In Search of Bounds on the Dimension of Ext between Irreducible Modules for Finite Groups of Lie Type

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0 Preface

In the second half of the 20th century, one of the biggest open problems in mathematics was the classification of the finite simple groups. Research which would eventually serve as the basis for this classification emerged in the 1950s, but there was no organized effort to classify the finite simple groups until the 1970s. A major breakthrough toward the classification came in 1963 with the Feit-Thompson Theorem, which showed that every finite group of odd order is solvable. In 1972, Daniel Gorenstein gave a series of lectures at the University of Chicago in which he outlined a program for the classification of the finite simple groups [27]. The mathematical community responded to Gorenstein's plan with great enthusiasm. In the years after Gorenstein presented his program, over 100 mathematicians across the world worked to implement his ideas. In 1981, Gorenstein announced that the proof of the classification theorem was complete. However, upon review, several gaps were discovered in the proof, and work on the classification theorem continued. The problems with the proof of the classification theorem were highly non-trivial; it took Aschbacher and Smith seven years (and two books) to resolve the last gap [27]. Ashbacher and Smith finished their work in 2004, and the consensus now is that the proof of the classification theorem is complete. (Gorenstein died in 1992, twelve years before the proof of the classification theorem was completed.)

The current proof of the classification theorem is immense - it consists of over 10,000 pages across about 500 journal articles [27]. Due to the scope of this work, few mathematicians understand the full proof of the classification theorem. In the 1980s, Gorenstein initiated an effort to address this issue and began working together with Lyons and Solomon to revise and shorten the proof of the classification theorem. After Gorenstein's death, Lyons and Solomon continued working on this "revisionism" project. The American Mathematical Society is publishing their work in a multi-volume series [32]; the first six volumes of this series were published between the years of 1994 and 2005 (the first three are introductory), and more are expected in the future [27].

According to the classification theorem, there are 18 infinite families of finite simple groups as well as 26 sporadic groups. In this dissertation, we will be concerned with one particular class of finite simple groups - the finite groups of Lie type (which are discussed in Chapter 3). Since most of the finite simple groups are either finite groups of Lie type or closely related to finite groups of Lie type [36, Theorem 1.1], the representation theory of finite groups of Lie type is of particular interest.

Let q be a power of a prime p, and let $G(q)$ be a finite group of Lie type. Let k be an algebraically closed field of characteristic r. There are three distinct cases to consider in the representation theory of $kG(q)$:

(1) $r = 0$ (the characteristic 0 case),

(2) $r = p$ (the defining characteristic case), and

(3) $r > 0$, $r \neq p$ (the non-defining characteristic, or cross-characteristic, case).

In each case, one area of interest is the parameterization of the irreducible $kG(q)$-modules. This task is almost complete in the characteristic 0 and in the defining characteristic cases [36]. However, only partial results exist for the alternating groups and the finite groups of Lie type [36, 4.2.2]. In the defining and non-defining characteristic cases, another area of interest is the homological algebra of $kG(q)$ (when $\mathrm{char}(k) = 0$, $kG(q)$ is semisimple by Maschke's theorem and there are no non-zero cohomology groups in positive degree). One particular problem in both the defining and non-defining characteristic cases is to find bounds on the dimension of the cohomology groups $H^i(G(q), V)$ $(i \geq 1)$, where V is any irreducible $kG(q)$-module.

In 2009, working in the defining characteristic, Cline, Parshall, and Scott [10] showed that the dimension of $H^1(G(q), V)$ (as a k-vector space) is bounded in the following sense.

Theorem 0.0.1. *([10]) Suppose that* $\mathrm{char}(k) = p > 0$*, let q be a power of p, and let $G(q)$ be a finite group of Lie type over $\mathbb{F}_q$. Then, there is a constant C which depends only on the rank of $G(q)$ such that* $\dim\, H^1(G(q), V) < C$ *for all irreducible $kG(q)$-modules V.*

After Cline, Parshall, and Scott discovered their bound on the dimension of $H^1(G(q), V)$ in defining characteristic, Guralnick and Tiep worked to find a bound on the dimension of $H^1(G(q), V)$ in non-defining characteristic. In 2011, Guralnick and Tiep published the following result.

Theorem 0.0.2. *([35]) Suppose that q is a power of p, $G(q)$ is a finite group of Lie type over $\mathbb{F}_q$, and* $\mathrm{char}(k) = r > 0$*, with $r \neq p$. Let W be the Weyl group of $G(q)$, and let e be the Lie rank of $G(q)$. If V is an irreducible kG-module, then*

$$\dim\, H^1(G(q), V) \leq \begin{cases} 1 & \text{if } V^B = 0 \\ |W| + e & \text{if } V^B \neq 0 \end{cases}$$

(where B is a Borel subgroup of $G(q)$).

Guralnick and Tiep's bounds on the dimension of $H^1(G(q), V)$ (where V is an irreducible $kG(q)$-module and $\mathrm{char}(k) = r \nmid q$) inspired much of the original research presented in this dissertation. To simplify notation, let $G(q) = G$. Our task was to generalize Guralnick and Tiep's results in order to find bounds on the dimension of $\mathrm{Ext}^i_{kG}(Y, V)$, $i \geq 1$, when $\mathrm{char}(k) = r > 0$, $r \nmid q$, and Y and V are irreducible kG-modules.[1]

In Part II of this thesis, we combine the techniques of Guralnick and Tiep [35] with techniques of modular Harish-Chandra theory to find bounds on the dimension of $\mathrm{Ext}^1_{kG}(Y, V)$

[1]Bendel, Nakano, Parshall, Pillen, Scott, and Stewart [2] showed that the dimensions of the Ext groups between irreducible kG-modules in the defining characteristic are bounded in all homological degrees (with the bounds depending on the homological degree and the root system Φ of G).

in cross characteristic in the case that G has a split BN-pair.[2] Let B be a Borel subgroup of G containing a maximal torus T. We consider the following cases.

1. $Y, V \in \mathrm{Irr}_k(G|B) = \mathrm{Irr}_k(G|(T,k))$ (where $\mathrm{Irr}_k(G|B)$ is the unipotent principal Harish-Chandra series, defined in Section 4.4)

   If Y and V are both unipotent principal series representations, we show that $\dim \mathrm{Ext}^1_{kG}(Y,V) \leq |W| + (\dim Y)e$ (Theorem 11.4.3). (In fact, by an argument analogous to that given in Corollary 13.3.4, it follows that $\dim \mathrm{Ext}^1_{kG}(Y,V) \leq |W| + \min(\dim \mathrm{Y}, \dim \mathrm{V})\mathrm{e})$. Here, the key idea is to work with the r-Sylow subgroup T_r of the abelian group T. Since T_r is an abelian r-group, it is possible to compute the exact dimension of $\mathrm{Ext}^1_{kT_r}(k,k)$. This, in turn, yields a bound on $\dim \mathrm{Ext}^1_{kT_r}(k,V)$ for any kT_r-module V (Lemma 11.4.2). Finally, to find a bound on $\dim \mathrm{Ext}^1_{kG}(Y,V)$, we use a long exact sequence in Ext to reduce the calculation of $\mathrm{Ext}^1_{kG}(Y,V)$ to a calculation of an Ext group over kT_r.

2. $Y \in \mathrm{Irr}_k(G|(T,X))$, $V \in \mathrm{Irr}_k(G|(T,X')$ (where X and X' are any irreducible kT-modules)

   We assume that Y and V are principal series representations (though not necessarily belonging to the unipotent principal series). When Y and V belong to distinct principal Harish-Chandra series, we show that $\dim \mathrm{Ext}^1_{kG}(Y,V) = 0$ (Corollary 13.2.3). And, when Y and V belong to the same principal series $\mathrm{Irr}_k(G|(T,X))$, we show that $\dim \mathrm{Ext}^1_{kG}(Y,V) \leq |W|^2 + \min(\dim \mathrm{Y}, \dim \mathrm{V})\mathrm{e}$ (Corollary 13.3.4). In the case that Y and V belong to distinct Harish-Chandra series, we generalize [35, Theorem 2.2] to show that $\dim \mathrm{Ext}^1_{kG}(Y,V)$ is equal to the number of times V occurs as a composition factor of a certain submodule of $R^G_T(X)$ (Theorem 13.2.1). We then use properties of Harish-Chandra induction and restriction to show that V cannot occur as a composition factor of $R^G_T(X)$ (Proposition 13.2.2), and the desired result follows. In the case that Y and V belong to the same principal series $\mathrm{Irr}_k(G|(T,X))$, we obtain the desired bound on $\dim \mathrm{Ext}^1_{kG}(Y,V)$ by reducing to the subgroup T_r of T.

3. $Y \in \mathrm{Irr}_k(G|B)$, $V \notin \mathrm{Irr}_k(G|B)$

We assume that Y is an irreducible kG-module belonging to the unipotent principal series, and that V is an irreducible kG-module outside of the unipotent principal series. We assume additionally that the pair (G,k) is such that every composition factor of $k|_B^G$ belongs to a Harish-Chandra series arising from a regular character of the unipotent radical U of B (in other words, the pair (G,k) has property (P), as defined in Chapter 14). In this case, we show that there is a subset $J \subseteq S$ (where S is a set of fundamental reflections in W) such that $\dim \mathrm{Ext}^1_{kG}(Y,V) \leq [W : W_J]$ (Theorem 14.3.3). To prove that this bound holds, we use the fact that $\dim \mathrm{Ext}^1_{kG}(Y,V)$ is equal to the number of times V occurs as a composition factor of a certain submodule of $k|_B^G$. In this case, though, the basic properties of Harish-

[2] "Split" BN-pairs are defined in Section 2.3. The assumption that G is split is needed in order to be able to use the results of modular Harish-Chandra theory given in [31, Section 4].

Chandra induction and restriction are not sufficient to find a bound on $[k|_B^G : V]$. Therefore, we introduce the Gelfand-Graev module for G and, using properties of the Gelfand-Graev module, find a bound on $[k|_B^G : V]$, which yields a bound on dim $\mathrm{Ext}^1_{kG}(Y, V)$.

In Part III of this thesis, we compute some specific examples of bounds on the dimension of Ext^1 in the case that $G = \mathrm{GL}_n(q)$ is the general linear group over the finite field $\mathbb{F}_q$ (as above, we assume $\mathrm{char}(k) = r > 0$ with $r \not| q$). There are three widely used parameterizations of the irreducible $k\mathrm{GL}_n(q)$-modules (we describe these parameterizations in Chapter 9). In our examples, we work with the parameterization given by Dipper and Du in [19, 4.2.11, (6)]. Using the results of their papers [18] and [19], Dipper and Du determined how the irreducible $k\mathrm{GL}_n(q)$-modules are partitioned into Harish-Chandra series. In [19, Section 4], Dipper and Du describe an algorithm which can be used to find the Harish-Chandra series containing an irreducible $k\mathrm{GL}_n(q)$-module indexed as in [19, 4.2.11, (6)]. Since our bounds on the dimension of Ext^1 between irreducible kG-modules depend on the Harish-Chandra series containing these irreducibles, Dipper and Du's indexing is an ideal choice for the computations of Part III.

Finally, in Part IV of this thesis, we turn to calculations of higher Ext groups. In this case, our main reference is [9], in which Cline, Parshall, and Scott connect H^i calculations for $G = \mathrm{GL}_n(q)$ in cross characteristic to Ext^i calculations over a q-Schur algebra.[3] Following [9], we assume that $r \not| q(q-1)$. The case of $i = 1$ is handled in [9, Theorem 10.1], where CPS compute $H^1(G, L)$ for an irreducible kG-module L. A key component of the proof of [9, Theorem 10.1] is the fact that k (the trivial irreducible kG-module) is in the head of the permutation module $k|_B^G$. But, any irreducible kG-module in the unipotent principal series $\mathrm{Irr}_k(G|B)$ is in the head of $k|_B^G$. This observation allows us to generalize [9, Theorem 10.1] in Chapter 18. We give our generalization in Theorem 18.2.1, where we compute $\mathrm{Ext}^1_{kG}(Y, V)$ between an irreducible kG-module Y belonging to the unipotent principal series and an irreducible kG-module V belonging to any Harish-Chandra series.

Cline, Parshall, and Scott's calculation of the higher cohomology groups $H^i(G, L)$ (where L is an irreducible kG-module) is given in [9, Theorem 12.4]. The proof of [9, Theorem 12.4] is significantly more challenging than the proof of [9, Theorem 10.1] and requires more background machinery. The key component of the proof of [9, Theorem 12.4] is the existence of a resolution of the trivial kG-module k which possesses certain desirable properties [9, Lemma 12.3]. We give our generalization of [9, Theorem 12.4] in Chapter 19. Using techniques of Parshall and Scott [44], we construct a resolution analogous to that of CPS in Lemma 19.2.5. And, in Theorem 19.2.6, we use this resolution to relate certain higher Ext groups for $\mathrm{GL}_n(q)$ in cross-characteristic to Ext groups over a q-Schur algebra.

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[3] Ext^i calculations over a q-Schur algebra can be translated to Ext^i calculations in the category of integrable modules for the quantum enveloping algebra $\tilde{U}_q(\mathfrak{gl}_n)$. So, the connection between Ext^i calculations for $\mathrm{GL}_n(q)$ and Ext^i calculations over a q-Schur algebra opens the possibility of using the theory of quantum groups to learn more about the structure of Ext groups for $\mathrm{GL}_n(q)$.

Part I

Background

1 The Ext Functor and Group Cohomology

1.1 The Ext Functor

In this section, we define the Ext functor, using [49, 2.5] as our main reference. Let R be a ring (with unit), and let $R-\text{mod}$ be the category of left R-modules. For any R-module M, the functor $\text{Hom}_R(M,-)$ is left exact. Since $R-\text{mod}$ has enough injectives, we can construct the right derived functors $R^i\text{Hom}_R(M,-)$ $(i \geq 0)$. Let $V \in R-\text{mod}$, and let

$$0 \to V \to I^0 \to I^1 \to I^2 \to \cdots$$

be an injective resolution of V by left R-modules. Consider the complex $\text{Hom}_R(M,I)$ obtained by applying the covariant left exact functor $\text{Hom}_R(M,-)$ to the injective resolution of V given above:

$$0 \to \text{Hom}_R(M,I^0) \to \text{Hom}_R(M,I^1) \to \text{Hom}_R(M,I^2) \to \cdots.$$

For $i \geq 0$, $R^i\text{Hom}_R(M,-)(V) = H^i(\text{Hom}_R(M,I))$ is the ith cohomology of this complex (where $H^0(\text{Hom}_R(M,I)) \cong \text{Hom}_R(M,V)$ and $H^i(\text{Hom}_R(M,I)) = \text{Ker}(\text{Hom}_R(M,I^i) \to \text{Hom}_R(M,I^{i+1}))/\text{Im}(\text{Hom}_R(M,I^{i-1}) \to \text{Hom}_R(M,I^i))$ for $i \geq 1$).

For $i \geq 0$, set $\text{Ext}^i_R(M,-) = R^i\text{Hom}_R(M,-)$. The functors $\text{Ext}^i_R(M,-)$ $(i \geq 0)$ are called the Ext functors. For any $V \in R-\text{mod}$ and $i \geq 0$, $\text{Ext}^i_R(M,V)$ is called an Ext group. (The definition of $\text{Ext}^i_R(M,V)$ is independent of the choice of injective resolution of V.)

Remark 1.1.1. If G is a finite group and k is a field, we can compute Ext groups over the group algebra $R = kG$. If M and V are any kG-modules, then $\text{Hom}_{kG}(M,V)$ has the structure of a k-vector space. Hence, the Ext groups $\text{Ext}^i_{kG}(M,V)$ $(i \geq 0)$ are also k-vector spaces.

It is also possible to compute $\text{Ext}^i_R(M,V)$ $(i \geq 0)$ using a projective resolution of M. Let $\cdots \to P^2 \to P^1 \to P^0 \to M \to 0$ be a projective resolution of M in $R-\text{mod}$. Applying the contravariant left exact functor $\text{Hom}_R(-,V)$ to this projective resolution of M, we obtain the complex

$$0 \to \text{Hom}_R(P^0,V) \to \text{Hom}_R(P^1,V) \to \text{Hom}_R(P^2,V) \to \cdots.$$

The ith cohomology of this complex gives the Ext group $\text{Ext}^i(M,V)$ for $i \geq 0$. (This definition of $\text{Ext}^i_R(M,V)$ is independent of the choice of projective resolution of M.)

Proposition 1.1.2. *([49, 2.5.1]) The following statements are equivalent.*

(1) V *is an injective* R*-module.*

(2) $\mathrm{Hom}_R(-, V)$ *is an exact functor.*

(3) $\mathrm{Ext}^1_R(M, V) = 0$ *for all* $M \in R - \mathrm{mod}$.

(4) If $i \geq 1$, $\mathrm{Ext}^i_R(M, V) = 0$ *for all* $M \in R - \mathrm{mod}$.

Proposition 1.1.3. *([49, 2.5.1]) The following statements are equivalent.*

(1) M *is a projective* R*-module.*

(2) $\mathrm{Hom}_R(M, -)$ *is an exact functor.*

(3) $\mathrm{Ext}^1_R(M, V) = 0$ *for all* $V \in R - \mathrm{mod}$.

(4) If $i \geq 1$, $\mathrm{Ext}^i_R(M, V) = 0$ *for all* $V \in R - \mathrm{mod}$.

1.2 Ext and Extensions

If $M, V \in R - \mathrm{mod}$, an extension of M by V is a short exact sequence of the form $0 \to V \to X \to M \to 0$ for some $X \in R - \mathrm{mod}$. Two extensions $0 \to V \to X \to M \to 0$ and $0 \to V \to X' \to M \to 0$ are said to be equivalent if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\cong} & & \downarrow{=} & & \\ 0 & \longrightarrow & V & \longrightarrow & X' & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

An extension is called split if it is equivalent to $0 \to V \to V \oplus M \to M \to 0$ (where $V \to V \oplus M$ is the natural inclusion map and $V \oplus M \to M$ is the natural projection map).

Proposition 1.2.1. *([49, 3.4.1]) If* $\mathrm{Ext}^1_R(M, V) = 0$, *then every extension of* M *by* V *is split.*

Proposition 1.2.2. *([49, 3.4.3]) There is a 1-1 correspondence between* $\mathrm{Ext}^1_R(M, V)$ *and equivalence classes of extensions of* M *by* V *(in which the split extension corresponds to* $0 \in \mathrm{Ext}^1_R(M, V)$*).*

The higher Ext groups can be analogously interpreted in terms of extensions. Given an integer $n > 0$, an exact sequence in $R - \mathrm{mod}$ of the form

$$0 \to V \to X_{n-1} \to \cdots \to X_0 \to M \to 0$$

is called an extension of degree n of M by V. If E and E' are two degree n extensions of M by V, we say that E is congruent to E' and write $E \vdash E'$ if there is a chain map $E \to E'$ such that the maps on M and V are the identity maps [7, pg. 14]. The congruence relation $\vdash$ is not an equivalence relation on the set of degree n extensions ($\vdash$ need not be symmetric). But, $\vdash$ can be used to define an equivalence relation $\sim$ on the set of degree n extensions of

M by V. We will say that E is equivalent to E' (and write $E \sim E'$) if there exists a sequence $F_0, F_1, \ldots, F_{2t}$ ($t \in \mathbb{N}$) of degree n extensions of M by V such that

$$E \dashv F_0 \vdash F_1 \dashv F_2 \vdash \cdots \dashv F_{2t} \vdash E'.$$

Proposition 1.2.3. *([7, pg. 14]) There is a 1-1 correspondence between $\mathrm{Ext}_R^n(M, V)$ ($n > 0$) and equivalence classes of degree n extensions of M by V (under the equivalence relation $\sim$ described above).*

1.3 Some Properties of the Ext Functor

Ext functors have been extensively studied, and there is a wide range of literature covering the properties and applications of Ext. In this section, we will describe several fundamental properties of the Ext functors which will be used in the original proofs of this dissertation.

1.3.1 The Long Exact Sequence in Ext

The long exact sequence in Ext was an indispensable tool in our search for bounds on the dimension of Ext groups.

Theorem 1.3.1. *([7, 1.5.6]) Suppose that $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence of R-modules. Given any R-module V, there is a long exact sequence*

$$0 \to \mathrm{Hom}_R(M_2, V) \to \mathrm{Hom}_R(M, V) \to \mathrm{Hom}_R(M_1, V) \to \mathrm{Ext}_R^1(M_2, V) \to \mathrm{Ext}_R^1(M, V) \to$$

$$\mathrm{Ext}_R^1(M_1, V) \to \cdots \to \mathrm{Ext}_R^{n-1}(M, V) \to \mathrm{Ext}_R^{n-1}(M_1, V) \to \mathrm{Ext}_R^n(M_2, V) \to \mathrm{Ext}_R^n(M, V) \to \cdots .$$

Theorem 1.3.2. *([7, 1.5.7]) Suppose that $0 \to V_1 \to V \to V_2 \to 0$ is a short exact sequence of R-modules. Given any R-module M, there is a long exact sequence*

$$0 \to \mathrm{Hom}_R(M, V_1) \to \mathrm{Hom}_R(M, V) \to \mathrm{Hom}_R(M, V_2) \to \mathrm{Ext}_R^1(M, V_1) \to \mathrm{Ext}_R^1(M, V) \to$$

$$\mathrm{Ext}_R^1(M, V_2) \to \cdots \to \mathrm{Ext}_R^{n-1}(M, V) \to \mathrm{Ext}_R^{n-1}(M, V_2) \to \mathrm{Ext}_R^n(M, V_1) \to \mathrm{Ext}_R^n(M, V) \to \cdots .$$

1.3.2 The Eckmann-Shapiro Lemma

Suppose that A is a subring of a ring R. Given an R-module M, we can restrict the action of R on M to an action of A on M and define the restricted A-module $M \downarrow_A^R$. Given an A-module N, we can define an induced R-module $N \uparrow_A^R = R \otimes_A N$. And, viewing R as an $A - R$-bimodule by left and right multiplication, we can define a co-induced R-module $N \Uparrow_A^R = \mathrm{Hom}_A(R, N)$.

Proposition 1.3.3. *([3, Lemma 2.8.2]) Suppose that A is a subring of R, M is an R-module, and N is an A-module. Then,* *(1)* $\mathrm{Hom}_A(N, M\downarrow_A^R) \cong \mathrm{Hom}_R(N\uparrow_A^R, M)$, *and*

(2) $\mathrm{Hom}_A(M\downarrow_A^R, N) \cong \mathrm{Hom}_R(M, N\Uparrow_A^R)$.

Therefore, the restriction functor $\downarrow_A$ has left adjoint $\uparrow^R$ and right adjoint $\Uparrow^R$.

Remark 1.3.4. Suppose that k is a field, G is a finite group, and $R = kG$ is the group algebra of G. If H is a subgroup of G, then kH is a subalgebra of kG. Given a kG-module M and a kH-module N, we write $M\downarrow_{kH}^{kG} = M\downarrow_H^G$, $N\uparrow_{kH}^{kG} = N\uparrow_H^G$, and $N\Uparrow_{kH}^{kG} = N\Uparrow_H^G$. In the case that G is a finite group, induction and coinduction coincide on $A-$mod, so $N\uparrow_H^G \cong N\Uparrow_H^G$ for any subgroup $H \leq G$ and kH-module N [3, 3.3]. Hence, Proposition 1.3.3 may be restated as follows.

Proposition 1.3.5. *Suppose that $H \leq G$ and k is a field. Then,*

(1) $\mathrm{Hom}_{kH}(N, M\downarrow_H^G) \cong \mathrm{Hom}_{kG}(N\uparrow_H^G, M)$, *and*

(2) $\mathrm{Hom}_{kH}(M\downarrow_H^G, N) \cong \mathrm{Hom}_{kG}(M, N\uparrow_H^G)$.

The isomorphism of Proposition 1.3.5 (1) is known as Frobenius reciprocity.

We now return to the more general setting in which R an arbitrary ring with unit and A a subring of R.

Proposition 1.3.6. *(Eckmann-Shapiro Lemma) Suppose that A is a subring of R, and that R is projective as an A-module. And, suppose that M is an R-module and N is an A-module. Then, for any $n \geq 0$,*

(1) $\mathrm{Ext}_A^n(N, M\downarrow_A^R) \cong \mathrm{Ext}_R^n(N\uparrow_A^R, M)$, *and*

(2) $\mathrm{Ext}_A^n(M\downarrow_A^R, N) \cong \mathrm{Ext}_R^n(M, N\Uparrow_A^R)$.

Proof. We will prove the statement of (1) (the statement of (2) can be proved using similar arguments). We follow Benson's proof in [3, 2.8.4]. Suppose that $\cdots \to P_2 \to P_1 \to P_0 \to N \to 0$ is a projective resolution of N as an A-module. Since R is projective as an A-module, the induction functor $\uparrow_A^R$ is exact. Therefore,

$$\cdots \to P_2\uparrow_A^R \to P_1\uparrow_A^R \to P_0\uparrow_A^R \to N\uparrow_A^R \to 0$$

is an exact sequence of R-modules with $P_i\uparrow_A^R$ projective for $i \geq 0$.

For $n \geq 0$, $\mathrm{Ext}_R^n(N\uparrow_A^R, M)$ is the nth cohomology of the complex

$$0 \to \mathrm{Hom}_R(P_0\uparrow_A^R, M) \to \mathrm{Hom}_R(P_1\uparrow_A^R, M) \to \mathrm{Hom}_R(P_2\uparrow_A^R, M) \to \cdots.$$

Now, $\mathrm{Ext}_A^n(N, M\downarrow_A^R)$ is the nth cohomology of the complex

$$0 \to \mathrm{Hom}_A(P_0, M\downarrow_A^R) \to \mathrm{Hom}_A(P_1, M\downarrow_A^R) \to \mathrm{Hom}_A(P_2, M\downarrow_A^R) \to \cdots.$$

But, for any $i \geq 0$, $\operatorname{Hom}_A(P_i, M \downarrow_A^R) \cong \operatorname{Hom}_R(P_i \uparrow_A^R, M)$ by Proposition 1.3.3 (1), and the desired result follows.

□

Remark 1.3.7. Suppose that G is a finite group, $H \leq G$, and k is a field. In this case, kG is projective (in fact, free) as a kH-module and we have the following version of the Eckmann-Shapiro Lemma.

Proposition 1.3.8. *If M is a kG-module, and N is a kH-module, then for any $n \geq 0$,*

(1) $\operatorname{Ext}^n_{kH}(N, M \downarrow_H^G) \cong \operatorname{Ext}^n_{kG}(N \uparrow_H^G, M)$*, and*

(2) $\operatorname{Ext}^n_{kH}(M \downarrow_H^G, N) \cong \operatorname{Ext}^n_{kG}(M, N \uparrow_H^G)$.

For $n = 0$, the statement of Proposition 1.3.8 (1) is simply the statement of Frobenius reciprocity. Thus, the Eckmann-Shapiro Lemma may be viewed as a generalization of Frobenius reciprocity to higher Ext groups.

1.3.3 Ext^1 between Irreducible Modules

Assume that R is an Artinian ring. The purpose of this section is to give an interpretation of $\dim \operatorname{Ext}^1_R(S, S')$ when S and S' are irreducible R-modules. We will follow [3, Section 2.4].

We define several important submodules and quotient modules of an R-module M. The socle of M (denoted $\operatorname{soc}(M)$) is the direct sum of all of the irreducible submodules of M. The radical of M (denoted $\operatorname{Rad}(M)$) is the intersection of the maximal submodules of M. And, the head of M is the quotient module $\operatorname{head}(M) = M/\operatorname{Rad}(M)$. M is completely reducible if and only if $\operatorname{Rad}(M) = 0$. In particular, since $\operatorname{Rad}(\operatorname{head}(M)) = 0$, $\operatorname{head}(M)$ is completely reducible.

Since R is Artinian, any finitely generated R-module M has a "minimal" projective resolution. (A projective resoluion $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ is minimal if each P_i is a projective cover of its image in P_{i-1}.)

Proposition 1.3.9. *Let S and S' be irreducible R-modules, and let P be a projective indecomposable module with* $\operatorname{head}(P) = S$. *Then,* $\operatorname{Ext}^1_R(S, S') \cong \operatorname{Hom}_R(\operatorname{Rad}(P)/\operatorname{Rad}^2(P), S')$.

Proof. Let $\cdots \stackrel{\partial_3}{\to} P_2 \stackrel{\partial_2}{\to} P_1 \stackrel{\partial_1}{\to} P_0 \stackrel{\partial_0}{\to} S \to 0$ be a minimal projective resolution of S. By minimality, $P_0 = P$ is the projective cover of S, and for all $i \geq 1$, P_i is the projective cover of $\operatorname{Im} \partial_i$. We apply the functor $\operatorname{Hom}_R(-, S')$ to the complex $\cdots \stackrel{\partial_3}{\to} P_2 \stackrel{\partial_2}{\to} P_1 \stackrel{\partial_1}{\to} P_0$ to obtain the complex

$$0 \to \operatorname{Hom}_R(P_0, S') \stackrel{\delta^0}{\to} \operatorname{Hom}_R(P_1, S') \stackrel{\delta^1}{\to} \operatorname{Hom}_R(P_2, S') \stackrel{\delta^2}{\to} \cdots . \tag{1.1}$$

By definition, $\mathrm{Ext}^1_R(S,S') = \mathrm{Ker}\,\delta^1/\mathrm{Im}\,\delta^0$.

We claim that $\mathrm{Ext}^1_R(S,S') = \mathrm{Hom}_R(P_1,S')$. In fact, we will show that all of the maps δ^i in the complex (1.1) are zero maps (from which it will follow that $\mathrm{Ext}^1_R(S,S') = \mathrm{Ker}\,\delta^1 = \mathrm{Hom}_R(P_1,S')$). For $i \geq 0$, the map $\delta^i : \mathrm{Hom}_R(P_i,S') \to \mathrm{Hom}_R(P_{i+1},S')$ is given by precomposition with $\partial_{i+1} : P_{i+1} \to P_i$. So, given $f \in \mathrm{Hom}_R(P_i,S')$, the element $\delta^i(f) \in \mathrm{Hom}_R(P_{i+1},S')$ is defined by $\delta^i(f) = f \circ \partial_{i+1}$. Note that $\mathrm{Im}\,\partial_{i+1} = \mathrm{Ker}\,\partial_i \subseteq \mathrm{Rad}(P_i)$, where the inclusion $\mathrm{Ker}\,\partial_i \subseteq \mathrm{Rad}(P_i)$ holds since P_i is the projective cover of Im ∂_i. Let $0 \neq f \in \mathrm{Hom}_R(P_i,S')$. Since S' is simple, f is surjectve. Thus, $\delta^i(f)(P_{i+1}) = (f \circ \partial_{i+1})(P_{i+1}) = f(\mathrm{Im}\,\partial_{i+1}) \subseteq f(\mathrm{Rad}(P_i)) \subseteq \mathrm{Rad}S' = 0$. So, $\delta^i = 0$, and the claim is proved.

Now, the module P_1 in the projective resolution of S is the projective cover of $\mathrm{Im}\,\partial_1 = \mathrm{Ker}\,\partial_0 = \mathrm{Rad}(P)$ (where we have used the fact that $P_0 = P$). To find the projective cover of $\mathrm{Rad}(P)$, we write the completely reducible module $\mathrm{Rad}(P)/\mathrm{Rad}^2(P)$ as a direct sum of simple modules. Suppose $\mathrm{Rad}(P)/\mathrm{Rad}^2(P) = \bigoplus_i S_i^{\oplus n_i}$, where each S_i is a simple R-module. For any i, let Q_i be the projective indecomposable R-module with head S_i. Then, the projective cover of $\mathrm{Rad}(P)$ is $\bigoplus_i Q_i^{\oplus n_i}$, which means that $P_1 = \bigoplus_i Q_i^{\oplus n_i}$ in the minimal projective resolution of S. Thus, by the claim of the previous paragraph, $\mathrm{Ext}^1_R(S,S') = \mathrm{Hom}_R(\bigoplus_i Q_i^{\oplus n_i}, S')$.

Finally, since S' is simple, we have $\mathrm{Hom}_R(\bigoplus_i Q_i^{\oplus n_i}, S') \cong \mathrm{Hom}_R(\bigoplus_i Q_i^{\oplus n_i}/\mathrm{Rad}(\bigoplus_i Q_i^{\oplus n_i}), S') \cong \mathrm{Hom}_R(\bigoplus_i S_i^{\oplus n_i}, S') = \mathrm{Hom}_R(\mathrm{Rad}(P)/\mathrm{Rad}^2(P), S')$.

□

Remark 1.3.10. It is possible to generalize Proposition 1.3.9 to higher Ext. Let M be an arbitrary finitely generated R-module, and let S be a simple R-module. Let $\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to 0$ be a minimal projective resolution of M. Set $\Omega^n(M) = \mathrm{Ker}(\partial_{n-1}) \cong \mathrm{Im}(\partial_n)$.

We claim that $\mathrm{Ext}^n_R(M,S) \cong \mathrm{Hom}_R(\Omega^n(M),S)$ for all $n \geq 1$. To see why this is the case, apply $\mathrm{Hom}_R(-,S)$ to the minimal projective resolution of M to obtain the complex

$$0 \to \mathrm{Hom}_R(P_0,S') \xrightarrow{\delta^0} \mathrm{Hom}_R(P_1,S') \xrightarrow{\delta^1} \mathrm{Hom}_R(P_2,S') \xrightarrow{\delta^2} \cdots .$$

An argument analogous to that given in the proof of Proposition 1.3.9 shows that $\delta^i = 0$ for all $i \geq 0$. Thus, for $n \geq 1$, $\mathrm{Ext}^n_R(M,S) \cong \mathrm{Hom}_R(P_n,S)$. Now, since P_n is the projective cover of $\mathrm{Im}(\partial_n)$, $\mathrm{Ker}(\partial_n)$ is contained in $\mathrm{Rad}(P_n)$. And, since $\mathrm{Rad}(S) = 0$, $\mathrm{Hom}_R(P_n,S) \cong \mathrm{Hom}_R(P_n/\mathrm{Ker}(\partial_n),S) \cong \mathrm{Hom}_R(\mathrm{Im}(\partial_n),S) \cong \mathrm{Hom}_R(\Omega^n(M),S)$.

Suppose now that R is an Artinian k-algebra. If S is an irreducible R-module and M is any finitely generated R-module, $\dim \mathrm{Hom}_R(M,S)$ is equal to the number of times that S appears in head(M). So, we have the following corollary to Proposition 1.3.9.

Corollary 1.3.11. *If R is a k-algebra and S, S' are irreducible R-modules, then* $\dim \mathrm{Ext}^1_R(S, S')$ *is the number of times that S' occurs in the head of* $\mathrm{Rad}(P)$.

Proof. By Proposition 1.3.9, $\dim \mathrm{Ext}^1_R(S, S') = \dim \mathrm{Hom}_R(\mathrm{Rad}(P)/\mathrm{Rad}^2(P), S')$, which is the number of times S' appears in $\mathrm{head}(\mathrm{Rad}(P)/\mathrm{Rad}^2(P)) = \mathrm{Rad}(P)/\mathrm{Rad}^2(P) = \mathrm{head}(\mathrm{Rad}(P))$. □

In [35, Theorem 2.2], Guralnick and Tiep give a result analogous to that of Corollary 1.3.11, proving that the dimension of $H^1(G, V)$ (where G is a finite group of Lie type, k is an algebraically closed field of characteristic different from the defining characteristic of G, and V is an irreducible kG-module with $V^B = 0$) is equal to the multiplicity of V in the head of a certain submodule of $k|_B^G$. In order to generalize Guralnick and Tiep's bounds on the dimension of Ext, it was necessary to generalize the result of [35, Theorem 2.2]. The fact that Corollary 1.3.11 has a homological proof suggested that [35, Theorem 2.2] should have a similar homological proof. This hypothesis turned out to be correct; in Proposition 12.2.1, we give a new homological proof of [35, Theorem 2.2], which can be generalized to extend Guralnick and Tiep's results.

1.4 Group Cohomology

1.4.1 Group Cohomology via the Standard Resolution

In this section, we will define group cohomology (following [3, Section 3.4]). Let G be a finite group, and let R be a fixed commutative ring. We view R as an RG-module with trivial G-action. For any $n \geq 0$, let F_n be the free R-module on $(n+1)$-tuples of elements of G with the action of G given by $g(g_0, \dots, g_n) = (gg_0, \dots, gg_n)$ when $g \in G$ and $(g_0, \dots, g_n) \in F_n$. There is a long exact sequence

$$\cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\mathrm{aug}} R \to 0,$$

with the boundary map ∂_n $(n \geq 1)$ defined by

$$\partial_n(g_0, \dots g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \dots, \widehat{g_i}, \dots g_n)$$

for any $(g_0, \dots, g_n) \in F_n$ (the notation $\widehat{g_i}$ indicates that g_i has been omitted). The map $\mathrm{aug} : \mathrm{F}_0 \to \mathrm{R}$ is the augmentation map, with $\mathrm{aug}(\sum_{\mathrm{g} \in \mathrm{G}} \mathrm{a_g g}) = \sum_{\mathrm{g} \in \mathrm{G}} \mathrm{a_g}$ for all $a_g \in R$. This resolution of R by RG-modules is called the standard (or bar) resolution.

Given an RG-module M, we can apply the contravariant Hom functor $\mathrm{Hom}_{RG}(-, M)$ to the standard resolution to obtain the complex

$$0 \to \mathrm{Hom}_{RG}(F_0, M) \xrightarrow{\delta^0} \mathrm{Hom}_{RG}(F_1, M) \xrightarrow{\delta^1} \mathrm{Hom}_{RG}(F_2, M) \xrightarrow{\delta^2} \cdots,$$

where $\delta^n : \mathrm{Hom}_{RG}(F_n, M) \to \mathrm{Hom}_{RG}(F_{n+1}, M)$ is given by $\delta^n(f) = f \circ \partial_{n+1}$ for any $f \in \mathrm{Hom}_{RG}(F_n, M)$.

Definition 1.4.1. For $n = 0$, let $H^0(G, M) = \mathrm{Ker}\,\delta^0$. And, for $n \geq 1$, let $H^n(G, M) = \mathrm{Ker}\,\delta^n / \mathrm{Im}\,\delta^{n-1}$. We say that $H^n(G, M)$ $(n \geq 0)$ is the nth cohomology group of G with coefficients in M.

From the definition, it is clear that

$$H^n(G, M) \cong \mathrm{Ext}^n_{RG}(R, M)$$

for any $n \geq 0$. In particular, this means that the properties of Ext discussed in the previous sections carry over to group cohomology. For instance, any short exact sequence of RG-modules yields a long exact sequence in group cohomology.

1.4.2 Group Cohomology via Cochains

It is possible to define group cohomology without reference to the standard resolution of R by free RG-modules. Given an RG-module M, let $C^0(G, M) = M$, and let $C^n(G, M)$ $(n \geq 1)$ be the collection of maps $G^n \to M$ (where G^n is the direct product of n copies of G). Then, $C^n(G, M)$ is an abelian group for all $n \geq 0$. ($C^0(G, M) = M$ is naturally an abelian group; and for $n \geq 1$, $C^n(G, M)$ is given the structure of an abelian group via pointwise addition of functions.) The elements of $C^n(G, M)$ are called n-cochains.

For any $n \geq 0$, there is an isomorphism of abelian groups $C^n(G, M) \cong \mathrm{Hom}_{RG}(F_n, M)$. With this identification, the the nth coboundary homomorphism $\delta^n : C^n(G, M) \to C^{n+1}(G, M)$ is given by

$$\delta^n(f)(g_1, \ldots, g_{n+1}) =$$

$$gf(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n)$$

for any $f \in C^n(G, M)$ and $(g_1, \ldots, g_{n+1}) \in G^{n+1}$.

For any $n \geq 0$, the elements of $\mathrm{Ker}\,\delta^n$ are called n-cocycles. For any $n \geq 1$, the elements of $\mathrm{Im}\,\delta^{n-1}$ are called n-coboundaries. As above, we can define $H^0(G, M) = \mathrm{Ker}\,\delta^0$ and $H^n(G, M) = \mathrm{Ker}\,\delta^n / \mathrm{Im}\,\delta^{n-1}$ for $n \geq 1$.

The following result is a basic consequence of the definition of $H^0(G, M)$.

Proposition 1.4.2. *If M is any RG-module, $H^0(G, M) \cong M^G$.*

1.4.3 Group Cohomology in Low Degree

The low degree cohomology groups $H^1(G, M)$ and $H^2(G, M)$ can be interpreted in terms of extensions. If G is a group and M is an RG-module, then an extension of G by M is a short exact sequence $1 \to M \to \hat{G} \to G \to 1$ (where M is viewed as a group under addition, written multiplicatively). The extension is said to be split if there is a copy of G in $\hat{G}$ such that M and G are complementary subgroups in $\hat{G}$ (that is, if $\hat{G} = MG$ and $M \cap G = \{1\}$).

Proposition 1.4.3. *([3, 3.7.2]) Let G be a group, let M be an RG-module, and let $\hat{G}$ be a group in which G and M are complementary subgroups (so that there is a split extension $1 \to M \to \hat{G} \to G \to 1$). The first cohomology group $H^1(G, M)$ parameterizes the conjugacy classes of complements to M in $\hat{G}$.*

Proposition 1.4.4. *([3, 3.7.3]) Let G be a group, and let M be an RG-module. The second cohomology group $H^2(G, M)$ parameterizes the isomorphism classes of extensions of G by M.*

1.4.4 The inflation homomorphism on group cohomology.

Suppose that N is a normal subgroup of G and M is an RG-module. Then, the set M^N of fixed points of M under the action of N has the structure of an $R(G/N)$-module. Now, let $\pi : G \twoheadrightarrow G/N$ be the natural projection map, and let $\iota : M^N \hookrightarrow M$ be the inclusion of M^N into M. The homomorphism $\pi : G \to G/N$ induces a map $\phi^n : G^n \to (G/N)^n$ for all $n \geq 1$. Therefore, for any $n \geq 1$, there is an induced homomorphism $C^n(G/N, M^N) \to C^n(G, M)$, given by $f \mapsto \iota \circ f \circ \phi^n$ for any $f \in C^n(G/N, M^N)$. And, when $n = 0$, the inclusion map ι gives a homomorphism $C^0(G/N, M^N) = M^N \to C^0(G, M) = M$. The maps $C^n(G/N, M^N) \to C^n(G, M)$ defined above induce homomorphisms on cohomology

$$\inf : H^n(G/N, M^N) \to H^n(G, M)$$

for all $n \geq 0$. The induced homomorphism inf on cohomology is called the inflation homomorphism.

The inflation homomorphism fits into a famous five-term exact sequence (called the inflation-restriction exact sequence).

Proposition 1.4.5. *If N is a normal subgroup of G and M is an RG-module, then there is a five-term exact sequence*

$$0 \to H^1(G/N, M^N) \stackrel{\inf}{\to} H^1(G, M) \to H^1(N, M)^{G/N} \to H^2(G/N, M^N) \stackrel{\inf}{\to} H^2(G, M).$$

It is possible to prove the existence of this five-term exact sequence using the standard resolutions for G and G/N along with the definition of group cohomology. However, this five-term exact sequence also follows as a consequence of a Lyndon-Hochschild-Serre spectral sequence.

1.5 The Restriction and Corestriction Homomorphisms on Group Cohomology

The material covered in this section can be found in [49, 6.7]. Let G be a finite group, and let R be a commutative ring.

1.5.1 The Restriction Homomorphism

Let M be an RG-module, and let H be a subgroup of G. By restriction, we can view M as an RH-module. We will define a restriction homomorphism on cohomology.

For $n \geq 0$, $H^n(G, M) \cong \operatorname{Ext}^n_{RG}(R, M)$ and $H^n(H, M) \cong \operatorname{Ext}^n_{RH}(R, M)$. Let $\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} R \to 0$ be a projective resolution of R as an RG-module. Since the restriction functor from the category of RG-modules to the category of RH-mod is exact, this is also a projective resolution of R as an RH-module. Applying $\operatorname{Hom}_{RG}(-, M)$ to this resolution, we obtain the complex

$$0 \to \operatorname{Hom}_{RG}(P_0, M) \xrightarrow{\delta^0} \operatorname{Hom}_{RG}(P_1, M) \xrightarrow{\delta^1} \operatorname{Hom}_{RG}(P_2, M) \xrightarrow{\delta^2} \cdots .$$

Similarly, we can apply $\operatorname{Hom}_{RH}(-, M)$ to obtain the complex

$$0 \to \operatorname{Hom}_{RH}(P_0, M) \xrightarrow{\delta^0} \operatorname{Hom}_{RH}(P_1, M) \xrightarrow{\delta^1} \operatorname{Hom}_{RH}(P_2, M) \xrightarrow{\delta^2} \cdots .$$

In both complexes, the coboundary map δ^n is given by precomposition with the morphism $\partial_{n+1} : P_{n+1} \to P_n$.

For all $n \geq 0$, we have inclusions $\operatorname{Hom}_{RG}(P_n, M) \hookrightarrow \operatorname{Hom}_{RH}(P_n, M)$, and the following commutative square.

$$\begin{array}{ccc} \operatorname{Hom}_{RG}(P_n, M) & \xrightarrow{\delta^n} & \operatorname{Hom}_{RG}(P_{n+1}, M) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{RH}(P_n, M) & \xrightarrow{\delta^n} & \operatorname{Hom}_{RH}(P_{n+1}, M) \end{array}$$

Therefore, the inclusions $\operatorname{Hom}_{RG}(P_n, M) \hookrightarrow \operatorname{Hom}_{RH}(P_n, M)$ give rise to a cochain map $\operatorname{Hom}_{RG}(P_*, M) \to \operatorname{Hom}_{RH}(P_*, M)$ and, consequently, induce a map on cohomology, called the restriction homomorphism. We denote this homomorphism by $\operatorname{Res} : H^n(G, M) \to H^n(H, M)$.

1.5.2 The Corestriction Homomorphism

Let M be an RG-module, and let H be a subgroup of G of index $m = [G : H]$ (since G is finite, the index m of H in G is necessarily finite). We will define the corestriction homomorphism Cor : $H^n(H, M) \to H^n(G, M)$ $(n \geq 0)$.

Let $g_1, \ldots, g_m$ be a system of representatives of the left cosets of H in G. Let $n \geq 0$, and define a map Cor : $\mathrm{Hom}_{RH}(P_n, M) \to \mathrm{Hom}_{RG}(P_n, M)$ by

$$\mathrm{Cor}(f)(x) = \sum_{i=1}^{m} g_i f(g_i^{-1}x)$$

for $f \in \mathrm{Hom}_{RH}(P_n, M)$ and $x \in P_n$. For any $f \in \mathrm{Hom}_{RH}(P_n, M)$, $h \in H$, $x \in P_n$, and $1 \leq i \leq m$, we have

$$g_i h f((g_i h)^{-1}x) = g_i h f(h^{-1}g_i^{-1}x) = g_i h h^{-1} f(g_i^{-1}x) = g_i f(g_i^{-1}x).$$

Thus, Cor is independent of the choice of coset representatives of H in G. We claim that $\mathrm{Cor}(f) : P_n \to A$ is a G-module homomorphism for any $f \in \mathrm{Hom}_{RH}(P_n, M)$. Let $g \in G$ and $x \in P_n$. Since G acts transitively on the set of left cosets of H in G, there is some permutation σ of the set $\{1, \ldots, m\}$ and elements $h_i \in H$ for all $1 \leq i \leq m$ such that $g^{-1}g_i = g_{\sigma(i)}h_i$. So, $\mathrm{Cor}(f)(gx) = \sum_{i=1}^{m} g_i f(g_i^{-1}gx) = \sum_{i=1}^{m} g_i f(h_i^{-1}g_{\sigma(i)}^{-1}x) = \sum_{i=1}^{m} g_i h_i^{-1} f(g_{\sigma(i)}^{-1}x) = \sum_{i=1}^{m} g g_{\sigma(i)} f(g_{\sigma(i)}^{-1}x) = g\sum_{i=1}^{m} g_{\sigma(i)} f(g_{\sigma(i)}^{-1}x) = g\mathrm{Cor}(f)(x)$.

To show that the homomorphisms Cor : $\mathrm{Hom}_{RH}(P_n, A) \to \mathrm{Hom}_{RG}(P_n, A)$ $(n \geq 0)$ induce a map on cohomology, we must check that the following square commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{RH}(P_n, A) & \xrightarrow{\delta^n} & \mathrm{Hom}_{RH}(P_{n+1}, A) \\ {\scriptstyle \mathrm{Cor}}\downarrow & & \downarrow{\scriptstyle \mathrm{Cor}} \\ \mathrm{Hom}_{RG}(P_n, A) & \xrightarrow{\delta^n} & \mathrm{Hom}_{RG}(P_{n+1}, A). \end{array}$$

Since $\partial_{n+1} : P_{n+1} \to P_n$ is an RG-module homomorphism, given $f \in \mathrm{Hom}_{RH}(P_n, M)$ and $x \in P_{n+1}$, we have $(\mathrm{Cor} \circ \delta^n)(f)(x) = \mathrm{Cor}(f \circ \partial_{n+1})(x) = \sum_{i=1}^{m} g_i f(\partial_{n+1}(g_i^{-1}x)) = \sum_{i=1}^{m} g_i f(g_i^{-1}\partial_{n+1}(x)) = \mathrm{Cor}(f)(\partial_{n+1}(x)) = \delta^n(\mathrm{Cor}(f))(x) = (\delta^n \circ \mathrm{Cor})(f)(x)$. Thus, the square above commutes, which means that Cor : $\mathrm{Hom}_{RH}(P_*, M) \to \mathrm{Hom}_{RG}(P_*, A)$ is a cochain map. Therefore, Cor induces a homomorphism on cohomology, called the corestriction homomorphism and denoted by Cor : $H^n(H, M) \to H^n(G, M)$ $(n \geq 0)$.

1.6 Composition of Restriction and Corestriction

As above, let H be a subgroup of G of finite index m, and let $g_1, \ldots, g_m$ be a system of representatives for the left cosets of H in G.

Proposition 1.6.1. *For any RG-module M, the composition* $\mathrm{Cor} \circ \mathrm{Res} : H^n(G, M) \to H^n(G, M)$ *is given by multiplication by $m = [G : H]$. That is, for all $n \geq 0$, given a cohomology class $c \in H^n(G, M)$,* $\mathrm{Cor}(\mathrm{Res}(c)) = mc$ *(where $mc = c + \cdots + c$, m times in $H^n(G, M)$).*

Proof. The homomorphism $\mathrm{Cor} \circ \mathrm{Res} : H^n(G, M) \to H^n(G, M)$ $(n \geq 0)$ on cohomology is induced by the composite cochain map

$$\mathrm{Hom}_{RG}(P_*, M) \overset{\mathrm{Res}}{\to} \mathrm{Hom}_{RH}(P_*, M) \overset{\mathrm{Cor}}{\to} \mathrm{Hom}_{RG}(P_*, M).$$

Given any $n \geq 0$, we explicitly compute the composition $\mathrm{Hom}_{RG}(P_n, M) \overset{\mathrm{Res}}{\to} \mathrm{Hom}_{RH}(P_n, M) \overset{\mathrm{Cor}}{\to} \mathrm{Hom}_{RG}(P_n, M)$. Let $f \in \mathrm{Hom}_{RG}(P_n, M)$. Then, $\mathrm{Res}(f) = f$, viewed as an RH-map by restriction. But, since f is an RG-module homomorphism, we have

$$\mathrm{Cor}(\mathrm{Res}(f))(x) = \sum_{i=1}^{m} g_i f(g_i^{-1} x) = \sum_{i=1}^{m} g_i g_i^{-1} f(x) = m f(x)$$

for any $x \in P_n$. Thus, $(\mathrm{Cor} \circ \mathrm{Res})(f) = mf$ for all $f \in \mathrm{Hom}_{RG}(P_n, M)$. It follows that the induced map on cohomology is given by $c \mapsto mc$ for a cohomology class $c \in H^n(G, M)$.

□

Suppose now that $R = k$ is a field of characteristic $r > 0$.

Corollary 1.6.2. *Let G be a finite group, and let H be a subgroup of G of index $[G : H] = m$. Suppose that k is a field of characteristic $r \neq 0$, and M is a kG-module. Then, if $r \nmid m$,* $\mathrm{Res} : H^n(G, M) \to H^n(H, M)$ *is an injective homomorphism.*

Proof. If $K = \mathrm{Ker}(\mathrm{Res})$, then we have $0 = \mathrm{Cor}(\mathrm{Res}(K)) = mK$. And, since $rH^n(G, M) = 0$, we also have $rK = 0$. Now, since $(r, m) = 1$, we can write $ar + bm = 1$ for some $a, b \in \mathbb{Z}$. Therefore, $K = (ar + bm)K = arK + bmK = 0 + 0 = 0$, and it follows that $\mathrm{Res} : H^n(G, M) \to H^n(H, M)$ is an injective homomorphism.

□

Similarly, given RG-modules M and M' and a subgroup H of G, it is possible to define a restriction map $\mathrm{Res} : \mathrm{Ext}^n_{RG}(M, M') \to \mathrm{Ext}^n_{RH}(M, M')$ for $n \geq 0$.

Proposition 1.6.3. *([3, 3.6.18]) Let G be a finite group, and let H be a subgroup of G of index $[G : H] = m$. Suppose that k is a field of characteristic $r \neq 0$, and let M and M' be kG-modules. Then, if $r \nmid m$,* $\mathrm{Res} : \mathrm{Ext}^n_{kG}(M, M') \to \mathrm{Ext}^n_{kH}(M, M')$ *is an injective homomorphism.*

We also mention that if G is a finite group such that $r \nmid |G|$ (where $r = \mathrm{char}(k)$), then the group algebra kG is semisimple by Maschke's Theorem and $\mathrm{Ext}^n_{kG}(M, M') = 0$ for all $n > 0$ and for all kG-modules M and M'.

2 Groups with a BN-pair

In this section, we will summarize certain basic features of groups with a BN-pair. Our main reference is Carter's "Finite Groups of Lie Type" [8].

2.1 The Axioms of a Group with a BN-pair

Let G be a group and let B and N be two subgroups of G. The subgroups B, N form a BN-pair if the following axioms hold:

(i) $G = \langle B, N \rangle$.

(ii) $T = B \cap N$ is a normal subgroup of N.

(iii) $N/T = W$ is generated by a set of elements s_i, $i \in I$ with $s_i^2 = 1$.

(iv) Let $\pi : N \twoheadrightarrow W$ be the natural quotient map. If $n_i \in N$ has image $s_i \in W$ under π, then $n_i B n_i \neq B$.

(v) For each $n \in N$ and each n_i $(i \in I)$, $n_i B n \subseteq B n_i n B \cup B n B$.

The group $W = N/T$ is called the Weyl group of the BN-pair.

Given an element $w \in W$, let n_w denote a preimage of w in N. We will write $BwB := Bn_wB$ (this definition is independent of the choice of preimage n_w of w by [8, Proposition 2.1.2]). The following result is an important consequence of the axioms of a group with a BN-pair.

Proposition 2.1.1. *([8, Proposition 2.1.2]) There exists a bijection $B \setminus G/B \leftrightarrow W$ between the set of double cosets of B in G and elements of W, with the double coset BnB corresponding to the element $\pi(n) \in W$.*

As a consequence, we have the Bruhat decomposition of G:

$$G = \bigcup_{w \in W} BwB,$$

where the union is disjoint.

By [8, Proposition 2.1.7], the Weyl group W corresponding to the BN-pair of G is a Coxeter group with generators s_i, $i \in I$. We can define a length function l on W, where $l(w)$ is the minimal length of an expression of w in the generators s_i, $i \in I$ for any $w \in W$. Some basic properties of the length function l are listed in the next proposition.

Proposition 2.1.2. *[8, Proposition 2.1.3] Let $w \in W$ and let $i \in I$. If $n, n_i \in N$ are such that $w = \pi(n)$ and $s_i = \pi(n_i)$, then*

(i) $l(s_i w) = l(w) \pm 1$,

(ii) if $l(s_i w) = l(w) + 1$ *then* $n_i B n \subseteq B n_i n B$, *and*

(iii) if $l(s_i w) = l(w) - 1$ *then* $n_i B n \nsubseteq B n_i n B$.

By [8, 2.2], it is possible to associate a root system Φ to the Weyl group W of a BN-pair of G. In the case that G is a finite group of Lie type, Φ is a root system in the sense of [40, III] (see Chapter 3 for further information concerning root systems associated to finite groups of Lie type).

2.2 Parabolic Subgroups

Let $W = \langle s_i \rangle_{i \in I}$ be the Weyl group of a BN-pair of G. Given a subset $J \subseteq I$, let $W_J = \langle s_i \rangle_{i \in J}$ be the subgroup of W generated by the s_i, $i \in J$. Then, W_J is the parabolic subgroup of W corresponding to the subset $J \subseteq I$. The parabolic W_J is a Coxeter group with respect to the generators s_i, $i \in J$. There are many well-known results concerning the cosets (and double cosets) of the parabolic subgroups in W. Let $J \subseteq I$, and let

$$^J W = \{w \in W | l(s_i w) > l(w) \text{ for all } i \in J\} \text{ and } W^J = (^J W)^{-1}.$$

Proposition 2.2.1. *([14, Proposition 4.16 (5)]) Multiplication $W^J \times W_J \to W$ defines a bijection of sets. For any $u \in W^J$ and $v \in W_J$, $l(uv) = l(u) + l(v)$. So, u is the unique element in the coset uW_J of minimal length.*

By [14, Proposition 4.16 (5)], the elements of W^J are the shortest left coset representatives of W_J in W. Similarly, the elements of $^J W$ are the shortest right coset representatives of W_J in W.

Given subsets $J, K \subseteq I$, let $^J W^K = {}^J W \cap W^K$. By [14, 4.3.2], the set $^J W^K$ gives a full set of shortest (W_J, W_K)-double coset representatives in W.

The parabolic subgroups of the Weyl group W corresponding to the BN-pair of G may be used to define parabolic subgroups of G. Given a subset $J \subseteq I$, let W_J be the parabolic subgroup of W corresponding to J. Let N_J be the subgroup of N satisfying $N_J/T = W_J$. Then, $P_J := BN_J B$ is a subgroup of G. Any subgroup of G conjugate to P_J for some $J \subseteq I$ is called a parabolic subgroup of G. We note that when $J = \emptyset$, $P_\emptyset = B$. and, when $J = I$, $P_I = G$.

Proposition 2.2.2. *[14, Theorem A.41]*

(1) For any $J, K \subseteq I$, the subset ${}^JW^K$ is a full set of (P_J, P_K)-double coset representatives in G. So, we have a disjoint union

$$G = \bigcup_{w \in {}^JW^K} P_J w P_K.$$

(2) For any $w \in {}^J W^K$, $P_J w P_K = BW_J w W_K B$.

2.3 Finite Groups with a Split BN-Pair

Let p be a prime number, and suppose that G is a finite group such that

(i) G has subgroups B and N, which form a BN-pair.

(ii) $B = UT$, where U is a normal p-subgroup of B and T is an abelian subgroup of order prime to p.

(iii) $\bigcap_{n \in N} nBn^{-1} = T$.

In this case, we will say that G has a split BN-pair of characteristic p.

Suppose that G has a split BN-pair. Let

$$U^- = U^{w_0},$$

$$X_i = U \cap (U^-)^{s_i} \text{ for } i \in I,$$
$$U_i = U \cap U^{s_i}, \text{ and}$$
$$U_w = U \cap U^{w_0 w} \text{ for any } w \in W.$$

Here, $U^{w_0} = w_0^{-1} U w_0 = n_0^{-1} U n_0$, where n_0 is a preimage of w_0 in N (this definition is independent of the choice of preimage n_0 of w_0). The subgroups $(U^-)^{s_i}$, U^{s_i}, and $U^{w_0 w}$ are defined similarly.

We will now define the root subgroups of a (finite) group G with a split BN-pair.

Proposition 2.3.1. *[8, Proposition 2.5.15] The set of subgroups nX_in^{-1} ($n \in N$, $i \in I$) of G is in bijective correspondence with the set Φ of roots. If n is a preimage of w, then the subgroup nX_in^{-1} corresponds to the root $w(\alpha_i)$.*

The subgroups nX_in^{-1} ($n \in N$) are called the root subgroups of G. (As above, we will write $nXn^{-1} = wXw^{-1}$ if w is the image of n in W.) If $w(\alpha_i) = \alpha \in \Phi$, we will denote the root subgroup wX_iw^{-1} by X_α.

Proposition 2.3.2. *[8, Corollary 2.1.17] $U = \prod_{\alpha \in \Phi^+} X_\alpha$ (any order).*

3 Finite groups of Lie type

Our main reference is [42, III]. (All of the proof strategies of this chapter are also due to Malle and Testerman.) In this chapter, we will discuss both the split and non-split finite groups of Lie type; but, since the original research presented in this dissertation deals only with split finite groups of Lie type, the reader may choose to skip the discussions of finite groups of Lie type arising from twisted and very twisted Steinberg endomorphisms. All algebraic groups considered in this section are assumed to be linear.

3.1 Steinberg endomorphisms

Let G be a connected reductive algebraic group over an algebraically closed field K of characteristic p, and let $\mathrm{GL}_n = \mathrm{GL}_n(K)$ for $n \geq 1$.

Let $q = p^f$ for some integer $f \geq 1$. Then, the group homomorphism $F_q : \mathrm{GL}_n \to \mathrm{GL}_n$ given by $(a_{ij}) \mapsto (a_{ij}^q)$ is called the standard Frobenius morphism of GL_n with respect to the finite field $\mathbb{F}_q$. The standard Frobenius morphism F_q is an automorphism of GL_n as an abstract group and an endomorphism of GL_n as an algebraic group, but not an automorphism of GL_n as an algebraic group. Note that the fixed point group of F_q is $(\mathrm{GL}_n)^{F_q} = \mathrm{GL}_n(\mathbb{F}_q)$.

The notion of a Frobenius morphism can be extended to an algebraic group G using the fact that G can be regarded as a closed subgroup of GL_n for some $n \geq 1$.

Definition 3.1.1. ([42, Definition 21.3]) An algebraic group endomorphism $F : G \to G$ is called a Steinberg endomorphism if there exists an integer $m \geq 1$ such that $F^m : G \to G$ is a Frobenius morphism.

Steinberg proved that endomorphisms of simple linear algebraic groups fall into precisely two categories, one of which consists of what are now referred to as the Steinberg endomorphisms.

Theorem 3.1.2. *([42, Theorem 21.5], due to Steinberg) Let G be a simple linear algebraic group with an endomorphism σ. Then, precisely one of the following holds:*

(1) σ is an automorphism of algebraic groups, or

(2) the group $G^\sigma = \{g \in G | \sigma(g) = g\}$ is finite.

An endomorphism $\sigma : G \to G$ falls into category (2) if and only if it is a Steinberg endomorphism.

Remark 3.1.3. If G is semisimple, it is still true that an endomorphism $\sigma : G \to G$ is a Steinberg endomorphism if and only if G^σ is finite.

The following fundamental theorem is the basis for many of the results we will state without proof in this chapter.

Theorem 3.1.4. *(Lang-Steinberg, [42, 21.2]) Let G be a connected linear algebraic group over an algebraically closed field k of characteristic $p > 0$, with a Steinberg endomorphism $F : G \to G$. Then, the morphism $L : G \to G$, $g \mapsto F(g)g^{-1}$ is surjective.*

Definition 3.1.5. Let G be a connected reductive linear algebraic group and let $F : G \to G$ be a Steinberg endomorphism. Then, G^F is called a finite group of Lie type.

To classify the finite groups of Lie type, one must classify the semisimple algebraic groups G and the Steinberg endomorphisms of these semisimple algebraic groups. The semisimple algebraic groups were classified by Chevalley. According to Chevalley's classification theorem, a semisimple algebraic group is determined uniquely by its root datum $(X(T), \Phi, Y(T), \Phi^\vee)$, where $T \leq G$ is a maximal torus, $X(T)$ is the character group of T, Φ is the root system with respect to T, $Y(T)$ is the cocharacter group, and $\Phi^\vee$ is the set of coroots. (The root datum is discussed in detail in [42, Chapter 9].)

We now discuss the task of classifying the Steinberg endomorphisms of the semisimple algebraic groups.

Definition 3.1.6. ([42, Definition 9.14]) Let G be a semisimple algebraic group with root datum $(X, \Phi, Y, \Phi^\vee)$, and let $\Omega = \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z})$. Since $X \cong \mathrm{Hom}(Y, \mathbb{Z})$ and $\Phi^\vee \subseteq Y$, restriction gives an injective homomorphism $X \to \Omega$. Therefore, X may be viewed as a subset of Ω, and we have $\mathbb{Z}\Phi \subseteq X \subseteq \Omega$. Now, let $\Lambda(G) = \Omega/X$ be the fundamental group of G. If $X = \Omega$ (which means that the fundamental group is trivial), G is of simply connected type. If $X = \mathbb{Z}\Phi$, G is of adjoint type.

Definition 3.1.7. ([42, 9.2]) Suppose $\phi : G \to H$ is a surjective homomorphism of algebraic groups. Then, ϕ is called an isogeny if $\mathrm{Ker}(\phi)$ is finite.

In particular, a Steinberg endomorphism $F : G \to G$ is an isogeny.

Theorem 3.1.8. *([42, Proposition 9.15]) Let G be semisimple, and let Φ be the root system of G. Then, there exits a simply connected group G_{sc} and an adjoint group G_{ad}, each with root system Φ, and isogenies $G_{sc} \xrightarrow{\pi_1} G$ and $G \xrightarrow{\pi_2} G_{ad}$.*

Proposition 3.1.9. *([42, Proposition 9.18]) Let G, G_{sc}, and $G_{sc} \xrightarrow{\pi_1} G$ be as in Theorem 3.1.8. Then, every isogeny $\sigma : G \to G$ can be lifted to an isogeny $\sigma_{sc} : G_{sc} \to G$ such that $\pi_1 \circ \sigma_{sc} = \sigma \circ \pi_1$.*

By Proposition 3.1.9, every Steinberg endomorphism of a semisimple algebraic group G is induced by a Steinberg endomorphism of a simply connected group. Therefore, to classify the Steinberg endomorphisms of the semisimple groups, it suffices to classify the Steinberg endomorphisms of the simply connected groups. In fact, it suffices to classify the Steinberg endomorphisms of the simple simply connected groups. Before discussing this classification, we give some necessary background theory.

3.2 The action of F on the character group of T

We will denote the root subgroup of G corresponding to $\alpha \in \Phi$ by U_α. For any $\alpha \in \Phi$, let $u_\alpha : \mathbb{G}_a \to U_\alpha$ be a fixed T-isomorphism. (With this notation, the element of U_α corresponding to a scalar $c \in K$ will be denoted by $u_\alpha(c)$.)

Lemma 3.2.1. *([42, Lemma 11.10]) Let G be a connected reductive algebraic group in characteristic $p > 0$, with a fixed maximal torus T contained in a Borel subgroup B. Let Φ, Φ^+, and Π be the corresponding root system, set of positive roots, and base of simple roots, respectively. And, let $\sigma : G \to G$ be an endomorphism stabilizing T and B. Then, there exists a permutation ρ of Φ^+ stabilizing Π such that for all $\alpha \in \Phi^+$, the following conditions hold:*

(a) There exists a positive integer q_α, equal to a power of $\mathrm{char}(k) > 0$ *such that $\sigma(\rho(\alpha)) := \rho(\alpha) \circ \sigma|_T = q_\alpha \alpha$.*

(b) There exists $a_\alpha \in k^\times$ such that $\sigma(u_\alpha(c)) = u_{\rho(\alpha)}(a_\alpha c^{q_\alpha})$ for all $c \in k$.

Proof. Since σ stabilizes B and T, σ permutes the root subgroups U_α, $\alpha \in \Phi^+$. Therefore, we can define a permutation ρ of Φ^+ by $\rho(U_\alpha) = U_{\rho(\alpha)}$. Since the base of Φ^+ is unique, ρ must preserve Π. Now, given $\alpha \in \Phi^+$ and $c \in K$, $\sigma(u_\alpha(c)) \in U_{\rho(\alpha)}$. Therefore, there exists an endomorphism ν_α of the additive group $\mathbb{G}_a$ of K such that $\sigma(u_\alpha(c)) = u_{\rho(\alpha)}(\nu_\alpha(c))$ for all $c \in K$. We claim that ν is a monomial.

Note that for any $t \in T$ and $c \in K$,

$$tu_\alpha(c)t^{-1} = u_\alpha(\alpha(t)c).$$

Applying σ to the left-hand side of the equation above, we have

$$\sigma(tu_\alpha(c)t^{-1}) = \sigma(t)\sigma(u_\alpha(c))\sigma(t)^{-1} = \sigma(t)u_{\rho(\alpha)}(\nu_\alpha(c))\sigma(t)^{-1} = u_{\rho(\alpha)}(\rho(\alpha)(\sigma(t))\nu(c)).$$

Applying σ to the right-hand side, we have $\sigma(u_\alpha(\alpha(t)c)) = u_{\rho(\alpha)}(\nu(\alpha(t)c))$. Therefore, $u_{\rho(\alpha)}(\rho(\alpha)(\sigma(t))\nu(c)) = u_{\rho(\alpha)}(\nu(\alpha(t)c))$, which means that

$$\rho(\alpha)(\sigma(t))\nu(c) = \nu(\alpha(t)c) \tag{3.1}$$

for all $c \in K$ and $t \in T$.

Now, since char $K = p > 0$, every algebraic group endomorphism of the additive group $\mathbb{G}_a$ of K is of the form $c \mapsto h(c)$, where $h = \sum_{i=0}^{n} a_i T^{p^i}$ ($a_i \in K$) is a polynomial in $K[T]$. In particular, $\nu(c) = \sum_{i=0}^{n} a_i c^{p^i}$ for some elements $a_i \in K$ and for all $c \in K$. Therefore, (3.1) can be expressed as

$$\sum_{i=0}^{n} a_i \rho(\alpha)(\sigma(t)) c^{p^i} = \sum_{i=0}^{n} a_i \alpha(t)^{p^i} c^{p^i}.$$

Since the expression above holds for all $c \in K$, we have

$$\sum_{i=0}^{n} a_i \rho(\alpha)(\sigma(t)) T^{p^i} = \sum_{i=0}^{n} a_i \alpha(t)^{p^i} T^{p^i},$$

in $K[T]$, which means that $a_i \rho(\alpha)(\sigma(t)) = a_i \alpha(t)^{p^i}$ for $0 \leq i \leq n$. In particular, for all i such that $a_i \neq 0$, we have $\rho(\alpha)(\sigma(t)) = \alpha(t)^{p^i}$. But, this is possible for only one i. Therefore, $\nu = a_i T^{p^i}$ for some $i \geq 0$.

Setting $q_\alpha = p^i$ and $a_\alpha = a_i$, we have proved part (b) of the lemma. To prove part (a), we substitute $c = 1$ in (3.1), getting $\rho(\alpha)(\sigma(t))\nu(1) = \nu(\alpha(t))$. Therefore, $\sigma(\rho(\alpha))(t) = (\rho(\alpha) \circ \sigma)(t) = \rho(\alpha)(\sigma(t)) = \dfrac{\nu(\alpha(t))}{\nu(1)} = \dfrac{a_\alpha \alpha(t)^{q_\alpha}}{a_\alpha} = \alpha(t)^{q_\alpha} = (q_\alpha \alpha)(t)$ for all $t \in T$.

□

For the remainder of this section, we fix an F-stable Borel subgroup B and an F-stable maximal torus T of G, with $T \leq B$ (such a pair (T, B) exists by [42, Corollary 21.12]). Let $X = X(T)$ be the character group of T, and $\Phi \subset X$ the root system of G with respect to T. Let Φ^+ be the set of positive roots corresponding to the choice of B. Suppose that the root system Φ has base Π.

We define an action of F on X as follows. Given $\chi \in X$, and $t \in T$, we let $F(\chi)(t) := \chi(F(t))$.

Lemma 3.2.2. *([42, Lemma 22.1]) Let T be an F-stable maximal torus of G. Then there exists $\delta \in \mathbb{N}$ and a power r of $p =$ char K such that $F^\delta|_{X(T)} = r 1_{X(T)}$ on $X(T)$ (where $1_{X(T)}$ is the identity on $X(T)$).*

Combining Lemmas 3.2.1 and 3.2.2, we have the following important result.

Proposition 3.2.3. *([42, Proposition 22.2]) Let G be a connected reductive algebraic group with a Steinberg endomorphism $F : G \to G$, and let T be an F-stable maximal torus of G*

contained in an F-stable Borel subgroup B. Let $X = X(T)$, and let Φ be the root system of G with respect to T and Φ^+ be the system of positive roots corresponding to B. Then,

(a) There is a permutation ρ of Φ^+ and, for each $\alpha \in \Phi^+$, a positive integer $q_\alpha > 1$ such that q_α is a power of $p = \operatorname{char} K$, and an element $a_\alpha \in K^\times$ such that $F(\rho(\alpha)) = q_\alpha \alpha$ and $F(u_\alpha(c) = u_{\rho(\alpha)}(a_\alpha c^{q_\alpha})$ for all $c \in K$.

(b) There exists an integer $\delta \geq 1$ such that $F^\delta|_X = q^\delta 1_X$ and $F = q\phi$ on $X_{\mathbb{R}}$ for some positive fractional power q of p (with q^δ a power of p) and some $\phi \in \operatorname{Aut}(X_{\mathbb{R}})$ of order δ which induces ρ^{-1} on Φ^+.

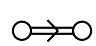
The automorphism ϕ of $X_{\mathbb{R}}$ defined in part (b) of the proposition above is key to the classification of the finite groups of Lie type, as we will see in the following section.

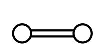
3.3 The classification of the finite groups of Lie type

As previously discussed, the task of classifying the finite groups of Lie type may be reduced to classifying the Steinberg endomorphisms of simple simply connected algebraic groups. Determining the Steinberg endomorphisms of a simple simply connected algebraic group is, in turn, closely related to the classification of the Dynkin diagram automorphisms and Coxeter diagram automorphisms of the base Π of the corresponding root system Φ. (We refer the reader to [40] for a full discussion of root systems; Dynkin diagrams and Coxeter graphs are covered in [40, 11.2].)

A Dynkin diagram automorphism is a permutation of the nodes of the Dynkin diagram which leaves the diagram invariant. A Coxeter diagram automorphism is a permutation of the nodes of the corresponding Coxeter diagram which leaves the diagram invariant. Every Dynkin diagram automorphism is also a Coxeter diagram automorphism. But, since the Coxeter diagram does not record root length, a root system which has both long and short simple roots may admit Coxeter diagram automorphisms which are not Dynkin diagram automorphisms.

Example 3.3.1. We consider the root system of type B_2. The Dynkin and Coxeter diagrams in type B_2 are shown below.

Dynkin diagram: 
1 2

Coxeter diagram: 
1 2

Since a Dynkin diagram automorphism must preserve root length, there are no non-trivial Dynkin diagram automorphisms in type B_2. However, there is one non-trivial Coxeter diagram automorphism, corresponding to flipping the diagram.

The only irreducible root systems having non-trivial Dynkin diagram automorphisms are in types A_n ($n \geq 2$), D_n ($n \geq 4$), and E_6. The automorphism group in type A_n ($n \geq 2$), D_n ($n \geq 5$), and E_6 is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The automorphism group in type D_4 is isomorphic to the symmetric group on 3 letters. The only irreducible root systems which admit a non-trivial Coxeter diagram automorphism are B_2, G_2, and F_4. In each case, there is one non-trivial Coxeter diagram automorphism, corresponding to flipping the diagram.

We are now ready to give a classification of the Steinberg endomorphisms of the simple simply connected groups. (Note that the following classification is due to Steinberg.)

Theorem 3.3.2. *([42, Theore 22.5]) Let G be a simple simply connected algebraic group (defined in characteristic $p > 0$) with root system Φ and base Π. Let Ω be the set consisting of all pairs (q, ρ) such that ρ is a Dynkin diagram automorphism of Π and q is an integral power of p or the root system of G is of type B_2, G_2, respectively F_4, ρ is the non-trivial Coxeter diagram automorphism of Π, and $q^2 = 2^{2f+1}, 3^{2f+1}$, respectively 2^{2f+1} for some $f \in \mathbb{N}_0$. Then, there is a one-to-one correspondence between Steinberg endomorphisms $F : G \to G$ and pairs $(q, \rho) \in \Omega$.*

If $F : G \to G$ is a Steinberg endomorphism, then $F = q\phi$ on $X_{\mathbb{R}}$ for some integral power q of p and for some $\phi \in \mathrm{Aut}(X_{\mathbb{R}})$. The automorphism ϕ of $X_{\mathbb{R}}$ induces a Coxeter diagram automorphism of Π. If we define ρ to be the inverse of this automorphism, then the pair (q, ρ) corresponds to F in the classification theorem above.

By Theorem 3.3.2, a finite group of Lie type is determined by its root system Φ, a power q of p, and the Coxeter diagram automorphism ρ of Π such that ρ^{-1} is induced by the automorphism $\phi \in \mathrm{Aut}(X_{\mathbb{R}})$ with $F = q\phi$ on $X_{\mathbb{R}}$. We observe that a diagram automorphism of Π is determined by its order: the root systems of type A_n ($n \geq 2$), D_n ($n \geq 5$), and E_6 admit exactly one non-trivial Dynkin diagram automorphism of order 2, D_4 admits non-trivial Dynkin diagram automorphisms of orders 2 and 3, and B_2, G_2, and F_4 admit a non-trivial Coxeter diagram automorphism of order 2. In particular, the diagram automorphism ρ of Theorem 3.3.2 is determined by the order δ of ϕ, which is either 1, 2, or 3.

Therefore, a finite group of Lie type is determined by Φ, a power q of p, and the order δ of ϕ. A classification of the finite groups of Lie type according to Φ, q, and δ is given in Table 22.1 of [42], which is reproduced in Table 1 below. In Table 1, G_{sc}^F denotes the finite group obtained by taking the F-fixed points of a simply connected group G_{sc} with the given root system, and G_{ad}^F denotes the finite group obtained by taking the F-fixed points of an adjoint group G_{ad} with the given root system.

The Steinberg endomorphisms F corresponding to the trivial diagram automorphisms (shown in the top section of Table 1) are called $\mathbb{F}_q$-*split*, which means that F acts as $q1_{X(T}$ on the character group $X(T)$ of an F-stable maximal torus T. The Steinberg endomorphisms

Table 1: Finite groups of Lie type (Table 22.1 in [MT])

Φ	δ	G_{sc}^F	G_{ad}^F	other isogeny types
A_{n-1}	1	$\mathrm{SL}_n(q)$	$\mathrm{PGL}_n(q)$	$\mathbb{Z}_{d/e}.\mathrm{PSL}_n(q).\mathbb{Z}_e$, where $e \mid d = (n, q-1)$
B_n	1	$\mathrm{Spin}_{2n+1}(q)$	$\mathrm{SO}_{2n+1}(q)$	
C_n	1	$\mathrm{Sp}_{2n}(q)$	$\mathrm{PCSp}_{2n}(q)$	
D_n	1	$\mathrm{Spin}_{2n}^+(q)$	$(\mathrm{PCO}_{2n}^o)^+(q)$	$\mathrm{SO}_{2n}^+(q)$, $\mathrm{HSpin}_{2n}^+(q)$
G_2	1	$G_2(q)$		
F_4	1	$F_4(q)$		
E_6	1	$(E_6)_{sc}(q)$	$(E_6)_{ad}(q)$	
E_7	1	$(E_7)_{sc}(q)$	$(E_7)_{ad}(q)$	
E_8	1	$E_8(q)$		
A_{n-1}	2	$\mathrm{SU}_n(q)$	$\mathrm{PGU}_n(q)$	$\mathbb{Z}_{d/e}.\mathrm{PSU}_n(q).\mathbb{Z}_e$, $e \mid d = (n, q+1)$
D_n	2	$\mathrm{Spin}_{2n}^-(q)$	$(\mathrm{PCO}_{2n}^o)^-(q)$	$\mathrm{SO}_{2n}^-(q)$
D_4	3	${}^3D_4(q)$		
E_6	2	$({}^2E_6)_{sc}(q)$	$({}^2E_6)_{ad}(q)$	
B_2	2	${}^2B_2(q^2) = \mathrm{Suz}(q^2)$, $q^2 = 2^{2f+1}$		
G_2	2	${}^2G_2(q^2)$, $q^2 = 3^{2f+1}$		
F_4	2	${}^2F_4(q^2) = \mathrm{Suz}(q^2)$, $q^2 = 2^{2f+1}$		

corresponding to the non-trivial Dynkin diagram automorphisms in types A_{n-1}, D_n, and E_6 (shown in the middle section of Table 1) are called *twisted*. A twisted Steinberg endomorphism is non-split; it is the product of an $\mathbb{F}_q$-split endomorphism with an algebraic group automorphism of G. Finally, the Steinberg endomorphisms corresponding to the non-trivial Coxeter diagram automorphisms in types B_2, G_2, and F_4 (shown in the bottom section of Table 1) are called *very twisted*.

The groups ${}^2B_2(q^2)$ corresponding to the very twisted Steinberg endomorphisms in type B_2 are called Suzuki groups, and the groups ${}^2G_2(q^2)$ and ${}^2F_4(q^2)$ corresponding to the very twisted Steinberg endomorphisms in types G_2 and F_4 are called Ree groups.

3.4 The Root System and Root Subgroups of the Finite Group of Lie Type G^F

In this section, we follow [42, Section 23.1]. Let G be a semisimple group with Steinberg endomorphism $F : G \to G$. Let T be an F-stable maximal torus of G, and let $N = N_G(T)$ be the normalizer of T in G. Then, the F-fixed point subset of W is given by $W^F = (N/T)^F \cong N_{G^F}(T)/T^F$. (Note that $N_{G^F}(T) \neq N_{G^F}(T^F)$ in general.)

Let B be an F-stable Borel subgroup of G containing the F-stable maximal torus T. Let Φ be the corresponding root system and Φ^+ be the corresponding system of positive roots. As discussed in the previous section, the Steinberg endomorphism F induces a permutation

ρ of Φ^+. Suppose $F|_{X_\mathbb{R}} = q\phi$ for some $\phi \in \mathrm{Aut}(X_\mathbb{R})$, where ϕ has (finite) order δ. Let $X_\mathbb{R}^\phi$ denote the fixed space of ϕ on $X_\mathbb{R}$, and define a projection $\pi : X_\mathbb{R} \to X_\mathbb{R}^\phi$ by $\chi \mapsto \frac{1}{\delta}\sum_{i=0}^{\delta-1}\phi^i(\chi)$.

The homomorphism $\pi : X_\mathbb{R} \to X_\mathbb{R}^\phi$ can be used to define an equivalence relation $\sim$ on the root system Φ of G, where $\alpha \sim \beta$ for $\alpha, \beta \in \Phi$ if and only if $\pi(\alpha)$ and $\pi(\beta)$ are positive multiplies of each other.

Given an equivalence class ω for the equivalence relation $\sim$, let α_ω be the maximal-length vector in the set $\{\pi(\alpha) \mid \alpha \in \omega\}$. Let Ω denote the set of equivalence classes ω for $\sim$. (By [42, Lemma 23.4], $\Omega = \{w(\Phi_I^+) \mid I \subseteq S$ is an orbit under F $, w \in W^F\}$.) Then, the set

$$\Phi_F = \{\alpha_\omega \mid \omega \in \Omega\}$$

is the root system of G^F. The root system Φ_F of G^F has base

$$\Pi_F = \{\alpha_\omega \mid \omega \subseteq \Pi\}.$$

We can now define the root subgroups of G^F. For any equivalence class $\omega \in \Omega$, let

$$U_\omega = \langle U_\alpha | \alpha \in \omega \rangle.$$

By [42, Proposition 23.7], U_ω ($\omega \in \Omega$) is F-stable and decomposes as $U_\omega = \prod_{\alpha\in\omega} U_\alpha$, with uniqueness of expression for a fixed order of the elements α in ω. Given any $\omega \in \Omega$, the subgroup U_ω^F of the U_ω is a root subgroup of G^F. The orders of the root subgroups of G^F are powers of q. By [42, Corollary 23.9], if G is simple and $\omega \in \Omega$, then $|U_\omega^F| = q^{|\omega|}$.

3.5 Construction of a BN-pair for the Finite Group of Lie Type G^F

Let G be a connected reductive linear algebraic group with a Steinberg endomorphism $F : G \to G$. Let T be an F-stable maximal torus of G, and let B be an F-stable Borel subgroup of G containing T. Let Φ and Φ^+ be the associated root system and set of positive roots, respectively, and let $W = N_G(T)/T$ be the Weyl group. Note that since T is F-stable, so is $N_G(T)$, which means that F acts on W.

The algebraic group G has Bruhat decomposition $G = \bigcup_{w\in W} BwB$, where the union is disjoint. For every $w \in W$, we have $BwB = U_w^- wB \cong U_w^- \times \{n_w\} \times B$, where n_w is a representative of the coset $w \in W$ and $U_w^- = \prod_{\substack{\alpha\in\Phi^+,\\ w.\alpha\in\Phi^-}} U_\alpha$.

The finite group G^F admits an analogous decomposition.

Theorem 3.5.1. *([42, Theorem 24.1]) Let G be a connected reductive algebraic group with Steinberg endomorphism $F : G \to G$, $T \leq B$ an F-stable maximal torus in an F-stable Borel subgroup of G, and W the Weyl group of G with respect to T. Then,*

$$G^F = \coprod_{w \in W^F} (U_w^-)^F w B^F,$$

with uniqueness of expression.

The Bruhat decomposition of G^F is key to proving the following theorem on the existence of a BN-pair in G^F.

Theorem 3.5.2. *([42, Theorem 24.10]) Let G be connected reductive with Steinberg endomorphism $F : G \to G$, $T \leq B$ an F-stable maximal torus in an F-stable Borel subgroup of G, and $N = N_G(T)$ the normalizer of T in G. Then, B^F, N^F is a BN-pair in G^F with Weyl group W^F.*

3.6 The Levi Decomposition of a Parabolic Subgroup of a Finite Group of Lie Type G with a Split BN-Pair

Suppose G is a finite group of Lie type with a split BN-pair (in characteristic p). Let W be the Weyl group associated to the BN-pair of G (with generators s_i, $i \in I$). When G is a finite group of Lie type, W is necessarily finite. Let Φ be the associated root system, and let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ be the set of simple roots in Φ. Given a subset $J \subseteq I$, let $\Pi_J = \{\alpha_i | i \in J\}$ and $\Phi_J = W_J(\Pi_J)$ (where W_J is the parabolic subgroup of W corresponding to J). And, define $L_J = \langle T, X_\alpha | \alpha \in \Phi_J \rangle$. The subgroup L_J of G is called a Levi subgroup.

Let B_J be the subgroup of L_J defined by $B_J = U_{w_{0J}} T$ (where w_{0J} is the longest element of W_J), and let N_J be the subgroup of L_J with $N_J/T = W_J$.

Proposition 3.6.1. *[8, Proposition 2.6.3] The group L_J has a split BN-pair (of characteristic p) corresponding to the subgroups B_J and N_J.*

For a subset $J \subseteq I$, let P_J be the parabolic subgroup of G corresponding to J. Define $U_J = U \cap U^{w_{0J}}$. Then, $U_J = O_p(P_J)$ is the largest normal p-subgroup of P_J (informally, we will refer to U_J as the unipotent radical of P_J). We have the decomposition,

$$P_J = U_J L_J = L_J U_J,$$

where $U_J \cap L_J = \{1\}$. The decomposition of P_J given above is called a Levi decomposition of P_J and L_J is called a Levi complement in P_J.

3.7 An Order Formula for a Finite Group of Lie Type Arising from a Simple Linear Algebraic Group

The order formulas for the finite groups of Lie type are well-known and can be found in many books and papers dealing with the structure of finite groups of Lie type. The account given here is based on [42, Section 24]. Let q be a power of a prime p. Let G be a simple linear algebraic group with Steinberg endomorphism $F : G \to G$, and let $T \leq B$ be an F-stable maximal torus in an F-stable Borel subgroup. Let Φ be the root system associated to B, and let W be the Weyl group. Then, B^F and $N_G(T)^F$ form a BN-pair in the finite group of Lie type G^F, with Weyl group W^F.

The order of the finite group of Lie type G^F is given by $|G^F| = |B^F| \sum_{w \in W^F} q^{l(w)}$, where $\sum_{w \in W^F} q^{l(w)}$ is the Poincaré polynomial of W^F evaluated at q, which gives the index $[G^F : B^F]$ of B^F in G^F. The Poincaré polynomial of W^F has a factorization of the form $\sum_{w \in W^F} x^{l(w)} = \sum_{w \in W^F} x^{l(w)} = \prod_{i=1}^{n} \frac{x^{d_i} - \varepsilon_i}{x - \varepsilon_i}$, where n is the rank of W^F, $d_1, \ldots, d_n$ are certain positive integers, and the ε_i are roots of unity (a more detailed description of the d_i and ε_i can be found in [42]). So, $[G^F : B^F] = \sum_{w \in W^F} q^{l(w)} = \prod_{i=1}^{n} \frac{q^{d_i} - \varepsilon_i}{q - \varepsilon_i}$. The order of the Borel subgroup B^F is $|B^F| = |U^F||T^F|$ (where U^F is the unipotent radical of B^F), and the order of U^F is $|U^F| = q^{|\Phi^+|}$. The order of T^F may be expressed in terms of the ε_i; in particular, $|T^F| = \prod_{i=1}^{n} (q - \varepsilon_i)$. Therefore, we have $|B^F| = q^{|\Phi^+|} \prod_{i=1}^{n} (q - \varepsilon_i)$ and

$$|G^F| = |B^F| \sum_{w \in W^F} q^{l(w)} = q^{|\Phi^+|} \prod_{i=1}^{n} (q - \varepsilon_i) \prod_{i=1}^{n} \frac{q^{d_i} - \varepsilon_i}{q - \varepsilon_i} = q^{|\Phi^+|} \prod_{i=1}^{n} (q^{d_i} - \varepsilon_i).$$

In the case that G is an arbitrary connected reductive algebraic group with a Steinberg endomorphism $F : G \to G$, the order of G^F is given by [42, Exercise 30.9]. Let $T \leq B$ be an F-stable maximal torus of G contained in an F-stable Borel subgroup. Suppose that G has root system Φ and positive root system Φ^+. Then,

$$|G^F| = \Big(\prod_{\alpha \in \Phi^+} q_\alpha \Big) |T^F| \sum_{w \in W^F} q_w,$$

where $q_w = \prod_{\alpha \in \Phi^+, w(\alpha) \in \Phi^-} q_\alpha$ for all $w \in W^F$ (q_α is defined as in [42, Lemma 11.10]).

4 Harish-Chandra Series

Our summary of Harish-Chandra theory is based on Section 4.2 of Geck and Jacon's "Representations of Hecke Algebras at Roots of Unity" [31]. Let G be a finite group of Lie type, defined in characteristic $p > 0$ (so, G is the fixed point subgroup of a connected reductive algebraic group $\mathbb{G}$ over $\bar{\mathbb{F}}_p$ under an endomorphism $F : \mathbb{G} \to \mathbb{G}$ such that some power of F is a Frobenius morphism). Assume, additionally, that G has subgroups B, N which form a split BN-pair of characteristic p, and let $T = B \cap N$. Let k be an algebraically closed field of characteristic $r > 0$, $r \neq p$.

4.1 Harish-Chandra Induction and Restriction

Given a subset $J \subseteq S$, let $P_J = \langle T, U_\alpha \mid \alpha \in \Phi^+ \cup \Phi_J \rangle$ be the standard parabolic subgroup of G corresponding to J. Let $U_{P_J} = O_p(P_J)$ (the subgroup U_{P_J} will be informally referred to as the unipotent radical of P_J), and let L_J denote the Levi subgroup of P_J. Using the notation of [31, 4.2.1], let $\mathscr{P}_G = \{{}^nP_J \mid J \subseteq S,\ n \in N\}$ and $\mathscr{L}_G = \{{}^nL_J \mid J \subseteq S,\ n \in N\}$.

Let $P \in \mathscr{P}_G$ and let $L \in \mathscr{L}_G$ be a Levi complement in P, so that $P = U_P \rtimes L$. Let $kL - \text{mod}$ denote the category of (finite dimensional) left kL-modules, and let $kG - \text{mod}$ denote the category of (finite dimensional) left kG-modules. Then, there is a Harish-Chandra induction functor

$$R^G_{L \subseteq P} : kL - \text{mod} \to kG - \text{mod}, \tag{4.1}$$

defined by $R^G_{L\subseteq P}(X) = \text{Ind}^G_P(\tilde{X})$ for all $X \in kL - \text{mod}$, where $\tilde{X}$ denotes the inflation of X from L to P via the surjective homomorphism $P \to L$ with kernel U_P.

There is also a Harish-Chandra restriction functor

$$^*R^G_{L \subseteq P} : kG - \text{mod} \to kL - \text{mod}. \tag{4.2}$$

Given a kG-module Y, $^*R^G_{L\subseteq P}(Y) = Y^{U_P}$ (which has the structure of a kL-module since U_P is a normal subgroup of P).

A key feature of Harish-Chandra induction is the following independence property, which was proved by Howlett and Lehrer [39] and Dipper and Du [18]. Let $L, M \in \mathscr{L}_G$, and suppose that L is a Levi complement of $P \in \mathscr{P}_G$ and M is a Levi complement of $Q \in \mathscr{P}_G$. Let $X \in kL - \text{mod}$ and $X' \in kM - \text{mod}$. If $M = {}^nL$ and $X' \cong {}^nX$ for some $n \in N$, then $R^G_{L\subseteq P}(X) \cong R^G_{M\subseteq Q}(X')$. As a particular application of this independence property, we have that the Harish-Chandra induction functor $R^G_{L\subseteq P}$ is independent of the choice of parabolic subgroup $P \in \mathscr{P}_G$ containing L. Similarly, the Harish-Chandra restriction functor $^*R^G_{L\subseteq P}$ is independent of the parabolic subgroup P containing L. Therefore, we will omit the parabolic subgroup P and write R^G_L and $^*R^G_L$ for the Harish-Chandra induction and restriction functors.

4.2 Some Properties of Harish-Chandra Induction and Restriction

Adjointness. Given a Levi subgroup $L \in \mathscr{L}_G$, the functors R_L^G and ${}^*R_L^G$ are exact. The functors R_L^G and ${}^*R_L^G$ are each other's two-sided adjoints. So, for any $Y \in kG-\text{mod}$ and $X \in kL-\text{mod}$, we have

$$\text{Hom}_{kG}(R_L^G(X), Y) \cong \text{Hom}_{kL}(X, {}^*R_L^G(Y)),$$

and

$$\text{Hom}_{kG}(Y, R_L^G(X)) \cong \text{Hom}_{kL}({}^*R_L^G(Y), X).$$

Transitivity. Harish-Chandra induction and restriction are transitive. Suppose that $L, M \in \mathscr{L}_G$ are such that $L \subseteq M$. If $X \in kL-\text{mod}$, then

$$R_L^G(X) \cong R_M^G(R_L^M(X)).$$

If $Y \in kG-\text{mod}$, then

$${}^*R_L^G(Y) \cong {}^*R_L^M({}^*R_M^G(Y)).$$

Mackey decomposition. As in the case of ordinary induction and restriction, we have a Mackey decomposition formula for Harish-Chandra induction and restriction. Suppose $L, M \in \mathscr{L}_G$ are Levi complements of the parabolic subgroups $P, Q \in \mathscr{P}_G$, respectively. Let X be a kL-module, and let $D(Q, P)$ denote a full set of (Q, P)-double coset representatives in G. Given an element $n \in D(Q, P)$, we can define a $k({}^nL)$-module structure on X by setting $nln^{-1}.x = l.x$ for any $l \in L$ and $x \in X$. The resulting $k({}^nL)$-module will be denoted by ${}^nX$. The Mackey formula provides the following direct sum decomposition of the kM-module ${}^*R_M^G(R_L^G(X))$:

$${}^*R_M^G(R_L^G(X)) \cong \bigoplus_{n \in D(Q,P)} R^M_{{}^nL \cap M}({}^*R^{{}^nL}_{{}^nL \cap M}({}^nX)). \tag{4.3}$$

Harish-Chandra induction and the linear dual. Let $L \in \mathscr{L}_G$ be the Levi complement of a parabolic subgroup $P \in \mathscr{P}_G$, and suppose that X is a left kL-module. Let X^* be the k-linear dual of X, viewed as a right kL-module. We claim that $R_L^G(X) \cong R_L^G(X^*)$. By [3, pg. 60], ordinary induction commutes with taking duals in the case of finite groups. So, we have $(R_L^G(X))^* = (\tilde{X}|_P^G)^* \cong (\tilde{X})^*|_P^G$. But, since the unipotent radical U_P of P acts trivially on $\tilde{X}$, U_P acts trivially on $\tilde{X}^*$, and it follows that $(\tilde{X})^* \cong \widetilde{X^*}$. Therefore, $(\tilde{X})^*|_P^G \cong \widetilde{X^*}|_P^G = R_L^G(X^*)$.

4.3 Cuspidal Modules and Harish-Chandra Series.

A kG-module Y is called cuspidal if ${}^*R_L^G(Y) = 0$ for all $L \in \mathscr{L}_G$ such that $L \subsetneq G$. (This definition extends to kL-modules for any $L \in \mathscr{L}_G$; a kL-module X is cuspidal if ${}^*R_{L'}^L(X) = 0$

for all $L' \in \mathscr{L}_G$ such that $L' \subsetneq L$.)

Let $\mathrm{Irr}_k(G)$ denote a full set of non-isomorphic irreducible kG-modules. (We will assume that all of the modules in $\mathrm{Irr}_k(G)$ are finite-dimensional over k.) Given a pair (L, X) with $L \in \mathscr{L}_G$ and X an irreducible cuspidal kL-module, let $\mathrm{Irr}_k(G|(L, X))$ be the subset of $\mathrm{Irr}_k(G)$ consisting of all $Y \in \mathrm{Irr}_k(G)$ such that $L \in \mathscr{L}_G$ is minimal with ${}^*R_L^G(Y) \neq 0$ and X is a composition factor of ${}^*R_L^G(Y)$. The set $\mathrm{Irr}_k(G|(L, X))$ is the Harish-Chandra series corresponding to the pair (L, X).

Following [31], let $\mathscr{L}_G^0 = \{(L, X)|L \in \mathscr{L}_G \text{ and } X \in \mathrm{Irr}_k(L) \text{ is cuspidal}\}$. There is an equivalence relation $\sim_N$ on $\mathscr{L}_G^0$ given by $(L, X) \sim_N (L', X')$ if and only if there exists some $n \in N$ such that $L' = {}^nL$ and $X' \cong {}^nX$. We will show that the Harish-Chandra series $\mathrm{Irr}_k(G|(L, X))$, $(L, X) \in \mathscr{L}_G^0/\sim_N$ partition $\mathrm{Irr}_k(G)$. We begin with two technical lemmas.

Lemma 4.3.1. *([31, Lemma 4.2.3]) Let Y be an irreducible kG-module, and let $L \in \mathscr{L}_G$ be minimal with ${}^*R_L^G(Y) \neq 0$. Then, every composition factor of ${}^*R_L^G(Y)$ is cuspidal.*

Proof. Let X be a composition factor of the kL-module ${}^*R_L^G(Y)$. Let M be a Levi subgroup of G with $M \subseteq L$ and suppose that ${}^*R_M^L(X) \neq 0$. Then, ${}^*R_M^L({}^*R_L^G(Y)) \neq 0$ since X is a composition factor of ${}^*R_L^G(Y)$. By transitivity of Harish-Chandra induction, we have

$$ {}^*R_M^G(Y) \cong {}^*R_M^L({}^*R_L^G(Y)) \neq 0. $$

So, by minimality of L, we must have $M = L$, which means that X is a cuspidal kL-module. ∎

Lemma 4.3.2. *([31, Lemma 4.2.4]) Let Y be an irreducible kG-module. Let $L, M \in \mathscr{L}_G$, $X \in \mathrm{Irr}_k(L)$, and $Z \in \mathrm{Irr}_k(M)$. And, assume that the following conditions are satisfied:*

(1) L is minimal such that ${}^*R_L^G(Y) \neq 0$ and X is a composition factor of ${}^*R_L^G(Y)$.

(2) Z is cuspidal and Y is in the head or socle of $R_M^G(Z)$.

Then, there is some $n \in N$ such that $L = {}^nM$ and $X \cong {}^nZ$.

In the next proposition, we give an alternate characterization of the Harish-Chandra series $\mathrm{Irr}_k(G|(L, X))$ (where $L \in \mathscr{L}_G$ and $X \in \mathrm{Irr}_k(L)$ is cuspidal).

Proposition 4.3.3. *([31, 4.2.6]) Let $(L, X) \in \mathscr{L}_G^0$. If Y is an irreducible kG-module, then the following statements are equivalent.*

(a) $Y \in \mathrm{Irr}_k(G|(L, X))$,

(b) Y is a composition factor of $\mathrm{head}(R_L^G(X))$, and

(c) Y is a composition factor of $\mathrm{soc}(R_L^G(X))$.

Proof. Suppose that $Y \in \mathrm{Irr}_k(G|(L,X))$. Then, ${}^*R_L^G(Y) \neq 0$, which means there is some irreducible kL-module X' in the socle of ${}^*R_L^G(Y)$. By adjointness of Harish-Chandra induction and restriction, we have

$$0 \neq \mathrm{Hom}_{kL}(X', {}^*R_L^G(Y)) \cong \mathrm{Hom}_{kG}(R_L^G(X'), Y),$$

which means that Y is in the head of the kG-module $R_L^G(X')$. Now, by Lemma 4.3.1, X' is a cuspidal irreducible kL-module. So, by Lemma 4.3.2, there exists an element $n \in N$ with $L = {}^nL$ and $X \cong {}^nX'$. Therefore, $R_L^G(X') \cong R_L^G(X)$ by the independence property of Harish-Chandra induction, and it follows that Y is in the head of $R_L^G(X)$. A similar argument shows that every irreducible kG-module $Y \in \mathrm{Irr}_k(G|(L,X))$ is in the socle of $R_L^G(X)$.

Suppose now that Y is in the socle or in the head of $R_L^G(X)$. If $Y \subseteq \mathrm{soc}(R_L^G(X))$, then adjointness of Harish-Chandra induction and restriction yields

$$0 \neq \mathrm{Hom}_{kG}(Y, R_L^G(X)) \cong \mathrm{Hom}_{kL}({}^*R_L^G(Y), X).$$

If $Y \subseteq \mathrm{head}(R_L^G(X))$, then adjointness yields

$$0 \neq \mathrm{Hom}_{kG}(R_L^G(X), Y) \cong \mathrm{Hom}_{kL}(X, {}^*R_L^G(Y)).$$

In either case, ${}^*R_L^G(Y) \neq 0$ and X is a composition factor of ${}^*R_L^G(Y)$. So, to show $Y \in \mathrm{Irr}_k(G|(L,X))$, it remains to check that L is minimal with ${}^*R_L^G(Y) \neq 0$. Suppose $M \subseteq L$ ($M \in \mathscr{L}_G$) is minimal with ${}^*R_M^G(Y) \neq 0$, and let $Z \in \mathrm{Irr}_k(M)$ be a composition factor of ${}^*R_M^G(Y)$. Then, Z is cuspidal by Lemma 4.3.1. So, by Lemma 4.3.2 (with the roles of L and M and the roles of X and Z reversed), $M = {}^nL$ for some $n \in N$. But, since $M \subseteq L$, we have $M = L$, as needed.

□

Theorem 4.3.4. *([31, 4.2.6]) The set* $\mathrm{Irr}_k(G)$ *is partitioned by the distinct Harish-Chandra series; there is a disjoint union*

$$\mathrm{Irr}_k(G) = \coprod_{(L,X) \in \mathscr{L}_G^0/\sim_N} \mathrm{Irr}_k(G|(L,X)).$$

Proof. Let $Y \in \mathrm{Irr}_k(G)$, and let L be minimal with ${}^*R_L^G(Y) \neq 0$. If X is any composition factor of ${}^*R_L^G(Y)$, then X is cuspidal by Lemma 4.3.1, which means that $Y \in \mathrm{Irr}_k(G|(L,X))$. Thus, every $Y \in \mathrm{Irr}_k(G)$ belongs to a Harish-Chandra series. Suppose now that Y belongs to $\mathrm{Irr}_k(G|(L,X))$ and $\mathrm{Irr}_k(G|(L',X'))$ for some pairs (L,X) and (L',X') in $\mathscr{L}_G^0$. Since $Y \in \mathrm{Irr}_k(G|(L,X))$, L is minimal with ${}^*R_L^G(Y) \neq 0$ and X is a composition factor of ${}^*R_L^G(Y)$. And, since $Y \in \mathrm{Irr}_k(G|(L',X'))$, Y is in the head of $R_{L'}^G(X')$. So, taking $Z = X'$ and $M = L'$ in Lemma 4.3.2, we have $L = {}^nL'$ and $X \cong {}^nX'$ for some $n \in N$. Therefore, $(L,X) \sim_N (L',X')$.

□

We note that the original proof of the fact that the distinct Harish-Chandra series partition $\mathrm{Irr}_k(G)$ is due to Hiss [37].

4.4 The Principal Series Representations

Since every kT-module is cuspidal, there is a Harish-Chandra series of the form $\mathrm{Irr}_k(G|(T,X))$ for every irreducible kT-module X. The irreducible representations of G belonging to a Harish-Chandra series of the form $\mathrm{Irr}_k(G|(T,X))$ are called *principal series representations*. (Since T is abelian, an irreducible kT-module X must be one-dimensional.)

The principal Harish-Chandra series corresponding to the pair (T,k) (where k is viewed as the trivial irreducible kT-module) is called the unipotent principal series and is denoted by $\mathrm{Irr}_k(G|B)$.[4] Since $R_T^G(k) \cong k|_B^G$, $\mathrm{Irr}_k(G|B)$ consists of the irreducible kG-modules which can be found in both the head and socle of the permutation module $k|_B^G$. If $V \in \mathrm{Irr}_k(G)$, then the multiplicity of V in the head of $k|_B^G$ is $[k|_B^G : V] = \dim \mathrm{Hom}_{kG}(k|_B^G, V) = \dim \mathrm{Hom}_{kB}(k,V) = \dim V^B$ (the second equality follows since $\mathrm{Hom}_{kG}(k|_B^G, V) \cong \mathrm{Hom}_{kB}(k,V)$ by Frobenius reciprocity). Similarly, $[\mathrm{soc}(k|_B^G) : V] = \dim V^B$. Thus, an irreducible kG-module V belongs to the unipotent principal series $\mathrm{Irr}_k(G|B)$ if and only if $V^B \neq 0$.

[4]The unipotent principal series representations are particularly important to the representation theory of finite groups of Lie type. According to results of Bonnafé and Rouquier [4], [5], the study of general blocks for finite groups of Lie type reduces to the study of unipotent blocks.

5 The Steinberg Module

The Steinberg module plays a key role in Geck's results in [29, Section 4], which are needed to prove the bound on the dimension of Ext^1 in Chapter 14 of this dissertation. While the bound of Chapter 14 is the most obvious application of the Steinberg module in this dissertation, the Steinberg module had a profound influence on all of the original research presented here. First and foremost, Geck's treatment of the Steinberg module in [29] inspired the idea to apply modular Harish-Chandra theory to the task of finding bounds on the dimension of Ext between irreducible modules for finite groups of Lie type in cross characteristic. It is also important to mention that the Steinberg module is in the background of many of the results of Cline, Parshall, and Scott [9] and of Dipper and James [21], which are used in the original proofs of Part IV of this dissertation.

5.1 Steinberg's Original Results on the "Steinberg Module"

5.1.1 A Basis for the Steinberg Module

Robert Steinberg originally introduced the Steinberg module in 1957 [47]; however, he did not refer to it as the "Steinberg module" in his own papers. In this section, we will give an overview of Steinberg's results in [47]. (All subsequent literature involving the Steinberg module is based on Steinberg's work in [47].)

Let G be a split finite group of Lie type, defined in characteristic p. Let B be a Borel subgroup of G, and let U denote the unipotent radical of B (so, U is a normal p-subgroup of B and $B = U \rtimes T$, where T is a maximal torus of G contained in B). Let $W = N_G(T)/T$ be the Weyl group of G with respect to T, and let S be the set of fundamental reflections generating W. Let Φ be the corresponding root system, and let Φ^+ denote the set of positive roots. Given a field k (of any characteristic)[5], we define an element $\mathfrak{e} \in kG$ by $\mathfrak{e} = \sum_{w \in W} (-1)^{l(w)} n_w \mathfrak{b}$, where $\mathfrak{b} = \sum_{b \in B} b$ and $n_w \in N_G(T)$ is a representative of the coset $w \in W = N_G(T)/T$. Then, the left ideal $\mathrm{St}_k := kG\mathfrak{e}$ of kG is the Steinberg module over k (this is Steinberg's original definition in [47], though Steinberg works with right rather than left kG-modules).

Given a root $\alpha \in \Phi$, let U_α denote the corresponding root subgroup of G. Given a closed subset $\Gamma \subseteq \Phi^+$, let U_Γ be the subgroup of G generated by the root subgroups U_α, $\alpha \in \Gamma$. Then, $U_\Gamma = \prod_{\alpha \in \Gamma} U_\alpha$. ($\Gamma$ is closed in the sense that for $\alpha, \beta \in \Gamma$ such that $\alpha + \beta$ is a root, $\alpha + \beta \in \Gamma$.) Now, if $w \in W$, then the set $\Phi_w = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}$ is a closed subset of Φ^+. We will denote the subgroup U_{Φ_w} of G by U_w. The complement of Φ_w is the closed subset $\Phi'_w = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^+\}$; we will denote the corresponding subgroup U_{Φ_w} by U'_w.

[5] The case with $k = \mathbb{C}$ is discussed further in the Appendix (Chapter 22).

The following technical lemma concerning the element $\mathfrak{e} \in kG$ is crucial to many of the properties of the Steinberg module.

Lemma 5.1.1. *([47, Lemma 1]) Fix a simple reflection $s \in S$, and let α be the simple root such that $s = s_\alpha$. Then,*

1. $n_s\mathfrak{e} = -\mathfrak{e}$.
2. *If $u_\alpha \in U_\alpha$ with $u_\alpha \neq 1$, then there exist some elements $\tilde{u}_\alpha, \bar{u}_\alpha \in U_\alpha$, and $t \in T$ such that*
$$n_s u_\alpha n_s^{-1} = \bar{u}_\alpha t n_s \tilde{u}_\alpha.$$
3. *With u_α and $\bar{u}_\alpha$ as in (2),*
$$(n_s u_\alpha - \bar{u}_\alpha + 1)\mathfrak{e} = 0.$$

Steinberg uses [47, Lemma 1] to find a k-basis for St_k, which is given in the next theorem.

Theorem 5.1.2. *([47, Theorem 1]) The set $\Omega = \{u\mathfrak{e} \mid u \in U\}$ is a k-vector space basis for St_k. Therefore, $\dim_k \mathrm{St}_k = |U|$.*

Corollary 5.1.3. *([47]) Let $x = \sum_{u \in U} \gamma_u u\mathfrak{e} \in \mathrm{St}_k$ (with $\gamma_u \in k$ for all u). Then, γ_u is $(-1)^{l(w_0)}$ times the coefficient of $un_{w_0}\mathfrak{b}$ in x. And, $\sum_{u \in U} \gamma_u$ is the coefficient of $\mathfrak{b}$ in x. If $x \in G$ and $x\mathfrak{e} = \sum_{u \in U} \gamma_u u\mathfrak{e}$, then $\gamma_u = 1$, -1, or 0, and at most $|W|$ of the γ_u are non-zero.*

5.1.2 An Irreducibility Criterion for the Steinberg Module

In the remainder of his paper, Steinberg proves the following irreducibility criterion for the Steinberg module St_k:

$$\mathrm{St}_k \text{ is irreducible if and only if } \mathrm{char}(k) \nmid [G : B].$$

Lemma 5.1.4. *([47, Lemma 2]) Let $m = [G : B]$, and, let $\mathfrak{u} = \sum_{u \in U} u$. Then, $(\sum_{w \in W} (-1)^{l(w)} n_w)\mathfrak{u}\mathfrak{e} = m\mathfrak{e}$.*

In the next two theorems, we will establish the irreducibility criterion for the Steinberg module (following [47]). Due to the importance of these theorems, we will present Steinberg's proofs of these results (providing extra details as necessary).

Theorem 5.1.5. *([47, Theorem 2])*

1. St_k *restricted to U is isomorphic to the left regular representation of U.*
2. *If $\mathrm{char}(k)$ is relatively prime to $m = [G : B]$, then St_k is irreducible.*

Proof. 1. The left regular representation of U is kU, viewed as a left kU-module via left multiplication. Since $\{u\mathfrak{e} \mid u \in U\}$ is a k-basis for St_k, there is a k-vector space isomorphism $\mathrm{St}_k \to kU$, $u\mathfrak{e} \mapsto u$. One can check that this map is a U-module homomorphism and therefore gives a kU-isomorphism $\mathrm{St}_k \cong kU$.

2. We will show that any non-zero element of St_k generates St_k as a left kG-module when m is relatively prime to $\mathrm{char}(k)$. Let $0 \neq \alpha \in \mathrm{St}_k$. To show that α generates St_k, it suffices to show that $\mathfrak{e} \in kG\alpha$. Since $\{u\mathfrak{e} \mid u \in U\}$ is a k-basis for St_k and each $u\mathfrak{e}$ is k-linear combination of elements of the form $x\mathfrak{b}$ ($x \in G$, $w \in W$), the same is true for α. So, if $\alpha \neq 0$, there must be some $x \in G$ such that the coefficient γ of $x\mathfrak{b}$ in α is not equal to zero. Then, the coefficient of $\mathfrak{b}$ in $x^{-1}\alpha$ is $\gamma \neq 0$. And, writing $x^{-1}\alpha = \sum_{u\in U} \gamma_u u\mathfrak{e}$ ($\gamma_u \in k$ for all $u \in U$), we have $\gamma = \sum_{u \in U} \gamma_u$ by Corollary 5.1.3.

Now, since γ and m are both invertible in k, we can carry out the following computation.

$$\begin{aligned}
\gamma^{-1}m^{-1}\Big(\sum_{w\in W}(-1)^{l(w)}w\Big)\mathfrak{u}x^{-1}\alpha &= \gamma^{-1}m^{-1}\Big(\sum_{w\in W}(-1)^{l(w)}w\Big)\mathfrak{u}\sum_{u\in U}\gamma_u u\mathfrak{e} \\
&= \gamma^{-1}m^{-1}\Big(\sum_{w\in W}(-1)^{l(w)}w\Big)\sum_{u\in U}\gamma_u(\mathfrak{u}u)\mathfrak{e} \\
&= \gamma^{-1}m^{-1}\Big(\sum_{w\in W}(-1)^{l(w)}w\Big)\sum_{u\in U}\gamma_u\mathfrak{u}\mathfrak{e} \\
&= \gamma^{-1}m^{-1}\Big(\sum_{w\in W}(-1)^{l(w)}w\Big)\gamma\mathfrak{u}\mathfrak{e} \\
&= m^{-1}\Big(\sum_{w\in W}(-1)^{l(w)}w\Big)\mathfrak{u}\mathfrak{e} \\
&= m^{-1}m\mathfrak{e} = \mathfrak{e}.
\end{aligned}$$

(The equality $(\sum_{w\in W}(-1)^{l(w)}w)\mathfrak{u}\mathfrak{e} = m\mathfrak{e}$ follows by Lemma 5.1.4.)

We have shown that $\mathfrak{e} \in kG\alpha$, completing the proof.

□

Finally, we show that the converse of part 2 of the previous theorem holds. So, in particular, St_k is irreducible if and only if $m = [G : B]$ is relatively prime to $\mathrm{char}(k)$.

Theorem 5.1.6. *([47, Theorem 3]) If* $\mathrm{char}(k)$ *divides* m*, then* St_k *is reducible.*

Proof. We will show that St_k has a non-zero proper submodule. Consider the element $\mathfrak{u}\mathfrak{e} \in \mathrm{St}_k$. We claim that $\mathfrak{u}\mathfrak{e} \neq 0$. We have $\mathfrak{u}\mathfrak{e} = \sum_{u\in U}\sum_{w\in W}(-1)^{l(w)}un_w\mathfrak{b}$. To show that $\mathfrak{u}\mathfrak{e} \neq 0$, it suffices to show that the coefficient of $n_{w_0}\mathfrak{b}$ in $\mathfrak{u}\mathfrak{e}$ is non-zero. But, suppose that $un_w\mathfrak{b} = n_{w_0}\mathfrak{b}$

for some $u \in U$, $w \in W$. Then, $un_wB = n_{w_0}B$, and by the Bruhat decomposition we must have $w = w_0$. But, $un_wB = n_{w_0}B$ implies that $u = 1$. Therefore, the coefficient of $n_{w_0}\mathfrak{b}$ in $\mathfrak{u}\mathfrak{e}$ is $(-1)^{l(w_0)} \neq 0$, which means that $\mathfrak{u}\mathfrak{e} \neq 0$. Thus, $kG\mathfrak{u}\mathfrak{e}$ is a non-zero kG-submodule of St_k.

To show that St_k is not irreducible, it remains to show that $kG\mathfrak{u}\mathfrak{e}$ is a proper submodule of St_k. We claim that $\mathfrak{e} \notin kG\mathfrak{u}\mathfrak{e}$. First, we will check that $\mathfrak{u}\mathfrak{e}(\sum_{w\in W}(-1)^{l(w)}n_w) = 0$. By Lemma 5.1.4, we have $(\sum_{w\in W}(-1)^{l(w)}n_w)\mathfrak{u}\mathfrak{e} = m\mathfrak{e}$. But, since $\mathrm{char}(k) \mid m$, $m = 0$ in k, so that $(\sum_{w\in W}(-1)^{l(w)}n_w)\mathfrak{u}\mathfrak{e} = 0$. Now, by definition of $\mathfrak{u}$ and $\mathfrak{e}$, we have $0 = (\sum_{w\in W}(-1)^{l(w)}n_w)\mathfrak{u}\mathfrak{e} = \sum_{\substack{w,w'\in W,\\ u\in U, b\in B}}(-1)^{l(w)}(-1)^{l(w')}n_wun_{w'}b$. Note that two elements $n_wun_{w'}b$ and $n_{\tilde{w}}\tilde{u}n_{\tilde{w}'}\tilde{b}$ in this sum are equal if and only if their inverses $(n_wun_{w'}b)^{-1}$ and $(n_{\tilde{w}}\tilde{u}n_{\tilde{w}'}\tilde{b})^{-1}$ are equal. So, setting $\tau = \sum_{t\in T} t$, we have

$$\begin{aligned}
0 &= \sum_{\substack{w,w'\in W,\\ u\in U, b\in B}} (-1)^{l(w)}(-1)^{l(w')}(n_wun_{w'}b)^{-1} \\
&= \sum_{\substack{w,w'\in W,\\ u\in U, b\in B}} (-1)^{l(w)}(-1)^{l(w')}b^{-1}n_{w'}^{-1}u^{-1}n_w^{-1} \\
&= (\sum_{w'\in W}(-1)^{l(w')}\mathfrak{b}n_{w'}^{-1}\mathfrak{u})(\sum_{w\in W}(-1)^{l(w)}n_w^{-1}) \\
&= (\sum_{w'\in W}(-1)^{l(w')}\mathfrak{u}\tau n_{w'}^{-1}\mathfrak{u})(\sum_{w\in W}(-1)^{l(w)}n_w^{-1}) \\
&= (\sum_{w'\in W}(-1)^{l(w')}\mathfrak{u}n_{w'}^{-1}\tau\mathfrak{u})(\sum_{w\in W}(-1)^{l(w)}n_w^{-1}) \\
&= (\sum_{w'\in W}(-1)^{l(w')}\mathfrak{u}n_{w'}^{-1}\mathfrak{b})(\sum_{w\in W}(-1)^{l(w)}n_w^{-1}) \\
&= (\sum_{w'\in W}(-1)^{l(w')}\mathfrak{u}n_{(w')^{-1}}\mathfrak{b})(\sum_{w\in W}(-1)^{l(w)}n_{w^{-1}}) \\
&= \mathfrak{u}\mathfrak{e}(\sum_{w\in W}(-1)^{l(w)}n_w).
\end{aligned}$$

(The second-to-last equality follows since n_w^{-1} is equal to $n_{w^{-1}}$ modulo T and $T \subseteq B$. The last equality follows since $l(w) = l(w^{-1})$ for any $w \in W$.)

Since $\mathfrak{u}\mathfrak{e}(\sum_{w\in W}(-1)^{l(w)}n_w) = 0$, every element of $kG\mathfrak{u}\mathfrak{e}$ is annihilated by right multiplication by $\sum_{w\in W}(-1)^{l(w)}n_w$. So, to show that $\mathfrak{e} \notin kG\mathfrak{u}\mathfrak{e}$, it suffices to show that $\mathfrak{e}(\sum_{w\in W}(-1)^{l(w)}n_w) \neq 0$. One of the properties satisfied by the unipotent radical U of B is that there is an element $u \in U$ such that $u \notin U'_w$ for all $w \neq 1$. We claim that u appears in $\mathfrak{e}(\sum_{w\in W}(-1)^{l(w)}n_w) =$

$\sum_{w',w\in W}(-1)^{l(w')}(-1)^{l(w')}n_{w'}\mathfrak{b}n_w$ with non-zero coefficient. The element u appears in one of the sums $n_{w'}\mathfrak{b}n_w$ ($w', w \in W$) if and only if $u \in n_{w'}Bn_w$ or, equivalently, $n_{w'}^{-1}u \in Bn_w$. If this is the case, $n_{w'}^{-1}u$ is contained in both Bn_wB and $Bn_{(w')^{-1}}B$ and, by the disjointness of the Bruhat decomposition, we must have $w = (w')^{-1}$. Therefore, $u \in n_w^{-1}Bn_w$, which means that $n_wun_w^{-1} \in U$, so that $u \in U'_w$. By assumption on u, we must have $w = 1$, and it also follows that $w' = 1$. So, u appears only in the summand $\mathfrak{b}$ of $\mathfrak{e}(\sum_{w\in W}(-1)^{l(w)}n_w)$. Since the coefficient of $\mathfrak{b}$ in $\mathfrak{e}(\sum_{w\in W}(-1)^{l(w)}n_w)$ is 1, the coefficient of u in $\mathfrak{e}(\sum_{w\in W}(-1)^{l(w)}n_w)$ is also 1. □

5.2 A New Proof of the Irreducibility Criterion for the Steinberg Module

5.2.1 Introduction

Let G be a finite group of Lie type, defined in characteristic p and having a split BN-pair, where B is a Borel subgroup of G containing a maximal torus T and $N = N_G(T)$ is the normalizer of T in G. Let U be the unipotent radical of B (U is a normal p-Sylow subgroup of B and $B = U \rtimes T$). Let $W = N_G(T)/T$ be the Weyl group of G with respect to T, and let S be the set of fundamental reflections generating W. And, let $\mathrm{St}_k := kG\mathfrak{e}$ be the Steinberg module over k.

In [47], Steinberg proves the following irreducibility criterion for St_k:

$$\mathrm{St}_k \text{ is irreducible if and only if } \mathrm{char}(k) \nmid [G:B].$$

Steinberg's original proof of the irreducibility criterion relies on a series of computations based on various properties of a group with a BN-pair (outlined in the previous section). In this section, we will provide a new (and more conceptual) proof of Steinberg's irreducibility criterion using the Hecke algebra.

5.2.2 The Proof of the Irreducibility Criterion

If $\mathrm{char}(k) = p$ (where p is the defining characteristic of G), then $p \nmid [G : B]$ since U is a p-Sylow subgroup of G, and it is well known that the Steinberg module is irreducible.[6] So, in what follows, we will assume that $\mathrm{char}(k) \neq p$. Additionally, we will assume that k is sufficiently large (meaning that it is a splitting field for G and all of its subgroups).

Let $k|_B^G$ be the permutation module on the cosets of B in G. Since $k|_B^G \cong kG\mathfrak{b}$, the Steinberg module St_k is a submodule of $k|_B^G$. Let $k|_B^G = M_1 \oplus \cdots \oplus M_n$ be a decomposition of

[6] We give a proof of the irreducibility of St_k in defining characteristic in the Appendix (Section 21).

$k|_B^G$ as a direct sum of indecomposable kG-modules. Each indecomposable direct summand M_i of $k|_B^G$ has a simple socle and a unique simple quotient (this was shown by Green in [34]). Now, by [29, Remark 2.4], there is a unique index i $(1 \le i \le n)$ such that $\mathrm{St}_k \subseteq M_i$. The indecomposable summand M_i of $k|_B^G$ containing the Steinberg module is called the Steinberg component of $k|_B^G$ and will be denoted by M_{St}.

The endomorphism algebra of $k|_B^G$ is the opposite of a Hecke algebra; specifically, we have $\mathrm{End}_{kG}(k|_B^G)^{\mathrm{op}} \cong \mathscr{H}_k$, where $\mathscr{H}_k$ is the Hecke algebra with k-basis $\{T_w | w \in W\}$, satisfying the relations

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w) \\ q^{c_s} T_{sw} + (q^{c_s} - 1) T_w & \text{if } l(sw) < l(w) \end{cases}$$

for $s \in S$ and $w \in W$ (with the positive integers c_s given by $|U_s| = q^{c_s}$ for all $s \in S$).[7]

There is a Hom functor

$$\mathfrak{F}_k : kG - \mathrm{mod} \to \mathscr{H}_k - \mathrm{mod}, \qquad V \mapsto \mathrm{Hom}_{kG}(k|_B^G, V).$$

By [29, Theorem 3.1], the Steinberg module St_k is mapped by $\mathfrak{F}_k$ to the sign representation SGN of $\mathscr{H}_k$. The trivial kG-module k is mapped by $\mathfrak{F}_k$ to the index representation IND of $\mathscr{H}_k$ [29, Example 3.3].

Let $\mathrm{Irr}_k(G)$ be a complete set of non-isomorphic irreducible kG-modules, and let $\mathrm{Irr}_k(G|B)$ denote the set of unipotent principal series representations of G. An irreducible kG-module Y is in $\mathrm{Irr}_k(G|B)$ if and only if Y is in $\mathrm{head}(k|_B^G)$, which is true if and only if Y is in $\mathrm{soc}(k|_B^G)$ [31, Theorem 4.2.6]. By [31, Theorem 4.2.9], the functor $\mathfrak{F}_k$ induces a bijection

$$\mathrm{Irr}_k(G|B) \overset{1-1}{\to} \mathrm{Irr}(\mathscr{H}_k)$$

from the unipotent principal series onto the set of irreducible $\mathscr{H}_k$-modules.

Lemma 5.2.1. *The irreducible socle of each indecomposable module M_i in the decomposition $k|_B^G = M_1 \oplus \cdots \oplus M_n$ is equal to its irreducible head.*

Proof. Fix an index i $(1 \le i \le n)$, and let $V = \mathrm{soc}(M_i)$ and $Y = \mathrm{head}(M_i)$. As remarked in the discussion above, both V and Y are irreducible. Furthermore, since M_i is a direct summand of $k|_B^G$, both V and Y belong to the unipotent principal series $\mathrm{Irr}_k(G|B)$. By [31, Corollary 4.1.10 (c)], the $\mathscr{H}_k$-module $\mathfrak{F}_k(M_i)$ is the projective indecomposable with head $\mathfrak{F}_k(Y)$. On the other hand, since the functor $\mathfrak{F}_k$ is left exact, we have $\mathfrak{F}_k(V) \subseteq \mathfrak{F}_k(M_i)$. Since $\mathfrak{F}_k$ establishes a bijection $\mathrm{Irr}_k(G|B) \to \mathrm{Irr}(\mathscr{H}_k)$, $\mathfrak{F}_k(V)$ is an irreducible $\mathscr{H}_k$-module contained in the socle of the projective indecomposable module $\mathfrak{F}_k(M_i)$. But, with the assumption that $\mathrm{char}(k) \ne p$, we have that $\mathscr{H}_k$ is a symmetric algebra. Thus, the socle of $\mathfrak{F}_k(M_i)$ is equal to its head, and it follows that $\mathfrak{F}_k(V) = \mathfrak{F}_k(Y)$. Since $\mathfrak{F}_k : \mathrm{Irr}_k(G|B) \to \mathrm{Irr}(\mathscr{H}_k)$ is a bijection,

[7] See Chapter 6 for further discussion of Hecke algebras.

we conclude that $V = Y$.

□

Theorem 5.2.2. St_k *is irreducible if and only if* $\mathrm{char}(k) \nmid [G : B]$

Proof. As remarked above, we may assume that $\mathrm{char}(k) \neq p$ (if $\mathrm{char}(k) = p$, then $p \nmid [G : B]$ and St_k is irreducible). Suppose first that $\mathrm{char}(k) \nmid [G : B]$. Then, by [31, Lemma 4.3.2], the permutation module $k|_B^G$ is completely reducible. Thus, every composition factor of $k|_B^G$ is contained in the unipotent principal series $\mathrm{Irr}_k(G|B)$. Since $\mathrm{St}_k \subseteq k|_B^G$, the same is true for all of the composition factors of St_k. But, by [29, Proposition 3.2], St_k has a unique composition factor belonging to $k|_B^G$. Therefore, St_k must be irreducible.

Conversely, suppose that St_k is an irreducible module. We claim that in this case, $M_{St} = \mathrm{St}_k$ (where M_{St} is the indecomposable Steinberg component of $k|_B^G$). Assume, for contradiction, that St_k is properly contained in M_{St}. By Lemma 5.2.1, M_{St} has St_k as both its head and socle. So, if $M_{St} \neq \mathrm{St}_k$, then St_k occurs at least twice as a composition factor of M_{St}, which means that $[k|_B^G : \mathrm{St}_k] \geq 2$ (where $[k|_B^G : \mathrm{St}_k]$ denotes the multiplicity of the Steinberg module as a composition factor of $k|_B^G$). Now, since $U \leq B$, there exists a surjective homomorphism $k|_U^G \twoheadrightarrow k|_B^G$. By Frobenius reciprocity, the multiplicity of St_k as a composition factor of $k|_U^G$ is equal to the multiplicity of k as a composition factor of the restriction $\mathrm{St}_k \downarrow_U$ of St_k to U. By [47, Theorem 1], the set $\{u\mathfrak{e} \mid u \in U\}$ forms a k-basis for St_k. Thus, the restriction of St_k to U is the regular representation kU of U, and $[\mathrm{St}_k \downarrow_U : k] = \dim_k(k) = 1$, which means that $[k|_U^G : \mathrm{St}_k] = 1$. Since $k|_B^G$ is a homomorphic image of $k|_U^G$, we must have $[k|_B^G : \mathrm{St}_k] \leq [k|_U^G : \mathrm{St}_k] = 1$, a contradiction.

By [31, Corollary 4.1.10 (c)], $\mathfrak{F}_k(M_{St})$ is the projective indecomposable $\mathscr{H}_k$-module with head $\mathfrak{F}_k(\mathrm{St}_k) = \mathrm{SGN}$. Thus, if St_k is irreducible, we have by the previous paragraph that $\mathfrak{F}_k(M_{St}) = \mathfrak{F}_k(\mathrm{St}_k) = \mathrm{SGN}$, which means that SGN is a projective $\mathscr{H}_k$-module. But, $\mathrm{IND} = \mathrm{SGN}^\Phi$, where SGN^Φ is the twist of the sign representation of $\mathscr{H}_k$ by the k-automorphism $\Phi : \mathscr{H}_k \to \mathscr{H}_k$, which is given by $T_w \mapsto (-1)^{l(w)} q_w T_{w^{-1}}^{-1}$ for any $w \in W$ (where $q_w = q^{c_{s_1}} \cdots q^{c_{s_m}}$ if $w = s_1 \cdots s_m$ is any reduced expression for w). The "twisting functor" $\mathscr{H}_k - \mathrm{mod} \to \mathscr{H}_k - \mathrm{mod}$, $M \mapsto M^\Phi$ is exact. So, since SGN is projective, the index representation IND of $\mathscr{H}_k$ is projective as well.

Now, since $k \in \mathrm{Irr}(G|B)$, there is an indecomposable component M_j of $k|_B^G$ having k as its irreducible head and socle. By [31, Corollary 4.1.10(c)], $\mathfrak{F}_k(M_j)$ is the projective cover of $\mathfrak{F}_k(k) = \mathrm{IND}$. Since IND is projective, we must have $\mathfrak{F}_k(M_j) = \mathrm{IND}$. But, by [31, Corollary 4.1.10 (b)], the multiplicity $[M_j : k]$ of k as a composition factor of M_j is equal to $\dim_k \mathrm{End}_{kG}(M_j)$. By [31, Lemma 4.1.2], $\dim_k \mathrm{End}_{kG}(M_j) = \dim_k \mathrm{End}_{\mathscr{H}_k}(\mathfrak{F}_k(M_j)) = \dim_k \mathrm{End}_{\mathscr{H}_k}(\mathrm{IND}) = 1$. Thus, $[M_j : k] = 1$ and, since M_j is indecomposable with $\mathrm{head}(M_j) \cong k$ and $\mathrm{soc}(M_j) \cong k$, we must have $M_j = k$. So, k is a direct summand of the permutation module $k|_B^G$, which can occur only if $\mathrm{char}(k) \nmid [G : B]$.

□

6 Hecke Algebras

6.1 The Hecke algebra associated to a Coxeter system

Let $\mathcal{Z} = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in an indeterminate t. Let (W, S) be a finite Coxeter system, and let $\widetilde{H} = \widetilde{H}(W, \mathcal{Z})$ denote the associated Hecke algebra over $\mathcal{Z}$. $\widetilde{H}$ is free over $\mathcal{Z}$, with basis $\{\tau_w\}_{w \in W}$, satisfying the relations

$$\tau_s \tau_w = \begin{cases} \tau_{sw} & \text{if } sw > w \\ t\tau_{sw} + (t-1)\tau_w & \text{otherwise} \end{cases}$$

for $s \in S$, $w \in W$ [9].

Let l denote the length function of W with respect to S. By [44, (2.0.3)], there is a $\mathcal{Z}$-involution $\Phi : \widetilde{H} \to \widetilde{H}$ with $\Phi(\tau_w) = (-t)^{l(w)}\tau_{w^{-1}}^{-1}$ for all $w \in W$. Given a right $\widetilde{H}$-module $\widetilde{M}$, let $\widetilde{M}^{\Phi}$ denote the $\widetilde{H}$-module obtained by twisting the action of $\widetilde{H}$ on $\widetilde{M}$ by Φ. The assignment $\widetilde{M} \mapsto \widetilde{M}^{\Phi}$ defines an exact functor $\Phi : \mathrm{mod} - \widetilde{H} \to \mathrm{mod} - \widetilde{H}$ (where $\mathrm{mod} - \widetilde{H}$ denotes the category of right $\widetilde{H}$-modules).

There is also a $\mathcal{Z}$-anti-involution $\iota : \widetilde{H} \to \widetilde{H}$ given by $\iota(\tau_w) = \tau_{w^{-1}}$ for any $w \in W$ [44, (2.0.5)]. Given a right $\widetilde{H}$-module $\widetilde{M}$, let $\widetilde{M}^{\iota}$ denote the left $\widetilde{H}$-module obtained by twisting the action of $\widetilde{H}$ on $\widetilde{M}$ by ι. The anti-automorphism ι is used primarily as a means to convert the left action of $\widetilde{H}$ on the dual of a right $\widetilde{H}$-module to a right action. Let $(-)^* = \mathrm{Hom}_{\mathcal{Z}}(-, \mathcal{Z})$ denote the linear dual on the category $\mathrm{mod} - \widetilde{H}$ of right $\widetilde{H}$-modules. Then, given a right $\widetilde{H}$-module $\widetilde{M} \in \mathrm{mod} - \widetilde{H}$, the dual $\widetilde{M}^*$ of $\widetilde{M}$ is a left $\widetilde{H}$-module. Thus, twisting the action of $\widetilde{H}$ on $\widetilde{M}^*$ by ι yields a right $\widetilde{H}$-module. For any $\widetilde{M} \in \mathrm{mod} - \widetilde{H}$, we will write $\widetilde{M}^{D_{\tilde{H}}} := \widetilde{M}^{*\iota}$.

The functor $D_{\widetilde{H}} : \mathrm{mod} - \widetilde{H} \to \mathrm{mod} - \widetilde{H}$, $\widetilde{M} \to \widetilde{M}^{D_{\tilde{H}}}$ is a contravariant duality functor. And, given any $\mathcal{Z}$-free right $\widetilde{H}$-module $\widetilde{M}$, we have $\widetilde{M}^{D^2_{\tilde{H}}} \cong \widetilde{M}$.

6.2 Parabolic Subalgebras of $\widetilde{H}$

Given a subset $\lambda \subseteq S$, let $W_\lambda = \langle s \rangle_{s \in \lambda}$ be the parabolic subgroup of W corresponding to λ. For every subset $\lambda \subseteq S$, we can define a parabolic subalgebra of $\widetilde{H}$ by setting $\widetilde{H}_\lambda = \langle \tau_s \rangle_{s \in \lambda}$.

Let $\mathrm{IND} : \widetilde{H} \to \mathcal{Z}$ be the linear character on $\widetilde{H}$ defined by $\mathrm{IND}(\tau_w) = t^{l(w)}$ for all $w \in W$. And, let $\mathrm{SGN} : \widetilde{H} \to \mathcal{Z}$ be the linear character on $\widetilde{H}$ defined by $\mathrm{SGN}(\tau_w) = (-1)^{l(w)}$ for all $w \in W$. The behavior of the $\widetilde{H}$-modules IND and SGN under the functors Φ and $D_{\widetilde{H}}$ is given in [44, (2.0.6)]:

$$\mathrm{IND}^{\Phi} \cong \mathrm{SGN},\ \mathrm{SGN}^{\Phi} \cong \mathrm{IND} \text{ and}$$
$$\mathrm{IND}^{D_{\widetilde{H}}} \cong \mathrm{IND},\ \mathrm{SGN}^{D_{\widetilde{H}}} \cong \mathrm{SGN}.$$

For any $\lambda \subseteq S$, let $\mathrm{IND}_\lambda = \mathrm{IND}|_{\widetilde{H}_\lambda}$ and $\mathrm{SGN}_\lambda = \mathrm{SGN}|_{\widetilde{H}_\lambda}$.

Given $\lambda \subseteq S$, define $x_\lambda = \sum_{w \in W_\lambda} \tau_w \in \widetilde{H}$ and $y_\lambda = \sum_{w \in W_\lambda} (-t)^{\lambda(w)} \tau_w \in \widetilde{H}$. For any $w \in W_\lambda$, the action of a basis element τ_w on the elements x_λ and y_λ of $\widetilde{H}$ is given in the following lemma.

Lemma 6.2.1. *([25, Lemma 1.1]) Let $w \in W_\lambda$. Then*

(1) $\tau_w x_\lambda = \mathrm{IND}_\lambda(\tau_w) x_\lambda$, and

(2) $\tau_w y_\lambda = \mathrm{SGN}_\lambda(\tau_w) y_\lambda$.

By Lemma 6.2.1, it follows that $\widetilde{H} x_\lambda \cong \mathrm{Ind}_{\widetilde{H}_\lambda}^{\widetilde{H}} \mathrm{IND}_\lambda$ and $\widetilde{H} y_\lambda \cong \mathrm{Ind}_{\widetilde{H}_\lambda}^{\widetilde{H}} \mathrm{SGN}_\lambda$ (where IND_λ and SGN_λ are the left $\widetilde{H}_\lambda$-modules defined by the linear characters IND_λ and SGN_λ, respectively). Similarly, if IND_λ and SGN_λ are the right $\widetilde{H}_\lambda$-modules defined by the linear characters IND_λ and SGN_λ, respectively, then $x_\lambda \widetilde{H} \cong \mathrm{Ind}_{\widetilde{H}_\lambda}^{\widetilde{H}} \mathrm{IND}_\lambda$ and $y_\lambda \widetilde{H} \cong \mathrm{Ind}_{\widetilde{H}_\lambda}^{\widetilde{H}} \mathrm{SGN}_\lambda$.

6.3 Specializing the Generic Hecke Algebra

If $\mathcal{Z}'$ is a commutative $\mathcal{Z}$-algebra, we write $\widetilde{H}' = \widetilde{H}_{\mathcal{Z}'} = \widetilde{H} \otimes_{\mathcal{Z}} \mathcal{Z}'$. The $\mathcal{Z}'$-algebra $\widetilde{H}'$ has $\mathcal{Z}$-basis consisting of elements $\tau_w \otimes 1$, $w \in W$, which satisfy relations analogous to the defining relations of $\widetilde{H}$. By abuse of notation, we will denote $\tau_w \otimes 1$ ($w \in W$) by τ_w. The $\mathcal{Z}$-automorphism Φ and the $\mathcal{Z}$-anti-automorphism ι of $\widetilde{H}$ extend to the specialized Hecke algebra $\widetilde{H}'$.

In the following lemma, we list several important results concerning the permutation modules $\widetilde{H}' x_\lambda$ and $\widetilde{H}' y_\lambda$.

Lemma 6.3.1. *([25, Lemma 1.1]) Let $\lambda, \mu \subseteq S$. Then,*

1. *$(x_\lambda \widetilde{H}')^* \cong \widetilde{H}' x_\lambda$ and $(y_\lambda \widetilde{H}')^* \cong \widetilde{H}' y_\lambda$ (where $(\ )^* = \mathrm{Hom}_{\mathcal{Z}'}(-, \mathcal{Z}')$ is the $\mathcal{Z}'$-linear dual).*
2. *$(x_\lambda \widetilde{H}')^{\Phi} \cong y_\lambda \widetilde{H}'$ and $\mathrm{Hom}_{\widetilde{H}'}(y_\lambda \widetilde{H}', y_\mu \widetilde{H}') \cong \mathrm{Hom}_{\widetilde{H}}(x_\lambda \widetilde{H}', x_\mu \widetilde{H}')$.*
3. *If the subgroups W_λ and W_μ are W-conjugate, then $x_\mu \widetilde{H}' \cong x_\lambda \widetilde{H}'$ and $y_\mu \widetilde{H}' \cong y_\lambda \widetilde{H}'$.*

6.4 The "Double Coset" Hecke Algebra

6.4.1 Definition of a Hecke Algebra given in [12]

In [12, 11.22], Curtis and Reiner give the following definition of a Hecke algebra. Let G be a finite group, and let B be any subgroup of G. Now, consider the group algebra KG, where K is a subfield of the complex numbers which is a splitting field for G and all of its subgroups. For a subset $X \subseteq G$, define an element $\underline{X}$ of KG by $\underline{X} = \sum_{x \in X} x$. Let $H(G, B)$ be the subset of KG consisting of all K-linear combinations of elements $\frac{1}{|B|}\underline{BgB}$, $g \in G$. $H(G, B)$ is called a Hecke algebra (we show in Proposition 6.4.1 below that $H(G, B)$ is a K-algebra).

For finite groups of Lie type, there is a useful modification of the definition of a Hecke algebra. Let $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in the indeterminate q. Let (W, S) be a finite Coxeter system, and let $\{c_s\}$ be a fixed system of index parameters for this Coxeter system, i.e. a system of integers c_s such that $c_s = c_t$ whenever s and t are W-conjugate. Given $w \in W$, let $w = s_1 \cdots s_m$ $(s_1, \ldots, s_m \in S)$ be a reduced expression for W. Then, define $q_w = q^{c_{s_1}} \cdots q^{c_{s_m}}$ (one can check that this definition is independent of the choice of reduced expression for w). Then, the generic Hecke algebra $\widetilde{H}$ over $\mathcal{Z}$ is the algebra with $\mathcal{Z}$-basis $\{\tau_w\}_{w \in W}$, and relations

$$\tau_s \tau_w = \begin{cases} \tau_{sw} & \text{if } l(sw) > l(w); \\ q_s \tau_{sw} + (q_s - 1)\tau_w & \text{if } l(sw) < l(w) \end{cases}$$

for all $s \in S$, $w \in W$. Additionally, if $\mathcal{Z}'$ is any commutative $\mathcal{Z}$-algebra, we can tensor $\widetilde{H}$ with $\mathcal{Z}'$ to obtain a Hecke algebra $\widetilde{H}_{\mathcal{Z}'}$ over $\mathcal{Z}'$. Then, $\widetilde{H}_{\mathcal{Z}'}$ has generating set $\{\tau_w \otimes 1\}_{w \in W}$, subject to relations analogous to those given above.

The "double coset" Hecke algebra $H(G, B)$ can be defined without reference to a Coxeter system. Therefore, at first glance, the "double coset" definition of a Hecke algebra seems unrelated to the "Coxeter system" definition. However, it can be shown that the "double coset" definition is, in fact, a generalization of the "Coxeter system" definition. We will demonstrate that this is the case using the example of a group with a BN-pair (which is naturally equipped with a Coxeter system). Specifically, Let G be a group with a BN-pair, and let $W = N/(B \cap N)$ be the Weyl group of G. Let $S = \{s_1, \ldots, s_n\}$ be a generating set for W as a Coxeter group. We will show that $H(G, B)$ admits a presentation by generators and relations analogous to the one given above.

6.4.2 The case of a group with a BN-pair

The following material is based on [12, 11D] and [13, 65A, 67A]. Let G be a finite group with a BN-pair, and let W denote the Weyl group of G.

If $n_w \in N$ is a representative of a coset $w \in W$, we write wB for the coset $n_w B$ (since $B \cap N \subset B$, one can check that the definition of wB is independent of the choice of representative n_w of w). We can decompose G as a disjoint union of (B, B)-double cosets $G = \bigcup\limits_{w \in W} BwB$ (Bruhat decomposition). In particular, every (B, B)-double coset in G is of the form BwB for some $w \in W$. Therefore, $H(G, B)$ (as defined in Curtis and Reiner) consists of K-linear combinations of elements $\frac{1}{|B|}\underline{BwB}$ of KG. In fact, by the Bruhat decomposition, the elements $\frac{1}{|B|}\underline{BwB}$, $w \in W$, form a K-basis of $H(G, B)$. For convenience, we will denote $\frac{1}{|B|}\underline{BwB}$ by a_w.

Proposition 6.4.1. *$H(G, B)$ is a K-algebra.*

Proof. First, we observe that the element $a_1 = \frac{1}{|B|}\underline{B} \in KG$ is idempotent. Therefore, $a_1 KG a_1$, is an algebra, with identity element a_1. We claim that $H(G, B)$ is equal to $a_1 KG a_1$. Note that $a_1 KG a_1$ is spanned by elements of the form $a_1 x a_1$, $x \in G$, and we have

$$a_1 x a_1 = \big(\frac{1}{|B|}\underline{B}\big)x\big(\frac{1}{|B|}\underline{B}\big) = \frac{1}{|B|^2}\sum_{b_1, b_2 \in B} b_1 x b_2. \tag{6.1}$$

Now, if $b_1 x b_2 = x$ for some $b_1, b_2 \in B$, then $b_1 = x b_2^{-1} x^{-1}$. Therefore, the number of times that x appears in the sum $\sum\limits_{b_1, b_2 \in B} b_1 x b_2$ is $|B \cap xBx^{-1}|$. Similarly, we can show that any element $b_1 x b_2 \in BxB$ appears $|B \cap xBx^{-1}|$ times in the sum $\sum\limits_{b_1, b_2 \in B} b_1 x b_2$.

Thus, (6.1) is equal to

$$\frac{1}{|B|^2}\sum_{b_1 x b_2 \in BxB} |B \cap xBx^{-1}|\ b_1 x b_2 = \frac{1}{|B|^2}|B \cap xBx^{-1}|\ \underline{BxB}.$$

Now, by the Bruhat decomposition, there exists some $w_x \in W$ such that $BxB = Bw_xB$. So, $\frac{1}{|B|^2}|B \cap xBx^{-1}|\ \underline{BxB} = \frac{1}{|B|^2}|B \cap xBx^{-1}|\ \underline{Bw_xB} = \frac{1}{|B|}|B \cap xBx^{-1}|\ a_{w_x}$. Consequently, we have $a_1 x a_1 = \frac{1}{|B|}|B \cap xBx^{-1}|\ a_{w_x}$. This calculation shows both that $H(G, B) \subseteq a_1 KG a_1$ and that $a_1 KG a_1 \subseteq H(G, B)$. Thus, $H(G, B) = a_1 KG a_1$, as claimed.

$\square$

In the next lemma, we establish a useful formula for the multiplication of basis elements a_w of $H(G,B)$. We will use without proof the fact that the group algebra KG can be identified with the set of K-valued functions on G, with multiplication given by the convolution product. Explicitly, an element $\sum_{x\in G}\alpha_x x \in KG$ can be identified with the function $f : G \to K$ given by $f(x) = \alpha_x$. If f and g are two K-valued functions on G, then their convolution product is given by $(f \cdot g)(x) = \sum_{y\in G} f(xy)g(y^{-1})$ for $x \in G$.

Lemma 6.4.2. *For $u, v \in W$, we have $a_u a_v = \sum_{w\in W} \mu_{u,v,w} a_w$, where $\mu_{u,v,w} = \frac{1}{|B|}|BuB \cap w(BvB)^{-1}|$.*

Proof. Since the elements a_w, $w \in W$, form a K-basis for $H(G,B)$, we can write

$$a_u a_v = \sum_{w\in W} \mu_{u,v,w} a_w,$$

where $\mu_{u,v,w}$ is some structure constant in K for $u, v, w \in W$. Fix an element $w' \in W$. Identifying KG with the set of K-valued functions on G, we have $(a_u a_v)(n_{w'}) = (\sum_{w\in W} \mu_{u,v,w} a_w)(n_{w'})$. By definition $(\sum_{w\in W} \mu_{u,v,w} a_w)(n_{w'})$ is the coefficient of $n_{w'}$ in $\sum_{w\in W} \mu_{u,v,w} a_w$. Since $n_{w'} \in Bw'B$ and the double cosets BwB $(w \in W)$ are disjoint, $n_{w'}$ occurs only in $a_{w'}$, with a coefficient of $\frac{1}{|B|}$. So,

$$(a_u a_v)(n_{w'}) = (\sum_{w\in W} \mu_{u,v,w} a_w)(n_{w'}) = \frac{1}{|B|}\mu_{u,v,w'}. \tag{6.2}$$

On the other hand, by definition of the convolution product, $(a_u a_v)(n_{w'}) = \sum_{y\in G} a_u(y) a_v(y^{-1} n_{w'})$. Since $a_u = \frac{1}{|B|}\underline{BuB}$, an element $y \in G$ occurs with non-zero coefficient in a_u if and only if $y \in BuB$. That is, $a_u(y) \neq 0$ if and only if $y \in BuB$. Similarly, $a_v(y^{-1}n_{w'}) \neq 0$ if and only if $y^{-1}n_{w'} \in BvB$, or, equivalently, $y \in n_{w'}(BvB)^{-1} = w'(BvB)^{-1}$. Thus, $a_u(y)a_v(y^{-1}n_{w'}) \neq 0$ if and only if $y \in BuB \cap w'(BvB)^{-1}$. Therefore,

$$(a_u a_v)(n_{w'}) = \sum_{y\in BuB\cap w'(BvB)^{-1}} a_u(y) a_v(y^{-1} n_{w'}). \tag{6.3}$$

Comparing the equations (6.2) and (6.3), we have $\frac{1}{|B|}\mu_{u,v,w'} = \sum_{y\in BuB\cap w'(BvB)^{-1}} a_u(y)a_v(y^{-1}n_{w'})$, so that $\mu_{u,v,w'} = |B| \sum_{y\in BuB\cap w'(BvB)^{-1}} a_u(y)a_v(y^{-1}n_{w'})$. And, since $a_u(y) = \frac{1}{|B|}$ and $a_v(y^{-1}n_{w'}) = \frac{1}{|B|}$ for $y \in BuB\cap w'(BvB)^{-1}$, $\mu_{u,v,w'} = |B| \sum_{y\in BuB\cap w'(BvB)^{-1}} \frac{1}{|B|}\frac{1}{|B|} = \frac{1}{|B|}|BuB\cap w(BvB)^{-1}|$.

□

Next, we will use Lemma 6.4.2, along with several properties of groups with a BN-pair, to compute $a_{si}a_w$ ($s_i \in S$, $w \in W$) when $l(s_iw) = l(w) + 1$ and $l(s_iw) = l(w) - 1$. We state without proof the properties of G which will be needed to carry out these computations.

Proposition 6.4.3. *Let G be a group with a BN-pair. Let W be the Weyl group of G associated with the BN-pair, and let $S = \{s_1, \ldots, s_n\}$ be the set of distinguished generators of W. The following statements hold.*

(a) ([13, 65.2]) For $s_i \in S$ and $w \in W$, $s_iBw \subseteq BwB \cup Bs_iwB$.

(b) ([13, 65.2]) If $l(s_iw) > l(w)$, then $s_iBw \subseteq Bs_iwB$.

We may extend the result of Proposition 6.4.3(a) as follows. Let $w, w' \in W$. Let $w' = s_{j(1)} \cdots s_{j(r)}$ be a reduced expression for w'. Then, by Proposition 6.4.3(a),

$$wBw' = (wBs_{j(1)}) \cdots s_{j(r)} \subseteq (BwB \cup Bs_{j(1)}wB)s_{j(2)} \cdots s_{j(r)}.$$

Repeated application of Proposition 6.4.3(a) shows that

$$wBw' \subseteq \cup Bws_{j(k_1)} \cdots s_{j(k_p)}B,$$

where the union is taken over all subsequences of $1 \leq k_1 < k_2 < \cdots < k_p \leq r$ of $1, \ldots, r$ ($1 \leq p \leq r$). (This result appears in the proof of [12, 65.8].)

Definition 6.4.4. Let G be a group with a BN-pair. Let W be the Weyl group of G, and let $S = \{s_1, \ldots, s_n\}$ be the set of distinguished generators of W. Then, for $1 \leq i \leq n$, the index parameter of s_i is $\text{ind}(s_i) = q_i = [B : B \cap s_iBs_i]$.

In fact, we may define an index parameter $[B : B \cap wBw^{-1}]$ for any $w \in W$. This definition gives rise to an "index" map $\text{ind} : H(G, B) \to \mathbb{C}$, given by $a_w \mapsto [B : B \cap wBw^{-1}]$. It can be shown that ind is an algebra homomorphism [12, Exercise 11.19].

Lemma 6.4.5. *For any $s_i \in S$, $a_{s_i}^2 = q_ia_1 + (q_i - 1)a_{s_i}$.*

Proof. By Proposition 6.4.3 (a), $(Bs_iB)(Bs_iB) = Bs_iBs_iB \subseteq B(B \cup Bs_iB)B = B \cup Bs_iB$. Now, $a_{s_i}^2 = \frac{1}{|B|^2}\underline{Bs_iB}\ \underline{Bs_iB} = \frac{1}{|B|^2}\sum_{x,y \in Bs_iB} xy$. In particular, every element xy appearing in $a_{s_i}^2$ is contained in $(Bs_iB)(Bs_iB) \subseteq B \cup Bs_iB$. Therefore, if we write $a_{s_i}^2 = \sum_{w \in W} \mu_{s_i,s_i,w}a_w$ as in Lemma 6.4.2, the only non-zero structure coefficients are $\mu_{s_i,s_i,1}$ and μ_{s_i,s_i,s_i}, and it follows that $a_{s_i}^2 = \mu_{s_i,s_i,1}a_1 + \mu_{s_i,s_i,s_i}a_{s_i}$.

Now, by Lemma 6.4.2, $\mu_{s_i,s_i,1} = \frac{1}{|B|}|Bs_iB \cap Bs_iB|$. And, $\frac{|Bs_iB|}{|B|} = \frac{|Bs_iBs_i|}{|B|} = |Bs_iBs_i/B| = |B/B \cap s_iBs_i| = [B : B \cap s_iBs_i] = q_i$ (where the third equality follows by the Second Isomorphism Theorem). Thus, $a_{s_i}^2 = q_i a_1 + \mu_{s_i,s_i,s_i} a_{s_i}$.

To compute μ_{s_i,s_i,s_i}, we use the homomorphism ind : $H(G,B) \to \mathbb{C}$. Applying ind to both sides of $a_{s_i}^2 = q_i a_1 + \mu_{s_i,s_i,s_i} a_{s_i}$, we have $q_i^2 = q_i(1) + q_i \mu_{s_i,s_i,s_i}$. So, $\mu_{s_i,s_i,s_i} = q_i - 1$, and $a_{s_i}^2 = q_i a_1 + (q_i - 1) a_{s_i}$, as needed.

□

Theorem 6.4.6. *Let $w \in W$ and $s_i \in S$. Then,*

$$a_{s_i} a_w = \begin{cases} a_{s_i w} & \text{if } l(s_i w) > l(w) \\ q_i a_{s_i w} + (q_i - 1) a_w & \text{if } l(s_i w) < l(w). \end{cases}$$

Proof. First, we consider the case of $l(s_i w) > l(w)$. By Lemma 6.4.2, we have $a_{s_i} a_w = \sum_{u \in W} \mu_{s_i,w,u} a_u$. Now, by Proposition 6.4.3(b), $s_i B w \subseteq B s_i w B$. So, $\mu_{s_i,w,u} = 0$ when $u \neq s_i w$, which means that $a_{s_i} a_w = \mu_{s_i,w,s_i w} a_{s_i w}$. So, to prove the theorem in the case of $l(s_i w) > l(w)$, it suffices to show that $\mu_{s_i,w,s_i w} = 1$.

But, by Lemma 6.4.2, we have

$$\mu_{s_i,w,s_i w} = \frac{1}{|B|} |B s_i B \cap s_i w (B w B)^{-1}| = \frac{1}{|B|} |B s_i B \cap s_i w B w^{-1} B| = \frac{1}{|B|} |s_i B s_i B \cap w B w^{-1} B|.$$

By Proposition 6.4.3 (a), $s_i B s_i \subseteq B \cup B s_i B$. So, since $B \subseteq w B w^{-1} B$, we have

$$s_i B s_i B \cap w B w^{-1} B \subseteq (B \cup B s_i B) \cap w B w^{-1} B = B \cup (B s_i B \cap w B w^{-1} B).$$

Thus, the claim that $\mu_{s_i,w,s_i w} = 1$ will follow if we we can show that $B s_i B \cap w B w^{-1} B = \emptyset$ (for, in that case, we will have $s_i B s_i B \cap w B w^{-1} B \subseteq B$, which means that $s_i B s_i B \cap w B w^{-1} B = B$, so that $\mu_{s_i,w,s_i w} = \frac{1}{|B|} |B s_i B \cap s_i w (B w B)^{-1}| = \frac{|B|}{|B|} = 1$).

We show that $B s_i B \cap w B w^{-1} B = \emptyset$ by contradiction. So, assume that $B s_i B \cap w B w^{-1} B \neq \emptyset$. Let $w = s_{j(1)} \cdots s_{j(r)}$ be a reduced expression for w. Then, $w^{-1} = s_{j(r)} \cdots s_{j(1)}$ is a reduced expression for w^{-1}. By the extended version of Proposition 6.4.3 (a), we have $w B w^{-1} \subseteq \cup B w s_{j(k_1)} \cdots s_{j(k_p)} B$, where the union is taken over all subsequences of $r \geq k_1 > k_2 > \cdots > k_p \leq 1$ of $1, \ldots, r$ $(1 \leq p \leq r)$. Since $B s_i B \cap w B w^{-1} B \neq \emptyset$, $B s_i B \cap (\cup B w s_{j(k_1)} \cdots s_{j(k_p)} B) \neq \emptyset$. By the disjointness of the double cosets appearing in the Bruhat decomposition of G, we must have $B s_i B = B w s_{j(k_1)} \cdots s_{j(k_p)} B$ for some subsequence $k_1, \ldots, k_p$ of $1, \ldots, r$. Thus, $s_i = w s_{j(k_1)} \cdots s_{j(k_p)}$, so that $w^{-1} s_i = s_{j(k_1)} \cdots s_{j(k_p)}$. And, $l(s_i w) = l(w^{-1} s_i) \leq p \leq r = l(w)$, which contradicts the assumption that $l(s_i w) > l(w)$.

Now, we consider the case of $l(s_i w) < l(w)$. Let $w' = s_i w$. Then, $l(s_i w') = l(w) > l(w')$. So, applying the first case, we have $a_{s_i} a_{w'} = a_w$. Using Lemma 6.4.5, $a_{s_i} a_w = a_{s_i}^2 a_{w'} = (q_i a_1 + (q_i - 1) a_{s_i}) a_{w'} = q_i a_{w'} + (q_i - 1) a_{s_i} a_{w'} = q_i a_{s_i w} + (q_i - 1) a_w$.

□

Corollary 6.4.7. *$H(G,B)$ is the K-algebra with generators a_w, $w \in W$, subject to the relations*

$$a_{s_i}a_w = \begin{cases} a_{s_iw} & \text{if } l(s_iw) > l(w) \\ q_ia_{s_iw} + (q_i - 1)a_w & \text{if } l(s_iw) < l(w). \end{cases}$$

for $s_i \in S$, $w \in W$.

6.5 Hecke Algebra Associated with a Harish Chandra Series

Let G be a finite group of Lie type defined in characteristic $p > 0$ (so, G is the fixed point subgroup of a connected reductive algebraic group $\mathbb{G}$ over $\bar{\mathbb{F}}_p$ under a Steinberg endomorphism), and assume that the BN-pair of G is split. Let U be the unipotent radical of B (i.e., the largest normal p-subgroup of B), so that $B = UT$ (where $T = B \cap N$). Let (W,S) be the Coxeter system corresponding to the BN-pair structure on G. Let k be an algebraically closed field of characteristic $r > 0$ with $r \neq p$. As in Section 4, let $\mathscr{P}_G = \{{}^nP_J \mid J \subseteq S,\ n \in N\}$ be the collection of parabolic subgroups in G, and let $\mathscr{L}_G = \{{}^nL_J \mid J \subseteq S,\ n \in N\}$ be the collection of Levi subgroups in G.

Following [31], we will work with left kG-modules in this section. Let $L \in \mathscr{L}_G$ and let X be a cuspidal irreducible (left) kL-module. Let R_L^G be the Harish-Chandra induction functor from the category of left kL-modules to the category of left kG-modules. We define

$$\mathscr{H}(L,X) := \mathrm{End}_{kG}(R_L^G(X))^{\mathrm{op}},$$

where $\mathrm{End}_{kG}(R_L^G(X))^{\mathrm{op}}$ denotes the opposite of the endomorphism algebra $\mathrm{End}_{kG}(R_L^G(X))$. The algebra $\mathscr{H}(L,X)$ is called the Hecke algebra associated with the pair (L,X).

The category of left kG-modules is related to the category of left $\mathscr{H}(L,X)$-modules via the Hom functor

$$\mathfrak{F}_{R_L^G(X)} : kG - \mathrm{mod} \to \mathscr{H}(L,X) - \mathrm{mod},\ Y \mapsto \mathrm{Hom}_{kG}(R_L^G(X), Y).$$

To simplify the notation, we will denote the Hom functor $\mathfrak{F}_{R_L^G(X)}$ by $\mathfrak{F}_k$.

The following theorem (which was first proved in [30]) describes the connection between the Harish-Chandra series $\mathrm{Irr}_k(G|(L,X))$ and the Hecke algebra $\mathscr{H}(L,X)$. We have chosen to include the proof of this result because some of the techniques used here are needed in the original proofs of this dissertation.

Theorem 6.5.1. *([31, Theorem 4.2.9]) Let $L \in \mathscr{L}^G$, and let X be a cuspidal irreducible (left) kL-module. Let $P \in \mathscr{P}_G$ be such that $P = U_P \rtimes L$. The functor $\mathfrak{F}_k$ induces a bijection*

$$\mathfrak{F}_k : \mathrm{Irr}_k(G|(L,X)) \overset{1-1}{\to} \mathrm{Irr}_k(\mathscr{H}(L,X)),$$

where $\mathrm{Irr}_k(\mathscr{H}(L,X))$ denotes the set of irreducible left $\mathscr{H}(L,X)$-modules, up to isomorphism.

Proof. We follow the proof given by Geck and Jacon [31, Theorem 4.2.9]. Let P_X be the projective cover of X (as a kL-module). Since the Harish-Chandra induction functor R_L^G is exact, the natural surjective kL-module homomorphism $P_X \twoheadrightarrow X$ induces a surjective kG-module homomorphism $R_L^G(P_X) \twoheadrightarrow R_L^G(X)$.

By [31, Lemma 4.1.7, Proposition 4.1.8], to prove the statement of the theorem, it suffices to prove that

$$\dim_k \operatorname{Hom}_{kG}(R_L^G(X), R_L^G(X)) = \dim_k \operatorname{Hom}_{kG}(R_L^G(P_X), R_L^G(X)).$$

First, using the adjointness of Harish-Chandra induction and restriction, we have

$$\operatorname{Hom}_{kG}(R_L^G(X), R_L^G(X)) \cong \operatorname{Hom}_{kL}(X, {}^*R_L^G(R_L^G(X))).$$

Now, by the Mackey decomposition,

$${}^*R_L^G(R_L^G(X)) \cong \bigoplus_{n\in D(P,P)} R^L_{{}^nL\cap L}({}^*R^{{}^nL}_{{}^nL\cap L}({}^nX)),$$

where $D(P,P)$ denotes a full set of (P,P)-double coset representatives in G. Therefore, we have

$$\operatorname{Hom}_{kG}(R_L^G(X), R_L^G(X)) \cong \bigoplus_{n\in D(P,P)} \operatorname{Hom}_{kL}(X, R^L_{{}^nL\cap L}({}^*R^{{}^nL}_{{}^nL\cap L}({}^nX))).$$

Since ${}^nX$ is a cuspidal k^nL-module for any $n \in D(P,P)$, ${}^*R^{{}^nL}_{{}^nL\cap L}({}^nX) = 0$ when ${}^nL \cap L \subsetneq {}^nL$. Now, for an element $n \in D(P,P)$, ${}^nL \cap L = L$ if and only if $n \in D(P,P) \cap N_G(L)$. Therefore,

$$\operatorname{Hom}_{kG}(R_L^G(X), R_L^G(X)) \cong \bigoplus_{n\in D(P,P)\cap N_G(L)} \operatorname{Hom}_{kL}(X, {}^nX).$$

A similar calculation shows that

$$\operatorname{Hom}_{kG}(R_L^G(P_X), R_L^G(X)) \cong \bigoplus_{n\in D(P,P)\cap N_G(L)} \operatorname{Hom}_{kL}(P_X, {}^nX).$$

But, since P_X is a projective cover of X, and ${}^nX$ is an irreducible kL-module when $n \in D(P,P) \cap N_G(L)$, $\operatorname{Hom}_{kL}(P_X, {}^nX) \cong \operatorname{Hom}_{kL}(X, {}^nX)$ and the desired result follows.

□

Geck and Jacon [31] also give the following semisimplicity criterion for $\mathscr{H}(L,X)$.

Proposition 6.5.2. *([31, Proposition 4.2.10]) Let L be a Levi subgroup in G and let X be a cuspidal irreducible (left) kL-module. Then, $R_L^G(X)$ is a semisimple kG-module if and only if $\mathscr{H}(L,X)$ is a semisimple algebra.*

Given a pair (L,X), where $L \in \mathscr{L}_G$ is a Levi subgroup of G and X is a cuspidal irreducible kG-module, let $\mathscr{W}(L,X) := \{n \in (N_G(L) \cap N)L \mid {}^nX \cong X\}/L$. Geck and Jacon [31] refer to $\mathscr{W}(L,X)$ as the inertia group of X.[8]

[8]Dipper and Du [19] refer to $\mathscr{W}(L,X)$ as the ramification group of $R_L^G(X)$.

Proposition 6.5.3. *Let $L \in \mathscr{L}_G$, and let X be a cuspidal irreducible (left) kL-module. Then,* $\dim_k \mathscr{H}(L,X) = |\mathscr{W}(L,X)|$.

We will now consider the special case with $L = T$ and $X = k$. Under these assumptions, we have $\mathrm{Irr}_k(G|(L,X)) = \mathrm{Irr}_k(G|(T,k)) = \mathrm{Irr}_k(G|B)$, which is the unipotent principal Harish-Chandra series. In this case, $R_L^G(X) = R_T^G(k) = k|_B^G$, and the associated Hecke algebra is $\mathscr{H}_k = \mathscr{H}(T,k) = \mathrm{End}_{kG}(k|_B^G)^{\mathrm{op}}$.

Note that the permutation module $k|_B^G$ is isomorphic to $kG\mathfrak{b}$, where $\mathfrak{b} = \sum_{b\in B} b$. Given an element $w \in W$, let $\dot{T}_w \in \mathscr{H}_k = \mathrm{End}_{kG}(k|_B^G)^{\mathrm{op}}$ be the kG-module homomorphism $\dot{T}_w : kG\mathfrak{b} \to kG\mathfrak{b}$ given by

$$\dot{T}_w(xB) = \sum_{yB\in kG\mathfrak{b} \text{ with } x^{-1}y\in BwB} yB$$

for any $x \in kG$. Now for any $s \in S$, let $q_s = |BsB/B|$, and let $\bar{q}_s = q_s 1_k$ be the image of the integer q_s in k. When $s,t \in S$ are W-conjugate, $\bar{q}_s = \bar{q}_t$ ([31, 4.3.1]). The Hecke algebra $\mathscr{H}_k$ has k-basis $\{\dot{T}_w \mid w \in W\}$, subject to the relations

$$\dot{T}_s\dot{T}_w = \begin{cases} \dot{T}_{sw} & \text{if } l(sw) > l(w), \\ \bar{q}_s\dot{T}_{sw} + (\bar{q}_s - 1)\dot{T}_w & \text{if } l(sw) < l(w) \end{cases}$$

for all $s \in S$ and $w \in W$ [31, 4.3.1]. (The presentation of $\mathscr{H}_k$ given above was originally found by Iwahori.)

We now give a modified presentation of $\mathscr{H}_k$. Since the field k is assumed to be large enough, there exist square roots $\bar{q}_s^{1/2} \in k$ for all $s \in S$, which satisfy $\bar{q}_s^{1/2} = \bar{q}_t^{1/2}$ when $s,t \in S$ are W-conjugate. Given any $s \in S$, let $T_s = \bar{q}_s^{-1/2}\dot{T}_s$. For an arbitrary element $w \in W$, let $T_w = T_{s_1}\cdots T_{s_n}$ when $w = s_1\cdots s_n$ ($s_1,\ldots,s_n \in S$) is a reduced expression for w. Then, the set $\{T_w \mid w \in W\}$ is a k-basis for $\mathscr{H}_k$, satisfying the relations

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (\bar{q}_s^{1/2} + \bar{q}_s^{-1/2})T_w & \text{if } l(sw) < l(w) \end{cases}$$

for all $s \in S$ and $w \in W$ [31, 4.3.1]. From the presentation of $\mathscr{H}_k$ given above, it follows that $\mathscr{H}_k$ is a specialization of a generic Hecke algebra, as defined in [25].

Remark 6.5.4. In [31], Geck and Jacon define the Hecke algebra $\mathscr{H}_k$ as $\mathrm{End}_{kG}(k_B^G)^{\mathrm{op}}$, where $k|_B^G$ is a left kG-module. With this definition, the Hom functor $\mathfrak{F}_k = \mathrm{Hom}_{kG}(k|_B^G, -)$ is a mapping from the category $kG-$ mod of left kG-modules to the category $\mathscr{H}_k-$ mod of left $\mathscr{H}_k$-modules. However, it is possible to define an analogous Hom functor relating the category of right kG-modules to the category of right modules for a Hecke algebra. Viewing $k|_B^G$ is a right kG-module, one can define the Hecke algebra as $\mathscr{H}_k = \mathrm{End}_{kG}(k|_B^G)$ (this is the definition used by Dipper and Du [19]). In this case, the Hom functor $\mathfrak{F}_k = \mathrm{Hom}_{kG}(k|_B^G, -)$

is a mapping from the category $\text{mod} - kG$ of right kG-modules to the category $\text{mod} - \mathscr{H}_k$ of right $\mathscr{H}_k$-modules.

7 Highest Weight Categories and Quasi-hereditary Algebras

7.1 Preliminaries

Let A be a finite dimensional algebra over an algebraically closed field k. Let $(\Lambda, \leq)$ be a finite poset which indexes the distinct isomorphism classes of irreducible objects in $A-\text{mod}$ (where $A-\text{mod}$ is the category of finite dimensional left A-modules). For $\lambda \in \Lambda$, let $L(\lambda) \in A-\text{mod}$ be an irreducible module in the class of λ. Let $P(\lambda)$ be the projective indecomposable module (PIM) with head $L(\lambda)$.

Definition 7.1.1. ([14, C.8]) $A-\text{mod}$, together with the poset structure $\leq$ on Λ, is called a *highest weight category* provided that the following conditions hold:

(HWC1) For $\lambda \in \Lambda$, there exists an object $\Delta(\lambda) \in A-\text{mod}$ with simple head $L(\lambda)$ and with the property that all composition factors $L(\mu)$ of the radical $\text{rad}(\Delta(\lambda))$ satisfy $\mu < \lambda$.

(HWC2) For $\lambda \in \Lambda$, there exists a filtration $P(\lambda) = F_0^\lambda \supset F_1^\lambda \supset \cdots \supset F_{t_\lambda}^\lambda = 0$ such that $F_0^l/F_1^\lambda \cong \Delta(\lambda)$ and, for $0 < i < t_\lambda$, $F_i^\lambda/F_{i+1}^\lambda \cong \Delta(\mu_i)$, for some $\mu_i \in \Lambda$ with $\mu_i > \lambda$.

If $A-\text{mod}$ is a highest weight category, the modules $\Delta(\lambda)$, $\lambda \in \Lambda$ are called the standard objects in $A-\text{mod}$. There is also a costandard object $\nabla(\lambda)$ in $A-\text{mod}$ for every element $\lambda \in \Lambda$ [14, pg. 707]. If $I(\lambda)$ is the injective envelope of $L(\lambda)$ in $A-\text{mod}$, then $I(\lambda)$ has a ∇-filtration with bottom section $\nabla(\lambda)$ and upper sections $\nabla(\mu)$, $\mu > \lambda$.

Example 7.1.2. Let A be a semisimple algebra over an algebraically closed field k, and let Λ be any set indexing the irreducible A-modules. We can define a partial order $\leq$ on Λ by setting $\leq = \{(\lambda,\lambda) | \lambda \in \Lambda\}$. For any $\lambda \in \Lambda$, we define $\Delta(\lambda) = \nabla(\lambda) = P(\lambda) = I(\lambda) = L(\lambda)$ (since A is semisimple, each irreducible A-module $L(\lambda)$ is projective and injective). Then, $A-\text{mod}$ (together with the poset structure $\leq$ on Λ is a highest weight category.

Proposition 7.1.3. *([14, C.2]) If $A-\text{mod}$ is a highest weight category with poset $(\Lambda, \leq)$ and A^{op} is the opposite algebra of A, then $A^{op}-mod$ is a highest weight category with the opposite poset Λ^{op} (obtained by reversing the relations in Λ).*

The category $A^{\text{op}}-\text{mod}$ may be identified with the category of finite dimensional right A-modules [14, pg. 707]. With this identification, the standard object of $A^{\text{op}}-\text{mod}$ corresponding to the weight $\lambda \in \Lambda$ is $\Delta^{\text{op}}(\lambda) = \nabla(\lambda)^* = \text{Hom}_k(\nabla(\lambda), k)$ (where $\nabla(\lambda)$ is the costandard object in $A-\text{mod}$ corresponding to λ). And, the costandard object of $A^{\text{op}}-\text{mod}$ corresponding to the weight $\lambda \in \Lambda$ is $\nabla^{\text{op}}(\lambda) = \Delta(\lambda)^* = \text{Hom}_k(\Delta(\lambda), k)$ (where $\Delta(\lambda)$ is the standard object in $A-\text{mod}$ corresponding to λ).

Proposition 7.1.4. *([14, Corollary C.11]) If A and B are finite dimensional k-algebras such that $A - \text{mod}$ and $B - \text{mod}$ are equivalent categories (i.e., if A and B are Morita equivalent k-algebras), then $A - \text{mod}$ is a highest weight category if and only if $B - \text{mod}$ is a highest weight category.*

Definition 7.1.5. A is a quasi-hereditary algebra if A-mod is a highest weight category.

Given the definition above, the result of [14, C.11] may be restated as follows.

Proposition 7.1.6. *If A and B are Morita equivalent finite-dimensional k-algebras, then A is a quasi-hereditary algebra if and only if B is a quasi-hereditary algebra.*

7.2 Homological Properties of the Standard and Costandard Objects

Let $A - \text{mod}$ be a highest weight category with poset Λ. We state without proof several homological properties of modules in $A - \text{mod}$.

Proposition 7.2.1. *([14, C.12]) Let $\lambda, \mu \in \Lambda$. If $\operatorname{Ext}^1_A(\Delta(\lambda), L(\mu)) \neq 0$, then $\mu > \lambda$. And, if $\operatorname{Ext}^1_A(L(\lambda), \nabla(\mu)) \neq 0$, then $\mu > \lambda$.*

Let $A - \text{mod}(\Delta)$ be the full subcategory of $A - \text{mod}$ consisting of all modules in $A - \text{mod}$ which have a Δ-filtration (i.e., modules which have a filtration in which each section is isomorphic to $\Delta(\lambda)$ for some $\lambda \in \Lambda$). Let $A - \text{mod}(\nabla)$ be the full subcategory of $A - \text{mod}$ consisting of all modules in $A - \text{mod}$ which have a ∇-filtration (i.e., modules which have a filtration in which each section is isomorphic to $\nabla(\lambda)$ for some $\lambda \in \Lambda$).

Proposition 7.2.2. *([23, A2.2])*

(1) A module $M \in A - \text{mod}$ has a Δ-filtration if and only if $\operatorname{Ext}^1_A(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$.

(2) A module $M \in A - \text{mod}$ has a ∇-filtration if and only if $\operatorname{Ext}^1_A(\Delta(\lambda), M) = 0$ for all $\lambda \in \Lambda$.

Proposition 7.2.3. *([14, C.13]) Let $A - \text{mod}$ be a highest weight category with poset Λ.*

(1) If $M \in A - \text{mod}(\Delta)$ and $N \in A - \text{mod}(\nabla)$, then $\operatorname{Ext}^n_A(M, N) = 0$ for all $n > 0$.

(2) Let $\lambda, \mu \in \Lambda$. If $\operatorname{Ext}^n_A(\Delta(\lambda), L(\mu)) \neq 0$ for some $n > 0$, then $\mu > \lambda$. So, if $\operatorname{Ext}^n_A(\Delta(\lambda), \Delta(\mu)) \neq 0$ for some $n > 0$, then $\mu > \lambda$.

(3) Let $\lambda, \mu \in \Lambda$. If $\operatorname{Ext}^n_A(L(\lambda), \nabla(\mu)) \neq 0$ for some $n > 0$, then $\lambda > \mu$. So, if $\operatorname{Ext}^n_A(\nabla(\lambda), \nabla(\mu)) \neq 0$ for some $n > 0$, then $\lambda > \mu$.

7.3 The Ringel Dual

The results of Sections 7.3-7.5 are not used directly in the original research presented in Parts III and IV of this dissertation, so the reader may choose to skip ahead to Chapter 8.

In this section, we follow [14, C.2]. Let $A - \text{mod}$ be a highest weight category with poset Λ. Let $A - \text{mod}(\Delta)$ be the subcategory of $A - \text{mod}$ consisting of modules which have a Δ-filtration, and let $A - \text{mod}(\nabla)$ be the subcategory of $A - \text{mod}$ consisting of modules which have a ∇-filtration. A module $M \in A{-}\text{mod}(\Delta) \cap A{-}\text{mod}(\nabla)$ is called a tilting module.

Given a weight $\lambda \in \Lambda$, there exists an indecomposable tilting module $X(\lambda) \in A - \text{mod}(\Delta) \cap A - \text{mod}(\nabla)$ such that $[X(\lambda) : L(\lambda)] = 1$ and for any composition factor $L(\mu)$ of $X(\lambda)$ with $\mu \neq \lambda$, $\mu < \lambda$. (Here, $[X(\lambda) : L(\lambda)]$ denotes the number of times that $L(\lambda)$ occurs as a composition factor of $X(\lambda)$). The indecomposable tilting module $X(\lambda)$ ($\lambda \in \Lambda$) is unique up to isomorphism. If $M \in A - \text{mod}(\Delta) \cap A - \text{mod}(\nabla)$ is any tilting module, then M has a decomposition as a direct sum of the indecomposable tilting modules $X(\lambda)$.

Let $X \in A - \text{mod}(\Delta) \cap A - \text{mod}(\nabla)$ be a tilting module which contains at least one direct summand isomorphic to $X(\lambda)$ for every weight $\lambda \in \Lambda$, and let $E = \text{End}_A(X)$. The algebra E is called the Ringel dual of A. The category $E - \text{mod}$ is a highest weight category with weight poset Λ^{op}.

7.4 Global Dimension of a Quasi-hereditary Algebra

Given a finite dimensional k-algebra A, let gl dim A denote the global dimension of A (i.e., the supremum of the set of projective dimensions of A-modules). And, given an A-module M, let $\text{pd}(M)$ denote the projective dimension of M.

Lemma 7.4.1. *If A is a finite dimensional k-algebra, then gl dim $A \leq 1$ if and only if A is a hereditary algebra (i.e., every submodule of a projective A-module is projective).*

Proof. Suppose first that gl dim $A \leq 1$. Let M be a projective A-module, and let N be a submodule of M. Then, there is a short exact sequence of A-modules $0 \to N \to M \to M/N \to 0$, which means that $\text{pd}(N) \leq \max(\text{pd}(M), \text{pd}(M/N) - 1)$. Now, since M is projective, $\text{pd}(M) = 0$. And, since gl dim $A \leq 1$, $\text{pd}(M/N) - 1 \leq 1 - 1 = 0$. Thus, $\text{pd}(N) = 0$, and it follows that N is projective.

Suppose now that A is a hereditary algebra. Let M be an A-module, and let $\phi : P \twoheadrightarrow M$ be a projective cover of M. Since $\text{Ker}(\phi)$ is a submodule of the projective A-module P and A is hereditary, $\text{Ker}(\phi)$ is a projective A-module, which means that $0 \to \text{Ker}(\phi) \to P \to M \to 0$ is a projective resolution of M. Thus, $\text{pd}(M) \leq 1$ for all A-modules M, and it follows that gl dim $A \leq 1$.

□

Proposition 7.4.2. *If A is a finite dimensional k-algebra with gl dim $A \leq 1$, then A is a quasi-hereditary algebra.*

Proof. We will show that $A-$ mod is a highest weight category. Since A is Artinian, we can write ${}_AA \cong P_1^{\oplus k_1} \oplus \cdots \oplus P_n^{\oplus k_n}$ for some positive integers k_i $(1 \leq i \leq n)$, where the P_i $(1 \leq i \leq n)$ are distinct projective indecomposable A-modules. For any i, $1 \leq i \leq n$, let $L_i = \text{head}(P_i)$. Then, the set $\{L(1), \ldots, L(n)\}$ is a full set of non-isomorphic irreducible A-modules. Let $\Lambda = \{1, \ldots, n\}$, and define a partial order $\leq'$ on Λ by $i \leq' j \Leftrightarrow P_i \subseteq P_j$. Given any $i \in \Lambda$, let $\Delta(i) = P(i)$. To show that A-mod is a highest weight category with poset Λ, it suffices to show that if $[\Delta(i) : L(j)] \neq 0$ for some $i, j \in \Lambda$, then $j \leq' i$. But, if $L(j)$ is a composition factor of $\Delta(i) = P_i$, then the projective A-module P_i has a submodule M with head $L(j)$. Since gl dim $A \leq 1$, A is a hereditary algebra, which means that M is projective. Therefore, $M \cong P_j$, and it follows that $P_j \subseteq P_i$ and $j \leq' i$. □

In fact, Dlab and Ringel [15] showed that a stronger result holds.

Proposition 7.4.3. *([15, Theorem 2]) If A is a finite dimensional k-algebra with gl dim $A \leq 2$, then A is a quasi-hereditary algebra.*

(Dlab and Ringel's result cannot be improved - when gl dim A=3, A need not be quasi-hereditary.)

Theorem 7.4.4. *If A is a quasi-hereditary algebra, then gl dim $A < \infty$.*

Proof. We follow Donkin's proof [23, A.2.3].

Suppose that the highest weight category $A-$ mod has finite weight poset Λ. Given $\lambda \in \Lambda$, let $l(\lambda)$ be the length of the longest chain $\lambda_0 < \lambda_1 < \cdots < \lambda_l = \lambda$ in Λ. And, let $l(\Lambda)$ be the maximum of the lengths $l(\lambda)$, $\lambda \in \Lambda$. We claim that $\text{Ext}^i_A(L(\lambda), L(\mu)) = 0$ for $\lambda, \mu \in \Lambda$ and $i > l(\lambda) + l(\mu)$. Following Donkin, we use induction on $l(\lambda) + l(\mu)$. For the base case, suppose $l(\lambda) + l(\mu) = 2$. Then, $l(\lambda) = 1$ and $l(\mu) = 1$, so $L(\lambda) = \Delta(\lambda)$ and $L(\mu) = \Delta(\mu)$. And, since $\mu \not> \lambda$, $\text{Ext}^i_A(\Delta(\lambda), \Delta(\mu)) = \text{Ext}^i_A(L(\lambda), L(\mu)) = 0$ for $i > l(\lambda) + l(\mu)$ by [14, C.13(2)].

For the inductive step, assume that $\text{Ext}^i_A(L(\lambda'), L(\mu')) = 0$ for all λ', μ' with $l(\lambda')+l(\mu') < l(\lambda)+l(\mu)$ and $i > l(\lambda')+l(\mu')$. Since $L(\lambda)$ is the head of $\Delta(\lambda)$, we have a short exact sequence $0 \to N \to \Delta(\lambda) \to L(\lambda) \to 0$. For every composition factor $L(\nu)$ of N, $\nu < \lambda$, which means that $l(\nu) < l(\lambda)$. Suppose $i > l(\lambda) + l(\mu)$. The long exact sequence in Ext induced by the short exact sequence $0 \to N \to \Delta(\lambda) \to L(\lambda) \to 0$ yields

$$\text{Ext}^{i-1}_A(N, L(\mu)) \to \text{Ext}^i_A(L(\lambda), L(\mu)) \to \text{Ext}^i_A(\Delta(\lambda), L(\mu)). \tag{7.1}$$

Suppose $L(\nu)$ is a composition factor of N. Then, $\nu < \lambda$, so that $l(\nu) + l(\mu) < l(\lambda) + l(\mu) < i$. And, since $l(\nu) \leq l(\lambda) - 1$, $i - 1 > l(\nu) + l(\mu)$. By the inductive hypothesis, $\mathrm{Ext}_A^{i-1}(L(\nu), L(\mu)) = 0$ for every composition factor $L(\nu)$ of N. Therefore, $\mathrm{Ext}_A^{i-1}(N, L(\mu)) = 0$, and (7.1) yields $\mathrm{Ext}_A^i(L(\lambda), L(\mu)) \cong \mathrm{Ext}_A^i(\Delta(\lambda), L(\mu))$. So, to prove that $\mathrm{Ext}_A^i(L(\lambda), L(\mu)) = 0$, it suffices to prove $\mathrm{Ext}_A^i(\Delta(\lambda), L(\mu)) = 0$.

Since $L(\mu)$ is the socle of $\nabla(\mu)$, we have a short exact sequence $0 \to L(\mu) \to \nabla(\mu) \to Q \to 0$, with every composition factor $L(\nu)$ of Q satisfying $\nu < \mu$. This short exact sequence induces a long exact sequence in Ext, which yields the exact sequence

$$\mathrm{Ext}_A^{i-1}(\Delta(\lambda), Q) \to \mathrm{Ext}_A^i(\Delta(\lambda), L(\mu)) \to \mathrm{Ext}_A^i(\Delta(\lambda), \nabla(\mu)). \tag{7.2}$$

If $L(\lambda')$ is a composition factor of $\Delta(\lambda)$ and $L(\nu)$ is a composition factor of Q, then $l(\lambda') + l(\nu) \leq l(\lambda) + l(\nu) < l(\lambda) + l(\mu)$. And, $l(\lambda) + l(\mu) < i$, so $l(\lambda') + l(\nu) < i - 1$ and $\mathrm{Ext}_A^{i-1}(L(\lambda'), L(\nu)) = 0$ by the inductive hypothesis. Therefore, $\mathrm{Ext}_A^{i-1}(\Delta(\lambda'), Q) = 0$. Also, $\mathrm{Ext}_A^i(\Delta(\lambda), \nabla(\mu)) = 0$ for $i > 0$ by [23, A.2.2(ii)]. Thus, (7.2) gives $\mathrm{Ext}_A^i(\Delta(\lambda), L(\mu)) = 0$.

We have proved that $\mathrm{Ext}_A^i(L(\lambda), L(\mu)) = 0$ for $\lambda, \mu \in \Lambda$ and $i > l(\lambda) + l(\mu)$. Therefore, we have gl dim $A \leq 2l(\Lambda) < \infty$. □

7.5 Heredity Ideals

In this section, we will discuss another characterization of a quasi-hereditary algebra.

Definition 7.5.1. An ideal $J \trianglelefteq A$ is idempotent if $J^2 = J$.

Theorem 7.5.2. *If $J \trianglelefteq A$ with $J^2 = J$, then there exists an idempotent e s.t. $J = AeA$.*

Definition 7.5.3. $J \trianglelefteq A$ is called a heredity ideal if

(1) $J = AeA$ for some idempotent e,

(2) ${}_AJ$ is projective, and

(3) eAe is a semisimple k-algebra.

Proposition 7.5.4. *Suppose J is a heredity ideal and M and N are A/J modules. Then, $\mathrm{Ext}_{A/J}^n(M, N) \cong \mathrm{Ext}_A^n(M, N)$.*

Proof. We follow Parshall's proof in [43, Lemma 1.3]. First, we construct a map $\mathrm{Ext}_{A/J}^n(M, N) \to \mathrm{Ext}_A^n(M, N)$. Let M be an A/J-module. The natural quotient map $A \to A/J$ defines an A-module structure on M, and we denote the resulting A-module by i_*M. A morphism $f : M \to N$ of A/J-modules defines a morphism $i_*f : i_*M \to i_*N$ of A-modules. The functor $i_* : A/J\text{-mod} \to A\text{-mod}$ is exact.

An element of $\mathrm{Ext}^n_{A/J}(M,N)$ can be represented by an equivalence class of n-extensions $0 \to N \to Q_1 \to \cdots \to Q_n \to M \to 0$ of A/J-modules. Applying i_* to an n-extension of A/J-modules yields an n-extension of A-modules, giving us a map $\mathrm{Ext}^n_{A/J}(M,N) \to \mathrm{Ext}^n_A(M,N)$. We will show that this map is an isomorphism.

We claim that it is enough to show that $\mathrm{Ext}^n_A(A/J,N) = 0$ for all A/J-modules N and for all $n > 0$. In this case, $\mathrm{Ext}^n_A(P,N) = 0$ for all projective A/J-modules P, all A/J-modules N, and all $n > 0$. Now, given an A/J-module M, fix a projective module P s.t. there is an epimorphism $P \twoheadrightarrow M$ and consider the resulting short exact sequence of A/J-modules $0 \to M' \to P \to M \to 0$. This short exact sequence induces the long exact sequences

$$\cdots \to \mathrm{Ext}^{n-1}_{A/J}(P,N) \to \mathrm{Ext}^{n-1}_{A/J}(M',N) \to \mathrm{Ext}^n_{A/J}(M,N) \to \mathrm{Ext}^n_{A/J}(P,N) = 0$$

$$\cdots \to \mathrm{Ext}^{n-1}_{A}(P,N) \to \mathrm{Ext}^{n-1}_{A}(M',N) \to \mathrm{Ext}^n_{A}(M,N) \to \mathrm{Ext}^n_{A}(P,N) = 0.$$

So, we have the following commutative diagram.

$$\begin{array}{ccccccc} \mathrm{Ext}^{n-1}_{A/J}(P,N) & \longrightarrow & \mathrm{Ext}^{n-1}_{A/J}(M',N) & \longrightarrow & \mathrm{Ext}^n_{A/J}(M,N) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Ext}^{n-1}_{A}(P,N) & \longrightarrow & \mathrm{Ext}^{n-1}_{A}(M',N) & \longrightarrow & \mathrm{Ext}^n_{A}(M,N) & \longrightarrow & 0 \end{array}$$

We proceed to show that $\mathrm{Ext}^n_{A/J}(M,N) \cong \mathrm{Ext}^n_A(M,N)$ by induction on n. We have $\mathrm{Hom}_{A/J}(M,N) \cong \mathrm{Hom}_A(M,N) = \mathrm{Hom}_A(i_*M, i_*N)$ by definition of the A-module structure of i_*M and i_*N. Suppose now that $n-1 \geq 0$ and $\mathrm{Ext}^{n-1}_{A/J}(M,N) \cong \mathrm{Ext}^{n-1}_A(M,N)$. We have $\mathrm{Hom}_{A/J}(P,N) \cong \mathrm{Hom}_A(P,N)$. And, for $n-1 > 0$, $\mathrm{Ext}^{n-1}_{A/J}(P,N) = 0$ and $\mathrm{Ext}^{n-1}_A(P,N) = 0$ since P is a projective A/J-module. It follows that the first two downward arrows of the commutative diagram above are isomorphisms, and we have $\mathrm{Ext}^n_{A/J}(M,N) \cong \mathrm{Ext}^n_A(M,N)$, as needed.

Thus, to prove the statement of the proposition, it remains to check that $\mathrm{Ext}^n_A(A/J,N) = 0$ for all A/J-modules N and for all $n > 0$. First, we show that $\mathrm{Ext}^n_A(J,N) = 0$ for all $n \geq 0$. In degree 0, we must check that $\mathrm{Hom}_A(J,N) = 0$. But, suppose $f \in \mathrm{Hom}_A(J,N)$. Since J is heredity, $J^2 = J$, so $f(J) = f(J^2) = Jf(J) = 0$ (here, $Jf(J) = 0$ since $f(J)$ is contained in the A/J-module N). And, for $n > 0$, $\mathrm{Ext}^n_A(J,N) = 0$ since J is projective as an A-module. Now, consider the short exact sequence $0 \to J \to A \to A/J \to 0$. For $n > 0$, the long exact sequence in Ext yields the exact sequence

$$0 = \mathrm{Ext}^{n-1}_A(J,N) \to \mathrm{Ext}^n_A(A/J,N) \to \mathrm{Ext}^n_A(A,N) = 0.$$

Therefore, $\mathrm{Ext}^n_A(A/J,N) = 0$.

□

Definition 7.5.5. A is a quasi-hereditary algebra if there is a sequence $0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = A$ of ideals s.t. J_i/J_{i-1} is heredity in A/J_{i-1} for $1 \leq i \leq n$.

8 The q-Schur algebra

Let $\mathcal{Z} = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials in an indeterminate t, and let $(W, S) = (\mathfrak{S}_m, S)$ be the Coxeter system in which $\mathfrak{S}_m$ is the symmetric group on m letters and $S = \{(1,2), (2,3), \ldots, (m-1, m)\}$ is the set of fundamental reflections. Let $\widetilde{H} = \widetilde{H}(\mathfrak{S}_m, \mathcal{Z})$ be the corresponding generic Hecke algebra (defined as in [9, pg. 24, (1)]).

Let V be a free $\mathcal{Z}$-module of rank $n > 0$ and let $\{v_1, \ldots, v_n\}$ be an ordered basis of V. Given a sequence $J = (j_1, \ldots, j_m)$ of integers with $1 \leq j_i \leq n$, let $v_J = v_{j_1} \otimes \cdots \otimes v_{j_m}$. The elements v_J give a basis of $V^{\otimes m}$. For $\sigma \in \mathfrak{S}_m$, let $J\sigma = (j_{\sigma^{-1}(1)}, \ldots, j_{\sigma^{-1}(m)})$. Then, for any $s \in S$, we can define an action of the generator τ_s of $\widetilde{H}$ on $V^{\otimes m}$ by

$$v_J \tau_s = \begin{cases} t v_{Js} & \text{if } j_i \leq j_{i+1} \\ v_{Js} + (t-1) v_J & \text{otherwise.} \end{cases}$$

This action extends to give a right action of $\widetilde{H}$ on $V^{\otimes m}$, and the endomorphism algebra $S_t(n, m) := \mathrm{End}_{\widetilde{H}}(V^{\otimes m})$ is the t-Schur algebra of bidegree (n, m) over $\mathcal{Z}$ (this is the definition given in [9, (8.2)]).

There is another description of the the tensor product $V^{\otimes m}$, involving partitions of m. Let $\Lambda(n, m)$ denote the set of compositions of m with at most n non-zero parts, and let $\Lambda^+(n, m)$ denote the set of partitions of m with at most n non-zero parts. Note that by rearranging parts, every composition λ corresponds to a unique partition Λ^+. To every composition $\lambda \in \Lambda(n, m)$, we may associate a subset $J(\lambda)$ of S ($J(\lambda)$ consists of those $s \in S$ which stabilize the rows of the standard tableau of shape λ described in [9, Section 8]. Then, as a right $\widetilde{H}$-module $V^{\otimes m} \cong \bigoplus_{\lambda \in \Lambda(n,m)} x_\lambda \widetilde{H}$. Now, $x_\lambda \widetilde{H} \cong x_\mu \widetilde{H}$ if $W_{J(\lambda)}$ and $W_{J(\mu)}$ are conjugate in W, which occurs if and only if $\lambda^+ = \mu^+$. In particular, for every $\lambda \in \Lambda(n, m)$, we have $x_\lambda \widetilde{H} \cong x_{\lambda^+} \widetilde{H}$. Therefore,

$$V^{\otimes m} \cong \bigoplus_{\lambda \in \Lambda^+(n,m)} x_\lambda \widetilde{H}^{\oplus k_\lambda} \tag{8.1}$$

for some non-negative integers k_λ, $\lambda \in \Lambda^+(n, m)$. And, it follows that the algebra $A = \mathrm{End}_{\widetilde{H}}(\bigoplus_{\lambda \in \Lambda^+(n,m)} x_\lambda \widetilde{H})$ is Morita equivalent to $S_t(n, m)$.

For a commutative ring R and a homomorphism $\mathcal{Z} \to R$, $t \mapsto q$, let $S_t(n, m) \otimes_{\mathcal{Z}} R = S_t(n, m)_R = S_q(n, m)$. Note that $S_q(n, m) \cong End_{\widetilde{H}(\mathfrak{S}_m, R, q)}(V_R^{\otimes m})$, where $\widetilde{H}(\mathfrak{S}_m, R, q)$ is the specialization of $\widetilde{H}$ to R. Moreover, if $R = k$ is a field and there is an algebra homomorphism $\mathcal{Z} \to k$, $t \mapsto q$, the category mod $- S_q(n, m)$ of right $S_q(n, m)$-modules is a highest weight category with poset $\Lambda^+(n, m)$ [24, Theorem 1]. The poset structure $\trianglelefteq$ on the set $\Lambda^+(n, m)$ is the dominance order, defined by $\lambda = (\lambda_1, \lambda_2, \ldots) \trianglelefteq \mu = (\mu_1, \mu_2, \ldots)$ if and only if $\lambda_1 \leq \mu_1, \lambda_1 + \lambda_2 \leq \mu_1 + \mu_2, \ldots$. For every $\lambda \in \Lambda^+(n, m)$, there exists a right $S_t(n, m)$-module

$\widetilde{\Delta}(\lambda)$ such that $\widetilde{\Delta}(\lambda)_k = \Delta(\lambda)$ is the standard object in mod $- S_q(n,m)$ corresponding to the partition λ. Similarly, for every $\lambda \in \Lambda^+(n,m)$, there exists a right $S_t(n,m)$-module $\widetilde{\nabla}(\lambda)$ such that $\widetilde{\nabla}(\lambda)_k = \nabla(\lambda)$ is the costandard object in mod $- S_q(n,m)$ corresponding to the partition λ. The irreducible $S_q(n,m)$-modules are indexed by $\Lambda^+(n,m)$. The irreducible $S_q(n,m)$-modules corresponding to a partition $\lambda \in \Lambda^+(n,m)$ will be denoted by $L^k(\lambda)$.

In this dissertation, we will work primarily with right $S_q(n,m)$-modules in order to make use of the Morita equivalence given in [9, Theorem 9.17]. The category $S_q(n,m) -$ mod of left $S_q(n,m)$-modules is also a highest weight category with poset $\Lambda^+(n,m)$. We will denote the standard object of $S_q(n,m) -$ mod corresponding to $\lambda \in \Lambda^+(n,m)$ by $\Delta^{\text{left}}(\lambda)$, and we will denote the costandard object of $S_q(n,m) -$ mod corresponding to $\lambda \in \Lambda^+(n,m)$ by $\nabla^{\text{left}}(\lambda)$. The standard objects of $S_q(n,m) -$ mod are related to the costandard objects of mod $- S_q(n,m)$ via the linear dual; for any $\lambda \in \Lambda^+(n,m)$, $\nabla(\lambda) = \Delta^{\text{left}}(\lambda)^*$. Similarly, the costandard objects of $S_q(n,m) -$ mod are related to the standard objects of mod $- S_q(n,m)$ via duality: for any $\lambda \in \Lambda^+(n,m)$, $\Delta(\lambda) = \nabla^{\text{left}}(\lambda)^*$ [14, pg. 707].

9 The Indexing of the Irreducible $k\mathrm{GL}_n(q)$-modules

Let q be a power of a prime p, and let $G = \mathrm{GL}_n(q)$ be the general linear group over the finite field $\mathbb{F}_q$ of q elements. Let B be the Borel subgroup of G consisting of all invertible upper triangular $n \times n$ matrices. Let U be the unipotent radical of B (that is, the subgroup of B consisting of invertible upper triangular matrices with 1's along the main diagonal), and let T be the maximal torus in B consisting of all invertible diagonal matrices.

Let k be an algebraically closed field of characteristic $r > 0$ such that $r \nmid q$. In this dissertation, it will be necessary to use results of Cline, Parshall, and Scott [9] together with those of Dipper [16], [17], Dipper and James [21], and Dipper and Du [19]. The authors of all of the papers referenced here work with right kG-modules. However, the indexing of the irreducible kG-modules is not consistent across these papers: in particular, the indexing used by CPS differs from the two indexings used by Dipper and James and Dipper and Du. In this section, we will present the three indexings of the irreducible kG-modules which are necessary to understand the results of CPS, DJ, and DD. Since the methods used by CPS in [9] are most relevant to the proofs appearing in Part IV of this dissertation, we will describe the indexing used by CPS in [9] in greater detail than the indexings used by DJ [21] and DD [19].

9.1 The First Indexing of the Irreducible $k\mathrm{GL}_n(q)$-Modules (CPS)

In this section, we will describe a Morita equivalence which Cline, Parshall, and Scott construct in Section 9 of [9] and describe how this Morita equivalence can be used to obtain an indexing of the irreducible right kG-modules. (CPS work with right, and not left, modules in [9].)

9.1.1 A Key Theorem

As above, let k be an algebraically closed field of characteristic $r > 0$. Let $(\mathcal{O}, K, k)$ be an r-modular system (so, $\mathcal{O}$ is a discrete valuation ring, K is the quotient field of $\mathcal{O}$, and k is the residue field). Assume that the quotient field K of $\mathcal{O}$ is large enough so that it is a splitting field for $\mathrm{GL}_n(q)$. We begin with the following key result.

Theorem 9.1.1. *([9, Theorem 9.2]) Let R be an $\mathcal{O}$-algebra which is free of finite rank over $\mathcal{O}$ and has the property that R_K is a semisimple algebra over K. Suppose that there is an exact sequence $0 \to N \to P \to M \to 0$ of right R-modules which are free of finite rank over $\mathcal{O}$, that the R_K-modules N_K and M_K have no composition factors in common, and that P is a projective R-module. Let $J = \mathrm{Ann}_R M$, and assume that every irreducible R/J-module lies in the head of the right R-module M. Then the functor $F(-) = \mathrm{Hom}_{R/J}(M, -)$ defines a Morita equivalence*

$$F : \mathrm{mod} - R/J \to \mathrm{mod} - \mathrm{End}_R(M)$$

of right module categories.

Following [9], we will construct an R-module M which satisfies the assumptions of Theorem 9.1.1.

9.1.2 A decomposition of $\mathcal{O}\mathrm{GL}_n(q)$ into sums of blocks

Let $\mathscr{C}$ denote a fixed set of representatives of the conjugacy classes in $\mathrm{GL}_n(q)$. The finite group $\mathrm{GL}_n(q)$ may be obtained as the set of fixed points of the algebraic group GL_n (over the algebraic closure of $\mathbb{F}_q$) under a Frobenius morphism. Let $\mathbb{G}_{ss}$ denote the set of semisimple elements in the algebraic group GL_n, and let $G_{ss} = \mathbb{G}_{ss} \cap \mathrm{GL}_n(q)$. We set $\mathscr{C}_{ss} = G_{ss} \cap \mathscr{C}$ (so, $\mathscr{C}_{ss}$ consists of the representatives of the conjugacy classes in $\mathrm{GL}_n(q)$ which are semisimple when viewed as elements of the algebraic group GL_n). Let $\mathscr{C}_{ss,r'}$ denote the set of elements in $\mathscr{C}_{ss}$ of order prime to r.

Given $s \in G_{ss}$, the centralizer of s is of the form $Z_G(s) \cong \prod_{i=1}^{m(s)} \mathrm{GL}_{n_i(s)}(q^{a_i(s)})$, where $\sum_{i=1}^{m(s)} a_i(s) n_i(s) = n$. For any $s \in G_{ss}$, we let $\underline{n}(s) = (n_1(s), \ldots, n_{m(s)}(s))$. Let $\Lambda^+(\underline{n}(s))$ be the set of multipartitions of $\underline{n}(s)$. An element $\lambda \in \Lambda^+(\underline{n}(s))$ is of the form $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m(s))})$, where $\lambda^{(1)} \vdash n_1(s)$, $\ldots$, $\lambda^{(m(s))} \vdash n_{m(s)}(s)$. The ordinary irreducible characters of $\mathrm{GL}_n(q)$ (which were first parameterized by Green) may be indexed by pairs (s, λ), where $s \in \mathscr{C}_{ss}$ and $\lambda \in \Lambda^+(\underline{n}(s))$ [9, pg. 29]. We will denote the irreducible character of $\mathrm{GL}_n(q)$ corresponding to the pair (s, λ) by $\chi_{s,\lambda}$.

The group algebra $\mathcal{O}\mathrm{GL}_n(q) = \mathcal{O}G$ decomposes as a direct sum of two-sided ideals called blocks, with every irreducible $\mathcal{O}G$-module belonging to a unique block. Given an element $s \in \mathscr{C}_{ss,r'}$, let $B_{s,G}$ be the sum of the blocks of $\mathcal{O}G$ which contain a character of the form $\chi_{st,\lambda}$, where $t \in Z_{\mathrm{GL}_n(q)}(s)$ is an r-element. $\mathcal{O}G$ has the direct sum decomposition

$$\mathcal{O}G = \bigoplus_{s \in \mathscr{C}_{ss,r'}} B_{s,G}.$$

9.1.3 An $\mathcal{O}G$-module which satisfies the hypotheses of the key theorem

Given an element $s \in \mathscr{C}_{ss,r'}$, the centralizer of s in $\mathrm{GL}_n(q)$ is of the form

$$Z_G(s) \cong \prod_{i=1}^{m(s)} \mathrm{GL}_{n_i(s)}(q^{a_i(s)}),$$

where $\sum_{i=1}^{m(s)} a_i(s)n_i(s) = n$. Let $L_s(q)$ be the subgroup of $G = \mathrm{GL}_n(q)$ defined by

$$L_s(q) = \prod_{i=1}^{m(s)} \mathrm{GL}_{a_i(s)n_i(s)}(q).$$

Let $H_s(q)$ be the subgroup of $L_s(q)$ defined by

$$H_s(q) = \prod_{i=1}^{m(s)} \mathrm{GL}_{a_i(s)}(q)^{n_i(s)}.$$

(So, $H_s(q)$ consists of block-diagonal matrices in $L_s(q)$.)

$L_s(q)$ is a Levi subgroup of a parabolic subgroup of G, and $H_s(q)$ is a Levi subgroup of a parabolic subgroup of $L_s(q)$. Therefore, we have Harish-Chandra induction functors

$$R_{H_s(q)}^{L_s(q)} : \mathrm{mod}-\mathcal{O}H_s(q) \to \mathrm{mod}-\mathcal{O}L_s(q), \text{ and}$$
$$R_{L_s(q)}^{\mathrm{GL}_n(q)} : \mathrm{mod}-\mathcal{O}L_s(q) \to \mathrm{mod}-\mathcal{O}\mathrm{GL}_n(q).$$

Now, to an element $s \in \mathscr{C}_{ss,r'}$, we can associate a $KH_s(q)$-module $C_K(s)$, which is a tensor product of certain irreducible cuspidal modules defined by Dipper and James (a more precise description of the module $C_K(s)$ can be found in [9, 9.6]). The representation $C_K(s)$ contains an $\mathcal{O}H_s(q)$-lattice $C_{\mathcal{O}}(s)$, and we can define a right $\mathcal{O}L_s(q)$-module associated to the element $s \in \mathscr{C}_{ss,r'}$ by

$$M_{s,L_s(q),\mathcal{O}} = R_{H_s(q)}^{L_s(q)} C_{\mathcal{O}}(s).$$

By [21, (2.17)], the endomorphism algebra of the $\mathcal{O}L_s(q)$-module $M_{s,L_s(q),\mathcal{O}}$ is a tensor product of Hecke algebras. Specifically,

$$\mathrm{End}_{\mathcal{O}L_s(q)}(M_{s,L_s(q),\mathcal{O}}) \cong \bigotimes_{i=1}^{m(s)} H(\mathfrak{S}_{n_i(s)}, \mathcal{O}, q^{a_i(s)}).$$

Now, given a multipartition $\lambda \vdash \underline{n}(s)$, let

$$y_\lambda = y_{\lambda^{(1)}} \otimes \cdots \otimes y_{\lambda^{(m(s))}} \in \bigotimes_{i=1}^{m(s)} H(\mathfrak{S}_{n_i(s)}, \mathcal{O}, q^{a_i(s)})$$

(the elements are defined in Section 6.2). Since $\mathrm{End}_{\mathcal{O}L_s(q)}(M_{s,L_s(q),\mathcal{O}}) \cong \bigotimes_{i=1}^{m(s)} H(\mathfrak{S}_{n_i(s)}, \mathcal{O}, q^{a_i(s)})$, the element $y_\lambda \in \bigotimes_{i=1}^{m(s)} H(\mathfrak{S}_{n_i(s)}, \mathcal{O}, q^{a_i(s)})$ acts on $M_{s,L_s(q),\mathcal{O}}$ as an $\mathcal{O}L_s(q)$-module endomorphism. In particular, $y_\lambda M_{s,L_s(q),\mathcal{O}}$ is an $\mathcal{O}$-submodule of $M_{s,L_s(q),\mathcal{O}}$. Let $\sqrt{y_\lambda M_{s,L_s(q),\mathcal{O}}}$ denote the smallest $\mathcal{O}$-submodule of $M_{s,L_s(q),\mathcal{O}}$ such that $y_\lambda M_{s,L_s(q),\mathcal{O}} \subseteq M_{s,L_s(q),\mathcal{O}}$ and $M_{s,L_s(q),\mathcal{O}}/\sqrt{y_\lambda M_{s,L_s(q),\mathcal{O}}}$ is $\mathcal{O}$-torsion free.

For any $1 \le i \le m(s)$, we have $\underset{\mu \in \Lambda(n_i(s),n_i(s))}{\oplus} x_\mu H \cong \underset{\mu \in \Lambda^+(n_i(s),n_i(s))}{\oplus} x_\mu H^{\oplus k_\mu(n_i(s),n_i(s))}$ for some positive integers $k_\mu(n_i(s), n_i(s))$ (the integers $k_\mu(n_i(s), n_i(s))$ are defined in (8.1)). Given a multipartition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m(s))}) \vdash \underline{n}(s)$, let $m_\lambda = \prod_{i=1}^{m(s)} k_{\lambda^{(i)}}(n_i(s), n_i(s))$ and let

$$\widehat{M}_{s,L_s(q),\mathcal{O}} = \underset{\lambda \vdash \underline{n}(s)}{\oplus} \sqrt{y_\lambda M_{s,L_s(q),\mathcal{O}}}^{\oplus m_\lambda}.$$

Then, $\widehat{M}_{s,L_s(q),\mathcal{O}}$ is an $\mathcal{O}L_s(q)$-module, and by [9, Lemma 9.11],

$$\mathrm{End}_{\mathcal{O}L_s(q)}(\widehat{M}_{s,L_s(q),\mathcal{O}}) \cong \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_{\mathcal{O}}.$$

We define an $\mathcal{O}\mathrm{GL}_n(q)$-module $\widehat{M}_{s,\mathrm{GL}_n(q),\mathcal{O}}$ by

$$\widehat{M}_{s,\mathrm{GL}_n(q),\mathcal{O}} = R_{L_s(q)}^{\mathrm{GL}_n(q)} \widehat{M}_{s,L_s(q),\mathcal{O}}. \tag{9.1}$$

(To simplify the notation, we will denote $\widehat{M}_{s,\mathrm{GL}_n(q),\mathcal{O}}$ simply by $\widehat{M}_{s,G,\mathcal{O}}$.) The endomorphism algebra of $\widehat{M}_{s,G,\mathcal{O}}$ over $\mathcal{O}\mathrm{GL}_n(q)$ is

$$\mathrm{End}_{\mathcal{O}\mathrm{GL}_n(q)}(\widehat{M}_{s,G,\mathcal{O}}) \cong \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_{\mathcal{O}}.$$

9.1.4 A Morita Equivalence for $\mathrm{GL}_n(q)$.

Cline, Parshall, and Scott [9] show that for any $s \in \mathscr{C}_{ss,r'}$, the $\mathcal{O}\mathrm{GL}_n(q)$-module $\widehat{M}_{s,G,\mathcal{O}}$ defined in (9.1) satisfies the hypotheses of Theorem 9.1.1 [9, Lemma 9.15]. Viewing $\widehat{M}_{s,G,\mathcal{O}}$ as a module for the subalgebra $B_{s,G}$ of $\mathcal{O}\mathrm{GL}_n(q)$, let $J_s = \mathrm{Ann}_{B_{s,G}}(\widehat{M}_{s,G,\mathcal{O}})$. Then, applying Theorem 9.1.1 with $R = B_{s,G}$, $J = J_s$, and $M = \widehat{M}_{s,G,\mathcal{O}}$, we see that the functor

$$F_s(-) = \mathrm{Hom}_{B_{s,G}/J_s}(\widehat{M}_{s,G,\mathcal{O}}, -) : \mathrm{mod} - B_{s,G}/J_s \to \mathrm{mod} - \mathrm{End}_{B_{s,G}}(\widehat{M}_{s,G,\mathcal{O}})$$

gives a Morita equivalence.

In fact, since $\mathrm{End}_{B_{s,G}}(\widehat{M}_{s,G,\mathcal{O}}) \cong \mathrm{End}_{\mathcal{O}\mathrm{GL}_n(q)}(\widehat{M}_{s,G,\mathcal{O}}) \cong \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_{\mathcal{O}}$, the functor F_s gives a Morita equivalence

$$\mathrm{mod} - B_{s,\mathrm{GL}_n(q)}/J_s(q) \xrightarrow{\sim} \mathrm{mod} - \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_{\mathcal{O}}. \tag{9.2}$$

Let $J = \sum_{s \in \mathscr{C}_{ss,r'}} J_s$. Since the group algebra $\mathcal{O}\mathrm{GL}_n(q)$ may be decomposed as

$$\mathcal{O}\mathrm{GL}_n(q) = \underset{s \in \mathscr{C}_{ss,r'}}{\oplus} B_{s,G},$$

it follows that $\mathcal{O}\mathrm{GL}_n(q)/J \cong \underset{s\in\mathscr{C}_{ss,r'}}{\oplus} B_{s,G}/J_s$. Therefore, taking direct sums over $s \in \mathscr{C}_{ss,r'}$ in (9.2), we have a Morita equivalence

$$F(-) = \mathrm{Hom}_{\mathcal{O}\mathrm{GL}_n(q)/J}(\underset{s\in\mathscr{C}_{ss,r'}}{\oplus} \widehat{M}_{s,G,\mathcal{O}}, -) :$$

$$\mathrm{mod} - \mathcal{O}\mathrm{GL}_n(q)/J \to \mathrm{mod} - \underset{s\in\mathscr{C}_{ss,r'}}{\oplus} \overset{m(s)}{\underset{i=1}{\otimes}} S_{q^{a_i(s)}}(n_i(s), n_i(s))_{\mathcal{O}}.$$

By [9, Theorem 9.17], this Morita equivalence remains valid upon base change to k; so, the category of right $k\mathrm{GL}_n(q)/J_k$-modules is Morita equivalent to the category of right $\underset{s\in\mathscr{C}_{ss,r'}}{\oplus} \overset{m(s)}{\underset{i=1}{\otimes}} S_{q^{a_i(s)}}(n_i(s), n_i(s))_k$-modules. More precisely, we have a Morita equivalence

$$\bar{F}(-) = \mathrm{Hom}_{k\mathrm{GL}_n(q)/J_k}(\underset{s\in\mathscr{C}_{ss,r'}}{\oplus} \widehat{M}_{s,G,k}, -) :$$

$$\mathrm{mod} - k\mathrm{GL}_n(q)/J_k \to \mathrm{mod} - \underset{s\in\mathscr{C}_{ss,r'}}{\oplus} \overset{m(s)}{\underset{i=1}{\otimes}} S_{q^{a_i(s)}}(n_i(s), n_i(s))_k$$

(where $\widehat{M}_{s,G,k} = \widehat{M}_{s,G,\mathcal{O}} \otimes_{\mathcal{O}} k$).

9.1.5 An Indexing of the Irreducible $k\mathrm{GL}_n(q)$-modules

By [9, Theorem 9.17], the algebras $k\mathrm{GL}_n(q)$ and $k\mathrm{GL}_n(q)/J_k$ have the same irreducible modules. In [9] (pg. 35), CPS use the Morita equivalence

$$\bar{F}(-) : \mathrm{mod} - k\mathrm{GL}_n(q)/J_k \to \mathrm{mod} - \underset{s\in\mathscr{C}_{ss,r'}}{\oplus} \overset{m(s)}{\underset{i=1}{\otimes}} S_{q^{a_i(s)}}(n_i(s), n_i(s))_k$$

to index the irreducible $k\mathrm{GL}_n(q)$-modules. Since the irreducible $S_{q^{a_i(s)}}(n_i(s), n_i(s))_k$-modules are indexed by the set $\Lambda^+(n_i(s))$ of partitions of $n_i(s)$, the irreducible $\overset{m(s)}{\underset{i=1}{\otimes}} S_{q^{a_i(s)}}(n_i(s), n_i(s))_k$-modules are indexed by the set of multipartitions $\Lambda^+(\underline{n}(s))$. Therefore, the irreducible $\underset{s\in\mathscr{C}_{ss,r'}}{\oplus} \overset{m(s)}{\underset{i=1}{\otimes}} S_{q^{a_i(s)}}(n_i(s), n_i(s))_k$-modules (and, consequently, the irreducible $k\mathrm{GL}_n(q)$-modules) are indexed by pairs (s, λ), where $s \in \mathscr{C}_{ss,r'}$ and λ is a multipartition of $\underline{n}(s)$. We will denote the irreducible $k\mathrm{GL}_n(q)$-module corresponding to the pair (s, λ) by $D(s, \lambda)$.

A special class of irreducible kG-modules is obtained by taking $s = 1$ above. Since $Z_{\mathrm{GL}_n(q)}(1) = \mathrm{GL}_n(q)$, $\underline{n}(1) = (n)$. Therefore, the irreducible kG-modules corresponding to the element $s = 1$ may be labeled as $D(1, \lambda)$, with λ a partition of n. By [45, Theorem 2.1], the irreducible kG-modules $D(1, \lambda)$, $\lambda \vdash n$, are precisely the composition factors of the permutation module $k|_B^G$. (An irreducible module $D(1, \lambda)$ may appear more than once as a composition factor of $k|_B^G$.) By [9, Remark 9.18(b)], the trivial module k may be parameterized as $D(1, (1^n))$, where (1^n) is the partition of n with each part equal to 1.

9.2 The Second Indexing of the Irreducible $k\mathrm{GL}_n(q)$-Modules

In this section, we will describe the indexing of the irreducible kG-modules given in [19, (4.2.3)] (where $G = \mathrm{GL}_n(q)$). This indexing is based on the work of Dipper and James [22] and Dipper [17].

For any semisimple element $s \in G$, let L_s denote the corresponding standard Levi subgroup of G (defined in [19, (4.1.2)]). By parts (3) and (4) of [19, 4.2.3], every Harish-Chandra series of irreducible right kG-modules is of the form $\mathrm{Irr}_k(G|(L_s, M_s))$, where M_s is the cuspidal irreducible right kL_s-module defined in [19, Section 4.1]. Given any semisimple element $s \in G$, let $W(L_s, M_s)$ be the ramification group of $R^G_{L_s}(M_s)$ and let $\mathscr{H}_s = \mathrm{End}_{kG}(R^G_{L_s}(M_s))$ be the Hecke algebra (over k) associated with $W(L_s, M_s)$. Suppose that L_s is of the form $L_s = \prod_i \mathrm{GL}_{d_i}(q)^{\times m_i}$ for some integers d_i and m_i, and let $\vec{m}_s = (m_1, m_2, \ldots)$. Let $\Lambda^+(\vec{m}_s)$ denote the set of multipartitions of $\vec{m}_s$. Also, let $\vec{l_s} = (l^{(1)}, l^{(2)}, \ldots)$, where $l^{(i)}$ is the minimal positive integer such that $1 + q^{d_i} + q^{2d_i} + \cdots + q^{(l^{(i)}-1)d_i} \equiv 0 \pmod{r}$.

By [19, (4.2.2,(1))], for every $\vec{\lambda} \in \Lambda^+(\vec{m}_s)$, there is a right $\mathscr{H}_s$-module $S_{\vec{\lambda}}$ called a Specht module. By [19, (4.2.2,(2))], if $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots)$ is $\vec{l_s}$-regular (meaning that λ_i is $l^{(i)}$-regular[9] for all i), then $S_{\vec{\lambda}}$ has a unique maximal submodule, with the corresponding factor module denoted by $D_{\vec{\lambda}}$. The set $\{D_{\vec{\lambda}} \mid \vec{\lambda} \vdash \vec{m}_s, \vec{\lambda} \text{ is } \vec{l_s} - \text{regular}\}$ is a complete set of non-isomorphic irreducible $\mathscr{H}_s$-modules.

Now, the category of right kG-modules is connected to the category of right $\mathscr{H}_s$-modules via the "Hecke functors." First, there is a functor $H_s : \mathrm{mod} - kG \to \mathrm{mod} - \mathscr{H}_s$, defined as follows. Let $\beta : \mathcal{Q} \to M_s$ be a fixed minimal projective cover of M_s (as a right kL_s-module), and let $\mathcal{P} = R^G_{L_s}(\mathcal{Q})$. Let J_β be the ideal of $\mathrm{End}_{kG}(\mathcal{P})$ consisting of the endomorphisms of $\mathcal{P}$ with image contained in the kernel of $R^G_{L_s}(\beta) : \mathcal{P} \to R^G_{L_s}(M_s)$. The functor H_s is defined as $H_s = \mathrm{Hom}_{kG}(\mathcal{P}, -)/\mathrm{Hom}_{kG}(\mathcal{P}, -)J_\beta$. (Given a right kG-module V, the right $\mathscr{H}_s$-module structure of $H_s(V)$ is described in [16, 2.2].) If V and V' are right kG-modules and $f : V \to V'$ is a right kG-module homomorphism, then there is a map $\mathrm{Hom}_{kG}(\mathcal{P}, V) \to \mathrm{Hom}_{kG}(\mathcal{P}, V')$ given by $\phi \to f\phi$ for any $\phi \in \mathrm{Hom}_{kG}(\mathcal{P}, V)$. As discussed in [16, 2.2], this map induces an $\mathscr{H}_s$-module homomorphism $H_s(f) : H_s(V) \to H_s(V')$.

The functor H_s has two right inverses, which are denoted by $\hat{H}$ and $\check{H}$ in [19]. The functor $\hat{H} : \mathrm{mod} - \mathscr{H}_s \to \mathrm{mod} - kG$ is defined by $\hat{H}(E) = E \otimes_{\mathscr{H}_s} R^G_{L_s}(M_s)$ for any $E \in \mathrm{mod} - \mathscr{H}_s$. The functor $\check{H} : \mathrm{mod} - \mathscr{H}_s \to \mathrm{mod} - kG$ is defined as follows. Given $E \in \mathrm{mod} - \mathscr{H}_s$, let $t_{\mathcal{P}}(E \otimes_{\mathscr{H}_s} R^G_{L_s}(M_s))$ denote the kG-submodule of $E \otimes_{\mathscr{H}_s} R^G_{L_s}(M_s)$ which is maximal with the property $\mathrm{Hom}_{kG}(\mathcal{P}, t_{\mathcal{P}}(E \otimes_{\mathscr{H}_s} R^G_{L_s}(M_s))) = 0$. With this definition, $\check{H}(E) = (E \otimes_{\mathscr{H}_s} R^G_{L_s}(M_s))/t_{\mathcal{P}}(E \otimes_{\mathscr{H}_s} R^G_{L_s}(M_s))$. Collectively, the functors H_s, $\hat{H}$, and $\check{H}$ are known as the Hecke functors.

[9] A partition λ of n is called l-regular if every part of λ appears less than l times.

Given a semisimple element $s \in G$ and multipartition $\vec{\lambda} \vdash \vec{m}_s$, there exists an irreducible right $\mathbb{C}G$-module $S_{\mathbb{C}}(s, \vec{\lambda})$ indexed by $(s, \vec{\lambda})$ (see [19, (4.2.3, (1))]). Let $S(s, \vec{\lambda})$ denote the reduction modulo r of $S_{\mathbb{C}}(s, \vec{\lambda})$. By [21, (3.1)], $S(s, \vec{\lambda})$ has a unique maximal submodule. The corresponding factor module, which we will denote by $D^1(s, \lambda)$, is an irreducible right kG-module.[10]

The modules described above behave particularly well under the Hecke functors. Given any semisimple element s in G and multipartition $\vec{\lambda} \vdash \vec{m}_s$, we have $H_s(S(s, \vec{\lambda})) = S_{\vec{\lambda}}$. If, in addition, $\vec{\lambda}$ is $\vec{l}_s$-regular, then $D^1(s, \vec{\lambda}) = \check{H}_s(D_{\vec{\lambda}})$.

Before we can give a full classification of the irreducible kG-modules, we must define an equivalence relation on the set of semisimple elements of G. Given such a semisimple element s, we can srite $s = s'y$, where s' has order prime to r and y is an r-element. As in [19], we will say that the semisimple elements s and t of G are r-equivalent (written $s \sim_r t$) if $s' = t'$ and $L_s = L_t$. Let $\mathscr{C}_r$ denote a full set of representatives of the equivalence classes of the semisimple elements of G under $\sim_r$. Then, by [19, (4.2.3, (4))],

$$\{D^1(s, \vec{\lambda}) \mid s \in \mathscr{C}_r,\ \lambda \vdash \vec{m}_s \text{ is } \vec{l}_s - \text{regular}\}$$

is a complete set of non-isomorphic irreducible kG-modules.

9.3 The Third Indexing of the Irreducible $k\mathrm{GL}_n(q)$-Modules

In this section, we will describe the indexing of the irreducible kG-modules given in [19, (4.2.11, (3))]. This indexing is based on the work of Dipper and James [22] and Dipper [17].

By [19, Theorem 4.2.7], there exits a q-Schur algebra $\mathscr{S}_s$ (over k) corresponding to the Hecke algebra $\mathscr{H}_s$ for every semisimple $s \in G$. The category of right kG-modules is related to the category of right $\mathscr{S}_s$-modules via the "q-Schur functors" $S_s : \mathrm{mod} - kG \to \mathrm{mod} - \mathscr{S}_s$, $\hat{S}_s : \mathrm{mod} - \mathscr{S}_s \to \mathrm{mod} - kG$, and $\check{S}_s : \mathrm{mod} - \mathscr{S}_s \to \mathrm{mod} - kG$. Given a semisimple element s in G, the irreducible right $\mathscr{S}_s$-modules are indexed by the multipartitions $\vec{\lambda}$ of $\vec{m}_s$; given a multipartition $\vec{\lambda} \vdash \vec{m}_s$, let $L^k(\lambda)$ denote the corresponding irreducible $\mathscr{S}_s$-module. For any $\vec{\lambda} \vdash \vec{m}_s$, $\check{S}_s(L^k(\lambda))$ is an irreducible kG-module, which we will denote by $D^2(s, \lambda')$.[11]

Let $\mathscr{C}_{ss,r'}$ denote a set of representatives of semisimple r-regular conjugacy classes in G. Then,

$$\{D^2(s, \vec{\lambda}) \mid s \in \mathscr{C}_{ss,r'},\ \vec{\lambda} \vdash \vec{m}_s\}$$

[10] In [19], the irreducible kG-module $D^1(s, \lambda)$ is denoted by $D(s, \lambda)$. We have chosen to use the notation $D^1(s, \lambda)$ instead in order to distinguish this indexing from that used by CPS in [9].

[11] In [19], $D^2(s, \lambda')$ is denoted by $D(s, \lambda')$.

is a complete set of non-isomorphic irreducible kG-modules.

9.4 Indexing of the Irreducible $k\mathrm{GL}_n(q)$-Modules belonging to the Unipotent Principal Series

In this section, we will index the irreducible right kG-modules belonging to the unipotent principal series $\mathrm{Irr}_k(G|B)$. Define an integer $l \in \mathbb{Z}^+$ by

$$l = \begin{cases} r & \text{if } |q \pmod r| = 1 \\ |q \pmod r| & \text{if } |q \pmod r| > 1, \end{cases}$$

where $|q \pmod r|$ denotes the multiplicative order of q mod r. By [19, (4.2.3, (5))], the irreducible kG-modules belonging to $\mathrm{Irr}_k(G|B)$ are indexed by partitions of n. In particular, $\mathrm{Irr}_k(G|B) = \{D^1(1,\lambda) \mid \lambda \vdash n,\ \lambda \text{ is } l-\text{regular}\}$. By [19, (4.2.11, (3))], $D^1(1,\lambda) = D^2(1,\lambda)$ for any l-regular partition $\lambda \vdash n$.

To prove the main results of Part IV of this dissertation, we need to determine the CPS parameterization of an irreducible kG-modules $D^1(1,\lambda)$ when $\lambda \vdash n$ is l-regular. To do this, we will fix an l-regular partition $\lambda \vdash n$ and apply the CPS functor $\bar{F}$ to the irreducible kG-module $D^1(1,\lambda)$. Since $D^1(1,\lambda) \in \mathrm{Irr}_k(G|B)$, $D^1(1,\lambda)$ is in the head of $k|_B^G$, which means that $\mathrm{Hom}_{kG}(k|_B^G, D^1(1,\lambda)) \neq 0$. Since $k|_B^G$ is a direct summand of the module $\widehat{M}_{1,G,k}$ (by [9, Remark 9.18 (b)]) and $\bar{F}(D^1(1,\lambda))$ is irreducible, we can compute

$$\begin{aligned} \bar{F}(D^1(1,\lambda)) &= \mathrm{Hom}_{kG/J_k}\Big(\bigoplus_{s \in \mathscr{C}_{ss,r'}} \widehat{M}_{s,G,k}, D^1(1,\lambda)\Big) \\ &\cong \mathrm{Hom}_{kG}\Big(\bigoplus_{s \in \mathscr{C}_{ss,r'}} \widehat{M}_{s,G,k}, D^1(1,\lambda)\Big) \\ &\cong \mathrm{Hom}_{kG}(k|_B^G, D^1(1,\lambda)). \end{aligned}$$

Since λ is l-regular, $D^1(1,\lambda) = D^2(1,\lambda)$ and $\mathrm{Hom}_{kG}(k|_B^G, D^1(1,\lambda)) \cong \mathrm{Hom}_{kG}(k|_B^G, D^2(1,\lambda))$. Tracing through the definition of the functor $S_1 : \mathrm{mod}-kG \to \mathrm{mod}-S_q(n,n)_k$ given in [17, Section 6] (and, using the fact that $S_1(D^2(1,\lambda))$ must be an irreducible $S_q(n,n)_k$-module), we find that $\mathrm{Hom}_{kG}(k|_B^G, D^2(1,\lambda)) \cong S_1(D^2(1,\lambda))$. But, the functor $\check{S}_1$ is a right inverse of S_1, which means that $S_1(D^2(1,\lambda)) \cong S_1(\check{S}_1(L^k(\lambda'))) \cong L^k(\lambda')$. Therefore, we have shown that $\bar{F}(D^1(1,\lambda)) \cong L^k(\lambda')$, and it follows that $D^1(1,\lambda) = D(1,\lambda')$ in the indexing of CPS when λ is an l-regular partition of n. In particular, $\mathrm{Irr}_k(G|B) = \{D(1,\lambda) \mid \lambda \text{ is l}-\text{restricted}\}$ in the CPS indexing.[12]

[12] A partition λ of n is called l-restricted when its dual partition λ' is l-regular.

Part II

Generalizing the Results of Guralnick and Tiep to Find Bounds on the Dimension of Ext^1 between Irreducible Modules

We start with a remark concerning our use of the terms “Borel subgroup,” “maximal torus,” and “unipotent radical” in the next three parts of this dissertation. Most readers will be familiar with these terms from the theory of algebraic groups; here, though, we will not be using this terminology in the sense of algebraic groups. Let G be a finite group of Lie type having a split BN-pair in characteristic p, let $T = B \cap N$, and let $U = O_p(B)$ be the largest normal p-subgroup of B. Following [35], we will refer to B as a Borel subgroup of G, T a maximal torus, and U a unipotent radical. B is not a Borel subgroup in the sense of algebraic groups any more than T is a maximal torus or U is a unipotent radical. However, this terminology has precedent in the literature and proves to be very convenient when discussing the cohomology of finite groups of Lie type.

10 A Summary of the Results of Guralnick and Tiep [35]

10.1 Introduction

Much of the original work of this dissertation is concerned with generalizing the results of Guralnick and Tiep [35]. This section summarizes these results. We will state many of the results of [35] without proof; however, we will provide (Guralnick and Tiep's) proofs of the results which were particularly useful to our generalization. In many cases, we will include details omitted in the original proofs of [35]. It is the hope of the author that this chapter will help readers gain a deeper understanding of [35]. Otherwise, the reader may choose to skip this chapter and proceed to Chapter 11.

Let k be an algebraically closed field of characteristic r. Let G be a finite group of Lie type over $\mathbb{F}_q$, where q is a power of a prime p with $r \neq p$. Let B be a Borel subgroup of G with unipotent radical Q and maximal torus T, and let $W = N_G(T)/T$ be the Weyl group of G with respect to T.[13] Let e denote the twisted rank of G. (As long as q is sufficiently large,

[13] To be consistent with Guralnick and Tiep's notation, we will denote the unipotent radical of B by Q in this section. In all other sections, we will denote the unipotent radical of B by U.

e is the rank of T as an abelian group.) We remark that in [35], Guralnick and Tiep work with left kG-modules. Hence, all kG-modules we consider in Part II of this dissertation will be assumed to be left kG-modules.

Remark 10.1.1. (An explanation of the notion of "rank")
In [33, Part I, Chapter 1, Section 11], Gorenstein, Lyons, and Solomon give the following description of the rank of a group of Lie type. Every group $G(q)$ of Lie type (q a power of a prime p) has a distinguished presentation (called the Steinberg presentation), with generators being certain elements of the root subgroups X_α, where α runs over the roots of the root system Σ of the associated complex Lie algebra. When $G(q)$ is an untiwsted group, the X_α are isomorphic to the additive group of the finite field $\mathbb{F}_q$. When $G(q)$ is twisted, the root subgroups X_α have more complicated structures, and the root system Σ may be non-reduced. The (Lie) rank of $G(q)$ is defined to be the rank of the root system Σ, i.e. the dimension of the Euclidean space $\mathbb{R}\Sigma$. The Lie rank of $G(q)$ is also referred to as the "twisted" rank of $G(q)$ (as is done in [35]). There is also a notion of an "untwisted" rank of a finite group $G(q)$ of Lie type. The untwisted rank of an untwisted group of Lie type is simply its (twisted) rank. And, the untwisted Lie rank of of a twisted group of Lie type is the Lie rank of the ambient algebraic group; i.e., it is the Lie rank of the untwisted group which was twisted to form $G(q)$.

The permutation module $\mathscr{L} = k|_B^G$ is a key ingredient in most of the results proved in [35]. We note several important features of $\mathscr{L}$ which will be useful to us. First, $\mathscr{L}$ is a self-dual module. By Frobenius reciprocity, the G-fixed point subspace of $\mathscr{L}$ is $\mathscr{L}^G \cong \mathrm{Hom}_{kG}(k, \mathscr{L}) \cong \mathrm{Hom}_{kB}(k, k) \cong k$ (since G is finite, induction and coinduction from B to G coincide, which means that Res_B^G is both right and left adjoint to Ind_B^G). We also claim that $\mathscr{L}$ has a unique submodule $\mathscr{L}^0$ of codimension 1. By Frobenius reciprocity, $\mathrm{Hom}_G(\mathscr{L}, k) \cong \mathrm{Hom}_B(k, k) = k$. Choose an element $\phi \in \mathrm{Hom}_G(\mathscr{L}, k)$ that generates $\mathrm{Hom}_G(\mathscr{L}, k)$ over k, and let $\mathscr{L}^0 = \mathrm{Ker}\,\phi$. Since $\phi \neq 0$ and k is an irreducible G-module, $\mathscr{L}^0$ is a maximal submodule of $\mathscr{L}$ of codimension 1. To show that $\mathscr{L}^0$ is unique, let M be any codimension 1 submodule of $\mathscr{L}$, so that $\mathscr{L}/M \cong k$. If $\phi' : \mathscr{L} \to k$ is the natural quotient map, then $\mathrm{t}M = \mathrm{Ker}(\phi')$. Since ϕ' is a nonzero element of $\mathrm{Hom}_G(\mathscr{L}, k)$, $\phi' = a\phi$ for some $a \in k^\times$. Thus. $M = \mathrm{Ker}(\phi) = \mathrm{Ker}(\phi') = \mathscr{L}^0$.

The purpose of the first part of [35] is to find a bound on dim $H^1(G, V)$ when V is an irreducible kG-module with $V^B \neq 0$. The case in which V is an irreducible kG-module with $V^B = 0$ is handled in the second part of the paper [35, Sections 4-6]. Somewhat surprisingly, finding a bound on dim $H^1(G, V)$ turns out to be a much more difficult task when $V^B = 0$. However, the bound obtained on dim $H^1(G, V)$ is much stronger in the case when $V^B = 0$. Guralnick and Tiep show that dim $H^1(G, V) \leq 1$ when $V^B = 0$; in the case that $V^B \neq 0$, though, they show only that dim $H^1(G, V) \leq \dfrac{|W| + e}{\dim(V^Q)}$, which (as they themselves state in the introduction) is "almost certainly not best possible."

10.2 Irreducible Modules V with $V^B \neq 0$

We begin with a basic (yet very important) lemma.

Lemma 10.2.1. *([35, Lemma 2.1]) Let $A := O_{r'}(B)$ be the largest normal subgroup of B of order prime to r. Then, for any kB-module V, the following statements are equivalent:*

(i) $V^B \neq 0$;

(ii) B *has trivial composition factors on* V;

(iii) $V^A \neq 0$;

(iv) $(V^*)^B \neq 0$.

Proof. (We follow the proof given in [35].) If $V^B \neq 0$, then V^B must contain k in its socle (k is the only irreducible kB-module on which B acts trivially), so (i) implies (ii). Suppose now that (ii) holds. Since $r \not| |A|$, V is completely reducible as a kA-module. Therefore, any copies of k appearing as composition factors of V as a kB-module become direct summands of V as a kA-module. Thus, $k = k^A \subseteq V^A$, so that $V^A \neq 0$ and (iii) holds. Now, suppose that (iii) holds, so that $V^A \neq 0$. Since A is a normal subgroup of B, V^A is a kB-module, which means that V^A is also a B/A-module. The r-group B/A has a non-zero fixed point on V^A since V^A is a non-zero vector space over a field of characteristic r (this is a standard fact in representation theory - see, for example [46, Proposition 26]). And, since $(V^A)^{B/A} \subseteq V^B$ (which follows because the action of B/A on V^A is defined by $bA.v = b.v$ for $bA \in B/A$, $v \in V^A$), we must have $V^B \neq 0$, which means that (i) holds.

We have proved that statements (i)-(iii) are equivalent. We will now show that statement (iii) is equivalent to (iv), which is sufficient to prove the lemma. Since V is completely reducible as a kA-module and k is the only irreducible trivial kA-module, $V^A \neq 0$ if and only if k is a direct summand of V as a kA-module. Since $k^* \cong k$, this is true if and only if k is a direct summand of V^*, which is true if and only if $(V^*)^A \neq 0$. But, by the equivalence of (i)-(iii), $(V^*)^A \neq 0$ if and only if $(V^*)^B \neq 0$.

□

Proposition 10.2.2. *([35, Proposition 3.1]) For any irreducible kG-module V, set $f_B(V) := \dim(V^B)$, $f_Q(V) := \dim(V^Q)$, and let $m_{\mathscr{L}}(V) := [\mathscr{L} : V]$ be the multiplicity of V as a composition factor of $\mathscr{L}$. Then, the following statements hold:*

1. $\dim \mathscr{L}^B = \dim \mathscr{L}^Q = |W|$
2. $X^Q = X^B$ *for any submodule* X *of* $\mathscr{L}$.
3. $m_{\mathscr{L}}(V) \geq f_B(V)$.
4. $\sum_{V \in \mathrm{Irr}_k(G)} m_{\mathscr{L}}(V) \cdot f_B(V) \leq \sum_{V \in \mathrm{Irr}_k(G)} m_{\mathscr{L}}(V) \cdot f_Q(V) = |W|$.
5. *If* $\dim(V^Q) = d > 0$, *then* $m_{\mathscr{L}}(V) \leq |W|/d \leq |W|$.

6. $m_{\mathscr{L}}(k) \leq \sum_{V\in \mathrm{Irr}_k(G),\ V^B\neq 0} m_{\mathscr{L}}(V) \leq |W|$.

Corollary 10.2.3. *([35, Corollary 3.2]) Let X be a submodule of $\mathscr{L}$. Then,* $\dim H^1(G,X) \leq |W| + e'$*, where e' is the r-rank of B/Q and is at most the twisted rank e of G.*

Corollary 10.2.4. *([35, Corollary 3.3])* $\sum_{V\in \mathrm{Irr}_k(G)} f_B(V)\cdot \dim\ H^1(G,V) \leq |W| + e$. *In particular,*

$$\sum_{V\in \mathrm{Irr}_k(G),\ V^B\neq 0} \dim\ H^1(G,V) \leq |W| + e,$$

and

$$\dim\ H^1(G,V) \leq \frac{|W|+e}{f_Q(V)}$$

if $V^B \neq 0$.

Proof. (We follow the proof given in [35].) Let $X = \mathrm{soc}(\mathscr{L})$. An irreducible module $V \in \mathrm{Irr}_k(G)$ embeds in X if and only if $\mathrm{Hom}_{kG}(V,\mathscr{L}) \neq 0$. But, $\mathrm{Hom}_{kG}(V,\mathscr{L}) \cong \mathrm{Hom}_{kB}(V,k) \cong \mathrm{Hom}_{kB}(k,V^*) \cong (V^*)^B$. And, by Lemma 2.1, $(V^*)^B \neq 0$ if and only if $V^B \neq 0$. Therefore, V embeds in X if and only if $V^B \neq 0$. And, if $V^B \neq 0$, the multiplicity of V as a composition factor of $X = \mathrm{soc}(\mathscr{L})$ is $[X : V] = \dim\ \mathrm{Hom}_{kG}(V,\mathscr{L}) = \dim\ \mathrm{Hom}_{kB}(V,k) = \dim\ \mathrm{Hom}_{kB}(k,V^*) = \dim(V^*)^B = \dim(V^*)^Q$, where we have used Frobenius reciprocity to conclude that $\dim\ \mathrm{Hom}_{kG}(V,\mathscr{L}) = \dim\ \mathrm{Hom}_{kB}(V,k)$ and [35, Proposition 3.1(ii)] to conclude that $\dim(V^*)^B = \dim(V^*)^Q$ (note that $V^B \neq 0$ implies that $(V^*)^B \neq 0$, so that $V^* \subseteq X$ is a submodule of $\mathscr{L}$ and [35, Proposition 3.1(ii)] applies). Now, since Q is a p-group and $\mathrm{char}(k) = r \neq p$, V is completely reducible as a kQ-module, with V^Q equal to the direct sum of all of the copies of k contained in the direct sum decomposition of V as a kQ-module. The same is true for $(V^*)^Q$, so that $V^Q \cong (V^*)^Q$. Therefore, $\dim(V^*)^Q = \dim V^Q = \dim V^B = f_B(V)$, where the second equality follows by [35, Proposition 3.1(ii)]. Combining this with the previous chain of equalities, we have $[X : V] = f_B(V)$ when $V^B \neq 0$.

By the arguments in the previous paragraph, $X \cong \oplus_{V\in \mathrm{Irr}_k(G),\ V^B\neq 0} V^{\oplus f_B(V)}$. In fact, since $f_B(V) = 0$ when $V^B = 0$, we can write $X \cong \oplus_{V\in \mathrm{Irr}_k(G)} V^{\oplus f_B(V)}$. Therefore, we have $H^1(G,X) \cong \oplus_{V\in \mathrm{Irr}_k(G),\ V^B\neq 0} H^1(G,V)^{\oplus f_B(V)}$ and $H^1(G,X) \cong \oplus_{V\in \mathrm{Irr}_k(G)} H^1(G,V)^{\oplus f_B(V)}$. And, since $\dim\ H^1(G,X) \leq |W| + e$ by [35, Corollary 3.2], the three desired inequalities hold. ∎

10.3 Irreducible Modules V with $V^B = 0$

Guralnick and Tiep [35] show that if V is an irreducible kG-module with $V^B = 0$, $\dim\ H^1(G,V) \leq 1$. In this section, we outline the results that lead to this bound on $\dim\ H^1(G,V)$.

Theorem 10.3.1. *([35, Theorem 2.2]) If V is an irreducible kG-module, with $V^B = 0$, then* $\dim H^1(G,V)$ *is the multiplicity of V in* $\mathrm{head}(\mathscr{L}^0)$.

The result of [35, Theorem 2.2] is crucial to our generalization of Guralnick and Tiep's work. We give a less group-theoretic proof of this theorem in Chapter 12.

Corollary 10.3.2. *([35, Corollary 2.3]) Let V be an irreducible kG-module. If V is not a composition factor of $\mathscr{L}$, then $H^1(G,V) = 0$.*

Theorem 10.3.3. *([35, Theorem 6.1]) Let V be a kG-module with $V^B = 0$. Let P_i, $1 \le i \le n$, denote the minimal parabolic subgroups of G containing B. If r does not divide $[P_i : B]$ for any i, then $H^1(G,V) = 0$.*

Proof. (We follow the proof given in [35].) Let $O_{r'}(B)$ denote the largest normal subgroup of B of order prime to r. Then, the Hochschild-Serre spectral sequence gives us the five-term inflation-restriction exact sequence in cohomology: $0 \to H^1(B/O_{r'}(B), V^{O_{r'}(B)}) \to H^1(B,V) \to H^1(O_{r'}(B),V)^{B/O_{r'}(B)} \to H^2(B/O_{r'}(B), V^{O_{r'}(B)}) \to H^2(B,V)$. Since $V^B = 0$, $V^{O_{r'}(B)} = 0$ by [35, Lemma 2.1]. Therefore, $H^1(B/O_{r'}(B), V^{O_{r'}(B)}) = 0$ and $H^2(B/O_{r'}(B), V^{O_{r'}(B)}) = 0$; so, by the exactness of the inflation-restriction sequence, we have $H^1(B,V) \cong H^1(O_{r'}(B),V)^{B/O_{r'}(B)}$. But, since $r \nmid |O_{r'}(B)|$, $H^1(O_{r'}(B),V) = 0$, and thus $H^1(B,V) = 0$.

Now, since r does not divide $[P_i : B]$ for any i, the restriction map $H^1(P_i,V) \to H^1(B,V)$ is injective for all i. It follows that $H^1(P_i,V) = 0$ for every minimal parabolic subgroup P_i of G. And, since $G = \langle P_1, \ldots, P_n \rangle$, the cohomology vanishing theorem of Alperin and Gorenstein [1] allows us to conclude that $H^1(G,V) = 0$.[14]

□

Remark 10.3.4. The following general result is needed to prove [35, Lemma 6.3].

Proposition 10.3.5. *Let G be a finite group and let H and R be subgroups of G. Assume that R is a normal subgroup, so that the product HR is a subgroup of G. Then, $k|_{HR}^G = (k|_H^G)^R$.*

Proof. Since G is finite, the induced module $k|_{HR}^G = kG \otimes_{kHR} k$ is isomorphic to the coinduced module $\mathrm{Hom}_{kHR}(kG,k)$. Similarly, $k|_H^G \cong \mathrm{Hom}_{kH}(kG,k)$. Now, since $H \le HR$, every HR-module homomorphism $kG \to k$ is also an H-module homomorphism, and it follows that $k|_{HR}^G \hookrightarrow k|_H^G$. We now check that $k|_{HR}^G = (k|_H^G)^R$. First, let $\phi \in k|_{HR}^G = \mathrm{Hom}_{HR}(kG,k)$ and let $r \in R$. Since R is a normal subgroup of G, given an element $g \in G$, there exists an element $r' \in R$ such that $gr = r'g$. Thus, $(r\phi)(g) = \phi(gr) = \phi(r'g) = r'\phi(g) = \phi(g)$, where the last equality follows since the action of G on k is trivial. Therefore, $\phi \in (k|_H^G)^R$, which shows that $k|_{HR}^G \subseteq (k|_H^G)^R$. Now, let $\phi \in (k|_H^G)^R$ and let $hr \in HR$. Given $g \in G$, let $r' \in R$ be

[14] In [1], Alperin and Gorenstein prove the following vanishing theorem. Suppose that a group G is generated by a collection $\mathcal{L}$ of subgroups such that $\mathcal{L}$ has a minimal element. If A is a G-module with $A^L = 0$ and $H^1(L,A) = 0$ for all $L \in \mathcal{L}$, then $A^G = 0$ and $H^1(G,A) = 0$.

such that $gr = r'g$. Then, $\phi(hrg) = h\phi(rg) = \phi(rg) = \phi(gr') = (r'\phi)(g) = \phi(g) = hr\phi(g)$. Therefore, $\phi \in k|_{HR}^{G}$, and $(k|_{H}^{G})^{R} \subseteq k|_{HR}^{G}$.

□

Lemma 10.3.6. *([35, Lemma 6.3]) Let P be a parabolic subgroup properly containing B, and let $R = R_u(P)$ be the unipotent radical of P. Let $W_0 \subset W$ be a full set of double coset representatives for $P\backslash G/B$ (the set of $P-B$-double cosets in G). Then, $\mathscr{L}^R \cong \bigoplus_{w \in W_0} k|_B^P$ as kP-modules.*

Proof. (We follow the proof given in [35].) By the Mackey decomposition, we have $\mathrm{Res}_P^G \ \mathrm{Ind}_B^G(k) = \bigoplus_{w\in W_0} \mathrm{Ind}_{P\cap B^w}^P \mathrm{Res}_{P\cap B^w}^{B^w}(k^w) = \bigoplus_{w\in W_0} \mathrm{Ind}_{P\cap B^w}^P(k)$. So, as a kP-module, $\mathscr{L} = k|_B^G$ decomposes as $\mathscr{L} = \bigoplus_{w\in W_0} k|_{P\cap B^w}^P$. Taking fixed points, $\mathscr{L}^R = \bigoplus_{w\in W_0} (k|_{P\cap B^w}^P)^R$. By Remark 10.3.4, $(k|_{P\cap B^w}^P)^R \cong k|_{(P\cap B^w)R}^P$, so $\mathscr{L}^R = \bigoplus_{w\in W_0} k|_{(P\cap B^w)R}^P$.

We claim that for any $w \in W$, $(P\cap B^w)R$ is a Borel subgroup of G. Note that $(P\cap B^w)R$ is, indeed, a subgroup of G: since $R \trianglelefteq P$, $(P \cap B^w)R$ is a subgroup of P, which menas that $(P\cap B^w)R$ is a subgroup of G. Now, since B^w is a Borel subgroup of G, B^w is solvable, which means that $P \cap B^w \leq B^w$ is solvable. And, since R is unipotent (and therefore solvable), the product $(P \cap B^w)R$ is a solvable subgroup of G. So, to show that $(P \cap B^w)R$ is a Borel subgroup, it suffices to check that $(P \cap B^w)R$ contains a maximal torus and exactly one of the root groups U_α and $U_{-\alpha}$ for every root α in the root system Φ of G.

Now, B^w and P are both parabolic subgroups of G, so their intersection $P \cap B^w$ contains a maximal torus, which must also be contained in $(P \cap B^w)R$. To show that one of $U_{\pm\alpha}$ is contained in $(P \cap B^w)R$ for every $\alpha \in \Phi$, we look closer at the structure of the parabolic subgroup P. Assume that $P = P_I$ for some subset I of the simple roots. Then, $P = P_I = L_I \rtimes R$, where L_I is the Levi subgroup of P_I. L_I is generated by the maximal torus $T \subset B$ and the root subgroups U_α for $\alpha \in \Phi_I$ (where $\Phi_I \subseteq \Phi$ denotes the set of roots that are linear combinations of the simple roots in I). And, R is generated by the root subgroups U_α with $\alpha \in \Phi^+ \setminus \Phi_I$. Now, let $\alpha \in \Phi$. If one of $U_{\pm\alpha}$ is contained in R, then it is contained in $(P \cap B^w)R$ and we are done. So, suppose that $U_\alpha \not\subseteq R$ and $U_{-\alpha} \not\subseteq R$. Then, both U_α and $U_{-\alpha}$ are contained in L_I. And, since B^w is a Borel subgroup of G, one of U_α and $U_{-\alpha}$ must lie in B^w. Therefore, one of U_α and $U_{-\alpha}$ lies in the intersection $L_I \cap B^w \subset P_I \cap B^w = P \cap B^w$, which means that one of U_α and $U_{-\alpha}$ lies in $(B^w \cap P)R$. Thus, $(P \cap B^w)R$ is a Borel subgroup of G, as claimed.

We have now shown that $\mathscr{L}^R = \bigoplus_{w\in W_0} (k|_{P\cap B^w}^P)^R = \bigoplus_{w\in W_0} k|_{(P\cap B^w)R}^P = \bigoplus_{w\in W_0} k|_{B'}^P$, where $B' = (P \cap B^w)R$ is a Borel subgroup of G contained in P. But, any two Borel subgroups of G are conjugate and, induction functors from conjugate subgroups are naturally isomorphic. Therefore, we have $k|_{B'}^P \cong k|_B^P$ and the statement of the lemma follows.

□

In [35], Guralnick and Tiep use the lemma above to prove the following theorem.

Theorem 10.3.7. *([35, Theorem 6.4]) Let P be a minimal parabolic subgroup containing B, with unipotent radical $R_u(P) = R$. Assume that r divides $[P : B]$. Let X be a submodule of $\mathscr{L}$ containing $\mathscr{L}^G$ such that $X^Q = \mathscr{L}^G$ and* $\dim X^R > 1$. *Then the following statements hold:*

(1) Either $\mathrm{soc}(X^R/X^G)$ *is an irreducible kP-module, or $r = 2$ and $[P, P]/R \cong SL_2(q)$ with q odd or $SU_3(q)$ with $q \equiv 1 (\mathrm{mod}\ 4)$. In the latter two cases,* $\mathrm{soc}(X^R/X^G)$ *is either irreducible or a direct sum of two nonisomorphic irreducible kP-modules.*

(2) If X/X^G is irreducible, then $\mathrm{Ext}^1_G(X, k) = 0$ *and* $\dim \mathrm{Ext}^1_G(X/k, k) = 1$.

A key step in the proof of this theorem is a reduction to the case of rank 1 groups. We will use this technique several times in the upcoming chapters as we generalize Guralnick and Tiep's bounds on the dimension of $H^1(G, V)$. Guralnick and Tiep's bound on $H^1(G, V)$ (where V is an irreducible kG-module with $V^B = 0$) follows as a corollary of [35, Theorem 6.4].

Corollary 10.3.8. *([35, Corollary 6.5(i)]) Let V be an irreducible kG-module with $V^B = 0$ and $H^1(G, V) \neq 0$. Then,* $\dim H^1(G, V) = 1$.

11 A Bound on the Dimension of Ext^1 between Irreducibles in the Unipotent Principal Series

11.1 Introduction

Let G be a finite group of Lie type, defined in characteristic $p > 0$ (so, G is the fixed point subgroup of a connected reductive algebraic group $\mathbb{G}$ over $\bar{\mathbb{F}}_p$ under an endomorphism $F : \mathbb{G} \to \mathbb{G}$ such that some power of F is a Frobenius morphism). Assume, additionally, that the BN-pair of G is split. ([35] handles all finite groups of Lie type; here, we restrict to split groups in order to use the methods of modular Harish-Chandra theory presented in [31, Section 4.2].) Let (W, S) be the Coxeter system corresponding to the BN-pair structure on G.

Let k be an algebraically closed field of characteristic $r > 0$, $r \neq p$, and let e denote the r-rank of the maximal torus T (that is, e is the maximal dimension of an elementary abelian r-subgroup of T)[15]. In [35], Guralnick and Tiep prove that dim $H^1(G, V) \leq |W| + e$ for any irreducible kG-module V with $V^B \neq 0$ (and, in the case that G is simple and $V^B = 0$, they prove the stronger result that dim $H^1(G, V) \leq 1$). Since an irreducible kG-module V satisfies $V^B \neq 0$ if and only if V belongs to the unipotent principal series $\mathrm{Irr}_k(G|B)$, Guralnick and Tiep's result may be restated as follows:

If V is an irreducible kG-module with $V \in \mathrm{Irr}_k(G|B)$, then dim $H^1(G, V) \leq |W| + e$.

In this section, we will present a partial generalization of Guralnick and Tiep's result. Specifically, we will prove that dim $\mathrm{Ext}^1_{kG}(V, V') \leq |W| + (\dim V)e$ if V and V' are irreducible kG-modules such that $V^B \neq 0$ and $(V')^B \neq 0$ (in the case that $V = k$ and $(V')^B \neq 0$, we recover the bound of [35]).

To generalize the result of [35], we will work with the induced module $k|_{T_r}^T$, where T_r is an r-Sylow subgroup of T (in [35], Guralnick and Tiep work with the permutation module $k|_B^G$ directly). Hence, we must break our proof into two cases, depending on whether or not $|B|$ is divisible by the characteristic r of k.

11.2 A Preliminary Result

Lemma 11.2.1. *If V is an irreducible kG-module in the unipotent principal series $\mathrm{Irr}_k(G|B)$, then $[k|_B^G : V] \leq |W|$ (where $[k|_B^G : V]$ denotes the number of times V occurs as a composition factor of $k|_B^G$).*

[15] Here, the dimension of an elementary abelian r-group refers its dimension as a vector space over the finite field $\mathbb{F}_r$ of r elements.

Proof. The proof strategy used here is based on that of [35, Proposition 3.1]. Let $\mathrm{Irr}_k(G)$ denote the set of all (non-isomorphic) irreducible kG-modules. Given an irreducible $Z \in \mathrm{Irr}_k(G)$, let $m_Z = [k|_B^G : Z]$ be the composition multiplicity of Z in $k|_B^G$. We consider the U-fixed point subspace $(k|_B^G)^U$ of $k|_B^G$. Since $r \nmid |U|$, every kU-module is completely reducible. Therefore, as a kU-module, $(k|_B^G)^U = \bigoplus_{Z \in \mathrm{Irr}_k(G)} (Z^U)^{\oplus m_Z}$, so that

$$\dim\, (k|_B^G)^U = \sum_{Z \in \mathrm{Irr}_k(G)} m_Z \dim(Z^U).$$

Now, since $V \in \mathrm{Irr}_k(G|B)$, $m_V \neq 0$ and ${}^*R_T^G(V) = V^U \neq 0$. Therefore, m_V occurs with non-zero coefficient in the sum giving $\dim\, (k|_B^G)^U$, and it follows that $m_V \leq \dim\, (k|_B^G)^U$.

But, $(k|_B^G)^U \cong \mathrm{Hom}_{kU}(k, k \uparrow_B^G \downarrow_U^G)$. By the Bruhat decomposition, W gives a full set of (B, B)-double coset representatives in G. Since every element of W normalizes T, we have $BwB = UTwB = UwTB = UwB$ for all $w \in W$. Therefore, W gives a full set of (U, B)-double coset representatives in G. So, applying the Mackey decomposition, we have

$$\mathrm{Hom}_{kU}(k, k \uparrow_B^G \downarrow_U^G) \cong \bigoplus_{w \in W} \mathrm{Hom}_{kU}(k, {}^w k \downarrow^{{}^w B}_{{}^w B \cap U} \uparrow^U_{{}^w B \cap U}). \tag{11.1}$$

Now, ${}^w k$ is the $k({}^w B)$-module with underlying space k and ${}^w B$-action given by $wbw^{-1}.a = b.a = a$ for any $b \in B$ and $a \in k$. Thus, ${}^w k$ is the trivial $k({}^w B)$-module k, and ${}^w k \downarrow^{{}^w B}_{{}^w B \cap U}$ is k, viewed as the trivial $k({}^w B \cap U)$-module. So, using (11.1) and the fact that induction is right adjoint to restriction in the case of finite groups, we have

$$\mathrm{Hom}_{kG}(k|_U^G, k|_B^G) \cong \bigoplus_{w \in W} \mathrm{Hom}_{kU}(k, k \uparrow^U_{{}^w B \cap U}) \cong \bigoplus_{w \in W} \mathrm{Hom}_{k({}^w B \cap U)}(k, k) \cong \bigoplus_{w \in W} k.$$

Therefore, $\dim\, (k|_B^G)^U = |W|$, which means that $[k|_B^G : V] = m_V \leq |W|$. □

11.3 Case I: $r \nmid |B|$

Assume that the characteristic r of k does not divide $|B|$.

Theorem 11.3.1. *Let V and V' be irreducible kG-modules in the unipotent principal series* $\mathrm{Irr}_k(G|B)$*. Then,* $\dim \mathrm{Ext}^1_{kG}(V, V') \leq |W|$*.*

Proof. We use the proof strategy of [35, Corollary 3.2]. Since $r \nmid |B|$, the group algebra kB is semisimple and every kB-module is projective and injective. In particular, k is an injective kB-module, which means that $k|_B^G$ is also injective (since induction from B to G is exact) and $\mathrm{Ext}^1_{kG}(V, k|_B^G) = 0$. Now, since V' is in $\mathrm{Irr}_k(G|B)$, V' is contained in the socle of $k|_B^G$. Therefore, we have a short exact sequence of kG-modules

$$0 \to V' \to k|_B^G \to Y \to 0$$

(where Y is a kG-module with $Y \cong k|_B^G/V'$). Since $\operatorname{Ext}^1_{kG}(V, k|_B^G) = 0$, this short exact sequence induces the exact sequence

$$0 \to \operatorname{Hom}_{kG}(V, V') \to \operatorname{Hom}_{kG}(V, k|_B^G) \to \operatorname{Hom}_{kG}(V, Y) \to \operatorname{Ext}^1_{kG}(V, V') \to 0.$$

Therefore, $\dim \operatorname{Ext}^1_{kG}(V, V') \leq \dim \operatorname{Hom}_{kG}(V, Y) = [\operatorname{soc}(Y) : V] \leq [Y : V] \leq [k|_B^G : V] \leq |W|$, where the last inequality follows by Lemma 11.2.1. □

11.4 Case II: $r \mid |B|$

Assume that $|B|$ is divisible by the characteristic r of k.

Let T_r be an r-Sylow subgroup of T. Since $r \mid |B| = |U||T|$ and U is a p-group, $r \mid |T|$. Hence, T_r is a non-trivial r-subgroup of T. Moreover, since T is abelian, T_r is a normal subgroup of T, which means that T/T_r is a group. So, the permutation module $k|_{T_r}^T$ is isomorphic to $k[T/T_r]$, which is semisimple by Maschke's Theorem. Thus, $k|_{T_r}^T$ is a completely reducible kT-module.

Lemma 11.4.1. *Let A be an abelian r-group or rank e. Then,* $\dim H^n(A, k) = \binom{n+e-1}{n}$ *for all $n \geq 0$.*

Proof. We proceed by induction on e. If $e = 1$, then $A = \mathbb{Z}/m\mathbb{Z}$ for some $m = r^d$ ($d \in \mathbb{Z}^+$), and $H^n(A, k) \cong k$ for all $n \geq 0$ since A is a cyclic r-group (a proof of this standard result can be found in the Appendix, Section 20). In particular, $\dim H^n(A, k) = 1 = \binom{n+1-1}{n}$, and the statement of the lemma holds. Suppose now that $e > 1$ and $\dim H^n(A', k) = \binom{n+e-2}{n}$ for all abelian r-groups A' of rank $e-1$ and all $n \geq 0$. Let $m_1, \ldots, m_e$ be positive integers such that $m_i = r^{d_i}$ ($d_i \in \mathbb{Z}^+$) for $1 \leq i \leq e$ and $A = \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \cdots \times \mathbb{Z}/m_e\mathbb{Z}$. By the Künneth formula,

$$\begin{aligned} H^n(A, k) &= H^n(\mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \cdots \times \mathbb{Z}/m_e\mathbb{Z}, k) \\ &\cong \bigoplus_{i=0}^{n} H^i(\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_{e-1}\mathbb{Z}, k) \otimes H^{n-i}(\mathbb{Z}/m_e\mathbb{Z}, k) \\ &\cong \bigoplus_{i=0}^{n} H^i(\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_{e-1}\mathbb{Z}, k) \otimes k \end{aligned}$$

By the inductive hypothesis, $\dim H^i(\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_{e-1}\mathbb{Z}, k) = \binom{i+e-2}{i}$ for $0 \leq i \leq n$. Therefore, $\dim H^n(A, k) = \sum_{i=0}^{n} \binom{i+e-2}{i} = \binom{n+e-1}{n}$.
(Note that the last step in the computation above is a consequence of the following sum

formula for binomial coefficients: $\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$.)

□

Lemma 11.4.2. *Let V be a kT_r-module and assume that* $\dim V = n$ *as a k-vector space. Then,* $\dim \operatorname{Ext}^1_{kT_r}(k, V) \leq ne$ *(where e is the r-rank of T).*

Proof. We proceed by induction on the dimension n of V. Throughout this proof, we will use the fact that the only irreducible module for an r-group in characteristic r is the trivial module k. When $n = 1$, we have $V = k$, so we must show that $\dim \operatorname{Ext}^1_{kT_r}(k, k) \leq e$. But, T_r is an abelian r-group of rank $e' \leq e$. So, by Lemma 11.4.1, $\dim \operatorname{Ext}^1_{kT_r}(k, k) = \dim H^1(T_r, k) = \binom{1+e'-1}{1} = e' \leq e$.

Suppose now that $\dim V = n > 1$ and that the statement of the lemma holds for all kT_r-modules V' of dimension $n-1$. Let $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ be a composition series for V as a kT_r-module. Note that every composition factor V_i/V_{i-1} $(1 \leq i \leq n)$ is isomorphic to k. Therefore, we have a short exact sequence $0 \to V_{n-1} \to V \to k \to 0$, which induces the following long exact Ext sequence:

$$\cdots \to \operatorname{Ext}^1_{kT_r}(k, k) \to \operatorname{Ext}^1_{kT_r}(V, k) \to \operatorname{Ext}^1_{kT_r}(V_{n-1}, k) \to \cdots .$$

By exactness, we have $\dim \operatorname{Ext}^1_{kT_r}(V, k) \leq \dim \operatorname{Ext}^1_{kT_r}(k, k) + \dim \operatorname{Ext}^1_{kT_r}(V_{n-1}, k)$. Now, $\dim \operatorname{Ext}^1_{kT_r}(k, k) \leq e$ and $\dim \operatorname{Ext}^1_{kT_r}(V_{n-1}, k) \leq (n-1)e$ by the base case and the inductive hypothesis, respectively. Thus, $\dim \operatorname{Ext}^1_{kT_r}(V, k) \leq e + (n-1)e = ne$, as needed.

□

Theorem 11.4.3. *Let V and V' be irreducible kG-modules in the unipotent principal series* $\operatorname{Irr}_k(G|B)$. *Then,* $\dim \operatorname{Ext}^1_{kG}(V, V') \leq |W| + (\dim V)e$.

Proof. Since induction is a two-sided adjoint of restriction in the case of finite groups, we have $\operatorname{Hom}_{kT}(k, k|_{T_r}^{T}) \cong \operatorname{Hom}_{kT_r}(k, k) \cong k$, from which it follows that k occurs exactly once as a composition factor of the completely reducible module $k|_{T_r}^{T}$. Thus, we have a short exact sequence of the form $0 \to k \to k|_{T_r}^{T} \to M \to 0$, where M is a completely reducible kT_r-module which does not contain k as a composition factor. Applying the Harish-Chandra induction functor R_T^G (defined in (4.1)), we obtain the short exact sequence

$$0 \to R_T^G(k) \to R_T^G(k|_{T_r}^{T}) \to R_T^G(M) \to 0. \tag{11.2}$$

Now, since $V' \in \operatorname{Irr}_k(G|B)$, V' is in the socle of $k|_B^G = R_T^G(k)$. By the short exact sequence (11.2), V' is in the socle of $R_T^G(k|_{T_r}^{T})$, so we obtain a short exact sequence

$$0 \to V' \to R_T^G(k|_{T_r}^{T}) \to Y \to 0$$

for some kG-module Y, which gives rise to the long exact sequence

$$0 \to \operatorname{Hom}_{kG}(V, V') \to \operatorname{Hom}_{kG}(V, R_T^G(k|_{T_r}^{T})) \to \operatorname{Hom}_{kG}(V, Y) \to \operatorname{Ext}^1_{kG}(V, V') \to$$
$$\operatorname{Ext}^1_{kG}(V, R_T^G(k|_{T_r}^{T})) \to \cdots .$$

By exactness, $\dim \operatorname{Ext}^1_{kG}(V,V') \le \dim \operatorname{Hom}_{kG}(V,Y) + \dim \operatorname{Ext}^1_{kG}(V, R^G_T(k|^T_{T_r}))$. So, to prove the theorem, it is enough to show $\dim \operatorname{Hom}_{kG}(V,Y) \le |W|$ and that $\dim \operatorname{Ext}^1_{kG}(V, R^G_T(k|^T_{T_r})) \le (\dim V)e$.

First, we show that $\dim \operatorname{Hom}_{kG}(V,Y) \le |W|$. We have $\dim \operatorname{Hom}_{kG}(V,Y) = [\operatorname{soc}(Y) : V] \le [Y : V] \le [R^G_T(k|^T_{T_r}) : V]$. By (11.2), $[R^G_T(k|^T_{T_r}) : V] = [R^G_T(k) : V] + [R^G_T(M) : V]$, so $\dim \operatorname{Hom}_{kG}(V,Y) \le [R^G_T(k) : V] + [R^G_T(M) : V]$. But, $[R^G_T(k) : V] = [k|^G_B : V] \le |W|$ by Lemma 11.2.1. So, to show that $\dim \operatorname{Hom}_{kG}(V,Y) \le |W|$, it suffices to show that $[R^G_T(M) : V] = 0$. Suppose, for contradiction, that $[R^G_T(M) : V] \neq 0$. Since W gives a full set of (B,B)-double coset representatives in G,

$$^*R^G_T(R^G_T(M)) \cong \bigoplus_{w\in W} R^T_{^wT\cap T}\ (^*R^{^wT}_{^wT\cap T}(^wM))$$

by the Mackey decomposition (4.3). For $w \in W$, ^wM is the ^wT-module with underlying space M and ^wT-action given by $wtw^{-1}.m = t.m$ for any $t \in T$, $m \in M$. Since $W = N_G(T)/T$, $^wT = wTw^{-1} = T$ for any $w \in W$. Therefore, the functors $R^T_{^wT\cap T}$ and $^*R^{^wT}_{^wT\cap T}$ are equal to the identity functor on $kT-\text{mod}$ for all $w \in W$, and it follows that $^*R^G_T(R^G_T(M)) \cong \bigoplus_{w\in W} M$. Since M is a completely reducible kT-module, $^*R^G_T(R^G_T(M))$ is a completely reducible kT-module. Thus,

$$^*R^G_T(R^G_T(M)) \cong \bigoplus_{Z\in \operatorname{Irr}_k(G)} {^*R^G_T(Z)}^{\oplus m_Z},$$

where $m_Z = [R^G_T(M) : Z]$ is the composition multiplicity of Z in $R^G_T(M)$ for any $Z \in \operatorname{Irr}_k(G)$. Now, since $V \in \operatorname{Irr}_k(G|B)$, $^*R^G_T(V) \neq 0$ and k is a composition factor of $^*R^G_T(V)$. But, we assumed $m_V = [R^G_T(M) : V] \neq 0$, so $^*R^G_T(V)$ is a submodule of $^*R^G_T(R^G_T(M)) = \bigoplus_{w\in W} M$. Since k is a composition factor of $^*R^G_T(V)$, k must also be a composition factor of M. This is a contradiction since M has no trivial composition factors; so, we must have $[R^G_T(M) : V] = 0$.

It remains to check that $\dim \operatorname{Ext}^1_{kG}(V, R^G_T(k|^T_{T_r})) \le (\dim V)e$. Using the definition of the functor R^G_T and the Eckmann-Shapiro Lemma, we have $\operatorname{Ext}^1_{kG}(V, R^G_T(k|^T_{T_r})) = \operatorname{Ext}^1_{kG}(V, (\widetilde{k|^T_{T_r}})|^G_B) \cong \operatorname{Ext}^1_{kB}(V, \widetilde{k|^T_{T_r}})$. Now, since $B = U \rtimes T$ and $r \nmid |U|$, the r-Sylow subgroup T_r of T is also an r-Sylow subgroup of B. So, in particular, $r \nmid [B : T]$, and the restriction map $\operatorname{Ext}^1_{kB}(V, \widetilde{k|^T_{T_r}}) \to \operatorname{Ext}^1_{kT}(V, \widetilde{k|^T_{T_r}}) = \operatorname{Ext}^1_{kT}(V, k|^T_{T_r})$ is injective. It follows that $\dim \operatorname{Ext}^1_{kG}(V, R^G_T(k|^T_{T_r})) = \dim \operatorname{Ext}^1_{kB}(V, \widetilde{k|^T_{T_r}}) \le \dim \operatorname{Ext}^1_{kT}(V, k|^T_{T_r})$. Applying the Eckmann-Shapiro Lemma once more, $\operatorname{Ext}^1_{kT}(V, k|^T_{T_r}) \cong \operatorname{Ext}^1_{kT_r}(V,k)$. Therefore, $\dim \operatorname{Ext}^1_{kG}(V, R^G_T(k|^T_{T_r})) \le \dim \operatorname{Ext}^1_{kT_r}(V,k) \le (\dim V)e$, where the last inequality holds by Lemma 11.4.2. □

12 A New Proof of Guralnick and Tiep's Bound on the Dimension of $H^1(G, V)$ in the case that $V^B = 0$

12.1 Introduction

Let G be a finite group of Lie type, defined over the finite field $\mathbb{F}_q$, where q is a power of a prime $p > 0$ (so, G is the fixed point subgroup of a connected reductive algebraic group $\mathbb{G}$ over $\bar{\mathbb{F}}_p$ under an endomorphism $F : \mathbb{G} \to \mathbb{G}$ such that some power of F is a Frobenius morphism). In order to use results of modular Harish-Chandra theory, we assume additionally that the BN-pair of G is split. Let (W, S) be the Coxeter system corresponding to the BN-pair structure on G.

Let k be an algebraically closed field of characteristic $r > 0$, $r \neq p$. In [35], Guralnick and Tiep prove that if V is an irreducible kG-module with $V^B = 0$, then $\dim H^1(G, V) \leq 1$. Since $V^B \neq 0$ if and only if $V \in \mathrm{Irr}_k(G|B)$, Guralnick and Tiep's result may be restated as follows.

Theorem 12.1.1. *If V is an irreducible kG-module with $V \notin \mathrm{Irr}_k(G|B)$, then $\dim H^1(G, V) \leq 1$.*

The purpose of this section is to present a new proof of this result. While many of the proof strategies used here differ from those used by Guralnick and Tiep, the idea of using induction on the rank of the Weyl group of G was inspired by a reading of [35].

12.2 A first result on the dimension of $H^1(G, V)$ when $V \notin \mathrm{Irr}_k(G|B)$.

By Frobenius reciprocity, $\mathrm{Hom}_{kG}(k|_B^G, k) \cong \mathrm{Hom}_{kB}(k, k) \cong k$. Therefore, there exists a surjective homomorphism $k|_B^G \to k$. Let $\mathscr{L}^0 \subset k|_B^G$ be the kernel of this homomorphism, so that the sequence $0 \to \mathscr{L}^0 \to k|_B^G \to k \to 0$ is exact (the notation $\mathscr{L}^0$ is consistent with [35]). [35, Theorem 2.2] establishes that the dimension of $H^1(G, V)$ is determined by the structure of the head of $\mathscr{L}^0$ when $V^B = 0$ (or, equivalently, when $V \notin \mathrm{Irr}_k(G|(T, k)) = \mathrm{Irr}_k(G|B)$). We give this crucial result in Proposition 12.2.1 below, along with an updated (less group-theoretic) proof.

Proposition 12.2.1. *Suppose that V is an irreducible kG-module which does not belong to the unipotent principal series. Then,* $\dim H^1(G, V) = [\mathrm{head}(\mathscr{L}^0) : V]$ *(where* $[\mathrm{head}(\mathscr{L}^0) : V]$ *denotes the multiplicity of V in* $\mathrm{head}(\mathscr{L}^0)$*).*

Proof. First, we claim that $H^1(B, V) = 0$. Let $A = O_{r'}(B)$ be the biggest normal subgroup of B of order prime to r. Then, B/A is an r-group. (The fact that B/A is an r-group can be seen as follows. Let T_r be the unique r-Sylow subgroup of the abelian group T, and let T' be the subgroup of T with $T \cong T_r \times T'$. then $A = U \rtimes T'$ is the biggest normal subgroup of B of order prime to r, and $B/A \cong T_r$ is an r-group.) Now, since $0 = V^B = (V^A)^{B/A}$, we have $V^A = 0$ (otherwise, the r-group B/A would have a non-zero fixed point on the k-vector space V^A).

Consider the five-term inflation-restriction exact sequence

$$0 \to H^1(B/A, V^A) \to H^1(B,V) \to H^1(A,V)^{B/A} \to H^2(B/A, V^A) \to H^2(B,V).$$

Since $V^A = 0$, $H^1(B/A, V^A) = 0$. Also, since $r \not| \, |A|$, kA is semisimple by Maschke's Theorem, from which it follows that $H^1(A,V) = 0$. Therefore, $H^1(B,V) = 0$ by the exactness of the sequence above.

To prove that dim $H^1(G,V) = [\text{head}(\mathscr{L}^0) : V]$, we use the short exact sequence $0 \to \mathscr{L}^0 \to k|_B^G \to k \to 0$, which induces the long exact sequence

$$0 \to \text{Hom}_{kG}(k,V) \to \text{Hom}_{kG}(k|_B^G, V) \to \text{Hom}_{kG}(\mathscr{L}^0, V) \to \text{Ext}^1_{kG}(k,V) \to \text{Ext}^1_{kG}(k|_B^G, V) \to \cdots.$$

Since $V^B = 0$, V is not in the head or socle of $k|_B^G$, so it follows that $\text{Hom}_{kG}(k,V) = 0$ and $\text{Hom}_{kG}(k|_B^G, V) = 0$. By the Eckmann-Shapiro Lemma, $\text{Ext}^1_{kG}(k|_B^G, V) \cong \text{Ext}^1_{kB}(k,V) \cong H^1(B,V) = 0$. So, by the exactness of the sequence above, $\text{Ext}^1_{kG}(k,V) \cong \text{Hom}_{kG}(\mathscr{L}^0, V)$. Thus, dim $H^1(G,V) = \dim \text{Ext}^1_{kG}(k,V) = \dim \text{Hom}_{kG}(\mathscr{L}^0, V) = [\text{head}(\mathscr{L}^0) : V]$.

□

We note that the idea to use the subgroup $A = O_{r'}(B)$ to prove that $H^1(B,V) = 0$ is due to Guralnick and Tiep [35].

12.3 Groups of Rank 1

In this section, we assume that the split group G has a Weyl group of rank 1, so that $S = \{s\}$ and $W = \{1, s\}$ for some simple reflection s. If α is the simple root corresponding to s, then the unipotent radical of B is of the form $U = U_\alpha$; in particular, U is an abelian group and $|U| = q$.

The purpose of this section is to show that dim $H^1(G,V) \leq 1$ when $V \notin \text{Irr}_k(G|B)$. We note that since G is of rank 1, T is the only proper Levi subgroup contained in G. Therefore, an irreducible kG-module Y is cuspidal if and only if ${}^*R_T^G(Y) = Y^U = 0$.

Theorem 12.3.1. *If G is of rank 1 and V is a non-cuspidal irreducible kG-module such that $V \notin \text{Irr}_k(G|B)$, then* dim $H^1(G,V) \leq 1$.

Proof. By Proposition 12.2.1, dim $H^1(G,V) = [\text{head}(\mathscr{L}^0) : V] \leq [k|_B^G : V]$ (where $\mathscr{L}^0$ is a submodule of $k|_B^G$ such that $0 \to \mathscr{L}^0 \to k|_B^G \to k \to 0$ is exact), so it suffices to show that

$[k|_B^G : V] \leq 1$. Since V is not cuspidal, $V^U \neq 0$. Now, since $r \nmid |U|$, $(k|_B^G)^U$ is a completely reducible kU-module and we have

$$(k|_B^G)^U \cong \bigoplus_{Y \in \mathrm{Irr}_k(G)} (Y^U)^{\oplus\ [k|_B^G:Y]}.$$

It follows that

$$\dim\ (k|_B^G)^U = \sum_{Y \in \mathrm{Irr}_k(G)} [k|_B^G : Y]\ \dim(Y^U). \tag{12.1}$$

We claim that $\dim\ (k|_B^G)^U = 2$. Since W gives a full set of (U, B)-double coset representatives in G, it follows by Frobenius reciprocity and the Mackey decomposition that

$$(k|_B^G)^U \cong \mathrm{Hom}_{kU}(k, k|_B^G) \cong \mathrm{Hom}_{kG}(k|_U^B, k|_B^G) \cong \mathrm{Hom}_{kU}(k, k \uparrow_B^G \downarrow_U^G) \cong$$

$$\mathrm{Hom}_{kU}(k, \bigoplus_{w \in W} {}^w k \downarrow^{{}^w B}_{{}^w B \cap U} \uparrow^U_{{}^w B \cap U}) \cong \bigoplus_{w \in W} \mathrm{Hom}_{kU}(k, {}^w k \uparrow^U_{{}^w B \cap U}). \tag{12.2}$$

Now, ${}^w k$ is k with ${}^w B$-action given by $wbw^{-1}.x = bx$ for any $b \in B$ and $x \in k$. Since B acts trivially on k, ${}^w k$ is the trivial one-dimensional $k({}^w B)$-module and we will denote ${}^w k$ by k. So, by (12.2) and Frobenius reciprocity, we have

$$(k|_B^G)^U \cong \bigoplus_{w \in W} \mathrm{Hom}_{kU}(k, k \uparrow^U_{{}^w B \cap U}) \cong \bigoplus_{w \in W} \mathrm{Hom}_{k({}^w B \cap U)}(k, k) \cong \bigoplus_{w \in W} k.$$

Since G is of rank 1, $|W| = 2$, and it follows that $\dim\ (k|_B^G)^U = 2$, as claimed.

Since $V^U \neq 0$ and $k^U \neq 0$, $[k|_B^G : V]$ and $[k|_B^G : k]$ appear with non-zero coefficient in (12.1). Also, $V \neq k$ since $k \in \mathrm{Irr}_k(G|B)$ and $V \notin \mathrm{Irr}_k(G|B)$. So, by (12.1),

$$[k|_B^G : V] + [k|_B^G : k] \leq \dim\ (k|_B^G)^U = 2.$$

But, k appears in the head of $k|_B^G$, so $[k|_B^G : k] \geq 1$, which means that $[k|_B^G : V] \leq 1$.

□

Theorem 12.3.2. *If G is of rank 1 and V is a cuspidal irreducible kG-module, then* $\dim\ H^1(G, V) \leq 1$.

Proof. By Proposition 12.2.1, $\dim\ H^1(G, V) = [\mathrm{head}(\mathscr{L}^0) : V] \leq [k|_B^G : V]$. Therefore, it suffices to show that $[k|_B^G : V] \leq 1$. Since V is cuspidal, $V^U = 0$, which means that k is not a direct summand of $V \downarrow_U$ (which is a completely reducible kU-module since $r \nmid |U|$). Now, since $U = U_\alpha$ is abelian, every irreducible kU-module is one-dimensional. Therefore, $V \downarrow_U$ has some non-trivial one-dimensional kU-module ψ as a direct summand. By Frobenius reciprocity, we have $0 \neq \mathrm{Hom}_{kU}(\psi, V) \cong \mathrm{Hom}_{kG}(\psi|_U^G, V)$, so V is in the head of $\psi|_U^G$.

Now, since $r \nmid |U|$, every kU-module is projective; in particular, ψ is a projective kU-module. Since induction from U to G is exact, $\psi|_U^G$ is a projective kG-module, which means that $\psi|_U^G$ is a direct sum of projective indecomposable kG-modules. Let P_V be the projective indecomposable kG-module with head V. Since $V \subseteq \text{head}(\psi|_U^G)$, P_V is a direct summand of $\psi|_U^G$. It follows that

$$[k|_B^G : V] = \dim \text{Hom}_{kG}(P_V, k|_B^G) \leq \dim \text{Hom}_{kG}(\psi|_U^G, k|_B^G). \tag{12.3}$$

Now, by Frobenius reciprocity, $\text{Hom}_{kG}(\psi|_U^G, k|_B^G) \cong \text{Hom}_{kU}(\psi, k \uparrow_B^G \downarrow_U^G)$. As shown in the proof of Theorem 12.3.1, $k \uparrow_B^G \downarrow_U^G \cong \bigoplus_{w \in W} k \uparrow_{{}^wB \cap U}^U$, so we have

$$\text{Hom}_{kG}(\psi|_U^G, k|_B^G) \cong \bigoplus_{w \in W} \text{Hom}_{kU}(\psi, k \uparrow_{{}^wB \cap U}^U) \cong \bigoplus_{w \in W} \text{Hom}_{k({}^wB \cap U)}(\psi, k).$$

Here, $W = \{1, s_\alpha\}$. When $w = 1$, ${}^wB \cap U = U$, and we have $\text{Hom}_{k({}^wB \cap U)}(\psi, k) = 0$ since $\psi|_U$ is non-trivial. When $w = s_\alpha$, ${}^wB \cap U = \{1\}$, and we have $\dim \text{Hom}_{k({}^wB \cap U)}(\psi, k) = \dim \text{Hom}_k(\psi, k) = \dim \psi = 1$. Therefore, $\dim \text{Hom}_{kG}(\psi|_U^G, k|_B^G) = 1$, and it follows by (12.3) that $[k|_B^G : V] \leq 1$.

□

12.4 Groups of Higher Rank

In this section, we consider (split) finite groups of Lie type G of higher rank.

Proposition 12.4.1. *Suppose that G is of rank (strictly) greater than 1. If V is a cuspidal irreducible kG-module, then $H^1(G, V) = 0$.*

Proof. Given a subset $I \subseteq S$, let P_I be the corresponding standard parabolic subgroup. Since $\text{rk}(G) > 1$, G is generated by the set $\mathcal{P}$ of proper standard parabolic subgroups P_I of G (i.e., those P_I such that $I \subsetneq S$). The set $\mathcal{P}$ contains B as a minimal element, and for any $P \in \mathcal{P}$, $V^P \subseteq V^B = 0$ ($V^B = 0$ since V is cuspidal and hence does not belong to $\text{Irr}_k(G|B)$). So, by the vanishing theorem of Alperin and Gorenstein [1], it is enough to show that $H^1(P, V) = 0$ for every parabolic subgroup $P \in \mathcal{P}$.

Let $P \in \mathcal{P}$, and let U_P and L be the unipotent radical and Levi subgroup of P, respectively. Since $P \subsetneq G$, it follows that $L \subsetneq G$, so that ${}^*R_L^G(V) = V^{U_P} = 0$ for the cuspidal kG-module V.

We now consider the five-term inflation-restriction exact sequence

$$0 \to H^1(P/U_P, V^{U_P}) \to H^1(P, V) \to H^1(U_P, V)^{P/U_P} \to H^2(P/U_P, V^{U_P}) \to H^2(P, V).$$

Since $V^{U_P} = 0$, $H^1(P/U_P, V^{U_P}) = 0$. Furthermore, since $r \nmid |U_P|$ (U_P is a p-group), kU_P is semisimple and $H^1(U_P, V)^{P/U_P} = 0$. So, by the exactness of the sequence above, we have $H^1(P, V) = 0$.

□

Proposition 12.4.2. *Suppose that V is an irreducible kG-module belonging to a Harish-Chandra series of the form $\mathrm{Irr}_k(G|(T,X))$, where X is a non-trivial irreducible kT-module and G has a Weyl group of any rank. Then, $H^1(G,V)=0$.*

Proof. Since $X \neq k$, the Harish-Chandra series $\mathrm{Irr}_k(G|(T,X))$ is not equal to the unipotent principal series $\mathrm{Irr}_k(G|B)$. Thus, if $V \in \mathrm{Irr}_k(G|(T,X))$, $\dim H^1(G,V) = [\mathrm{head}(\mathscr{L}^0) : V] \leq [\mathscr{L}^0 : V] \leq [k|_B^G : V]$ by Theorem 12.2.1 (where the last inequality follows since $\mathscr{L}^0$ is a submodule of $k|_B^G$). So, to show that $H^1(G,V)=0$, it suffices to show that $[k|_B^G : V]=0$.

Since $V \in \mathrm{Irr}(G|(T,X))$, V is in the head and socle of $R_T^G(X)$ and ${}^*R_T^G(V) \neq 0$. Now, since W gives a full set of (B,B)-double coset representatives in G,

$$ {}^*R_T^G(R_T^G(X)) \cong \bigoplus_{w\in W} R^T_{{}^wT\cap T}\, {}^*R^{{}^wT}_{{}^wT\cap T}({}^wX) $$

by the Mackey decomposition (for $w \in W$, ${}^wX$ is the ${}^wT$-module with underlying space X and ${}^wT$-action given by $wtw^{-1}.x = t.x$ for any $t \in T$, $x \in X$). Since $W = N_G(T)/T$, ${}^wT = wTw^{-1} = T$ for any $w \in W$. Therefore, the functors $R^T_{{}^wT\cap T}$ and ${}^*R^{{}^wT}_{{}^wT\cap T}$ are equal to the identity functor on $kT-\mathrm{mod}$ for all $w \in W$, and it follows that

$$ {}^*R_T^G(R_T^G(X)) \cong \bigoplus_{w\in W} X. $$

Since ${}^*R_T^G(R_T^G(X))$ is completely reducible, ${}^*R_T^G(V)$ is a (non-zero) direct summand of ${}^*R_T^G(R_T^G(X))$. Thus, ${}^*R_T^G(V)$ is a direct sum of copies of X as a kT-module.

Assume, for contradiction, that V is a composition factor of $k|_B^G = R_T^G(k)$. Applying the Mackey decomposition as in the paragraph above, we have

$$ {}^*R_T^G(R_T^G(k)) \cong \bigoplus_{w\in W} R^T_{{}^wT\cap T}\, {}^*R^{{}^wT}_{{}^wT\cap T}({}^wk) \cong \bigoplus_{w\in W} k. $$

So, if V is a composition factor of $R_T^G(k)$, the non-zero kT-module ${}^*R_T^G(V)$ is a direct sum of copies of k. This is impossible since $k \neq X$, so we conclude that $[k|_B^G : V]=0$.

□

Theorem 12.4.3. *Let G be a finite group of Lie type (of arbitrary finite rank) having a split BN-pair of characteristic p, and let V be an irreducible kG-module such that $V \notin \mathrm{Irr}_k(G|B)$. Then,* $\dim H^1(G,V) \leq 1$.

Proof. We proceed by induction on the rank of the Weyl group of G. The base case (i.e., the case of rank 1 groups) has been handled in Section 12.3.

For the inductive step, we assume that $\mathrm{rk}(G) > 1$ and that the statement of the theorem holds for all groups of rank strictly less than the rank of G. Suppose that V belongs to the Harish-Chandra series $\mathrm{Irr}_k(G|(L,X))$ for some cuspidal irreducible kL-module X. If $L=G$ or $L=T$, the theorem is proved by Propositions 12.4.1 and 12.4.2, respectively. Thus, we may assume that $T \subsetneq L \subsetneq G$.

Since $V \subseteq \mathrm{soc}(R_L^G(X))$, we have a short exact sequence of the form $0 \to V \to R_L^G(X) \to M \to 0$ for some kG-module M. We consider the induced long exact sequence

$$0 \to \mathrm{Hom}_{kG}(k, V) \to \mathrm{Hom}_{kG}(k, R_L^G(X)) \to \mathrm{Hom}_{kG}(k, M) \to \mathrm{Ext}^1_{kG}(k, V) \to \mathrm{Ext}^1_{kG}(k, R_L^G(X)) \to \cdots .$$

By exactness, we have

$$\dim \mathrm{Ext}^1_{kG}(k, V) \leq \dim \mathrm{Hom}_{kG}(k, M) + \dim \mathrm{Ext}^1_{kG}(k, R_L^G(X)). \tag{12.4}$$

We claim that $\dim \mathrm{Hom}_{kG}(k, M) = 0$. We have $\dim \mathrm{Hom}_{kG}(k, M) = [\mathrm{soc}(M) : k] \leq [M : k] \leq [R_L^G(X) : k]$ (where the last inequality follows since M is a quotient of $R_L^G(X)$). Therefore, it suffices to show that $[R_L^G(X) : k] = 0$. To prove that this is the case, we consider the kL-module ${}^*R_L^G(R_L^G(X))$. Assume that $L = L_I$ is the Levi complement of the standard parabolic subgroup $P = P_I$ ($\emptyset \neq I \subsetneq S$). Then, the subset ${}^IW^I = {}^IW \cap W^I$ of W is a full set of (P, P)-double coset representatives in G (here, ${}^IW$ denotes the set of shortest right coset representatives of W_I in W and W^I denotes the set of shortest left coset representatives of W_I in W). So, by the Mackey decomposition,

$${}^*R_L^G(R_L^G(X)) \cong \bigoplus_{w \in {}^IW^I} R^L_{{}^wL \cap L} \, {}^*R^{{}^wL}_{{}^wL \cap L}({}^wX).$$

But, since X is a cuspidal kL-module, ${}^wX$ is a cuspidal $k({}^wL)$-module. Therefore, the only non-zero terms in the direct sum above correspond to those $w \in {}^IW^I$ for which ${}^wL \cap L = {}^wL$ or, equivalently, for which $w \in N_G(L)$. Now, for all $w \in {}^IW^I \cap N_G(L)$, $R^L_{{}^wL \cap L}$ and ${}^*R^{{}^wL}_{{}^wL \cap L}$ are the identity functors on the category of left kL-modules. So, we have

$${}^*R_L^G(R_L^G(X)) \cong \bigoplus_{w \in {}^IW^I \cap N_G(L)} X.$$

Since $L \neq T$, k is not a cuspidal kL-module. Therefore, $k \neq X$, which means that k cannot be a composition factor of $R_L^G(X)$ (otherwise, $k = {}^*R_L^G(k)$ would be a direct summand of the completely reducible module ${}^*R_L^G(R_L^G(X))$, which is impossible since $k \neq X$). Hence, $[R_L^G(X) : k] = 0$, as needed.

Since $\dim \mathrm{Hom}_{kG}(k, M) = 0$, (12.4) yields $\dim \mathrm{Ext}^1_{kG}(k, V) \leq \dim \mathrm{Ext}^1_{kG}(k, R_L^G(X))$. But, by the Eckmann-Shapiro Lemma, $\mathrm{Ext}^1_{kG}(k, R_L^G(X)) = \mathrm{Ext}^1_{kG}(k, \tilde{X}|_P^G) \cong \mathrm{Ext}^1_{kP}(k, \tilde{X})$ (where P is the parabolic subgroup with Levi complement L). Since $[P : L] = |U_P|$ is a power of p (and, hence, not divisible by $\mathrm{char}(k) = r$), the restriction homomorphism $\mathrm{Ext}^1_{kP}(k, \tilde{X}) \to \mathrm{Ext}^1_{kL}(k, X)$ is injective. Therefore, $\dim \mathrm{Ext}^1_{kP}(k, \tilde{X}) \leq \dim \mathrm{Ext}^1_{kL}(k, X)$. Now, since X is a cuspidal irreducible kL-module, $\dim \mathrm{Ext}^1_{kL}(k, X) = \dim H^1(L, X) \leq 1$ by the inductive hypothesis, and it follows that

$$\dim H^1(G, V) = \dim \mathrm{Ext}^1_{kG}(k, V) \leq \dim \mathrm{Ext}^1_{kL}(k, X) \leq 1.$$

$\square$

13 A Bound on the Dimension of Ext^1 Between Principal Series Irreducible Modules in Cross Characteristic

13.1 Introduction

Let G be a finite group of Lie type, defined in characteristic $p > 0$, and assume that the BN-pair of G is split. Let (W, S) be the Coxeter system corresponding to the BN-pair structure on G.

Let k be an algebraically closed field of characteristic $r > 0$, $r \neq p$, and let e denote the r-rank of the maximal torus T (that is, e is the maximal dimension of an elementary abelian r-subgroup of T). In [35], Guralnick and Tiep prove that dim $H^1(G, V) \leq |W| + e$ if V is an irreducible kG-module with $V^B \neq 0$, and that dim $H^1(G, V) \leq 1$ if V is an irreducible kG-module with $V^B = 0$. An irreducible kG-module V satisfies the condition $V^B \neq 0$ if and only if V belongs to the unipotent principal Harish-Chandra series $\mathrm{Irr}_k(G|(T, k)) = \mathrm{Irr}_k(G|B)$. Since $H^1(G, V) \cong \mathrm{Ext}^1_{kG}(k, V)$ for any kG-module V, Guralnick and Tiep's result may be restated as follows. Given an irreducible kG-module V,

$$\dim\, \mathrm{Ext}^1_{kG}(k, V) \leq \begin{cases} 1 & \text{if } V \notin \mathrm{Irr}_k(G|B) \\ |W| + e & \text{if } V \in \mathrm{Irr}_k(G|B). \end{cases}$$

The purpose of this section is to present a partial generalization of this result. Let Y and V be two principal series representations (in the sense of [31, Example 4.2.8]), with Y belonging to the Harish-Chandra series $\mathrm{Irr}_k(G|(T, X))$ and V belonging to the Harish-Chandra series $\mathrm{Irr}_k(G|(T, X'))$ for some (necessarily one-dimensional) irreducible kT-modules X and X'. We will show that

(1) dim $\mathrm{Ext}^1_{kG}(Y, V) = 0$ if $\mathrm{Irr}_k(G|(T, X)) \neq \mathrm{Irr}_k(G|(T, X'))$, and

(2) dim $\mathrm{Ext}^1_{kG}(Y, V) \leq |W|^2 + \min(\dim Y, \dim V)e$ if $\mathrm{Irr}_k(G|(T, X)) = \mathrm{Irr}_k(G|(T, X'))$.

Remark 13.1.1. If Y and V both belong to the unipotent principal series $\mathrm{Irr}_k(G|B) = \mathrm{Irr}_k(G|(T, k))$, the bound in (2) can be improved. By Theorem 11.4.3, dim $\mathrm{Ext}^1_{kG}(Y, V) \leq |W| + \min(\dim Y, \dim V)e$ when Y and V belong to $\mathrm{Irr}_k(G|B)$.[16] In particular, if we take $Y = k \in \mathrm{Irr}_k(G|B)$, we recover Guralnick and Tiep's bound.

[16]By Theorem 11.4.3, dim $\mathrm{Ext}^1_{kG}(Y, V) \leq |W| + (\dim V)e$ when $Y, V \in \mathrm{Irr}_k(G|B)$. But, an argument analogous to that given in Corollary 13.3.4 shows that dim $\mathrm{Ext}^1_{kG}(Y, V) \leq |W| + \min(\dim Y, \dim V)e$ when $Y, V \in \mathrm{Irr}_k(G|B)$.

13.2 A Bound on the Dimension of $\mathrm{Ext}^1_{kG}(Y,V)$ when Y and V belong to distinct Principal Series

Throughout this section, we will assume that the irreducible kG-module Y belongs to the principal series $\mathrm{Irr}_k(G|(T,X))$ and that the irreducible kG-module V belongs to the principal series $\mathrm{Irr}_k(G|(T,X'))$, with $\mathrm{Irr}_k(G|(T,X)) \neq \mathrm{Irr}_k(G|(T,X'))$. Our goal is to prove that $\mathrm{Ext}^1_{kG}(Y,V) = 0$. Since $Y \in \mathrm{Irr}_k(G|(T,X))$, Y is in the head of $R^G_T(X)$. Let $\mathscr{L}^0$ denote a maximal submodule of $R^G_T(X)$ with $R^G_T(X)/\mathscr{L}^0 \cong Y$. In the next result, we generalize [35, Theorem 2.2] to show that the dimension of $\mathrm{Ext}^1_{kG}(Y,Z)$ is determined by the structure of $\mathrm{head}(\mathscr{L}^0)$ when Z is an irreducible kG-module which does not belong to the Harish-Chandra series $\mathrm{Irr}_k(G|(T,X))$.

Theorem 13.2.1. *Let Y be an irreducible kG-module in the principal series $\mathrm{Irr}_k(G|(T,X))$ (where X is a one-dimensional kT-module). Then, if Z is an irreducible kG-module such that $Z \notin \mathrm{Irr}_k(G|(T,X))$, $\dim \mathrm{Ext}^1_{kG}(Y,Z) = [\mathrm{head}(\mathscr{L}^0) : Z]$ (where $[\mathrm{head}(\mathscr{L}^0) : Z]$ denotes the multiplicity of Z as a composition factor of $\mathrm{head}(\mathscr{L}^0)$).*

Proof. By definition of $\mathscr{L}^0$, we have a short exact sequence of kG-modules $0 \to \mathscr{L}^0 \to R^G_T(X) \to Y \to 0$, which induces the long exact sequence

$$0 \to \mathrm{Hom}_{kG}(Y,Z) \to \mathrm{Hom}_{kG}(R^G_T(X),Z) \to \mathrm{Hom}_{kG}(\mathscr{L}^0,Z) \to \mathrm{Ext}^1_{kG}(Y,Z) \to \mathrm{Ext}^1_{kG}(R^G_T(X),Z) \to \cdots .$$

Now, since Z does not belong to the principal series $\mathrm{Irr}_k(G|(T,X))$, $Z \neq Y$ and Z is not in the head of $R^G_T(X)$. Therefore, $\mathrm{Hom}_{kG}(Y,Z) = 0$ and $\mathrm{Hom}_{kG}(R^G_T(X),Z) = 0$, which means that the exact sequence above reduces to

$$0 \to \mathrm{Hom}_{kG}(\mathscr{L}^0,Z) \to \mathrm{Ext}^1_{kG}(Y,Z) \to \mathrm{Ext}^1_{kG}(R^G_T(X),Z) \to \cdots .$$

Therefore, to prove the statement of the theorem, it is enough to show that $\mathrm{Ext}^1_{kG}(R^G_T(X),Z) = 0$. For, if $\mathrm{Ext}^1_{kG}(R^G_T(X),Z) = 0$, then $\mathrm{Ext}^1_{kG}(Y,Z) \cong \mathrm{Hom}_{kG}(\mathscr{L}^0,Z)$ by the exactness of the sequence above, which means that $\dim \mathrm{Ext}^1_{kG}(Y,Z) = \dim \mathrm{Hom}_{kG}(\mathscr{L}^0,Z) = [\mathrm{head}(\mathscr{L}^0) : Z]$.

We will now prove that $\mathrm{Ext}^1_{kG}(R^G_T(X),Z) = 0$. By definition of $R^G_T(X)$ and by the Eckmann-Shapiro Lemma, we have $\mathrm{Ext}^1_{kG}(R^G_T(X),Z) = \mathrm{Ext}^1_{kG}(\tilde{X}|^G_B,Z) \cong \mathrm{Ext}^1_{kB}(\tilde{X},Z)$ (where $\tilde{X}$ is the inflation of X from T to B via the surjective homomorphism $B \twoheadrightarrow T$ with kernel U). Thus, it suffices to prove that $\mathrm{Ext}^1_{kB}(\tilde{X},Z) = 0$.

Let $A = O_{r'}(B)$ be the biggest normal subgroup of B of order prime to $r = \mathrm{char}(k)$. Then, B/A is an r-group. We claim that $\mathrm{Hom}_{kA}(\tilde{X},Z) = 0$. Since $Z \notin \mathrm{Irr}_k(G|(T,X))$, $Z \not\subseteq \mathrm{head}(R^G_T(X))$. Thus, $0 = \mathrm{Hom}_{kG}(R^G_T(X),Z) = \mathrm{Hom}_{kG}(\tilde{X}|^G_B,Z) \cong \mathrm{Hom}_{kB}(\tilde{X},Z)$ (where the isomorphism $\mathrm{Hom}_{kG}(\tilde{X}|^G_B,Z) \cong \mathrm{Hom}_{kB}(\tilde{X},Z)$ follows by Frobenius reciprocity). Now, the group B acts on the k-vector space $\mathrm{Hom}_k(\tilde{X},Z)$ (given an element $b \in B$ and a k-vector space homomorphism $\phi \in \mathrm{Hom}_k(\tilde{X},Z)$, $b.\phi \in \mathrm{Hom}_k(\tilde{X},Z)$

is defined by $(b.\phi)(x) = b\phi(b^{-1}x)$ for any $x \in \tilde{X}$), and we have an isomorphism $(\mathrm{Hom}_k(\tilde{X}, Z))^B \cong \mathrm{Hom}_{kB}(\tilde{X}, Z)$. So, since $\mathrm{Hom}_{kB}(\tilde{X}, Z) = 0$, it follows that $(\mathrm{Hom}_k(\tilde{X}, Z))^B = 0$. Now, since A is a normal subgroup of B, we have $0 = (\mathrm{Hom}_k(\tilde{X}, Z))^B = [(\mathrm{Hom}_k(\tilde{X}, Z))^A]^{B/A}$. But, B/A is an r-group and $\mathrm{char}(k) = r$, so this is possible only if $(\mathrm{Hom}_k(\tilde{X}, Z))^A = 0$ (otherwise, the r-group B/A would have a non-zero fixed point on the k-vector space $(\mathrm{Hom}_k(\tilde{X}, Z))^A$). Therefore, $\mathrm{Hom}_{kA}(\tilde{X}, Z) \cong (\mathrm{Hom}_k(\tilde{X}, Z))^A = 0$.

Now, applying the five-term inflation-restriction exact sequence on cohomology to the kB-module $\tilde{X}^* \otimes_k Z$ (where $\tilde{X}^*$ is the k-linear dual of $\tilde{X}$), we have:

$0 \to H^1(B/A, (\tilde{X}^* \otimes_k Z)^A) \to H^1(B, \tilde{X}^* \otimes_k Z) \to H^1(A, \tilde{X}^* \otimes_k Z)^{B/A} \to H^2(B/A, (\tilde{X}^* \otimes_k Z)^A) \to H^2(B, \tilde{X}^* \otimes_k Z)$.

Since $r \nmid |A|$, kA is semisimple by Maschke's Theorem, which means that $H^1(A, \tilde{X}^* \otimes_k Z)^{B/A} = 0$. Since $\tilde{X}^* \otimes_k Z \cong \mathrm{Hom}_k(\tilde{X}, Z)$, we have $(\tilde{X}^* \otimes_k Z)^A \cong (\mathrm{Hom}_k(\tilde{X}, Z))^A \cong \mathrm{Hom}_{kA}(\tilde{X}, Z) = 0$, so that $H^1(B/A, (\tilde{X}^* \otimes_k Z)^A) = 0$. We conclude that $H^1(B, \tilde{X}^* \otimes_k Z) = 0$ by exactness of the sequence above. But, $H^1(B, \tilde{X}^* \otimes_k Z) \cong \mathrm{Ext}^1_{kB}(k, \tilde{X}^* \otimes_k Z) \cong \mathrm{Ext}^1_{kB}(\tilde{X}, Z)$, so it follows that $\mathrm{Ext}^1_{kB}(\tilde{X}, Z) = 0$, as needed.

□

Proposition 13.2.2. *Let Y be an irreducible kG-module in the principal series $\mathrm{Irr}_k(G|(T, X))$, and let V be an irreducible kG-module in the principal series $\mathrm{Irr}_k(G|(T, X'))$, with $\mathrm{Irr}_k(G|(T, X)) \neq \mathrm{Irr}_k(G|(T, X'))$. Then, $[R_T^G(X) : V] = 0$.*

Proof. Since $V \in \mathrm{Irr}_k(G|(T, X'))$, V is a composition factor of the head and socle of $R_T^G(X')$ and ${}^*R_T^G(V) \neq 0$. Now, since W gives a full set of (B, B)-double coset representatives in G,

$$ {}^*R_T^G(R_T^G(X')) \cong \bigoplus_{w \in W} R^T_{{}^wT \cap T} \, {}^*R^{{}^wT}_{{}^wT \cap T}({}^wX') $$

by the Mackey decomposition (for $w \in W$, ${}^wX'$ is the ${}^wT$-module with underlying space X' and ${}^wT$-action given by $wtw^{-1}.x = t.x$ for any $t \in T$, $x \in X'$). But, since $W = N_G(T)/T$, ${}^wT = wTw^{-1} = T$ for any $w \in W$. Therefore, the functors $R^T_{{}^wT \cap T}$ and ${}^*R^{{}^wT}_{{}^wT \cap T}$ are equal to the identity functor on $kT - \mathrm{mod}$ for all $w \in W$, and it follows that

$$ {}^*R_T^G(R_T^G(X')) \cong \bigoplus_{w \in W} {}^wX'. $$

For any $w \in W$, ${}^wX'$ is an irreducible kT-module, which means that ${}^*R_T^G(R_T^G(X'))$ is completely reducible. Since ${}^*R_T^G$ is exact, ${}^*R_T^G(V)$ is a non-zero kT-submodule of ${}^*R_T^G(R_T^G(X'))$. Thus, there must be some subset $\Omega \subseteq W$ such that

$$ {}^*R_T^G(V) \cong \bigoplus_{w \in \Omega} {}^wX'. \tag{13.1} $$

Suppose, for contradiction, that $[R_T^G(X) : V] \neq 0$. Applying the Mackey decomposition as in the paragraph above, we have ${}^*R_T^G(R_T^G(X)) \cong \bigoplus_{w \in W} {}^wX$. So, if V is a composition factor of

$R_T^G(X)$, then ${}^*R_T^G(V)$ is a non-zero submodule of ${}^*R_T^G(R_T^G(X))$ and ${}^wX \subseteq {}^*R_T^G(V)$ for some $w \in W$. By (13.1), this means that ${}^wX \cong {}^{w'}X'$ for some $w' \in \Omega$, so that $X' \cong {}^{(w')^{-1}w}X$. But, if X' is a twist of X by an element of W, then $\mathrm{Irr}_k(G|(T,X)) = \mathrm{Irr}_k(G|(T,X'))$, contradicting the assumption in the statement of the proposition.

□

Corollary 13.2.3. *Let Y be an irreducible kG-module in the principal series $\mathrm{Irr}_k(G|(T,X))$, and let V be an irreducible kG-module in the principal series $\mathrm{Irr}_k(G|(T,X'))$, with $\mathrm{Irr}_k(G|(T,X)) \neq \mathrm{Irr}_k(G|(T,X'))$. Then, $\mathrm{Ext}^1_{kG}(Y,V) = 0$.*

Proof. As above, let $\mathscr{L}^0$ be a submodule of $R_T^G(X)$ with $R_T^G(X)/\mathscr{L}^0 \cong Y$. Then, by Theorem 13.2.1 and Proposition 13.2.2, $\dim \mathrm{Ext}^1_{kG}(Y,V) = [\mathrm{head}(\mathscr{L}^0) : V] \leq [\mathscr{L}^0 : V] \leq [R_T^G(X) : V] = 0$.

□

13.3 A Bound on the Dimension of $\mathrm{Ext}^1_{kG}(Y,V)$ when Y and V belong to the same Principal Series

The results of this section generalize the results of Chapter 11. From this point on, we will assume that Y and V belong to the same principal series $\mathrm{Irr}_k(G|(T,X))$ (where X is any one-dimensional kT-module). Our goal is to prove that $\dim \mathrm{Ext}^1_{kG}(Y,V) \leq |W|^2 + \min(\dim Y, \dim V)e$ (where e is the r-rank of T). We begin with a lemma which generalizes Lemma 11.2.1 and provides a bound on the number of times an irreducible representation in the principal series $\mathrm{Irr}_k(G|(T,X))$ can appear as a composition factor of the induced module $R_T^G(X)$.

Lemma 13.3.1. *If Z is an irreducible module in the principal series $\mathrm{Irr}_k(G|(T,X))$, then $[R_T^G(X) : Z] \leq |W|$.*

Proof. Since W gives a full set of (B,B)-double coset representatives in G and ${}^wT = T$ for any $w \in W$, the Mackey decomposition yields

$$ {}^*R_T^G(R_T^G(X)) \cong \bigoplus_{w\in W} R^T_{{}^wT\cap T}\, {}^*R^{{}^wT}_{{}^wT\cap T}({}^wX) \cong \bigoplus_{w\in W} {}^wX. $$

Now, ${}^wX$ is an irreducible one-dimensional kT-module for every $w \in W$, which means that the kT-module ${}^*R_T^G(R_T^G(X))$ is completely reducible and $\dim {}^*R_T^G(R_T^G(X)) = |W|$. On the other hand, since ${}^*R_T^G$ is exact, ${}^*R_T^G(R_T^G(X))$ has the following direct sum decomposition as a kT-module:

$$ {}^*R_T^G(R_T^G(X)) \cong \bigoplus_{Z'\in \mathrm{Irr}_k(G)} {}^*R_T^G(Z')^{\oplus[R_T^G(X):Z']}. $$

Therefore,

$$ \dim {}^*R_T^G(R_T^G(X)) = \sum_{Z'\in \mathrm{Irr}_k(G)} [R_T^G(X) : Z'] \dim {}^*R_T^G(Z'). \tag{13.2} $$

By assumption, the irreducible kG-module Z belongs to the principal series $\mathrm{Irr}_k(G|(T,X))$, which means that ${}^*R_T^G(Z) \neq 0$. Thus, $[R_T^G(X) : Z]$ appears with non-zero coefficient in (13.2), and it follows that $[R_T^G(X) : Z] \leq \dim {}^*R_T^G(R_T^G(X)) = |W|$.

□

To establish the desired bound on $\mathrm{Ext}^1_{kG}(Y,V)$, we will work with the unique Sylow r-subgroup T_r of the abelian group T. Hence, we will have to break our proof into two cases, depending on whether or not $|B|$ is divisible by the characteristic r of k.

13.3.1 Case I: $r \nmid |B|$

Assume that the characteristic r of k does not divide $|B|$.

Theorem 13.3.2. *Let Y and V be irreducible kG-modules in the principal series* $\mathrm{Irr}_k(G|(T,X))$. *Then,* $\dim \mathrm{Ext}^1_{kG}(Y,V) \leq |W|$.

Proof. Since $r \nmid |B|$, the group algebra kB is semisimple and every kB-module is projective and injective. In particular, $\tilde{X}$ is an injective kB-module, which means that $R_T^G(X) = \tilde{X}|_B^G$ is also injective since induction from B to G is exact. Now, since $V \in \mathrm{Irr}_k(G|(T,X))$, V is contained in the socle of $R_T^G(X)$. Therefore, we have a short exact sequence of kG-modules

$$0 \to V \to R_T^G(X) \to M \to 0$$

(where $M \cong R_T^G(X)/V$). Since $R_T^G(X)$ is injective, $\mathrm{Ext}^1_{kG}(Y, R_T^G(X)) = 0$, so the short exact sequence above induces the exact sequence

$$0 \to \mathrm{Hom}_{kG}(Y,V) \to \mathrm{Hom}_{kG}(Y, R_T^G(X)) \to \mathrm{Hom}_{kG}(Y,M) \to \mathrm{Ext}^1_{kG}(Y,V) \to 0.$$

Therefore, $\dim \mathrm{Ext}^1_{kG}(Y,V) \leq \dim \mathrm{Hom}_{kG}(Y,M) = [\mathrm{soc}(M) : Y] \leq [M : Y] \leq [R_T^G(X) : Y] \leq |W|$, where the last inequality follows by Lemma 13.3.1. □

13.3.2 Case II: $r \mid |B|$

Assume that $|B|$ is divisible by the characteristic r of k. As above, let T_r be the (unique) r-Sylow subgroup of T. Since $r \mid |B| = |U||T|$ and U is a p-group, $r \mid |T|$. Hence, T_r is a non-trivial (abelian) r-subgroup of T. A key step in proving that $\dim \mathrm{Ext}^1_{kG}(Y,V) \leq |W|^2 + \min(\dim Y, \dim V)e$ will be to reduce the problem to finding a bound on the dimension of a cohomology group of T_r. We will use the results on the cohomology of T_r developed in Chapter 11.

As in Chapter 11, we will work with the completely reducible permutation module $k|_{T_r}^T$. Let Z be any irreducible kT-module. Then, Z is one-dimensional, which means that the restriction $Z\downarrow_{T_r}$ of Z to T_r is irreducible. Since the only irreducible module for an r-group in characteristic r is the trivial module k, it follows that $Z\downarrow_{T_r} = k$. So, by Frobenius reciprocity, we have $\mathrm{Hom}_{kT}(k|_{T_r}^T, Z) \cong \mathrm{Hom}_{kT_r}(k, Z\downarrow_{T_r}) = \mathrm{Hom}_{kT_r}(k,k) \cong k$, from which it follows that Z occurs exactly once as a composition factor of the completely reducible module $k|_{T_r}^T$. Thus, $k|_{T_r}^T$ contains every irreducible kT-module as a direct summand exactly once.

Theorem 13.3.3. *Let Y and V be irreducible kG-modules in the principal series $\mathrm{Irr}_k(G|(T,X))$. Then,* $\dim \mathrm{Ext}^1_{kG}(Y,V) \le |W|^2 + (\dim V)e$.

Proof. Since ${}^wT = T$ for any $w \in W$, the $k({}^wT)$-module ${}^wX$ is also a kT-module for each $w \in W$. Therefore, the set $\widetilde{\mathcal{I}} = \{{}^wX \mid w \in W\}$ is a subset of $\mathrm{Irr}_k(T)$ consisting of at most $|W|$ distinct irreducible kT-modules. Let $\mathcal{I}$ be a subset of $\widetilde{\mathcal{I}}$ consisting of one representative of each isomorphism class of irreducible kT-modules appearing in $\widetilde{\mathcal{I}}$. By the independence property of Harish-Chandra induction, $R_T^G(X) \cong R_T^G(X')$ for all $X' \in \mathcal{I}$.

Since every irreducible kT-module occurs exactly once as a composition factor of the completely reducible kT-module $k|_{T_r}^T$, we have a direct sum decomposition

$$k|_{T_r}^T \cong M \oplus \big(\bigoplus_{X' \in \mathcal{I}} X'\big),$$

where M is a completely reducible kT-module which does not contain X' as a composition factor for any $X' \in \mathcal{I}$. Applying the (exact) Harish-Chandra induction functor R_T^G, we can write

$$R_T^G(k|_{T_r}^T) \cong R_T^G(M) \oplus \big(\bigoplus_{X' \in \mathcal{I}} R_T^G(X')\big). \tag{13.3}$$

Now, since $Y \in \mathrm{Irr}_k(G|(T,X))$, Y is in the head of $R_T^G(X)$, which means that Y is in the head of $R_T^G(k|_{T_r}^T)$ by (13.3). Therefore, we have a short exact sequence of kG-modules

$$0 \to M' \to R_T^G(k|_{T_r}^T) \to Y \to 0$$

(where M' is a kG-submodule of $R_T^G(k|_{T_r}^T)$ such that $R_T^G(k|_{T_r}^T)/M' \cong Y$), which gives rise to the long exact sequence

$$0 \to \mathrm{Hom}_{kG}(Y,V) \to \mathrm{Hom}_{kG}(R_T^G(k|_{T_r}^T), V) \to \mathrm{Hom}_{kG}(M',V) \to \mathrm{Ext}^1_{kG}(Y,V) \to \mathrm{Ext}^1_{kG}(R_T^G(k|_{T_r}^T), V) \to \cdots .$$

By exactness, $\dim \mathrm{Ext}^1_{kG}(Y,V) \le \dim \mathrm{Hom}_{kG}(M',V) + \dim \mathrm{Ext}^1_{kG}(R_T^G(k|_{T_r}^T), V)$. So, to prove the theorem, it is enough to show that $\dim \mathrm{Hom}_{kG}(M',V) \le |W|^2$ and that $\dim \mathrm{Ext}^1_{kG}(R_T^G(k|_{T_r}^T), V) \le (\dim V)e$.

First, we show that $\dim \mathrm{Hom}_{kG}(M',V) \le |W|^2$. We have $\dim \mathrm{Hom}_{kG}(M',V) = [\mathrm{head}(M') : V] \le [M' : V] \le [R_T^G(k|_{T_r}^T) : V]$. By (13.3),

$$[R_T^G(k|_{T_r}^T) : V] = [R_T^G(M) : V] + \sum_{X' \in \mathcal{I}} [R_T^G(X') : V],$$

so $\dim \operatorname{Hom}_{kG}(M', V) \leq [R_T^G(M) : V] + \sum_{X' \in \mathcal{I}} [R_T^G(X') : V]$. But, for all $X' \in \mathcal{I}$, $R_T^G(X') \cong R_T^G(X)$ by the independence property. Since $V \in \operatorname{Irr}_k(G|(T, X))$, $[R_T^G(X) : V] \leq |W|$ by Lemma 13.3.1, which means that

$$\sum_{X' \in \mathcal{I}} [R_T^G(X') : V] = \sum_{X' \in \mathcal{I}} [R_T^G(X) : V] \leq \sum_{X' \in \mathcal{I}} |W| \leq |\mathcal{I}|\ |W| \leq |W|^2.$$

So, to show that $\dim \operatorname{Hom}_{kG}(M', V) \leq |W|^2$, it suffices to show that $[R_T^G(M) : V] = 0$. Now, since M is a completely reducible kT-module which does not contain X' as a composition factor for any $X' \in \mathcal{I}$, M has a decomposition $M \cong \bigoplus_{i=0}^{m} Z_i$ such that each Z_i $(1 \leq i \leq m)$ is an irreducible kT-module with $Z_i \ncong X'$ for all $X' \in \mathcal{I}$. So, for all i, $1 \leq i \leq m$, $Z_i \ncong {}^wX$ for all $w \in W$, which means that the Harish-Chandra series $\operatorname{Irr}_k(G|(T, Z_i))$ and $\operatorname{Irr}_k(G|(T, X))$ are distinct. Therefore, by Proposition 13.2.2, $[R_T^G(Z_i) : V] = 0$ for all i, and it follows that $[R_T^G(M) : V] = [\bigoplus_{i=0}^{m} R_T^G(Z_i) : V] = \sum_{i=0}^{m} [R_T^G(Z_i) : V] = 0$.

It remains to check that $\dim \operatorname{Ext}^1_{kG}(R_T^G(k|_{T_r}^T), V) \leq (\dim V)e$. Using the definition of the functor R_T^G and the Eckmann-Shapiro Lemma, we have $\operatorname{Ext}^1_{kG}(R_T^G(k|_{T_r}^T), V) = \operatorname{Ext}^1_{kG}((\widetilde{k|_{T_r}^T})|_B^G, V) \cong \operatorname{Ext}^1_{kB}(\widetilde{k|_{T_r}^T}, V)$. Now, since $B = U \rtimes T$ and $r \nmid |U| = [B : T]$, the restriction map $\operatorname{Ext}^1_{kB}(\widetilde{k|_{T_r}^T}, V) \to \operatorname{Ext}^1_{kT}(\widetilde{k|_{T_r}^T, V)} = \operatorname{Ext}^1_{kT}(k|_{T_r}^T, V)$ is injective. It follows that $\dim \operatorname{Ext}^1_{kG}(R_T^G(k|_{T_r}^T), V) = \dim \operatorname{Ext}^1_{kB}(\widetilde{k|_{T_r}^T}, V) \leq \dim \operatorname{Ext}^1_{kT}(k|_{T_r}^T, V)$. Applying the Eckmann-Shapiro Lemma once more, $\operatorname{Ext}^1_{kT}(k|_{T_r}^T, V) \cong \operatorname{Ext}^1_{kT_r}(k, V)$. Therefore, $\dim \operatorname{Ext}^1_{kG}(R_T^G(k|_{T_r}^T), V) \leq \dim \operatorname{Ext}^1_{kT_r}(k, V) \leq (\dim V)e$, where the last inequality holds by Lemma 11.4.2.

□

Corollary 13.3.4. *Let Y and V be irreducible kG-modules in the principal series* $\operatorname{Irr}_k(G|(T, X))$. *Then,* $\dim \operatorname{Ext}^1_{kG}(Y, V) \leq |W|^2 + \min(\dim \mathrm{Y}, \dim \mathrm{V})\mathrm{e}$.

Proof. By Theorem 13.3.3, $\dim \operatorname{Ext}^1_{kG}(Y, V) \leq |W|^2 + (\dim V)e$. Since Y and V belong to the principal series $\operatorname{Irr}_k(G|(T, X))$, Y and V occur in both the head and the socle of $R_T^G(X)$. Hence, the k-linear duals Y^* and V^* occur in both the head and the socle of $(R_T^G(X))^* \cong R_T^G(X^*)$. Thus, Y^* and V^* both belong to the principal series $\operatorname{Irr}_k(G|(T, X^*))$, and $\dim \operatorname{Ext}^1_{kG}(Y, V) = \dim \operatorname{Ext}^1_{kG}(V^*, Y^*) \leq |W|^2 + (\dim Y^*)e = |W|^2 + (\dim Y)e$ (where the inequality $\dim \operatorname{Ext}^1_{kG}(V^*, Y^*) \leq |W|^2 + (\dim Y^*)e$ follows by another application of Theorem 13.3.3).

□

14 A Bound on the Dimension of Ext^1 between a Unipotent Principal Series Representation and an Irreducible Outside the Unipotent Principal Series

14.1 Introduction

Let G be a finite group of Lie type, defined in characteristic $p > 0$, and assume that the BN-pair of G is split. Let (W, S) be the Coxeter system corresponding to the BN-pair structure on G. Let Φ be the corresponding root system, Φ^+ the set of positive roots, and Π the set of simple roots in Φ^+. Given a root $\alpha \in \Phi$, let U_α denote the corresponding root subgroup of G.

Let k be an algebraically closed field of characteristic $r > 0$, $r \neq p$, and let e denote the r-rank of the maximal torus T (that is, e is the maximal dimension of an elementary abelian r-subgroup of T). In [35], Guralnick and Tiep prove that if V is an irreducible kG-module, then

$$\dim \mathrm{Ext}^1_{kG}(k, V) \leq \begin{cases} 1 & \text{if } V \notin \mathrm{Irr}_k(G|B) \\ |W| + e & \text{if } V \in \mathrm{Irr}_k(G|B). \end{cases}$$

In this chapter, we will assume certain additional conditions on the group G in order to find a bound on $\dim \mathrm{Ext}^1_{kG}(Y, V)$ when $Y \in \mathrm{Irr}_k(G|B)$ and $V \notin \mathrm{Irr}_k(G|B)$. Several key ideas behind the proof of the main theorem (Theorem 14.3.3) can be found in [29], so we begin with a summary of the relevant results of [29].

14.2 Some Preliminaries from [29, Section 4]: The Steinberg Module and Harish-Chandra Series

14.2.1 Introduction

Let G be a finite group of Lie type with a split BN-pair and associated Coxeter system (W, S). Let Φ be the corresponding root system, Φ^+ the set of positive roots, and Π the set of simple roots. Let k be a field, and let St_k be the Steinberg module of G. In [29], Geck studies the composition factors of the Steinberg module. It is well known that St_k is irreducible if and only if $\mathrm{char}(k) \nmid [G : B]$. In the case that $\mathrm{char}(k) \mid [G : B]$ and St_k is not irreducible, Tinberg [48] proved that the socle of St_k is simple. However, many open questions remain concerning the structure of St_k when $\mathrm{char}(k) \mid [G : B]$. For instance, the length of a composition series of St_k is not known. This question is of particular interest to Geck; in [29], he works toward finding a bound on the length of a composition series of St_k.

Geck [29] uses the Hecke algebra associated to the Coxeter system (W, S) to prove that $\mathrm{soc}(\mathrm{St}_k)$ is irreducible (Tinberg uses the endomorphism algebra $\mathrm{End}_{kG}(k|_U^G)$ in his original proof). Geck's proof yields new information on the structure of St_k. For example, in the case that $G = \mathrm{GL}_n(q)$ (where q is a power of a prime p), Geck identifies the partition that parameterizes the socle of St_k [29, 3.6]. In [29, Section 4], Geck studies the relationship between St_k and Harish-Chandra series of irreducible kG-modules, which allows him to determine the composition length of St_k in certain cases [29, 4.7-4.14]. While the composition length of St_k is not needed to prove any of the original results presented in this dissertation, the relationship between St_k and Harish-Chandra series is crucial to the proof of Theorem 14.3.3.

14.2.2 The BN-Pair of G

Since G is a finite group of Lie type, there exists a connected reductive algebraic group $\mathbb{G}$ over the algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$ and a Steinberg endomorphism F of $\mathbb{G}$ such that $G = \mathbb{G}^F$. Let $\mathbb{B}$ be an F-stable Borel subgroup of $\mathbb{G}$, and let $\mathbb{T}$ be an F-stable maximal torus contained in $\mathbb{B}$. Then, the subgroups $B = \mathbb{B}^F$ and $N = N_{\mathbb{G}}(\mathbb{T})^F$ form a BN-pair in G. If $\mathbb{U}$ is the unipotent radical of $\mathbb{B}$, then $U = \mathbb{U}^F$ is the unipotent radical of $\mathbb{B}$.

Let $[\mathbb{U}, \mathbb{U}]$ be the commutator of $\mathbb{U}$ (by [29, Remark 2.5], $[\mathbb{U}, \mathbb{U}]$ is an F-stable closed connected normal subgroup of $\mathbb{U}$), and define $U^* = [\mathbb{U}, \mathbb{U}]^F$. (It is always true that $[U, U] \subseteq U^*$; however, the reverse containment does not hold in certain small characteristics.)

14.2.3 An r-modular system.

Let k be a field of characteristic $r > 0$, $r \neq p$. We will work with an r-modular system $(\mathcal{O}, K, k)$, where $\mathcal{O}$ is a discrete valuation ring with residue field k and field of fractions K (with $\mathrm{char}(K) = 0$). We will assume that the fields k and K are both large enough, meaning that they are splitting fields for G and all of its subgroups.

An $\mathcal{O}G$-module M will be called a lattice if M is finitely generated and free over $\mathcal{O}$. Given a lattice M, we let $KM := K \otimes_{\mathcal{O}} M$ and $\overline{M} := k \otimes_{\mathcal{O}} M$. It is a standard fact in modular representation theory that given a projective $\mathcal{O}G$-lattice M and an $\mathcal{O}G$-lattice M' (with M' not necessarily projective), $\dim \mathrm{Hom}_{KG}(KM, KM') = \dim \mathrm{Hom}_{kG}(\overline{M}, \overline{M'})$ [46, 15.4].

As recorded in [29] (pg. 14), Harish-Chandra induction and restriction are compatible with the r-modular system in the following sense. Let $J \subseteq S$ and let L_J denote the Levi complement of the standard parabolic subgroup P_J of G. Then, if X is an $\mathcal{O}L_J$-lattice,

$$KR_{L_J}^G(X) \cong R_{L_J}^G(KX) \text{ and } \overline{R_{L_J}^G(X)} \cong R_{L_J}^G(\overline{X}).$$

If Y is an $\mathcal{O}G$-lattice,

$$K^*R_{L_J}^G(Y) \cong {}^*R_{L_J}^G(KY) \text{ and } \overline{{}^*R_{L_J}^G(Y)} \cong {}^*R_{L_J}^G(\overline{Y}).$$

14.2.4 An $\mathcal{O}G$-lattice which yields the Steinberg module

Let $\mathrm{St}_{\mathcal{O}}=\mathcal{O}G\mathfrak{e}$ be the Steinberg module over $\mathcal{O}G$ (we will call $\mathrm{St}_{\mathcal{O}}$ the Steinberg lattice). Then, $K\mathrm{St}_{\mathcal{O}} \cong \mathrm{St}_K$ and $\overline{\mathrm{St}_{\mathcal{O}}} = \mathrm{St}_k$. Since $\mathrm{char}(K) = 0$, the Steinberg module St_K is irreducible. In [29, Section 4], Geck constructs another $\mathcal{O}G$-lattice which yields St_K upon base change to K. In the remainder of this section, we will describe this alternate lattice.

Let $\sigma : U \to K^{\times}$ be a group homomorphism, and define $\mathfrak{u}_\sigma := \sum_{u\in U} \sigma(u)u \in KG$. Since U is a p-subgroup of G, $r \nmid |U|$ and $\sigma(u) \in \mathcal{O}^{\times}$ for all $u \in U$, which means $\mathfrak{u}_\sigma \in \mathcal{O}G$. We have

$$\mathfrak{u}_\sigma^2 = \sum_{u,u'\in U} \big(\sigma(u)u\big)\big(\sigma(u')u'\big) = \sum_{u\in U}\sum_{u'\in U}\sigma(uu')uu' = \sum_{u\in U}\mathfrak{u}_\sigma = |U|\mathfrak{u}_\sigma.$$

Since $|U|$ is a unit in $\mathcal{O}$, $\frac{1}{|U|}\mathfrak{u}_\sigma$ is an idempotent in $\mathcal{O}G$ and the $\mathcal{O}G$-lattice $\Gamma_\sigma := \mathcal{O}G\mathfrak{u}_\sigma$ is projective.

By [29, 4.1], $\dim \mathrm{Hom}_{\mathcal{O}G}(\Gamma_\sigma, \mathrm{St}_{\mathcal{O}}) = 1$ and $\dim \mathrm{Hom}_{KG}(K\Gamma_\sigma, \mathrm{St}_K) = 1$.

Proposition 14.2.1. *([29, Proposition 4.2]) For any group homomorphism $\sigma : U \to K^{\times}$, there exists a unique $\mathcal{O}G$-sublattice $\Gamma'_\sigma \subseteq \Gamma_\sigma$ such that $K(\Gamma_\sigma/\Gamma'_\sigma) \cong \mathrm{St}_K$. If $\mathscr{S}_\sigma = \Gamma_\sigma/\Gamma'_\sigma$, the kG-module $D_\sigma := \overline{\mathscr{S}_\sigma}/\mathrm{rad}(\overline{\mathscr{S}}_\sigma)$ is simple and occurs exactly once as a composition factor of* St_k.

14.2.5 The Gelfand-Graev module

We will now assume that the center of $\mathbb{G}$ is connected. Let $\sigma : U \to K^{\times}$ be a fixed regular character (i.e., σ is a group homomorphism such that $U^* \subseteq \mathrm{Ker}(\sigma)$ and $\sigma|_{U_\alpha}$ is non-trivial for all $\alpha \in \Pi$). In this case, the projective $\mathcal{O}G$-module Γ_σ is called a Gelfand-Graev module for G (the Gelfand-Graev module is unique up to isomorphism when the center of $\mathbb{G}$ is connected). Since Γ_σ is a projective $\mathcal{O}G$-lattice, the r-modular reduction $\overline{\Gamma}_\sigma$ of Γ_σ is a projective kG-module.

For any $J \subseteq S$, let L_J be the corresponding Levi subgroup of G. By [29, 4.3], there is a Gelfand-Graev module Γ_σ^J for $\mathcal{O}L^J$. Therefore, [29, Proposition 4.2] yields an $\mathcal{O}L_J$-lattice $\mathscr{S}_\sigma^J = \Gamma_\sigma^J/(\Gamma_\sigma^J)'$ such that $D_\sigma^J := \overline{\mathscr{S}}_\sigma^J/\mathrm{rad}(\overline{\mathscr{S}}_\sigma^J)$ is a simple kL_J-module and $K\mathscr{S}_\sigma^J \cong \mathrm{St}_K^J$ (where St_K^J is the Steinberg module for KL_J).

The $\mathcal{O}G$-modules Γ_σ and $\mathscr{S}_\sigma$ behave particularly well with respect to Harish-Chandra restriction. For any $J \subseteq S$, we have the following isomorphisms of $\mathcal{O}L_J$-modules [29, Lemma 4.4]:

$$^*R_{L_J}^G(\Gamma_\sigma) \cong \Gamma_\sigma^J \text{ and } ^*R_{L_J}^G(\mathscr{S}_\sigma) \cong \mathscr{S}_\sigma^J.$$

(The isomorphism ${}^*R^G_{L_J}(\Gamma_\sigma) \cong \Gamma^J_\sigma$ is due to Rodier.)

Since the Harish-Chandra restriction functor is compatible with the r-modular system $(\mathcal{O}, K, k)$, we also have ${}^*R^G_{L_J}(K\Gamma_\sigma) \cong K\Gamma^J_\sigma$, ${}^*R^G_{L_J}(K\mathscr{S}_\sigma) \cong K\mathscr{S}^J_\sigma$, ${}^*R^G_{L_J}(\overline{\Gamma}_\sigma) \cong \overline{\Gamma}^J_\sigma$, and ${}^*R^G_{L_J}(\overline{\mathscr{S}}_\sigma) \cong \overline{\mathscr{S}}^J_\sigma$.

14.2.6 Harish-Chandra series arising from the regular character σ

Let $\mathscr{P}^*_\sigma = \{J \subseteq S \mid D^J_\sigma \text{ is a cuspidal } kL_J - \text{module}\}$. For every $J \in \mathscr{P}^*_\sigma$, D^J_σ is an irreducible cuspidal kL_J-module, which means that there is a Harish-Chandra series of the form $\mathrm{Irr}_k(G|(L_J, D^J_\sigma))$. If $J \in \mathscr{P}^*_\sigma$, the Harish-Chandra series $\mathrm{Irr}_k(G|(L_J, D^J_\sigma))$ consists of the irreducible kG-modules in the head of $R^G_{L_J}(D^J_\sigma)$ (equivalently, $\mathrm{Irr}_k(G|(L_J, D^J_\sigma))$ consists of the irreducible kG-modules in the socle of $R^G_{L_J}(D^J_\sigma)$). Given $I, J \in \mathscr{P}^*_\sigma$, the series $\mathrm{Irr}_k(G|(L_I, D^I_\sigma))$ and $\mathrm{Irr}_k(G|(L_J, D^J_\sigma))$ contain the same irreducible kG-modules if and only if $J = wIw^{-1}$ for some $w \in W$.

We include Geck's proof of the next proposition because some of the techniques used in this proof are needed to prove Theorem 14.3.3.

Proposition 14.2.2. *([29, Proposition 4.6] If $J \in \mathscr{P}^*_\sigma$, then St_k has a unique composition factor which belongs to the Harish-Chandra series $\mathrm{Irr}_k(G|(L_J, D^J_\sigma))$.*

Proof. Since $\mathrm{St}_K \cong K\mathscr{S}_\sigma$, the kG-modules St_k and $\overline{\mathscr{S}}_\sigma$ have the same composition factors, with multiplicity. Therefore, it suffices to show that the statement of the proposition holds for $\overline{\mathscr{S}}_\sigma$. First, we show that $\overline{\mathscr{S}}_\sigma$ has a composition factor which belongs to $\mathrm{Irr}_k(G|(L_J, D^J_\sigma))$ for every $J \in \mathscr{P}^*_\sigma$. Let $J \in \mathscr{P}^*_\sigma$ be fixed. Since Harish-Chandra induction and restriction are adjoint functors,

$$\mathrm{Hom}_{kG}(\overline{\mathscr{S}}_\sigma, R^G_{L_J}(D^J_\sigma)) \cong \mathrm{Hom}_{kL_J}({}^*R^G_{L_J}(\overline{\mathscr{S}}_\sigma), D^J_\sigma).$$

Since ${}^*R^G_{L_J}(\overline{\mathscr{S}}_\sigma) \cong \overline{\mathscr{S}}^J_\sigma$ by [29, Lemma 4.4], we have

$$\mathrm{Hom}_{kL_J}({}^*R^G_{L_J}(\overline{\mathscr{S}}_\sigma), D^J_\sigma) \cong \mathrm{Hom}_{kL_J}(\overline{\mathscr{S}}^J_\sigma, D^J_\sigma).$$

But, by definition of D^J_σ, $\mathrm{Hom}_{kL_J}(\overline{\mathscr{S}}^J_\sigma, D^J_\sigma) \neq 0$, and it follows that $\mathrm{Hom}_{kG}(\overline{\mathscr{S}}_\sigma, R^G_{L_J}(D^J_\sigma)) \neq 0$. Therefore, some composition factor of $\overline{\mathscr{S}}_\sigma$ is in the socle of $R^G_{L_J}(D^J_\sigma)$. Since every irreducible summand of $\mathrm{soc}(R^G_{L_J}(D^J_\sigma))$ is in the Harish-Chandra series $\mathrm{Irr}_k(G|(L_J, D^J_\sigma))$, $\overline{\mathscr{S}}_\sigma$ has a composition factor which belongs to $\mathrm{Irr}_k(G|(L_J, D^J_\sigma))$.

Next, we show that given $J \in \mathscr{P}^*_\sigma$, only one irreducible kG-module in $\mathrm{Irr}_k(G|(L_J, D^J_\sigma))$ can be a composition factor of $\overline{\mathscr{S}}_\sigma$. If V is an irreducible kG-module with $V \in \mathrm{Irr}_k(G|(L_J, D^J_\sigma))$, then $V \subseteq \mathrm{head}(R^G_{L_J}(D^J_\sigma))$. Now, since there is a surjective homomorphism $\overline{\Gamma}_\sigma^J \twoheadrightarrow D_\sigma^J$ and the Harish-Chandra induction functor $R_{L_J}^G$ is exact, there is a surjective homomorphism $R_{L_J}^G(\overline{\Gamma}_\sigma^J) \twoheadrightarrow R_{L_J}^G(D_\sigma^J)$. Therefore, for every $V \in \mathrm{Irr}_k(G|(L_J, D_\sigma^J))$, $V \subseteq \mathrm{head}(R_{L_J}^G(\overline{\Gamma}_\sigma^J))$. Since $R_{L_J}^G(\overline{\Gamma}_\sigma^J)$ is a projective kG-module, the projective cover P_V of V is a direct summand of $R_{L_J}^G(\overline{\Gamma}_\sigma^J)$ for every $V \in \mathrm{Irr}_k(G|(L_J, D_\sigma^J))$.

Given the results of the previous paragraph, the total number of composition factors of $\overline{\mathscr{S}}_\sigma$ belonging to the Harish-Chandra series $\mathrm{Irr}_k(G|(L_J, D_\sigma^J))$ ($J \in \mathscr{P}_\sigma^*$) is

$$\begin{aligned}\sum_{V\in \mathrm{Irr}_k(G|(L_J,D_\sigma^J))} [\overline{\mathscr{S}}_\sigma : V] &= \sum_{V\in \mathrm{Irr}_k(G|(L_J,D_\sigma^J))} \dim \mathrm{Hom}_{kG}(P_V, \overline{\mathscr{S}}_\sigma) \\ &= \dim \mathrm{Hom}_{kG}(\bigoplus_{V\in \mathrm{Irr}_k(G|(L_J,D_\sigma^J))} P_V, \overline{\mathscr{S}}_\sigma) \\ &\leq \dim \mathrm{Hom}_{kG}(R_{L_J}^G(\overline{\Gamma}_\sigma^J), \overline{\mathscr{S}}_\sigma).\end{aligned}$$

Now, since the Harish-Chandra induction functor $R_{L_J}^G$ respects the r-modular system and $R_{L_J}^G(\Gamma_\sigma^J)$ is a projective $\mathcal{O}G$-lattice, we have

$$\dim \mathrm{Hom}_{kG}(R_{L_J}^G(\overline{\Gamma}_\sigma^J), \overline{\mathscr{S}}_\sigma) = \dim \mathrm{Hom}_{KG}(R_{L_J}^G(K\Gamma_\sigma^J), K\mathscr{S}_\sigma).$$

Using the adjointness of Harish-Chandra induction and restriction along with [29, Lemma 4.4], we have $\dim \mathrm{Hom}_{KG}(R_{L_J}^G(K\Gamma_\sigma^J), K\mathscr{S}_\sigma) = \dim \mathrm{Hom}_{KL_J}(K\Gamma_\sigma^J, {}^*R_{L_J}^G(K\mathscr{S}_\sigma)) = \dim \mathrm{Hom}_{KL_J}(K\Gamma_\sigma^J, K\mathscr{S}_\sigma^J)$. Since $K\mathscr{S}_\sigma^J \cong \mathrm{St}_K$, it follows that $\dim \mathrm{Hom}_{KL_J}(K\Gamma_\sigma^J, K\mathscr{S}_\sigma^J) = \dim \mathrm{Hom}_{KL_J}(K\Gamma_\sigma^J, \mathrm{St}_K) = 1$.

□

Proposition 14.2.3. *([29, Proposition 4.7]) Assume that every composition factor of $k|_B^G$ belongs to $\mathrm{Irr}_k(G|(L_J, D_\sigma^J))$ for some $J \in \mathscr{P}_\sigma^*$. Then, St_k is multiplicity-free and the length of a composition series of St_k is equal to the number of subsets $J \in \mathscr{P}_\sigma^*$, up to W-conjugacy.*

The result of [29, Proposition 4.7] inspired the following definition.

Definition 14.2.4. We will say that the pair (G, k) satisfies property (P) if every composition factor of $k|_B^G$ belongs to a Harish-Chandra series of the form $\mathrm{Irr}_k(G|(L_J, D_\sigma^J))$ for some $J \in \mathscr{P}_\sigma^*$.

When the field k is clear from context we will simply say that G has property (P). There are many examples of finite groups of Lie type G such that (G, k) has property (P). If q is a power of p, then the pair $(\mathrm{GL}_n(q), k)$ has property (P) [29, Example 4.9]. If r is a linear prime,[17] then the following pairs have property (P) [29, 4.14]:

1. $(SO_n(q), k)$, n odd and q odd, and
2. $(Sp_n(q), k)$, n even and q a power of 2.

[17] r is a linear prime for $SO_n(q)$ and $Sp_n(q)$ if $q^{i-1} \not\equiv -1 \bmod r$ for all $i \geq 1$.

14.3 A Bound on $\dim \operatorname{Ext}^1_{kG}(Y,V)$

In this section, we will find a bound on $\dim \operatorname{Ext}^1_{kG}(Y,V)$ when the pair (G,k) has property (P) and Y and V are irreducible kG-modules such that $Y \in \operatorname{Irr}_k(G|B)$ and $V \notin \operatorname{Irr}_k(G|B)$. Since $Y \in \operatorname{Irr}_k(G|B)$, Y is in the head of the permutation module $k|_B^G$. Therefore, there exists a kG-submodule $\mathscr{L}^0$ of $k|_B^G$ such that $k|_B^G/\mathscr{L}^0 \cong Y$. When $V \notin \operatorname{Irr}_k(G|B)$, the dimension of $\operatorname{Ext}^1_{kG}(Y,V)$ is determined by the structure of the head of $\mathscr{L}^0$.

Proposition 14.3.1. *If $V \notin \operatorname{Irr}_k(G|B)$, $\dim \operatorname{Ext}^1_{kG}(Y,V) = [\operatorname{head}(\mathscr{L}^0) : V]$ (where $[\operatorname{head}(\mathscr{L}^0) : V]$ denotes the number of times that V occurs as a composition factor of $\operatorname{head}(\mathscr{L}^0)$).*

Proof. This result is a special case of Theorem 13.2.1. □

Corollary 14.3.2. *Suppose that $Y \in \operatorname{Irr}_k(G|B)$ and $V \notin \operatorname{Irr}_k(G|B)$. If V is not a composition factor of $k|_B^G$, then $\operatorname{Ext}^1_{kG}(Y,V) = 0$.*

Proof. By Proposition 14.3.1, $\dim \operatorname{Ext}^1_{kG}(Y,V) = [\operatorname{head}(\mathscr{L}^0) : V] \leq [k|_B^G : V] = 0$. □

By Corollary 14.3.2, it suffices to find a bound on $\dim \operatorname{Ext}^1_{kG}(Y,V)$ in the case that V is a composition factor of $k|_B^G$. Since (G,k) has property (P), we can assume that the irreducible kG-module V (which is a composition factor of $k|_B^G$) belongs to a Harish-Chandra series of the form $\operatorname{Irr}_k(G|(L_J, D_\sigma^J))$ for some $J \in \mathscr{P}_\sigma^*$.

Theorem 14.3.3. *Suppose that the pair (G,k) has property (P), and let Y and V be irreducible kG-modules such that $Y \in \operatorname{Irr}_k(G|B)$ and $V \notin \operatorname{Irr}_k(G|B)$. Assume that V is a composition factor of $k|_B^G$ and that V belongs to the Harish-Chandra series $\operatorname{Irr}_k(G|(L_J, D_\sigma^J))$ ($J \in \mathscr{P}_\sigma^*$). Then, $\dim \operatorname{Ext}^1_{kG}(Y,V) \leq [W : W_J]$, where W_J is the parabolic subgroup of W generated by J.*

Proof. By Proposition 14.3.1, $\dim \operatorname{Ext}^1_{kG}(Y,V) = [\operatorname{head}(\mathscr{L}^0) : V] \leq [k|_B^G : V]$, so it suffices to prove that $[k|_B^G : V] \leq [W : W_J]$. Since $V \in \operatorname{Irr}_k(G|(L_J, D_\sigma^J))$, V is in the head of the kG-module $R_{L_J}^G(D_\sigma^J)$. By definition, D_σ^J is in the head of the r-modular reduction $\overline{\Gamma}_\sigma^J$ of the Gelfand-Graev module Γ_σ^J for L_J, which means that there is a surjective kL_J-module homomorphism $\overline{\Gamma}_\sigma^J \twoheadrightarrow D_\sigma^J$. Since the Harish-Chandra induction functor $R_{L_J}^G$ is exact, there is a surjective kG-module homomorphism $R_{L_J}^G(\overline{\Gamma}_\sigma^J) \twoheadrightarrow R_{L_J}^G(D_\sigma^J)$. So, since $V \subseteq \operatorname{head}(R_{L_J}^G(D_\sigma^J))$, it follows that $V \subseteq \operatorname{head}\big(R_{L_J}^G(\overline{\Gamma}_\sigma^J)\big)$.

Let P_V denote the projective indecomposable kG-module with $\operatorname{head}(P_V) = V$. Since $\overline{\Gamma}_\sigma^J$ is a projective kL_J-module and $R_{L_J}^G$ is exact, $R_{L_J}^G(\overline{\Gamma}_\sigma^J)$ is a projective kG-module. So, since

$V \subseteq \mathrm{head}\big(R_{L_J}^G(\overline{\Gamma}_\sigma^J)\big)$, P_V is a direct summand of $R_{L_J}^G(\overline{\Gamma}_\sigma^J)$ and we have

$$[k|_B^G : V] = \dim \mathrm{Hom}_{kG}(P_V, k|_B^G) \leq \dim \mathrm{Hom}_{kG}(R_{L_J}^G(\overline{\Gamma}_\sigma^J), k|_B^G).$$

Now, $R_{L_J}^G(\overline{\Gamma}_\sigma^J) \cong \overline{R_{L_J}^G(\Gamma_\sigma^J)}$ is the reduction of the $\mathcal{O}G$-lattice $R_{L_J}^G(\Gamma_\sigma^J)$, and $k|_B^G \cong \overline{\mathcal{O}|_B^G}$ is the reduction of the $\mathcal{O}G$-lattice $\mathcal{O}|_B^G$. So, since $KR_{L_J}^G(\Gamma_\sigma^J) \cong R_{L_J}^G(K\Gamma_\sigma^J)$ and $K\mathcal{O}|_B^G \cong K|_B^G = R_T^G(K)$, we have

$$\dim \mathrm{Hom}_{kG}(R_{L_J}^G(\overline{\Gamma}_\sigma^J), k|_B^G) = \dim \mathrm{Hom}_{KG}(R_{L_J}^G(K\Gamma_\sigma^J), K|_B^G) = \dim \mathrm{Hom}_{KG}(R_{L_J}^G(K\Gamma_\sigma^J), R_T^G(K)).$$

(The first equality above holds because $R_{L_J}^G(\Gamma_\sigma^J)$ is a projective $\mathcal{O}G$-lattice.)

We will now compute $\dim \mathrm{Hom}_{KG}(R_{L_J}^G(K\Gamma_\sigma^J), R_T^G(K))$. First, since the functor ${}^*R_{L_J}^G$ is right adjoint to $R_{L_J}^G$, we have

$$\mathrm{Hom}_{KG}(R_{L_J}^G(K\Gamma_\sigma^J), R_T^G(K)) \cong \mathrm{Hom}_{KL_J}(K\Gamma_\sigma^J, {}^*R_{L_J}^G R_T^G(K)).$$

Let P_J be the standard parabolic subgroup of G containing L_J, and let ${}^JW$ denote the set of shortest right coset representatives of W_J in W. Then, ${}^JW$ gives a full set of (P_J, B)-double coset representatives in G and it follows by the Mackey decomposition that

$$\begin{aligned}\mathrm{Hom}_{KL_J}(K\Gamma_\sigma^J, {}^*R_{L_J}^G R_T^G(K)) &\cong \mathrm{Hom}_{KL_J}(K\Gamma_\sigma^J, \bigoplus_{w\in {}^JW} R_{{}^wT\cap L_J}^{L_J}\, {}^*R_{{}^wT\cap L_J}^{{}^wT}({}^wK)) \\ &\cong \bigoplus_{w\in {}^JW} \mathrm{Hom}_{KL_J}(K\Gamma_\sigma^J, R_{{}^wT\cap L_J}^{L_J}\, {}^*R_{{}^wT\cap L_J}^{{}^wT}({}^wK)).\end{aligned}$$

Now, ${}^wK$ is the $K({}^wT)$-module with underlying vector space K and ${}^wT$-action given by $wtw^{-1}.x = tx$ for all $t \in T$ and $x \in K$. So, since K is the trivial one-dimensional KT-module, ${}^wK$ is the trivial one-dimensional $K({}^wT)$-module. Since ${}^wT = wTw^{-1} = T$ for all $w \in W$, ${}^*R_{{}^wT\cap L_J}^{{}^wT} = {}^*R_{T\cap L_J}^T = {}^*R_T^T$ is the identity functor on KT-mod, ${}^wK = K$, and $R_{{}^wT\cap L_J}^{L_J} = R_T^{L_J}$ for all $w \in W$. Therefore, continuing the calculation above, we have

$$\bigoplus_{w\in {}^JW} \mathrm{Hom}_{KL_J}(K\Gamma_\sigma^J, R_{{}^wT\cap L_J}^{L_J}\, {}^*R_{{}^wT\cap L_J}^{{}^wT}({}^wK)) \cong \bigoplus_{w\in {}^JW} \mathrm{Hom}_{KL_J}(K\Gamma_\sigma^J, R_T^{L_J}(K)),$$

and since $R_T^{L_J}$ is right adjoint to ${}^*R_T^{L_J}$,

$$\bigoplus_{w\in {}^JW} \mathrm{Hom}_{KL_J}(K\Gamma_\sigma^J, R_T^{L_J}(K)) \cong \bigoplus_{w\in {}^JW} \mathrm{Hom}_{KT}({}^*R_T^{L_J}(K\Gamma_\sigma^J), K).$$

Now, since $T = L_\emptyset$, ${}^*R_T^{L_J}(K\Gamma_\sigma^J) \cong K\Gamma_\sigma^\emptyset$ by Rodier's result on the restriction of Gelfand-Graev modules in characteristic 0. Since T has a trivial unipotent radical, the Gelfand-Graev module $K\Gamma_\sigma^\emptyset$ for T is equal to the group algebra KT. Therefore,

$$\bigoplus_{w\in {}^JW} \mathrm{Hom}_{KT}({}^*R_T^{L_J}(K\Gamma_\sigma^J), K) \cong \bigoplus_{w\in {}^JW} \mathrm{Hom}_{KT}(KT, K).$$

K appears once as a direct summand of the completely reducible KT-module KT, so $\dim \mathrm{Hom}_{KT}(KT, K) = 1$ and, following our chain of calculations, we have

$$\dim \mathrm{Hom}_{KG}(R^G_{L_J}(K\Gamma^J_\sigma), R^G_T(K)) = |{}^J W|.$$

Thus, $[k|^G_B : V] \leq \dim \mathrm{Hom}_{kG}(R^G_{L_J}(\overline{\Gamma}^J_\sigma), k|^G_B) = \dim \mathrm{Hom}_{KG}(R^G_{L_J}(K\Gamma^J_\sigma), R^G_T(K)) = |{}^J W| = [W : W_J]$, as needed.

□

The bound of Theorem 14.3.3 is particularly strong in the case that V is a cuspidal irreducible kG-module.

Corollary 14.3.4. *Suppose that the pair (G, k) has property (P), and let V be a cuspidal irreducible kG-module. Then,* $\dim \mathrm{Ext}^1_{kG}(Y, V) \leq 1$ *for any irreducible kG-module $Y \in \mathrm{Irr}_k(G|B)$.*

Proof. If V is not a composition factor of $k|^G_B$, then $\mathrm{Ext}^1_{kG}(Y, V) = 0$ by Corollary 14.3.2. Suppose now that V is a composition factor of $k|^G_B$. Since V is a cuspidal kG-module and (G, k) has property (P), the Harish-Chandra series containing V is of the form $\{V\} = \mathrm{Irr}_k(G|(G, D^S_\sigma))$. So, by Theorem 14.3.3, $\dim \mathrm{Ext}^1_{kG}(Y, V) \leq [W : W] = 1$. □

Part III

Explicit Computations of Bounds on the Dimension of Ext^1 Between Irreducible Modules for $\mathrm{GL}_n(q)$ in Cross Characteristic

15 Introduction

Let G be a finite group of Lie type defined over a finite field $\mathbb{F}_q$, where q is a power of a prime $p > 0$ (so, G is the fixed point subgroup of a connected reductive algebraic group $\mathbb{G}$ over $\bar{\mathbb{F}}_p$ under an endomorphism $F : \mathbb{G} \to \mathbb{G}$ such that some power of F is a Frobenius morphism). Assume that the BN-pair of G is split, and let (W, S) be the Coxeter system corresponding to the BN-pair structure on G.

Let k be an algebraically closed field of characteristic $r > 0$, $r \neq p$, and suppose that the pair (G, k) satisfies property (P) (see Definition 14.2.4). Let Y and V be irreducible kG-modules with Y belonging to the unipotent principal series $\mathrm{Irr}_k(G|B)$ and V belonging to a Harish-Chandra series $\mathrm{Irr}_k(G|(L_J, X))$ $(J \subseteq S)$ with $\mathrm{Irr}_k(G|(L_J, X)) \neq \mathrm{Irr}_k(G|B)$. In Chapter 14, we showed that

$$\begin{cases} \dim \mathrm{Ext}^1_{kG}(Y, V) = 0 & \text{if } [k|_B^G : V] = 0 \\ \dim \mathrm{Ext}^1_{kG}(Y, V) \leq \dfrac{|W|}{|W_J|} & \text{if } [k|_B^G : V] \neq 0 \end{cases}$$

(where $[k|_B^G : V]$ denotes the multiplicity of V as a composition factor of $k|_B^G$). In particular, if V is a composition factor of $k|_B^G$ and V belongs to the Harish-Chandra series $\mathrm{Irr}_k(G|(L_J, X))$, $\mathrm{Irr}_k(G|(L_J, X)) \neq \mathrm{Irr}_k(G|B)$, the bound on $\dim \mathrm{Ext}^1_{kG}(Y, V)$ depends only on the Levi subgroup L_J (for, W_J is the Weyl group of L_J).

If the irreducible kG-module V belongs to the Harish-Chandra series $\mathrm{Irr}_k(G|(L_J, X))$, we will say that L_J is a Harish-Chandra vertex of V, and that X is a Harish-Chandra source of V (this terminology is used in [18] and [19]). The Harish-Chandra vertex of V is unique up to conjugation by elements of $N = N_G(T)$. (If L is a Levi subgroup of a parabolic subgroup of G and X is a cuspidal kL-module, then the Harish-Chandra series $\mathrm{Irr}_k(G|(L, X))$ and $\mathrm{Irr}_k(G|({}^nL, {}^nX))$ are equal for any $n \in N$. Thus, if L is a Harish-Chandra vertex of an irreducible kG-module V, then ${}^nL$ is as well.) Given this terminology, we may restate the last assertion of the paragraph above as follows: if V is a composition factor of $k|_B^G$ and V belongs to the Harish-Chandra series $\mathrm{Irr}_k(G|(L_J, X))$, $\mathrm{Irr}_k(G|(L_J, X)) \neq \mathrm{Irr}_k(G|B)$, then the bound on dim $\text{Ext}^1_{kG}(Y,V)$ found in Chapter 14 depends only on the Harish-Chandra vertex of V.

We will explicitly demonstrate the bounds on dim $\text{Ext}^1_{kG}(Y,V)$ given in Chapter 14 in a series of examples. In these examples, we will work with the general linear group $G = \text{GL}_n(q)$ over the finite field $\mathbb{F}_q$. By [29, 4.9], the pair $(\text{GL}_n(q), k)$ satisfies property (P), and consequently the bounds of Chapter 14 apply.

We will use the parameterization of $k\text{GL}_n(q)$-modules given in [19, (4.2.11, (6))] (see Chapter 9). In this labeling, the complete set of irreducible $k\text{GL}_n(q)$-modules is given by $\{D^2(s,\vec{\lambda}) \mid s \in \mathscr{C}_{ss,r'}, \vec{\lambda} \vdash \vec{m}_s\}$. In the next section, we will present an algorithm of Dipper and Du [19] which determines the Harish-Chandra vertex of any composition factor of $k|_B^G$ (where $G = \text{GL}_n(q)$). Then, we will apply Dipper and Du's algorithm to certain irreducible $k\text{GL}_n(q)$-modules in order to obtain explicit bounds on the dimension of Ext^1 between these irreducibles.

16 The Harish-Chandra Vertex of the Module $D^2(1,\lambda)$

In [19, Section 4.3], Dipper and Du present an algorithm which may be used to determine a Harish-Chandra vertex of an irreducible $k\text{GL}_n(q)$-module $D^2(s,\lambda)$ for any $s \in \mathscr{C}_{ss,r'}$ and multipartition $\lambda \vdash \vec{m}_s$. In our examples, we will work only with the irreducible $k\text{GL}_n(q)$-modules $D^2(1,\lambda)$, where $\lambda \vdash n$. Therefore, we will outline Dipper and Du's algorithm in the special case of $s = 1$.

As above, let $\text{char}(k) = r > 0$, with $r \nmid q$. Let $|q \pmod r|$ denote the multiplicative order of q modulo r, and define an integer $l \in \mathbb{Z}^+$ by

$$l = \begin{cases} r & \text{if } |q \pmod r| = 1 \\ |q \pmod r| & \text{if } |q \pmod r| > 1. \end{cases}$$

Let $\lambda \vdash n$ and let $D^2(1,\lambda)$ be an irreducible $k\text{GL}_n(q)$-module (as indexed in [19, (4.2.3 (11))]). A Harish-Chandra vertex of $D^2(1,\lambda)$ may be determined as follows.

Let $\lambda' = \lambda'_{-1} + l\lambda'_0 + lr\lambda'_1 + lr^2\lambda'_2 + \cdots$ be the $l-r$-adic decomposition of λ', meaning that $\lambda'_{-1} \vdash n_{-1}$ is l-restricted and $\lambda'_a \vdash n_a$ is r-restricted for $a \geq 0$. [18] Since $\lambda' \vdash n$, the integers $n_a \in \mathbb{Z}_{\geq 0}$ must satisfy the relation

$$n = n_{-1} + \sum_{a \geq 0} lr^a n_a.$$

[18] A partition $\mu \vdash n$ is called l-restricted if its dual μ' is l-regular, meaning that every part of μ' occurs less than l times.

If λ' has the $l-r$-adic decomposition given above, a Harish-Chandra vertex of the irreducible $k\mathrm{GL}_n(q)$-module $D^2(1,\lambda)$ is the Levi subgroup

$$L = \mathrm{GL}_1(q)^{\times n_{-1}} \times \mathrm{GL}_l(q)^{\times n_0} \times \mathrm{GL}_{lr}(q)^{\times n_1} \times \mathrm{GL}_{lr^2}(q)^{\times n_2} \times \cdots$$

of $\mathrm{GL}_n(q)$. (Any conjugate of L by an element $n \in N$ gives another Harish-Chandra vertex of $D^2(1,\lambda)$.)

Example 16.0.1. Let $\mathrm{char}(k) = r = 2$ and let $G = \mathrm{GL}_2(q)$. Assume that $2 \nmid q$ (since $[G:B] = q+1$ in this case, we must have $2 \mid [G:B]$). The only partitions of 2 are (2) and $(1,1) = (1^2)$. Therefore, the irreducible kG-modules which appear as composition factors of $k|_B^G$ are $D^2(1,(2))$ and $D^2(1,(1^2))$. Here, $D^2(1,(2)) = k \in \mathrm{Irr}_k(G|B) = \mathrm{Irr}_k(G|(T,k))$. Therefore, the Harish-Chandra vertex of $D^2(1,(2))$ is the maximal torus T.

We will apply Dipper and Du's algorithm to determine a Harish-Chandra vertex of the irreducible kG-module $D^2(1,(1^2))$. Since $r = 2$ and $2 \nmid q$, $|q \pmod r| = |q \pmod 2| = 1$. Therefore, we set $l = r = 2$. To find a Harish-Chandra vertex of $D^2(1,(1^2))$, we must find the $2-2$-adic decomposition of (2).

The $2-2$-adic decomposition of (2) is of the form

$$(2) = \lambda_{-1} + 2\lambda_0 + 2(2)\lambda_1 + 2(2)^2\lambda_2 + \cdots,$$

where $\lambda_{-1} \vdash n_{-1}$ is $l = 2$-restricted and $\lambda_a \vdash n_a$ is $r = 2$-restricted for $a \geq 0$. Since $n = 2$, the integers $n_a \in \mathbb{Z}_{\geq 0}$, $a \geq -1$, must satisfy the equation

$$2 = n_{-1} + \sum_{a \geq 0} 2(2^a)n_a.$$

By inspection of this equation, only two cases are possible:

1. $n_{-1} = 2$, $n_a = 0$ for $a \geq 0$, or
2. $n_{-1} = 0$, $n_0 = 1$, $n_a = 0$ for $a \geq 1$.

But, since $(2)' = (1,1)$ is not $l = 2$-regular, (2) is not 2-restricted, which eliminates the first possibility. Therefore, the second possibility must hold - that is, we must have $\lambda_{-1} = (0)$, $\lambda_0 = (1)$, and $\lambda_a = (0)$ for $a \geq 1$, which means that $(2) = (0) + 2(1) = 2(1)$ is the $2-2$-adic decomposition of (2). Since $\lambda_0 = (1) \vdash 1$, $n_0 = 1$. And, since $\lambda_a = (0)$ for all $a \neq 0$, $n_a = 0$ for $a \neq 0$. Thus, the Harish-Chandra vertex of $D^2(1,(1^2))$ is $\mathrm{GL}_2(q) = G$, which means that $D^2(1,(1^2))$ is cuspidal.

Remark 16.0.2. It is possible to verify that $D^2(1,(1^2))$ is cuspidal in the set-up of the previous example using Geck's results on the structure of the Steinberg module St_k [29]. By assumption, $2 \mid [G:B] = q+1$. Thus, the Steinberg module St_k is not irreducible and $\mathrm{head}(\mathrm{St}_k)$ is an irreducible kG-module which does not belong to the unipotent principal series. So, since $\mathrm{St}_k \subseteq k|_B^G$ and all of the composition factors of $k|_B^G$ are isomorphic to either $D^2(1,(2)) = k$ or $D^2(1,(1^2))$, we must have $\mathrm{head}(\mathrm{St}_k) = D^2(1,(1^2)) \notin \mathrm{Irr}_k(G|B)$. Now, since $G = \mathrm{GL}_2(q)$ is of rank 1, the only proper Levi subgroup of G is T, which means that every irreducible kG-module is either a principal series representation or cuspidal. But, the only principal series irreducibles appearing as composition factors of $k|_B^G$ are those belonging to $\mathrm{Irr}_k(G|B)$. Thus, it follows that $D^2(1,(1^2))$ must be cuspidal.

17 Some Examples of Bounds on the Dimension of Ext^1

Let $G = \mathrm{GL}_n(q)$ and let k be an algebraically closed field of characteristic $r > 0$, $r \nmid q$. By [45, Theorem 2.1] and [19, (4.2.11, (6))], the irreducible constituents of $k|_B^G$ are of the form $D^2(1,\lambda)$, $\lambda \vdash n$. Thus, the bounding result of Chapter 14 may be stated as follows.

Theorem 17.0.1. *Let $\lambda \vdash n$ and assume that the irreducible kG-module $D^2(1,\lambda)$ belongs to the unipotent principal series $\mathrm{Irr}_k(G|B)$.*

1. *If $s \in \mathscr{C}_{ss,r'}$ with $s \neq 1$ and $\mu \vdash \vec{m}_s$, then $\mathrm{Ext}^1_{kG}(D^2(1,\lambda), D^2(s,\mu)) = 0$.*
2. *Let $\mu \vdash n$ and suppose that $D^2(1,\mu)$ has Harish-Chandra vertex $L_J \neq T$ ($\emptyset \neq J \subseteq S$). Then,* $\dim \mathrm{Ext}^1_{kG}(D^2(1,\lambda), D^2(1,\mu)) \leq \dfrac{|W|}{|W_J|}$. [19]

Note that the first part of Theorem 17.0.1 is a basic consequence of the fact that $D^2(s,\mu)$ and $D^2(1,\lambda)$ belong to different blocks when $s \neq 1$ (see [19, (4.2.11)]).

In the remainder of this section, we will provide several explicit examples of the bound given in Theorem 17.0.1 (2).

Example 17.0.2. $G = \mathrm{GL}_3(q)$, $\mathrm{char}(k) = r > 0$, $r \nmid q$

In this case, $W = \mathfrak{S}_3$, the symmetric group on the set $\{1,2,3\}$, and S is the set $\{(1,2),(2,3)\}$ of fundamental reflections in W. There are three partitions of 3: (3), (2, 1), and (1^3). Therefore, the irreducible kG-modules which occur as composition factors of $k|_B^G$ are $D^2(1,(3))$, $D^2(1,(2,1))$, and $D^2(1,(1^3))$. Since $D^2(1,(3)) = k$, $D^2(1,(3)) \in \mathrm{Irr}_k(G|B)$ for any r ($r \nmid q$).

Let

$$l = \begin{cases} r & \text{if } |q \pmod r| = 1 \\ |q \pmod r| & \text{if } |q \pmod r| > 1. \end{cases}$$

[19] In the case that $\lambda = (n)$ and $D^2(1,\lambda) = k$, this bound can be improved. As was shown by Guralnick and Tiep, $\dim \mathrm{Ext}^1_{kG}(k, D^2(1,\mu)) = \dim H^1(G, D^2(1,\mu)) \leq 1$ when $D^2(1,\mu) \notin \mathrm{Irr}_k(G|B)$.

By definition, $l > 1$. So, since $(2,1)' = (2,1)$, $(2,1)$ is both an l-regular and an l-restricted partition for any r, q with $r \nmid q$. Thus, $(2,1)$ is its own $l-r$-adic decomposition for any choice of r ($r \nmid q$), and Dipper and Du's algorithm yields $\mathrm{GL}_1(q)^{\times 3} = T$ as the Harish-Chandra vertex of $D^2(1,(2,1))$. Since the only principal series representations which occur as composition factors of $k|_B^G$ are those in $\mathrm{Irr}_k(G|B)$, we must have $D^2(1,(2,1)) \in \mathrm{Irr}_k(G|B)$ for any r, $r \nmid q$.

Thus, $D^2(1,(1^3))$ is the only composition factor of $k|_B^G$ whose Harish-Chandra vertex varies with r. We will compute the Harish-Chandra vertex of $D^2(1,(1^3))$ and find the corresponding Ext bounds in several cases.

1. $\mathrm{GL}_3(q)$, $r = 2 \nmid q$

   Since $r = 2 \nmid q$, $|q \pmod 2| = 1$. Thus, we set $l = r = 2$. The $2-2$-adic decomposition of (3) is $(3) = (1) + 2(1)$. So, the Harish-Chandra vertex of $D^2(1,(1^3))$ is $L = \mathrm{GL}_1(q) \times \mathrm{GL}_2(q)$. Now, L is conjugate to the standard Levi subgroup $L_{J(2,1)} = \mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$ of $\mathrm{GL}_3(q)$. The subset $J(2,1)$ of S is defined as follows. Let $\mathscr{Y}(2,1)$ be the Young diagram of shape $(2,1)$. Fill in the boxes of the first row of this Young diagram with the integers $1,2$, and fill the box of the last row with the integer 3. The result of this process is a standard tableau $\mathfrak{t}^{(2,1)}$ of shape $(2,1)$. $J(2,1)$ is then the subset of S consisting of those elements $s \in S$ which stabilize the rows of $\mathfrak{t}^{(2,1)}$.

   Since L is conjugate to $L_{J(2,1)}$, $L_{J(2,1)}$ is another Harish-Chandra vertex of $D^2(1,(1^3))$. The Weyl group $W_{J(2,1)}$ of $L_{J(2,1)}$ consists of those $w \in W$ which stabilize the rows of $\mathfrak{t}^{(2,1)}$. Thus, $W_{J(2,1)} = \mathrm{Sym}(\{1,2\}) \times \mathrm{Sym}(\{3\})$ and $|W_{J(2,1)}| = 2! \cdot 1! = 2$. So, Theorem 17.0.1 (2) yields the following bounds:

   $$\dim \mathrm{Ext}^1_{kG}\big(k, D^2(1,(1^3))\big) = \dim H^1\big(G, D^2(1,(1^3))\big) \le \frac{|W|}{|W_{J(2,1)}|} = \frac{3!}{2} = 3, \text{ and}$$
   $$\dim \mathrm{Ext}^1_{kG}\big(D^2(1,(2,1)), D^2(1,(1^3))\big) \le 3.$$

2. $\mathrm{GL}_3(q)$, $r = 3 \nmid q$

   In this case, $|q \pmod 3|$ is equal to 1 or 2. Repeating the same analysis as above, we obtain the following results.

   If $|q \pmod 3| = 1$, then $l = 3$ and the $3-3$-adic decomposition of (3) is $(3) = 3(1)$. Thus, the Harish-Chandra vertex of $D^2(1,(1^3))$ is $\mathrm{GL}_3(q) = G$, and it follows that $D^2(1,(1^3))$ is cuspidal. Since G has Weyl group W, Theorem 17.0.1 (2) yields the following bounds:

dim $H^1\big(G, D^2(1,(1^3))\big) \le \dfrac{W}{W} = 1,$ and
dim $\mathrm{Ext}^1_{kG}\big(D^2(1,(2,1)), D^2(1,(1^3))\big) \le 1.$

If $|q \pmod 3| = 2$, then $l = 2$ and the $2-3$-adic decomposition of (3) is $(3) = (1)+2(1)$. So, a Harish-Chandra vertex of $D^2(1,(1^3))$ is $\mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$, and Theorem 17.0.1 (2) yields the bounds:

dim $H^1\big(G, D^2(1,(1^3))\big) \le 3,$ and
dim $\mathrm{Ext}^1_{kG}\big(D^2(1,(2,1)), D^2(1,(1^3))\big) \le 3.$

Remark 17.0.3. There are infinitely many choices of r for which the module $D^2(1,(1^3))$ belongs to the unipotent principal series $\mathrm{Irr}_k(G|B)$. By [31, Lemma 4.3.2], the permutation module $k|_B^G$ is completely reducible when $r \nmid [G:B]$. Thus, when $r \nmid [G:B]$, every composition factor of $k|_B^G$ is a direct summand and therefore belongs to $\mathrm{Irr}_k(G|B)$. In particular, $D^2(1,(1^3)) \in \mathrm{Irr}_k(G|B)$ when $r \nmid [G:B]$.

Example 17.0.4. $G = \mathrm{GL}_4(q)$, $r = 2 \nmid q$

In this case, $W = \mathfrak{S}_4$, and $l = 2$. The partitions of (4) are: (4), $(3,1)$, $(2,2)$, $(2,1,1)$, and (1^4). Thus, the composition factors of $k|_B^G$ are $D^2(1,(4))$, $D^2(1,(3,1))$, $D^2(1,(2,2))$, $D^2(1,(2,1,1))$, and $D^2(1,(1^4))$. The approach of the previous example yields the following results.

$\mathrm{Irr}_k(G|B) = \{D^2(1,(4)) = k, D^2(1,(3,1))\}$

$\lambda \vdash n$	$2-2$-adic decomposition of λ	Harish-Chandra vertex L_J of $D^2(1,\lambda')$	$\lvert W_J\rvert$	$\dfrac{\lvert W\rvert}{\lvert W_J\rvert}$
(4)	$(4) = 4(1)$	G	$4!$	$4!/4! = 1$
$(3,1)$	$(3,1) = (1,1) + 2(1)$	$\mathrm{GL}_2(q) \times \mathrm{GL}_1(q)^{\times 2}$	2	$4!/2 = 12$
$(2,2)$	$(2,2) = 2(1,1)$	$\mathrm{GL}_2(q)^{\times 2}$	4	$4!/4 = 6$

Thus, Theorem 17.0.1 (2) yields the bounds:
dim $H^1\big(G, D^2(1,(1^4))\big) \le 1,$
dim $H^1\big(G, D^2(1,(2,1,1))\big) \le 12,$
dim $H^1\big(G, D^2(1,(2,2))\big) \le 6,$
dim $\mathrm{Ext}^1_{kG}\big(D^2(1,(3,1)), D^2(1,(1^4))\big) \le 1,$
dim $\mathrm{Ext}^1_{kG}\big(D^2(1,(3,1)), D^2(1,(2,1,1))\big) \le 12,$ and
dim $\mathrm{Ext}^1_{kG}\big(D^2(1,(3,1)), D^2(1,(2,2))\big) \le 6.$

Part IV

Generalizing the Results of Cline, Parshall, and Scott [9] to Calculate Ext Groups between Irreducible $k\mathrm{GL}_n(q)$-Modules

18 Self-Extensions of Irreducible Modules for $\mathrm{GL}_n(q)$ in Certain Cross Characteristics

18.1 Introduction

Let q be a power of a prime p, and let $G = \mathrm{GL}_n(q)$ be the general linear group over the finite field $\mathbb{F}_q$ of q elements. Let B be the Borel subgroup of G consisting of all invertible upper triangular $n \times n$ matrices. Let U be the unipotent radical of B (that is, the subgroup of B consisting of invertible upper triangular matrices with 1's along the main diagonal), and let T be the maximal torus in B consisting of all invertible diagonal matrices.

Let k be an algebraically closed field of characteristic $r > 0$ such that $r \nmid q$. For the remainder of this dissertation, we will work with right kG-modules. We will show that under the additional assumption $r \nmid (q-1)$, $\mathrm{Ext}^1_{kG}(Y, V)$ (where Y is an irreducible kG-module belonging to the unipotent principal series $\mathrm{Irr}_k(G|B)$ and V is an arbitrary irreducible kG-module) is isomorphic to an Ext^1 group over a q-Schur algebra. As a consequence, we will show that there are no non-split self-extensions of irreducible kG-modules belonging to the unipotent principal series. In other words, we will show that given any irreducible kG-module Y belonging to the unipotent principal series $\mathrm{Irr}_k(G|B)$, $\mathrm{Ext}^1_{kG}(Y, Y) = 0$.[20] The proof of this result relies on the techniques and methods of Cline, Parshall, and Scott [9].

In [9], Cline, Parshall, and Scott prove that there is an ideal $J_k \trianglelefteq kG$ such there is a Morita equivalence

$$\bar{F} : \mathrm{mod} - kG/J_k \to \mathrm{mod} - \bigoplus_{s \in \mathscr{C}_{ss,r'}} \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_k.$$

Since kG and kG/J_k have the same irreducible modules [9, 9.17], the Morita equivalence $\bar{F}$ yields an indexing of the irreducible kG-module. In this indexing, the full set of irreducible

[20] Using the methods of Guralnick and Tiep, it is only possible to show that dim $\mathrm{Ext}^1_{kG}(Y, Y) \leq |W| = n!$ (where $W = S_n$ is the Weyl group of $\mathrm{GL}_n(q)$).

kG-modules is given by

$$\{D(s,\lambda) \mid s \in \mathscr{C}_{ss,r'},\ \lambda \vdash \underline{n}(s)\}$$

(see Section 9.1). The irreducible constituents of the permutation module $k|_B^G$ are indexed by $D(1,\lambda)$, $\lambda \vdash n$. If

$$l = \begin{cases} r & \text{if } |q \pmod r| = 1 \\ |q \pmod r| & \text{if } |q \pmod r| > 1, \end{cases}$$

then the irreducible kG-modules belonging to the unipotent principal series $\mathrm{Irr}_k(G|B)$ are indexed by $D(1,\lambda)$, where $\lambda \vdash n$ is l-restricted.

18.2 $\mathrm{Ext}^1_{kG}(D(1,\lambda), D(1,\lambda))$ in the case that $\lambda \vdash n$ is l-restricted.

The q-Schur algebra $S_q(n,n)_k$ has weight poset $\Lambda^+(n) = \{\lambda|\lambda \vdash n\}$ (the poset structure on $\Lambda^+(n)$ is given by the dominance order). Therefore, the irreducible $S_q(n,n)_k$-modules are indexed by partitions λ of n. Given a partition $\lambda \vdash n$, let $L^k(\lambda)$ denote the corresponding irreducible $S_q(n,n)_k$-module. By construction, $\bar{F}(D(1,\lambda)) = L^k(\lambda)$ for any $\lambda \vdash n$.

In [9], CPS establish a connection between H^1-calculations for $G = \mathrm{GL}_n(q)$ and Ext^1 calculations for the q-Schur algebra $S_q(n,n)_k$. By [9, Theorem 10.1], under the additional assumption that $\mathrm{char}(k) = r \nmid (q-1)$ (so that $r \nmid q(q-1)$),

$$H^1(G, D(s,\mu)) \cong \begin{cases} \mathrm{Ext}^1_{S_q(n,n)_k}(L^k((1^n)), L^k(\mu)) \text{ if } s = 1 \\ 0 \text{ if } s \neq 1 \end{cases}$$

for any $s \in \mathscr{C}_{ss,r'}$ and any multipartition $\mu \vdash \underline{n}(s)$. Since $k = D(1,(1^n))$, the result of [9, Theorem 10.1] may be restated as follows: if $\mathrm{char}(k) = r \nmid q(q-1)$,

$$\mathrm{Ext}^1_{kG}(D(1,(1^n)), D(s,\mu)) \cong \begin{cases} \mathrm{Ext}^1_{S_q(n,n)_k}(L^k((1^n)), L^k(\mu)) \text{ if } s = 1 \\ 0 \text{ if } s \neq 1 \end{cases}$$

for any $s \in \mathscr{C}_{ss,r'}$ and any multipartition $\mu \vdash \underline{n}(s)$.

The key component of the proof of [9, Theorem 10.1] is the observation that k is in the head of the permutation module $k|_B^G$. But, k is not the only irreducible kG-module contained in $\mathrm{head}(k|_B^G)$; any irreducible kG-module of the form $D(1,\lambda)$, where λ is an l-restricted partition of n, is contained in $\mathrm{head}(k|_B^G)$ (this follows since $\lambda \vdash n$ is l-restricted if and only if $D(1,\lambda) \in \mathrm{Irr}_k(G|B)$). Hence, the proof of [9, Theorem 10.1] holds if we replace $k = D(1,(1^n))$ by an irreducible kG-module $D(1,\lambda)$ with $\lambda \vdash n$ l-restricted.

Theorem 18.2.1. *Suppose that $r \nmid q(q-1)$ and that λ is an l-restricted partition of n. Then, for any $s \in \mathscr{C}_{ss,r'}$ and any multipartition $\mu \vdash \underline{n}(s)$,*

$$\mathrm{Ext}^1_{kG}(D(1,\lambda), D(s,\mu)) \cong \begin{cases} \mathrm{Ext}^1_{S_q(n,n)_k}(L^k(\lambda), L^k(\mu)) \text{ if } s = 1 \\ 0 \text{ if } s \neq 1 \end{cases}$$

Proof. (The following proof is due to CPS [9, 10.1] - no modifications are necessary when k is replaced by an irreducible kG-module $D(1,\lambda)$ with $\lambda \vdash n$ l-restricted.) Let $s \in \mathscr{C}_{ss,r'}$, let $\mu \vdash \underline{n}(s)$, and let $D(s,\mu)$ be the corresponding irreducible kG-module.

Since $\lambda \vdash n$ is l-restricted, $D(1,\lambda) \in \mathrm{Irr}_k(G|B)$, which means that $D(1,\lambda) \subseteq \mathrm{head}(k|_B^G)$. Therefore, there exists a short exact sequence of kG-modules

$$0 \to \mathscr{L} \to k|_B^G \to D(1,\lambda) \to 0 \tag{18.1}$$

for some submodule $\mathscr{L}$ of $k|_B^G$. Now, since $r \not| q(q-1)$, we have $r \not| |T| = (q-1)^n$ and $r \not| |U|$. Therefore, $r \not| |B| = |T||U|$, which means that every kB-module is projective. In particular, the trivial kB-module k is projective. Since induction from B to G is exact, it follows that $k|_B^G$ is a projective kG-module, so that $\mathrm{Ext}^1_{kG}(k|_B^G, D(s,\mu)) = 0$. Therefore, the short exact sequence (18.1) induces the exact sequence

$$\mathrm{Hom}_{kG}(k|_B^G, D(s,\mu)) \to \mathrm{Hom}_{kG}(\mathscr{L}, D(s,\mu)) \to \mathrm{Ext}^1_{kG}(D(1,\lambda), D(s,\mu)) \to 0. \tag{18.2}$$

Now, by [9, Theorem 9.17], any irreducible kG-module $D(s,\tau)$ ($\tau \vdash \underline{n}(s)$) is also an irreducible kG/J_k-module. By [9, Remark 9.18(c)], $k|_B^G$ is a kG/J_k-direct summand of a projective kG/J_k-module and thus is itself a projective kG/J_k-module. It follows that (18.1) is a short exact sequence of kG/J_k-modules in which $k|_B^G$ is a projective kG/J_k-module. Therefore, (18.1) also induces the exact sequence

$$\mathrm{Hom}_{kG/J_k}(k|_B^G, D(s,\mu)) \to \mathrm{Hom}_{kG/J_k}(\mathscr{L}, D(s,\mu)) \to \mathrm{Ext}^1_{kG/J_k}(D(1,\lambda), D(s,\mu)) \to 0. \tag{18.3}$$

The exact sequences (18.2) and (18.3) give rise to a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_{kG}(k|_B^G, D(s,\mu)) & \longrightarrow & \mathrm{Hom}_{kG}(\mathscr{L}, D(s,\mu)) & \longrightarrow & \mathrm{Ext}^1_{kG}(D(1,\lambda), D(s,\mu)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{kG/J_k}(k|_B^G, D(s,\mu)) & \to & \mathrm{Hom}_{kG/J_k}(\mathscr{L}, D(s,\mu)) & \to & \mathrm{Ext}^1_{kG/J_k}(D(1,\lambda), D(s,\mu)) & \to & 0
\end{array}$$

which has exact rows. The two left vertical arrows of the commutative diagram above are isomorphisms, and it follows that the third vertical arrow is also an isomorphism. Therefore, $\mathrm{Ext}^1_{kG}(D(1,\lambda), D(s,\mu)) \cong \mathrm{Ext}^1_{kG/J_k}(D(1,\lambda), D(s,\mu))$. Since $\bar{F}(D(1,\lambda)) = L^k(\lambda)$, it follows by [9, Theorem 9.17] that $\mathrm{Ext}^1_{kG/J_k}(D(1,\lambda), D(s,\mu)) \cong \mathrm{Ext}^1_{S_q(n,n)_k}(L^k(\lambda), \bar{F}(D(s,\mu)))$.[21] Therefore,

$$\mathrm{Ext}^1_{kG}(D(1,\lambda), D(s,\mu)) \cong \mathrm{Ext}^1_{S_q(n,n)_k}(L^k(\lambda), \bar{F}(D(s,\mu))). \tag{18.4}$$

[21] The irreducible $S_q(n,n)_k$-module $L^k(\lambda)$ has the structure of a $\bigoplus_{s\in\mathscr{C}_{ss,r'}} \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_k$-module via the natural quotient map $\bigoplus_{s\in\mathscr{C}_{ss,r'}} \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_k \to S_q(n,n)_k$ ($S_q(n,n)_k$ is the summand of $\bigoplus_{s\in\mathscr{C}_{ss,r'}} \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_k$ corresponding to $s = 1$). So, for $s \neq 1$, the tensor product $\bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_k$ acts as 0 on $L^k(\lambda)$, and it follows that $\mathrm{Ext}^1_{kG/J(q)_k}(D(1,\lambda), D(s,\mu)) \cong \mathrm{Ext}^1_{\bigoplus_{s\in\mathscr{C}_{ss,r'}} \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s),n_i(s))_k}(L^k(\lambda), \bar{F}(D(s,\mu))) \cong \mathrm{Ext}^1_{S_q(n,n)_k}(L^k(\lambda), \bar{F}(D(s,\mu)))$.

Now, if $s \neq 1$, $D(1,\lambda)$ and $D(s,\mu)$ belong to different blocks, which means that

$$\mathrm{Ext}^1_{kG}(D(1,\lambda), D(s,\mu)) = 0.$$

If $s = 1$, then μ is a partition of n, $\bar{F}(D(s,\mu)) = \bar{F}(D(1,\mu)) = L^k(\mu)$, and the isomorphism (18.4) gives $\mathrm{Ext}^1_{kG}(D(1,\lambda), D(s,\mu)) \cong \mathrm{Ext}^1_{S_q(n,n)_k}(L^k(\lambda), L^k(\mu))$.

□

To prove that there are no non-split self-extensions of irreducible kG-modules belonging to the unipotent principal series $\mathrm{Irr}_k(G|B)$, we will require the following result on self-extensions of irreducible modules for the q-Schur algebra $S_q(n,n)_k$.

Proposition 18.2.2. *If λ is a partition of n and $L^k(\lambda)$ is the corresponding irreducible $S_q(n,n)_k$-module, then $\mathrm{Ext}^1_{S_q(n,n)_k}(L^k(\lambda), L^k(\lambda)) = 0$.*

Proof. Let $\Delta(\lambda)$ denote the standard module in mod-$S_q(n,n)_k$ corresponding to the weight λ. $\Delta(\lambda)$ has simple top $L^k(\lambda)$, which means that there is a short exact sequence

$$0 \to \mathrm{rad}(\Delta(\lambda)) \to \Delta(\lambda) \to L^k(\lambda) \to 0.$$

This short exact sequence gives rise to the long exact sequence $0 \to \mathrm{Hom}_{S_q(n,n)_k}(L^k(\lambda), L^k(\lambda)) \to \mathrm{Hom}_{S_q(n,n)_k}(\Delta(\lambda), L^k(\lambda)) \to \mathrm{Hom}_{S_q(n,n)_k}(\mathrm{rad}(\Delta(\lambda)), L^k(\lambda)) \to \mathrm{Ext}^1_{S_q(n,n)_k}(L^k(\lambda), L^k(\lambda)) \to \mathrm{Ext}^1_{S_q(n,n)_k}(\Delta(\lambda), L^k(\lambda)) \to \cdots$.

By definition of $\Delta(\lambda)$, all composition factors $L^k(\mu)$ of $\mathrm{rad}(\Delta(\lambda))$ satisfy $\mu < \lambda$, which means that $\mathrm{Hom}_{S_q(n,n)_k}(\mathrm{rad}(\Delta(\lambda)), L^k(\lambda)) = 0$. Also, by [14, Lemma C.12], $\mathrm{Ext}^1_{S_q(n,n)_k}(\Delta(\lambda), L^k(\lambda)) = 0$. So, by exactness of the sequence above, we have $\mathrm{Ext}^1_{S_q(n,n)_k}(L^k(\lambda), L^k(\lambda)) = 0$.

□

Corollary 18.2.3. *If λ is an l-restricted partition of n (so that $D(1,\lambda) \in \mathrm{Irr}_k(G|B)$), then $\mathrm{Ext}^1_{kG}(D(1,\lambda), D(1,\lambda)) = 0$.*

Proof. By Theorem 18.2.1 and Proposition 18.2.2, we have

$$\mathrm{Ext}^1_{kG}(D(1,\lambda), D(1,\lambda)) \cong \mathrm{Ext}^1_{S_q(n,n)_k}(L^k(\lambda), L^k(\lambda)) = 0.$$

□

19 Calculations of Higher Ext Groups Between Certain Modules for $\mathrm{GL}_n(q)$ in Cross Characteristic

19.1 Introduction

Let q be a power of a prime p, and let $G = \mathrm{GL}_n(q)$ be the general linear group over the finite field $\mathbb{F}_q$ of q elements. Let B be the Borel subgroup of G consisting of all invertible upper triangular $n \times n$ matrices. Let U be the unipotent radical of B (that is, the subgroup of B consisting of invertible upper triangular matrices with 1's along the main diagonal), and let T be the maximal torus in B consisting of all invertible diagonal matrices.

Let k be an algebraically closed field of characteristic $r > 0$ such that $r \not| q$. In [9], Cline, Parshall, and Scott prove that there is an ideal $J_k \trianglelefteq kG$ such there is a Morita equivalence

$$\bar{F} : \mathrm{mod} - kG/J_k \to \mathrm{mod} - \bigoplus_{s \in \mathscr{C}_{ss,r'}} \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_k.$$

CPS [9] use this Morita equivalence to relate H^i-calculations for $\mathrm{GL}_n(q)$ to Ext^i-calculations for a q-Schur algebra. By [9, Theorem 10.1], if $r \not| q(q-1)$ and V is a right kG-module with $J_k \subseteq \mathrm{Ann}_{kG}(V)$,

$$H^1(G, V) \cong \mathrm{Ext}^1_{S_q(n,n)_k}(L^k((1^n)), \bar{F}(V))$$

(here, $L^k((1^n))$ is the irreducible $S_q(n,n)_k$-module corresponding to the partition (1^n)). By [9, Theorem 12.4], if $r \not| q \prod_{j=1}^{m+1} (q^j - 1)$ for some integer $m \geq 0$ and V is a right kG-module with $J_k \subseteq \mathrm{Ann}_{kG}(V)$,

$$H^i(G, V) \cong \mathrm{Ext}^i_{S_q(n,n)_k}(L^k((1^n)), \bar{F}(V))$$

for $0 \leq i \leq m+1$.

The result of [9, Theorem 10.1] was generalized in the previous chapter. The purpose of this chapter is to obtain a generalization of the result of [9, Theorem 12.4].

19.2 A Generalization of [9, Theorem 12.4]

Going forward, we will assume that $r \not| q(q-1)$. When $r \not| q(q-1)$, $r \not| |B|$ (for, $|B| = |T||U|$, where $|T|$ is a power of $(q-1)$ and $|U|$ is a power of p). Thus, k is a projective right kB-module, which means that $k|_B^G$ is a projective right kG-module.

Let $\mathfrak{S}_n$ be the symmetric group on n letters, and let S be the generating set of fundamental reflections in $\mathfrak{S}_n$. Let $\widetilde{H} = \widetilde{H}(\mathfrak{S}_n, S)$ denote the generic Hecke algebra over the Laurent polynomial ring $\mathcal{Z} = \mathbb{Z}[t, t^{-1}]$ corresponding to the pair $(\mathfrak{S}_n, S)$. Let $\widetilde{H}_{\mathcal{O}}$ denote the $\mathcal{O}$-algebra obtained by base change to $\mathcal{O}$, and let $\widetilde{H}_k$ denote the k-algebra obtained by base change to k. As in [44], we will denote $\widetilde{H}_k$ by H. (For consistency, we will also denote the q-Schur algebra $S_q(n,n)_k$ by $S_q(n,n)$.)

Since $H \cong \mathrm{End}_{kG}(k_B^G)$ and $k|_B^G$ is projective, the functor $H_1 : \mathrm{mod} - kG \to -\mathrm{mod} - H$ of [19, Section 4.1] is given by $H_1 = \mathrm{Hom}_{kG}(k|_B^G, -)$. Thus, we will adopt the notation of Geck and Jacon [31] and denote the functor H_1 by $\mathfrak{F}_k$. The functor $\mathfrak{F}_k$ has a right inverse $\mathfrak{G}_k : \mathrm{mod} - H \to \mathrm{mod} - kG$, which is given by $\mathfrak{G}_k(E) = E \otimes_H k|_B^G$ for any right H-module E (here, the right kG-module $k|_B^G$ is naturally a left module for the endomorphism algebra $H = \mathrm{End}_{kG}(k|_B^G)$). In the notation of [19], $\mathfrak{G}_k = \hat{H}_1$. Since $k|_B^G$ is projective, $\mathfrak{G}_k$ is a two-sided inverse of of $\mathfrak{F}_k$ on the full subcategory of $\mathrm{mod} - kG$ consisting of all $V \in \mathrm{mod} - kG$ such that

1. every non-zero submodule of V has a composition factor in $\mathrm{Irr}_k(G|B)$, and
2. every non-zero quotient module of V has a composition factor in $\mathrm{Irr}_k(G|B)$

[29, 4.1.14].

Lemma 19.2.1. *If V is a kG-module in the image of the functor $\mathfrak{G}_k : \mathrm{mod}{-}H \to \mathrm{mod}{-}kG$, then V is annihilated by J_k.*

Proof. Let E be a right H-module such that $\mathfrak{G}_k(E) = E \otimes_{kG} k|_B^G \cong V$. Given $\alpha \in kG$ and $e \otimes x \in E \otimes_{kG} k|_B^G$ (where $e \in E$ and $x \in k|_B^G$), we have $(e \otimes x).\alpha = e \otimes (x.\alpha)$. But, since $k|_B^G$ is a direct summand of the right kG/J_k-module $\widehat{M_{1,G,k}}$, $k|_B^G$ is annihilated by J_k. Thus, $e \otimes (x.\alpha) = e \otimes 0 = 0$. It follows that $E \otimes_{kG} k|_B^G$ is annihilated by J_k, which means that the same statement holds for V.

□

Before we proceed, we must define a certain $S_q(n,n) - H$ bimodule which is used in [44] to link the representation theory of H to the representation theory of $S_q(n,n)$. As in Chapter 8, let V be a free $\mathcal{Z} = \mathbb{Z}[t,t^{-1}]$-module of rank n. Given $\lambda \vdash n$, let $k_\lambda \in \mathbb{Z}$ be such that $V^{\otimes n} \cong \bigoplus_{\lambda \in \Lambda^+(n)} (x_\lambda \widetilde{H})^{k_\lambda}$ (see (8.1)). We define

$$\widetilde{T} = \bigoplus_{\lambda \in \Lambda^+(n)} (x_\lambda \widetilde{H})^{k_\lambda}.$$

$\widetilde{T}$ is a right $\widetilde{H}$-module by definition; and, since $S_t(n,n) = \mathrm{End}_{\widetilde{H}}(V^{\otimes n}) \cong \mathrm{End}_{\widetilde{H}}(\widetilde{T})$, there is a natural left action of $S_t(n,n)$ on $\widetilde{T}$. Let $T = \widetilde{T}_k$ be the $S_q(n,n) - H$ bimodule obtained by reduction of $\widetilde{T}$ to k.

Since $k|_B^G$ is a right kG-module, $k|_B^G$ is a left module for $H = \mathrm{End}_{kG}(k|_B^G)$, which means that the dual $(k|_B^G)^*$ has the structure of a right H-module.

Lemma 19.2.2. *Suppose that $r \not| q(q-1)$, and let $\widehat{M}_{1,G,k} \otimes_{kG} (k|_B^G)^*$ be the right H-module defined via the right action of H on $(k|_B^G)^*$.*[22] *There is an isomorphism $\widehat{M}_{1,G,k} \otimes_{kG} (k|_B^G)^* \cong T^\Phi$ of right H-modules (where T^Φ denotes the right H-module obtained by twisting the action of H on T by the involution Φ of H defined in [44, (2.0.3)]).*

Proof. Since $r \not| q(q-1)$, $r \not| |B|$, which means that $k|_B^G$ is a projective right kG-module. Now, since $k|_B^G$ is self-dual and projective, there is an isomorphism of right H-modules $\widehat{M}_{1,G,k} \otimes_{kG} (k|_B^G)^* \cong \mathrm{Hom}_{kG}(k|_B^G, \widehat{M}_{1,G,k})$ by [16, Remark 2.3], with the right action of H on $\mathrm{Hom}_{kG}(k|_B^G, \widehat{M}_{1,G,k})$ induced by the left action of H on $k|_B^G$. We also have an isomorphism $\mathrm{Hom}_{kG}(k|_B^G, \widehat{M}_{1,G,k}) \cong \mathrm{Hom}_{\mathcal{O}G}(\mathcal{O}|_B^G, \widehat{M}_{1,G,\mathcal{O}})_k$ since $k|_B^G$ is projective. By definition, $\widehat{M}_{1,G,\mathcal{O}} = \bigoplus_{\lambda \vdash n} \sqrt{y_\lambda \mathcal{O}|_B^G}^{\,k_\lambda}$ (see (9.1)); so,

$$\mathrm{Hom}_{\mathcal{O}G}(\mathcal{O}|_B^G, \widehat{M}_{1,G,\mathcal{O}})_k \cong \bigoplus_{\lambda \vdash n} \mathrm{Hom}_{\mathcal{O}G}\big(\mathcal{O}|_B^G, \sqrt{y_\lambda \mathcal{O}|_B^G}\big)_k^{k_\lambda}.$$

By [21, Lemmas 2.22, 2.15 (ii) and 2.8], $\mathrm{Hom}_{\mathcal{O}G}(\mathcal{O}|_B^G, \sqrt{y_\lambda \mathcal{O}|_B^G}) \cong y_\lambda H_{\mathcal{O}}$.[23] Thus,

$$\mathrm{Hom}_{kG}(k|_B^G, \widehat{M}_{1,G,k}) \cong \mathrm{Hom}_{\mathcal{O}G}(\mathcal{O}|_B^G, \widehat{M}_{1,G,\mathcal{O}})_k \cong \bigoplus_{\lambda \vdash n} (y_\lambda H_{\mathcal{O}})_k^{k_\lambda} \cong (\bigoplus_{\lambda \vdash n} (y_\lambda H_{\mathcal{O}})^{k_\lambda})_k \cong \widetilde{T}_k^\Phi \cong T^\Phi,$$

where the isomorphism $\bigoplus_{\lambda \vdash n} (y_\lambda H_{\mathcal{O}})^{k_\lambda} \cong \widetilde{T}^\Phi$ follows by [25, Lemma 1.1 (c)].

□

Before we proceed to the next result, we must compare the indexing of Specht modules for the Hecke algebra used by Du, Parshall, and Scott [25] and Parshall and Scott [44] with that used by Dipper and James [21]. In the work of DPS and PS, the Specht module in mod $-\, \widetilde{H}_{\mathcal{O}}$ corresponding to a partition $\lambda \vdash n$ is denoted by $\widetilde{S}_\lambda$. The definition of $\widetilde{S}_\lambda$ can be found in [44, Section 2.2]. Given a partition $\lambda \vdash n$, $\mathrm{Hom}_{\widetilde{H}_{\mathcal{O}}}(y_{\lambda'} \widetilde{H}_{\mathcal{O}}, x_\lambda \widetilde{H}_{\mathcal{O}}) \cong \mathcal{O}$. Therefore, there exist indecomposable summands $\widetilde{Y}_\lambda^\natural$ of $y_{\lambda'} \widetilde{H}_{\mathcal{O}}$ and $\widetilde{Y}_\lambda$ of $x_\lambda \widetilde{H}_{\mathcal{O}}$ such that $\mathrm{Hom}_{\widetilde{H}_{\mathcal{O}}}(\widetilde{Y}_\lambda^\natural, \widetilde{Y}_\lambda) \cong \mathcal{O}$. (For $\lambda \vdash n$, $\widetilde{Y}_\lambda^\natural$ is called a twisted Young module, and $\widetilde{Y}_\lambda$ is called a Young module.) Let $\zeta_\lambda : \widetilde{Y}_\lambda^\natural \to \widetilde{Y}_\lambda$ be a generator of $\mathrm{Hom}_{\widetilde{H}_{\mathcal{O}}}(\widetilde{Y}_\lambda^\natural, \widetilde{Y}_\lambda)$. Then, the Specht module $\tilde{S}_\lambda$ corresponding to λ is defined to be the image of the homomorphism ζ_λ.

In the work of Dipper and James [20], [21] (and, in the work of Dipper and Du [19]), the Specht module corresponding to the partition $\lambda \vdash n$ is denoted by $\widetilde{S}^\lambda$ and defined as the right ideal $\widetilde{S}^\lambda = x_\lambda \tau_{w_{0\lambda}} y_{\lambda'} \widetilde{H}_{\mathcal{O}}$ of $\widetilde{H}_{\mathcal{O}}$, where $w_{0\lambda}$ is the longest element of the parabolic subgroup W_λ of W [20, Section 4].

---

[22] See (9.1) for the definition of $\widehat{M}_{1,G,k}$.

[23] In the notation of Dipper and James, $\mathcal{O}|_B^G = M$ and the lemmas of [21] apply as follows. Let $\lambda \vdash n$. By [21, Lemma 2.22], $\sqrt{y_\lambda \mathcal{O}|_B^G} = \sqrt{y_\lambda M} \cong M_{y_\lambda} = \{u \in M \mid l(y_\lambda H_{\mathcal{O}})u = 0\}$, where $l(y_\lambda H_{\mathcal{O}})$ is the left annihilator of $y_\lambda H_{\mathcal{O}}$ in $H_{\mathcal{O}}$. By [21, Lemma 2.15 (ii)], $\mathrm{Hom}_{\mathcal{O}G}(\mathcal{O}|_B^G, M_{y_\lambda}) = rl(y_\lambda H_{\mathcal{O}})$, where $rl(y_\lambda H_{\mathcal{O}})$ denotes the right annihilator of $l(y_\lambda H_{\mathcal{O}})$ in $H_{\mathcal{O}}$. And, by [21, Lemma 2.8], $rl(y_\lambda H_{\mathcal{O}}) = y_\lambda H_{\mathcal{O}}$. Combining these results, we see that $\mathrm{Hom}_{\mathcal{O}G}(\mathcal{O}_B^G, \sqrt{y_\lambda \mathcal{O}|_B^G}) \cong y_\lambda H_{\mathcal{O}}$.

Lemma 19.2.3. *The DPS labeling of the Specht modules for $H_{\mathcal{O}}$ is consistent with the DJ labeling. That is, for any $\lambda \vdash n$, $\widetilde{S}_\lambda \cong \widetilde{S}^\lambda$ as right $\widetilde{H}_{\mathcal{O}}$-modules.*

Proof. By [20, Corollary 4.2], $\widetilde{S}^\lambda = x_\lambda \widetilde{H}_{\mathcal{O}} y_{\lambda'} \widetilde{H}_{\mathcal{O}}$. By parts (h) and (f) of [25, Lemma 1.1], $\widetilde{S}_\lambda$ (which is defined as the image of a generator of $\mathrm{Hom}_{\widetilde{H}_{\mathcal{O}}}(y_{\lambda'}\widetilde{H}_{\mathcal{O}}, x_\lambda \widetilde{H}_{\mathcal{O}})$) is isomorphic to $x_\lambda \widetilde{H}_{\mathcal{O}} y_{\lambda'} \widetilde{H}_{\mathcal{O}}$ as well.

□

The PS indexing of the irreducible H-modules is also consistent with the DJ indexing. Let $S_\lambda = \widetilde{S}_{\lambda k}$ denote the Specht module for H obtained by base change to k. As before, let

$$l = \begin{cases} r & \text{if } |q \pmod r| = 1 \\ |q \pmod r| & \text{if } |q \pmod r| > 1. \end{cases}$$

When λ is an l-regular partition of n, $D_\lambda := S_\lambda/\mathrm{rad}(S_\lambda)$ is an irreducible H-module. A full set of irreducible H-modules is given by $\{D_\lambda \mid \lambda \text{ is } l-\text{regular}\}$. In the work of Dipper and James, the irreducible H-module corresponding to an l-regular partition λ of n is denoted by D^λ. But, by Lemma 19.2.3, we have $D^\lambda \cong D_\lambda$ for all l-regular partitions λ of n.

As in Section 9.2, given a partition μ of n, let $S(1,\mu)$ denote the indecomposable right kG-module defined by Dipper and James with $\mathrm{head}(S(1,\mu)) = D^1(1,\mu)$ (where $D^1(1,\mu)$ is indexed according to [19, (4.2.3)]). When $\lambda \vdash n$ is l-restricted, the right kG-module $D(1,\lambda)$ (in the CPS indexing) is isomorphic to $D^1(1,\lambda')$ (see Section 9.4). Thus, when $\lambda \vdash n$ is l-restricted, the indecomposable right kG-module $S(1,\lambda')$ of Dipper and James has the irreducible $D(1,\lambda)$ (in the CPS indexing) in its head. We can now prove the following proposition, which is key to our generalization of [9, Theorem 12.4].

Proposition 19.2.4. *If $l > 2$, λ is an l-restricted partition of n, and $r \not| \, q(q-1)$, then $\bar{F}(S(1,\lambda')) \cong \Delta(\lambda)$ (where $\Delta(\lambda)$ is the standard object corresponding to λ in the category of right $S_q(n,n)$-modules).*

Proof. By [21, (3.1)], any composition factor of $S(1,\lambda')$ is of the form $D^1(1,\mu)$ for some partition $\mu \vdash n$. By [19, (4.2.3), (4)], $D^1(1,\mu)$ belongs to $B_{1,G}$ for any $\mu \vdash n$ (where $B_{1,G}$ is the sum of the unipotent blocks of G). Now, as shown in the proof of [9, Theorem 9.17], all composition factors of the head of the kG/J_k-module $\widehat{M}_{s,G,k}$ (where $s \in \mathscr{C}_{ss,r'}$) belong to $B_{s,G}$. Therefore, when $s \neq 1$, the head of $\widehat{M}_{s,G,k}$ has no irreducible constituents belonging to $B_{1,G}$. In particular, when $s \neq 1$, the head of $\widehat{M}_{s,G,k}$ has no irreducible constituents of the form $D^1(1,\mu)$, $\mu \vdash n$. So, viewing $\widehat{M}_{s,G,k}$ as a kG-module via the natural quotient map $kG \twoheadrightarrow kG/J_k$, we have $\mathrm{Hom}_{kG}(\widehat{M}_{s,G,k}, S(1,\lambda')) = 0$ when $s \neq 1$.

Now, since $r \not| \, q(q-1)$, $k|_B^G$ is a projective right kG-module. Thus, by [19, (4.2.3, (1))] and [29, (4.1.4)], $S(1,\lambda') \cong \mathfrak{G}_k(\mathfrak{F}_k(S(1,\lambda'))) \cong \mathfrak{G}_k(S_{\lambda'})$. Since $S(1,\lambda')$ is in the image of $\mathfrak{G}_k$, $S(1,\lambda')$ is a kG/J_k-module by Lemma 19.2.1, which means that we can apply the CPS functor $\bar{F}$ to $S(1,\lambda')$. Using the arguments of the first paragraph, we can compute

$$\bar{F}(S(1,\lambda')) = \mathrm{Hom}_{kG/J_k}(\bigoplus_{s\in\mathscr{C}_{ss,r'}} \widehat{M}_{s,G,k}, S(1,\lambda'))$$
$$\cong \mathrm{Hom}_{kG}(\bigoplus_{s\in\mathscr{C}_{ss,r'}} \widehat{M}_{s,G,k}, S(1,\lambda'))$$
$$\cong \mathrm{Hom}_{kG}(\widehat{M}_{1,G,k}, S(1,\lambda')),$$

where $\mathrm{Hom}_{kG}(\widehat{M}_{1,G,k}, S(1,\lambda'))$ is viewed as a right $S_q(n,n)_k$-module via the natural left action of $S_q(n,n)_k \cong \mathrm{End}_{kG}(\widehat{M}_{1,G,k})$ on the right kG-module $\widehat{M}_{1,G,k}$.

As shown in the previous paragraph, $S(1,\lambda') \cong \mathfrak{G}_k(S_{\lambda'}) \cong S_{\lambda'} \otimes_H k|_B^G$ (here, we have used Lemma 19.2.3 to identify the Dipper-James Specht module $S^{\lambda'}$ with the CPS Specht module $S_{\lambda'}$). We claim that there is an isomorphism $S_{\lambda'} \otimes_H k|_B^G \cong \mathrm{Hom}_H((k|_B^G)^*, S_{\lambda'})$ of right kG-modules. (The right action of kG on $\mathrm{Hom}_H((k|_B^G)^*, S_{\lambda'})$ is given by $(\phi.\alpha)(f) = \phi(\alpha.f)$ for any $\alpha \in k|_B^G$, $\phi \in \mathrm{Hom}_H((k|_B^G)^*, S_{\lambda'})$, and $f \in (k|_B^G)^*$.) It is known that there is a vector space isomorphism $\Omega : S_{\lambda'} \otimes_H k|_B^G \to \mathrm{Hom}_H((k|_B^G)^*, S_{\lambda'})$, given by $\Omega(s \otimes x) = (f \mapsto f(x)s)$ for any $s \in S_{\lambda'}$ and $x \in k|_B^G$. So, to prove the claim, it suffices to show that Ω respects the right action of kG. But, given any $\alpha \in kG$, $s \in S_{\lambda'}$, $x \in k|_B^G$, and $f \in (k|_B^G)^*$, $\Omega((s\otimes x).\alpha)(f) = \Omega(s \otimes x\alpha)(f) = f(x\alpha)s = (\alpha.f)(x)s = \Omega(s\otimes x)(\alpha.f) = (\Omega(s\otimes x).\alpha)(f)$. Thus, Ω is a right kG-module isomorphism and the claim follows.

Since $S_{\lambda'} \otimes_H k|_B^G \cong \mathrm{Hom}_H((k|_B^G)^*, S_{\lambda'})$ (as right kG-modules),

$$\mathrm{Hom}_{kG}(\widehat{M}_{1,G,k}, S(1,\lambda')) \cong \mathrm{Hom}_{kG}(\widehat{M}_{1,G,k}, \mathrm{Hom}_H((k|_B^G)^*, S_{\lambda'})) \cong \mathrm{Hom}_H(\widehat{M}_{1,G,k} \otimes_{kG} (k|_B^G)^*, S_{\lambda'})$$

as right $S_q(n,n)$-modules (the third isomorphism in the chain of isomorphisms above follows by tensor-hom adjunction, which preserves the right $S_q(n,n)$-module structure of $\mathrm{Hom}_{kG}(\widehat{M}_{1,G,k}, \mathrm{Hom}_H((k|_B^G)^*, S_{\lambda'}))$.

By Lemma 19.2.2, $\widehat{M}_{1,G,k} \otimes_{kG} (k|_B^G)^* \cong T^\Phi$ as right H-modules. Thus, tracing through the calculations above, we have

$$\bar{F}(S(1,\lambda')) \cong \mathrm{Hom}_H(T^\Phi, S_{\lambda'}),$$

with the right action of $S_q(n,n)$ on $\mathrm{Hom}_H(T^\Phi, S_{\lambda'})$ defined via the left action of $S_q(n,n)$ on T^Φ. Now, it follows from the proof of [25, Theorem 7.7] that the right $S_q(n,n)_{\mathcal{O}}$-module $\mathrm{Hom}_{H_{\mathcal{O}}}(\widetilde{T}^\Phi, \widetilde{S}_{\lambda'})$ identifies with $\widetilde{\Delta}^{\mathrm{left}}(\lambda)^{\tilde{\beta}}$, where $\widetilde{\Delta}^{\mathrm{left}}(\lambda)$ is the standard object corresponding to the partition λ in the category of left $S_q(n,n)_{\mathcal{O}}$-modules and $\widetilde{\Delta}^{\mathrm{left}}(\lambda)^{\tilde{\beta}}$ is the right $S_q(n,n)_{\mathcal{O}}$-module obtained by converting the left action of $S_q(n,n)_{\mathcal{O}}$ on $\widetilde{\Delta}^{\mathrm{left}}(\lambda)$ to a right action via the anti-automorphism $\tilde{\beta}$ defined in [25, Lemma 2.2]. But, since

$$(\widetilde{\Delta}^{\mathrm{left}}(\lambda))^{*\tilde{\beta}} = (\widetilde{\Delta}^{\mathrm{left}}(\lambda))^{D_{S_q(n,n)_{\mathcal{O}}}} \cong \widetilde{\nabla}^{\mathrm{left}}(\lambda)$$

(where $D_{S_q(n,n)}$ is the duality on $\text{mod} - S_q(n,n)_{\mathcal{O}}$), we have $\widetilde{\Delta}^{\text{left}}(\lambda)^{\widetilde{\beta}} \cong \widetilde{\nabla}^{\text{left}}(\lambda)^*$ and

$$\text{Hom}_{H_{\mathcal{O}}}(\widetilde{T}^{\Phi}, \widetilde{S}_{\lambda'}) \cong \widetilde{\Delta}^{\text{left}}(\lambda)^{\widetilde{\beta}} \cong \widetilde{\nabla}^{\text{left}}(\lambda)^*.$$

Since $\widetilde{\nabla}^{\text{left}}(\lambda)^* \cong \widetilde{\Delta}(\lambda)$ (where $\widetilde{\Delta}(\lambda)$ is the standard object corresponding to λ in the category of right $S_q(n,n)_{\mathcal{O}}$-modules), it follows that

$$\text{Hom}_{H_{\mathcal{O}}}(\widetilde{T}^{\Phi}, \widetilde{S}_{\lambda'}) \cong \widetilde{\Delta}(\lambda).$$

Finally, when $l > 2$, an argument analogous to that given in the second part of [44, Lemma 2.4] shows that the isomorphism $\text{Hom}_{H_{\mathcal{O}}}(\widetilde{T}^{\Phi}, \widetilde{S}_{\lambda'}) \cong \widetilde{\Delta}(\lambda)$ holds upon base change to k. Therefore, when $l > 2$,

$$\bar{F}(S(1,\lambda')) \cong \text{Hom}_H(T^{\Phi}, S_{\lambda'}) \cong \Delta(\lambda).$$

□

In order to generalize [9, Theorem 12.4], we must construct a suitable resolution of $S(1,\lambda')$ when λ is l-restricted. To construct this resolution, we will use the functors $\mathfrak{N}_i : \text{mod} - \widetilde{H} \to \text{mod} - \widetilde{H}$ defined in [44, Section 3.1]. As above, let S denote the set of fundamental reflections in the symmetric group $\mathfrak{S}_n$. Since $|S| = n-1$, the functor $\mathfrak{N}_i$ is defined for $0 \le i \le n-1$ by

$$\mathfrak{N}_i = \prod_{J \subseteq S, |J|=i} {}^{\otimes}\text{Ind}_{\widetilde{H}_J}^{\widetilde{H}} \circ \text{Res}_{\widetilde{H}_J}^{\widetilde{H}} : \text{mod} - \widetilde{H} \to \text{mod} - \widetilde{H}.$$

Lemma 19.2.5. *Suppose that $r \not| \, q(q-1)$ and λ is an l-restricted partition of n. Then, there exists an exact sequence of right kG-modules of the form $0 \to M_{n-1} \to \cdots \to \cdots M_1 \to M_0 \to S(1,\lambda') \to 0$ in which every module is annihilated by J_k and M_i is projective as both a kG and a kG/J_k-module for $0 \le i \le l-2$.*

Proof. By [44, Theorem 3.4], there exists an exact sequence of right $\widetilde{H}_{\mathcal{O}}$-modules of the form

$$0 \to \widetilde{S_\lambda}^{\Phi} \to \mathfrak{N}_0(\widetilde{S_\lambda}) \to \mathfrak{N}_1(\widetilde{S_\lambda}) \to \cdots \to \mathfrak{N}_{n-1}(\widetilde{S_\lambda}) \to 0.$$

By [44, Remark 3.5], this sequence remains exact upon base change to k, yielding the exact sequence

$$0 \to S_\lambda^{\Phi} \to \mathfrak{N}_0(S_\lambda) \to \mathfrak{N}_1(S_\lambda) \to \cdots \to \mathfrak{N}_{n-1}(S_\lambda) \to 0$$

of right H-modules. Now, by [44, Remark 3.10], $\mathfrak{N}_i(S_\lambda)$ is a projective right H-module for $i \le l-2$. Applying the contravariant duality functor $D_H : \text{mod} - H - \text{mod} - H$ to the exact sequence above, we obtain the exact sequence

$$0 \to \mathfrak{N}_{n-1}(S_\lambda)^{D_H} \to \cdots \to \mathfrak{N}_1(S_\lambda)^{D_H} \to \mathfrak{N}_0(S_\lambda)^{D_H} \to (S_\lambda^{\Phi})^{D_H} \to 0$$

of right H-modules, in which $\mathfrak{N}_i(S_\lambda)^{D_H}$ is projective for $i \le l-2$. Since $(S_\lambda^{\Phi})^{D_H} \cong S_{\lambda'}$, the exact sequence above can be re-written as

$$0 \to \mathfrak{N}_{n-1}(S_\lambda)^{D_H} \to \cdots \to \mathfrak{N}_1(S_\lambda)^{D_H} \to \mathfrak{N}_0(S_\lambda)^{D_H} \to S_{\lambda'} \to 0.$$

Now, since $r \nmid q(q-1)$, $k|_B^G$ is a projective right kG-module and the functor $\mathfrak{G}_k(-) = - \otimes_H k|_B^G$ is exact. Applying $\mathfrak{G}_k$ to the exact sequence above and using the isomorphism $\mathfrak{G}_k(S_{\lambda'}) \cong S(1, \lambda')$, we obtain the exact sequence

$$0 \to \mathfrak{G}_k(\mathfrak{N}_{n-1}(S_\lambda)^{D_H}) \to \cdots \to \mathfrak{G}_k(\mathfrak{N}_1(S_\lambda)^{D_H}) \to \mathfrak{G}_k(\mathfrak{N}_0(S_\lambda)^{D_H}) \to S(1, \lambda') \to 0 \quad (19.1)$$

of right kG-modules.

By Lemma 19.2.1, each of the right kG-modules in the exact sequence (19.1) is annihilated by J_k, which means that (19.1) is also an exact sequence of right kG/J_k-modules. For $0 \le i \le n-1$, let $M_i = \mathfrak{G}_k(\mathfrak{N}_i(S_\lambda)^{D_H})$. Since $\mathfrak{N}_i(S_\lambda)^{D_H}$ is a projective right H-module for $i \le l-2$ and $\mathfrak{G}_k$ is exact, M_i is a projective right kG-module for $i \le l-2$. So, since every projective kG-module which is annihilated by J_k is also a projective kG/J_k-module, it follows that M_i is a projective kG/J_k-module for $i \le l-2$.

□

We are now ready to prove our generalization of [9, Theorem 12.4].

Theorem 19.2.6. *Suppose that $r \nmid q(q-1)$, $l > 2$, and λ is an l-restricted partition of n. If V is a right kG-module with $J_k \subseteq Ann_{kG}(V)$, then* $\mathrm{Ext}_{kG}^i(S(1,\lambda'), V) \cong \mathrm{Ext}_{S_q(n,n)}^i(\Delta(\lambda), \bar{F}(V))$ *for $0 \le i \le l-1$.*

Proof. Given the results of Proposition 19.2.4 and Lemma 19.2.5, the proof of [9, Theorem 12.4] goes through with virtually no change.

Let $0 \to M_{n-1} \to \cdots \to \cdots M_1 \to M_0 \to S(1, \lambda') \to 0$ be the exact sequence obtained in Lemma 19.2.5. This sequence is exact in both the category of right kG-modules and the category of right kG/J_k-modules. For $0 \le i \le l-2$, M_i is projective for both kG and kG/J_k. Let $R = kG$ or kG/J_k, and let $C_{\bullet\bullet}$ denote the double complex obtained by applying the functor $\mathrm{Hom}_R(-, V)$ to a Cartan-Eilenberg resolution of the complex $M_\bullet$. Filtering $C_{\bullet\bullet}$ by columns $C_{i\bullet}$ leads to the spectral sequence

$$\mathrm{Ext}_R^t(M_i, V) \Rightarrow \mathrm{Ext}_R^{i+t}(S(1, \lambda'), V).$$

But, M_i is projective for $0 \le i \le l-2$, so $\mathrm{Ext}_R^t(M_i, V) = 0$ for $t > 0$ and $0 \le i \le l-2$. Since $\mathrm{Hom}_{kG}(-,-)$ and $\mathrm{Hom}_{kG/J_k}(-,-)$ are equivalent bifunctors on mod $-\, kG/J_k$, we have

$$\mathrm{Ext}_{kG}^i(S(1, \lambda'), V) \cong \mathrm{Ext}_{kG/J_k}^i(S(1, \lambda'), V) \text{ for } 0 \le i \le l-1. \quad (19.2)$$

Finally, since $\bar{F}(S(1, \lambda')) \cong \Delta(\lambda)$ when $l > 2$ (by Proposition 19.2.4) and $\bar{F}$ is a Morita equivalence, we have $\mathrm{Ext}_{kG/J_k}^i(S(1, \lambda'), V) \cong \mathrm{Ext}_{S_q(n,n)}^i(\Delta(\lambda), \bar{F}(V))$ for all i. Combining this isomorphism with the isomorphism of (19.2), we have $\mathrm{Ext}_{kG}^i(S(1, \lambda'), V) \cong \mathrm{Ext}_{S_q(n,n)}^i(\Delta(\lambda), \bar{F}(V))$ for $0 \le i \le l-1$.

□

Remark 19.2.7. When $\lambda = (1^n)$, $\lambda' = (n)$ and $S(1,\lambda') = S(1,(n)) = D^1(1,(n)) = D(1,(1^n)) = k$. So, in this case, Theorem 19.2.6 yields $H^i(G,V) \cong \mathrm{Ext}^i_{kG}(k,V) \cong \mathrm{Ext}^i_{S_q(n,n)}(L^k((1^n)), \bar{F}(V))$ for $0 \leq i \leq l-1$.

Theorem 19.2.6 allows us to use known Ext results in the category of modules for the q-Schur algebra to obtain new Ext results in the category of right kG-module. For example, we can use the fact that mod $- S_q(n,n)$ is a highest weight category to prove the following corollary.

Corollary 19.2.8. *Suppose that $r \not| q(q-1)$, $l > 2$, and λ is an l-restricted partition of n.*

(a) If $\mu \vdash n$ is such that $\mu \trianglelefteq \lambda$, then $\mathrm{Ext}^i_{kG}(S(1,\lambda'), D(1,\mu)) = 0$ for $1 \leq i \leq l-1$.

(b) If $\mu \vdash n$ is l-restricted and $\mu \trianglelefteq \lambda$, then $\mathrm{Ext}^i_{kG}(S(1,\lambda'), S(1,\mu')) = 0$ for $1 \leq i \leq l-1$.

Proof. (a) By Theorem 19.2.6,

$$\mathrm{Ext}^i_{kG}(S(1,\lambda'), D(1,\mu)) \cong \mathrm{Ext}^i_{S_q(n,n)}(\Delta(\lambda), \bar{F}(D(1,\mu))) = \mathrm{Ext}^i_{S_q(n,n)}(\Delta(\lambda), L^k(\mu))$$

for $1 \leq i \leq l-1$. Since $\mu \trianglelefteq \lambda$, $\mathrm{Ext}^i_{S_q(n,n)}(\Delta(\lambda), L^k(\mu)) = 0$ for $1 \leq i \leq l-1$ by [14, Proposition C.13 (2)].

(b) By Proposition 19.2.4, $\bar{F}(S(1,\mu')) \cong \Delta(\mu)$. Thus, Theorem 19.2.6 yields $\mathrm{Ext}^i_{kG}(S(1,\lambda'), S(1,\mu')) \cong \mathrm{Ext}^i_{S_q(n,n)}(\Delta(\lambda), \Delta(\mu))$ for $1 \leq i \leq l-1$. Since $\mu \trianglelefteq \lambda$, $\mathrm{Ext}^i_{S_q(n,n)}(\Delta(\lambda), \Delta(\mu)) = 0$ for $1 \leq i \leq l-1$ by [14, Proposition C.13 (2)].

□

Part V

Conclusion

Let q be a power of a prime number p, let $G = G(q)$ be a finite group of Lie type with a split BN-pair, and let k be an algebraically closed field of characteristic $r > 0$ with $r \neq p$. The aim of this dissertation was to study Ext groups between irreducible kG-modules. In Part II we generalized the work of Guralnick and Tiep [35] to find bounds on $\dim \mathrm{Ext}^1_{kG}(Y, V)$ in the following cases:

(1) $Y, V \in \mathrm{Irr}_k(G|B)$ (Chapter 11),

(2) $Y \in \mathrm{Irr}_k(G|(T, X))$, $V \in \mathrm{Irr}_k(G|(T, X'))$ (Chapter 13), and

(3) (G, k) has property (P), $Y \in \mathrm{Irr}_k(G|B)$, $V \notin \mathrm{Irr}_k(G|B)$ (Chapter 14).

In Part IV, we generalized the work of Cline, Parshall, and Scott [9] to compute Ext groups between irreducible modules for $\mathrm{GL}_n(q)$ in cross characteristic. In Chapter 18, we related Ext^1 groups for $\mathrm{GL}_n(q)$ to Ext^1 groups over a q-Schur algebra, and in Chapter 19, we related higher Ext groups for $\mathrm{GL}_n(q)$ to higher Ext groups over a q-Schur algebra. (Both of the generalizations were done under the assumption that $\mathrm{char}(k) = r \not| \, q(q-1)$.)

We have made progress toward a more complete understanding of Ext groups for kG in non-defining characteristic, but there are still many directions for future research. First, there are several other cases to consider when computing bounds on $\dim \mathrm{Ext}^1_{kG}(Y, V)$ (where Y, V are irreducible kG-modules.) In this dissertation, we have assumed that $Y \in \mathrm{Irr}_k(G|(T, X))$ is a principal series representation. But, what if Y belongs to a Harish-Chandra series of the form $\mathrm{Irr}_k(G|(L, X))$ where $T \subsetneq L$? A natural question to ask is whether we can find bounds on $\dim \mathrm{Ext}^1_{kG}(Y, V)$ analogous to those of Chapters 13 and 14 when Y is not a principal series representation. Another question to consider when generalizing the results of Chapter 14 is whether we can drop some of the additional assumptions on G. Specifically, is there a bound analogous to that found in Chapter 14 when the pair (G, k) does not satisfy property (P)?

Another possibility for future research is to extend the bounds on the dimension of Ext^1 given in this dissertation to a wider range of groups. In Part II, we worked only with finite groups of Lie type which have a split BN-pair. (We chose to focus on the split case in order to take advantage of the results on modular Harish-Chandra theory given in [31, Section 4] and [29].) However, Guralnick and Tiep's bounds on the dimension of $H^1(G, V)$ (where V is an irreducible kG-module) apply both to split and non-split finite groups of Lie type [35]. This suggests that there is likely a way to extend our bounds on the dimension of Ext^1 to include the non-split finite groups of Lie type.

There are also many ways in which we could generalize and extend the results of Part

IV of this dissertation. We have related certain Ext groups over $k\mathrm{GL}_n(q)$ to Ext groups over a q-Schur algebra and used homological properties of the q-Schur algebra to obtain Ext vanishing results for $k\mathrm{GL}_n(q)$ (Corollaries 18.2.3 and 19.2.8). However, we have not yet used the fact that Ext calculations over the q-Schur algebra translate to Ext calculations in a category of integrable modules over a quantum group. Perhaps, we can find bounds on the dimension of Ext in the category of integrable modules for the quantum group, which could in turn yield bounds on the dimension of Ext over $k\mathrm{GL}_n(q)$. In particular, it would be interesting to use the quantum group to search for lower bounds on the dimension of Ext over $k\mathrm{GL}_n(q)$. Combining such lower bounds on the dimension of Ext with the upper bounds found in this dissertation could yield the exact dimension of certain Ext groups over $k\mathrm{GL}_n(q)$. (Or, at the very least, we would know that the dimensions of certain Ext groups lie within a range of possible values.)

We could also try to generalize the result of Theorem 19.2.6 to even higher Ext groups. Using the machinery developed by Parshall and Scott in [44], we were able calculate Ext^i only for $i \leq l-1$ (where $l = |q \bmod(r)|$ if $|q \bmod(r)| > 1$ and $l = r$ if $|q \bmod(r)| = 1$). However, it may be possible to generalize Parshall and Scott's work in [44] and construct resolutions which would allow us to compute Ext^i for $i > l-1$. Additionally, much of Parshall and Scott's work in [44] is done under the assumption that $l > 2$. Perhaps, we can find analogous results in the case that $l = 2$ and thus obtain Ext calculations when $\mathrm{char}(k) = 2$. Once this is done, we can work to generalize our Ext calculations to an even wider range of characteristics. As in [9], we assumed that $\mathrm{char}(k) = r \nmid q(q-1)$. The assumption that $r \nmid (q-1)$ is needed to ensure that the permutation module $k|_B^G$ is projective. However, it may be possible to find a projective module (or a collection of projective modules) which could be used in place of $k|_B^G$, allowing us to drop the assumption that $r \nmid (q-1)$.

Finally, it may be possible to extend the Ext calculations of Part IV of this dissertation to more general finite groups of Lie type. Since the q-Schur algebra is defined in Type A, Cline, Parshall, and Scott work only with the finite general linear group $\mathrm{GL}_n(q)$ in [9]. Therefore, our Ext calculations in Chapters 18 and 19 (which generalize Cline, Parshall, and Scott's work in [9]) apply only in Type A. However, using recent work of Du, Parshall, and Scott [26], it may be possible to generalize our Ext calculations to other types. In [26], Du, Parshall, and Scott construct an analog of the q-Schur algebra outside of Type A. Replacing the q-Schur algebra with the new endomorphism algebra of [26] may yield cross characteristic Ext calculations for finite groups of Lie type other than $\mathrm{GL}_n(q)$.

Part VI

Appendices

20 Cohomology of Cyclic r-Groups

Proposition 20.0.1. *Let k be a field of characteristic $r > 0$, and let $G = \langle g \rangle$ be any cyclic r-group, with $|g| = r^l$ for some $l \geq 1$. Then, $H^n(G, k) \cong k$ for $n \geq 0$ (where k is viewed as a G-module with trivial action).*

Proof. (Following [28]) We begin by constructing a projective resolution for k as a kG-module. Let aug : $kG \to k$, $\sum_{i=0}^{r^l-1} a_i g^i \mapsto \sum_{i=0}^{r^l-1} a_i$ (with $a_i \in k$ for $0 \leq i \leq r^l - 1$), be the augmentation map. Then, the augmentation ideal Ker(aug) is generated by the element $g - 1 \in kG$. Now, let $T = 1 + g + g^2 + \cdots + g^{r^l-1} \in kG$. We claim that $T = (g-1)^{r^l-1}$. To show that this is true, we consider the polynomial ring $k[t]$ in an indeterminate t. The polynomial $t^{r^l} - 1$ has factorization $t^{r^l} - 1 = (t-1)(1 + t + \cdot + t^{r^l-1})$. On the other hand, since char $k = r$, $t^{r^l} - 1 = (t-1)^{r^l}$. Therefore, $(t-1)^{r^l} = (t-1)(1 + t + \cdot + t^{r^l-1})$, from which it follows that $(t-1)^{r^l-1} = 1 + t + \cdot + t^{r^l-1}$ by the uniqueness of factorization in $k[t]$. Since $kG \cong k[t]/(t^{r^l} - 1)$, we have $T = (g-1)^{r^l-1}$ in kG.

Now, we consider the following sequence of kG-modules:

$$\cdots \stackrel{(g-1)}{\to} kG \stackrel{T}{\to} kG \stackrel{(g-1)}{\to} kG \stackrel{\text{aug}}{\to} k \to 0,$$

where $kG \stackrel{(g-1)}{\to} kG$ is the map given by multiplication by $(g-1)$, and $kG \stackrel{T}{\to} kG$ is the map given by multiplication by T. For convenience, we will replace the module kG in the n^{th} position of the sequence above by the isomorphic module kGx_n, the free kG-module on one generator x_n. With this notation, the sequence above can be written as

$$\cdots \stackrel{(g-1)}{\to} kGx_2 \stackrel{T}{\to} kGx_1 \stackrel{(g-1)}{\to} kGx_0 \stackrel{\text{aug}}{\to} k \to 0. \tag{20.1}$$

Note that every one of the modules kGx_i in the sequence (20.1) is a projective (in fact, free) kG-module. So, to prove that (20.1) is a projective resolution of k, we need only show that it is exact. Exactness at kGx_0 follows because Ker(aug) $= (g-1)kG$. To prove exactness at all of the other terms in the sequence, it suffices to to show that $\text{Im}(g-1) = \text{Ker}(T)$ and $\text{Im}(T) = \text{Ker}(g-1)$.

First, note that $T(g-1) = g^{r^l} - 1 = 0$. So, $\text{Im}(g-1) \subseteq \text{Ker}(T)$ and $\text{Im}(T) \subseteq \text{Ker}(g-1)$. It remains to check that $\text{Ker}(T) \subseteq \text{Im}(g-1)$ and $\text{Ker}(g-1) \subseteq \text{Im}(T)$. To show that the first containment holds, let $\alpha = \sum_{j=0}^{r^l-1} a_j g^j \in \text{Ker}(T)$ (where $a_j \in k$ for all j). Then,

$$0 = T\alpha = (\sum_{i=0}^{r^l-1} g^i)(\sum_{j=0}^{r^l-1} a_j g^j) = \sum_{j=0}^{r^l-1} a_j g^j (\sum_{i=0}^{r^l-1} g^i) = \sum_{j=0}^{r^l-1} a_j (\sum_{i=0}^{r^l-1} g^{i+j}) = \sum_{j=0}^{r^l-1} a_j (\sum_{i=0}^{r^l-1} g^i) = (\sum_{j=0}^{r^l-1} a_j)T,$$

where the next-to-last equality follows because $g^{r^l} = 1$, which means that $g^{i+j} = g^{i+j \ (\text{mod} r^l)}$. Since $0 = (\sum_{j=0}^{r^l-1} a_j)T$, we must have $\sum_{j=0}^{r^l-1} a_j = 0$, which means that $\alpha \in \text{Ker}(\text{aug}) = \text{Im}(g-1)$.

It remains to prove that $\text{Ker}(g-1) \subseteq \text{Im}(T)$. Suppose $\alpha = \sum_{i=0}^{r^l-1} a_i g^i \in \text{Ker}(g-1)$. Then,

$$\begin{aligned}
0 = (g-1)\alpha = (g-1)\sum_{i=0}^{r^l-1} a_i g^i \\
&= \sum_{i=0}^{r^l-1} a_i (g^{i+1} - g^i) \\
&= a_0(g-1) + a_1(g^2 - g) + a_2(g^3 - g^2) + \cdots + a_{r^l-1}(1 - g^{r^l-1}) \\
&= (a_{r^l-1} - a_0)1 + (a_0 - a_1)g + \cdots + (a_{r^l-2} - a_{r^l-1})g^{r^l-1}.
\end{aligned}$$

Since $\{1, g, \ldots, g^{r^l-1}\}$ is a k-basis for kG, we must have $a_{r^l-1} - a_0 = a_0 - a_1 = \cdots = a_{r^l-2} - a_{r^l-1} = 0$, from which it follows that $a_0 = a_1 = a_2 = \cdots = a_{r^l-1}$. Therefore, $\alpha = a_0 \sum_{i=0}^{r^l-1} g^i = a_0 T \in \text{Im}(T)$. □

21 The Steinberg Module in Defining Characteristic

Let G be a finite group of Lie type, defined over a finite field $\mathbb{F}_q$ (q a power of a prime p) and having a split BN-pair, where B is a Borel subgroup of G containing a maximal torus T and $N = N_G(T)$ is the normalizer of T in G. So, G is the fixed point subgroup of a connected reductive algebraic group $\mathbb{G}$ over $\bar{\mathbb{F}}_p$ under an endomorphism $F : \mathbb{G} \to \mathbb{G}$ such that some power of F is a Frobenius morphism. We assume that the algebraic group $\mathbb{G}$ is simple and of simply connected type.

Let U be the unipotent radical of B (U is a normal p-subgroup of B and $B = U \rtimes T$), and let $W = N_G(T)/T$ be the Weyl group of G with respect to T. Given a field k (of any characteristic), we define an element $\mathfrak{e} \in kG$ by $\mathfrak{e} = \sum_{w \in W} (-1)^{l(w)} n_w \mathfrak{b}$, where $\mathfrak{b} = \sum_{b \in B} b$ and $n_w \in N_G(T)$ is a representative of the coset $w \in W = N_G(T)/T$. Then, the left ideal $\mathrm{St}_k := kG\mathfrak{e}$ of kG is the Steinberg module over k.

From now on, we will assume that $k = \bar{\mathbb{F}}_p$, the algebraically closed field of characteristic equal to the defining characteristic of G. We will show that St_k is an irreducible kG-module, and we will classify St_k according to its highest weight. The ideas behind the proofs in this section can be found in [41, Section 2], though the details are left to the reader in [41].

The irreducibility of St_k follows by the following result of Brauer and Nesbitt (as stated in the introduction of [41]).

Theorem 21.0.1. *([6, Theorem 1]) Let G be a group of order $p^a b$, where p is prime and $(p, b) = 1$. An ordinary irreducible representation of degree divisible by p^a remains irreducible after reduction modulo p.*

The Steinberg module is irreducible in characteristic 0, and St_k can be obtained as a reduction mod p of the ordinary Steinberg representation. By [47, Theorem 1], $\dim_k \mathrm{St}_k = |U|$. Since U is a p-Sylow subgroup of G, $|U|$ is the biggest power of p dividing G. Thus, St_k is irreducible by Brauer-Nesbitt.

Now, an irreducible module for the simply connected algebraic group $\mathbb{G}$ (over the field k) can be classified by its highest weight; there is a bijective correspondence between the dominant weights λ and irreducible $k\mathbb{G}$-modules $L(\lambda)$. As shown by Steinberg, the irreducible kG-modules can be obtained by restricting certain irreducible $k\mathbb{G}$-modules $L(\lambda)$ to G. Given a dominant weight λ, we can write $\lambda = \sum c_i \varpi_i$, where each ϖ_i is a fundamental dominant weight and each c_i is a nonnegative integer. Steinberg proved that a complete set of nonisomorphic irreducible kG-modules is obtained by restricting those $L(\lambda)$ for which c_i lies in $[0, q-1]$ for all i [36, Theorem 2.5].

Since St_k is an irreducible kG-module, St_k must be the restriction of some $L(\lambda)$, where λ is a dominant weight with coordinates lying in $[0, q-1]$. Let $\rho = \sum \varpi_i$ be the sum of the fundamental dominant weights (equivalently, ρ is the half-sum of the positive roots for the root system of $\mathbb{G}$).

Theorem 21.0.2. $\mathrm{St}_k = L((q-1)\rho)$.

Proof. Each irreducible $k\mathbb{G}$-module $L(\lambda)$ is a quotient of a Weyl module $V(\lambda)$. Thus, $\dim_k L(\lambda) \leq \dim_k V(\lambda)$ for each dominant weight λ. Let $\lambda = \sum c_i \varpi_i$ be such that c_i is in $[0, q-1]$ for all i. By Weyl's dimension formula, we have

$$\dim_k V(\lambda) = \frac{\prod_{\alpha \in \Phi^+} (\lambda + \rho, \alpha)}{\prod_{\alpha \in \Phi^+} (\rho, \alpha)} = \prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)},$$

where Φ is the root system for $\mathbb{G}$ and Φ^+ is the set of positive roots.

We will find a bound on each term $\frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}$ $(\alpha \in \Phi^+)$ in the product, which will in turn give us a bound on $\dim_k V(\lambda)$.

First, we observe that $\frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)} = \frac{\frac{2}{(\alpha,\alpha)}(\lambda + \rho, \alpha)}{\frac{2}{(\alpha,\alpha)}(\rho, \alpha)} = \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}$.

We will denote the simple roots giving a base for Φ by α_i (with the simple root α_i corresponding to the fundamental weight ϖ_i). The duals $\alpha_i^\vee$ of the simple roots α_i form a base for the dual root system $\Phi^\vee$. Therefore, given an element $\alpha \in \Phi^+$, we can write $\alpha^\vee = \sum d_i \alpha_i^\vee$ for some nonnegative integers d_i. So, $\langle \lambda + \rho, \alpha \rangle = (\lambda + \rho, \alpha^\vee) = \sum d_i (\lambda + \rho, \alpha_i^\vee) = \sum d_i \langle \lambda + \rho, \alpha_i \rangle$. Now, for each i, $\langle \lambda + \rho, \alpha_i \rangle$ is the coefficient of ϖ_i in $\lambda + \rho$. And, since $\lambda + \rho = \sum c_i \varpi_i + \sum \varpi_i = \sum (c_i + 1) \varpi_i$, we have $\langle \lambda + \rho, \alpha_i \rangle = c_i + 1$ for all i. By our assumption on λ, $c_i \leq q - 1$ for all i, from which it follows that $\langle \lambda + \rho, \alpha \rangle = \sum d_i (c_i + 1) \leq \sum d_i q$ (with equality occuring if and only if $c_i = q - 1$ for all i or, equivalently, $\lambda = (q-1)\rho$). Now, $\langle \rho, \alpha \rangle = (\rho, \alpha^\vee) = \sum d_i (\rho, \alpha_i^\vee) = \sum d_i$. Therefore, $\frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \leq \frac{(\sum d_i) q}{\sum d_i} = q$. And, $\frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = q$ if and only if $\lambda = (q-1)\rho$.

By the discussion of the previous paragraph, we have $\dim_k V(\lambda) \leq q^{|\Phi^+|}$ for all dominant weights λ with coordinates in $[0, q-1]$, with equality occurring if and only if $\lambda = (q-1)\rho$. But, $q^{|\Phi^+|} = |U| = \dim_k \mathrm{St}_k$. Thus, St_k cannot be a quotient of $V(\lambda)$ (λ with coordinates in $[0, q-1]$) unless $\lambda = (q-1)\rho$. Hence, we must have $\mathrm{St}_k = L((q-1)\rho)$ (which, by dimension considerations, is also equal to $V((q-1)\rho)$. $\square$

22 The Character of the Steinberg Module in Characteristic 0

22.1 Introduction

Let G be a finite group of Lie type, defined in characteristic p. Let B be a Borel subgroup of G, and let U denote the unipotent radical of B (so, U is a normal p-subgroup of B and $B = U \rtimes T$, where T is a maximal torus of G contained in B). Let $W = N_G(T)/T$ be the Weyl group of G with respect to T, and let S be the set of fundamental reflections generating W. Let Φ be the corresponding root system, and let Φ^+ denote the set of positive roots. Given a field k, we define an element $\mathfrak{e} \in kG$ by $\mathfrak{e} = \sum_{w\in W} (-1)^{l(w)} n_w \mathfrak{b}$, where $\mathfrak{b} = \sum_{b\in B} b$. Then, $\mathrm{St}_k := kG\mathfrak{e}$ is the Steinberg module over k. Steinberg showed that a k-basis for St_k is given by the set $\{u\mathfrak{e} \mid u \in U\}$, so that $\dim_k(\mathrm{St}_k) = |U|$.

Note that the Steinberg module may be defined over a field of any characterstic. However, the structure of St_k can vary greatly depending on the characteristic of k. In particular, Steinberg showed that the module St_k is irreducible if and only if $\mathrm{char}(k)$ does not divide $[G : B]$.

In this section, we will assume that $k = \mathbb{C}$, the field of complex numbers. Since we are in characteristic 0, the Steinberg module $\mathrm{St}_{\mathbb{C}}$ is irreducible. We will show that the character of $\mathrm{St}_{\mathbb{C}}$ is given by the formula $\sum_{J\subseteq S} (-1)^{|J|} 1_{P_J}^G$, where P_J denotes the standard parabolic subgroup of G corresponding to the subset $J \subseteq S$, and $1_{P_J}^G$ is the trivial character on P_J induced to G. This character formula was originally proved by Curtis in 1965 [11]. In what follows, we will use ideas from Curtis's paper [11] as well as from [8, Chapter 6].

22.2 A generalized character of G

Following [8, 6.2], we will show that the generalized character $\chi := \sum_{J\subseteq S} (-1)^{|J|} 1_{P_J}^G$ is, in fact, an irreducible character of G. We will use the following standard facts in our proof.

(1) Let H and K be subgroups of G, and let R be a set of double coset representatives of G with respect to H and K. Let ϕ be a character of H, and let ψ be a character of K. Then, the scalar product of the induced characters ϕ^G and ψ^G is given by the formula $(\phi^G, \psi^G) = \sum_{r\in R} (\phi, {}^r\psi)_{H\cap {}^rK}$, where ${}^r\psi$ is the character of ${}^rK$ defined by ${}^r\psi({}^rx) = \psi(x)$ for $x \in K$. (This result is due to Mackey.)

(2) Given a subset $J \subseteq S$, let $W_J = \langle s \rangle_{s \in J}$ be the parabolic subgroup of W corresponding to J. Then, the generalized character $\varepsilon = \sum_{J \subseteq S} (-1)^{|J|} 1_{W_J}^W$ of W is given by $\varepsilon(w) = (-1)^{l(w)}$ for $w \in W$. Therefore, ε is the irreducible character of W corresponding to the sign representation.

Lemma 22.2.1. *Let $I, J \subseteq S$. And, let ${}^I W^J = {}^I W \cap W^J$ (where ${}^I W$ is a set of shortest right coset representatives of W_I in W, and W^J is a set of shortest left coset representatives of W_J in W). Then, $(1_{P_I}^G, 1_{P_J}^G) = |{}^I W^J|$ and $(1_{W_I}^W, 1_{W_J}^W) = |{}^I W^J|$.*

Proof. Observe that ${}^I W^J$ is both a set of (P_I, P_J) double coset representatives in G and a set of (W_I, W_J) double coset representatives in W. By Mackey's formula, we have

$$\begin{aligned}
(1_{P_I}^G, 1_{P_J}^G) &= \sum_{w \in {}^I W^J} (1, {}^w 1)_{P_I \cap {}^w P_J} \\
&= \sum_{w \in {}^I W^J} \frac{1}{|P_I \cap {}^w P_J|} \sum_{p \in P_I \cap {}^w P_J} 1(p) \overline{{}^w 1(p)} \\
&= \sum_{w \in {}^I W^J} \frac{1}{|P_I \cap {}^w P_J|} \sum_{p \in P_I \cap {}^w P_J} 1 \\
&= \sum_{w \in {}^I W^J} \frac{1}{|P_I \cap {}^w P_J|} |P_I \cap {}^w P_J| \\
&= \sum_{w \in {}^I W^J} 1 \\
&= |{}^I W^J|.
\end{aligned}$$

The proof that $(1_{W_I}^W, 1_{W_J}^W) = |{}^I W^J|$ is analogous.

□

Proposition 22.2.2. $(\chi, \chi) = 1$.

Proof. By Lemma 22.2.1, we have

$$\begin{aligned}
(\chi, \chi) &= \sum_{I \subseteq S} \sum_{J \subseteq S} (-1)^{|I|} (-1)^{|J|} (1_{P_I}^G, 1_{P_J}^G) \\
&= \sum_{I \subseteq S} \sum_{J \subseteq S} (-1)^{|I|} (-1)^{|J|} |{}^I W^J| \\
&= \sum_{I \subseteq S} \sum_{J \subseteq S} (-1)^{|I|} (-1)^{|J|} (1_{W_I}^W, 1_{W_J}^W) \\
&= (\varepsilon, \varepsilon) = 1,
\end{aligned}$$

where the last equality follows since ε is an irreducible character of W.

□

Using Proposition 22.2.2, we conclude that $\pm\chi$ must be an irreducible character of G.

Proposition 22.2.3. $(1_B^G, \chi) = 1$.

Proof. Since $B = P_\emptyset$, Lemma 22.2.1 gives $(1_B^G, 1_{P_J}^G) = |W^J|$ for any subset $J \subseteq S$. Therefore, we can compute

$$(1_B^G, \chi) = \sum_{J \subseteq S} (-1)^{|J|} (1_B^G, 1_{P_J}^G) = \sum_{J \subseteq S} (-1)^{|J|} |W^J| = \sum_{J \subseteq S} (-1)^{|J|} (1_1^W, 1_{W_J}^W) = (1_1^W, \varepsilon).$$

Now, by Frobenius reciprocity, $(1_1^W, \varepsilon) = (1, \varepsilon|_1)_1$. Since the restriction of the sign character to the trivial subgroup $\{1\}$ of W is the trivial character 1, we have $(1, \varepsilon|_1)_1 = (1, 1)_1 = 1$. □

Now, 1_B^G is a a character of G (and not just a generalized character). Since the coefficient of χ in 1_B^G is $(1_B^G, \chi) = 1$, χ must be an irreducible character of G.

We end this section with another computation, which will be useful in the next section.

Proposition 22.2.4. *Let $I \subseteq S$, with $I \neq \emptyset$. Then, $(1_{P_I}^G, \chi) = 0$.*

Proof. We have $(1_{P_I}^G, \chi) = \sum_{J \subseteq S} (-1)^{|J|} (1_{P_I}^G, 1_{P_J}^G) = \sum_{J \subseteq S} (-1)^{|J|} |{}^I W^J| = \sum_{J \subseteq S} (-1)^{|J|} (1_{W_I}^W, 1_{W_J}^W) = (1_{W_I}^W, \varepsilon)$. By Frobenius reciprocity, $(1_{W_I}^W, \varepsilon) = (1, \varepsilon_{W_I})_{W_I}$, where ε_{W_I} is the sign representation of W_I. Now, since $I \neq \emptyset$, W_I is a non-trivial subgroup of W. Therefore, ε_{W_I} is an irreducible character of W_I distinct from 1, which means that $(1, \varepsilon_{W_I})_{W_I} = 0$. □

22.3 The character of $\mathrm{St}_{\mathbb{C}}$

In this section, we follow Curtis's arguments in [11]. Curtis gives the following characterization of the character $\chi = \sum_{J \subseteq S} (-1)^{|J|} 1_{P_J}^G$ [11, Theorem 2 (d)].

Lemma 22.3.1. *χ is the unique irreducible character of G such that χ is a component of 1_B^G but not a component of $1_{P_J}^G$ for any $J \neq \emptyset$.*

Proof. We showed already that χ possesses the desired properties. It remains to check that χ is unique with these properties. But, suppose that χ' is another such irreducible character. In particular, $(\chi', 1_B^G) \neq 0$ and $(\chi', 1_{P_J}^G) = 0$ for any $J \neq \emptyset$. Therefore, $(\chi', \chi) = \sum_{J \subseteq S} (-1)^{|J|} (\chi', 1_{P_J}^G) = (\chi', 1_B^G) \neq 0$. Since χ and χ' are both irreducible characters, we must have $\chi = \chi'$. □

We can now prove that the irreducible Steinberg module $\mathrm{St}_{\mathbb{C}} = \mathbb{C}G\mathfrak{e}$ has character χ [11, Theorem 3]. We will use without proof the following results proved by Steinberg [47].

Lemma 22.3.2. *Given $s \in S$, let n_s denote a representative of s in $N_G(T)$. And, let α be the simple root such that $s = s_\alpha$.*

1. $n_s\mathfrak{e} = -\mathfrak{e}$.
2. *For any element $u_\alpha \neq 1$ in the root subgroup U_α, there exists an element $\bar{u}_\alpha \in U_\alpha$ such that $n_s^{-1}u_\alpha\mathfrak{e} = (\bar{u}_\alpha - 1)\mathfrak{e}$.*

Theorem 22.3.3. *The character χ is afforded by the module* $\mathrm{St}_{\mathbb{C}}$.

Proof. First, we note that the character 1_B^G is afforded by the permutation module $\mathbb{C}|_B^G \cong \mathbb{C}G\mathfrak{b}$, and the character $1_{P_J}^G$ ($J \subseteq S$) is afforded by the permutation module $\mathbb{C}|_{P_J}^G \cong \mathbb{C}G\mathfrak{p}_J$, where $\mathfrak{p}_J = \sum_{p \in P_J} p \in \mathbb{C}P_J$. Over the complex numbers, representations are determined by their characters. Therefore, by Lemma 22.3.1, the module which affords χ is the unique irreducible $\mathbb{C}G$-module which occurs in $\mathbb{C}G\mathfrak{b}$ but not in $\mathbb{C}G\mathfrak{p}_J$ for any non-empty subset J of S. So, to show that $\mathrm{St}_{\mathbb{C}}$ has character χ, it suffices to check that $\mathrm{St}_{\mathbb{C}}$ occurs in $\mathbb{C}G\mathfrak{b}$ and not in $\mathbb{C}G\mathfrak{p}_J$ for any $J \neq \emptyset$; in fact, since all $\mathbb{C}G$-modules are completely reducible, it suffices to check that $\mathrm{St}_{\mathbb{C}} \subseteq \mathbb{C}G\mathfrak{b}$ and that $\mathrm{St}_{\mathbb{C}} \nsubseteq \mathbb{C}G\mathfrak{p}_J$ for any $J \neq \emptyset$.

The fact that $\mathrm{St}_{\mathbb{C}} \subseteq \mathbb{C}G\mathfrak{b}$ follows directly from the definition of $\mathfrak{e}$. Now, let J be a non-empty subset of S and assume, for contradiction, that $\mathrm{St}_{\mathbb{C}} \subseteq \mathbb{C}G\mathfrak{p}_J$. Observe that $\mathfrak{p}_J^2 = \sum_{p \in P_J}\sum_{p' \in P_J} pp' = \sum_{p \in P_J} \mathfrak{p}_J = |P_J|\mathfrak{p}_J$. Therefore, if $\mathrm{St}_{\mathbb{C}} \subseteq \mathbb{C}G\mathfrak{p}_J$, then $\mathrm{St}_{\mathbb{C}}\mathfrak{p}_J = \mathrm{St}_{\mathbb{C}}$. Now, since $\frac{1}{|P_J|}\mathfrak{p}_J$ is an idempotent in $\mathbb{C}G$, we have an isomorphism $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}G\mathfrak{p}_J) \to \mathfrak{p}_J\mathbb{C}G\mathfrak{p}_J{}^{\mathrm{op}}$, given by $\phi \mapsto \phi(\mathfrak{p}_J)$ for any $\phi \in \mathrm{End}_{\mathbb{C}G}(\mathbb{C}G\mathfrak{p}_J)$. And, since all representations over the complex numbers are completely reducible, $\mathrm{St}_{\mathbb{C}}\mathfrak{p}_J$ is a direct summand of $\mathbb{C}G\mathfrak{p}_J$. Thus, $\mathrm{End}_{\mathbb{C}G}(\mathrm{St}_{\mathbb{C}}\mathfrak{p}_J) \subseteq \mathrm{End}_{\mathbb{C}G}(\mathbb{C}G\mathfrak{p}_J) \cong \mathfrak{p}_J\mathbb{C}G\mathfrak{p}_J{}^{\mathrm{op}}$. In the algebra $\mathfrak{p}_J\mathbb{C}G\mathfrak{p}_J{}^{\mathrm{op}}$, $\mathrm{End}_{\mathbb{C}G}(\mathrm{St}_{\mathbb{C}}\mathfrak{p}_J)$ may be identified with $\mathfrak{p}_J\mathrm{St}_{\mathbb{C}}\mathfrak{p}_J{}^{\mathrm{op}}$. And, since $\mathrm{St}_{\mathbb{C}}\mathfrak{p}_J = \mathrm{St}_{\mathbb{C}} \neq 0$, $\mathrm{End}_{\mathbb{C}G}(\mathrm{St}_{\mathbb{C}}\mathfrak{p}_J) \neq 0$, which means that $\mathfrak{p}_J\mathrm{St}_{\mathbb{C}} = \mathfrak{p}_J\mathrm{St}_{\mathbb{C}}\mathfrak{p}_J \neq 0$.

We claim that $\mathfrak{u}\mathfrak{e} \in \mathfrak{p}_J\mathrm{St}_{\mathbb{C}}$. We begin by showing that $\mathfrak{b}\mathrm{St}_{\mathbb{C}} = \langle \mathfrak{u}\mathfrak{e}\rangle_{\mathbb{C}}$, the $\mathbb{C}$-span of the element $\mathfrak{u}\mathfrak{e} \in \mathrm{St}_{\mathbb{C}}$ (where $\mathfrak{u} = \sum_{u \in U} u$). First, we check that $\langle \mathfrak{u}\mathfrak{e}\rangle_{\mathbb{C}} \subseteq \mathfrak{b}\mathrm{St}_{\mathbb{C}}$ (where $\mathfrak{b} = \sum_{b \in B} b$). Since $B = U \rtimes T$, any element $b \in B$ may be written as $b = ut$, with $u \in U$, $t \in T$. Now, since $T \subseteq N_G(U)$, $n_w \in N_G(T)$ for all $w \in W$, and $u\mathfrak{u} = \mathfrak{u}$, we have $b\mathfrak{u}\mathfrak{e} = ut\mathfrak{u}\mathfrak{e} = \mathfrak{u}ut\mathfrak{e} = \mathfrak{u}t\sum_{w \in W}(-1)^{l(w)}n_w\mathfrak{b} = \mathfrak{u}\sum_{w \in W}(-1)^{l(w)}n_w t\mathfrak{b} = \mathfrak{u}\sum_{w \in W}(-1)^{l(w)}n_w\mathfrak{b} = \mathfrak{u}\mathfrak{e}$ (where the equality $t\mathfrak{b} = \mathfrak{b}$ follows because $t \in B$). Therefore, $\mathfrak{b}\mathfrak{u}\mathfrak{e} = |B|\mathfrak{u}\mathfrak{e}$, so that $\mathfrak{u}\mathfrak{e} = \dfrac{1}{|B|}\mathfrak{b}\mathfrak{u}\mathfrak{e} \in \mathfrak{b}\mathrm{St}_{\mathbb{C}}$ and $\langle \mathfrak{u}\mathfrak{e}\rangle_{\mathbb{C}} \subseteq \mathfrak{b}\mathrm{St}_{\mathbb{C}}$. Now, we check that $\mathfrak{b}\mathrm{St}_{\mathbb{C}} \subseteq \langle \mathfrak{u}\mathfrak{e}\rangle_{\mathbb{C}}$. Since $U \subseteq B$, $\mathfrak{b}\mathrm{St}_{\mathbb{C}} = \mathrm{St}_{\mathbb{C}}^B \subseteq \mathrm{St}_{\mathbb{C}}^U = \mathfrak{u}\mathrm{St}_{\mathbb{C}}$. And, since $\mathfrak{u}u\mathfrak{e} = \mathfrak{u}\mathfrak{e}$ for any $u \in U$ and the set $\{u\mathfrak{e} | u \in U\}$ is a $\mathbb{C}$-basis for $\mathrm{St}_{\mathbb{C}}$, we have $\mathfrak{u}\mathrm{St}_{\mathbb{C}} = \langle \mathfrak{u}\mathfrak{e}\rangle_{\mathbb{C}}$. Thus, $\mathfrak{b}\mathrm{St}_{\mathbb{C}} \subseteq \langle \mathfrak{u}\mathfrak{e}\rangle_{\mathbb{C}}$, as needed. Now, since

$\mathfrak{p}_J\mathrm{St}_{\mathbb{C}} = \mathrm{St}_{\mathbb{C}}^{P_J}$, $\mathfrak{b}\mathrm{St}_{\mathbb{C}} = \mathrm{St}_{\mathbb{C}}^{B}$, and $B \subseteq P_J$, we have $\mathfrak{p}_J\mathrm{St}_{\mathbb{C}} \subseteq \mathfrak{b}\mathrm{St}_{\mathbb{C}} = \langle \mathfrak{u}\mathfrak{e}\rangle_{\mathbb{C}}$. But, $\mathfrak{p}_J\mathrm{St}_{\mathbb{C}} \neq 0$ and $\langle \mathfrak{u}\mathfrak{e}\rangle_{\mathbb{C}}$ is one-dimensional; it follows that $\mathfrak{p}_J\mathrm{St}_{\mathbb{C}} = \langle \mathfrak{u}\mathfrak{e}\rangle_{\mathbb{C}}$, so that $\mathfrak{u}\mathfrak{e} \in \mathfrak{p}_J\mathrm{St}_{\mathbb{C}}$.

Now, for any $s \in J$, a representative n_s of $s \in W = N/T$ is contained in P_J, which means that $n_s\mathfrak{p}_J = \mathfrak{p}_J$. And, since $\mathfrak{u}\mathfrak{e} \in \mathfrak{p}_J\mathrm{St}_{\mathbb{C}}$, we must have $n_s\mathfrak{u}\mathfrak{e} = \mathfrak{u}\mathfrak{e}$ for all $s \in J$. We now fix a simple reflection $s \in J$. Let α be the simple root with $s = s_\alpha$. We can write $U = U'_\alpha U_\alpha$, where U_α is the root subgroup corresponding to α and $U'_\alpha = \prod_{\beta\in\Phi^+\setminus\{\alpha\}} U_\beta$. Since every element $u \in U$ can be expressed uniquely in the form $u = u'_\alpha u_\alpha$ with $u'_\alpha \in U'_\alpha$ and $u_\alpha \in U_\alpha$, we have $\mathfrak{u} = \sum_{u\in U} u = \sum_{\substack{u'_\alpha\in U'_\alpha,\\ u_\alpha\in U_\alpha}} u'_\alpha u_\alpha = \sum_{u_\alpha\in U_\alpha} (\sum_{u'_\alpha\in U'_\alpha} \mathfrak{u}'_\alpha) u_\alpha = \sum_{u_\alpha\in U_\alpha} \mathfrak{u}' u_\alpha$, where $\mathfrak{u}' := \sum_{u'_\alpha\in U'_\alpha} \mathfrak{u}'_\alpha$. Now, using the statements of Lemma 173, we carry out the following computation. For the simple reflection $s \in J$,

$$\begin{aligned}
n_s^{-1}\mathfrak{u}\mathfrak{e} &= n_s^{-1}(\sum_{u_\alpha\in U_\alpha} \mathfrak{u}'u_\alpha)\mathfrak{e} = n_s^{-1}(\sum_{\substack{u_\alpha\in U_\alpha,\\ u_\alpha\neq 1}} \mathfrak{u}'u_\alpha + \mathfrak{u}')\mathfrak{e} = n_s^{-1}\mathfrak{u}'(\sum_{\substack{u_\alpha\in U_\alpha,\\ u_\alpha\neq 1}} u_\alpha)\mathfrak{e} + n_s^{-1}\mathfrak{u}'\mathfrak{e} \\
&= n_s^{-1}\mathfrak{u}'n_s n_s^{-1}(\sum_{\substack{u_\alpha\in U_\alpha,\\ u_\alpha\neq 1}} u_\alpha)\mathfrak{e} + n_s^{-1}\mathfrak{u}'\mathfrak{e} = n_s^{-1}\mathfrak{u}'n_s(\sum_{\substack{u_\alpha\in U_\alpha,\\ u_\alpha\neq 1}} n_s^{-1}u_\alpha\mathfrak{e}) + n_s^{-1}\mathfrak{u}'\mathfrak{e} \\
&= n_s^{-1}\mathfrak{u}'n_s(\sum_{\substack{u_\alpha\in U_\alpha,\\ u_\alpha\neq 1}} (\bar{u}_\alpha - 1)\mathfrak{e}) + n_s^{-1}\mathfrak{u}'\mathfrak{e} \\
&= n_s^{-1}\mathfrak{u}'n_s(\sum_{\substack{u_\alpha\in U_\alpha,\\ u_\alpha\neq 1}} \bar{u}_\alpha)\mathfrak{e} - (|U_\alpha| - 1)n_s^{-1}\mathfrak{u}'n_s\mathfrak{e} + n_s^{-1}\mathfrak{u}'\mathfrak{e}.
\end{aligned}$$

Since s permutes the elements of $\Phi^+ \setminus \alpha$, we have $n_s^{-1}U'_\alpha n_s = \prod_{\beta\in\Phi^+\setminus\{\alpha\}} U_{s(\beta)} = U'_\alpha$. Therefore, $n_s^{-1}\mathfrak{u}'n_s = \mathfrak{u}'$. Also, since the assignment $U_\alpha \setminus \{1\} \to U_\alpha \setminus \{1\}$, $u_\alpha \to \bar{u}_\alpha$ is bijective, we have $\sum_{\substack{u_\alpha\in U_\alpha,\\ u_\alpha\neq 1}} \bar{u}_\alpha = \sum_{\substack{u_\alpha\in U_\alpha,\\ u_\alpha\neq 1}} u_\alpha$. Continuing the calculation above, we have

$$
\begin{aligned}
(^*) &= \mathfrak{u}'(\sum_{\substack{u_\alpha \in U_\alpha,\\ u_\alpha \neq 1}} \bar{u}_\alpha)\mathfrak{e} - (|U_\alpha| - 1)\mathfrak{u}'\mathfrak{e} + n_s^{-1}\mathfrak{u}'\mathfrak{e} \\
&= \mathfrak{u}'(\mathfrak{u} - 1)\mathfrak{e} - (|U_\alpha| - 1)\mathfrak{u}'\mathfrak{e} + n_s^{-1}\mathfrak{u}'\mathfrak{e} \\
&= \mathfrak{u}\mathfrak{e} - \mathfrak{u}'\mathfrak{e} - (|U_\alpha| - 1)\mathfrak{u}'\mathfrak{e} - n_s^{-1}\mathfrak{u}' n_s\mathfrak{e} \\
&= \mathfrak{u}\mathfrak{e} - |U_\alpha|\mathfrak{u}'\mathfrak{e} - \mathfrak{u}'\mathfrak{e} \\
&= \mathfrak{u}\mathfrak{e} - (|U_\alpha| + 1)\mathfrak{u}'\mathfrak{e} \\
&\neq \mathfrak{u}\mathfrak{e}.
\end{aligned}
$$

(The last assertion follows since the elements $u'_\alpha\mathfrak{e}$, $u'_\alpha \in U'_\alpha \subseteq U$ are part of the $\mathbb{C}$-basis $\{u\mathfrak{e} \mid u \in U\}$ for $\mathrm{St}_{\mathbb{C}}$, which means that $(|U_\alpha| + 1)\mathfrak{u}'\mathfrak{e} = \sum_{u'_\alpha \in U'_\alpha} (|U_\alpha| + 1)u'_\alpha\mathfrak{e} \neq 0$.)

Our calculation has shown that $n_s^{-1}\mathfrak{u}\mathfrak{e} \neq \mathfrak{u}\mathfrak{e}$, which means $n_s\mathfrak{u}\mathfrak{e} \neq \mathfrak{u}\mathfrak{e}$, contradicting the statement that $n_s\mathfrak{u}\mathfrak{e} = \mathfrak{u}\mathfrak{e}$ for all $s \in J$. Thus, we conclude that $\mathrm{St}_{\mathbb{C}} \nsubseteq \mathbb{C}G\mathfrak{p}_J$, as needed.

□

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