The Great Inequality of Jupiter and Saturn.

A Dissertation for the Degree of Doctor of Philosophy, Presented by

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To the Faculty of the University of Virginia.

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THE GREAT INEQUALITY OF *JUPITER* AND *SATURN*,

By EDGAR ODELL LOVETT.

The Great Inequality of *Jupiter* and *Saturn* is a long-period inequality depending upon the difference between twice the mean motion of *Jupiter* and five times that of *Saturn*, in virtue of which the line of conjunctions slowly advances, and the planets return to their original configurations after periods of about 929 years.

As early as 1625 Kepler remarked, on comparing the observations of Tycho Brahe with those of Ptolemy, that the observed places of *Jupiter* and *Saturn* could not be reconciled with the admitted values of their mean motions (*Keplerus et Berneggerus, Epistolae mutuae,* p. 70; 12° Argentorati, 1672; also, *Keplerus,* Opera VI, p. 617, 1866). The errors of both planets were found to increase continually in the same direction, with this difference, that the tables made the mean motion of *Jupiter* too slow, and that of *Saturn* too rapid.

In his memoir on the aphelia and eccentricities (*Phil. Trans.,* 1676, p. 683) Halley was led to the belief that the anomalous irregularities of the two planets were due to their mutual attraction. He also attempted to determine the magnitude of the inequality for each planet, and concluded from his researches that in 2000 years the acceleration of *Jupiter* amounted to 3° 49′, and the retardation of *Saturn* to 9° 16′. In his tables of the planets he represented the errors by two secular equations, increasing as the square of the time, the one being additive to the mean motion of *Jupiter*, and the other subtractive from the mean motion of *Saturn* (see Grant's *History of Physical Astronomy,* pp. 47 et seq.; and Houzeau's *Vade-mecum de l'Astronomie,* § 246).

By comparing tables made at different epochs, Flamsteed confirmed the opinion that *Jupiter* was being steadily accelerated, and *Saturn* retarded (Flamsteed, J., *Exact Account of the Three Late Conjunctions of Jupiter and Saturn, London Phil. Trans.,* 1685, p. 244).

The most startling conclusions were drawn from these variations in the planetary motions. It was known that when the angular velocity of a body increases from century to century it must be approaching the center of motion; on the other hand a diminution in this velocity would indicate a recession of the planet from the sun. Hence it was inferred that the solar system would in the course of ages lose two of its most prominent members — that *Jupiter* would fall into the sun, while *Saturn* would be driven away into the depths of space. Euler and Lagrange had searched in vain for the cause of the anomalous behavior of *Jupiter* and *Saturn*, which appeared to be inconsistent with the law of gravitation; but in their researches they had neglected terms of the perturbations depending upon the cubes and higher powers of the eccentricities. In 1785 Laplace discovered from other considerations, that if we assume an acceleration of *Jupiter* a retardation of *Saturn* will necessarily result, and that in the very proportion observed; hence he set himself to examine the terms depending on the cubes of the eccentricities, and at once encountered the argument, $5n't-2nt+\text{const.}$ Taking account of the fact that five times the mean motion of *Saturn* is nearly equal to twice the mean motion of *Jupiter*, and that in virtue of the double integration for the perturbation in longitude, these terms are affected by the small divisor, $5n'-2n$, in the second power, he did not hesitate to conclude that the long inequality depends on terms having this argument. Computation at once confirmed this hypothesis, and made known the cause and law of the great inequality. From the nature of the cause thus made known Laplace immediately perceived that there would arise in other parts of our system similar long inequalities depending on near approach to commensurability in the mean motions. In the *Mécanique Céleste* and subsequent works on Physical Astronomy these long-period inequalities have been determined. The explanation of these long-period perturbations by the law of gravitation was justly regarded by Laplace as one of the strongest proofs of the exactness of this law. The theorems relative to the stability of the eccentricities, inclinations, and major axes of the planetary orbits previously enunciated, could of course be regarded as valid only after the discovery that the long-period inequalities result from the theory of universal gravitation.

The principal coefficient of the great inequality, as it affects the mean longitude of *Jupiter*, has been determined as follows:

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1785. LAPLACE (*Mémoires de l'Academie des Sciences*, 1785, p. 33; continued in the volume for 1786); Epoch 1750.

$$1249''.5 - 0''.047233\,t$$

1789. DELAMBRE (*Tables de Jupiter et de Saturne;* Paris). The value of the inequality as given above by LAPLACE is repeated, except that the coefficient is stated 1249″.15.

1799. BURCKHARDT (*Von Zach, Allgemeine geographische Ephemeriden*, 8°, Weimar, vol. III, p. 401); Epoch 1750.

1802. LAPLACE (Laplace, *Méc. Cél.*, III; book VI, ch. XII, no. 33); Epoch 1750.

$$1265''.25 - 0''.0468\,t + 0''.000037\,t^2$$

1808. A. BOUVARD (*Nouvelles tables de Jupiter et de Saturne;* 4°, Paris); Epoch 1800.

$$1203''.68 - 0''.0327\,t + 0''.000036\,t^2$$

1821. A. BOUVARD (*Tables astronomiques contenant les tables de Jupiter;* 4°, Paris); Epoch 1800.

$$1186''.619 - 0''.03470\,t + 0''.0000334\,t^2$$

1831. HANSEN (*Untersuch. der gegenseitigen Störungen des Jupiter und Saturns;* 4°, Berlin). The results of HANSEN are not given in this table, since his computations are given under a different form, and the numbers will not be directly comparable.

1834. F. T. SCHUBERT (*Traité d'Astronomie theorique;* t. III, p. 422); Epoch 1800.

$$1183''. - 0''.047108\,t + 0''.00000663\,t^2$$

1834. PONTÉCOULANT (*Théorie analytique du Système du Monde*, t. III, p. 450; 4 vol. 8°, Paris); Epoch 1800.

$$1187''.247 - 0''.04845\,t + 0''.00000226\,t^2$$

1876. LEVERRIER (*Ann. de l'Obs. de Paris*, XII, 33); Epoch 1850.

$$1205''.96 - 0''.05536\,t + 0''.00009577\,t^2 + 0''.00000000501\,t^3$$

1890. HILL (A New Theory of *Jupiter* and *Saturn*, Astron. Papers of American Ephemeris and Nautical Almanac, vol. IV, Washington, 1890). The method of HANSEN is used. Dr. HILL has reduced LEVERRIER's coefficient of the inequality of *Saturn* to HANSEN's form, and finds it larger than his own by 36″.95.

In all these previous determinations of the great inequalities the aim has been to present them in the form

$$(A_s^{(0)} + A_s^{(1)}t + A_s^{(2)}t^2 + \ldots)\sin(5l' - 2l) + (A_c^{(0)} + A_c^{(1)}t + A_c^{(2)}t^2 + \ldots)\cos(5l' - 2l)$$

Some time ago Dr. G. W. HILL proposed that the inequalities be developed in a form which does not involve t in the coefficients, and suggested the method of treatment adopted in this paper. Accordingly the writer desires to record his great indebtedness to this illustrious geometer. Acknowledgements are also due Professor STONE for his interest, enthusiasm, and helpful suggestions; to Dr. T. J. J. SEE, for a critical examination of the manuscript; and to Mr. H. Y. BENEDICT, for computing some of the deviations in the last of the appended tables.

The periodic terms, that is, those depending on the mean longitudes of the planets, result from the integration of differentials of the form

$$m'Z\,{}^{\cos}_{\sin}(i'l' + il + j'\pi' + j\pi + k'\theta' + k\theta) \qquad (1)$$

where m' is the mass of the perturbing body; Z a function of the eccentricities, inclinations, and mean distances of the two planets; l, l' the mean longitudes of the planets; π, π' the longitudes of their perihelia; θ, θ' the longitudes of their nodes; i, j, k, integral numerical coefficients subject to the condition

$$i + i' + j + j' + k + k' = 0$$

The coefficient Z is of the form

$$Ae^j e'^{j'} \varphi^k \varphi'^{k'} (1 + A_1 e^2 + A_2 e'^2 + \ldots),$$

in which the As are functions of the mean distances only; while the circular function of which it is a coefficient may be put in the form

$$\begin{aligned} &{}^{\cos}_{\sin}(j\pi + j'\pi' + k\theta + k'\theta')\cos(i'l' + il) \\ \pm\, &{}^{\sin}_{\cos}(j\pi + j'\pi' + k\theta + k'\theta')\sin(i'l' + il) \end{aligned}$$

By the integration of these equations, as effected heretofore (see NEWCOMB's General Integrals of Planetary Motion, *Smiths. Contrib.*, p. 281), the different elements are developed in powers of the time, and we are thus led to expressions of the form

$$(a + a't + a''t^2 + \ldots)\,{}^{\cos}_{\sin}(i'l' + il)$$

Now it is well known that, if we neglect quantities of the third order with respect to the eccentricities and inclinations, the integration of the equations which give the secular variations of those elements, and of the longitudes of the perihelia and of the nodes, leads to the general expressions

$$\left.\begin{aligned} e\sin\pi &= \sum_0^n N_i \sin(g_i t + \beta_i) \\ e\cos\pi &= \sum_0^n N_i \cos(g_i t + \beta_i) \\ \varphi\sin\theta &= \sum_0^n M_i \sin(k_i t + \delta_i) \\ \varphi\cos\theta &= \sum_0^n M_i \cos(k_i t + \delta_i) \end{aligned}\right\} \qquad (2)$$

$N+1$ being the number of the planets; N_i, M_i, g_i, k_i being functions of the eccentricities at a given epoch, and of the mean distances; while β_i and δ_i are angles depending upon the positions of the perihelia and nodes at the given epoch.

If we expand (1) in terms of $e\sin\pi$, $e\cos\pi$, etc., substitute the values of $e\sin\pi$, $e\cos\pi$, etc. from (2), and integrate the resulting expressions, we come upon terms of the form

$$C\sin(i'l' + il + kt + \alpha) \qquad (3)$$

in which C and α are absolute constants, and t appears only in the argument of the sine or cosine.

A precise determination of the long-period inequalities of *Jupiter* and *Saturn* by the above substitution would be quite lengthy and more elaborate than the formula (2) will warrant. A first approximation, however, is free from these objections.

Accordingly, we may put in (2) the values of the constants N_i, g_i, β_i, M_i, etc. for the solar system as developed from the investigations of LeVerrier or Stockwell, and thus take into consideration the perturbations of all eight planets, but only to terms of the first order of the masses, since expressions (2) lose their simple form when higher powers of the masses are taken into account; and, for a more practical reason, since Stockwell's, the last, determination of the constants N_i, etc., includes only terms of the first order. It will be seen from the sequel, that when new values are given to the constants N_i, g_i, β_i, etc., so as to include higher powers of the masses, a rigorous determination of the inequalities to any degree of approximation may be readily obtained, by substitution in the forms of the approximation reached by this process.

Assuming Hill's elements of *Jupiter* and *Saturn* (*New Theory of Jupiter and Saturn*, p. 19), LeVerrier's expansion of the perturbative function in terms of the form (1) (*Ann. de l'Obs. de Paris*, t. X, pp. 37 et seq.), Stockwell's values of N_i, g_i, M_i, etc. in (2) (*Mem. on Sec. Var., Smiths. Contrib.*, p. 232), it is proposed to determine the variation of the mean longitude of *Jupiter*, and also of that of *Saturn*, to terms including the first powers of the masses, the second powers of the inclinations, and the third powers of the eccentricities.

LeVerrier's notation is used where it is not stated otherwise.

The values of the elements and masses as used by Hill are as follows (see *New Theory of Jupiter and Saturn*, pp. 19–20):

Epoch 1850, January $0^d.0$, Greenwich M.T.

Elements of *Jupiter*.		Elements of *Saturn*.	
L =	159° 56′ 26″.60	L' =	14° 49′ 34″.04
π =	11 56 9.33	π' =	90 6 46.22
θ =	98 56 19.79	θ' =	112 20 49.05
φ =	1 18 42.10	φ' =	2 29 40.19
e =	0.04824277	e' =	0.05605688
n =	109256″.55563	n' =	43996″.07844

Masses of the Eight Principal Planets.

Mercury,	$m = \frac{1}{5000000}$	*Jupiter*,	$m^{iv} = \frac{1}{1047.879}$
Venus,	$m' = \frac{1}{425000}$	*Saturn*,	$m^{v} = \frac{1}{3501.6}$
Earth,	$m'' = \frac{1}{322800}$	*Uranus*,	$m^{vi} = \frac{1}{21000}$
Mars,	$m''' = \frac{1}{3093500}$	*Neptune*,	$m^{vii} = \frac{1}{19700}$

For the periodic part, $a'R_{(0,1)}$, of the perturbative function we have

$$a'R_{(0,1)} = He^h e'^{h'} \eta^f \cos(5l'-2\lambda+k'\varpi'+k\omega+u\tau')$$

where

$h = 0,1,2,3;\quad h' = 0,1,2,3;\quad f = 0,2;\quad h+h'+f = 3$

$k' = -1,-2,-3;\ k = -1,-2,-3;\ u = 0,-2;\ k'+k+u = -3$

Putting V for $5l'-2\lambda$, the following are the terms to be considered:

$$\begin{array}{lr} H_1\, e^3 \cos(V-3\omega) & \text{(I)} \\ H_2\, e^2 e' \cos(V-\varpi'-2\omega) & \text{(II)} \\ H_3\, e e'^2 \cos(V-2\varpi'-\omega) & \text{(III)} \\ H_4\, e'^3 \cos(V-3\varpi') & \text{(IV)} \\ H_5\, e\eta^2 \cos(V-\varpi-2\tau') & \text{(V)} \\ H_6\, e'\eta^2 \cos(V-\omega'-2\tau') & \text{(VI)} \end{array}$$

where $\omega = \varpi+\tau'-\tau$.

τ and τ' are determined by means of the following formula (see *Ann. de l'Obs. de Paris*, t. X, p. 15):

$$\left.\begin{aligned} \frac{\sin\gamma\sin\tau}{\cos\varphi\cos\varphi'} &= p-p'+2\sin^2\frac{\varphi}{2}\frac{\tan\varphi'}{\cos\varphi}\sin(\theta-\theta')\cos\theta \\ \frac{\sin\gamma\cos\tau}{\cos\varphi\cos\varphi'} &= q-q'-2\sin^2\frac{\varphi}{2}\frac{\tan\varphi'}{\cos\varphi}\sin(\theta-\theta')\sin\theta \end{aligned}\right\} \quad (4)$$

together with two expressions involving τ' similar to the above, and written therefrom at once by putting $\cos\theta'$, $\sin\theta'$, φ, φ', in place of $\cos\theta$, $\sin\theta$, φ', φ, respectively.

With the assumed elements these relations give $\tau'-\tau = 23''.88$, a small quantity which may be neglected in the present discussion; an assumption that will simplify to a great extent the expanded forms of (I)-(VI).

$$\begin{aligned} \text{Put}\quad h &= e\sin\varpi = e\sin\omega = \sum_0^7 N_i\sin(g_it+\beta_i) \\ h' &= e'\sin\varpi' = e'\sin\omega' = \sum_0^7 N_i'\sin(g_it+\beta_i) \\ u &= e\cos\varpi = e\cos\omega = \sum_0^7 N_i\cos(g_it+\beta_i) \\ u' &= e'\cos\varpi' = e'\cos\omega' = \sum_0^7 N_i'\cos(g_it+\beta_i) \end{aligned} \quad (5)$$

$$\begin{aligned} \text{then}\quad e^2 &= h^2+u^2 = \sum_0^7{}^2 N_i\sin(g_it+\beta_i)+\sum_0^7{}^2 N_i\cos(g_it+\beta_i) \\ e'^2 &= h'^2+u'^2 = \sum_0^7{}^2 N_i'\sin(g_it+\beta_i)+\sum_0^7{}^2 N_i'\cos(g_it+\beta_i) \end{aligned}$$

Expanding and substituting from the above, we have

$$\begin{aligned} e^3\cos(V-3\omega) &= e^3(\cos 3\omega\cos V+\sin 3\omega\sin V) \\ &= e^3(\cos^3\omega-3\sin^2\omega\cos\omega)\cos V \\ &\qquad +e^3(3\sin\omega\cos^2\omega-\sin^3\omega)\sin V \\ &= (u^3-3h^2u)\cos V+(3hu^2-h^3)\sin V \\ &= \left\{\sum_0^7{}^3 N_i\cos(g_it+\beta_i)-3\sum_0^7{}^2 N_i\sin(g_it+\beta_i)\right. \\ &\qquad \left.\sum_0^7 N_i\cos(g_it+\beta_i)\right\}\cos V \\ &\quad +\left\{3\sum_0^7 N_i\sin(g_it+\beta_i)\sum_0^7{}^2 N_i\cos(g_it+\beta_i)\right. \\ &\qquad \left.-\sum_0^7{}^3 N_i\sin(g_it+\beta_i)\right\}\sin V \end{aligned}$$

This last form may now be expanded by means of the product

$$\left(\sum_0^n a_i\, b_i\right)\left(\sum_0^n a_i\, b_i'\right)\left(\sum_0^n a_i' b_i''\right)$$

which product is seen to include as particular cases,

$\sum_0^7{}^3 N_i\cos(g_it+\beta_i)$, $\sum_0^7{}^2 N_i\sin(g_it+\beta_i)$, $\sum_0^7 N_i\cos(g_it+\beta_i)$, etc.

Collecting the terms, after the expansion, the above expression assumes the following convenient form:

$$e^3\cos(V-3\omega) = \sum_0^7 N_i^3\cos(V-3g_it-3\beta_i)$$
$$+3\sum_0^7 N_i\sum_0^7 N_j^2\cos(V-g_{ijj}t-\beta_{ijj})$$
$$+6\sum_0^7 N_i\sum_0^7 N_j\sum_0^7 N_k\cos(V-g_{ijk}t-\beta_{ijk}),\ i\pm j\pm k$$

where, for brevity, $g_{ijk} = g_i+g_j+g_k$, $\beta_{ijk} = \beta_i+\beta_j+\beta_k$ $N_{ijk} = N_iN_jN_k$. It should be noted that g_{2ij} in these abbreviations is equivalent to $g_2+g_i+g_j$, and not to $2g_i+g_j$; hence if the same subscript occurs more than once it should be repeated and not given as a numerical coefficient.

By writing N' in place of N in the above we have at once the transformed expression for $e'^3\cos(V-3\omega')$.

Similarly,

$$e^2e'\cos(V-\varpi'-2\omega)$$
$$= e^2e'[\cos(\varpi'+2\omega)\cos V+\sin(\varpi'+2\omega)\sin V]$$
$$= e^2e'(2\cos^2\omega\cos\varpi'-\cos\varpi'-2\sin\omega\cos\omega\sin\varpi')\cos V$$
$$+e^2e'(2\cos^2\omega\sin\varpi'-\sin\varpi'+2\sin\omega\cos\omega\cos\varpi')\sin V$$
$$= [2u^2u'-(h^2+u^2)u'-2huh']\cos V$$
$$+[2u^2h'-(h^2+u^2)h'+2huu']\sin V$$
$$= (u^2u'-h^2u'-2huh')\cos V+(u^2h'-h^2h'+2huu')\sin V$$
$$= \Big\{\sum_0^7{}^2 N_i\cos(g_it+\beta_i)\sum_0^7 N_i'\cos(g_it+\beta_i)$$
$$-\sum_0^7{}^2 N_i\sin(g_it+\beta_i)\sum_0^7 N_i'\cos(g_it+\beta_i)$$
$$-2\sum_0^7 N_i\sin(g_it+\beta_i)\sum_0^7 N_i\cos(g_it+\beta_i)$$
$$\sum_0^7 N_i\sin(g_it+\beta_i)\Big\}\cos V$$
$$+\Big\{\sum_0^7{}^2 N_i\cos(g_it+\beta_i)\sum_0^7 N_i'\sin(g_it+\beta_i)$$
$$-\sum_0^7{}^2 N_i\sin(g_it+\beta_i)\sum_0^7 N_i'\sin(g_it+\beta_i)$$
$$+2\sum_0^7 N_i\sin(g_it+\beta_i)\sum_0^7 N_i\cos(g_it+\beta_i)$$
$$\sum_0^7 N_i'\cos(g_it+\beta_i)\Big\}\sin V$$
$$= 2\Big\{\sum_0^7 N_i'\sum_{i+1}^7 N_j\sum_{i+2}^7 N_k+\sum_0^7 N_i\sum_{i+1}^7 N_j'\sum_{i+2}^7 N_k$$
$$+\sum_0^7 N_i\sum_{i+1}^7 j\sum_{i+2}^7 N_k'\Big\}\cos(V-g_{ijk}-\beta_{ijk})$$
$$+\Big\{2\sum_0^7 N_iN_i'\sum_{j+1}^7 N_j+\sum_0^7 N_i^2\sum_{i+1}^7 N_j\Big\}\cos(V-g_{iij}-\beta_{iij})$$
$$+\Big\{2\sum_0^7 N_i\sum_{i+1}^7 N_jN_j'+\sum_0^7 N_i'\sum_{i+1}^7 N_j^2\Big\}\cos(V-g_{ijj}-\beta_{ijj})$$
$$+\sum_0^7 N_i^2N_i'\cos(V-g_{iii}-\beta_{iii})$$

Omitting the Hs, (II) becomes (III) by interchanging N and N' in the above expression. It will also be noted that omitting the primes in (II) we obtain (I), and using primes throughout we obtain (IV). This is an independent check on the expansions, and might have been expected, since in the original forms (II) and (III) become (I) or (IV) according as we use primes not at all or altogether in their expressions.

It now remains to develop (V) and (VI) in a manner similar to that followed in the development of (I)—(IV).

Expanding we have

$$e\eta^2\cos(V-\omega-2\tau') = e\eta^2[\cos(\omega+2\tau')\cos V+\sin(\omega+2\tau')\sin V]$$
$$= e\eta^2(\cos\omega\cos 2\tau'-\sin\omega\sin 2\tau')\cos V$$
$$+e\eta^2(\sin\omega\cos 2\tau'+\cos\omega\sin 2\tau')\sin V$$
$$= (uf-hd)\cos V+(hf+ud)\sin V$$

where $d = \eta^2\sin 2\tau'$, and $f = \eta^2\cos 2\tau'$.

Putting

$$\left.\begin{aligned} \tan\varphi\sin\theta &= p = \sum_0^7 M_i\sin(k_it+\delta_i)\\ \tan\varphi\cos\theta &= q = \sum_0^7 M_i\cos(k_it+\delta_i)\\ \tan\varphi'\sin\theta' &= p' = \sum_0^7 M_i'\sin(k_it+\delta_i)\\ \tan\varphi'\cos\theta' &= q' = \sum_0^7 M_i'\cos(k_it+\delta_i)\end{aligned}\right\}\quad(6)$$

and neglecting terms of higher order, equations (5) become

$$\left.\begin{aligned}\sin\gamma\sin\tau' &= p-p'\\ \sin\gamma\cos\tau' &= q-q'\end{aligned}\right\}\quad(7)$$

whence

$$\sin^2\gamma\sin\tau'\cos\tau' = \tfrac{1}{2}\sin^2\gamma\sin 2\tau'$$
$$= 2\sin^2\tfrac{1}{2}\gamma(1-\sin^2\tfrac{1}{2}\gamma)\sin 2\tau'$$
$$= 2\eta^2(1-\eta^2)\sin 2\tau'$$
$$= (p-p')(q-q')$$
$$\therefore\ \eta^2\sin 2\tau' = \frac{(p-p')(q-q')}{2(1-\eta^2)} = \tfrac{1}{2}(p-p')(q-q') = d$$

In like manner

$$\sin^2\gamma(\cos^2\tau'-\sin^2\tau') = (q-q')^2-(p-p')^2$$

or

$$4\eta^2(1-\eta^2)\cos 2\tau' = (q-q')^2-(p-p')^2$$

whence

$$\eta^2\cos 2\tau' = \frac{(q-q')^2-(p-p')^2}{4(1-\eta^2)}$$
$$= \tfrac{1}{4}[(q-q')^2-(p-p')^2] = f$$

Then

$$d = \eta^2\sin 2\tau' = \tfrac{1}{2}(p-p')(q-q')$$
$$= \tfrac{1}{2}\sum_0^7 W_i\sin(k_it+\delta_i)\sum_0 W_i\cos(k_it+\delta_i)$$

and

$$f = \eta^2\cos 2\tau' = \tfrac{1}{4}\{(q-q')^2-(p-p')^2\}$$
$$= \tfrac{1}{4}\Big[\sum_0^7{}^2 W_i\cos(k_it+\delta_i)-\sum_0{}^2 W_i\sin(k_it+\delta_i)\Big]$$

where

$$W_i = M_i-M_i'$$

whence $e\eta^2\cos(V-\omega-2\tau') = (uf-hd)\cos V+(hf+ud)\sin V$

$$= \Big\{\tfrac{1}{4}\sum_0^7 N_i\cos(g_it+\beta_i)\Big[\sum_0^7{}^2 W_i\cos(k_it+\delta_i)-\sum_0^7{}^2 W_i\sin(k_it+\delta_i)\Big]$$
$$-\tfrac{1}{2}\sum_0^7 N_i\sin(g_it+\beta_i)\sum_0^7 W_i\sin(k_it+\delta_i)\sum_0^7 W_i\cos(k_it+\delta_i)\Big\}\cos V$$
$$+\Big\{\tfrac{1}{4}\sum_0^7 N_i\sin(g_it+\beta_i)\Big[\sum_0^7{}^2 W_i\cos(k_it+\delta_i)-\sum_0^7{}^2 W_i\sin(k_it+\delta_i)\Big]$$
$$+\tfrac{1}{2}\sum_0^7 N_i\cos(g_it+\beta_i)\sum_0^7 W_i\sin(k_it+\delta_i)\sum_0^7 W_i\cos(k_it+\delta_i)\Big\}\sin V$$

$$= \tfrac{1}{4}\left\{\sum_0^7 N_i\cos(g_it+\beta_i)\cos V+\sum_0^7 N_i\sin(g_it+\beta_i)\sin V\right\}\left\{\sum_0^{7}{}^2 W_i\cos(k_it+\delta_i)-\sum_0^{7}{}^2 W_i\sin(k_it+\delta_i)\right\}$$

$$-\tfrac{1}{2}\left\{\sum_0^7 N_i\sin(g_it+\beta_i)\cos V-\sum_0^7 N_i\cos(g_it+\beta_i)\sin V\right\}\left\{\sum_0^7 W_i\sin(k_it+\delta_i)\sum_0^7 W_i\cos(k_it+\delta_i)\right\}$$

Expanding and reducing, this last expression becomes the following, a form easily checked by comparison with the expansion of $\left(\sum_0^7 a_ib_i\right)\left(\sum_0^7 a_ib_i'\right)\left(\sum_0^7 a_i'b_i''\right)$ and convenient for the numerical computation of its several terms:

$e\eta^2\cos(V-\omega-2\tau')$

$$= \tfrac{1}{2}\sum_0^7 N_i\sum_1^7 W_j\sum_2^7 W_k\cos\{V-(g_i-k_j-k_k)t-(\beta_i-\delta_j-\delta_k)\}$$

$$+\tfrac{1}{2}\sum_0^7 W_i\sum_1^7 N_j\sum_2^7 W_k\cos\{V-(g_j-k_i-k_k)t-(\beta_j-\delta_i-\delta_k)\}$$

$$+\tfrac{1}{2}\sum_0^7 W_i\sum_1^7 W_j\sum_2^7 N_k\cos\{V-(g_k-k_i-k_j)t-(\beta_k-\delta_i-\delta_j)\}$$

$$+\tfrac{1}{2}\sum_0^7 N_i\sum_0^7 W_i\sum_1^7 N_j\cos\{V-(g_i-k_i-k_j)t-(\beta_i-\delta_i-\delta_j)\}$$

$$+\tfrac{1}{4}\sum_0^7 N_i\sum_1^7 W_j^2\cos\{V-(g_i-2k_j)t-(\beta_i-2\delta_j)\}$$

$$+\tfrac{1}{4}\sum_0^7 W_i^2\sum_1^7 N_j\cos\{V-(g_j-2k_i)t-(\beta_j-2\delta_i)\}$$

$$+\tfrac{1}{4}\sum_0^7 W_i^2\sum_0^7 N_i\cos\{V-(g_i-2k_i)t-(\beta_i-2\delta_i)\}$$

$e'\eta^2\cos(V-\omega-2\tau')$ may be transformed in like manner, and the resulting expression differs from the above only in that N' takes the place of N.

(I), (II), (III), (IV), (V), (VI) have now been replaced by a series of expressions, such as

$$(8)\qquad H\Pi\cos(V-\mathrm{s}t-\mathrm{D})$$

where H is the numerical factor of the perturbative function; Π a product of Ns, Ms and Ws; s a sum or difference of gs and ks; D a sum or difference of βs and δs.

It may be remarked that I, II, III and IV have the same arguments, s and D, and that VI has the same arguments as V. Hence it is possible to add the coefficients of I, II, III, and IV that correspond to the same arguments, and thus reduce the number of terms in the series to one-fourth its present magnitude. The same is true of V and VI, and the number of these terms might be reduced one-half. But in view of an independent check on the numerical work afforded by a direct comparison of the results with the integrals obtained by LeVerrier's theory (*Ann. de l'Obs. de Paris*, t. X, pp. 108, et seq.), the identity of I–VI will be preserved throughout. It will also be noted that H, Π, s, and D are constants.

In order to be as nearly as possible consistent with the notation of LeVerrier (*Ann. de l'Obs. de Paris*, t. X, p. 104), put

$$V-\mathrm{s}t-\mathrm{D} = D, \qquad \frac{m'a}{\mu} = B$$

Also, let us put,

$$(a)\quad \frac{d^2A}{dt^2} = -3\frac{m'an^2}{\mu}\frac{dR_{(0,1)}}{d\varepsilon} = +6Bn^2H\Pi\sin D$$

$$\frac{dA}{dt} = -\frac{2m'an}{\mu}\left(a\frac{dR_{(0,1)}}{da}\right) = -2Bna\frac{dH}{da}\Pi\cos D \qquad (b)$$

$$\frac{d\Phi}{dt} = \tan\tfrac{1}{2}\phi\frac{dF}{dt} = +\tan\tfrac{1}{2}\phi\,\frac{m'an}{\mu}\cos\phi\,\frac{dR_{(0,1)}}{de}$$

$$= +\tan\tfrac{1}{2}\phi\cos\phi\, hBn\frac{H\Pi}{\sin\phi}\cos D$$

$$= \frac{\cos\phi}{1+\cos\phi}hnBH\Pi\cos D$$

$$= hnBH\Pi\cos D(\tfrac{1}{2}-\tfrac{1}{8}e^2+\ldots)$$

$$= \tfrac{1}{2}hnBH\Pi\cos D \qquad (c)$$

$$\frac{dG}{dt} = +\frac{m'an}{2\mu}\frac{\cos\frac{1}{2}\gamma}{\cos\phi}\frac{dR_{(0,1)}}{d\eta}$$

$$= \frac{fB\cos\frac{1}{2}\gamma}{2\cos\phi}nHe\eta^{f-1}\cos D' \qquad (d)$$

$$\frac{dT}{dt} = -\frac{m'an}{2\mu}\frac{1}{\cos\phi\cos\frac{1}{2}\gamma}\frac{1}{\eta}\left[\frac{dR_{(0,1)}}{d\tau'}\right]$$

$$= \frac{uB}{2\cos\phi\cos\frac{1}{2}\gamma}nHe\eta^{f-1}\sin D' \qquad (e)$$

in which $f = 0, 2$ and $u = 0, -2$. The D' in (d) and (e) is LeVerrier's D. These last two expressions will be taken up later and reduced to their equivalent forms.

The differential equations to be integrated for the variation of the mean longitude of *Jupiter* are

$$\frac{d^2\rho}{dt^2} = \frac{d^2A}{dt^2}$$

$$\frac{d\varepsilon}{dt} = \frac{dA}{dt}+\frac{d\Phi}{dt}+\frac{d\Psi}{dt}$$

where $\frac{d\Psi}{dt} = \tan\frac{1}{2}\varphi\sin\varphi\frac{\delta\theta}{dt}$, and $\sin\varphi\frac{\delta\theta}{dt} = G\cos(\tau-\theta)+T\sin(\tau-\theta)$, to terms of an order not higher than the third.

Integrating (a), (b), (c) we have

$$A = -\frac{6Bn^2}{(5n'-2n-\mathrm{s})^2}H\Pi\sin D$$

$$A = -\frac{2Bn}{5n'-2n-\mathrm{s}}a\frac{dH}{da}\sin D$$

$$\Phi = +\frac{hBn}{2(5n'-2n-\mathrm{s})}H\Pi\sin D$$

The integrals of (d) and (e) in their present form give rise to terms containing $e\eta$. It is necessary to transform them to corresponding forms in which $e\eta$ and $e'\eta$ are not allowed to appear. This is most conveniently done at this point, since G and T occur in $\sin\varphi\frac{\delta\theta}{dt}$, and after substitution the latter is readily reducible to an equivalent form $cH\Pi\sin D$.

It will be noticed that the numerical coefficients of the integrals of $\frac{dG}{dt}$ and $\frac{dT}{dt}$ are the same to terms of the third order, and that G contains $\sin D'$ as a factor, while T contains $\cos D'$; hence we may write

$$\Psi = \tan\tfrac{1}{2}\varphi \sin\varphi\, \delta\theta = B' \tan\tfrac{1}{2}\varphi \{G\cos(\tau-\theta) + T\sin(\tau-\theta)\}$$
$$= B'\tan\tfrac{1}{2}\varphi\, e\eta \sin Y$$

remembering that $\qquad D' = V-\omega-2\tau,$

and putting $\qquad V-\omega-\tau-\theta = Y$

Therefore,

$$\Psi = \tfrac{1}{4}\{u[q(q-q')-p(p-p')]-h[p(q-q')+q(p-p')]\}\sin V$$
$$-\tfrac{1}{4}\{h[q(q-q')-p(p-p')]+u[p(q-q')+q(p-p')]\}\cos V$$

because of the relations (7), and the smallness of φ and γ, by virtue of which we may put $\tan\varphi = 2\tan\frac{1}{2}\varphi$ and $\sin\gamma = 2\sin\frac{1}{2}\gamma$. Substituting for p, q, h, and u their values, we have, finally, (VII)

$$\Psi = \frac{B'n}{4(5n'-2n-\mathrm{s})} H\left(\sum_0^7 M_i \sum_0^7 W_j \sum_0^7 N_k + \sum_0^7 W_i \sum_0^7 M_j \sum_0^7 N_k\right)$$
$$\sin\{V-(k_i+k_j+g_k)-(\delta_i+\delta_j+\beta_k)\} = \frac{Bn}{4(5n'-2n+\mathrm{s})} H\Pi \sin D$$

where the H is the H of (V), and Π and D are characteristic of (VII) itself. By writing N' in place of N the corresponding expression in which $e'\eta$ occurs is given; it is of the same form, and differs only in that its H is that of (VI). It will be referred to hereafter as (VIII). The symbols (VII) and (VIII) are here made to do double service — they refer to both the unintegrated and the integrated forms. This is done to save space. These insignificant terms have already taken undue space, but are essential to the completeness of the theory.

Finally, $\quad \delta l = \delta\rho + \delta\varepsilon \qquad \delta\varepsilon = A + \Phi + \Psi$

$\qquad \delta\rho = \delta\Lambda \qquad \therefore\ \delta l = \delta\Lambda + A + \Phi + \Psi$

Putting $\quad \Lambda = \Sigma L \sin D \qquad \Phi = \Sigma P \sin D$

$\qquad A = \Sigma C \sin D \qquad \Psi = \Sigma Q \sin D$

the values of the several integrals whose sum gives δl are given in the tables at the end of this paper in the columns headed L, C, etc.

To determine $\delta l'$ we have, if $B' = \dfrac{m\nu}{\mu'}$, $\nu = \dfrac{n'}{n}$ (*Ann. de l'Obs. de Paris*, t. X, p. 143),

$$a'R_{(1,0)} = H'e^{k}e'^{k'}\eta'\cos(5l'-2\lambda+k'\varpi'+k\omega+u\tau')$$
$$\Lambda' = \int\left(-\frac{3m'an'^2}{\mu'}\right)\frac{dR_{(1,0)}}{dl'}\,dt^2$$
$$= \frac{-15B'\nu n^2 i'}{(5n'-2n-\mathrm{s})^2} H'\Pi\sin D$$
$$A' = \int\left(-\frac{2ma'^2n'}{\mu'}\right)\frac{dR_{(1,0)}}{da}\,dt$$
$$= \frac{+2B'n}{5n'-2n-\mathrm{s}}\left(a\frac{dH'}{da}+H'\right)\Pi\sin D$$
$$\Phi' = \tan\frac{\varphi'}{2}F' = \int\frac{ma'n'\cos\varphi'}{\mu'}\,\frac{dR_{(1,0)}}{de'}\,dt$$
$$= \frac{+h'B'n}{5n'-2n-\mathrm{s}} H'\Pi\sin D$$
$$\Psi' = \frac{B'n}{4(5n'-m-\mathrm{s})} H'\Pi\sin D$$

In these expressions for Λ', A', Φ', and Ψ' the Πs, Ds, and ss are the same as those that occur in the corresponding terms of Λ, A, etc. Λ and Λ' differ only in having different Hs. The same is true of A and A', etc.

We have, as before,

$$\delta l' = \Lambda' + A' + \Phi' + \Psi'.$$

Putting $\quad \Lambda' = \Sigma L' \sin D \qquad \Phi' = \Sigma P' \sin D$

$\qquad A' = \Sigma C' \sin D \qquad \Psi' = \Sigma Q' \sin D$

the value of $\delta l'$ may be written from the tables at the end of the paper, where under the columns headed L', C', etc., will be found the values of its several terms. The tables include all terms whose coefficients outside the sine form are equal to and greater than 0″.0005.

We have now $\quad \delta l = \Sigma C_i \sin(V-\kappa_i t-\alpha_i)$

$\qquad \delta l' = \Sigma C_i' \sin(V-\kappa_i' t-\alpha_i')$

in which the Cs, C's, κs, κ's, αs, and α's are constants, and V varies directly with the time. In general, the κ's and α's are the same as the κs and αs; but since C is to C' about as two to five, $\delta l'$ will contain more terms equal to or above any particular limiting fraction of a second of arc than δl. Hence κs and αs not found in δl will be introduced into $\delta l'$, though all angles found in δl will appear in $\delta l'$. For this reason κ and α are written with primes in the summation for $\delta l'$.

It will be noted from the tables that the inequality for *Jupiter* is represented very closely by four principal terms, and likewise the perturbation of *Saturn* by four terms corresponding severally to those of *Jupiter*. Yet, if we take only the principal terms of the above summations, the series are still too unwieldy for practical purposes. It is proposed now to put δl in the form

$$\delta l = J\sin(V'+J') + 2\,\Sigma C_i\cos\{V'+h_i't-\alpha_i\}\sin\tfrac{1}{2}h_i't \qquad (9)$$

and $\delta l'$ under the similar form

$$\delta l' = S\sin(V'+S') + 2\Sigma C_i'\cos\{V'+h_i't-\alpha_i'\}\sin\tfrac{1}{2}h_i't \qquad (10)$$

where the first parts $J\sin(V'+J')$, $S\sin(V'+S')$ are periodic, and represent the inequalities within less than 1″ for a period of a hundred years near the epoch. Furthermore, (9) and (10) are in a very convenient form for a rigorously exact representation of the inequalities, since the summations are easily computed and readily tabulated, the deviations from the strict periodic form being slight, as $\frac{1}{2}(h-\kappa)$ is a small angle. In (9) and (10) J and S are absolute constants, J essentially positive, and S essentially negative; J' and S' are also constants, are nearly equal, since the ratios of the Cs to the corresponding C's, although not exactly equal to one another, are all very nearly as two to five, and $\alpha = \alpha'$, $V' = V - ht$ where h is an arbitrary constant to be so chosen that the forms resulting hereafter may be best adapted for computation; h' is a variable, and equal to $\frac{1}{2}(h-\kappa)$.

Putting $\qquad V' = V - ht$

we have $\qquad V-\kappa t = V' + (h-\kappa)t$

whence $\delta l = \Sigma C_i \sin[V'+(h-\kappa_i)t-\alpha_i]$

$$= \Sigma C_i \sin(V'-\alpha_i) + 2\Sigma C_i \cos[V'+\tfrac{1}{2}(h-\kappa_i)t-\alpha_i]\sin\tfrac{1}{2}(h-\kappa_i)t$$

$$\Sigma C_i \sin(V'-\alpha_i) = J\sin(V'+J')$$

if we determine J and J' by the equations

$$J\sin J' = \Sigma C_i \sin\alpha_i$$
$$J\cos J' = \Sigma C_i \cos\alpha_i$$

Hence we get

$$\delta l = J\sin(V'+J')+2\Sigma C_i\cos\{V'+\tfrac{1}{2}(h-\kappa_i)t-\alpha_i\}\sin\tfrac{1}{2}(h-\kappa_i)t$$

If we substitute S for J, S' for J', C' for C, and α' for α, we obtain the expression for $\delta l'$.

It is obvious that h should be made equal to the κ corresponding to the largest C. The largest C corresponds to the largest C'; hence the same h should be used for both terms, $h = 67''.50927$, hence

$$V' = 1399''.77167\,t$$

The expressions of $J\sin(V'+J')$, $S\sin(V'+S')$, are

$$J\sin(V'+J') = 1160''.359\sin(V'+270°\,12'\,19''.91)$$
$$S\sin(V'+S') = -2862''.135\sin(V'+270°\,11'\,54''.76)$$

The differences between these and the true inequalities have been computed for different epochs in an interval of five thousand years, and entered in the last table.

Of the numerical data used the elements and masses have already been given. LeVerrier's values of the numerical coefficients of the perturbative function were employed, and are as follows. (See *Ann. de l'Obs. de Paris*, t. X, pp. 86–88):

	$\log H$	$\log\left(a\frac{dH}{da}\right)$	$\log\left(a\frac{dH}{da}+H\right)$
(I)	0.06578*n*	0.82185*n*	0.89202*n*
(II)	0.7643927	1.43806	1.52156
(III)	0.982959*n*	1.555067*n*	1.658134*n*
(IV)	0.719908	1.159654	1.294224
(V) (VII)	0.12390*n*	0.9539*n*	1.0138*n*
(VI) (VIII)	0.40780	1.1678	1.2374

Stockwell has given N_i, g_i, β_i, etc., in the form

$$N = A+\sum_0^7 B_i\mu_i, \text{ etc.}$$

and the masses in the form $\dfrac{1+\mu_i}{C_i}$; where A, B, and C are constants; the μs are known as soon as the masses are assumed, by equating the value of the mass to $\dfrac{1+\mu}{C}$ and solving for μ. The Ns, gs, etc. were obtained from Stockwell's memoir (*Smiths. Contrib.*, p. 232), the values there given being modified by the introduction of Hill's masses by means of the above formulas. The resulting values are tabulated below:

Subscript	$\log N$	$\log N'$	$\log M$	$\log M'$	$\log W$
0	*n*4.9095560	*n*4.8692317	*n*5.3494718	*n*5.4331295	4.6766936
1	5.0285713	5.0429691	5.1975562	5.3340514	*n*4.7649230
2	*n*3.9867717	*n*4.8369567	*n*4.4712917	*n*5.3034121	5.2342641
3	3.8388491	4.9294189	*n*3.3802112	*n*4.6669857	4.6442416
4	5.8318698	5.8823537	$-\infty$	$-\infty$	$-\infty$
5	7.3178169	7.2788245	7.1556973	7.1169562	6.0868579
6	8.6365279	8.5328158	7.3883783	6.9282268	7.2035469
7	8.1931101	*n*8.6841779	7.8003183	*n*8.1972861	8.3436915

Subscript	g	k	β	δ
	″	″	° ′ ″	° ′ ″
0	5.3946525	− 5.1489278	86 18 14.5	17 50 38.6
1	7.3942869	− 6.6302715	12 9 1.3	134 6 18.3
2	17.3885210	17.0803393	332 28 32.3	296 51 52.7
3	18.0567791	−19.1588596	136 15 28.9	254 30 21.1
4	0.6599087	0.0	72 15 0.9	106 22 29.1
5	2.7034487	− 0.7048689	108 54 41.3	21 12 25.7
6	3.7460135	− 2.9230301	28 1 4.0	133 59 48.4
7	22.5030888	−25.9744026	307 50 48.7	306 21 4.2

All terms less than $0''.0005$ have been omitted; all others included. All results have been checked by duplicate computations. The integrals for δl were first computed and checked, then those of $\delta l'$; afterwards, as a final check, those of *Saturn* were computed from those of *Jupiter* by determining the ratios of corresponding terms.

Two of the mistakes made are not without interest. A mistake in copying made N_7' positive instead of negative. As a result the inequality was reduced to one-tenth of its real value. In computing $J\sin(V+J)$ the co-named functions for the arguments of the inclination terms were taken. The resulting value of J differed less than 1″ from its true value.

Tables I to VIII contain the integrals arising from the expressions (I) to (VIII). In the first column on the left are given as indices the products of Ns, Ms, and Ws occurring as factors in the coefficients. In table IX only one of these products is given for each argument, which serves to identify that argument. Table X contains the values of δl and $\delta l'$ in their final forms.

I. Inequalities Arising from $H_1e^3\cos(V-3\omega)$, $\delta l=\Sigma\sin(V-st-\text{D})$, $\delta l'=\Sigma'\sin(V-st-\text{D})$.

	L	C	P	L'	C'	Σ	Σ′	S	D
	″	″	″	″	″	″	″	″	° ′ ″
N_5^3	— 0.011	0.000	0.000	+ 0.028	0.000	— 0.011	+ 0.028	8.11035	326 44 3.9
N_6^3	—102.458	+2.595	— .341	+254.697	—7.533	—100.204	+247.164	11.23804	84 3 12.0
N_7^3	— 5.182	+0.126	— .017	+ 12.883	—0.366	— 5.073	+ 12.517	67.50927	203 32 26.1
$N N_6^2$	+ 0.058	— .001	.000	— 0.144	+ .004	+ 0.057	— 0.140	12.88668	142 20 22.5
$N N_7^2$	+ .008	.000	.000	— .020	+ .001	+ .008	— .019	50.40083	341 59 51.9
$N_1N_6^2$	— .076	+ .002	.000	+ .189	— .006	— .074	+ .183	14.88631	69 11 9.3
$N_1N_7^2$	— .010	.000	.000	+ .026	— .001	— .010	+ .025	52.40046	267 50 38.7
$N_2N_6^2$	+ .007	.000	.000	— .017	+ .001	+ .007	— .016	24.88055	28 30 40.3
$N_2N_7^2$	+ .001	.000	.000	— .002	.000	+ .001	— .002	62.39470	228 10 9.7
$N_3N_6^2$	— .005	.000	.000	+ .012	.000	— .005	+ .012	25.54881	192 17 36.9
$N_3N_7^2$	— .001	.000	.000	+ .002	.000	— .001	+ .002	63.06295	31 57 6.3
$N_4N_5^2$	— .001	.000	.000	+ 0.003	.000	— .001	+ 0.003	6.06680	290 4 23.5
$N_4N_6^2$	— .481	+ .012	— .002	+ 1.193	— .035	— .471	+ 1.158	8.15194	128 17 8.9
$N_4N_7^2$	— 0.066	+ .002	.000	+ 0.163	—0.004	— 0.064	+ 0.159	45.66609	327 56 38.3
$N_5N_6^2$	— 14.734	+ .374	— .049	+ 36.627	—1.084	— 14.409	+ 35.543	10.19548	164 56 49.3
$N_5N_7^2$	— 2.014	+0.050	— .006	+ 5.008	—0.144	— 1.970	+ 4.864	47.70963	4 36 18.7
$N_6N_7^2$	— 42.025	+1.037	— .136	+104.468	—3.010	— 41.124	+101.458	48.75219	283 42 41.4
$N_4^2N_6$	+ 0.001	0.000	.000	+ 0.002	.000	+ 0.001	+ 0.002	5.06583	172 31 5.8
$N_4^2N_7$	.000	.000	.000	+ 0.001	.000	.000	+ 0.001	23.82291	92 20 50.5
$N_5^2N_6$	+ .706	— .018	+ .002	+ 1.766	— .052	— .690	+ 1.714	9.15291	245 50 26.6
$N_5^2N_7$	— 0.261	+0.006	— .001	+ 0.649	—0.019	— 0.256	+ 0.630	27.90999	165 40 11.3
$N_6^2N_7$	—113.635	+2.842	— .374	+282.481	—8.247	—111.167	+274.234	29.99512	3 52 56.7
NN_5N_6	+ 0.006	0.000	.000	— 0.014	0.000	+ 0.006	— 0.014	11.84411	223 13 59.8
NN_5N_7	+ .002	.000	.000	— .005	.000	+ .002	— .005	30.60119	143 3 44.5
NN_6N_7	+ .043	— .001	.000	— .106	+ .003	+ .042	— .103	31.64375	62 10 7.2
$N_1N_4N_6$	.000	.000	.000	+ .001	.000	.000	+ .001	11.80021	112 25 6.2
$N_1N_5N_6$	— .007	.000	.000	+ .018	— .001	— .007	+ .017	13.84375	149 4 46.6
$N_1N_5N_7$	— .003	.000	.000	+ .007	.000	— .003	+ .007	32.60082	68 54 31.3
$N_1N_6N_7$	— .056	+ .001	.000	+ .140	— .004	— .055	+ .136	33.64339	348 0 54.0
$N_2N_5N_6$	+ .001	.000	.000	— .002	.000	+ .001	— .002	23.83798	109 24 17.6
$N_2N_5N_7$	.000	.000	.000	— .001	.000	.000	— .001	42.59506	29 14 2.3
$N_2N_6N_7$	+ .005	.000	.000	— .013	.000	+ .005	— .013	43.63762	308 20 25.0
$N_3N_5N_6$	.000	.000	.000	+ .001	.000	.000	+ .001	24.50624	273 11 14.2
$N_3N_6N_7$	— .004	.000	.000	+ .009	.000	— .004	+ .009	44.30588	112 7 31.6
$N_4N_5N_6$	— .046	+ .001	.000	+ .114	— .003	— .045	+ .111	7.10937	209 10 45.2
$N_4N_5N_7$	— .017	.000	.000	+ .042	— .001	— .017	+ .041	25.86645	129 0 30.9
$N_4N_6N_7$	— 0.355	+ .009	— .001	+ 0.882	— .026	— 0.347	+ 0.856	26.90901	48 6 53.6
$N_5N_6N_7$	— 10.891	+0.273	—0.036	+ 27.074	—0.791	— 10.654	+ 26.283	28.75255	84 46 34.0

II. Inequalities Arising from $H_2e^2e'\cos(V-\varpi'-2\omega)$.

	L	C	P	L'	C'	P'	Σ	Σ′	S	D
	″	″	″	″	″	″	″	″	″	° ′ ″
$N_5^2N_5'$	+ 0.052	— 0.001	0.000	— 0.128	+ 0.003	0.000	+ 0.051	— 0.125	8.11035	326 44 3.9
$N_6^2N_6'$	+403.132	— 8.447	+ .896	—1002.132	+25.281	+1.106	+395.581	—975.744	11.23804	84 3 12.0
$N_7^2N_7'$	— 80.206	+ 1.626	—0.171	+ 199.381	— 4.836	—0.211	— 78.751	+194.331	67.50927	203 32 26.1
NN_5N_5'	0.000	0.000	.000	+ 0.001	0.000	.000				
$N'N_5^2$	.000	.000	.000	+ .001	.000	.000	0.000	+ 0.002	10.80155	304 7 37.1
NN_6N_6'	— .152	+ .003	.000	+ .377	— .009	.000				
$N'N_6^2$	— .088	+ .002	.000	+ .218	— .005	.000	— .235	+ .583	12.88668	142 20 22.5
NN_7N_7'	+ .081	— .002	.000	— .203	+ .005	.000				
$N'N_7^2$	— .012	.000	.000	+ .030	— .001	.000	+ .067	— .169	50.40083	341 59 51.9
$N_1N_5N_5'$	+ .001	.000	.000	— .001	.000	.000				
$N_1'N_5^2$	.000	.000	.000	— .001	.000	.000	+ .001	— .002	12.80118	229 58 23.9
$N_1N_6N_6'$	+ .200	— .004	.000	— .497	+ .012	+0.001				
$N_1'N_6^2$	+ .131	— .003	.000	— .326	+ .008	.000	+ .324	— .802	14.88631	69 11 9.3
$N_1N_7N_7'$	— .108	+ .002	.000	+ .268	— .007	.000				
$N_1'N_7^2$	+ .018	.000	.000	— .045	+ .001	.000	— .088	+ .217	52.40046	267 50 38.7
$N_2N_6N_6'$	— .018	.000	.000	+ .046	— .001	.000				
$N_2'N_6^2$	— 0.083	+ 0.002	0.000	+ 0.206	— 0.005	0.000	— 0.099	+ 0.246	24.88055	28 30 40.3

	L	C	P	L′	C′	P′	Σ	Σ′	S	D
	″	″	″	″	″	″	″	″	″	° ′ ″
$N_2N_7N_7'$	+ 0.010	0.000	0.000	— 0.025	+ 0.001	0.000				
$N_2'N_7^2$	— .011	.000	.000	+ .028	— .001	.000	— 0.001	+ 0.003	62.39470	228 10 0.7
$N_3N_6N_6'$	+ .013	.000	.000	— .033	+ .001	.000				
$N_3'N_6^2$	+ .103	— .002	.000	— .255	+ .006	.000	+ .114	— .281	25.54881	192 17 36.9
$N_3N_7N_7'$	— .007	.000	.000	+ .018	.000	.000				
$N_3'N_7^2$	+ .014	.000	.000	— .035	+ .001	.000	+ .007	— .016	63.06296	31 57 6.3
$N_4N_5N_5'$	+ .003	.000	.000	— .008	.000	.000				
$N_4'N_5^2$	+ 0.002	.000	.000	— 0.005	.000	.000	+ 0.005	— 0.013	6.06680	290 4 23.5
$N_4N_6N_6'$	+ 1.260	— .026	+ .003	— 3.129	+ .079	+ .004				
$N_4'N_6^2$	+ 0.898	— .019	+ .002	— 2.232	+ .056	+ .002	+ 2.118	— 5.220	8.15194	128 17 8.9
$N_4N_7N_7'$	— .677	+ .014	— .001	+ 1.683	— .042	.000				
$N_4'N_7^2$	+ 0.123	— .003	.000	— 0.305	+ 0.008	+ .106	— 0.544	+ 1.343	45.66609	327 56 38.3
$N_5N_6N_6'$	+ 38.650	— .810	+ .086	— 96.077	+ 2.426	+ .062				
$N_5'N_6^2$	+ 22.430	— .470	+ .050	— 55.759	+ 1.408	— .056	+ 59.936	—147.834	10.19548	164 56 49.3
$N_5N_7N_7'$	— 20.785	+ .425	— .045	+ 51.668	— 1.273	+0.008				
$N_5'N_7^2$	+ 3.067	— 0.063	+ .006	— 7.622	+ 0.188	—1.158	— 17.395	+ 42.913	47.70963	4 36 18.7
$N_6N_7N_7'$	—433.605	+ 8.852	— .938	+1077.883	—26.492	+0.147				
$N_6'N_7^2$	+ 55.117	— 1.125	+ .119	— 137.013	+ 3.368	.000	—371.580	+916.735	48.75219	283 42 41.4
$N_4N_4'N_6$	+ 0.003	0.000	.000	— 0.007	0.000	.000				
$N_4^2N_6'$	+ .001	.000	.000	— .002	.000	.000	+ 0.004	— 0.009	5.06583	172 31 5.8
$N_4N_4'N_7$	+ .001	.000	.000	— .003	.000	.000				
$N_4^2N_7'$	— 0.001	.000	.000	+ 0.004	.000	.000	0.000	+ 0.001	23.82291	92 20 50.5
$N_5N_5'N_6$	+ 2.150	— 0.045	+ .005	— 5 346	+ .135	+ .006				
$N_5^2N_6'$	+ 0.926	— .019	+ .002	— 2.303	+ .058	+ .003	+ 3.019	— 7.447	9.15291	245 50 26.6
$N_5N_5'N_7$	+ 0.795	— .016	+ .002	— 1.976	+ .049	+ .002				
$N_5^2N_7'$	— 1.347	+ 0.028	— .003	+ 3.348	— .084	— .004	— 0.541	+ 1.335	27.90999	165 40 11.3
$N_6N_6'N_7$	+298.073	— 6.174	+0.656	— 740.975	+18.452	+0.771				
$N_6^2N_7'$	—586.236	+12.126	—1.285	+1457.303	—36.290	—1.516	—282.840	+697.745	29.99512	3 52 56.7
$N'N_4N_6$	0.000	0.000	0.000	+ 0.001	0.000	.000				
$NN_4'N_6$	.000	.000	.000	+ .001	.000	.000				
NN_4N_6'	.000	.000	.000	+ .001	.000	.000	0.000	+ 0.003	9.80057	186 34 19.4
$N'N_4N_7$	.000	.000	.000	.000	.000	.000				
$NN_4'N_7$	.000	.000	.000	.000	.000	.000				
NN_4N_7'	.000	.000	.000	— .001	.000	.000	.000	— .001	28.55765	106 24 4.1
$N'N_5N_6$	— .008	.000	.000	+ .021	— .001	.000				
$NN_5'N_6$	— .008	.000	.000	+ .021	— .001	.000				
NN_5N_6'	— .007	.000	.000	+ .018	.000	.000	— .023	+ .058	11.84411	223 13 59.8
$N'N_5N_7$	— .003	.000	.000	+ .008	.000	.000				
$NN_5'N_7$	— .003	.000	.000	+ .008	.000	.000				
NN_5N_7'	+ .010	.000	.000	— .026	+ .001	.000	+ .004	— .009	30.60119	143 3 44.5
$N'N_6N_7$	— .065	+ .001	.000	+ .161	— .004	.000				
$NN_6'N_7$	— .056	+ .001	.000	+ .139	— .003	.000				
NN_6N_7'	+ .220	— .004	.000	— .548	+ .014	.000	+ .097	— .241	31.64375	62 10 7.2
$N_1'N_4N_6$	.000	.000	.000	— .001	.000	.000				
$N_1N_4'N_6$	.000	.000	.000	— .001	.000	.000				
$N_1N_4N_6'$	.000	.000	.000	— .001	.000	.000	.000	— .003	11.80021	112 25 6.2
$N_1'N_4N_7$	.000	.000	.000	.000	.000	.000				
$N_1N_4'N_7$	.000	.000	.000	.000	.000	.000				
$N_1N_4N_7'$	.000	.000	.000	+ .001	.000	.000	.000	+ .001	30.55728	32 14 50.9
$N_1'N_5N_6$	+ .013	.000	.000	— .031	+ .001	.000				
$N_1N_5'N_6$	+ .011	.000	.000	— .028	+ .001	.000				
$N_1N_5N_6'$	+ .010	.000	.000	— .024	+ .001	.000	+ .034	— .080	13.84375	149 4 46.6
$N_1'N_5N_7$	+ .005	.000	.000	— .012	.000	.000				
$N_1N_5'N_7$	+ .004	.000	.000	— .010	.000	.000				
$N_1N_5N_7'$	— .014	.000	.000	+ .034	— .001	.000	— .005	+ .011	32.60082	68 54 31.3
$N_1'N_6N_7$	+ .097	— .002	.000	— .241	+ .006	.000				
$N_1N_6'N_7$	+ .074	— .002	.000	— .184	+ .004	.000				
$N_1N_6N_7'$	— .291	+ .006	.000	+ .723	— .018	— .001	— .118	+ .289	33.64339	348 0 54.0
$N_2'N_5N_6$	— .008	.000	.000	+ .020	— .001	.000				
$N_2N_5'N_6$	— .001	.000	.000	+ .002	.000	.000				
$N_2N_5N_6'$	— .001	.000	.000	+ .002	.000	.000	— 0.010	+ 0.023	23.83798	109 24 17.6
$N_2'N_5N_7$	— 0.003	0.000	0.000	+ 0.007	0.000	0.000				

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	L	C	P	L′	C′	P′	Σ	Σ′	S	D
	″	″	″	″	″	″	″	″	″	° ′ ″
$N_2N_5'N_7$	0.000	0.000	0.000	+ 0.001	0.000	0.000				
$N_2N_5N_7'$	+ .001	.000	.000	— .003	.000	.000	— 0.002	+ 0.005	42.59506	29 14 2.3
$N_2'N_6N_7$	— .061	+ .001	.000	+ .152	— .004	.000				
$N_2N_6'N_7$	— .007	.000	.000	+ .017	.000	.000				
$N_2N_6N_7'$	+ .027	— .001	.000	— .066	+ .002	.000	— .041	+ .101	43.63762	308 20 25.0
$N_3'N_5N_6$	+ .010	.000	.000	— .024	+ .001	.000				
$N_3N_5'N_6$	+ .001	.000	.000	— .002	.000	.000				
$N_3N_5N_6'$	+ .001	.000	.000	— .002	.000	.000	+ 0.12	— .027	24.50624	273 11 14.2
$N_3'N_5N_7$	+ .004	.000	.000	— .009	.000	.000				
$N_3N_5'N_7$	.000	.000	.000	— .001	.000	.000				
$N_3N_5N_7'$	— .001	.000	.000	+ .002	.000	.000	+ .003	— .008	43.26332	193 0 58.9
$N_3'N_6N_7$	+ .076	— .001	.000	— .188	+ .005	.000				
$N_3N_6'N_7$	+ .005	.000	.000	— .012	.000	.000				
$N_3N_6N_7'$	— .019	.000	.000	+ .047	— .001	.000	+ .061	— .149	44.30588	112 7 31.6
$N_4'N_5N_6$	+ .086	— .002	.000	— .214	+ .006	.000				
$N_4N_5'N_6$	+ .070	— .001	.000	— .174	+ .005	.000				
$N_4N_5N_6'$	+ .060	— .001	.000	— .150	+ .004	.000	+ .202	— .523	7.10937	209 10 45.2
$N_4'N_5N_7$	+ .032	— .001	.000	— .079	+ .002	.000				
$N_4N_5'N_7$	+ .026	— .001	.000	— .064	+ .002	.000				
$N_4N_5N_7'$	— .088	+ .002	.000	+ 0.218	— .005	.000	— .030	+ 0.074	25.86645	129 0 30.9
$N_4'N_6N_7$	+ .664	— .014	+ .001	— 1.650	+ .041	+ .002				
$N_4N_6'N_7$	+ 0.465	— .010	+ .001	— 1.157	+ .029	+ .001				
$N_4N_6N_7'$	— 1.830	+ .038	— .004	+ 4.550	— 0.114	— .005	— .689	+ 1.696	26.90901	48 6 53.6
$N_5'N_6N_7$	+ 16.579	— .343	+ .036	— 41.215	+ 1.033	+ .045				
$N_5N_6'N_7$	+ 14.282	— 0.296	+ .031	— 35.509	+ 0.885	+ .039				
$N_5N_6N_7'$	— 56.187	+ 1.163	—0.123	+ 139.675	— 3.481	—0.152	— 24.858	+ 61.320	28.75225	84 46 34.0

III. Inequalities Arising from $H_3ee'^2 \cos(V-2\varpi'-\omega)$.

	L	C	P	L′	C′	P′	Σ	Σ′	S	D
	″	″	″	″	″	″	″	″	″	° ′ ″
$N_5'^2N_5$	— 0.078	+ 0.001	0.000	+ 0.194	— 0.004	0.000	— 0.077	+ 0.190	8.11035	326 44 3.9
$N_6'^2N_6$	— 525.172	+ 8.710	— .583	+1305.507	—27.268	—2.880	— 517.045	+1275.359	11.23804	84 3 12.0
$N_7'^2N_7$	— 411.000	+ 6.553	— .438	+1021.691	—20.515	—2.167	— 404.885	+ 999.009	67.50927	203 32 26.1
$N'N_5N_5'$	0.000	0.000	.000	— 0.002	0.000	0.000				
$NN_5'^2$	.000	.000	.000	— .001	.000	.000	0.000	— 0.003	10.80155	304 7 37.1
$N'N_6N_6'$	+ .228	— .004	.000	— .568	+ .012	+ .001				
$NN_6'^2$	+ .099	— .002	.000	— .245	+ .005	+ .001	+ .321	— .794	12.88668	142 20 22.5
$N'N_7N_7'$	— .123	+ .002	.000	+ .305	— .006	— .001				
$NN_7'^2$	+ .209	— .003	.000	— .519	+ .011	+ .001	+ .085	— .209	50.40083	341 59 51.9
$N_1'N_5N_5'$	— .001	.000	.000	+ .002	.000	.000				
$N_1N_5'^2$	.000	.000	.000	+ .001	.000	.000	— .001	+ 0.003	12.80118	229 58 23.9
$N_1'N_6N_6'$	— .342	+ .006	— .001	+ .849	— .018	— .002				
$N_1N_6'^2$	— .130	+ .002	.000	+ .324	— .007	— .001	— .465	+ 1.145	14.88631	69 11 9.3
$N_1'N_7N_7'$	+ .185	— .003	.000	— .459	+ .009	+ .001				
$N_1N_7'^2$	— .277	+ .004	.000	+ .688	— .014	— .001	— .091	+ 0.224	52.40046	267 50 38.7
$N_2'N_5N_5'$	+ .001	.000	.000	— .001	.000	.000				
$N_2N_5'^2$	.000	.000	.000	.000	.000	.000	+ .001	— .001	22.79542	190 17 54.9
$N_2'N_6N_6'$	+ .216	— .004	.000	— .536	+ .011	+ .001				
$N_2N_6'^2$	+ .012	.000	.000	— .030	+ .001	.000	+ .224	— .553	24.88055	28 30 40.3
$N_2'N_7N_7'$	— .116	+ .002	.000	+ .288	— .006	— .001				
$N_2N_7'^2$	+ .025	.000	.000	— .063	+ .001	.000	— .089	+ .219	62.39470	228 10 0.7
$N_3'N_5N_5'$	+ .001	.000	.000	+ .002	.000	.000				
$N_3N_5'^2$	.000	.000	.000	.000	.000	.000	+ .001	+ .002	23.46368	354 4 51.5
$N_3'N_6N_6'$	— .267	+ .004	.000	+ .664	— .014	— .001				
$N_3N_6'^2$	— .009	.000	.000	+ .021	.000	.000	— .272	+ .670	25.54881	192 17 36.9
$N_3'N_7N_7'$	+ .143	— .002	.000	— .357	+ .007	+ .001				
$N_3N_7'^2$	— .018	.000	.000	+ .045	— .001	.000	+ .123	— .305	63.06296	31 57 6.3
$N_4'N_5N_5'$	— .006	.000	.000	+ .016	.000	.000				
$N_4N_5'^2$	— 0.003	.000	.000	+ 0.006	.000	.000	— 0.009	+ 0.022	6.06680	290 4 23.5
$N_4'N_6N_6'$	— 2.339	+ .039	— .005	+ 5.813	— .122	— .013				
$N_4N_6'^2$	— 0.820	+ 0.013	—0.002	+ 2.038	— 0.043	—0.005	— 3.114	+ 7.668	8.15194	128 17 8.9

	L	C	P	L′	C′	P′	Σ	Σ	S	D
	″	″	″	″	″	″	″	″	″	° ′ ″
$N_4'N_7N_7'$	+ 1.258	— 0.020	+0.003	— 3.127	+ 0.064	+0.007				
$N_4N_7'^2$	— 1.734	+ .028	— .004	+ 4.311	— 0.088	— .009	— 0.469	+ 1.158	45.66609	327 56 38.3
$N_5'N_6N_6'$	— 58.441	+ .970	— .065	+ 145.277	— 3.037	— .319				
$N_5N_6'^2$	— 25.175	+ .418	— .028	+ 62.581	— 1.308	— .139	— 82.321	+ 203.055	10.19548	164 56 49.3
$N_5'N_7N_7'$	+ 31.428	— .508	+ .034	— 78.126	+ 1.591	+ .168				
$N_5N_7'^2$	— 53.254	+ 0.861	— .058	+ 132.382	— 2.696	—0.285	— 21.497	+ 53.034	47.70963	11 36 18.7
$N_6'N_7N_7'$	+ 564.870	— 9.126	+0.611	—1404.189	+28.573	+3.019				
$N_6N_7'^2$	—1110.959	+17.951	—1.202	+2761.694	—56.197	—5.936	— 537.855	+1326.964	48.75219	283 42 41.4
$N_4'N_4N_6'$	— 0.004	0.000	0.000	+ 0.009	0.000	0.000				
$N_4'^2N_6$	— .003	.000	.000	+ .006	.000	.000	— 0.007	+ 0.015	5.06583	172 31 5.8
$N_4N_4'N_7$	+ .005	.000	.000	— .013	.000	.000				
$N_4'^2N_7$	— 0.001	.000	.000	+ 0.002	.000	.000	+ 0.004	— 0.011	23.82291	92 20 50.5
$N_5'N_5N_6'$	— 2.801	+ .047	— .006	+ 6.964	— .145	— .015				
$N_5'^2N_6$	— 1.626	+ .027	— .004	+ 4.041	— .084	— .009	— 4.363	+ 10.752	9.15291	245 50 26.6
$N_5'N_5N_7'$	+ 4.074	— .067	+ .009	— 10.127	+ .209	+ .022				
$N_5'^2N_7$	— 0.601	+ 0.009	—0.001	+ 1.494	— 0.031	—0.003	+ 3.423	— 8.436	27.90999	165 40 11.3
$N_6'N_6N_7'$	+1527.412	—25.006	+1.674	—3796.938	+78.285	+8.270				
$N_6'^2N_7$	— 194.155	+ 3.179	—0.213	+ 482.641	— 9.951	—1.051	+1312.891	—3238.744	29.99512	3 52 56.7
$NN_4'N_6'$	0.000	— 0.000	.000	— 0.001	0.000	0.000				
$N'N_4N_6'$	.000	.000	.000	— .001	.000	.000				
$N'N_4'N_6$	+ .001	.000	.000	— .001	.000	.000	+ 0.001	— 0.003	9.80057	186 34 19.4
$NN_4'N_7'$	— .001	.000	.000	+ .002	.000	.000				
$N'N_4N_7'$	.000	.000	.000	+ .001	.000	.000				
$N'N_4'N_7$	.000	.000	.000	.000	.000	.000	— .001	+ .003	28.55765	106 24 4.1
$NN_5'N_6'$	+ .011	.000	.000	— .027	+ .001	.000				
$N'N_5N_6'$	+ .011	.000	.000	— .027	+ .001	.000				
$N'N_5'N_6$	+ .013	.000	.000	— .032	+ .001	.000	+ .035	— .083	11.84411	223 13 59.8
$NN_5'N_7'$	— .016	.000	.000	+ .040	— .001	.000				
$N'N_5N_7'$	— .016	.000	.000	+ .040	— .001	.000				
$N'N_5'N_7$	+ .005	.000	.000	— .011	.000	.000	— .027	+ 0.067	30.60119	143 3 44.5
$NN_6'N_7'$	— .287	+ .004	.000	+ .713	— .015	— .002				
$N'N_6N_7'$	— .332	+ .005	.000	+ .826	— .017	— .002				
$N'N_6'N_7$	+ .084	— .001	.000	— .210	+ .004	.000	— .527	+ 1.299	31.64375	62 10 7.2
$N_1N_4'N_6'$	— .001	.000	.000	+ .001	.000	.000				
$N_1'N_4N_6'$	— .001	.000	.000	+ .001	.000	.000				
$N_1'N_4'N_6$	— .001	.000	.000	+ .002	.000	.000	— .003	+ 0.004	11.80021	112 25 6.2
$N_1N_4'N_7'$	+ .001	.000	.000	— .002	.000	.000				
$N_1'N_4N_7'$	+ .001	.000	.000	— .002	.000	.000				
$N_1'N_4'N_7$	.000	.000	.000	+ .001	.000	.000	+ .002	— .003	30.55728	32 14 50.9
$N_1N_5'N_6'$	— .014	.000	.000	+ .036	— .001	.000				
$N_1'N_5N_6'$	— .016	.000	.000	+ .041	— .001	.000				
$N_1'N_5'N_6$	— .019	.000	.000	+ .047	— .001	.000	— .049	+ .121	13.84375	149 4 46.6
$N_1N_5'N_7'$	+ .021	.000	.000	— .052	+ .001	.000				
$N_1'N_5N_7'$	+ .024	.000	.000	— .059	+ .001	.000				
$N_1'N_5'N_7$	— .007	.000	.000	+ .018	.000	.000	+ .038	— 0.091	32.60082	68 54 31.3
$N_1N_6'N_7'$	+ .378	— .006	.000	— 0.941	+ .019	+ .002				
$N_1'N_6N_7'$	+ .497	— .008	+ .001	— 1.235	+ .025	+ .003				
$N_1'N_6'N_7$	— .126	+ .002	.000	+ 0.314	— .006	— .001	+ .738	— 1.820	33.64339	348 0 54.0
$N_2N_4'N_7'$	.000	.000	.000	.000	.000	.000				
$N_2'N_4N_7'$	.000	.000	.000	+ .001	.000	.000				
$N_2'N_4'N_7$	.000	.000	.000	.000	.000	.000	.000	+ 0.001	40.55152	352 34 21.9
$N_2N_5'N_6'$	+ .001	.000	.000	— .003	.000	.000				
$N_2'N_5N_6'$	+ .010	.000	.000	— .026	+ .001	.000				
$N_2'N_5'N_6$	+ .012	.000	.000	— .030	+ .001	.000	+ .023	— .057	23.83798	109 24 17.6
$N_2N_5'N_7'$	— .002	.000	.000	+ .005	.000	.000				
$N_2'N_5N_7'$	— .015	.000	.000	+ .037	— .001	.000				
$N_2'N_5'N_7$	+ .004	.000	.000	— .011	.000	.000	— .013	+ .030	42.59506	29 14 2.3
$N_2N_6'N_7'$	— .035	+ .001	.000	+ .087	— .002	0.000				
$N_2'N_6N_7'$	— .314	+ .005	.000	+ .780	— .016	— .002				
$N_2'N_6'N_7$	+ .080	— .001	.000	— .198	+ .004	.000	—0.264	+ 0.653	43.63762	308 20 25.0
$N_3N_4'N_7'$	.000	.000	.000	.000	.000	.000				
$N_3'N_4N_7'$	0.000	0.000	0.000	— 0.001	.000	0.000				

	L	*C*	*P*	*L′*	*C′*	*P′*	Σ	Σ′	S	D
	″	″	″	″	″	″	″	″	″	° ′ ″
$N_3'N_4'N_7$	0.000	0.000	0.000	+ 0.001	0.000	0.000	0.000	0.000	41.21978	156 21 18.5
$N_3N_5'N_6'$	— .001	.000	.000	+ .002	.000	.000				
$N_3'N_5N_6'$	— .013	.000	.000	+ .032	— .001	.000				
$N_3'N_5'N_6$	— .015	.000	.000	+ .037	— .001	.000	— .029	+ .069	24.50624	273 11 14.2
$N_3N_5'N_7'$	+ .001	.000	.000	— .003	.000	.000				
$N_3'N_5N_7'$	+ .019	.000	.000	— .046	+ .001	.000				
$N_3'N_5'N_7$	— .005	.000	.000	+ .014	.000	.000	+ .015	— .034	43.26332	193 0 58.9
$N_3N_6'N_7'$	+ .025	.000	.000	— .062	+ .001	.000				
$N_3'N_6N_7'$	+ .388	— .006	.000	— .965	+ .019	+ .002				
$N_3'N_6'N_7$	— .099	+ .002	.000	+ .245	— .005	— .001	+ .310	— .766	44.30588	112 7 31.6
$N_4N_5'N_6'$	— .091	+ .001	.000	+ .227	— .004	— .001				
$N_4'N_5N_6'$	— .112	+ .002	.000	+ .279	— .006	— .001				
$N_4'N_5'N_6$	— .130	+ .002	.000	+ .324	— .007	— .001	— .328	+ .811	7.10937	209 10 45.2
$N_4N_5'N_7'$	+ .133	— .002	.000	— .330	+ .007	+ .001				
$N_4'N_5N_7'$	+ .163	— .003	.000	— .405	+ .008	+ .001				
$N_4'N_5'N_7$	— 0.048	+ .001	.000	+ 0.119	— .002	.000	+ 0.244	— 0.601	25.86645	129 0 30.9
$N_4N_6'N_7'$	+ 2.384	— .039	+ .003	— 5.928	+ .122	+ .013				
$N_4'N_6N_7'$	+ 3.401	— .056	+ .004	— 8.455	+ .175	+ .018				
$N_4'N_6'N_7$	— 0.865	+ 0.014	— .001	+ 2.149	— 0.044	— .006	+ 4.845	— 11.956	26.90901	48 6 53.6
$N_5N_6'N_7'$	+ 73.197	— 1.199	+ .080	— 181.958	+ 3.755	+ .397				
$N_5'N_6N_7'$	+ 84.960	— 1.392	+ .093	— 211.199	+ 4.358	+ .460				
$N_5'N_6'N_7$	— 21.599	+ 0.354	—0.019	+ 53.692	— 1.108	—0.117	+ 134.475	— 331.720	28.75255	84 46 34.0

IV. Inequalities Arising from $H_4 e'^3 \cos(V-3\omega')$.

	L	*C*	*L′*	*C′*	*P′*	Σ	Σ′	S	D
	″	″	″	″	″	″	″	″	° ′ ″
$N_5'^3$	+ 0.039	0.000	— 0.097	+ 0.002	0.000	+ 0.039	— 0.095	8.11035	326 44 3.9
$N_6'^3$	+ 225.703	— 2.760	— 561.068	+ 9.290	+ 1.857	+ 222.943	— 549.921	11.23804	84 3 12.0
$N_7'^3$	— 694.800	+ 8.172	+1727.175	—27.494	— 5.495	— 686.628	+1694.186	67.50927	203 32 26.1
$N'N_5'^2$	0.000	0.000	+ 0.001	0.000	0.000	0.000	+ 0.001	10.80155	304 7 37.1
$N'N_6'^2$	— .147	+ .002	+ .366	— .006	— .001	— .145	+ .359	12.88668	142 20 22.5
$N'N_7'^2$	— .311	+ .004	+ .775	— .012	— .002	— .307	+ .761	50.40083	341 59 51.9
$N_1'N_5'^2$	+ .001	.000	— .002	.000	.000	+ .001	— .002	12.80118	229 58 23.9
$N_1'N_6'^2$	+ .220	— .003	— 0.548	+ .009	+ .002	+ .217	— 0.537	14.88631	69 11 9.3
$N_1'N_7'^2$	+ .468	— .005	— 1.164	+ .019	+ .004	+ .463	— 1.141	52.40046	267 50 38.7
$N_2'N_5'^2$	.000	.000	+ 0.001	.000	.000	.000	+ 0.001	22.79542	190 17 54.9
$N_2'N_6'^2$	— .139	+ .002	+ .345	— .006	— .001	— .137	+ .338	24.88055	28 30 40.3
$N_2'N_7'^2$	— .294	+ .004	+ .731	— .012	— .002	— .290	+ .717	62.39470	228 10 0.7
$N_3'N_5'^2$	.000	.000	— .001	.000	.000	.000	— .001	23.46368	354 4 51.5
$N_3'N_6'^2$	+ .172	— .002	— .428	+ .007	+ .001	+ .170	— .420	25.54881	192 17 36.9
$N_3'N_7'^2$	+ .364	— .004	— .906	+ .014	+ .003	+ .360	— .889	63.06296	31 57 6.3
$N_4'N_5'^2$	+ 0.005	.000	— 0.012	.000	.000	+ 0.005	— 0.012	6.06680	290 4 23.5
$N_4'N_6'^2$	+ 1.508	— .018	— 3.748	+ .062	+ .012	+ 1.490	— 3.674	8.15194	128 17 8.9
$N_4'N_7'^2$	+ 3.189	— .038	— 7.929	+ 0.128	+ .026	+ 3.151	— 7.775	45.66609	327 56 38.3
$N_5'N_6'^2$	+ 37.674	— .461	— 93.653	+ 1.552	+ .310	+ 37.213	— 91.791	10.19548	164 56 49.3
$N_5'N_7'^2$	+ 79.694	— 0.950	— 198.099	+ 3.198	+ 0.639	+ 78.744	— 194.262	47.70963	4 36 18.7
$N_6'N_7'^2$	+1432.374	—17.064	—3560.683	+57.439	+11.480	+1415.310	—3491.764	48.75219	283 42 41.4
$N_4'^2N_6'$	+ 0.003	0.000	— 0.008	0.000	0.000	+ 0.003	— 0.008	5.06583	172 31 5.8
$N_4'^2N_7'$	— 0.005	.000	+ 0.012	.000	.000	— 0.005	+ 0.012	23.82291	92 20 50.5
$N_5'^2N_6'$	+ 2.096	— .026	— 5.211	+ .087	+ .017	+ 2.070	— 5.107	9.15291	245 50 26.6
$N_5'^2N_7'$	— 3.048	+ 0.037	+ 7.577	— 0.121	— 0.025	— 3.011	+ 7.431	27.90999	165 40 11.3
$N_6'^2N_7'$	— 984.657	+11.885	+2447.720	—40.007	— 7.996	— 972.772	+2399.717	29.99512	3 52 56.7
$N'N_4'N_6'$	— 0.001	+ 0.000	+ 0.002	0.000	0.000	— 0.001	+ 0.002	9.80057	186 34 19.4
$N'N_4'N_7'$	+ .001	.000	— .002	.000	.000	+ .001	— .002	28.55765	106 24 4.1
$N'N_5'N_6'$	— .016	.000	+ .041	— .001	.000	— .016	+ .040	11.84411	223 13 59.8
$N'N_5'N_7'$	+ .024	.000	— 0.059	+ .001	.000	+ .024	— 0.058	30.60119	143 3 44.5
$N'N_6'N_7'$	+ .428	— .005	— 1.065	+ .017	+ .004	+ .423	— 1.044	31.64375	62 10 7.2
$N_1'N_4'N_6'$	+ .001	.000	— 0.002	.000	.000	+ .001	— 0.002	11.80021	112 25 6.2
$N_1'N_4'N_7'$	— .001	.000	+ .002	.000	.000	— .001	+ .002	30.55728	32 14 50.9
$N_1'N_5'N_6'$	+ 0.024	0.000	— 0.061	+ 0.001	0.000	+ 0.024	— 0.060	13.84375	149 4 46.6

	L	C	L'	C'	P	Σ	Σ'	s	D
	″	″	″	″	″	″	″	″	° ′ ″
$N_1'N_5'N_7'$	— 0.036	0.000	+ 0.089	— 0.001	0.000	— 0.036	+ 0.088	32.60082	68 54 31.3
$N_1'N_6'N_7'$	— .641	+ .008	+ 1.592	— .026	— .005	— .633	+ 1.561	33.64339	348 0 54.0
$N_2'N_4'N_6'$	— .001	.000	+ 0.001	.000	.000	— .001	+ 0.001	21.79444	72 44 37.2
$N_2'N_4'N_7'$	+ .001	.000	— .001	.000	.000	+ .001	— .001	40.55152	352 34 21.9
$N_2'N_5'N_6'$	— .015	.000	+ .038	— .001	.000	— .015	+ .037	23.83798	109 24 17.6
$N_2'N_5'N_7'$	+ .022	.000	— 0.056	+ .001	.000	+ .022	— .055	42.59506	29 14 2.3
$N_2'N_6'N_7'$	+ .404	— .005	— 1.005	+ .016	+ .004	+ .399	— .985	43.63762	308 20 25.0
$N_3'N_4'N_6'$	+ .001	.000	— 0.001	.000	.000	+ .001	— .001	22.46270	236 31 33.8
$N_3'N_4'N_7'$	— .001	.000	+ .001	.000	.000	— .001	+ .001	41.21978	156 21 18.5
$N_3'N_5'N_6'$	+ .019	.000	— .048	+ .001	.000	+ .019	— .047	24.50624	273 11 14.2
$N_3'N_5'N_7'$	— .028	.000	+ 0.069	— .001	.000	— .028	+ 0.068	43.26332	193 0 58.9
$N_3'N_6'N_7'$	— .501	+ .006	+ 1.245	— .020	— .004	— .495	+ 1.221	44.30588	112 7 31.6
$N_4'N_5'N_6'$	+ .168	— .002	— 0.417	+ .007	+ .001	+ .166	— 0.409	7.10937	209 10 45.2
$N_4'N_5'N_7'$	— 0.244	+ .003	+ 0.607	— .007	— .002	— 0.241	+ 0.598	25.86645	129 0 30.9
$N_4'N_6'N_7'$	— 4.385	+ 0.053	+ 10.901	— 0.178	— .036	— 4.332	+ 10.687	26.90901	45 6 53.6
$N_5'N_6'N_7'$	— 109.540	+ 1.323	+ 272.301	— 4.455	— 0.890	— 108.217	+ 266.956	28.75255	84 46 34.0

V. Inequalities Arising from $H_5 e\eta'^2 \cos(V-\varpi-2\tau')$.

	L	C	P	L'	C'	Σ	Σ'	s	D
	″	″	″	″	″	″	″	″	° ′ ″
$N_5 W_6^2$	—0.002	0.000	0.000	+ 0.005	0.000	—0.002	+ 0.005	8.54915	200 55 4.5
$N_6 W_6^2$	— .040	+ .001	.000	+ .099	— .004	— .039	+ .095	9.59207	120 1 27.2
$N_7 W_6^2$	— .015	.000	.000	+ .036	— .002	— .015	+ .034	28.34915	39 51 11.9
$N W_7^2$	+ .001	.000	.000	— .001	.000	+ .001	— .001	57.34346	193 36 6.3
$N_1 W_7^2$	— .002	.000	.000	+ .005	.000	— .002	+ .005	59.34309	119 26 53.1
$N_4 W_7^2$	— .012	.000	.000	+ .031	— .001	— .012	+ .030	52.60871	179 32 52.5
$N_5 W_7^2$	—0.388	+ .011	— .002	+ 0.964	— .039	—0.379	+ 0.925	54.65225	216 12 32.9
$N_6 W_7^2$	—8.088	+ .236	— .035	+20.107	— .821	—7.889	+19.286	55.69482	135 18 55.6
$N_7 W_7^2$	—2.993	+ .086	— .012	+ 7.439	— .300	—2.919	+ 7.139	74.45189	54 8 40.3
$N_6 W W_7$	—0.004	.000	.000	+ 0.008	.000	—0.004	+ 0.008	34.86934	63 49 21.2
$N_7 W W_7$	— .001	.000	.000	+ .003	.000	— .001	+ .003	53.62642	343 39 5.9
$N_6 W_1 W_7$	+ .004	.000	.000	— .010	.000	+ .004	— .010	36.35068	307 30 11.4
$N_7 W_1 W_7$	+ .001	.000	.000	— .002	.000	+ .001	— .002	55.10776	227 23 26.2
$N_6 W_2 W_7$	— .012	.000	.000	+ .029	— .001	— .012	+ .028	12.64008	144 48 7.1
$N_7 W_2 W_7$	— .004	.000	.000	+ .011	.000	— .004	+ .011	31.39715	64 37 51.8
$N_6 W_3 W_6$	.000	.000	.000	+ .001	.000	.000	+ .001	25.82790	359 30 54.5
$N_6 W_3 W_7$	— .003	.000	.000	+ .008	.000	— .003	+ .008	48.87928	187 9 38.7
$N_7 W_3 W_7$	— .001	.000	.000	+ .003	.000	— .001	+ .003	67.63635	106 59 23.4
$N_5 W_5 W_6$	.000	.000	.000	+ .001	.000	.000	+ .001	6.33135	313 42 27.2
$N_6 W_5 W_6$	— .005	.000	.000	+ .015	— .001	— .005	+ .014	7.37391	232 48 49.9
$N_7 W_5 W_6$	— .002	.000	.000	+ .006	.000	— .002	+ .006	26.13099	152 38 34.6
$N_5 W_5 W_7$	— .004	.000	.000	— .008	.000	— .004	— .008	29.38272	141 21 11.4
$N_6 W_5 W_7$	— .086	+ .002	.000	+ .215	— .009	— .084	+ .206	30.42538	60 27 34.1
$N_7 W_5 W_7$	— .032	+ .001	.000	+ .080	— .003	— .031	+ .077	49.18235	340 16 18.8
$N_4 W_6 W_7$	— .002	.000	.000	+ .004	.000	— .002	+ .004	29.55734	351 54 8.3
$N_5 W_6 W_7$	—0.054	+ .001	.000	+ .135	— .006	—0.053	+ .129	31.60088	28 33 48.7
$N_6 W_6 W_7$	—1.134	+ .032	— .006	+ 2.820	— .117	—1.108	+ 2.703	32.64345	307 30 11.4
$N_7 W_6 W_7$	—0.420	+0.012	—0.002	+ 1.042	—0.043	—0.410	+ 0.999	51.40052	226 19 56.1

VI. Inequalities Arising from $H_6 e'\eta'^2 \cos(V-\varpi'-2\tau')$.

	L	C	L'	C'	P'	Σ	Σ'	s	D
	″	″	″	″	″	″	″	″	° ′ ″
$N_5' W_6^2$	+ 0.003	0.000	— 0.008	0.000	0.000	+ 0.003	— 0.008	8.54951	200 55 4.5
$N_6' W_6^2$	+ .060	— .001	— .150	+ .005	.000	+ .059	— .145	9.59207	120 1 27.2
$N_7' W_6^2$	— .088	+ .002	+ .218	— .006	.000	— .086	+ .212	28.34915	39 51 11.9
$N' W_7^2$	— .003	.000	+ .007	.000	.000	— .003	+ .007	57.34345	193 36 6.1
$N_1' W_7^2$	+ .004	.000	— .010	.000	.000	+ .004	— .010	59.34309	119 26 52.9
$N_2' W_7^2$	— 0.003	0.000	+ 0.006	0.000	0.000	— 0.003	+ 0.006	69.33733	79 46 23.9

	L	C	L'	C'	P'	Σ	Σ′	S	D
	″	″	″	″	″	″	″	″	° ′ ″
$N_3' W_7^2$	+ 0.003	0.000	— 0.008	0.000	0.000	+ 0.003	— 0.008	70.00559	243 33 20.5
$N_4' W_7^2$	+ .027	— .001	— 0.068	+ .002	.000	+ .026	— 0.066	52.60871	179 32 52.5
$N_5' W_7^2$	+ 0.682	— .017	— 1.694	+ .049	+ .002	+ 0.665	— 1.643	54.65225	216 12 32.9
$N_6' W_7^2$	+12.248	— .304	—30.446	+0.880	+ .033	+11.944	—29.533	55.69482	135 18 55.6
$N_7' W_7^2$	—17.825	+ .436	+44.311	—1.263	— .047	—17.389	+43.001	74.45189	54 8 40.3
$N_6' W W_7$	+ 0.005	.000	— 0.012	0.000	.000	+ 0.005	— 0.012	34.86934	63 49 21.2
$N_7' W W_7$	— .008	.000	+ .018	.000	.000	— .008	+ .018	53.62642	343 39 5.9
$N_6' W_1 W_7$	— .006	.000	+ .016	.000	.000	— .006	+ .016	36.35068	307 33 41.5
$N_7' W_1 W_7$	+ .009	.000	— .023	+ .001	.000	+ .009	— .023	55.10776	227 23 26.2
$N_6' W_2 W_7$	+ .018	.000	— .044	+ .001	.000	+ .018	— .043	12.64008	144 48 7.1
$N_7' W_2 W_7$	— .026	+ .001	+ .065	— .002	.000	— .025	+ .063	31.39715	64 37 51.8
$N_6' W_3 W_6$	.000	.000	— .001	.000	.000	.000	— .001	25.82790	359 30 54.5
$N_7' W_3 W_6$	.000	.000	+ .001	.000	.000	.000	+ .001	44.58498	279 20 39.2
$N_5' W_3 W_7$	.000	.000	— .001	.000	.000	0.00	— .001	47.83671	268 3 16.0
$N_6' W_3 W_7$	+ .005	.000	— .012	.000	.000	+ .005	— .012	48.87928	187 9 38.7
$N_7' W_3 W_7$	— .007	.000	+ .018	.000	.000	— .007	+ .018	67.63635	106 59 23.4
$N_5' W_5 W_6$	+ .001	.000	— .001	.000	.000	+ .001	— .001	6.33135	313 42 27.2
$N_6' W_5 W_6$	+ .008	.000	— .023	+ .001	.000	+ .008	— .022	7.37391	232 48 49.9
$N_7' W_5 W_6$	— .014	.000	+ .033	— .001	.000	— .014	+ .032	26.13099	152 38 34.6
$N_5' W_5 W_7$	+ .007	.000	— .014	.000	.000	+ .007	— .014	29.38272	141 21 11.4
$N_6' W_5 W_7$	+ .131	— .003	— .326	+ .010	.000	+ .128	— .316	30.42538	60 27 34.1
$N_7' W_5 W_7$	— .190	+ .005	+ .473	— .014	— .001	— .185	+ .458	49.18235	340 16 18.8
$N_4' W_6 W_7$	+ .004	.000	— .010	.000	.000	+ .004	— .010	29.55734	351 54 8.3
$N_5' W_6 W_7$	+ 0.096	— .003	— 0.238	+ .007	.000	+ 0.093	— 0.231	31.60088	28 33 48.7
$N_6' W_6 W_7$	+ 1.717	— .043	— 4.270	+ .125	+ .005	+ 1.674	— 4.140	32.64345	307 30 11.4
$N_7' W_6 W_7$	— 2.498	+0.062	+ 6.210	—0.180	—0.007	— 2.436	+ 6.023	51.40052	226 19 56.1

VII. Inequalities Arising from Formula (VII).

	Q	Q'	S	D
	″	″	″	° ′ ″
$N_6 M_5 W_7$	−0.001	+0.003	−22.93326	355 37 30.9
$N_7 M_5 W_7$	.000	+ .001	− 4.17618	275 24 18.6
$N_6 M_6 W_7$	− .002	+ .004	−25.15142	108 21 56.6
$N_7 M_6 W_7$	− .001	+ .002	− 6.39434	27 11 41.3
$N_6 M_7 W_6$	.000	+ .001	−25.15142	108 21 56.6
$N_5 M_7 W_7$	.000	+ .001	−49.24536	1 36 49.7
$N_6 M_7 W_7$	− .005	+ .012	−48.20280	280 43 12.4
$N_7 M_7 W_7$	−0.002	+0.004	−29.44572	200 32 57.1

VIII. Inequalities Arising from Formula (VIII).

	Q	Q'	S	D
	″	″	″	° ′ ″
$N_6' M_5 W_7$	+0.002	−0.004	−22.93326	355 37 30.9
$N_7' M_5 W_7$	− .002	+ .006	− 4.17618	275 24 18.6
$N_6' M_6 W_7$	.000	− .006	−25.15142	108 21 56.6
$N_7' M_6 W_7$	− .004	+ .009	− 6.39434	27 11 41.3
$N_6' M_7 W_6$	+ .003	− .001	−25.15142	108 21 56.6
$N_7' M_7 W_6$	− .001	+ .002	− 6.39434	27 11 41.3
$N_5' M_7 W_7$	.000	− .001	−49.24536	1 36 49.7
$N_6' M_7 W_7$	+ .007	− .017	−48.20280	280 43 12.4
$N_7' M_7 W_7$	−0.010	+0.025	−29.44572	200 32 57.1

IX. Summation of Preceding Inequalities.

	Σ	Σ′	S	D		Σ	Σ′	S	D
	″	″	″	° ′ ″		″	″	″	° ′ ″
N_7^3	−1175.337	+2900.043	+67.50927	203 32 26.1	$N_5^2 N_7$	— 0.385	+ 0.960	+27.90999	165 40 11.3
$N_6 N_7^2$	+ 464.751	−1146.607	48.75219	283 42 41.4	$N_2 N_7^2$	— 0.379	+ 0.937	62.39470	228 10 9.7
$N_6^2 N_7$	— 53.888	+ 132.952	29.99512	3 52 56.7	$N_5 W_7^2$	+ 0.286	— 0.718	54.65225	216 12 32.9
$N_5 N_7^2$	+ 37.882	— 93.451	47.70963	4 36 18.7	$N_1 N_7^2$	+ 0.274	— 0.675	52.40046	267 50 38.7
$N_7 W_7^2$	— 20.308	+ 50.140	74.45189	54 8 40.3	$N_7 W_5 W_7$	— 0.216	+ 0.535	49.18235	340 16 18.8
$N_5 N_6 N_7$	— 9.254	+ 22.839	28.75225	84 46 34.0	$N N_7^2$	— 0.147	+ 0.364	50.40083	341 59 51.9
$N_6 W_7^2$	+ 4.055	— 10.247	55.69482	135 18 55.6	$N_3 N_6 N_7$	— 0.128	+ 0.315	44.30588	112 7 31.6
$N_7 W_6 W_7$	— 2.846	+ 7.022	51.40052	226 19 56.1	$N_7 W_6^2$	— 0.101	+ 0.246	28.34915	39 51 11.9
$N_4 N_7^2$	+ 2.074	— 5.115	45.66609	327 56 38.3	$N_2 N_6 N_7$	+ 0.099	— 0.244	43.63762	308 20 25.0
N_6^3	+ 1.275	— 3.142	11.23804	84 3 12.0	$N_1 N_6 N_7$	— 0.068	+ 0.166	33.64339	348 0 54.0
$N_6 W_6 W_7$	+ 0.566	— 1.437	32.64345	307 30 11.4	$N_6 W_5 W_7$	+ 0.044	— 0.110	30.42538	60 27 34.1
$N_4 N_6 N_7$	— 0.523	+ 1.283	26.90901	48 6 53.6	$N_4 N_5 N_7$	— 0.044	+ 0.112	25.86645	129 0 30.9
$N_3 N_7^2$	+ 0.489	— 1.208	63.06295	31 57 6.3	$N_5 W_6 W_7$	+ 0.040	— 0.102	31.60088	28 33 48.7
$N_5 N_6^2$	+ 0.419	— 1.027	+10.19548	164 56 49.3	$N_5^2 N_6$	+ 0.036	— 0.088	+ 9.15291	245 50 26.6

	Σ	Σ′	S	D		Σ	Σ′	S	D
	″	″	″	° ′ ″		″	″	″	° ′ ″
NN_6N_7	+0.035	−0.089	+31.64375	62 10 7.2	NN_5N_6	+0.002	+0.001	+11.84411	223 13 59.8
$N_7W_2W_7$	−0.029	+0.074	31.39715	64 37 51.8	$N_1N_5N_6$	+0.002	−0.002	13.84375	149 4 46.6
$N_4N_6^2$	+0.023	−0.068	8.15194	128 17 8.9	$N_3N_5N_6$	+0.002	−0.004	24.50624	273 11 14.2
$N_6W_6^2$	+0.020	−0.050	9.59207	120 1 27.2	$N_1W_7^2$	+0.002	−0.005	59.34309	119 26 53.1
$N_7W_5W_6$	−0.016	+0.038	26.13099	152 38 34.6	$N_6W_3W_7$	+0.002	−0.004	48.87928	187 9 38.7
$N_4W_7^2$	+0.014	−0.036	+52.60871	179 32 52.5	$N_4W_6W_7$	+0.002	−0.006	29.55734	351 54 8.3
$N_7M_7W_7$	−0.012	+0.029	−29.44572	200 32 57.1	$N_1N_6^2$	+0.002	−0.011	+14.88631	69 11 9.3
$N_7W_1W_7$	+0.010	−0.025	+55.10776	227 23 26.2	$N_7M_5W_7$	−0.002	+0.007	− 4.17618	275 24 18.6
$N_3N_5N_7$	−0.010	+0.026	43.26332	193 0 58.9	NN_6^2	−0.002	+0.008	+12.88668	142 20 22.5
N_7WW_7	−0.009	+0.021	53.62642	343 39 5.9	NW_7^2	−0.002	+0.006	57.34346	193 36 6.3
$N_7W_3W_7$	−0.008	+0.021	67.63635	106 59 23.4	$N_6W_1W_7$	−0.002	+0.006	36.35069	307 33 41.5
$N_3N_6^2$	+0.007	−0.019	25.54881	192 17 36.9	$N_1N_4N_6$	−0.002	0.000	11.80021	112 25 6.2
$N_2N_5N_7$	+0.007	−0.021	42.59506	29 14 2.3	$N_3N_5^2$	+0.001	+0.001	23.46368	354 4 51.5
$N_6W_2W_7$	+0.006	−0.015	12.64008	144 48 7.1	$N_2N_5^2$	+0.001	0.000	22.79542	190 17 54.9
$N_1N_5N_7$	−0.006	+0.015	+32.60082	68 54 31.3	$N_4^2N_6$	+0.001	0.000	5.06583	172 31 5.8
$N_7M_6W_7$	−0.005	+0.013	− 6.39434	27 11 4.3	$N_1N_4N_7$	+0.001	0.000	30.55728	32 14 50.9
$N_2N_6^2$	−0.005	+0.015	+24.88055	28 30 40.3	$N_2N_4N_7$	+0.001	0.000	40.55152	352 34 21.9
NN_5N_7	−0.005	+0.013	30.60119	143 3 44.5	$N_5W_6^2$	+0.001	−0.003	8.54951	200 55 4.5
$N_4N_5N_6$	−0.005	+0.010	7.10937	209 10 45.2	$N_3N_4N_6$	+0.001	−0.001	22.46270	236 31 33.8
$N_6W_5W_6$	+0.003	−0.008	7.37391	232 48 49.9	N_6WW_7	+0.001	−0.004	34.86934	63 49 21.2
$N_3W_7^2$	+0.003	−0.008	70.00559	243 33 20.5	$N_5W_5W_6$	+0.001	0.000	6.33135	313 42 27.2
$N_5W_5W_7$	+0.003	−0.006	29.38272	141 21 11.4	$N_1N_5^2$	+0.001	−0.001	12.80118	229 58 23.9
$N_2W_7^2$	−0.003	+0.006	+69.33733	79 46 23.9	$N_2N_5N_6$	−0.001	+0.001	23.83798	109 24 17.6
$N_6M_7W_7$	+0.002	−0.005	−48.20280	280 43 12.4	$N_4^2N_7$	−0.001	+0.003	23.82291	92 20 50.5
$N_6M_7W_6$	+0.002	−0.003	−25.15142	108 21 56.6	$N_2N_4N_6$	−0.001	−0.001	21.79444	72 44 37.2
N_5^3	+0.002	−0.002	+ 8.11035	326 44 3.9	$N_3N_4N_7$	−0.001	+0.001	+41.21978	156 21 18.5

X. Δ and Δ′. $\delta l = 1160''.36 \sin(V'+270°\ 12'\ 19''.9)$. $\delta l' = -2862''.14 \sin(V'+270°\ 11'\ 54''.8)$. $V' = 1399''.77167\,t$
Epoch 1850.

t	Δ	Δ′	t	Δ	Δ′	t	Δ	Δ′
	″	″		″	″		″	″
−2500	−102.68	+253.25	−50	−1.00	+2.46	+ 60	− 0.71	+ 1.75
−2000	− 80.45	+198.41	−40	−0.69	+1.70	+ 70	− 1.03	+ 2.55
−1500	+ 51.45	−126.06	−30	−0.43	+1.08	+ 80	− 1.40	+ 3.46
−1000	− 25.80	+ 63.43	−20	−0.23	+0.57	+ 90	− 1.82	+ 4.48
− 500	+ 7.80	− 18.99	−10	−0.07	+0.17	+ 100	− 2.27	+ 5.62
− 100	− 3.18	+ 7.86	+10	+0.01	−0.02	+ 500	+ 3.88	− 9.58
− 90	− 2.68	+ 6.60	+20	0.00	0.00	+1000	−19.04	+ 46.98
− 80	− 2.20	+ 5.42	+30	−0.08	+0.22	+1500	+41.62	−102.68
− 70	− 1.76	+ 4.33	+40	−0.24	+0.59	+2000	+73.05	−180.35
− 60	− 1.35	+ 3.35	+50	−0.44	+1.10	+2500	+98.21	−242.31

Leander McCormick Observatory, University of Virginia, 1895 *May* 1.