AN EXAMPLE IN PERIODIC ORBITS

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AN EXAMPLE IN PERIODIC ORBITS, THE SECOND-ORDER PERTURBATIONS OF *JUPITER* AND *SATURN* INDEPENDENT OF THE ECCENTRICITIES AND OF THE MUTUAL INCLINATION,

BY JAMES PARK McCALLIE.

INTRODUCTORY.

A periodic solution is a particular integral in the problem of three bodies. It is possible only under certain restricting conditions which do not exist in nature. Yet such solutions are of great beauty and interest, and often possess real value in assisting us to obtain a more general solution of the differential equations of motion; as, for example, the periodic orbit used as an intermediary by Dr. G. W. HILL in his "*Researches in the Lunar Theory.*"[1]

Analytically a solution is said to be periodic when the coordinates, referred to axes rotating with a uniform angular velocity, may be expressed in series periodic with respect to the time, while geometrically a periodic orbit is one in which a body, referred to the same rotating axes, returns periodically to the same position with reference to the other two bodies.

LAGRANGE in a very elegant manner discovered the first periodic solutions in the problem of three bodies, which however are without much practical value, since to obtain them he assumed the mutual distances as always being in a constant ratio to each other. These are the straight line and equilateral triangle solutions.

The next periodic solutions were obtained by G. W. HILL.[1] By neglecting the lunar inclination and the solar parallax and eccentricity he finds a particular integral of the equations for the moon's motion about the earth under the influence of the disturbing force of the sun. The curve corresponding to this particular integral is closed when referred to rotating axes, and is what is known as the *variational* orbit of the moon. This is used by Dr. HILL as an intermediary instead of the ellipse or modified ellipse of other lunar theorists.

POINCARÉ has shown that there are an infinite number of such solutions that are really distinct. Since POINCARÉ's wholly analytical treatment, the subject of periodic orbits has attracted many astronomers and mathematicians, and a number of memoirs, both analytical and numerical, have been produced. The whole field of periodic orbits is recognized as a fertile one, though by no means easy of entrance.

Of the memoirs on the subject may be mentioned one by C. V. L. CHARLIER,[1] in which he obtains analytically some of the results found by DARWIN in his extensive numerical work on periodic orbits.[2] In the majority of memoirs one mass is assumed infinitesimal, but periodic orbits exist, whether the mass be infinitesimal or not. It is in the case where none of the masses are infinitesimal that I have selected the following numerical example in periodic orbits. The case is purely an ideal one, but it was in the hope that the results might be of some interest to astronomers that the work was undertaken.

The suggestion of the problem is due to Dr. G. W. HILL, and I desire to express my great indebtedness to him, and also my appreciation to Prof. ORMOND STONE for his encouragement and helpful suggestions, and to Mr. T. McN. SIMPSON, Jr., for checking some of the numerical work.

Example. If two masses, small relatively to a third mass, revolve around the latter in coplanar orbits, having no proper eccentricities, they will have symmetrical conjunctions and oppositions, *i.e.*, their conjunctions and oppositions will be symmetrically placed with regard to their mutually perturbed orbits, which will cut the line of syzygies perpendicularly. Let us take the time of such a symmetrical conjunction as the origin of time, and the longitude of this conjunction as the origin of longitudes. The differential equations of the two bodies will then have particular integrals, or periodic solutions, as is shown by HILL and POINCARÉ. Assume the masses of the two planets to be, for the inner, the mass of *Jupiter*, and for the outer, the mass of *Saturn*, with periods also respec-

[1] *The American Journal of Mathematics*, Vol. 1.

[1] *Meddelanden fran Lunds Astronomiska Observatorium*, No. 18.

[2] *Acta Mathematica*, Vol. 21.

tively equal to those of *Jupiter* and *Saturn*, while the mass of the largest body is the mass of the sun. The problem in hand is to find the expressions for the coordinates of the two small bodies as far as the terms proportional to the squares and products of the masses. These terms have been found before, as for instance in HILL'S "*New Theory of Jupiter and Saturn*," but they are there mixed up with terms involving the eccentricities, etc., and it is the present purpose to determine them entirely separate from such influences, and in the light of a periodic solution. It may be of some interest to know just how large these terms of the second order are.

Coordinates. I shall refer to the three bodies in question as the *Sun*, *Jupiter*, and *Saturn*. That the latter two may have the same expression for their perturbative functions it is necessary only to use symmetrical differential-equations as explained in TISSERAND, Vol. I, Chap. IV. *Jupiter* is referred to the center of the *Sun* as origin, while *Saturn* is referred to the center of mass of *Jupiter* and the *Sun*. Allowing the subscripts 0, 1, 2 to refer to the *Sun*, *Jupiter*, and *Saturn* respectively, and denoting the masses severally by m_i $(i = 0, 1, 2)$, we have for the heliocentric coordinates of *Jupiter* and *Saturn*

$$\xi_1 = x_1 \quad , \quad \xi_2 = x_2 + \kappa_1 x_1$$
$$\eta_1 = y_1 \quad , \quad \eta_2 = y_2 + \kappa_1 y_1$$

where

$$\kappa_i = \frac{m_i}{\mu_i} \quad , \quad \mu_i = m_0 + m_1 + \ldots + m_i$$

If r_i, v_i $(i = 1, 2)$ represent the radii vectores and true longitudes of *Jupiter* and *Saturn* respectively, then

$$x_1 = \mathrm{r}_1 \cos v_1 \ , \ y_1 = \mathrm{r}_1 \sin v_1 \ ; \ x_2 = \mathrm{r}_2 \cos v_2 \ , \ y_2 = \mathrm{r}_2 \sin v_2$$

PERTURBATIVE FUNCTION.

The potential of the system is

$$U = \frac{m_0 m_1}{\Delta_{0,1}} + \frac{m_0 m_2}{\Delta_{0,2}} + \frac{m_1 m_2}{\Delta_{1,2}}$$
$$= \frac{m_0 m_1}{\mathrm{r}_1} + \frac{m_0 m_2}{\mathrm{r}_2} + m_1 m_2 F$$

where

$$m_1 m_2 F = m_0 m_2 \left[\frac{1}{\Delta_{0,2}} - \frac{1}{\mathrm{r}_2}\right] + \frac{m_1 m_2}{\Delta_{1,2}}$$
$$= m_0 m_2 \left[\left\{\mathrm{r}_2^2 + \kappa_1^2 \mathrm{r}_1^2 + 2\kappa_1 \mathrm{r}_1 \mathrm{r}_2 \cos(v_2 - v_1)\right\}^{-\frac{1}{2}} - \frac{1}{\mathrm{r}_2}\right] + \frac{m_1 m_2}{\Delta_{1,2}}$$

If we put

$$\frac{\mu_1}{m_0}\mathrm{r}_2 = r_2 \ , \ \mathrm{r}_1 = r_1 \quad , \quad v_2 - v_1 = \theta \ , \ \frac{\mu_1}{\mu_0} = \mathrm{m}$$

F has the approximate expression

$$F = \frac{\mathrm{m}}{r_2}\left[\left\{1 - 2\frac{r_1}{r_2}\cos\theta + \left(\frac{r_1}{r_2}\right)^2\right\}^{-\frac{1}{2}} - \frac{r_1}{r_2}\cos\theta + \tfrac{1}{4}\frac{m_1}{m_0}\left(\frac{r_1}{r_2}\right)^2 \left\{1 + 3\cos 2\theta\right\}\right]$$
$$= F_0 + F_1$$

in which F_1 is the part having as a factor the small mass m_1. Since the planets have no proper eccentricities, and lie in the same plane, the perturbations will depend on the single argument $v_2 - v_1$, or the elongation. Hence it is sufficient to put in the function F, as a *first approximation,*

$$r_1 = a_1 \ , \ r_2 = a_2 \ , \ v_2 = l_2 = n_2 t, \ v_1 = l_1 = n_1 t \ , \ \theta_0 = (n_2 - n_1) t$$

Then F may be written separately in its two parts,

$$\left.\begin{aligned} F_0 &= \mathrm{m}\left[\tfrac{1}{2}\sum_{i}^{+\infty}{}_{-\infty} A^i \cos i\theta_0 - \frac{a_1}{a_2^2}\cos\theta_0 + \tfrac{1}{2}A^0\right] \\ F_1 &= \tfrac{1}{4}\,\mathrm{m}\, m_1 \frac{a_1^2}{a_2^3}\left[1 + 3\cos 2\theta_0\right] \end{aligned}\right\} \quad (1)$$

where

$$\tfrac{1}{2}\sum_{i}^{+\infty}{}_{-\infty} A^i \cos i\theta_0 = \frac{1}{a_2}\left[1 - 2\alpha\cos\theta_0 + \alpha^2\right]^{-\frac{1}{2}}$$

In F_0 the value of A^i for $i = 0$ has been taken from under the sign Σ, and so hereafter.

DIFFERENTIAL EQUATIONS OF MOTION.

1. *For Jupiter.* The equations for *Jupiter* in rectangular coordinates are

$$\mu_0 \kappa_1 \frac{d^2 x_1}{dt^2} = \frac{\partial U}{\partial x_1} \quad , \quad \mu_0 \kappa_1 \frac{d^2 y_1}{dt^2} = \frac{\partial U}{\partial y_1}$$

These equations expressed in the polar coordinates r_1, v_1 after the manner of DEPONTECOULANT'S equations in the "*Lunar Theory*,"[1] are

$$\left.\begin{aligned} \tfrac{1}{2}\frac{d^2}{dt^2}(r_1^2) - \frac{\mu_1}{r_1} + \frac{\mu_1}{a_1} &= \mathrm{m}_2\left[r_1\frac{\partial F}{\partial r_1} + 2\int d'F + 2\mathrm{m} g_1\right] \\ \frac{dv_1}{dt} &= \frac{1}{r_1^2}\left[h_1 + \mathrm{m}_2\int\frac{\partial F}{\partial v_1}dt\right] \end{aligned}\right\} \quad (2)$$

In these equations the new expressions introduced have the following significance:

$$d'F = \left\{\frac{\partial F}{\partial r_1}\frac{dr_1}{dt} + \frac{\partial F}{\partial v_1}\frac{dv_1}{dt}\right\} dt \quad , \quad \mathrm{m}_2 = \mathrm{m}\, m_2$$

$$-\frac{\mu_1}{2\mathrm{m}_2 a_1} + \mathrm{m}\, g_1 = \text{constant of integration attached to } \int d'F$$

$$h_1 = \text{constant of integration.}$$

2. *For Saturn.* The equations for *Saturn* formed in the same way are (3)

$$\left.\begin{aligned} \tfrac{1}{2}\frac{d^2}{dt^2}(r_2^2) - \frac{\mu_2 \mathrm{m}^2}{r_2} + \frac{\mu_2 \mathrm{m}^2}{a_2} &= \frac{\mu_2 \mathrm{m}\, m_1}{m_0}\left[r_2\frac{\partial F}{\partial r_2} + 2\int d''F + 2\mathrm{m} g_2\right] \\ \frac{dv_2}{dt} &= \frac{1}{r_2^2}\left(h_2 + \frac{\mu_2 \mathrm{m}\, m_1}{m_0}\int\frac{\partial F}{\partial v_2}dt\right) \end{aligned}\right\}$$

where the corresponding terms have an exactly similar meaning to those employed in *Jupiter's* equations.

3. *Equations Connecting Constants.* The above equations for *Jupiter*, or *Saturn*, are of the second and first

[1] See BROWN'S *Lunar Theory*, pp. 16, 17.

order respectively, and are sufficient to determine three arbitrary constants, besides h_i and a_i $(i=1 \text{ or } 2)$. But since the orbits have no inclinations or nodes there are only four constants, e_i, π_i, n_i, ϵ_i $(i=1 \text{ or } 2)$, to be determined for each body, and therefore we must have another equation connecting the constants. Three of the constants are immediately determined by the special conditions of the problem. For since the orbits of the planets have no eccentricity other than that caused by their mutual perturbations, and hence their perihelia are indeterminate, we have

$$e_i \sin \pi_i = 0$$
$$e_i \cos \pi_i = 0$$

By reason of the way in which we have chosen our origins of longitudes and of time, ϵ_1 and ϵ_2 are zero. Hence the only two independent constants are n_1 and n_2. The equations above referred to are, for *Jupiter* and *Saturn*, respectively,[1]

$$(4) \quad \begin{cases} \dfrac{1}{r_1}\dfrac{d^2 r_1}{dt^2} - \left(\dfrac{dv_1}{dt}\right)^2 + \dfrac{\mu_1}{r_1^3} = \dfrac{\mathrm{m}_2}{r_1}\dfrac{\partial F}{\partial r_1} \\[2ex] \dfrac{1}{r_2}\dfrac{d^2 r_2}{dt^2} - \left(\dfrac{dv_2}{dt}\right)^2 + \dfrac{\mu_2 \mathrm{m}^2}{r_2^3} = \dfrac{\mu_2 \mathrm{m} m_1}{m_0}\dfrac{1}{r_2}\dfrac{\partial F}{\partial r_2} \end{cases}$$

Units Employed. Let us take m_0, the mass of the *Sun*, as our unit of mass, and let the mean distance of the earth from the *Sun* be the unit of length. Then that k, the *Gaussian Constant*, may also be unity, the unit of time must be 58.13245 mean solar days. Hence we may put

$$\mu_1 = 1 + m_1 = n_1^2 a_1^3 \quad , \quad \mu_2 \mathrm{m}^2 = (1+m_1+m_2)(1+m_1)^2 = n_2^2 a_2^3$$

$$\alpha = \frac{a_1}{a_2} = [(1+m_1+m_2)(1+m_1)]^{-\frac{1}{3}} \left(\frac{n_2}{n_1}\right)^{\frac{2}{3}}$$

The values of m_1, m_2, n_1, n_2 are taken from p. 558 of HILL'S "*New Theory of Jupiter and Saturn*," and are

$$m_1 = \tfrac{1}{1047.879} \quad , \quad n_1 = 109256''.62552$$
$$m_2 = \tfrac{1}{3501.6} \quad , \quad n_2 = 43996''.21506$$

The above mean motions are for a sidereal year. Taking as our values for the mass and mean motion (in a sidereal year) of the earth,

$$m' = \tfrac{1}{327000} \quad , \quad n' = 1295977''.41516$$

from the equation $a' = (1+m')^{\frac{1}{3}} n'^{-\frac{2}{3}}$ we obtain the numerical value of a', which, used as the unit of distance, gives

$$\log a_1 = 0.716237409 \qquad \log \nu = \log \frac{n_2}{n_1} = 9.604967534$$

$$\log a_2 = 0.979909852 \qquad \log \alpha = 9.736327557$$

Integration of Equations of Motion. In order to solve equations (3) and (4) it seems best to put

$$r_1^2 = a_1^2(1+u_1+\delta u_1) \quad , \quad \frac{dv_1}{dt} = n_1 + z_1 + \delta z_1$$

$$r_2^2 = a_2^2(1+u_2+\delta u_2) \quad , \quad \frac{dv_2}{dt} = n_2 + z_2 + \delta z_2$$

[1] BROWN'S *Lunar Theory*, pp. 16, 17.

where u_1, z_1, u_2, z_2 represent perturbations of the first order with respect to the masses and δu_1, δz_1, δu_2, δz_2 are of the second order.

1. *First Order Terms for Jupiter.* The radius-vector equation for *Jupiter* becomes, to terms of the first order,

$$\frac{d^2 u_1}{dt^2} + n_1^2 u_1 = 2 n_1^2 a_1 m_2 \left[a_1 \frac{\partial F_0}{\partial a_1} + 2 n_1 \int \frac{\partial F_0}{\partial l_1} dt + 2 \mathrm{m}\, g_1 \right] \quad (5)$$

This linear differential equation of the second order may be solved by indeterminate coefficients. Since its right member is a cosine function of the elongation, θ_0, only, we put

$$u_1 = 2 \sum_{-\infty}^{+\infty} \mathrm{a}_i \cos i\, \theta_0$$

Substituting this value of u_1 in the above equation, and equating coefficients of the same argument on either side we have

$$\mathrm{a}_0 = \mathrm{m}_2 \left[\tfrac{1}{2} a_1^2 \frac{\partial A^0}{\partial a_1} + 2 a_1 g_1 \right]$$

$$\mathrm{a}_{-1} = \mathrm{a}_1 = \frac{\mathrm{m}_2}{\nu(1-\nu)(2-\nu)} \left[\frac{1-\nu}{2} a_1^2 \frac{\partial A^1}{\partial a_1} + a_1 A^1 - \frac{3-\nu}{2} \alpha^2 \right]$$

$$\mathrm{a}_i = \frac{\mathrm{m}_2}{(1-\nu)\{1-i^2(1-\nu)^2\}} \left[\frac{1-\nu}{2} a_1^2 \frac{\partial A^i}{\partial a_1} + a_1 A^i \right] (i = \pm 2, \ldots)$$

To the same order the longitude equation is

$$n_1 + z_1 - \frac{h'_1}{a_1^2} = \left[h'_1 + \delta_1 h_1 + \mathrm{m}_2 \int \frac{\partial F_0}{\partial l_1} dt \right] \frac{1-u_1}{a_1^2} - \frac{h'_1}{n_1^2}$$

where h_1 has been replaced by $h'_1 + \delta_1 h_1$. In the circular orbit $h_1 = h'_1 = n_1 a_1^2$. Hence $\delta_1 h_1$ is a small constant of the order of the masses. Then

$$z_1 = \frac{\delta_1 h_1}{a_1^2} - n_1 u_1 + \frac{\mathrm{m}_2}{\mathrm{m}} n_1^2 a_1 \int \frac{\partial F_0}{\partial l_1} dt \quad (6)$$

Putting

$$\delta_1 v_1 = \int z_1\, dt = \sum_{-\infty}^{+\infty} \alpha_i \sin i\, \theta_0$$

we find

$$-\alpha_{-1} = \alpha_1 = \frac{\mathrm{m}_2}{2\nu(1-\nu)^2(2-\nu)} \left[2(1-\nu) a_1^2 \frac{\partial A^1}{\partial a_1} + (\nu^2 - 2\nu + 4) a_1 A^1 - (\nu^2 - 4\nu + 6) \alpha^2 \right]$$

$$\alpha_i = \frac{\mathrm{m}_2}{2(1-\nu)\, i \{1-i^2(1-\nu)^2\}} \left[2(1-\nu) a_1^2 \frac{\partial A^i}{\partial a_1} + \{3 + i^2(1-\nu)^2\} a_1 A^i \right] (i = \pm 2, \ldots)$$

The constant term of z_1 is $\dfrac{\delta_1 h_1}{a_1^2} - 2 n_1 \mathrm{a}_0$; but we shall define n_1 as the mean motion of *Jupiter* in disturbed as well as in undisturbed orbit, and it will be obtained directly from observation. Hence

$$\frac{\delta_1 h_1}{a_1^2} - 2 n_1 \mathrm{a}_0 = 0$$

Since the arbitraries e_1 and π_1 of the general solution of the problem are zero in this case, and a_1 is not independent of n_1, all the arbitrary constants have now been fixed, for ϵ_1 is zero by the conditions laid down. Hence g_1 is not independent of the other arbitraries, and we find it by means of the first of equations (4). This equation will also enable us to verify the preceding work, inasmuch as the coefficients of $\cos i\,\theta_0$ on each side of the equation should be identical. To terms of the first order the equation is

$$\frac{d^2u_1}{dt^2} - 4n_1 z_1 - 3n_1^2 u_1 = 2\,\frac{\mathrm{m}_2}{\mathrm{m}}\,n_1^2 a_1^2 \frac{\partial F_0}{\partial a_1}$$

Substituting in this the above values of u_1 and z_1 we find

$$g_1 = -\tfrac{1}{3}\,a_1 \frac{\partial A^0}{\partial a_1}\ , \quad \text{or} \quad \mathrm{a}_0 = -\tfrac{1}{6}\,\mathrm{m}_2\,a_1^2\,\frac{\partial A^0}{\partial a_1}$$

2. *First Order Terms for Saturn.* The radius-vector equation for *Saturn* is

$$(7)\qquad \frac{d^2u_2}{dt^2} + n_2^2u_2 = 2\frac{m_1}{\mathrm{m}}\,n_2^2 a_2\left[a_2\frac{\partial F_0}{\partial a_2} + 2\int\frac{\partial F_0}{\partial l_2}\,dt + 2\mathrm{m}\,g_2\right]$$

Let

$$u_2 = 2\sum_{-\infty}^{+\infty} \mathrm{b}_i \cos i\,\theta_0$$

In forming $a_2\partial F_0/\partial a_2$ we make use of the relation

$$a_2\frac{\partial F_0}{\partial a_2} + a_1\frac{\partial F_0}{\partial a_1} = -F_0$$

Then

$$\mathrm{b}_0 = \frac{m_1}{2}\left[4a_2g_2 - a_1a_2\frac{\partial A^0}{\partial a_1} - a_2A^0\right]$$

$$\mathrm{b}_{-1} = \mathrm{b}_1 = \frac{m_1\nu^2}{(1-\nu)(1-2\nu)}\left[\frac{1-\nu}{2}\,a_1a_2\frac{\partial A^1}{\partial a_1} + \frac{1+\nu}{2}\,a_2A^1 - \alpha\right]$$

$$\mathrm{b}_i = \frac{m_1\nu^2}{(1-\nu)\{i^2(1-\nu)^2-\nu^2\}}\left[\frac{1-\nu}{2}\,a_1a_2\frac{\partial A^i}{\partial a_1} + \frac{1+\nu}{2}\,a_2A^i\right]$$

The differential equation for longitude of *Saturn* is, to terms of first order,

$$(8)\qquad z_2 = \frac{\delta_1 h_2}{a_2^2} - n_2u_2 + \frac{m_1}{\mathrm{m}}\,n_2^2a_2\int\frac{\partial F_0}{\partial l_2}\,dt$$

Putting

$$\delta_1 v_2 = \int z_2\,dt = \sum_{-\infty}^{+\infty}\beta_i \sin i\,\theta_0$$

we find

$$-\beta_{-1} = \beta_1 = \frac{m_1\nu^2}{2(1-\nu)^2(1-2\nu)}\Big[2\nu(1-\nu)a_1a_2\frac{\partial A^1}{\partial a_1} + (1+2\nu^2)\,a_2A^1 - (1+2\nu)\alpha\Big]$$

$$\beta_i = \frac{m_1\nu^2}{2(1-\nu)^2 i\{i^2(1-\nu)^2-\nu^2\}}\Big[2\nu(1-\nu)\,a_1a_2\frac{\partial A^i}{\partial a_1} + \{2\nu+\nu^2+i^2(1-\nu)^2\}a_2A^i\Big]$$

and since the constant term in $\delta_1 v_2$ is zero

$$\frac{\delta_1 h_2}{a_2^2} - 2n_2\mathrm{b}_0 = 0$$

The equation determining the constant term in u_2 is to terms of first order,

$$\frac{d^2u_2}{dt^2} - 4n_2z_2 - 3n_2^2u_2 = 2\,\frac{m_1}{\mathrm{m}}\,n_2^2a_2^2\frac{\partial F_0}{\partial a_2}$$

This gives

$$\mathrm{b}_0 = \tfrac{1}{2}m_1a_2g_2 = \tfrac{1}{6}\,m_1\left[a_1a_2\frac{\partial A^0}{\partial a_1} + a_2A^0\right]$$

Differential Equations Including Second Order Terms.

Having now solved the differential equations as far as the terms proportional to the masses we are prepared to push our approximation still further and include all terms proportional to the squares and products of the masses. It is well known that the form of the solution remains unchanged in all the successive approximations of including the squares, cubes, and higher powers of the masses, and hence our differential equations preserve the same form, and are solved precisely in the same way as before.

1. *For Jupiter.—a) Radius Vector Equation.* When we extend the radius vector equation to terms of the second order, and omit all terms of the first order, we get

$$\frac{d^2}{dt^2}\delta u_1 + n_1^2\delta u_1 = \tfrac{3}{4}n_1^2u_1^2 + 2\,\frac{\mathrm{m}_2}{\mathrm{m}}\,n_1^2a_1\Big[\delta\Big(r_1\frac{\partial F}{\partial r_1}\Big) + 2\delta\int d'F + a_1\frac{\partial F_1}{\partial a_1} + 2\int d'F_1 + 2\mathrm{m}\,\delta g_1\Big]$$

In this equation F_1 has the value given above, and $\mathrm{m}\,\delta g_1$ is the constant of integration attached to $\delta\int d'F$, and is of the second order. We shall proceed to express fully the right member of this equation.

Since

$$r_1 = a_1(1 + \tfrac{1}{2}u_1 + \tfrac{1}{2}\delta u_1 - \tfrac{1}{8}u_1^2)$$

we have

$$\frac{\delta_1 r_1}{a_1} = \tfrac{1}{2}u_1\ \ ;\ \ \text{also}\ \ \frac{\delta_1 r_2}{a_2} = \tfrac{1}{2}u_2$$

Also since F_0 is a function of $l_2 - l_1$ we have

$$\frac{\partial F_0}{\partial l_2} = -\frac{\partial F_0}{\partial l_1}$$

With these relations and that given above, with reference to $a_2\partial F_0/\partial a_2$ we may easily express $\delta(r_1\partial F/\partial r_1)$. We also have

$$\delta\int d'F = \int\delta\left[\frac{\partial F}{\partial r_1}\frac{dr_1}{dt} + \frac{\partial F}{\partial v_1}\frac{dv_1}{dt}\right]dt$$

in which

$$\delta\Big(\frac{dr_1}{dt}\Big) = \frac{d}{dt}(\delta_1 r_1) = \frac{a_1}{2}\frac{du_1}{dt}$$

$$\delta\Big(\frac{dv_1}{dt}\Big) = \frac{d}{dt}(\delta_1 v_1) = z_1$$

Then the above differential equation in its expanded form is

(9)

$$\frac{d^2}{dt^2}(\delta u_1) + n_1{}^2 \delta u_1 = n_1{}^2 \Bigg[\tfrac{3}{4} u_1{}^2 + 2 \frac{\mathrm{m}_2}{\mathrm{m}} a_1 \Big\{ \Big(a_1 \frac{\partial F_0}{\partial a_1} + a_1^2 \frac{\partial^2 F_0}{\partial a_1{}^2} \Big) \frac{u_1}{2}$$
$$- \Big(2a_1 \frac{\partial F_0}{\partial a_1} + a_1{}^2 \frac{\partial^2 F_0}{\partial a_1{}^2} \Big) \frac{u_2}{2} + a_1 \frac{\partial^2 F_0}{\partial l_1 \partial a_1} (\delta_1 v_1 - \delta_1 v_2) + a_1 \frac{\partial F_1}{\partial a_1}$$
$$+ 2 \int d' F_1 + 2\mathrm{m}\, \delta g_1$$
$$+ 2n_1 \int \Big[\frac{a_1 \partial^2 F_0}{\partial a_1 \partial l_1} \Big(\frac{u_1}{2} - \frac{u_2}{2} \Big) - \frac{\partial F_0}{\partial l_1} \frac{u_2}{2} + \frac{\partial^2 F_0}{\partial l_1{}^2} (\delta_1 v_1 - \delta_1 v_2)$$
$$+ \frac{1}{n_1} \Big(\frac{a_1}{2} \frac{\partial F_0}{\partial a_1} \frac{du_1}{dt} + \frac{\partial F_0}{\partial l_1} z_1 \Big) \Big] dt \Big\} \Bigg]$$

We see immediately that the right member is composed of products of series, either cosine by cosine, sine by sine, or, underneath the integral sign, cosine by sine. In every case we get, after multiplication, and integration of the last mentioned products, a cosine series. If now we attach the factor $2\mathrm{m}_2 a_1/\mathrm{m}$ above to F_0, every coefficient in each factor is of the order of the masses. We shall designate the coefficients of cosine series by Roman letters, of sine series by Greek letters. In each series the subscript i has every integral value from $-\infty$ to $+\infty$ including zero. For cosine series we may put $\mathrm{a}_i = \mathrm{a}_{-i}$, for sine series $\alpha_i = -\alpha_{-i}$. Hence we may write

$$\Sigma_i \mathrm{a}_i \cos i\,\theta_0 \times \Sigma_j \mathrm{b}_j \cos j\,\theta_0 = \Sigma_i \Sigma_j \mathrm{a}_i \mathrm{b}_j \cos (i+j)\,\theta_0$$
$$\Sigma_i \mathrm{a}_i \cos i\,\theta_0 \times \Sigma_j \alpha_j \sin j\,\theta_0 = \Sigma_i \Sigma_j \mathrm{a}_i \alpha_j \sin (i+j)\,\theta_0$$
$$\Sigma_i \alpha_i \sin i\,\theta_0 \times \Sigma_j \beta_j \sin j\,\theta_0 = -\Sigma_i \Sigma_j \alpha_i \beta_j \cos (i+j)\,\theta_0$$

whence the equation for δu_1 becomes

$$\frac{d^2}{dt^2}(\delta u_1) + n_1{}^2 \delta u_1 = n_1{}^2 [\Sigma_i \Sigma_j S_{i+j} \cos (i+j)\,\theta_0 + 2\sigma \mathrm{m}\, \delta g_1]$$

the solution of which gives

$$\delta u_1 = \Sigma_i \Sigma_j \frac{S_{i+j}}{1-(i+j)^2(1-\nu)^2} \cos (i+j)\,\theta_0 + 2\sigma\,\mathrm{m}\, \delta g_1$$

where $\sigma = 2\mathrm{m}_2 a_1/\mathrm{m}$ and

$$S_{i+j} = 3\,\mathrm{a}_i \mathrm{a}_j + \mathrm{e}_i \mathrm{a}_j - \mathrm{f}_i \mathrm{b}_j - \epsilon_i \gamma_j + \mathrm{k}_{i+j} + \mathrm{l}_{i+j}$$
$$+ \frac{2}{(i+j)(1-\nu)} [\epsilon_i \mathrm{c}_j - \zeta_i \mathrm{b}_j + \mathrm{g}_i \gamma_j + \mathrm{h}_i \delta_j + \zeta_i \mathrm{d}_j]$$

These letters express in order the coefficients of the various factors just as they occur in the right-hand member of the expanded equation (9) given above.

b) Longitude Equation. To terms of the second order this equation becomes, when we put $h_1 = h'_1 + \delta_1 h_1 + \delta_2 h_1$ and omit terms of the first order,

(10)

$$\delta z_1 = -u_1 z_1 - n_1 \delta u_1 + \frac{\delta_2 h_1}{a_1{}^2} + \tfrac{1}{2} \sigma n_1 \int \Big\{ a_1 \frac{\partial^2 F_0}{\partial l_1 \partial a_1} \Big(\frac{u_1}{2} - \frac{u_2}{2} \Big)$$
$$- \frac{\partial F_0}{\partial l_1} \frac{u_2}{2} + \frac{\partial^2 F_0}{\partial l_1{}^2} (\delta_1 v_1 - \delta_1 v_2) + \frac{\partial F_1}{\partial l_1} \Big\} dt$$
$$= n_1 \Sigma_i \Sigma_j P_{i+j} \cos (i+j)\,\theta_0$$

where

$$P_{i+j} = -2\mathrm{a}_i \mathrm{d}_j - \frac{1}{1-(i+j)^2(1-\nu)^2} S_{i+j}$$
$$+ \frac{1}{2\,(i+j)(1-\nu)} [\epsilon_i \mathrm{c}_j - \zeta_i \mathrm{b}_j + \mathrm{g}_i \gamma_j] + \tfrac{1}{2} \mathrm{l}_{i+j} - 2\sigma \mathrm{m} \delta g_1 + \frac{\delta_2 h_1}{n_1 a_1{}^2}$$

where the whole constant part is included in P_0, which for reasons given above must be equated to zero.

Then

$$\delta_2 v_1 = \int \delta z_1\, dt = \Sigma_i \Sigma_j \frac{P_{i+j}}{(i+j)(\nu-1)} \sin (i+j)\,\theta_0$$

c) Equation Determining Constant Part of δu_1. The first of equations (4) extended to terms of the second order is sufficient to determine the constant δg_1 which occurs in δu_1, and at the same time to verify our equations for δu_1 and δz_1. For on summing the coefficients of like cosines we shall find that they vanish identically, and only a constant term is left. If we let

$$\dot{u}_1 = \frac{du_1}{dt} \quad , \quad \ddot{u}_1 = \frac{d^2 u_1}{dt^2}$$

this equation is (11)

$$\frac{\delta u_1}{n_1{}^2} - 3\delta u_1 - 4 \frac{\delta z_1}{n_1} - \frac{u_1 \ddot{u}_1}{n_1{}^2} - \tfrac{1}{2} \frac{\dot{u}_1{}^2}{n_1{}^2} - 2 \frac{z_1{}^2}{n_1{}^2} + \tfrac{15}{4} u_1{}^2$$
$$= \tfrac{1}{2} \sigma \Big[-a_1 \frac{\partial F_0}{\partial a_1} u_1 + 2a_1 \delta \Big(\frac{\partial F}{\partial r_1} \Big) + 2a_1 \frac{\partial F_1}{\partial a_1} \Big]$$

Substituting in this equation the expressions for δu_1, $\delta \ddot{u}_1$, δz_1, and making use of equations (5) and (6) we arrive at the equation

$$\tfrac{3}{2} u_1{}^2 - \tfrac{1}{2} \frac{\dot{u}_1{}^2}{n_1{}^2} - 2 \frac{z_1{}^2}{n_1{}^2} + 2\sigma \mathrm{m}\, \delta g_1 - 4 \frac{\delta_2 h_1}{n_1 a_1{}^2}$$
$$+ 2\sigma \int \Big\{ a_1 \frac{\partial F_0}{\partial a_1} \frac{u_1}{2} + \frac{\partial F_0}{\partial l_1} z_1 \Big\} dt = 0$$

We shall find a different expression for the last term. Equation (6) is

$$\frac{z_1}{n_1} + u_1 = \frac{\delta_1 h_1}{n_1 a_1{}^2} + \frac{\sigma}{2} n_1 \int \frac{\partial F_0}{\partial l_1} dt$$

By means of this relation and its derivative we find

$$2\sigma \int z_1 \frac{\partial F_0}{\partial l_1} dt = 2 \frac{z_1{}^2}{n_1{}^2} + 4 \frac{\delta_1 h_1}{n_1 a_1{}^2} u_1 - 2u_1{}^2$$
$$+ \int 2\sigma n_1 \dot{u}_1 \Big\{ \int \frac{\partial F_0}{\partial l_1} dt \Big\} dt$$

By adding

$$2\sigma \int a_1 \frac{\partial F_0}{\partial a_1} \frac{u_1}{2} dt$$

to each member of this equation, the integral in the right member becomes

$$\sigma \int \Big\{ a_1 \frac{\partial F_0}{\partial a_1} + 2n_1 \int \frac{\partial F_0}{\partial l_1} dt \Big\} \dot{u}_1\, dt$$

which by means of equation (5) may be completely integrated. Hence we obtain

$$2\sigma \int \Big\{ a_1 \frac{\partial F_0}{\partial a_1} \frac{u_1}{2} + \frac{\partial F_0}{\partial l_1} z_1 \Big\} dt = \Big[2 \frac{z_1{}^2}{n_1{}^2} - \tfrac{3}{2} u_1{}^2 + \tfrac{1}{2} \frac{\dot{u}_1{}^2}{n_1{}^2} \Big]_{-0}$$

where [$]_{-0}$ means that the constant term is absent. The equation under consideration then gives for the constant in δu_1

$$\sigma \mathrm{m} \delta g_1 = 2\,\frac{\delta_2 h_1}{n_1 a_1^2} + \left[\frac{z_1^2}{n_1^2} - \tfrac{3}{4} u_1^2 + \tfrac{1}{4}\frac{\dot{u}_1^2}{n_1^2}\right]_0$$

Here [$]_0$ means that only the constant part is present. Thus it is seen that all periodic terms identically vanish, and the equations for δu_1 and δz_1 are verified. We shall use this same equation to verify the numerical work.

Since the constant term in δz_1 is zero we have

$$\frac{\delta_2 h_1}{n_1 a_1^2} = 2\sigma \mathrm{m}\,\delta g_1 + 2\,[\mathrm{a}_i \mathrm{d}_j]_0 + S_0$$

Hence

$$-3\sigma \mathrm{m}\,\delta g_1 = 4\,[\mathrm{a}_i \mathrm{d}_j]_0 + 2\,S_0 + \left[\frac{z_1^2}{n_1^2} - \tfrac{3}{4} u_1^2 + \tfrac{1}{4}\frac{\dot{u}_1^2}{n_1^2}\right]_0$$

2. *For Saturn.* — *a*) *Radius Vector Equation.* The equations for *Saturn*, being formed in a manner exactly similar to that pursued in forming *Jupiter's* equations, may simply be written down. The first is

$$(12)\qquad \frac{d^2}{dt^2}(\delta u_2) + n_2^2\,\delta u_2$$

$$= n_2^2\Bigg[\tfrac{3}{4} u_2^2 - 2\,\frac{m_1}{\mathrm{m}}\,a_2\,\Bigg\{\left(2a_1\frac{\partial F_0}{\partial a_1} + a_1^2\frac{\partial^2 F_0}{\partial a_1^2}\right)\left(\frac{u_1}{2} - \frac{u_2}{2}\right)$$

$$-\left(F_0 + a_1\frac{\partial F_0}{\partial a_1}\right)\frac{u_2}{2} + \left(\frac{\partial F_0}{\partial l_1} + a_1\frac{\partial^2 F_0}{\partial l_1 \partial a_1}\right)(\delta_1 v_1 - \delta_1 v_2)$$

$$- a_2\frac{\partial F_1}{\partial a_2} - 2\int d'' F_1 - 2\mathrm{m}\delta g_2$$

$$+ 2n_2 \int\left[a_1\frac{\partial^2 F_0}{\partial l_1 \partial a_1}\left(\frac{u_1}{2} - \frac{u_2}{2}\right) - \frac{\partial F_0}{\partial l_1}\frac{u_2}{2} + \frac{\partial^2 F_0}{\partial l_1^2}(\delta_1 v_1 - \delta_1 v_2)\right.$$

$$\left. + \left(F_0 + a_1\frac{\partial F_0}{\partial a_1}\right)\frac{\dot{u}_2}{2n_2} + \frac{\partial F_0}{\partial l_1}\frac{z_2}{n_2}\right] dt\Bigg\}\Bigg]$$

We see that, as in *Jupiter's* radius vector equation, the right member is composed of the products of series, all of which result in cosine series. Many of the individual series are the same as those entering *Jupiter's* equation, except for the constant factor $2m_1 a_2/\mathrm{m}$. Denoting this constant by ω, we can put

$$\omega = \frac{\omega}{\sigma}\cdot\sigma$$

and we can then use the same letters as before to denote the same coefficients here. New letters will be used where we have new coefficients, and arranging them in exactly the order in which they occur above, we may write the equation for δu_2,

$$\frac{d^2}{dt^2}(\delta u_2) + n_2^2\,\delta u_2 = n_2^2[\Sigma_i \Sigma_j R_{i+}\ \cos(i+j)\,\theta_0 + 2\omega\mathrm{m}\,\delta g_2]$$

the solution of which is

$$\delta u_2 = \Sigma_i \Sigma_j \frac{\nu^2}{\nu^2 - (i+j)^2(1-\nu)^2} R_{i+j}\cos(i+j)\,\theta_0 + 2\omega\mathrm{m}\,\delta g_2$$

where

$$R_{i+j} = 3\mathrm{b}_i\mathrm{b}_j - \frac{\omega}{\sigma}[\mathrm{f}_i\mathrm{c}_j - \mathrm{q}_i\mathrm{b}_j - \theta_i\gamma_j - \mathrm{m}_{i+j} - 2\mathrm{n}_{i+j}$$

$$+ \frac{2\nu}{(i+j)(1-\nu)}\left\{\epsilon_i\mathrm{c}_j - \zeta_i\mathrm{b}_j + \mathrm{g}_i\gamma_j + \mathrm{q}_i\eta_j + \zeta_i\mathrm{p}_j\right\}\Big]$$

b) *Longitude Equation.* To terms of the second order this is (13)

$$\delta z_2 = -u_2 z_2 - n_2\delta u_2 + \frac{\delta_2 h_2}{a_2^2}$$

$$+ \tfrac{1}{2}\,\omega\int d'' F_0 - \tfrac{1}{2}\,n_2\,\omega\int\Bigg\{a_1\frac{\partial^2 F_0}{\partial l_1 \partial a_1}\left(\frac{u_1}{2} - \frac{u_2}{2}\right)$$

$$- \frac{\partial F_0}{\partial l_1}\frac{u_2}{2} + \frac{\partial^2 F_0}{\partial l_1^2}(\delta_1 v_1 - \delta_1 v_2)\Bigg\}\,dt$$

$$= n_2\,\Sigma_i \Sigma_j K_{i+j}\cos(i+j)\,\theta_0$$

where

$$K_{i+j} = -2\mathrm{b}_i\mathrm{p}_j - \frac{\nu^2}{\nu^2 - (i+j)^2(1-\nu)^2}R_{i+j} - 2\omega\mathrm{m}\,\delta g_2 + \frac{\delta_2 h_2}{n_2 a_2^2}$$

$$+ \tfrac{1}{2}\,\frac{\omega}{\sigma}\,n_{i+j} - \tfrac{1}{2}\,\frac{\omega}{\sigma}\,\frac{\nu}{(i+j)(1-\nu)}\,[\epsilon_i\mathrm{c}_j - \zeta_i\mathrm{b}_j + \mathrm{g}_i\gamma_j]$$

Then

$$\delta_2 v_2 = \int \delta z_2\,dt = \Sigma_i \Sigma_j \frac{\nu}{(i+j)(\nu-1)}K_{i+j}\sin(i+j)\,\theta_0$$

c) *Equation Determining Constant Part of* δu_2. The second of equations (4), expressed to terms of the second order, is

$$\frac{\delta \ddot{u}_2}{n_2^2} - 3\,\delta u_2 - 4\frac{\delta z_2}{n_2} - \frac{u_2 \ddot{u}_2}{n_2^2} - \tfrac{1}{2}\frac{\dot{u}_2^2}{n_2^2} - 2\frac{z_2^2}{n_2^2} + \tfrac{15}{4}u_2^2 \qquad (14)$$

$$= -\omega\left[A\left(\frac{u_1}{2} - \frac{u_2}{2}\right) - 3B\,\frac{u_2}{2} + \frac{\partial B}{\partial l_1}(\delta_1 v_1 - \delta_1 v_2) - a_2\frac{\partial F_1}{\partial a_2}\right]$$

where

$$A = 2a_1\frac{\partial F_0}{\partial a_1} + a_1^2\frac{\partial^2 F_0}{\partial a_1^2}\quad,\quad B = F_0 + a_1\frac{\partial F_0}{\partial a_1}$$

From this equation we get, exactly as in the equation for *Jupiter*,

$$-3\omega\mathrm{m}\,\delta g_2 = 4\,[\mathrm{b}_i\mathrm{p}_j]_0 + 2R_0 + \left[\frac{z_2^2}{n_2^2} - \tfrac{3}{4}u_2^2 + \tfrac{1}{4}\frac{\dot{u}_2^2}{n_2^2}\right]_0$$

which is exactly similar to the expression for δg_1.

Reference of Coordinates of *Saturn* to Center of *Sun*.

Let r_2', v_2' be the polar coordinates of *Saturn* referred to the center of the *Sun* as origin. Then in the triangle of *Sun*, *Saturn*, mass-center of *Sun* and *Jupiter*, the angles are respectively $v_2' - v_1$, q, and $\pi - (v_2 - v_1)$, and the sides opposite r_2, $\kappa_1\mathrm{r}_1$, and r_2'. If we put

$$v_2 - v_1 = l_2 - l_1 + \delta_1 v_2 - \delta_1 v_1 + \ldots = \theta_0 + \theta_1 + \ldots,$$

we get $\quad \mathrm{r}_2' = [\mathrm{r}_2^2 + \kappa_1^2\mathrm{r}_1^2 + 2\kappa_1\mathrm{r}_1\mathrm{r}_2\cos(v_2 - v_1)]^{\frac{1}{2}}$

or, approximately

$$\mathrm{r}_2' = \mathrm{r}_2 + \kappa_1\mathrm{r}_1\cos(\theta_0 + \theta_1) + \tfrac{1}{4}\kappa_1^2\frac{\mathrm{r}_1^2}{\mathrm{r}_2}[1 - \cos 2(\theta_0 + \theta_1)]$$

$$= \mathrm{a}_2\Bigg[1 + \tfrac{1}{2}u_2 + \tfrac{1}{2}\,\delta u_2 - \tfrac{1}{8}u_2^2 + \tfrac{1}{4}\kappa_1^2\frac{\mathrm{a}_1^2}{\mathrm{a}_2^2}(1 - \cos 2\theta_0)$$

$$+ \kappa_1\frac{\mathrm{a}_1}{\mathrm{a}_2}\left\{(1 + \tfrac{1}{2}u_1)\cos\theta_0 - \theta_1\sin\theta_0\right\}\Bigg]$$

since $\theta_1 = \delta_1 v_2 - \delta_1 v_1$ is a very small angle. As far as to terms of the first order

$$r_2' = a_2\left[1+\tfrac{1}{2}u_2+\kappa_1\frac{a_1}{a_2}\cos\theta_0\right]$$

so that to terms of this order r_2' differs from r_2 only in the term of argument θ_0. It is also seen that r_2' has the same mean value as has r_2. When terms of the second order are included this ceases to be true.

In the same triangle as mentioned above we have

$$v_2' = v_2 - \varphi$$

and $$\frac{\sin\varphi}{\sin(v_2-v_1)} = \frac{\kappa_1 r_1}{r_2'} = \tfrac{1}{2000}, \quad \text{approximately.}$$

Hence

$$\sin\varphi = \varphi = \frac{\kappa_1 r_1}{r_2'}\sin(v_2-v_1)$$

and therefore

$$v_2' = v_2 - \kappa_1\frac{a_1}{a_2}\left[1+\frac{u_1}{2}-\frac{u_2}{2}-\kappa_1\frac{a_1}{a_2}\cos\theta_0\right]\left[\sin\theta_0+\theta_1\cos\theta_0\right]$$

$$= v_2 - \kappa_1\frac{a_1}{a_2}\left[\sin\theta_0+\theta_1\cos\theta_0 + \left(\frac{u_1}{2}-\frac{u_2}{2}\right)\sin\theta_0 - \tfrac{1}{2}\kappa_1\frac{a_1}{a_2}\sin 2\theta_0\right]$$

It is seen that v_2' has the same mean rate of increase, n_2, as has v_2, being as much less than the latter in the first and second quadrants as greater in the third and fourth.

Computation of First-Order Terms.

It is necessary first to obtain the values of the functions A^i entering into the perturbative function. Let

$$[1-2\alpha\cos\theta_0+\alpha^2]^{-\frac{1}{2}} = \tfrac{1}{2}\sum_{-\infty}^{+\infty} b^i\cos i\,\theta_0$$

Hence if we compute b^i, $\alpha\dfrac{db^i}{d\alpha}$, $\alpha^2\dfrac{d^2b^i}{d\alpha^2}$ we can obtain from them A^i, $a_1\dfrac{\partial A^i}{\partial a_1}$, $a_1^2\dfrac{\partial^2 A^i}{\partial a_1^2}$ by well known relations. These quantities may be computed in several ways, all well known, and it is unnecessary here to reproduce the formulas. By glancing at the perturbations under consideration as given by DePontécoulant, "*Théorie Analytique du Système du Monde,*" it is seen that several coefficients are quite large; for instance, 196″ is the coefficient of $\sin 2\theta_0$ in $\delta_1 v_1$. For this and similar terms nine-place logarithms are necessary, but only a few terms demand so many figures. In general seven-place logarithms suffice for terms of the first order, while five-, and for one or two terms, six-place logarithms, will give the same accuracy for the second-order terms. The b^i and their derivatives have been computed for $\log\alpha = 9.736327557$, and the computations were checked twice, and in some cases, three times by recomputation.

The values found for b^i, $\alpha\dfrac{db^i}{d\alpha}$, $\alpha^2\dfrac{d^2b^i}{d\alpha^2}$ are

i	b^i	$\alpha\dfrac{db^i}{d\alpha}$	$\alpha^2\dfrac{d^2b^i}{d\alpha^2}$
0	0.338438916	9.643539018	9.930590
1	9.792423038	9.907211461	9.878787
2	9.410262287	9.779191774	0.018692
3	9.07072475	9.59673039	0.020155
4	8.7510906	9.3914979	9.948196
5	8.4430357	9.173599	9.833742
6	8.1425680	8.947617	9.691950
7	7.847463	8.715983	9.53118
8	7.556353	8.480187	9.35651
9	7.268330	8.24120	9.17124
10	6.98277	7.99967	8.97758
11	6.69922	7.75609	8.7772
12	6.4174	7.5105	8.5716

From these data we immediately compute the first-order terms of *Jupiter* given below. The coefficients are expressed in abstract numbers for $\delta_1 r_1/a_1$, in seconds of arc for $\delta_1 v_1$.

$$\frac{\delta_1 r_1}{a_1} = \left\{\begin{array}{ll}
-0.00001\,14252 & \\
+0.00012\,45421 & \cos\theta_0 \\
-0.00053\,33873 & \cos 2\theta_0 \\
-0.00005\,55968 & \cos 3\theta_0 \\
-0.00001\,43934 & \cos 4\theta_0 \\
-0.00000\,47600 & \cos 5\theta_0 \\
-0.00000\,17772 & \cos 6\theta_0 \\
-0.00000\,07141 & \cos 7\theta_0 \\
-0.00000\,03016 & \cos 8\theta_0 \\
-0.00000\,01320 & \cos 9\theta_0 \\
-0.00000\,00593 & \cos 10\theta_0 \\
-0.00000\,00273 & \cos 11\theta_0 \\
-0.00000\,00127 & \cos 12\theta_0
\end{array}\right\}$$

$$\delta_1 v_1 = \left\{\begin{array}{ll}
+\ 79''.24829 & \sin\theta_0 \\
-195.77043 & \sin 2\theta_0 \\
-\ 16.33180 & \sin 3\theta_0 \\
-\ \ 3.75436 & \sin 4\theta_0 \\
-\ \ 1.15702 & \sin 5\theta_0 \\
-\ \ 0.41297 & \sin 6\theta_0 \\
-\ \ 0.16100 & \sin 7\theta_0 \\
-\ \ 0.06656 & \sin 8\theta_0 \\
-\ \ 0.02868 & \sin 9\theta_0 \\
-\ \ 0.01275 & \sin 10\theta_0 \\
-\ \ 0.00581 & \sin 11\theta_0 \\
-\ \ 0.00269 & \sin 12\theta_0
\end{array}\right\}$$

The corresponding values for *Saturn* are

$$\frac{\delta_1 r_2}{a_2} = \left\{\begin{array}{ll} +0.00041\,67147 & \\ +0.00034\,91670 & \cos\theta_0 \\ +0.00014\,74618 & \cos 2\theta_0 \\ +0.00003\,40816 & \cos 3\theta_0 \\ +0.00001\,05662 & \cos 4\theta_0 \\ +0.00000\,37863 & \cos 5\theta_0 \\ +0.00000\,14794 & \cos 6\theta_0 \\ +0.00000\,06123 & \cos 7\theta_0 \\ +0.00000\,02639 & \cos 8\theta_0 \\ +0.00000\,01173 & \cos 9\theta_0 \\ +0.00000\,00534 & \cos 10\theta_0 \\ +0.00000\,00247 & \cos 11\theta_0 \\ +0.00000\,00116 & \cos 12\theta_0 \end{array}\right\}$$

$$\delta_1 v_2 = \left\{\begin{array}{ll} +103''.82924 & \sin\theta_0 \\ +\ \ 32.01024 & \sin 2\theta_0 \\ +\ \ \ \ 6.66903 & \sin 3\theta_0 \\ +\ \ \ \ 1.99553 & \sin 4\theta_0 \\ +\ \ \ \ 0.70687 & \sin 5\theta_0 \\ +\ \ \ \ 0.27562 & \sin 6\theta_0 \\ +\ \ \ \ 0.11428 & \sin 7\theta_0 \\ +\ \ \ \ 0.04944 & \sin 8\theta_0 \\ +\ \ \ \ 0.02206 & \sin 9\theta_0 \\ +\ \ \ \ 0.01008 & \sin 10\theta_0 \\ +\ \ \ \ 0.00469 & \sin 11\theta_0 \\ +\ \ \ \ 0.00222 & \sin 12\theta_0 \end{array}\right\}$$

We have shown that in order to reduce $\delta_1 r_2 / a_2$ and $\delta_1 v_2$ to $\delta_1 r_2' / a_2$ and $\delta_1 v_2'$ respectively, it is necessary to change the coefficient of argument θ_0 only, adding $\kappa_1 a_1 / a_2$ in the first case, and subtracting it in the second. This amounts to

$$\text{Red. to } \frac{\delta_1 r_2'}{a_2} = +0.00052\,00157 \qquad \text{Red. to } \delta_1 v_2' = -107''.26093$$

Computation of Second-Order Terms.

With the values obtained for A^i, $a_1 \dfrac{\partial A^i}{\partial a_1}$, $a_1^2 \dfrac{\partial^2 A^i}{\partial a_1^2}$ were computed the coefficients $a_i, \ldots, q_i$ and $\alpha_i, \ldots, \theta_i$. In order then to find the numerical values of δu_1, δz_1, δu_2, δz_2 it was necessary to multiply together series having the above as coefficients. This multiplication was performed by the method of special values as set forth in Hansen's "*Auseinandersetzung*," pp. 159–164, or in Tisserand's "*Mécanique Céleste*," Tome IV. The semi-circumference was divided into twelve equal parts, and to θ_0 were given the thirteen equidistant values 0°, 15°, 30°, ... , 180°. It is important in these computations to take advantage of any checks that may present themselves. When no checks were available the computations were repeated. After all the products had been computed equation (11), determining the constant part of the radius-vector, was employed as a partial verification of the work.

1. *Computation of* δu_1 *and* $\delta z_1 / n_1$. The numerical values of the coefficients entering into δu_1 and δz_1 are tabulated below in terms of their logarithms. It will be denoted whether the series (which is a product of two other series) is a cosine or a sine series, and by the numbers $i+j$ at the left what is the multiple of the argument θ_0 whose coefficient is opposite. By multiplying by two each of the coefficients $a_i, \ldots, \alpha_i, \ldots,$ except when $i = 0$, we may regard $i+j$ as always positive.

$i+j$	cosine $a_i a_j$	cosine $e_i a_j$	cosine $f_i b_j$	cosine $-\varepsilon_i \gamma_j$
0	3.18127	3.13068n	3.39200	3.37675n
1	2.588179n	3.260253n	3.681298	3.506168n
2	2.322029	3.261497n	3.741587	3.331918n
3	2.80822n	3.10171n	3.70378	2.97340n
4	3.13372	3.21001n	3.61656	3.07604
5	2.44947	3.15498n	3.48959	3.22051
6	1.9421	3.05398n	3.33996	3.19793
7	1.501	2.9239n	3.17452	3.10977
8	1.098	2.7740n	2.99747	2.98594
9	0.718	2.6055n	2.8098	2.8358
10	0.35	2.4241n	2.6133	2.6738
11	9.95	2.1817n	2.3949	2.5558
12	9.7	2.0110n	2.1544	2.3660

$i+j$	sine $\varepsilon_i c_j$	sine $\zeta_i b_j$	sine $g_i \gamma_j$	sine $h_i d_j$
1	3.228778n	2.624453	3.264226n	1.597713
2	3.502550n	2.976050	3.256116n	2.420552n
3	3.54078n	2.95902	3.04924n	2.19131n
4	3.59282n	2.84670	2.96747	2.73957n
5	3.52926n	2.67205	3.15861	2.64359n
6	3.42139n	2.4729	3.15214	2.48659n
7	3.28530n	2.2601	3.07239	2.3024n
8	3.12894n	2.0379	2.9536	2.1014n
9	2.9597n	1.812	2.8096	1.8935n
10	2.7762n	1.577	2.7506	1.680n
11	2.5576n	1.33	2.5291	1.472n
12	2.3456n	1.06	2.3477	1.249n

$i+j$	sine $\zeta_i d_j$	cosine $a_i d_j$	cosine k_{i+j}	cosine l_{i+j}
0	. . . .	3.50648n	2.64482	. . . .
1	2.349305	2.796439	. . . .	. . . .
2	2.340550	2.514531n	3.121940	3.044708
3	2.04846	3.10381	. . . .	. . . .
4	2.95404	3.46114n	. . . .	. . . .
5	2.85854	2.81880n	. . . .	. . . .
6	2.69805	2.35268n	. . . .	. . . .
7	2.5099	1.9517n	. . . .	. . . .
8	2.3051	1.5856n	. . . .	. . . .
9	2.0929	1.245n	. . . .	. . . .
10	1.872	0.926n	. . . .	. . . .
11	1.613	0.573n	. . . .	. . . .
12	1.395	0.28n	. . . .	. . . .

In order to find the constant δg_1 which enters into δu_1, and at the same time verify the preceding calculations, it is necessary to compute the additional products in equation (11), namely,

$$\frac{\dot{u}_1^2}{4n_1^2},\ \frac{\dot{z}_1^2}{n_1^2},\ \frac{u_1 \ddot{u}_1}{4n_1^2},\ \text{and } \sigma a_1 \frac{\partial F_0}{\partial a_1} \frac{u_1}{2}$$

the numerical values of which are tabulated below. The same nomenclature is used as before, and the tabulation is in the same order in which the terms are here written.

$i+j$	cosine $-\delta_i\delta_j$	cosine $d_i d_j$	cosine $a_i o_j$	cosine $a_i r_j$
0	3.32522	3.83271	3.32522*n*	2.55100*n*
1	2.30343	2.96510*n*	2.12056*n*	2.97100*n*
2	2.12794	2.66130	2.45334*n*	2.89591*n*
3	2.73772	3.39792*n*	2.68638	2 88204*n*
4	3.28384*n*	3.78892	3.29248*n*	2.76675*n*
5	2.77321*n*	3.18525	2.82099*n*	2.82824*n*
6	2.39858*n*	2.75328	2.49363*n*	2.80472*n*
7	2.0656*n*	2.38169	2.20678*n*	2.72459*n*
8	1.7530*n*	2.0415	1.9372*n*	2.60879*n*
9	1.451*n*	1.7187	1.673*n*	2.4579*n*
10	1.158*n*	1.407	1.412*n*	2.301*n*
11	0.886*n*	1.087	1.147*n*	2.068*n*
12	0.60*n*	0.80	0.89*n*	1.911*n*

It was found that the last three or four coefficients (except the twelfth) obtained by the method of special values did not satisfy the checks, whereas the same coefficients computed by direct multiplication of series did. Hence all these coefficients were thus recomputed. In this way were obtained the twelfth coefficients in the sine series, which are not given by the method of special values. In the cosine series the twelfth coefficient is the same for both ways of computing.

From the above data we get for δu_1 and $\delta z_1 / n_1$,

$i+j$	cosine δu_1	cosine $\delta z_1 / n_1$
0	3.78191	. . . .
1	4.559802*n*	4.501489
2	4.554736	4.569894
3	3.84237	4.04265*n*
4	2.84305	3.63491
5	2.6374	2.6812
6	2.2726	1.713
7	1.909	1.125*n*
8	1.555	1.232*n*
9	1.211	1.076*n*
10	0.86	0.78*n*
11	0.09	0.46
12	9.4	0.45

We have

$$\frac{\delta_2 r_1}{a_1} = \tfrac{1}{2}\left[\delta u_1 - \tfrac{1}{4} u_1^2\right] \quad , \quad \delta_2 v_1 = \int \delta z_1 \, dt$$

The numerical values of these quantities are here given.

$$\frac{\delta_2 r_1}{a_1} = \left\{\begin{array}{ll} +0.00000\,02267 & \\ -0.00000\,17952 & \cos\theta_0 \\ +0.00000\,17830 & \cos 2\theta_0 \\ +0.00000\,03800 & \cos 3\theta_0 \\ -0.00000\,00332 & \cos 4\theta_0 \\ +0.00000\,00076 & \cos 5\theta_0 \\ +0.00000\,00050 & \cos 6\theta_0 \\ +0.00000\,00025 & \cos 7\theta_0 \\ +0.00000\,00012 & \cos 8\theta_0 \\ +0.00000\,00006 & \cos 9\theta_0 \\ +0.00000\,00003 & \cos 10\theta_0 \end{array}\right\}$$

$$\delta_2 v_1 = \left\{\begin{array}{ll} -1.''09575 & \sin\theta_0 \\ +0.64134 & \sin 2\theta_0 \\ +0.12699 & \sin 3\theta_0 \\ -0.03725 & \sin 4\theta_0 \\ -0.00332 & \sin 5\theta_0 \\ -0.00030 & \sin 6\theta_0 \\ +0.00007 & \sin 7\theta_0 \\ +0.00007 & \sin 8\theta_0 \\ +0.00005 & \sin 9\theta_0 \\ +0.00002 & \sin 10\theta_0 \\ -0.00001 & \sin 11\theta_0 \\ -0.00001 & \sin 12\theta_0 \end{array}\right\}$$

These values of $\delta_2 r_1 / a_1$ and $\delta_2 v_1$ constitute the solution of the problem for *Jupiter's* coordinates, but, that the expressions for $\frac{r_1}{a_1}$ and v_1 may be complete, we add the first- and second-order terms, thus forming the tables

$$\frac{r_1}{a_1} = \left\{\begin{array}{ll} 1-0.00001\,11985 & \\ +0.00012\,27470 & \cos\theta_0 \\ -0.00053\,16043 & \cos 2\theta_0 \\ -0.00005\,52168 & \cos 3\theta_0 \\ -0.00001\,44266 & \cos 4\theta_0 \\ -0.00000\,47524 & \cos 5\theta_0 \\ -0.00000\,17722 & \cos 6\theta_0 \\ -0.00000\,07116 & \cos 7\theta_0 \\ -0.00000\,03004 & \cos 8\theta_0 \\ -0.00000\,01314 & \cos 9\theta_0 \\ -0.00000\,00591 & \cos 10\theta_0 \\ -0.00000\,00272 & \cos 11\theta_0 \\ -0.00000\,00127 & \cos 12\theta_0 \end{array}\right\}$$

$$v_1 = n_1 t + \left\{\begin{array}{ll} +\ 78.''15254 & \sin\theta_0 \\ -195.12909 & \sin 2\theta_0 \\ -\ 16.20481 & \sin 3\theta_0 \\ -\ \ 3.79161 & \sin 4\theta_0 \\ -\ \ 1.16033 & \sin 5\theta_0 \\ -\ \ 0.41327 & \sin 6\theta_0 \\ -\ \ 0.16093 & \sin 7\theta_0 \\ -\ \ 0.06649 & \sin 8\theta_0 \\ -\ \ 0.02863 & \sin 9\theta_0 \\ -\ \ 0.01273 & \sin 10\theta_0 \\ -\ \ 0.00582 & \sin 11\theta_0 \\ -\ \ 0.00270 & \sin 12\theta_0 \end{array}\right\}$$

2. *Computation of δu_2 and $\delta z_2 / n_2$.* Several of the series entering into δu_2 and $\delta z_2 / n_2$ have already been computed as they enter also into δu_1 and $\delta z_1 / n_1$. The remaining coefficients, in terms of their logarithms, are tabulated below.

$i+j$	cosine $b_i b_j$	cosine $f_i c_j$	cosine $q_i b_j$	cosine $-\theta_i \gamma_j$
0	3.39116	3.63496n	3.32535	3.52548n
1	3.541489	3.849053n	3.415321	3.622736n
2	3.29549	3.89548n	3.36101	3.42446n
3	2.92530	3.81759n	3.19688	3.01129n
4	2.52038	3.79771n	3.02462	3.29467
5	2.09649	3.68966n	2.81428	3.36085
6	1.69460	3.55232n	2.59122	3.30380
7	1.3130	3.39590n	2.36141	3.19518
8	0.948	3.22480n	2.12730	3.05590
9	0.595	3.0439n	1.8916	2.89760
10	0.25	2.8505n	1.647	2.7281
11	9.91	2.6250n	1.395	2.5953
12	9.60	2.4080n	1.119	2.4079

$i+j$	sine $q_i \eta_j$	sine $\zeta_i p_j$	cosine m_{i+j}	cosine n_{i+j}
0	. . .	. . . .	2.82091n	. . .
1	3.169687	2.848613n	. . .	. . .
2	3.19792	2.79111n	3.29803n	2.64967n
3	3.09005	3.00085n	. . .	. . .
4	3.05805	3.00635n	. . .	. . .
5	2.91879	2.88634n	. . .	. . .
6	2.74412	2.72391n	. . .	. . .
7	2.55063	2.53883n	. . .	. . .
8	2.34727	2.3401n	. . .	. . .
9	2.1352	2.1324n	. . .	. . .
10	1.9187	1.9153n	. . .	. . .
11	1.7057	1.684n	. . .	. . .
12	1.503	1.428n	. . .	. . .

There is one additional product needed for $\delta z_2 / n_2$ and three for the numerical expression of equation (14). These are, respectively,

$$\frac{u_2 z_2}{2n_2}, \quad \frac{u_2^2}{4n_2^2}, \quad \frac{z_2^2}{n_2^2}, \quad \frac{u_2 u_2}{4n_2^2}$$

The coefficients of these products are given below in the same order in which they occur here. Then δu_2 and $\delta z_2 / n_2$ have the values given.

$i+j$	cosine $b_i p_j$	cosine $-\eta_j \gamma_j$	cosine $p_i p_j$	cosine $s_i b_j$
0	3.22296n	3.38700	3.59894	3.38700n
1	3.669376n	3.48381	3.62334	3.83723n
2	3.56529n	2.35849n	3.62027	3.92742n
3	3.32728n	3.25359n	3.60254	3.81581n
4	3.01265n	3.18388n	3.37820	3.63415n
5	2.66516n	2.94751n	3.08329	3.40628n
6	2.33235n	2.68587n	2.78977	3.16908n
7	2.00986n	2.41787n	2.50202	2.92815n
8	1.7018n	2.1475n	2.2185	2.6852n
9	1.3983n	1.8775n	1.9386	2.4422n
10	1.106n	1.6090n	1.6602	2.192n
11	0.80n	1.3485n	1.378	1.937n
12	0.46n	1.0995n	1.080	1.655n

$i+j$	cosine δu_2	cosine $\delta z_2/n_2$
0	4.20364n	. . .
1	5.005758n	5.074954
2	4.04903n	4.36740
3	3.59051n	4.07674
4	3.08096n	3.71344
5	2.65093n	3.37455
6	2.25643n	3.0540
7	1.8850n	2.7447
8	1.5294n	2.4442
9	1.1877n	2.150
10	0.835n	1.852
11	0.13n	1.34
12	9.43n	0.89

As in the case of r_1 and v_1 we have

$$\frac{\delta_1 r_2}{a_2} = \tfrac{1}{2}\left[\delta u_2 - \tfrac{1}{4} u_2^2\right] \quad , \quad \delta_2 v_2 = \int \delta z_2 \, dt$$

In order to determine $\frac{\delta_2 r_2'}{a_2}$ and $\delta_2 v_2'$, the second-order perturbations of *Saturn's* coordinates when referred to the center of the sun, we must apply to the former the following reductions respectively: —

$$+\tfrac{1}{4}\kappa_1^2 \frac{a_1^2}{a_2^2}(1-\cos 2\theta_0) + \kappa_1 \frac{a_1}{a_2}\left[\frac{u_1}{2}\cos\theta_0 + (\delta_1 v_1 - \delta_1 v_2)\sin\theta_0\right]$$

and

$$+\tfrac{1}{2}\kappa_1^2 \frac{a_1^2}{a_2^2}\sin 2\theta_0 - \kappa_1 \frac{a_1}{a_2}\left[\left(\frac{u_1}{2} - \frac{u_2}{2}\right)\sin\theta_0 - (\delta_1 v_1 - \delta_1 v_2)\cos\theta_0\right]$$

In order to show the amount of these reductions the values of $\frac{\delta_2 r_2}{a_2}$ and $\delta_2 v_2$ are placed beside them below. In $\frac{\delta_2 r_2}{a_2}$ and its reduction the numbers are expressed in units of the tenth decimal.

$i+j$	cosine $\frac{\delta_2 r_2}{a_2}$	Reduction	sine $\delta_2 v_2$ ″	Reduction ″
0	− 9222	+ 690	. . . .	. . . .
1	−52407	−4318	−1.65252	−0.04982
2	− 6585	− 477	−0.16202	+0.02275
3	− 2368	+1374	−0.05531	−0.02554
4	− 768	+ 109	−0.01797	−0.00211
5	− 286	+ 22	−0.00659	−0.00051
6	− 115	+ 6	−0.00262	−0.00017
7	− 49	+ 2	−0.00110	−0.00007
8	− 21	0	−0.00048	−0.00003
9	− 10	0	−0.00022	−0.00001
10	− 4	0	−0.00010	0.00000
11	− 1	0	−0.00003	0.00000
12	0	0	−0.00001	0.00000

We can now form the tables for r_2' / a_2 and v_2'.

$$\frac{r_2'}{a_2} = \left\{\begin{array}{l} 1+0.00041\,58615 \\ \quad +0.00086\,35112\ \cos\theta_0 \\ \quad +0.00014\,67556\ \cos 2\theta_0 \\ \quad +0.00003\,39822\ \cos 3\theta_0 \\ \quad +0.00001\,05003\ \cos 4\theta_0 \\ \quad +0.00000\,37599\ \cos 5\theta_0 \\ \quad +0.00000\,14685\ \cos 6\theta_0 \\ \quad +0.00000\,06076\ \cos 7\theta_0 \\ \quad +0.00000\,02618\ \cos 8\theta_0 \\ \quad +0.00000\,01163\ \cos 9\theta_0 \\ \quad +0.00000\,00530\ \cos 10\theta_0 \\ \quad +0.00000\,00246\ \cos 11\theta_0 \\ \quad +0.00000\,00116\ \cos 12\theta_0 \end{array}\right\}$$

$$v_2' = n_2 t + \left\{\begin{array}{l} -\ \ 5.\!''13402\ \sin\theta_0 \\ +31.87097\ \sin 2\theta_0 \\ +\ \ 6.58817\ \sin 3\theta_0 \\ +\ \ 1.97545\ \sin 4\theta_0 \\ +\ \ 0.69977\ \sin 5\theta_0 \\ +\ \ 0.27283\ \sin 6\theta_0 \\ +\ \ 0.11311\ \sin 7\theta_0 \\ +\ \ 0.04893\ \sin 8\theta_0 \\ +\ \ 0.02183\ \sin 9\theta_0 \\ +\ \ 0.00998\ \sin 10\theta_0 \\ +\ \ 0.00466\ \sin 11\theta_0 \\ +\ \ 0.00221\ \sin 12\theta_0 \end{array}\right\}$$

Thus we get

$$\frac{r_1}{a_1} = 1 + \sum_1^\infty A_i \cos i\,(l_2 - l_1)$$

$$v_1 = l_1 + \sum_1^\infty B_i \sin i\,(l_2 - l_1)$$

$$\frac{r_2'}{a_2} = 1 + \sum_1^\infty A_i' \cos i\,(l_2 - l_1)$$

$$v_2' = l_2 + \sum_1^\infty B_i' \sin i\,(l_2 - l_1)$$

Leander McCormick Observatory, 1903 *May* 15.

If we refer the coordinates to axes intersecting in the *Sun*, and rotating in the direction of motion with the uniform velocity n_1, it is evident that we may write

$$v_1 - l_1 = w_1 = \sum_1^\infty B_i \sin ikt$$

$$v_2' - l_1 = w_2 = kt + \sum_1^\infty B_i' \sin ikt$$

where $k = n_2 - n_1$. Then all the coordinates are periodic with respect to the time.