SCHUR DUALITIES ARISING FROM QUANTUM SYMMETRIC PAIRS

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Abstract

The Schur-Jimbo duality is one of the most fundamental topics in representation theory, bridging the irreducible representations of a Hecke algebra with those of a Drinfeld-Jimbo quantum group. Evolving alongside the advancements in the field, the Schur-Jimbo duality has been extended in tandem with the emergence of $\imath$quantum groups, which are a natural generalization of quantum groups arising from the theory of quantum symmetric pairs.

In this dissertation, we construct various $\imath$Schur dualities stemming from quantum symmetric pairs of types AI, AII, and AIII. Particularly, the $\imath$Schur duality of type AIII, accommodating black nodes in its Satake diagram, presents a unified extension of Jimbo-Schur duality and Bao-Wang's quasi-split $\imath$Schur duality.

Moreover, expanding the classic works of Kazhdan-Lusztig and Deodhar, we establish bar involutions and canonical bases on quasi-permutation modules over the type B Hecke algebra, where the bases are parameterized by cosets of (possibly non-parabolic) reflection subgroups of the Weyl group of type B. The quasi-parabolic KL bases on quasi-permutation Hecke modules are shown to match with the $\imath$canonical basis on the tensor space.

Finally, we establish two specific families of quantum supersymmetric pairs, denoted as type AIII and type AI-II, respectively. We elucidate their fundamental properties, including the coideal algebra property and the quantum Iwasawa decomposition, which ensure that the $\imath$quantum supergroups attain the expected sizes. Within the framework of these quantum supersymmetric pairs, we provide super generalizations of the aforementioned dualities.

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Contents

Chapter 1

Introduction

1.1 Background

Schur-Jimbo duality

Let $\mathfrak{sl}_m$ be the type A simple Lie algebra and $\mathfrak{S}_n$ be the symmetric group on n letters. In the classical Schur duality, the actions of the enveloping algebra $\mathbf{U}(\mathfrak{sl}_m)$ and the group algebra $\mathbb{C}[\mathfrak{S}_n]$ on the tensor space $(\mathbb{C}^m)^{\otimes n}$ commute with each other and satisfy the double centralizer property. In this way, the Schur duality connects the irreducible representations of $\mathfrak{sl}_m$ and $\mathfrak{S}_n$.

Independently in the mid-1980s, Drinfeld and Jimbo introduced the quantum groups, which are q-deformations of complex simple Lie algebras. The universal R-matrix introduced by Drinfeld [Dr86] provides solutions to the Yang-Baxter equation. The quantum groups have led to many advances in mathematical physics, representation theory, algebraic geometry, and algebraic combinatorics.

Inspired by Ringel's Hall algebra realization of half a quantum group [R90], Lusztig introduced the canonical basis [Lus90] arising from quantum groups (see also Kashiwara [Ka91] for another approach). Additionally, Lusztig developed canonical bases for tensor products of modules in [Lus92].

A quantum analog of the Schur duality [Jim86] is naturally provided by the quantum group $\mathbf{U}_q(\mathfrak{sl}_m)$ and the type A Iwahori-Hecke algebra $\mathscr{H}_{\mathfrak{S}_n}$. In this context, the vector space $\mathbb{C}^m$ is replaced by the natural representation $\mathbb{V}$ of $\mathbf{U}_q(\mathfrak{sl}_m)$. The actions of $\mathscr{H}_{\mathfrak{S}_n}$ are realized by the R-matrix. Moreover, the type A (parabolic) Kazhdan-Lusztig basis ([KL79], [De87]) can be identified with the canonical basis on $\mathbb{V}^{\otimes n}$ when viewed as a direct sum of permutation modules over the Hecke algebra, see [FKK98] (cf. [LW20]).

Quantum symmetric pairs and $\imath$Schur dualities

Let $\mathfrak{g}$ be a finite-dimensional semisimple or reductive Lie algebra over $\mathbb{C}$, and let θ be an involution on $\mathfrak{g}$. The classification of irreducible symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\theta)$ is given by the Satake diagrams [Ara62], which are bi-colored Dynkin diagrams $I = I_\bullet \cup I_\circ$ together with a diagram involution τ.

The theory of quantum symmetric pairs $(\mathbf{U}_q(\mathfrak{g}), \mathbf{U}^\imath)$, which provides a quantization of the symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$, was systematically developed by Letzter [Let99, Let02]. In this context, $\mathbf{U}_q(\mathfrak{g})$ represents the Drinfeld-Jimbo quantum group associated with $\mathfrak{g}$, while $\mathbf{U}^\imath$ a coideal subalgebra of $\mathbf{U}_q(\mathfrak{g})$, nowadays known as an $\imath$quantum group. Kolb [Ko14] further expanded and generalized this theory to cover the Kac-Moody case.

Generalizing Lusztig's approach on canonical basis in [Lus92, Lus93], Bao and Wang [BW18a, BW18b] have developed a theory of $\imath$canonical basis for $\imath$quantum groups arising from quantum symmetric pairs. They showed that any based module M of a quantum group of finite type (cf. [Lus93, Chapter 27]) when viewed as a module over an $\imath$quantum group with suitable parameters can be endowed with a new bar map $\psi_\imath$ and a distinguished $\psi_\imath$-invariant basis (called $\imath$canonical basis).

In [BW18a], the authors demonstrate that the Hecke algebra of type B, denoted by $\mathscr{H}_{B_d}$, and the $\imath$quantum group of type AIII satisfy a double centralizer property (see also [Bao17]).

Furthermore, it was shown that the Kazhdan-Lusztig basis of type B coincides with the $\imath$canonical bases [BW18b] arising from tensor product modules of $\imath$quantum groups. This result was later generalized to a multi-parameter setting [BWW18].

Bao and Wang in [BW18a, §2.3] also introduced the quasi K-matrix for a quantum symmetric pair as an analogue of the quasi R-matrix; a proof for the existence of quasi K-matrix for general quantum symmetric pairs was given in [BK19]. This quasi K-matrix serves as the intertwiner between the embedding of the $\imath$quantum group into the underlying quantum group and its bar-conjugated embedding, subject to certain parameter conditions (cf. [BK19]). In the quasi-split $\imath$Schur duality, the action of the additional generator H_0 of $\mathscr{H}_{B_d}$ was realized via the K-matrix by Bao and Wang [BW18a, Theorems 2.18, 5.4] (this is the first construction of a K-matrix built on the notion of quasi K-matrix therein).

Brauer algebra

In [Br37] Brauer introduced the so-called Brauer algebra, and established the double centralizer property between it and the orthogonal group O_m or symplectic group Sp_{2m}. The Brauer algebra was further studied in [Br56a], [Br56b] and so on. The Birman-Murakami-Wenzl algebra (or BMW algebra for short), as a two-parameter deformation of the Brauer algebra, was algebraically defined by Birman and Wenzl [BW89], and independently by Murakami [Mu87].

In the Schur-Jimbo duality, when $\mathbf{U}_q(\mathfrak{sl}_m)$ is replaced by $\mathbf{U}_q(\mathfrak{o}_m)$ or $\mathbf{U}_q(\mathfrak{sp}_{2m})$, the role of $\mathscr{H}_{\mathfrak{S}_n}$ was played by the BMW algebra with the parameters being appropriately specialized; see [CP94] or [Ha92]. In [FG95] a canonical basis of the BMW algebra has been constructed and the associated cell structure has been studied. However, the Iwahori-Hecke algebra $\mathscr{H}_{\mathfrak{S}_n}$ is not naturally a subalgebra of the BMW algebra while the algebras $\mathbf{U}_q(\mathfrak{o}_m)$ and $\mathbf{U}_q(\mathfrak{sp}_{2m})$ are not isomorphic to subalgebras of the type A quantum groups either.

Besides the BMW algebra, another multi-parameter deformation of the Brauer algebra, which depends on two indeterminates q and z, was introduced by Molev [M03]; moreover, Molev showed that the action of his algebra (specializing z to q^m) on $\mathbb{V}^{\otimes n}$ commutes with that of the twisted quantized enveloping algebra $\mathbf{U}_q^{\mathrm{tw}}(\mathfrak{so}_m)$ introduced by Noumi in [No96], where $\mathbb{V}$ is the natural representation of $\mathbf{U}_q(\mathfrak{sl}_m)$ (also cf. [We12b]).

Later on, in [We12a], Wenzl defined a quotient of Molev's algebra called the *q-Brauer algebra*, which will be the main object of Part II. Many properties of the q-Brauer algebra have been studied by Nguyen. For example, in [N14], Nguyen constructed a standard basis for the q-Brauer algebra which is labeled by a natural basis of Brauer algebras; the q-Brauer algebra contains $\mathscr{H}_{\mathfrak{S}_n}$ as a natural subalgebra under the standard basis. Moreover, in [N14] and [N18], it was shown that the q-Brauer algebra is a cellular algebra and its irreducible representations can be classified using the general theory of cellular algebras in [GL96].

Lie superalgebra and quantum supersymmetric pairs

Now suppose $\mathfrak{g}$ is a basic Lie superalgebra of any finite type. In general, the fundamental systems of the root system Φ associated with $\mathfrak{g}$ are not conjugated under the Weyl group actions due to the existence of odd isotropic roots. Consequently, the Dynkin diagrams associated with $\mathfrak{g}$ depend on the choice of positive roots Φ^+.

In [Ya94], Yamane has constructed quantized enveloping algebras $\mathbf{U} = \mathbf{U}_q(\mathfrak{g})$ as well as their universal R-matrices associated with arbitrary Dynkin diagrams. In his subsequent work [Ya99], he further quantized the odd reflections into algebra isomorphisms of $\mathbf{U}$ associated with different presentations, providing a super analogue of Lusztig's braid group operators.

In this thesis, we will be interested in quantum symmetric pairs for Lie superalgebras in Part III. Recently, examples of quantum symmetric pairs for Lie superalgebras has been constructed. Kolb and Yakimov's work in [KY20] studied Drinfeld doubles of pre-Nichols algebras of diagonal type, which include as an example the quantum supersymmetric pair of type AIII, with $I_\bullet = \varnothing$. Moreover, Chung [Ch19] has studied quantum symmetric pairs

$(\mathbf{U}_\pi, \mathbf{U}^\imath_\pi)$ for quantum covering algebras $\mathbf{U}_\pi$ which is introduced in [CHW13] and specializes to the Lusztig quantum group when $\pi = 1$ and quantum supergroups of anisotropic type when $\pi = -1$; see also [Ch21].

The super analogue of the Schur duality is given by Sergeev [S85] between $\mathfrak{gl}(m|n)$ and the symmetric group. Moreover, it was shown in [Mi06] that the type A quantum supergroup associated to the standard Dynkin diagram and the Hecke algebra of type A satisfy a double centralizer property.

1.2 Goal

In this dissertation, we construct and study in depth various $\imath$Schur dualities emerging from $\imath$quantum groups of different types. We also study the canonical bases arising from such dualities. Additionally, we construct two families of quantum supersymmetric pairs (for general constructions of basic types, refer to [SW24]) and establish their fundamental properties, including the quasi K-matrix. As an application, we also extend the aforementioned dualities to the super setting using the quantum supersymmetric pairs.

The work presented in this thesis relies extensively on the following framework, originating from [BW18a],

$$\begin{array}{ccccc} \mathbf{U} & \curvearrowright & \mathbb{V}^{\otimes n} & \curvearrowleft & \mathscr{H}_{\mathfrak{S}_n} \\ \uparrow & & & & \downarrow \\ \mathbf{U}^\imath & \curvearrowright & \mathbb{V}^{\otimes n} & \curvearrowleft & \mathscr{H}_{B_d} \end{array} \tag{1.1}$$

where $\mathbf{U}^\imath$ of quasi-split type AIII ($I_\bullet = \varnothing$) is a natural subalgebra of $\mathbf{U}$ while $\mathscr{H}_{\mathfrak{S}_n}$ is a natural subalgebra of $\mathscr{H}_{B_d}$.

In Part I, we extend the $\imath$quantum group $\mathbf{U}^\imath$ in (1.1) by allowing black nodes to appear in the corresponding Satake diagram of type AIII. The $\imath$Schur duality we formulate in this case can be viewed as a common generalization of Jimbo-Schur duality and Bao-Wang's quasi-split $\imath$Schur duality. The results presented in this part have appeared in a joint paper with Wang

[SW23].

In Part II, we focus on $\mathbf{U}^\imath$ of type AI and AII. The type AI $\imath$quantum group serves as a quantization of the special orthogonal Lie algebra, while the type AII $\imath$quantum group quantizes the symplectic Lie algebra. In this case, the place of $\mathscr{H}_{B_d}$ in (1.1) is replaced by Wenzl's q-Brauer algebra. Additionally, we construct a bar involution and a KL-type basis for the q-Brauer algebra. The results presented in this part have appeared in the joint paper with W. Cui [CS22].

In Part III, we delve into the study of quantum supersymmetric pairs. We first explore a super generalization of the type AIII quantum symmetric pairs, allowing black nodes. For suitable Satake diagrams associated with the type A Lie superalgebra (7.12), we construct the pair $(\mathbf{U}, \mathbf{U}^\imath)$ along with its fundamental properties, including a quantum Iwasawa decomposition and quasi K-matrix. Within the framework (1.1), we establish an $\imath$Schur duality between the $\imath$quantum supergroup $\mathbf{U}^\imath$ and $\mathscr{H}_{B_d}$.

Additionally, we consider a super Satake diagram (9.1) that combines the Satake diagrams of type AI and AII. In this scenario, we construct the pair $(\mathbf{U}, \mathbf{U}^\imath)$ of super type AI-II, where the classical limit of $\mathbf{U}^\imath$ corresponds to the ortho-symplectic Lie superalgebra. Under the framework (1.1), we establish an $\imath$Schur duality of super type AI-II, which merges the $\imath$Schur dualities of type AI and AII discussed in Part II. The results presented in this part on super type AIII have appeared in [Sh22] and on super type AI-II will appear in [SW24].

1.3 Main results for Part I

In this part we establish bar involutions and canonical bases on quasi-permutation modules over the type B Hecke algebra. We also formulate an $\imath$Schur duality between an $\imath$quantum group of type AIII (allowing black nodes in its Satake diagram) and the Hecke algebra of type B.

Type B Kazhdan-Lusztig, expanded

Let $W = W_d$ be the Weyl group of type B_d generated by the simple reflections $s_0, s_1, \ldots, s_{d-1}$, which contains the symmetric group S_d naturally as a subgroup. Let $\mathscr{H}_{B_d}$ be its associated Hecke algebra generated by $H_0, H_1, \ldots, H_{d-1}$ in 2 parameters q, p, which contains the Hecke algebra $\mathscr{H}_{S_d}$ as a subalgebra. (In the introduction, we shall assume that p is an integer power of q; a reader can take $p = q$.)

Consider reflection subgroups of W_d of the form

$$W_f = W_{m_1} \times \ldots \times W_{m_k} \times S_{m_{k+1}} \times \ldots \times S_{m_l}. \tag{1.2}$$

where $m_1 + \ldots + m_l = d$, $k \leq l$ and all m_i are positive. Clearly, W_f is a parabolic subgroup of W_d if and only if $k \leq 1$. For $k \leq 1$, there exists a right $\mathscr{H}_{B_d}$-module $\mathbb{M}_f$, the induced module from the trivial module of the subalgebra $\mathscr{H}_{W_f}$, parameterized by the set ${}^fW$ of right minimal length representatives of W_f. The celebrated Kazhdan-Lusztig (KL) basis on the regular representation of $\mathscr{H}_{B_d}$ (see [KL79] for $p = q$, and [Lus03] for $p \in q^{\mathbb{Z}}$) admits a parabolic generalization in terms of $\mathbb{M}_f$ (see Deodhar [De87]); that is, $\mathbb{M}_f$ admits a bar involution and a distinguished bar-invariant basis, known as the parabolic KL basis.

Our first main result is to extend the above classic works of Kazhdan, Lusztig and Deodhar to construct canonical bases (also called quasi-parabolic KL bases) of type B associated to arbitrary reflection subgroups W_f of the form (1.2). By definition, our modules $\mathbb{M}_f$ depend only on the reflection subgroup W_f of W_d, and each $\mathbb{M}_f$ comes with a standard basis $\{M_{f\cdot\sigma}\}$, where σ runs over the set ${}^fW$ of minimal length representatives of right cosets of W_f in W_d. We denote by $<$ the Chevalley-Bruhat order on ${}^fW$.

Theorem A (Proposition 3.2.12, Theorem 3.2.13). (1) There exists an anti-linear bar involution $\psi_\imath$ on $\mathbb{M}_f$ such that $\psi_\imath(M_f) = M_f$, which is compatible with the bar operator on $\mathscr{H}_{B_d}$, i.e., $\psi_\imath(xh) = \psi_\imath(x)\bar{h}$, for all $x \in \mathbb{M}_f,\ h \in \mathscr{H}_{B_d}$.

(2) The module $\mathbb{M}_f$ admits a canonical basis $\{C_\sigma | \sigma \in {}^fW\}$ such that C_σ is bar invariant

and $C_\sigma \in M_{f\cdot\sigma} + \sum_{w\in {}^fW, w<\sigma} q^{-1}\mathbb{Z}[q^{-1}]M_{f\cdot w}$.

The module $\mathbb{M}_f$ also admits a dual canonical basis $\{C^*_\sigma | \sigma \in {}^fW\}$ such that C^*_σ is bar invariant and $C^*_\sigma \in M_{f\cdot\sigma} + \sum_{w\in {}^fW, w<\sigma} q\mathbb{Z}[q]M_{f\cdot w}$; see Proposition 3.2.14.

Theorem **A** is totally unexpected when W_f is not parabolic, given the fundamental importance of Kazhdan-Lusztig bases and how well they have been studied from various viewpoints since 1970's. We are led to the formulation of this result from a new $\imath$Schur duality and the corresponding $\imath$canonical bases, which we shall explain below momentarily.

As W_f may not be parabolic, the Hecke algebra $\mathscr{H}(W_f)$ is not a subalgebra of $\mathscr{H}_{B_d}$ in any natural manner, and hence $\mathbb{M}_f$ is not an induced module from an $\mathscr{H}(W_f)$-module in general. Accordingly, it is more difficult to establish a key property (see Theorem 3.2.6) concerning the action of the simple reflections s_i on the poset ${}^fW$, generalizing the parabolic case in [De77, De87]. This leads to explicit formulas (see Proposition 3.2.8) for the actions of the generators H_i of $\mathscr{H}_{B_d}$ on the standard basis of $\mathbb{M}_f$ parametrized by the minimal length coset representatives for $W_f\backslash W$; remarkably, these formulas look identical to those for W_f parabolic. The self-contained proof of Theorem **A** (which is independent of $\imath$Schur duality below) will occupy Section 3.2.

The canonical bases in Theorem **A** include parabolic KL bases of type A (besides those of type B) as special cases. For example, consider the non-parabolic subgroup $W_f = W_1 \times \ldots \times W_1$ (generated by the d sign reflections). In this case, ${}^fW = S_d$, and the canonical basis of $\mathbb{M}_f$ in Theorem **A** is identified with the KL basis of $\mathscr{H}_{S_d}$. See Example 3.2.15(2) where an arbitrary parabolic KL basis of type A arises as a canonical basis of type B.

$\imath$Schur duality

Let $\mathbb{V}$ be the natural representation of the Drinfeld-Jimbo quantum group $\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_{2r+m})$. Let $(\mathbf{U}, \mathbf{U}^\imath)$ be the quantum symmetric pair of type AIII (§ 2.2), where $\mathbf{U}^\imath$ is a coideal subalgebra of $\mathbf{U}$ whose $q \mapsto 1$ limit is the enveloping algebra of $\mathfrak{sl}(r+m) \oplus \mathfrak{gl}(r)$.

When $\mathbb{V}$ is viewed as a representation of $\mathbf{U}^\imath$, its standard basis $\{v_i | i \in \mathbb{I}_{r|m|r}\}$ is naturally bicolored (where the m indices in the middle are colored as $\bullet$, while the remaining $2r$ indices are colored as $\circ$). When $m = 0$ or 1, $\mathbf{U}^\imath$ is quasi-split, and on the other extreme when $r = 0$, we have $\mathbf{U}^\imath = \mathbf{U}$.

We endow the tensor space $\mathbb{V}^{\otimes d}$ with a (right) $\mathscr{H}_{B_d}$-module structure. The aforementioned $\mathscr{H}_{B_d}$-modules $\mathbb{M}_f$ arise as direct summands of the tensor module $\mathbb{V}^{\otimes d}$ of $\mathscr{H}_{B_d}$, and are called *quasi-permutation modules*. Each $\mathbb{M}_f$ is spanned by a standard basis M_g where g runs over a W_d-orbit. (We have chosen to parametrize $\mathbb{M}_f$ by "anti-dominant weights" f.)

Our second main result is the following.

Theorem B (Theorem 4.1.6). The actions of $\mathbf{U}^\imath$ and $\mathscr{H}_{B_d}$ on $\mathbb{V}^{\otimes d}$ commute with each other, and form double centralizers.

The $\imath$quantum group $\mathbf{U}^\imath$ comes with parameters § 2.2, and for our purpose, the parameters are fixed once for all by the double centralizer property in Theorem **B**.

Note that in the extreme case when $r = 0$ and $\mathbf{U}^\imath = \mathbf{U}$, we (somewhat surprisingly) claim to have an action on $\mathbb{V}^{\otimes d}$ by $\mathscr{H}_{B_d}$, not by $\mathscr{H}_{S_d}$ which one is familiar with. The puzzle is resolved when we note that the action of the generator H_0 of $\mathscr{H}_{B_d}$ reduces to $p \cdot \mathrm{Id}$, and we recover Jimbo duality [Jim86] (q-Schur duality of type A) in disguise in this extreme case. On the other hand, when $m = 0$ or 1, $(\mathbf{U}, \mathbf{U}^\imath)$ is quasi-split, and we recover the (quasi-split) $\imath$Schur duality due to [BW18a] for $p = q$ (and generalized to $p = 1$ in [Bao17] and to general p in [BWW18]). The action of H_0 in general is a suitable mixture of the actions in the 2 special cases.

Recall that in the quasi-split $\imath$Schur duality [BW18a], the action of the additional generator H_0 of $\mathscr{H}_{B_d}$ was realized via the K-matrix. We show that the action of H_0 in the setting of Theorem **B** is again realized by a K-matrix, which has been available in greater generality in Balagovic-Kolb [BK19]. This can be viewed as a distinguished example that the K-matrix provides solutions to the reflection equation, a property of the K-matrix in general as established in [BK19].

Compatible canonical bases

Apply the general constructions of $\imath$canonical bases in [BW18b] to the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ of type AIII, and $M = \mathbb{V}^{\otimes d}$, as in the setting of Theorem **B**. Denote by $\{C_g \mid g \in \mathbb{I}^d_{r|m|r}\}$ and $\{C^*_g \mid g \in \mathbb{I}^d_{r|m|r}\}$ the $\imath$canonical and dual $\imath$canonical basis on $\mathbb{V}^{\otimes d}$.

Theorem C (Proposition 4.2.7, Theorem 4.2.9). (1) There exists a bar involution on $\mathbb{V}^{\otimes d}$ which is compatible with the bar involutions on $\mathbf{U}^\imath$ and $\mathscr{H}_{B_d}$.

(2) The (dual) $\imath$canonical basis on $\mathbb{V}^{\otimes d}$ viewed as a $\mathbf{U}^\imath$-module coincide with the (dual) quasi-parabolic KL basis on $\mathbb{V}^{\otimes d}$ viewed as an $\mathscr{H}_{B_d}$-module (see Theorem **A**.

In the extreme case when $r = 0$ and $\mathbf{U}^\imath = \mathbf{U}$ (i.e., in the setting of [Jim86]), Theorem **C** recovers the main result of I. Frenkel, Khovanov and Kirillov [FKK98]. In the special case when $m = 0$ or 1, it reduces to the (quasi-split) $\imath$Schur duality in [BW18a] (as well as the generalizations in [Bao17, BWW18]). In the general case (for arbitrary r and m), the $\imath$canonical basis elements in $\mathbb{V}^{\otimes d}$ parameterized by all black nodes $\bullet$ (respectively, by all white nodes $\circ$) can be identified with parabolic KL of type A (respectively, B), but there are other $\imath$canonical basis elements of mixed colors without such identifications.

An inversion formula

An inversion formula for KL polynomials originated in [KL79] and was subsequently generalized to the parabolic setting by Douglass [Do90]; also see [So97] for an exposition. In type A, the inversion formula can be reformulated and reproved using a symmetric bilinear form on the tensor product $\mathbf{U}$-module $\mathbb{V}^{\otimes d}$; see Brundan [Br06] and Cao-Lam [CL16]. We generalize the approach in [CL16] via the $\imath$Schur duality by formulating a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{V}^{\otimes d}$ as a $\mathbf{U}^\imath$-module.

Theorem D (Theorems 4.3.7–4.3.8). (1) The bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{V}^{\otimes d}$ is symmetric.

(2) The $\imath$canonical basis and dual $\imath$canonical basis on $\mathbb{V}^{\otimes d}$ are dual with respect to $\langle\cdot,\cdot\rangle$, i.e., $\langle C_g, C^*_{-h}\rangle = \delta_{g,h}$, for $g,h \in f\cdot W_d$.

Theorem **D** can be reformulated as a duality between (dual) quasi-parabolic KL polynomials; see Corollary 4.3.9. It can be extended easily to a useful duality between super KL polynomials introduced in [BW18a]; see Remark 4.3.10. The proof of Theorem **D**(1) uses some old and new properties of the quasi R-matrix $\Theta^\imath$ introduced in [BW18a] (and generalized by Kolb [Ko20]) and an anti-involution σ_i on $\mathbf{U}^\imath$ in [BW21].

1.4 Main results for Part II

In this part, we construct a bar involution and the canonical basis on the q-Brauer algebra introduced by Wenzl. We also formulate $\imath$Schur dualities between the q-Brauer algebra and the $\imath$quantum groups of type AI and AII respectively.

Canonical bases of the q-Brauer algebra

Fix an integer $N \in \mathbb{Z}\setminus\{0\}$. Let $D_n(N)$ denote the Brauer algebra which is a $\mathbb{Z}$-algebra with a linear basis consisting of all partitions of the set

$$\{1,2,\ldots,n,1',2',\ldots,n'\}$$

into two-element subsets.

Let I_n denote the set of all basis diagrams of the Brauer algebra. Proposition 5.2.6 gives a standard basis of the q-Brauer algebra $\mathfrak{B}_n(q,z)$ that is labeled by the basis diagrams, denoted by $\{H_d \mid d \in I_n\}$.

Theorem E. (1)(Lemma 5.3.2) There is a unique involutive homomorphism $\overline{\cdot}$ on $\mathfrak{B}_n(q,z)$ which is $\mathbb{Q}$-linear and satisfies $\overline{q} = q^{-1}, \overline{z} = z^{-1}, \overline{H_i} = H_i^{-1}$ and $\overline{e} = e$.

(2)(Theorems 5.4.7) The q-Brauer algebra admits a canonical basis $\{C_d \mid d \in I_n\}$ such that C_d is bar invariant and $C_d \in H_d + \sum_{d' \in I_n, \ell(d') < \ell(d)} q^{-1}\mathbb{Z}[q^{-1}]H_{d'}$.

The bar involution $\overline{\cdot}$ is shown to be compatible with the one on its natural subalgebra $\mathscr{H}_{\mathfrak{S}_n}$. A direct consequence of the compatibility of the bar involutions is that the usual type A Kazhdan-Lusztig basis is a part of the canonical basis we obtain. Moreover, one can see that the coefficients, when expanding the canonical basis as a sum of the standard basis elements, are polynomials in q, which do not depend on z. A similar phenomenon was found in [FG95, §5.2].

$\imath$Schur duality of type AI and AII

Let $\mathbf{U}^\imath(\mathfrak{so}_m)$ denote the $\imath$quantum group of type AI and $\mathbf{U}^\imath(\mathfrak{sp}_{2m})$ denote the $\imath$quantum group of type AII; cf. § 2.2. Let $\mathbb{V}$ be the natural representation of $\mathbf{U}_q(\mathfrak{sl}_m)$ and $\mathbb{W}$ be the one of $\mathbf{U}_q(\mathfrak{sl}_{2m})$. Enlightened by the pioneering work about $\imath$Schur dualities in [BWW18], we construct explicit actions of the q-Brauer algebra $\mathfrak{B}_n(q,z)$ on the tensor modules $\mathbb{V}^{\otimes n}$ and $\mathbb{W}^{\otimes n}$ respectively; cf. Proposition 6.2.2 and Proposition 6.4.1.

Theorem F (Theorems 6.2.4–6.4.2). (1) The left action of $\mathbf{U}^\imath(\mathfrak{so}_m)$ on $\mathbb{V}^{\otimes n}$ commutes with the right action of $\mathfrak{B}_n(q,q^m)$. Moreover, when m is odd or m is even with $m-1 \geqslant 2n$, they form double centralizers.

(2) The left action of $\mathbf{U}^\imath(\mathfrak{sp}_{2m})$ on $\mathbb{W}^{\otimes n}$ commutes with the right action of $\mathfrak{B}_n(-q^{-1}, q^{2m})$ and they form double centralizers.

The commuting action in the case of type AI was also formulated in [M03] with a restriction on the parameters of the $\imath$quantum group and using the R-matrix presentation of the quantum group $\mathbf{U}_q(\mathfrak{gl}_m)$. Both commuting actions were also discovered in [ST19, (7.10)–(7.11)] through the Web category but not explicitly constructed.

1.5 Main results for Part III

In this part we introduce the framework of quantum supersymmetirc pairs associated with Lie superalgebras by constructing two explicit families of them. Moreover, we formulate various $\imath$Schur dualities in the super setting, providing a generalization of the dualities in Part I and II.

Quantum supersymmetric pair of type AIII

Let $(\mathfrak{g}, \mathfrak{g}^\theta)$ be a *supersymmetric pair* of type AIII; see (7.12). To define a quantum supersymmetric pair associated with $(\mathfrak{g}, \mathfrak{g}^\theta)$, we start with a diagram X of the form (7.12) that satisfies the conditions (7.14), where the index set is denoted as $I = I_\circ \cup I_\bullet$. Taking into account the presence of odd reflections, we can mimic the non-super case to define a longest element $w_\bullet$ associated with the Weyl groupoid of the Levi subalgebra corresponding to $I_\bullet$ (cf. [HY08]). Applying $w_\bullet$ to X results in another diagram Y that satisfies (7.14) and is indexed by I as well. The algebra $\mathbf{U}(Y)$ (resp. $\mathbf{U}(X)$) is generated by E_j, F_j (resp. $E_j^{\mathtt{X}}, F_j^{\mathtt{X}}$) along with the Cartan part. The $\imath$quantum group $\mathbf{U}^\imath(Y)$ in the pair $(\mathbf{U}(Y), \mathbf{U}^\imath(Y))$ is generated by E_j, F_j $(j \in I_\bullet)$,

$$B_j = F_j + \varsigma_j T_{w_\bullet}(E_j^{\mathtt{X}}) K_j^{-1}, \quad \text{for } j \in I_\circ$$

together with certain Cartan elements. Set $\mathbf{U}^\imath = \mathbf{U}^\imath(Y)$ and $\mathbf{U} = \mathbf{U}(Y)$.

Theorem G. (1)(Proposition 7.2.6) $\mathbf{U}^\imath$ is a right coideal subalgebra of $\mathbf{U}$ and $(\mathbf{U}, \mathbf{U}^\imath)$ forms a quantum supersymmetric pair.

(2)(Theorem 7.2.17) The quantum Iwasawa decomposition holds for $(\mathbf{U}, \mathbf{U}^\imath)$ of type AIII. (Hence $\mathbf{U}^\imath$ has the expected size.)

The coideal subalgebra property of $\mathbf{U}^\imath$ in $\mathbf{U}$ generalizes the non-super constructions in [Let99, Let02, Ko14]. We note that the original methods used in [Let99, Let02, Ko14] do not directly apply to the super case. As a result, we provide a new proof specifically tailored for the super type AIII case.

Subsequently, we establish new quantum $\imath$Serre relations (Proposition 7.2.14) by employing the projection technique introduced in [Ko14]. This allows us to obtain a natural filtration on $\mathbf{U}^\imath$ (Proposition 7.2.16), where the associated graded algebra $gr\mathbf{U}^\imath$ is essentially isomorphic to a parabolic subalgebra of $\mathbf{U}$ modulo the Cartan part. Hence we obtain the quantum Iwasawa decomposition of $\mathbf{U}$ with respect to $\mathbf{U}^\imath$ in Theorem 7.2.17.

Type AIII $\imath$Schur duality revisited

Sergeev [S85] has extended the Schur duality in the setting of $\mathfrak{gl}(m|n)$. The *quantum supergroup* $\mathbf{U}$, as a Drinfeld-Jimbo quantization of $\mathfrak{g}$, has been defined in [Ya94] associated to any Dynkin diagram of $\mathfrak{g}$. Moreover, it was shown in [Mi06] that the type A quantum supergroup associated to the standard Dynkin diagram and the Hecke algebra of type A satisfy a double centralizer property.

Having established the quantum supersymmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$, we proceed to establish a multi-parameter $\imath$Schur duality of type AIII between $\mathbf{U}^\imath$ and $\mathscr{H}_{B_d}$ in the same flavor of Theorem **B**. Let $\mathbb{W}$ be the natural representation of $\mathbf{U}$. We show that $\mathbb{W}^{\otimes d}$ possesses a right $\mathscr{H}_{B_d}$-module structure (Proposition 8.1.1) and a left $\mathbf{U}^\imath$-module structure via the comultiplication in the same time.

Theorem H (Theorem 8.1.8). Under the assumption on the parameters (4.4), the actions of $\mathbf{U}^\imath$ and $\mathscr{H}_{B_d}$ on $\mathbb{W}^{\otimes d}$ commute with each other and form a double centralizer property.

In the extreme case when $\mathbf{U}^\imath = \mathbf{U}$, the duality in Theorem **H** recovers the Schur-Sergeev duality for the quantum supergroup of type A. On the other hand, when $I_\bullet = \varnothing$, we obtain a super analogue of the two-parameter $\imath$Schur duality due to [BWW18].

Construction of the (quasi) K-matrix

For the construction of the quasi K-matrix Υ and the K-matrix $\mathcal{T}$ we impose one more condition (8.4); i.e. $I_\bullet$ consists of even simple roots only. We also define $B_i := F_i$ for $i \in I_\bullet$.

Theorem I (Theorem 8.2.18). There exists a quasi-K matrix $\Upsilon = \sum \Upsilon_\mu$ in the completion of $\mathbf{U}$ with $\Upsilon_0 = 1$ and $\Upsilon_\mu \in \mathbf{U}^+_\mu$, such that the equality

$$B_i \Upsilon = \Upsilon(\tau \circ \sigma(B_{\tau i}))$$

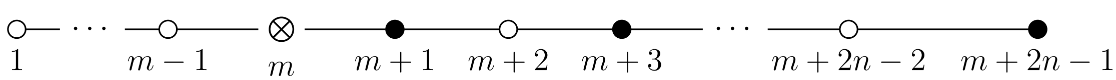
holds for all $i \in I$.

The construction of Υ follows from intertwining relations of [WZ22] and strategies of [BW18a, BK19]. With the quasi K-matrix Υ being established, we impose one more constraint on the parameters and construct a unique bar involution $\psi_\imath$ on $\mathbf{U}^\imath$ (Corollary 8.2.19), which is a super analogue of the bar involution established in [BW18b, Ko22]. The bar involution ψ on the quantum supergroup $\mathbf{U}$ and $\psi_\imath$ on $\mathbf{U}^\imath$ is intertwined by Υ such that $\psi_\imath(x)\Upsilon = \Upsilon\psi(x)$, for all $x \in \mathbf{U}^\imath$.

Finally, following the construction presented in [BW18b], we formulate the K-matrix $\mathcal{T}$. In Proposition 8.3.6, we demonstrate that $\mathcal{T}$ induces an $\mathbf{U}^\imath$-isomorphism on $\mathbb{W}$ and compute its action on $\mathbb{W}$, which coincides with the H_0-action. Consequently, the H_0-action on $\mathbb{W}^{\otimes d}$ is realized by $\mathcal{T} \otimes 1^{\otimes d-1}$.

$\imath$Schur duality of type AI-II

Consider the following Satake diagram I:



where $I_{\bar{1}} = \{m\}$, $I_\bullet = \{m+2a-1 \mid 1 \leq a \leq n\}$ and $I_\circ = I \backslash I_\bullet$. In the case $n=0$, we obtain a Satake diagram of type AI; when $m=0$, we obtain a Satake diagram of type AII.

A theory of quantum supersymmetric pairs $(\mathbf{U}, \mathbf{U}^\imath)$ associated with super Satake diagrams was developed in [SW24] for most of the Lie superalgebras of basic types. Here, we refer to [SW24] for details regarding the construction of quantum supersymmetric pairs associated with this specific (super) Satake diagram and their fundamental properties as outlined below.

Theorem J (Proposition 9.1.2). Let $(\mathbf{U}, \mathbf{U}^\imath)$ be the quantum supersymmetric pair of type AI-II. Then we have

1. $\mathbf{U}^\imath$ is a right coideal subalgebra of $\mathbf{U}$.

2. There exists a quantum Iwasawa decomposition of $\mathbf{U}$ with respect to $\mathbf{U}^\imath$.

3. There exists a unique quasi K-matrix for $(\mathbf{U}, \mathbf{U}^\imath)$.

The classical limit of $\mathbf{U}^\imath$ in this case goes back to the ortho-symplectic Lie superalgebra, which forms a Schur type duality together with the Brauer algebra.

Let $\mathbb{V}$ denote the natural representation of $\mathbf{U}$.

Theorem K (Theorem 9.3.1). With a suitable parameter, the actions of $\mathbf{U}^\imath$ and the q-Brauer algebra $\mathfrak{B}_d(q, q^{m-2n})$ on $\mathbb{V}^{\otimes d}$ commute with each other. Moreover, when $\mathfrak{B}_d(q, q^{m-2n})$ is semisimple, they form a double centralizer property.

This duality can be viewed as a common super generalization of the $\imath$Schur dualities of type AI and AII in Part II.

1.6 Notations

We list the notations which are often used throughout the dissertation.

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}$–sets of non-negative integers, integers, rational and complex numbers

- $\triangleright$ $(Y, X, \langle\cdot,\cdot\rangle, \cdots)$–root datum of finite type $(I, \cdot)$
- $\triangleright$ $(\cdot,\cdot)$–symmetric bilinear form on $\mathbb{Z}[I]$
- $\triangleright$ (a_{ij})–Cartan matrix
- $\triangleright$ $\mathfrak{S}_n$–Symmetric group on n letters
- $\triangleright$ W_d–type B Coxeter group on d-letters
- $\triangleright$ $\mathscr{H}_{\mathfrak{S}_n}$–Hecke algebra of type A
- $\triangleright$ $\mathscr{H}_{B_d}$–Hecke algebra of type B
- $\triangleright$ $\mathbb{M}_f$–quasi-permutation module
- $\triangleright$ W_f–fixed point subgroup of weight f in W_d
- $\triangleright$ ${}^fW$–the set of minimal length right coset representative of W_f in W_d
- $\triangleright$ $(I = I_\bullet \cup I_\circ, \tau)$– admissible pairs (aka Satake diagrams)
- $\triangleright$ T_i–Braid group operators
- $\triangleright$ $W_\bullet$–Weyl group associated to the Levi subalgebra corresponding to $I_\bullet$
- $\triangleright$ $w_\bullet$–longest element in $W_\bullet$
- $\triangleright$ $\mathbf{U} = \mathbf{U}_q(\mathfrak{g})$–quantum group
- $\triangleright$ Δ, ϵ, S–comultiplication, counit, antipode
- $\triangleright$ $\mathbb{V}, \mathbb{W}$–natural representations of $\mathbf{U}$
- $\triangleright$ $\mathbf{U}^\imath$–$\imath$quantum group
- $\triangleright$ ς_i, κ_i–parameters of $\mathbf{U}^\imath$

- ▷ Υ–quasi K-matrix

- ▷ ψ–a bar involution on $\mathbf{U}$

- ▷ $\psi_\imath$–a bar involution on $\mathbf{U}^\imath$; see Lemma 4.2.1 and Corollary 8.2.19

- ▷ $\Theta^\imath$–quasi R-matrix associated to $(\mathbf{U}, \mathbf{U}^\imath)$; see (4.21)

- ▷ $D_n(N)$–Brauer algebra

- ▷ $\mathfrak{B}_n(q, z)$–q-Brauer algebra

- ▷ I_n–set of basis Brauer diagrams

- ▷ $I_{n,k}$–set of basis Brauer diagrams with exactly k pairs of horizontal edges

- ▷ $B_k, B_k^*, B_{k,n}, B_{k,n}^*$–specific subsets of I_n

- ▷ $\mathbf{U}^\imath(\mathfrak{so}_m)$–$\imath$quantum group of type AI

- ▷ $\mathbf{U}^\imath(\mathfrak{sp}_{2m})$–$\imath$quantum group of type AII

- ▷ $p(\cdot)$–parity function

- ▷ ϱ–parity operator of the quantum supergroup

- ▷ ○, ●, ◐ and $\bigotimes$–white even roots, black even roots, black odd roots and white odd roots

- ▷ ${}_ir, r_i$–skew derivations; see (8.7)

Chapter 2

Preliminaries

In this chapter, we set up notations for quantum groups, and quantum symmetric pairs in the finite dimensional setting following [BW18b].

2.1 Quantum groups

Let $(Y, X, \langle\cdot,\cdot\rangle, \cdots)$ be a root datum of finite type $(I, \cdot)$; cf. [Lus93, §2.2]. We have a symmetric bilinear form (ν, ν') on $\mathbb{Z}[I]$. We also have an embedding $I \subset X$ $(i \mapsto i')$, an embedding $I \subset Y$ $(i \mapsto i)$, and a perfect pairing $\langle\cdot,\cdot\rangle : Y \times X \to \mathbb{Z}$ such that $\langle i, j'\rangle = \frac{2(i,j)}{(i,i)}$ for $i, j \in I$. The matrix $(\langle i, j'\rangle) = (a_{ij})$ is the corresponding Cartan matrix. The Weyl group W is generated by the simple reflections $s_i : \mathbb{Z}[I] \to \mathbb{Z}[I]$, for $i \in I$.

Let q be an indeterminate and $\mathbb{Q}(q)$ be the field of rational functions in q with coefficients in $\mathbb{Q}$, the field of rational numbers. For any $i \in I$, we set $q_i = q^{\frac{(i,i)}{2}}$. For $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, we define

$$[a]_i = \frac{q_i^a - q_i^{-a}}{q_i - q_i^{-1}}, \quad [b]_i! = \prod_{h=1}^{b} [h]_i, \quad \begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a]_i!}{[b]_i![a-b]_i!}.$$

When $q_i = q$, we often omit the lower script i.

We denote by $\mathbf{U}$ the associated quantum group. By definition, $\mathbf{U}$ is the associative algebra over $\mathbb{Q}(q)$ with generators E_i, F_i for $i \in I$ and K_μ for $\mu \in Y$, subject to the following relations:

$$K_0 = 1,\ K_\mu K'_\mu = K_{\mu+\mu'} \qquad \text{for all } \mu, \mu' \in Y,$$

$$K_\mu E_j = q^{\langle \mu, j' \rangle} E_j K_\mu, \quad K_\mu F_j = q^{-\langle \mu, j' \rangle} F_j K_\mu,$$

$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_j E_i^{1-a_{ij}-s} E_j E_i^s = 0 \qquad \text{for } i \neq j,$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_j F_i^{1-a_{ij}-s} F_j F_i^s = 0 \qquad \text{for } i \neq j.$$

Let $\mathbf{U}^+, \mathbf{U}^0$ and $\mathbf{U}^-$ be the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}$ generated by E_i $(i \in I)$, K_μ, $(\mu \in Y)$ and F_i, $(i \in I)$ respectively. Then $\mathbf{U}$ admits the triangular decomposition $\mathbf{U} = \mathbf{U}^+\mathbf{U}^0\mathbf{U}^-$.

It is well known that $\mathbf{U}$ is a Hopf algebra with the comultiplication Δ as follows:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$

$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

$$\Delta(K_\mu) = K_\mu \otimes K_\mu.$$

We also recall the braid group action $T_i = T''_{i,+1} : \mathbf{U} \to \mathbf{U}$ and its inverse from [Lus93, 5.2.1], whose the action on $\mathbf{U}^+$ is given as follows: for $i \neq j \in I$,

$$\begin{aligned} T_i(E_i) &= -F_i K_i, \qquad T_i(E_j) = \sum_{r+s=-a_{ij}} (-1)^r q_i^{-r} E_i^{(s)} E_j E_i^{(r)}; \\ T_i^{-1}(E_i) &= -K_i^{-1} F_i, \qquad T_i^{-1}(E_j) = \sum_{r+s=-a_{ij}} (-1)^r q_i^{-r} E_i^{(r)} E_j E_i^{(s)}. \end{aligned} \tag{2.1}$$

For any Weyl group element w, an automorphism T_w of $\mathbf{U}$ is defined via a reduced expression of w. This applies in particular to the longest element in the Weyl group.

As an extension of a bar involution on $\mathbb{Q}(q)$ such that $\overline{q} = q^{-1}$, there exists a bar involution ψ on the algebra $\mathbf{U}$ given by $\psi(q) = q^{-1}$, $\psi(E_i) = E_i$, $\psi(F_i) = F_i$, $\psi(K_\mu) = K_{-\mu}$.

2.2 Quantum symmetric pairs

Let $(Y, X, \langle \cdot, \cdot \rangle, \cdots)$ be a root datum of finite type $(I, \cdot)$. A permutation τ of the set I is an involution of the Cartan datum $(I, \cdot)$ if $\tau^2 = id$ and $(\tau(i), \tau(j)) = (i, j)$ for all $i, j \in I$. We further assume that τ extends to an involution on both X and Y such that the perfect pairing is invariant under the involution τ.

Given a subset $I_\bullet \subset I$, let $W_{I_\bullet}$ denote the parabolic subgroup of W generated by simple reflections s_i with $i \in I_\bullet$. Let $w_\bullet$ denote the longest element in $W_{\mathbb{I}_\bullet}$. Let $R^\vee_\bullet$ denote the set of coroots associated to the simple coroots $I_\bullet \hookrightarrow Y$, and let $R_\bullet$ denote the set of roots associated to the simple roots $I_\bullet \hookrightarrow X$. Let $\rho^\vee_\bullet$ (resp. $\rho_\bullet$) denote the half sum of all positive coroots (resp. roots) in the set $R^\vee_\bullet$ (resp. $R_\bullet$). We shall write $I_\circ = I \backslash I_\bullet$.

An admissible pair $(I_\bullet, \tau)$ (cf. [Ko14]) consists of a partition $I = I_\circ \cup I_\bullet$ and an involution τ of $(I, \cdot)$ (where $\tau = id$ is allowed) such that

1. $\tau(I_\bullet) = I_\bullet$;

2. $-w_\bullet \circ \tau = id$ on $I_\bullet$;

3. If $j \in I_\circ$ and $\tau(j) = j$, then $\langle \rho^\vee_\bullet, j' \rangle \in \mathbb{Z}$.

We define

$$\begin{aligned} X_\imath &= X \big/ \{\mu + w_\bullet \tau(\mu) \mid \mu \in X\}, \\ Y^\imath &= \{\nu - w_\bullet \tau(\nu) \mid \nu \in Y\}. \end{aligned} \tag{2.2}$$

We call an element in $X_\imath$ an $\imath$-weight and $X_\imath$ the $\imath$-weight lattice. Also define $\theta = -w_\bullet \circ \tau$ and

$$I_{ns} := \{i \in I_\circ \mid \tau(i) = i,\ \langle i, j' \rangle = 0 \text{ for all } j \in I_\bullet\}.$$

According to [Ko14], the admissible pairs of finite type are in bijection with the Satake diagrams [Ara62] arising from classification of real simple Lie algebras. We refer to [BW18b, Table 4] for a complete list of Satake diagrams.

The $\imath$quantum group $\mathbf{U}^\imath$ associated to the Satake diagram $(I_\circ \cup I_\bullet, \tau)$ with parameters $\varsigma_i, \in \mathbb{Q}(q), \kappa_i \in \mathbb{Q}(q)$ is the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}$ generated by the following elements:

$$\begin{aligned} F_i + \varsigma_i T_{w_\bullet}(E_{\tau i}) K_i^{-1} + \kappa_i K_i^{-1},\ (i \in I_\circ), \\ K_\mu\ (\mu \in Y^\imath),\ F_i, E_i,\ (i \in I_\bullet). \end{aligned}$$

The parameters are required to satisfy the following conditions:

$$\begin{aligned} &\kappa_i = 0 \text{ unless } i \in I_{ns} \text{ and } \langle i, j' \rangle \in 2\mathbb{Z} \text{ for all } j \in I_{ns} \backslash \{i\}, \\ &\varsigma_i = \varsigma_{\tau i} \text{ if } (i, \theta(i)) = 0. \end{aligned}$$

The theory of quantum symmetric pairs $(\mathbf{U}, \mathbf{U}^\imath)$, as developed by Letzter in [Let99, Let02], offers a natural quantization of these symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\theta)$. Kolb further extended Letzter's work to symmetric pairs related to symmetrizable Kac-Moody Lie algebras; see [Ko14].

2.3 Quantum supergroups of type A

In Part III we will be mainly interested in Lie superalgebras and their quantum analogues, especially in type A. Hence we We adopt basic notations from [CW12] here.

The general linear Lie superalgebra

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace such that $End(V)$ is an associative superalgebra. Then $End(V)$, equipped with the supercommutator, forms a Lie superalgebra, called the general linear Lie superalgebra and is denoted by $\mathfrak{gl}(V)$ or $\mathfrak{gl}(m|n)$ where $dim V_{\bar{0}} = m,\ dim V_{\bar{1}} = n$.

Choose bases for $V_{\bar{0}}$ and $V_{\bar{1}}$ such that they combine to a homogeneous basis of V. We will make it a convention to parameterize such a basis by the set

$$I(m|n) = \{\overline{1}, \ldots, \overline{m}, \underline{1}, \ldots, \underline{n}\} \tag{2.3}$$

with total order

$$\overline{1} < \cdots < \overline{m} < 0 < \underline{1} < \cdots \underline{n}.$$

Here 0 is inserted for convention. With such an ordered basis, $\mathfrak{gl}(m|n)$ can be realized as $(m+n) \times (m+n)$ complex matrices of the block form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.4}$$

where a, b, c and d are respectively $m \times m, m \times n, n \times m$ and $n \times n$ matrices.

The even subalgebra $\mathfrak{g}_{\bar{0}}$ consists of matrices of the form (2.4) with $b = c = 0$, while the odd subspace $\mathfrak{g}_{\bar{1}}$ consists of those with $a = d = 0$. For each element g, we define the supertrace to be

$$str(g) = tr(a) - tr(d).$$

The supertrace str on the general linear Lie superalgebra gives rise to a non-degenerate supersymmetric bilinear form

$$(\cdot, \cdot) : \mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n) \to \mathbb{C}, \quad (x, y) = str(xy).$$

Root system of $\mathfrak{gl}(m|n)$

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathfrak{h}$ be the Cartan subalgebra of diagonal matrices.

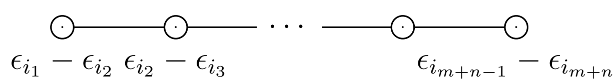
Restricting the supertrace to the Cartan subalgebra $\mathfrak{h}$, we obtain a non-degenerate symmetric bilinear form on it. Denote by $\{\epsilon_a\}_{a\in I(m|n)}$ the basis of $\mathfrak{h}^*$ dual to the set of standard matrices $\{E_{a,a}\}_{a\in I(m|n)}$. Its root system $\Phi = \Phi_{\bar{0}} \oplus \Phi_{\bar{1}}$ is given by

$$\begin{aligned} \Phi_{\bar{0}} =& \{\epsilon_a - \epsilon_b \mid a \neq b \in I(m|n), a, b > 0 \text{ or } a, b < 0\}, \\ \Phi_{\bar{1}} =& \{\pm(\epsilon_a - \epsilon_b) \mid a < 0 < b\}. \end{aligned} \tag{2.5}$$

A fundamental system of $\mathfrak{gl}(m|n)$ consists of $m + n - 1$ roots

$$\epsilon_{i_1} - \epsilon_{i_2}, \ldots, \epsilon_{i_{m+n-1}} - \epsilon_{i_{m+n}},$$

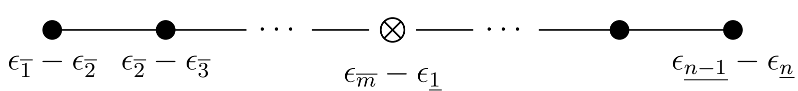
where $\{i_1, \ldots, i_{m+n}\} = I(m|n)$. We denote even simple roots by $\bullet$ and odd simple roots by $\bigotimes$. Then the corresponding Dynkin diagram is of the form



(2.6)

where $\bigodot$ is either $\bullet$ or $\bigotimes$.

Example 2.3.1. *The standard Dynkin diagram is given by*



Given a Dynkin diagram of the form (2.6). Let

$$\Pi = \{\alpha_j = \epsilon_{i_j} - \epsilon_{i_{j+1}} \mid j = 1, \cdots, m + n - 1\}$$

denote the set of simple roots with the index set $I = \{1, \ldots, m+n-1\}$. We see that I is a disjoint union of two subsets $I = I_{\bar{0}} \cup I_{\bar{1}}$ where $I_{\bar{0}}$ (resp. $I_{\bar{1}}$) consists of all even (resp. odd) simple roots. Let p be the parity function on $I(m|n)$ such that

$$p(\epsilon_a) := p(a) = \begin{cases} 0 & \text{if } a > 0,\ a \in I(m|n), \\ 1 & \text{if } a < 0,\ a \in I(m|n). \end{cases} \tag{2.7}$$

We define the weight lattice $P = \oplus_{b \in I(m|n)} \mathbb{Z}\epsilon_b$ while the symmetric bilinear form on P is given by

$$(\epsilon_a, \epsilon_{a'}) = \begin{cases} 1 & \text{if } a = a' < 0, \\ -1 & \text{if } a = a' > 0, \\ 0 & \text{else.} \end{cases} \tag{2.8}$$

Then the parity function p extends to a function on P linearly. We also define

$$p(k) := p(\alpha_k) = \begin{cases} 0 & \text{if } k \in I_{\bar{0}}, \\ 1 & \text{if } k \in I_{\bar{1}}. \end{cases}$$

We define the coweight lattice $P^\vee = \oplus_{b \in I(m|n)} \mathbb{Z}\epsilon_b^\vee$ and we have the pairing $\langle \cdot, \cdot \rangle : P^\vee \times P \to \mathbb{Z}$ with $\langle \epsilon_a^\vee, \epsilon_b \rangle = \delta_{a,b}$. Then $\Pi^\vee = \{h_j \mid j \in I\}$, the set of simple coroots, is given by

$$h_j = \epsilon_{i_j}^\vee - (-1)^{p(j)} \epsilon_{i_{j+1}}^\vee. \tag{2.9}$$

The generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ associated with $\mathfrak{g}$ is defined by $a_{ij} = \langle h_j, \alpha_i \rangle$. We observe that A is symmetrizable, meaning that there exist non-zero integers ℓ_j satisfying

$$\ell_j \langle h_j, \lambda \rangle = (\alpha_j, \lambda) \quad \text{for any } \lambda \in P. \tag{2.10}$$

When $p(j) = 0$, we see that $\ell_j = \frac{(\alpha_j, \alpha_j)}{2}$.

Quantum supergroup of type A

Following [Ya94], we define a quantum supergroup associated to any fixed Dynkin diagram of the form (2.6).

It will be convenient for us to introduce the following notation. We will say $i, j \in I$ are *connected* if $i = j \pm 1$ and write $i \sim j$. Likewise, we say *not connected* if $i \neq j, j \pm 1$ and write $i \nsim j$.

Let $K_i = q^{\ell_i h_i}$, we recall the definition of $\mathbf{U}_q(\mathfrak{gl}(m|n))$ to be the unital associative algebra over $\mathbb{Q}(q)$ with generators q^h $(h \in P^\vee), E_i, F_i$ $(i \in I)$ which satisfy the following defining relations:

$$
\begin{aligned}
&(R1)\ q^h = 1, \quad \text{for } h = 0,\\
&(R2)\ q^{h_1} q^{h_2} = q^{h_1 + h_2},\\
&(R3)\ q^h E_j = q^{\langle h, \alpha_j \rangle} E_j q^h \quad \text{for } j \in I,\\
&(R4)\ q^h F_j = q^{-\langle h, \alpha_j \rangle} F_j q^h, \quad \text{for } j \in I,\\
&(R5)\ [E_j, F_k] = E_j F_k - (-1)^{p(j)p(k)} F_k E_j = \delta_{j,k} \frac{K_j - K_j^{-1}}{q^{\ell_j} - q^{-\ell_j}}, \quad \text{for } j, k \in I,\\
&(R6)\ E_j^2 = F_j^2 = 0, \quad \text{for } j \in I_{\bar{1}},\\
&(R7)\ E_j E_k = (-1)^{p(j)p(k)} E_k E_j, \quad F_j F_k = (-1)^{p(j)p(k)} F_k F_j, \text{ for } j \nsim k,\\
&(R8)\ E_j^2 E_k - [2] E_j E_k E_j + E_k E_j^2 = 0, \quad \text{for } j \sim k, p(j) = 0,\\
&(R9)\ F_j^2 F_k - [2] F_j F_k F_j + F_k F_j^2 = 0, \quad \text{for } j \sim k, p(j) = 0,\\
&(R10)\ S_{p(k),p(\ell)}(E_k, E_j, E_\ell) = 0, \quad \text{for } k \sim j \sim \ell, k < \ell, p(j) = 1,\\
&(R11)\ S_{p(k),p(\ell)}(F_k, F_j, F_\ell) = 0, \quad \text{for } k \sim j \sim \ell, k < \ell, p(j) = 1.
\end{aligned}
\tag{2.11}
$$

where $S_{t_1,t_2}(x_1, x_2, x_3) \in \mathbb{Q}(q)\langle x_1, x_2, x_3 \rangle$ is the polynomial in three non-commuting variables for $t_1, t_2 \in \{\overline{0}, \overline{1}\}$ given by

$$\begin{aligned} S_{t_1,t_2}(x_1,x_2,x_3) &= [2]x_2x_3x_1x_2 - [((-1)^{t_1}x_2x_3x_2x_1 + (-1)^{t_1+t_1t_2}x_1x_2x_3x_2) \\ &+ ((-1)^{t_1t_2+t_2}x_2x_1x_2x_3 + (-1)^{t_2}x_3x_2x_1x_2)]. \end{aligned} \tag{2.12}$$

Moreover, let $q_j := q^{\ell_j}$, we define maps $\sigma, \wp, \bar{\cdot}$ on $\mathbf{U}_q(\mathfrak{gl}(m|n))$ satisfying:

$$\begin{aligned} &\sigma(E_j) = E_j, \quad \sigma(F_j) = F_j, \quad \sigma(K_j) = (-1)^{p(j)}K_j^{-1}, \quad \sigma(xy) = \sigma(y)\sigma(x), \\ &\wp(E_j) = q_jK_jF_j, \quad \wp(F_j) = q_j^{-1}E_jK_j^{-1}, \quad \wp(K_j) = K_j, \quad \wp(xy) = \wp(y)\wp(x), \\ &\overline{E_j} = E_j, \quad \overline{F_j} = F_j, \quad \overline{K_j} = K_j^{-1}, \quad \overline{q} = q^{-1}, \quad \overline{xy} = \overline{x}\cdot\overline{y}. \end{aligned} \tag{2.13}$$

In general, $\mathbf{U}_q(\mathfrak{gl}(m|n))$ is a Hopf superalgebra (cf. [C16, Lemma 2.1]) but not a Hopf algebra. We define an involutive operator ϱ of parity 0 on it by

$$\varrho(q^h) = q^h, \quad \varrho(E_j) = (-1)^{p(j)}E_j \text{ and } \varrho(F_j) = (-1)^{p(j)}F_j, \quad \forall j \in I. \tag{2.14}$$

Let

$$\mathbf{U} = \mathbf{U}_q(\mathfrak{gl}(m|n)) \oplus \mathbf{U}_q(\mathfrak{gl}(m|n))\varrho.$$

Then $\mathbf{U}$ is an algebra with the additional multiplication law given by

$$\varrho^2 = 1, \quad \varrho^{-1}x\varrho = \varrho(x) \text{ for any } x \in \mathbf{U}_q(\mathfrak{gl}(m|n)). \tag{2.15}$$

As established in [Ya94], $\mathbf{U}$ is a Hopf algebra whose comultiplication Δ, counit ϵ, antipode S

are given by

$$
\begin{aligned}
&\Delta(q^h) = q^h \otimes q^h \quad \text{ for } h \in P^\vee,\\
&\Delta(E_j) = E_j \otimes 1 + \varrho^{p(j)} K_j \otimes E_j \quad \text{ for } j \in I,\\
&\Delta(F_j) = F_j \otimes K_j^{-1} + \varrho^{p(j)} \otimes F_j \quad \text{ for } j \in I,\\
&\Delta(\varrho) = \varrho \otimes \varrho, \qquad\qquad (2.16)\\
&\epsilon(\varrho) = \epsilon(q^h) = 1, \text{ for } h \in P^\vee, \quad \epsilon(E_j) = \epsilon(F_j) = 0, \quad \text{ for } j \in I,\\
&S(\varrho) = \varrho, \quad S(q^h) = q^{-h} \quad \text{ for } h \in P^\vee,\\
&S(F_j) = -\varrho^{p(j)} F_j K_j, \quad S(E_j) = -\varrho^{p(j)} K_j^{-1} E_j \quad \text{ for } j \in I.
\end{aligned}
$$

We naturally extend the maps in (2.13) from $\mathbf{U}_q(\mathfrak{gl}(m|n))$ to $\mathbf{U}$ by setting

$$\sigma(\varrho) = \wp(\varrho) = \overline{\varrho} = \varrho.$$

Remark 2.3.2. The choice of the comultiplication we made here is different from [Mi06]. When $\mathfrak{g}$ is a Lie algebra, our Δ is compatible with the comultipication in [Lus93].

As in [Lus93], the multiplication map gives a triangular decomposition of $\mathbf{U}$:

$$\mathbf{U} \cong \mathbf{U}^+ \otimes \mathbf{U}^0 \otimes \mathbf{U}^-. \qquad (2.17)$$

where $\mathbf{U}^+$ (resp. $\mathbf{U}^-$) denotes the subalgebra of $\mathbf{U}$ generated by E_j (resp. F_j), $j \in I$ and $\mathbf{U}^0$ denotes the subalgebra of $\mathbf{U}$ generated by $\{q^\mu, \varrho \mid j \in I, \mu \in P^\vee\}$.

Part I

$\imath$Schur duality and Kazhdan-Lusztig bases

Chapter 3

Quasi-permutation modules

In this chapter, we extend the seminal works of Kazhdan-Lusztig and Deodhar to establish bar involutions and canonical bases, termed quasi-parabolic KL bases, on quasi-permutation modules over the type B Hecke algebra. These bases are characterized by their parameterization through cosets of reflection subgroups of the Weyl group of type B, which may not necessarily be parabolic. Moreover, both type A and type B (parabolic) KL bases emerge as special cases of our quasi-parabolic KL bases.

3.1 Modules over Hecke algebra of type B

In this section we introduce the Hecke algebra $\mathscr{H}_{B_d}$ of type B and its action on a tensor space. This leads to quasi-permutation modules of $\mathscr{H}_{B_d}$.

3.1.1 Weyl group and Hecke algebra of type B

The Weyl group $W = W_d$ of type B_d is generated by s_i, for $0 \leq i \leq d-1$, subject to the Coxeter relations: $s_i^2 = 1, (s_i s_{i+1})^3 = 1, (s_0 s_1)^4 = 1$, and $(s_i s_j)^2 = 1$ $(|i-j| > 1)$. The symmetric group $\mathfrak{S}_d$ is a subgroup of W_d generated by s_i, for $1 \leq i \leq d-1$. The length function $l : W_d \to \mathbb{N}$ is defined such that $l(\sigma) = k$ if σ has a reduced expression $\sigma = s_{i_1} \cdots s_{i_k}$.

For a real number $x \in \mathbb{R}$ and $m \in \mathbb{N}$, we denote $[x, x+m] = \{x, x+1, \dots, x+m\}$. For $a \in \mathbb{Z}_{\geq 1}$, we denote by

$$\mathbb{I}_a = \left[\frac{1-a}{2}, \frac{a-1}{2}\right].$$

For $r, m \in \mathbb{N}$ (not both zero), we introduce a new notation for $\mathbb{I}_{2r+m}$ to indicate a fixed set partition:

$$\mathbb{I}_{r|m|r} := \mathbb{I}_{2r+m}, \qquad \mathbb{I}_{r|m|r} = \mathbb{I}_\circ^- \cup \mathbb{I}_\bullet \cup \mathbb{I}_\circ^+ \tag{3.1}$$

where the subsets

$$\mathbb{I}_\circ^+ = \left[\frac{m+1}{2}, r + \frac{m-1}{2}\right]. \qquad \mathbb{I}_\bullet = \left[\frac{1-m}{2}, \frac{m-1}{2}\right], \qquad \mathbb{I}_\circ^- = -\mathbb{I}_\circ^+, \tag{3.2}$$

have cardinalities r, m, r, respectively.

We view $f \in \mathbb{I}^d_{r|m|r}$ as a map $f : \{1, \dots, d\} \to \mathbb{I}_{r|m|r}$, and identify $f = (f(1), \dots, f(d))$, with $f(i) \in \mathbb{I}_{r|m|r}$. We define a right action of the Weyl group W_d on $\mathbb{I}^d_{r|m|r}$ such that, for $f \in \mathbb{I}^d_{r|m|r}$ and $0 \leq j \leq d-1$,

$$f^{s_j} = f \cdot s_j = \begin{cases} (\cdots, f(j+1), f(j), \cdots), & \text{if } j > 0; \\ (-f(1), f(2), \cdots, f(d)), & \text{if } j = 0,\ f(1) \in \mathbb{I}_\circ^- \cup \mathbb{I}_\circ^+; \\ (f(1), f(2), \cdots, f(d)), & \text{if } j = 0,\ f(1) \in \mathbb{I}_\bullet. \end{cases} \tag{3.3}$$

The only nontrivial relation $(s_0 s_1)^4 = 1$ can be verified by case-by-case inspection depending on whether or not $f(1), f(2) \in \mathbb{I}_\bullet$. We sometimes write

$$f^\sigma = f \cdot \sigma = (f(\sigma(1)), \cdots, f(\sigma(d))),$$

where it is understood that

$$f(\sigma(i)) = \begin{cases} f(\sigma(i)), & \text{if } \sigma(i) > 0; \\ f(-\sigma(i)), & \text{if } \sigma(i) < 0, f(-\sigma(i)) \in \mathbb{I}_{\bullet}; \\ -f(-\sigma(i)), & \text{if } \sigma(i) < 0, f(-\sigma(i)) \in \mathbb{I}_{\circ}^{-} \cup \mathbb{I}_{\circ}^{+}. \end{cases}$$

Let p, q be two indeterminates. We denote $q_i = q$ for $1 \le i \le d-1$ and $q_0 = p$. The Iwahori-Hecke algebra of type B, denoted by $\mathscr{H}_{B_d}$, is a $\mathbb{Q}(p,q)$-algebra generated by $H_0, H_1, \cdots, H_{d-1}$, subject to the following relations:

$$\begin{aligned} &(H_i - q_i)(H_i + q_i^{-1}) = 0, && \text{for } i \ge 0; \\ &H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1}, && \text{for } i \ge 1; \\ &H_i H_j = H_j H_i, && \text{for } |i-j| > 1; \\ &H_0 H_1 H_0 H_1 = H_1 H_0 H_1 H_0. && \end{aligned}$$

The subalgebra generated by H_i, for $1 \le i \le d-1$, can be identified with Hecke algebra $\mathscr{H}_{\mathfrak{S}_d}$ associated to the symmetric group $\mathfrak{S}_d$. If $\sigma \in W_d$ has a reduced expression $\sigma = s_{i_1} \cdots s_{i_k}$, we denote $H_\sigma = H_{i_1} \cdots H_{i_k}$. It is well known that $\{H_\sigma \mid \sigma \in W_d\}$ form a basis for $\mathscr{H}_{B_d}$, and $\{H_\sigma \mid \sigma \in \mathfrak{S}_d\}$ form a basis for $\mathscr{H}_{\mathfrak{S}_d}$.

3.1.2 A tensor module of $\mathscr{H}_{B_d}$

Consider the $\mathbb{Q}(p,q)$-vector space

$$\mathbb{V} = \bigoplus_{a \in \mathbb{I}_{r|m|r}} \mathbb{Q}(p,q) v_a. \tag{3.4}$$

Given $f = (f(1), \dots, f(d)) \in \mathbb{I}^d_{r|m|r}$, we denote

$$M_f = v_{f(1)} \otimes v_{f(2)} \otimes \dots \otimes v_{f(d)}.$$

We shall call f a weight and $\{M_f \mid f \in \mathbb{I}^d_{r|m|r}\}$ the standard basis for $\mathbb{V}^{\otimes d}$.

In cases $|\mathbb{I}_\bullet| = 0$ or 1 (i.e., $m = 0$ or 1), the following lemma reduces to [BW18a, (6.8)] or [BWW18, (4.4)] in different notations.

Lemma 3.1.1. *There is a right action of the Hecke algebra $\mathscr{H}_{B_d}$ on $\mathbb{V}^{\otimes d}$ as follows:*

$$M_f \cdot H_i = \begin{cases} M_{f\cdot s_i} + (q - q^{-1})M_f, & \text{if } f(i) < f(i+1),\ i > 0; \\ M_{f\cdot s_i}, & \text{if } f(i) > f(i+1),\ i > 0; \\ qM_f, & \text{if } f(i) = f(i+1),\ i > 0; \\ M_{f\cdot s_i} + (p - p^{-1})M_f, & \text{if } f(1) \in \mathbb{I}^+_\circ,\ i = 0; \\ M_{f\cdot s_i}, & \text{if } f(1) \in \mathbb{I}^-_\circ,\ i = 0, \\ pM_f, & \text{if } f(1) \in \mathbb{I}_\bullet,\ i = 0. \end{cases}$$

Proof. It is a well known result of Jimbo [Jim86] that the first 3 formulas above for H_i with $i > 0$ define a right action of Hecke algebra $\mathscr{H}_{\mathfrak{S}_d}$.

It is clear that $(H_0 - p)(H_0 + p^{-1}) = 0$ and $H_0H_i = H_iH_0$, for $i \geq 2$.

Hence, it remains to verify the braid relation $H_0H_1H_0H_1 = H_1H_0H_1H_0$. To that end, we only need to consider the case $d = 2$ and verify the braid relation when acting on $v_i \otimes v_j$.

If $i, j \in \mathbb{I}_\bullet$, then H_0 acts on the span of $v_i \otimes v_j$ and $v_j \otimes v_i$ as $p \cdot \mathrm{Id}$, and so the braid relation $H_0H_1H_0H_1 = H_1H_0H_1H_0$ trivially holds.

Assume now that at most one of i, j lies in $\mathbb{I}_\bullet$. If we formally regard this possible index in $\mathbb{I}_\bullet$ as 0, then we are basically reduced to the setting of the action of Hecke algebra $\mathscr{H}_{B_d}$ [BW18a, (6.8)] or [BWW18, (4.4)] (except a different partial ordering on $\mathbb{I}^d_{r|m|r}$ was used therein, and q, p here correspond to q^{-1}, p^{-1} therein). In any case, the braid relation can be verified directly case-by-case, and we provide some details below.

For $i < j \in \mathbb{I}_\circ^-$, we have

$$\begin{aligned}(v_i \otimes v_j)H_0H_1H_0H_1 &= v_{-i} \otimes v_{-j} + (q - q^{-1})v_{-j} \otimes v_{-i} \\ &= (v_i \otimes v_j)H_1H_0H_1H_0.\end{aligned}$$

For $i \in \mathbb{I}_\circ^-, j \in \mathbb{I}_\bullet$, we have

$$\begin{aligned}(v_i \otimes v_j)H_0H_1H_0H_1 &= pv_{-i} \otimes v_j + p(q - q^{-1})v_j \otimes v_{-i} \\ &= (v_i \otimes v_j)H_1H_0H_1H_0.\end{aligned}$$

For $i \in \mathbb{I}_\circ^-, j \in \mathbb{I}_\circ^+$ such that $-i > j$, we have

$$\begin{aligned}&(v_i \otimes v_j)H_0H_1H_0H_1 \\ &= v_{-i} \otimes v_{-j} + (q - q^{-1})v_{-j} \otimes v_{-i} + (p - p^{-1})v_{-i} \otimes v_j + (p - p^{-1})(q - q^{-1})v_j \otimes v_{-i} \\ &= (v_i \otimes v_j)H_1H_0H_1H_0.\end{aligned}$$

The remaining cases are similar and skipped. □

3.1.3 Quasi-permutation modules

Recall $\mathbb{I}^d_{r|m|r}$ from (3.1). A weight $f \in \mathbb{I}^d_{r|m|r}$ is called *anti-dominant* if

$$\frac{m-1}{2} \geq f(1) \geq f(2) \geq \cdots \geq f(d). \tag{3.5}$$

Note that $f(j) \in \mathbb{I}_\circ^- \cup \mathbb{I}_\bullet$, for $1 \leq j \leq d$, if f is anti-dominant. We denote

$$\mathbb{I}^{d,-}_{r|m|r} = \{f \in \mathbb{I}^d_{r|m|r} \mid f \text{ is anti-dominant}\}.$$

We can decompose $\mathbb{V}^{\otimes d}$ into a direct sum of cyclic submodules generated by M_f, for anti-dominant weights f, as follows:

$$\mathbb{V}^{\otimes d} = \bigoplus_{f \in \mathbb{I}^{d,-}_{r|m|r}} \mathbb{M}_f, \qquad \text{where } \ \mathbb{M}_f = M_f \mathscr{H}_{B_d}. \tag{3.6}$$

Denote by $\mathcal{O}_f$ the orbit of f under the action of W_d on $\mathbb{I}^d_{r|m|r}$. The following is immediate from the formulas for the action of $\mathscr{H}_{B_d}$ in Lemma 3.1.1.

Lemma 3.1.2. *The right $\mathscr{H}_{B_d}$-module $\mathbb{M}_f$ admits a $\mathbb{Q}(q)$-basis $\{M_g \mid g \in \mathcal{O}_f\}$. (It will be called the* standard basis.*)*

By (3.5), we can suppose that $f \in \mathbb{I}^{d,-}_{r|m|r}$ is of the form

$$f = (\underbrace{a_1, \ldots, a_1}_{m_1}, \ldots, \underbrace{a_k, \ldots, a_k}_{m_k}, \underbrace{a_{k+1}, \ldots, a_{k+1}}_{m_{k+1}}, \ldots, \underbrace{a_l, \ldots, a_l}_{m_l}), \tag{3.7}$$

where $a_1 > \ldots > a_k > a_{k+1} > \ldots > a_l$, $\{a_1, \ldots, a_k\} \subset \mathbb{I}_\bullet$, $\{a_{k+1}, \ldots, a_l\} \subset \mathbb{I}^-_\circ$, and $m_1 + \ldots + m_l = d$. The stabilizer subgroup of f in W_d is

$$W_f = W_{m_1} \times \ldots \times W_{m_k} \times S_{m_{k+1}} \times \ldots \times S_{m_l}. \tag{3.8}$$

Note the stabilizer subgroup W_f is not a parabolic subgroup of W_d when 2 or more of the integers $m_1, \ldots, m_k$ are positive. (This phenomenon does not occur in the setting of [BW18a, BWW18].) We shall call the summand $\mathbb{M}_f$ in (3.6) *quasi-permutation modules.* Clearly, for $f, f' \in \mathbb{I}^{d,-}_{r|m|r}$, we have

$$\mathbb{M}_f \cong \mathbb{M}_{f'}, \qquad \text{if } \ W_f = W_{f'}.$$

If W_f is not parabolic, $\mathbb{M}_f$ is in general not an induced module as those considered in parabolic Kazhdan-Lusztig theory [De87]; see [So97, LW20].

Remark 3.1.3. The quasi-permutation modules have appeared earlier in different formulations in [DJM98] and [DS00] independently. In our setting it is straightforward to write down the Hecke action and bases for the quasi-permutation modules $\mathbb{M}_f$ starting from $\mathbb{V}^{\otimes d}$, but it takes some nontrivial efforts to achieve this in [DJM98, DS00]. In their approaches, the q-permutation modules are cyclic submodules of the right regular representation of $\mathscr{H}_{B_d}$ with generators constructed by Jucys-Murphy elements. The quasi-permutation modules here are isomorphic to those *loc. cit.*; this follows by comparing the formulas in Lemma 3.1.1 and (3.6) with those in [DJM98, Lemmas 3.9, 3.11].

3.2 Canonical bases on quasi-permutation modules

In this section, the minimal length representatives of the reflection subgroup W_f of W_d are studied. We construct a bar involution on the quasi-permutation modules $\mathbb{M}_f$ which are compatible with the bar involution on $\mathscr{H}_{B_d}$. Then we construct a canonical basis on $\mathbb{M}_f$.

3.2.1 Basic properties of W_d

There is a natural left action of the Weyl group W_d on the set

$$[\pm d] := \{-d, \dots, -2, -1, 1, 2, \dots, d\}.$$

such that

$$\sigma(-i) = -\sigma(i), \qquad \forall \sigma \in W_d,\ i \in [\pm d].$$

In one line notation we write

$$\sigma = [\sigma(1), \dots, \sigma(d)].$$

Let $f \in \mathbb{I}^{d,-}_{r|m|r}$. The stabilizer of f in the symmetric group $\mathfrak{S}_d$ is always a parabolic subgroup generated by some subset $J(f) \subset \{s_1, \dots, s_{d-1}\}$. We continue the notation (3.7) for

$f \in \mathbb{I}^{d,-}_{r|m|r}$. Denote

$$d_\bullet = m_1 + \ldots + m_k, \qquad d_\circ = d - d_\bullet. \tag{3.9}$$

That is, among $f(j)$, for $1 \le j \le d$, the first $d_\bullet$ of them belong to $\mathbb{I}_\bullet$. Denote

$$t_1 = s_0, \qquad t_i = s_{i-1}t_{i-1}s_{i-1}, \qquad \text{for } 1 \le i \le d. \tag{3.10}$$

Then t_i is the swap (sign change) of i and $-i$ while fixing $j \in [\pm d]$ with $j \neq \pm i$.

Lemma 3.2.1. *Let $f \in \mathbb{I}^{d,-}_{r|m|r}$. Then the stabilizer W_f in W_d is generated by*

$$J_f := \{t_i \mid 1 \le i \le d_\bullet\} \cup J(f).$$

Proof. Recall f from (3.7). The lemma follows since elements in W_f are compositions of permutations in $\mathfrak{S}_d$ that fix f and sign changes that fix each $a_j, 1 \le j \le k$. □

For $\sigma \in W_d$, the type B inversion number $\mathrm{inv}_B(\sigma)$ is defined to be (cf. [BB05])

$$\mathrm{inv}_B(\sigma) = \mathrm{inv}(\sigma) + n_B(\sigma), \tag{3.11}$$

where

$$\mathrm{inv}(\sigma) = \#\{(i,j) \mid 1 \le i < j \le d, \sigma(i) > \sigma(j)\}; \tag{3.12}$$

$$n_B(\sigma) = - \sum_{\{1 \le j \le d \mid \sigma(j) < 0\}} \sigma(j). \tag{3.13}$$

For $\sigma \in \mathfrak{S}_d$, $\mathrm{inv}_B(\sigma) = \mathrm{inv}(\sigma)$ coincides with the inversion number of $\mathfrak{S}_d$.

Lemma 3.2.2. *[BB05, Proposition 8.1.1] For any $\sigma \in W_d$, we have $l(\sigma) = inv_B(\sigma)$.*

3.2.2 Minimal length representatives

Let $f \in \mathbb{I}^{d,-}_{r|m|r}$. Recall the stabilizer subgroup W_f (3.8) of W_d is a (not-necessarily parabolic) reflection subgroup in general.

Lemma 3.2.3. *[Lus84, Lemma 1.9] [DS00, Theorem 2.2.5] Every right coset of W_f in the Weyl group W_d has a unique minimal length representative.*

Denote by ${}^fW$ the set of minimal length right coset representatives for W_f in W_d, for $f \in \mathbb{I}^{d,-}_{r|m|r}$. We shall establish a basic property for ${}^fW$.

Lemma 3.2.4. *Let $1 \le i \le d$ and $\sigma \in {}^fW$. If $|\sigma(i)| \le d_\bullet$, then $\sigma(i) > 0$.*

Proof. We prove by contradiction. Suppose this were not true, then there exists $1 \le i_\bullet \le d$ such that $\sigma(i_\bullet) < 0$ and $u_\bullet = |\sigma(i_\bullet)| \le d_\bullet$. By Lemma 3.2.1 we have $t_{u_\bullet} \in W_f$ and thus $t_{u_\bullet}\sigma \in W_f\sigma$. Now by (3.13) we have $n_B(t_{u_\bullet}\sigma) = n_B(\sigma) - u_\bullet$. On the other hand, since there are at most $u_\bullet - 1$ indices less than $u_\bullet$, we have $\mathrm{inv}(w_{u_\bullet}\sigma) \le \mathrm{inv}(\sigma) + u_\bullet - 1$. Hence by the above 2 identities, (3.11) and Lemma 3.2.2, we have

$$\begin{aligned} l(t_{u_\bullet}\sigma) &= \mathrm{inv}(t_{u_\bullet}\sigma) + n_B(t_{u_\bullet}\sigma) \\ &\le \mathrm{inv}(\sigma) + n_B(\sigma) - 1 = l(\sigma) - 1, \end{aligned}$$

which is a contradiction to the minimal length property of σ. □

Example 3.2.5. *If W_f is non-parabolic, the equality $l(ww') = l(w)l(w')$ may fail for $w \in W_f$ and $w' \in {}^fW$. For example, take $W_f = \langle s_0, s_1s_0s_1\rangle \subset W_{B_2}$ and s_1 is the minimal length representative of W_fs_1. Note $(s_1s_0s_1)s_1 = s_1s_0$, but $l(s_1s_0s_1) + l(s_1) = 4 \neq 2 = l(s_1s_0)$.*

The example above indicates [De87, Lemma 2.1(i)-(ii)] may fail for non-parabolic reflection subgroups. The next theorem, which is a generalization of [De87, Lemma 2.1(iii)] to reflection subgroups, is more difficult to establish. It will play a key role in constructing the bar involution and canonical bases for quasi-permutation modules.

Theorem 3.2.6. *Let $\sigma \in {}^fW$, and $0 \leq i \leq d-1$. Then exactly one of the following possibilities occurs:*

(i) $l(\sigma s_i) < l(\sigma)$. In this case, $\sigma s_i \in {}^fW$;

(ii) $l(\sigma s_i) > l(\sigma)$ and $\sigma s_i \in {}^fW$;

(iii) $l(\sigma s_i) > l(\sigma)$ and $\sigma s_i \notin {}^fW$, for $i \neq 0$. In this case, $\sigma s_i = s'\sigma$, for some $s' \in J(f)$;

(iii$_0$) $l(\sigma s_0) > l(\sigma)$ and $\sigma s_0 \notin {}^fW$. In this case, $\sigma s_0 = t\sigma$, for some $t \in J_f \backslash J(f)$.

(More precisely, in case (iii), we have $f(\sigma(i)) = f(\sigma(i+1))$ and $s' = (|\sigma(i)|, |\sigma(i+1)|)$; in case (iii$_0$), $\sigma(1) > 0$ and $t = t_{\sigma(1)}$.)

Proof. We shall compare $\sigma \in {}^fW$ with σs_i. Our argument below uses the action of W_d on $\mathbb{V}^{\otimes d}$ crucially. We separate the proof into 2 cases depending on whether or not $i = 0$.

(1) Assume $i = 0$. We separate into 3 subcases (i$_0$)-(iii$_0$) below by the range of $f^\sigma(1)$.

(i$_0$) $\underline{f^\sigma(1) \in \mathbb{I}_\circ^+ \Rightarrow \text{Case (i) for } i = 0}$.

In this case, we have $\sigma(1) < 0$ since $f(\sigma(1)) = f^\sigma(1) \in \mathbb{I}_\circ^+$ while $f(j) \notin \mathbb{I}_\circ^+$ (for $1 \leq j \leq d$) thanks to f being anti-dominant.

Claim 1. $l(\sigma s_0) = l(\sigma) - 1$.

Indeed, by Lemma 3.2.2 it suffices to show that $\mathrm{inv}_B(\sigma s_0) < \mathrm{inv}_B(\sigma)$. Note that $\sigma s_0(j) = \sigma(j)$, for $2 \leq j \leq d$, and $\sigma s_0(1) > 0 > \sigma(1)$. By (3.13) we have $n_B(\sigma s_0) = n_B(\sigma) + \sigma(1)$. On the other hand, we have $\mathrm{inv}(\sigma s_0) \leq \mathrm{inv}(\sigma) - \sigma(1) - 1$ since there are at most $(-\sigma(1) - 1)$ indices smaller than $-\sigma(1)$. Hence by (3.11), $\mathrm{inv}_B(\sigma s_0) \leq \mathrm{inv}_B(\sigma) - 1$, and Claim 1 follows.

It remains to verify that $\sigma s_0 \in {}^fW$. If this were not true, there exists $\tau \in W_f \sigma s_0$ such that $l(\tau) < l(\sigma s_0) = l(\sigma) - 1$. Hence $l(\tau s_0) \leq l(\tau) + 1 < l(\sigma)$; this is a contradiction since $\tau s_0 \in W_f \sigma$ and σ is a minimal length representative of $W_f \sigma$.

(ii$_0$) $\underline{f^\sigma(1) \in \mathbb{I}_\circ^- \Rightarrow \text{Case (ii) for } i = 0}$.

In this case, $f^{\sigma s_0}(1) \in \mathbb{I}_\circ^+$, and $\sigma(1) > 0$, thanks to f being anti-dominant. Arguing as in (i_0) for Claim 1, we have $l(\sigma s_0) = l(\sigma) + 1$. It remains to verify that $\sigma s_0 \in {}^f W$. If this were not true, we choose the minimal length representative $\tau \in W_f \sigma s_0$. Since $\tau \in {}^f W$ and $f^\tau(1) \in \mathbb{I}_\circ^+$, by ($\mathrm{i}_0$) we know that $l(\tau s_0) = l(\tau) - 1 < l(\sigma s_0) - 1 = l(\sigma)$; this is a contradiction since $\tau s_0 \in W_f \sigma$ and σ is a minimal length representative of $W_f \sigma$.

(iii_0) $\underline{f^\sigma(1) \in \mathbb{I}_\bullet \Rightarrow \text{Case } (\mathrm{iii}_0)}$.

Thanks to $f^\sigma(1) \in \mathbb{I}_\bullet$, we obtain $f^\sigma = f^{\sigma s_0}$, that is, $\sigma s_0 \in W_f \sigma$. Then $l(\sigma s_0) > l(\sigma)$ and $\sigma s_0 \notin {}^f W$, since σ is a minimal length representative in $W_f \sigma$. Also, we have $\sigma s_0 \sigma^{-1} = t_{|\sigma(1)|}$, and thus, $\sigma s_0 = t_{|\sigma(1)|}\sigma$; cf. (3.10). Since $f^\sigma(1) \in \mathbb{I}_\bullet$, we have $|\sigma(1)| \leq d_\bullet$; cf. (3.9). By Lemma 3.2.4, we know that $\sigma(1) > 0$. Hence, $t_{\sigma(1)} \in J_f \backslash J(f)$.

(2) Assume $i > 0$. We compare $\sigma \in {}^f W$ with σs_i. By using inversion numbers, we see that $l(\sigma s_i) > l(\sigma)$ if and only if $f^\sigma(i) \geq f^\sigma(i+1)$. We separate into 3 subcases (i)-(iii) below depending on whether $f^\sigma(i) - f^\sigma(i+1)$ is negative, positive or zero.

(i) $\underline{(f^\sigma(i) < f^\sigma(i+1)) \Rightarrow \text{Case (i) for } i > 0}$.

In this case, $l(\sigma s_i) < l(\sigma)$. It remains to verify that $\sigma s_i \in {}^f W$. If this were not true, then there exists $\tau \in W_f \sigma s_i$ such that $l(\tau) < l(\sigma s_i) = l(\sigma) - 1$. Thus $l(\tau s_i) \leq l(\tau) + 1 < l(\sigma)$; this is a contradiction since σ has the minimal length and $\tau s_i \in W_f \sigma$.

(ii) $\underline{(f^\sigma(i) > f^\sigma(i+1)) \Rightarrow \text{Case (ii) for } i > 0}$.

In this case, $l(\sigma s_i) > l(\sigma)$. Let us verify $\sigma s_i \in {}^f W$. If this were not true, choose the minimal length representative $\tau \in W_f \sigma s_i$. Since $f^\tau(i) < f^\tau(i+1)$, by (i) we have $l(\tau s_i) = l(\tau) - 1 < l(\sigma s_i) - 1 \leq l(\sigma)$, which is again a contradiction.

(iii) $\underline{(f^\sigma(i) = f^\sigma(i+1)) \Rightarrow \text{Case (iii)}}$.

In this case, $f^{\sigma s_i} = f^\sigma$, and $\sigma s_i \in W_f \sigma$. Without loss of generality we assume that $|\sigma(i)| < |\sigma(i+1)|$. It follows from the anti-dominance of f that $\sigma(i)$ and $\sigma(i+1)$ have the same sign if $f^\sigma(i) = f^\sigma(i+1) \in \mathbb{I}_\circ^- \cup \mathbb{I}_\circ^+$; On the other hand, if $f^\sigma(i) = f^\sigma(i+1) \in \mathbb{I}_\bullet$, then

$\sigma(i)$ and $\sigma(i+1)$ have the same + sign by Lemma 3.2.4.

Therefore, we have $f(|\sigma(i)|) = f(|\sigma(i+1)|)$, and thus,

$$\sigma s_i \sigma^{-1} = (|\sigma(i)|, |\sigma(i)|+1), \tag{3.14}$$

that is,

$$\sigma s_i = s_{|\sigma(i)|} s_{|\sigma(i)|+1} \cdots s_{|\sigma(i+1)|-1} \cdots s_{|\sigma(i)|+1} s_{|\sigma(i)|} \sigma \in W_f \sigma. \tag{3.15}$$

Since f is anti-dominant (cf. (3.5)), we must have

$$\{s_{|\sigma(i)|}, s_{|\sigma(i)|+1}, \cdots, s_{|\sigma(i+1)|-1}\} \subset J(f).$$

Claim. We have $|\sigma(i+1)| = |\sigma(i)| + 1$.

Let us prove the Claim. Let $\sigma = s'_1 s'_2 \cdots s'_k$ be a reduced expression. Assume to the contrary that $|\sigma(i+1)| > |\sigma(i)| + 1$. Then we can reduce the length of the RHS of (3.15) by deleting a pair of simple reflections, at least one of which is some s'_i from σ; otherwise, it would contradict the identity (3.14). Now the element in the RHS of (3.15) after the deletion contradicts the minimality of σ as a representative of $W_f\sigma$. Thus the Claim holds.

Hence, setting $s' = (|\sigma(i)|, |\sigma(i+1)|) \in J(f)$, we have $s'\sigma = \sigma s_i$. □

Remark 3.2.7. The conditions in Theorem 3.2.6 have their counterparts in terms of f^σ listed in the proof above, and they are useful in later applications. For instance, for $\sigma \in {}^fW$ and $i > 0$, we have $\sigma s_i \in {}^fW$ if and only if $f^\sigma(i) \neq f^\sigma(i+1)$.

3.2.3 The Hecke modules $\mathbb{M}_f$ revisited

Recall the action of Hecke algebra on $\mathbb{V}^{\otimes d}$ from Lemma 3.1.1 and hence on $\mathbb{M}_f$ from (3.6). Applying Theorem 3.2.6 and its proof, we shall obtain explicit descriptions for the action of the Hecke generators H_i on the standard basis $\{M_{f\cdot\sigma} \mid \sigma \in {}^fW\}$ for $\mathbb{M}_f$, which is independent of the tensor module $\mathbb{V}^{\otimes d}$. Clearly, the length inequalities in Theorem 3.2.6 can be replaced by the Chevalley-Bruhat order $\leq$ on W_d.

Proposition 3.2.8. *Let $f \in \mathbb{I}^{d,-}_{r|m|r}$, $\sigma \in {}^fW$, and $0 \leq i \leq d-1$. Then*

$$M_{f\cdot\sigma}H_i = \begin{cases} M_{f\cdot\sigma s_i} + (q_i - q_i^{-1})M_{f\cdot\sigma}, & \textit{if } \ \sigma s_i < \sigma; \\ M_{f\cdot\sigma s_i}, & \textit{if } \ \sigma s_i > \sigma \textit{ and } \sigma s_i \in {}^fW; \\ qM_{f\cdot\sigma}, & \textit{if } \ i \neq 0, \ \sigma s_i > \sigma \textit{ and } \sigma s_i \notin {}^fW; \\ pM_{f\cdot\sigma}, & \textit{if } \ i = 0, \ \sigma s_0 > \sigma \textit{ and } \sigma s_0 \notin {}^fW. \end{cases}$$

Proof. In this proof we label the four cases in the proposition as (i), (ii), (iii), (iii$_0$), as they exactly correspond to the 4 cases in the same labelings in Theorem 3.2.6.

We first assume $i \neq 0$. Then the cases (i), (ii), (iii) here match with the cases (i), (ii), (iii) in the proof of Theorem 3.2.6 in the same order, which correspond to the 3 conditions $f^\sigma(i) < f^\sigma(i+1)$, $f^\sigma(i) > f^\sigma(i+1)$, and $f^\sigma(i) = f^\sigma(i+1)$ therein, respectively. Hence, the formulas in the proposition (with $i \neq 0$) follow by the first 3 formulas in Lemma 3.1.1.

Now we assume $i = 0$. Then the cases (i), (ii), (iii$_0$) here match with the cases (i$_0$), (ii$_0$), (iii$_0$) in the proof of Theorem 3.2.6 in the same order, which correspond to the 3 conditions $f^\sigma(1) \in \mathbb{I}_\circ^+$, $f^\sigma(1) \in \mathbb{I}_\circ^-$, and $f^\sigma(1) \in \mathbb{I}_\bullet$ therein, respectively. Hence the formulas in the proposition (with $i = 0$) follow by the last 3 formulas in Lemma 3.1.1. □

Remark 3.2.9. The formulas in Proposition 3.2.8 miraculously take the same form as in the parabolic case [De87, So97]. However, in contrast to *loc. cit.* it seems difficult to verify directly these formulas define a representation of $\mathscr{H}_{B_d}$ in such a general reflection subgroup setting. The proof of Theorem 3.2.6 provides us a crucial identification as *posets* between the orbit $f \cdot W_d$ (used in Lemma 3.1.1) and the set of minimal length representatives ${}^fW$ for $W_f \backslash W_d$ (used in Proposition 3.2.8).

3.2.4 The bar involution on $\mathbb{M}_f$

We prepare some lemmas toward the construction of the bar involution on $\mathbb{M}_f$.

Lemma 3.2.10. *For $f \in \mathbb{I}^{d,-}_{r|m|r}$ and $\sigma \in {}^fW$, we have $M_f H_\sigma = M_{f\cdot\sigma}$.*

Proof. We use induction on $l(\sigma)$. The case for $l(\sigma) = 0$ is trivially true. If $l(\sigma) = 1$, then $\sigma = s_i$ for some i. If $i = 0$, we have $f(1) \in \mathbb{I}^-_\circ$, as otherwise we would have $s_0 \in W_f$ (contradicting $\sigma = s_0 \in {}^fW$). Hence, $M_f H_0 = M_{f\cdot s_0}$, by Lemma 3.1.1. If $\sigma = s_i$ for $i > 0$, we must have $f(i) > f(i+1)$. Thus $M_f H_i = M_{f\cdot s_i}$, again by Lemma 3.1.1.

Suppose $l(\sigma) > 0$. We have a reduced expression $\sigma = s_{i_1} \cdots s_{i_k}$. Denote $\sigma' = s_{i_1} \cdots s_{i_{k-1}}$, and note $l(\sigma') < l(\sigma)$. By Theorem 3.2.6(i), $\sigma' \in {}^fW$. By the inductive assumption, $M_f H_{\sigma'} = M_{f\cdot\sigma'}$. Now if $s_{i_k} = s_0$, then this only happens when $f^{\sigma'}(1) \in \mathbb{I}^-_\circ$, by case ($\mathrm{i}_0$) in the proof of Theorem 3.2.6. Thus, we have $M_f H_\sigma = M_f H_{\sigma'} H_0 = M_{f\cdot\sigma'} H_0 = M_{f\cdot\sigma}$, by Lemma 3.1.1. If $s_{i_k} = s_j$ for some $j \geq 1$, similarly we must have $f^{\sigma'}(j) > f^{\sigma'}(j+1)$, by case (i) in the proof of Theorem 3.2.6. Thus we have $M_f H_\sigma = M_f H_{\sigma'} H_j = M_{f\cdot\sigma'} H_j = M_{f\cdot\sigma}$, again by Lemma 3.1.1. □

Lemma 3.2.11. *Suppose that $\sigma \in {}^fW$ satisfies that $1 \neq |\sigma(1)| \leq d_\bullet$. Then $\sigma(1) > 1$, and σ must have a reduced expression which starts with $s_{\sigma(1)-1} s_{\sigma(1)-2} \cdots s_2 s_1$.*

Proof. Lemma 3.2.4 is applicable by the assumption, and so we must have $\sigma(1) > 0$, and then $\sigma(1) > 1$, thanks to the assumption $1 \neq |\sigma(1)|$.

Set $u = \sigma(1)$. We prove the lemma by induction on the length of σ. If $l(\sigma) = 1$, then $\sigma = s_1$ (thanks to $\sigma(1) > 1$), and the lemma holds trivially.

Now suppose that $l(\sigma) > 1$. There exists $1 \leq a \leq d$ such that $\sigma(a) = u-1$ by Lemma 3.2.4. Then we have $s_{u-1}\sigma(1) = u-1, s_{u-1}\sigma(a) = u$ and thus

$$l(s_{u-1}\sigma) = \mathrm{inv}_B(s_{u-1}\sigma) = \mathrm{inv}_B(\sigma) - 1 = l(\sigma) - 1.$$

By the inductive assumption, $s_{u-1}\sigma$ has a reduced expression which starts with $s_{\sigma(1)-2}\cdots s_2s_1$. Therefore, σ has a reduced expression which starts with $s_{\sigma(1)-1}s_{\sigma(1)-2}\cdots s_2s_1$. □

The bar involution on $\mathscr{H}_{B_d}$, denoted by $^-$, is the $\mathbb{Q}$-algebra automorphism such that

$$\bar{H}_i = H_i^{-1},\ \bar{q} = q^{-1},\ \bar{p} = p^{-1},\ \forall 0 \leq i \leq d-1.$$

(We shall refer to a map such that $q^m \mapsto q^{-m}$ and $p^m \mapsto p^{-m}$ *anti-linear*.)

Let $f \in \mathbb{I}^{d,-}_{r|m|r}$. We define a $\mathbb{Q}$-linear map $\psi_\imath$ on the module $\mathbb{M}_f$ (which has a basis $M_{f\cdot\sigma}$, for $\sigma \in {}^fW$) by

$$\psi_\imath(q) = q^{-1}, \quad \psi_\imath(p) = p^{-1}, \quad \psi_\imath(M_{f\cdot\sigma}) = M_f\bar{H}_\sigma, \quad \forall \sigma \in {}^fW. \tag{3.16}$$

Now we can establish the existence of bar involution on $\mathbb{M}_f$, generalizing the parabolic case [De87, So97].

Proposition 3.2.12. *Let $f \in \mathbb{I}^{d,-}_{r|m|r}$. The map $\psi_\imath$ on $\mathbb{M}_f$ in (3.16) is compatible with the bar operator on the Hecke algebra, i.e.,*

$$\psi_\imath(xh) = \psi_\imath(x)\overline{h}, \qquad \textit{for all } x \in \mathbb{M}_f,\ h \in \mathscr{H}_{B_d}. \tag{3.17}$$

In particular, $\psi_\imath^2 = Id$. (We shall call $\psi_\imath$ the bar involution on $\mathbb{M}_f$.)

Proof. Note $\psi_\imath(M_f) = M_f$, by definition (3.16).

A simple induction on $l(w)$ reduces the proof of (3.17), for $h = H_w$ with $w \in W_d$, to proving the following formula:

$$\psi_\imath(xH_i) = \psi_\imath(x)\bar{H}_i, \qquad \text{for all } x \in \mathbb{M}_f, 0 \leq i \leq d-1. \tag{3.18}$$

It suffices to verify (3.18) for the basis elements of $\mathbb{M}_f$, $x = M_fH_\sigma$ (that is, $x = M_{f\cdot\sigma}$ by

Lemma 3.2.10), for $\sigma \in {}^fW$. We proceed case-by-case following Theorem 3.2.6.

(i) Assume $l(\sigma s_i) < l(\sigma)$. In this case $\sigma s_i \in {}^fW$, and thus

$$\begin{aligned}\psi_\imath(M_f H_\sigma H_i) &= \psi_\imath(M_f H_{\sigma s_i} + (q_i - q_i^{-1}) M_f H_\sigma) \\ &= M_f \bar{H}_{\sigma s_i} + (q_i^{-1} - q_i) M_f \bar{H}_\sigma \\ &= M_f \overline{(H_{\sigma s_i} + (q_i - q_i^{-1}) H_\sigma)} = M_f \bar{H}_\sigma \bar{H}_i = \psi_\imath(M_f H_\sigma) \bar{H}_i.\end{aligned}$$

(ii) If $l(\sigma s_i) > l(\sigma)$ and $\sigma s_i \in {}^fW$, then

$$\psi_\imath(M_f H_\sigma H_i) = \psi_\imath(M_f H_{\sigma s_i}) = \psi_\imath(M_f) \bar{H}_{\sigma s_i} = \psi_\imath(M_f) \bar{H}_\sigma \bar{H}_i = \psi_\imath(M_f H_\sigma) \bar{H}_i.$$

(iii) Assume $l(\sigma s_i) > l(\sigma)$ and $\sigma s_i \notin {}^fW$, for $i > 0$. In this case, we have $\sigma s_i = s'\sigma$ for some $s' \in J(f)$, and $M_f H_{s'} = q M_f$ by Lemma 3.1.1. Thus, we have

$$\psi_\imath(M_f H_\sigma H_i) = \psi_\imath(M_f H_{\sigma s_i}) = \psi_\imath(M_f H_{s'\sigma}) = \psi_\imath(q M_f H_\sigma) = q^{-1} M_f \bar{H}_\sigma.$$

On the other hand, we have

$$\psi_\imath(M_f H_\sigma) \bar{H}_i = M_f \bar{H}_\sigma \bar{H}_i = M_f \bar{H}_{\sigma s_i} = M_f \bar{H}_{s'\sigma} = M_f H_{s'}^{-1} \bar{H}_\sigma = q^{-1} M_f \bar{H}_\sigma.$$

Hence (3.18) holds for $x = M_f H_\sigma$ in this case.

(iii_0) Assume $i = 0$, $l(\sigma s_0) > l(\sigma)$, and $\sigma s_0 \notin {}^fW$. By Theorem 3.2.6(iii_0) and its proof in case (iii_0), we have $f^\sigma(1) \in \mathbb{I}_\bullet$ and thus $|\sigma(1)| \leq d_\bullet$. By Lemma 3.2.4, $\sigma(1) > 0$. We separate into 2 subcases (iii_0-1) and (iii_0-2).

Subcase (iii_0-1): $\sigma(1) = 1$. Then $f(1) \in \mathbb{I}_\bullet$ and $s_0\sigma = \sigma s_0$, by Theorem 3.2.6(iii_0) and its proof in case (iii_0). Thus we have

$$\psi_\imath(M_f H_\sigma H_0) = \psi_\imath(M_f H_{\sigma s_0}) = \psi_\imath(M_f H_{s_0\sigma}) = \psi_\imath(M_f H_0 H_\sigma) = p^{-1} M_f \bar{H}_\sigma.$$

On the other hand, $\psi_\imath(M_f H_\sigma)\bar{H}_0 = M_f \bar{H}_{\sigma s_0} = M_f \bar{H}_{s_0\sigma} = p^{-1}M_f\bar{H}_\sigma$. So $\psi_\imath(M_f H_\sigma H_0) = \psi_\imath(M_f H_\sigma)\bar{H}_0$, proving (3.18) for $x = M_f H_\sigma$ in this case.

<u>Subcase (iii$_0$-2): $\sigma(1) > 1$</u>. Set $u = \sigma(1) \le d_\bullet$. We have $\sigma s_0 = t_u\sigma$ by Theorem 3.2.6(iii$_0$); see (3.10) for t_u. By Lemma 3.2.11, σ has a reduced expression of the form

$$\sigma = s_{u-1}s_{u-2}\cdots s_2 s_1 s_{i_1}\cdots s_{i_m}.$$

Hence, $t_u\sigma = s_{u-1}\cdots s_1 s_0 s_{i_1}\cdots s_{i_m}$, also a reduced expression for length reason. Thus

$$\begin{aligned}
\psi_\imath(M_f H_\sigma H_0) &= \psi_\imath(M_f H_{\sigma s_0}) = \psi_\imath(M_f H_{t_u\sigma}) \\
&= \psi_\imath(M_f H_{s_{u-1}\cdots s_1} H_0 H_{s_{i_1}\cdots s_{i_m}}) \\
(u \le d_\bullet, \text{Lemma 3.1.1 for } H_0) \Rightarrow \quad &= p^{-1}\psi_\imath(M_f H_{s_{u-1}\cdots s_1} H_{s_{i_1}\cdots s_{i_m}}) \\
&= p^{-1}M_f\bar{H}_\sigma.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\psi_\imath(M_f H_\sigma)\bar{H}_0 &= M_f\bar{H}_\sigma\bar{H}_0 = M_f\bar{H}_{\sigma s_0} = M_f\bar{H}_{t_u\sigma} \\
&= M_f\overline{H_{s_{u-1}\cdots s_1}}\,\bar{H}_0\overline{H_{s_{i_1}\cdots s_{i_m}}} \\
(u \le d_\bullet, \text{Lemma 3.1.1 for } H_0) \Rightarrow \quad &= p^{-1}M_f\overline{H_{s_{u-1}\cdots s_1}}\,\overline{H_{s_{i_1}\cdots s_{i_m}}} \\
&= p^{-1}M_f\bar{H}_\sigma.
\end{aligned}$$

Therefore, the proof of (3.18) is completed, for all $x = M_f H_\sigma$.

Finally, we have $\psi_\imath^2(M_f H_\sigma) = M_f\bar{\bar{H}}_\sigma = M_f H_\sigma$, i.e., $\psi_\imath^2 = \mathrm{Id}$. □

3.2.5 Canonical basis on $\mathbb{M}_f$

For the formulation of canonical basis on $\mathbb{M}_f$, we shall specialize to a one-parameter setting. Our assumption below that $p \in q^{\mathbb{Z}}$ below amounts to choosing distinguished weight functions à la Lusztig [Lus03]. (The general weight functions therein work here too, but it would require additional notations to set up properly.)

Suppose $p \in q^{\mathbb{Z}}$. Then $\mathscr{H}_{B_d}$ becomes a $\mathbb{Q}(q)$-algebra, and $\mathbb{M}_f$ becomes a $\mathbb{Q}(q)$-vector space and an $\mathscr{H}_{B_d}$-module. The bar involution $\psi_\imath$ on $\mathbb{M}_f$ remain valid. With Proposition 3.2.8 and Proposition 3.2.12 at our disposal, the proof of the next theorem follows by standard arguments.

Theorem 3.2.13. *Suppose $p \in q^{\mathbb{Z}}$, and let $f \in \mathbb{I}^{d,-}_{r|m|r}$. Then for each $\sigma \in {}^fW$, there exists a unique element $C_\sigma \in \mathbb{M}_f$ such that*

(i) $\psi_\imath(C_\sigma) = C_\sigma$;

(ii) $C_\sigma \in M_{f\cdot\sigma} + \sum\limits_{w\in {}^fW} q^{-1}\mathbb{Z}[q^{-1}]M_{f\cdot w}$.

Moreover, we have

(ii′) $C_\sigma \in M_{f\cdot\sigma} + \sum\limits_{w\in {}^fW, w<\sigma} q^{-1}\mathbb{Z}[q^{-1}]M_{f\cdot w}$.

The set $\{C_\sigma | \sigma \in {}^fW\}$ is called a *canonical basis or quasi-parabolic KL basis* for $\mathbb{M}_f$.

Proof. Let $\sigma \in {}^fW$. Assume $p \in q^{\mathbb{Z}_{>0}}$, and set $b_i = H_i + q_i^{-1}$, which is bar invariant. Proposition 3.2.8 can be rewritten as

$$M_{f\cdot\sigma}b_i = \begin{cases} M_{f\cdot\sigma s_i} + q_i M_{f\cdot\sigma}, & \text{if } \ \sigma s_i < \sigma; \\ M_{f\cdot\sigma s_i} + q_i^{-1} M_{f\cdot\sigma}, & \text{if } \ \sigma s_i > \sigma \text{ and } \sigma s_i \in {}^fW; \\ (q+q^{-1})M_{f\cdot\sigma}, & \text{if } \ \sigma s_i > \sigma \text{ and } \sigma s_i \notin {}^fW, i \neq 0; \\ (p+p^{-1})M_{f\cdot\sigma}, & \text{if } \ \sigma s_0 > \sigma \text{ and } \sigma s_0 \notin {}^fW. \end{cases} \tag{3.19}$$

Now the existence of C_σ satisfying Conditions (i) and (ii′) can be proved using (3.19) by an induction on the Chevalley-Bruhat order for σ, following exactly the same argument as for [So97, Theorem 3.1].

(For $p \in q^{\mathbb{Z}_{<0}}$, one reruns the argument therein by using a variant of (3.19) with $b_0 = H_0 - p$; for $p = 1$, one uses $b_0 = H_0$ instead.)

The uniqueness of the basis $\{C_\sigma\}$ follows from the following (cf. [So97]).

Claim. Suppose $z = \sum_{w \in {}^f W} h_w M_{f \cdot w}$ with all $h_w \in q^{-1}\mathbb{Z}[q^{-1}]$ satisfies $\psi_\imath(z) = z$. Then $z = 0$.

Indeed, if $z \neq 0$, we can choose w' with maximal length such that $h_{w'} \neq 0$. Then it follows by the existence of $\{C_\sigma\}$ satisfying (i) and (ii$'$) above and $z = \psi_\imath(z)$ that $h_{w'} = \bar{h}_{w'}$, which forces $h_{w'} = 0$ (since $h_{w'} \in q^{-1}\mathbb{Z}[q^{-1}]$), which is a contradiction. The Claim follows. $\square$

Set $b'_i = H_i - q_i$. Proposition 3.2.8, for $f \in \mathbb{I}^{d,-}_{r|m|r}, \sigma \in {}^f W$, can be rewritten as

$$M_{f \cdot \sigma} b'_i = \begin{cases} M_{f \cdot \sigma s_i} - q_i^{-1} M_{f \cdot \sigma}, & \text{if } \ \sigma s_i < \sigma; \\ M_{f \cdot \sigma s_i} - q_i M_{f \cdot \sigma}, & \text{if } \ \sigma s_i > \sigma \text{ and } \sigma s_i \in {}^f W; \\ 0, & \text{if } \ \sigma s_i > \sigma \text{ and } \sigma s_i \notin {}^f W, i \neq 0; \\ 0, & \text{if } \ \sigma s_0 > \sigma \text{ and } \sigma s_0 \notin {}^f W. \end{cases} \tag{3.20}$$

The following counterpart of Theorem 3.2.13 (with q^{-1} replaced by q) can be proved in the same way using (3.20).

Proposition 3.2.14. *Suppose $p \in q^{\mathbb{Z}}$. There exists a basis $\{C^*_\sigma | \sigma \in {}^f W\}$ (called dual canonical basis) for $\mathbb{M}_f$ which is characterized by $\psi_\imath(C^*_\sigma) = C^*_\sigma$ and $C^*_\sigma \in M_{f \cdot \sigma} + \sum_{w \in {}^f W} q\mathbb{Z}[q] M_{f \cdot w}$. Moreover, we have $C^*_\sigma \in M_{f \cdot \sigma} + \sum_{\substack{w \in {}^f W \\ w < \sigma}} q\mathbb{Z}[q] M_{f \cdot w}$.*

The set $\{C^*_\sigma | \sigma \in {}^f W\}$ is called a *dual canonical or dual quasi-parabolic KL basis* for $\mathbb{M}_f$.

Example 3.2.15.

1. *If $f \in \mathbb{I}^{d,-}_{r|m|r}$ satisfies $f(i) \in \mathbb{I}^-_\circ$, for all $1 \leq i \leq d$ (or more generally, if $k \leq 1$ in (3.7)–(3.8)), then the subgroup ${}^f W$ is parabolic. In this case, the canonical basis of $\mathbb{M}_f$ is exactly the parabolic Kazhdan-Lusztig basis of type B [KL79, De87].*

2. *If $f \in \mathbb{I}^{d,-}_{r|m|r}$ satisfies $f(i) \in \mathbb{I}_\bullet$, for all $1 \le i \le d$, then the action of H_0 is given by $p \cdot Id$ on $\mathbb{M}_f$, and the $\mathscr{H}_{B_d}$-module $\mathbb{M}_f$ essentially reduces to an $\mathscr{H}_{\mathfrak{S}_d}$-module. In this case, $W_f = B_{m_1} \times \ldots \times B_{m_k}$ with $m_1 + \ldots + m_k = d$, the canonical basis of $\mathbb{M}_f$ is identified with the parabolic KL basis of $\mathscr{H}_{\mathfrak{S}_d}$ associated to $(S_{m_1} \times \ldots \times S_{m_k}) \backslash \mathfrak{S}_d$. (This follows by the uniqueness of a canonical basis, since $\mathbb{M}_f$ as an $\mathscr{H}_{B_d}$-module and as an $\mathscr{H}_{\mathfrak{S}_d}$-module has the same standard basis and the same bar map.)*

Example 3.2.16. *For non-parabolic W_f, the canonical basis on $\mathbb{M}_f$ may not be a (usual) KL basis. Consider $\mathbb{V}^{\otimes 3}$ for $\mathbb{V}$ of dimension 5 with standard basis $\{v_i\}_{-2 \le i \le 2}$, where $\mathbb{I}_\bullet = \{-1, 0, 1\}$ (i.e., $m = 3, r = 1$ and $d = 3$). We consider $f = (0, -1, -2)$ and $W_f = B_1 \times B_1 = \langle s_0, s_{101} \rangle$; here and below we shall write $s_i s_j s_k \cdots = s_{ijk\cdots}$. Then*

$$^fW = \{e, s_1, s_2, s_{12}, s_{21}, s_{121}, s_{210}, s_{2101}, s_{1210}, s_{12101}, s_{21012}, s_{121012}\}.$$

We have the following 12 canonical basis elements in $\mathbb{M}_f$ (as linear combinations of the 12 standard basis elements $M_{f \cdot \sigma}$, for $\sigma \in {}^fW$):

$$
\begin{aligned}
C_f &= M_f, \quad C_{f\cdot s_1} = M_{f\cdot s_1} + q^{-1}M_f, \quad C_{f\cdot s_2} = M_{f\cdot s_2} + q^{-1}M_f,\\
C_{f\cdot s_{12}} &= M_{f\cdot s_{12}} + q^{-1}M_{f\cdot s_1} + q^{-1}M_{f\cdot s_2} + q^{-2}M_f,\\
C_{f\cdot s_{21}} &= M_{f\cdot s_{21}} + q^{-1}M_{f\cdot s_2} + q^{-1}M_{f\cdot s_1} + q^{-2}M_f,\\
C_{f\cdot s_{121}} &= M_{f\cdot s_{121}} + q^{-1}M_{f\cdot s_{12}} + q^{-1}M_{f\cdot s_{21}} + q^{-2}M_{f\cdot s_1} + q^{-2}M_{f\cdot s_2} + q^{-3}M_f,\\
C_{f\cdot s_{210}} &= M_{f\cdot s_{210}} + q^{-1}M_{f\cdot s_{21}} + q^{-2}M_{f\cdot s_2} + q^{-2}M_{f\cdot s_1} + (q^{-3} - q^{-1})M_f,\\
C_{f\cdot s_{2101}} &= M_{f\cdot s_{2101}} + q^{-1}M_{f\cdot s_{210}} + q^{-2}M_{f\cdot s_{21}}\\
&\quad + (q^{-3} - q^{-1})M_{f\cdot s_1} + q^{-3}M_{f\cdot s_2} + (q^{-4} - q^{-2})M_f,\\
C_{f\cdot s_{1210}} &= M_{f\cdot s_{1210}} + q^{-1}M_{f\cdot s_{210}} + q^{-1}M_{f\cdot s_{121}} + q^{-2}M_{f\cdot s_{21}} + q^{-2}M_{f\cdot s_{12}}\\
&\quad + q^{-3}M_{f\cdot s_1} + q^{-3}M_{f\cdot s_2} + q^{-4}M_f,\\
C_{f\cdot s_{21012}} &= M_{f\cdot s_{21012}} + q^{-1}M_{f\cdot s_{2101}} + q^{-1}M_{f\cdot s_{1210}} + q^{-2}M_{f\cdot s_{210}} + q^{-2}M_{f\cdot s_{121}}\\
&\quad + q^{-3}M_{f\cdot s_{21}} + (q^{-3} - q^{-1})M_{f\cdot s_{12}} + (q^{-4} - q^{-2})M_{f\cdot s_1} + q^{-4}M_{f\cdot s_2} + q^{-5}M_f,\\
C_{f\cdot s_{12101}} &= M_{f\cdot s_{12101}} + q^{-1}M_{f\cdot s_{1210}} + q^{-1}M_{f\cdot s_{2101}} + q^{-2}M_{f\cdot s_{210}} + q^{-2}M_{f\cdot s_{121}}\\
&\quad + q^{-3}M_{f\cdot s_{21}} + q^{-3}M_{f\cdot s_{12}} + q^{-4}M_{f\cdot s_2} + q^{-4}M_{f\cdot s_1} + q^{-5}M_f,\\
C_{f\cdot s_{121012}} &= M_{f\cdot s_{121012}} + q^{-1}M_{f\cdot s_{21012}} + q^{-1}M_{f\cdot s_{12101}} + q^{-2}M_{f\cdot s_{2101}} + q^{-2}M_{f\cdot s_{1210}}\\
&\quad + q^{-3}M_{f\cdot s_{210}} + q^{-3}M_{f\cdot s_{121}} + q^{-4}M_{f\cdot s_{21}} + q^{-4}M_{f\cdot s_{12}}\\
&\quad + q^{-5}M_{f\cdot s_2} + q^{-5}M_{f\cdot s_1} + q^{-6}M_f.
\end{aligned}
$$

Note that some polynomials in q^{-1} above do not have positive coefficients.

Chapter 4

$\imath$Schur duality of type AIII and $\imath$canonical bases

In this chapter, we formulate a double centralizer property for the actions of $\mathbf{U}^\imath$ of type AIII and $\mathscr{H}_{B_d}$ on the tensor space $\mathbb{V}^{\otimes d}$. The quasi-parabolic KL bases on quasi-permutation Hecke modules are shown to match with the $\imath$canonical basis on the tensor space. An inversion formula for quasi-parabolic KL polynomials is established via the $\imath$Schur duality.

Within this chapter, fix $r, m \in \mathbb{N}$ (as in the previous sections), it is convenient to introduce

$$n = \frac{m}{2} \in \frac{1}{2}\mathbb{N},$$

and denote

$$I := \mathbb{I}_{2r+2n-1} = [1-n-r, n+r-1]\,.$$

4.1 $\imath$Schur duality of type AIII

Recall the basic set up about quantum groups in § 2.1. Since the underlying Dynkin diagram of a type AIII Satake diagram is of type A (see [BW18b]), we have $q_i = q$ for all $i \in I$ and hence we omit the lower script i whenever there it is clear in the context.

Denote the set of simple roots and the weight lattice for $\mathfrak{sl}_{2r+m}$ by

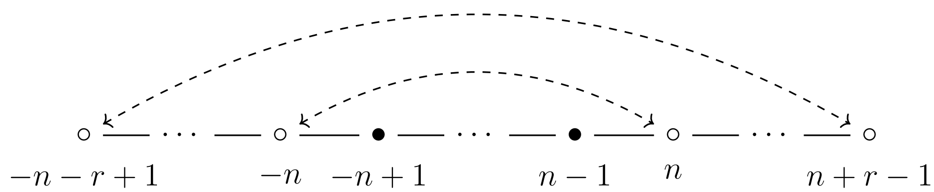
$$\Pi = \{\alpha_i = \epsilon_{i-\frac{1}{2}} - \epsilon_{i+\frac{1}{2}} \mid i \in I\}, \qquad P = \bigoplus_{i \in \mathbb{I}_{r|m|r}} \mathbb{Z}\epsilon_i.$$

Define the symmetric bilinear form on P, $(\cdot,\cdot) : P \times P \to \mathbb{Z}$, such that $(\epsilon_i, \epsilon_j) = \delta_{ij}$.

We also recall the braid group action $T_i = T''_{i,+1} : \mathbf{U} \to \mathbf{U}$ and its inverse from § 2.1.

4.1.1 $\imath$Quantum group of type AIII

We consider the Satake diagram of type AIII with $m-1 = 2n-1$ black nodes and r pairs of white nodes, together with a diagram involution τ:



(In case $n = 0$, the black nodes are dropped; the nodes n and $-n$ are identified and fixed by τ.) The involution τ on I sends $i \mapsto \tau(i) = -i$, for all i, and it induces an involution of $\mathbf{U}$, denoted again by τ, by permuting the indices of its generators $E_i, F_i, K_i^{\pm 1}$.

Let

$$I_\bullet = [1-n, n-1]$$

be the set of all black nodes in I so that

$$I = I_\bullet \cup I_\circ, \qquad \text{where } I_\circ := I \backslash I_\bullet.$$

Denote by $w_\bullet$ the longest element in the Weyl group of the Levi subalgebra associated to $I_\bullet$. Recall from § 2.2 that the $\imath$quantum group of type AIII, denoted by $\mathbf{U}^\imath$, depends on the parameters $\varsigma_i \in \mathbb{Q}(q)$, for $i \in I_\circ$, which satisfy the conditions $\varsigma_i = \varsigma_{-i}$, for $i \in I_\circ \backslash \{\pm n\}$.

More precisely, $\mathbf{U}^\imath$ is the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}$ generated by K_μ $(\mu \in Y^\imath)$, E_i $(i \in I_\bullet)$, and

$$B_i = F_i + \varsigma_i T_{w_\bullet}(E_{\tau(i)})K_i^{-1}, \quad \text{for } i \in I_\circ. \tag{4.1}$$

(In case $n = 0$, B_0 will be allowed to take a more general form $B_0 = F_0 + \varsigma_0 E_0 K_0^{-1} + \kappa_0 K_0^{-1}$, for an additional parameter $\kappa_0 \in \mathbb{Q}(q)$.)

Moreover, the algebra $\mathbf{U}^\imath$ satisfies the relations

$$\begin{aligned}
&K_\mu B_i = q^{-(\mu,\alpha_i)} B_i K_\mu, \ \forall i \in I_\circ,\\
&K_\mu F_i = q^{-(\mu,\alpha_i)} F_i K_\mu, \quad K_\mu E_i = q^{(\mu,\alpha_i)} E_i K_\mu, \ \forall i \in I_\bullet, \mu \in Y^\imath,
\end{aligned}$$

and additional Serre type relations.

4.1.2 $\imath$Schur duality

In this subsection we will construct an $\imath$Schur duality between type B Hecke algebra with two parameters p, q and $\mathbf{U}^\imath$. To avoid considering a field extension of $\mathbb{Q}(q)$, we shall assume $p \in \mathbb{Q}(q)$. Then $\mathscr{H}_{B_d}$ is a $\mathbb{Q}(q)$-algebra. The $\mathbb{Q}(q)$-vector space $\mathbb{V} = \oplus_{a \in \mathbb{I}_{r|m|r}} \mathbb{Q}(q) v_a$ from (3.4) can be identified with the natural representation of $\mathbf{U}$, where

$$\begin{aligned}
&E_i v_a = \delta_{i+1,a} v_{a-1}, \qquad F_i v_a = \delta_{i,a} v_{a+1},\\
&\quad K_a v_a = q v_a, \quad K_a v_{a+1} = q^{-1} v_{a+1}, \quad K_a v_b = v_b \ (b \neq a, a+1).
\end{aligned} \tag{4.2}$$

The tensor product $\mathbb{V}^{\otimes d}$ is naturally a $\mathbf{U}$-module via the comultiplication Δ. Recall $\mathbb{V}^{\otimes d}$ is a right $\mathscr{H}_{B_d}$-module (and hence a right $\mathscr{H}_{\mathfrak{S}_d}$-module) from Lemma 3.1.1.

Proposition 4.1.1. *[Jim86] The actions of $\mathbf{U}$ and $\mathscr{H}_{\mathfrak{S}_d}$ on $\mathbb{V}^{\otimes d}$ commute with each other, and their images in End $(\mathbb{V}^{\otimes d})$ form double centralizers.*

We shall compute explicitly the action of B_i, for $i \in I_\circ$, on $\mathbb{V}$ in the following 2 lemmas.

Recall $m = 2n \in \mathbb{N}$.

Lemma 4.1.2. *For $a \in \mathbb{I}_{r|m|r}$ and $i \in I_\circ = [1-n-r, -n] \cup [n, n+r-1]$, we have*

$$T_{w_\bullet}(E_{\tau(i)})(v_a) = \begin{cases} E_{-i}(v_a), & |i| > n; \\ E_{-n+1}E_{-n+2}\cdots E_{n-1}E_n(v_a), & i = -n; \\ (-1)^{m-1}q^{-m+1}E_{-n}E_{-n+1}\cdots E_{n-2}E_{n-1}(v_a), & i = n. \end{cases}$$

Proof. For $i < -n$ and $i > n$, we have $T_{w_\bullet}(E_{\tau(i)}) = E_{-i}$.

Let $i = -n$. We choose the following reduced expression of $w_\bullet$:

$$w_\bullet = (s_{-n+1}s_{-n+2}\cdots s_{n-1})(s_{-n+1}s_{-n+2}\cdots s_{n-2})\cdots(s_{-n+1}s_{-n+2})(s_{-n+1}).$$

Thus we compute

$$\begin{aligned} T_{w_\bullet}(E_{\tau(-n)})(v_a) &= T_{s_{-n+1}}\cdots T_{s_{n-1}}(E_n)(v_a) \\ &= T_{s_{-n+1}}\cdots T_{s_{n-2}}(E_{n-1}E_n - q^{-1}E_nE_{n-1})v_a \\ &= T_{s_{-n+1}}\cdots T_{s_{n-2}}(E_{n-1})E_n(v_a) - q^{-1}T_{s_{-n+1}}\cdots T_{s_{n-2}}(E_nE_{n-1})v_a. \end{aligned} \tag{4.3}$$

The second term on the RHS (4.3) vanishes since $T_w(E_nE_{n-1})v_a = zT_w(E_nE_{n-1}v_{w(a)})$, for some scalar z, and $E_nE_{n-1}v_{w(a)} = 0$ by (4.2), for any w, a. Thus we derive that

$$T_{w_\bullet}(E_{\tau(-n)})(v_a) = T_{s_{-n+1}}\cdots T_{s_{n-1}}(E_n)(v_a) = T_{s_{-n+1}}\cdots T_{s_{n-2}}(E_{n-1})E_n(v_a).$$

Hence by a simple induction on n we obtain

$$T_{w_\bullet}(E_{\tau(-n)})(v_a) = E_{-n+1}E_{-n+2}\cdots E_{n-1}E_n(v_a).$$

Similarly, using another reduced expression

$$w_\bullet = (s_{n-1}s_{n-2}\cdots s_{-n+1})\cdots(s_{n-1}s_{n-2})(s_{n-1}),$$

we compute $T_{w_\bullet}(E_{\tau(n)})(v_a)$ as follows:

$$\begin{aligned}
T_{w_\bullet}(E_{\tau(n)})(v_a) &= T_{s_{n-1}}\cdots T_{s_{-n+1}}(E_{-n})(v_a)\\
&= T_{s_{n-1}}\cdots T_{s_{-n+2}}(E_{-n+1}E_{-n} - q^{-1}E_{-n}E_{-n+1})v_a\\
&= -q^{-1}E_{-n}T_{s_{n-1}}\cdots T_{s_{-n+2}}(E_{-n+1})(v_a).
\end{aligned}$$

Again by induction on n, recalling $m = 2n$ we have

$$T_{w_\bullet}(E_{\tau(n)})(v_a) = (-1)^{m-1}q^{-m+1}E_{-n}E_{-n+1}\cdots E_{n-2}E_{n-1}(v_a).$$

The lemma is proved. $\square$

Lemma 4.1.2 together with the formula for B_i in (4.1) immediate imply the following.

Lemma 4.1.3. *Let $a \in \mathbb{I}_{r|m|r}$ and $i \in I_\circ$. The action of B_i on $\mathbb{V}$ is given by:*

$$B_{-n}(v_a) = \begin{cases} v_{-n+\frac{1}{2}}, & \textit{if } a = -n - \frac{1}{2};\\ \varsigma_{-n}v_{-n+\frac{1}{2}}, & \textit{if } a = n + \frac{1}{2};\\ 0, & \textit{else}, \end{cases}$$

$$B_i(v_a) = \begin{cases} v_{i+\frac{1}{2}}, & \textit{if } a = i - \frac{1}{2};\\ \varsigma_i v_{-i-\frac{1}{2}}, & \textit{if } a = -i + \frac{1}{2};\\ 0, & \textit{else}, \end{cases} \qquad \textit{for } |i| > n,$$

and (recall $m = 2n$)

$$B_n(v_a) = \begin{cases} v_{n+\frac{1}{2}} + (-1)^{m-1} q^{-m} \varsigma_n v_{-n-\frac{1}{2}}, & \textit{if } a = n - \frac{1}{2}; \\ 0, & \textit{else}. \end{cases}$$

From now on, we shall fix the parameters to be

$$\begin{cases} \varsigma_i = 1, \quad \text{if } i \neq \pm n, \\ \varsigma_{-n} = p, & \text{if } m = 2n \in \mathbb{Z}_{\geq 1}, \\ \varsigma_n = (-1)^{m-1} q^m p^{-1}, \end{cases} \tag{4.4}$$

and

$$\begin{cases} \varsigma_i = 1, \quad \text{if } i \neq 0, \\ \varsigma_0 = q^{-1}, & \text{if } m = 0. \\ \kappa_0 = \dfrac{p - p^{-1}}{q - q^{-1}}, \end{cases} \tag{4.5}$$

That is, for $m = 0$, we take $B_0 = F_0 + q^{-1} E_0 K_0^{-1} + \frac{p-p^{-1}}{q-q^{-1}} K_0^{-1}$, following [BWW18].

Introduce the $\mathbb{Q}(q)$-subspaces of $\mathbb{V}$:

$$\mathbb{V}_- = \bigoplus_{a \in \mathbb{I}_\circ^+} \mathbb{Q}(q)(v_a - p v_{-a}), \qquad \mathbb{V}_\bullet = \bigoplus_{a \in \mathbb{I}_\bullet} \mathbb{Q}(q) v_a,$$
$$\mathbb{V}_+ = \bigoplus_{a \in \mathbb{I}_\circ^+} \mathbb{Q}(q)(v_a + p^{-1} v_{-a}).$$

Lemma 4.1.4. *Assume* (4.4)–(4.5)*. Then $\mathbb{V}_-$ and $\mathbb{V}_\bullet \oplus \mathbb{V}_+$ are $\mathbf{U}^\imath$-submodules of $\mathbb{V}$. Hence, we have a $\mathbf{U}^\imath$-module decomposition $\mathbb{V} = (\mathbb{V}_\bullet \oplus \mathbb{V}_+) \oplus \mathbb{V}_-$.*

Proof. Follows by a direct computation using the formulas (4.2) and Lemma 4.1.3. □

The decomposition of $\mathbb{V}$ above is also compatible with the H_0-action.

Lemma 4.1.5. *The Hecke generator H_0 acts on $\mathbb{V}_-$ as $(-p^{-1})Id$ and acts on $\mathbb{V}_\bullet \oplus \mathbb{V}_+$ as $p \cdot Id$.*

Proof. Follows by Lemma 3.1.1. □

Theorem 4.1.6. *Suppose the parameters satisfy* (4.4)–(4.5). *Then the actions of* $\mathbf{U}^\imath$ *and* $\mathscr{H}_{B_d}$ *on* $\mathbb{V}^{\otimes d}$ *commutes with each other:*

$$\mathbf{U}^\imath \overset{\Psi}{\curvearrowright} \mathbb{V}^{\otimes d} \overset{\Phi}{\curvearrowleft} \mathscr{H}_{B_d}.$$

Moreover, $\Psi(\mathbf{U}^\imath)$ *and* $\Phi(\mathscr{H}_{B_d})$ *form double centralizers in End* $(\mathbb{V}^{\otimes d})$.

Proof. As the case for $m = 0$ was covered in [BWW18], we shall assume $m \geq 1$ below.

By the Jimbo duality (see Proposition 4.1.1), we know that the action of $\mathbf{U}$ commutes with the action of H_i, for $1 \leq i \leq d-1$. Thus, to show the commuting actions of $\mathbf{U}^\imath$ and $\mathscr{H}_{B_d}$, it remains to check the commutativity of the actions of H_0 and the generators of $\mathbf{U}^\imath$.

To that end, it suffices to consider $d = 1$ (thanks to the coideal property of $\mathbf{U}^\imath$ and the fact that the action of H_0 depends solely on the first tensor factor). In this case, the commutativity between $\mathbf{U}^\imath$-action and H_0-action on $\mathbb{V}$ follows directly from Lemmas 4.1.4 and 4.1.5.

The double centralizer property is equivalent to a multiplicity-free decomposition of $\mathbb{V}^{\otimes d}$ as an $\mathbf{U}^\imath \otimes \mathscr{H}_{B_d}$-module, which reduces by a deformation argument to the $q = 1$ setting. At the specialization $q \mapsto 1$, $\mathbf{U}^\imath$ becomes the enveloping algebra of $\mathfrak{sl}(r+m) \oplus \mathfrak{gl}(r)$, $\mathbb{V} = (\mathbb{V}_\bullet \oplus \mathbb{V}_+) \oplus \mathbb{V}_-$ becomes the natural representation of $\mathfrak{sl}(r+m) \oplus \mathfrak{gl}(r)$, on which $s_0 \in W_d$ acts as $(\mathrm{Id}_{\mathbb{V}_\bullet \oplus \mathbb{V}_+}, -\mathrm{Id}_{\mathbb{V}_-})$. The multiplicity-free decomposition of $\mathbb{V}^{\otimes d}$ at $q = 1$ can be established by a standard approach where the simples are parameterized by ordered pairs of partitions (λ, μ) such that $l(\lambda) \leq r + m, l(\mu) \leq r$ and $|\lambda| + |\mu| = d$. □

Remark 4.1.7. Theorem 4.1.6 is a common generalization of q-Schur dualities of type A and B. It specializes to Jimbo duality (Proposition 4.1.1) when $r = 0$. (In this case, $\mathbf{U}^\imath = \mathbf{U}$, and H_0 acts as $p \cdot \mathrm{Id}$ and so the action of $\mathscr{H}_{B_d}$ reduces to the action of $\mathscr{H}_{\mathfrak{S}_d}$.)

On the other hand, for $m = 0, 1$, Theorem 4.1.6 reduces to [BW18a, Theorems 5.4, 6.27] (for $p = q$), [Bao17, Theorem 3.4] (for $p = 1$), and [BWW18, Theorems 2.6, 4.4] for general p. The conventions *loc. cit.* are consistent with each other, while a different comultiplication for $\mathbf{U}$ is used in this part; this has led to a different partial ordering on $\mathbb{I}^d_{r|m|r}$ and a switch of q, p from *loc. cit.* to q^{-1}, p^{-1} for the action of Hecke algebra; cf. Lemma 3.1.1.

4.1.3 Realizing H_0 via K-matrix

For quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ of quasi-split type AIII, an $\mathbf{U}^\imath$-module isomorphism $\mathcal{T}$ on any weight $\mathbf{U}$-module M was constructed [BW18a, Theorem 2.18] by twisting the quasi K-matrix Υ by a weight function $\xi : X \to \mathbb{C}$. This construction has been generalized to general quantum symmetric pairs [BK19, Corollary 7.7], who referred to it as a K-matrix and changed the notation to be $\mathcal{K}$. Let us quickly review it.

Let $\gamma : \mathbb{I} \to \mathbb{Q}(q)$ be a function defined by

$$\gamma(i) = \begin{cases} 1, & \text{if } i \in I_\bullet \\ -\varsigma_i, & \text{if } i \in I_\circ. \end{cases}$$

Define a function $\xi \colon X \to \mathbb{Q}(q)$ by the following recursion:

$$\xi(\mu + \alpha_i) = \gamma(i) q^{(\alpha_i, w_\bullet \tau(\alpha_i)) - (\mu, \alpha_i - w_\bullet \tau(\alpha_i))} \xi(\mu), \quad \forall \mu \in X, i \in \mathbb{I}. \tag{4.6}$$

The function ξ induces a linear map $\widetilde{\xi}$ on any weight module $M = \sum_{\mu \in X} M_\mu$ by letting

$$\widetilde{\xi}(z) = \xi(\lambda) z, \quad \text{for } z \in M_\lambda.$$

From now on, we fix the function ξ with $\xi(\epsilon_{n+r-\frac{1}{2}}) = 1$.

Lemma 4.1.8. *Let $\xi(\epsilon_{n+r-\frac{1}{2}}) = 1$. Then we have*

$$\xi(\epsilon_a) = \begin{cases} (-q)^{n+r-\frac{1}{2}-a}, & a \leq -n - \dfrac{1}{2}; \\ (-q)^{m+r-1}p^{-1}, & -n + \dfrac{1}{2} \leq a \leq n + \dfrac{1}{2}; \\ (-q)^{n+r-\frac{1}{2}-a}, & a \geq n + \dfrac{3}{2}. \end{cases}$$

Proof. The function ξ is completely determined by the recursion (4.6) and the fixed value for $\xi(\epsilon_{n+r-\frac{1}{2}})$. Note that $\xi(\epsilon_a) = \xi(\epsilon_{a+1} + \alpha_{a+\frac{1}{2}})$. Thus by (4.6), for $a \leq -n - \frac{3}{2}$, we have

$$\xi(\epsilon_a) = \gamma(a + \frac{1}{2}) q^{(\alpha_{a+\frac{1}{2}}, w_\bullet \tau(\alpha_{a+\frac{1}{2}})) - (\epsilon_{a+1}, \alpha_{a+\frac{1}{2}} - w_\bullet \tau(\alpha_{a+\frac{1}{2}}))} \xi(\epsilon_{a+1}) = -q\xi(\epsilon_{a+1}).$$

The remaining cases of the recursion can be similarly made explicit. $\square$

Proposition 4.1.9. *[BW18a, Theorem 2.18] [BK19, Corollary 7.7] For any finite dimensional* $\mathbf{U}$*-module M and any ξ which satisfies the recursion in* (4.6)*, the element $\mathcal{K} = \Upsilon \widetilde{\xi} T_{w_\bullet}^{-1} T_{w_0}^{-1}$ defines an* $\mathbf{U}^\imath$*-module isomorphism:*

$$\mathcal{K} \colon M \longrightarrow M, \qquad z \mapsto \Upsilon \circ \widetilde{\xi} \circ T_{w_\bullet}^{-1} T_{w_0}^{-1}(z).$$

We compute the action of $\mathcal{K}$ on the natural $\mathbf{U}$-module $\mathbb{V}$.

Lemma 4.1.10. *The* $\mathbf{U}^\imath$*-isomorphism $\mathcal{K}$ on $\mathbb{V}$ acts as $(-p)Id$ on the submodule $\mathbb{V}_-$ and as $p^{-1}Id$ on $\mathbb{V}_+ \oplus \mathbb{V}_\bullet$.*

Proof. First one computes that the actions of T_{w_0} and $T_{w_\bullet}$ on $\mathbb{V}$ are given by

$$T_{w_0}(v_a) = (-q)^{r+m-a-n-\frac{1}{2}} v_{-a}, \quad \forall a \in \mathbb{I}_{r|m|r},$$
$$T_{w_\bullet}(v_a) = \begin{cases} (-q)^{m-a-n-\frac{1}{2}} v_{-a}, & \text{if } a \in \mathbb{I}_\bullet; \\ v_a, & else. \end{cases}$$

Hence by a direct computation using these 2 formulas and Lemma 4.1.8 we have

$$\widetilde{\xi} \circ T_{w_\bullet}^{-1} T_{w_0}^{-1}(v_a) = \begin{cases} v_{-a}, & a \in \mathbb{I}_\circ^- \cup \mathbb{I}_\circ^+; \\ p^{-1} v_a, & a \in \mathbb{I}_\bullet. \end{cases} \tag{4.7}$$

By Lemma 4.2.5 we have

$$\mathcal{K}(v_{n+\frac{1}{2}} - p v_{-n-\frac{1}{2}}) = -p(v_{n+\frac{1}{2}} - p v_{-n-\frac{1}{2}}),$$
$$\mathcal{K}(v_{n+\frac{1}{2}} + p^{-1} v_{-n-\frac{1}{2}}) = p^{-1}(v_{n+\frac{1}{2}} + p^{-1} v_{-n-\frac{1}{2}}).$$

Again by Lemma 4.2.5 we have $\mathcal{K}(v_a) = p^{-1} v_a,\ \forall a \in \mathbb{I}_\bullet$. Now the lemma follows. □

The action of the generators H_i for $\mathscr{H}_{\mathfrak{S}_d}$, for $1 \le i \le d-1$, on $\mathbb{V}^{\otimes d}$ are realized via R-matrix [Jim86] (also see [LW20]). This has the following generalization for the generator H_0 in $\mathscr{H}_{B_d}$.

Proposition 4.1.11. *The action of H_0^{-1} on $\mathbb{V}^{\otimes d}$ in Lemma 3.1.1 is realized via the K-matrix as $\mathcal{K} \otimes Id^{\otimes d-1}$.*

In case $m = 0$ or 1, Proposition 4.1.11 is established in [BW18a, BWW18]. The property of a K-matrix in Proposition 4.1.9 also provides a conceptual explanation for the commutativity of H_0 and $\mathbf{U}^\imath$ acting on $\mathbb{V}^{\otimes d}$.

4.2 $\imath$Canonical basis on the tensor module

In this section, we fix the parameters ς_i ($i \in I_\circ$) as in (4.4)–(4.5) as for Theorem 4.1.6, and further assume that $p \in q^{\mathbb{Z}}$. We show that the bar involution on the tensor space is compatible with the bar involutions on the algebras $\mathbf{U}^\imath$ and $\mathscr{H}_{B_d}$. We further show that the $\imath$canonical bases on the tensor space arising from the $\imath$quantum group and from Hecke algebra coincide.

4.2.1 Generalities of $\imath$canonical bases

In this subsection we review several constructions in the theory of $\imath$canonical basis [BW18a, BW18b].

A bar involution $\psi_\imath$ on $\mathbf{U}^\imath$ was given in [BW18a] of the quasi-split type AIII (i.e., $m = 0, 1$); it was stated therein that a bar involution exists for general $\imath$quantum groups, and this was subsequently established in [BK15]. In any case, the existence of the bar involution for $\mathbf{U}^\imath$ of type AIII under the assumption on parameters (4.4)–(4.5) can be checked directly from the known presentation of $\mathbf{U}^\imath$.

Lemma 4.2.1. *There is a unique bar involution on $\mathbf{U}^\imath$, denoted by $\psi_\imath$, such that*

$$\psi_\imath(q) = q^{-1},\ \psi_\imath(B_j) = B_j,\ \psi_\imath(E_i) = E_i,\ \psi_\imath(F_i) = F_i,\ \psi_\imath(K_\mu) = K_{-\mu},$$

for $j \in I_\circ, i \in I_\bullet$, and $\mu \in Y^\imath$.

Note that $\psi_\imath(p) = p^{-1}$ as $p \in q^{\mathbb{Z}}$. The two bar maps on $\mathbf{U}^\imath$ and $\mathbf{U}$ are not compatible under the inclusion map $\mathbf{U}^\imath \to \mathbf{U}$. As a generalization of quasi R-matrix [Lus93, 4.1.2], a notion of quasi K-matrix (also known earlier as intertwiner), denoted by Υ, was formulated in [BW18a]; a proof in greater generality was subsequently given in [BK19]; also cf. [BW18b].

Proposition 4.2.2. *[BW18a, BK19, BW18b] There exists a unique family of elements $\Upsilon_\mu \in \mathbf{U}^+_\mu$, such that $\Upsilon_0 = 1$ and $\Upsilon = \sum_\mu \Upsilon_\mu$ satisfies*

$$\psi_\imath(u)\Upsilon = \Upsilon\psi(u), \quad \forall u \in \mathbf{U}^\imath.$$

Moreover, $\Upsilon_\mu = 0$ unless $w_\bullet\tau(\mu) = \mu$.

Given based $\mathbf{U}$-modules M_i $(i = 1, 2)$ with bar involution $^-$, Lusztig [Lus93, 27.3.1] defined a bar involution on $\psi : M_1 \otimes M_2 \to M_1 \otimes M_2$ by $\psi(x_1 \otimes x_2) = \Theta(\bar{x}_1 \otimes \bar{x}_2)$, where Θ is the quasi-R matrix. The natural representation $\mathbb{V}$ of $\mathbf{U}$ admits a bar involution such that $\bar{v}_i = v_i$, for all i. Inductively, we obtain a bar involution ψ on $\mathbb{V}^{\otimes d}$.

The $\mathbf{U}$-weight of $f \in \mathbb{I}^d_{r|m|r}$ is defined to be $\mathrm{wt}(f) = \sum_{i=1}^d \epsilon_{f(i)}$. Recall the $\imath$weight lattice $X_\imath$ from (2.2). Define the $\mathbf{U}^\imath$-weight of f to be

$$\mathrm{wt}_\imath(f) = \sum_{i=1}^{d} \bar{\epsilon}_{f(a)} \in X_\imath,$$

which is the image of $\mathrm{wt}(f)$ in $X_\imath$. Following [BW18b, (5.2)] we define the following partial order $\preceq_\imath$ on $\mathbb{I}^d_{r|m|r}$:

$$g \preceq_\imath f \Leftrightarrow \mathrm{wt}_\imath(g) = \mathrm{wt}_\imath(f) \text{ and } \mathrm{wt}(g) - \mathrm{wt}(f) \in \mathbb{N}[I] \cap \mathbb{N}[w_\bullet I]. \tag{4.8}$$

We also write $g \prec_\imath f$ if $g \preceq_\imath f$ and $g \neq f$. A $\mathbf{U}^\imath$-module M equipped with a bar involution $\psi_\imath$ is called *$\imath$-involutive* if

$$\psi_\imath(uz) = \psi_\imath(u)\psi_\imath(z), \quad \forall u \in \mathbf{U}^\imath, z \in M.$$

Proposition 4.2.3. *[BW18b] The $\mathbf{U}$-module $\mathbb{V}^{\otimes d}$ is an $\imath$-involutive $\mathbf{U}^\imath$-module with the bar involution*

$$\psi_\imath := \Upsilon \circ \psi. \tag{4.9}$$

Moreover, for $f \in \mathbb{I}^d_{r|m|r}$, we have

$$\Upsilon(M_f) \in M_f + \sum_{g \prec_\imath f} \mathbb{Z}[q, q^{-1}] M_g. \tag{4.10}$$

Proof. The first statement is a special case of [BW18b, Proposition 5.1]. The formula (4.10) follows by Proposition 4.2.2 and the definition of the partial order $\preceq_\imath$ in (4.8). □

Below is a very special case of [BW18b, Theorem 5.7] concerning about $\mathbb{V}^{\otimes d}$.

Proposition 4.2.4. *(1) The $\mathbf{U}^\imath$-module $\mathbb{V}^{\otimes d}$ admits a unique $\imath$canonical basis* $\{C_g | g \in \mathbb{I}^d_{r|m|r}\}$

which is characterized by 2 properties: (i) C_g is $\psi_\imath$-invariant; (ii) C_g is of the form:

$$C_g \in M_g + \sum_{g' \in \mathbb{I}^d_{r|m|r}} q^{-1}\mathbb{Z}[q^{-1}]M_{g'}. \tag{4.11}$$

(2) The $\mathbb{V}^{\otimes d}$ admits a unique dual ıcanonical basis $\{C^*_g | g \in \mathbb{I}^d_{r|m|r}\}$ such that (i) C^*_g is $\psi_\imath$-invariant; (ii) $C^*_g \in M_g + \sum_{g' \in \mathbb{I}^d_{r|m|r}} q\mathbb{Z}[q]M_{g'}$.

It was then shown that the C_g satisfy a stronger property: $C_g \in M_f + \sum_{g' \prec_\imath g} q^{-1}\mathbb{Z}[q^{-1}]M_{g'}$.

4.2.2 ıCanonical basis on $\mathbb{V}$

Recall the notations $\mathbb{I}_\circ^-, \mathbb{I}_\circ^+, \mathbb{I}_\bullet$ from (3.2) and $m = 2n$.

Lemma 4.2.5. *We have*

$$\psi_\imath(v_a) = \Upsilon(v_a) = v_a, \quad a \in \mathbb{I}_\circ^- \cup \mathbb{I}_\bullet; \tag{4.12}$$

$$\psi_\imath(v_a) = \Upsilon(v_a) = v_a + (p^{-1} - p)v_{-a}, \quad a \in \mathbb{I}_\circ^+. \tag{4.13}$$

Proof. As v_a is bar invariant (i.e., ψ-invariant), the equality $\psi_\imath(v_a) = \Upsilon(v_a)$, for all a, follows by definition $\psi_i = \Upsilon\psi$ in (4.9).

Let $a \in \mathbb{I}_\circ^- \cup \mathbb{I}_\bullet$. The equality $\Upsilon(v_a) = v_a$ is a direct consequence of (4.10).

It remains to prove the formula (4.13), for $a \in \mathbb{I}_\circ^+$ (i.e., $a \in \mathbb{I}_{r|m|r}$ with $a \geq n + \frac{1}{2}$). By a simple induction on a, we have

$$B_{a-\frac{1}{2}} \cdots B_{n+1}B_n(v_{n-\frac{1}{2}}) = v_a + p^{-1}v_{-a}. \tag{4.14}$$

The element (4.14) is $\psi_\imath$-invariant, since the B_k's are $\psi_\imath$-invariant by Lemma 4.2.1, $v_{n-\frac{1}{2}}$ is $\psi_\imath$-invariant by (4.12), and $\mathbb{V}$ is ı-involutive by Proposition 4.2.3. On the other hand, thanks to $-a \in \mathbb{I}_\circ^-$, we have v_{-a} is $\psi_\imath$-invariant by (4.12). Hence, it follows that

$$\begin{aligned}\psi_\imath(v_a) &= \psi_\imath\big((v_a + p^{-1}v_{-a}) - p^{-1}v_{-a}\big) \\ &= (v_a + p^{-1}v_{-a}) - pv_{-a} \\ &= v_a + (p^{-1} - p)v_{-a}.\end{aligned}$$

This proves the lemma. □

Proposition 4.2.6. *The ıcanonical basis of $\mathbb{V}$ is given by*

1. $\{v_a \mid a \in \mathbb{I}_\circ^- \cup \mathbb{I}_\bullet\} \cup \{v_a + p^{-1}v_{-a},\ a \in \mathbb{I}_\circ^+\}$, *if* $p = q^{\mathbb{Z}_{>0}}$;
2. $\{v_a \mid a \in \mathbb{I}_{r|m|r}\}$, *if* $p = 1$;
3. $\{v_a \mid a \in \mathbb{I}_\circ^- \cup \mathbb{I}_\bullet\} \cup \{v_a - pv_{-a},\ a \in \mathbb{I}_\circ^+\}$, *if* $p = q^{\mathbb{Z}_{<0}}$.

Proof. It follows by Lemma 4.2.5 that these elements are $\psi_\imath$-invariant, and they are clearly of the form (4.11). Hence the proposition follows by the characterization of ıcanonical basis in Proposition 4.2.4. □

4.2.3 Compatible bar involutions and canonical bases

We formulate a compatibility between several bar involutions, which generalizes [BW18a, Theorem 5.8]; the same proof therein carries over.

Proposition 4.2.7. *There exists a unique anti-linear bar involution $\psi_\imath \colon \mathbb{V}^{\otimes d} \to \mathbb{V}^{\otimes d}$ such that $\psi_\imath(M_f) = M_f$, for $f \in \mathbb{I}_{r|m|r}^{d,-}$, and it is compatible with the bar involutions on $\mathscr{H}_{B_d}$ and $\mathbf{U}^\imath$; that is, for $u \in \mathbf{U}^\imath$, $v \in \mathbb{V}^{\otimes d}$, and $h \in \mathscr{H}_{B_d}$,*

$$\psi_\imath(uvh) = \psi_\imath(u)\psi_\imath(v)\bar{h}.$$

Remark 4.2.8. Thanks to the compatibility with the bar map on $\mathscr{H}_{B_d}$ and $\overline{M}_f = M_f$, the bar map $\psi_\imath$ on $\mathbb{V}^{\otimes d}$ when restricted to $\mathbb{M}_f$, for anti-dominant f, coincides with $\psi_\imath$ in Proposition 3.2.12.

Recall from (3.6) that $\mathbb{V}^{\otimes d}$ is a direct sum of the quasi-permutation modules $\mathbb{M}_f$ of $\mathscr{H}_{B_d}$. The union of the (dual) quasi-parabolic KL bases on the direct summands $\mathbb{M}_f$ (see Theorem 3.2.13 and Proposition 3.2.14) provide us a (dual) KL basis on $\mathbb{V}^{\otimes d}$.

Theorem 4.2.9. *The (dual) ıcanonical bases on $\mathbb{V}^{\otimes d}$ (viewed as a $\mathbf{U}^\imath$-module) coincides with the (dual) KL bases on $\mathbb{V}^{\otimes d} = \oplus_f \mathbb{M}_f$ (viewed as an $\mathscr{H}_{B_d}$-module). More precisely, we have the identifications of bases in $\mathbb{M}_f$: $C_{f\cdot\sigma} = C_\sigma$ and $C^*_{f\cdot\sigma} = C^*_\sigma$, for $f \in \mathbb{I}^{d,-}_{r|m|r}$ and $\sigma \in {}^fW$.*

(See Theorem 3.2.13, Proposition 3.2.14 and Proposition 4.2.4 for notations.)

Proof. We only need to consider the ıcanonical basis as the dual version follows by the same argument. Both bases are invariant under the same bar map $\psi_\imath$ (thanks to Proposition 4.2.7) and are of the form $C_g \in M_g + \sum_{g' \in \mathbb{I}^d_{r|m|r}} q^{-1}\mathbb{Z}[q^{-1}]M_{g'}$. Now by the uniqueness in Proposition 4.2.4 the ıcanoical basis coincides with the KL basis. The precise formula $C_{f\cdot\sigma} = C_\sigma$ follows as both sides have the same leading term $M_{f\cdot\sigma}$. □

Remark 4.2.10.

1. In case $m = 0$ (the case $m = 1$ is similar), Proposition 4.2.7 and Theorem 4.2.9 reduce to [BW18b, Theorem 5.8, Remark 5.9] and [BWW18, Proposition 3.9, Theorem 3.10]. Here we choose not to use general weight functions as in [BWW18] to avoid clumsy notations thought there is no difficulty in setting up in such a generality.

2. In case $r = 0$, the ıSchur duality reduces to Jimbo duality by Remark 4.1.7. Accordingly Proposition 4.2.7 and Theorem 4.2.9 recover the main results in [FKK98].

3. The ıcanonical basis on the space $\mathbb{V}_\bullet^{\otimes d}$ coincides with Lusztig's canonical basis. By Theorem 4.2.9 and Example 3.2.15, parts of the ıcanonical basis on $\mathbb{V}^{\otimes d}$ can be identified with (parabolic) Kazhdan-Lusztig bases of type A or type B, but not always.

4.3 An inversion formula for quasi-parabolic KL polynomials

In this section we prove an inversion formula for quasi-parabolic KL polynomials, generalizing [KL79] and [Do90]; also cf. [So97]. Inspired by the type A works [Br06] and [CL16], our approach is based on the tensor module formulation and uses the $\imath$Schur duality.

4.3.1 Symmetries ϱ, $\sigma'_\imath$ and $\sigma_\imath$

Let $(\cdot,\cdot)$ denote the standard symmetric bilinear form on $\mathbb{V}^{\otimes d}$ defined by

$$(M_f, M_g) = \delta_{f,g}, \ \forall f, g \in \mathbb{I}^d_{r|m|r}. \tag{4.15}$$

We recall several symmetries of $\mathbf{U}$; cf. [Lus93].

Lemma 4.3.1. *(1) There is an anti-involution ϱ of $\mathbf{U}$ such that, for $i \in I, \mu \in Y$,*

$$\varrho(E_i) = q^{-1}F_iK_i, \quad \varrho(F_i) = q^{-1}E_iK_i^{-1}, \quad \varrho(K_\mu) = K_\mu. \tag{4.16}$$

(2) There is an anti-involution σ of $\mathbf{U}$ such that, for $i \in I, \mu \in Y$,

$$\sigma(E_i) = E_i, \quad \sigma(F_i) = F_i, \quad \sigma(K_\mu) = K_{-\mu}. \tag{4.17}$$

The bilinear form $(\cdot,\cdot)$ on $\mathbb{V}^{\otimes d}$ defined by (4.15) satisfies (cf. [Lus93])

$$(ux, y) = (x, \varrho(u)y), \tag{4.18}$$

for all $x, y \in \mathbb{V}^{\otimes d}$, and $u \in \mathbf{U}$.

Following [BW21, §3.6.2], we consider an anti-linear anti-involution $\sigma_\imath'$ of $\mathbf{U}$ such that

$$\sigma_\imath' = \sigma \circ \tau \circ \psi. \tag{4.19}$$

Note the (anti-)involutions σ, τ, and ψ commute with each other.

Lemma 4.3.2. *The maps $\sigma_\imath'$ and ϱ are coalgebra morphisms, that is,*

$$(\sigma_\imath' \otimes \sigma_\imath')\Delta(u) = \Delta(\sigma_\imath'(u)),$$
$$(\varrho \otimes \varrho)\Delta(u) = \Delta(\varrho(u)), \quad \textit{for all } u \in \mathbf{U}.$$

Proof. It is straightforward to check on generators $u \in \mathbf{U}$ that

$$(\sigma\psi \otimes \sigma\psi)\Delta(u) = \Delta(\sigma\psi(u)),$$
$$(\tau \otimes \tau)\Delta(u) = \Delta(\tau(u)).$$

Hence these 2 identities hold for all $u \in \mathbf{U}$ since $\sigma\psi$ and τ are (anti-)involutions on $\mathbf{U}$. The lemma now follows from by definition of $\sigma_\imath' = \sigma\psi\tau$ in (4.19) and these identities.

The (well known) statement that ϱ is a coalgebra morphism (cf. [CL16]) can also be checked on the generators of $\mathbf{U}$ directly. $\square$

By the proof of [BW21, Proposition 3.13], $\sigma_\imath'$ defined in (4.19) preserves the subalgebra $\mathbf{U}^\imath$ of $\mathbf{U}$. Note that $\psi_\imath$ and $\sigma_\imath'$ commute on $\mathbf{U}^\imath$.

Lemma 4.3.3. *[BW21, Proposition 3.13] We have an anti-linear anti-involution $\sigma_\imath'$ of $\mathbf{U}^\imath$ by restriction and a $\mathbb{Q}(q)$-linear anti-involution $\sigma_\imath$ of $\mathbf{U}^\imath$ given by*

$$\sigma_\imath = \psi_\imath \circ \sigma_\imath'. \tag{4.20}$$

4.3.2 Quasi R-matrix $\Theta^\imath$

Recall the quasi K-matrix Υ from Proposition 4.2.2. As in [BW18a, (3.1)], we define the quasi R-matrix $\Theta^\imath$ associated to the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ by

$$\Theta^\imath = \Delta(\Upsilon)\Theta(\Upsilon^{-1} \otimes 1). \tag{4.21}$$

We also define

$$\begin{aligned} \overline{\Delta} :&\mathbf{U}^\imath \longrightarrow \mathbf{U}^\imath \otimes \mathbf{U}, \\ \overline{\Delta}(u) &= (\psi_\imath \otimes \psi)\Delta(\psi_\imath(u)), \ \forall u \in \mathbf{U}^\imath. \end{aligned} \tag{4.22}$$

The fundamental properties of $\Theta^\imath$ in Proposition 4.3.4 (1)-(2) below were established in [BW18a, Propositions 3.2, 3.5] and generalized in [Ko20, Propositions 3.9-3.10]. The uniqueness below can be found in the proof of [BW18a, Propositions 3.7], and in general can be derived from a variant of the interwining property given by [Ko20, (3.28)].

Proposition 4.3.4. *(cf. [BW18a, Ko20])*

1. *We have $\Theta^\imath = \sum_{\mu \in \mathbb{N}I} \Theta^\imath_\mu$, where $\Theta^\imath \in \mathbf{U}^\imath \otimes \mathbf{U}^+_\mu$ and $\Theta^\imath_0 = 1 \otimes 1$.*

2. *$\Theta^\imath$ satisfies that $\Delta(u)\Theta^\imath = \Theta^\imath \overline{\Delta}(u)$.*

Moreover, an element $\Theta^\imath$ of the form (1) satisfying the intertwining property (2) is unique.

The following new property of $\Theta^\imath$ is actually valid for a general quantum symmetric pair as in [BW21]. It will play a role in the proof of Theorem 4.3.7 below.

Lemma 4.3.5. *We have $(\sigma_\imath \otimes \sigma\tau)(\Theta^\imath) = \Theta^\imath$.*

Proof. Denote $\check{\Theta}^\imath = (\sigma_\imath \otimes \sigma\tau)(\Theta^\imath)$, which is well defined thanks to Lemma 4.3.3 and Proposition 4.3.4(1).

Applying the *anti*-involution $\sigma_\imath \otimes \sigma\tau$ to the identity $\Delta(u)\Theta^\imath = \Theta^\imath\overline{\Delta}(u)$ (see Proposition 4.3.4), we obtain

$$\check{\Theta}^\imath\ (\sigma_\imath \otimes \sigma\tau)\Delta(u) = (\sigma_\imath \otimes \sigma\tau)\overline{\Delta}(u)\ \check{\Theta}^\imath,$$

which can be rewritten as

$$\check{\Theta}^\imath\ (\psi_\imath \otimes \psi)(\sigma'_\imath \otimes \sigma'_\imath)\Delta(u) = (\sigma'_\imath \otimes \sigma'_\imath)\Delta(\psi_\imath(u))\ \check{\Theta}^\imath.$$

Applying Lemma 4.3.2 to the above identity, we obtain

$$\check{\Theta}^\imath\ (\psi_\imath \otimes \psi)\Delta(\sigma'_\imath(u)) = \Delta(\sigma'_\imath\psi_\imath(u))\ \check{\Theta}^\imath.$$

Setting $x = \sigma'_\imath\psi_\imath(u) = \psi_\imath\sigma'_\imath(u)$, the above identity can be read in the notation of (4.22) as

$$\check{\Theta}^\imath\ \overline{\Delta}(x) = \Delta(x)\ \check{\Theta}^\imath,$$

that is, $\check{\Theta}^\imath$ satisfies the intertwining property in Proposition 4.3.4(2). Clearly, $\check{\Theta}^\imath$ also satisfies Proposition 4.3.4(1). It follows by the uniqueness (see Proposition 4.3.4) that $\check{\Theta}^\imath = \Theta^\imath$. $\square$

4.3.3 A bilinear form $\langle\cdot,\cdot\rangle$

We introduce an *anti*-linear map

$$\begin{aligned} \Im\colon \mathbb{V}^{\otimes d} &\longrightarrow \mathbb{V}^{\otimes d}, \\ \Im(M_f) = M_{-f},&\ \text{ for } f \in \mathbb{I}^d_{r|m|r}. \end{aligned} \tag{4.23}$$

We define a new bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{V}^{\otimes d}$ in terms of the standard one $(\cdot, \cdot)$ in (4.15) by letting

$$\langle x, y \rangle := (x, \Im \circ \psi_\imath(y)), \ \forall x, y \in \mathbb{V}^{\otimes d}. \tag{4.24}$$

The following lemma will also be used in the proof of Theorem 4.3.7.

Lemma 4.3.6. *For all $x \in \mathbb{V}^{\otimes d}$ and $u \in \mathbf{U}$, we have $\Im(ux) = \varrho(\sigma_\imath'(u))\Im(x)$.*

Proof. The formula in case of $d = 1$ can be verified directly on u being generators and $x = v_a$. The formula in general follows by induction on d by noting by Lemma 4.3.2 that ϱ and $\sigma_\imath'$ are coalgebra morphisms. $\square$

Theorem 4.3.7. *The bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{V}^{\otimes d}$ given in (4.24) is symmetric.*

Proof. For $d = 1$, by definition (4.24) and using the formulas $\psi_\imath(v_a)$ in Lemma 4.2.5, we compute that $\langle v_a, v_{-a} \rangle = 1$, for all $a \in \mathbb{I}$; $\langle v_a, v_a \rangle = 1$, for all $a \in \mathbb{I}_\circ^+$; and otherwise $\langle v_a, v_b \rangle = 0$. Therefore, $\langle \cdot, \cdot \rangle$ is symmetric on $\mathbb{V}$.

We proceed by induction on d. Given $f, g \in \mathbb{I}^d_{r|m|r}$, write $f' = (f(1), \cdots, f(d-1))$, $f'' = (f(d))$ and similarly for g', g''. Hence $M_g = M_{g'} \otimes M_{g''}$. We use $\overline{}$ to denote ψ and $\overline{}^\imath$ to denote $\psi_\imath$ below. The bar map $\psi_\imath$ on a tensor product $\mathbf{U}$-module such as $\mathbb{V}^{\otimes d}$ can be defined inductively via $\Theta^\imath$ as (cf. [BW18a, (3.17), Remark 3.14])

$$\psi_\imath(M_g) = \Theta^\imath(\overline{M_{g'}}^\imath \otimes \overline{M_{g''}}). \tag{4.25}$$

Denote $\Theta^\imath = \sum a' \otimes a''$ with $a' \in \mathbf{U}^\imath, a'' \in \mathbf{U}$. Then we have

$$\begin{aligned} \langle M_f, M_g \rangle &= \big(M_{f'} \otimes M_{f''}, \Im(\Theta^\imath(\overline{M_{g'}}^\imath \otimes \overline{M_{g''}}))\big) \\ &= \sum \big(M_{f'}, \Im(a'\overline{M_{g'}}^\imath)\big) \big(M_{f''}, \Im(a''\overline{M_{g''}})\big). \end{aligned} \tag{4.26}$$

By Lemma 4.3.6 and the adjunction formula (4.18), we have

$$\begin{aligned}\big(M_{f'}, \Im(a'\overline{M_{g'}}^{\imath})\big) &= \big(M_{f'}, \varrho\sigma_\imath'(a')\Im(\overline{M_{g'}}^{\imath})\big)\\ &= \big(\sigma_\imath'(a')M_{f'}, \Im(\overline{M_{g'}}^{\imath})\big)\\ &= \langle \sigma_\imath'(a')M_{f'}, M_{g'}\rangle,\end{aligned}$$

which, thanks to the symmetry of $\langle\cdot,\cdot\rangle$ on $\mathbb{V}^{\otimes d-1}$ by the inductive assumption and Proposition 4.2.7, is equal to

$$\big(M_{f'}, \Im(a'\overline{M_{g'}}^{\imath})\big) = \langle M_{g'}, \sigma_\imath'(a')M_{f'}\rangle = \big(M_{g'}, \Im\circ\psi_\imath\sigma_\imath'(a')(\overline{M_{f'}}^{\imath})\big). \tag{4.27}$$

Similarly, we have

$$\big(M_{f''}, \Im(a''\overline{M_{g''}})\big) = \big(M_{g''}, \Im\circ\sigma\tau(a'')(\overline{M_{f''}})\big). \tag{4.28}$$

The formula (4.28) on $\mathbb{V}$ can be verified directly by definitions for a'' being generators of $\mathbf{U}$. (Such a formula is valid in general on $\mathbb{V}^{\otimes d}$; cf. [CL16, Proposition 3.3] and its proof.)

Plugging (4.27)–(4.28) into (4.26), we obtain

$$\begin{aligned}\langle M_f, M_g\rangle &= \sum \big(M_{g'}, \Im\circ\psi_\imath\sigma_\imath'(a')(\overline{M_{f'}}^{\imath})\big)\big(M_{g''}, \Im\circ\sigma\tau(a'')(\overline{M_{f''}})\big)\\ &= \Big(M_{g'}\otimes M_{g''}, \Im\sum(\psi_\imath\sigma_\imath'(a')\otimes\sigma\tau(a''))(\overline{M_{f'}}^{\imath}\otimes\overline{M_{f''}})\Big)\\ &= \Big(M_g, \Im(\sigma_\imath\otimes\sigma\tau)(\Theta^\imath)(\overline{M_{f'}}^{\imath}\otimes\overline{M_{f''}})\Big),\end{aligned}$$

which, by Lemma 4.3.5 and (4.25), can be rewritten as

$$\begin{aligned}\langle M_f, M_g\rangle &= \big(M_g, \Im\Theta^\imath(\overline{M_{f'}}^{\imath}\otimes\overline{M_{f''}})\big)\\ &= \big(M_g, \Im\psi_\imath(M_{f'}\otimes M_{f''})\big) = \langle M_g, M_f\rangle.\end{aligned}$$

This completes the proof of the theorem. □

4.3.4 An inversion formula

By Proposition 4.2.4 (also see Theorem 4.2.9), we can write

$$C_g = \sum_{y \in \mathbb{I}^d_{r|m|r}} l_{y,g}(q) M_y, \tag{4.29}$$

for $l_{y,g}(q) \in \mathbb{Z}[q^{-1}]$; these polynomials $l_{y,g}(q)$ are called *(quasi-parabolic) KL polynomials.* Note $l_{g,g} = 1$, and $l_{y,g} = 0$ unless $y \preceq_\imath g$.

Similarly, we have

$$C^*_g = \sum_{y \in \mathbb{I}^d_{r|m|r}} l^*_{y,g}(q) M_y, \tag{4.30}$$

for $l^*_{y,g}(q) \in \mathbb{Z}[q]$; these polynomials $l^*_{y,g}$ are called *(quasi-parabolic) dual KL polynomials.* Note $l^*_{g,g} = 1$, and $l^*_{y,g} = 0$ unless $y \preceq g$.

Theorem 4.3.8. *We have $\langle C_g, C^*_{-h} \rangle = \delta_{g,h}$, for $g, h \in f \cdot W_d$.*

Proof. Since C^*_{-h} is $\psi_\imath$-invariant, by (4.30) we have

$$C_g = \sum_{y \in \mathbb{I}^d_{r|m|r}} l_{y,g}(q) M_y, \quad C^*_{-h} = \sum_{-y \in \mathbb{I}^d_{r|m|r}} l^*_{-y,-h}(q^{-1}) \psi_\imath(M_y). \tag{4.31}$$

Similarly, since C_g is $\psi_\imath$-invariant, we have

$$C^*_{-h} = \sum_{y \in \mathbb{I}^d_{r|m|r}} l^*_{-y,-h}(q) M_{-y}, \quad C_g = \sum_{y \in \mathbb{I}^d_{r|m|r}} l_{y,g}(q^{-1}) \psi_\imath(M^*_{-y}). \tag{4.32}$$

By definition of $\langle \cdot, \cdot \rangle$ we have

$$\langle M_y, \psi_\imath(M_{-y'}) \rangle = (M_y, M_{y'}) = \delta_{y,y'}.$$

Therefore, by (4.31) and (4.32) we obtain

$$\langle C_g, C^*_{-h} \rangle = \sum_y l_{y,g}(q) l^*_{-y,-h}(q^{-1}) \equiv \delta_{g,h} \pmod{q^{-1}\mathbb{Z}[q^{-1}]},$$
$$\langle C^*_{-h}, C_g \rangle = \sum_y l^*_{-y,-h}(q) l_{y,g}(q^{-1}) \equiv \delta_{g,h} \pmod{q\mathbb{Z}[q]}.$$

By Theorem 4.3.7, $\langle C_g, C^*_{-h} \rangle = \langle C^*_{-h}, C_g \rangle$, and so the above two congruence identities imply that $\langle C_g, C^*_{-h} \rangle = \delta_{g,h}$. $\square$

We obtain the following inversion formula for quasi-parabolic KL polynomials as a reformulation of Theorem 4.3.8; this generalizes [KL79, Do90].

Corollary 4.3.9. *For all $g, h \in \mathbb{I}^d_{r|m|r}$, we have*

$$\sum_{y \in \mathbb{I}^d_{r|m|r}} l_{y,g}(q) l^*_{-y,-h}(q^{-1}) = \delta_{g,h}.$$

Remark 4.3.10. The bilinear form $\langle \cdot, \cdot \rangle$ defined by (4.24) still makes sense for a $\mathbf{U}$-module $\mathbb{V}^{\otimes m} \otimes \mathbb{V}^{*\otimes n}$ as studied in [BW18a]. Theorem 4.3.7 and a version of Corollary 4.3.9 remain valid in such a generality, and it provides an inversion formula for the super Kazhdan-Lusztig polynomials of $\mathfrak{osp}$ type *loc. cit.* This generalizes the results in super type A in [CL16].

Part II

Canonical bases of the q-Brauer algebra and $\imath$Schur dualities of type AI and AII

Chapter 5

Canonical bases of the q-Brauer algebra

In this chapter we study the q-Brauer algebra $\mathfrak{B}_n(q,z)$ and define a bar involution on it. The bar involution is shown to be compatible with the one on its natural subalgebra $\mathscr{H}_{\mathfrak{S}_n}$. Applying the bar involution to the standard basis of $\mathfrak{B}_n(q,z)$ constructed in [N14], we are able to construct a Kazhdan-Lusztig-type basis (called the canonical basis) on $\mathfrak{B}_n(q,z)$ through a standard approach due to Lusztig. A direct consequence of the compatibility of the bar involutions is that the usual type A Kazhdan-Lusztig basis is a part of the canonical basis we obtain.

5.1 Brauer algebras

Recall $D_n(N)$ to be the Brauer algebra with a linear basis consisting of all partitions of the set

$$\{1, 2, \ldots, n, 1', 2', \ldots, n'\}$$

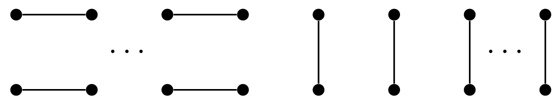
into two-element subsets. As usual, we can represent each basis element by a diagram with two rows, where the top row has n vertices marked by $1, 2, \ldots, n$, and the bottom row is numbered by $1', 2', \ldots, n'$; the vertex i is joined to j by an edge if they are in the same subset. We will call an edge *horizontal* if it connects two vertices on the same row, and *vertical* otherwise. Two diagrams d_1 and d_2 are multiplied by concatenation, that is, $d_1 \cdot d_2$ is defined to be $N^{\gamma(d_1,d_2)}d$, where $\gamma(d_1, d_2)$ counts the number of cycles produced by forming the concatenation and d is the resulting diagram after removing all cycles.

In fact, we have the following presentation for the Brauer algebra $D_n(N)$.

Definition 5.1.1. (cf. [N14, §2.1.1]) The Brauer algebra $D_n(N)$ is the unital associative $\mathbb{Z}$-algebra generated by $s_1, \ldots, s_{n-1}$, together with elements $e_{(1)}, e_{(2)}, \ldots, e_{(\lfloor \frac{n}{2} \rfloor)}$, which satisfy the following relations:

$$
\begin{array}{ll}
(S_1)\ s_i^2 = 1 & \text{for } 1 \le i \le n-1, \\
(S_2)\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } 1 \le i \le n-2, \\
(S_3)\ s_i s_j = s_j s_i & \text{for } |i-j| \ge 2, \\
(1)\ e_{(k)} e_{(i)} = e_{(i)} e_{(k)} = N^i e_{(k)} & \text{for } 1 \le i \le k, \\
(2)\ e_{(i)} s_{2j} e_{(k)} = e_{(k)} s_{2j} e_{(i)} = N^{i-1} e_{(k)} & \text{for } 1 \le j \le i \le k, \\
(3)\ s_{2i+1} e_{(k)} = e_{(k)} s_{2i+1} = e_{(k)} & \text{for } 0 \le i < k, \\
(4)\ s_i e_{(k)} = e_{(k)} s_i & \text{for } i \ge 2k+1, \\
(5)\ s_{2i-1} s_{2i} e_{(k)} = s_{2i+1} s_{2i} e_{(k)} & \text{for } 1 \le i < k, \\
(6)\ e_{(k)} s_{2i} s_{2i-1} = e_{(k)} s_{2i} s_{2i+1} & \text{for } 1 \le i < k, \\
(7)\ e_{(k+1)} = e_{(1)} s_2 \cdots s_{2k+1} s_1 \cdots s_{2k} e_{(k)} & \text{for } 1 \le k < \lfloor \frac{n}{2} \rfloor,
\end{array}
$$

Observe that the subalgebra of $D_n(N)$ generated by $s_1, \ldots, s_{n-1}$ is isomorphic to $\mathbb{Z}\mathfrak{S}_n$. It is spanned by the basis diagrams which only have vertical edges. In [Br37, §2] Brauer points out that each basis diagram in $D_n(N)$ which has exactly $2k$ horizontal edges can be obtained in the form $w_1 e_{(k)} w_2$, where w_1 and w_2 are two permutations in S_n and $e_{(k)}$ is a diagram of the following form:



where each row has exactly k horizontal edges.

For each $w \in \mathfrak{S}_n$, let $\ell(w)$ be the smallest integer $r \in \mathbb{Z}_{\geqslant 0}$ such that $w = s_{i_1}s_{i_2}\cdots s_{i_r}$; we then say that $s_{i_1}s_{i_2}\cdots s_{i_r}$ is a *reduced expression* of w and $\ell(w)$ is the length of w. By using the length function on $\mathfrak{S}_n$, Wenzl [We12a, §1.4] defined a length function on $D_n(N)$ as follows: for each basis diagram $d \in D_n(N)$ with exactly $2k$ horizontal edges, the length $\ell(d)$ of it is defined by

$$\ell(d) = \min\{\ell(\omega_1) + \ell(\omega_2) \mid d = \omega_1 e_{(k)} \omega_2, \omega_1, \omega_2 \in \mathfrak{S}_n\}.$$

For $1 \leq i, j \leq n-1$, let

$$s_{i,j} = \begin{cases} s_i s_{i+1} \cdots s_j & \text{if } i \leq j, \\ s_i s_{i-1} \cdots s_j & \text{if } i > j. \end{cases}$$

It is easy to prove that

$$\mathfrak{S}_n = \mathfrak{S}_{n-1} \bigsqcup \Big(\bigsqcup_{r=1}^{n-1} s_{r,n-1}\mathfrak{S}_{n-1}\Big) \qquad \text{(a disjoint union)},$$

and moreover, $\ell(s_{r,n-1}w) = \ell(s_{r,n-1}) + \ell(w)$ for any $w \in \mathfrak{S}_{n-1}$. Hence, we see that for any $w \in \mathfrak{S}_n$, there exist unique elements $t_{n-1}, t_{n-2}, \ldots, t_1$ such that $w = t_{n-1}t_{n-2}\cdots t_1$, where $t_j = 1$ or $t_j = s_{i_j,j}$ with $1 \leq j \leq n-1$ and $1 \leq i_j \leq j$, and moreover, $\ell(w) = \ell(t_{n-1}) + \ell(t_{n-2}) + \cdots + \ell(t_1)$. For each $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, we set

$$B_k^* = \{t_{n-1}t_{n-2}\cdots t_{2k}t_{2k-2}t_{2k-4}\cdots t_2 \mid \forall\ j,\ t_j = 1 \text{ or } t_j = s_{i_j,j} \text{ for some } 1 \leq i_j \leq j\}, \tag{5.1}$$

where B_0^* is understood as the entire symmetric group $\mathfrak{S}_n$. We set $B_k := \{\omega^{-1} \mid \omega \in B_k^*\}$. Observe that B_k^* has $\frac{n!}{2^k k!}$ elements (cf. [N14, Remark 2.1(3)]).

5.2 q-Brauer algebras

Let q and z be two invertible indeterminates.

Definition 5.2.1. (cf. [We12a, Definition 3.1], [N14, Definition 3.1]) Fix $n \in \mathbb{Z}_{\geqslant 2}$. We define the q-Brauer algebra $\mathfrak{B}_n(q,z)$ over $\mathbb{Q}(q,z)$ with generators $H_1, \ldots, H_{n-1}, e$ and the following relations:

$$(Q1)\ (H_i - q)(H_i + q^{-1}) = 0,$$

$$(Q2)\ H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1},$$

$$(Q3)\ H_i H_j = H_j H_i \quad \text{for } |i-j| > 1,$$

$$(Q4)\ e^2 = \frac{z - z^{-1}}{q - q^{-1}} e,$$

$$(Q5)\ H_1 e = e H_1 = q e,$$

$$(Q6)\ e H_2 e = z e,$$

$$(Q7)\ H_i e = e H_i \quad \text{for } i > 2,$$

$$(Q8)\ H_2 H_3 H_1^{-1} H_2^{-1} e_{(2)} = e_{(2)} = e_{(2)} H_2 H_3 H_1^{-1} H_2^{-1},$$

$$\text{where } e_{(2)} = e(H_2 H_3 H_1^{-1} H_2^{-1}) e.$$

The following proposition gives the dimension of the q-Brauer algebra $\mathfrak{B}_n(q,z)$.

Proposition 5.2.2. ([We12a, Theorem 3.8]) *The q-Brauer algebra $\mathfrak{B}_n(q,z)$ is a free $\mathbb{Q}(q,z)$-module of rank $(2n-1)!! = (2n-1)(2n-3)\cdots 1$.*

Let

$$H_{l,r}^{+} = \begin{cases} H_l H_{l+1} \cdots H_r & \text{if } l \leq r, \\ H_l H_{l-1} \cdots H_r & \text{if } l > r, \end{cases}$$

and

$$H^-_{l,r} = \begin{cases} H_l^{-1}H_{l+1}^{-1}\cdots H_r^{-1} & \text{if } l \leq r, \\ H_l^{-1}H_{l-1}^{-1}\cdots H_r^{-1} & \text{if } l > r, \end{cases}$$

for $1 \leq l, r \leq n$.

We make the convention that $e_{(0)} = 1$. For each $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, we define the elements $e_{(k)}$ in $\mathfrak{B}_n(q,z)$ inductively by

$$e_{(1)} = e \quad \text{and} \quad e_{(k+1)} = eH^+_{2,2k+1}H^-_{1,2k}e_{(k)} \text{ for } k \geq 1.$$

Remark 5.2.3. We will abuse the notation by denoting $e_{(k)}$ both a basis diagram in the Brauer algebra $D_n(N)$ and an element in the q-Brauer algebra $\mathfrak{B}_n(q,z)$.

In the following lemma we shall collect a few identities in $\mathfrak{B}_n(q,z)$ which will be used in the sequel.

Lemma 5.2.4. (cf. [We12a, Lemmas 3.2-3.3], [N14, Lemmas 3.3-3.4], [N18, Lemma 3.1], [N14, Remark 3.10(1)]) *In $\mathfrak{B}_n(q,z)$ we have*

(1) $e_{(2)} = e(H_2^{-1}H_1^{-1}H_3H_2)e = e(H_2^{-1}H_3^{-1}H_1H_2)e$,

(2) $H_{2j+1}e_{(k)} = e_{(k)}H_{2j+1} = qe_{(k)}$ *and* $H_{2j+1}^{-1}e_{(k)} = e_{(k)}H_{2j+1}^{-1} = q^{-1}e_{(k)}$ *for* $0 \leq j < k$,

(3) $e_{(k)}H_{2j}H_{2j-1} = e_{(k)}H_{2j}H_{2j+1}$ *and* $e_{(k)}H_{2j}^{-1}H_{2j-1}^{-1} = e_{(k)}H_{2j}^{-1}H_{2j+1}^{-1}$ *for* $1 \leq j < k$,

(4) $H_{2j-1}H_{2j}e_{(k)} = H_{2j+1}H_{2j}e_{(k)}$ *and* $H_{2j-1}^{-1}H_{2j}^{-1}e_{(k)} = H_{2j+1}^{-1}H_{2j}^{-1}e_{(k)}$ *for* $1 \leq j < k$,

(5) $\left(\frac{z-z^{-1}}{q-q^{-1}}\right)^{j-1} e_{(k+1)} = e_{(j)}H^+_{2j,2k+1}H^-_{2j-1,2k}e_{(k)}$ *for* $1 \leq j < k$,

(6) $e_{(j)}e_{(k)} = e_{(k)}e_{(j)} = \left(\frac{z-z^{-1}}{q-q^{-1}}\right)^{j} e_{(k)}$ *for* $1 \leq j \leq k$,

(7) $e_{(j)}H_{2j}e_{(k)} = e_{(k)}H_{2j}e_{(j)} = z\left(\frac{z-z^{-1}}{q-q^{-1}}\right)^{j-1} e_{(k)}$ *for* $1 \leq j \leq k$,

(8) $e_{(i)}H_j = H_je_{(i)}$ *for* $i \geq 1$ *and* $j \geq 2i+1$.

Let $w \in \mathfrak{S}_n$ and let $w = s_{i_1}\cdots s_{i_r}$ be a reduced expression of w. It is well-known that the element $H_w := H_{i_1}\cdots H_{i_r}$ does not depend on the choice of the reduced expression of w. Let

$\mathfrak{S}_{2k+1,n}$ be the subgroup of $\mathfrak{S}_n$ generated by elements $s_{2k+1}, s_{2k+2}, \ldots, s_{n-1}$. In [N14, §3.2] it has been show that each basis diagram $d \in D_n(N)$ with exactly $2k$ horizontal edges can be uniquely represented by a triple $(\omega_1, \omega_{(d)}, \omega_2)$ with $\omega_1 \in B_k^*, \omega_2 \in B_k$ and $\omega_{(d)} \in \mathfrak{S}_{2k+1,n}$ such that $N^k d = \omega_1 e_{(k)} \omega_{(d)} e_{(k)} \omega_2$ and $\ell(d) = \ell(\omega_1) + \ell(\omega_{(d)}) + \ell(\omega_2)$. We call such a unique triple a *reduced expression* of d.

Definition 5.2.5. ([N14, Definition 3.12]) For each diagram d of $D_n(N)$, we define a corresponding element H_d in $\mathfrak{B}_n(q,z)$ as follows: if d has exactly $2k$ horizontal edges and $(\omega_1, \omega_{(d)}, \omega_2)$ is a reduced expression of d with $\omega_1 \in B_k^*, \omega_2 \in B_k$ and $\omega_{(d)} \in \mathfrak{S}_{2k+1,n}$, then we define

$$H_d := H_{\omega_1} e_{(k)} H_{\omega_{(d)}} H_{\omega_2}.$$

If the diagram d has no horizontal edge, then d is regarded as a permutation $\omega_{(d)}$ of $\mathfrak{S}_n$, and in this case, we define $H_d := H_{\omega_{(d)}}$.

Let I_n denote the set of all basis diagrams of the Brauer algebra $D_n(N)$. The next proposition gives a standard basis of $\mathfrak{B}_n(q,z)$ that is labeled by the basis diagrams of $D_n(N)$, which can be used to define a cellular structure on $\mathfrak{B}_n(q,z)$.

Proposition 5.2.6. ([N14, Theorem 3.13]) *The set $\{H_d \mid d \in I_n\}$ forms a basis of $\mathfrak{B}_n(q,z)$ over $\mathbb{Q}(q,z)$.*

Let $\mathcal{D}_{k,n}^*$ be the set of all diagrams d^* satisfying the following three properties:

(1) d^* has exactly k horizontal edges on each row,

(2) the bottom row of d^* is the same as that of $e_{(k)}$,

(3) there is no crossing between any two vertical edges of d^*.

Set

$$B_{k,n}^* := \{\omega \in B_k^* \mid d^* = \omega e_{(k)} \in \mathcal{D}_{k,n}^* \text{ and } \ell(d^*) = \ell(\omega)\},$$

and

$$B_{k,n} := \{\omega^{-1} \mid \omega \in B^*_{k,n}\}.$$

Note that $B^*_{k,n}$ has $\frac{n!}{2^k(n-2k)!k!}$ elements (cf. [N14, Remark 3.18(1)]).

The following lemma gives a decomposition of each element in B^*_k in terms of $B^*_{k,n}$ and $\mathfrak{S}_{2k+1,n}$.

Lemma 5.2.7. ([N14, Corollary 4.3 and Lemma 4.4]) *Let σ be a permutation of B^*_k. Then there exist unique elements $\omega' \in B^*_{k,n}$ and $\pi' \in \mathfrak{S}_{2k+1,n}$ such that $\sigma = \omega'\pi'$ and $\ell(\sigma) = \ell(\omega') + \ell(\pi')$. Similarly, for each element $\varrho \in B_k$, there exist unique elements $\tau' \in \mathfrak{S}_{2k+1,n}$ and $\varpi' \in B_{k,n}$ such that $\varrho = \tau'\varpi'$ and $\ell(\varrho) = \ell(\tau') + \ell(\varpi')$.*

For each $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, let $I_{k,n}$ denote the set of all diagrams in I_n which has exactly k horizontal edges both on the top and bottom rows. By [N14, Lemmas 4.1 and 4.7], we obtain the following result.

Lemma 5.2.8. *There exists a bijection $\rho : B^*_{k,n} \times \mathfrak{S}_{2k+1,n} \times B_{k,n} \to I_{k,n}$. Under this bijection, if $(\omega_1, \omega_{(d)}, \omega_2) \in B^*_{k,n} \times \mathfrak{S}_{2k+1,n} \times B_{k,n}$ and $d \in I_{k,n}$ are such that $\rho((\omega_1, \omega_{(d)}, \omega_2)) = d$, then we have $H_{\omega_1} e_{(k)} H_{\omega_{(d)}} H_{\omega_2} = H_d$, and moreover, $\ell(d) = \ell(\omega_1) + \ell(\omega_{(d)}) + \ell(\omega_2)$.*

5.3 A bar involution

The following lemma provides an involutive anti-automorphism on $\mathfrak{B}_n(q, z)$, which is necessary for establishing its cellularity.

Lemma 5.3.1. ([N14, Proposition 3.14]) *The map $\jmath$ which is defined by*

$$\jmath(e) = e \quad \text{and} \quad \jmath(H_w) = H_{w^{-1}} \text{ for each } w \in \mathfrak{S}_n$$

can be uniquely extended to a $\mathbb{Q}(q,z)$-linear involutive anti-automorphism on $\mathfrak{B}_n(q,z)$. Moreover, it satisfies that $\jmath(e_{(k)}) = e_{(k)}$ for each k.

The following lemma provides an involutive automorphism $\bar{\cdot}$, called the bar involution, on $\mathfrak{B}_n(q,z)$, which is necessary for constructing its canonical basis.

Lemma 5.3.2. *There is a unique involutive homomorphism $\bar{\cdot}$ on $\mathfrak{B}_n(q,z)$ which is $\mathbb{Q}$-linear and satisfies $\overline{q} = q^{-1}, \overline{z} = z^{-1}, \overline{H_i} = H_i^{-1}$ and $\overline{e} = e$.*

Proof. It is easy to check that the homomorphism $\bar{\cdot}$ preserves the relations except (Q8) in Definition 5.2.1. Thus, if suffices to prove that

$$e(H_2^{-1}H_3^{-1}H_1H_2)e = e(H_2^{-1}H_3^{-1}H_1H_2)eH_2^{-1}H_3^{-1}H_1H_2, \tag{5.2}$$

and

$$e(H_2^{-1}H_3^{-1}H_1H_2)e = H_2^{-1}H_3^{-1}H_1H_2e(H_2^{-1}H_3^{-1}H_1H_2)e.$$

We only prove (5.2), and the second one can be proved similarly.

Since $H_2H_3H_1^{-1}H_2^{-1}e_{(2)} = e_{(2)}$, by Lemma 5.3.1 we have

$$e(H_2^{-1}H_1^{-1}H_3H_2)e = e(H_2^{-1}H_1^{-1}H_3H_2)eH_2^{-1}H_1^{-1}H_3H_2.$$

By Lemma 5.2.4(1), we have

$$e_{(2)} = e(H_2^{-1}H_1^{-1}H_3H_2)e = e(H_2^{-1}H_3^{-1}H_1H_2)e.$$

In order to prove (5.2), it suffices to show that

$$e(H_2^{-1}H_3^{-1}H_1H_2)eH_2^{-1}H_3^{-1}H_1H_2 = e(H_2^{-1}H_3^{-1}H_1H_2)eH_2^{-1}H_1^{-1}H_3H_2. \tag{5.3}$$

We have

$$
\begin{aligned}
& e(H_2^{-1}H_3^{-1}H_1H_2)eH_2^{-1}H_3^{-1}H_1H_2 \\
=& e_{(2)}H_2^{-1}(H_3+(q^{-1}-q))(H_1^{-1}+(q-q^{-1}))H_2 \\
=& e_{(2)}H_2^{-1}(H_1^{-1}H_3+(q-q^{-1})H_3+(q^{-1}-q)H_1^{-1}+(q^{-1}-q)(q-q^{-1}))H_2 \\
=& e_{(2)}H_2^{-1}H_1^{-1}H_3H_2+(q-q^{-1})e_{(2)}H_2^{-1}H_3H_2+ \\
& \qquad\qquad (q^{-1}-q)e_{(2)}H_2^{-1}H_1^{-1}H_2-(q-q^{-1})^2e_{(2)} \\
=& e_{(2)}H_2^{-1}H_1^{-1}H_3H_2+(q-q^{-1})e_{(2)}H_3H_2H_3^{-1}+(q^{-1}-q)e_{(2)}H_1H_2^{-1}H_1^{-1} \\
& \qquad\qquad -(q-q^{-1})^2e_{(2)}.
\end{aligned}
$$

Therefore, in order to prove (5.3) it suffices to show that

$$
(q-q^{-1})e_{(2)}H_3H_2H_3^{-1}+(q^{-1}-q)e_{(2)}H_1H_2^{-1}H_1^{-1}-(q-q^{-1})^2e_{(2)}=0. \tag{5.4}
$$

By Lemma 5.2.4(2), we have $e_{(2)}H_3 = e_{(2)}H_1 = qe_{(2)}$. By Lemma 5.2.4(3), we have $e_{(2)}H_2H_3 = e_{(2)}H_2H_1$. Therefore we have

$$
\begin{aligned}
& (q-q^{-1})e_{(2)}H_3H_2H_3^{-1}+(q^{-1}-q)e_{(2)}H_1H_2^{-1}H_1^{-1}-(q-q^{-1})^2e_{(2)} \\
=& (q^2-1)e_{(2)}H_2H_3^{-1}+(1-q^2)e_{(2)}H_2^{-1}H_1^{-1}-(q-q^{-1})^2e_{(2)} \\
=& (q^2-1)e_{(2)}H_2(H_3+(q^{-1}-q))+(1-q^2)e_{(2)}(H_2+(q^{-1}-q))(H_1+(q^{-1}-q)) \\
& -(q-q^{-1})^2e_{(2)} \\
=& (q^2-1)e_{(2)}H_2H_3-q^{-1}(1-q^2)^2e_{(2)}H_2+(1-q^2)e_{(2)}H_2H_1+q^{-1}(1-q^2)^2e_{(2)}H_2 \\
& +(1-q^2)(q^{-1}-q)e_{(2)}H_1+(1-q^2)(q^{-1}-q)^2e_{(2)}-(q-q^{-1})^2e_{(2)} \\
=& 0.
\end{aligned}
$$

Thus, (5.4) holds and we are done. $\square$

Lemma 5.3.3. *For each $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, we have $\overline{e_{(k)}} = e_{(k)}$.*

Proof. We first prove that

$$e_{(l+1)} = \left(\frac{q-q^{-1}}{z-z^{-1}}\right)^{l-1} e_{(l)} H_{2l} H_{2l+1} H_{2l-1}^{-1} H_{2l}^{-1} e_{(l)} \tag{5.5}$$

for $l \geq 1$. We shall prove (5.5) by induction on l. When $l = 1$, (5.5) holds by definition. We assume (5.5) holds for $l-1$. By Lemma 5.2.4(5) we have

$$e_{(l+1)} = \left(\frac{q-q^{-1}}{z-z^{-1}}\right)^{l-2} e_{(l-1)} H_{2l-2} H_{2l-1} H_{2l} H_{2l+1} H_{2l-3}^{-1} H_{2l-2}^{-1} H_{2l-1}^{-1} H_{2l}^{-1} e_{(l)}. \tag{5.6}$$

By Lemma 5.2.4(8) we have $e_{(i)} H_j = H_j e_{(i)}$ for $j \geq 2i+1$. Moreover, by Lemma 5.2.4(6) we have $e_{(l)} = \left(\frac{q-q^{-1}}{z-z^{-1}}\right)^{l-1} e_{(l-1)} e_{(l)}$. Therefore, by (5.6) and the assumption that (5.5) holds for $l-1$, we have

$$\begin{aligned} & e_{(l+1)} \\ =& \left(\frac{q-q^{-1}}{z-z^{-1}}\right)^{l-1} \left(\frac{q-q^{-1}}{z-z^{-1}}\right)^{l-2} e_{(l-1)} H_{2l-2} H_{2l-1} H_{2l-3}^{-1} H_{2l-2}^{-1} e_{(l-1)} \\ & \times H_{2l} H_{2l+1} H_{2l-1}^{-1} H_{2l}^{-1} e_{(l)} \\ =& \left(\frac{q-q^{-1}}{z-z^{-1}}\right)^{l-1} e_{(l)} H_{2l} H_{2l+1} H_{2l-1}^{-1} H_{2l}^{-1} e_{(l)}. \end{aligned}$$

Next we prove the lemma by induction on k. By Lemma 5.3.2 and Lemma 5.2.4(1), we have $\overline{e_{(1)}} = e_{(1)}$and $\overline{e_{(2)}} = e_{(2)}$, that is, the lemma holds for $k = 1, 2$. We assume that it is true for k and want to show that $\overline{e_{(k+1)}} = e_{(k+1)}$. By (5.5) we have

$$\overline{e_{(k+1)}} = \left(\frac{q-q^{-1}}{z-z^{-1}}\right)^{k-1} e_{(k)} H_{2k}^{-1} H_{2k+1}^{-1} H_{2k-1} H_{2k} e_{(k)}.$$

By Lemma 5.3.1 and (5.5) we have

$$e_{(k+1)} = \jmath(e_{(k+1)}) = \Big(\frac{q-q^{-1}}{z-z^{-1}}\Big)^{k-1} e_{(k)} H_{2k}^{-1} H_{2k-1}^{-1} H_{2k+1} H_{2k} e_{(k)}.$$

Therefore, in order to prove that $\overline{e_{(k+1)}} = e_{(k+1)}$, it suffices to show that

$$e_{(k)} H_{2k}^{-1} H_{2k+1}^{-1} H_{2k-1} H_{2k} e_{(k)} = e_{(k)} H_{2k}^{-1} H_{2k+1} H_{2k-1}^{-1} H_{2k} e_{(k)}. \tag{5.7}$$

We have

$$\begin{aligned}
& e_{(k)} H_{2k}^{-1} H_{2k+1}^{-1} H_{2k-1} H_{2k} e_{(k)} \\
=& e_{(k)} H_{2k}^{-1} (H_{2k+1} + (q^{-1}-q))(H_{2k-1}^{-1} + (q-q^{-1})) H_{2k} e_{(k)} \\
=& e_{(k)} H_{2k}^{-1} H_{2k+1} H_{2k-1}^{-1} H_{2k} e_{(k)} + (q-q^{-1}) e_{(k)} H_{2k}^{-1} H_{2k+1} H_{2k} e_{(k)} \\
& + (q^{-1}-q)) e_{(k)} H_{2k}^{-1} H_{2k-1}^{-1} H_{2k} e_{(k)} - (q-q^{-1})^2 e_{(k)}^2 \\
=& e_{(k)} H_{2k}^{-1} H_{2k+1} H_{2k-1}^{-1} H_{2k} e_{(k)} + (q-q^{-1}) e_{(k)} H_{2k+1} H_{2k} H_{2k+1}^{-1} e_{(k)} \\
& + (q^{-1}-q)) e_{(k)} H_{2k-1} H_{2k}^{-1} H_{2k-1}^{-1} e_{(k)} - (q-q^{-1})^2 e_{(k)}^2.
\end{aligned}$$

By Lemma 5.2.4(7) and (8), we have

$$\begin{aligned}
e_{(k)} H_{2k+1} H_{2k} H_{2k+1}^{-1} e_{(k)} =& H_{2k+1} e_{(k)} H_{2k} e_{(k)} H_{2k+1}^{-1} \\
=& z\Big(\frac{z-z^{-1}}{q-q^{-1}}\Big)^{k-1} H_{2k+1} e_{(k)} H_{2k+1}^{-1} \\
=& e_{(k)} H_{2k} e_{(k)}.
\end{aligned}$$

By Lemma 5.2.4(2), we have

$$e_{(k)} H_{2k-1} H_{2k}^{-1} H_{2k-1}^{-1} e_{(k)} = e_{(k)} H_{2k}^{-1} e_{(k)}.$$

Therefore, we have

$$
\begin{aligned}
& e_{(k)}H_{2k}^{-1}H_{2k+1}^{-1}H_{2k-1}H_{2k}e_{(k)} \\
=& e_{(k)}H_{2k}^{-1}H_{2k+1}H_{2k-1}^{-1}H_{2k}e_{(k)} + (q-q^{-1})e_{(k)}(H_{2k}-H_{2k}^{-1})e_{(k)} - (q-q^{-1})^2 e_{(k)}^2 \\
=& e_{(k)}H_{2k}^{-1}H_{2k+1}H_{2k-1}^{-1}H_{2k}e_{(k)}.
\end{aligned}
$$

Thus, (5.7) holds and we are done. □

5.4 Canonical bases

In this subsection we shall construct a Kazhdan-Lusztig-type basis on the q-Brauer algebra $\mathfrak{B}_n(q,z)$.

Lemma 5.4.1. ([We12a, Lemma 1.2(a)]) *For any $w \in \mathfrak{S}_n$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, there exists a unique element $\sigma \in B_k^*$ such that $we_{(k)} = \sigma e_{(k)}$ and $\ell(\sigma e_{(k)}) = \ell(\sigma) \leq \ell(w)$.*

In fact, the element $\sigma \in B_k^*$ in Lemma 5.4.1 can be constructed as follows (refer to the proof of [We12a, Lemma 1.2(a)]). We set $d = we_{(k)}$. Using exactly the same arguments as those before (5.1), we see that there exist unique elements $t_{n-1}, t_{n-2}, \ldots, t_{2k}$ such that $d' = (t_{n-1}t_{n-2}\cdots t_{2k})^{-1}d$ is a diagram in $\mathfrak{S}_{2k}e_{(k)}$, and moreover, $\ell(t_{n-1}t_{n-2}\cdots t_{2k}y) = \ell(t_{n-1}) + \ell(t_{n-2}) + \cdots + \ell(t_{2k}) + \ell(y)$ for any $y \in \mathfrak{S}_{2k}$. Let i_{2k-2} be the label of the vertex of d' which is connected with the $2k$-th vertex on the top row. If $i_{2k-2} = 2k-1$, we set $t_{2k-2} = 1$; if $i_{2k-2} \leq 2k-2$, then we set $t_{2k-2} = s_{i_{2k-2},2k-2}$. Then in the diagram $d'' = t_{2k-2}^{-1}d'$, the $(2k-1)$-st and $2k$-th vertices on the top row are connected by a horizontal edge. Proceeding in this way, we see that there exist some elements $t_{2k-2}, t_{2k-4}, \ldots, t_2$ such that $e_{(k)} = t_2^{-1}\cdots t_{2k-4}^{-1}t_{2k-2}^{-1}d'$, that is, $d' = t_{2k-2}t_{2k-4}\cdots t_2 e_{(k)}$. Set

$$\sigma := t_{n-1}t_{n-2}\cdots t_{2k}t_{2k-2}t_{2k-4}\cdots t_2.$$

Then σ is just the required element in Lemma 5.4.1, that is, $\sigma \in B_k^*$ is such that $we_{(k)} = \sigma e_{(k)}$ and $\ell(\sigma e_{(k)}) = \ell(\sigma) \leq \ell(w)$. From the above process, we see that the choices of the elements $t_{2k-2}, t_{2k-4}, \ldots, t_2$ depend only on the defining relations (S_1)-(S_3), (3) and (5) in Definition 5.1.1 (refer to the last paragraph on [N14, p. 1385]).

In an analogous way, by using the corresponding relations $(Q1)$-$(Q3)$ on the generators H_i in Definition 5.2.1 as well as two relations (2) and (4) in Lemma 5.2.4, we see that the element $H_w e_{(k)}$ transforms into the form

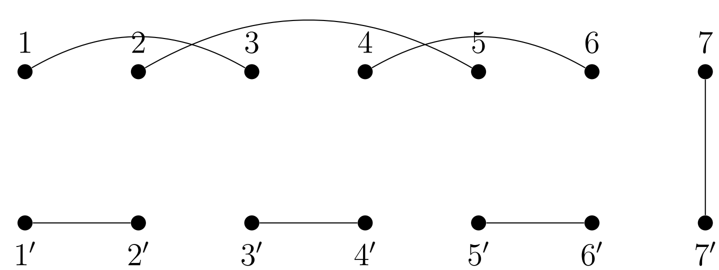
$$\sum_{\substack{\sigma' \in B_k^* \\ \ell(\sigma') \leq \ell(w)}} r_{\sigma',w} H_{\sigma'} e_{(k)}$$

for some $r_{\sigma',w} \in \mathbb{Z}[q, q^{-1}]$ (refer to the proof of [N14, Lemma 4.10]).

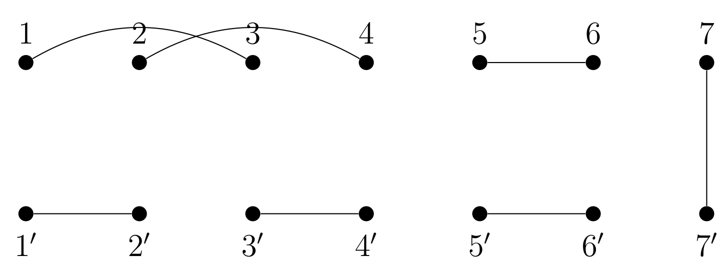
Let us look at an example.

Example 5.4.2. *Fix $n = 7$ and $k = 3$. Assume that $w = s_6 s_{1,5} s_{2,4} s_2 \in \mathfrak{S}_7$.*

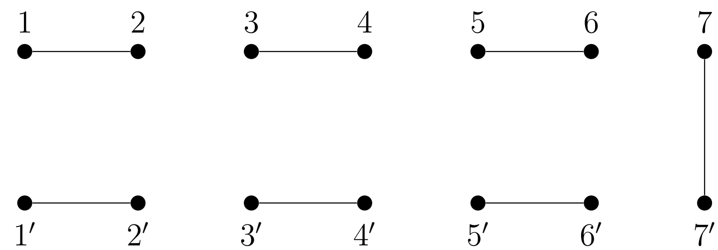
We set $t_6 = s_6$. Then $t_6^{-1} we_{(3)} \in \mathfrak{S}_6 e_{(3)}$ and

$t_6^{-1} we_{(3)} =$ 

In the diagram $t_6^{-1} we_{(3)}$, we see that the label of its vertex which is connected with the 6-th vertex on the top row is 4. Thus, we set $t_4 = s_{4,4} = s_4$. Then we have

$t_4^{-1} t_6^{-1} we_{(3)} =$ 

In the diagram $t_4^{-1}t_6^{-1}we_{(3)}$*, the* 5*-th and* 6*-th vertices on the top row are connected by a horizontal edge and the label of its vertex which is connected with the* 4*-th vertex on the top row is* 2*. Thus we set* $t_2 = s_{2,2} = s_2$*. Then we have*

$t_2^{-1}t_4^{-1}t_6^{-1}we_{(3)} =$ 

Therefore $t_2^{-1}t_4^{-1}t_6^{-1}we_{(3)} = e_{(3)}$*. We set* $\sigma = t_6t_4t_2 = s_6s_4s_2$*. Then,* $\sigma \in B_3^*$ *satisfies that* $we_{(3)} = \sigma e_{(3)}$ *and* $\ell(\sigma e_{(3)}) = \ell(\sigma) < \ell(w)$*.*

We can give an equivalent description of the above procedure using relations (S_1)*-*(S_3)*,* (3) *and* (5) *in Definition 5.1.1. We have*

$$\begin{aligned} we_{(3)} &\overset{(S_3)}{=} s_6s_{1,4}s_{2,3}s_2(s_5s_4e_{(3)}) \overset{(5)}{=} s_6s_{1,4}s_{2,3}s_2(s_3s_4e_{(3)}) \overset{(S_2)}{=} s_6s_{1,4}s_2(s_2s_3s_2)s_4e_{(3)} \\ &\overset{(S_1),(S_3)}{=} s_6s_{1,3}(s_4s_3s_4)s_2e_{(3)} \overset{(S_1),(S_2)}{=} s_6s_{1,2}s_4(s_3s_2e_{(3)}) \overset{(S_3),(5)}{=} s_6s_4s_1s_2(s_1s_2e_{(3)}) \\ &\overset{(S_1),(S_2)}{=} s_6s_4s_2(s_1e_{(3)}) \overset{(3)}{=} s_6s_4s_2e_{(3)}. \end{aligned}$$

We set $\sigma = s_6s_4s_2$*. Then* $\sigma \in B_3^*$ *is the desired element.*

In an analogous way, in $\mathfrak{B}_7(q,z)$ we have

$$\begin{aligned}
&H_w e_{(3)} = H_6 H^+_{1,5} H^+_{2,4} H_2 e_{(3)} \overset{(Q3)}{=} H_6 H^+_{1,4} H^+_{2,3} H_2 (H_5 H_4 e_{(3)}) \\
&\overset{Lemma\ 5.2.4(4)}{=} H_6 H^+_{1,4} H^+_{2,3} H_2 (H_3 H_4 e_{(3)}) \overset{(Q2)}{=} H_6 H^+_{1,4} H_2 (H_2 H_3 H_2) H_4 e_{(3)} \\
&\overset{(Q2),(Q3)}{=} H_6 H^+_{1,3} H_2^2 (H_3 H_4 H_3) H_2 e_{(3)} \\
&\overset{(Q1),Lemma\ 5.2.4(4)}{=} H_6 H^+_{1,3} ((q-q^{-1}) H_2 + 1) H_3 H_4 (H_1 H_2 e_{(3)}) \\
&= (q-q^{-1}) H_6 H^+_{1,2} (H_2 H_3 H_2) H_4 H_1 H_2 e_{(3)} + H_6 H^+_{1,2} ((q-q^{-1}) H_3 + 1) H_4 H_1 H_2 e_{(3)} \\
&= (q-q^{-1}) H_6 H_1 ((q-q^{-1}) H_2 + 1) H_3 H_4 H_1 H_2 H_1 e_{(3)} \\
&\quad + (q-q^{-1}) H_6 H^+_{1,4} H^+_{1,2} e_{(3)} + H_6 H_4 ((q-q^{-1}) H_1 + 1) H_2 H_1 e_{(3)} \\
&\overset{Lemma\ 5.2.4(2)}{=} q(q-q^{-1})^2 H_6 H^+_{1,4} H^+_{1,2} e_{(3)} + q(q-q^{-1}) H_6 H^+_{3,4} ((q-q^{-1}) H_1 + 1) H_2 e_{(3)} \\
&\qquad + (q-q^{-1}) H_6 H^+_{1,4} H^+_{1,2} e_{(3)} + q(q-q^{-1}) H_6 H_4 H^+_{1,2} e_{(3)} + q H_6 H_4 H_2 e_{(3)} \\
&= q^2 (q-q^{-1}) H_6 H^+_{1,4} H^+_{1,2} e_{(3)} + q(q-q^{-1})^2 H_6 H^+_{3,4} H^+_{1,2} e_{(3)} \\
&\quad + q(q-q^{-1}) H_6 H^+_{3,4} H_2 e_{(3)} + q(q-q^{-1}) H_6 H_4 H^+_{1,2} e_{(3)} + q H_6 H_4 H_2 e_{(3)}.
\end{aligned}$$

Thus, we see that the element $H_w e_{(3)}$ can be written as a $\mathbb{Z}[q,q^{-1}]$-linear combination of elements $H_{\sigma_j} e_{(3)}$ $(1 \le j \le 5)$, where each σ_j satisfies that $\sigma_j \in B_3^*$ and $\ell(\sigma_j) < \ell(w)$.

Summarizing, we obtain the following lemma.

Lemma 5.4.3. *For any $w \in \mathfrak{S}_n$, we have*

$$H_w e_{(k)} = \sum_{\substack{\sigma' \in B_k^* \\ \ell(\sigma') \le \ell(w)}} r_{\sigma',w} H_{\sigma'} e_{(k)}$$

for some $r_{\sigma',w} \in \mathbb{Z}[q,q^{-1}]$.

Applying Lemma 5.3.1, we immediately get the following lemma.

Lemma 5.4.4. *For any $y \in \mathfrak{S}_n$, we have*

$$e_{(k)}H_y = \sum_{\substack{\varpi' \in B_k \\ \ell(\varpi') \leq \ell(y)}} s_{\varpi',y} e_{(k)} H_{\varpi'}$$

for some $s_{\varpi',y} \in \mathbb{Z}[q, q^{-1}]$.

Lemma 5.4.5. *For each $w, y \in \mathfrak{S}_n$ and $\omega_{(d)} \in \mathfrak{S}_{2k+1,n}$, we have*

$$H_w e_{(k)} H_{\omega_{(d)}} H_y = \sum_{\substack{a \in I_{k,n} \\ \ell(a) \leq \ell(w) + \ell(\omega_{(d)}) + \ell(y)}} r_a H_a$$

for some $r_a \in \mathbb{Z}[q, q^{-1}]$.

Proof. By Lemma 5.2.4(8) we see that $e_{(k)}H_w = H_w e_{(k)}$ for any $w \in \mathfrak{S}_{2k+1,n}$. By Lemma 5.2.4(6), we have $e_{(k)}^2 = \left(\frac{z-z^{-1}}{q-q^{-1}}\right)^k e_{(k)}$. Thus, by Lemmas 5.2.7, 5.4.3 and 5.4.4, we have

$$\begin{aligned} H_w e_{(k)} H_{\omega_{(d)}} H_y = &\left(\frac{q-q^{-1}}{z-z^{-1}}\right)^k \times \sum_{\substack{(\omega',\pi') \in B^*_{k,n} \times \mathfrak{S}_{2k+1,n} \\ \ell(\omega') + \ell(\pi') \leq \ell(w)}} r_{\omega',\pi'} H_{\omega'} H_{\pi'} e_{(k)} \\ &\times H_{\omega_{(d)}} \times \sum_{\substack{(\tau',\varpi') \in \mathfrak{S}_{2k+1,n} \times B_{k,n} \\ \ell(\tau') + \ell(\varpi') \leq \ell(y)}} s_{\tau',\varpi'} e_{(k)} H_{\tau'} H_{\varpi'} \end{aligned}$$

for some $r_{\omega',\pi'}, s_{\tau',\varpi'} \in \mathbb{Z}[q, q^{-1}]$.

For any $\pi', \tau' \in \mathfrak{S}_{2k+1,n}$ as above, we have

$$H_{\pi'} H_{\omega_{(d)}} H_{\tau'} = \sum_{\substack{\chi \in \mathfrak{S}_{2k+1,n} \\ \ell(\chi) \leq \ell(\pi') + \ell(\omega_{(d)}) + \ell(\tau')}} t^{\chi}_{\pi',\tau'} H_\chi$$

for some $t^{\chi}_{\pi',\tau'} \in \mathbb{Z}[q,q^{-1}]$. Thus, we have

$$H_w e_{(k)} H_{\omega_{(d)}} H_y = \sum_{\substack{(\omega',\chi,\varpi') \in B^*_{k,n} \times \mathfrak{S}_{2k+1,n} \times B_{k,n} \\ \ell(\omega')+\ell(\chi)+\ell(\varpi') \leq \ell(w)+\ell(\omega_{(d)})+\ell(y)}} r_{\omega'} s_{\varpi'} t^{\chi} H_{\omega'} e_{(k)} H_{\chi} H_{\varpi'}$$

for some $r_{\omega'}, s_{\varpi'}, t^{\chi} \in \mathbb{Z}[q,q^{-1}]$.

By Lemma 5.2.8, we see that $H_w e_{(k)} H_{\omega_{(d)}} H_y = \sum_{\substack{a \in I_{k,n} \\ \ell(a) \leq \ell(w)+\ell(\omega_{(d)})+\ell(y)}} r_a H_a$ for some $r_a \in \mathbb{Z}[q,q^{-1}]$. □

Lemma 5.4.6. *For each diagram $d \in I_{k,n}$, we have*

$$\overline{H_d} = H_d + \sum_{\substack{d' \in I_{k,n} \\ \ell(d') < \ell(d)}} r_{d',d} H_{d'}$$

for some $r_{d',d} \in \mathbb{Z}[q,q^{-1}]$.

Proof. For $k = 0$, that is, d has no horizontal edge, it is well known. Assume $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. By Lemma 5.2.8, if $(\omega_1, \omega_{(d)}, \omega_2) \in B^*_{k,n} \times \mathfrak{S}_{2k+1,n} \times B_{k,n}$ is such that $\rho((\omega_1, \omega_{(d)}, \omega_2)) = d$, then we have $H_d = H_{\omega_1} e_{(k)} H_{\omega_{(d)}} H_{\omega_2}$ and $\ell(d) = \ell(\omega_1) + \ell(\omega_{(d)}) + \ell(\omega_2)$. We have

$$\begin{aligned}\overline{H_d} =& \overline{H_{\omega_1}} e_{(k)} \overline{H_{\omega_{(d)}} H_{\omega_2}} \\ =& \Bigg(H_{\omega_1} + \sum_{\omega_1'; \ell(\omega_1') < \ell(\omega_1)} r_{\omega_1',\omega_1} H_{\omega_1'} \Bigg) e_{(k)} \Bigg(H_{\omega_{(d)}} + \sum_{\substack{\omega_{(d')} \in \mathfrak{S}_{2k+1,n} \\ \ell(\omega_{(d')}) < \ell(\omega_{(d)})}} r_{\omega_{(d')},\omega_{(d)}} H_{\omega_{(d')}} \Bigg) \\ & \times \Bigg(H_{\omega_2} + \sum_{\omega_2'; \ell(\omega_2') < \ell(\omega_2)} r_{\omega_2',\omega_2} H_{\omega_2'} \Bigg).\end{aligned}$$

By Lemma 5.4.5, we obtain the desired result. □

By Proposition 5.2.6, Lemma 5.4.6 and Lusztig's lemma (cf. [Lus93, Lemma 24.2.1]), we obtain the canonical and dual canonical basis for $\mathfrak{B}_n(q,z)$ over $\mathbb{Q}(q,z)$.

Theorem 5.4.7. *There exists a unique basis $\{C_d \mid d \in I_{k,n}, 0 \le k \le \lfloor \frac{n}{2} \rfloor\}$ of $\mathfrak{B}_n(q,z)$ over $\mathbb{Q}(q,z)$, called the canonical basis, such that*

$$(1)\ \ \overline{C_d} = C_d,$$

$$(2)\ \ C_d = H_d + \sum_{\substack{d' \in I_{k,n} \\ \ell(d') < \ell(d)}} p_{d',d} H_{d'},\ \ \textit{where } p_{d',d} \in q^{-1}\mathbb{Z}[q^{-1}].$$

Theorem 5.4.8. *There exists a unique basis $\{C^*_d \mid d \in I_{k,n}, 0 \le k \le \lfloor \frac{n}{2} \rfloor\}$ of $\mathfrak{B}_n(q,z)$ over $\mathbb{Q}(q,z)$, called the dual canonical basis, such that*

$$(1)\ \ \overline{C^*_d} = C^*_d,$$

$$(2)\ \ C^*_d = H_d + \sum_{\substack{d' \in I_{k,n} \\ \ell(d') < \ell(d)}} p^*_{d',d} H_{d'},\ \ \textit{where } p^*_{d',d} \in q\mathbb{Z}[q].$$

Remark 5.4.9. Note that in the above theorems, the coefficients $p_{d',d}$ (resp. $p^*_{d',d}$) are polynomials in q^{-1} (resp. q), which do not depend on z (compare with [FG95, §5.2]).

Remark 5.4.10. Fix some $0 \le k \le \lfloor \frac{n}{2} \rfloor$ and $d \in I_{k,n}$. Assume by Lemma 5.2.8, $(\omega_1, \omega_{(d)}, \omega_2)$ is the unique element in $B^*_{k,n} \times \mathfrak{S}_{2k+1,n} \times B_{k,n}$ such that $\rho((\omega_1, \omega_{(d)}, \omega_2)) = d$. It is clear from the definition that $(\omega_2^{-1}, \omega_{(d)}^{-1}, \omega_1^{-1})$ also belongs to $B^*_{k,n} \times \mathfrak{S}_{2k+1,n} \times B_{k,n}$, and we can assume that $d' \in I_{k,n}$ is such that $\rho((\omega_2^{-1}, \omega_{(d)}^{-1}, \omega_1^{-1})) = d'$.

It is easy to check that the bar involution $\bar{\cdot}$ on $\mathfrak{B}_n(q,z)$ commutes with the anti-involution $\jmath$, and moreover, $\jmath$ is $\mathbb{Q}(q,z)$-linear. Therefore, we have $\jmath(C_d) = C_{d'}$ and $\jmath(C^*_d) = C^*_{d'}$.

Finally, let us look at some examples.

Example 5.4.11. *(1) When $n = 2$, the canonical basis of $\mathfrak{B}_2(q,z)$ is given by $\{1, e, H_1 + q^{-1}\}$.*

(2) When $n=3$, the canonical basis of $\mathfrak{B}_3(q,z)$ is given by

$$
\begin{aligned}
&C_0 = 1, \quad C_1 = H_1 + q^{-1}, \quad C_2 = H_2 + q^{-1},\\
&C_{12} = H_1H_2 + q^{-1}H_1 + q^{-1}H_2 + q^{-2},\\
&C_{21} = H_2H_1 + q^{-1}H_1 + q^{-1}H_2 + q^{-2},\\
&C_{121} = H_1H_2H_1 + q^{-1}H_1H_2 + q^{-1}H_2H_1 + q^{-2}H_1 + q^{-2}H_2 + q^{-3},\\
&C_e = e, \quad C_{2e} = H_2e + q^{-1}e, \quad C_{e2} = eH_2 + q^{-1}e,\\
&C_{2e2} = H_2eH_2 + q^{-1}H_2e + q^{-1}eH_2 + q^{-2}e,\\
&C_{12e} = H_1H_2e + q^{-1}H_2e + q^{-2}e,\\
&C_{e21} = eH_2H_1 + q^{-1}eH_2 + q^{-2}e,\\
&C_{12e2} = H_1H_2eH_2 + q^{-1}H_2eH_2 + q^{-1}H_1H_2e + q^{-2}H_2e + q^{-2}eH_2 + q^{-3}e,\\
&C_{2e21} = H_2eH_2H_1 + q^{-1}H_2eH_2 + q^{-1}eH_2H_1 + q^{-2}H_2e + q^{-2}eH_2 + q^{-3}e,\\
&C_{12e21} = H_1H_2eH_2H_1 + q^{-1}H_1H_2eH_2 + q^{-1}H_2eH_2H_1 + q^{-2}H_1H_2e\\
&\qquad +q^{-2}H_2eH_2 + q^{-2}eH_2H_1 + q^{-3}H_2e + q^{-3}eH_2 + q^{-4}e.
\end{aligned}
$$

Moreover, we can compute the structure constants of $\mathfrak{B}_3(q,z)$ with respect to the above basis. For example, we have

$$
\begin{aligned}
C_1 \cdot C_{2e} &= C_{12e} + C_e,\\
C_{2e} \cdot C_{e2} &= \frac{z - z^{-1}}{q - q^{-1}} C_{2e2},\\
C_e \cdot C_{12e} &= \frac{q^2 z - q^{-2} z^{-1}}{q - q^{-1}} C_e.
\end{aligned}
$$

Chapter 6

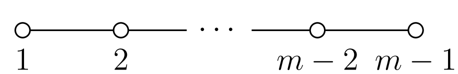
$\imath$Schur dualities of type AI and AII

As the title suggested, in this chapter we develop the duality between the q-Brauer algebra and the $\imath$quantum group of type AI and AII, respectively.

Recall again the basic set up about quantum groups in § 2.1. Since the underlying Dynkin diagram of a type AI and AII Satake diagrams are still of type A (see [BW18b]), we have $q_i = q$ for all $i \in I$ and hence we omit the lower script i whenever there it is clear in the context.

6.1 $\imath$quantum group of type AI

In this section we fix $m \in \mathbb{Z}_{\geq 2}$ and focus on the quantum symmetric pair of type AI with the Satake diagram as below (cf. [BW18b, Table 4]):



Let $\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_m)$ denote the quantum group of type A_{m-1}. According to § 2.2, we have

Definition 6.1.1. The $\imath$quantum group $\mathbf{U}^\imath(\mathfrak{so}_m)$ of type AI, with a set of parameters $\{\varsigma_i \mid$

$1 \leq i \leq m-1\} \subset \mathbb{Z}[q, q^{-1}]$, is the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}$ generated by the following elements:

$$B_i = F_i + \varsigma_i E_i K_i^{-1} \quad \text{for } 1 \leq i \leq m-1. \tag{6.1}$$

Remark 6.1.2. Suppose $\varsigma_i = -1$ for $1 \leq i \leq m-1$. When taking the $q \to 1$ limit in $\mathbf{U}^\imath(\mathfrak{so}_m)$, we see that the generator B_i specializes to $E_{i+1,i} - E_{i,i+1}$, where $E_{j,k}$'s are the $m \times m$ elementary matrices. Therefore, $\mathbf{U}^\imath(\mathfrak{so}_m)$ specializes to the enveloping algebra $\mathbf{U}(\mathfrak{so}_m)$ of the special orthogonal Lie algebra $\mathfrak{so}_m$.

6.2 $\imath$Schur duality of type AI

Let $\mathbb{V} = \sum_{i=1}^m \mathbb{Q}(q) v_i$ be the natural representation of $\mathbf{U}$ with the action of the generators as follows:

$$\begin{aligned}
E_i \cdot v_r &= \delta_{r,i+1} v_{r-1}, \\
F_i \cdot v_r &= \delta_{r,i} v_{r+1}, \\
K_i \cdot v_r &= \begin{cases} q v_i & \text{if } r = i, \\ q^{-1} v_{i+1} & \text{if } r = i+1, \\ v_r & \text{else.} \end{cases}
\end{aligned}$$

Therefore, the action of B_i on $\mathbb{V}$ can be computed by (6.1):

$$B_i \cdot v_r = \begin{cases} v_{i+1} & \text{if } r = i, \\ q\varsigma_i v_i & \text{if } r = i+1, \\ 0 & \text{else.} \end{cases}$$

Lemma 6.2.1. *$\mathbb{V}^{\otimes n}$ is a left $\mathbf{U}^\imath(\mathfrak{so}_m)$-module via Δ.*

For $i = 2, \ldots, m$, we set

$$\tau_i := \prod_{j=1}^{i-1} (-\varsigma_j)$$

and $\tau_1 = 1$. Then we have the following lemma.

Proposition 6.2.2. $\mathbb{V}^{\otimes n}$ *is a right* $\mathfrak{B}_n(q, q^m)$*-module with the action given by*

$$\begin{aligned}
&v_{a_1} \otimes \cdots \otimes v_{a_n} \cdot H_j \\
&= \begin{cases} q v_{a_1} \otimes \cdots \otimes v_{a_n} & \text{if } a_j = a_{j+1}, \\ \cdots \otimes v_{a_{j+1}} \otimes v_{a_j} \otimes \cdots & \text{if } a_j > a_{j+1}, \\ \cdots \otimes v_{a_{j+1}} \otimes v_{a_j} \otimes \cdots + (q - q^{-1}) v_{a_1} \otimes \cdots \otimes v_{a_n} & \text{if } a_j < a_{j+1}, \end{cases} \\
&v_{a_1} \otimes v_{a_2} \otimes \cdots \otimes v_{a_n} \cdot e = \delta_{a_1, a_2} \tau_{a_1} \left(\sum_{i=1}^{m} \tau_i^{-1} q^{m-2i+1} v_i \otimes v_i \right) \otimes v_{a_3} \otimes \cdots \otimes v_{a_n}.
\end{aligned}$$

Proof. By [Jim86], the action of H_i satisfies relations (Q1)-(Q3) in Definition 5.2.1. In order to verify the relation (Q4), noting that the action of e depends solely on the first two tensor factors, it suffices to show that

$$v_{a_1} \otimes v_{a_2} \cdot e^2 = \frac{q^m - q^{-m}}{q - q^{-1}} v_{a_1} \otimes v_{a_2} \cdot e.$$

We have

$$
\begin{aligned}
& v_{a_1} \otimes v_{a_2} \cdot e^2 \\
=& \delta_{a_1,a_2} \tau_{a_1} \left(\sum_{i=1}^{m} \tau_i^{-1} q^{m-2i+1} v_i \otimes v_i \right) \cdot e \\
=& \delta_{a_1,a_2} \tau_{a_1} \sum_{i=1}^{m} \tau_i^{-1} q^{m-2i+1} \left(\tau_i \sum_{j=1}^{m} \tau_j^{-1} q^{m-2j+1} v_j \otimes v_j \right) \\
=& \delta_{a_1,a_2} \tau_{a_1} \sum_{j=1}^{m} \tau_j^{-1} q^{m-2j+1} \left(\sum_{i=1}^{m} q^{m-2i+1} \right) v_j \otimes v_j \\
=& \frac{q^m - q^{-m}}{q - q^{-1}} v_{a_1} \otimes v_{a_2} \cdot e.
\end{aligned}
$$

The relation (Q5) can be easily verified. In order to verify the relation (Q6), it suffices to show that

$$
v_{a_1} \otimes v_{a_2} \otimes v_r \cdot e H_2 e = q^m v_{a_1} \otimes v_{a_2} \otimes v_r \cdot e.
$$

We have

$$
\begin{aligned}
& v_{a_1} \otimes v_{a_2} \otimes v_r \cdot eH_2e \\
=& \delta_{a_1,a_2}\tau_{a_1} \left(\sum_{i=1}^{m} \tau_i^{-1} q^{m-2i+1} v_i \otimes v_i \otimes v_r \right) \cdot H_2 e \\
=& \delta_{a_1,a_2}\tau_{a_1} \sum_{i=1}^{r-1} \tau_i^{-1} q^{m-2i+1} \big(v_i \otimes v_r \otimes v_i + (q-q^{-1}) v_i \otimes v_i \otimes v_r \big) \cdot e \\
& + \delta_{a_1,a_2}\tau_{a_1}\tau_r^{-1} q^{m-2r+1} \cdot q v_r \otimes v_r \otimes v_r \cdot e \\
& + \delta_{a_1,a_2}\tau_{a_1} \sum_{i=r+1}^{m} \tau_i^{-1} q^{m-2i+1} v_i \otimes v_r \otimes v_i \cdot e \\
=& \delta_{a_1,a_2}\tau_{a_1} \sum_{i=1}^{r-1} \tau_i^{-1} q^{m-2i+1} (q-q^{-1}) \left(\sum_{j=1}^{m} \tau_i \tau_j^{-1} q^{m-2j+1} v_j \otimes v_j \otimes v_r \right) \\
& + \delta_{a_1,a_2}\tau_{a_1}\tau_r^{-1} q^{m-2r+2} \left(\sum_{j=1}^{m} \tau_r \tau_j^{-1} q^{m-2j+1} v_j \otimes v_j \otimes v_r \right) \\
=& \delta_{a_1,a_2} \sum_{j=1}^{m} \tau_{a_1}\tau_j^{-1} q^{m-2j+1} \left(\sum_{i=1}^{r-1} q^{m-2i+1}(q-q^{-1}) + q^{m-2r+2} \right) v_j \otimes v_j \otimes v_r \\
=& q^m v_{a_1} \otimes v_{a_2} \otimes v_r \cdot e.
\end{aligned}
$$

The relation (Q7) can be easily verified. From the action of H_j we can easily obtain the action of H_j^{-1} as follows:

$$
\begin{aligned}
& v_{a_1} \otimes \cdots \otimes v_{a_n} \cdot H_j^{-1} \\
=& \begin{cases}
q^{-1} v_{a_1} \otimes \cdots \otimes v_{a_n}, & \text{if } a_j = a_{j+1}, \\
\cdots \otimes v_{a_{j+1}} \otimes v_{a_j} \otimes \cdots + (q^{-1}-q) v_{a_1} \otimes \cdots \otimes v_{a_n}, & \text{if } a_j > a_{j+1}, \\
\cdots \otimes v_{a_{j+1}} \otimes v_{a_j} \otimes \cdots, & \text{if } a_j < a_{j+1}.
\end{cases}
\end{aligned}
$$

In order to verify $e(H_2H_3H_1^{-1}H_2^{-1})e = e(H_2H_3H_1^{-1}H_2^{-1})eH_2H_3H_1^{-1}H_2^{-1}$, it suffices to show that

$$v_{a_1} \otimes v_{a_1} \otimes v_k \otimes v_l \cdot e(H_2H_3H_1^{-1}H_2^{-1})eH_2H_1v_{a_1} \otimes v_{a_1} \otimes v_k \otimes v_l \cdot e(H_2H_3H_1^{-1}H_2^{-1})eH_2H_3. \quad (6.2)$$

When $k < l$, we have

$$\begin{aligned}
&v_{a_1} \otimes v_{a_1} \otimes v_k \otimes v_l \cdot e(H_2H_3H_1^{-1}H_2^{-1})e \\
=&\left(\sum_{i=1}^{m} \tau_{a_1}\tau_i^{-1}q^{m-2i+1}v_i \otimes v_i \otimes v_k \otimes v_l\right) \cdot H_2H_3H_1^{-1}H_2^{-1}e \\
=&\sum_{i=1}^{k-1} \tau_{a_1}\tau_i^{-1}q^{m-2i+1}\left(v_i \otimes v_k \otimes v_i \otimes v_l + (q-q^{-1})v_i \otimes v_i \otimes v_k \otimes v_l\right) \cdot H_3H_1^{-1}H_2^{-1}e \\
&+ \tau_{a_1}\tau_k^{-1}q^{m-2k+1} \cdot qv_k \otimes v_k \otimes v_k \otimes v_l \cdot H_3H_1^{-1}H_2^{-1}e \\
&+ \sum_{i=k+1}^{m} \tau_{a_1}\tau_i^{-1}q^{m-2i+1}v_i \otimes v_k \otimes v_i \otimes v_l \cdot H_3H_1^{-1}H_2^{-1}e \\
=&\tau_{a_1}\tau_k^{-1}q^{m-2k+1} \cdot q(q-q^{-1})q^{-2}v_k \otimes v_k \otimes v_k \otimes v_l \cdot e \\
&+ \sum_{i=k+1}^{l-1} \tau_{a_1}\tau_i^{-1}q^{m-2i+1}(q-q^{-1})(q^{-1}-q)v_i \otimes v_i \otimes v_k \otimes v_l \cdot e \\
&+ \tau_{a_1}\tau_l^{-1}q^{m-2l+1} \cdot q(q^{-1}-q)v_l \otimes v_l \otimes v_k \otimes v_l \cdot e \\
=&\sum_{j=1}^{m} \tau_{a_1}\tau_j^{-1}q^{m-2j+1}v_j \otimes v_j \otimes v_k \otimes v_l \\
&\times \left(q^{m-2k}(q-q^{-1}) - (q-q^{-1})^2 \sum_{i=k+1}^{l-1} q^{m-2i+1} + q^{m-2l+2}(q^{-1}-q)\right) \\
=&0.
\end{aligned}$$

When $k > l$, it can be similarly shown that

$$v_{a_1} \otimes v_{a_1} \otimes v_k \otimes v_l \cdot e(H_2H_3H_1^{-1}H_2^{-1})e = 0.$$

When $k = l$, we have

$$
\begin{aligned}
& v_{a_1} \otimes v_{a_1} \otimes v_k \otimes v_k \cdot e(H_2 H_3 H_1^{-1} H_2^{-1}) e \\
= & \left(\sum_{i=1}^{m} \tau_{a_1} \tau_i^{-1} q^{m-2i+1} v_i \otimes v_i \otimes v_k \otimes v_k \right) \cdot H_2 H_3 H_1^{-1} H_2^{-1} e \\
= & \sum_{i=1}^{k-1} \tau_{a_1} \tau_i^{-1} q^{m-2i+1} v_k \otimes v_k \otimes v_i \otimes v_i \cdot e + \tau_{a_1} \tau_k^{-1} q^{m-2k+1} v_k \otimes v_k \otimes v_k \otimes v_k \cdot e \\
& + \sum_{i=k+1}^{m} \tau_{a_1} \tau_i^{-1} q^{m-2i+1} v_k \otimes v_k \otimes v_i \otimes v_i \cdot e \\
= & \sum_{i=1}^{m} \tau_{a_1} \tau_i^{-1} s q^{m-2i+1} v_k \otimes v_k \otimes v_i \otimes v_i \cdot e \\
= & \sum_{i,j=1}^{m} \tau_{a_1} \tau_k \tau_i^{-1} \tau_j^{-1} q^{2m-2i-2j+2} v_j \otimes v_j \otimes v_i \otimes v_i.
\end{aligned}
$$

It is straightforward to show that

$$
\begin{aligned}
& \sum_{i,j=1}^{m} \tau_{a_1} \tau_k \tau_i^{-1} \tau_j^{-1} q^{2m-2i-2j+2} v_j \otimes v_j \otimes v_i \otimes v_i \cdot H_2 H_1 \\
= & \sum_{i,j=1}^{m} \tau_{a_1} \tau_k \tau_i^{-1} \tau_j^{-1} q^{2m-2i-2j+2} v_j \otimes v_j \otimes v_i \otimes v_i \cdot H_2 H_3.
\end{aligned}
$$

Therefore, (6.2) holds. The equality

$$
H_2 H_3 H_1^{-1} H_2^{-1} e (H_2 H_3 H_1^{-1} H_2^{-1}) e = e (H_2 H_3 H_1^{-1} H_2^{-1}) e
$$

can be proved similarly. We are done. $\square$

Remark 6.2.3. Let O_m and SO_m denote the orthogonal group and special orthogonal group, respectively. As shown in [Br37], the Brauer algebra surjects onto $End_{O_m}(V^{\otimes k})$ for all $k \in \mathbb{Z}_{>0}$, where V is the natural representation of O_m. But if one replaces O_m by SO_m, then we have the following result (see [LZ06, §5.1.3]):

If m is odd, then $End_{O_m}(V^{\otimes k}) = End_{SO_m}(V^{\otimes k})$ for all k.

If m is even, then $End_{O_m}(V^{\otimes k}) = End_{SO_m}(V^{\otimes k})$ if and only if $m-1 \geqslant 2k$.

Theorem 6.2.4. (1) *The left action of* $\mathbf{U}^\imath(\mathfrak{so}_m)$ *on* $\mathbb{V}^{\otimes n}$ *commutes with the right action of* $\mathfrak{B}_n(q,q^m)$ *defined in Proposition 6.2.2:*

$$\mathbf{U}^\imath(\mathfrak{so}_m) \overset{\Psi}{\curvearrowright} \mathbb{V}^{\otimes n} \overset{\Phi}{\curvearrowleft} \mathfrak{B}_n(q,q^m).$$

(2) *When* m *is odd or* m *is even with* $m-1 \geqslant 2n$*, the following double centralizer property holds:*

$$\Psi(\mathbf{U}^\imath(\mathfrak{so}_m)) = End_{\ \mathfrak{B}_n(q,q^m)}(\mathbb{V}^{\otimes n}),$$
$$\Phi(\mathfrak{B}_n(q,q^m)) = End_{\ \mathbf{U}^\imath(\mathfrak{so}_m)}(\mathbb{V}^{\otimes n}).$$

Proof. (1) By the Jimbo duality in [Jim86], we know that the action of $\mathbf{U}$ commutes with the action of H_i for $1 \leq i \leq n-1$. Thus, to show the commuting actions of $\mathbf{U}^\imath(\mathfrak{so}_m)$ and $\mathfrak{B}_n(q,q^m)$, it remains to check the commutativity of the actions of B_i $(1 \leq i \leq m-1)$ and e.

Thanks to $\Delta(B_i) = B_i \otimes K_i^{-1} + 1 \otimes B_i$ and the fact that the action of e depends solely on the first two tensor factors, it suffices to consider $n=2$. By a direct calculation, it can be shown that

$$B_i \cdot (v_{a_1} \otimes v_{a_2} \cdot e) = 0 = (B_i \cdot v_{a_1} \otimes v_{a_2}) \cdot e.$$

We omit the details.

(2) The double centralizer property is equivalent to the multiplicity-free decomposition of $\mathbb{V}^{\otimes n}$ as an $\mathbf{U}^\imath(\mathfrak{so}_m)$-$\mathfrak{B}_n(q,q^m)$-bimodule. According to [We12a, §4.5] or [N18, §4], the q-

Brauer algebra $\mathfrak{B}_n(q,q^m)$ is semisimple when q is generic; moreover, when taking the $q \to 1$ limit, the cell module of $\mathfrak{B}_n(q,q^m)$ recovers the cell module of the classical Brauer algebra defined in [GL96]. Thus, the proof of the double centralizer property reduces by a deformation argument to the $q = 1$ setting. When taking the $q \to 1$ limit and $\varsigma_i = -1$ $(1 \le i \le m-1)$, $\mathbf{U}^\imath(\mathfrak{so}_m)$ becomes the enveloping algebra of the special orthogonal Lie algebra $\mathfrak{so}_m$, $\mathbb{V}$ becomes its natural representation. By lifting, $\mathbb{V}$ can also be regarded as a representation of the special orthogonal group SO_m. Moreover, according to Remark 6.2.3, when m is odd or m is even with $m-1 \geqslant 2n$, we have

$$End_{\mathfrak{so}_m}(\mathbb{V}^{\otimes n}) = End_{SO_m}(\mathbb{V}^{\otimes n}) = End_{O_m}(\mathbb{V}^{\otimes n}).$$

The multiplicity-free decomposition of $\mathbb{V}^{\otimes n}$ in this case has been established in [Br37], [Br56a] and [Br56b]. We are done. $\square$

Remark 6.2.5. The condition on m required in Theorem 6.2.4(2) can be removed if we enlarge the $\imath$quantum group to an algebra generated by $\mathbf{U}^\imath(\mathfrak{so}_m)$ and ϱ over $\mathbb{Q}(q)$ with the following relations:

$$\varrho^2 = 1, \quad \varrho B_i = (-1)^{\delta_{1,i}} B_i \varrho \quad \text{for } 1 \le i \le m-1.$$

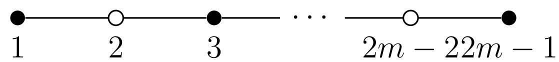
We put the action of ϱ on $\mathbb{V}$ by

$$\varrho \cdot v_r = \begin{cases} -v_1 & \text{if } r = 1, \\ v_r & \text{if } r > 1. \end{cases}$$

One can show this action commutes with the q-Brauer algebra action and when taking the $q \to 1$ limit, the new algebra specializes to $\mathbf{U}(\mathfrak{so}_m) \oplus \mathbf{U}(\mathfrak{so}_m)\varrho$.

6.3 $\imath$quantum group of type AII

In this section we fix $m \in \mathbb{Z}_{\geq 1}$ and focus on the quantum symmetric pair of type AII with the Satake diagram as below (cf. [BW18b, Table 4]):



Let $\mathbf{U}_q(\mathfrak{sl}_{2m})$ denote the corresponding quantum group over $\mathbb{Q}(q)$. According to § 2.2, we have

Definition 6.3.1. The $\imath$quantum group $\mathbf{U}^\imath(\mathfrak{sp}_{2m})$ of type AII, with a set of parameters $\{\varsigma_i \mid i = 2, 4, \ldots, 2m-2\} \subset \mathbb{Z}[q, q^{-1}]$, is the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}_q(\mathfrak{sl}_{2m})$ generated by the following elements:

$$B_i = F_i + \varsigma_i T_{i-1} T_{i+1}(E_i) K_i^{-1} \quad \text{for } i = 2, 4, \ldots, 2m-2,$$

$$E_j, F_j, K_j^{\pm 1} \quad \text{for } j = 1, 3, \ldots, 2m-1.$$

Remark 6.3.2. Let $E_{i,j}$ denote the $2m \times 2m$ elementary matrices and M be a $2m \times 2m$ skew-symmetric quasi-diagonal matrix $M = diag\{J, \ldots, J\}$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Suppose $\varsigma_i = -1$ (i even). When taking the $q \to 1$ limit in $\mathbf{U}^\imath(\mathfrak{sp}_{2m})$, we see that the generators E_j and F_j (j odd) specialize to matrices $E_{j,j+1}$ and $E_{j+1,j}$ respectively. Moreover, B_i (i even) specializes to $E_{i+1,i} + E_{i-1,i+2}$.

Therefore, $\mathbf{U}^\imath(\mathfrak{sp}_{2m})$ indeed specializes to the enveloping algebra of the symplectic Lie algebra $\mathfrak{sp}_{2m}$, which is characterized as a Lie algebra consisting of all $2m \times 2m$ matrices X satisfying the condition $X^t M + MX = 0$.

Let $\mathbb{W} = \sum_{i=1}^{2m} \mathbb{Q}(q) v_i$ be the natural representation of $\mathbf{U}_q(\mathfrak{sl}_{2m})$. By a direct calculation we see that the action of B_{2l} ($l = 1, 2, \ldots, m-1$) on $\mathbb{W}$ is given by

$$B_{2l} \cdot v_r = \begin{cases} v_{2l+1} & \text{if } r = 2l, \\ -q^{-1}\varsigma_{2l} v_{2l-1} & \text{if } r = 2l+2, \\ 0 & \text{else.} \end{cases}$$

6.4 $\imath$Schur duality of type AII

For $i = 2, \ldots, m$, we set

$$\eta_i := \prod_{j=1}^{i-1} (-\varsigma_{2j})$$

and $\eta_1 = 1$.

Then the following lemma gives a right $\mathfrak{B}_n(-q^{-1}, q^{2m})$-module structure on $\mathbb{W}^{\otimes n}$.

Proposition 6.4.1. *There is a right action of $\mathfrak{B}_n(-q^{-1}, q^{2m})$ on $\mathbb{W}^{\otimes n}$ via*

$$\begin{aligned} & v_{a_1} \otimes \cdots \otimes v_{a_n} \cdot H_k \\ = & \begin{cases} q v_{a_1} \otimes \cdots \otimes v_{a_n} & \text{if } a_k = a_{k+1}, \\ \cdots \otimes v_{a_{k+1}} \otimes v_{a_k} \otimes \cdots & \text{if } a_k > a_{k+1}, \\ \cdots \otimes v_{a_{k+1}} \otimes v_{a_k} \otimes \cdots + (q - q^{-1}) v_{a_1} \otimes \cdots \otimes v_{a_n} & \text{if } a_k < a_{k+1}, \end{cases} \end{aligned}$$

and

$$
v_{a_1} \otimes v_{a_2} \otimes \cdots \otimes v_{a_n} \cdot e
= \begin{cases}
\sum_{j=1}^{m} \eta_i \eta_j^{-1} q^{2m+1-3i-j} \big(v_{2j-1} \otimes v_{2j} - q v_{2j} \otimes v_{2j-1}\big) \otimes v_{a_3} \otimes \cdots \otimes v_{a_n} & \\
 & \textit{if } a_1 = 2i-1, a_2 = 2i, \\
(-q) v_{a_2} \otimes v_{a_1} \otimes v_{a_3} \otimes \cdots \otimes v_{a_n} \cdot e & \textit{if } a_1 = 2i, a_2 = 2i-1, \\
0 & \textit{else},
\end{cases}
$$

where $i = 1, 2, \ldots, m$.

Proof. Noting that the action of e depends solely on the first two tensor factors, in order to verify the relation (Q4) in Definition 5.2.1 it suffices to show that

$$
v_1 \otimes v_2 \cdot e^2 = \frac{q^{2m} - q^{-2m}}{q - q^{-1}} v_1 \otimes v_2 \cdot e.
$$

We have

$$
\begin{aligned}
v_1 \otimes v_2 \cdot e^2 =& \sum_{j=1}^{m} \eta_1 \eta_j^{-1} q^{2m-2-j} \big(v_{2j-1} \otimes v_{2j} - q v_{2j} \otimes v_{2j-1}\big) \cdot e \\
=& \sum_{j=1}^{m} \eta_1 \eta_j^{-1} q^{2m-2-j} (1+q^2) (v_{2j-1} \otimes v_{2j}) \cdot e \\
=& \sum_{j=1}^{m} \eta_1 \eta_j^{-1} q^{2m-2-j} (1+q^2) \eta_j \eta_1^{-1} q^{-3(j-1)} (v_1 \otimes v_2) \cdot e \\
=& (1+q^2) \left(\sum_{j=1}^{m} q^{2m+1-4j} \right) (v_1 \otimes v_2) \cdot e \\
=& \frac{q^{2m} - q^{-2m}}{q - q^{-1}} (v_1 \otimes v_2) \cdot e.
\end{aligned}
$$

The relation (Q5) can be easily verified. In order to verify the relation (Q6), it suffices to show that

$$v_1 \otimes v_2 \otimes v_r \cdot eH_2e = q^{2m} v_1 \otimes v_2 \otimes v_r \cdot e.$$

When $r = 2k-1$, we have

$$\begin{aligned}
& v_1 \otimes v_2 \otimes v_{2k-1} \cdot eH_2e \\
=& \sum_{j=1}^{m} \eta_1 \eta_j^{-1} q^{2m-2-j} (v_{2j-1} \otimes v_{2j} \otimes v_{2k-1} - q v_{2j} \otimes v_{2j-1} \otimes v_{2k-1}) \cdot H_2 e \\
=& \sum_{j=1}^{k-1} q^{2m+1-4j} (q - q^{-1}) v_1 \otimes v_2 \otimes v_{2k-1} \cdot e \\
& - q \sum_{j=1}^{k-1} q^{2m+1-4j} (q - q^{-1})(-q) v_1 \otimes v_2 \otimes v_{2k-1} \cdot e \\
& - q^{2m+2-4k} (-q^2) v_1 \otimes v_2 \otimes v_{2k-1} \cdot e \\
=& \left((q - q^{-1})(1 + q^2) \sum_{j=1}^{k-1} q^{2m+1-4j} + q^{2m+4-4k} \right) v_1 \otimes v_2 \otimes v_{2k-1} \cdot e \\
=& q^{2m} v_1 \otimes v_2 \otimes v_r \cdot e.
\end{aligned}$$

When $r = 2k$, it can be proved similarly. (Q7) can be easily verified.

In order to verify $e(H_2H_3H_1^{-1}H_2^{-1})e = e(H_2H_3H_1^{-1}H_2^{-1})eH_2H_3H_1^{-1}H_2^{-1}$, it suffices to show that

$$v_1 \otimes v_2 \otimes v_p \otimes v_q \cdot e(H_2H_3H_1^{-1}H_2^{-1})eH_2H_1 = v_1 \otimes v_2 \otimes v_p \otimes v_q \cdot e(H_2H_3H_1^{-1}H_2^{-1})eH_2H_3. \quad (6.3)$$

When $p = 2k$ and $q = 2l$ with $k < l$, we have

$$
\begin{aligned}
&v_1 \otimes v_2 \otimes v_{2k} \otimes v_{2l} \cdot e(H_2 H_3 H_1^{-1} H_2^{-1})e \\
=&\sum_{j=1}^{m} \eta_1 \eta_j^{-1} q^{2m-2-j} (v_{2j-1} \otimes v_{2j} \otimes v_{2k} \otimes v_{2l} - \\
&\qquad\qquad\qquad\qquad q v_{2j} \otimes v_{2j-1} \otimes v_{2k} \otimes v_{2l}) \cdot H_2 H_3 H_1^{-1} H_2^{-1} e \\
=&-(q-q^{-1})^2 \sum_{j=k+1}^{l-1} \eta_1 \eta_j^{-1} q^{2m-2-j} v_{2j-1} \otimes v_{2j} \otimes v_{2k} \otimes v_{2l} \cdot e \\
&+ \eta_1 \eta_l^{-1} q^{2m-2-l} \cdot q(q^{-1} - q) v_{2l-1} \otimes v_{2l} \otimes v_{2k} \otimes v_{2l} \cdot e \\
&- \eta_1 \eta_k^{-1} q^{2m-1-k} (q - q^{-1}) q^{-1} v_{2k} \otimes v_{2k-1} \otimes v_{2k} \otimes v_{2l} \cdot e \\
&- \eta_1 \eta_k^{-1} q^{2m-1-k} (q - q^{-1})^2 q^{-1} v_{2k-1} \otimes v_{2k} \otimes v_{2k} \otimes v_{2l} \cdot e \\
&+ (q-q^{-1})^2 \sum_{j=k+1}^{l} \eta_1 \eta_j^{-1} q^{2m-1-j} v_{2j} \otimes v_{2j-1} \otimes v_{2k} \otimes v_{2l} \cdot e \\
=&A v_1 \otimes v_2 \otimes v_{2k} \otimes v_{2l} \cdot e,
\end{aligned}
$$

where

$$
\begin{aligned}
A =&-(q-q^{-1})^2 \sum_{j=k+1}^{l-1} q^{2m+1-4j} + (q^{-1} - q) q^{2m+2-4l} \\
&+ (q - q^{-1}) q^{2m+2-4k} - (q-q^{-1})^2 q^{2m+1-4k} - (q-q^{-1})^2 \sum_{j=k+1}^{l} q^{2m+3-4j} \\
=&-(q-q^{-1})^2 (1+q^2) \sum_{j=k+1}^{l-1} q^{2m+1-4j} \\
&\qquad\qquad\qquad\qquad + (q - q^{-1}) q^{2m-4k} + (q^{-1} - q) q^{2m+4-4l} \\
=&0.
\end{aligned}
$$

Therefore, in this case we have $v_1 \otimes v_2 \otimes v_{2k} \otimes v_{2l} \cdot e(H_2 H_3 H_1^{-1} H_2^{-1})e = 0$.

In a similar way, we can show that $v_1 \otimes v_2 \otimes v_{2k} \otimes v_{2l} \cdot e(H_2 H_3 H_1^{-1} H_2^{-1})e = 0$ when

$k \geq l$, $v_1 \otimes v_2 \otimes v_{2k-1} \otimes v_{2l-1} \cdot e(H_2H_3H_1^{-1}H_2^{-1})e = 0$ for any k, l, and $v_1 \otimes v_2 \otimes v_{2k} \otimes v_{2l-1} \cdot e(H_2H_3H_1^{-1}H_2^{-1})e = 0 = v_1 \otimes v_2 \otimes v_{2k-1} \otimes v_{2l} \cdot e(H_2H_3H_1^{-1}H_2^{-1})e$ when $k \neq l$.

When $p = 2k$ and $q = 2k-1$, we have

$$\begin{aligned}
& v_1 \otimes v_2 \otimes v_{2k} \otimes v_{2k-1} \cdot e(H_2H_3H_1^{-1}H_2^{-1})e \\
=& \sum_{j=1}^{k-1} \eta_1\eta_j^{-1}q^{2m-2-j} v_{2k} \otimes v_{2k-1} \otimes v_{2j-1} \otimes v_{2j} \cdot e \\
& + \eta_1\eta_k^{-1}q^{2m-2-k} v_{2k} \otimes v_{2k-1} \otimes v_{2k-1} \otimes v_{2k} \cdot e \\
& + \sum_{j=k+1}^{m} \eta_1\eta_j^{-1}q^{2m-2-j} v_{2k} \otimes v_{2k-1} \otimes v_{2j-1} \otimes v_{2j} \cdot e \\
& - \sum_{j=1}^{k-1} \eta_1\eta_j^{-1}q^{2m-1-j} v_{2k} \otimes v_{2k-1} \otimes v_{2j} \otimes v_{2j-1} \cdot e \\
& - \eta_1\eta_k^{-1}sq^{2m-1-k} v_{2k} \otimes v_{2k-1} \otimes v_{2k} \otimes v_{2k-1} \cdot e \\
& - \sum_{j=k+1}^{m} \eta_1\eta_j^{-1}q^{2m-1-j} v_{2k} \otimes v_{2k-1} \otimes v_{2j} \otimes v_{2j-1} \cdot e \\
=& \sum_{j=1}^{m} \eta_1\eta_j^{-1}q^{2m-2-j} v_{2k} \otimes v_{2k-1} \otimes (v_{2j-1} \otimes v_{2j} - qv_{2j} \otimes v_{2j-1}) \cdot e \\
=& B \cdot \sum_{i,j=1}^{m} \eta_1^2\eta_i^{-1}\eta_j^{-1}q^{4m-4-i-j} (v_{2i-1} \otimes v_{2i} - qv_{2i} \otimes v_{2i-1}) \otimes \\
& \qquad (v_{2j-1} \otimes v_{2j} - qv_{2j} \otimes v_{2j-1}),
\end{aligned}$$

where $B = -q^{4-3k}\eta_k\eta_1^{-1}$. By a direct calculation, we can show that

$$\begin{aligned}
& \sum_{i,j=1}^{m} \eta_1^2\eta_i^{-1}\eta_j^{-1}q^{4m-4-i-j} (v_{2i-1} \otimes v_{2i} - qv_{2i} \otimes v_{2i-1}) \otimes \\
& \qquad (v_{2j-1} \otimes v_{2j} - qv_{2j} \otimes v_{2j-1}) \cdot H_2H_1 \\
=& \sum_{i,j=1}^{m} \eta_1^2\eta_i^{-1}\eta_j^{-1}q^{4m-4-i-j} (v_{2i-1} \otimes v_{2i} - qv_{2i} \otimes v_{2i-1}) \otimes \\
& \qquad (v_{2j-1} \otimes v_{2j} - qv_{2j} \otimes v_{2j-1}) \cdot H_2H_3.
\end{aligned}$$

Therefore, (6.3) holds when $p = 2k$ and $q = 2k-1$. Similarly, we can show that (6.3) holds when $p = 2k-1$ and $q = 2k$. The equality

$$H_2H_3H_1^{-1}H_2^{-1}e(H_2H_3H_1^{-1}H_2^{-1})e = e(H_2H_3H_1^{-1}H_2^{-1})e$$

can be proved similarly. We are done. $\square$

Theorem 6.4.2. *The left action of* $\mathbf{U}^\imath(\mathfrak{sp}_{2m})$ *on* $\mathbb{W}^{\otimes n}$ *commutes with the right action defined in Proposition 6.4.1:*

$$\mathbf{U}^\imath(\mathfrak{sp}_{2m}) \overset{\Psi'}{\curvearrowright} \mathbb{W}^{\otimes n} \overset{\Phi'}{\curvearrowleft} \mathfrak{B}_n(-q^{-1}, q^{2m}).$$

Moreover, the following double centralizer property holds:

$$\Psi'(\mathbf{U}^\imath(\mathfrak{sp}_{2m})) = End_{\mathfrak{B}_n(-q^{-1},q^{2m})}(\mathbb{W}^{\otimes n}),$$
$$\Phi'(\mathfrak{B}_n(-q^{-1}, q^{2m})) = End_{\mathbf{U}^\imath(\mathfrak{sp}_{2m})}(\mathbb{W}^{\otimes n}).$$

Proof. By the Jimbo duality in [Jim86], we know that the action of $\mathbf{U}_q(\mathfrak{sl}_{2m})$ commutes with the action of H_k for $1 \leq k \leq n-1$. Thus, to show the commuting actions of $\mathbf{U}^\imath(\mathfrak{sp}_{2m})$ and $\mathfrak{B}_n(-q^{-1}, q^{2m})$, it remains to check the commutativity of the actions of the generators of $\mathbf{U}^\imath(\mathfrak{sp}_{2m})$ and e. Noting that the action of e depends solely on the first two tensor factors, it suffices to consider $n = 2$.

We have $E_1 \cdot v_1 \otimes v_2 = qv_1 \otimes v_1$, $E_1 \cdot v_2 \otimes v_1 = v_1 \otimes v_1$ and $E_1 \cdot v_k \otimes v_l = 0$ for $\{k, l\} \neq \{1, 2\}$, which imply that

$$E_1 \cdot (v_{a_1} \otimes v_{a_2} \cdot e) = 0 = (E_1 \cdot v_{a_1} \otimes v_{a_2}) \cdot e.$$

Similarly, we can show that

$$E_j \cdot (v_{a_1} \otimes v_{a_2} \cdot e) = 0 = (E_j \cdot v_{a_1} \otimes v_{a_2}) \cdot e,$$

$$F_j \cdot (v_{a_1} \otimes v_{a_2} \cdot e) = 0 = (F_j \cdot v_{a_1} \otimes v_{a_2}) \cdot e,$$

$$K_j^{\pm 1} \cdot (v_{a_1} \otimes v_{a_2} \cdot e) = v_{a_1} \otimes v_{a_2} \cdot e = (K_j^{\pm 1} \cdot v_{a_1} \otimes v_{a_2}) \cdot e$$

for $j = 1, 3, \ldots, 2m-1$.

According to [Ko14, Example 7.9] we have

$$\begin{aligned}
&\Delta(B_{2l})|_{\mathbb{W}^{\otimes 2}} \\
=&B_{2l} \otimes K_{2l}^{-1} + 1 \otimes F_{2l} + \varsigma_{2l}(q - q^{-1})E_{2l+1} \otimes E_{2l-1}E_{2l} \\
&- \varsigma_{2l}(q^{-1} - q^{-3})E_{2l-1} \otimes E_{2l}E_{2l+1} - \varsigma_{2l}q^{-1}K_{2l-1}K_{2l+1} \otimes E_{2l-1}E_{2l}E_{2l+1}.
\end{aligned}$$

Therefore, we have $B_2 \cdot v_1 \otimes v_2 = v_1 \otimes v_3$, $B_2 \cdot v_2 \otimes v_1 = v_3 \otimes v_1$, $B_2 \cdot v_3 \otimes v_4 = -\varsigma_2 v_3 \otimes v_1$ and $B_2 \cdot v_4 \otimes v_3 = -\varsigma_2 v_1 \otimes v_3 + \varsigma_2(q - q^{-1})v_3 \otimes v_1$. Thus,

$$\begin{aligned}
B_2 \cdot (v_1 \otimes v_2 \cdot e) =&B_2 \cdot (q^{2m-3}(v_1 \otimes v_2 - qv_2 \otimes v_1) - \varsigma_2^{-1}q^{2m-4}(v_3 \otimes v_4 - qv_4 \otimes v_3)) \\
=&0,
\end{aligned} \tag{6.4}$$

and $(B_2 \cdot v_1 \otimes v_2) \cdot e = v_1 \otimes v_3 \cdot e = 0$. By (6.4), we have $B_2 \cdot (v_{a_1} \otimes v_{a_2} \cdot e) = 0$. By a direct calculation, we can show that $(B_2 \cdot v_{a_1} \otimes v_{a_2}) \cdot e = 0$. Similarly, we can show that

$$B_{2l} \cdot (v_{a_1} \otimes v_{a_2} \cdot e) = 0 = (B_{2l} \cdot v_{a_1} \otimes v_{a_2}) \cdot e,$$

for any $l = 2, \ldots, m-1$. We omit the details.

The proof of the double centralizer property is almost identical to the proof of Proposition 6.2.4, which is equivalent to the multiplicity-free decomposition of $\mathbb{W}^{\otimes n}$ as an $\mathbf{U}^\imath(\mathfrak{sp}_{2m})$-$\mathfrak{B}_n(-q^{-1}, q^{2m})$-bimodule. The proof of the double centralizer property reduces by a deformation argument to the $q = 1$ setting. When taking the $q \to 1$ limit and $\varsigma_i = -1$, $\mathbf{U}^\imath(\mathfrak{sp}_{2m})$ becomes the enveloping algebra of the symplectic Lie algebra $\mathfrak{sp}_{2m}$, $\mathbb{W}$ becomes its natural representation, and the multiplicity-free decomposition of $\mathbb{W}^{\otimes n}$ in this case has been established in [Br37], [Br56a] and [Br56b]. We are done. $\square$

Part III

Quantum supersymmetric pairs and $\imath$Schur dualities

Chapter 7

Quantum supersymmetric pairs of type AIII

Recall basic set ups of $\mathfrak{g} = \mathfrak{gl}(m|n)$ and its quantum analogue $\mathbf{U}$ from § 2.3. In this chapter we construct quantum supersymmetric pairs $(\mathbf{U}, \mathbf{U}^\imath)$ of type AIII and elucidate their fundamental properties. An $\imath$Schur duality between the $\imath$quantum supergroup $\mathbf{U}^\imath$ and the Hecke algebra of type B acting on a tensor space is established, providing a super generalization of the $\imath$Schur duality of type AIII in Part I.

7.1 Braid group operators

In order to define quantum supersymmetric pairs, we need to study the braid group operators on $\mathbf{U}$, especially the ones associated with odd simple roots.

7.1.1 Odd reflections

As noted in [Ya99], when fixing a simple root α, the braid operator associated with it extends the action of s_α on the weight data. The key distinction between odd and even reflections lies in the fact that odd reflections change the generalized Cartan matrix A, while even reflections do not; see also [C16, §4].

The fundamental systems of the root system Φ associated to $\mathfrak{gl}(m|n)$ are not conjugated under the Weyl group actions because of the existence of odd roots (cf. [CW12, §1.3.6]). In fact, we have the following lemma,

Lemma 7.1.1. [Ya99, Proposition 2.2.1], [CW12, Lemma 1.26] *Let α be an odd simple root of $\mathfrak{gl}(m|n)$ in a positive system Φ^+. Then,*

$$\Phi^+_\alpha := \{-\alpha\} \cup \Phi^+ \backslash \{\alpha\}$$

is a new positive system, whose corresponding fundamental system Π_α is given by

$$\Pi_\alpha = \{\beta \in \Pi \mid (\beta, \alpha) = 0, \beta \neq \alpha\} \cup \{\beta + \alpha \mid \beta \in \Pi, (\beta, \alpha) \neq 0\} \cup \{-\alpha\}. \tag{7.1}$$

The operation of obtaining Π_α from Π is denoted by s_α and referred to as an odd reflection. When $\beta \in \Pi$ is an even simple root, we abuse the notation s_β to denote the even reflection associated to β. For a diagram as in (2.6), we let $s_j := s_{\alpha_j}$ for all $j \in I$.

Let $\mathcal{D}_{m,n}$ denote the set of all possible Dynkin diagrams for $\mathfrak{gl}(m|n)$. The following lemma provides information on how the reflections change parities, which enables us to determine the matrix units of A. For any diagram $X \in \mathcal{D}_{m,n}$, we denote by $p_{\mathtt{X}}$ the corresponding parity function.

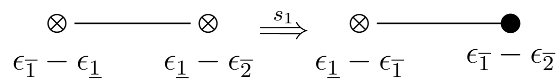
Lemma 7.1.2. *If $j, k, \ell \in I$ with $j \sim k$ and $j \nsim \ell$. Then for any $X \in \mathcal{D}_{m,n}$ we have*

$$p_{s_j(X)}(j) = p_X(j), \quad p_{s_j(X)}(k) = p_X(k) + p_X(j) \mod 2, \quad p_{s_j(X)}(\ell) = p_X(\ell).$$

Proof. We always have $s_j(\alpha_j) = -\alpha_j$, $s_j(\alpha_\ell) = \alpha_\ell$ and $s_j(\alpha_k) = \alpha_k + \alpha_j$ for a diagram as in (2.6). $\square$

From Lemma 7.1.2 we see that even reflections will not change the parity of any simple root while odd reflections change the parities of the ones adjacent to it.

Example 7.1.3.



More precisely, for any two fundamental systems Π and Π' of a basic Lie superalgebra of any classical type, there exists a sequence consisting of even and odd reflections $s_1, \ldots, s_k$ such that $s_1 \cdots s_k(\Pi) = \Pi'$. (cf. [CW12])

7.1.2 Generalized braid group operators

To avoid confusion, X and Y in this chapter represents diagrams in $\mathcal{D}_{m,n}$ rather than the set of roots and coroots for a root datum.

For each $X \in \mathcal{D}_{m,n}$, we can associate a quantum enveloping algebra $\mathbf{U}(X)$ with generators $E_i^{\mathtt{X}}$, $F_i^{\mathtt{X}}$, q^μ and $\varrho_{\mathtt{X}}$ as in § 2.3. Equipped with this family of algebras, the braid group operators were constructed in [Ya99, Proposition 7.5.1]. In [C16, Theorem 4.5], an equivalent reformulation of these operators was given in the case of $\mathfrak{gl}(m|1)$. In Theorem 7.1.4, we adopt the notations from [C16, Theorem 4.5] and restate the results of [Ya99, Proposition 7.5.1] specifically for $\mathfrak{gl}(m|n)$.

Theorem 7.1.4. *Let $i \in I$, $X \in \mathcal{D}_{m,n}$, $e = \pm 1$ and set $Y = s_i(X)$. There exist $\mathbb{Q}(q)$-linear algebra isomorphisms $T'_{i,e}, T''_{i,e} : \mathbf{U}(X) \to \mathbf{U}(Y)$ satisfying*

$$T'_{i,-e}(E_j^{\mathtt{X}}) = \begin{cases} -(-1)^{p_Y(i)} K_{\mathtt{Y},i}^{-e} F_{\mathtt{Y},i}, & \textit{if} \quad j = i, \\ E_{\mathtt{Y},j} E_{\mathtt{Y},i} - (-1)^{p_Y(i)p_Y(j)} q^{e(\alpha_{\mathtt{Y},i}, \alpha_{\mathtt{Y},j})} E_{\mathtt{Y},i} E_{\mathtt{Y},j} & \textit{if} \quad j \sim i, \\ E_{\mathtt{Y},j} & \textit{if} \quad j \nsim i. \end{cases} \tag{7.2}$$

$$T'_{i,-e}(F_j^{\boldsymbol{X}}) = \begin{cases} -(-1)^{p_Y(i)} E_{\boldsymbol{Y},i} K^e_{Y,i}, & \textit{if} \quad j = i, \\ F_{\boldsymbol{Y},i} F_{\boldsymbol{Y},j} - (-1)^{p_Y(i)p_Y(j)} q^{-e(\alpha_{Y,i},\alpha_{Y,j})} F_{\boldsymbol{Y},j} F_{\boldsymbol{Y},i} & \textit{if} \quad j \sim i, \\ F_{\boldsymbol{Y},j} & \textit{if} \quad j \nsim i. \end{cases} \tag{7.3}$$

$$T'_{i,-e}(K_j^{\boldsymbol{X}}) = \begin{cases} (-1)^{p_Y(i)} K^{-1}_{\boldsymbol{Y},i}, & \textit{if} \quad j = i, \\ (-1)^{p_Y(i)p_Y(j)} K_{\boldsymbol{Y},i} K_{\boldsymbol{Y},j} & \textit{if} \quad j \sim i, \\ K_{\boldsymbol{Y},j} & \textit{if} \quad j \nsim i. \end{cases} \tag{7.4}$$

$$T'_{i,-e}(\varrho_{\boldsymbol{X}}) = \varrho_{\boldsymbol{Y}}. \tag{7.5}$$

and

$$T''_{i,e}(E_j^{\boldsymbol{X}}) = \begin{cases} -F_{\boldsymbol{Y},i} K^e_{\boldsymbol{Y},i}, & \textit{if} \quad j = i, \\ E_{\boldsymbol{Y},i} E_{\boldsymbol{Y},j} - (-1)^{p_Y(i)p_Y(j)} q^{e(\alpha_{\boldsymbol{Y},i},\alpha_{\boldsymbol{Y},j})} E_{\boldsymbol{Y},j} E_{\boldsymbol{Y},i} & \textit{if} \quad j \sim i, \\ E_{\boldsymbol{Y},j} & \textit{if} \quad j \nsim i. \end{cases} \tag{7.6}$$

$$T''_{i,e}(F_j^{\boldsymbol{X}}) = \begin{cases} -K^{-e}_{\boldsymbol{Y},i} E_{\boldsymbol{Y},i}, & \textit{if} \quad j = i, \\ F_{\boldsymbol{Y},j} F_{\boldsymbol{Y},i} - (-1)^{p_Y(i)p_Y(j)} q^{-e(\alpha_{\boldsymbol{Y},i},\alpha_{\boldsymbol{Y},j})} F_{\boldsymbol{Y},i} F_{\boldsymbol{Y},j} & \textit{if} \quad j \sim i, \\ F_{\boldsymbol{Y},j} & \textit{if} \quad j \nsim i. \end{cases} \tag{7.7}$$

$$T''_{i,e}(K_{\boldsymbol{Y},j}) = \begin{cases} (-1)^{p_Y(i)} K^{-1}_{\boldsymbol{Y},i}, & \textit{if} \quad j = i, \\ (-1)^{p_Y(i)p_Y(j)} K_{\boldsymbol{Y},i} K_{\boldsymbol{Y},j} & \textit{if} \quad j \sim i, \\ K_{\boldsymbol{Y},j} & \textit{if} \quad j \nsim i. \end{cases} \tag{7.8}$$

$$T''_{i,e}(\varrho_{\boldsymbol{X}}) = \varrho_{\boldsymbol{Y}}. \tag{7.9}$$

To ensure self-consistency, we will now present the proof of Theorem 7.1.4 in the remaining part of this subsection. To do so succinctly, recall σ and $\bar{\cdot}$ from (2.13), we observe that

$$T'_{i,-e} = \sigma T''_{i,e}\sigma, \quad T'_{i,-e} = \bar{\ }T'_{i,e}\bar{\ }, \quad T''_{i,e} = (T'_{i,-e})^{-1}. \tag{7.10}$$

One can check these identities on the generators of $\mathbf{U}(X)$. Thus to establish Theorem 7.1.4, it is sufficient to focus on the case of $T'_{j,-1}$. Specifically, we need to demonstrate that the images of the generators of $\mathbf{U}(X)$ under $T'_{j,-1}$ satisfy the relations in (2.11) and (2.15). To ensure the clarity of the proof, we will break down the verification into lemmas and make reference to relevant results from [C16]. Given the complexity of the calculations involved, we will omit the subscripts on the generators of $\mathbf{U}(Y)$ for readability. Additionally, we will consistently omit the subscript on ϱ since it is evident from the context which algebra it belongs to.

First we take a look at (2.15), we have

Lemma 7.1.5. *If $j, k \in I$, then*

$$\begin{aligned}
\varrho T'_{j,-1}(E_k^{\mathtt{X}})\varrho^{-1} &= T'_{j,-1}(\varrho(E_k^{\mathtt{X}})) = (-1)^{p_X(k)}T'_{j,-1}(E_k^{\mathtt{X}}),\\
\varrho T'_{j,-1}(F_k^{\mathtt{X}})\varrho^{-1} &= T'_{j,-1}(\varrho(F_k^{\mathtt{X}})) = (-1)^{p_X(k)}T'_{j,-1}(F_k^{\mathtt{X}}),\\
\varrho T'_{j,-1}(K_k^{\mathtt{X}})\varrho^{-1} &= T'_{j,-1}(\varrho(K_k^{\mathtt{X}})) = (-1)^{p_X(k)}T'_{j,-1}(K_k^{\mathtt{X}}).
\end{aligned}$$

Proof. We prove for E and the other two are similar.

When $j = k$ or $j \nsim k$, by a direct computation we have $p(k) = p_X(k)$ and

$$\varrho T'_{j,-1}(E_k^{\mathtt{X}})\varrho^{-1} = (-1)^{p(k)}T'_{j,-1}(E_k^{\mathtt{X}}) = (-1)^{p_X(k)}T'_{j,-1}(E_k^{\mathtt{X}}).$$

When $j \sim k$, by a direct computation we have $p(k) = p_X(j)+p_X(k)$ and thus $p(k)+p(j) = p_X(k)$. Hence

$$\varrho T'_{j,-1}(E_k^{\mathtt{X}})\varrho^{-1} = (-1)^{p(k)+p(j)}T'_{j,-1}(E_k^{\mathtt{X}}) = (-1)^{p_X(k)}T'_{j,-1}(E_k^{\mathtt{X}}).$$

This proves the lemma. $\square$

Recall the defining relations of $\mathbf{U}$ from (2.11). The relations (R1)–(R4) can be verified directly. For the relation (R6), we have

Lemma 7.1.6. *If $j, k \in I$ such that $p_X(k) = 1$, then*

$$T'_{j,-1}(E_k^{\mathbf{x}})^2 = T'_{j,-1}(F_k^{\mathbf{x}})^2 = 0.$$

Proof. It follows from the same argument as in [C16, Lemma 4.7]. $\square$

To verify the relation (R5), we split into two cases.

Lemma 7.1.7. *If $j = k \in I$, then*

$$\begin{aligned} &T'_{j,-1}(E_k^{\mathbf{x}})T'_{j,-1}(F_k^{\mathbf{x}}) - (-1)^{p_X(k)}T'_{j,-1}(F_k^{\mathbf{x}})T'_{j,-1}(E_k^{\mathbf{x}}) \\ =&\frac{T'_{j,-1}(K_k^{\mathbf{x}}) - T'_{j,-1}(K_k^{\mathbf{x}})^{-1}}{q^{\ell_k} - q^{-\ell_k}}. \end{aligned}$$

Proof. It follows from the same argument as in [C16, Lemma 4.9]. $\square$

To verify the relation (R5) when $k \neq l$, note that we need the following lemma.

Lemma 7.1.8. *If $k = j - 1$ and $\ell = j + 1$, then we have*

$$p_X(k)p_X(\ell) + p(k)p(\ell) + p(j)p(\ell) + p(j)p(k) \equiv p(j) \mod 2. \tag{7.11}$$

Proof. This lemma follows from (7.1.2) and a direct computation. $\square$

Lemma 7.1.9. (Compare [C16, Lemma 4.8]) *If $j, k, \ell \in I$ with $k \neq \ell$, then*

$$T'_{j,-1}(E_k^{\mathbf{x}})T'_{j,-1}(F_\ell^{\mathbf{x}}) = (-1)^{p_X(k)p_X(\ell)}T'_{j,-1}(F_\ell^{\mathbf{x}})T'_{j,-1}(E_k^{\mathbf{x}})$$

Proof. Let $c_{k,\ell} = T'_{j,-1}(E_k^{\mathtt{X}})T'_{j,-1}(F_\ell^{\mathtt{X}}) - (-1)^{p_X(k)p_X(\ell)}T'_{j,-1}(F_\ell^{\mathtt{X}})T'_{j,-1}(E_k^{\mathtt{X}})$. We want to prove $c_{k,\ell} = 0$ for all $k \neq \ell$.

If one of them is not connected to j, let us say $j \nsim k$, then $T'_{j,-1}(E_k^{\mathtt{X}}) = E_k$ and $p(j) = p_X(j)$. On the other hand, $T'_{j,-1}(E_\ell^{\mathtt{X}})$ is a polynomial in the elements K_j, F_j, F_ℓ and F_j with $p(T'_{j,-1}(F_\ell^{\mathtt{X}})) = p_X(\ell)$. Since E_k super-commutes with all of those elements, the statement follows.

The remaining cases involve situations where both k and ℓ are either equal to or connected with j. For the case where one of them is connected to j and the other is equal to j, the verification has already been conducted in [C16, Lemma 4.8].

When k and ℓ are both connected to j, without loss of generality, we assume that $k = j-1$ and $\ell = j+1$. Using (7.11) and the relation (R5) in (2.11) repeatedly we get

$$
\begin{aligned}
c_{k,\ell} =& (E_kE_j - (-1)^{p(j)p(k)}q^{(\alpha_j,\alpha_k)}E_jE_k)(F_jF_\ell - (-1)^{p(j)p(\ell)}q^{-(\alpha_j,\alpha_\ell)}F_\ell F_j) \\
& - (-1)^{p_X(k)p_X(\ell)}(F_jF_\ell - (-1)^{p(j)p(\ell)}q^{-(\alpha_j,\alpha_\ell)}F_\ell F_j) \\
& \qquad (E_kE_j - (-1)^{p(j)p(k)}q^{(\alpha_j,\alpha_k)}E_jE_k) \\
=& E_k[E_j,F_j]F_\ell - q^{(\alpha_j,\alpha_k)}[E_j,F_j]E_kF_\ell - q^{-(\alpha_j,\alpha_\ell)}E_kF_\ell[E_j,F_j] \\
& + (-1)^{p(k)p(\ell)}q^{(\alpha_j,\alpha_k)-(\alpha_j,\alpha_\ell)}F_\ell[E_j,F_j]E_k \\
=& \frac{1}{q^{\ell_j} - q^{-\ell_j}}E_k((1 - q^{2(\alpha_j,\alpha_k)} - 1 + q^{2(\alpha_j,\alpha_k)})K_j \\
& \qquad - (1 - 1 - q^{-2(\alpha_j,\alpha_\ell)} + q^{-2(\alpha_j,\alpha_\ell)})K_j^{-1})F_\ell = 0.
\end{aligned}
$$

This proves the lemma. $\square$

The next lemma checks the relation (R7).

Lemma 7.1.10. *If $j, k, \ell \in I$ such that $k \nsim \ell$, then*

$$T'_{j,-1}(E_k^{\mathtt{X}})T'_{j,-1}(E_\ell^{\mathtt{X}}) = (-1)^{p_X(k)p_X(\ell)}T'_{j,-1}(E_{\mathtt{X},\ell})T'_{j,-1}(E_k^{\mathtt{X}}),$$

$$T'_{j,-1}(F_k^{\mathtt{X}})T'_{j,-1}(F_\ell^{\mathtt{X}}) = (-1)^{p_X(k)p_X(\ell)}T'_{j,-1}(F_{\mathtt{X},\ell})T'_{j,-1}(F_k^{\mathtt{X}}).$$

Proof. We only prove for E. If either k or ℓ is not connected to j, we are done. So we suppose that $k = j-1$ and $\ell = j+1$. We break the proof into two cases:

(Case-1) Assume $p_X(j) = 0$. Observe that in this case we must have $(\alpha_j, \alpha_k) = (\alpha_j, \alpha_\ell)$. Without loss of generality we assume $(\alpha_j, \alpha_k) = (\alpha_j, \alpha_\ell) = 1$. Then we have $T'_{j,-1}(E_k^{\mathtt{X}}) = E_kE_j - qE_jE_k$ and $T'_{j,-1}(E_\ell^{\mathtt{X}}) = E_\ell E_j - qE_jE_\ell$ and thus

$$T'_{j,-1}(E_k^{\mathtt{X}})T'_{j,-1}(E_\ell^{\mathtt{X}}) = E_kE_jE_\ell E_j - qE_jE_kE_\ell E_j - qE_kE_j^2E_\ell + q^2E_jE_kE_jE_\ell,$$
$$T'_{j,-1}(E_\ell^{\mathtt{X}})T'_{j,-1}(E_k^{\mathtt{X}}) = E_\ell E_jE_kE_j - qE_jE_\ell E_kE_j - qE_\ell E_j^2E_k + q^2E_jE_\ell E_jE_k.$$

First we see that $E_jE_kE_\ell E_j = (-1)^{p(k)p(\ell)}E_jE_\ell E_kE_j = (-1)^{p_X(k)p_X(\ell)}E_jE_\ell E_kE_j$. Thus, by applying (R8) repeatedly we get

$$\begin{aligned}
&T'_{j,-1}(E_k^{\mathtt{X}})T'_{j,-1}(E_\ell^{\mathtt{X}}) - (-1)^{p_X(k)p_X(\ell)}T'_{j,-1}(E_{\mathtt{X},\ell})T'_{j,-1}(E_k^{\mathtt{X}})\\
=&E_kE_jE_\ell E_j - q(E_kE_j^2 - qE_jE_kE_j)E_\ell\\
&\qquad - (-1)^{p_X(k)p_X(\ell)}[E_\ell E_jE_kE_j - q(E_\ell E_j^2 - qE_jE_\ell E_j)E_k]\\
=&E_kE_jE_\ell E_j - q(q^{-1}E_jE_kE_j - E_j^2E_k)E_\ell\\
&\quad - (-1)^{p_X(k)p_X(\ell)}[E_\ell E_jE_kE_j - q(q^{-1}E_jE_\ell E_j - E_j^2E_\ell)E_k]\\
=&\frac{1}{q+q^{-1}}[E_k(E_j^2E_\ell + E_\ell E_j^2) - (E_kE_j^2 + E_j^2E_k)E_\ell\\
&\quad - (-1)^{p_X(k)p_X(\ell)}E_\ell(E_kE_j^2 + E_j^2E_k) + (-1)^{p_X(k)p_X(\ell)}(E_j^2E_\ell + E_\ell E_j^2)E_k]\\
=&0.
\end{aligned}$$

(Case-2) Assume $p_X(j) = 1$. In this case we always have $(\alpha_j, \alpha_k) = -(\alpha_j, \alpha_\ell)$. Again we may assume that $(\alpha_j, \alpha_k) = 1$. Then $(\alpha_j, \alpha_\ell) = -1$. Thus we have $T'_{j,-1}(E_k^{\mathtt{X}}) = E_kE_j -$

$(-1)^{p(k)}qE_jE_k$ and $T'_{j,-1}(E_\ell^{\mathtt{X}}) = E_\ell E_j - (-1)^{p(\ell)}q^{-1}E_jE_\ell$ and thus

$$T'_{j,-1}(E_k^{\mathtt{X}})T'_{j,-1}(E_\ell^{\mathtt{X}}) = E_kE_jE_\ell E_j - (-1)^{p(k)}qE_jE_kE_\ell E_j + (-1)^{p(k)+p(\ell)}E_jE_kE_jE_\ell,$$
$$T'_{j,-1}(E_\ell^{\mathtt{X}})T'_{j,-1}(E_k^{\mathtt{X}}) = E_\ell E_jE_kE_j - (-1)^{p(\ell)}q^{-1}E_jE_\ell E_kE_j + (-1)^{p(k)+p(\ell)}E_jE_\ell E_jE_k.$$

By taking a difference of the above two equations and unravelling the relation (R10) we can conclude that

$$T'_{j,-1}(E_k^{\mathtt{X}})T'_{j,-1}(E_\ell^{\mathtt{X}}) = (-1)^{p_X(k)p_X(\ell)}T'_{j,-1}(E_\ell^{\mathtt{X}})T'_{j,-1}(E_k^{\mathtt{X}}).$$

This proves the lemma. □

The verification process for the relations (R8) and (R9) is no different from that in the $\mathfrak{gl}(m|1)$ case. Hence we have

Lemma 7.1.11. [C16, Lemma 4.11] *If $j,k,\ell \in I$ such that $p_X(k)=0$ and $k \sim \ell$, then*

$$\begin{aligned}
&T'_{j,-1}(E_k^{X})^2T'_{j,-1}(E_\ell^{X}) - (q+q^{-1})T'_{j,-1}(E_k^{X})T'_{j,-1}(E_\ell^{X})T'_{j,-1}(E_k^{X})\\
&\qquad + T'_{j,-1}(E_\ell^{X})T'_{j,-1}(E_k^{X})^2 = 0,\\
&T'_{j,-1}(F_k^{X})^2T'_{j,-1}(F_\ell^{X}) - (q+q^{-1})T'_{j,-1}(F_k^{X})T'_{j,-1}(F_\ell^{X})T'_{j,-1}(F_k^{X})\\
&\qquad + T'_{j,-1}(F_\ell^{X})T'_{j,-1}(F_k^{X})^2 = 0.
\end{aligned}$$

Finally we need to verify the relations (R10) and (R11).

Lemma 7.1.12. *Let $z,k,j,\ell \in I$ with $k \sim j \sim \ell$, $k < \ell$ and $p_X(j)=1$, then*

$$S_{p_X(k),p_X(\ell)}(T'_{z,-1}(E_k^{X}), T'_{z,-1}(E_j^{X}), T'_{z,-1}(E_\ell^{X})) = 0,$$
$$S_{p_X(k),p_X(\ell)}(T'_{z,-1}(F_k^{X}), T'_{z,-1}(F_j^{X}), T'_{z,-1}(F_\ell^{X})) = 0.$$

Proof. We only prove the first equality as the second one can be proved similarly. If none of k,j,ℓ is connected or equal to z, there is nothing to prove.

Now we first suppose that $j \nsim z$ and $k \sim z$. In this case we have $T'_{z,-1}(E^{\mathtt{X}}_k) = E_k E_z - (-1)^{p(z)p(k)} q^{(\alpha_k,\alpha_z)} E_z E_k,\ T'_{z,-1}(E^{\mathtt{X}}_j) = E_j,\ T'_{z,-1}(E^{\mathtt{X}}_\ell) = E_\ell$. Thus

$$
\begin{aligned}
&S_{p_X(k),p_X(\ell)}(T'_{z,-1}(E^{\mathtt{X}}_k), T'_{z,-1}(E^{\mathtt{X}}_j), T'_{z,-1}(E^{\mathtt{X}}_\ell)) \\
=&S_{p_X(k),p_X(\ell)}(E_k E_z, E_j, E_\ell) - (-1)^{p(z)p(k)} q^{(\alpha_z,\alpha_k)} S_{p_X(k),p_X(\ell)}(E_z E_k, E_j, E_\ell)
\end{aligned}
$$

When $p(z) = 0$, E_z commutes with E_j and E_ℓ. Also we have $p_X(k) = p(k),\ p_X(\ell) = p(\ell)$. Hence

$$
\begin{aligned}
S_{p_X(k),p_X(\ell)}(E_k E_z, E_j, E_\ell) &= S_{p(k),p(\ell)}(E_k, E_j, E_\ell) E_z = 0, \\
S_{p_X(k),p_X(\ell)}(E_z E_k, E_j, E_\ell) &= E_z S_{p_X(k),p_X(\ell)}(E_k, E_j, E_\ell) = 0.
\end{aligned}
$$

When $p(z) = 1$, since $E_z E_j = (-1)^{p(z)} E_j E_z,\ E_z E_\ell = (-1)^{p(z)p(\ell)} E_\ell E_z,\ p(k) = p_X(k) + 1$ and $p(\ell) = p_X(\ell)$, again we have

$$
\begin{aligned}
S_{p_X(k),p_X(\ell)}(E_k E_z, E_j, E_\ell) &= S_{p(k),p(\ell)}(E_k, E_j, E_\ell) E_z = 0, \\
S_{p_X(k),p_X(\ell)}(E_z E_k, E_j, E_\ell) &= E_z S_{p_X(k),p_X(\ell)}(E_k, E_j, E_\ell) = 0.
\end{aligned}
$$

Note that the case when $j \nsim h,\ \ell \sim h$ is similar.

Next, suppose $z \sim j$ and without loss of generality that $z = k$. We further assume that $(\alpha_j, \alpha_k) = -1$, thus $(\alpha_j, \alpha_\ell) = -1$. Note that when $p(k)p(\ell) = 0$, the proof is already given in [C16, Lemma 4.12]. So we only need to consider the case when $p(z) = p(k) = p(\ell) = 1$. In this case we have $p_X(k) = p_X(\ell) = 1$, $p(j) = 0$ and $T'_{z,-1}(E^{\mathtt{X}}_j) = E_j E_z - q^{-1} E_z E_j,\ T'_{z,-1}(E^{\mathtt{X}}_\ell) = E^{\mathtt{X}}_\ell$ and $T'_{z,-1}(E^{\mathtt{X}}_k) = K_k^{-1} F_k$. Let

$$
\eth_{abcd} = T'_{z,-1}(E^{\mathtt{X}}_a) T'_{z,-1}(E^{\mathtt{X}}_b) T'_{z,-1}(E^{\mathtt{X}}_c) T'_{z,-1}(E^{\mathtt{X}}_d).
$$

The goal is to prove that

$$(q+q^{-1})\eth_{j\ell kj} = -\eth_{j\ell jk} + \eth_{kj\ell j} + \eth_{jkj\ell} - \eth_{\ell jkj}.$$

Note the identities

$$T'_{z,-1}(E_k^{\mathtt{X}})T'_{z,-1}(E_\ell^{\mathtt{X}}) = -T'_{z,-1}(E_\ell^{\mathtt{X}})T'_{z,-1}(E_k^{\mathtt{X}}),$$
$$T'_{z,-1}(E_j^{\mathtt{X}})T'_{z,-1}(E_k^{\mathtt{X}}) = -q^{-1}T'_{z,-1}(E_k^{\mathtt{X}})T'_{z,-1}(E_j^{\mathtt{X}}) + q^{-1}E_j.$$

With the above identities and Lemma 7.1.6 we see that

$$\eth_{j\ell jk} = -q^{-1}\eth_{j\ell kj} + q^{-1}E_jE_kE_\ell E_j - q^{-2}E_kE_jE_\ell E_j,$$
$$\eth_{kj\ell j} = -q\eth_{jk\ell j} + E_jE_\ell E_jE_k - q^{-1}E_jE_kE_\ell E_j,$$
$$\eth_{jkj\ell} = E_jE_kE_jE_\ell,$$
$$\eth_{\ell jkj} = E_\ell E_jE_kE_j$$

Thus using Serre relation (R8) repeatedly we have

$$(q+q^{-1})\eth_{j\ell kj} = -\eth_{j\ell jk} + \eth_{kj\ell j} + \eth_{jkj\ell} - \eth_{\ell jkj}.$$

Finally we suppose that $z = j$. Again when $p(k)p(\ell) = 0$ the proof is given in [C16, Lemma 4.12]. So we only need to consider the case when $p(k) = p(\ell) = 1$. Thus we have $p_X(k) = p_X(\ell) = 0$. Without loss of generality we assume that $(\alpha_k, \alpha_j) = -(\alpha_j, \alpha_\ell) = -1$. Then $T'_{z,-1}(E_k^{\mathtt{X}}) = E_kE_j + q^{-1}E_jE_k$, $T'_{z,-1}(E_\ell^{\mathtt{X}}) = E_\ell E_j + qE_jE_\ell$ and $T'_{z,-1}(E_j^{\mathtt{X}}) = K_j^{-1}F_j$. Note that we have the identities

$$T'_{j,-1}(E_k^{\mathtt{X}})T'_{j,-1}(E_j^{\mathtt{X}}) = q^{-1}T'_{j,-1}(E_j^{\mathtt{X}})T'_{j,-1}(E_k^{\mathtt{X}}) + q^{-1}E_k,$$
$$T'_{j,-1}(E_\ell^{\mathtt{X}})T'_{j,-1}(E_j^{\mathtt{X}}) = qT'_{j,-1}(E_j^{\mathtt{X}})T'_{j,-1}(E_\ell^{\mathtt{X}}) - qE_\ell.$$

Thus we have

$$\eth_{j\ell jk} = -K_j^{-1}F_j(qE_\ell E_k E_j + E_\ell E_j E_k),$$
$$\eth_{kj\ell j} = -E_k E_\ell - K_j^{-1}F_j(E_k E_j E_\ell + q^{-1}E_j E_k E_\ell),$$
$$\eth_{jkj\ell} = K_j^{-1}F_j(q^{-1}E_k E_\ell E_j + E_k E_j E_\ell),$$
$$\eth_{\ell jkj} = -E_\ell E_k + K_j^{-1}F_j(E_\ell E_j E_k + qE_j E_\ell E_k),$$
$$\eth_{jk\ell j} = \eth_{j\ell kj} = K_j F_j^{-1}(E_k E_\ell E_j - E_j E_k E_\ell).$$

Then we conclude that

$$(q+q^{-1})\eth_{j\ell kj} = \eth_{j\ell jk} + \eth_{kj\ell j} + \eth_{jkj\ell} + \eth_{\ell jkj}.$$

This proves the lemma. $\square$

We have now proved that $T'_{j,e}$ and $T''_{j,e}$ are algebra isomorphisms for all $j \in I$ and $e = \pm 1$. The next proposition states that the braid group operators in Theorem 7.1.4 satisfy the type A braid relations.

Proposition 7.1.13. *Let $j, k, \ell \in I$ and $X \in \mathcal{D}_{m,n}$.*

(1) If $j \nsim k$, then $T'_{j,e}T'_{k,e} = T'_{k,e}T'_{j,e}$ and $T''_{j,e}T''_{k,e} = T''_{k,e}T''_{j,e}$.

(2) If $j \sim k$ and $Y = s_j s_k(X)$, then

$$T'_{j,-e}T'_{k,-e}(E_j^{\mathbf{X}}) = T''_{j,e}T''_{k,e}(E_j^{\mathbf{X}}) = E_k^{\mathbf{Y}},$$

$$T'_{j,-e}T'_{k,-e}(F_j^{\mathbf{X}}) = T''_{j,e}T''_{k,e}(F_j^{\mathbf{X}}) = F_k^{\mathbf{Y}},$$

$$T'_{j,-e}T'_{k,-e}(K_j^{\mathbf{X}}) = T''_{j,e}T''_{k,e}(K_j^{\mathbf{X}}) = K_k^{\mathbf{Y}}.$$

(3) If $j \sim k$, then $T'_{j,e}T'_{k,e}T'_{j,e} = T'_{k,e}T'_{j,e}T'_{k,e}$ and $T''_{j,e}T''_{k,e}T''_{j,e} = T''_{k,e}T''_{j,e}T''_{k,e}$.

Proof. It follows from [Ya99, Lemma 8.1.1]; see also [H10, §6.3]. □

From now on we denote by T_i the braid operator $T''_{i,1}$ defined in Theorem 7.1.4. The next lemma can be proved similarly as in [Jan95, §8.18–§8.20].

Lemma 7.1.14. *Let $w \in W$, $X \in \mathcal{D}_{m,n}$, $Y = w(X)$ and $\alpha \in \Pi_X$. If $w(\alpha) > 0$ in the root system associated to X, then $T_w(E_\alpha^{\mathbf{X}}) \in \mathbf{U}(Y)^+$. If $w(\alpha) \in \Pi_X$, then $T_w(E_\alpha^{\mathbf{X}}) = E_{w(\alpha)}^{\mathbf{Y}}$.*

7.2 Quantum supersymmetric pair of type AIII

In this section we define the quantum supersymmetric pairs and the corresponding ıquantum supergroups of type AIII.

7.2.1 Definition and notations

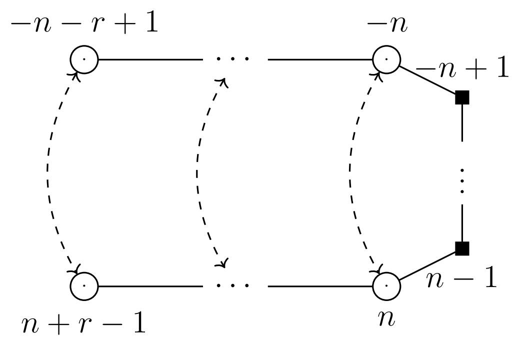
Recall the notation $[x, x+m] = \{x, x+1, \ldots, x+m\}$ and $\mathbb{I}_a$ for any real number $x \in \mathbb{R}$ and $m \in \mathbb{N}$ from § 3.1.1.

Fix

$$n = \frac{m}{2} \in \frac{1}{2}\mathbb{N}.$$

We consider the super Satake diagram of type AIII with $m - 1 = 2n - 1$ black nodes and r

pairs of white nodes, together with a diagram involution τ indicated by the dashed arrows:



(7.12)

where $\odot$ stands for white dots and $\blacksquare$ stands for black dots. We will denote the white even roots, black even roots, black odd roots and white odd roots respectively by $\bigcirc$, $\newmoon$, $\LEFTcircle$ and $\otimes$.

For any Satake diagram in $\mathcal{D}_{m,n}$ of the form (7.12), we denote the index set by

$$I = \mathbb{I}_{m+2r-1} = I_{\circ} \cup I_{\bullet}, \qquad (m+2r = m+n), \tag{7.13}$$

where $I_{\circ}$ is the collection of white dots and $I_{\bullet}$ is the collection of black dots. Switching to this notation has the advantage of easily identifying the diagram involution τ with -1 on the index set of the simple roots.

Both white and black dots allow different parities under the following assumption:

$$\begin{aligned} &\#\{p(j) = 1 \mid j \in I_{\bullet}\} \equiv 0 \mod 2, \\ &p(j) = p(\tau(j)), \quad \forall i \in I_{\circ}, \\ &i \in I_{\overline{0}} \text{ if } \tau i = i \text{ and } i \in I_{\circ}. \end{aligned} \tag{7.14}$$

where $I_{\bullet} = [1-n, n-1]$, $I_{\circ} = I \backslash I_{\bullet}$. (In case $n = 0$, the black nodes are dropped; the nodes n and $-n$ are identified and fixed by τ.)

Let $\mathfrak{S}_{m-1}$ denote the symmetric group associated with $m-1$ letters in $I_{\bullet} = [1-n, n-1]$, and let $w_\bullet$ represent the longest element of $\mathfrak{S}_{m-1}$. For any reduced expression $w_\bullet = s_{i_1} \cdots s_{i_\ell}$ as a product of simple generators, we regard s_{i_t} as the simple reflection $s_{\alpha_{i_t}}$. Consequently, we can view $w_\bullet$ as a product of even and odd reflections. It follows from [HY08] that $w_\bullet$ is independent of the choice of the reduced expression.

Following [BW18b], we further assume that τ extends to an involution on P and $P^\vee$, respectively, such that the bilinear pairing $\langle \cdot, \cdot \rangle$ is invariant under τ. Then we define

$$\begin{aligned} P_\imath &= P \big/ \{\mu + w_\bullet \tau(\mu) \mid \mu \in P\}, \\ P_\imath^\vee &= \{\nu - w_\bullet \tau(\nu) \mid \nu \in P^\vee\}. \end{aligned} \tag{7.15}$$

For any Satake diagram X in the form of (7.12), without considering the diagram involution τ, the diagram X corresponds to a Lie superalgebra $\mathfrak{gl}(m|n)$ for certain non-negative integers m and n, where $m + n = 2r + m$. Recall $I(m|n)$ from (2.3). The simple roots of X are given by

$$\Pi_{\mathtt{X}} = \{\alpha_{\mathtt{X},k} = \epsilon^{\mathtt{X}}_{k-\frac{1}{2}} - \epsilon^{\mathtt{X}}_{k+\frac{1}{2}} \mid k \in I\}$$

where $\{\epsilon^{\mathtt{X}}_{k\pm\frac{1}{2}} \mid k \in I\} = \{\epsilon_a \mid a \in I(m|n)\}$.

In the remaining part of this section, we fix a diagram $X \in \mathcal{D}_{m,n}$ of the form (7.12) satisfying (7.14). Furthermore, we recall the definition of ℓ_j from equation (2.10). In addition, we provide two lemmas that will be useful for future reference.

Lemma 7.2.1. *We have*

$$\ell_j = (-1)^{p(j)} \ell_{-j}, \ \forall j \in I.$$

Proof. By observation we have $\ell_j = (-1)^{p\left(\epsilon^{\mathtt{X}}_{j-\frac{1}{2}}\right)}$. Moreover, since $p(j) = p(-j)$ for all $j \in I$, we have $\ell_j = (-1)^{p(j)} \ell_{-j}, \ \forall j \in I$. □

Lemma 7.2.2. *Suppose that $Y = w_\bullet(X)$, then $Y \in \mathcal{D}_{m,n}$ and satisfies* (7.14).

Proof. For each $k \in I$, we have

$$\alpha_{\mathtt{Y},k} := w_\bullet(\alpha_{\mathtt{X},k}) = \begin{cases} \epsilon^{\mathtt{X}}_{k-\frac{1}{2}} - \epsilon^{\mathtt{X}}_{k+\frac{1}{2}} & \text{if } |k| > n, \\ \epsilon^{\mathtt{X}}_{-n+\frac{1}{2}} - \epsilon^{\mathtt{X}}_{n+\frac{1}{2}} & \text{if } k = n, \\ \epsilon^{\mathtt{X}}_{-k+\frac{1}{2}} - \epsilon^{\mathtt{X}}_{-k-\frac{1}{2}} & \text{if } -n < k < n, \\ \epsilon^{\mathtt{X}}_{-n-\frac{1}{2}} - \epsilon^{\mathtt{X}}_{n-\frac{1}{2}} & \text{if } k = -n. \end{cases} \tag{7.16}$$

From (7.16) we can see that $\alpha_{\mathtt{X},k} = \alpha_{\mathtt{Y},k}$ if $|k| > n$ and so is the parity. For $|k| = n$, suppose $p(\alpha_{\mathtt{X},-n}) = p(\alpha_{\mathtt{X},n}) = 0$, then $\epsilon^{\mathtt{X}}_{-n-\frac{1}{2}}$ and $\epsilon^{\mathtt{X}}_{-n+\frac{1}{2}}$ have the same parity while $\epsilon^{\mathtt{X}}_{n-\frac{1}{2}}$ and $\epsilon^{\mathtt{X}}_{n+\frac{1}{2}}$ have the same parity. Thus $p(\alpha_{\mathtt{Y},n}) = p(\alpha_{\mathtt{Y},-n})$. It can be checked similarly that when $p(\alpha_{\mathtt{X},-n}) = p(\alpha_{\mathtt{X},n}) = 1$, we still have $p(\alpha_{\mathtt{Y},n}) = p(\alpha_{\mathtt{Y},-n})$.

For $-n < k < n$, we see that $\alpha_{\mathtt{Y},k} = -\alpha_{\mathtt{X},-k}$. Thus the number of black odd roots stays unchanged. Moreover, if $\tau i = i$ and $i \in I_\circ$, then we have $I_\bullet = \varnothing$. Hence $Y = X$. □

Let $Y := w_\bullet(X)$. According to (7.16), we see that $\alpha_{\mathtt{Y},k} = \epsilon^{\mathtt{Y}}_{k-\frac{1}{2}} - \epsilon^{\mathtt{Y}}_{k+\frac{1}{2}}$ where

$$\epsilon^{\mathtt{Y}}_t = \begin{cases} \epsilon^{\mathtt{X}}_t, & \text{if } t > n - \frac{1}{2} \text{ or } t \leqslant -n - \frac{1}{2}, \\ \epsilon^{\mathtt{X}}_{-t}, & \text{if } -n - \frac{1}{2} < t \leqslant n - \frac{1}{2} \end{cases} \tag{7.17}$$

Let $\mathbf{U}(Y)$ represent the quantum supergroup associated with generators $\varrho, E^{\mathtt{Y}}_j, F^{\mathtt{Y}}_j, q^\mu$, where $j \in I$ and $\mu \in P^\vee$, corresponding to the Dynkin diagram Y. Similarly, let $\mathbf{U}(X)$ denote the same algebra with generators $\varrho, E^{\mathtt{X}}_j, F^{\mathtt{X}}_j, q^\mu$, where $j \in I$ and $\mu \in P^\vee$, but with a different presentation corresponding to the Dynkin diagram X. We note that the comultiplication Δ is dependent on the chosen presentation, as shown in equation (2.16). For simplicity, we use the same notation Δ and the parity function p for different presentations, and we omit the script Y unless necessary.

The $\imath$quantum supergroup of type AIII, denoted by $\mathbf{U}^\imath = \mathbf{U}^\imath(Y)$, is the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}(Y)$ generated by q^{μ} $(\mu \in P_{\imath}^{\vee})$, E_j, F_j $(j \in I_{\bullet})$, ϱ and

$$B_j = F_j + \varsigma_j T_{w_\bullet}(E^{\mathtt{X}}_{\tau j})K_j^{-1}, \quad \text{for } j \in I_\circ. \tag{7.18}$$

where parameters $\varsigma_j \in \mathbb{Q}(q)$, for $j \in I_\circ$ satisfy the conditions $\varsigma_j = \varsigma_{-j}$, for $j \in I_\circ \backslash \{\pm n\}$ [Let02] (also cf. [BK15, BW21]). (When $n = 0$, B_0 will be allowed to take a more general form $B_0 = F_0 + \varsigma_0 E_0 K_0^{-1} + \kappa_0 K_0^{-1}$, for an additional parameter $\kappa_0 \in \mathbb{Q}(q)$.)

For each reduced expression $w_\bullet = s_{j_1} \cdots s_{j_l}$, we can write $T_{w_\bullet} = T_{j_1} \cdots T_{j_l}$. By Proposition 7.1.13, $T_{w_\bullet}$ is a well-defined operator as a product of braid operators associated to both odd and even simple roots in $I_\bullet$.

Now $(\mathbf{U}(Y), \mathbf{U}^{\imath}(Y))$ forms a quantum supersymmetric pair of type AIII [Let99, Let02] (cf. [BW18a, BK19]). The algebra $\mathbf{U}^{\imath}$ satisfies the following relations

$$\begin{aligned}
&q^{\mu} B_j = q^{-\langle \mu, \alpha_j \rangle} B_j q^{\mu}, \ \forall j \in I_\circ, \\
&q^{\mu} F_j = q^{-\langle \mu, \alpha_j \rangle} F_j q^{\mu}, \quad q^{\mu} E_j = q^{\langle \mu, \alpha_j \rangle} E_j q^{\mu}, \ \forall j \in I_\bullet, \mu \in P_{\imath}^{\vee}, \\
&\varrho(B_j) = (-1)^{p(j)} B_j, \quad \forall j \in I.
\end{aligned}$$

and additional Serre type relations. By definition we see the following relation holds in $\mathbf{U}^{\imath}$.

Lemma 7.2.3. *For any $j \in I_\bullet$, $k \in I$ we have*

$$E_j B_k - (-1)^{p(j)p(k)} B_k E_j = \delta_{jk} \frac{K_j - K_j^{-1}}{q^{\ell_j} - q^{-\ell_j}}. \tag{7.19}$$

For future use, we let $\mathbf{U}_\bullet$ denote the subalgebra of $\mathbf{U}$ generated by $\{E_j, F_j, K_j^{\pm 1}, \varrho \mid j \in I_\bullet\}$. Let $\mathbf{U}^{\imath 0}$ denote the subalgebra of $\mathbf{U}^{\imath}$ generated by $\{q^{\mu}, \varrho \mid \mu \in P_{\imath}^{\vee}\}$.

The next lemma will help us pin down one of the conditions on the parameters.

Lemma 7.2.4. *If $\varsigma_j \neq \varsigma_{-j}$ for $j \in I_\circ \backslash \{\pm n\}$, then $(K_j^{-1} K_{-j}^{-1}) \in \mathbf{U}^{\imath 0}$.*

Proof. This claim follows from the relation

$$B_jB_{-j} - (-1)^{p(j)}B_{-j}B_j = -\varsigma_{-j}(-1)^{p(j)}\frac{K_j - K_j^{-1}}{q^{\ell_j} - q^{-\ell_j}}K_{-j}^{-1} + \varsigma_j\frac{K_{-j} - K_{-j}^{-1}}{q^{\ell_{-j}} - q^{-\ell_{-j}}}K_j^{-1}$$

According to Lemma 7.2.1 we have $\ell_j = (-1)^{p(j)}\ell_{-j}$. Thus $q^{\ell_{-j}} - q^{-\ell_{-j}} = (-1)^{p(j)}(q^{\ell_j} - q^{-\ell_j})$. Hence the lemma follows. □

By the above lemma we see that $\mathbf{U}^\imath \cap \mathbf{U}^0 = \mathbf{U}^{\imath 0}$ can only be satisfied if the parameters satisfy $\varsigma_j = \varsigma_{-j}$ for $j \in I_\circ \backslash \{\pm n\}$. From now on we assume the parameters $\{\varsigma_j\}$ always satisfy this condition.

Furthermore, we determine the action of $T_{w_\bullet}$ on $\mathbf{U}_\bullet$.

Lemma 7.2.5. *For all $j \in I_\bullet$, we have*

$$\begin{aligned} T_{w_\bullet}(E_j^{\mathbf{X}}) &= -F_{-j}K_{-j}, \quad T_{w_\bullet}(F_j^{\mathbf{X}}) = -K_{-j}^{-1}E_{-j}, \quad T_{w_\bullet}(K_j^{\mathbf{X}}) = K_{-j}^{-1}, \\ T_{w_\bullet}^{-1}(E_j^{\mathbf{X}}) &= -K_{-j}^{-1}F_{-j}, \quad T_{w_\bullet}^{-1}(F_j^{\mathbf{X}}) = -E_{-j}K_{-j}, \quad T_{w_\bullet}^{-1}(K_j^{\mathbf{X}}) = K_{-j}^{-1}. \end{aligned} \tag{7.20}$$

Proof. The proof of this lemma follows from the same argument in [Ko14, Lemma 3.4] and Lemma 7.1.14. □

7.2.2 Coideal subalgebra property

One of the key properties of the $\imath$quantum group is that it is a coideal subalgebra of the underlying Hopf algebra rather than a Hopf subalgebra. Here we observe such a structure for $\mathbf{U}^\imath$ as well.

Proposition 7.2.6. $\mathbf{U}^\imath$ *is a right coideal subalgebra of* $\mathbf{U}$.

Proof. It is not hard to show that $\mathbf{U}_\bullet$ and $\mathbf{U}^{\imath 0}$ are Hopf subalgebras of $\mathbf{U}$. Thus it suffices to show that

$$\Delta(B_f) \in \mathbf{U}^\imath \otimes \mathbf{U},\ \forall f \in I_\circ. \tag{7.21}$$

Recall Δ from (2.16). It is straightforward to compute for $f \in I_\circ \backslash \{\pm n\}$ that

$$\Delta(B_f) - B_f \otimes K_f^{-1} \in \mathbf{U}_\bullet^+ \mathbf{U}^{\imath 0} \otimes \mathbf{U}. \tag{7.22}$$

Now suppose $f = -n$. For any reduced expression $\underline{w}_\bullet^{(y)} = s_{y_1} \cdots s_{y_\ell}$ of $w_\bullet$, we define

$$Y_t^{(y)} = s_{y_t} \cdots s_{y_\ell}(X), \quad 1 \leqslant t \leqslant \ell.$$

More specifically, in this case we choose

$$\underline{w}_\bullet^{(y)} = (s_{n-1}s_{n-2} \cdots s_{-n+1}) \cdots (s_{n-1}s_{n-2})(s_{n-1}).$$

For convention we drop (y) in the following proof and we define

$$\alpha_k^{\mathtt{D}} := \alpha_{\mathtt{D},k}, \qquad \text{for any } D \in \mathcal{D}_{m,n}.$$

Then we observe that

$$\begin{aligned} &T_{w_\bullet}(E_{-n}^{\mathtt{X}}) = T_{s_{n-1}} \cdots T_{s_{-n+1}}(E_{-n}^{\mathtt{Y}_{2n}}) \\ =&T_{s_{n-1}} \cdots T_{s_{-n+2}}(E_{-n+1}^{\mathtt{Y}_{2n-1}} E_{-n}^{\mathtt{Y}_{2n-1}} \\ &\quad - (-1)^{p(\alpha_{-n+1}^{\mathtt{Y}_{2n-1}})p(\alpha_{-n}^{\mathtt{Y}_{2n-1}})} q^{(\alpha_{-n+1}^{\mathtt{Y}_{2n-1}}, \alpha_{-n}^{\mathtt{Y}_{2n-1}})} E_{-n}^{\mathtt{Y}_{2n-1}} E_{-n+1}^{\mathtt{Y}_{2n-1}}) \\ =&T_{s_{n-1}} \cdots T_{s_{-n+2}}(E_{-n+1}^{\mathtt{Y}_{2n-1}})E_{-n} - zE_{-n}T_{s_{n-1}} \cdots T_{s_{-n+2}}(E_{-n+1}^{\mathtt{Y}_{2n-1}}) \end{aligned} \tag{7.23}$$

where $z = (-1)^{p(\alpha_{-n+1}^{\mathtt{Y}_{2n-1}})p(\alpha_{-n}^{\mathtt{Y}_{2n-1}})} q^{(\alpha_{-n+1}^{\mathtt{Y}_{2n-1}}, \alpha_{-n}^{\mathtt{Y}_{2n-1}})}$.

By (7.23) we see that in order to prove (7.22) for $f = -n$, it suffices to prove that

$$\Delta(T_{s_{n-1}} \cdots T_{s_{-n+1}}(E_{-n}^{\mathtt{Y}_{2n}})) \in T_{s_{n-1}} \cdots T_{s_{-n+1}}(E_{-n}^{\mathtt{Y}_{2n}}) \otimes 1 + \mathbf{U}_\bullet^+ \mathbf{U}^0 K_n \otimes \mathbf{U}. \tag{7.24}$$

We prove (7.24) by proving the following claim.

Claim: For any $1 \leqslant k \leqslant 2n-1$, we have

$$\begin{aligned} &\Delta(T_{s_{n-1}} \cdots T_{s_{-n+k}}(E_{-n+k-1}^{\Upsilon_{2n+1-k}})) \\ &\quad \in T_{s_{n-1}} \cdots T_{s_{-n+k}}(E_{-n+k-1}^{\Upsilon_{2n+1-k}}) \otimes 1 \\ &\quad + \mathbf{U}_{\bullet}^{+}\mathbf{U}^{0}\varrho^{p(-n+k-1)}K_{-n+k-1} \otimes \mathbf{U}. \end{aligned} \tag{7.25}$$

We prove (7.24) through induction on $r = 2n-1-k$. We see that $0 \leq r \leq 2n-2$.

When $r = 0$, we have $k = 2n-1$ and

$$\begin{aligned} \Delta(T_{s_{n-1}}(E_{n-2}^{\Upsilon_2})) =& \Delta(E_{n-1}E_{n-2} - (-1)^{p(\alpha_{n-1})p(\alpha_{n-2})}q^{(\alpha_{n-1},\alpha_{n-2})}E_{n-2}E_{n-1}) \\ \in& T_{s_{n-1}}(E_{n-2}^{\Upsilon_2}) \otimes 1 + \mathbf{U}_{\bullet}^{+}\mathbf{U}^{0}\varrho^{p(n-2)}K_{n-2} \otimes \mathbf{U}. \end{aligned}$$

Now suppose the claim is true for $r = j$, that is $k = 2n-1-j$ and

$$\begin{aligned} &\Delta(T_{s_{n-1}} \cdots T_{s_{n-1-j}}(E_{n-2-j}^{\Upsilon_{j+2}})) \\ &\quad = T_{s_{n-1}} \cdots T_{s_{n-1-j}}(E_{n-2-j}^{\Upsilon_{j+2}}) \otimes 1 + \sum_{\ell} x_{\ell}\varrho^{p(n-2-j)}K_{n-2-j} \otimes y_{\ell}, \end{aligned} \tag{7.26}$$

for some $x_\ell \in \mathbf{U}_{\bullet}^{+}\mathbf{U}^{0},\ y_\ell \in \mathbf{U}$.

In view of (2.16), (7.23) and (7.26), we see that

$$\begin{aligned} &\Delta(T_{s_{n-1}} \cdots T_{s_{n-1-j}}(E_{n-2-j}^{\Upsilon_{j+2}}))\Delta(E_{n-3-j}) \\ \in& T_{s_{n-1}} \cdots T_{s_{n-1-j}}(E_{n-2-j}^{\Upsilon_{j+2}})E_{n-3-j} \otimes 1 \\ &\quad + \sum_{\ell} x_{\ell}\varrho^{p(n-2-j)}K_{n-2-j}E_{n-3-j} \otimes y_{\ell} + \mathbf{U}_{\bullet}^{+}\mathbf{U}^{0}\varrho^{p(n-3-j)}K_{n-3-j} \otimes \mathbf{U}, \\ &\Delta(E_{n-3-j})\Delta(T_{s_{n-1}} \cdots T_{s_{n-1-j}}(E_{n-2-j}^{\Upsilon_{j+2}})) \\ \in& E_{n-3-j}T_{s_{n-1}} \cdots T_{s_{n-1-j}}(E_{n-2-j}^{\Upsilon_{j+2}}) \otimes 1 \\ &\quad + \sum_{\ell} E_{n-3-j}x_{\ell}\varrho^{p(n-2-j)}K_{n-2-j} \otimes y_{\ell} + \mathbf{U}_{\bullet}^{+}\mathbf{U}^{0}\varrho^{p(n-3-j)}K_{n-3-j} \otimes \mathbf{U}. \end{aligned} \tag{7.27}$$

By comparing both sides of (7.26) we see that x_ℓ is a monomial of the form $a_{(i)}Z_{i_1}\cdots Z_{i_{j+1}}$ where

$$Z_{i_t}\in\{E_{i_t},\varrho^{p(i_t)}K_{i_t}\},\quad a_{(i)}\in\mathbb{Q}(q).$$

and $\{i_1,\dots,i_{j+1}\}=\{n-1,\dots,n-1-j\}$. Hence we have:

$$\begin{aligned}E_{n-3-j}x_\ell\varrho^{p(n-2-j)}&=(-1)^{p(n-3-j)\cdot[p(n-1)+\cdots+p(n-1-j)]}x_\ell E_{n-3-j}\varrho^{p(n-2-j)}\ell\\&=(-1)^{p(n-3-j)\cdot[p(n-1)+\cdots+p(n-2-j)]}x_\ell\varrho^{p(n-2-j)}E_{n-3-j}.\end{aligned}\tag{7.28}$$

Moreover, we compute directly that

$$\begin{aligned}&p(\alpha^{\mathtt{Y}_{j+2}}_{n-2-j})=p(s_{n-j-2}(s_{n-1}\cdots s_{n-j-1})\alpha^{\mathtt{X}}_{n-j-2})=p(n-1)+\cdots p(n-2-j),\\&p(\alpha^{\mathtt{Y}_{j+2}}_{n-3-j})=p(\alpha_{n-3-j})=p(n-3-j),\\&(\alpha^{\mathtt{Y}_{j+2}}_{n-2-j},\alpha^{\mathtt{Y}_{j+2}}_{n-3-j})=(\alpha_{n-2-j},\alpha_{n-3-j}).\end{aligned}\tag{7.29}$$

Now consider the case $r=j+1$, that is to compute $\Delta(T_{w_\bullet}(E^{\mathtt{X}}_{n-3-j}))$, from (7.23), (7.27), (7.28) and (7.29) we have

$$\begin{aligned}&\Delta(T_{w_\bullet}(E^{\mathtt{X}}_{n-3-j}))=\Delta(T_{s_{n-1}}\cdots T_{s_{n-j-2}}(E^{\mathtt{Y}_{j+3}}_{n-3-j}))\\=&\Delta(T_{s_{n-1}}\cdots T_{s_{n-1-j}}(E^{\mathtt{Y}_{j+2}}_{n-2-j}))\Delta(E_{n-3-j})\\&\quad-(-1)^{p(\alpha^{\mathtt{Y}_{j+2}}_{n-j-2})p(\alpha^{\mathtt{Y}_{j+2}}_{n-j-3})}q^{(\alpha^{\mathtt{Y}_{j+2}}_{n-j-2},\alpha^{\mathtt{Y}_{j+2}}_{n-j-3})}\Delta(E_{n-j-3})\Delta(T_{s_{n-1}}\cdots T_{s_{n-j-1}}(E^{\mathtt{Y}_{j+2}}_{n-j-2}))\\&\in T_{w_\bullet}(E^{\mathtt{X}}_{n-j-3})\otimes 1+\mathbf{U}^+_\bullet\mathbf{U}^0\varrho^{p(n-j-3)}K_{n-j-3}\otimes\mathbf{U}.\end{aligned}\tag{7.30}$$

Thus the claim is proved and (7.24) is exactly the case when $r=2n-2$. Similarly for $f=n$ we also have

$$\Delta(T_{w_\bullet}(E^{\mathtt{X}}_n))\in T_{w_\bullet}(E^{\mathtt{X}}_n)\otimes 1+\mathbf{U}^+_\bullet\mathbf{U}^0K_n\otimes\mathbf{U}\tag{7.31}$$

Thus we conclude from (7.22), (7.30) and (7.31) that

$$\Delta(B_f) - B_f \otimes K_f^{-1} \in \mathbf{U}_\bullet^+ \mathbf{U}^{\imath 0} \otimes \mathbf{U} \quad \forall f \in I_\circ. \tag{7.32}$$

This proves the proposition. □

7.2.3 Quantum $\imath$Serre relations

Recall the Serre relations $(R5) - (R11)$ from (2.11). In this subsection we explore the Serre relations of $\mathbf{U}^\imath$. For convention, we extend the definition of B_j by setting $B_j = F_j$ for $j \in I_\bullet$.

The triangular decomposition (2.17) implies an isomorphism between vector spaces

$$\mathbf{U}^+ \otimes \mathbf{U}^0 \otimes S(\mathbf{U}^-) \cong \mathbf{U}. \tag{7.33}$$

This leads to a direct sum decomposition

$$\mathbf{U} = \bigoplus_{\mu \in P^\vee} \mathbf{U}^+ K_\mu S(\mathbf{U}^-) \oplus \mathbf{U}^+ K_\mu \varrho S(\mathbf{U}^-). \tag{7.34}$$

For any $\mu \in P^\vee$, let $P_\mu : \mathbf{U} \to \mathbf{U}^+ K_\mu S(\mathbf{U}^-) \oplus \mathbf{U}^+ K_\mu \varrho S(\mathbf{U}^-)$ denote the projection with respect to (7.34). We also use the symbol P_λ for $\lambda \in Q$ to denote the projection $P_\lambda : \mathbf{U} \to \mathbf{U}^+ K_\lambda S(\mathbf{U}^-) \oplus \mathbf{U}^+ K_\lambda \varrho S(\mathbf{U}^-)$ as above.

On the other hand, let $Q^+ := \mathbb{N}\Pi$, we also have the decomposition

$$\mathbf{U} = \bigoplus_{\alpha, \beta \in Q^+} U_\alpha^+ \mathbf{U}^0 \mathbf{U}_{-\beta}^- \tag{7.35}$$

We let $\pi_{\alpha,\beta} : \mathbf{U} \to U_\alpha^+ \mathbf{U}^0 \mathbf{U}_{-\beta}^-$ denote the projection with respect to (7.35).

The fact that

$$\Delta \circ P_\mu(x) = (id \otimes P_\mu)\Delta(x), \ \forall \mu \in P^\vee, x \in \mathbf{U} \tag{7.36}$$

implies the following lemma.

Lemma 7.2.7. [Ko14, Lemma 5.9] *We have* $\mathbf{U}^\imath = \bigoplus_{\mu \in P^\vee} P_\mu(\mathbf{U}^\imath)$.

The following lemma gives the first Serre type relation of $\mathbf{U}^\imath$.

Lemma 7.2.8. *For* $j \in I_{\bar{1}}$, *we have* $B_j^2 = 0$.

Proof. It suffices to check for $j \in I_\circ \cap I_{\bar{1}}$. Because of (7.14), we always have $j \neq -j$. Hence

$$\begin{aligned} B_j^2 =& (F_j + \varsigma_j T_{w_\bullet}(E_{-j}^{\mathtt{X}})K_j^{-1})^2 \\ =& F_j^2 + \varsigma_j [F_j, T_{w_\bullet}(E_{-j}^{\mathtt{X}})] K_j^{-1} + \varsigma_j^2 T_{w_\bullet}(E_{-j}^{\mathtt{X}})^2 K_j^{-2} \\ =& \, 0. \end{aligned}$$

This proves the lemma. □

We define two special weights $\lambda_{j,k} = (1 + |(\alpha_j, \alpha_k)|)\alpha_j + \alpha_k$ and $\lambda_j = 2\alpha_j + \alpha_{j-1} + \alpha_{j+1}$ in order to apply the projection technique in [Ko14]. Furthermore, we define

$$S(x_1, x_2) = x_1^2 x_2 - [2] x_1 x_2 x_1 + x_2 x_1^2, \quad \forall x_1, x_2 \in \mathbf{U}$$

and recall $S_{t_1,t_2}(x_1, x_2, x_3)$ from (2.12).

Lemma 7.2.9. *(1) Assume* $j \in I_\bullet$, $k \in I$ *and* $j \nsim k \in I$, *then we have* $[B_j, B_k] = B_j B_k - (-1)^{p(j)p(k)} B_k B_j = 0$.

(2) For $j \in I_\bullet \cap I_{\bar{0}}$ *and* $k \sim j$, *we have* $S(B_j, B_k) = 0$.

(3) For $j \in I_\bullet \cap I_{\bar{1}}$ *and* $k \sim j \sim \ell$, *we have* $S_{p(k),p(\ell)}(B_k, B_j, B_\ell) = 0$.

Proof. (1) In general we have

$$[B_j, B_k] = [F_j, F_k] + \varsigma_k [F_j, T_{w_\bullet}(E_{-k}^{\mathtt{X}}) K_k^{-1}].$$

Now if $k \in I_\bullet$, there is nothing to prove. If k is in $I_\circ$ then in this case we can rewrite F_j as $-T_{w_\bullet}(E^{\mathtt{X}}_{-j})K_j^{-1}$ according to Lemma 7.2.5 and one computes that

$$[T_{w_\bullet}(E^{\mathtt{X}}_{-j})K_j^{-1}, T_{w_\bullet}(E^{\mathtt{X}}_{-k})K_k^{-1}] = T_{w_\bullet}([E^{\mathtt{X}}_{-j}K^{\mathtt{X}}_{-j}, E^{\mathtt{X}}_{-k}K^{\mathtt{X}}_{-k}]) = 0.$$

(2) If $k \in I_\bullet$, there is nothing to prove. Thus we can assume that $k \in I_\circ$. In this case we have

$$\begin{aligned} S(B_j, B_k) =& S(F_j, \varsigma_k T_{w_\bullet}(E^{\mathtt{X}}_{-k})K_k^{-1}) \\ =& \varsigma_k S(T_{w_\bullet}(E^{\mathtt{X}}_{-j})K_j^{-1}, T_{w_\bullet}(E^{\mathtt{X}}_{-k})K_k^{-1}) \\ =& \varsigma_k z S(T_{w_\bullet}(E^{\mathtt{X}}_{-j}), T_{w_\bullet}(E^{\mathtt{X}}_{-k}))K_j^{-2}K_k^{-1} \end{aligned}$$

for some $z \in \mathbb{Z}[q, q^{-1}]$. Since $T_{w_\bullet}$ is an algebra homomorphism, we see that $S(B_j, B_k) = 0$.

(3) Without loss of generality, we assume $k = j - 1$ and $\ell = j + 1$. If both $k, \ell \in I_\bullet$, then $B_k = F_k, B_j = F_j$ and $B_\ell = F_\ell$ hence there is nothing to prove. Otherwise, we note that only one of k and ℓ can belong to $I_\circ$. Suppose that $k \in I_\bullet,\ \ell \in I_\circ$, then we have

$$\begin{aligned} & S_{p(k),p(\ell)}(B_k, B_j, B_\ell) \\ =& \varsigma_\ell S_{p(k),p(\ell)}(F_k, F_j, T_{w_\bullet}(E^{\mathtt{X}}_{-\ell})K_\ell^{-1}) \\ =& -\varsigma_\ell S_{p(k),p(\ell)}(T_{w_\bullet}(E^{\mathtt{X}}_{-k})K_k^{-1}, T_{w_\bullet}(E^{\mathtt{X}}_{-j})K_j^{-1}, T_{w_\bullet}(E^{\mathtt{X}}_{-\ell})K_\ell^{-1}) \\ =& -\varsigma_\ell z' S_{p(k),p(\ell)}(T_{w_\bullet}(E^{\mathtt{X}}_{-k}), T_{w_\bullet}(E^{\mathtt{X}}_{-j}), T_{w_\bullet}(E^{\mathtt{X}}_{-\ell}))K_k^{-1}K_j^{-2}K_\ell^{-1} = 0 \end{aligned}$$

for some $z' \in \mathbb{Z}[q, q^{-1}]$. The case $k \in I_\circ,\ \ell \in I_\bullet$ can be proved similarly. $\square$

The following technical lemmas provide key steps in the proof of Lemma 7.2.12.

Lemma 7.2.10. *(1) For any $j \nsim k \in I$, we have $\pi_{0,0}([B_j, B_k]) \in \mathbf{U}^{\imath 0}$.*

(2) For any $j \sim k$ where $j \in I_{\overline{0}}$, we have $\pi_{0,0}(S(B_j, B_k)) = 0$.

(3) For any $k \sim j \sim \ell$ where $j \in I_{\overline{1}}$, we have $\pi_{0,0}(S_{p(k),p(\ell)}(B_k, B_j, B_\ell)) = 0$.

Proof. (1) If one of j, k is in $I_\bullet$, then $[B_j, B_k] = 0$. If $j, k \in I_\circ$ and $j \neq \tau k$, then we also have $[B_j, B_k] = 0$. If both $j = \tau k \neq \pm n \in I_\circ$, then $k = -j$ and $\varsigma_j = \varsigma_{-j}$. Moreover, we see that

$$B_j B_{-j} - (-1)^{p(j)} B_{-j} B_j = -\varsigma_{-j}(-1)^{p(j)} \frac{K_j - K_j^{-1}}{q^{\ell_j} - q^{-\ell_j}} K_{-j}^{-1} + \varsigma_j \frac{K_{-j} - K_{-j}^{-1}}{q^{\ell_{-j}} - q^{-\ell_{-j}}} K_j^{-1}$$

According to Lemma 7.2.1 we have $\ell_j = (-1)^{p(j)} \ell_{-j}$. Thus $q^{\ell_{-j}} - q^{-\ell_{-j}} = (-1)^{p(j)}(q^{\ell_j} - q^{-\ell_j})$ and $B_j B_{-j} - (-1)^{p(j)} B_{-j} B_j \in \mathbf{U}^{\imath 0}$. Now suppose that $j = -n = -k$. Since $j \nsim k$, we must have $I_\bullet \neq \varnothing$. Hence for weight reason we have $\pi_{0,0}([B_j, B_k]) = 0$.

(2) The case when $j \in I_\bullet$ follows from Lemma 7.2.9. When $j \in I_\circ$, for weight reason we always have $\pi_{0,0}(S_{p(k),p(\ell)}(B_k, B_j, B_\ell)) = 0$.

(3) Suppose $k < j < \ell$. The case when $j \in I_\bullet$ follows from Lemma 7.2.9. When $j \in I_\circ$, at least one of k and ℓ lies in $I_\circ$, hence for weight reason we have $\pi_{0,0}(S_{p(k),p(\ell)}(B_k, B_j, B_\ell)) = 0$. □

Lemma 7.2.11. *Assume $k \sim j \sim \ell$ and $j \in I_{\bar{1}}$. Let $\alpha, \beta \in Q^+$, if $\pi_{\alpha,\beta}(S_{p(k),p(\ell)}(B_k, B_j, B_\ell)) \neq 0$, then $\lambda_j - \alpha \notin P_\imath$ and $\lambda_j - \beta \notin P_\imath$.*

Proof. By Lemma 7.2.9, there is nothing to show if $j \in I_\bullet$. Hence we may assume that $j \in I_\circ$. Without loss of generality we can suppose that $k < j < \ell$ and $k \in I_\circ$. Consider first the case $\ell \in I_\bullet$. We see that

$$\begin{aligned} &S_{p(k),p(\ell)}(T_{w_\bullet}(E^{\mathtt{X}}_{-k})K_k^{-1}, T_{w_\bullet}(E^{\mathtt{X}}_{-j})K_j^{-1}, F_\ell) \\ =&S_{p(k),p(\ell)}(T_{w_\bullet}(E^{\mathtt{X}}_{-k})K_k^{-1}, T_{w_\bullet}(E^{\mathtt{X}}_{-j})K_j^{-1}, T_{w_\bullet}(E^{\mathtt{X}}_{-\ell})K_\ell^{-1}) = 0 \end{aligned}$$

Hence if $\pi_{\alpha,\beta}(S_{p(k),p(\ell)}(B_k, B_j, B_\ell)) \neq 0$, we have $0 \leq \beta \leq \lambda_j - \alpha_j$ and $0 \leq \alpha \leq -\Theta(\lambda_j - \alpha_j)$. This implies that $\lambda_j - \beta \notin P_\imath$ and $\lambda_j - \alpha \notin P_\imath$. The case when $\ell \in I_\circ$ can be proved similarly as in [Ko14, Lemma 5.14]. □

For any $J = (j_1, \ldots, j_r) \in I^r$ define $wt(J) = \sum_{i=1}^r \alpha_{j_i}$ and

$$E_J = E_{j_1} \cdots E_{j_r}, \quad F_J = F_{j_1} \cdots F_{j_r}, \quad B_J = B_{j_1} \cdots B_{j_r}. \tag{7.37}$$

In this case we also define $|J| = r$.

Lemma 7.2.12. *(1) Assume $j \nsim k \in I$, then we have $P_{-\lambda_{j,k}}([B_j, B_k]) = 0$.*

(2) Assume $j \sim k$ and $j \in I_{\bar{0}}$, then we have $P_{-\lambda_{j,k}}(S(B_j, B_k)) = 0$.

(3) Assume $k \sim j \sim \ell$ and $j \in I_{\bar{1}}$, then we have $P_{-\lambda_j}(S_{p(k),p(\ell)}(B_k, B_j, B_\ell)) = 0$.

Proof. The proofs for all three cases follow the strategy presented in [Ko14, Proposition 5.16]. Therefore, we will provide the proof for case (3) only since the proofs for cases (1) and (2) can be derived similarly from Lemma 7.2.9, Lemma 7.2.10, and [Ko14, Lemma 5.14].

Assume now $k \sim j \sim \ell$ and $j \in I_{\bar{1}}$. By Lemma 7.2.9 we can assume that $j \in I_\circ$.

Set $\Xi = S_{p(k),p(\ell)}(B_k, B_j, B_\ell)$ and $Z = P_{-\lambda_j}(\Xi)$. It follows from (7.19) and (7.32) that

$$\Delta(\Xi) \in \Xi \otimes K_{-\lambda_j} + \sum_{\{J | wt(J) < \lambda_j\}} \mathbf{U}^+_\bullet \mathbf{U}^{\imath 0} B_J \otimes \mathbf{U}. \tag{7.38}$$

Moreover, relations (7.36) and (7.38) imply

$$\Delta(Z) \in \Xi \otimes K_{-\lambda_j} + \sum_{\{J | wt(J) < \lambda_j\}} \mathbf{U}^+_\bullet \mathbf{U}^{\imath 0} B_J \otimes P_{-\lambda}(\mathbf{U}). \tag{7.39}$$

Assume now that $Z \neq 0$. Choose $\alpha \in Q^+$ maximal such that $\pi_{\alpha,\beta}(Z) \neq 0$ for some $\beta \in Q^+$. In this case by (2.16) we have

$$0 \neq (id \otimes \pi_{\alpha,0})\Delta(Z) \in S(U^-)K_{-\lambda+\alpha} \otimes U^+_\alpha K_{-\lambda} \oplus S(U^-)\varrho K_{-\lambda+\alpha} \otimes U^+_\alpha K_{-\lambda}. \tag{7.40}$$

Now if $\alpha \neq 0$, the relations (7.39) and (7.40) imply that $K_{-\lambda+\alpha} \in \mathbf{U}^\imath$, which is in contradiction to Lemma 7.2.11. $\square$

Remark 7.2.13. The assumption (7.14) is required for Lemma 7.2.9 and Lemma 7.2.10.

Let $\mathcal{J}$ denote a fixed subset of $\cup_{s\in\mathbb{Z}_{\geqslant 0}} I^s$ such that $\{F_J \mid J \in \mathcal{J}\}$ is a basis of $\mathbf{U}^-$. Now we can apply the projection technique to conclude that

Proposition 7.2.14. *In $\mathbf{U}^\imath$ one has the relation*

$$\begin{aligned}
&(1)[B_j, B_k] \in \sum_{\{J\in\mathcal{J}|wt(J)<\lambda_{jk}\}} \mathbf{U}^+_\bullet \mathbf{U}^{\imath 0} B_J, && \textit{for all } j \nsim k \in I,\\
&(2)S(B_j, B_k) \in \sum_{\{J\in\mathcal{J}|wt(J)<\lambda_{j,k}\}} \mathbf{U}^+_\bullet \mathbf{U}^{\imath 0} B_J, && \textit{for all } j \sim k \in I,\ j \in I_{\bar{0}},\\
&(3)S_{p(k),p(\ell)}(B_k, B_j, B_\ell) \in \sum_{\{J\in\mathcal{J}|wt(J)<\lambda_j\}} \mathbf{U}^+_\bullet \mathbf{U}^{\imath 0} B_J, && \textit{for all } k \sim j \sim \ell \in I,\ j \in I_{\bar{1}}.
\end{aligned} \tag{7.41}$$

Proof. We only proof for (3) and the proof for (1) and (2) is similar.

Since $P_{-\lambda_j}(\Xi) = 0$ according to Lemma 7.2.12, by applying the counit to the second tensor factor in (7.39) we get $S_{p(k),p(\ell)}(B_k, B_j, B_\ell) \in \sum_{\{J|wt(J)<\lambda_j\}} \mathbf{U}^+_\bullet \mathbf{U}^{\imath 0} B_J$. $\square$

7.2.4 Quantum Iwasawa decomposition

Define a filtration $\mathcal{F}^*$ of $\mathbf{U}^-$ by

$$\mathcal{F}^t(\mathbf{U}^-) = span\{F_J \mid J \in I^s,\ s \leqslant t\}, \quad t \in \mathbb{Z}_{\geqslant 0}.$$

As the quantum Serre relations for $\mathbf{U}$ are homogeneous, the set $\{F_J \mid J \in \mathcal{J},\ |J| \leqslant t\}$ forms a basis for $\mathcal{F}^t(\mathbf{U}^-)$.

Proposition 7.2.15. *The set $\{B_J \mid J \in \mathcal{J}\}$ is a basis of the left $\mathbf{U}^+\mathbf{U}^0$-module $\mathbf{U}$.*

Proof. First for any $J \in \mathcal{J}$ such that $|J| = t$ we have $F_J - B_J \in \mathbf{U}^+\mathbf{U}^0\mathcal{F}^{t-1}(\mathbf{U}^-)$. Thus by induction on t we can conclude that each F_J is contained in the left $\mathbf{U}^+\mathbf{U}^0$-module generated by $\{B_J \mid J \in \mathcal{J}\}$.

It remains to show $\{B_J \mid J \in \mathcal{J}\}$ is linearly independent. Assume there exists a non-empty finite subset $\mathcal{J}' \subset \mathcal{J}$ such that $\sum_{J\in\mathcal{J}'} a_J B_J = 0$. Let $t_0 = max\{|J| \mid J \in \mathcal{J}'\}$. In view of the definition of B_j, we have

$$\sum_{J\in\mathcal{J}',|J|=t_0} a_J F_J = 0.$$

The linear independence of $\{F_J \mid J \in \mathcal{J}\}$ implies $a_J = 0$ for all $J \in \mathcal{J}', |J| = t_0$. Then through induction we conclude the desired result. $\square$

By Proposition 7.2.15 any element in $\mathbf{U}^\imath$ can be written as a linear combination of elements in $\{B_J \mid J \in \mathcal{J}\}$ with coefficients in $\mathbf{U}^+\mathbf{U}^0$. We want to further show that the coefficients are from $\mathbf{U}^+_\bullet\mathbf{U}^{\imath 0}$.

Proposition 7.2.16. *The set $\{B_J \mid J \in \mathcal{J}\}$ is a basis of the left $\mathbf{U}^+_\bullet\mathbf{U}^{\imath 0}$-module $\mathbf{U}^\imath$.*

Proof. First of all, since $\{B_J \mid J \in \mathcal{J}\}$ is linearly independent over $\mathbf{U}^+\mathbf{U}^0$, it is also independent over $\mathbf{U}^+_\bullet\mathbf{U}^{\imath 0}$.

Secondly, let $L \in I^t$. One can apply the Serre relations (R9) and (R11) in (2.11) repeatedly to write

$$F_L = \sum_{J\in\mathcal{J},|J|=t} a_J F_J$$

for some $a_J \in \mathbb{C}(q)$. According to Proposition 7.2.14 and Lemma 7.2.8 one sees that

$$B_L - \sum_{J\in\mathcal{J},|J|=t} a_J B_J \in \sum_{s<t}\sum_{J\in I^s} \mathbf{U}^+_\bullet\mathbf{U}^{\imath 0} B_J.$$

Thus through induction we see that $\{B_J \mid J \in \mathcal{J}\}$ spans the left $\mathbf{U}^+_\bullet\mathbf{U}^{\imath 0}$-module $\mathbf{U}^\imath$. $\square$

Define a subspace of $\mathbf{U}^\imath$ by

$$\mathbf{U}^\imath_{\mathcal{J}} := \sum_{J\in\mathcal{J}} \mathbb{C}(q) B_J. \tag{7.42}$$

Then Proposition 7.2.16 can be reformulated by saying that the multiplication map

$$\mathbf{U}_{\bullet}^{+} \otimes \mathbf{U}^{\imath 0} \otimes \mathbf{U}_{\mathcal{J}}^{\imath} \to \mathbf{U}^{\imath}$$

is an isomorphism of vector spaces.

Fix a subset $I_\tau \subset I_\circ$ consists of exactly one element of each τ-orbit within $I_\circ$. Let $\mathbf{U}_{\tau}^{\imath 0}$ denote the subalgebra generated by $\{K_i^{\pm 1} \mid i \in I_\tau\}$. Then we have the following algebra isomorphism

$$\mathbf{U}_{\tau}^{\imath 0} \otimes \mathbf{U}^{\imath 0} \cong \mathbf{U}^{0}. \tag{7.43}$$

Define $V_{\bullet}^{+}$ to be the subalgebra generated by the elements of all the finite dimensional subspace $\mathrm{ad}(\mathbf{U}_{\bullet})(E_i)$ for $i \in I_\circ$. It is proved in [K99] that

$$\mathbf{U}^{+} \cong V_{\bullet}^{+} \otimes \mathbf{U}_{\bullet}^{+}. \tag{7.44}$$

The following proposition gives the quantum Iwasawa decomposition of $\mathbf{U}$ associated with $\mathbf{U}^{\imath}$.

Theorem 7.2.17. *The multiplication map gives an isomorphism of vector spaces*

$$V_{\bullet}^{+} \otimes \mathbf{U}_{\tau}^{\imath 0} \otimes \mathbf{U}^{\imath} \cong \mathbf{U}.$$

Proof. Combining (7.44) with (7.43) we have an isomorphism

$$\mathbf{U}^{+}\mathbf{U}^{0} \cong V_{\bullet}^{+} \otimes \mathbf{U}_{\tau}^{\imath 0} \otimes \mathbf{U}_{\bullet}^{+} \otimes \mathbf{U}^{\imath 0}$$

of vector spaces. Thus Proposition 7.2.16 implies the desired result. □

Chapter 8

$\imath$Schur duality of type AIII in the super setting

In this chapter, an $\imath$Schur duality between the $\imath$quantum supergroup $\mathbf{U}^\imath$ and the Hecke algebra of type B acting on a tensor space is established, providing a super generalization of the $\imath$Schur duality of type AIII. Additionally, we construct a (quasi) K-matrix for arbitrary parameters, which facilitates the realization of the Hecke algebra action on the tensor space.

8.1 $\imath$Schur duality revisited

In this section, we explore the fundamental representation $\mathbb{W}$ of $\mathbf{U}$ and establish a commuting action between the $\imath$quantum supergroup $\mathbf{U}^\imath$ and the Hecke algebra of type B on $\mathbb{W}^{\otimes d}$.

8.1.1 Bimodule structure

Recall from (7.16) and (7.17) that

$$\Pi = w_\bullet(\Pi_{\mathtt{X}}) = \{\alpha_i = \epsilon^{\mathtt{Y}}_{i-\frac{1}{2}} - \epsilon^{\mathtt{Y}}_{i+\frac{1}{2}} \mid i \in I\}$$

is the set of simple roots of Y. Recall in (7.13) we switch the index set to $I = \mathbb{I}_{m+n-1}$. Another advantage of this notation is that we can naturally parameterize the natural representations of $\mathbf{U}$ by $\mathbb{I}_{m+n} = \mathbb{I}_{2r+m}$.

Let $\mathbb{W}$ denote the natural representation of $\mathbf{U}$. We recall the notations $\mathbb{I}_{r|m|r}, \mathbb{I}_\circ^-, \mathbb{I}_\bullet, \mathbb{I}_\circ^+$ as in § 3.1.1.

With these notations, the natural representation $\mathbb{W}$ is a vector superspace with an ordered basis $\{w_a \mid a \in \mathbb{I}_{r|m|r}\}$ such that

$$wt(w_a) = \epsilon_a^{\mathtt{Y}}, \quad E_j(w_a) = \delta_{a,j+\frac{1}{2}} w_{a-1}, \quad F_j(w_a) = \delta_{a,j-\frac{1}{2}} w_{a+1}, \quad \varrho(w_a) = (-1)^{p(w_a)} w_a \tag{8.1}$$

for all $a \in \mathbb{I}_{r|m|r}$ and $j \in I$. Note that $p(w_a) := p(\epsilon_a^{\mathtt{Y}})$.

Recall basic set ups from § 3.1.1 for the type B Weyl group W_d, Hecke algebra $\mathscr{H}_{B_d}$ and the right action of W_d on $\mathbb{I}^d_{r|m|r}$. In this section, we replace the parameter p in the definition of $\mathscr{H}_{B_d}$ with Q to avoid confusion with the parity function $p(\cdot)$.

The following proposition is a multi-parameter version of [CL22, Proposition 2.10].

Proposition 8.1.1. *There is an right action of $\mathscr{H}_{B_d}$ on $\mathbb{W}^{\otimes d}$ as follows:*

$$M_f \cdot H_i = \begin{cases} (-1)^{p(w_{f(i)})p(w_{f(i+1)})} M_{f\cdot s_i} + (q-q^{-1})M_f, & \textit{if } f(i) < f(i+1),\ i>0; \\ (-1)^{p(w_{f(i)})p(w_{f(i+1)})} M_{f\cdot s_i}, & \textit{if } f(i) > f(i+1),\ i>0; \\ \dfrac{(-1)^{p(w_{f(i)})}(q+q^{-1}) + q - q^{-1}}{2} M_f, & \textit{if } f(i) = f(i+1),\ i>0; \\ (-1)^{p(w_{f(1)})} M_{f\cdot s_i} + (Q-Q^{-1})M_f, & \textit{if } f(1) \in \mathbb{I}_\circ^+,\ i=0; \\ (-1)^{p(w_{f(1)})} M_{f\cdot s_i}, & \textit{if } f(1) \in \mathbb{I}_\circ^-,\ i=0, \\ QM_f, & \textit{if } f(1) \in \mathbb{I}_\bullet,\ i=0. \end{cases}$$

Proof. It has been established in [Mi06] that this defines an action of $\mathscr{H}_{S_d}$ on $\mathbb{W}^{\otimes d}$. The remaining nontrivial relation to verify is $H_0H_1H_0H_1 = H_1H_0H_1H_0$, which can be confirmed through a case-by-case check. $\square$

8.1.2 $\imath$Schur(-Sergeev) duality

We first recall results from [Mi06] which establish a type A Schur duality between the quantum supergroup and the Hecke algebra of type A.

We let ${}^{st}\mathbf{U}_q^{\varrho}(\mathfrak{gl}(m|n))$ denote the quantum supergroup corresponding to the standard Dynkin diagram as in Example 2.3.1. The actions we define in Proposition 8.1.1 coincides with [Mi06, (3.1)(3.2)]. We denote by Φ^{st} (resp. Φ) the homomorphism from ${}^{st}\mathbf{U}_q^{\varrho}(\mathfrak{gl}(m|n))$ (resp. $\mathbf{U}$) to $End(\mathbb{W}^{\otimes d})$. Both images of Φ and Φ^{st} equal to the centralizer of $\mathscr{H}_{S_d}$-actions within $End(\mathbb{W}^{\otimes d})$, hence we have $\Phi(\mathbf{U}) = \Phi^{st}({}^{st}\mathbf{U}_q^{\varrho}(\mathfrak{gl}(m|n)))$. Moreover, we have the following theorem.

Theorem 8.1.2. *[Mi06] The actions of $\mathbf{U}$ and $\mathscr{H}_{S_d}$ on $\mathbb{W}^{\otimes d}$ commute with each other:*

$$\mathbf{U} \overset{\Phi}{\curvearrowright} \mathbb{W}^{\otimes d} \overset{\Psi}{\curvearrowleft} \mathscr{H}_{S_d}.$$

Moreover, $\Phi(\mathbf{U})$ and $\Psi(\mathscr{H}_{S_d})$ form double centralizers in End $(\mathbb{W}^{\otimes d})$.

Following the strategy of [SW23], we develop a type B $\imath$Schur duality between $\mathbf{U}^\imath$ and $\mathscr{H}_{B_d}$. For any reduced expression $\underline{w}_\bullet^{(y)} = s_{y_1} \cdots s_{y_\ell}$ of $w_\bullet$, as in the proof of Proposition 7.2.6, we define

$$Y_t^{(y)} = s_{y_t} \cdots s_{y_\ell}(X), \quad 1 \leqslant t \leqslant \ell.$$

Note that $Y = Y_1^{(y)}$ for any $\underline{w}_\bullet^{(y)}$.

In the next two lemmas we compute explicitly the actions of B_j ($j \in I_\circ$) on $\mathbb{W}$.

Lemma 8.1.3. *For $a \in \mathbb{I}_{r|m|r}$ and $i \in I_\circ = [1-n-r, -n] \cup [n, n+r-1]$, we have*

$$T_{w_\bullet}(E^{\mathtt{X}}_{\tau(i)})(w_a) = \begin{cases} E_{-i}(w_a), & |i| > n; \\ E_{-n+1}E_{-n+2}\cdots E_{n-1}E_n(w_a), & i = -n; \\ \hbar_m E_{-n}E_{-n+1}\cdots E_{n-2}E_{n-1}(w_a), & i = n. \end{cases}$$

where

$$\hbar_m = (-1)^{m-1+p(\alpha^{Y_{2n-1}}_{-n+1})p(\alpha^{Y_{2n-1}}_{-n})+\cdots+p(\alpha^{Y_1}_{n-1})p(\alpha^{Y_1}_{n-2})} q^{(\alpha^{\mathtt{X}}_{-n+1},\alpha^{\mathtt{X}}_{-n})+\cdots+(\alpha^{\mathtt{X}}_{n-1},\alpha^{\mathtt{X}}_{n-2})}. \tag{8.2}$$

Proof. The computation follows similarly as in [SW23, Lemma 4.2].

Take $i = n$ for example. We choose

$$\underline{w}^{(y)}_\bullet = (s_{n-1}s_{n-2}\cdots s_{-n+1})\cdots(s_{n-1}s_{n-2})(s_{n-1}).$$

For convention we drop (y) in the following proof. Then we compute $T_{w_\bullet}(E^{\mathtt{X}}_{-n})(w_a)$ as follows:

$$\begin{aligned}
&T_{w_\bullet}(E^{\mathtt{X}}_{-n})(w_a) \\
=&T_{s_{n-1}}\cdots T_{s_{-n+1}}(E^{Y_{2n}}_{-n})(w_a) \\
=&T_{s_{n-1}}\cdots T_{s_{-n+2}}(E^{Y_{2n-1}}_{-n+1}E^{Y_{2n-1}}_{-n} \\
&\quad -(-1)^{p(\alpha^{Y_{2n-1}}_{-n+1})p(\alpha^{Y_{2n-1}}_{-n})} q^{(\alpha^{Y_{2n-1}}_{-n+1},\alpha^{Y_{2n-1}}_{-n})} E^{Y_{2n-1}}_{-n}E^{Y_{2n-1}}_{-n+1})w_a \\
=&-(-1)^{p(\alpha^{Y_{2n-1}}_{-n+1})p(\alpha^{Y_{2n-1}}_{-n})} q^{(\alpha^{\mathtt{X}}_{-n+1},\alpha^{\mathtt{X}}_{-n})} E_{-n}T_{s_{n-1}}\cdots T_{s_{-n+2}}(E^{Y_{2n-1}}_{-n+1})(w_a)
\end{aligned}$$

By induction on n, we have

$$T_{w_\bullet}(E^{\mathtt{X}}_{-n})(w_a) = \hbar_m E_{-n}E_{-n+1}\cdots E_{n-2}E_{n-1}(w_a).$$

This proves the lemma. $\square$

Lemma 8.1.3 together with the formula for B_j immediately imply the following.

Lemma 8.1.4. *Let $a \in \mathbb{I}_{r|m|r}$ and $j \in I_\circ$. The action of B_j on $\mathbb{W}$ is given by:*

$$B_{-n}(w_a) = \begin{cases} w_{-n+\frac{1}{2}}, & \text{if } a = -n - \frac{1}{2}; \\ \varsigma_{-n} w_{-n+\frac{1}{2}}, & \text{if } a = n + \frac{1}{2}; \\ 0, & \text{else}, \end{cases}$$

$$B_i(w_a) = \begin{cases} w_{i+\frac{1}{2}}, & \text{if } a = i - \frac{1}{2}; \\ \varsigma_i w_{-i-\frac{1}{2}}, & \text{if } a = -i + \frac{1}{2}; \\ 0, & \text{else}, \end{cases} \qquad \text{for } |i| > n,$$

and (recall $m = 2n$)

$$B_n(w_a) = \begin{cases} w_{n+\frac{1}{2}} + q^{-(-1)^{p\left(w_{n-\frac{1}{2}}\right)}} \hbar_m \varsigma_n w_{-n-\frac{1}{2}}, & \text{if } a = n - \frac{1}{2}; \\ 0, & \text{else}. \end{cases}$$

Remark 8.1.5. When $p(j) = 0$ for all $j \in I_\bullet$, the computations can be greatly simplified and we have ($p(w_a) = 0$ for all $a \in \mathbb{I}_{r|m|r}$)

$$\hbar_m = (-1)^{m-1} q^{1-m}.$$

For the rest of this section we fix the parameters to be

$$\begin{cases} \varsigma_j = (-1)^{p(j)}, \quad \text{if } j \neq \pm n, \\ \varsigma_{-n} = (-1)^{p\left(w_{n+\frac{1}{2}}\right)} Q, \\ \varsigma_n = (-1)^{p\left(w_{n+\frac{1}{2}}\right)} q^{(-1)^{p\left(w_{n-\frac{1}{2}}\right)}} Q^{-1} \hbar_m^{-1}. \end{cases} \qquad \text{where } m = 2n \in \mathbb{Z}_{\geq 1}, \tag{8.3}$$

Introduce the $\mathbb{Q}(Q,q)$-subspaces of $\mathbb{W}$:

$$\mathbb{W}_- = \bigoplus_{a\in\mathbb{I}_\circ^+} \mathbb{Q}(Q,q)(w_a - (-1)^{p(w_{-a})}Qw_{-a}), \qquad \mathbb{W}_\bullet = \bigoplus_{a\in\mathbb{I}_\bullet} \mathbb{Q}(Q,q)w_a,$$
$$\mathbb{W}_+ = \bigoplus_{a\in\mathbb{I}_\circ^+} \mathbb{Q}(Q,q)(w_a + (-1)^{p(w_{-a})}Q^{-1}w_{-a}).$$

Lemma 8.1.6. *Assume* (8.3). *Then* $\mathbb{W}_-$ *and* $\mathbb{W}_\bullet \oplus \mathbb{W}_+$ *are* $\mathbf{U}^\imath$*-submodules of* $\mathbb{W}$. *Hence, we have a* $\mathbf{U}^\imath$*-module decomposition* $\mathbb{W} = (\mathbb{W}_\bullet \oplus \mathbb{W}_+) \oplus \mathbb{W}_-$.

Proof. It follows by a direct computation using the formulas (8.1) and Lemma 8.1.4. □

The decomposition of $\mathbb{W}$ above is also compatible with the H_0-action.

Lemma 8.1.7. *The Hecke generator* H_0 *acts on* $\mathbb{W}_-$ *as* $(-Q^{-1})Id$ *and acts on* $\mathbb{W}_\bullet \oplus \mathbb{W}_+$ *as* $Q \cdot Id$.

Proof. It follows from a direct computation on the basis vectors. □

Theorem 8.1.8. *Suppose the parameters satisfy* (8.3). *Then the actions of* $\mathbf{U}^\imath$ *and* $\mathscr{H}_{B_d}$ *on* $\mathbb{W}^{\otimes d}$ *commute with each other:*

$$\mathbf{U}^\imath \overset{\Psi}{\curvearrowright} \mathbb{W}^{\otimes d} \overset{\Phi}{\curvearrowleft} \mathscr{H}_{B_d}.$$

Moreover, $\Psi(\mathbf{U}^\imath)$ *and* $\Phi(\mathscr{H}_{B_d})$ *form double centralizers in End* $(\mathbb{W}^{\otimes d})$.

Proof. By Theorem 8.1.2, we know that the actions of $\mathbf{U}$ commute with the action of H_i, for $1 \leq i \leq d-1$. Thus, to show the commuting actions of $\mathbf{U}^\imath$ and $\mathscr{H}_{B_d}$, it remains to check the commutativity of the actions of H_0 and the generators of $\mathbf{U}^\imath$.

To that end, it suffices to consider $d = 1$ (thanks to the coideal property of $\mathbf{U}^\imath$ and the fact that the action of H_0 depends solely on the first tensor factor). In this case, the commutativity between $\mathbf{U}^\imath$-action and H_0-action on $\mathbb{W}$ follows directly from Lemmas 8.1.6 and 8.1.7.

The double centralizer property is equivalent to a multiplicity-free decomposition of $\mathbb{W}^{\otimes d}$ as an $\mathbf{U}^\imath \otimes \mathscr{H}_{B_d}$-module, which reduces by a deformation argument to the $q = 1$ setting. At the specialization $q, Q \mapsto 1$, $\mathbf{U}^\imath$ becomes the enveloping algebra of a direct sum of two type A Lie superalgebras (cf. [Se83]), $\mathbb{W} = (\mathbb{W}_\bullet \oplus \mathbb{W}_+) \oplus \mathbb{W}_-$ becomes the natural representation of it, on which $s_0 \in W_d$ acts as $(\mathrm{Id}_{\mathbb{W}_\bullet \oplus \mathbb{W}_+}, -\mathrm{Id}_{\mathbb{W}_-})$. The multiplicity-free decomposition of $\mathbb{W}^{\otimes d}$ at $q = Q = 1$ can be established by a standard approach as in [CW12, Theorem 3.9]. $\square$

8.2 Quasi K-matrix

From this section on we impose one extra condition on the Satake diagrams we are working with:

$$p(j) = 0, \ \forall j \in I_\bullet. \tag{8.4}$$

Under the assumption (8.4), the braid group operators T_i for $i \in I_\bullet$ reduce to the ones of Lusztig and we do not need to work with different presentations of $\mathbf{U}$ anymore. Hence the scripts standing for the underlying Dynkin diagrams will be omitted.

In this section, we follow [BK19] and [Ko22, §3.2] to construct the quasi K-matrix under the assumption (8.4).

8.2.1 Preparation

Suppose $Y \in \mathcal{D}_{m,n}$ is of the form (7.12) and satisfies (7.14) and (8.4). Again we let $\mathbf{U}(Y)$ denote the quantum supergroup with generators $\varrho, E_j, F_j, K_j, j \in I$ associated to Y. Recall that $\mathbf{U}^\imath(Y)$ is the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}(Y)$ generated by q^μ $(\mu \in P_\imath^\vee)$, E_j, F_j $(j \in I_\bullet)$, ϱ and

$$B_j = F_j + \varsigma_j T_{w_\bullet}(E_{\tau j}) K_j^{-1}, \quad \text{for } j \in I_\circ. \tag{8.5}$$

Abusing the notation τ, the diagram involution τ gives rise to the following algebra homomorphism on $\mathbf{U}$:

Proposition 8.2.1. *Under the assumption* (7.14), *there is an involution* τ *on* $\mathbf{U}$ *such that*

$$\tau(E_j) = E_{\tau j}, \quad \tau(F_j) = F_{\tau j}, \quad \tau(K_j) = (-1)^{p(j)} K_{\tau j}, \quad \tau(\varrho) = \varrho \tag{8.6}$$

for all $j \in I$.

Proof. The proof follows from checking on the generators and Lemma 7.2.1. $\square$

The super skew derivations (cf. [CHW13, §1.5]) ${}_ir$ and r_i on $\mathbf{U}^+$ satisfy the following relations $r_i(E_j) = \delta_{i,j}$, ${}_ir(E_j) = \delta_{i,j}$, and

$$\begin{aligned} {}_ir(xy) &= (-1)^{p(y)p(i)} {}_ir(x)y + q^{(\alpha_i,\mu)} x {}_ir(y), \\ r_i(xy) &= (-1)^{p(y)p(i)} q^{(\alpha_i,\upsilon)} r_i(x)y + x r_i(y) \end{aligned} \tag{8.7}$$

for all $x \in \mathbf{U}^+_\mu$, $y \in \mathbf{U}^+_\upsilon$.

Let $\mathbf{U}^{\geq}$ (resp. $\mathbf{U}^{\leq}$) denote the Hopf subalgebra of $\mathbf{U}$ generated by $\mathbf{U}^0$ and $\mathbf{U}^+$ (resp. $\mathbf{U}^-$). According to [Ya94, §2.4], there is a non-degenerated bilinear pairing $\langle \cdot, \cdot \rangle$ on $\mathbf{U}^{\leq} \times \mathbf{U}^{\geq}$ such that for all $x, x' \in \mathbf{U}^{\geq}$, $y, y' \in \mathbf{U}^{\leq}$, $\mu, \upsilon \in P$ and $a, b \in \{0, 1\}$, we have

$$\begin{aligned} &\langle y, xx' \rangle = \langle \Delta(y), x' \otimes x \rangle, && \langle yy', x \rangle = \langle y \otimes y', \Delta(x) \rangle, \\ &\langle q^\mu \varrho^a, q^\upsilon \varrho^b \rangle = (-1)^{ab} q^{-(\mu,\upsilon)}, && \langle F_j, E_k \rangle = \delta_{j,k}, \\ &\langle q^\mu \varrho^a, E_j \rangle = 0, && \langle F_j, q^\mu \varrho^a \rangle = 0 \end{aligned} \tag{8.8}$$

The next lemma is a super analogue of [Jan95, §6.14].

Lemma 8.2.2. *For all* $x \in \mathbf{U}^+$, $y \in \mathbf{U}^-$ *and* $j \in I$ *one has*

$$\langle F_i y, x \rangle = (-1)^{p(x)p(j)} \langle F_i, E_i \rangle \langle y, {}_ir(x) \rangle, \quad \langle y F_i, x \rangle = \langle F_i, E_i \rangle \langle y, r_i(x) \rangle. \tag{8.9}$$

Proof. Suppose $x \in \mathbf{U}^+_\mu$, then we have that

$$\Delta(x) = x \otimes 1 + \sum_{i\in I} r_i(x)\varrho^{p(i)} K_i \otimes E_i + (rest)_1,$$
$$\Delta(x) = \varrho^{p(x)} K_\mu \otimes x + \sum_{i\in I} (-1)^{p(x)p(i)} E_i \varrho^{p(\mu-\alpha_i)} K_{\mu-\alpha_i} \otimes {}_i r(x) + (rest)_2$$

where $(rest)_1, (rest)_2 \in \varrho\mathbf{U}^+_{\mu-\upsilon} K_\upsilon \otimes \mathbf{U}^+_\upsilon$ or $\mathbf{U}^+_{\mu-\upsilon} K_\upsilon \otimes \mathbf{U}^+_\upsilon$ with $\upsilon > 0,\ \upsilon \notin \Pi$. Hence the lemma follows from (8.8). □

The next lemma is a crucial ingredient to construct the quasi K-matrix.

Lemma 8.2.3. *For all $x \in \mathbf{U}^+$, we have*

$$[x, F_j] = xF_j - (-1)^{p(x)p(j)} F_j x = \frac{1}{q^{\ell_j} - q^{-\ell_j}} (r_j(x) K_j - K_j^{-1} {}_j r(x)). \tag{8.10}$$

Proof. We induct on $ht(x)$. When $x = E_j$ for some $j \in I$, (8.10) follows from the definition. Now if $x = uv$ where $ht(u) < ht(x)$ and $ht(v) < ht(x)$, then we have

$$\begin{aligned}
uvF_j =& (-1)^{p(v)p(i)} uF_j v + \frac{ur_j(v)K_j - uK_j^{-1}{}_j r(v)}{q^{\ell_j} - q^{-\ell_j}} \\
=& (-1)^{p(uv)p(j)} F_j uv + \frac{ur_j(v)K_j - q^{(\alpha_j, |u|)} K_j^{-1} u_j r(v)}{q^{\ell_j} - q^{-\ell_j}} \\
&+ (-1)^{p(v)p(j)} \frac{r_j(u)K_j v - K_j^{-1}{}_j r(u) v}{q^{\ell_j} - q^{-\ell_j}} \\
=& (-1)^{p(uv)p(j)} F_j uv + \frac{r_j(uv)K_j - K_j^{-1}{}_i r(uv)}{q^{\ell_j} - q^{-\ell_j}}.
\end{aligned}$$

This proves the lemma. □

8.2.2 A recursive formula

Define $Q^+_0 := \{\alpha \in Q^+ \mid p(\alpha) = 0\}$. Extending (7.18), we write $B_i = F_i$ for $i \in I_\bullet$. Following [BK19, §6], we establish the following lemma to give equivalent conditions on the existence of the quasi K-matrix.

Let $\rho_\bullet$ denote the half sum of positive roots of the Levi subalgebra associated with $I_\bullet \subset I$.

Lemma 8.2.4. *Let $\Upsilon = \sum_{\mu \in Q_0^+} \Upsilon_\mu$ with $\Upsilon_\mu \in \mathbf{U}_\mu^+$ be an element in the completion of $\mathbf{U}$, then the following are equivalent.*

(1) For all $i \in I$, we have (cf. [WZ22, (3.20)])

$$B_i \Upsilon = \Upsilon \tau \circ \sigma(B_{\tau i}). \tag{8.11}$$

(2) For all $i \in I$, we have

$$B_i \Upsilon = \Upsilon(F_i + (-1)^{(2\rho_\bullet, \alpha_i)} q^{(\alpha_i, 2\rho_\bullet + w_\bullet \alpha_{\tau i})} \varsigma_{\tau i} \overline{T_{w_\bullet}(E_{\tau i})} K_i). \tag{8.12}$$

(3) The element Υ satisfy the following relations:

$$\begin{aligned} r_i(\Upsilon_\mu) &= -(q^{\ell_i} - q^{-\ell_i}) \Upsilon_{\mu - \alpha_i - w_\bullet(\alpha_{\tau i})} (-1)^{(2\rho_\bullet, \alpha_i)} q^{(\alpha_i, 2\rho_\bullet + w_\bullet \alpha_{\tau i})} \varsigma_{\tau i} \overline{T_{w_\bullet}(E_{\tau i})}, \\ {}_i r(\Upsilon_\mu) &= -(q^{\ell_i} - q^{-\ell_i}) q^{(\alpha_i, w_\bullet \alpha_{\tau i})} \varsigma_i T_{w_\bullet}(E_{\tau i}) \Upsilon_{\mu - \alpha_i - w_\bullet \alpha_{\tau i}}. \end{aligned} \tag{8.13}$$

Moreover, if these relations hold then additionally we have

$$x\Upsilon = \Upsilon x \text{ for all } x \in \mathbf{U}^{\imath 0} \mathbf{U}_\bullet. \tag{8.14}$$

and

$$\Upsilon_\mu = 0 \text{ unless } w_\bullet \tau(\mu) = \mu. \tag{8.15}$$

Proof. Note that

$$\tau \circ \sigma(B_{\tau i}) = F_i + \varsigma_{\tau i} K_i T_{w_\bullet}^{-1}(E_{\tau i}).$$

Thus the equivalence of (1) and (2) follows from [BW18b, Lemma 4.17]. The equivalence of (2) and (3) follows from Lemma 8.2.3. Moreover, (8.15) follows from an induction argument on $ht(\mu)$; cf. [BK19, Proposition 6.1]. $\square$

The following lemma states that every non-vanishing term of the quasi K-matrix is expected to have parity 0.

Lemma 8.2.5. *For $\mu \in Q^+$, if $w_\bullet\tau(\mu) = \mu$, then $p(\mu) = 0$.*

Proof. Let's use induction on $ht(\mu)$. Now we can write $\mu = \sum_{t=1}^{\ell} a_t\alpha_{j_t}$ where $a_t > 0$ for $1 \leqslant t \leqslant \ell$. Now if all α_{j_t} are even roots, then we have $p(\mu) = 0$. On the other hand, suppose $p(\alpha_{j_1}) = 1$. Since $w_\bullet\tau(\mu) = \mu$, we have $\mu' = \mu - (\alpha_{j_1} + w_\bullet\tau(\alpha_{j_1})) \in Q^+$ and $w_\bullet\tau(\mu') = \mu'$ and $ht(\mu') < ht(\mu)$. Thus by the inductive hypothesis we have $p(\mu') = 0$. Also, according to (7.14), we have $p(\alpha_{j_1}) = p(w_\bullet\tau(\alpha_{j_1}))$. Thus we have $p(\mu) = 0$ as well. $\square$

Lemma 8.2.6. *For any $i, j \in I$, we have*

$$r_i \circ {}_jr(x) = (-1)^{p(i)p(j)}\, {}_jr \circ r_i(x), \quad \forall x \in \mathbf{U}^+.$$

Proof. If $u = E_k$ or 1, then we certainly have $r_i \circ {}_jr(u) = (-1)^{p(i)p(j)}\, {}_jr \circ r_i(u)$. Thus It is enough to show that $r_i \circ {}_jr(xy) = {}_jr \circ r_i(xy)$ for any $x \in \mathbf{U}^+_\mu,\ y \in \mathbf{U}^+_\upsilon$.

We have

$$\begin{aligned}
&r_i \circ {}_jr(xy)\\
=&r_i((-1)^{p(y)p(j)}\,{}_jr(x)y + q^{(\alpha_j,\mu)}x\,{}_jr(y))\\
=&(-1)^{p(y)p(j)}[(-1)^{p(y)p(i)}q^{(\alpha_i,\upsilon)}r_i \circ {}_jr(x)y + {}_jr(x)r_i(y)]\\
&\quad + q^{(\alpha_j,\mu)}[(-1)^{p({}_jr(y))p(i)}q^{(\alpha_i,\upsilon-\alpha_j)}r_i(x)\,{}_jr(y) + xr_i \circ {}_jr(y)],\\
&{}_jr \circ r_i(xy)\\
=&{}_jr((-1)^{p(y)p(i)}q^{(\alpha_i,\upsilon)}r_i(x)y + xr_i(y))\\
=&(-1)^{p(y)p(i)}q^{(\alpha_i,\upsilon)}[(-1)^{p(y)p(j)}\,{}_jr \circ r_i(x)y + q^{(\alpha_j,\mu-\alpha_i)}r_i(x)\,{}_jr(y)]\\
&\quad + [(-1)^{p({}_ir(y))p(j)}\,{}_jr(x)r_i(y) + q^{(\alpha_j,\mu)}x\,{}_jr \circ r_i(y)].
\end{aligned}$$

Now since $p({}_k r(y)) = p(y) \pm p(k)$ for any $k \in I$, we have $r_i \circ {}_j r = (-1)^{p(i)p(j)} {}_j r \circ r_i$. $\square$

The system of equations (8.13) for all $i \in I$ provides an equivalent condition for the existence of Υ, and our objective is to solve it recursively using the following proposition.

Proposition 8.2.7. *(cf. [BK19, Proposition 6.3]) Let $\mu \in Q^+_{\bar{0}}$ with $ht(\mu) \geqslant 2$ and fix A_i, ${}_iA \in \mathbf{U}^+_{\mu-\alpha_i}$ for all $i \in I$. The following are equivalent.*

(1) There exists an element $\Xi \in \mathbf{U}^+_\mu$ such that

$$r_i(\Xi) = A_i, \quad {}_ir(\Xi) = {}_iA, \quad \forall i \in I.$$

(2) The elements A_i and ${}_iA$ satisfy the following properties.

(2a) For all $i, j \in I$, we have

$$r_i({}_jA) = (-1)^{p(i)p(j)} {}_ir(A_j). \tag{8.16}$$

(2b) For all $i \in I_{\bar{1}}$, we have

$$\langle F_i, A_i \rangle = 0. \tag{8.17}$$

(2c) For all $i \nsim j \in I$, we have

$$\langle F_i, A_j \rangle = (-1)^{p(i)p(j)} \langle F_j, A_i \rangle. \tag{8.18}$$

(2d) For all $i \in I_{\bar{0}}$ and $j \sim i$, we have

$$\langle F_i^2, A_j \rangle - [2] \langle F_iF_j, A_i \rangle + \langle F_jF_i, A_i \rangle = 0. \tag{8.19}$$

(2e) For all $i \in I_{\bar{1}}$ and $j \sim i \sim k$, we have

$$\begin{aligned} [2]\langle F_iF_kF_j, A_i\rangle = (-1)^{p(j)}\langle F_iF_kF_i, A_j\rangle + (-1)^{p(j)+p(j)p(k)}\langle F_jF_iF_k, A_i\rangle \\ +(-1)^{p(k)}\langle F_iF_jF_i, A_k\rangle + (-1)^{p(k)+p(j)p(k)}\langle F_kF_iF_j, A_i\rangle. \end{aligned} \tag{8.20}$$

Proof. The proposition follows by a rerun of proof of [BK19, Proposition 6.3]. □

8.2.3 Technical Lemmas

Define

$$\varsigma_i' = (-1)^{(2\rho_\bullet,\alpha_i)} q^{(\alpha_i, 2\rho_\bullet + w_\bullet \alpha_{\tau i})} \varsigma_{\tau i}, \text{ for all } i \in I. \tag{8.21}$$

Thus we can rewrite (8.13) as

$$\begin{aligned} A_i &= -(q^{\ell_i} - q^{-\ell_i})\Upsilon_{\mu-\alpha_i-w_\bullet\alpha_{\tau i}} \varsigma_i' \overline{T_{w_\bullet}(\ E_{\tau i})}, \\ {}_iA &= -(q^{\ell_i} - q^{-\ell_i}) q^{(\alpha_i, w_\bullet \alpha_{\tau i})} \varsigma_i T_{w_\bullet}(E_{\tau i})\Upsilon_{\mu-\alpha_i-w_\bullet\alpha_{\tau i}} \end{aligned} \tag{8.22}$$

In order to construct the quasi K-matrix Υ recursively, it suffices to show (8.22) satisfies relations (8.16)–(8.20) for all $i \in I$. Following the strategy from [BK19] we develop several lemmas as follows.

Lemma 8.2.8. *For all $u \in \mathbf{U}^+_\mu$, we have*

$$\sigma \circ {}_ir(u) = (-1)^{p(i)(p(u)+1)} r_i \circ \sigma(u). \tag{8.23}$$

Proof. We prove by induction on $ht(\mu)$. When $u = E_j$ or $u = 1$ the equality holds by definition. Now suppose $x \in \mathbf{U}^+_{\mu_1},\ y \in \mathbf{U}^+_{\mu_2}$ where $u = xy,\ \mu = \mu_1 + \mu_2$ and $\mu_1, \mu_2 > 0$. Then

we have

$$\begin{aligned}\sigma \circ {}_ir \circ \sigma(xy) =& \sigma((-1)^{p(x)p(i)}{}_ir \circ \sigma(y)\sigma(x) + q^{(\alpha_i,\mu_2)}\sigma(y){}_ir \circ \sigma(x)) \\ =& (-1)^{p(x)p(i)}x\sigma \circ {}_ir \circ \sigma(y) + q^{(\alpha_i,\mu_2)}\sigma \circ {}_ir \circ \sigma(x)y \\ =& (-1)^{p(xy)p(i)+p(i)}r_i(xy).\end{aligned}$$

This proves the lemma. □

Lemma 8.2.9. *For all $x \in \mathbf{U}^+_\mu$, we have*

$$\overline{r_i}(x) = q^{(\alpha_i,\alpha_i-\mu)}{}_ir(x). \tag{8.24}$$

Proof. We prove by induction on $ht(\mu)$. When $u = E_j$ or $u = 1$ the equality holds by definition. Now suppose $x \in U^+_{\mu_1}$, $y \in \mathbf{U}^+_{\mu_2}$ where $u = xy$, $\mu = \mu_1 + \mu_2$ and $\mu_1, \mu_2 > 0$. Then we have

$$\begin{aligned}\overline{r_i}(xy) =& x\overline{r_i}(y) + (-1)^{p(y)p(i)}q^{-(\alpha_i,\mu_2)}\overline{r_i}(x)y \\ =& q^{(\alpha_i,\alpha_i-\mu_2)}x_ir(y) + (-1)^{p(y)p(i)}q^{(\alpha_i,\alpha_i-\mu_1-\mu_2)}{}_ir(x)y \\ =& q^{(\alpha_i,\alpha_i-\mu)}[q^{(\alpha_i,\mu_1)}{}_ir(y) + (-1)^{p(y)p(i)}{}_ir(x)y] \\ =& q^{(\alpha_i,\alpha_i-\mu)}{}_ir(xy).\end{aligned}$$

This proves the lemma. □

Lemma 8.2.10. *For all $i \in I_\circ$, we have*

$$\overline{r_i(T_{w_\bullet}(E_i))} = (-1)^{(\alpha_i,2\rho_\bullet)}q^{(\alpha_i,\alpha_i-w_\bullet\alpha_i-2\rho_\bullet)}\sigma \circ \tau(r_{\tau i}(T_{w_\bullet}(E_{\tau i}))). \tag{8.25}$$

Proof. Follow from a rerun of the proof of [BK15, Lemma 2.9]. □

Lemma 8.2.11. *For all $i \in I_\circ$, we have*

$$\sigma \circ \tau(r_i(T_{w_\bullet}(E_i))) = r_i(T_{w_\bullet}(E_i)). \tag{8.26}$$

Proof. Follow from a rerun of the proof of [BK15, Proposition 2.3]. □

Combining Lemma 8.2.10 and Lemma 8.2.11 we get

Corollary 8.2.12. *For all $i \in I_\circ$, we have*

$$\overline{r_i(T_{w_\bullet}(E_i))} = (-1)^{(\alpha_i, 2\rho_\bullet)} q^{(\alpha_i, \alpha_i - w_\bullet \alpha_i - 2\rho_\bullet)} r_{\tau i}(T_{w_\bullet}(E_{\tau i})). \tag{8.27}$$

8.2.4 Construction of Υ

Now we are ready to check that (8.22) for all $i \in I$ indeed satisfy relations (8.16)–(8.20).

Lemma 8.2.13. *The relation $r_i({}_jA) = (-1)^{p(i)p(j)} {}_jr(A_i)$ holds for all $i, j \in \mathbb{I}$.*

Proof. We calculate that

$$
\begin{aligned}
&\frac{1}{-(q^{\ell_j}-q^{-\ell_j})} r_i({}_jA)\\
=&q^{(\alpha_j,w_\bullet\alpha_{\tau j})}\varsigma_j[(-1)^{p(i)p(\mu)}q^{(\alpha_i,\mu-\alpha_j-w_\bullet\alpha_{\tau j})}r_i(T_{w_\bullet}(E_{\tau j}))\Upsilon_{\mu-\alpha_j-w_\bullet\alpha_{\tau j}}\\
&\qquad + T_{w_\bullet}(E_{\tau j})r_i(\Upsilon_{\mu-\alpha_j-w_\bullet\alpha_{\tau j}})]\\
=&q^{(\alpha_j,w_\bullet\alpha_{\tau j})}\varsigma_j[(-1)^{p(i)p(\mu)}q^{(\alpha_i,\mu-\alpha_j-w_\bullet\alpha_{\tau j})}r_i(T_{w_\bullet}(E_{\tau j}))\Upsilon_{\mu-\alpha_j-w_\bullet\alpha_{\tau j}}\\
&\qquad - (q^{\ell_i}-q^{-\ell_i})T_{w_\bullet}(E_{\tau j})\Upsilon_{\mu-\alpha_j-w_\bullet\alpha_{\tau j}-\alpha_i-w_\bullet\alpha_{\tau i}}\varsigma_i'\overline{T_{w_\bullet}(E_{\tau i})}],\\
&\frac{1}{-(q^{\ell_i}-q^{-\ell_i})} {}_jr(A_i)\\
=&(-1)^{p(w_\bullet\alpha_{\tau i})p(j)}{}_jr(\Upsilon_{\mu-\alpha_i-w_\bullet\alpha_{\tau i}})\varsigma_i'\overline{T_{w_\bullet}(E_{\tau i})}\\
&\qquad + q^{(\alpha_j,\mu-\alpha_i-w_\bullet\alpha_{\tau i})}\varsigma_i'\Upsilon_{\mu-\alpha_i-w_\bullet\alpha_{\tau i}}{}_jr(\overline{T_{w_\bullet}(E_{\tau i})})\\
=&-(-1)^{p(i)p(j)}(q^{\ell_j}-q^{-\ell_j})q^{(\alpha_j,w_\bullet\alpha_{\tau j})}\varsigma_jT_{w_\bullet}(E_{\tau j})\cdot\\
&\Upsilon_{\mu-\alpha_j-w_\bullet\alpha_{\tau j}-\alpha_i-w_\bullet\alpha_{\tau i}}\varsigma_i'\overline{T_{w_\bullet}(E_{\tau i})}\\
&\qquad + q^{(\alpha_j,\mu-\alpha_i-w_\bullet\alpha_{\tau i})}\varsigma_i'\Upsilon_{\mu-\alpha_i-w_\bullet\alpha_{\tau i}}{}_jr(\overline{T_{w_\bullet}(E_{\tau i})})
\end{aligned}
$$

Recall that Υ_μ vanishes whenever $p(\mu)=1$. By comparing the two equations we see that the relation $r_i({}_jA)=(-1)^{p(i)p(j)}{}_jr(A_i)$ holds if and only if

$$
\begin{aligned}
&q^{(\alpha_j,w_\bullet\alpha_{\tau j})}\varsigma_jq^{(\alpha_i,\mu-\alpha_j-w_\bullet\alpha_{\tau j})}r_i(T_{w_\bullet}(E_{\tau j}))\Upsilon_{\mu-\alpha_j-w_\bullet\alpha_{\tau j}}\\
=&(-1)^{p(i)p(j)}\frac{q^{\ell_j}-q^{-\ell_j}}{q^{\ell_i}-q^{-\ell_i}}q^{(\alpha_j,\mu-\alpha_i-w_\bullet\alpha_{\tau i})}\varsigma_i'\Upsilon_{\mu-\alpha_i-w_\bullet\alpha_{\tau i}}{}_jr(\overline{T_{w_\bullet}(E_{\tau i})})
\end{aligned}
\tag{8.28}
$$

We may assume $i=\tau j$, otherwise both sides of (8.28) vanish. According to (8.24) we have

$$
{}_{\tau i}r(\overline{T_{w_\bullet}(E_{\tau i})}) = q^{(\alpha_{\tau i},w_\bullet\alpha_{\tau i}-\alpha_{\tau i})}\overline{r_{\tau i}(T_{w_\bullet}(E_{\tau i})}.
$$

Substituting this together with Lemma 7.2.1 we see that (8.28) is equivalent to

$$\begin{aligned}&\varsigma_{\tau i} q^{(\alpha_i,\mu-\alpha_{\tau i}+w_\bullet\alpha_{\tau i}-w_\bullet\alpha_i)} r_i(T_{w_\bullet}(E_i))\Upsilon_{\mu-\alpha_{\tau i}-w_\bullet\alpha_i}\\ =&q^{(\alpha_{\tau i},\mu-\alpha_i-\alpha_{\tau i})}\varsigma_i'\Upsilon_{\mu-\alpha_i-w_\bullet\alpha_{\tau i}}\overline{r_{\tau i}(T_{w_\bullet}(E_{\tau i}))}\end{aligned} \tag{8.29}$$

Observe that $\mu-\alpha_{\tau i}-w_\bullet\alpha_i=\mu-\alpha_i-w_\bullet\alpha_{\tau i}$ and $w_\bullet\alpha_i-w_\bullet\alpha_{\tau i}=\alpha_i-\alpha_{\tau i}$. We may further assume that $\Upsilon_{\mu-\alpha_{\tau i}-w_\bullet\alpha_i}\neq 0$, thus $w_\bullet\tau(\mu)=\mu$ and hence $(\alpha_i-\alpha_{\tau i},\mu)=0$. Thus we see (8.29) is equivalent to

$$\begin{aligned}&q^{(\alpha_{\tau i},\alpha_i)}\varsigma_{\tau i} r_i(T_{w_\bullet}(E_i))=\varsigma_i'\overline{r_{\tau i}(T_{w_\bullet}(E_{\tau i}))}\\ \overset{(8.27)}{=}&\varsigma_i'(-1)^{(\alpha_{\tau i},2\rho_\bullet)}q^{(\alpha_{\tau i},\alpha_{\tau i}-w_\bullet\alpha_{\tau i}-2\rho_\bullet)}r_i(T_{w_\bullet}(E_i)),\end{aligned} \tag{8.30}$$

which follows from the definition of ς_i' (8.21). □

The next lemma verifies the relation (8.17).

Lemma 8.2.14. *For all $i\in I$, we have*

$$\langle F_i, A_i\rangle=0.$$

Proof. Since $wt(A_i)=\mu-\alpha_i$. We see that $\langle F_i,A_i\rangle$ is zero unless $\mu=2\alpha_i$. Moreover, we see that $A_i=0$ if $i\in I_\bullet$. Assume that $i\in I_\circ$, in this case we always have $\mu-\alpha_i-w_\bullet\alpha_{\tau i}\notin Q^+$ since $i\neq\tau i$. Hence $A_i=0$. □

To verify the relation (8.18), we have

Lemma 8.2.15. *For all $i\nsim j\in I$, we have*

$$\langle F_i, A_j\rangle=(-1)^{p(i)p(j)}\langle F_j,A_i\rangle.$$

Proof. According to [BK19, Lemma 6.4], we can assume that $j=\tau i\in I_\circ\backslash\{\pm n\}$ and $\mu=\alpha_i+\alpha_j$, otherwise all terms vanish. In this case we have $\varsigma_i=\varsigma_j$, $A_i=-(q^{\ell_i}-q^{-\ell_i})\varsigma_i' E_{\tau i}$, and

$A_j = -(q^{\ell_j} - q^{-\ell_j})\varsigma_j' E_{-j}$. Thus we have

$$\langle F_i, A_{\tau i}\rangle = -(q^{\ell_{\tau i}} - q^{-\ell_{\tau i}})\varsigma_{\tau i}' = -(-1)^{p(i)}(q^{\ell_i} - q^{-\ell_i})\varsigma_i' = (-1)^{p(i)}\langle F_{\tau i}, A_i\rangle.$$

This proves the lemma. $\square$

To verify the relation (8.19), we have

Lemma 8.2.16. *For all $i \in I_{\bar{0}}$ and $j \sim i$, we have*

$$\langle F_i^2, A_j\rangle - [2]\langle F_iF_j, A_i\rangle + \langle F_jF_i, A_i\rangle = 0.$$

Proof. We can assume that $\mu = 2\alpha_i + \alpha_j$, otherwise all terms in the above sum vanish. But by [BK19, Lemma 6.4] in this case we have $w_\bullet \circ \tau(\mu) \neq \mu$ for all $j \sim i \in I$. Hence all terms still vanish. $\square$

To verify the relation (8.20), we have

Lemma 8.2.17. *For all $i \in I_{\bar{1}}$ and $j \sim i \sim k$, we have*

$$\begin{aligned}[2]\langle F_iF_kF_j, A_i\rangle = (-1)^{p(j)}\langle F_iF_kF_i, A_j\rangle + (-1)^{p(j)+p(j)p(k)}\langle F_jF_iF_k, A_i\rangle \\ +(-1)^{p(k)}\langle F_iF_jF_i, A_k\rangle + (-1)^{p(k)+p(j)p(k)}\langle F_kF_iF_j, A_i\rangle.\end{aligned}$$

Proof. Again we may assume that $\mu = 2\alpha_i + \alpha_k + \alpha_j$ otherwise all terms vanish. But in this case we see that $w_\bullet \circ \tau(\mu) \neq \mu$ unless $\tau j = k,\ \tau i = i$ and $i, j, k \in I_\circ$. However, this is excluded by (7.14). Hence all terms still vanish. $\square$

Therefore, we conclude that

$$\begin{aligned}A_i &= -(q - q^{-1})\Upsilon_{\mu-\alpha_i-w_\bullet\alpha_{\tau i}}\varsigma_i'\overline{T_{w_\bullet}(E_{\tau i})},\\ {}_iA &= -(q - q^{-1})q^{(\alpha_i, w_\bullet\alpha_{\tau i})}\varsigma_i T_{w_\bullet}(E_{\tau i})\Upsilon_{\mu-\alpha_i-w_\bullet\alpha_{\tau i}}\end{aligned}$$

for all $i \in I$ satisfy relations (8.16)–(8.20).

Thus we can conclude the main result of this section.

Theorem 8.2.18. *There exists a uniquely determined element* $\Upsilon = \sum_{\mu \in Q_{\bar{0}}^+} \Upsilon_\mu$ *in the completion of* $\mathbf{U}$ *with* $\Upsilon_0 = 1$ *and* $\Upsilon_\mu \in \mathbf{U}_\mu^+$*, such that the equality*

$$B_i \Upsilon = \Upsilon(\tau \circ \sigma(B_{\tau i}))$$

holds for all $i \in I$*.*

Moreover, $\Upsilon_\mu = 0$ *unless* $w_\bullet \tau(\mu) = \mu$*.*

Once Υ is constructed, we can define a unique bar involution on $\mathbf{U}^\imath$ with certain assumption on the parameters as follows.

Corollary 8.2.19. *Under the assumption that* $\overline{\varsigma_j} = \varsigma_j'$ *for all* $j \in I$*, there is a unique bar involution* $\psi_\imath$ *on* $\mathbf{U}^\imath$*, defined by*

$$\psi_\imath(x) = \Upsilon \overline{x} \Upsilon^{-1}, \text{ for all } x \in \mathbf{U}^\imath$$

and such that

$$\psi_\imath(q) = q^{-1}, \ \psi_\imath(B_j) = B_j, \ \psi_\imath(E_k) = E_k, \ \psi_\imath(F_k) = F_k,$$

for $j \in I_\circ, k \in I_\bullet$*.*

Proof. For all $i \in I$, it follows from Lemma 8.2.4 that $B_i \Upsilon = \Upsilon(\tau \circ \sigma(B_{\tau i}))$ is equivalent to (8.12). Under the assumption $\varsigma_i' = \overline{\varsigma}_i$ we see that (8.12) is equivalent to

$$B_i \Upsilon = \Upsilon \overline{B_i}.$$

This concludes the proof. $\square$

Remark 8.2.20. One can construct Υ associated to more general Satake diagrams. For example, one can replace (8.4) by a weaker condition:

$$p(j) = p(\tau j), \quad \forall j \in I_{\bullet}.$$

Under this assumption formally we still have $w_{\bullet}(Y) = Y$. Thus $T_{w_{\bullet}}$ can still be treated as an automorphism on $\mathbf{U}(Y)$ although it is a composition of both even and odd braid group operators.

In the last of this subsection we give an example of Υ.

Example 8.2.21. *Consider the following Satake diagram*

$$\underset{-\frac{1}{2}}{\otimes} \overset{\longleftrightarrow}{\text{——}} \underset{\frac{1}{2}}{\otimes}$$

We have $(\alpha_{\frac{1}{2}}, \alpha_{-\frac{1}{2}}) = 1$, $\ell_{-\frac{1}{2}} = 1 = -\ell_{\frac{1}{2}}$ *and*

$$\begin{aligned} B_{\frac{1}{2}} &= F_{\frac{1}{2}} + \varsigma_{\frac{1}{2}} E_{-\frac{1}{2}} K_{\frac{1}{2}}^{-1}, \\ B_{-\frac{1}{2}} &= F_{-\frac{1}{2}} + \varsigma_{-\frac{1}{2}} E_{\frac{1}{2}} K_{-\frac{1}{2}}^{-1}. \end{aligned}$$

In this case, following the constructions in this section we get

$$\Upsilon = (\sum_{k \geqslant 0} \frac{(\varsigma_{\frac{1}{2}})^k}{\{k\}!})(E_{\frac{1}{2}} E_{-\frac{1}{2}} + q E_{-\frac{1}{2}} E_{\frac{1}{2}})^k)(\sum_{k \geqslant 0} \frac{(\varsigma_{-\frac{1}{2}})^k}{\{k\}!})(E_{-\frac{1}{2}} E_{\frac{1}{2}} + q E_{\frac{1}{2}} E_{-\frac{1}{2}})^k),$$

where $\{k\} = q^{k-1}[k]$ *and* $\{k\}! = \{k\} \cdots \{1\}$.

8.3 K-matrix and the H_0-action

In this section we follow [BW18b] (also cf. [BK19]) to construct a $\mathbf{U}^\imath$-module intertwiner (or K-matrix).

8.3.1 K-matrix

Recall the assumption (8.4). We review several basic lemmas from [BW18b] below. Recall σ and $\wp$ from (2.13).

Lemma 8.3.1. *For all $i \in I_\bullet, j \in I$ and $e = \pm 1$, we have*

$$\begin{aligned} \wp(T''_{i,e}(E_j)) &= (-q)^{e(\alpha_i,\alpha_j)} T'_{i,-e}(\wp(E_j)), \\ \wp(T'_{i,e}(E_j)) &= (-q)^{-e(\alpha_i,\alpha_j)} T''_{i,-e}(\wp(E_j)). \end{aligned} \tag{8.31}$$

Proof. It follows from a rerun of the proof of [BW18b, Lemma 4.4]. □

Lemma 8.3.2. *For $i \in I_\circ$ and $e = \pm 1$, we have*

$$\begin{aligned} \wp(T''_{w_\bullet,e}(E_i)) &= (-1)^{(2\rho_\bullet,\alpha_i)} q^{e(2\rho_\bullet,\alpha_i)} T'_{w_\bullet,-e}(\wp(E_i)), \\ \wp(T'_{w_\bullet,e}(E_i)) &= (-1)^{(2\rho_\bullet,\alpha_i)} q^{-e(2\rho_\bullet,\alpha_i)} T''_{w_\bullet,-e}(\wp(E_i)). \end{aligned} \tag{8.32}$$

Proof. It follows from (8.31) and a rerun of the proof of [BW18b, Corollary 4.5] . □

Recall $q_i = q^{\ell_i}$. Following [BW18b, §4.5], under the assumption (7.14), we define the following automorphism of $\mathbf{U}$ obtained by composition $\vartheta = \sigma \circ \wp \circ \tau$ such that

$$\begin{aligned} \vartheta(E_j) = (-1)^{p(j)} q_{\tau j} F_{\tau j} K_{\tau j}^{-1}, \quad \vartheta(F_j) = (-1)^{p(j)} q_{\tau j}^{-1} K_{\tau j} E_{\tau j}, \\ \vartheta(K_j) = K_{\tau j}^{-1}, \quad \vartheta(\varrho) = \varrho, \quad \text{for all } j \in I. \end{aligned} \tag{8.33}$$

For any finite-dimensional $\mathbf{U}$-module M, we define a $\mathbf{U}$-module ${}^\vartheta M$ twisted by ϑ as follows:

▷ ${}^\vartheta M = M$ as an $\mathbb{Q}(q)$-vector space,

▷ We denote a vector in ${}^{\vartheta}M$ by ${}^{\vartheta}m$ for $m \in M$,

▷ the action of $u \in \mathbf{U}$ on ${}^{\vartheta}M$ is given by $\vartheta(u)^{\vartheta}m = {}^{\vartheta}(um)$.

Let

$$g : P \to \mathbb{Q}(q) \tag{8.34}$$

be a function such that for all $\mu \in P$, we have the following two recursive relations of g:

$$g(\mu) = -q_j q^{2(\alpha_j,\mu)} g(\mu + \alpha_j), \quad \forall j \in I_\bullet. \tag{8.35}$$

$$g(\mu) = g(\mu - \alpha_j)(-1)^{p(j)} \varsigma_j (-1)^{(2\rho_\bullet,\alpha_j)} q^{(2\rho_\bullet,\alpha_j)} q_j q^{(\alpha_{\tau j}, w_\bullet \mu)} q^{-(\alpha_j,\mu)}, \quad \forall j \in I_\circ. \tag{8.36}$$

Such a function g exists; cf. [BW18b, (4.15)]. Note that under our assumption (8.4), we have $q_j = q_{\tau j}$ for all $j \in I_\bullet$.

Lemma 8.3.3. *For any $\mu \in P$, we have*

$$g(\mu) = g(\mu - w_\bullet \alpha_j)(-1)^{p(j)} \varsigma_j q^{(\alpha_{\tau j},\mu)} q_j q^{-(\alpha_j, w_\bullet \mu)}, \quad \forall j \in I_\circ. \tag{8.37}$$

Proof. Recall the following identity [BW18b, (4.18)]:

$$g(\mu - \alpha_j) = g(\mu - w_\bullet \alpha_j)(-1)^{(2\rho_\bullet,\alpha_j)} q^{-(2\rho_\bullet,\alpha_j)} q^{2(\alpha_i - w_\bullet \alpha_i, \mu)}, \quad \forall j \in I_\circ. \tag{8.38}$$

Then applying (8.36) to (8.38) we get (8.37). □

The function g induces a $\mathbb{Q}(q)$-linear map on any finite dimensional $\mathbf{U}$-module M:

$$\tilde{g} \colon M \to M, \quad \tilde{g}(m) = g(\mu) m, \quad m \in M_\mu.$$

In the next theorem we construction the K-matrix.

Theorem 8.3.4. (cf. [BW18b, Theorem 4.18]) *For any finite-dimensional* $\mathbf{U}$*-module* M*, we have the following isomorphism of* $\mathbf{U}^\imath$*-modules*

$$\mathcal{T} := \Upsilon \circ \tilde{g} \circ T_{w_\bullet}^{-1} : M \to {}^{\vartheta}M.$$

Proof. It suffices to verify that $\mathcal{T}$ defines a homomorphism of $\mathbf{U}^\imath$-modules. We shall prove the following identity

$$\mathcal{T}(\vartheta(u) \cdot m) = u \cdot \mathcal{T}(m), \qquad \text{for } u \in \mathbf{U}^\imath,\ m \in M_\mu. \tag{8.39}$$

It is straightforward to check (8.39) for $u = K_\mu,\ \varrho$. Also, for $u = F_j,\ E_j\ (j \in I_\bullet)$, the proof are essentially the same as those of [BW18b, Case(2)-(3),Theorem 4.18]. Thus we only verify for $u = B_j\ (j \in I_\circ)$ as below.

For $u = B_i (i \in I_\circ)$, first we see that

$$\begin{aligned}
\vartheta(B_i) =&(-1)^{p(i)} q_{\tau i}^{-1} K_{\tau i} E_{\tau i} + \varsigma_i \sigma \circ \varrho \circ \tau \circ T_{w_\bullet}(E_{\tau i}) K_{\tau i} \\
\overset{(8.32)}{=}&(-1)^{p(i)} q_{\tau i}^{-1} K_{\tau i} E_{\tau i} + \varsigma_i (-1)^{(2\rho_\bullet, \alpha_i)} q^{(2\rho_\bullet, \alpha_i)} T_{w_\bullet}(\sigma \circ \varrho(E_i)) K_{\tau i} \\
=&(-1)^{p(i)} q_{\tau i}^{-1} K_{\tau i} E_{\tau i} + (-1)^{p(i)} \varsigma_i (-1)^{(2\rho_\bullet, \alpha_i)} q^{(2\rho_\bullet, \alpha_i)} q_i T_{w_\bullet}(F_i) T_{w_\bullet}(K_i^{-1}) K_{\tau i}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
T_{w_\bullet}^{-1} \circ \vartheta(B_i) =&(-1)^{p(i)} q_{\tau i}^{-1} T_{w_\bullet}^{-1}(K_{\tau i}) T_{w_\bullet}^{-1}(E_{\tau i}) \\
&+ \varsigma_i (-1)^{p(i)} (-1)^{(2\rho_\bullet, \alpha_i)} q^{(2\rho_\bullet, \alpha_i)} q_i F_i K_i^{-1} T_{w_\bullet}^{-1}(K_{\tau i}).
\end{aligned}$$

Now we have

$$
\begin{aligned}
&\mathcal{T}(\vartheta(B_i)(m)) = \Upsilon \circ \tilde{g} \circ T_{w_\bullet}^{-1}(\vartheta(B_i)m) = \Upsilon \circ \tilde{g} \left(T_{w_\bullet}^{-1} \circ \vartheta(B_i)(T_{w_\bullet}^{-1}(m)) \right) \\
=&\Upsilon \circ \tilde{g}((-1)^{p(i)} q_{\tau i}^{-1} T_{w_\bullet}^{-1}(K_{\tau i}) T_{w_\bullet}^{-1}(E_{\tau i}) T_{w_\bullet}^{-1}(m) \\
&\quad + (-1)^{p(i)} \varsigma_i (-1)^{(2\rho_\bullet, \alpha_i)} q^{(2\rho_\bullet, \alpha_i)} q_i F_i K_i^{-1} T_{w_\bullet}^{-1}(K_{\tau i}) T_{w_\bullet}^{-1}(m)) \\
=&\Upsilon(g(w_\bullet \mu + w_\bullet \alpha_{\tau i})(-1)^{p(i)} q_{\tau i}^{-1} T_{w_\bullet}^{-1}(K_{\tau i}) T_{w_\bullet}^{-1}(E_{\tau i}) T_{w_\bullet}^{-1}(m) \\
&\quad + (-1)^{p(i)} g(w_\bullet \mu - \alpha_i) \varsigma_i (-1)^{(2\rho_\bullet, \alpha_i)} q^{(2\rho_\bullet, \alpha_i)} q_i F_i K_i^{-1} T_{w_\bullet}^{-1}(K_{\tau i}) T_{w_\bullet}^{-1}(m)) \\
=&\Upsilon(g(w_\bullet \mu + w_\bullet \alpha_{\tau i})(-1)^{p(i)} q_{\tau i}^{-1} q^{(\alpha_{\tau i}, \mu + \alpha_{\tau i})} T_{w_\bullet}^{-1}(E_{\tau i}) T_{w_\bullet}^{-1}(m) \\
&\quad + (-1)^{p(i)} g(w_\bullet \mu - \alpha_i) \varsigma_i (-1)^{(2\rho_\bullet, \alpha_i)} q^{(2\rho_\bullet, \alpha_i)} q_i q^{(\alpha_{\tau i}, \mu)} q^{-(\alpha_i, w_\bullet \mu)} F_i T_{w_\bullet}^{-1}(m)).
\end{aligned}
$$

On the other hand, we have

$$
B_i \Upsilon = \Upsilon B_{\tau i}^{\sigma\tau} = \Upsilon(F_i + \varsigma_{\tau i} K_i T_{w_\bullet}^{-1}(E_{\tau i})).
$$

Thus

$$
\begin{aligned}
B_i \cdot \mathcal{T}(m) =& B_i \left(\Upsilon \circ \tilde{g} \circ T_{w_\bullet}^{-1}(m) \right) \overset{(8.11)}{=} \Upsilon(F_i(\tilde{g} \circ T_{w_\bullet}^{-1}(m)) + \varsigma_{\tau i} K_i T_{w_\bullet}^{-1}(E_{\tau i})(\tilde{g} \circ T_{w_\bullet}^{-1}(m))) \\
=&\Upsilon(g(w_\bullet \mu) F_i T_{w_\bullet}^{-1}(m) \\
&\quad + g(w_\bullet \mu) \varsigma_{\tau i} q^{(\alpha_i, w_\bullet \mu + w_\bullet \alpha_{\tau i})} T_{w_\bullet}^{-1}(E_{\tau i}) T_{w_\bullet}^{-1}(m)).
\end{aligned}
$$

Now the identity (8.39) for $u = B_j$ follows from comparing the coefficients using (8.36) and (8.37). □

8.3.2 Realizing H_0 via K-matrix

In this subsection the goal is to realize H_0-action on $\mathbb{W}$ as in Proposition 8.1.1 via the K-matrix $\mathcal{T}$.

We assume the parameters satisfying (8.3) so that the $\imath$Schur duality holds between $\mathbf{U}^\imath$ and $\mathscr{H}_{B_d}$, and moreover, Υ and $\psi_\imath$ in Corollary 8.2.19 uniquely exist. We also assume that

$Q \in q^{\mathbb{Z}}$ thus $\psi_\imath(Q) = Q^{-1}$.

Given a $\mathbf{U}$-module M, a $\mathbf{U}$-module ${}^{\vartheta}M$ is simple if and only if M is simple itself. Let λ be a dominant integral weight and $L(\lambda)$ be the unique irreducible highest weight module with highest weight vector η_λ. Moreover, we define a lowest weight $\mathbf{U}$-module ${}^{\omega}L(\lambda)$ of weight $-\lambda$ which has the same underlying vector space as $L(\lambda)$ but with the action twisted by the automorphism ω where

$$\omega(E_j) = F_j, \quad \omega(F_j) = (-1)^{p(j)}E_j, \quad \omega(K_\mu) = K_{-\mu}. \tag{8.40}$$

When we consider η_λ as a vector in ${}^{\omega}L(\lambda)$, we shall denote it by $\xi_{-\lambda}$. We check by definition that

$${}^{\vartheta}L(\lambda) \cong {}^{\omega}L(\tau\lambda).$$

A basic example of $L(\lambda)$ is our fundamental representation $\mathbb{W} = L(\epsilon_{-n-r+\frac{1}{2}})$. We check by definition that

$${}^{\vartheta}\mathbb{W} = {}^{\vartheta}L(\epsilon_{-n-r+\frac{1}{2}}) \cong {}^{\omega}L(-\epsilon_{n+r-\frac{1}{2}}) = L(\epsilon_{-n-r+\frac{1}{2}}). \tag{8.41}$$

Recall $\mathcal{T} = \Upsilon \circ \tilde{g} \circ T_{w_\bullet}^{-1}$ and Theorem 8.3.4. Together with (8.41) we see that $\mathcal{T}$ induces an $\mathbf{U}^\imath$-automorphism on $\mathbb{W}$ and send $T_{w_\bullet}^{-1}(\eta_{\epsilon_{-n-r+\frac{1}{2}}})$ to $\xi_{\epsilon_{n+r-\frac{1}{2}}}$, cf. [BW18b, Theorem 4.18]. Thus we have the following corollary:

Corollary 8.3.5. *The K-matrix $\mathcal{T}$ is an $\mathbf{U}^\imath$-module automorphism of $\mathbb{W}$:*

$$\mathcal{T} : \mathbb{W} \to \mathbb{W}, \qquad w_{-n-r+\frac{1}{2}} \mapsto (-1)^{p\left(w_{-n-r+\frac{1}{2}}\right)} w_{n+r-\frac{1}{2}}.$$

Proposition 8.3.6. *The action of H_0 on $\mathbb{W}^{\otimes d}$ in Proposition 8.1.1 is realized via the K-matrix as $\mathcal{T} \otimes Id^{\otimes d-1}$.*

Proof. According to Corollary 8.3.5, we have $\mathcal{T}(w_{-n-r+\frac{1}{2}}) = (-1)^{p\left(w_{-n-r+\frac{1}{2}}\right)} w_{n+r-\frac{1}{2}}$. Recall the parameters satisfy (8.3).

Suppose $a \in \mathbb{I}_\circ^-$, a simple induction on a shows that

$$\begin{aligned}\mathcal{T}(w_a) =&\mathcal{T}(B_{a-\frac{1}{2}}B_{a-\frac{3}{2}}\cdots B_{-n-r+1}w_{-n-r+\frac{1}{2}})\\ =&B_{a-\frac{1}{2}}B_{a-\frac{3}{2}}\cdots B_{-n-r+1}\mathcal{T}(w_{-n-r+\frac{1}{2}}) = (-1)^{p(w_a)}w_{-a} = w_a \cdot H_0.\end{aligned}$$

Now suppose $a = -n - \frac{1}{2}$, we have

$$\begin{aligned}\mathcal{T}(w_{-n+\frac{1}{2}}) =&\mathcal{T}(B_{-n}w_{-n-\frac{1}{2}}) = B_{-n}\mathcal{T}(w_{-n-\frac{1}{2}}) = (-1)^{p\left(w_{n+\frac{1}{2}}\right)}B_{-n}w_{n+\frac{1}{2}}\\ =&Qw_{-n+\frac{1}{2}} = w_{-n+\frac{1}{2}} \cdot H_0.\end{aligned}$$

Thus for any $a \in \mathbb{I}_\bullet$, we have

$$\begin{aligned}\mathcal{T}(w_a) =&\mathcal{T}(F_{a-\frac{1}{2}}F_{a-\frac{3}{2}}\cdots F_{-n+1}w_{-n+\frac{1}{2}}) = F_{a-\frac{1}{2}}F_{a-\frac{3}{2}}\cdots F_{-n+1}\mathcal{T}(w_{-n+\frac{1}{2}})\\ =&Qw_a = w_a \cdot H_0.\end{aligned}$$

Next we suppose $a = n + \frac{1}{2}$, we have

$$\begin{aligned}\mathcal{T}(w_{n+\frac{1}{2}}) =&\mathcal{T}(B_n(w_{n-\frac{1}{2}}) - (-1)^{p\left(w_{n+\frac{1}{2}}\right)}Q^{-1}w_{-n-\frac{1}{2}})\\ =&QB_n(w_{n-\frac{1}{2}}) - Q^{-1}w_{n+\frac{1}{2}}\\ =&(-1)^{p\left(w_{n+\frac{1}{2}}\right)}w_{-n-\frac{1}{2}} + (Q - Q^{-1})w_{n+\frac{1}{2}} = w_{n+\frac{1}{2}} \cdot H_0.\end{aligned}$$

Thus for any $a \in \mathbb{I}_\circ^+$, another simple induction on a shows that

$$\begin{aligned}\mathcal{T}(w_a) =&\mathcal{T}(B_{a-\frac{1}{2}}B_{a-\frac{3}{2}}\cdots B_{n+1}w_{n+\frac{1}{2}}) = B_{a-\frac{1}{2}}B_{a-\frac{3}{2}}\cdots B_{n+1}\mathcal{T}(w_{n+\frac{1}{2}})\\ =&(-1)^{p(w_a)}w_{-a} + (Q - Q^{-1})w_a = w_a \cdot H_0.\end{aligned}$$

This completes the proof. $\square$

In case $m = 0$ or 1, the non-super specialization of Proposition 8.3.6 is established in [BW18a, BWW18]. The property of a K-matrix $\mathcal{T}$ in Corollary 8.3.5 also provides a conceptual explanation for the commutativity of H_0 and $\mathbf{U}^\imath$ acting on $\mathbb{W}^{\otimes d}$.

Chapter 9

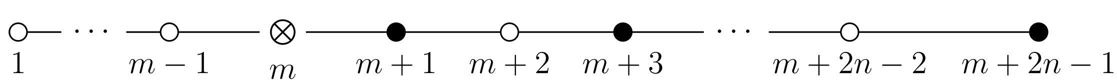
$\imath$Schur duality of type AI-II

In [SW24], we have formulated super Satake diagrams and the corresponding supersymmetric pairs, building on arbitrary Dynkin diagrams for basic Lie superalgebras. We develop a theory of quantum supersymmetric pairs associated to these super Satake diagrams.

The case of type AIII in previous chapters (when $I_\bullet$ only contains even simple roots) is one family of these quantum supersymmetric pairs. In this chapter we introduce another interesting family of type AI-II. We also formulate an $\imath$Schur duality between the $\imath$quantum supergroup of type AI-II and the q-Brauer algebra.

9.1 The $\imath$quantum supergroups of type AI-II

We consider the following Satake diagram I:



(9.1)

where $I_{\bar{1}} = \{m\}$, $I_\bullet = \{m + 2a - 1 \mid 1 \le a \le n\}$ and $I_\circ = I \backslash I_\bullet$. In the case $n = 0$, we obtain a Satake diagram of type AI; when $m = 0$, we obtain a Satake diagram of type AII.

In [SW24, Definition 2.3], we formulate super admissible conditions. A pair $(I = I_\bullet \cup I_\circ, \tau)$

satisfying these conditions are referred to as a super admissible pair.

Lemma 9.1.1. *[SW24, Lemma 8.1] The pair* $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau = id)$ *forms a super admissible pair.*

Recall ℓ_j from (2.10), for the super Satake diagram (9.1) we have

$$\ell_j = \begin{cases} 1 & 1 \leq j \leq m, \\ -1 & m+1 \leq j \leq m+2n-1. \end{cases}$$

The underlying Dynkin diagram of (9.1) corresponds to a (standard) fundamental system of the Lie superalgebra $\mathfrak{gl}(m|2n)$. As in § 2.3, we choose bases for $V_{\bar{0}}$ and $V_{\bar{1}}$ such that they combine to a homogeneous basis of V. Such a basis is parameterized by the set $I(m|2n)$.

Let $\mathbf{U}$ denote the type A quantum supergroup associated to (9.1). We define $\mathbf{U}^\imath$ to be the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}$ generated by

$$B_j = F_j + \varsigma_j T_{w_\bullet}(E_j) K_j^{-1}, \quad \text{for } j \in I_\circ.$$

together with $K_j^{\pm 1}$, E_j, F_j $(j \in I_\bullet)$, ϱ. In the case $m = 0$, our $\mathbf{U}^\imath$ specializes to the $\imath$quantum group of type AII. In the case $n = 0$, our $\mathbf{U}^\imath$ specializes to the $\imath$quantum group of type AI.

In [SW24], it has been established that quantum supersymmetric pairs $(\mathbf{U}, \mathbf{U}^\imath)$ associated with super admissible pairs possess the desired properties of quantum symmetric pairs. Specifically, the proof strategy for these properties of quantum supersymmetric pairs of type AI-II aligns with § 7.2. Therefore, we will not repeat the proof but only list the results here.

Proposition 9.1.2. [SW24]

1. $\mathbf{U}^\imath$ *is a right coideal subalgebra of* $\mathbf{U}$.

2. *There exists a quantum Iwasawa decomposition of* $\mathbf{U}$ *with respect to* $\mathbf{U}^\imath$.

3. *There exists a unique quasi* K*-matrix for* $(\mathbf{U}, \mathbf{U}^\imath)$.

9.2 Action of the q-Brauer algebra

Proposition 9.2.1. [RSS22, Theorem A] *For any $d \geq 2$, the q-Brauer algebra $\mathfrak{B}_d(q,z)$ is (split) semisimple if and only if $z^2 \neq q^{2a}$ for $a \in \clubsuit$ where*

$$\clubsuit = \{i \in \mathbb{Z} \mid 4-2d \leq i \leq d-2\} \setminus \{i \in \mathbb{Z} \mid 4-2n < i < 3-d,\ 2 \nmid i\}\,. \tag{9.2}$$

Let $\mathbb{V}$ be the natural representation of $\mathbf{U}$, i.e. $\mathbb{V}$ is $\mathbb{Q}(q)$-vector superspace with an ordered basis $\{v_i \mid i \in I(m|2n)\}$. Let $|\cdot|$ denote the parity function on $\mathbb{V}$ where $|v_i| = 1$ for all $i < 0$ and $|v_i| = 1$ otherwise. Recall from Proposition 8.1.1 that $\mathbb{V}^{\otimes d}$ endows a right $\mathscr{H}_{\mathfrak{S}_d}$-module structure.

For $i = 2, \dots, m$ and $j = 2, \dots, n$, we set

$$\tau_i := \prod_{j=1}^{i-1}(-\varsigma_j), \quad \zeta_j := \prod_{k=1}^{j-1}(-\varsigma_{m+2\mathrm{j}}),\ \tau_1 = \zeta_1 = 1.$$

A $\mathbb{Q}(q)$-linear operator Ξ on $\mathbb{V} \otimes \mathbb{V}$ is defined by

$$\begin{aligned}
\Xi(v_{\bar{1}} \otimes v_{\bar{1}}) &= q^{2n}\left(\sum_{i=1}^{m} \tau_i^{-1} q^{m-2i+1} v_{\bar{i}} \otimes v_{\bar{i}}\right) \\
&\quad - \tau_m^{-1} q^{2n-m}\left(\sum_{j=1}^{n} \zeta_j^{-1} q^{-4j+3}\left(v_{\underline{2j-1}} \otimes v_{\underline{2j}} - q^{-1} v_{\underline{2j}} \otimes v_{\underline{2j-1}}\right)\right), \\
\Xi(v_{\bar{i}} \otimes v_{\bar{i}}) &= \tau_i \Xi(v_{\bar{1}} \otimes v_{\bar{1}}), \quad \text{for all } 2 \leq i \leq m, \\
\Xi(v_{\underline{1}} \otimes v_{\underline{2}}) &= \tau_m \Xi(v_{\bar{1}} \otimes v_{\bar{1}}), \quad \Xi(v_{\underline{2}} \otimes v_{\underline{1}}) = (-q^{-1})\Xi(v_{\underline{1}} \otimes v_{\underline{2}}), \\
\Xi(v_{\underline{2j-1}} \otimes v_{\underline{2j}}) &= \zeta_j \Xi(v_{\underline{1}} \otimes v_{\underline{2}}), \quad \text{for all } 2 \leq j \leq n, \\
\Xi(v_{\underline{2j}} \otimes v_{\underline{2j-1}}) &= (-q^{-1})\Xi(v_{\underline{2j-1}} \otimes v_{\underline{2j}}), \quad \text{for all } 2 \leq j \leq n, \\
\Xi(v_a \otimes v_b) &= 0, \quad \text{if } (a,b) \notin \{(\bar{i}, \bar{i}), (\underline{2j-1}, \underline{2j}), (\underline{2j}, \underline{2j-1}) \mid 1 \leq i \leq m, 1 \leq j \leq n\}.
\end{aligned} \tag{9.3}$$

Proposition 9.2.2. *For $d \geq 2$, $\mathbb{V}^{\otimes d}$ is a right $\mathfrak{B}_d(q, q^{m-2n})$-module by letting $\mathscr{H}_{\mathfrak{S}_d}$ act as in*

Proposition 8.1.1 and e act as $\Xi \otimes 1^{\otimes d-2}$.

Proof. By [Mi06], the action of H_i satisfies relations (Q1)-(Q3) in Definition 5.2.1. The verification of the relation (Q4)-(Q7) is very similar to the proof of [CS22, Prpoposition 4.4 and 5.3]. We prove the relation (Q4) for both statements as an example. Noting that the action of e depends solely on the first two tensor factors, hence it suffices to show that

$$v_{\overline{1}} \otimes v_{\overline{1}} \cdot e^2 = \frac{q^{m-2n} - q^{-m+2n}}{q - q^{-1}} v_{\overline{1}} \otimes v_{\overline{1}} \cdot e.$$

Indeed we have

$$\begin{aligned}
&v_{\overline{1}} \otimes v_{\overline{1}} \cdot e^2 \\
=&q^{-2n} \left(\sum_{i=1}^{m} \tau_i^{-1} q^{m-2i+1} v_{\overline{i}} \otimes v_{\overline{i}} \cdot e \right) \\
&\qquad - \tau_m^{-1} q^{2n-m} \left(\sum_{j=1}^{n} \zeta_j^{-1} q^{-4j+3} (v_{\underline{2j-1}} \otimes v_{\underline{2j}} - q^{-1} v_{\underline{2j}} \otimes v_{\underline{2j-1}}) \cdot e \right) \\
=& \left[q^{-2n} \left(\sum_{i=1}^{m} q^{m-2i+1} \right) - q^{2n-m} \left(\sum_{j=1}^{n} q^{-4j+3} (1 + q^{-2}) \right) \right] v_{\overline{1}} \otimes v_{\overline{1}} \cdot e \\
=& \left(q^{-2n} \frac{q^m - q^{-m}}{q - q^{-1}} - q^{-m} \frac{q^{2n} - q^{-2n}}{q - q^{-1}} \right) v_{\overline{1}} \otimes v_{\overline{1}} \cdot e \\
=&\frac{q^{m-2n} - q^{2n-m}}{q - q^{-1}} v_{\overline{1}} \otimes v_{\overline{1}} \cdot e.
\end{aligned}$$

This conclude the proof of the relation (Q4). $\square$

9.3 $\imath$Schur duality of type AI-II

By direct calculation we obtain that

$$B_i \cdot v_a = \begin{cases} v_{\overline{i+1}} & \text{if } 1 \le i \le m-1, a = \bar{i}, \\ q\varsigma_i v_{\bar{i}} & \text{if } 1 \le i \le m-1, a = \overline{i+1}, \\ v_{\underline{1}} & \text{if } i = m, a = \overline{m}, \\ -q\varsigma_m v_{\overline{m}} & \text{if } i = m, a = \underline{2}, \\ v_{\underline{2k+1}} & \text{if } i = m+2k, a = \underline{2k}, \\ -q\varsigma_{m+2k} v_{\underline{2k-1}} & \text{if } i = m+2k, a = \underline{2k+2}, \\ 0 & \text{else}. \end{cases} \tag{9.4}$$

Via the comultiplication Δ, we naturally view $\mathbb{V}^{\otimes d}$ as a left $\mathbf{U}^\imath$-module.

Theorem 9.3.1. (1) *If* $\varsigma_m = q^{-4n+3}$. *then the left action of* $\mathbf{U}^\imath$ *on* $\mathbb{V}^{\otimes d}$ *commutes with the right action of* $\mathfrak{B}_d(q, q^{m-2n})$ *defined in Proposition 9.2.2:*

$$\mathbf{U}^\imath \overset{\Psi}{\curvearrowright} \mathbb{V}^{\otimes d} \overset{\Phi}{\curvearrowleft} \mathfrak{B}_d(q, q^{m-2n}).$$

(2) *The following double centralizer property holds if* $m-2n, 2n-m \notin \clubsuit$ (9.2)*:*

$$\Psi(\mathbf{U}^\imath) = End_{\mathfrak{B}_d(q,q^{m-2n})}(\mathbb{V}^{\otimes d}),$$
$$\Phi(\mathfrak{B}_d(q, q^{m-2n})) = End_{\mathbf{U}^\imath}(\mathbb{V}^{\otimes d}).$$

Proof. (1): By [Mi06], we know that the actions of $\mathbf{U}$ commute with the action of H_i, for $1 \le i \le d-1$. Moreover, by [CS22], we know that the actions of B_i for $i \ne m$ and $\mathbf{U}_\bullet$ commutes with the action of $\mathfrak{B}_d(q, q^{m-2n})$. Therefore, to prove (1)&(2) it suffices to show that the action of B_m commutes with the action of e. Since e acts on $\mathbb{V}^{\otimes d}$ by $\Xi \otimes 1^{\otimes d-2}$, it suffices to verify the commuting actions on the first two tensor factors.

By definition we have $B_m = F_m + q\varsigma_m K_m^{-1} T_{m+1}(E_m)$. We first observe that

$$\begin{aligned} F_m[\Xi(v_{\overline{1}} \otimes v_{\overline{1}})] &= (F_m \otimes K_m^{-1} + \varrho \otimes F_m)[\Xi(v_{\overline{1}} \otimes v_{\overline{1}})] \\ =&\tau_m^{-1} q^{-m \pm 2n+1}(q^{-1} v_{\underline{1}} \otimes v_{\overline{m}} + v_{\overline{m}} \otimes v_{\underline{1}}) \end{aligned} \tag{9.5}$$

Secondly we compute that

$$\begin{aligned} E_m E_{m+1}[\Xi(v_{\overline{1}} \otimes v_{\overline{1}})] &= -\tau_m^{-1} q^{2n-m-1} E_m E_{m+1}(v_{\underline{1}} \otimes v_{\underline{2}} - q^{-1} v_{\underline{2}} \otimes v_{\underline{1}}) = 0, \\ E_{m+1} E_m[\Xi(v_{\underline{1}} \otimes v_{\underline{1}})] &= -\tau_m^{-1} q^{2n-m-1} E_{m+1} E_m(v_{\underline{1}} \otimes v_{\underline{2}} - q^{-1} v_{\underline{2}} \otimes v_{\underline{1}}) \\ =& -\tau_m^{-1} q^{2n-m-1}(q^{-1} v_{\underline{1}} \otimes v_{\overline{m}} + v_{\overline{m}} \otimes v_{\underline{1}}). \end{aligned} \tag{9.6}$$

Combine (9.5) and (9.6) we see that

$$\begin{aligned} B_m[\Xi(v_{\overline{1}} \otimes v_{\overline{1}})] =&\tau_m^{-1}(q^{-2n-m+1} - q^{2n-m-2}\varsigma_m)(q^{-1} v_{\underline{1}} \otimes v_{\overline{m}} + v_{\overline{m}} \otimes v_{\underline{1}}), \\ \Xi[B_m \cdot (v_{\overline{1}} \otimes v_{\overline{1}})] &= 0. \end{aligned}$$

Therefore by the assumptions on ς_m we see that B_m and e commute on the basis vector $v_{\overline{1}} \otimes v_{\overline{1}}$. The verification on other basis vectors are similar.

(2) The double centralizer property is synonymous with the multiplicity-free decomposition of $\mathbb{V}^{\otimes d}$ as an $\mathbf{U}^\imath$-$\mathfrak{B}_d(q, q^{m-2n})$-bimodule. Proposition 9.2.1 affirms that the q-Brauer algebra $\mathfrak{B}_d(q, q^{m-n})$ is semisimple given our assumption. Consequently, proving the double centralizer property is reduced, through a deformation argument, to the case where $q = 1$. In the limit as q tends to 1 and ς_i takes on -1 for $1 \leq i \leq m-1$, $\mathbf{U}^\imath$ transforms into the enveloping algebra of the orthosymplectic Lie algebra $osp(m|2n)$, while $\mathbb{V}$ becomes its natural representation. The multiplicity-free decomposition of $\mathbb{V}^{\otimes d}$ in this scenario has already been established in [ES16]. This concludes the proof. □

Remark 9.3.2. The duality presented in Theorem 9.3.1 merges the $\imath$Schur duality of types AI and AII, as established in Part II, by incorporating an odd isotropic simple root $\bigotimes$ in between; cf. [SW24]. This type of duality can be extended to encompass super Satake diagrams formed by an arbitrary number of alternating Satake diagrams of types AI and AII, interconnected by odd isotropic simple roots in the same way as above.

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