algebra journey. It all started with courses in India under Manish Kumar, Mrinal Kanti Das and Neena Gupta. Thank you David Wu for helping me in understanding Cryptography which helped immensely whenever I felt very demotivated in mathematics. I am very grateful to all my teachers in school and also professors in college who believed in me (and more to the ones who did not). I would like to convey sincerest thanks to all my thesis committee members as well.

Let me move on beyond the mathematics part. As John Von Neumann had pointed out: "If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is." I am indebted to my parents and my brother Monosij. Without the latter's voice of reason, I would not have stayed sane and rational (it played a big impact in me developing a deep interest in teaching as well). You introduced me to Inner Engineering and Sadhguru and the experience has changed my life. I will not apologize for the numerous intellectual and spiritual debates (sometimes really heated ones) that I have had with you and hope to continue having. My mother's efforts at trying to provide us the best resources that she could manage during my school years while simultaneously maintaining a stable state to my father's declining health conditions, continue to inspire me. This thesis is dedicated to you, my lovely mother.

Speaking of friends, it would be highly irrational if I attempt to list all the names. But Debanjan, Ujjwal and Subham: I really appreciate the fact that you are still actively in my life even after such a long time has passed since we first met. Peter, my office mate and batchmate, thank you for being my closest international math friend. Badri and Ross, thank you for joining this tiny math club in the last two years. Zia (my first friend in Charlottesville), Parijat, Yaseswini, Shreyashi, Khairul, Asanka, Rachna: you guys have played significant roles in my life and have made my stay in Charlottesville healthy and lively. Besides being emotionally supportive, you have always very warmly fed me as well. It turns out that food is quite important for research ideas to take shape, and this thesis would not have been possible without your cooking.

Rucha, thank you for bringing so much light and happiness to my life in the final year of my PhD. I look forward to whatever awaits in the journey ahead.

I am thankful to the universe (possibly multiverse) for providing all the necessary and sufficient conditions for me to stay alive happily and peacefully.

Abstract

Let k be a perfect field and R be a one dimensional complete local k-algebra which is a domain. Under these hypotheses, there is a long-standing conjecture of R.W. Berger which concerns the universally finite module of differentials, $\Omega_{R/k}$. The torsion submodule of $\Omega_{R/k}$ is conjectured to be non-zero if and only if R is a regular local ring. In this thesis, we study this problem via two approaches. The conductor ideal of R turns out to be central in both the attacks.

The first and main approach is to introduce and study a numerical invariant, $\mathrm{h}(M)$, where M is a finitely generated module with positive rank. This definition is based on a partial trace ideal of this module and captures information regarding the torsion part of M whenever M has rank one. We establish a relationship with the conductor ideal of R. We then apply this to $\Omega_{R/k}$ to prove the truth of the conjecture provided $\mathrm{h}(\Omega_{R/k})$ has a suitable upper bound depending on the embedding dimension of R. In particular, we generalize an old result of G. Scheja and also illustrate with examples that some further cases of the conjecture can be settled using this approach. We provide an efficient algorithm to compute $\mathrm{h}(\Omega_{R/k})$.

The second approach, which is joint work with Craig Huneke and Vivek Mukundan, is to study the ring via valuations induced from the integral closure. This approach leads to settling further new cases of the conjecture in terms of the valuations of the conductor ideal of R. In particular, we extend the work of Güttes.

Chapter 0

Introduction

Given a finitely generated module M over a Noetherian integral domain, the study of the vanishing of the torsion submodule of M has been a topic of interest among algebraists. There exist quite a few questions in specific setups; for instance, the Huneke-Wiegand Conjecture [HW94], the Berger Conjecture [Ber63], etc. Though these questions have been studied extensively and have been proven in many special cases, a complete answer is still elusive.

This dissertation is primarily aimed at discussing an approach to attack the problem of vanishing of torsion for rank one modules over a one dimensional Noetherian local domain. We first interpret the torsion submodule as the kernel of any non-zero homomorphism in the dual of the module. This naturally leads us towards introducing an invariant for any finitely generated module M, in terms of the *colengths* of the partial images of M in the underlying one dimensional domain R. The collection of all such images forms the *trace ideal* of M and consequently, we refer to such a partial image as a *partial trace ideal* of M. We establish an upper bound on this invariant that helps us to conclude the non-vanishing of the torsion submodule. We prove various properties of this invariant and link it with the conductor ideal of R.

We then apply this study to establish some cases of Berger's Conjecture. It centers around the *universally finite module of differentials*. In order to get an initial idea about the importance of the module of differentials, we can resort to the following statements made by E. Kunz [Käh11, Why Kähler Differentials]:

> "Right from the start, F. K. Schmidt taught us the 'universal' module of differentials which we soon called in our discussions 'Kähler's' module of differentials. Kähler had constructed the differentials of a commutative algebra R/K ... A different approach to the universal module of differentials of an algebra R/K was used by Cartier ... If one takes $R \otimes_K R \to R$ ($a \otimes b \mapsto ab$), then I/I^2 with the derivation d given by $da = a \otimes 1 - 1 \otimes a + I^2$ for $a \in R$ is the universal module of differentials of R/K. ... I am not in the position to trace back the sequence of ideas which may have led Kähler to his algebraic differential calculus. ... The desire for such a calculus, purely in algebraic terms, but allowing to transfer differentiation processes of analysis to algebra, has a long history and has led to many approaches in various degrees of generality ..."

We should mention here that Kunz's book *Kähler Differentials* [Kun86] is one of the classic texts to learn about the module of differentials in generality. We refer the reader to Appendix A.2 of this dissertation for some more quick details regarding the *universally finite module of differentials.*

Around 1963, R. Berger conjectured that the torsion submodule of the universally finite module of differentials is zero if and only if the underlying one dimensional reduced k-algebra is regular (see Chapter 3 for the statement). In his initial paper [Ber63], R. Berger motivates his conjecture as well as the importance of the module of differentials in the following way (when roughly translated from German):

> "Since E. Kähler has introduced the module of differentials, this notion has proved to be extremely useful in many parts of commutative algebra and algebraic geometry. While the theory of singularity-free algebraic manifolds has received new impulses, so far differential modules for algebraic manifolds with singularities, in particular differential modules of non-regular local rings have not been examined in detail. The only general results are the knowledge of the minimum number of generators of the differential module of any local ring and the resulting criterion: A geometric local ring is regular if and only if its differential module is free ... The question immediately arises, which structure this differential module has in the non-regular case. Given the well-known difficulties that one always encounters when singularities occur, a general, exhaustive answer to this question is not to be expected any time soon, and it makes sense to start our investigations with the simplest case, namely the local rings

> of singular points of algebraic curves. ... Numerous examples suggest that the non-regularity of R in the case of dimension 1 is coupled with the occurrence of torsion in differential module."

We discuss this conjecture in more detail in Chapter 3 and also quickly survey some of the established cases, before we apply our techniques to get some further cases. However, the question still stands open even after almost 60 years have passed since its origin.

The dissertation also discusses another approach to this conjecture with the conductor ideal again playing a critical role. We establish further new cases via this new attack.

Some of the results in Chapters 2 and 3 appear in [Mai22]. Most of the results in Chapter 4 appear in [HMM21].

0.1 Structure of the dissertation

We now briefly describe the structure of this thesis. Chapter 1 and Appendix A collect the basic background results that we use throughout this thesis. New results can be found Chapter 2 onwards.

0.1.1 Chapter 1

In Chapter 1, we collect some background results that we frequently refer to throughout the thesis. We skip the proofs and provide references for the interested reader to delve into.

0.1.2 Chapter 2

In Chapter 2, we discuss the notion of a torsion submodule of a finitely generated module M over a one dimensional Noetherian local domain. We provide an interpretation to the torsion submodule whenever we restrict ourselves to modules of rank one, and this provides the primary motivation to define an invariant $\mathrm{h}(M)$ of M (see Definition 2.1) based on *partial trace ideals* of M. More precisely,

$$\mathrm{h}(M) := \min\{\lambda(R/J) \mid M \to J \to 0, J \subseteq R\}.$$

We say that J realizes M when $\mathrm{h}(M)$ is achieved by a J as above. We also refer to such a J as a partial trace ideal of M. This chapter is the crux of this thesis in a lot of ways. (We should mention here that the above definition is closely related to [Gre84, Definition 1.1], which we found after the work in this thesis was completed. The latter definition is restricted only to ideals in a one dimensional local domain.)

We prove a general upper bound on this invariant which enables us to capture information about vanishing of torsion (see Theorem 2.5).

Theorem A. Let $(S, \mathfrak{n}, k)$ be a regular local ring and $R = S/I$ be a one dimensional domain with maximal ideal $\mathfrak{m}$ and embedding dimension n. Let M be a rank one R-module with $\mu(M) \geq 2$. Assume that $I \subseteq \mathfrak{n}^{s+1}$ for some $s \geq 1$ and $J \subseteq \mathfrak{m}^s$ for some ideal J that realizes M. If

$$\mathrm{h}(M) < \frac{-\dim_k \operatorname{Tor}_1^R(M,k) + \mu(I)}{n} + \binom{n+s}{s}\left(\frac{s}{s+1}\right),$$

then $\tau(M) \neq 0$.

We further study the invariant $\mathrm{h}(\cdot)$ for arbitrary ideals (and hence, in effect for any positive rank module) of a one dimensional analytically unramified local domain (i.e., the $\mathfrak{m}$-adic completion is reduced) and prove the following result (see Theorem 2.18).

Theorem B. [Mai22, Theorem 2.5] Let $(R, \mathfrak{m}, k)$ be a one dimensional analytically unramified Noetherian local domain with integral closure $\overline{R}$ and fraction field K. Further assume that $\overline{R}$ is a DVR and the canonical module ω_R exists. Then for any ideal J of R, the following statements are equivalent:

(*a*) $\mathrm{h}(J) = \lambda(R/J)$ where $\mathrm{h}(J) := \min\{\lambda(R/I) \mid I \cong J\}$ and $\lambda(\cdot)$ denotes length.

(*b*) $R :_K J \subseteq \overline{R}$.

In fact, (*b*) $\implies$ (*a*) holds even without the DVR assumption. The second condition is a key finding as in general it does not hold for ideals. This is also crucial as the conductor ideal and all ideals containing the conductor ideal will satisfy the above (Corollary 2.20). More importantly it establishes the relationship of the colength of the conductor ideal with $\mathrm{h}(M)$ (Proposition 2.26). In particular, we study another invariant of the ring, namely $\mathrm{h}(\omega_R)$, and establish bounds in terms of the colength of the conductor ideal, multiplicity and type of R in the non-Gorenstein scenario (see Theorem 2.29)).

Finally in Section 2.3.3, we discuss the natural relationship of this invariant with the *trace ideal* of a module, which justifies the name partial trace ideal.

0.1.3 Chapter 3

In Chapter 3, we state and discuss Berger's Conjecture in detail. We apply our previous study to the numerical invariant, $\mathrm{h}(\Omega_{R/k})$, of the module of differentials and prove the following (see Theorem 3.12).

Theorem C. [Mai22, Theorem 4.7] Let $R = \frac{k[[X_1,\ldots,X_n]]}{I}$ be a complete, local one dimensional non-regular domain, with k perfect field and $I \subseteq (X_1,\ldots,X_n)^{s+1}$ for $s \geq 1$. Then Berger's Conjecture is true if

$$\mathrm{h}(\Omega_{R/k}) < \binom{n+s}{s}\frac{s}{s+1}.$$

This provides a generalization of an old result of G. Scheja and also provides some further cases of the conjecture (see Corollary 3.15).

We further discuss an algorithm to compute $\mathrm{h}(\Omega_{R/k})$ using software which given any specific example (with some added condition on characteristic of k), provides an efficient way of verifying the truth of the conjecture using our approach (see Theorem 3.19).

Finally, we establish a relationship between the notion of *quasi homogeneity* and the valuation of the trace ideal of $\Omega_{R/k}$ (see Theorem 3.28).

Theorem D. Let $R = \frac{k[[X_1,\dots,X_n]]}{I}$ be a complete, local one dimensional non-regular domain, and $I \subseteq (X_1, \dots, X_n)^2$. Assume that k is algebraically closed of characteristic 0. Let $\overline{R} = k[[t]]$ with valuation function v. Let $x_i(t)$ denote $X_i + I$ in terms of the parameter t and $x'(t) = \frac{d}{dt}x(t)$. Finally let $D = Rx_1'(t) + \cdots + Rx_n'(t)$ and $v(D^{-1}) = \min\{v(\alpha) \mid \alpha \in K, \alpha D \subseteq R\}$. Then R is quasi homogeneous if and only if $v(\operatorname{tr}_R(\Omega_{R/k})) = v(\mathfrak{m})$.

Here, the valuation function v is a discrete valuation on $k((t))$ which takes a power series in t and outputs the least exponent of t.

0.1.4 Chapter 4

Chapter 4 is based on joint work with Craig Huneke and Vivek Mukundan. Here we study Berger's Conjecture from the point of view of valuations that are induced on the ring from its integral closure (we saw a brief glimpse of it in Theorem D above). We assume that k is algebraically closed of characteristic 0 in this section.

We already saw in Chapter 2 that the conductor ideal plays a critical role. We define the conductor ideal in more detail in this chapter and subsequent investigations quickly lead us to the following result (see Theorems 4.5 and 4.18).

Theorem E. [HMM21, Theorem 3.1, Corollary 5.9] Let $R = \frac{k[[X_1,\dots,X_n]]}{I}$ be a complete, local one dimensional non-regular domain with $I \subseteq (X_1, \dots, X_n)^2$, and let $\mathfrak{m}$

denote the maximal ideal. Assume that k algebraically closed of characteristic 0. Let $\mathfrak{C}$ denote the conductor ideal.

1. If $\mathfrak{C} \not\subseteq \mathfrak{m}^2$, then Berger's Conjecture is true.

2. Assume that R is Gorenstein with embedding dimension $n \geq 6$. If $\mathfrak{m}^6 \subseteq x_1 R$, then $\tau(\Omega_{R/k}) \neq 0$.

The second part in the above theorem extends the work of Güttes [Güt90]. It provides credence to the approach of studying torsion elements by first moving into an over-ring of R, which we construct using $\mathfrak{C}$, and then trying to pull back a torsion to $\Omega_{R/k}$ (see Section 4.4 for further details).

We also provide a partial answer to Berger's Conjecture by analyzing the various valuations of R and that of the conductor (see Theorem 4.9). This result was obtained in joint work with Vivek Mukundan.

0.1.5 Appendix

In Appendix A, we introduce the notions of derivations and build up to the definition of the Module of Kähler Differentials. In Appendix A.1, we discuss its existence, construction, provide examples and discuss techniques to compute this. In Appendix A.2, we look at the universally finite module of differentials and discuss its existence. This is the main object of study in this thesis. We provide known results about its rank and number of generators. There are no new results in this appendix.

Chapter 1

Preliminaries

In this very brief chapter, we make a list of preliminary results that we shall refer to in later chapters. We omit the proofs of these and provide references wherever possible for the convenience of the reader.

1.1 Background Results

Rings in this thesis will be commutative rings with unity. By $(R, \mathfrak{m}, k)$ we denote a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. We denote the $\mathfrak{m}$-adic completion of R by $\widehat{R}$. For any module M, we will denote by $\lambda(M)$ the length of M (possibly ∞), $\mu(M)$ will denote the minimal number of generators of M and $\operatorname{rank}_R(M)$ denotes the rank of M (whenever rank exists; see the discussion preceeding Proposition 1.3). All modules M considered here will be finitely generated.

An important invariant of a module M is its *depth*. It is well-known ([BH98, Theorem 1.2.8]) that

$$\operatorname{depth}(M) = \inf\{i \mid \operatorname{Ext}^i_R(k, M) \neq 0\}.$$

A module M is Maximal Cohen Macaulay (MCM) if $\operatorname{depth}(M) = \dim(M) = \dim(R)$. R is called Cohen-Macaulay if it is so as a module over itself. We often use $\operatorname{edim}(R)$ to denote the embedding dimension of R, i.e., $\mu(\mathfrak{m})$. (For any arbitrary dimensional local ring, we have $\operatorname{edim}(R) \geq \dim(R)$ with equality if and only if R is a regular local ring.)

The following theorem is very well-known and extremely useful in handling complete k-algebras where k is a field. In fact, the conjecture of interest that we will apply our results to eventually will highly center around representing our ring R as the quotient of a power series ring.

Theorem 1.1 (Cohen Structure Theorem)**.** *Let $(R, \mathfrak{m}, k)$ be a k-algebra which is a complete local d dimensional ring of embedding dimension n. Then*

$$R = \frac{k[[X_1, X_2, ..., X_n]]}{I}$$

for some ideal I in $S = k[[X_1, ..., X_n]]$ of height $n - d$ with $I \subseteq (X_1, \ldots, X_n)^2$.

With the added condition of R being reduced or domain, the ideal I above becomes a radical or prime ideal respectively.

We will use the notion and properties of the Hilbert-Samuel multiplicity in this dissertation to quite an extent.

Definition 1.2. *For an $\mathfrak{m}$-primary ideal I of $(R, \mathfrak{m}, k)$ and a finitely generated R-module M, the* ***Hilbert-Samuel multiplicity*** *of M with respect to I can be defined as:*

$$e(I, M) = \lim_{t \to \infty} \frac{d! \lambda(M/I^t M)}{t^d}$$

where $d =$ dim R.

Of great importance is the case that $M = R$ and $I = \mathfrak{m}$, where we just write $e(R)$ for the multiplicity $e(\mathfrak{m}, R)$. Some of the important properties of multiplicity and reductions that we shall use are listed below. The proofs can be found in sources like [Ser65],[Ser97], [BH98] and[SH06].

We need some standard properties of Hilbert-Samuel multiplicities. In order to list these, we need the following. Let R be a ring (not necessarily local), M an R-module and Q be the total ring of fractions of R. Then M has *rank* r if $M \otimes_R Q$ is a free Q-module of rank r. If $\phi : M \to N$ is a homomorphism of R-modules, then ϕ has rank r if $\operatorname{Im} \phi$ has rank r.

Proposition 1.3 (Multiplicity Standard Properties)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Let I be an $\mathfrak{m}$-primary ideal and M a finitely generated R-module.*

1. *$e(I; R) \geq e(R)$.*

2. *$e(I, M) = e(\bar{I}, M)$ where $\bar{I}$ is the integral closure of I.*

3. *If M is MCM, R is Cohen Macaulay, and $\mu(I) = d$, then $e(I, M) = \lambda(M/IM)$.*

4. If M has a rank, then $e(I,M) = e(I,R)\operatorname{rank}_R(M)$.

Reductions of ideals turn out to be crucial in the study of multiplicities. They have the same integral closure as the original ideal itself. We recall the notion here and state an important existence result.

Definition 1.4. *Let R be a Noetherian ring. Let $J \subseteq I$ be ideals. J is said to be a* ***reduction*** *of I if there exists a non-negative integer n such that $I^{n+1} = JI^n$. A reduction J of I is called* ***minimal*** *if no ideal strictly contained in J is a reduction of I.*

Theorem 1.5. *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring with infinite residue field, I an $\mathfrak{m}$-primary ideal. Then every minimal reduction of I is minimally generated by exactly d elements. In particular, every reduction of I contains a reduction generated by d elements.*

For the following two fundamental notions, we follow the description provided in [BH98, Definition 1.2.15, Definition 1.2.18]. We start with a local ring $(R, \mathfrak{m}, k)$. For any R-module M of depth t (i.e., $\inf\{i \mid \operatorname{Ext}^i_R(k, M) \neq 0\} = t$), the *type* of M is defined as

$$\operatorname{type}(M) := \dim_k \operatorname{Ext}^t_R(k, M)$$

whereas the socle of M is $\operatorname{Soc}(M) := 0 :_M \mathfrak{m} = \{m \in M \mid m\mathfrak{m} = 0\}$. These two notions are related via the relation

$$\operatorname{type}(M) = \dim_k \operatorname{Soc}(M/\underline{x}M)$$

for any maximal M-regular sequence $\underline{x}$ (i.e., a maximal sequence in R such that x_1 is regular on M, x_2 is regular on M/x_1M and so on) [BH98, Lemma 1.2.19].

As we discussed earlier, we are interested in studying the torsion submodule of a module M. The torsion submodule can be interpreted as the *zeroth local cohomology of* M *with respect to the maximal ideal* $\mathfrak{m}$. There is a fundamental duality that enables us to equivalently study this module in terms of suitable Ext modules. The following paragraphs discuss the relevant tools and results.

We let E denote the injective hull of the residue field k of R. The Matlis dual of M, denoted $M^\vee$, is defined to be the R-module

$$M^\vee := \operatorname{Hom}_R(M, E).$$

For suitably chosen modules over a Cohen Macaulay ring, there are critical identities between the Matlis dual and dualization with respect to the canonical module ω_R (provided it exists) via local cohomology. For any module M, the i^{th} local cohomology of M with respect to any ideal I, denoted $H^i_I(M)$, is defined to be

$$\mathrm{H}^i_I(M) := \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/I^n, M).$$

It can also be treated as the right-derived functor of the covariant left exact functor $\Gamma_I(M) = \bigcup_{n\geq 0}(0 :_M I^n)$.

It is well-known that ω_R exists and can be identified with a height one ideal of R whenever R is a complete local Cohen-Macaulay k-algebra which is generically Gorenstein, i.e., R_P is Gorenstein for all minimal prime ideals P of R. For further details, we refer the reader to [BH98, Chapter 3]. The following are important properties.

Theorem 1.6 (Properties of Matlis Dual and the Canonical Module)**.** *Let* $(R, \mathfrak{m}, k)$ *be Cohen-Macaulay of dimension* d *with canonical module* ω_R*.*

1. $\operatorname{Ann}_R(M) = \operatorname{Ann}_R(M^\vee)$ *for any finitely generated module* M*. Further, if* M *has finite length, then* $\lambda(M) = \lambda(M^\vee)$*.*
2. *For any regular sequence* $\underline{x}$*,* $\omega_{R/\underline{x}R} \cong \omega_R/\underline{x}\omega_R$*.*
3. *If* $d = 0$*, then* $E \cong \omega_R$*, where* E *is the injective hull of* k*.*
4. $\mu(\omega_R) = \operatorname{type}(R)$*.*
5. *If* M *is MCM, then* $M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega_R), \omega_R)$ *and* $\operatorname{Ext}^i_R(M, \omega_R) = 0$ *for all* $i > 0$*.*
6. $\operatorname{Hom}_R(\omega_R, \omega_R) \cong R$*.*

We will require the following major result which can be found in [BH98, Theorem 3.5.8].

Theorem 1.7 (Local Duality)**.** *Let* $(R, \mathfrak{m}, k)$ *be a Cohen Macaulay complete local ring of dimension* d*. Then for all finite* R*-modules* M *and for all integers* i*, there exist natural isomorphisms*

$$\operatorname{Ext}^i_R(M, \omega_R) \cong \left(\mathrm{H}^{d-i}_{\mathfrak{m}}(M)\right)^\vee, \quad \mathrm{H}^i_{\mathfrak{m}}(M) \cong \left(\operatorname{Ext}^{d-i}_R(M, \omega_R)\right)^\vee.$$

The second one holds even without the completeness assumption as long as ω_R *exists.*

Our object of interest is the module of differentials of R. For a detailed description of this module, we refer the reader to [Kun86]. We provide sufficient details on $\Omega_{R/k}$ in Appendices A.1 and A.2 for the interested reader. We shall refer to Appendix A for the properties of $\Omega_{R/k}$.

In particular, the 'universally finite module of differentials' and the module of differentials are not the same in general. However, for the purpose of this dissertation, we can work with the following definition.

Definition 1.8. *Let $R = S/I$ where $S = k[[X_1, \ldots, X_n]]$ or $S = k[X_1, \ldots, X_n]$ where k is any field. The universally finite module of differentials of R, denoted $\Omega_{R/k}$, is the finitely generated R-module which has the following presentation:*

$$R^{\mu(I)} \xrightarrow{A} R^n \to \Omega_{R/k} \to 0$$

where A is given as follows: if $I = (f_1, \ldots, f_{\mu(I)})$, then $A = \left(\frac{\partial f_j}{\partial X_i}\right)_{\substack{1 \le j \le \mu(I) \\ 1 \le i \le n}}$, considered as a matrix over R.

Remark 1.9. Taking into account the formal partial derivations dX_i, we can also describe $\Omega_{R/k}$ as follows: if $I = (f_1, \ldots, f_m)$, then

$$\Omega_{R/k} = \frac{\bigoplus_{i=1}^{n} R dX_i}{U}$$

where U is the R-module generated by the elements $df_i = \sum_{i=1}^{n} \frac{\partial f_j}{\partial X_i} dX_i, j = 1, \ldots, m$.

Theorem 1.10. *Let $R = k[\![X_1, \ldots, X_n]\!]/I$ be a domain. Assume further that k is a perfect field. Then we have the following.*

1. $\mu(\Omega_{R/k}) = \mu(\mathfrak{m})$.

2. $\operatorname{rank}_R(\Omega_{R/k}) = n - \operatorname{ht}(I) = \dim(R)$.

The proof of the above statement is provided in Appendix A (Theorem A.32).

Chapter 2

An Invariant Based On Partial Trace Ideals

The main goal of this chapter is to introduce and study an invariant of any module with non-zero rank over a one dimensional Noetherian local domain. This invariant will help in capturing information regarding the torsion submodule of any rank one module over such a ring. We restrict our study to the case of ideals and establish a suitable criterion between this invariant being achieved by an ideal itself and the R-dual of the ideal being inside the integral closure of the ring. We prove bounds on this invariant by establishing relationship with the conductor ideal. Using these techniques, we establish a general bound on another invariant of the ring (see Section 2.3.2) We also establish the relationship with trace ideals which justifies the nomenclature 'partial trace ideals'.

2.1 Motivation: An Interpretation of Torsion Submodule

Let $(R, \mathfrak{m}, k)$ be a one dimensional Noetherian local domain and let M be a finitely generated R-module. Denote the torsion submodule of M by $\tau(M)$. By definition,

$$\tau(M) := \{m \in M \mid xm = 0, 0 \neq x \in R\}.$$

In order to have discussions regarding $\tau(M)$, a natural question to ask is how do we interpret the torsion beyond the above definition. Note that if Q denotes the fraction field of R, then

$$\tau(M) = \ker(M \to M \otimes_R Q).$$

In addition, if M has rank one, then we get the following interpretation.

Let J be an ideal to which M surjects. Since both J and M have rank one, we get the following exact sequence:

$$0 \to \tau(M) \to M \to J \to 0.$$

Thus, $J \cong \frac{M}{\tau(M)}$ for any ideal J which appears as the image of an element in $\mathrm{Hom}_R(M, R)$. Thus studying such ideals J is a natural approach to capture some information about $\tau(M)$. This precisely motivates the definition in the following discussion.

2.2 An Invariant that captures torsion

Definition 2.1. *Let R be a local Noetherian one dimensional domain. For any R-module M, define*

$$\mathrm{h}(M) := \min\{\lambda(R/J) \mid M \to J \to 0, J \subseteq R\}$$

where $\lambda(\cdot)$ denotes the length as an R-module.

We say that an ideal J ***realizes*** *M if M surjects to J and $\mathrm{h}(M) = \lambda(R/J)$.*

The assumption that R is a one dimensional domain ensures that $\mathrm{h}(M)$ is finite whenever M is not a torsion-module. As we shall see in Section 2.3.3, the ideal J can be thought of as partial trace ideal of M.

Remark 2.2. Notice that if $\mathrm{h}(M) = 0$, then we have

$$0 \to K \to M \to R \to 0$$

(K is the kernel) and hence M has a free summand. So if $\mathrm{rank}(M) = 1$ (in this case, $K = \tau(M)$) and $\mu(M) \geq 2$, then it is clear that $\tau(M) \neq 0$. So to study torsion of a rank one module which is generated by at least two elements, it is naturally more interesting to look at the case $\mathrm{h}(M) \geq 1$. Thus, for the rest of this section, we will assume that $\mathrm{h}(\cdot)$ is a positive integer. Note that this necessarily implies that any surjective image of such an M is not a principal ideal: To see this, note that a principal ideal is isomorphic to R and thus we will have a surjection of M to R implying $\mathrm{h}(M) = 0$, a contradiction to the assumption.

We shall explore further properties of $\mathrm{h}(\cdot)$ independently in the subsequent sections without having this positivity assumption.

Remark 2.3. We should mention here that Definition 2.1 is closely related to [Gre84, Definition 1.1] which we found after the work in [Mai22] was published. As stated before, a lot of results in this thesis already appear in [Mai22]. We also point out that the definition in [Gre84] is restricted only to ideals in a one dimensional local domain, whereas we study it for any finitely generated module.

We provide an important lemma now that will help in establishing how this invariant captures information about the vanishing of the torsion submodule of some rank one module M. The proof of this lemma is motivated by the proof of [BH98, Theorem 2.3.2]. It is simply a higher power (of the maximal ideal) analogue of the aforementioned proof under suitable modifications to the hypothesis.

Before stating it, we recall the following notion: The *associated graded ring* corresponding to the maximal ideal $\mathfrak{n}$ in a local ring $(S, \mathfrak{n}, k)$ is defined to be

$$\mathrm{gr}_{\mathfrak{n}}(S) := \oplus_{i\geq 0} \frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}.$$

For any element $s \in S$, we choose the greatest integer ℓ such that $s \in \mathfrak{n}^\ell$, and define the leading term of s to be $s^* := s$ modulo $\mathfrak{n}^{\ell+1} \in \frac{\mathfrak{n}^\ell}{\mathfrak{n}^{\ell+1}} \subseteq \mathrm{gr}_{\mathfrak{n}}(S)$. If S is a regular local ring, then $\mathrm{gr}_{\mathfrak{n}}(S)$ is isomorphic to a polynomial ring over k with $\mu(\mathfrak{n})$ variables [BH98, Theorem 1.1.8]. For further details, see [Eis13, 5.1].

Lemma 2.4. *[Mai22, Proposition 4.5] Let $(S, \mathfrak{n}, k)$ be a regular local ring of embedding* *dimension n and let $R = S/I$ for an ideal I in S. Let $\mathfrak{m}$ be the maximal ideal of R. Further assume that $I \subseteq \mathfrak{n}^{s+1}$ for some $s \geq 1$. Then*

$$\dim_k \left(\operatorname{Tor}_1^R(\mathfrak{m}^s, k)\right) = s\binom{n+s-1}{s+1} + \mu(I).$$

Proof. Choose $x_1, \dots, x_n$ to be a minimal system of generators of $\mathfrak{m}$. Let $y_1, y_2, \dots, y_n$ be a regular system of parameters in S such that $\overline{y_i} = x_i$ where $\bar{\cdot}$ denotes going modulo I.

Using the short exact sequence $0 \to \mathfrak{m}^s \to R \to R/\mathfrak{m}^s \to 0$, we get $\operatorname{Tor}_1^R(\mathfrak{m}^s, k) \cong \operatorname{Tor}_2^R(R/\mathfrak{m}^s, k)$. In order to get the desired conclusion, we shall make use of the minimal resolutions of $R/\mathfrak{m}^s$ and $S/\mathfrak{n}^s$. Note that since $I \subseteq \mathfrak{n}^{s+1}$, we have $\mu(\mathfrak{n}^s) = \mu(\mathfrak{m}^s) = \binom{n+s-1}{s}$.

Let F denote the free module S^{n+s-1}, G denote the free module S^s and $L : F \to G$ be given by the $s \times (n+s-1)$ matrix

$$\begin{bmatrix} y_1 & y_2 & \dots & y_n & 0 & \dots & 0 \\ 0 & y_1 & y_2 & \dots & y_n & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & y_1 & y_2 & \dots & y_n \end{bmatrix}.$$

By the discussion following Remark 2.13 in [BV06], the ideal of $s \times s$ minors of L, denoted $I_s(L)$, is $\mathfrak{n}^s$. Since S is a regular local ring and $\operatorname{height}(I_s(L)) = \operatorname{height}(\mathfrak{n}^s) = n = \operatorname{rank}_S(F) - \operatorname{rank}_S(G) + 1$ (for further details, on height of an ideal, we refer the reader to [BH98, Appendix]), we get that the Eagon-Northcott complex

$$\mathbb{EN}(L)_\bullet : \dots \to G^* \otimes \wedge^{s+1} F \xrightarrow{d_2} \wedge^s F \xrightarrow{\wedge^s L} \wedge^s G \cong S \to 0$$

resolves $S/I_s(L)$ [Eis05, Theorem A2.60].

Let $\mu(I) = m$ and I be minimally generated by $c_1, \ldots, c_m$. Fix $p = \binom{n+s-1}{s} = \operatorname{rank}_S(\wedge^s F)$ and $q = s\binom{n+s-1}{s+1} = \operatorname{rank}_S(G^* \otimes \wedge^{s+1} F)$. Using the standard bases of $\wedge^s F \cong S^p$ and $\wedge^s G \cong S$, we can consider the map $\wedge^s L$ as the matrix $[Y_1 \cdots Y_p]$ where $Y_i \in \mathfrak{n}$ are the $s \times s$ minors of L. Notice that these minimally generate $I_s(L)$. Since $I \subseteq \mathfrak{n}^{s+1} \subseteq \mathfrak{n}^s = I_s(L)$ we can write $c_i = \sum_{k=1}^{p} b_{ki} Y_k$ with $b_{ki} \in \mathfrak{n}$. Writing this in row format and denoting $C = [c_1 \cdots c_m]$, we get that

$$C = \begin{bmatrix} Y_1 \cdots Y_p \end{bmatrix} B = (\wedge^s L) B \tag{2.4.1}$$

where $B := (b_{ki})_{\substack{1 \le k \le p \\ 1 \le i \le m}}$ is a $p \times m$ matrix with entries in $\mathfrak{n}$. Let $\{e_i\}$ denote the standard basis of S^m. We have $Ce_i = c_i$ and identifying $\wedge^s G$ with S, we get the commutative diagram

$$\begin{array}{ccc} \wedge^s F & \xrightarrow{\wedge^s L} & \wedge^s G \\ {\scriptstyle B}\uparrow & & \wr\vert \\ S^m & \xrightarrow[C]{} & S \end{array}$$

Let $v_i = Be_i$, the columns of B. Thus we obtain the exact sequence

$$0 \to Z_1 \to \wedge^s F \otimes_S R \xrightarrow{g} \wedge^s G \otimes_S R \to R/\mathfrak{m}^s \to 0$$

where $g = \wedge^s L \otimes_S R$ and $Z_1 = \ker g$.

Note that $\dim_k \operatorname{Tor}_1^R(\mathfrak{m}^s, k) = \dim_k \operatorname{Tor}_2^R(R/\mathfrak{m}^s, k) = \mu(Z_1)$. Let $\bar{b} \in Z_1$ where $b \in \wedge^s F$. So, $(\wedge^s L)(b) \in I$ and hence, $(\wedge^s L)(b) = \sum_{i=1}^m s_i c_i$, $s_i \in S$. Now using Equation (2.4.1), we get

$$(\wedge^s L)(b) = \sum_{i=1}^{m} s_i c_i = \sum_{i=1}^{m} s_i (Ce_i) = \sum_{i=1}^{m} s_i (\wedge^s L)(Be_i) = (\wedge^s L)(\sum_{i=1}^{m} s_i v_i)$$

and thus $b-\sum_{i=1}^{m} s_iv_i \in \ker(\wedge^s L) = \mathrm{Im}(d_2)$, where $G^* \otimes \wedge^{s+1} F \xrightarrow{d_2} \wedge^s F$ is the second differential of $\mathbb{EN}(L)_\bullet$. Denoting the standard basis of $G^* \otimes \wedge^{s+1} F$ by $\{w_i\}, 1 \leq i \leq q$, we can write $b - \sum_{i=1}^{m} s_iv_i = d_2(\sum_{i=1}^{q} s_i' w_i)$, $s_i' \in S$. So

$$\bar{b} = \sum_{i=1}^{q} \overline{s_i' d_2(w_i)} + \sum_{i=1}^{m} \overline{s_i v_i}.$$

Since $(\wedge^s L)(d_2(w_i)) = 0$, $d_2(w_i) \in \ker(\wedge^s L)$ and hence $\overline{d_2(w_i)} \in Z_1$ for $1 \leq i \leq q$. Moreover, $c_j = Ce_j = (\wedge^s L)(Be_j) = (\wedge^s L)v_j$. So, $g(\overline{v_j}) = 0$ (as $g = \wedge^s L \otimes_S R$) and hence $\overline{v_j}$ is in Z_1 for $1 \leq j \leq m$. Thus we obtain that

$$\mu(Z_1) \leq q + m = s\binom{n+s-1}{s+1} + \mu(I).$$

We now show that $\{\overline{d_2(w_i)}, \overline{v_j}\}$ for $1 \leq i \leq q, 1 \leq j \leq m$ form a minimal generating set of Z_1.

Suppose $\sum\limits_{i=1}^{q} \overline{\alpha_i d_2(w_i)} + \sum\limits_{j=1}^{m} \overline{\beta_j v_j} = 0$ where $\alpha_i, \beta_j \in S$. We are done if we show that all the $\overline{\alpha_i}$ and $\overline{\beta_j}$ are in $\mathfrak{m}$, i.e. α_i, β_j are in $\mathfrak{n}$. The relation under consideration implies that

$$\sum_{i=1}^{q} \alpha_i d_2(w_i) + \sum_{j=1}^{m} \beta_j v_j \in I(\wedge^s F). \tag{2.4.2}$$

Applying $\wedge^s L$, we have

$$\begin{aligned}\sum_{i=1}^{q} \alpha_i (\wedge^s L)(d_2(w_i)) + \sum_{j=1}^{m} \beta_j (\wedge^s L)(v_j) &= \sum_{j=1}^{m} \beta_j (\wedge^s L)(Be_j) \\ &= \sum_{j=1}^{m} \beta_j (Ce_j) = \sum_{j=1}^{m} \beta_j c_j\end{aligned}$$

where the first equality follows from the following: $(\wedge^s L) \circ d_2 = 0$ and $Be_j = v_j$. The second equality is a consequence of Equation (2.4.1). We conclude that

$$\sum_{j=1}^{m} \beta_j c_j \in I(\wedge^s L)(\wedge^s F).$$

As $(\wedge^s L)(\wedge^s F) \subseteq \mathfrak{n}^s$ (recall that $\wedge^s G \cong S$), we obtain that $\sum_{j=1}^{m} \beta_j c_j \in I\mathfrak{n}^s \subseteq \mathfrak{n}I$. Since c_j's minimally generate I, this implies that $\beta_j \in \mathfrak{n}$. Since the entries of B are in $\mathfrak{n}$, $v_j = Be_j \in \mathfrak{n}(\wedge^s F)$. Therefore $\sum_{j=1}^{m} \beta_j v_j \in \mathfrak{n}^2(\wedge^s F)$. Now from (2.4.2) we obtain that

$$\sum_{i=1}^{q} \alpha_i d_2(w_i) \in I(\wedge^s F) + \mathfrak{n}^2(\wedge^s F) \subseteq \mathfrak{n}^{s+1}(\wedge^s F) + \mathfrak{n}^2(\wedge^s F) \subseteq \mathfrak{n}^2(\wedge^s F) \qquad (2.4.3)$$

where the first inclusion is due to $I \subseteq \mathfrak{n}^{s+1}$.

We are trying to show that $\alpha_i \in \mathfrak{n}$. Therefore, assume without loss of generality that $\alpha_{i_1}, \ldots, \alpha_{i_\ell} \notin \mathfrak{n}$ and the remaining α_j's are in $\mathfrak{n}$. Recall that the entries of d_2 are linear forms in the entries of L (see discussions in [Eis13, A2.6.1] and [Eis05, Example A2.69]). From (2.4.3), we conclude that

$$\sum_{t=1}^{\ell} \alpha_{i_t} d_2(w_{i_t}) \in \mathfrak{n}^2(\wedge^s F). \qquad (2.4.4)$$

Now recall that $\mathrm{gr}_{\mathfrak{n}}(S)$ is isomorphic to a polynomial ring in n variables over k. Note that the Eagon-Northcott complex over $\mathrm{gr}_{\mathfrak{n}}(S)$ corresponding to the $s \times (n+s-1)$ matrix

$$L' = \begin{bmatrix} y_1^* & y_2^* & \dots & y_n^* & 0 & \dots & 0 \\ 0 & y_1^* & y_2^* & \dots & y_n^* & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & y_1^* & y_2^* & \dots & y_n^* \end{bmatrix}$$

where the y_i^* are the leading forms of y_i (recall that the y_i's minimally generate $\mathfrak{n}$), is also exact since the y_i^* form a regular sequence and generate the homogeneous maximal ideal, say $\mathfrak{N}$, of $\operatorname{gr}_{\mathfrak{n}}(S)$. Let us call the differentials d_i'.

By abuse of notation, we still use F, G to denote the corresponding free modules of ranks $n+s-1$ and s respectively, over $\operatorname{gr}_{\mathfrak{n}}(S)$. We also use $\{w_i\}$ to denote the canonical basis of $G^* \otimes_{\operatorname{gr}_{\mathfrak{n}}(S)} \wedge^{s+1} F$. Again, for $i \geq 2$, the entries of d_i' are linear forms in the entries of L'. Thus upon comparing degrees, from (2.4.4) we get that $\sum_{t=1}^{\ell} \alpha_{i_t}^* d_2'(w_{i_t}) = 0$ where the $\alpha_{i_t}^* \in k \setminus \{0\}$ are the leading forms of α_{i_t}. In other words,

$$\sum_{t=1}^{\ell} \alpha_{i_t}^* w_{i_t} \in \ker(d_2') = \operatorname{Im}(d_3') \subseteq \mathfrak{N}(G^* \otimes_{\operatorname{gr}_{\mathfrak{n}}(S)} \wedge^{s+1} F).$$

This contradicts the fact that the $\{w_i\}$ are a basis. Thus, $\alpha_{i_t}^* = 0$ for all $0 \leq t \leq \ell$. This in turn implies that $\alpha_i \in \mathfrak{n}$ for all i, thereby finishing the proof. □

Theorem 2.5. *Let $(S, \mathfrak{n}, k)$ be a regular local ring and $R = S/I$ be a one dimensional domain with maximal ideal $\mathfrak{m}$ and embedding dimension n. Let M be a rank one R-module with $\mu(M) \geq 2$. Assume that $I \subseteq \mathfrak{n}^{s+1}$ for some $s \geq 1$ and $J \subseteq \mathfrak{m}^s$ for some ideal J that realizes M. If*

$$\operatorname{h}(M) < \frac{-\dim_k \operatorname{Tor}_1^R(M,k) + \mu(I)}{n} + \binom{n+s}{s}\left(\frac{s}{s+1}\right),$$

then $\tau(M) \neq 0$.

Proof. We have that $\operatorname{h}(M) = \lambda(R/J)$ since J realizes M. Using additivity of length and the assumption $J \subseteq \mathfrak{m}^s$, we have

$$\lambda\left(\mathfrak{m}^s/J\right) = \lambda(R/J) - 1 - \sum_{i=1}^{s-1} \mu(\mathfrak{m}^i).$$

Using a composition series for $\mathfrak{m}^s/J$, we get

$$\dim_k \operatorname{Tor}_1^R\left(\frac{\mathfrak{m}^s}{J}, k\right) \leq \lambda\left(\mathfrak{m}^s/J\right) \dim_k \operatorname{Tor}_1^R(k,k)$$

(each simple module in the composition series is isomorphic to k (see Remark 2.7(b) for the detailed statement)). Thus we get,

$$\dim_k \operatorname{Tor}_1^R\left(\frac{\mathfrak{m}^s}{J}, k\right) \leq \mu(\mathfrak{m}) \left(\lambda(R/J) - 1 - \sum_{i=1}^{s-1} \mu(\mathfrak{m}^i)\right) \tag{2.2.1}$$

Tensoring the exact sequence $0 \to J \to \mathfrak{m}^s \to \mathfrak{m}^s/J \to 0$ with k, we get

$$\operatorname{Tor}_1^R(J,k) \to \operatorname{Tor}_1^R(\mathfrak{m}^s,k) \to \operatorname{Tor}_1^R\left(\frac{\mathfrak{m}^s}{J}, k\right) \to J \otimes k \to \mathfrak{m}^s \otimes k \to \frac{\mathfrak{m}^s}{J} \otimes k \to 0$$

as part of a long exact sequence. Hence,

$$\dim_k \operatorname{Tor}_1^R\left(\frac{\mathfrak{m}^s}{J}, k\right) \geq -\dim_k \operatorname{Tor}_1^R(J,k) + \dim_k \operatorname{Tor}_1^R(\mathfrak{m}^s,k) + \mu(J) - \mu(\mathfrak{m}^s) + \mu\left(\mathfrak{m}^s/J\right) \tag{2.2.2}$$

Combining Equations (2.2.1) and (2.2.2), we get

$$\lambda(R/J) \geq \frac{-\dim_k \operatorname{Tor}_1^R(J,k) + \dim_k \operatorname{Tor}_1^R(\mathfrak{m}^s,k) + \mu(J) - \mu(\mathfrak{m}^s) + \mu\left(\frac{\mathfrak{m}^s}{J}\right)}{\mu(\mathfrak{m})} + 1 + \sum_{i=1}^{s-1} \mu(\mathfrak{m}^i)$$

Recall that we have the exact sequence

$$0 \to \tau(M) \to M \to J \to 0$$

Suppose, on the contrary, that $\tau(M) = 0$, then $J \cong M$ and hence $\dim_k \operatorname{Tor}_1^R(J,k) = \beta_1(M)$ (this is the *first betti number* of M and it is defined to be $\dim_k \operatorname{Tor}_1^R(M,k)$) and $\mu(J) = \mu(M)$. Further, since $I \subseteq \mathfrak{n}^{s+1}$, we have that $\mu(\mathfrak{m}^i) = \binom{n+i-1}{i}$ for $1 \leq i \leq s$.

So, $1+\sum_{i=1}^{s-1}\mu(\mathfrak{m}^i)=\sum_{i=0}^{s-1}\binom{n-1+i}{i}=\binom{n+s-1}{s-1}$. Combining these observations along with Lemma 2.4, we get that

$$\begin{aligned}\lambda(R/J) &\geq \frac{-\beta_1(M)+\mu(I)+s\binom{n+s-1}{s+1}+\mu(M)-\mu(\mathfrak{m}^s)+\mu\left(\frac{\mathfrak{m}^s}{J}\right)}{n}+\binom{n+s-1}{s-1} \\ &= \frac{-\beta_1(M)+\mu(I)+\mu(M)-\mu(\mathfrak{m}^s)+\mu\left(\frac{\mathfrak{m}^s}{J}\right)}{n}+X\end{aligned}$$

where

$$\begin{aligned}X &= \frac{s\binom{n+s-1}{s+1}}{n}+\binom{n+s-1}{s-1}=\frac{s(n+s-1)!}{n(s+1)!(n-2)!}+\frac{(n+s-1)!}{(s-1)!n!} \\ &= \frac{(n+s-1)!}{(s-1)!n(n-2)!}\left(\frac{1}{s+1}+\frac{1}{n-1}\right)=\frac{(n+s-1)!}{(s-1)!n!}\left(\frac{n+s}{s+1}\right) \\ &= \frac{(n+s)!}{s!n!}\left(\frac{s}{s+1}\right)=\binom{n+s}{s}\left(\frac{s}{s+1}\right)\end{aligned}$$

So we conclude that

$$\lambda(R/J)\geq \frac{-\beta_1(M)+\mu(I)+\mu(M)-\mu(\mathfrak{m}^s)+\mu\left(\frac{\mathfrak{m}^s}{J}\right)}{n}+\binom{n+s}{s}\left(\frac{s}{s+1}\right). \tag{2.2.3}$$

Since $\mu(\mathfrak{m}^s/J)\geq\mu(\mathfrak{m}^s)-\mu(J)=\mu(\mathfrak{m}^s)-\mu(M)$, we get that

$$\lambda(R/J)\geq\frac{-\beta_1(M)+\mu(I)}{n}+\binom{n+s}{s}\left(\frac{s}{s+1}\right).$$

But this a contradiction to the hypothesis on $\mathrm{h}(M)$. Hence, $\tau(M)\neq 0$. □

Corollary 2.6. *Let $(S,\mathfrak{n},k)$ be a regular local ring and $R=S/I$ be a non-regular one dimensional domain with embedding dimension n. Assume that $I\subseteq\mathfrak{n}^2$. Let M be a rank one R-module with $\mu(M)\geq 2$. Then $\tau(M)\neq 0$ if*

$$\mathrm{h}(M)<\frac{-\beta_1(M)+\mu(I)}{n}+\frac{n+1}{2}.$$

Proof. If $\mathrm{h}(M)=0$, then we are done. So we assume that $\mathrm{h}(M)\geq 1$. Putting $s=1$ in Theorem 2.5 gives us the desired conclusion. □

Remark 2.7. We jot down a few observations in connection to the proof of Theorem 2.5.

a) As in the proof of Theorem 2.5, we can tensor the short exact sequence, in Equation (2.3.5), with k and then truncate the long exact sequence to obtain that $\mu(\tau(M)) \geq \mu(M) - \mu(J)$. So, if M surjects to an ideal J which is generated by less than $\mu(M)$ elements, then $\tau(M) \neq 0$.

b) We can repeat the proof of Theorem 2.5 up to Equation (2.2.3) and plug in $\mu(\mathfrak{m}^s) = \binom{n+s-1}{s}$ directly to get

$$\begin{aligned}\lambda(R/J) &\geq \frac{-\beta_1(M) + \mu(I) + \mu(M) + \mu(\mathfrak{m}^s/J)}{n} + \binom{n+s}{s}\left(\frac{s}{s+1}\right) - \frac{\binom{n+s-1}{s}}{n} \\ &= \frac{-\beta_1(M) + \mu(I) + \mu(M) + \mu(\mathfrak{m}^s/J)}{n} + \binom{n+s-1}{s-1}\frac{s^2 + s(n-1) - 1}{s(s+1)}.\end{aligned}$$

Hence, $\tau(M) \neq 0$ if

$$\mathrm{h}(M) < \frac{-\beta_1(M) + \mu(I) + \mu(M) + \mu(\mathfrak{m}^s/J)}{n} + \binom{n+s-1}{s-1}\frac{s^2 + s(n-1) - 1}{s(s+1)}.$$

c) In the proof of Theorem 2.5, we used a special case of a more general statement:

$$\dim_k \operatorname{Tor}_1^R(M,k) \leq \lambda(M)(n-1) + \mu(M)$$

for any finite length R-module M. Here, $n = \mu(\mathfrak{m}) = \dim_k \operatorname{Tor}_1^R(k,k)$.

Proof. Start with a composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_{\lambda(M)-1} \subset M_{\lambda(M)} = M$. Tensor with k the short exact sequence $0 \to M_{\lambda(M)-1} \to M \to M_{\lambda(M)}/M_{\lambda(M)-1} \cong k \to 0$ to get a long exact sequence of Tors which gives us

$$\dim_k \operatorname{Tor}_1^R(M,k) \leq \dim_k \operatorname{Tor}_1^R(M_{\lambda(M)-1}) + \dim_k \operatorname{Tor}_1^R(k,k) + \mu(M) - \mu(M_{\lambda(M)-1}) - 1.$$

Keep repeating this procedure for $M_{\lambda(M)-1}, \ldots, M_1$ successively to conclude that

$$\dim_k \operatorname{Tor}_1^R(M,k) \leq \lambda(M) \dim_k \operatorname{Tor}_1^R(k,k) + \mu(M) - \lambda(M).$$

This finishes the proof. □

2.3 Properties of the Invariant

In this section, we make a general study of the invariant $\mathrm{h}(\cdot)$ defined above. But before we do that, let us recall the notion of the conductor ideal and canonical module.

For any reduced ring R with total ring of fractions K, let $\mathfrak{C}$ denote the conductor of R in $\overline{R}$ where the integral closure is taken in K. Let ω_R denote the canonical module of R (which exists if we assume that R is the quotient of a regular local ring as above) and can be identified with some ideal of R. (Chapter 3, [BH98]). We also recall the correspondence between colon ideals and Hom sets.

Remark 2.8. Let R be any ring with total ring of quotients K. For any two $R-$submodules I_1, I_2 of K such that I_2 has a non-zero divisor on R, we know by [SH06, Lemma 2.4.2],

$$I_1 :_K I_2 \cong \operatorname{Hom}_R(I_2, I_1)$$

where the isomorphism is as R-modules. The idea behind this isomorphism is as follows: notice that for any $x, y \in I_2$, we get that $y\phi(x) = x\phi(y)$. Hence, if there exists a non-zero divisor x in I_2, we get for any $r \in I_2$, $\phi(r) = r\frac{\phi(x)}{x}$.

In this thesis, we are going to identify these two R-submodules of K and use them interchangeably. We denote $R :_K I$ by I^{-1}.

Remark 2.9. $\mathfrak{C}$ is the largest common ideal of R and $\overline{R}$; $\mathfrak{C} = R :_K \overline{R}$. By Remark 2.8, we have $\mathfrak{C} = \mathrm{Hom}_R(\overline{R}, R)$. We shall deal with the conductor in much greater detail in Chapter 3. Notice that if R is a complete local one dimensional domain, then $\mathfrak{C}$ is non-zero. This follows because $\bar{R}$ is finitely generated over R [SH06, Corollary 4.6.2]. Also note that discussions on the conductor are relevant mainly when $\bar{R}$ is finitely generated as an R-module. This is ensured for instance when R is analytically unramified, i.e. the completion $\widehat{R}$ is reduced.

We start by recording some lemmas which will come in handy. The first lemma is well-known.

Lemma 2.10. *Let R be any Noetherian domain of dimension one, with finite integral closure $\overline{R}$ and conductor $\mathfrak{C}$. Then $\overline{R} = R :_K \mathfrak{C}$. In particular by Remark 2.8, we can write $\overline{R} = \mathrm{Hom}_R(\mathfrak{C}, R)$.*

Proof. $\overline{R} \subseteq R :_K \mathfrak{C}$ since $\mathfrak{C}\overline{R} \subseteq R$. For the other inclusion, note that

$$R :_K \mathfrak{C} = R :_K \mathfrak{C}\overline{R} = (R :_K \overline{R}) :_K \mathfrak{C} = \mathfrak{C} :_K \mathfrak{C}.$$

By the determinant trick, we have $\mathfrak{C} :_K \mathfrak{C} \subseteq \overline{R}$ finishing the proof. □

Lemma 2.11. *Let R be a one dimensional complete local domain with canonical module ω_R, conductor ideal $\mathfrak{C}$ and integral closure $\overline{R}$. We have*

$$\left(\frac{\overline{R}}{R}\right)^{\vee} \cong \frac{\omega_R}{\mathfrak{C}\omega_R},$$

where $^\vee$ *denotes the Matlis dual. Hence, we have* $\lambda\left(\dfrac{\overline{R}}{R}\right) = \lambda\left(\dfrac{\omega_R}{\mathfrak{C}\omega_R}\right)$.

Proof. We break the proof into two parts. The first part establishes the isomorphism. The second part will help in understanding the length equality via identifications of Hom with colons as in Remark 2.8. We also repeatedly use Theorem 1.6 and Theorem 1.7.

PART 1: Applying $\operatorname{Hom}_R(\ ,\omega_R)$ to the exact sequence $0 \to R \to \overline{R} \to \overline{R}/R \to 0$ we get $0 \to \operatorname{Hom}_R(\overline{R}, \omega_R) \to \omega_R \to \operatorname{Ext}^1_R(\overline{R}/R, \omega_R) \to 0$ using the properties of ω_R:

- $\overline{R}/R$ is torsion whereas ω_R is torsion-free, and hence $\operatorname{Hom}_R(\overline{R}/R, \omega_R) = 0$;
- Since $\overline{R}$ is MCM, we get $\operatorname{Ext}^1_R(\overline{R}, \omega_R) = 0$ (see Theorem 1.6).

Next, by local duality (Theorem 1.7), we have $\operatorname{Ext}^1_R(\overline{R}/R, \omega_R) \cong \left(\overline{R}/R\right)^\vee$. Hence, it is enough to show that $\operatorname{Hom}_R(\overline{R}, \omega_R) \cong \mathfrak{C}\omega_R$. Using Lemma 2.10, Theorems 1.6 and 1.7, we have

$$\begin{aligned}
\operatorname{Hom}_R(\overline{R}, \omega_R) &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\mathfrak{C}, R), \omega_R) \\
&\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\mathfrak{C}, \operatorname{Hom}_R(\omega_R, \omega_R)), \omega_R) && \text{[BH98, Theorem 3.3.10]} \\
&\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\mathfrak{C} \otimes_R \omega_R, \omega_R), \omega_R) && \text{(Hom-tensor adjointness)} \\
&\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\mathfrak{C}\omega_R, \omega_R), \omega_R) \\
&\cong \mathfrak{C}\omega_R
\end{aligned}$$

The second to last isomorphism follows since the kernel of the natural map from $\mathfrak{C} \otimes_R \omega_R$ onto $\mathfrak{C}\omega_R$ is a torsion module, and any homomorphism $\mathfrak{C} \otimes_R \omega_R \to \omega_R$

automatically sends torsion elements to zero. This completes the proof for the isomorphism in the statement.

Part 2: We start again with the exact sequence $0 \to R \to \overline{R} \to \frac{\overline{R}}{R} \to 0$ and observe that all these maps are the natural maps. Hence, applying $\mathrm{Hom}_R(,\omega_R)$ and using identification with colons we obtain

$$0 \to \omega_R :_K \overline{R} \to \omega_R :_K R \to \mathrm{Ext}^1_R\left(\frac{\overline{R}}{R}, \omega_R\right) \to 0$$

where the second map is induced via inclusion. Now note that by Lemma 2.10 and using the fact that $\omega_R :_K \omega_R = R$, we obtain

$$\overline{R} = R :_K \mathfrak{C} = (\omega_R :_K \omega_R) :_K \mathfrak{C} = \omega_R :_K \mathfrak{C}\omega_R$$

and hence,

$$\omega_R :_K \overline{R} = \omega_R :_K (\omega_R :_K \mathfrak{C}\omega_R) = \mathfrak{C}\omega_R$$

by using duality. Similarly, we get

$$\omega_R :_K R = \omega_R :_K (\omega_R :_K \omega_R) = \omega_R.$$

Thus we can rewrite the short exact sequence above as

$$0 \to \mathfrak{C}\omega_R \to \omega_R \to \mathrm{Ext}^1_R\left(\frac{\overline{R}}{R}, \omega_R\right)$$

where the second map is induced by inclusion. This gives us

$$\lambda\left(\frac{\omega_R}{\mathfrak{C}\omega_R}\right) = \lambda\left(\mathrm{Ext}^1_R\left(\frac{\overline{R}}{R}, \omega_R\right)\right).$$

Since taking Matlis duals preserves length, we get the desired equality by using Theorem 1.7 and the fact that $\mathrm{H}^0_{\mathfrak{m}}\left(\overline{R}/R\right) = \overline{R}/R$, the latter being a torsion module. □

Remark 2.12. The proof of the last statement of Lemma 2.11 appears as a part of proof in [BH92, Theorem 3] and also [Del94, Proposition 2.1].

Lemma 2.13. *[Mai22, Lemma 2.9] Let $(R, \mathfrak{m}, k)$ be a one dimensional Cohen Macaulay ring such that ω_R exists and J be an ideal such that it contains a non-zero divisor. Then*

$$\mathrm{Ann}_R\left(\frac{\omega_R}{J\omega_R}\right) = x :_R (x :_R J)$$

for every non zero divisor x in J.

Proof. First choose a non-zero divisor $x \in J$. Then we have

$$\frac{\omega_R}{J\omega_R} \cong \frac{\omega_{R/xR}}{\overline{J}\omega_{R/xR}} \cong \frac{E}{\overline{J}E}$$

where E denotes the injective hull of k as an R/xR module (Theorem 1.6). Hence we can reduce the problem to R being zero dimensional. It is enough to show now that the annihilator is $0 : (0 : J)$ (we are using J again instead of writing $\overline{J}$, for convenience).

$$\begin{aligned}
\operatorname{Ann}\left(\frac{E}{JE}\right) &= \operatorname{Ann}\left(\left(\frac{E}{JE}\right)^{\vee}\right) && \text{(Theorem 1.6)}\\
&= \operatorname{Ann}\left(\operatorname{Hom}_R(E \otimes R/J, E)\right)\\
&= \operatorname{Ann}\left(\operatorname{Hom}_R(R/J, \operatorname{Hom}_R(E, E)\right) && \text{(Hom-tensor adjointness)}\\
&= \operatorname{Ann}(\operatorname{Hom}_R(R/J, R))\\
&= 0 : (0 : J)
\end{aligned}$$

This finishes the proof. □

2.3.1 Relationship With The Conductor

It is clear that we can restrict our study of h(·) to ideals. More explicitly, for any ideal $\mathcal{I}$ in a one dimensional Noetherian local domain R with quotient field K, we have

$$\mathrm{h}(\mathcal{I}) := \min\{\lambda(R/J) \mid \mathcal{I} \cong J\}.$$

Remark 2.14. Two ideals I, J are isomorphic means that there exists $\alpha \in K$ such that $I = \alpha J$.

Proof. By Remark 2.8, any such isomorphism $\phi : I \to J$ must be multiplication by some element of $\alpha \in K$. In fact, one can note that for any $x, y \in I$, by R-linearity we have,

$$y\phi(x) = x\phi(y) \implies \frac{\phi(x)}{x} = \frac{\phi(y)}{y}.$$

Thus, we can choose $\alpha = \frac{\phi(x)}{x}$ for any nonzero divisor $x \in I$. $\square$

Proposition 2.15. *For any R-module M, if J realizes M for some ideal J, then*

$$\lambda(R/J) = \mathrm{h}(J) = \mathrm{h}(M).$$

In particular, J realizes itself.

Proof. Suppose $\mathrm{h}(J) = \lambda(R/I)$ for some $I \cong J$ with $\lambda(R/I) < \lambda(R/J)$. Let $\phi : J \to I$ be the isomorphism. Also let $M \xrightarrow{f} J \to 0$ be the surjection to J. Then

$$M \xrightarrow{\phi \circ f} I \to 0$$

and this implies that $\mathrm{h}(M) = \lambda(R/I) < \lambda(R/J)$, a contradiction to the hypothesis that J realizes M. $\square$

Before moving on, we should point out that isomorphic ideals can have different lengths as the following example shows.

Example 2.16. $R = k[[x, y]]/(x^2 + y^3)$. Note that $I_1 = (x, y)R$ and $I_2 = (x^2, xy)R$ are isomorphic since, $xI_1 = I_2$. However, $\lambda(R/I_2) = \lambda\left(\frac{k[[x,y]]}{(x^2,xy,y^3)}\right) = 4$ whereas $\lambda(R/I_1) = 1$.

The discussion before the example shows that it is actually enough to restrict the study of $\mathrm{h}(\cdot)$ to ideals. For what follows, we shall use $e(R)$ to denote the Hilbert-Samuel multiplicity of R. See Proposition 1.3 for the properties that we will use here.

Lemma 2.17. *[Mai22, Lemma 2.4] Let $(R, \mathfrak{m}, k)$ be a one dimensional Noetherian local domain with integral closure $\overline{R}$, both having the same field of fractions. Further assume that R is analytically unramified, i.e., $\widehat{R}$ is reduced, and let $x \in R$. Then*

$$\lambda\left(\frac{R}{xR}\right) = \lambda\left(\frac{\overline{R}}{x\overline{R}}\right).$$

Proof. Under the hypothesis, $\overline{R}$ is module finite over R by [SH06, Theorem 4.3.4]. Also note that since R and $\overline{R}$ have the same fraction field, $\operatorname{rank}_R(\overline{R}) = 1$. Since $\overline{R}$ is MCM over R, using Proposition 1.3 we get that

$$\lambda\left(\frac{\overline{R}}{x\overline{R}}\right) = e(x, \overline{R}) = e(x, R)\operatorname{rk}_R \overline{R} = e(x; R) = \lambda\left(\frac{R}{xR}\right). \qquad \square$$

Clearly the above holds as long as we choose x to be a non zero divisor. So the domain assumption can be removed.

Theorem 2.18. *[Mai22, Theorem 2.5] Let $(R, \mathfrak{m}, k)$ be a one dimensional analytically unramified Noetherian local domain with integral closure $\overline{R}$ and fraction field K. Further assume that $\overline{R}$ is a DVR and ω_R exists. Then for any ideal J of R, the following statements are equivalent:*

(a) $\mathrm{h}(J) = \lambda(R/J)$.

(b) $R :_K J \subseteq \overline{R}$.

Proof.

$(b) \implies (a)$**:** First, let $R :_K J \subseteq \overline{R}$. Suppose on the contrary that

$$\mathrm{h}(J) = \lambda(R/I) < \lambda(R/J)$$

for some ideal $I \cong J$. By Remark 2.14, there exists $a, b \in R$ such that $I = \frac{a}{b}J$, or equivalently $bI = aJ$. Now,

$$\frac{a}{b}J = I \subseteq R,$$

and hence by assumption we have $\frac{a}{b} \in \overline{R}$. Hence we get,

$$\lambda\left(\frac{\overline{R}}{a\overline{R}}\right) \geq \lambda\left(\frac{\overline{R}}{b\overline{R}}\right). \tag{2.3.1}$$

Next note that we have the following exact sequences coming from natural inclusion maps.

$$0 \to \frac{aR}{aI} \to \frac{R}{aI} \to \frac{R}{aR} \to 0,$$

$$0 \to \frac{bR}{bJ} \to \frac{R}{bJ} \to \frac{R}{bR} \to 0.$$

Using the fact that a, b are non zero divisors and also using additivity of length on these exact sequences, we get

$$\lambda(R/J) + \lambda(R/aR) = \lambda(R/aJ) = \lambda(R/bI) = \lambda(R/I) + \lambda(R/bR) \tag{2.3.2}$$

Since $\lambda(R/I) < \lambda(R/J)$, we have $\lambda(R/aR) < \lambda(R/bR)$ and thus by Lemma 2.17, we have

$$\lambda\left(\frac{\overline{R}}{a\overline{R}}\right) < \lambda\left(\frac{\overline{R}}{b\overline{R}}\right)$$

but this is a contradiction to Equation (2.3.1). This finishes one direction of the proof.

$(a) \implies (b)$**:** Suppose $\mathrm{h}(J) = \lambda(R/J)$ and let $\frac{a}{b} \in R :_K J$. Suppose $a \notin b\overline{R}$. Since $\overline{R}$

is a DVR, we have $b\overline{R} \subsetneq a\overline{R}$ and hence

$$\lambda\left(\frac{\overline{R}}{b\overline{R}}\right) > \lambda\left(\frac{\overline{R}}{a\overline{R}}\right). \tag{2.3.3}$$

Set $I = \dfrac{a}{b}J$. By assumption we have $\lambda\left(R/I\right) \geq \lambda\left(R/J\right)$. Hence, from Equation (2.3.2), we get $\lambda\left(R/aR\right) \geq \lambda\left(R/bR\right)$ which by Lemma 2.17, is a contradiction to Equation (2.3.3). This completes the proof. $\square$

By Remark 2.8, we can equivalently refer to statement (b) in Theorem 2.18 as $\mathrm{Hom}_R(J, R) \subseteq \overline{R}$ as R-submodules of K. $\mathrm{Hom}_R(J, R)$ is denoted in the literature as J^{-1} as well as J^*. We actually have now identified the ideals J whose inverses reside in $\overline{R}$.

Remark 2.19. Note that in the above proof, the direction (b) $\implies$ (a) did not require that $\overline{R}$ is a DVR. This helps us in recovering [Gre84, Lemma 3.1] as the next result shows.

Corollary 2.20. *[Mai22, Corollary 2.6] Let $(R, \mathfrak{m}, k)$ be a one dimensional analytically unramified domain. If an ideal J contains $\mathfrak{C}$, then*

$$\mathrm{h}(J) = \lambda\left(R/J\right).$$

Proof. Note that since $\widehat{R}$ is reduced, $\bar{R}$ is finitely generated over R and $\mathfrak{C}$ contains a non-zero divisor. Thus, R/J indeed has finite length. Now note that

$$\mathfrak{C} \subseteq J \implies R :_K J \subseteq R :_K \mathfrak{C} = \overline{R}$$

where the last equality follows from Lemma 2.10. By [$(b) \implies (a)$] part of Theorem 2.18 and by Remark 2.19, we are done. ☐

We can in fact make a stronger statement as the following theorem shows with the added assumption that $\overline{R}$ is a DVR.

Corollary 2.21. *[Mai22, Corollary 2.7] Let R be as in Theorem 2.18 and let J be an ideal that realizes itself. Then for any ideal $\mathcal{I}$ which contains J, we have* $\mathrm{h}(\mathcal{I}) = \lambda\,(R/\mathcal{I})$.

Proof. By assumption $\mathrm{h}(J) = \lambda(R/J)$. By Theorem 2.18, we have $R :_K J \subseteq \overline{R}$. Since $J \subseteq \mathcal{I}$, we have $R :_K I \subseteq R :_K J \subseteq \overline{R}$. Another application of Theorem 2.18 finishes the proof. ☐

The case when the conductor $\mathfrak{C}$ becomes equal to some power of the maximal ideal $\mathfrak{m}$ is often interesting. For instance, extensive discussions of such cases exist in [Ore81]. In such cases, we can sometimes describe the invariant $\mathrm{h}(\cdot)$ for high enough powers of $\mathfrak{m}$.

Lemma 2.22. *Let $(R, \mathfrak{m}, k)$ be a one dimensional local domain with finite integral closure $\overline{R}$. Then for any non-zero ideal I with a principal reduction z, we have*

$$z\overline{R} = I\overline{R}.$$

Proof. Since $z \subseteq I$, we get

$$z\overline{R} \cap R \subseteq I\overline{R} \cap R \subseteq \overline{I\overline{R}} \cap R.$$

By [SH06, Proposition 1.5.2], we have $z\overline{R} = \overline{z\overline{R}}$. Hence applying [SH06, Proposition 1.6.1] to the above containments, we immediately obtain

$$\overline{z} \subseteq I\overline{R} \cap R \subseteq \overline{I}.$$

But $\overline{I} = \overline{z}$ since z is a reduction of I. Hence, $z\overline{R} \cap R = I\overline{R} \cap R$. The result follows. □

Proposition 2.23. *Let $(R, \mathfrak{m}, k)$ be a one dimensional analytically unramified domain with infinite residue field. If $\mathfrak{C} = I^N$ for some N, then*

$$\mathrm{h}(I^\ell) = \lambda(R/\mathfrak{C})$$

for all $\ell \geq N$.

Proof. Let z be a minimal reduction of I which exists by Theorem 1.5. By Lemma 2.22, $I\overline{R} = z\overline{R}$. Since $\mathfrak{C}\overline{R} = \mathfrak{C}$, we immediately obtain that

$$\mathfrak{C}I = z\mathfrak{C},$$

i.e., $\mathfrak{C}I \cong \mathfrak{C}$. Since $\mathfrak{C} = I^N$, we get $I^\ell \cong \mathfrak{C}$ for all $\ell \geq N$. The proof is now complete by Corollary 2.20. □

Corollary 2.24. *Let $(R, \mathfrak{m}, k)$ be a one dimensional analytically unramified local domain with infinite residue field. If $\mathfrak{C} = \mathfrak{m}^N$ for some N and $\mu(\mathfrak{m}^i) = \binom{n+i-1}{i}$ for $1 \leq i \leq N-1$ where $n = \mu(\mathfrak{m})$, then*

$$\mathrm{h}(\mathfrak{m}^\ell) = \binom{N+n-2}{n}$$

for all $\ell \geq N$.

Proof. By Proposition 2.23, $\mathrm{h}(\mathfrak{m}^\ell) = \lambda(R/\mathfrak{C})$ for all $\ell \geq N$. Note that

$$\lambda(R/\mathfrak{m}^N) = 1 + \sum_{i=1}^{N-1} \mu(\mathfrak{m}^i) = \sum_{i=0}^{N-1} \binom{n+i-1}{i} = \binom{N+n-1}{n}.$$

□

Note that Corollary 2.24 recovers [Gre84, Lemma 5.5] by putting $n = 2$.

Theorem 2.25. *[Mai22, Theorem 2.10] Let R be as in the hypothesis of Theorem 2.18. For any ideal J of R, the following statements are equivalent.*

(a) $\mathrm{h}(J) = \lambda\left(R/J\right)$;

(b) $\mathfrak{C} \subseteq x :_R (x :_R J)$ *for some* $x \in J$;

(c) $\mathfrak{C}\omega_R \subseteq J\omega_R$.

Proof.

[(a) $\implies$ (b)]**:** By Theorem 2.18, $R :_K J \subseteq \overline{R}$. Let $a \in x :_R J$. Now, $aJ \subseteq (x)$ implies that $\frac{a}{x}J \subseteq R$ and hence by assumption, $\frac{a}{x} \in \overline{R}$. Thus by Lemma 2.10, we have

$$\frac{a}{x}\mathfrak{C} \subseteq R$$

which in turn implies statement (b).

[(b) $\implies$ (c)]**:** By Lemma 2.13, we get

$$\mathfrak{C} \subseteq \mathrm{Ann}_R\left(\frac{\omega_R}{J\omega_R}\right).$$

Hence, $\mathfrak{C}\omega_R \subseteq J\omega_R$.

[$(c) \implies (a)$]**:** Note that we have $\omega_R :_K J\omega_R \subseteq \omega_R :_K \mathfrak{C}\omega_R$ and this implies

$$(\omega_R :_K \omega_R) :_K J \subseteq (\omega_R :_K \omega_R) :_K \mathfrak{C}$$

by properties of colons. Since $R = \omega_R :_K \omega_R$ as submodules of K, we have

$$R :_K J \subseteq R :_K \mathfrak{C}.$$

This implies statement (a) by Theorem 2.18 and Lemma 2.10. This finishes the proof. □

Proposition 2.26. *Let $(R, \mathfrak{m}, k)$ be a one dimensional analytically unramified local domain with integral closure $\overline{R}$ and fraction field K. Further assume that $\overline{R}$ is a DVR and ω_R exists. Identifying ω_R with some ideal of R, we have*

$$\mathrm{h}(M) \leq \mathrm{h}(\mathfrak{C}) + \lambda\left(\frac{\mathfrak{C}}{\mathfrak{C}\omega_R}\right) = \lambda\left(\frac{R}{\mathfrak{C}\omega_R}\right)$$

for any R-module M which has non-zero rank.

Proof. Let J realize M. Then by Proposition 2.15, J realizes itself. So, by Theorem 2.25, we have $\mathfrak{C}\omega_R \subseteq J\omega_R$. Now combining all this data we get,

$$\begin{aligned}
\mathrm{h}(M) &= \lambda\left(R/J\right) \\
&= \lambda\left(R/J\omega_R\right) - \lambda\left(J/J\omega_R\right) \\
&= \lambda\left(R/\mathfrak{C}\omega_R\right) - \lambda\left(J\omega_R/\mathfrak{C}\omega_R\right) - \lambda\left(J/J\omega_R\right) \\
&= \lambda\left(R/\mathfrak{C}\right) + \lambda\left(\mathfrak{C}/\mathfrak{C}\omega_R\right) - \left(\lambda\left(J\omega_R/\mathfrak{C}\omega_R\right) + \lambda\left(J/J\omega_R\right)\right) \\
&= \lambda\left(R/\mathfrak{C}\right) + \lambda\left(\mathfrak{C}/\mathfrak{C}\omega_R\right) - \lambda\left(J/\mathfrak{C}\omega_R\right) \\
&= \mathrm{h}\left(\mathfrak{C}\right) + \lambda\left(\mathfrak{C}/\mathfrak{C}\omega_R\right) - \lambda\left(J/\mathfrak{C}\omega_R\right)
\end{aligned}$$

where we used Corollary 2.20 to conclude $\mathrm{h}(\mathfrak{C}) = \lambda(R/\mathfrak{C})$. This finishes the proof. □

Corollary 2.27. *Let $(R, \mathfrak{m}, k)$ be an analytically unramified one dimensional Gorenstein local domain with integral closure $\overline{R}$ and fraction field K. Further assume that $\overline{R}$ is a DVR. Then*

$$\mathrm{h}(M) \leq \lambda\left(\frac{R}{\mathfrak{C}}\right)$$

for any R-module M which has non-zero rank.

Proof. Since R is Gorenstein, ω_R exists and can be chosen to be R itself. The proof now follows immediately from Proposition 2.26. □

We can also link $\mathrm{h}(\cdot)$ with another invariant of the ring as the following proposition shows.

Proposition 2.28. *Let $(R, \mathfrak{m}, k)$ be a one dimensional analytically unramified local domain with integral closure $\overline{R}$ and fraction field K. Further assume that $\overline{R}$ is a DVR and ω_R exists. Then*

$$\mathrm{h}(M) \leq \mathrm{h}(\omega_R) + \lambda\left(\frac{\overline{R}}{R}\right)$$

for any R-module M that has non-zero rank.

Proof. Let J realize M and identify ω_R with some ideal of R. We proceed exactly as in Proposition 2.26.

$$\begin{aligned}
\mathrm{h}(M) &= \lambda\left(R/J\right) \\
&= \lambda\left(R/J\omega_R\right) - \lambda\left(J/J\omega_R\right) \\
&= \lambda\left(R/\mathfrak{C}\omega_R\right) - \lambda\left(J\omega_R/\mathfrak{C}\omega_R\right) - \lambda\left(J/J\omega_R\right) \\
&= \lambda\left(R/\omega_R\right) + \lambda(\omega_R/\mathfrak{C}\omega_R) - (\lambda(J\omega_R/\mathfrak{C}\omega_R) + \lambda(J/J\omega_R)) \\
&= \lambda\left(R/\omega_R\right) + \lambda\left(\overline{R}/R\right) - \lambda\left(J/\mathfrak{C}\omega_R\right)
\end{aligned}$$

where in the last line we used Lemma 2.11. Since this is satisfied by all possible identifications of ω_R inside R, the proof is complete by Definition 2.1. □

2.3.2 The Invariant $\mathrm{h}(\omega_R)$

We point out that $\mathrm{h}(\omega_R)$ is an invariant of the ring and by itself can be an interest for further inquiry. For example, $\mathrm{h}(\omega_R) = 0$ if and only if R is Gorenstein. Thus it is natural to ask if $\mathrm{h}(\omega_R)$ can be used as a tool in classification problems.

Theorem 2.29. *Let $(R, \mathfrak{m}, k)$ be a one dimensional analytically unramified local domain with infinite residue field k. Assume that $\overline{R}$ is a DVR, the canonical module exists and can be identified with some ideal of R. Then the following statements hold.*

1. $\mathrm{h}(\omega_R) = 0$ *if and only if R is Gorenstein.*

2. *If R is not Gorenstein, then*

$$e(J;R) - \mu(\omega_R) + 1 \geq \mathrm{h}(\omega_R) \geq \lambda(R/\mathfrak{C}) + e(R) - \lambda(\overline{R}/R)$$

where J is any ideal that realizes the canonical module.

Proof. Statement (1) is clear. For statement (2), we follow the proof of Proposition 2.26 with M being the canonical module. Let J be the ideal that realizes the canonical module and let us denote this J by ω_R. So, we get

$$\mathrm{h}(\omega_R) = \lambda(R/\omega_R) = \lambda(R/\mathfrak{C}) + \lambda(\mathfrak{C}/\mathfrak{C}\omega_R) - \lambda(\omega_R/\mathfrak{C}\omega_R).$$

Choose z to be a minimal reduction of ω_R which exists by Theorem 1.5. Then by Lemma 2.22, we have $\omega_R\overline{R} = z\overline{R}$. Since $\mathfrak{C} = \mathfrak{C}\overline{R}$, we get $\mathfrak{C}\omega_R = z\mathfrak{C}$. Thus using Lemma 2.11, we obtain that

$$\mathrm{h}(\omega_R) = \lambda(R/\mathfrak{C}) + \lambda(\mathfrak{C}/z\mathfrak{C}) - \lambda(\overline{R}/R). \tag{2.3.4}$$

Since $\mathfrak{C}$ is MCM of rank 1, Proposition 1.3 gives us

$$\mathrm{h}(\omega_R) = \lambda(R/\mathfrak{C}) + e(z;R) - \lambda(\overline{R}/R).$$

For the upper bound, first notice that $e(z;R) = e(\omega_R;R)$ [SH06, Proposition 11.2.1]. Then using [SH06, Proposition 12.2.3], we get

$$\lambda(R/\mathfrak{C}) - \lambda(\overline{R}/R) \leq 1 - \mu(\omega_R).$$

Combining these data, Equation (2.3.4) immediately yields the required upper bound.

Further, since R is not Gorenstein, we have $e(z;R) \geq e(R)$. So, Equation (2.3.4) immediately gives the lower bound

$$\mathrm{h}(\omega_R) \geq \lambda(R/\mathfrak{C}) + e(R) - \lambda(\overline{R}/R). \qquad \square$$

The second part of Theorem 2.29 answers a question that was raised by S. Greco in an informal conversation. He had asked if $\min\{\lambda(R/\omega_R)\} \leq \lambda(R/\mathfrak{C}) + e(R) - \lambda(\overline{R}/R)$. Our computations show that this is not the case. Moreover, the following example illustrates that the inequalities in Theorem 2.29 can be strict. It also shows that $\mathfrak{C}\omega_R \subseteq J\omega_R$ does not imply $\mathfrak{C} \subseteq J$ in Theorem 2.25. For more details regarding the computations here, we refer the reader to Theorem 3.19.

Example 2.30. Let $R = k[[t^5, t^6, t^8]] \cong \dfrac{k[[x,y,z]]}{(y^3 - x^2z, x^2y - z^2, x^4 - y^2z)}$ where k is algebraically closed of characteristic 0. Then $e(R) = 5, \overline{R} = k[[t]]$ and $\mathfrak{C} = t^{10}\overline{R} = (x^2, xy, y^2, xz, yz)$. So, $\lambda(R/\mathfrak{C}) = 4$ (notice that the missing valuations from $\mathfrak{C}$ which are present in R are $\{0, 5, 6, 8\}$). We also have that $\lambda(\overline{R}/R) = 6$ (the valuations missing from R but are present in $\overline{R}$ are $\{1, 2, 3, 4, 7, 9\}$. Thus,

$$\lambda(R/\mathfrak{C}) + e(R) - \lambda(\overline{R}/R) = 4 + 5 - 6 = 3.$$

Moreover, we can choose the canonical ideal to be $\omega_R = (t^6, t^8) = (y, z)$. With this choice, notice that $\mathfrak{C}\omega_R \subseteq \omega_R^2$ and hence by Theorem 2.25, we get that $\mathrm{h}(\omega_R) = \lambda(R/\omega_R) = \lambda\left(k[[x,y,z]]/(y, z, x^4)\right) = 4$. Thus,

$$\mathrm{h}(\omega_R) > \lambda(R/\mathfrak{C}) + e(R) - \lambda(\overline{R}/R)$$

in this case.

Next, notice that $\mu(\omega_R) = 2, e(\omega_R; R) = 6$. So,

$$e(\omega_R; R) - \mu(\omega_R) + 1 = 6 - 2 + 1 = 5 > 4 = \mathrm{h}(\omega_R).$$

Finally, notice that $\mathfrak{C} \not\subseteq \omega_R$ even though $\mathfrak{C}\omega_R \subseteq \omega_R^2$. This shows that without the Gorenstein assumption, in general we may not have $\mathfrak{C}$ contained inside a realizing ideal of a module M.

Remark 2.31. In an unpublished work, J. Sally had shown that $\omega_R \ncong \mathfrak{m}$ in a one dimensional non-regular local domain, where ω_R exists. Indeed if such an isomorphism exists, then $\operatorname{Hom}_R(\mathfrak{m}, \mathfrak{m}) \cong R$ (Theorem 1.6(6)). But it is well known that if R is not regular, then $\operatorname{Hom}_R(\mathfrak{m}, \mathfrak{m}) = \operatorname{Hom}_R(\mathfrak{m}, R)$. This is because if some homomorphism g does not map $\mathfrak{m}$ into $\mathfrak{m}$, then g is a surjection from $\mathfrak{m}$ to R. But then R splits off $\mathfrak{m}$, which is not possible under the hypothesis. Thus, we get that $\operatorname{Hom}_R(\mathfrak{m}, R) \cong R$. Using Remark 2.8, we get that $R :_K \mathfrak{m} = R$. Take any non-zero-divisor $x \in \mathfrak{m}$, then using [SH06, Lemma 2.4.3], we see that

$$R = R :_K \mathfrak{m} = \frac{1}{x}(xR :_R \mathfrak{m}).$$

This implies that $xR = xR :_R \mathfrak{m}$ which in turn, shows that $\mathfrak{m} = (x)$. But this is a contradiction to the assumption that R is non-regular.

Thus, $\omega_R \cong \mathfrak{m}$ implies that $\mathrm{h}(\omega_R) = 0$. So, we conclude that $\mathrm{h}(\omega_R) \geq 2$ if R is non-Gorenstein.

2.3.3 Relationship With The Trace Ideal

As the name suggests, in this section we are going to briefly venture into trace ideals. We shall show that if J realizes M, we can establish a relationship of J to the trace ideal of M. The fact that there is a relationship is actually immediate as soon as one recalls the definition of trace ideal. Here we let R be any Noetherian ring with total ring of fractions K.

Definition 2.32. *The trace ideal of an R-module M, denoted $\operatorname{tr}_R(M)$, is the ideal $\sum \alpha(M)$ as α ranges over $M^* := \operatorname{Hom}_R(M, R)$.*

Notice that any J to which a module M surjects, is the image of a map in M^*. Thus we can think of any such J as a ***partial trace ideal*** of M. In particular, any ideal which realizes M is a partial trace ideal in the sense we just discussed.

The trace ideal can also be thought of as coming from a *trace map* which serves as sort of a generalization of the ordinary trace map that we encounter in linear algebra. We make this notion explicit.

Remark 2.33. The *trace map* is the following:

$$\tau_M : \; M \otimes_R M^* \to R$$

$$\tau_M(m \otimes \alpha) = \alpha(m)$$

Note that the image of the trace map is $\operatorname{tr}_R(M)$. This can be seen as follows:

$$\operatorname{Im}(\tau_M) = \left\{ \sum_i \alpha_i m_i \;\middle|\; \alpha_i \in M^*, m_i \in M \right\} = \sum_{\alpha \in M^*} \alpha(M) = \operatorname{tr}_R(M).$$

An ideal I is called a trace ideal if $I = \operatorname{tr}_R(M)$ for some R-module M. We list some of the properties that we shall use. As mentioned, the proofs can be found in a variety of sources.

Proposition 2.34. *[Lin17, Proposition 2.8],[KT19, Proposition 2.4] Let M, N be R-modules. Let I be any ideal that contains a non-zero divisor. The following statements hold.*

1. *If $M \cong N$, then $\operatorname{tr}_R(M) = \operatorname{tr}_R(N)$.*

2. *$\operatorname{tr}_R(I) = I(R :_K I)$.*

3. *$\operatorname{tr}_R(I) = I$ if and only if $I :_K I = R :_K I$.*

The following remark was originally made by W. Vasconcelos [Vas91, Remark 3.3]. One can also find a detailed proof in Herzog, Hibi and Stamate[HHS19, Proposition 3.1]. It gives an algorithmic way of computing trace ideals.

Remark 2.35. One can calculate the trace ideal of a module from its presentation matrix as follows. Suppose Φ is a presentation matrix for a module M and Ψ is a matrix whose columns generate the kernel of Φ^t. Then there is an equality:

$$\operatorname{tr}_R(M) = I_1(\Psi)$$

where $I_1(\Psi)$ is the ideal generated by the entries of Ψ.

Example 2.36. Let $R = k[x, y, z]$. Let $I = (xy, yz)$. The resolution of I looks like

$$R \xrightarrow{\begin{bmatrix} -z \\ x \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} xy & yz \end{bmatrix}} R \to 0$$

Following notation of Remark 2.35, $\Phi^t = [-z \;\; x]$; so $\Psi = [x \; z]^t$. Hence, $\mathrm{tr}_R(I) = (x, z)$. In fact, this is an example where $I \neq \mathrm{tr}_R(I)$ and hence is not a trace ideal by Proposition 2.34; however, $I \cong \mathrm{tr}_R(I)$.

We include a Macaulay 2 code (see Figure 2.1) that computes trace ideals of ideals in quotients of polynomial rings.

```
traceIdeal = J -> (
        C := res J;
        A := C.dd_2;
        trJ = minors(1,syz(transpose A));
        return trJ
        )
```

Figure 2.1: Code to Compute Trace Ideal

The algorithm in Remark 2.35 is very easy to implement as is evident from the code above.

For the time being let us run the same example again that we computed before to verify that the code works (see Figure 2.2). We compute another example before we link this to our invariant (see Figure 2.3).

We are now ready to establish the relationship between trace ideals and the invariant. It is quite easy as the following observations show.

Proposition 2.37. *[Mai22, Proposition 3.4] Let R be a one dimensional analytically unramified local domain. For any trace ideal I,*

$$\mathrm{h}(I) = \lambda(R/I).$$

```
i1 : traceIdeal = J -> (
         C := res J;
         A := C.dd_2;
         trJ = minors(1,syz(transpose A));
         return trJ
         )

o1 = traceIdeal

o1 : FunctionClosure

i2 : R=QQ[x,y,z]

o2 = R

o2 : PolynomialRing

i3 : I=ideal(x*y,y*z)

o3 = ideal (x*y, y*z)

o3 : Ideal of R

i4 : traceIdeal(I)

o4 = ideal (x, z)

o4 : Ideal of R
```

Figure 2.2: Example 1

```
i1 : traceIdeal = J -> (
         C := res J;
         A := C.dd_2;
         trJ = minors(1,syz(transpose A));
         return trJ
         )

o1 = traceIdeal

o1 : FunctionClosure

i2 : R=QQ[x,y,z]

o2 = R

o2 : PolynomialRing

i3 : I=ideal(x^2*y*z+y^2,z^3)

             2       2   3
o3 = ideal (x y*z + y , z )

o3 : Ideal of R

i4 : traceIdeal(I)

             3   4   2       2   3    2 2
o4 = ideal (z , y , x y*z + y , y z, y z )

o4 : Ideal of R
```

Figure 2.3: Example 2

Proof. The proof follows from Proposition 2.34(3) and Theorem 2.18. □

Proposition 2.38. *[Mai22, Proposition 3.4] Let R, M be as in Definition 2.1 with M having non-zero rank. Then* $\mathrm{h}(M) \geq \mathrm{h}\left(\mathrm{tr}_R(M)\right)$.

Proof. Choose an ideal J which realizes M. Let $f_J : M \to J$ be the corresponding surjection. By Remark 2.33, $J = \mathrm{Im}(f_J) \subseteq \mathrm{tr}_R(M)$. This observation along with Proposition 2.37 finishes the proof. □

Proposition 2.39. *[Mai22, Proposition 3.5] Let R, M be as in Definition 2.1 and assume that M has rank one. Then for any partial trace ideal J of M, we have*

$$\mathrm{tr}_R(J) = \mathrm{tr}_R(M).$$

Proof. For any ideal J such that M surjects to J, we have the following exact sequence:

$$0 \to \tau(M) \to M \xrightarrow{f_J} J \to 0 \tag{2.3.5}$$

where $\tau(M)$ is the torsion submodule of M. It is clear from this that $\operatorname{tr}_R(J) \subseteq \operatorname{tr}_R(M)$ as we can precompose any map $J \to R$ with f_J to get a map $M \to R$. Applying $\operatorname{Hom}_R(\ \ , R)$ to (2.3.5), we have the isomorphism $f_J^* : J^* \to M^*$. For the other containment, observe that

$$\operatorname{tr}_R(M) = \sum_{\alpha \in M^*} \alpha(M) = \sum_{\substack{g \in J^* \\ \alpha = f_J^*(g)}} f_J^*(g)(M) = \sum_{\substack{g \in J^* \\ \alpha = f_J^*(g)}} g(f_J(M)) = \sum_{\substack{g \in J^* \\ \alpha = f_J^*(g)}} g(J) \subseteq \operatorname{tr}_R(J).$$

□

Chapter 3

Partial Trace Ideals and Berger's Conjecture

The main goal of this chapter is to establish some new cases of a conjecture due to R. W. Berger. We apply the study of partial trace ideals and the length invariant $\mathrm{h}(\cdot)$ to $\Omega_{R/k}$ to achieve this goal. We show for instance that if $\mathrm{h}(\Omega_{R/k})$ is less than $\frac{\mathrm{edim}(R)+1}{2}$, then the conjecture always holds (see Theorem 3.12 for the general statement). This provides a generalization to G. Scheja's proof which, in our notations, is the case when $\mathrm{h}(\Omega_{R/k})$ equals 1. We further provide an algorithm to compute $\mathrm{h}(\Omega_{R/k})$ (see Theorem 3.19,Corollary 3.21 and Section 3.3.1). This in turn gives a new classification of quasi homogeneous rings in terms of the valuation of the trace ideal of $\Omega_{R/k}$ (see Theorem 3.28).

3.1 A Conjecture of R. W. Berger

Around 1963, R. W. Berger in his paper *Differentialmoduln eindimensionaler lokaler Ringe* [Ber63], investigated the relationship between the regularity of a one dimensional complete local domain which is an algebra over a perfect field k and the torsion submodule of the universally finite module of differentials (Definition 1.8). Based on the result he established (we shall state it soon), he made the following conjecture.

Berger's Conjecture. Let k be a perfect field and $(R, \mathfrak{m}, k)$ be a complete reduced local one dimensional k-algebra. Then R is regular if and only if the universally finite differential module $\Omega_{R/k}$ is torsion-free.

Remark 3.1. Let R be as in the conjecture. Then by Theorem 1.1,

$$R = \frac{k[[X_1, X_2, ..., X_n]]}{I}$$

for some radical ideal I in $S = k[[X_1, ..., X_n]]$ with I in the square of the maximal ideal of S and $\mathrm{ht}(I) = n - 1$. Note that we have an explicit description of the presentation of $\Omega_{R/k}$ as an R-module using Definition 1.8. Hence if R is regular, then $n = 1$, and using the presentation we get that $\Omega_{R/k}$ is free. This shows that the main question is the following:

Question 3.2. If $\Omega_{R/k}$ is torsion-free, is R regular?

This conjecture has been studied to quite an extent using different methods. In all these cases, the conjecture has been shown to be true.

1. The conjecture actually arose when R. W. Berger in [Ber63] proved the cases of *deviation* at most 1 under the assumption that R is a domain (deviation of ideal I is defined to be the difference between $\mu(I)$ and height of the ideal). This was generalized to deviation at most 3 in the reduced case by B. Ulrich [Ulr81]. Another such discussion can be found in R. Hübl's work [H9̈0].

2. J. Herzog [Her78, Satz 3.2 and Satz 3.3] showed that the conjecture is true for R a domain with embedding dimension 3 and also for R Gorenstein domain of embedding dimension 4.

3. S. Isogawa [Iso91, Theorem 1] provided a proof for the case when the Hilbert-Samuel multiplicity of R is at most $\binom{n}{2} - 1$ where n is the embedding dimension. A more general result can be found in the work of Güttes [Güt90, Satz 5 and Satz 5′]. In particular, it was shown in [Güt90, Satz 6] that the conjecture is true for R a domain of multiplicity at most 9 and also if R is Gorenstein domain of multiplicity at most 13.

4. The conjecture has also been studied from the point of view of *linkage* by J. Herzog and R. Waldi. The conjecture was shown to be true if R is in the linkage class of a complete intersection (i.e., if the defining ideal of R is licci) and k is algebraically closed ([HW84, Corollary 3], [HW86, Theorem 4.5]). In fact, this result generalizes (2) above.

5. In [CnGW98], the authors study an Artinian version of Berger's conjecture and

prove that this version implies the original conjecture [CnGW98, Main Theorem 0.1]). The paper also proves the conjecture in the case where the cube of the maximal ideal of R is inside the conductor of R [CnGW98, Theorem 2.3]. In [CnK99], the authors discuss cases when the Artinian Berger Conjecture is true.

6. The conjecture is known to be true when R is a graded domain and $\text{char}(k) = 0$. This was proved by G. Scheja [Sch70, Satz 9.8]. We shall soon generalize this result and in fact, this result is the main motivation behind this chapter's techniques.

Other techniques also have been used such as *smoothability* ([Bas77], [BG80], [HW86], [Koc83]), *maximal torsion* [Poh91, Theorem 1], *equisingularity* [Poh89], *quadratic transforms* [Ber88] etc. A very nice summary of most of the the above mentioned results, along with the main ideas of proofs, can be found in [Ber94].

We should mention here that Berger's Conjecture does not hold in higher dimensions. We illustrate this using the following example.

Example 3.3. Let $R = \dfrac{\mathbb{C}[[X,Y,Z]]}{(X^3 - YZ)}$. This is a two dimensional non-regular, normal local domain with maximal ideal $(x, y, z) = (X, Y, Z)R$ and perfect residue field $\mathbb{C}$.

Now using Definition 1.8, we get the exact sequence,

$$R \xrightarrow[A=\begin{bmatrix}3x^2\\ -z\\ -y\end{bmatrix}]{} R^3 \to \Omega_{R/k} \to 0.$$

Notice that A is injective: for any $r \in R$, $Ar = 0$ implies that $3x^2r = zr = yr = 0$; since R is a domain with maximal ideal (x, y, z), $r = 0$. Thus, the projective dimension of $\Omega_{R/k}$, $\mathrm{pdim}_R(\Omega_{R/k}) = 1$ (see [Rot09, Chapter 8] for further details on pdim).

Hence, by Auslander-Buchsbaum formula ([Eis13, Theorem 19.9]), we get

$$\mathrm{depth}(\Omega_{R/k}) = 2 - 1 = 1.$$

This shows that $\mathfrak{m}$ is not an associated prime of $\Omega_{R/k}$ (it is well-known that P is an associated prime of M if and only if $\mathrm{depth}(M_P) = 0$).

Now suppose that $0 \neq \tau \in \tau(\Omega_{R/k})$, the torsion submodule of $\Omega_{R/k}$. Then there exists $0 \neq r \in R$ such that $r\tau = 0$. Let P be an associated prime of $\Omega_{R/k}$ containing r ([Eis13, Theorem 3.1(b)]). Then by our discussion above, P must have height 1 and $\mathrm{depth}((\Omega_{R/k})_P) = 0$. However since R is normal, R_P is a one dimensional regular local ring (see [SH06, Theorem 4.5.3]), and hence $(\Omega_{R/k})_P$ is free. So we get that

$$0 = \mathrm{depth}((\Omega_{R/k})_P) = \mathrm{depth}(R_P) = 1,$$

a contradiction. Thus, $\tau(\Omega_{R/k}) = 0$.

In fact, as discussed in [Her94, 3], G. Scheja in [Sch70, Satz 9.4] had shown that in a reduced analytic k-algebra R, where k is a perfect field, if

$$\operatorname{depth}(\Omega_{R/k}) \geq \sup\{\dim(R/P) \mid P \in \operatorname{Spec}(R), R_P \text{ is non-regular}\} + 1,$$

then $\tau(\Omega_{R/k}) = 0$.

3.1.1 Setup, Notations and Main Goal

In this thesis, we are going to concentrate on the case when R, as in Berger's Conjecture, is a domain. For notational convenience, we are going to isolate the main situation which we will refer to frequently throughout this thesis.

Set up 3.4. We say that $(R, \mathfrak{m}, k)$ is a ***local singular complete curve*** if

- k is a perfect field.
- $R = P/I$ where $P = k[[X_1, ..., X_n]]$, $n \geq 2$.
- I is a prime ideal in P with $I \subseteq \mathfrak{n}^2$ where $\mathfrak{n}$ is the maximal ideal of P,
- $\operatorname{height}(I) = n - 1$,

Let $K = Frac(R)$ and $\overline{R}$ be the integral closure of R in K. We shall denote the conductor ideal by $\mathfrak{C}$ (this was defined in Chapter 2) and the canonical module by ω_R. Note that ω_R exists under the hypothesis that R is a local singular complete curve and can be identified with some ideal of R (Chapter 3, [BH98]).

Remark 3.5. Note that we are putting $n \geq 2$ because we are mainly interested in the contrapositive of Question 3.2.

Let $\Omega_{R/k}$ denote the universally finite module of differentials (Definition 1.8) and $\tau(\Omega_{R/k})$ be its torsion submodule.

MAIN QUESTION: Let $(R, \mathfrak{m}, k)$ be a local singular complete curve. Is $\tau(\Omega_{R/k}) \neq 0$?

3.1.2 Interpretation of Non-Zero Torsion

In Section 2.1, we discussed the idea of capturing torsion of a finitely generated rank one module M over a one dimensional local domain by looking at its partial trace ideals. When R is a local singular complete curve, we get that $\Omega_{R/k}$ is a finitely generated rank one module by Theorem 1.10. As such we can apply our study of the invariant $\mathrm{h}(\cdot)$ to $\Omega_{R/k}$. Let us briefly try to understand what it means to be a non-zero element of $\tau(\Omega_{R/k})$ under our setup.

A WAY TO GET THE TORSION SUBMODULE $\tau(\Omega_{R/k})$: If $\Omega_{R/k}$ surjects on an ideal J, then by rank calculations we get the following exact sequence:

$$0 \to \tau(\Omega_{R/k}) \to \Omega_{R/k} \to J \to 0.$$

MEANING OF "NON-ZERO TORSION": Here is an algorithmic approach of concluding whether a torsion element is non-zero or not.

STEP 1: Get a column vector which is in the kernel of a surjection as above.

STEP 2: Check whether this column vector can be written in terms of columns of

$A = \text{Jac}(I)$ (recall that A is the presentation matrix of $\Omega_{R/k}$ as in Definition 1.8).

We provide an example to explain the above procedure.

Example 3.6. Let $R = \mathbb{Q}[[t^3, t^4, t^5]] = \frac{\mathbb{Q}[[X,Y,Z]]}{I}$, $I = (Y^2 - XZ, X^2Y - Z^2, X^3 - YZ)$. Let $(x, y, z) = (X, Y, Z)R$. By Definition 1.8, we have $\Omega_{R/k} = \dfrac{R^3}{< \text{columns of } A >}$ where

$$A = \begin{bmatrix} -z & 2xy & 3x^2 \\ 2y & x^2 & -z \\ -x & -2z & -y \end{bmatrix}.$$

Equivalently using Remark 1.9 , $\Omega_{R/k} \cong \dfrac{RdX \oplus RdY \oplus RdZ}{U}$ where U is given by

$$u_1 = -zdX + 2ydY - xdZ,$$

$$u_2 = 2xydX + x^2dY - 2zdZ,$$

$$u_3 = 3x^2dX - zdY - ydZ$$

Here $\Omega_{R/k} \twoheadrightarrow (X, Y, Z)R = (x, y, z)$ via the lifting of the Euler derivation map (see Example A.2 (3)). Explicitly, the surjection is as follows.

$$dX \mapsto 3x, dY \mapsto 4y, dZ \mapsto 5z$$

So, $-4zdX + 3ydY$ is a torsion element as it is in the kernel. Moreover, it is non-zero as it can't be written in the span of the columns of A.

[**Alternatively**, one can see that $-4zdX + 3ydY$ is a torsion element since it is killed by z as the following computation shows.

$$y^2 = xz \qquad 2ydY = xdZ + zdX$$

$$x^2y = z^2 \qquad 2xydX + x^2dY = 2zdZ$$

$$x^3 = yz \qquad 3x^2dX = ydZ + zdY$$

$$\alpha = z(-4zdX + 3ydY) = z(-8ydY + 4xdZ + 3ydY) = -5yzdY + 4xzdZ$$

$$= -5yzdY + 4x^2ydX + 2x^3dY = -5yzdY + 4z^2dX + 2yzdY$$

$$= -3yzdY + 4z^2dX = -\alpha \implies \alpha = 0]$$

Remark 3.7. Rings with a surjection of $\Omega_{R/k}$ onto the maximal ideal were termed ***quasi homogeneous*** by Scheja[Sch70]. Kunz and Ruppert [KR77] showed that any quasi homogeneous analytic k-algebra is the completion of a graded (not necessarily standard) k-algebra.

Scheja proved Berger's conjecture when R is quasi homogeneous [Sch70, Satz 9.8]. As we shall soon see, our result will generalize Scheja's result. In fact, the motivation for Definition 2.1 came primarily from analyzing Scheja's proof.

3.2 Cases of Berger's Conjecture using $\mathrm{h}(\Omega_{R/k})$

Equipped with the invariant $\mathrm{h}(\Omega_{R/k})$, we now direct our efforts to establish some cases of Berger's Conjecture. So in this section, $(R, \mathfrak{m}, k)$ will be a local singular complete curve. We begin with the following very important result whose proof we skip.

Theorem 3.8 ([Tei76],[Hir83]). *Let $(R, \mathfrak{m}, k)$ be as in Berger's Conjecture. Suppose* $\operatorname{char}(k) = 0$*. If* $\mathrm{h}(\Omega_{R/k}) = 0$*, then R is regular.*

This can be found, for instance in the Nordic Summer School Lecture notes of B. Teissier: [Tei76, discussion on pages 586-587]. This is essentially a generalization of the proof with $k = \mathbb{C}$ which can be found for instance in the works of H. Hironaka [Hir83, Lemma 5]. Both the authors attribute the proofs to Zariski.

Thus, the above proof shows that in characteristic 0, $\mathrm{h}(\Omega_{R/k}) \neq 0$ whenever R is a local singular complete curve. We state the next proposition to take care of the case $\mathrm{h}(\Omega_{R/k}) = 0$ for complete curves in any characteristic.

Proposition 3.9. *Let $(R, \mathfrak{m}, k)$ be a local singular complete curve. The following statements hold.*

1. *If* $\operatorname{char}(k) = 0$*, then* $\mathrm{h}(\Omega_{R/k}) \geq 1$*.*

2. *Suppose* $\operatorname{char}(k) \neq 0$*. If* $\mathrm{h}(\Omega_{R/k}) = 0$*, then* $\tau(\Omega_{R/k}) \neq 0$*.*

Proof. The proof of (1) is clear from Theorem 3.8, whereas (2) follows from Remark 2.2 using the fact that $\mu(\Omega_{R/k}) = n \geq 2$ (Theorem 1.10). □

We also summarize the various properties of $\mathrm{h}(\Omega_{R/k})$ as direct consequences of the results in Chapter 2. We use the notations from Section 3.1.1.

Theorem 3.10. *[Mai22, Proposition 4.1] Let $(R, \mathfrak{m}, k)$ be a local singular complete curve. Then the following statements hold.*

(*a*) $\mathfrak{C} \subseteq x :_R (x :_R J)$ *for some* $x \in J$ *for any ideal* J *that realizes* $\Omega_{R/k}$.

(*b*) $\mathfrak{C}\omega_R \subseteq J\omega_R$ *for any ideal* J *that realizes* $\Omega_{R/k}$.

(*c*) $\mathrm{h}(\Omega_{R/k}) \leq \lambda\left(\frac{R}{\mathfrak{C}}\right) + \lambda\left(\frac{\mathfrak{C}}{\mathfrak{C}\omega_R}\right) = \lambda\left(\frac{R}{\mathfrak{C}\omega_R}\right)$, *where* ω_R *is identified with an ideal in* R.

(*d*) $\mathrm{h}(\Omega_{R/k}) \leq \mathrm{h}(\omega_R) + \lambda\left(\frac{\overline{R}}{R}\right)$.

(*e*) $\mathrm{h}(\Omega_{R/k}) \geq \mathrm{h}\left(\mathrm{tr}_R\left(\Omega_{R/k}\right)\right) = \lambda\left(\frac{R}{\mathrm{tr}_R(\Omega_{R/k})}\right)$.

(*f*) *Assume further that* R *is Gorenstein. Then we have the following.*

(*i*) $\mathfrak{C} \subseteq J$ *for any ideal* J *that realizes* $\Omega_{R/k}$.

(*ii*) $\mathrm{h}(\Omega_{R/k}) \leq \lambda\left(\frac{R}{\mathfrak{C}}\right)$.

Proof. The proof follows immediately from Theorem 2.25, Proposition 2.26, Corollary 2.27, Proposition 2.28, Proposition 2.38 and Proposition 2.39. □

The next lemma will be crucial to the proof of the main theorem.

Lemma 3.11. *[Mai22, Lemma 4.4] Let* $(R, \mathfrak{m}, k)$ *be a local singular complete curve. Suppose* J *realizes* $\Omega_{R/k}$. *If* $I \subseteq \mathfrak{n}^{s+1}$ *for some* $s \geq 1$ *and* $\tau(\Omega_{R/k}) = 0$, *then* $J \subseteq \mathfrak{m}^s$.

Proof. We have the exact sequence

$$0 \to \tau(\Omega_{R/k}) \to \Omega_{R/k} \to J \to 0.$$

Hence, by assumption, $\Omega_{R/k} \cong J$. By Definition 1.8, any relation between the generators of J must come from the Jacobian matrix of I. Suppose if possible, there exists

$y_1, y_2, \ldots, y_n$, generators of J, such that $y_1 \notin \mathfrak{m}^s$. There exists the Koszul relation, $-y_2 y_1 + y_1 y_2 = 0$. Hence, the column vector $[-y_2 \quad y_1 \quad 0 \quad \ldots \quad 0]^t$ must be written as an R-linear combination of the columns of the Jacobian matrix of I (denoted A as in Definition 1.8). By assumption, the entries of A are in $\mathfrak{n}^s$, which is a contradiction to the choice of y_1. □

We can now state the main result of this section.

Theorem 3.12. *[Mai22, Theorem 4.7] Let $(R, \mathfrak{m}, k)$ be a local singular complete curve. Suppose $I \subseteq \mathfrak{n}^{s+1}$ for some $s \geq 1$. If*

$$\mathrm{h}(\Omega_{R/k}) < \binom{n+s}{s}\left(\frac{s}{s+1}\right),$$

then $\tau(\Omega_{R/k}) \neq 0$. In particular, Berger's Conjecture is true.

Proof. Assume that $\tau(\Omega_{R/k}) = 0$. Choose an ideal J which realizes $\Omega_{R/k}$; by Lemma 3.11, $J \subseteq \mathfrak{m}^s$. Also, note that from Notation A.19 and Theorem A.34, we have the following exact sequence

$$0 \to \frac{I}{I^{(2)}} \to R^n \to \Omega_{R/k} \to 0.$$

Tensoring this sequence with k, we get

$$0 \to \mathrm{Tor}_1^R\left(\Omega_{R/k}, k\right) \to \frac{I}{\mathfrak{m}I + I^{(2)}} \to k^n \to \Omega_{R/k} \otimes k \to 0.$$

Using additivity of length and by Theorem 1.10 (1), we immediately get

$$\dim_k \mathrm{Tor}_1^R\left(\Omega_{R/k}, k\right) \leq \mu(I). \tag{3.2.1}$$

Using Equation (3.2.1) and the assumption that $\tau(\Omega_{R/k}) = 0$, we get

$$\mathrm{h}(\Omega_{R/k}) \geq \binom{n+s}{s} \frac{s}{s+1}$$

by Theorem 2.5. However this is a contradiction to the hypothesis on $\mathrm{h}(\Omega_{R/k})$. Hence, $\tau(\Omega_{R/k}) \neq 0$. □

Corollary 3.13. *Let $(R, \mathfrak{m}, k)$ be a local singular complete curve. If*

$$\mathrm{h}(\Omega_{R/k}) < \frac{n+1}{2},$$

then Berger's Conjecture is true.

Proof. The proof follows immediately by putting $s = 1$ in Theorem 3.12. □

Proposition 3.14. *Let $(R, \mathfrak{m}, k)$ be a local singular complete curve.*

a) If $\Omega_{R/k}$ surjects onto an ideal J which is generated by less than $\mathrm{edim}(R) = n$ elements, then $\tau(\Omega_{R/k}) \neq 0$.

b) Suppose $I \subseteq \mathfrak{n}^{s+1}$ for some $s \geq 1$. If $\tau(\Omega_{R/k}) = 0$, then

$$\mathrm{h}(\Omega_{R/k}) \geq \binom{n+s-1}{s-1} \frac{s^2 + s(n-1) - 1}{s(s+1)} + 1 + \frac{1}{n}\mu\left(\frac{\mathfrak{m}^s}{J}\right).$$

Proof. The first statement follows directly from Remark 2.7(a).

For the second statement, first note that by Equation (3.2.1) we have that $\beta_1(\Omega_{R/k}) \leq \mu(I)$. By part ($a$), we can safely assume that $\mu(J) = \mu(\Omega_{R/k}) = n$ for all ideals J which are surjective images of $\Omega_{R/k}$. The statement now follows from Remark 2.7(b). □

From existing results in the literature, we can get the following results upon combining with the above discussion.

Corollary 3.15. *[Mai22, Corollary 4.9] Let* $(R, \mathfrak{m}, k)$ *be a local singular complete curve.*

(*a*) $\tau(\Omega_{R/k}) \neq 0$ *if* $\mathrm{h}(\Omega_{R/k}) = 1, 2$.

(*b*) *If* R *is Gorenstein, then* $\tau(\Omega_{R/k}) \neq 0$ *if* $\mathrm{h}(\Omega_{R/k}) = 1, 2, 3$.

Proof. We can always assume that $n \geq 4$ due to [Her78, Satz 3.3]. Also, we can safely assume that $\mu(J) = n$ for all surjective images J of M in R by Proposition 3.14(a).

(*a*) Follows immediately from Corollary 3.13.

(*b*) By [Her78, Satz 3.2], we can assume $n \geq 5$ in this case. From (a), we only need to take care of $\mathrm{h}(\Omega_{R/k}) = 3$. Take $s = 1, n = 5$ in Proposition 3.14 and note that $J \neq \mathfrak{m}$ by assumption. This finishes the proof. □

As an immediate corollary, we recover G. Scheja's proof of Berger's Conjecture.

Corollary 3.16. *[Sch70, Satz 9.8] Let* R *be a positively graded singular domain and assume that* $\mathrm{char}(k) = 0$. *Then* $\tau(\Omega_{R/k}) \neq 0$.

Proof. The image of the Euler derivation $\delta : R \to R$ is the graded maximal ideal $\mathfrak{m}$ and hence we have an induced surjection $\delta_* : \Omega_{R/k} \to \mathfrak{m}$: the Euler homomorphism by the universal property of $\Omega_{R/k}$. Hence, $\mathrm{h}(\Omega_{R/k}) = 1$. The proof now follows from Corollary 3.15. □

Corollary 3.17. *Suppose R is a Gorenstein complete curve and also assume that $I \subseteq \mathfrak{n}^{s+1}$ for $s \geq 1$. If*

$$\lambda\left(\frac{R}{\mathfrak{C}}\right) < \binom{n+s-1}{s-1}\frac{s^2+s(n-1)-1}{s(s+1)} + 1,$$

then $\tau(\Omega_{R/k}) \neq 0$.

Proof. The proof follows from Theorem 3.10$(f)(ii)$ and Proposition 3.14. □

However, in the examples we tried on Macaulay 2, the stated condition on the colength of the conductor ideal was never met. So this naturally raises the question if such an upper bound can actually occur.

3.3 Explicit Computation of $\mathrm{h}(\Omega_{R/k})$

We shall now discuss an explicit way of computing $\mathrm{h}(\Omega_{R/k})$ which can help in providing examples of classes of rings where $\mathrm{h}(\Omega_{R/k})$ is relatively small compared to the embedding dimension and hence Theorem 3.12 applies.

Remark 3.18. From now on, we assume that k is algebraically closed of characteristic 0. The algebraically closed condition can be relaxed with a field of characteristic 0, but we keep it in the hypothesis in the spirit of the discussion as in [Ber94]. We need the following observations.

1. $\Omega_{R/k}$ surjects to D where

$$D = Rx_1'(t) + \cdots + Rx_n'(t)$$

is the fractional ideal (i.e. an R-submodule of K) of derivatives of the maximal ideal

$$\mathfrak{m} = (x_1(t), x_2(t), \ldots, x_n(t))$$

of R when identified inside $\overline{R} = k[[t]]$. This follows from the chain rule of derivatives (see, for example, the discussion at the beginning of [Ulr81, Section 3]). Multiplying by a suitable power series in t, we can make D an ideal of R and hence we have constructed an ideal of R to which $\Omega_{R/k}$ surjects.

2. It follows from [Ber94, 2.2], that

$$\lambda\left(\frac{\overline{R}}{R}\right) \geq \lambda\left(\frac{\overline{R}}{D}\right).$$

In fact, if equality holds, then Berger's Conjecture is true as is shown in [Poh91, Theorem 1]. This is the case which is referred to as *maximal torsion*.

Theorem 3.19. *[Mai22, Theorem 5.2] Let $(R, \mathfrak{m}, k)$ be a local singular complete curve. Further assume that k is algebraically closed of characteristic 0. Let $\overline{R} = k[[t]]$ with valuation function v. Let $x_i(t)$ denote $X_i + I$ in terms of the parameter t and $x'(t) = \frac{d}{dt}x(t)$. Finally let $D = Rx_1'(t) + \cdots + Rx_n'(t)$ and $v(D^{-1}) := \min\{v(\alpha) \mid \alpha \in K, \alpha D \subseteq R\}$. Then we get*

$$\mathrm{h}(\Omega_{R/k}) = \lambda\left(\frac{\overline{R}}{D}\right) - \lambda\left(\frac{\overline{R}}{R}\right) + v(D^{-1}) \leq v(D^{-1}).$$

If the rightmost equality holds, then Berger's Conjecture is true.

Proof. Let J be any ideal to which $\Omega_{R/k}$ surjects. Because of rank considerations we know that any such ideal is isomorphic to D (considered as R-submodules of K). Let $J = \frac{a}{b}D$ for some $a, b \in R$. Note that

$$\lambda(R/bJ) = \lambda(R/J) + \lambda(R/bR)$$

$$\lambda(R/aD) = \lambda(\overline{R}/aD) - \lambda(\overline{R}/R).$$

Since $bJ = aD$, we get

$$\begin{aligned}\lambda(R/J) &= \lambda(\overline{R}/aD) - \lambda(\overline{R}/R) - \lambda(R/bR)\\ &= \lambda(\overline{R}/D) + \lambda(D/aD) - \lambda(\overline{R}/R) - \lambda(R/bR).\end{aligned}$$

Since D is maximal Cohen-Macaulay of rank 1 over R, we get $\lambda(D/aD) = \lambda(R/aR)$ (Proposition 1.3) and hence we obtain

$$\lambda(R/J) = \lambda(\overline{R}/D) - \lambda(\overline{R}/R) + \lambda(R/aR) - \lambda(R/bR).$$

Using Lemma 2.17 we thus obtain

$$\lambda(R/J) = \lambda(\overline{R}/D) - \lambda(\overline{R}/R) + v(a) - v(b).$$

From Definition 2.1 and Remark 3.18(2), the proof is now complete. □

Remark 3.20. Since R is complete and $\overline{R} = k[[t]]$, we can always write the conductor $\mathfrak{C} = (t^i)_{i \geq c}$ where c is the least integer such that $t^{c-1} \notin R$ but $t^i \in R$ for all $i \geq c$ (see [SH06, Chapter 12]). We shall explore this in detail in Chapter 3. For the time being, we simply notice from Remark 3.18 that:

$\Omega_{R/k}$ surjects to the ideal $t^c D$.

Corollary 3.21. *Let* $(R, \mathfrak{m}, k)$ *be a local singular complete curve. Assume that* k *is algebraically closed of characteristic* 0*. Let* $\overline{R} = k[[t]]$ *with valuation function* v*. Let* $x_i(t)$ *denote* $X_i + I$ *in terms of the parameter* t *and* $x'(t) = \frac{d}{dt}x(t)$*. Finally let* $D = Rx_1'(t) + \cdots + Rx_n'(t)$ *and* $v(D^{-1}) = \min\{v(\alpha) \mid \alpha \in K, \alpha D \subseteq R\}$*. Then*

$$\mathrm{h}(\Omega_{R/k}) = \lambda\left(\frac{R}{t^c D}\right) - c + v(D^{-1}).$$

Also, we get that $\lambda(R/t^c D) \leq c$*.*

Proof. The ideal $t^c D$ is relevant by Remark 3.20. Now apply the same argument as in the proof of Theorem 3.19 to $\frac{t^c}{1}D$ to get

$$\lambda(R/t^c D) = \lambda(\overline{R}/D) - \lambda(\overline{R}/R) + c.$$

Thus, for any image $J = \frac{a}{b}D$ of $\Omega_{R/k}$, we get

$$\lambda(R/J) = \lambda(R/t^c D) + (v(a) - v(b)) - c.$$

Hence,

$$\mathrm{h}(\Omega_{R/k}) = \lambda(R/t^c D) - c + v(D^{-1}).$$

The last statement follows from Remark 3.18(2) and the above computation. □

Corollary 3.22. *[Mai22, Corollary 5.3] Let* $(R, \mathfrak{m}, k)$ *be a local singular complete curve. Assume that* k *is algebraically closed of characteristic* 0*. Let* $\overline{R} = k[[t]]$ *with valuation function* v*. Let* $x_i(t)$ *denote* $X_i + I$ *in terms of the parameter* t *and* $x'(t) = \frac{d}{dt}x(t)$*. Finally let* $D = Rx_1'(t) + \cdots + Rx_n'(t)$ *and* $v(D^{-1}) = \min\{v(\alpha) \mid \alpha \in K, \alpha D \subseteq$

$R\}$. *If*

$$v(D^{-1}) < \binom{n+s}{s}\frac{s}{s+1} + \lambda\left(\frac{\overline{R}}{R}\right) - \lambda\left(\frac{\overline{R}}{D}\right),$$

then $\tau(\Omega_{R/k}) \neq 0$.

Proof. The proof is immediate from Theorem 3.12 and Theorem 3.19. □

3.3.1 Ways to make the above computations feasible on software

For the following proposition, we use the notation that for any fractional ideal I, (i.e. an R-submodule of K),

$$v(I) = \min\{v(\alpha) \mid \alpha \in I\}$$

where v is the valuation function on $\overline{R} = k[[t]]$. Recall that $I^{-1} = R :_K I$.

Proposition 3.23. *Let* $(R, \mathfrak{m}, k)$ *be a local singular complete curve. Assume that* k *is algebraically closed of characteristic* 0. *Let* $\overline{R} = k[[t]]$ *with valuation function* v. *Let* $x_i(t)$ *denote* $X_i + I$ *in terms of the parameter* t *and* $x'(t) = \frac{d}{dt}x(t)$. *Finally let* $D = Rx_1'(t) + \cdots + Rx_n'(t)$. *Then*

$$v(D^{-1}) = v(\mathrm{tr}_R(\Omega_{R/k}) - v(D) = v(\mathrm{tr}_R(\Omega_{R/k})) - v(x_1'(t)).$$

Proof. The last equality follows due to assumption that k is characteristic 0 and the fact that $v(\mathfrak{m}) = v(x_1(t))$ (this choice can always be made suitably).

By Proposition 2.39, $\mathrm{tr}_R(D) = \mathrm{tr}_R(\Omega_{R/k})$. By Proposition 2.34(2), we get

$$v(\mathrm{tr}_R(D)) = v(D) + v(D^{-1}).$$

This finishes the proof. □

We can now make an efficient algorithm to implement Theorem 3.19 or Corollary 3.21 to compute $\mathrm{h}(\Omega_{R/k})$ using software.

1. $\lambda(\overline{R}/R)$ and $\lambda(\overline{R}/D)$ can be computed using *suitable valuation semi-groups* – significant discussions on this exist in the literature (see [HK71],[Yos86],[Iso91]). Namely, we need to count the missing valuations from the valuation semi-groups of R and D, respectively.

2. Most of the times on Macaulay 2, the length can be computed as well as the conductor valuation c.

3. Compute $\mathrm{tr}_R(\Omega_{R/k})$ using Remark 2.35. Read off the valuation of the ideal.

4. Compute $v(D^{-1})$ using Proposition 3.23.

5. Compute $\mathrm{h}(\Omega_{R/k})$.

Thus we can efficient check whether Theorem 3.12 can be applied to a given instance of verifying Berger's Conjecture as well as simply computing $\mathrm{h}(\Omega_{R/k})$ or $v(D^{-1})$. The following examples which were computed using Macaulay 2 illustrate this.

Example 3.24.

$$R = \mathbb{C}[[t^8+t^9, 64t^{10}-81t^{12}, 8t^{12}-9t^{13}, t^{14}, t^{15}, t^{16}, t^{17}]] = \mathbb{C}[[x,y,z,w,u,v,p]]/I$$

where I is the defining ideal which has 21 generators. Moreover,

$$\operatorname{tr}_R(\Omega_{R/k}) = (p, v, u, w, z, y, x^2)$$

and so $v(\operatorname{tr}_R(\Omega_{R/k})) = 10$ given by the element y. Hence,

$$v(D^{-1}) = 10 - v(x_1'(t)) = 10 - 7 = 3 \leq \frac{7+1}{2}$$

which shows that $\tau \neq 0$ by Corollary 3.22.

In fact, here $c = 14$ and Macaulay 2 computations on the ideal

$$\begin{aligned} t^c D =&(z\,p, y\,p, w\,v, z\,v, 71\,y\,v - 2048\,x\,p, 11360\,x\,u - 11360\,x\,v - 3237\,x\,p,\\ &355\,y^2 - 181760\,x\,z + 2249280\,x\,w - 2329155\,x\,v - 597051\,x\,p) \end{aligned}$$

show that $\lambda(R/t^c D) = 14$. Hence, $\operatorname{h}(\Omega_{R/k}) = 3$ using Corollary 3.21.

For the next example, we include a sample code and its outputs to show how Macaulay 2 computes the length.

Example 3.25. Let $R = \mathbb{C}[[t^9, t^{14} + t^{15}, t^{17}, t^{29}]] = \mathbb{C}[[x, y, z, w]]/I$ where

$$\begin{aligned} I =\Big(& y^2z^2 - x\,z^3 - x^2y\,w - x^2z\,w + y\,z\,w - 2\,z^2w, y^3z + x\,y\,z^2 - x\,z^3 - x^2y\,w\\ &- y^2w - 2\,y\,z\,w + w^2, x^3z + x\,z^2 - y\,w, y^4 - x\,z^3 - x^3w - 4\,y^2w + 2\,w^2,\\ &x\,y^3 - x\,y^2z + x^3w - z^3 - 3\,x\,y\,w + 2\,x\,z\,w, z^5 - x^3w^2, y\,z^4 - x\,y^2z\,w\\ &+ x^2z^2w + x^3w^2 - x\,y\,w^2 + 2\,x\,z\,w^2, x\,z^4 - x^2y^2w + 2\,x^2w^2 + z\,w^2, x^2y^2z - y\,z^3\\ &- z^4 + x\,y^2w - 2\,x^2z\,w - z^2w - 2\,x\,w^2, x^3y^2 - x\,y\,z^2 - 2\,x^3w - x\,z\,w + w^2,\\ &x^4y - x^2y^2 - x^2y\,z + y\,z^2 - z^3 + 2\,x^2w + z\,w, x^5 - y^2z + x^2w + 2\,z\,w\Big). \end{aligned}$$

Here $\Omega_{R/k}$ surjects to the ideal

$$t^c D = (x^3 w, z^4, 15\,x\,y^2 z - 14\,x^2 z^2 + 14\,x\,y\,w - 30\,x\,z\,w, x^2 y^2 - 2\,x^2 w - z\,w)$$

whose colength is 37. This can be computed on Macaulay 2 using any of the following two methods.

Treat this as an ideal of $\mathbb{C}[[x, y, z, w]]$ and add I to it.

$$\begin{aligned} I_1 = t^c D + I = &(x^3 w, z^4, y^2 z^2 - x\,z^3 - x^2 y\,w - x^2 z\,w + y\,z\,w - 2\,z^2 w, x\,y\,z^2 - w^2, \\ &y^3 z + x\,y\,z^2 - x\,z^3 - x^2 y\,w - y^2 w - 2\,y\,z\,w + w^2, \\ &15\,x\,y^2 z - 14\,x^2 z^2 + 14\,x\,y\,w - 30\,x\,z\,w, x^3 z + x\,z^2 - y\,w, \\ &y^4 - x\,z^3 - 4\,y^2 w + 2\,w^2, x\,y^3 - x\,y^2 z - z^3 - 3\,x\,y\,w + 2\,x\,z\,w, \\ &x^2 y^2 - 2\,x^2 w - z\,w, x\,y^2 z\,w - x^2 z^2 w + x\,y\,w^2 - 2\,x\,z\,w^2, \\ &x^4 y - x^2 y\,z + y\,z^2 - z^3, x^5 - y^2 z + x^2 w + 2\,z\,w). \end{aligned}$$

Let $n_1 = (x, y, z, w)\mathbb{C}[[x, y, z, w]]$. To compute $\lambda(R/t^c D) = \lambda(\mathbb{C}[[x, y, z, w]]/I_3)$:

Method 1: Compute $\lambda(R/t^c D) = 1 + \sum_{i \geq 1} \frac{n_1^i + I_1}{n_1^{i+1} + I_1}$.

```
i22 : mingens(trim(n1/(n1^2+I1)))

o22 = | w z y x |

1       4

o22 : Matrix R  <--- R

i23 : mingens(trim((n1^2+I1)/(n1^3+I1)))

o23 = | xw z2 yz xz y2 xy x2 |
```

```
1       7
o23 : Matrix R  <--- R
i24 : mingens(trim((n1^3+I1)/(n1^4+I1)))
o24 = | zw yw xyz x2z y3 xy2 x2y x3 |
1       8
o24 : Matrix R  <--- R
i25 : mingens(trim((n1^4+I1)/(n1^5+I1)))
o25 = | xyw 2x2w+zw z3 yz2 xz2-yw x3y x4 |
1       7
o25 : Matrix R  <--- R
i26 : mingens(trim((n1^5+I3)/(n1^6+I3)))
o26 = | w2 yzw xzw y2w y2z-x2w-2zw x2yz-yz2+z3 |
1       6
o26 : Matrix R  <--- R
i27 : mingens(trim((n1^6+I1)/(n1^7+I1)))
o27 = | x2zw x2yw-y2w-yzw+2w2 x2z2-xyw |
1       3
o27 : Matrix R  <--- R
i28 : mingens(trim((n1^7+I1)/(n1^8+I1)))
o28 = | xw2 |
1       1
o28 : Matrix R  <--- R
```

```
i29 : mingens(trim((n1^8+I1)/(n1^9+I1)))

o29 = 0

1

o29 : Matrix R  <--- 0
```

Once it hits 0, then it continue to be 0. Hence,

$$\lambda(R/t^c D) = 1 + 4 + 7 + 8 + 7 + 6 + 3 + 1 = 37.$$

<u>Method 2</u>: Use the `degree` command on Macaulay 2.

```
i30 : degree I1

o30 = 37
```

Moreover,

$$\mathrm{tr}_R(\Omega_{R/k}) = (w, z^2, yz, y^2 + 14xz, x^2z, x^2y, x^3 + xz)$$

whose valuation is 26 given by the element $x^3 + xz$. Thus,

$$v(D^{-1}) = 26 - 8 = 18, \quad \text{and } \mathrm{h}(\Omega_{R/k}) = 37 - 40 + 18 = 15.$$

Example 3.26. Let

$$R = \mathbb{C}[[t^4 + t^5, t^9]] = \frac{\mathbb{C}[[x, y]]}{(x^9 - 9\,x^7y + 27\,x^5y^2 - 30\,x^3y^3 + 9\,x\,y^4 - y^5 - y^4)}.$$

We get

$$\begin{aligned}\mathrm{tr}_R(\Omega_{R/k}) = (&51\,x^4 + 35\,x^3y + 25\,x^2y^2 + 16\,x^3 - 178\,x^2y - 90\,x\,y^2 - 25\,y^3 - 12\,x\,y + 76\,y^2,\\ &125\,x^3y^2 + 16\,x^4 - 109\,x^3y - 20\,x^2y^2 - 250\,x\,y^3 - 28\,x^2y + 218\,x\,y^2 + 20\,y^3 - 4\,y^2,\\ &25\,x^4y - 16\,x^4 + 9\,x^3y - 55\,x^2y^2 + 28\,x^2y - 18\,x\,y^2 + 5\,y^3 + 4\,y^2,\\ &25\,x^5 + 29\,x^4 - 55\,x^3y - 16\,x^3 - 62\,x^2y + 5\,x\,y^2 + 12\,x\,y + 4\,y^2).\end{aligned}$$

So, $v(\mathrm{tr}_R(\Omega_{R/k})) = 12$ given by $25\,x^5 + 29\,x^4 - 55\,x^3y - 16\,x^3 - 62\,x^2y + 5\,x\,y^2 + 12\,x\,y + 4\,y^2$. Consequently,

$$v(D^{-1}) = 12 - 3 = 9.$$

Next we compute the valuations present in the semi-groups of R and D. Let us call these semi-groups $S(R)$ and $S(D)$ respectively. We make a chart of the valuations, with the corresponding elements. We can compute the conductor valuation to be $c = 24$ and it is given by

$$t^{24} = \frac{x^6 - 6x^4y - x^3y^2 + 9x^2y^2 + 3xy^3 - 2y^3}{(1-y)}.$$

So beyond 24 all valuations are present.

$S(R)$:

Valuation	Element
$4, 8, 12, 16, 20$	$(t^4 + t^5)^i$, $i = 1, 2, 3, 4, 5$
$9, 18$	$(t^9)^i$, $i = 1, 2$
$13, 17, 21, 22$	$(t^4 + t^5)^i(t^9)^j$, $i = 1, 2, 3$, $j = 1, 2$
$[24, \infty)$	✓

Thus, $\lambda(\overline{R}/R) = 12$ because the missing valuations are

$$\{1, 2, 3, 5, 6, 7, 10, 11, 14, 15, 19, 23\}.$$

One can similarly compute $S(D)$. Note that $D = R(4t^3+5t^4)+Rt^8$. Also, recall that $c = 24$.

$S(D)$:

Valuation	Element
$3, 7, 11, 15, 19, 23$	$(t^4+t^5)^i(4t^3+5t^4),\ 0 \le i \le 5$
$8, 12, 16, 20, 24$	$(t^4+t^5)^i t^8,\ 0 \le i \le 4$
21	$(t^9)^2(4t^3+5t^4)$
$17, 26$	$(t^9)^i t^8,\ i = 1, 2$
25	$(t^9)(t^4+t^5)^2 t^8$
13	$4(t^4+t^5)t^8 - t^9(4t^3+5t^4)$
18	$\big(4(t^4+t^5)^2+t^9\big)t^8 - t^9(t^4+t^5)(4t^3+5t^4)$
22	$(t^4+t^5)\big(4(t^4+t^5)^2+t^9\big)t^8 - t^9(t^4+t^5)^2(4t^3+5t^4)$
$[27, \infty)$	$t^{c+i}(4t^3+5t^4), i \ge 0$

Thus the missing valuations are

$$\{0, 1, 2, 4, 5, 6, 9, 10, 14\}.$$

So, $\lambda(\overline{R}/D) = 9$. Thus, from Theorem 3.19, we get

$$\mathrm{h}(\Omega_{R/k}) = 9 - 12 + 9 = 6.$$

Example 3.27. $R = \mathbb{C}[[t^4+t^5, t^6, t^8, t^9]]$. Then $R = \dfrac{\mathbb{C}[[x,y,z,w]]}{I}$ where

$$I = \Big(y^2 - x\,z - y\,z + x\,w, z^3 - y\,w^2, y\,z^2 + y\,z\,w - x\,w^2, x\,z^2 + x^2 w - x\,y\,w$$
$$- x\,z\,w - y\,z\,w + x\,w^2 - z\,w - 2\,w^2, x\,y\,z - x^2 w + z\,w + w^2,$$
$$x^2 z - z^2 - 2\,z\,w - w^2, x^2 y - y\,z - z^2 - 2\,y\,w, x^3 - x\,z - y\,z - 2\,x\,w - y\,w\Big).$$

We get that

$$\operatorname{tr}_R(\Omega_{R/k}) = \left(w, z, y, x^2\right)$$

and hence its valuation is 6. Thus $\tau \neq 0$ by Corollary 3.22 since

$$v(D^{-1}) = 6 - 3 = 3 \leq \frac{4+1}{2} + 1 = 3.5$$

Here clearly $c = 8$ as all valuations above it are present. One easily notes that the missing valuations from $S(R)$ are $\{1, 2, 3, 5, 7\}$. Next note that $D = R(4t^3 + 5t^4) + Rt^5 + Rt^7 + Rt^8$ and so

$S(D)$:

Valuation	Element
$3, 5, 7, 8$	1 times the generators of D
9	$t^6(4t^3 + 5t^4)$
10	$t^6(4t^3 + 5t^4) - 4(t^4 + t^5)t^5$
$[11, \infty)$	$t^j(4t^3 + 5t^4),\ j \geq 8$

So the missing valuations are $\{0, 1, 2, 4, 6\}$. Thus, $\operatorname{h}(\Omega_{R/k}) = 5 - 5 + 3 = 3$.

3.3.2 Quasi Homogeneity using $\mathbf{tr}_R(\Omega_{R/k})$

Recall from Remark 3.7 that a local singular complete curve is quasi homogeneous if there exists a surjection from $\Omega_{R/k}$ to $\mathfrak{m}$. This makes it clear that $\operatorname{tr}_R(\Omega_{R/k})$ must be $\mathfrak{m}$. However, using $\operatorname{h}(\Omega_{R/k})$, we can answer the converse question as well. We make it explicit in the following theorem.

Theorem 3.28. *Let $(R, \mathfrak{m}, k)$ be a local singular complete curve. Assume that k is algebraically closed of characteristic 0. Let $\overline{R} = k[[t]]$ with valuation function v.*

Let $x_i(t)$ *denote* $X_i + I$ *in terms of the parameter* t *and* $x'(t) = \frac{d}{dt}x(t)$*. Finally let* $D = Rx_1'(t) + \cdots + Rx_n'(t)$ *and* $v(D^{-1}) = \min\{v(\alpha) \mid \alpha \in K, \alpha D \subseteq R\}$*. Then* R *is quasi homogeneous if and only if* $v(\mathrm{tr}_R(\Omega_{R/k})) = v(\mathfrak{m})$*.*

Proof. If R is quasi homogeneous, then we get $\mathrm{tr}_R(\Omega_{R/k}) = \mathrm{tr}_R(\mathfrak{m})$ using Proposition 2.39. Since R is non-regular, we get that $\mathrm{tr}_R(\mathfrak{m}) = \mathfrak{m}$. Hence, $v(\mathrm{tr}_R(\Omega_{R/k})) = v(\mathfrak{m})$.

Conversely, assume that $v(\mathrm{tr}_R(\Omega_{R/k})) = v(\mathfrak{m})$. By Proposition 3.23, we get that

$$v(D^{-1}) = v(\mathfrak{m}) - (v(\mathfrak{m}) - 1) = 1.$$

Hence by Theorem 3.19, we obtain that

$$\mathrm{h}(\Omega_{R/k}) = \lambda(\overline{R}/D) - \lambda(\overline{R}/R) + 1.$$

This implies that

$$\mathrm{h}(\Omega_{R/k}) - 1 = \lambda(\overline{R}/D) - \lambda(\overline{R}/R).$$

By Proposition 3.9, the LHS is non-negative whereas by Remark 3.18(2), the RHS is non-positive. Hence, the only possibility is $\mathrm{h}(\Omega_{R/k}) = 1$ which implies that $\mathfrak{m}$ realizes $\Omega_{R/k}$. Thus R is quasi homogeneous. □

3.4 Discussion on $\mathrm{h}(\Omega_{R/k}) = 2$

We already discussed the case $\mathrm{h}(\Omega_{R/k}) = 1$, i.e., the case when R is a singular quasi-homogeneous complete curve. In this section, we briefly try to understand the case when $\mathrm{h}(\Omega_{R/k}) = 2$.

Proposition 3.29. *Let $(R, \mathfrak{m}, k)$ be a local singular complete curve. Assume that k is algebraically closed of characteristic 0. Let $\overline{R} = k[[t]]$ with valuation function v. Suppose the indexing is chosen such that $v(x_i) \leq v(x_{i+1})$ for $1 \leq i \leq n-1$. Let J realize $\Omega_{R/k}$. If $\mathrm{h}(\Omega_{R/k}) = 2$, then $J = (x_1^2, x_2, \ldots, x_n)$.*

Proof. Suppose $x_1 \in J$. Then $v(J) = v(x_1) = v(\mathfrak{m})$. But this implies that R is quasi homogeneous by Theorem 3.28. This contradicts $\mathrm{h}(\Omega_{R/k}) = 2$. Hence, $x_1 \notin J$.

Notice that $\lambda(R/J) = 2$ implies that $\lambda(\mathfrak{m}/J) = \mu(\mathfrak{m}/J) = 1$. So, $\mathfrak{m}^2 + J = J$. So, $x_1^2 \in J$. Since $\lambda((x_1, \ldots, x_n))/J) = 1$, we immediately obtain that $J = (x_1^2, \ldots, x_n)$. □

Corollary 3.30. *Let $(R, \mathfrak{m}, k)$ be a local singular complete curve. Assume that k is algebraically closed of characteristic 0. Let $\overline{R} = k[[t]]$ with valuation function v and let $x_i(t)$ denote $X_i + I$ in terms of the parameter t. Set $x'(t) = \frac{d}{dt}x(t)$. Suppose the indexing is chosen such that $v(x_i) \leq v(x_{i+1})$ for $1 \leq i \leq n-1$. Let $D = Rx_1'(t) + \cdots + Rx_n'(t)$. If $\mathrm{h}(\Omega_{R/k}) = 2$, then*

$$\lambda(\overline{R}/R) - \lambda(\overline{R}/D) + 1 = \min\{v(x_1), v(x_2) - v(x_1)\}.$$

Proof. Let $v(x_i) = a_i$. By Proposition 3.29, we know $J = (x_1^2, \ldots, x_n)$ realizes $\Omega_{R/k}$. Hence $v(J) = \min\{2a_1, a_2\}$. Now note that if $\mathrm{tr}_R(J) = \mathfrak{m}$, then by Proposition 2.39, $v(\mathrm{tr}_R(\Omega_{R/k})) = v(\mathrm{tr}_R(J)) = v(\mathfrak{m})$. But this contradicts $\mathrm{h}(\Omega_{R/k}) = 2$ by Theorem 3.28. Hence, $v(\mathrm{tr}_R(\Omega_{R/k})) = v(J) = \min\{2a_1, a_2\}$.

Using Theorem 3.19 and Proposition 3.23, we immediately obtain that

$$2 = \lambda(\overline{R}/D) - \lambda(\overline{R}/R) + \min\{2a_1, a_2\} - a_1 + 1$$

$$\implies \lambda(\overline{R}/R) - \lambda(\overline{R}/D) + 1 = \min\{a_1, a_2 - a_1\}$$

finishing the proof. □

Chapter 4

Berger's Conjecture and An Approach Via Conductor

In this chapter, we discuss another approach to Berger's Conjecture with the conductor ideal as the primary tool and the added assumption that k is algebraically closed. We establish that the conjecture is true if we have a Gorenstein domain with algebraically closed residue field of characteristic zero and where the sixth power of the maximal ideal resides inside a principal ideal (see Theorem 4.18). This extends work of Güttes ([Güt90]). We also establish the conjecture whenever the conductor ideal is not in the square of the maximal ideal (see Theorem 4.5). These results were obtained in joint work with Craig Huneke and Vivek Mukundan [HMM21]. Theorem 4.9 was obtained in joint work with Vivek Mukundan.

4.1 Basic Setup and Properties of the Conductor Ideal

Throughout the rest of this chapter, we are going to assume that $(R, \mathfrak{m}, k)$ is a local singular complete curve (as in Chapter 2) with k algebraically closed and $\text{char}(k) = 0$. Recall that $\overline{R}$ is a DVR and let t be the uniformizing parameter. With this notation, we have $\overline{R} = k[\![t]\!], K = k((t))$.

We define the *order valuation* v on $\overline{R}$ given by

$$v(p(t)) = a$$

for $p(t) \in k[\![t]\!]$, $p(t) = t^a\alpha$ where α is a unit in $k[\![t]\!]$. Using this, we can write the elements of R in terms of t. More precisely, suppose $\mathfrak{m} = (x_1, \ldots, x_n)$ where $x_i = X_i + I$, and let in $\bar{R}$, $v(x_i) = a_i$ for $1 \leq i \leq n$. Then we have the following description of R: $R = k[\![\alpha_1 t^{a_1}, \ldots, \alpha_n t^{a_n}]\!]$ where α_i's are units in $\overline{R}$. Further we can choose the ordering and indexing suitably to make $a_1 \leq a_2 \leq \cdots \leq a_n$.

Recall that the conductor ideal is $\mathfrak{C} = R :_K \overline{R}$. Since $\overline{R} = k[\![t]\!]$ and $\mathfrak{C}$ is an ideal of $\bar{R}$ as well, we have that

$$\mathfrak{C} = (t^i)_{i \geq c}$$

where c is the least integer such that $t^{c-1} \notin R$, and $t^{c+i} \in R$ for all $i \geq 0$. Henceforth, we denote by c, the least valuation in $\mathfrak{C}$. This, by definition, is $v(\mathfrak{C})$.

Remark 4.1. It is clear from the above discussion that there cannot be any element

$r \in R$, such that $v(r) = c - 1$. We shall explore this observation by adjoining t^{c-1} to R which in turn will help in proving a new case of the conjecture.

Example 4.2. Let $R = k[\![t^4 + t^5, t^7, t^8, t^9]\!]$. Then the conductor $\mathfrak{C} = (t^7, t^8, \dots)$. In fact, $\mathfrak{C} = (t^7, t^8, t^9, t^{10})$ as the following proposition will show.

Proposition 4.3. *Let $(R, \mathfrak{m}, k)$ be a local singular complete curve with algebraically closed residue field of characteristic* 0. *Let $\overline{R} = k[[t]]$ with valuation function v, and conductor ideal be $\mathfrak{C}$. Then*

$$\mu(\mathfrak{C}) = e(R) = v(x_1).$$

Proof. For a minimal reduction x of $\mathfrak{m}$, we have $\mathfrak{m}\bar{R} = x\bar{R} \implies \mathfrak{m}\mathfrak{C} = x\mathfrak{C}$. The claim now follows using Proposition 1.3:

$$\lambda(\mathfrak{C}/\mathfrak{m}\mathfrak{C}) = e(x, \mathfrak{C}) = e(x, R)\mathrm{rank}_R(\mathfrak{C}) = e(R).$$

The last part of the proof follows since $a_1 = v(x_1)$ is the least valuation of an element in R and hence

$$\mathfrak{m}\bar{R} = (t^{a_1})\bar{R},$$

i.e., x_1 can always be considered as a minimal reduction of $\mathfrak{m}$. □

4.2 Quick Recap To Computing $\tau(\Omega_{R/k})$

We have the following commutative diagram using the functorial universal properties of the module of differentials and the associated universal derivations.

$$\begin{array}{ccc} \Omega_{R/k} & \xrightarrow{f} & \Omega_{\overline{R}/k} \\ {\scriptstyle d}\uparrow & & \uparrow{\scriptstyle d} \\ R & \overset{i}{\hookrightarrow} & \overline{R} \end{array}$$

Here i is the natural inclusion. We use the same symbols d without loss of generality. Since $\operatorname{rank}(\Omega_{R/k}) = \operatorname{rank}(\Omega_{\overline{R}/k}) = 1$, we get that the $\tau(\Omega_{R/k}) = \ker f$.

Also note that by commutativity of the diagram,

$$f(dx_i) = \frac{dx_i}{dt} dt.$$

Thus ignoring dt, we see that $\Omega_{R/k}$ surjects to the R-submodule $\displaystyle\sum_{i=1}^{n} R\frac{dx_i}{dt}$ of $\overline{R} = k[\![t]\!]$. So, we get that $\tau(\Omega_{R/k})$ consists of the tuples $\begin{bmatrix} r_1 & \dots & r_n \end{bmatrix}^t$ such that $\displaystyle\sum_{i=1}^{n} r_i \frac{dx_i}{dt} = 0$. The torsion submodule $\tau(\Omega_{R/k})$ is non-zero precisely when these tuples $\begin{bmatrix} r_1 & \dots & r_n \end{bmatrix}^t$ are not in the image of the presentation matrix J (jacobian matrix of I) of $\Omega_{R/k}$. (We write A^t to denote the transpose of a matrix A.)

This will be the approach we shall follow in this chapter to investigate $\tau(\Omega_{R/k})$. We are now in a position take care of a few cases of Berger's Conjecture immediately.

In the discussion that follows, we will need to check that some torsion element ω in $\Omega_{R/k}$ is nero onzero. The methodology we apply is along the lines of the discussion in the work of Cortiñas, Geller and Weibel [CnGW98]. Mainly, we will look at the image of ω in $\Omega_{R'/k}$ where $R' = R/\mathfrak{m}_R^2$ (we specify here $\mathfrak{m}_R$ to emphasize that we will vary the ring). Note that R' is a finite local k-algebra, i.e., it is a finitely generated k-algebra and has finite length. If the image of ω in this quotient ring is nonzero, then it will be nonzero in $\Omega_{R/k}$ since we have a natural map $\Omega_{R/k} \to \Omega_{R'/k}$ induced by the surjection $R \to R/\mathfrak{m}_R^2$. We state the two main results that we will use repeatedly.

Lemma 4.4. *[CnGW98, Proposition 2.6, Corollary 2.7] Let $(A, \mathfrak{m}_A)$ be a finite local k-algebra where k is algebraically closed of characteristic 0. Let $\mathfrak{m}_A = (x_1, \ldots, x_n)$ with the x_i being a set of minimal generators. Then the following statements hold.*

1. *Assume $\mathfrak{m}_A^2 = 0$. Then $x_i dx_j, i < j$ are k-linearly independent elements in $\Omega_{A/k}$.*

2. *Suppose $xy = 0$ for two elements $x, y \in \mathfrak{m}_A$ which are linearly independent mod $\mathfrak{m}_A^2$. Then xdy is nonzero in $\Omega_{A/k}$.*

4.3 Berger's Conjecture using bounds on valuations

We first settle the case $\mathfrak{C} \not\subseteq \mathfrak{m}^2$ by showing that this condition always leads to nonzero torsion in $\Omega_{R/k}$.

Theorem 4.5. *[HMM21, Theorem 3.1] Let $(R, \mathfrak{m}, k)$ be a local singular complete curve with integral closure $\overline{R} = k[[t]]$, where k is algebraically closed of characteristic zero. If $\mathfrak{C} \not\subseteq \mathfrak{m}^2$, then $\tau(\Omega_{R/k}) \neq 0$. In particular, Berger's Conjecture is true.*

Proof. Write $R = k[\![\alpha_1 t^{a_1}, \ldots, \alpha_n t^{a_n}]\!]$ with conductor $\mathfrak{C} = (t^c)\overline{R}$. We first monomialize the r^{th} term as follows: by multiplying by a nonzero element of k, we may assume that the constant term of the unit α_r is 1. By Hensel's lemma [Eis13, Theorem 7.3], there exists an element $\beta \in R$ such that $\beta^{a_r} = \alpha_r$. (Here we use that the characteristic of k is 0.) We write $\beta = 1 + \beta_1 t + \cdots$. Consider the change of variables $s = \beta t$. Under this change of variables, notice that $k[\![t]\!] = k[\![s]\!]$. Now

$$s = \beta t = t + \beta_1 t^2 + \beta_2 t^3 + \cdots .$$

Note that $s^{a_r} = (\beta t)^{a_r} = \alpha_r t^{a_r} \in R$. Furthermore, $\alpha_i t^{a_i} = \alpha_i' s^{a_i}, 1 \leq i \neq r \leq n$, where $\alpha_i' \in \overline{R}$ are units. Then

$$R = k[\![\alpha_1 t^{a_1}, \ldots, \alpha_n t^{a_n}]\!] = k[\![\alpha_1' s^{a_1}, \ldots, \alpha_{r-1}' s^{a_{r-1}}, s^{a_r}, \alpha_{r+1}' s^{a_{r+1}}, \ldots, \alpha_n' s^{a_n}]\!].$$

We apply this change of variables with $r = 1$ to assume without loss of generality, for the remainder of this proof, that $R = k[\![t^{a_1}, \alpha_2 t^{a_2}, \cdots, \alpha_n t^{a_n}]\!]$.

Since $\mathfrak{C}$ is not contained in $\mathfrak{m}^2$, we must have that $a_n \geq c_R$. Write $\alpha_n = \alpha_{n0} + tb$, where $\alpha_{n0} \neq 0$ is in k and $b \in \overline{R}$. Then $\alpha_n t^{a_n} = \alpha_{n0} t^{a_n} + t^{a_n+1} b$, where $b \in \overline{R}$. However, since $a_n \geq c_R$, it follows that $t^{a_n+1} b \in \mathfrak{C} \subset R$. Hence, $t^{a_n} = \alpha_{n0}^{-1}(\alpha_n t^{a_n} - t^{a_n+1} b) \in R$ as well, and then $R = k[\![t^{a_1}, \alpha_2 t^{a_2}, \ldots, \alpha_{n-1} t^{a_{n-1}}, t^{a_n}]\!]$. Now consider the element

$$\omega = a_n x_n dx_1 - a_1 x_1 dx_n \in \Omega_{R/k}$$

and the exact sequence

$$0 \to \tau(\Omega_{R/k}) \to \Omega_{R/k} \xrightarrow{f} R\frac{dx_1}{dt} + \cdots + R\frac{dx_n}{dt} \to 0$$

where the map f is the R-module map given by $f(dx_i) = \frac{dx_i}{dt}, 1 \leq i \leq n$. Under this map,

$$f(\omega) = a_n t^{a_n} \frac{dt^{a_1}}{dt} - a_1 t^{a_1} \frac{dt^{a_n}}{dt} = (a_1 a_n t^{a_1+a_n-1} - a_n a_1 t^{a_1+a_n-1})dt = 0.$$

Thus $\omega \in \tau(\Omega_{R/k})$. It remains to see that it is nonzero.

Let $R' = \frac{R}{\mathfrak{m}^2}$. Consider the image $\overline{w} = \overline{a_n x_n dx_1 - a_1 x_1 dx_n}$ in $\Omega_{R'/k}$. As $x_1 x_n \in \mathfrak{m}^2$, it follows that in $\Omega_{R'/k}$, $\overline{x_1 dx_n + x_n dx_1} = 0$. Hence

$$\overline{\omega} = (a_1 + a_n)\overline{x_1}d(\overline{x_n})$$

in $\Omega_{R'/k}$. Now using Lemma 4.4, we have $\overline{\omega} \neq 0$ in $\Omega_{R'/k}$. Thus $\omega \neq 0$ in $\Omega_{R/k}$. □

Example 4.6. Let $R = k[\![t^4 + t^5, t^7 + t^{10}, t^8 + t^{10}, t^9 + t^{10}]\!]$. Macaulay2 computations show that the conductor is $\mathfrak{C} = (t^c)\overline{R} = (t^7)\overline{R}$. Since $a_2 \geq c_R$, we have $\tau(\Omega_{R/k}) \neq 0$ using the previous result.

In fact, if α_i, α_j are units in R for some $i \neq j$, then using the same exact method as in the proof of Theorem 4.5, we can show that $\tau(\Omega_{R/k})$ is nonzero.

Proposition 4.7. *[HMM21, Remark 3.3] Let $(R, \mathfrak{m}, k)$ be a local singular complete curve with integral closure $\overline{R} = k[[t]]$, where k is algebraically closed of characteristic zero. If α_i, α_j are units in R for some $i \neq j$, then $\tau(\Omega_{R/k})$ is nonzero.*

Proof. Assume $i < j$. After multiplying by suitable inverses, we can safely assume that

$$R = k[\![\alpha_1 t^{a_1}, \alpha_2 t^{a_2}, \dots, t^{a_i}, \alpha_{i+1} t^{a_{i+1}}, \dots, t^{a_j}, \dots, \alpha_n t^{a_n}]\!].$$

Now we have the exact sequence

$$0 \to \tau(\Omega_{R/k}) \to \Omega_{R/k} \xrightarrow{f} R\frac{dx_1}{dt} + \dots + R\frac{dx_n}{dt} \to 0$$

where the map f is R-module map given by $f(dx_i) = \frac{dx_l}{dt}, 1 \leq l \leq n$. Consider the element $w := a_j x_j dx_i - a_i x_i dx_j \in \Omega_{R/k}$ and let us look at its image under the map f:

$$f(w) = a_j(t^{a_j})\frac{d(t^{a_i})}{dt} - a_i t^{a_i}\frac{d(t^{a_j})}{dt} = (a_i a_j t^{a_i+a_j-1} - a_j a_i t^{a_i+a_j-1})dt = 0.$$

Thus $w \in \tau(\Omega_{R/k})$. To show $w \neq 0$, consider its image $\overline{w}$ in $\Omega_{\frac{R}{\mathfrak{m}^2}/k}$. Let $R' = R/\mathfrak{m}^2$. In R', we have $x_i x_j = 0$ and so, $x_i dx_j = -x_j dx_i$. Thus

$$\overline{\omega} = (a_i + a_j)\overline{x_i}d\overline{x_j}$$

in $\Omega_{R'/k}$. The proof is now complete using Lemma 4.4. □

Remark 4.8. In light of Theorem 4.5, in order to investigate Berger's Conjecture it is enough to focus on the case when $\mathfrak{C} \subseteq \mathfrak{m}^2$. Also note that, we can safely assume that x_1 is a monomial to begin with using the same 'monomialization trick' that we used in Theorem 4.5. Thus, for the rest of this section, we can assume that

$$R = k[\![t^{a_1}, \alpha_2 t^{a_2}, \ldots, \alpha_n t^{a_n}]\!]$$

where embedding dimension of R is n.

Theorem 4.9. *Let $(R, \mathfrak{m}, k)$ be a local singular complete curve with integral closure $\overline{R} = k[[t]]$, where k is algebraically closed of characteristic zero. Let $\mathfrak{C} = (t^c)\overline{R}$ and the valuation function be denoted by v. Then $\tau(\Omega_{R/k}) \neq 0$ if*

$$v(x_n) + v(x_{n-1}) \geq c + v(x_1).$$

Proof. First, we reduce to the case $\mathfrak{C} \subset \mathfrak{m}^2$ and $x_1 = t^{a_1}$ following Remark 4.8. In $\overline{R} = k[\![t]\!]$, we have

$$\frac{dx_n}{dt}\left(\frac{dx_1}{dt}\right) - \frac{dx_1}{dt}\left(\frac{dx_n}{dt}\right) = 0$$
$$\left(a_n\alpha_n t^{a_n-1} + t^{a_n}\frac{d(\alpha_n)}{dt}\right)\left(\frac{dx_1}{dt}\right) - a_1 t^{a_1-1}\left(\frac{dx_n}{dt}\right) = 0 \tag{4.3.1}$$

Multiplying (4.3.1) by $\alpha_{n-1}t^{a_{n-1}-a_1+1}$, we get

$$\left(a_n\alpha_n\alpha_{n-1}t^{a_n-1+a_{n-1}-a_1+1} + \alpha_{n-1}t^{a_n+a_{n-1}-a_1+1}\frac{d(\alpha_n)}{dt}\right)\left(\frac{dx_1}{dt}\right) - a_1\alpha_{n-1}t^{a_1-1+a_{n-1}-a_1+1}\left(\frac{dx_n}{dt}\right) = 0$$
$$\left(a_n\alpha_n\alpha_{n-1}t^{a_n+a_{n-1}-a_1} + \alpha_{n-1}t^{a_n+a_{n-1}-a_1+1}\frac{d(\alpha_n)}{dt}\right)\left(\frac{dx_1}{dt}\right) - a_1\alpha_{n-1}t^{a_n-1}\left(\frac{dx_n}{dt}\right) = 0$$
$$\left(a_n\alpha_n\alpha_{n-1}t^{a_n+a_{n-1}-a_1} + \alpha_{n-1}t^{a_n+a_{n-1}-a_1+1}\frac{d(\alpha_n)}{dt}\right)\left(\frac{dx_1}{dt}\right) - a_1x_{n-1}\left(\frac{dx_n}{dt}\right) = 0 \tag{4.3.2}$$

Note that

$$v\left(a_n\alpha_n\alpha_{n-1}t^{a_n+a_{n-1}-a_1} + \alpha_{n-1}t^{a_n+a_{n-1}-a_1+1}\frac{d(\alpha_n)}{dt}\right) \geq a_n + a_{n-1} - a_1.$$

Since $a_n + a_{n-1} - a_1 \geq c$, we get

$$w = a_n\alpha_n\alpha_{n-1}t^{a_n+a_{n-1}-a_1} + \alpha_{n-1}t^{a_n+a_{n-1}-a_1+1}\frac{d(\alpha_n)}{dt} \in \mathfrak{C} \subset R.$$

Thus (4.3.2), as an element of $\Omega_{R/k}$, looks like

$$\tau = \begin{bmatrix} w & 0 & \cdots & 0 & a_1x_{n-1} \end{bmatrix}^t \tag{4.3.3}$$

Consider the torsion element τ in $\Omega_{R'/k}$ where $R' = R/\mathfrak{m}^2$. Since $\mathfrak{C} \subset \mathfrak{m}^2$, τ takes the from $a_1\overline{x_{n-1}}d\overline{x_n}$ in $\Omega_{R'/k}$. This is nonzero due to Lemma 4.4. $\square$

Example 4.10. Consider $R = k[\![t^{10}, t^{11} + t^{13}, t^{15} + t^{19}, t^{25} + t^{26}]\!]$. Here

$$a_1 = 10, a_{n-1} = 15, a_n = 25$$

and also $\mathfrak{C} = (t^{30}, \ldots, t^{39})$. Thus, $\tau(\Omega_{R/k}) \neq 0$ by Theorem 4.9.

4.4 Berger's Conjecture Using $R[t^{c-1}]$

Throughout this section, we assume that $\mathfrak{C} \subseteq \mathfrak{m}^2$ and $x_1 = t^{a_1}$ as discussed in Remark 4.8. Recall that $v(\mathfrak{C}) = c$. We construct the ring $S := R[t^{c-1}]$. We shall eventually restrict to the case when R is Gorenstein. In that case, it will turn out that S is the same as adjoining $\frac{\mathfrak{C}}{x_1}$ to R, i.e. all valuations in the interval $[c-a_1, c-1]$ get added to the valuation semi-group of R by adjoining t^{c-1}.

We first collect some observations about S. The following theorem is a special case of [HMM21, Theorem 4.6].

Theorem 4.11. *Let $R = \frac{k[\![X_1,\ldots,X_n]\!]}{I}$ be a local singular complete curve of embedding dimension n with $\mathfrak{C} \subseteq \mathfrak{m}^2$. Set $P = k[\![X_1, ..., X_n]\!]$. Then there exists a presentation of S as follows:*

$$S = R[t^{c-1}] = \frac{k[\![X_1, \ldots, X_n, T]\!]}{I + (X_iT - g_i(X_1, \ldots, X_n), T^2 - h(X_1, \ldots, X_n))_{1 \le i \le n}}$$

where $g_i(X_1, \ldots, X_n), h(X_1, \ldots, X_n) \in (X_1, \ldots, X_n)^2P$ for all i.

Moreover, $g_i(x_1, \ldots, x_n), h(x_1, \ldots, x_n) \in \mathfrak{C}$.

Proof. Define $\Psi : k[\![X_1, \ldots, X_n, T]\!] \twoheadrightarrow S$ where

$$\Psi(X_i) = \alpha_i t^{a_i}, 1 \le i \le n, \qquad \Psi(T) = t^{c-1}.$$

By reading off valuations, we see that the images of X_iT, T^2 are all in $\mathfrak{C} \subset \mathfrak{m}^2$. Hence there exists $g_i(X_1, \ldots, X_n), h(X_1, \ldots, X_n) \in (X_1, \ldots, X_n)^2P$ such that

$g_i(x_1, \ldots, x_n), h(x_1, \ldots, x_n) \in \mathfrak{C}$ and

$$X_iT - g_i(X_1, \ldots, X_n), T^2 - h(X_1, \ldots, X_n) \in \ker \Psi.$$

Set $J = \big(X_iT - g_i(X_1, \ldots, X_n), T^2 - h(X_1, \ldots, X_n)\big)_{1 \le i \le n}$. By the above discussion, $J \subset \ker \Psi$.

Conversely, let $p(X_1, \ldots, X_n, T) \in \ker \Psi$. Modulo the ideal J, we can write

$$p(X_1, \ldots, X_n, T) \equiv p'(X_1, \ldots, X_n) + \beta T$$

where $\beta \in k$ and $p'(X_1, \ldots, X_n) \in P$. Since $J \subset \ker \Psi$, we have $p'(X_1, \ldots, X_n) + \beta T \in \ker \Psi$. Thus, $\beta t^{c-1} = \Psi(\beta T) = \Psi(-p'(X_1, \ldots, X_n)) \in R$. By the choice of c's, we immediately obtain that $\beta t^{c-1} = 0$. Thus $\beta = 0$ and hence $p'(X_1, \ldots, X_n) \in I$. This shows that $\ker \Psi = \ker \Phi + J$. □

Remark 4.12. Using the defining ideal of the S in the previous theorem and using $I = (f_1, \ldots, f_m)$, we get the following presentation of $\Omega_{S/k}$:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} & T - \frac{\partial g_1}{\partial x_1} & -\frac{\partial g_2}{\partial x_1} & \cdots & -\frac{\partial h}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_2} & -\frac{\partial g_1}{\partial x_2} & T - \frac{\partial g_2}{\partial x_2} & \cdots & -\frac{\partial h}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} & -\frac{\partial g_1}{\partial x_n} & \cdots & T - \frac{\partial g_n}{\partial x_n} & -\frac{\partial h}{\partial x_n} \\ 0 & \cdots & 0 & x_1 & \cdots & x_n & 2T \end{bmatrix} \overbrace{\begin{matrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \\ dT \end{matrix}}^{\text{basis}} \tag{4.4.1}$$

Note that the embedding dimension of S is $n+1$. Finally by observing the valuations, we get that $\mathfrak{m}S \subset \mathfrak{C} \subset R$ i.e. S/R is a k-vector space. Moreover, note that this dimension is 1 as the only basis element that remains after modding out S by R is T.

Theorem 4.13. *[HMM21, Theorem 4.9] Let* $(R, \mathfrak{m}, k)$ *be a local singular complete curve with algebraically closed residue field of characteristic* 0*. Assume that* $\mathfrak{C} \subseteq \mathfrak{m}^2$*. Construct* $S = R[t^{c-1}]$ *as in Theorem 4.11. Let* $\tau(\Omega_{R/k}), \tau(\Omega_{S/k})$ *represent the torsion submodules of* $\Omega_{R/k}, \Omega_{S/k}$ *respectively. If* $\lambda(\tau(\Omega_{S/k})) \geq n+1$ *and all these torsion elements have non-units in the last row (corresponding to* dT*), then a* k*-linear combination of these torsion elements can be pulled back to a nonzero torsion element in* $\tau(\Omega_{R/k})$*. In particular,* $\tau(\Omega_{R/k}) \neq 0$*.*

Proof of Theorem 4.13. Since $\lambda(\tau(\Omega_S)) \geq n+1$, let $\tau_1, \ldots, \tau_{n+1}$ denote these k-linearly independent torsion elements with non units in the last row (corresponding to dT). Using the presentation from Remark 4.12, we can rewrite $\tau_1, \ldots, \tau_{n+1}$ as $\rho_1, \ldots, \rho_{n+1}$ which are k-linearly independent and have zeroes in the last row.

Let $B = [\rho_1 \ \ldots \ \rho_{n+1}]$ be the $(n+1) \times (n+1)$ matrix obtained by concatenating the column vectors $\rho_i, 1 \leq i \leq n+1$. Since the last row of B is zero, it is effectively an $n \times (n+1)$ matrix. Our goal is to pullback a k-linear combination of the columns of B to Ω_R. Passing to the vector space S/R gives an $n \times (n+1)$ matrix of linear forms in T. Writing the coefficients of each linear form as its own 1×1 column, we obtain an $n \times (n+1)$ matrix over k. By elementary column operations over k it follows that we can obtain a column of zeroes. Performing the same operations on the matrix B gives us a nonzero torsion element whose last row is zero, and whose entries are in R. This necessarily also is a torsion element in $\Omega_{R/k}$, since it is a syzygy of $\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}$. If this torsion element were zero in $\Omega_{R/k}$, it would also be zero in $\Omega_{S/k}$, since the presentation of $\Omega_{S/k}$ contains the Jacobian matrix associated to R. $\square$

Proposition 4.14. *[HMM21, Corollary 5.2] Suppose $S = R[t^{c-1}]$ is as in the discussion above. For every $\tau = \sum r_i dx_i + r_{n+1} dT \in 0 :_{\Omega_S} x_1$, $r_i \in S$, we have that $r_{n+1} \in S$ cannot be a unit.*

Proof. Suppose that $r_{n+1} \in S$ is a unit. Let $J = \langle x_1^2, x_1 T, T^2, x_2, \ldots, x_n \rangle$. Writing $\overline{()}$ for images in $\Omega_{\frac{S}{J}/k}$, we get $\overline{x_1 \tau} = \overline{x_1 r_1} dx_1 + \overline{x_1 r_{n+1}} dT$. Since $x_1^2 \in J$, we have $\overline{x_1} d\overline{x_1} = 0$. Thus we have $\overline{x_1 \tau} = \overline{x_1 r_{n+1}} dT$ which is nonzero in $\Omega_{\frac{S}{J}/k}$ by Lemma 4.4. $\square$

Lemma 4.15 ([Güt90])**.** *Let $(R, \mathfrak{m}, k)$ be a one dimensional complete local reduced k-algebra with* $\operatorname{char}(k) = 0$ *and embedding dimension $n \geq 3$. Suppose y is a non-zero divisor such that $edim(R/R.y) = n - 1$*

a) If $\mathfrak{m}^4 \subseteq R.y$, then

$$\lambda(0 :_{\Omega_R} y) \geq \frac{(n-2)(n-1)}{2}.$$

b) If $\mathfrak{m}^5 \subseteq R.y$, then

$$\lambda(0 :_{\Omega_R} y) \geq \frac{(n-2)(n-1)}{2} - \operatorname{type}(R).$$

Proof. For a), we refer the reader to the proof of [Güt90, Satz 4]. For (b), see the proof of [Güt90, Anmerkung, Page 506-507]. $\square$

Lemma 4.16. *[HMM21, Lemma 5.3] Let $S = R[t^{c-1}]$ as in the discussion above. Then* $\operatorname{type}(S) \leq \operatorname{type}(R) + n - 1$. *In particular, if R is Gorenstein, then* $\operatorname{type}(S) \leq n$.

Proof. Using Remark 4.12, we get the following short exact sequence $0 \to R \to S \to \frac{S}{R} \cong k \to 0$. Dualizing it into the canonical module, gives $0 \to \omega_S \to \omega_R \to \operatorname{Ext}^1_R(k, \omega_R) \cong k \to 0$ (we used Theorems 1.6 and 1.7). Now tensoring this sequence with k, we get

$$\operatorname{Tor}_1^R(k,k) \to \omega_S \otimes_R k \to \omega_R \otimes_R k \to k \to 0.$$

Comparing the length of the modules appearing in this short exact sequence now yields, $\dim_k \operatorname{Tor}_1^R(k,k) + \operatorname{type}(R) \geq \operatorname{type}(S) + 1$ The proof is now complete using [BH98, Theorem 3.3.11].

The last statement follows because $\operatorname{type}(R) = 1$ when R is Gorenstein. $\square$

Remark 4.17. If we assume further that R is Gorenstein, then adding t^{c-1} to R is the same as adding all elements of the form $\frac{c}{x_1}$, where $c \in \mathfrak{C}$. To see this, first we note that the new valuations being added to the valuation semi-group of R in the latter scenario are precisely $c - a_1, \ldots, c - 1$. Now since R is Gorenstein, by [Kun70, Theorem] we get that the valuation semi-group of R already contains $c - a_1, c - a_1 + 1, \ldots, c - 2$ to begin with (as it is *symmetric* (see [Kun70, Definition]) around $c - 1$ as the proof of [Kun70, Theorem] shows); so, we are adding only t^{c-1}. Hence, $S = R[t^{c-1}] = R[\frac{\mathfrak{C}}{x_1}]$. Thus, $x_1 S = (x_1, \mathfrak{C})R$.

The next theorem extends the work of Güttes ([Güt90]), who proved that if either $\mathfrak{m}^4 \subseteq x_1 R$ or R is Gorenstein and $\mathfrak{m}^5 \subseteq x_1 R$, then Berger's conjecture holds.

Theorem 4.18. *[HMM21, Corollary 5.9] Let $(R, \mathfrak{m}, k)$ be a local Gorenstein singular* *complete curve with embedding dimension $n \geq 6$. If $\mathfrak{m}^6 \subseteq x_1R$, then $\tau(\Omega_{R/k}) \neq 0$.*

Proof. We may assume that $\mathfrak{C} \subseteq \mathfrak{m}^2$, because we have already shown that the torsion is nonzero in case the inclusion does not hold (Theorem 4.5). The condition that $\mathfrak{m}^6 \subseteq x_1R$ implies that $\mathfrak{m}^5 \subseteq (x_1R : \mathfrak{m})$. However, it is always true that $(\mathfrak{C}, x_1)R \subseteq x_1 : \mathfrak{m}$, since $x_1\mathfrak{C} = \mathfrak{m}\mathfrak{C}$. Since R is Gorenstein, we have that $\frac{x_1:\mathfrak{m}}{x_1}$ is a one dimensional k-vector space. Further from Corollary 2.20, we get that the conductor is never inside x_1R. Hence we obtain that $(\mathfrak{C}, x_1)R = x_1 : \mathfrak{m}$. Therefore, $\mathfrak{m}^5 \subseteq (\mathfrak{C}, x_1)R$ and so $\mathfrak{m}^5 \subseteq x_1S$ by Remark 4.17. Also from Theorem 4.11, we note that $\mathfrak{m} \subseteq \mathfrak{m}_S$ and $\mathfrak{m}\mathfrak{m}_S \subseteq \mathfrak{m}_S^2 = \mathfrak{m}^2$. Hence,

$$\mathfrak{m}_S^5 = (\mathfrak{m}_S^2)^2\mathfrak{m}_S = (\mathfrak{m}^2)^2\mathfrak{m}_S = \mathfrak{m}^3\mathfrak{m}\mathfrak{m}_S \subseteq \mathfrak{m}^3\mathfrak{m}^2 = \mathfrak{m}^5 \subseteq \mathfrak{m}_S^5.$$

So from the preceding discussion, we conclude that $\mathfrak{m}_S^5 \subseteq x_1S$.

Now using Lemma 4.15(b), Lemma 4.16 and the fact that $\operatorname{edim}(S) = n+1$, we get

$$\lambda(\tau(\Omega_S)) \geq \lambda(0 :_{\Omega_S} x_1) \geq \frac{(n+1-2)(n+1-1)}{2} - \operatorname{type}(S) \geq \binom{n}{2} - n.$$

Since $n \geq 6$, we obtain that

$$\lambda(\tau(\Omega_S)) \geq \lambda(0 :_{\Omega_S} x_1) \geq \binom{n}{2} - n \geq n+1.$$

By Proposition 4.14, the last row cannot be a unit for all these torsion elements. Hence, the proof is now complete using Theorem 4.13. $\square$

Appendix A

The Module of Kähler Differentials

In this chapter, we provide background to the *module of Kähler differentials*, denoted $\Omega_{R/k}$, where k is a subring of a commutative ring R (sometimes we simply say *the module of differentials*). The basic definitions and results in this section are taken mainly from notes by Craig Huneke, Chapter 16 of [Eis13] and also Jürgen Herzog's lecture notes in Trieste[Her94]. For the detailed treatment, one can refer to the classic book of Ernst Kunz [Kun86].

A.1 Part I

In this section, we introduce the notion of derivations and then define the module of differentials. We shall also discuss some of the main techniques that are used in computing it. We provide examples wherever possible to illustrate the use of these techniques.

A.1.1 Derivations

Definition A.1. *Let $k \subset R$ be inclusion of rings, M an R-module. A k-**derivation** into M is a map $d : R \to M$ such that for each $r, s \in R$,*

1. *$d(r + s) = dr + ds$.*

2. *$d(rs) = rds + sdr$.*

3. *d is k-linear.*

(For notational convenience, we are writing dr to mean $d(r)$. We shall use the second notation if the situation demands it to be explicit.)

We say d is a derivation to mean that d is a $\mathbb{Z}$-derivation.

Example A.2.

1. The map $d : R \to M$ with $d(r) = 0$ for all r is a derivation and is called the *trivial derivation*.

2. Let k be a ring. If $R = k[x_1, \ldots, x_n]$ or $R = k[[x_1, \ldots, x_n]]$ then

$$\frac{\partial}{\partial x_i} : R \to R, \qquad F \mapsto \frac{\partial F}{\partial x_i}$$

(formal partial differentiation) is a k-derivation into R. Let $\{dx_1, \ldots, dx_n\}$ be a set of indeterminates and let $M := Rdx_1 \oplus \cdots \oplus Rdx_n$ be the free R-module with basis $(dx_1, \ldots, dx_n)$, the module of *formal differentials*. Then the map

$$d : R \to M, \qquad F \mapsto \frac{\partial F}{\partial x_1} dx_1 + \cdots + \frac{\partial F}{\partial x_n} dx_n$$

is a k-derivation of R into M. We call dF the *formal differential* of F.

3. Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded ring. For $r \in R$, let $r = \sum_{n\in\mathbb{Z}} r_n$ be the decomposition into homogeneous elements $r_n \in R_n$. The ***Euler derivation*** of R is the map

$$\delta : R \to R, \left(r \mapsto \sum_{n\in\mathbb{Z}} n r_n \right).$$

Remark A.3. Let $d : R \to M$ be a $\mathbb{Z}$-derivation. The following statements hold.

a) $d(1) = 0$.

b) For $k \subset R$, d is k-linear if and only if $d(a) = 0$ for all $a \in k$.

c) $d^{-1}(0)$ is a subring of R, hence it is the largest subring over which d is linear.

d) $d(r^n) = nr^{n-1}dr$ for all $n \in \mathbb{N}$.

e) If $\mathrm{char}(R) = p$, then d is an R^p-derivation, where R^p denotes the image of R under the Frobenius map $F : R \to R$, $r \mapsto r^p$.

f) If $\phi : M \to N$ is an R-module homomorphism, then $\phi \circ d : R \to N$ is a derivation into N.

Note that adding two derivations or multiplying a derivation by an element in R yields another derivation. Thus the set of all k-derivations from R to M naturally forms an R-module. We denote this by $\mathrm{Der}_k(R, M)$. It is in general not very easy to compute this module.

Notation A.4. For a derivation $\delta : R \to M$, let $R\delta R$ denote the R-submodule of M generated by $(\delta r)_{r\in R}$. Thus

$$R\delta R = \Big\{ \sum_{i=1}^{n} r_i \delta r_i' \mid r_i, r_i' \in R, n \in \mathbb{N} \Big\}.$$

A.1.2 Module of Differentials

Let $k \to R$ be a map of rings.

Definition A.5. *The **module of Kähler differentials** or simply, module of differentials of R over k, denoted $\Omega_{R/k}$, is an R-module M which satisfies the following.*

1. *There exists a k-derivation $d_{R/k} : R \to \Omega_{R/k}$, called the **universal k-derivation**;*

2. *Given any k-derivation $D : R \to M$, there exists a unique R-module homomorphism $f : \Omega_{R/k} \to M$ such that the following diagram commutes.*

$$\begin{array}{ccc} R & \xrightarrow{d_{R/k}} & \Omega_{R/k} \\ & \searrow^{D} & \downarrow{\exists ! f} \\ & & M \end{array}$$

It is clear that if such a module exists, then it is unique up to isomorphism. In particular, we get the following isomorphism.

$$\mathrm{Hom}_R(\Omega_{R/k}, M) \cong \mathrm{Der}_k(R, M).$$

We mention two constructions that guarantee the existence of $\Omega_{R/k}$.

First Construction

Consider the R-linear map

$$\mathcal{E} : R \otimes_k R \to R$$

$$r \otimes s \mapsto rs$$

Lemma A.6. *Let* $\mathcal{I} = \ker(\mathcal{E})$*. We have the following.*

1. *As a left or right* R*-module,* $\mathcal{I}$ *is generated by* $\{1 \otimes r - r \otimes 1\}_{r \in R}$.

2. *The two* R*-module structures on* $\mathcal{I}$ *agree modulo* $\mathcal{I}^2$.

3. *The map* $d : R \to \mathcal{I}/\mathcal{I}^2$ *given by* $d(r) = r \otimes 1 - 1 \otimes r$ *is a* k*-derivation.*

Proposition A.7. $\Omega_{R/k} = (\mathcal{I}/\mathcal{I}^2)$ *(up to isomorphism) with* $d : R \to \Omega_{R/k}$ *given as in Lemma A.6.*

Remark A.8. Let d be as in Lemma A.6. Then we have

$$RdR = \Omega_{R/k}$$

because of Lemma A.6 (1).

We now give another purely formal construction. It turns out that this will be really helpful in computations as we shall soon see.

Second Construction

Let $F = \bigoplus_{r \in R} Rd_r$, the free module with basis $\{d_r \mid r \in R\}$. Let $H \subset F$ be the R-submodule of generated by

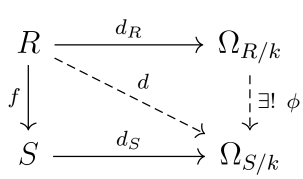
$$\{d_{r+s} - d_r - d_s\}_{r,s \in R} \cup \{d_{\alpha r} - \alpha d_r\}_{\substack{\alpha \in k \\ r \in R}} \cup \{d_{rs} - rd_s - sd_r\}_{r,s \in R}.$$

Then $\Omega_{R/k} := F/H$ with $d_{R/k} : R \to \Omega_{R/k}, \quad r \mapsto d_r$ works as the universal derivation.

A.1.3 Some Techniques For Computation

In this section, we shall discuss various results that help in computing $\Omega_{R/k}$. We shall discuss proofs of some of the results which are very constructive and involve only chasing definitions carefully.

Note that if $f : R \to S$ be a k-algebra morphism and if d_R, d_S denote their respective universal derivations, then there is an unique induced R-linear map $\Omega_{R/k} \to \Omega_{S/k}$ by applying the definition. We make it explicit using the diagram below. Note that $d := d_S \circ f : R \to \Omega_{S/k}$ is a derivation.



This in particular shows that the association of a k-algebra R (or equivalently a map $k \to R$ of rings) to the R-module $\Omega_{R/k}$ and the derivation $d : R \to \Omega_{R/k}$ is a functor. Finally note that we have also found an S-linear map $S \otimes_R \Omega_{R/k} \to \Omega_{S/k}$

using

$$\mathrm{Hom}_S(S \otimes_R \Omega_{R/k}, \Omega_{S/k}) \cong \mathrm{Hom}_R(\Omega_{R/k}, \Omega_{S/k}).$$

Proposition A.9. *Let $R = k[X_\lambda]_{\lambda\in\Lambda}$, a polynomial algebra over k. Then $\Omega_{R/k} \cong \bigoplus_{\lambda\in\Lambda} RdX_\lambda$, free on basis $\{dX_\lambda\}_{\lambda\in\Lambda}$. Here the universal derivation is*

$$d = d_{R/k} : R \to \bigoplus_{\lambda\in\Lambda} RdX_\lambda, \quad d(f) = \sum_{\lambda\in\Lambda} \frac{\partial f}{\partial X_\lambda} dX_\lambda.$$

Proof. Define $D : R \to F := \bigoplus_{\lambda\in\Lambda} RdX_i$ as

$$D(f) = \sum_{\lambda\in\Lambda} \frac{\partial f}{\partial X_\lambda} dX_\lambda.$$

This lands in $\bigoplus_{\lambda\in\Lambda} RdX_i$ since f is a polynomial. Moreover d is a k-derivation since each $\frac{\partial}{\partial X_\lambda}$ is a k-derivation. Also note that $D(X_\lambda) = dX_\lambda$. Since, F is free, we can define a well-defined unique R-map $\phi : F \to \Omega_{R/k}$ as $\phi(dX_\lambda) = d_{R/k}(X_\lambda)$. In order to achieve the conclusion, it is enough to check $\phi \circ D = d_{R/k}$. Moreover, since ϕ is R-linear and $\phi \circ D = d_{R/k}$ on $\{X_\lambda\}_{\lambda\in\Lambda}$, it is enough to check equality on monomials $f \in R$. So let $f = \prod_{i=1}^{n} X_{\lambda_i}^{a_i}$. Then

$$d_{R/k}(f) = \sum_{i=1}^{n} \Big(a_i X_{\lambda_i}^{a_i - 1} \prod_{\substack{jj=1 \\ j\neq i}}^{n} X_{\lambda_j}^{a_j} \Big) d_{R/k}(X_{\lambda_i}).$$

This finishes the proof. □

Remark A.10. Although it is very tempting to repeat the construction as in Proposition A.9 for power series rings, it turns out that we get different results. In fact, if we replace R by $R = k[\![x_1, \ldots, x_n]\!]$ above, then $\delta : R \to \bigoplus_{i=1}^{n} Rdx_i$ is not a universal derivation unlike what we had in the polynomial algebra setup. We shall need some further assumptions on R and k to reproduce an identical result.

Theorem A.11. *Let R be a k-algebra and $I \subset R$ and let $R' = R/I$. Then*

$$\Omega_{R'/k} \cong \frac{\Omega_{R/k}}{RdI + I\Omega_{R/k}}$$

where $RdI = \sum_{i \in I} Rdi$ where $d : R \to \Omega_{R/k}$ is the universal derivation. Moreover,

$$d_{R'/k} : R' \to \Omega_{R'/k},\ \overline{r} \mapsto q(dr)$$

where q is the composition of the natural maps

$$\Omega_{R/k} \twoheadrightarrow \Omega_{R/k}/I\Omega_{R/k} \twoheadrightarrow \Omega_{R/k}/(I\Omega_{R/k} + RdI).$$

One can actually see easily that $I\Omega_{R/k} \subseteq RdI$ as follows: If $r \in R, i \in I$, then $idr = d(ri) - rdi \in RdI$. Hence, $I\Omega_{R/k} \subseteq RdI$. We still write it in the summation form above as such a description will help in the cases that will be the most relevant to us.

Example A.12. Let $R = k[X_\lambda]_{\lambda \in \Lambda}$ be a polynomial ring and $I = (f_1, \ldots, f_m)$ be an ideal of polynomials. Let $R' = R/I$ and $\overline{r} := r + I$. We can now use Theorem A.11 to give a description of $\Omega_{R'/k}$. First notice that by Proposition A.9 we have $\Omega_{R/k} \cong \bigoplus_{\lambda \in \Lambda} RdX_\lambda$. Next, formal partial differentiation as in Example A.2(2) give $RdI + I\Omega_{R/k}$. Finally going modulo I, from the description in Theorem A.11 we obtain that

$$\Omega_{R'/k} \cong \bigoplus_{\lambda \in \Lambda} R'dX_\lambda \Big/ \left(\sum_{\lambda \in \Lambda} \overline{\frac{\partial f_i}{\partial X_\lambda}} dX_\lambda \right)_{1 \le i \le m}$$

where the parenthesis in the denominator means the span as a submodule of the numerator.

Remark A.13. Let $I = (f_1, \ldots, f_m)$ be an ideal in $R = k[X_1, \ldots, X_n]$, a polynomial ring in n variables over some Noetherian ring k. Let $\dfrac{\partial}{\partial x_i} : R \to R$ be as in Example A.2(2).

$$A = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_2}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_1} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial f_1}{\partial x_n} & \dfrac{\partial f_2}{\partial x_n} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

A is an $n \times m$ matrix over R. It is called the ***Jacobian matrix*** of I over R.

We have thus shown that in the situation of Example A.12 with $|\Lambda| < \infty$ and k Noetherian, the Jacobian matrix of I, modulo I (i.e. over R'), forms a presentation matrix for $\Omega_{R'/k}$.

Example A.14. Suppose $S = k[X]/(f)$ for some $f \in k[X]$. Then $\Omega_{R/k} \cong \dfrac{SdX}{Sd(f(X))}$. Next note that $d((f(X))) = f'(X)dX$ and $d(Xf) = Xdf + f'dX$. Thus we obtain that

$$\Omega_{R/k} \cong \frac{k[X]}{(f, f')}.$$

Corollary A.15. *Suppose $k \subseteq l$ is a simple separable extension of fields. Then $\Omega_{l/k} = 0$.*

Proof. $l = k[X]/(f)$. Thus, by Example A.14, $\Omega_{l/k} \cong \dfrac{k[X]}{(f, f')}$. Using separability, we have $(f, f') = k[X]$, thus we are done. □

We now state some important sequences concerning the module of differentials of ring extensions. These are frequently used in computations. In fact, some of the above results can also be recovered using these.

Theorem A.16 (**Conormal Sequence**). *Let R be a k-algebra, $I \subset R$. Let $R' = R/I$. Then there exists an exact sequence*

$$\frac{I}{I^2} \xrightarrow{\alpha} R' \otimes_R \Omega_{R/k} \xrightarrow{\beta} \Omega_{R'/k} \to 0$$

where $\alpha(i + I^2) = \bar{1} \otimes di$, $\beta(\bar{r} \otimes da) = r d_{R'/k}(\bar{a})$ for $i \in I$ and $r, a \in R$.

Example A.17. $R = k[X,Y]/I$ where k is a field containing $1/3$ and $I = (X^2 + Y^3 - 1)$. Let $(x, y) = (X, Y)R$. We know by Example A.12 that

$$\Omega_{R/k} = \frac{RdX \oplus RdY}{(2xdX + 3y^2dY)}.$$

Since, I is generated by a regular element, I/I^2 is free of rank 1. Thus, by Theorem A.16, there is an exact sequence with the following mappings

$$R \cong I/I^2 \longrightarrow RdX \oplus RdY \longrightarrow \Omega_{R/k} \longrightarrow 0$$

$$1 \mapsto x^2 + y^3 - 1 \mapsto 2xdX + 3y^2dY$$

Since, R is a domain, $1 \mapsto 2xdX + 3y^2dY$ is injective. Hence, we can rewrite the sequence again with standard basis

$$0 \to R \xrightarrow{B=\begin{bmatrix} 2x \\ 3y^2 \end{bmatrix}} R^2 \to \Omega_{R/k} \to 0.$$

If char$(k) = 2$ or 3, this shows that pdim$(\Omega_{R/k}) = 1$.

If char$(k) \neq 2, 3$, define $A : R^2 \to R$ by $A = \begin{bmatrix} \frac{x}{2}, \frac{y}{3} \end{bmatrix}$. Since, $AB = 1$, we get that the sequence above splits. Thus, $\Omega_{R/k} \oplus R \cong R^2$ i.e. $\Omega_{R/k}$ is projective of rank 1.

Example A.18. $R = k[X,Y]/I$ where k is a field containing $1/3$ and $I = (X^3, Y^3)$. Let $(x, y) = (X, Y)R$. Note that $I/I^2 \cong R^2$ as it is generated by a regular sequence of length 2. By Theorem A.16, we have

$$R^2 \cong I/I^2 \longrightarrow RdX \oplus RdY \longrightarrow \Omega_{R/k} \to 0$$

$$(1, 0) \mapsto x^3 \mapsto 3x^2 dX$$

$$(0, 1) \mapsto y^3 \mapsto 3y^2 dY$$

Thus we get the following sequence,

$$R^2 \xrightarrow{\begin{pmatrix} 3x^2 & 0 \\ 0 & 3y^2 \end{pmatrix}} R^2 \to \Omega_{R/k} \to 0$$

Since $1/3 \in k$, we get $\Omega_{R/k} \cong R/(x^2) \oplus R/(y^2)$.

Notation A.19. Let $\Gamma_{R'/k} := \ker \alpha$ where R', α are as in Theorem A.16. So, we have an exact sequence

$$0 \to \Gamma_{(R/I)/k} \to I/I^2 \to R/I \otimes_R \Omega_{R/k} \to \Omega_{R'/k} \to 0.$$

Theorem A.20 (**Jacobi Zariski Sequence**). *Let $k \to R \to S$ be ring homomorphisms. Then there exists an exact sequence*

$$\Gamma_{S/k} \to \Gamma_{S/R} \to S \otimes_R \Omega_{R/k} \to \Omega_{S/k} \to \Omega_{S/R} \to 0.$$

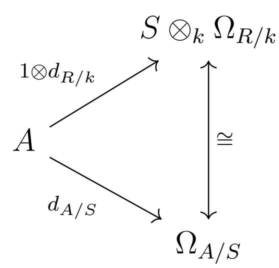
Furthermore, if $\Omega_{R/k}$ is flat over R, then we can add the term $S \otimes_R \Gamma_{R/k}$ on the left.

Restricting the above sequence to only the last three terms on the right gives us what is known as the ***Relative Cotangent Sequence***.

We end this section by listing some further tools that make working with modules of differentials convenient.

Proposition A.21. *The following statements hold.*

1. *(**Base Change**) Let R, S be k-algebras. Let $A = S \otimes_k R$. There is a commutative diagram*



2. *(**Direct Limits**) Let $\{R_\lambda\}_{\lambda \in \Lambda}$ be a direct system of k-algebras and $R = \varinjlim R_\lambda$. Then*

$$\Omega_{R/k} \cong \varinjlim \Omega_{R_\lambda/k}.$$

3. *(**Localization**) Let R be a k-algebra and $W \subset R$ be a multiplicatively closed*

set. Then

$$\Omega_{W^{-1}R/k} \cong W^{-1}R \otimes_R \Omega_{R/k}$$

and the universal derivation is obtained by uniquely extending $d_{R/k}$ *via the quotient rule, i.e. for* $r \in R, w \in W$

$$d(r/w) = \frac{wd_{R/k}(r) - rd_{R/k}(w)}{w^2}.$$

4. *(**Direct Products**) If* $R_1, \ldots, R_n$ *are* k*-algebras and* $R = \prod_i R_i$*, then*

$$\Omega_{R/k} = \prod_i \Omega_{R_i/k}.$$

One can find detailed proofs of these above statements, for instance, in [Eis13, Chapter 16]. In the next section, we shall establish a few more results that will be pertinent to our goal of discussing a conjecture of R.W. Berger.

A.2 Part II

In this brief section, we shall mainly discuss the properties of the module of differentials when we consider analytic algebras over fields. Note that Remark A.10 had already indicated that we need some extra assumptions to still have a nice description of the module of differentials. We shall first venture a bit into field extensions and then apply the results in the later sections. Most of the results here can be found in [Kun86].

A.2.1 Field Extensions and Differentials

Lemma A.22 (**MacClane-Cartier Equality**). *Let $k \subseteq L$ be such that L is finitely generated as a field over k. Then*

$$\dim_L(\Omega_{L/k}) = \mathrm{Trdeg}(L/k) + \dim_L(\Gamma_{L/k}).$$

The proof involves first choosing a transcendence basis and then writing down L as an algebra over the field generated by k and the transcendence basis. Finally one uses the conormal sequence.

Theorem A.23. *If $k \subset L$ is separable, then $\Gamma_{L/k} = 0$.*

The proof involves first reducing to finitely generated field extension case using Proposition A.21. Next we bootstrap by first assuming that L is a simple extension. This case is taken care by Corollary A.15 and again the conormal sequence. Then, the cases of purely transcendental extension and the case where there exists a separating transcendence basis are take care of respectively. This proof is somewhat involved but in spirit, the Jacobi-Zariski sequence plays the crucial role.

Corollary A.24. *Let $k \subset L$ be such that L is separable and finitely generated as a field over k. Then*

$$\dim_L(\Omega_{L/k}) = \mathrm{Trdeg}(L/k).$$

Proof. The proof follows immediately from the above Theorem A.23 and Lemma A.22.

□

As an immediate consequence, we can see that $\Omega_{R/k}$ need not be finitely generated module for $k \subseteq R$ though the examples in Appendix A all seemed to suggest otherwise.

Example A.25. Let k be a field with $\mathrm{char}(k) = 0$. Let $R = k[\![X]\!]$. If $\Omega_{R/k}$ is a finitely generated module, then by Proposition A.21(3), $\Omega_{L/k}$ will have finite dimension over L, where $L = Q(R)$. However, note that $L = k((X))$ and $\mathrm{Trdeg}(L/k) = \infty$, a contradiction by Corollary A.24. Thus $\Omega_{R/k}$ cannot be finitely generated.

This example suggests that we have to modify our definition for $\Omega_{R/k}$ if we want to assure 'finite generation as a module' condition for it. In fact, as we shall see soon, the conjecture of major interest in this thesis involves such a modified version of $\Omega_{R/k}$. So in a way, this example is apt in motivating the study of 'universally finite module of differentials'.

A.2.2 Universally Finite Module of Differentials

We can talk about the construction of this module as has been done in [Kun86]. However, for such a general discussion we need to talk about the *DeRham algebra* which is the exterior algebra of $\Omega_{R/k}$ and from there one can construct further universal notions which will eventually lead to the object of interest. Instead, we are going to follow the approach as in [Her94].

Definition A.26. *For a k-algebra R, the* ***universally finite module of differentials*** *of R over k, again denoted $\Omega_{R/k}$, is a finitely generated R-module that satisfies* *the following universal property: for any finitely generated R-module M,*

$$\mathrm{Der}_k(R, M) \cong \mathrm{Hom}_R(\Omega_{R/k}, M).$$

The derivation $d_{R/k}$ that comes associated with it is called the ***universally finite derivation****.*

Here, we keep the notations unchanged from Appendix A, as this will be the object of study for us.

Note that the universally finite module of differentials need not exist in general. One can find such an example in [Kun86, Example 11.2] but as mentioned before we have to do a global study in order to explicitly describe the example. Hence, we skip it.

It is also clear from the universal property that if the module of Kähler differentials is finitely generated then the two coincide.

Important Remark.

1. Henceforth, in this thesis, by $\Omega_{R/k}$ we shall mean the universally finite module of differentials, and just for convenience of discussion, we are going to again refer to $\Omega_{R/k}$ as the module of differentials. We shall also refer to the universally finite derivation as the universal derivation.

2. If $\Omega_{R/k}$ exists, then all the results of Appendix A.1 carry through. An interested reader can follow the arguments in [Kun86, Chapter 11].

Existence

As mentioned earlier, $\Omega_{R/k}$ need not exist. However, as a consequence of results in Theorem A.11 and Proposition A.21(3), we immediately get the following statement.

Theorem A.27. *If R is the localization of an affine k-algebra, then $\Omega_{R/k}$ exists.*

We will eventually be interested in quotients of power series rings over fields which are primary examples of the so called *analytic k-algebras* where k is the underlying field. Thus, we now discuss a result which will guarantee the existence of $\Omega_{R/k}$ over rings which are of primary interest to us.

Theorem A.28. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring which is an R_0-algebra. Let $\hat{R}$ denote the completion of R in the $\mathfrak{m}$-adic topology. If Ω_{R/R_0} exists, then*

$$\Omega_{\hat{R}/R_0} = \widehat{\Omega_{R/R_0}}$$

where the latter refers to the completion of Ω_{R/R_0} in the $\mathfrak{m}$-adic topology.

This theorem appears as Corollary 12.10 in [Kun86]. The proof is dependent on the fact that each R_0-derivation is continuous in the $\mathfrak{m}$-adic topology. Further by Krull's Intersection Theorem, we get that Ω_{R/R_0} is separated in the $\mathfrak{m}$-adic topology as well. Then using continuity, we extend the universal derivation of R over R_0 to the completion of Ω_{R/R_0}. Finally, we need to check the universal property.

A.2.3 Analytic k-algebras

Let k be a valuated field and $S = k[\![X_1, \dots, X_n]\!]$ be the power series ring over k. An algebra of the form

$$R = \frac{k[\![X_1, \dots, X_n]\!]}{I}$$

for some ideal I in S will be called an ***analytic*** k***-algebra***.

This forms an important (in fact, the most important for our work) class of examples where $\Omega_{R/k}$ exists and we can explicitly describe it.

Theorem A.29. *Let R be an analytic k-algebra as above. Let $I = (f_1, \dots, f_m)$ where $f_i \in S$. Then*

$$\Omega_{R/k} \cong \bigoplus_{i=1}^{n} R dX_i \Big/ \left(\sum_{j=1}^{n} \overline{\frac{\partial f_i}{\partial X_j}} dX_j \right)_{1 \leq i \leq m}.$$

Proof. (Sketch)

Let $S_1 = k[X_1, \dots, X_n]$ and $\mathfrak{m} := (X_1, \dots, X_n)$. Then $S = \widehat{S_1}$ in the $\mathfrak{m}$-adic topology. By Theorem A.28, we have

$$\Omega_{S/k} = \widehat{\Omega_{S_1/k}} = S \otimes_{S_1} \Omega_{S_1/k} = S \otimes_{S_1} (\oplus_{i=1}^{n} S_1 dX_i) = \oplus_{i=1}^{n} S dX_i.$$

The rest of the proof is exactly follows the same reasoning as the proof of Theorem A.11 and can be found in standard texts. □

Remark A.30. By Theorem A.29, the derivation constructed in Remark A.10 is the universally finite derivation.

Remark A.31. By Theorem A.29, $\Omega_{R/k}$ has a presentation given by the Jacobian matrix of I over R in the same way as was observed in Remark A.13. Note also that the conormal sequence gives a natural starting point when one tries to find resolution of $\Omega_{R/k}$ over R in this case.

We want to conclude this section with a result involving the rank and minimum number of generators of $\Omega_{R/k}$ when R is an analytic k-algebra with the added assumption of k being perfect.

Theorem A.32. *Let $R = k[\![X_1, \ldots, X_n]\!]/I$ be a domain. Assume further that k is a perfect field. Then we have the following.*

1. $\mu(\Omega_{R/k}) = \mu(\mathfrak{m})$.

2. $\operatorname{rk}(\Omega_{R/k}) = n - \operatorname{ht}(I) = \dim(R)$.

Proof. Since $k[\![X_1, \ldots, X_n]\!]$ is regular, we have $\dim(R) = n - \operatorname{ht}(I)$. This takes care of the last part of (2).

1. Let $\mathfrak{m}$ be the unique maximal ideal of R. We apply the conormal sequence (Theorem A.16) to the map $R \to k = R/\mathfrak{m}$ to get

$$\mathfrak{m}/\mathfrak{m}^2 \cong k \otimes_R \Omega_{R/k}$$

Thus $\mu(\mathfrak{m}) = \mu(\Omega_{R/k})$.

2. Since R is domain and k is perfect, we get

$$\begin{aligned}
\operatorname{rk}(\Omega_{R/k}) &= \dim_{Q(R)} \big(Q(R) \otimes_R \Omega_{R/k}\big) \\
&= \dim_{Q(R)} \big(\Omega_{Q(R)/k}\big) && \text{(Proposition A.21(2))} \\
&= \operatorname{Trdeg}(Q(R)/k) && \text{(Corollary A.24)} \\
&= \dim(R)
\end{aligned}$$

□

We should mention that modified versions of Theorem A.32 exist in much broader contexts. As always, [Kun86] is an excellent source to find these results. For our purposes, we are satisfied with this restricted setup only.

Finally we want to mention another result without proof which will be of importance to us. We recall the following definition.

Definition A.33. *Let R be a Noetherian ring and I be an ideal in R with no embedded primes. Then the second symbolic power of I is the ideal defined by*

$$I^{(2)} = \bigcap_{\operatorname{Ass}(R/I)} (I^2 R_P \cap R).$$

Theorem A.34. *Let $R = k[\![X_1, \dots, X_n]\!]/I$ be reduced. Assume further that k is a perfect field. Then*

$$\Gamma_{(R/I)/k} = \frac{I^{(2)}}{I^2}.$$

Bibliography

[Bas77] Richard Bassein. On smoothable curve singularities: local methods. *Math. Ann.*, 230(3):273–277, 1977.

[Ber63] Robert Berger. Differentialmoduln eindimensionaler lokaler Ringe. *Math. Z.*, 81:326–354, 1963.

[Ber88] Robert W. Berger. On the torsion of the differential module of a curve singularity. *Arch. Math. (Basel)*, 50(6):526–533, 1988.

[Ber94] Robert W. Berger. Report on the Torsion of Differential Module of an Algebraic Curve. In *Algebraic Geometry and its Applications*, pages 285–303. Springer, 1994.

[BG80] Ragnar-Olaf Buchweitz and Gert-Martin Greuel. The Milnor number and deformations of complex curve singularities. *Invent. Math.*, 58(3):241–281, 1980.

[BH92] William C. Brown and Jürgen Herzog. One-dimensional local rings of maximal and almost maximal length. *J. Algebra*, 151(2):332–347, 1992.

[BH98] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay Rings*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 1998.

[BV06] Winfried Bruns and Udo Vetter. *Determinantal rings*, volume 1327. Springer, 2006.

[CnGW98] Guillermo Cortiñas, Susan C. Geller, and Charles A. Weibel. The Artinian Berger conjecture. *Math. Z.*, 228(3):569–588, 1998.

[CnK99] Guillermo Cortiñas and Fabiana Krongold. Artinian algebras and differential forms. *Comm. Algebra*, 27(4):1711–1716, 1999.

[Del94] Donatella Delfino. On the inequality $\lambda(\overline{R}/R) \leq t(R)\lambda(R/\mathfrak{C})$ for one-dimensional local rings. *J. Algebra*, 169(1):332–342, 1994.

[Eis05] David Eisenbud. *The geometry of syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry.

[Eis13] David Eisenbud. *Commutative Algebra: with a view toward Algebraic Geometry*, volume 150. Springer Science & Business Media, 2013.

[Gre84] S Greco. On the postulation of finite subschemes of projective integral curves. *Queens Papers Pure Appl. Mat*, 67, 1984.

[Güt90] Klaus Güttes. Zum Torsionsproblem bei Kurvensingularitäten. *Archiv der Mathematik*, 54(5):499–510, May 1990.

[H9̈0] Reinhold Hübl. A note on the torsion of differential forms. *Arch. Math. (Basel)*, 54(2):142–145, 1990.

[Her78] Jürgen Herzog. Ein Cohen-Macaulay-Kriterium mit Anwendungen auf den Konormalenmodul und den Differentialmodul. *Math. Z.*, 163(2):149–162, 1978.

[Her94] Jürgen Herzog. The module of differentials. In *lecture notes from Workshop on Commutative Algebra and its Relation to Combinatorics and Computer Algebra (Trieste, 1994)*, 1994.

[HHS19] Jürgen Herzog, Takayuki Hibi, and Dumitru I. Stamate. The trace of the canonical module. *Israel J. Math.*, 233(1):133–165, 2019.

[Hir83] Heisuke Hironaka. On Nash blowing-up. In *Arithmetic and geometry*, pages 103–111. Springer, 1983.

[HK71] Jürgen Herzog and Ernst Kunz. Die Wertehalbgruppe eines lokalen Rings der Dimension 1. In *Die Wertehalbgruppe eines lokalen Rings der Dimension I*, pages 3–43. Springer, 1971.

[HMM21] Craig Huneke, Sarasij Maitra, and Vivek Mukundan. Torsion in differentials and Bergers conjecture. *Research in the Mathematical Sciences*, 8(4):1–15, 2021.

[HW84] Jürgen Herzog and Roland Waldi. Differentials of linked curve singularities. *Archiv der Mathematik*, 42:335–343, 1984.

[HW86] Jürgen Herzog and Rolf Waldi. Cotangent functors of curve singularities. *Manuscripta Math.*, 55(3-4):307–341, 1986.

[HW94] Craig Huneke and Roger Wiegand. Tensor products of modules and the rigidity of Tor. *Mathematische Annalen*, 299(3):449–476, 1994.

[Iso91] Satoru Isogawa. On Berger's conjecture about one-dimensional local rings. *Arch. Math. (Basel)*, 57(5):432–437, 1991.

[Käh11] Erich Kähler. Mathematische Werke/Mathematical works. In *Mathematische Werke/Mathematical Works*. de Gruyter, 2011.

[Koc83] Josef Koch. *Über die Torsion des Differentialmoduls von Kurvensingularitäten*, volume 5 of *Regensburger Mathematische Schriften [Regensburg Mathematical Publications]*. Universität Regensburg, Fachbereich Mathematik, Regensburg, 1983.

[KR77] Ernst Kunz and Walter Ruppert. Quasihomogene Singularitäten algebraischer Kurven. *manuscripta mathematica*, 22(1):47–61, 1977.

[KT19] Toshinori Kobayashi and Ryo Takahashi. Rings whose ideals are isomorphic to trace ideals. *Mathematische Nachrichten*, 292(10):2252–2261, 2019.

[Kun70] Ernst Kunz. The value-semigroup of a one-dimensional Gorenstein ring. *Proceedings of the American Mathematical Society*, 25(4):748–751, 1970.

[Kun86] Ernst Kunz. *Kähler differentials*. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1986.

[Lin17] Haydee Lindo. Trace ideals and centers of endomorphism rings of modules over commutative rings. *J. Algebra*, 482:102–130, 2017.

[Mai22] Sarasij Maitra. Partial trace ideals and Berger's conjecture. *Journal of Algebra*, 598:1–23, 2022.

[Ore81] Ferruccio Orecchia. Points in generic position and conductors of curves with ordinary singularities. *Journal of the London Mathematical Society*, 2(1):85–96, 1981.

[Poh89] Thomas Pohl. *Über die Torsion des Differentialmoduls von Kurvensingularitäten*. PhD thesis, Verlag nicht ermittelbar, 1989.

[Poh91] Thomas Pohl. Differential modules with maximal torsion. *Arch. Math. (Basel)*, 57(5):438–445, 1991.

[Rot09] Joseph J. Rotman. *An introduction to homological algebra*, volume 2. Springer, 2009.

[Sch70] Günter Scheja. Differentialmoduln lokaler analytischer Algebren, Schriftenreihe Math. *Inst. Univ. Fribourg, Univ. Fribourg, Switzerland*, 1970.

[Ser65] Jean-Pierre Serre. Algèbre locale. Multiplicités, volume 11 of cours au collège de france, 1957–1958, rédigé par pierre gabriel. seconde édition, 1965. *Lecture Notes in Mathematics. Springer-Verlag, Berlin*, 6, 1965.

[Ser97] Jean-Pierre Serre. *Algèbre locale, multiplicités: cours au Collège de France, 1957-1958*, volume 11. Springer Science & Business Media, 1997.

[SH06] Irena Swanson and Craig Huneke. *Integral closure of ideals, rings, and modules*, volume 336 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

[Tei76] Bernard Teissier. The hunting of invariants in the geometry of discriminants. *Real and complex singularities, Oslo*, pages 565–677, 1976.

[Ulr81] Bernd Ulrich. Torsion des Differentialmoduls und Kotangentenmodul von Kurvensingularitäten. *Arch. Math. (Basel)*, 36(6):510–523, 1981.

[Vas91] Wolmer V. Vasconcelos. Computing the integral closure of an affine domain. *Proc. Amer. Math. Soc.*, 113(3):633–638, 1991.

[Yos86] Yuji Yoshino. Torsion of the differential modules and the value semigroups of one dimensional local rings. *Mathematics reports*, 9:83–96, 1986.