Exploring the Use of STEM-Related Hands-On Instructional Activities to Support Students' Understanding of Algebra and Geometry

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by

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ABSTRACT

Through a series of three manuscripts, this dissertation explores how STEM-related hands-on instructional activities can be use to support students' understanding of algebra and geometry. For the first manuscript, fifth-grade students participated in a digital-fabrication augmented surface area unit in which they fabricated their own rectangular prisms using computer-aided design software. Digital fabrication provided students with the opportunity to develop a conceptual understanding of surface area, as well as two problem solving strategies (*Keeping Track* and *Seeing What's Not Visible*). For the second and third manuscripts, students experimentally derived Ampere's Law, which relates three independent variables (number of wraps of wire, length, and electric current) to a single dependent variable (magnetic field strength generated by a solenoid). Rising eighth-grade students participated in a four-part version of the *Deriving Ampere's Law* activity in which they developed intermediate models analyzing each independent variable separately before developing the final model. Pre-service mathematics teachers and pre-service science teachers participated in a holistic version of the *Deriving Ampere's Law* activity in which they were asked to only derive the final model. Both the rising eighth-grade students and the pre-service teachers were able to successfully derive Ampere's Law and their participation in the activity revealed different modeling strategies and applications of prior knowledge.

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APPROVAL OF THE DISSERTATION

This dissertation, "Exploring the Use of STEM-Related Hands-On Instructional Activities to Support Students' Understanding of Algebra and Geometry," has been approved by the Graduate Faculty of the Curry School of Education in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Date

TABLE OF CONTENTS

DEDICATION

I dedicate this work to my parents, Chunsim and Douglas Corum. Your unconditional love and support have made all of this possible. Thanks for always being in my corner.

엄마! 다했다! 아자 아자 파이팅! ~ 엄마 아빠 많이 사랑해요.

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First and foremost, I would like to acknowledge my program advisor and committee chair, Joe Garofalo. As a student in your secondary mathematics methods course in the fall of 2007, I could have never imagined that my journey as a teacher would have brought me here. I am so grateful for all that you have done to help me achieve my greatest accomplishment to date.

I would also like to thank my committee members, Robert Berry, Glen Bull, and Gregory Fairchild. Your expertise, guidance, and encouragement have meant so much.

I owe a great deal of gratitude to Nigel Standish, who always found the time to help me design, print, and wrap solenoids.

And finally, I would like to thank my friends and family. To Kim, thank you for inspiring me to always make sure that my work makes a difference. To Anna, Chris, and Danielle, thank you for making New York feel like home and for helping me become the person I am today. To Will, thank you for everything.

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

Exploring the Use of STEM-Related Hands-On Instructional Activities to Support Students' Understanding of Algebra and Geometry

In their review of the literature regarding the educational effectiveness of animations and simulations, Plass, Homer, and Hayward (2009) suggest eleven principles, classified as either visual design principles or interaction design principles, to improve the design of dynamic visualizations to support student learning. While these principles were suggested specific to multimedia learning, their applications can be extended to the design and integration of manipulatives in the teaching and learning of mathematics, specifically the principles of task appropriateness (alignment between the use of manipulatives and the specified learning goals), learner control of pacing, and manipulation of content. This dissertation explores the effectiveness of using hands-on instructional activities to support students' understanding of mathematics. The instructional activities presented in these three manuscripts are aligned with the above principles and each of the papers discusses the learning of mathematics in a STEM-related (science, technology, engineering, mathematics) environment where students manipulated artifacts to develop deeper understandings of algebra and geometry.

The first paper explored the implementation of a digital fabrication-augmented unit to support fifth-grade students' understanding of surface area. Digital fabrication is defined as "the process of translating a digital design developed on a computer into a physical object" (Berry et al., 2010, p. 168). Students regularly used multiple technologies (i.e., computer-aided design software, die cutters) to fabricate prisms. Having access to both virtual and physical manipulatives not only supported students' development of a conceptual understanding of surface area, but it also facilitated students' development of two important problem solving strategies when solving surface area tasks (*Keeping Track* and *Seeing What's Not Visible*).

The second and third papers explored students' strategies when developing a mathematical model relating three independent variables to a single dependent variable. Students were asked to experimentally derive Ampere's Law, which relates the magnetic field strength generated by a solenoid to the number of wraps of wire, solenoid length, and electric current. The number of wraps of wire and electric current is directly related to magnetic field strength, while the solenoid length is inversely related to magnetic field strength. The solenoid is an integral component in a number of historic and modern-day inventions and technologies (e.g., speaker, telephone, electric guitar pickups).

There are two versions of the *Deriving Ampere's Law* activity. For the second paper, rising eighth-grade students completed a four-part version of the activity where they were asked to develop models relating each independent variable to the dependent variable prior to developing the final model. These students were able to apply their prior knowledge from their algebra coursework to develop their models. For the third paper, pre-service mathematics teachers and pre-service science teachers completed a more holistic version of the activity where the only direction given was to develop the final model. While there were similarities in the pre-service teachers' modeling strategies, their difficulties with the activity were influenced by their beliefs about the nature of science and mathematics in schools.

Designing Hands-On Instructional Activities

All three papers involved the use of manipulatives to support students' understanding of mathematics. In the first paper, students had access to physical and virtual manipulatives through the design and construction of rectangular prisms using computer-aided design software and die cutters. In the second and third papers, students were provided with a set of solenoids, a variable power supply, and a magnetic field sensor. While a solenoid does not necessarily align with the traditional definition of mathematical manipulatives, students needed to measure the different parameters of the solenoids in order to complete the *Deriving Ampere's Law* activity. Instead of using traditional measurement tools (e.g., ruler, protractor), students used a magnetic field sensor to measure the strength of the different magnetic fields. The use of manipulatives was *task appropriate* because they were purposefully selected to support students' learning of mathematics concepts (i.e., surface area, direct/inverse variation, mathematical modeling). The structure of the instructional activities allowed for *learner control of pacing* and *manipulation of content* because the students were given control of the use of the manipulatives (e.g., constructing their own prisms, deciding how to manage data collection) and the manipulatives provided them with the opportunity to interact with the content (e.g., varying parameters when fabricating prisms, manipulating independent variables to identify relationships).

The instructional activities developed for this dissertation are also consistent with the findings in National Research Council's *How People Learn* (Donovan, Bransford, &

Pellegrino, 1999). These findings recommend three practices to support and facilitate student learning. First, students' prior knowledge and preconceptions need to be engaged. Second, students need to develop both a foundation of factual knowledge and a conceptual framework to understand how facts and ideas are related within a given subject area and have the ability to organize their knowledge so that it can be retrieved and applied. Third, students need to be provided with opportunities to develop the necessary metacognitive strategies to allow them to define their own learning goals and monitor their own progress. These insights about how students learn should be used to inform teaching practices. The three instructional activities developed for this dissertation attended to students' prior knowledge (e.g., relating two-dimensional area measurement to surface area, recognizing slope and linearity when analyzing data, applying their understanding of direct and inverse variation when developing complex mathematical models) and how this knowledge could be applied to solve new problems. The instructional activities also required students to regularly monitor their own progress.

Learning Mathematics in a STEM-Related Environment

The National Council of Teachers of Mathematics (NCTM) has advocated in favor of the importance of helping students make connections between mathematics and other disciplines. In its *Principles and Standards for School Mathematics* (2002), NCTM recommends "school mathematics experiences at all levels should include opportunities to learn about mathematics by working on problems arising in contexts outside of mathematics" (pp. 65-66). And while there are applications of mathematics in a variety of subject areas, the NCTM Connections Standard specifies the value of connecting mathematics and science since "the link between mathematics and science is not only through content but also through process" (p. 66). The scientific process "can inspire an approach to solving problems that applies to the study of mathematics" (p. 66). NCTM (2002) also recommends six principles for school mathematics; one of these six principles is the Technology Principle. According to this principle, "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (p. 24). Exploring how science, engineering, and technology tasks can be utilized to support students' understanding of mathematics closely aligns with the recommendations outlined by NCTM.

The three papers in this dissertation provide examples of learning mathematics in a STEM-related environment. In the first paper, the primary goal was to support students' understanding of surface area. The secondary goal of the unit was to introduce students to manufacturing and design. Throughout the unit, students used several technologies (e.g., computer-aided design software, die cutters) that are often used in engineering design and manufacturing. In the second and third papers, students participated in a truly integrated STEM activity. The *Deriving Ampere's Law* activity incorporates both science (electromagnetism) and mathematics (direct and inverse variation, modeling) content. In order to measure the magnetic field strength of the different solenoids, students use a variety of technologies (variable power supply, sensors, data collection software). The final model they develop can then be used when designing and manufacturing solenoids to achieve a specified goal (engineering).

While the benefit of using manipulatives to support students' understanding of mathematics has been well documented, this dissertation provides empirical evidence of how to utilize Plass et al.'s (2009) principles for effective design of visualizations when developing hands-on instructional activities to support students' learning of algebra and geometry. This dissertation also provides examples of how STEM-related activities can be used to support mathematics instruction. In their *Principles to Action* (2014), the National Council of Teachers of Mathematics outlined several challenges to improve mathematics achievement for all students in the United States. One of these challenges is to "increase the number of high school graduates, especially those from traditionally underrepresented groups, who are interested in, and prepared for, STEM careers" (p. 2). And while the use of STEM as an acronym implies an integration of its four component content areas, the nature of K-12 classrooms makes this integration nearly impossible. The instructional activities presented in this dissertation provide examples of how mathematics instruction can be situated within a STEM-related environment.

CHAPTER 2

Learning about Surface Area through a Digital Fabrication-Augmented Unit[1]

Surface area and volume are consistently identified as curriculum standards for K-12 students by both national and state education boards. The National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (NCTM, 2000) specifies that students beginning in upper elementary and extending through 12th grade should understand surface area and volume. The Common Core State Standards (2012) indicate that students should not only be able to use formulas to compute surface area and volume measurements, but should also be able to explain the derivation of the formulas. State boards expect students to calculate the surface area and volume for regular polyhedral (e.g. Virginia Department of Education, 2010; NYS Board of Regents, 2005). In particular, the Virginia Standards of Learning lists surface area and volume as grade level standards for 5th grade, 8th grade, and geometry. Even though surface area and volume regularly appear in content standards and also on national and international assessments, there continues to be a lack of empirical research on the learning of these topics. The few relevant studies that exist indicate that students struggle with related topics (see below). The growing use of digital fabrication, defined as "the process of translating a digital design developed on a computer into a physical object," (Berry et al.,

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[1] Corum, K., & Garofalo, J. (2016) Learning about surface area through a digital fabrication augmented unit. *Journal of Computers in Mathematics and Science Teaching*, *35*, 33-59.

2010, p. 168), in classrooms raises the question of whether or not this technology can be used to improve students' understanding of surface area and volume.

Relevant Literature

While there is not much research focusing on students' understanding of surface area and volume, there is research on students' performance on perimeter and area tasks. After analyzing National Assessment of Educational Progress (NAEP) results from 1990 to 2000, Strutchens, Martin, and Kenny (2003) concluded that length measurement, perimeter, and area continue to be areas of difficulty, arguing that "students have basic knowledge of measuring length, but that their knowledge tends to be superficial" (p. 197). The 2013 NAEP results further indicate students' difficulty with measurement tasks. There were four separate tasks on the 4^{th} grade exam where students had to measure length, and the percentage correct on these tasks ranged from 12% to 38%. While 64% of 4^{th} graders tested were able to determine the perimeter of a rectangle, only 23% were able to correctly identify which rectangle had the greatest area. Even though 81% of 8^{th} grade students correctly measured the length of a line segment, many struggled with area; less than half of the students were able to determine the area of a rectangle given one side length and the perimeter of the figure (National Center for Educational Statistics, 2013). Strutchens et al. (2003) warn that prematurely introducing students to area formulas can hinder their conceptual understandings. This warning extends to measuring surface area and volume. An overemphasis of formulas can result in conceptual misunderstandings, such as using formulas when they are inappropriate or inefficient, or believing that irregular plane figures do not have area because there is not a corresponding formula (Zacharos, 2006).

One approach, long advocated by many, to helping students develop conceptual understanding of mathematics includes the use of manipulatives. Sowell's (1989) meta-analysis of the effectiveness of manipulatives in mathematics instruction across grade levels revealed that the incorporation of manipulatives significantly improved student achievement if there was long-term (at least one full school year) use of manipulatives. In a subsequent meta-analysis of the effect of mathematics instruction using concrete manipulatives, Carbonneau, Marley, and Selig (2012) found that overall, "using manipulatives in mathematics instruction produces a small- to medium-sized effect on student learning when compared with instruction that uses abstract symbols alone" (p. 396). They also found that concrete manipulatives had a larger effect on students who were coded as "concrete operational" or students between ages 7-11. However, these two meta-analyses are inconsistent with each other in regard to the impact of the length of instruction with concrete manipulatives on achievement, and more importantly, they do not identify the conditions under which instruction using manipulatives is most effective.

Unfortunately, there is not a research base on the effectiveness of concrete manipulatives on students' understanding of surface area. However, Hwang and Hu (2013) examined the effectiveness of virtual manipulatives on 5^{th} grade students learning of surface area and volume. Virtual manipulatives are defined as "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer, Bolyard, & Spikell, 2002, p. 373). Students in the experimental group had access to a virtual platform that allowed them to construct and manipulate virtual solids, and collaborate with their peers. Students in the control group were provided traditional pen and paper materials, where three-dimensional solids were represented in two dimensions. The experimental group performed significantly better on the posttest than the control group. Hwang and Hu found that "Interaction among multiple representations, including manipulation of the 3-D shapes and literacy on the white board, encourages students to interpret mathematical meanings from different viewpoints" (p. 317). Their findings indicate that the virtual "hands-on" learning experience combined with the opportunity for peer collaboration had the greatest effect (Hwang & Hu, 2013).

The positive effect of students working with multiple representations is consistent with several learning frameworks, such as the notion of concept image (Tall & Vinner, 1981), and Mayer's (2009) theory of multimedia learning. A student's *concept image* refers to "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). Mental pictures include "any kind of representation-picture, symbolic form, diagram, graph, etc." (Vinner & Dreyfus, 1989, p. 356). Mayer's (2009) theory specifies that learning involves building connections among pieces of verbal knowledge to create a coherent verbal model, and building connections among elements of pictorial knowledge to create a coherent pictorial model. A crucial step "involves a change from having two separate representations – a pictorial model and a verbal model – to having an integrated model in which corresponding elements and relations from one model are mapped onto the other" (2009, p. 74). Working with manipulatives can enhance and connect the mental representations students construct and thus can lead to improved concept images.

The above literature suggests that there may be a benefit to using either concrete or virtual manipulatives when teaching surface area. Digital fabrication provides students with *both* virtual and concrete manipulatives, by building concrete manipulatives from their virtual representations. Personal computers can be transformed into personal fabrication systems through the use of 3D design software programs along with die cutters or 3D printers, making digital fabrication accessible in many classrooms (Berry et al., 2010, Bull & Garofalo, 2009).

The curriculum unit on which this paper is based was designed to provide students with opportunities to develop conceptual understandings of surface area and volume, using digital fabrication, prior to being introduced to formulas. The specific question we explore in this paper is: How did participation in a digital fabrication-augmented surface area unit affect 5th grade students' ability to solve surface area tasks?

Methodology

This effort was undertaken to gain insight into the effectiveness of digital fabrication-augmented units addressing surface area and volume. Fifth-grade students used *FabLab ModelMaker* (Aspex Software) and *Silhouette* die cutters to create three-dimensional cubes and rectangular prisms from cardstock, which were then used during classroom instruction to help students develop conceptual understanding of surface area and volume to aid in solving non-contextual and contextual tasks. Students were assessed with a project-made pretest and a posttest consisting of open-ended tasks. The posttest was given at the conclusion of both units (see the results section for specific examples). This paper focuses on students' performance on the surface area tasks.

School Setting

The units were taught in an elementary school located in central Virginia which serves about 325 students. The school is 20 miles outside of a state university and as a result, the geographic area can be categorized as partially suburban/partially rural. As of the 2012-2013 school year, the student body was 86% White, 4% African American, 4% Hispanic, 1% Asian, and 5% Multi-Racial, with approximately 20% of the students receiving free or reduced lunch.

The school received an overall pass rate of 90% on the 2011-2012 Virginia Standards of Learning (SOL) exams, which was adjusted to account for transfer students and ELLs at the school, meaning the school is fully accredited. On the Math SOL exams, the pass rates were 58%, 65%, and 74%, for grades 3, 4, and 5 respectively, with less than 10% of students in grades 3, 4, and 5 passing at an advanced level.

Students

Students in two separate 5th grade mathematics classes, both taught by the same teacher, participated in the digital fabrication-augmented surface unit during the 2011-2012 school year. Both classes were taught in the morning. In terms of academic achievement, there was a slight difference between the two groups; one class was classified by the school as students of average ability, while the other class had a mix of students of average and slightly above average ability. Students who were classified as high ability were placed with a gifted resource teacher. As a result, there were 14 students in the average ability class and 16 students in the average/above average class.

Digital Fabrication-Augmented Unit

The surface area unit consisted of five 50-minute class periods over the course of a month. Due to disruptions from daily SOL review sessions and school assemblies, as well as several issues with the digital fabrication technology, classroom instruction during these five periods actually ranged from 30 minutes to 50 minutes. There was also a class period that was used to introduce volume prior to the completion of the surface area activities. A premade sequence of lessons loosely guided classroom instruction, which included teacher-led discussions, teacher-led demonstrations of the digital fabrication software, and a series of student hands-on tasks, including the use of the digital fabrication software to construct cubes and rectangular prisms. Both sections followed the same instructional sequence and the specific instructional activities that occurred on each day of the treatment are outlined below.

Day 1 (October 19th). Students were introduced to surface area by looking at plastic three-dimensional shapes that were provided by the classroom teacher. The teacher opened the classroom discussion by asking students about the number of faces, edges, and vertices of each shape and what information they would need to know to determine how to "cover" the shapes. There was no discussion of square units or a formal discussion of area, but students recalled what they learned in fourth grade about area and perimeter measurement and some students suggested (length x width) or (length + width).

Day 2 (November 2nd). The teacher demonstrated both the ModelMaker software and the Silhouette die cutter. ModelMaker was used to create a printable net (see Figure 1) and the die cutter was used to score the printed nets. Students used the technology to create their own 1-inch cardstock cubes. Prior to working with ModelMaker, the teacher reminded students of the purpose by asking, "What are we trying to figure out?" to which students responded "perimeter," "height," "area," and "surface area." The majority of the period was dedicated to the use of the software and hardware. At the end of the class period, the teacher collected the students' 1-inch cubes.

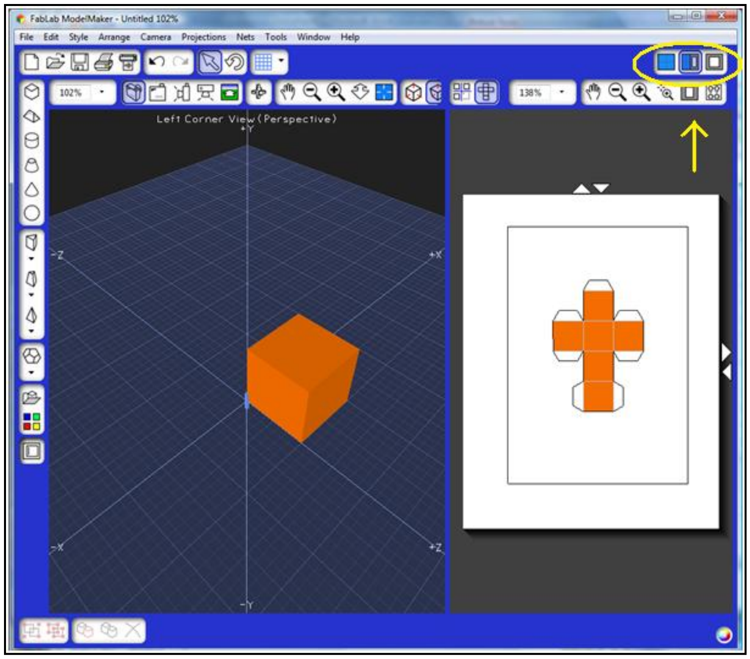


Figure 1. Screenshot of creating a cube using the ModelMaker software.

Day 3 (November 3rd). The teacher asked students, “What is surface area? How would you define it for a fourth grader?” Student responses varied, depending on whether or not they had their cubes (see the vignette under Findings for details). Following this discussion in both classes, the teacher provided the students with the formal definition of surface area: “the sum of the areas of all the faces of a solid.” At the end of the period, the teacher also demonstrated how to use ModelMaker to find the perimeter and area of a 1-inch cube, which are defined as parameters in the software.

Day 4 (November 4th). To explore the relationship between a two-dimensional net and a three-dimensional shape, the teacher utilized the ModelMaker software to guide students to identify the relationship between their nets and folded cubes. Using the software, the teacher was able to rotate the cube to show students the visible and non-visible faces and edges, as well as fold and unfold the net. The teacher asked students, “If we need to find the surface area of the cube, could we use the net to do that?” and one student responded that you “just need the faces.” The teacher asked students to begin thinking about formulas for surface area. Some students hypothesized different formulas, such as “six times the edges,” “1 + 1 + 1 + 1 + 1 + 1,” and “six times the area of each face.” These hypothesized formulas were an application of the definition of surface area that the students had created the day before. Given the unique properties of a 1-inch cube, the teacher asked students to construct a 1-inch by 2-inches by 3-inches rectangular prism using the digital fabrication technology. Once students had physical models, the teacher asked students again to think of a formula for surface area. This influenced some students to ask about the surface areas of other solids, such as cones, pyramids, and spheres.

Day 5 (November 18th). The teacher reviewed the definition of surface area. Using ModelMaker, the teacher projected a 1-inch by 2-inches by 4-inches prism and its corresponding net and asked a student to pick a face (2-in. by 4-in.) and color for that face (red). She had students calculate the area of that face. A student commented that the opposite face ‘is the same thing, the same area…do it again or times two…16” and the teacher colored it red. This process was repeated for the other faces, with opposite faces being colored the same as each other (i.e., red, blue, green). As the teacher and the students continued coloring the remaining faces, the teacher suggested keeping a running total of face areas on the side of the figure, which was a strategy she modeled to help students keep track of their work. Following this review of surface area, students attempted to solve a series of surface area tasks from the premade lesson sequence, such as finding the surface area of irregular solids. Note that between Days 4 and 5 the teacher introduced students to volume, asking students what is volume and providing students with the formula for volume.

Data Collection

Prior to the unit, students were given a project-designed pretest that not only aligned with the Virginia SOL content standards for surface area and volume, but also extended beyond them. All of the tasks on the pretest were open-ended and included tasks on calculating area, surface area, volume, identifying the number of faces, edges, and vertices on both a cube and an L-shaped prism, and identifying a rectangular prism based on its net. During the unit, field notes were recorded daily by the second author. At the conclusion of the unit, students were assessed with a posttest that included the same or similar tasks as the pretest as well as two multiple-choice questions taken from a prior Virginia 7th Grade Mathematics SOL exam (to provide some comparison information). The posttest was administered at the conclusion of both the surface area and volume units; thus students were not tested immediately following the surface area portion. We were only concerned with performance on the individual tasks and did not calculate any overall test scores.

Data Analysis

Our primary intent was to see how participation in the digital fabrication-augmented unit affected students' strategies. For this paper, four tasks were selected to analyze. The three open-ended surface area tasks were asked on both the pretest and the posttest and included contextual and non-contextual tasks; an SOL item was asked only on the posttest to give some comparison to state-wide student performance.

First we analyzed student performance on the open-ended tasks, using the rubric below, to get an overall sense of change in student performance. The open-ended tasks were scored independently by two doctoral students in mathematics education, using a weighted six-point rubric that prioritized the conceptual knowledge required for solving surface area tasks. One of those students is the first author. Neither scorer was part of the data collection and instruction. Initial inter-scorer agreement was 94%, but after discussing their scores, the two scorers came to agreement on all point allocations.

Table 1

Rubric for Scoring Surface Area Tasks

Category	Point Value	Description
Recognition	3	Recognized the need to find the area of the six faces in order to determine the surface area.
Set Up	1	Set up appropriate calculations needed to correctly determine the surface area.
Computations	1	Correctly carried appropriate computations.
Units	1	Used appropriate units.

More importantly, we analyzed student written work on each of these tasks to identify strategies that they may have used in order to solve them. The first author, with the rubric in mind, made notes on each student's solution to each task. Then she revisited her notes to find common strategies and solution methods. After that, the second author reviewed students' work and the first author's notes and strategy interpretations. For the most part, it was not difficult for the two authors to reach the same interpretations, largely because of the clearness of the student work. However, there was one solution not immediately interpretable, but both authors ultimately reached a consensus on an interpretation. Unfortunately, there was one solution that neither author could fully interpret.

Findings

Of the 30 students who participated in the unit, there were 28 students who completed both the pretest and the posttest. Their pretest and posttest performance on the four surface area tasks is described below, followed by a vignette from Day 3 of the instruction which gives a glimpse of how students responded to having three-dimensional prisms.

Task 1: Calculating Surface Area

For this task (see Figure 2), students were given a diagram of a rectangular prism and were asked to find the surface area:

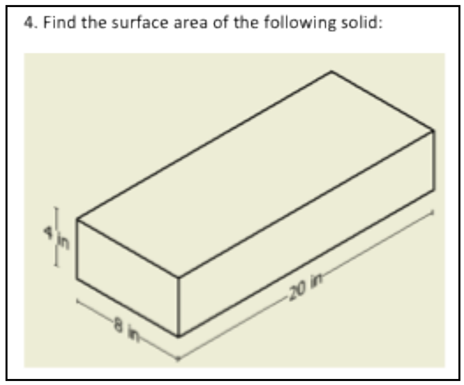


Figure 2. Non-contextual surface area task from pretest/posttest assessment.

Scores. Not one of the students in either class received a full score for this task on the pretest. When considering the individual components of the task, none of the students recognized the need to calculate the area of the six faces or which dimensions were required to calculate surface area on the pretest. The mean score for this task on the pretest was 0.04 (out of 6 possible points) and ten of the students left the problem blank. In contrast, on the posttest, 21 students recognized the need to determine the areas of the six faces and of these, 15 students received a full score and four just left out units. The mean score on this problem was 4.3 points and none of the students left the problem blank. Table 2 summarizes the scores on this task.

Table 2

Distribution of Students' Scores on Task 1

	0 – 1 Points	4 – 4.5 Points	5 – 6 Points
Pretest	28	0	0
Posttest	7	2	19

Strategies. There were three common *incorrect* strategies students used for this task on the pretest. Of the 18 students who attempted to complete the task on the pretest, five calculated the volume and two calculated the area for only one of the faces. The third common incorrect strategy was to double, quadruple, or square the dimensions and then find the sum. Figure 3 is an example of this incorrect strategy, where Isaac approached the task by computing $2W + 2L + 4H$.

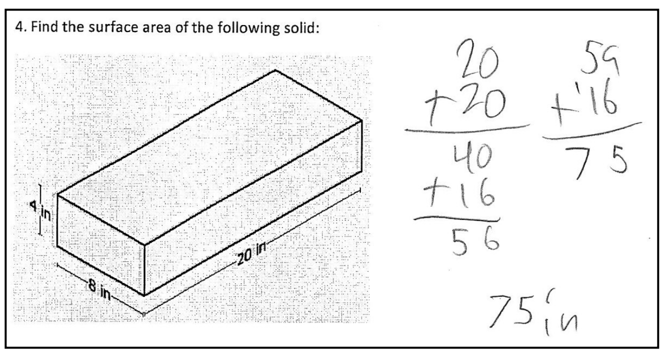


Figure 3. Isaac's pretest solution for Task 1.

A total of nine students attempted some variation of this procedure. It appears that many students tried to synthesize formulas for area and perimeter in an attempt to calculate surface area.

Student performance on this task was substantially better on the posttest. Of the 19 students who successfully completed the task on the posttest (received a score of 5 or 6), there were two common *correct* strategies. The first common strategy was computing the sum of the areas of all six faces (visible and not visible). Twelve students employed this first strategy and kept track of their work by either labeling the faces on the diagram or listing the areas of each face. Isaac, whose pretest solution was presented in Figure 3, solved this task on the posttest using a listing strategy (Figure 4). The second common strategy was doubling the areas of the visible faces. Six students employed this strategy, an example of which can be seen in Figure 5. The *incorrect* strategies from the pretest appeared again, with five students manipulating the dimensions prior to finding the sum and two students calculating volume.

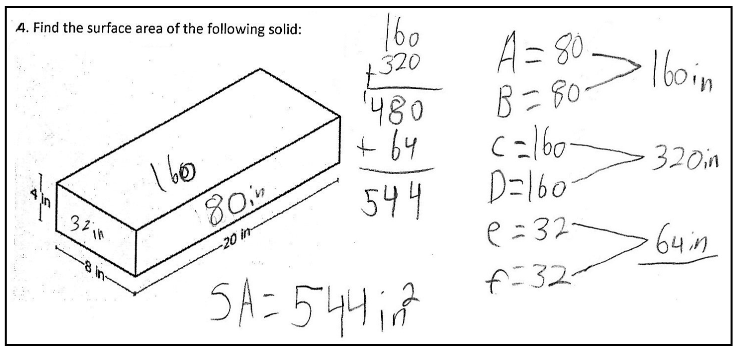


Figure 4. Isaac's posttest solution for Task 1 (listing areas of each face).

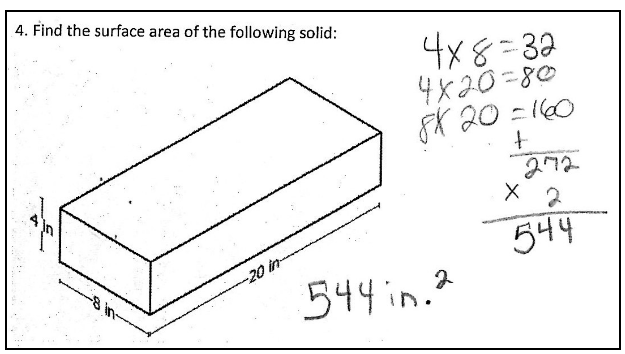


Figure 5. Evelyn's posttest solution for Task 1 (doubling areas of each face).

Task 2: Wrapping a Box

For this contextualized task (Figure 6) students were provided with a diagram of a rectangular prism but instead of being asked to find the surface area, they were asked to find the minimum amount of wrapping paper needed to wrap a pencil box.

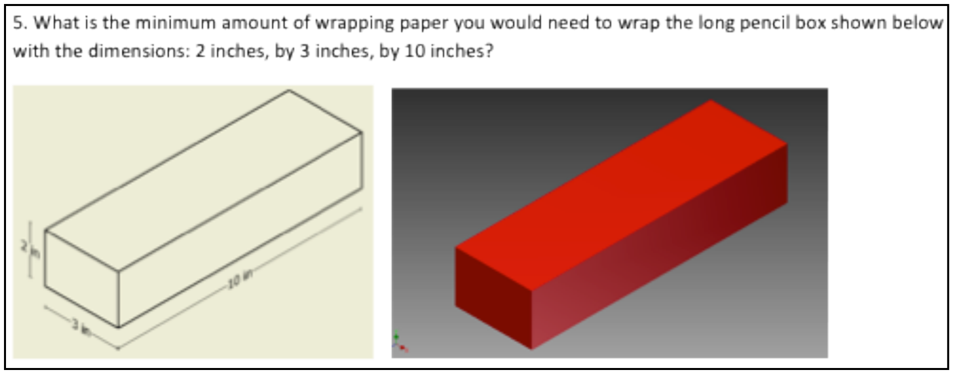


Figure 6. Contextual surface area task from pretest/posttest assessment (wrapping a box).

Scores. On the pretest, students performed worse on this task than on the previous surface area task. Similar to the first task, not one of the students received a full score and none of the students recognized the need to calculate the area of the six faces or which dimensions were required to calculate surface area. The mean score was 0 and twelve students left the problem blank. Students again made substantial gains when assessed with this task on the posttest. Twenty of the students recognized the need to determine the area of the six faces. Of these 20 students, 12 students received a full score and seven either left out units or made a minor computation error. The mean score was 4.2 points (out of 6) and none of the students left the problem blank. Table 3 summarizes student scores on this task.

Table 3

Distribution of Students' Scores on Task 2

	0 – 1 Points	4.75 Points	5 – 6 Points
Pretest	28	0	0
Posttest	8	1	19

Strategies. There were two common *incorrect* strategies for this task on the pretest. Of those 16 students who attempted a solution on the pretest, seven calculated volume. Another common incorrect strategy for the wrapping task on the pretest was doubling or quadrupling the dimensions and then finding their sum, similar to the strategy used for the surface area task on the pretest. Nine students attempted this strategy, including Evan, whose solution is presented in Figure 7. Evan's solution shows that he attempted to compute the minimum amount of wrapping paper needed by calculating $2L + 2W + 4H$. Again, it appears Evan's solution is an attempt to apply the formula for perimeter of a rectangle to calculate surface area.

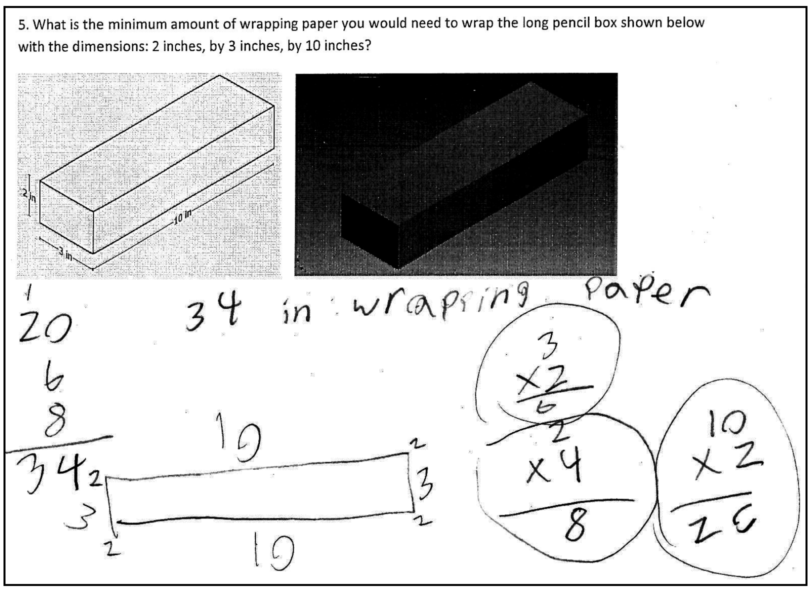


Figure 7. Evan's pretest solution for Task 2.

Students were much more successful with completing this task on the posttest. The common *correct* strategies for this task on the posttest were similar to the strategies used on the posttest for Task 1 – students kept track of their work by either listing the areas of all six faces (Figure 8) or doubling the area of the three visible faces prior to computing the sum (Figure 9). These strategies that were used for both the surface area task (Task 1) and the wrapping task (Task 2) were similar to the coloring strategy that the students explored using the ModelMaker software, which allowed them to keep track of which faces they had already computed the areas of and allowed them to see faces in pairs by assigning corresponding faces the same color.

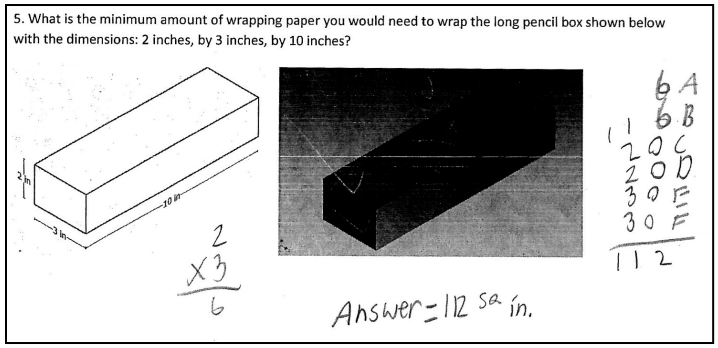


Figure 8. Evan's posttest solution for Task 2 (listing areas of each face).

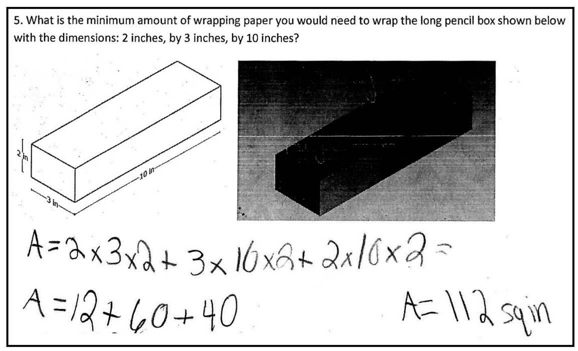


Figure 9. Leah's posttest solution for Task 2 (doubling areas of visible faces).

Six students employed the common *incorrect* strategies from the pretest on the posttest – three students manipulated the dimensions prior to finding the sum and the other three calculated volume. Of the remaining three students who incorrectly completed this task on the posttest, one student's solution was indiscernible, another correctly found the sum of the areas of the six faces, but used the wrong dimensions, and the third student halved the sum of the visible faces.

Task 3: Wrapping Stacked Boxes

For the third surface area task, students were asked to determine the minimum amount of wrapping paper needed to wrap two stacked pencil boxes (Figure 10). The pencil boxes in this task had the same dimensions as the pencil box in Task 2.

7. What is the minimum amount of wrapping paper you would need to wrap 2 long pencil boxes, if one is stacked on top of the other, like below, and then wrapped together?

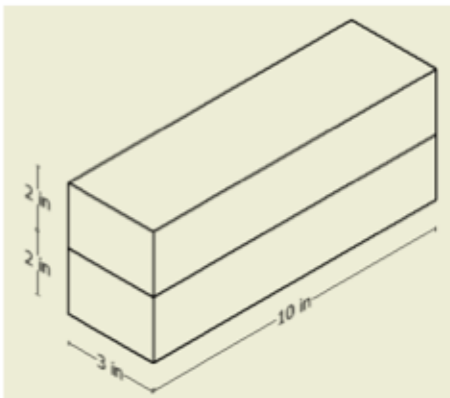


Figure 10. Contextual surface area task from pretest/posttest assessment (wrapping stacked boxes).

Scores. As with the first wrapping task, not one of the students received a full score on the pretest and none of the students recognized the need to calculate the area of the six faces or which dimensions were required to calculate surface area. The mean score was 0 and ten students left the problem blank. On the posttest, 17 students recognized the need to determine the area of the six faces, with six of these students receiving a full score and ten either left out units or made a minor computation error. The mean score was 3.6 points (out of 6) and none of the students left the problem blank. A summary of student scores on this task is presented in Table 4.

Table 4

Distribution of Students' Scores on Task 3

	0 – 1 Points	4.75 Points	5 – 6 Points
Pretest	28	0	0
Posttest	11	1	16

Strategies. There were two common *incorrect* strategies for this task on the pretest. Of the 18 students who attempted a solution to this task on the pretest, seven calculated volume. The other common incorrect strategy was doubling or quadrupling the dimensions and then finding the sum. An example of this strategy is shown in Figure 11. Bridget attempted to solve this task on the pretest by computing $4L + 4W + 4H$. Nine students, including Bridget, employed some format of this incorrect strategy on the pretest. Slightly more than half of the students who did not leave the problem blank recognized that the height of the stacked boxes was 4 inches.

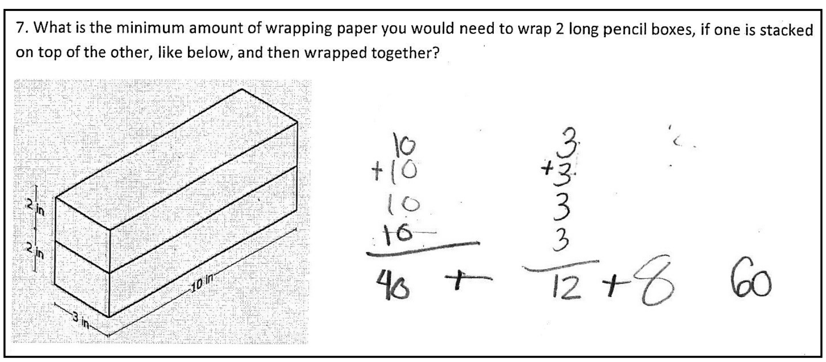


Figure 11. Bridget's pretest solution for Task 3.

Again, students were much more successful with completing this task on the posttest. Similar to the first two surface area tasks, there were two common *correct* strategies – listing the areas of all six faces (Figure 12) or doubling the area of the three visible faces prior to computing the sum (Figure 13). Bridget's posttest solution (Figure 12) is an example of listing the areas of each individual face as a means of keeping track, whereas Ellie's solution (Figure 13) is an example of thinking of the faces in pairs as a means of keeping track.

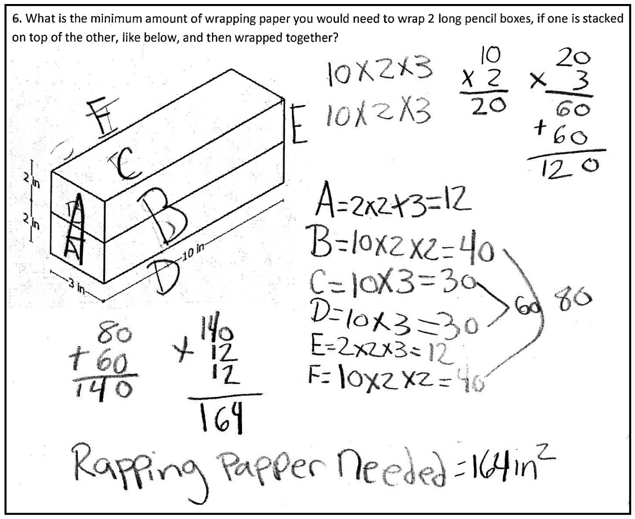


Figure 12. Bridget's posttest solution for Task 3 (listing areas of each face).

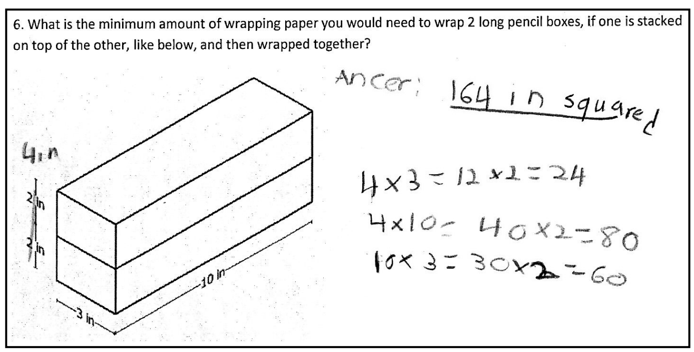


Figure 13. Ellie's posttest solution for Task 3 (doubling areas of visible faces).

Four students attempted to solve this task by using their solutions from the first wrapping task. Based on her numerical values, it appears that Lily doubled the amount of wrapping paper she calculated for Task 2 (Figure 14) and then subtracted 60 square inches of wrapping paper to account for the bottom of the first box and the top of the second box since those would not require wrapping paper (Figure 15). While her solutions were incorrect, Lily's work demonstrates her understanding of the relationship between Task 2 and Task 3.

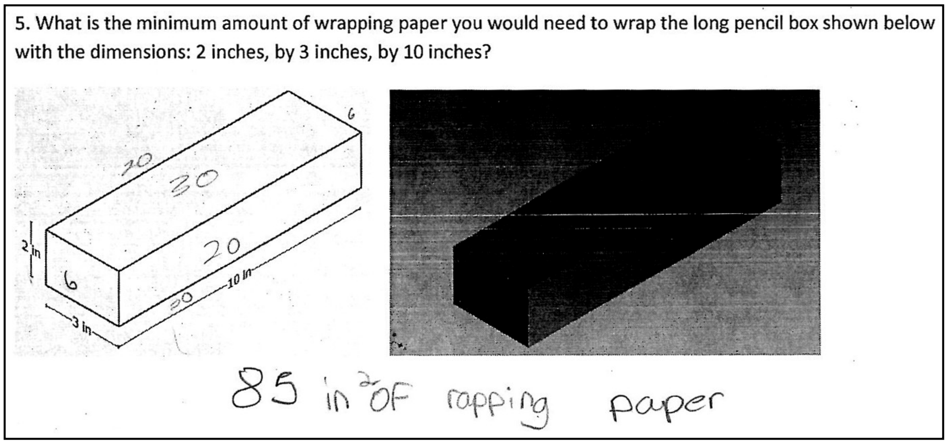


Figure 14. Lily's posttest solution for Task 2.

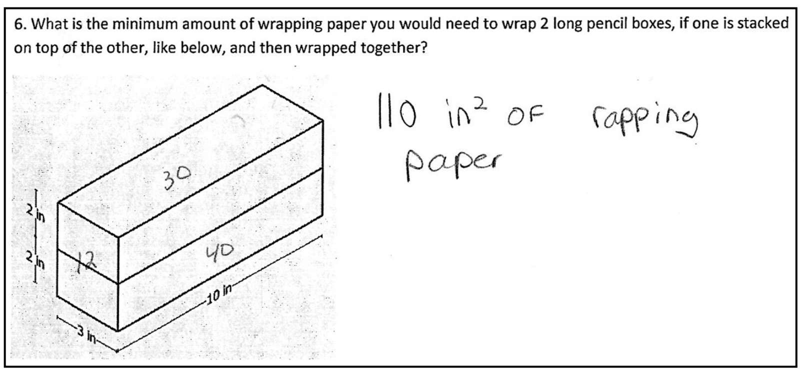


Figure 15. Lily's posttest solution for Task 3.

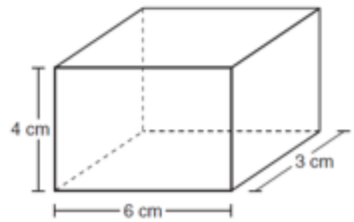
Of the students who did not successfully complete this task on the posttest, four students manipulated the dimensions prior to finding the sum and one student calculated volume, both of which were common *incorrect* strategies from the pretest. Three students doubled their solution from Task 2 and two students' solution was indiscernible. Similar to the posttest solutions for Task 2, one student correctly found the sum of the areas of the six faces, but used the wrong dimensions and another student halved the sum of the visible faces.

Task 4: 7th Grade Virginia SOL Item

The posttest included a paper-covering task (Figure 16) that was taken from the Virginia Department of Education Standards of Learning Grade 7 Mathematics

Examination. Students were asked to determine the minimum amount of paper needed to wrap a rectangular prism.

22 **Chelsea wants to cover a rectangular prism-shaped box with paper. Which is closest to the minimum amount of paper Chelsea needs?**



F 26 cm²
G 54 cm²
H 72 cm²
J 108 cm²

Figure 16. Contextual surface area task from 7th Grade Standards of Learning task (wrapping a rectangular prism).

This task was a multiple-choice question and included distracters, such as the volume of the rectangular prism (72 cm^3) or the sum of the areas of the three unique faces (54 cm^2).

Scores. Eighteen students correctly chose J as their answer on the posttest, while seven students chose H, which is the volume of the rectangular prism. Only two students left the problem blank. This 64% success rate for these 5th graders compares favorably to the 53% statewide success rate for 7th graders, who were prepped for the test.

Strategies. Students employed the same common strategies in correctly solving this task as they did with the other surface area tasks. Students' track keeping strategies included listing the areas of all six faces (Figure 17) or doubling the area of the three visible faces (Figure 18).

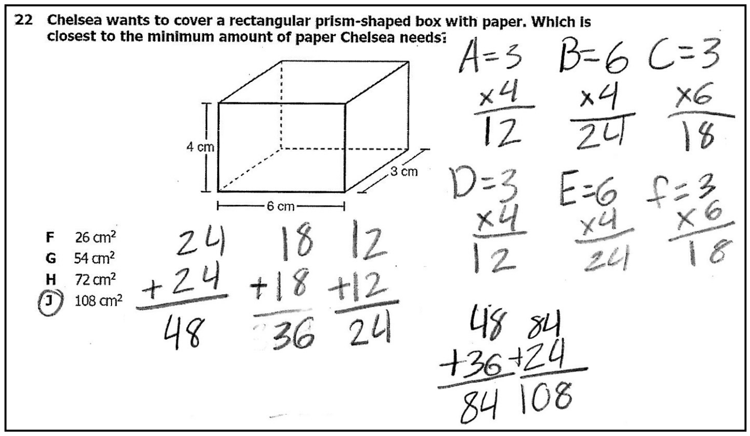


Figure 17. Caroline's solution for the 7th Grade SOL task (listing areas of each face).

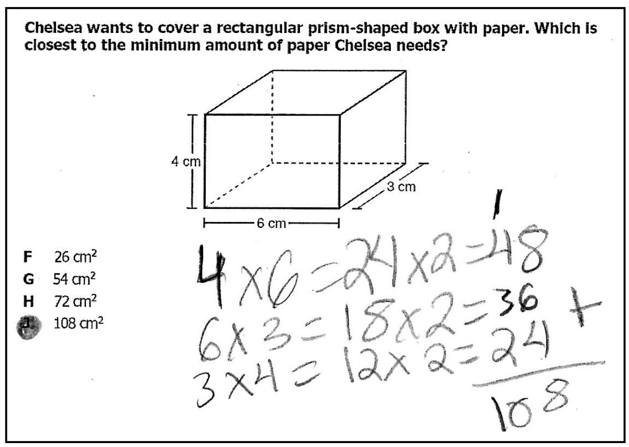


Figure 18. Alaina's solution for the 7th Grade SOL task (doubling areas of visible faces).

Vignette: Discussions during Day 3

The pretest-posttest differences in strategies for the tasks above show that students developed an understanding of surface area over the course of the unit. The vignette below (as documented in the field notes) illustrates the usefulness of students exploring physical prisms. Recall that on Day 3, the teacher asked students, "What is surface area? How would you define it for a fourth grader?" This question was initially posed to the first class *without* providing the students with the 1-inch cubes they had constructed prior. Student responses to these questions included, "length," "amount of how many squares," "area of the surface of something," and "area of the face of the cube." At this point, the teacher distributed the 1-inch cubes the students made the previous day and *instructed them to play with their cubes and think about how they would show someone surface area*. After manipulating their cubes, students provided accurate explanations of surface area, describing the process of calculating surface area by referencing the sum of the areas of the faces. Some student definitions of surface area after having access to their cubes were, "one of the faces…you can add or times it by how many faces…add it all up," and "find the area and add all the faces together." Prior to holding their manipulatives, the students struggled with defining surface area; their initial confusion was immediately resolved once the teacher provided students with their cubes.

The sequence of events varied for the second class that same day; the teacher began by first distributing the students' cubes and giving students time to play with their cubes before asking them how they would define surface area to a fourth-grader. Without hesitation, students began describing surface area in terms of the sum of the areas of each face. Student responses included, "add the area of each face, get the dimensions of it," and "the added up area of every face of the cube." Unlike the first class, students in the second class initially had access to their cubes, and thus were able to immediately describe a process for finding surface area.

Near the end of the second class that day, a student posed another more challenging task; "a hard problem would be to find the surface area of a cone… You have to find the area of the bottom and with the vertex, it's hard to know how many edges." With access to physical models and the opportunity to explore properties of cubes students were encouraged to hypothesize about properties of different three-dimensional figures.

Discussion

Students showed marked improvements in their ability to complete surface area tasks following participation in the digital fabrication-augmented unit. Recall that none of the 28 students were able to correctly solve any of the three open-ended tasks on the pretest whereas 19 students, 19 students, and 16 students earned full or nearly full credit on Task 1, Task 2, and Task 3, respectively, on the posttest. Additionally, there was a 64% success rate on the SOL task, compared to the 53% success rate of 7th grade students who were prepped for the exam. The unit provided students with opportunities to develop effective strategies that allowed them to recognize qualities of three-dimensional figures that cannot be seen in a two-dimensional representation ("Seeing What's Not Visible") and to effectively carry out multi-step processes ("Keeping Track"). These strategies, discussed below, enabled students to be more successful with completing surface area tasks on the posttest.

Seeing What's Not Visible

Finding surface area when viewing two-dimensional representations of three-dimensional figures requires students to visualize faces that are not observable in a diagram. Students are only able to see the top face, front face, and a side face of a rectangular prism. In order to determine the surface area of a rectangular prism, students must be able to recognize that there is a corresponding face to each of the visible faces. The strategies students used when successfully completing the four surface area tasks on the posttest show that they were aware of the faces that were not visible from the diagram. By listing the areas they calculated for each of the six faces, students were thinking about all of the faces (both visible and not visible), or when listing the areas of the three visible faces and then doubling them, students were first addressing what can be seen in the diagram and then are accounting for the faces that were not visible.

There were three activities in the surface area unit that gave students experiences with non-visible faces. First, the teacher and students used the software to rotate two-dimensional representations of prisms to make the initially invisible faces visible. Second, the software showed two-dimensional representations of solids, along with their corresponding nets, which displayed all of the faces (see Figure 1). These two activities were carried out as part of the fabrication process. Finally, students physically explored their three-dimensional fabricated prisms daily. The noted difference in the students' ability to define and discuss surface area based on their interaction with their 1-inch cubes (as described in the analytic vignette) further highlights the benefit of the three-dimensional fabricated prisms.

The consideration of all faces shows that students had developed appropriate mental models for these prisms; their concept images for cubes and prisms included the invisible faces. As Vinner and Dreyfus (1989) argue, "the student's image is a result of his or her experience with examples and nonexamples of the concept" (p. 356).

Keeping Track

Those successful students who were able to consider invisible faces when solving surface area tasks also used strategies to keep track of their work. These strategies included *listing* the areas of each of the six faces (e.g., Figure 8) or listing the areas of the three visible faces (e.g., Figure 5), *labeling* faces with letters to make sure each was accounted for (e.g., Figure 12), and *annotating* faces with calculated areas (e.g., Figure 4).

Some aspects of the digital fabrication software may have facilitated students' keeping track strategies. Students used the ModelMaker software to rotate representations and color opposite faces as they computed the areas of each face. The teacher also encouraged students to keep track when she was demonstrating use of the software. Students also kept track of faces when holding physical prisms. Some students used their fingers as calipers to hold and count opposite faces in pairs, and some labeled counted faces with a mark, letter, or area value. These experiences allowed students to develop strategies for *keeping track*, which were apparent in students' solutions for the four surface area tasks on the posttest.

Keeping track of one's work is an important aspect of mathematical problem solving. Shoenfeld (1992) points out that such self-regulation and metacognitive actions facilitate problem solving. Students were able to implement their "*seeing what's not visible*" ability by *"keeping track"* of their work.

Transferring *Seeing What's Not Visible* and *Keeping Track* to More Complex Tasks

Students were able to use both their "*seeing what's not visible"* and "*keeping track"* strategies beyond simple prisms, providing further evidence that students developed meaningful strategies that they could apply to unfamiliar tasks. As part of the unit, students explored the number of faces, edges, and vertices of different prisms. While many students were able to determine the number of faces, edges, and vertices of a cube on both the pretest and posttest, *all* of the students struggled with completing the L-shaped prism task below (Figure 19) on the pretest (only counting what they could see), but many showed improvement on the posttest. Even though students did not work with L-shaped prisms during the surface area and volume units, some were still able to apply their "*seeing what's not visible*" and "*keeping track*" strategies in order to determine the number of faces, edges, and vertices of this prism.

One "*seeing what's not visible*" strategy that some students used was to draw the edges and vertices that were not visible in the diagram to help them count (see Figure 19). One "*keeping track*" strategy that students used to count the number of faces, edges, and vertices of the L-shaped prism was to number the faces or use hash marks to help aid their counting (see Figure 20). Both of these strategies appear to be extensions of the track keeping strategies students used when coloring the faces of a rectangular prism using the ModelMaker software.

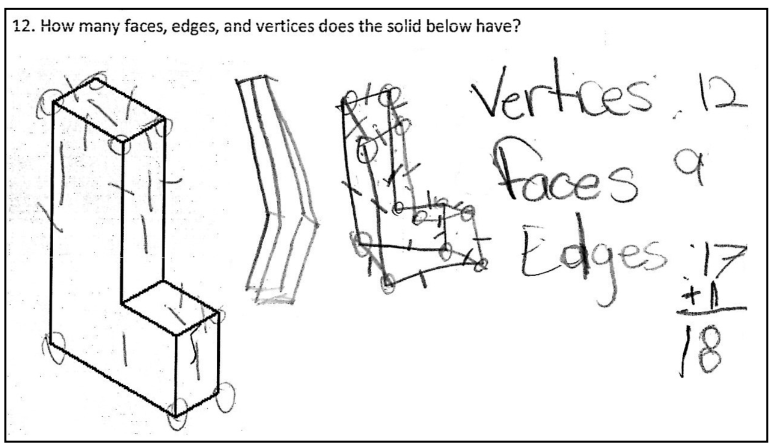


Figure 19. Natalie's method of "*seeing what's not visible*" when counting the faces, edges, and vertices of an L-shaped prism.

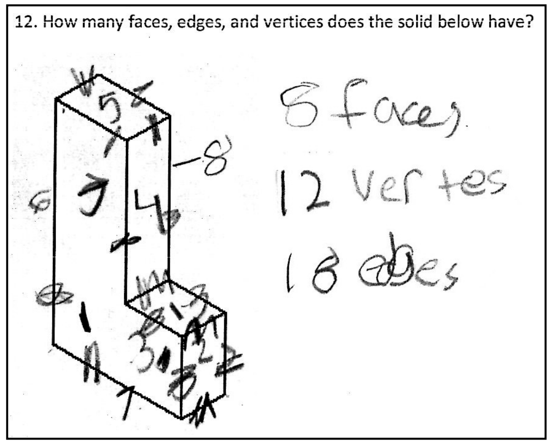


Figure 20. Noah's method of "*keeping track*" when counting the faces, edges, and vertices of an L-shaped prism.

Reducing the "Volume Error" on Surface Area Tasks

There were some students who calculated the volume instead of surface area on either the pretest or posttest. This "volume error" could be attributed to either problem solving errors and/or conceptual errors. On the pretest, five students calculated the volume of the rectangular prisms for all three tasks. There were also two other students who calculated volume for Tasks 2 and 3. Of these seven students who committed a volume error on the pretest, five of them subsequently correctly solved all four tasks on the posttest. Another student correctly solved Task 1 (non-contextual), but incorrectly solved Tasks 2, 3, and 4 (contextual), without committing a volume error. The remaining student, who committed the volume error on every pretest task, only made a volume error on the multiple choice task on the posttest. Overall, the fabrication unit activities may have helped these seven students eliminate most or all volume errors.

On the posttest, 24 of the 28 students did not commit a volume error on any of the first three tasks. However, seven students chose the volume of the rectangular prism as their solution to the multiple-choice SOL task (Task 4). These results suggest that for some students, volume errors on the posttest may have been problem solving errors (e.g., reading), but for several students volume errors may have been conceptual.

Limitations

There are several limitations to this exploration. Data collection occurred in two classes taught by the same teacher within the same school and the data collected was limited to student work on project-developed tests. No student interviews were conducted, making it difficult to fully understand the specific features of the unit activities that led to improvement and the reasons why some misconceptions remained for those students who were unable to successfully complete the surface area tasks on the posttest. Additionally, there was no control group, thereby making it impossible to compare the effectiveness of unit to other types of instruction on the topic. However, even with these limitations, there is ample suggestive evidence of the growth that resulted from participation in the unit.

Conclusion

Students were introduced to surface area through the use of physical models that they created using digital fabrication software. It was during the third day of instruction that the definition of surface area, as the sum of the areas of all of the faces of a prism, was informally verbalized by students after exploring their prisms, and subsequently formalized by the teacher. On the fourth day students began calculating the surface areas of simple prisms, by adding the areas of all faces of a prism or doubling those of opposite faces then adding the results. Students did not ask for a formula. This understanding of surface area was applied to more complex prisms on the last day.

Despite the limitations listed above, the amount of growth students exhibited at the end of the digital fabrication-augmented unit is very promising. These results support and extend the findings reported in the research cited earlier (e.g. Carbonneau, Marley, and Selig, 2012; Hwang and Hu, 2013) that manipulation of three-dimensional shapes can lead to improved students' performance on surface area tasks. Teachers incorporating concrete and virtual manipulatives into their instruction can facilitate their students' development of conceptual understanding and problem solving strategies that can be applied to both non-routine tasks and traditional assessments.

CHAPTER 3

Middle School Students' Mathematical Modeling with Three Independent Variables: The Derivation of Ampere's Law

Algebra has consistently been identified as a gatekeeper subject because of the role success in algebra has on students' ability to graduate from high school, their readiness for college-level mathematics, and their opportunities for employment (Loveless, 2013; Rech & Harrington, 2000). The importance of algebra has led to efforts to reform mathematics curriculum in schools across all grade levels so that algebraic reasoning is encouraged beginning in early elementary grades (e.g., NCTM, 2000; National Governors Association Center for Best Practices, 2010).

In addition to developing students' algebraic reasoning early, greater emphasis has been placed on the importance of mathematical modeling. The National Mathematics Advisory Panel (2008) identified "fitting simple mathematical models to data" (p. 16) as one of the major topics of school algebra. Mathematical modeling can be found in algebra curricula standards at both the national and local levels. For example, the National Council of Teachers of Mathematics (2000) specifies the importance of mathematical modeling in the Algebra Strand of its *Principles and Standards for School Mathematics*: "One of the most powerful uses of mathematics is the mathematical modeling of phenomena. Students at all levels should have opportunities to model a wide variety of phenomena mathematically in ways that are appropriate to their level," (p. 39).

Similarly, one of the algebra standards in the Virginia Mathematics Standards of Learning (Virginia Department of Education, 2009) is that "given a real-world context, [the student] will analyze a relation to determine whether a direct or inverse variation exists, and represent a direct variation algebraically and graphically and an inverse variation algebraically" (p. 25).

Relevant Literature

Mathematical modeling is the process of representing real-world situations using mathematics as a way to understand and solve a specified problem (Daher & Shahbari, 2015; Bliss & Libertini, 2016) and the model itself is the mathematical description of the real-world situation (Lesh & Lehrer, 2003). This definition is deceptively simple since mathematical modeling requires one to be able to move fluidly between the real world and the mathematized world. When creating a mathematical model, one needs to interpret the real-world problem, decide how it should be mathematized, and determine what information in the real-world problem is relevant to the model and which mathematical techniques are relevant (Crouch & Hanes, 2004).

Mathematical proficiency is required in order to develop mathematical models. According to the National Research Council (2001), mathematical proficiency is comprised of five interwoven strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Students who have mastery of the five strands of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) are able to identify connections within their existing mathematical knowledge base and discern how their mathematical knowledge can be utilized to solve problems (National

Research Council, 2001). In order for students to develop mathematical proficiency, these strands cannot be addressed in isolation.

However, national and international assessments consistently document that students' mathematical proficiency is limited to procedural fluency. The results of the 2012 Programme for International Student Assessment (PISA) indicated that only 8.8% of students in the United States tested at Level 5 or higher, which is the proficiency level at which students should be able to develop and engage with mathematical models (OECD, 2013). Moreover, student performance on both the PISA and the National Assessment of Educational Progress (NAEP) showed that students' understanding of mathematics was limited. Only 33% of those students tested on the 2015 NAEP Exam scored at or above the *Proficient* level, where *Proficient* is defined as the ability to apply conceptual understanding and procedural fluency to solve complex problems across the five mathematics content areas (Kena et al., 2016). In fact, 29% of the eighth grade students tested scored below the *Basic* level, meaning these students lacked even partial mastery of the prerequisite knowledge and fundamental skills required of grade-level proficiency. The 2012 PISA results tell a similar tale. One in four students in the United States scored below the baseline for mathematics literacy (Level 2), indicating difficulty with applying basic mathematical procedures, interpreting situations that only require direct inference, and interpreting mathematical results (OECD, 2013).

Given the disconnect between the emphasis on modeling in algebra curriculum standards and many students' lack of mathematical proficiency, it is important to understand how students utilize their prior mathematics knowledge and which strategies they use when developing algebraic models to represent real-world phenomena.

Modeling Cycle

Mathematical modeling is a cyclical process (see Figure 1). As mathematical models are developed, they are tested and revised and the initial real-world problem itself is revisited and reinterpreted as the model is amended (Delice & Kertil, 2015; Bliss & Libertini, 2016).

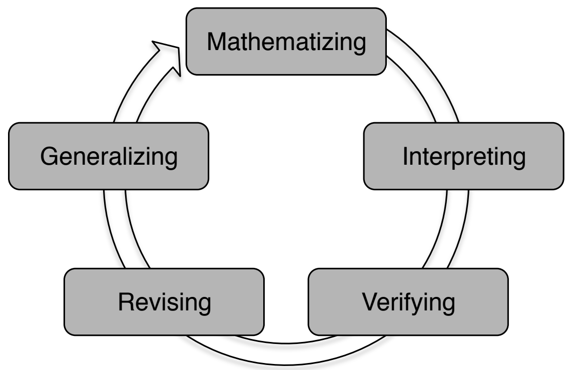


Figure 1. Mathematical modeling cycle (Delice & Kertil, 2015).

Kaiser and Sriraman (2006) proposed a way of classifying types of mathematical modeling based on the central aims of each different approach: realistic, contextual, educational, socio-critical, epistemological, and cognitive. Of these different modeling perspectives, contextual modeling, or solving subject-specific word problems, has historically been the most common in traditional classroom environments. However, the current push for the use of modeling as a way to solve real-world problems and to encourage cross-disciplinary studies is most closely aligned with realistic modeling, which emphasizes understanding the world through modeling (Kaiser & Sriraman, 2006).

Students' Difficulties

The National Mathematics Advisory Panel (2008) found that many middle and high school students are underprepared to study algebra, noting that lack of basic arithmetic fluency and difficulty with the structural nature of algebra are two of the most common hindrances to students' success. The transition to algebra in the early secondary grades is particularly difficult for students "because it introduces more abstract representations and more complex relationships between quantities" (Booth et al., 2014, p. 10). The middle school grades in particular are a critical time to prepare students for studying algebra, as this is the time when they are transitioning from concrete to abstract representations of mathematics (Bush & Karp, 2013).

Making sense of mathematical symbols has often been cited as a major area of difficulty for students (e.g., Koedinger & Nathan, 2004; Kaput, Blanton, & Moreno, 2008; Alibali, Stephens, Brown, Kao, & Nathan, 2014; Wagner, 1993). These symbols include mathematical operations (e.g., +, –, =) and the presence of letters in mathematics. Understanding how the use of mathematical symbols varies in algebra as compared to arithmetic can be especially demanding for students as they transition to more abstract representations (Koedinger & Nathan, 2004). Having facility with mathematical symbols requires students to be able to *look at* and *look through* symbols (Kaput et al., 2008). Looking at symbols is recognizing the symbols themselves, while looking through symbols is connecting the symbols with the mathematical concepts they are meant to represent.

Conceptual gaps in students' understanding of algebraic symbols could also hinder their ability to interpret equations and translate situations into equations (Alibali et al., 2014). In the classic "Students and Professors" problem, students are asked to write an equation that represents the statement, "There are six times as many students as professors at this university" (Clement, Lochhead, & Monk, 1981, p. 288). This was posed to a group of 150 calculus-level students and of the 37% of students who missed the problem, the most common error was to write the equation $6S = P$. Instead of thinking about the variables S and P as representing the number of students and the number of professors as quantities, S and P represented students and professors as concrete objects (Rosnick, 1981). The symbols for mathematical operations (i.e., $6S$, =) were also misapplied in this incorrect translation. A lack of facility with mathematical symbols makes modeling even more challenging for students and can lead to literal translations rather than mathematical models.

Model-Eliciting Activities

Model-eliciting activity (MEA) is "a problem solving activity constructed using specific principles of instructional design in which students make sense of meaningful situations, and invent, extend, and refine their own mathematical constructs" (Kaiser & Sriraman, 2006, p. 306). Students are expected to develop a realistic model that describes a real-world situation (Delice & Kertil, 2015). The purpose of MEAs is the modeling process itself rather than the application of known procedures to produce a final solution. As a result, MEAs can accomplish several goals. By emphasizing the modeling process rather than applying known procedures, MEAs encourage students to think mathematically and provide students with the opportunity to showcase their mathematical understanding and capabilities (Daher & Shahbari, 2015). MEAs also provide students with multiple entry points to a problem because they encourage authentic problem solving. Problem solving is defined as "engaging in a task for which the solution method is not known in advance" (National Council of Teachers of Mathematics, 2000, p. 52). Because there is not a prescribed procedure for MEAs, mathematical modeling tasks are open-ended and the final models themselves can vary (Bliss & Libertini, 2016).

When meaningfully incorporated into classroom instruction, MEAs can support students' ability to transition between abstract representations in algebra and applications of algebraic reasoning to real-world problems. The model-eliciting activity on which this study is based was designed to provide students with the opportunity to develop a mathematical model that related three independent variables to a single dependent variable. The specific dependent variable in this context is magnetic field strength and the three independent variables are those connected with the attributes of a *solenoid* (further explained below). The research questions we explore in this paper are: (1) What conceptual and procedural knowledge did students who had already taken algebra utilize when developing mathematical models involving both direct and inverse variation? (2) What problem solving strategies do these students use when participating in model-eliciting activities?

Methodology

The realistic model-eliciting activity used in this study, *Deriving Ampere's Law*, involved solenoids and magnetic fields. A solenoid is a coil of conductive wire; when electric current flows through the wire, the coil generates a magnetic field (Figure 2).

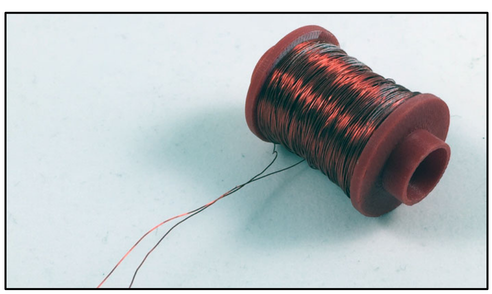


Figure 2. Example of a solenoid

The strength of the magnetic field produced by a solenoid (*B*) is dependent on the number of wraps of wire that comprise the solenoid (*N*), the length of the solenoid (*L*), and the electrical current (I). This relationship is known as Ampere's Law, $B = \mu\left(\frac{N \cdot I}{L}\right)$, where μ is some constant that is dependent on the magnetic constant, μ_0, and the relative permeability of the solenoid core.

Setting

The *Deriving Ampere's Law* activity took place in June 2016 during the Summer Engineering Academy, an annual enrichment program hosted at the K-12 Engineering Design Lab in the Curry School of Education at the University of Virginia. A total of twelve rising eighth-grade students from two different local middle schools were selected by their principals to participate in the two-week long academy. These students worked alongside their science and engineering teachers, Curry faculty, and doctoral students to build a solenoid, a generator, a motor, and a speaker in order to understand the science behind these historic inventions. The culminating activity of the Summer Engineering Academy was for students to exhibit their recreations at the Smithsonian Natural History Museum's Draper Spark!Lab (Breen, 2016).

Participants

Of the twelve rising eighth-grade students who participated in the 2016 Summer Engineering Academy, six students were purposefully selected to participate in the *Deriving Ampere's Law* activity. These six students were recommended by their science and engineering teachers based on their prior experiences and interest in mathematics. The students were grouped based on the mathematics coursework they had completed during seventh grade and the groups included students from both of the two local middle schools. The three students in the first group had already completed algebra (algebra group) and the three students in the second group had not yet taken algebra (pre-algebra group). The algebra group consisted of one male student (Brent) and two female students (Caitlin and Erin). The pre-algebra group consisted of two male students (Carter and Jamal) and one female student (Kayla).

The *Deriving Ampere's Law* activity was scheduled for the last two days of the first week of the Summer Engineering Academy. Earlier in the week, students in the academy had explored electromagnetism by constructing their own solenoids. Those participating in the *Deriving Ampere's Law* activity were pulled from the morning session of the academy to complete the activity. The algebra group completed the activity on the first scheduled day and the pre-algebra group completed the activity on the second scheduled day. Students spent approximately two-and-a-half hours on the activity, which included time spent debriefing the activity itself. While both groups of students successfully derived Ampere's Law, this paper focuses on the strategies of the algebra group.

Task Description

The impetus for the *Deriving Ampere's Law* activity was to explore how solenoids could be utilized as a hands-on manipulative in the teaching and learning of mathematics. The two authors[2] met to research how different parameters affected the magnetic field strength of a solenoid, which inspired the question of whether or not Ampere's Law could be derived experimentally. To test this theory, the authors decided to create a calibrated set of solenoids. Over the course of several months in early 2016, the first author developed multiple sets of solenoids and used a variety of different methods to measure magnetic field strength before the activity was finalized.

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[2] J. Garofalo (second author)

To complete the *Deriving Ampere's Law* activity, students were provided with a set of pre-made solenoids that vary in both the number of wraps of wire and the length of the solenoid tubes. Students were also provided with a variable DC power supply and a magnetic field sensor. In order to generate magnetic field measurements that would allow a wide range of students to be successful with this activity, the pre-wrapped solenoids were calibrated during task development (Table 1). The power supply used for this activity consistently and reliably held 3.16 A, which is why the solenoids were calibrated using this current value.

Table 1

Solenoid Data Collected under Laboratory Conditions

Number of Wraps (N)	Solenoid Length (L)	Electric Current (I)	Field Strength (B)	Constant (μ)
50	2 in	3.16 A	35.97 G	0.455
100	2 in	3.16 A	71.87 G	0.455
150	2 in	3.16 A	107.8 G	0.455
50	1 in	3.16 A	71.87 G	0.455
50	2 in	3.16 A	35.97 G	0.455
50	3 in	3.16 A	24.20 G	0.459
50	4 in	3.16 A	18.10 G	0.458
50	2 in	0.79 A	9.0 G	0.456
50	2 in	1.58 A	17.97 G	0.455
50	2 in	3.16 A	35.97 G	0.455

The task was presented as four separate open-ended investigations. In the first investigation, students were asked to relate the strength of the magnetic field produced by the solenoid to the *number of wraps of wire*, with the other independent variables held constant. Similarly, in the second investigation, students were asked to relate the magnetic field strength to the *length of the solenoid*. In the third investigation, students were asked to relate the magnetic field strength to the *electric current*. For each of the first three investigations, students were asked to develop a model that could be used to predict magnetic field strength. Once students had developed separate models for each of the independent variables, the fourth investigation challenged students to create a new model that related magnetic field strength to the three independent variables.

Investigation 1: Relating Wraps of Wire to Magnetic Field Strength. For this investigation, students were provided with a set of three solenoids. The length of the solenoid (2 inches) and the electric current (3.16 A) were held constant, but the solenoids varied in the number of wraps of wire (50 wraps, 100 wraps, and 150 wraps). There is a direct relationship between the number of wraps of wire of a solenoid and the strength of the magnetic field produced. When graphed, the data generates a straight line (Figure 3) and resulting equation is of the form $B = k_1 N$. Using the data collected under laboratory conditions (see Table 1), $k_1 = \frac{36}{50}$.

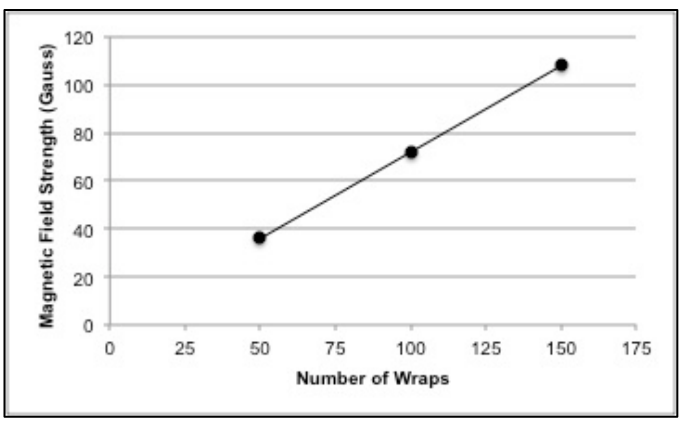

Figure 3. Relationship between the number of wraps of wire of a solenoid and the strength of its magnetic field.

Investigation 2: Relating Solenoid Length to Magnetic Field Strength. For this investigation, students were provided with a set of four solenoids. The number of wraps of wire (50 wraps) and the electric current (3.16 A) were held constant, but the solenoids varied in length (1 inch, 2 inches, 3 inches, 4 inches). There is an inverse relationship between solenoid length and the strength of the magnetic field produced. When graphed, the data generates a curved line (Figure 4) and the resulting equation is of the form $B = \frac{k_2}{L}$. Using the data collected under laboratory conditions (see Table 1), $k_2 = 72$.

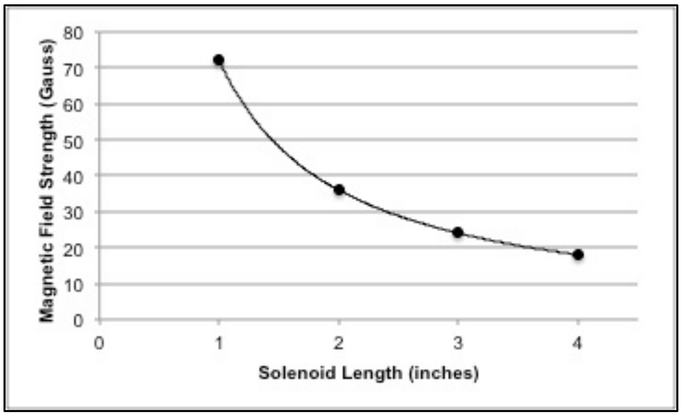

Figure 4. Relationship between the length of a solenoid and the strength of its magnetic field.

Investigation 3: Relating Electric Current to Magnetic Field Strength. For this investigation, students were provided with a variable power supply to measure the magnetic field strength of a single solenoid at varying levels of electric current. The other two independent variables (number of wraps of wire and solenoid length) were held constant. There is a direct relationship between electric current and the strength of the magnetic field produced. When graphed, the data generates a straight line (Figure 5) and the resulting equation is of the form $B = k_3 I$. Using the data collected under laboratory conditions, $k_3 = \frac{900}{79}$.

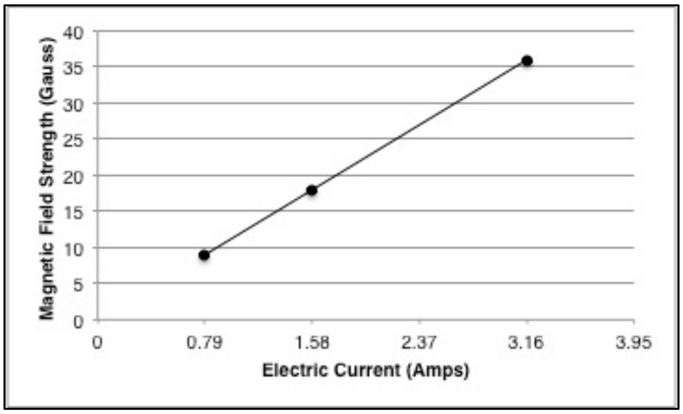

Figure 5. Relationship between electric current and the strength of a solenoid's magnetic field.

Investigation 4: Developing a Final Model for Ampere's Law. For this investigation, students were asked to review the models generated from the previous investigations. Using the structure of the previous models as their guide, students generated a final model that incorporates two direct variations and one inverse variation. They then calculated the constant based on their previously collected data. This final model related the three independent variables (number of wraps of wire, solenoid length, and electric current) to a single dependent variable (magnetic field strength).

Data Collection

Students were video recorded while working on the *Deriving Ampere's Law* activity and the audio was transcribed. Students' written work was also collected for analysis. While students worked on the activity, they engaged in discussion with each other regarding their problem solving strategies. Both authors recorded field notes throughout the activity session to further capture students' conversations and their written work. At certain points when it seemed like the students had lost sight of the task itself or were distracted by computational errors, the authors interjected reminders and posed questions to facilitate students' progress. These interjections are described in more detail in the *Findings* section. Upon completing the activity, the students participated in a debriefing interview to further explore their opinions about the activity and how this activity compared to their classroom experiences.

Data Analysis

The primary goals of this project were to understand how students' applied their prior conceptual and procedural knowledge when developing mathematical models and which problem solving strategies they utilized. Immediately following data collection, both authors met to discuss their observations and to explore commonalities between their recorded field notes and observational inferences. The first author then analyzed the transcript and students' written work for each of the four investigations separately from the second author to better understand the students' strategies. During preliminary data analysis, the first author attempted to code the transcripts based on students' application of prior knowledge (e.g., slope, variables, direct variation, linear equations). However, differentiating among applications of prior knowledge is difficult when mathematical concepts are interconnected (e.g., slope and linear equations) and coding efforts resulted in the data becoming disjointed. The first author then analyzed the data more holistically by reading through the transcript multiple times, utilizing the mathematical modeling framework (see Figure 1) as a way to interpret students' progression through the task, and connecting students' problem solving strategies to their prior knowledge as indicated by the transcript and students' written work.

After completing the initial round of data analysis, the first author met with the second author to confirm her interpretation. The two authors reread parts of the transcript, reanalyzed students' written work, and reviewed their separately collected field notes.

During this meeting, both authors regularly revisited their multiple data sources to ensure that their analysis and interpretations were warranted. The first author then prepared narrative descriptions of students' work on the first investigation, making sure to note the different strategies, both productive and unproductive, that the students employed. The two authors met to triangulate the narrative description of the first investigation with the observational field notes, audio transcripts, and students' written work. The authors came to a consensus that the narrative accurately captured what the students had done to complete the first investigation. This process continued for the remaining three investigations.

Findings

Brent, Caitlin, and Erin successfully developed mathematical models for all four investigations in the *Deriving Ampere's Law* activity. Aside from a few computational errors, these students had little difficulty with developing separate models relating magnetic field strength to number of wraps of wire (Investigation 1) and solenoid length (Investigation 2). However, the students struggled with developing the model relating magnetic field strength to electric current (Investigation 3), partially due to an arithmetic error made early in the investigation that hindered their progress. Relating a single dependent variable to three independent variables (Investigation 4) was something that the students had never been asked to do prior to this activity. With scaffolding in the form of questions and reminders from the two authors, the students were able to successfully derive Ampere's Law.

Investigation 1: Relating Wraps of Wire to Magnetic Field Strength

To begin this investigation, Brent suggested that they measure the strength of the magnetic field for each of the provided solenoids. He verified the attributes that were held constant for each set of solenoids. Erin confirmed with the group that the first solenoid they measured was two-inches in length and had 50 wraps of wire and that the resulting magnetic field strength was 36 G. They then measured the two-inch, 100-wrap solenoid and found that the rounded value of the resulting magnetic field strength was 72 G. Erin recorded the collected data in a table (Figure 6).

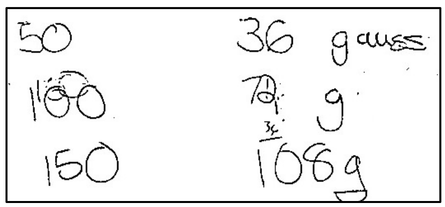

Figure 6. Erin's data table for the first investigation.

Based on the first two solenoids they measured, Erin predicted that the two-inch, 150-wrap solenoid would produce a magnetic field strength of 108 G, which they confirmed by measuring the solenoid. At this point, Erin suggested that the incremental change of 36 G would be related to slope and Brent interjected that there was also another incremental change of 50 wraps of wire. Caitlin reminded the group that the measurements they collected were not exact and that they were working with rounded values and Brent reminded her that rounded measurements were acceptable.

Brent verbalized the collected measurements once more, noting that the magnetic field strength increased by 36 each time. Erin asked if that meant the slope was 36 and Brent suggested that they graph the data. Caitlin began setting up the axes for the graph and the group discussed how they should scale the axes and which were the independent and dependent variables. Brent offered (50, 36) as a possible coordinate pair for the graph, which led him to discuss with Erin which variable they should represent with *x* and which they should represent with *y*. Erin suggested that the number of wraps should be represented by *x* and the magnetic field strength should be represented by *y*. While Brent and Erin discussed the variables, Caitlin was still deciding how to scale the graph's axes. Erin verbally repeated the *x*-coordinates for their collected data (i.e., 50, 100, and 150) and suggested scaling the *x*-axis by five. Brent suggested scaling the *x*-axis by 50 and then verbally repeated the *y*-coordinates (i.e., 36, 72, 108). Erin commented on the trend of the data, stating, "It's going up, so we know it's a positive –," before returning her attention to Caitlin's graph. Caitlin decided to start the graph over and while she was doing this, Erin sketched a quick graph of the group's data (Figure 7).

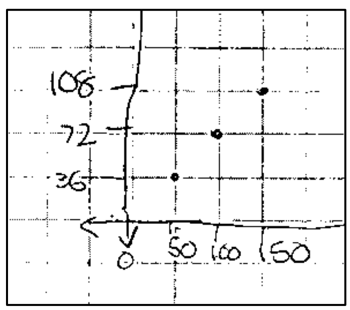

Figure 7. Erin's graph of the data collected in the first investigation.

Brent suggested that the group's next step should be to try writing an equation for their graph. His first step for writing the equation was to consider the slope of the graph:

> ***Caitlin:*** *What's the slope? The slope is, let me see [the graph.] Slope is going up 36. It's going over 50. So, 36-50ths.*

Brent then asked his group for the simplified form of 36/50, which Caitlin incorrectly claimed was 12/25. Using this value for the slope and zero for the y-intercept, Brent proposed that the equation should be $y = \frac{12}{25}x$. Erin then applied this equation to their collected data:

> ***Erin:*** *So, 12 over 25 times 50 and it would somehow have to add or subtract to 36 basically, right?*

Brent argued that, assuming their simplifying of the original fraction was correct, multiplying 50 by 12/25 should equal 36. Erin did not agree that this multiplication resulted in 36, so Brent asked his group to simplify 36/50 again. Caitlin incorrectly confirmed that the fraction simplified to 12/25, so Brent asserted that the equation must be $y = \frac{12}{25}x$. Erin suggested that they double-check their equation with the two-inch, 100-wrap solenoid. Brent wanted to check their equation against a solenoid whose wraps did not vary by 50 wraps, which was something the students did not have access to. As both Brent and Erin tested their equation against the solenoids that they had already measured, Erin interrupted Brent and questioned how they had simplified $\frac{36}{50}$.

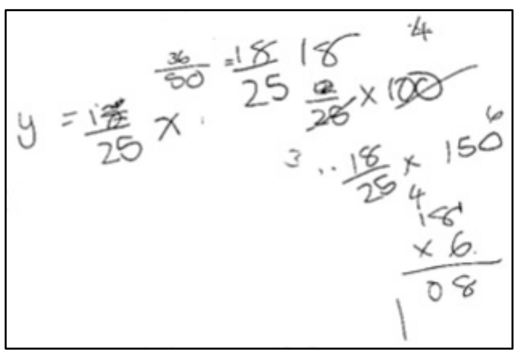

Figure 8. Erin's verification of their equation for the first investigation.

Erin initially wrote the leftmost equation as $y = \frac{12}{25}x$ (Figure 8). Directly to the right of that equation, she set up the expression $\frac{12}{25} \times 100$ to verify the strength of the magnetic field for a 100-wrap solenoid. She then noticed that it did not yield what they had found experimentally (e.g., 72 G). She corrected the simplification of $\frac{36}{50}$ to $\frac{18}{25}$, as seen in the top center of the figure. Then, Erin changed the 12 to 18 in both the leftmost equation and the expression and verified the calculated strength with what was measured experimentally. Prompted by Brent, she then verified that the strength for a 150-wrap solenoid is 108 G. After this verification, all three students confidently settled on the equation $y = \frac{18}{25}x$.

The prior conceptual knowledge that students utilized for this investigation was their understanding of slope and determining a linear equation using slope. These students also recognized that the equation of the line through the scatterplot would be their mathematical model. Their problem solving strategies included testing their model using their collected data, which uncovered an arithmetical error, revising their model, and re-verifying.

Investigation 2: Relating Length of Solenoid to Magnetic Field Strength

Brent, Caitlin, and Erin knew that they would also be asked to relate the length of the solenoid to the magnetic field strength, so they chose to measure the remaining solenoids in the given set before analyzing the data for the first investigation. After measuring the two-inch solenoids with varying wraps of wire, they then measured the one-inch, three-inch, and four-inch, 50-wrap solenoids. They found that these additional solenoids produced magnetic field strengths of 72 G, 24 G, and 18 G, respectively. Erin recorded the collected data sequentially based on solenoid length, as seen in Figure 9.

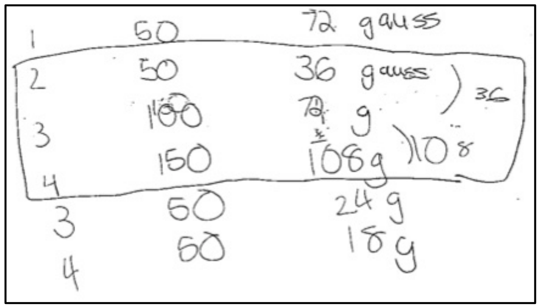

Figure 9. Erin's data table for first two investigations.

Having successfully completed the first investigation, the students returned to the data they collected for the second investigation. Brent reminded his group of the relationship between solenoid length and magnetic field strength that he observed during data collection, which was:

> ***Brent:*** *So for these, first you subtract one-half, then you subtract one-third, then you subtract one-fourth from it…See what I'm saying? So for one, you're subtracting from 72, and 36 is half of 72. Then you have 36 and 24, and 24 is two-thirds of 36. You're subtracting one-third. You subtract six, which is one-fourth of 24 to get that.*

He then proposed that the group approach this investigation in a similar fashion as the first investigation, suggesting they graph the data using the length of the solenoid as *x* and the magnetic field strength as *y*. While Caitlin worked on graphing the group's data, Brent and Erin continued to discuss the relationships they observed in the data.

Brent and Erin tried to use the pattern he observed in the data to predict the magnetic field strength that would result from a five-inch, 50-wrap solenoid. Brent again noted that the change from the one-inch solenoid to the two-inch solenoid was 36 G and the change from the two-inch solenoid to the three-inch solenoid was 12 G. He also observed that 12 is one-third of 36. Similarly, he noted that the change from three-inch solenoid to the four-inch solenoid was 6 G and that 6 is one-half of 12. However, neither Brent nor Erin was sure of how to extend this pattern. Brent commented that being able to measure a five-inch, 50-wrap solenoid would be beneficial to confirming the pattern he observed. Erin suggested that the field strength of the five-inch solenoid might decrease by 3 G, even though this did not match the pattern.

At this point, Caitlin finished graphing the data and she showed her scatterplot to Brent and Erin, who turned their attention to the graph (Figure 10). Caitlin commented that the relationship between solenoid length and magnetic field strength was "not direct variation."

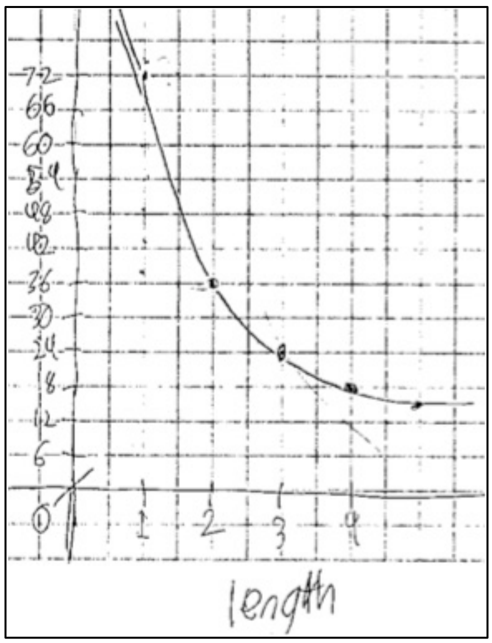

Figure 10. Caitlin's graph of data collected in the second investigation.

Both Brent and Erin looked at Caitlin's graph and they suggested drawing a line through the plotted points. Brent asked for a graphing calculator and explained that he had never been asked to develop an equation for non-linear data without using a calculator. Erin suggested that they could try to write an equation for the "general line" by finding the average of the slopes between successive data points. Caitlin disagreed with her reasoning of using a linear equation to fit non-linear data:

> ***Caitlin:*** *That might not work so well because if you look at [the graph], there's a pretty big fluctuation between where the points are. If you're just gonna draw a straight line, you're gonna miss most of them entirely.*

Brent used the graph to once again try to predict the magnetic field strength produced by a five-inch solenoid. He noted that the scale along the horizontal axes was two unit squares for every inch and he added a point to the curve at the five-inch mark and approximated this to be 15 G. He struggled with consolidating the magnetic field strength as predicted by the graph and the pattern he had previously observed. Erin explained that it looked like the graph would "even out." Brent asked his group members whether or not they thought the graph might be a parabola, but Erin said that they could not know for sure without having other solenoids of greater lengths to measure. Caitlin later explained that a parabola would not fit the shape of the data because she did not believe it made sense for the magnetic field strength to begin to increase as the solenoid length increased. Erin also noted that the relationship between the solenoid length and the magnetic field strength was decreasing.

Caitlin and Erin continued to analyze the graph together while Brent began thinking out loud. While he initially thought that the graph's curve resembled a parabola, he also recalled that indirect variations also resulted in a curved graph. Thinking out loud, he said:

> ***Brent:*** *Is it an indirect variation? … It is an indirect variation because you increase this, this goes down. So you just have to figure out, wait so, indirect variation. That's xy = a, I think. Is that the equation for indirect variation?*

At this point, Caitlin and Erin refocused their attention on Brent. Caitlin did not recall studying indirect variation in school and Erin recalled hearing the term in class. Brent continued to develop the equation for the second investigation:

Brent: *K over x equals y, so that means x times y should equal…it was two times 36, 72, three times 24 is 72…four times 18 is 72. K is 72, so that means y = 72 over x. That's the equation.*

After sharing his equation with the group, Erin verified the equation by checking to see if the magnetic field strengths predicted by the equation matched their collected data. After confirming Brent's equation worked, all three students agreed on the equation $y = \frac{72}{x}$.

The prior conceptual knowledge that students utilized for this investigation was their ability to recognize and extend patterns, identify non-linear relationship, differentiate between direct and inverse variation. These students also recognized that their data could not be modeled with a quadratic function based on their understanding of parabolas. Their problem solving strategies included verifying and confirming their model using their collected data.

Investigation 3: Relating Electric Current to Magnetic Field Strength

Having completed the first two investigations, Brent decided to consolidate the group's data and the derived equations into a single table (Figure 11).

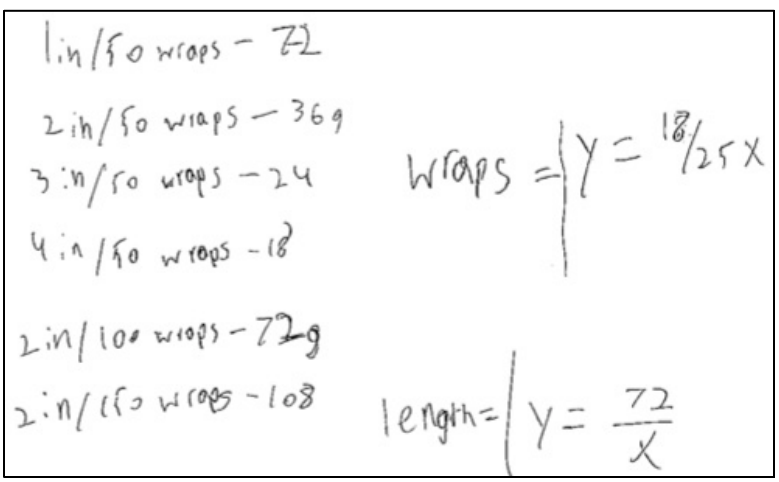

Figure 11. Brent's data and equations from the first two investigations.

Unlike the first two investigations that utilized pre-wrapped and calibrated solenoids, the students were required to manipulate the independent variable (i.e., electric current) themselves for this third investigation. The students chose the two-inch, 50-wraps solenoid for their data collection and Caitlin was tasked with adjusting the current on the variable power supply. The students collected their previous data at 3.16 A, which was the maximum current for these solenoids. Using this as their starting point, Caitlin then decreased the current at intervals of 0.5 A and the students collected the following data (which they rounded to the nearest whole number):

Table 2

Students' First Round of Data Collection for the Third Investigation

Current	3.16 A	2.66 A	2.16 A	1.66 A	1.16 A	0.66 A
Field Strength	36 G	29 G	23 G	17 G	11 G	6 G

The students immediately observed that there was not an exact constant rate of change between these values for current and magnetic field strength. Even with a constant change in current of 0.5 A, the change in magnetic field strength fluctuated from 7 G to 6 G to 5 G. Brent suggested that the group should approximate the change in magnetic field strength to be 6 G and that they should change their readings at 3.16 A and 0.66 A to 35 G and 5 G, respectively. With this change, Caitlin commented that the relationship between electric current and magnetic field strength was direct variation. Caitlin started graphing the group's data, while Erin commented that while "a visual representation always helps," they could determine the equation without using a graph. In her initial attempt to graph the data, Caitlin placed magnetic field strength along the *x*-axis and electric current along the *y*-axis.

While Caitlin continued to work on the graph, Brent told his group that the slope was “0.5 over six” (which represented the $\frac{\Delta x}{\Delta y}$ rather than $\frac{\Delta y}{\Delta x}$). The students then tried to simplify this so that there was not a decimal value in the numerator. Brent first suggested incorrectly that $\frac{0.5}{6}$ simplified to $\frac{1}{3}$, but then second-guessed his work. Caitlin first suggested that $\frac{0.5}{6}$ simplified to $\frac{1}{12}$, but would later change her calculation. Brent then suggested that the slope was $\frac{1}{6}$ because there are “two 0.5’s in one,” but then realized he needed to perform the same operation on the denominator, which resulted in $\frac{1}{12}$. Erin asked the group several times to wait so that she could perform the simplification herself, but without pause, Caitlin disagreed with Brent’s simplification. She argued that to simplify $\frac{0.5}{6}$, they would need to multiply the fraction by 2, which resulted in $\frac{0.5}{6} \cdot \frac{2}{1} = \frac{1}{6}$. Brent then responded, “How is point-five…whatever,” and both Brent and Erin accepted Caitlin’s simplification as correct.

Knowing that the relationship between electric current and magnetic field strength is direct variation and using $\frac{1}{6}$ as their slope, Brent suggested that the equation for the third investigation should be $y = \frac{1}{6}x$. Erin and Caitlin tried to verify Brent’s proposed equation, but they both realized this equation did not work with their collected data. Unfortunately, none of the students thought to revisit their initial slope calculation of $\frac{1}{12}$ and instead tried to manipulate numbers to find a slope that would work. Neither Erin nor Caitlin used units while talking through their computations. For example:

Erin: *Wait, so does one-sixth times 35…*
Caitlin: *Can I test it on your calculator?*
Erin: *Yeah.*
Caitlin: *Let’s do it with five and 0.16*

Brent: *So, what's x?*
Caitlin: *x is five. Let's start with five.*
Erin: *Yeah, does it equal two-thirds?*
Caitlin: *No.*
Erin: *Oh, okay. Can you flip it then? No, the increments would get smaller.*

At this point, Brent interrupted them and asked them to clarify what five represents, to which Erin responded, "Five was the Gauss, or 0.66, right?" Instead of questioning their data analysis, Caitlin attempted to measure the magnetic field strength at 0.16 A again, but struggled to adjust the variable power supply. Erin asked her group members if this meant that they would need to recollect all of their data and Brent agreed.

In order to encourage the students to reanalyze their existing data rather than recollect, the first author interjected and asked the students to restate the variables in this investigation. Caitlin said that the variables were Gauss and current, which led the first author to ask a follow-up question regarding which variable was represented by *x* and which was represented by *y*. Brent replied that the current is *x* and the Gauss is *y*, which led Erin to tell her group that they needed to "reverse it," referring to the slope. Caitlin asked her group to confirm that the equation for the third investigation was $y = \frac{1}{6}x$ and Brent agreed, but Erin once again asked if their slope was incorrect. Brent said that the *y* variable was Gauss, which seemed to confuse Caitlin. Erin then suggested that the slope should be six, not one-sixth, but she then realized that also did not work with their collected data. Brent tried testing the equation again, but he was still using one-sixth as the slope.

The first author then asked the students how they arrived upon one-sixth to encourage the students to revisit their calculations. Erin said that it was from $\frac{0.5}{6}$ and Caitlin explained that it was because they decreased the current by 0.5 A for each reading, at which point Brent asserted that the slope should be negative. Caitlin argued that the slope was in fact positive. Erin then asked if there would be a *y*-intercept for their equation. At this point, Brent suggested that they recollect all of their data and that they should graph the data. Caitlin had previously struggled with scaling her axes and had abandoned her initial attempt at graphing the group's data from the initial round of data collection.

The students then collected their data for the second time. They again chose to change the current in increments of 0.5 A and they recorded their first measurement at 0.16 A. Brent instructed Erin to record the data without rounding (Table 2) and he would record the data rounded to the nearest whole number.

Table 3

Students' Second Round of Data Collection for the Third Investigation

Current	0.16 A	0.66 A	1.16 A	1.66 A	2.16 A	2.66 A	3.15 A
Field Strength	2.7 G	8.1 G	14 G	19.6 G	25.2 G	30.6 G	36 G

Brent noticed that once again, there was not a constant change in magnetic field strength in their data. Caitlin suggested using an average to determine the change in magnetic field strength. Erin described this as finding the "general slope," but Brent questioned this approach. Caitlin then suggested collecting their data again, but that they take their first measurement at 3.0 A rather than 0.16 A (Table 3).

Table 4

Students' Third Round of Data Collection for the Third Investigation

Current	3.0 A	2.5 A	2.0 A	1.5 A	1.0 A	0.5 A
Field Strength	34 G	27.9 G	21.9 G	16.1 G	10 G	4.5 G

Brent commented that with this set of data, the change in magnetic field strength was 6 G except from 1.0 A to 0.5 A, where the change was 5 G. He suggested that the group graph their data. Caitlin erased the scale from her first graph, but kept magnetic field strength as her *x*-axis and electric current as her *y*-axis. Erin then suggested another approach (see Figure 11):

> ***Erin:*** *Couldn't we just work it out exactly like a math problem for slope? So, it's gonna be 6.1 over five-ish. Do you want to do increments of six as the general slope?*

Brent agreed with approximating the change in magnetic field strength as 6 G. Erin then set up her slope calculation as $\frac{6}{-0.5} = -12$, but then questioned if she had accidentally flipped her fraction. Caitlin disagreed that the slope would be negative, noting that there was "a positive correlation" in the data. Caitlin returned to her graph and Erin checked her slope calculation and then asserted that the slope should be $\frac{1}{12}$ (Figure 12).

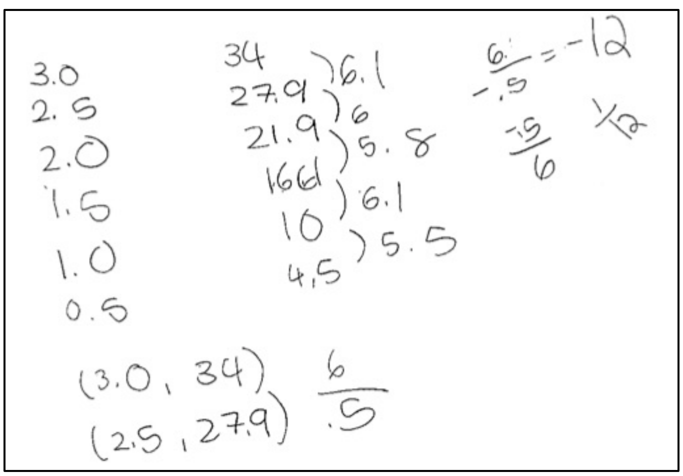

Figure 12. Erin's slope calculation using data from the third round of data collection.

Caitlin continued trying to graph the group's most recently collected data, but struggled with scaling her axes. She attributed her difficulty to the fact that she needed a different scale for her *x*-axis and her *y*-axis. The first author asked the group which variable the students should assign to their *x*-axis. Brent initially said Gauss, which Caitlin agreed with, but he then asked why Gauss would be along the *x*-axis. Erin said that she thought Gauss was the dependent variable, but Caitlin argued that it was the independent variable. Brent disagreed and reasoned, “Gauss isn’t the thing we’re changing. It’s voltage [*sic*] that we’re changing.” Caitlin then changed her axes once again and finished the graph (Figure 13).

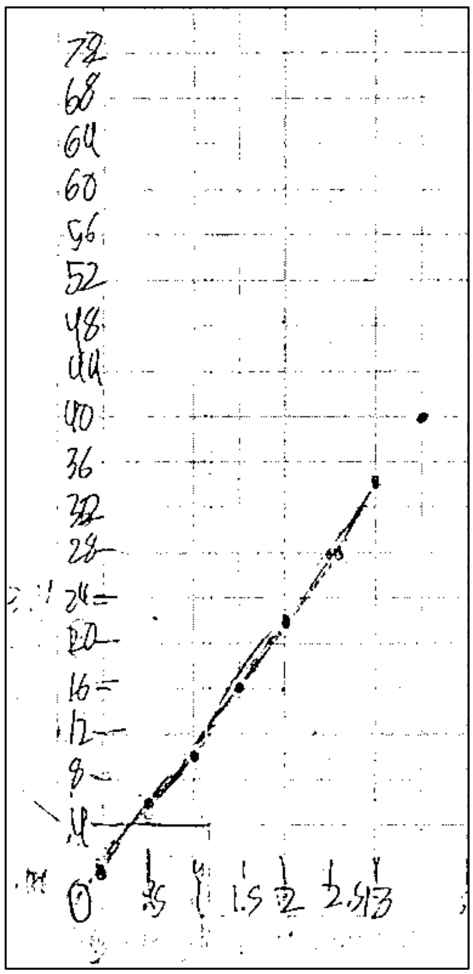

Figure 13. Caitlin’s graph of the data collected in the third round of data collection.

Using Caitlin’s graph, the three students once again try to develop an equation to model the relationship between electric current and magnetic field strength. Erin noticed that the *y*-intercept was zero. Brent concluded that as long as they assumed the change in magnetic field strength from 1.0 A to 0.5 A was 6 G, this was an example of direct variation. The three students again attempted to find the slope of the line. Brent and Caitlin's approach to calculating slope was to use the graph to determine the change in *y* and the change in *x*; however, they read the scale along the axes differently. Caitlin believed that the slope was $\frac{1.5}{6}$, but Brent argued that the slope was $\frac{4}{0.5}$. Brent then corrected himself and divided the *y*-value by the *x*-value in their data set to arrive at a slope of $\frac{34}{3}$, which he approximated as 11.33. However, he noted he arrived at different values when using other data points.

Erin then proposed that they consider their data as ordered pairs and to calculate slope using $\frac{y_2-y_1}{x_2-x_1}$. Caitlin noticed that she had miscounted the units along her graph and that the slope should have been $\frac{6}{0.5}$, which Brent simplified as 12, but Caitlin simplified as three. Brent corrected her and Caitlin agreed that six divided by 0.5 equals 12. Brent questions whether or not 12 is the correct slope, referring back to $\frac{34}{3}$. At this point, the students had been working on the third investigation for approximately 40 minutes. The second author asked if $\frac{34}{3}$ was close to 12 and Brent agreed that the two values were close to each other. The second author then reminded students of the variation they observed during data collection and that using rounded values would be acceptable.

At this point, all three students agreed that the slope was 12 and Brent proposed that the equation should be $y = 12x$. Erin proposed that, based on their collected data, the equation should be $y = 12x - 2$. Caitlin and Erin test the equation $y = 12x - 2$ against their data and both confirm that this second equation fits better than $y = 12x$. Caitlin then proposes that they use their equation to predict the magnetic field strength at a current value that they have not yet measured. Using x = 1.25 A, Brent calculates that the magnetic field strength should be 13 G. Caitlin then set the variable power supply to 1.25 A and measured the magnetic field strength of the two-inch, 50-wrap solenoid and found that the field strength was 15 G. Erin suggested that Caitlin adjust the current until the field strength was 13 G so that they could see the associated current value. Brent noticed that at 1.05 A, the magnetic field strength was 13 G, but at 1.0 A, the magnetic field strength was 10 G. Given the variation they had observed in their data collection, all three students agreed that the equation $y = 12x - 2$ fit their collected data.

The first author asked the students to connect their equation to the context of the investigation. Caitlin explained that the slope represents the relationship between current and Gauss. Erin explained that when they set the current to 0 A, the magnetic field sensor picked up a reading of –2.2 G, which is why the y-intercept is –2. The second author then asked what the y-intercept would be if there were not any ambient magnetic field in the room and Caitlin said 0. The students were encouraged to disregard the ambient field reading, at which point they settled on the equation $y = 12x$.

The prior conceptual knowledge that students utilized for this investigation was their understanding of slope, linear equations, and direct variation. Their problem solving strategies included verifying their model using their collected data. However, an initial incorrect slope calculation (representing slope as $\frac{\Delta x}{\Delta y}$ rather than $\frac{\Delta y}{\Delta x}$) and arithmetic errors when simplifying their slope calculation was a significant hindrance to students' ability to develop a model for this investigation. Students exhibited a lack of problem solving strategies when they recollected their data rather than revisiting their data analysis.

Investigation 4: Developing a Final Model

For this final investigation, the students were asked to develop a single equation that related wraps of wire, solenoid length, and electric current to magnetic field strength. Caitlin suggested that the group should either review their equations from the first three investigations or review their previously collected data. Brent proposed collecting more data, but when reminded of the time constraints, he agreed that reviewing the data they had already collected would be a good starting point. Brent summarized the relationships they observed in the first three investigations:

> ***Brent:*** *Well, for the length, the longer the tube is, the less Gauss you're gonna have. And then for the other thing [wraps of wire and electric current], the more of each thing you have, the more Gauss you're gonna have.*

Caitlin commented that they would be unable to make a graph, so Erin suggested organizing their data into a single table. Erin began setting up the group's data table when Caitlin realized that they would need to rename the variables in their previous equations because they could not represent all three independent variables with *x*. Caitlin proposed that they define the electric current as *a*, the solenoid length as *b*, and number of wraps of wire as *c*.

While Erin continued to organize the group's data, Caitlin proposed that they could combine their previous equations, but would need to "average it out." She explained that they could *add* their equations from the first three investigations and then *divide the sum by three* (see Figure 11). Brent suggested that they use the data from one of their previously measured solenoids to test Caitlin's equation. Using the data collected from the one-inch, 50-wrap solenoid measured at 3.16 A, Caitlin set up her equation (Figure 14).

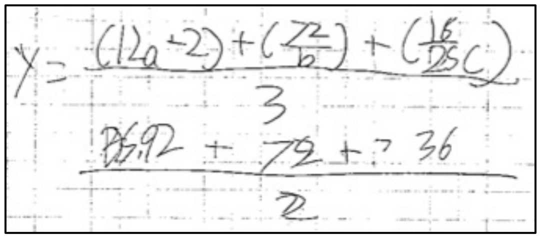

Figure 14. Caitlin's first approach with developing a final model.

Caitlin's equation resulted in a magnetic field strength of 47.97 G, which did not match their measured field strength of 72 G. Erin noticed that dividing by two instead of three would result in a predicted field strength of 71.96 G. This revision to Caitlin's equation can be seen in the second expression in Figure 7. Using the revised equation, $y = \frac{(12a-2)+\left(\frac{72}{b}\right)+\left(\frac{18}{25}c\right)}{2}$, the students calculated the magnetic field strength for a two-inch, 50-wrap solenoid measured at 3.16 A. The equation resulted in a magnetic field strength of 53.96 A, which again did not match their measured field strength of 36 G.

At this point, the two authors provided the students with additional scaffolding for this investigation because this was the first time these students had been asked to develop a mathematical model that involved three independent variables. The students were motivated to complete the task, but were unsure of how to move forward. The scaffolding that was provided consisted of a suggestion, a few reminders, and some questioning. The first author suggested that the students revisit the types of variation they observed during each of the investigations. The students recalled that the relationships between the number of wraps and magnetic field strength and electric current and magnetic field strength were both direct variation, while the relationship between solenoid length and magnetic field strength was indirect variation. Caitlin suggested that instead of trying to combine all three equations at once, they should instead focus on first combining the direct variations and proposed that they subtract these two equations.

The second author then asked the students how they would put the two direct variations together and Erin commented that there were only four ways to combine the equations, referring to the four basic mathematical operations. Brent recalled that in the third investigation, the group had agreed that the final equation should be $y = 12x$ because subtracting two would no longer result in direct variation. The second author asked the students to restate the equation for a direct variation and Brent responded, "y equals k x." The second author then asked a follow-up question, "If you had two variables that had a direct relationship, what do you think that [equation] would look like?"

Brent asked if the equation would be "y equals 2K" and Erin followed with the equation "K squared and x squared." [Joe] reminded them that K could be any number. Returning to the general form of a direct variation equation proposed by Brent ($y = Kx$), the second author asked the students where they would put the next variable that had a direct variation. Erin suggested "after," but did not specify an operation. Caitlin suggested that they "add it to both sides." Brent suggested that the equation would be "y equals K times the sum of W plus C," where W represented the number of wraps of wire and C represented the electric current. The second author asked if this equation still represented direct variation and Erin replied that it did not and then stated that they would need to multiply the two variables and Caitlin agreed that it would still be direct variation. The second author then asked how they would include the variable that has an indirect variation and Brent responded that they would need to divide. Brent then wrote the equation $y = K\left(\frac{WC}{L}\right)$.

Brent shared his equation with Caitlin and Erin. Erin asked, "What is *x* and what is *y*?" and Brent explained that *W*, *C*, and *L* represented the *x* variables from their previous equations. Caitlin asked what *K* represented and Brent said that *K* was a constant. The first author asked the students how they could figure out a numerical value for *K*. Erin asked if it would be the slope, since the constant in direct variation equations is the slope. Caitlin reminded Erin that they were unable to graph the data for this equation, so they would not be able to determine the slope. Brent suggested that they could multiply the constants from the previous three equations. This gave him a constant of 622, which he immediately questioned. The first author then reminded the students of a comment Caitlin had made during the first investigation. Caitlin explained that the equation they developed relating wraps of wire to magnetic field strength would only work for two-inch solenoids because all of their data was from a two-inch solenoid. The first author asked if they could use their previously derived constants to determine the constant for this new equation and Brent responded that there needed to be a new constant.

Brent then suggested using the data for one of their solenoids to calculate the constant. Using the one-inch, 50-wrap solenoid measured at 3.16 A, Brent calculated a value of 158. The first author asked Brent what that represented in his equation and he said, "it'll get us the constant." Brent then restated the equation,

Brent: *The wraps and the currents divided by L times the constant would give you y. We don't know what the constant is.*

The second author then asked if they had values for *y* and Brent responded that they did and that they could use that to determine a value for *K*. Brent then returned to the equation $y = K\left(\frac{WC}{L}\right)$ and the three students attempted to solve the equation for *K*.

Caitlin:	*Would we need to complete the square to get rid of the curve, like you do for the quadratic proof?*
Erin:	*Complete the square? I don't see the square in here.*
Caitlin:	*There might be one. No, that wouldn't work.*
Erin:	*I think you're overcomplicating it.*
Caitlin:	*I think you're right.*
Brent:	*We just need to find K.*

At this point, Brent had divided both sides of the equation by *K*, but was then unsure of what to do next. He knew that he wanted to isolate *K* on one side of the equation. The first author asked the students if it would be helpful for them to use their data rather than working strictly with variables. Brent then set up the equation again using the one-inch, 50-wrap solenoid measured at 3.16 A. After working through the calculations, he was left with the equation $72 = K \cdot 158$, which he was able to solve for *K* (Figure 15).

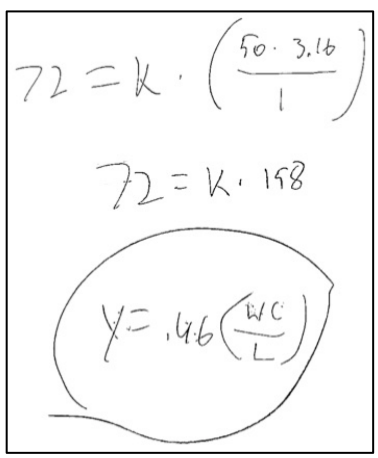

Figure 15. Brent's method for determining the value of *K*.

Using *K* = 0.46, Brent then tested his final equation using his data for the two-inch, 50-wrap solenoid measured at 3.0 A and saw that the equation worked. Caitlin and Erin also tested this equation using a calculator. After having tested the equation against at least three different sets of data, Brent, Caitlin, and Erin were all convinced that this equation worked and the three students agreed that the final equation was $y = 0.46\left(\frac{WC}{L}\right)$, which was the correct model for their set of solenoids.

The prior conceptual knowledge that students utilized for this investigation was their understanding of variables, knowledge of the structure of direct and inverse variations, and solving an equation for an unknown value. These students also recognized that this relationship could not be modeled graphically because their model incorporated three independent variables. Their problem solving strategies included verifying and confirming their model using their collected data.

Discussion

The research questions for this paper focus on the prior mathematical knowledge, both conceptual and procedural, students utilized when developing mathematical models and students' problem solving strategies. While there was some application of prior knowledge and problem solving strategies that were common across all four investigations, there were also instances where the prior knowledge accessed and strategies used were unique to each investigation. There were also instances where misapplication of prior knowledge (e.g., reversing slope, confusing variables) and a lack of problem solving strategies (e.g., not verifying computations) delayed the students' ability to develop their models.

Prior Mathematical Knowledge

For all four investigations, students relied on their knowledge of generating graphs from data tables and analyzing relationships between variables based on graphical representations. The students also consistently utilized pattern recognition and extension to help them develop their models. The students' prior knowledge of direct and inverse variation helped them recognize the models for the first three investigations and allowed them to develop their final model relating a single dependent variable to three independent variables.

Investigation 1. The prior mathematical knowledge Brent, Caitlin, and Erin utilized to complete the first investigation included identifying independent and dependent variables, graphing data, recognizing and extending linear patterns, and calculating slope. Immediately after measuring the two-inch, 50-wrap solenoid and the two-inch, 100-wrap solenoid, Erin recognized that the magnetic field strength increased by 36 G and she predicted that the two-inch, 150-wrap solenoid would produce a magnetic field strength of 108 G. Brent realized that the incremental change of 36 G corresponded with an incremental change of 50 wraps, resulting in the slope $\frac{36}{50}$. When the students graphed their data, they discussed which variable was the independent variable and which was the dependent variable. They also recognized from their graph that the y-intercept for their model should be zero, which resulted in $y = kx$ as the structure for their model.

Investigation 2. Brent, Caitlin, and Erin again utilized their prior mathematical knowledge to complete the second investigation. The students were able to eliminate different types of relationships based on their graph. For example, the students immediately recognized that the data was not linear and it was not direct variation. Caitlin also knew that the data was not quadratic because at no point would the graph turn. As soon as Brent recalled that the shape of an indirect variation graph was also curved, he immediately recalled the definition of indirection variation and the equation's structure.

Investigation 3. The prior knowledge Brent, Caitlin, and Erin recalled for the third investigation was similar to the first two investigations. They graphed data, calculated slope (using both the graph and the slope formula), and recognized direct variation. However, the third investigation was much more challenging for the students. Unlike the first two investigations, the students were required to manipulate the independent variable themselves by changing the current values using the variable power supply. The third investigation also highlighted gaps in the students' prior knowledge. When they first attempted to determine the slope using their initially collected data, they calculated as $\frac{\Delta x}{\Delta y}$ rather than $\frac{\Delta y}{\Delta x}$, which resulted in $\frac{0.5}{6}$. Inverting slope was an error that was also observed in previous pilot testing iterations of *Deriving Ampere's Law* activity; two high school students who completed the activity also inverted their initial slope calculation.

In addition to inverting the slope, Brent, Caitlin, and Erin also struggled with rewriting this fraction so that there was not a decimal value in the numerator, which indicated a lack of both conceptual and procedural knowledge. Caitlin first simplified $\frac{0.5}{6}$ as $\frac{1}{6}$ without recognizing that $\frac{1}{6}$ must be greater than $\frac{0.5}{6}$. This initial arithmetic error proved to be catastrophic for the students and led students to recollect their data two more times. The students also misinterpreted the slope's direction; because the magnetic field strength was decreasing, the students believed that the slope was negative without recognizing that they were also decreasing the electric current. It was not until Erin recommended using the formula for slope that the students were able to correctly calculate slope and develop their model.

Investigation 4. Developing a model that related a single dependent variable to three independent variables was something that Brent, Caitlin, and Erin had never done prior to this activity. However, the students were able to utilize prior mathematical knowledge to help them make sense of this task. The students recognized the need to rename the variables in their final model. In the previous three investigations, they represented the independent variable with x and the dependent variable with y. Since the final model involved three independent variables, the students realized that they could not represent all three with the same letter. After renaming the variables, the students first attempt at writing their final model was to average the equations from the first three investigations. This approach was also observed during previous pilot testing iterations of the activity; both high school students and undergraduate mathematics majors who had completed the *Deriving Ampere's Law* activity first tried to find the average of the first three models. After realizing that finding the average would not work, Brent, Caitlin, and Erin then returned to their understanding of the structure of direct and inverse variations to help them develop the general structure of their final model. They also used their knowledge of how to solve an equation for an unknown value in order to determine the constant for their final model.

Problem Solving Strategies

The students used similar problem solving strategies for all four investigations. The two problem solving strategies that seemed to be the most valuable for students were developing and analyzing multiple representations of their data (i.e., tables, graphs, and equations) and verifying the appropriateness of their model. Once a model was developed, the students consistently used their collected data to confirm their model worked.

Investigation 1. Brent, Caitlin, and Erin effectively used several problem solving strategies to complete the first investigation, including using multiple representations (e.g., data table, graph), making predictions, and verifying their equation using the collected data. It is important to reiterate that the students were not instructed to graph their data; the students felt that making a graph would help them recognize the relationship between the independent and dependent variables and the fact that there was not an additive constant in their model. While they did not initially check the accuracy of their calculations, verifying their equation using the collected data helped the students uncover an arithmetic error they had made early in the investigation (incorrectly simplifying $\frac{36}{50}$ as $\frac{12}{25}$).

Investigation 2. The students used many of the same strategies as they did in the first investigation. They used multiple representations, looked for patterns, and made predictions. They also verified their equation using the collected data. Analyzing the graph of their data was a particularly useful strategy for this investigation. From the shape of the graph, the students were able to limit the types of relationships that could be used for their model. The most difficult aspect of the second investigation was generating an equation for non-linear data without using a calculator.

Investigation 3. Again, the problem solving strategies the students successfully used for this investigation included generating multiple representations and verifying their model using collected data. However, prior to developing their final model, there was an apparent lack of strategy in the students' problem solving approach. In particular, the students did not revisit their initial slope calculation to confirm that their computations were correct. Instead of checking their work, the students recollected their data. Even after collecting their data a third time, the students did not immediately revisit their slope calculations and continued to think of slope as $\frac{\Delta x}{\Delta y}$. It was not until the students decided to abandon their calculations that they were able to correctly determine the slope.

Investigation 4. The students quickly realized that the relationship for the fourth investigation could not be modeled graphically, making one of their most used problem solving strategies not applicable. However, the students did make use of other problem solving strategies. The students organized their data from the first three investigations into a single table. The students also approached this task by analyzing a simpler problem (i.e., creating a model that only involved the two direct variation relationships). The students also verified their model by using their collected data to confirm their equation was correct.

Implications

The *Deriving Ampere's Law* activity attends to the five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (National Research Council, 2001). As a result of completing the *Deriving Ampere's Law* activity, the students developed a better understanding of direct and inverse variation. The activity provided students with opportunities to observe real-world examples of direct and inverse variation and to apply their prior mathematical knowledge of these concepts to develop their models. Brent, Caitlin, and Erin also learned how to model relationships that involved multiple independent variables. After completing the fourth investigation, Brent asked if he would be able to combine even more independent variables using a similar structure (i.e., identifying the types of variation and using that to determine which variables are in the numerator and which are in the denominator). Throughout the entire activity session, the three students remained engaged and were determined to develop a correct model. As a model-eliciting activity, the *Deriving Ampere's Law* activity provided the students with an opportunity to engage in the mathematical modeling process (see Figure 1); they had to regularly verify, revise, and adapt their models throughout each of the investigations.

The findings from this study reveal several implications for classroom instruction. First, the concept of slope should be taught alongside independent and dependent variables and it should be taught within some sort of real-world context. Brent, Caitlin, and Erin utilized their prior knowledge of slope in both the first and the third investigation. In the first investigation, their discussion of slope was rooted in the context of the task; the students spoke about slope as the incremental change in Gauss that results from an incremental change in wraps of wire. However, in the third investigation, the students regularly lost sight of the context and spoke about slope strictly in terms of numeric values. Second, application tasks need to be more authentic. For the first two investigations, the students measured a set of pre-wrapped solenoids that had been calibrated prior to the activity. In the third investigation, the students were asked to manipulate the independent variable themselves, which resulted in "messier" data. Analyzing the data from the third investigation was much more difficult for the students. Finally, problem solving strategies and control strategies need to be explicitly integrated into classroom instruction. The students' lack of control strategies came to light during the third investigation. Instead of checking their work for computation errors, the students assumed the error was in the data they collected. Had they revisited their initial slope calculation, the students most likely would have realized that their errors were mainly computational.

The students' affective responses also indicated their preference for activities such as this. They shared that they found the *Deriving Ampere's Law* activity engaging and unlike what they had previously experienced in school. Erin explained that interdisciplinary work in school is rare: "Ever since elementary school, teachers would draw a really big distinction between math and science. It's like there's math and there's science and there isn't any correlation." Caitlin appreciated the fact that there was not a prescribed procedure: "It was really interesting because we didn't have a textbook and we didn't have any experience with these equations. It was really cool to go in completely blind, not knowing what we're doing, and come out with a proof of Ampere's Law." Brent enjoyed the ability to authentically engage in the scientific process: "We figured out an equation without really any help. We went through the same process that [Ampere] went through…and we're seventh graders. That's pretty cool."

Limitations

There are three potential limitations to this research study. The first limitation is that researcher bias may have influenced data analysis. The first author developed the materials for the *Deriving Ampere's Law* task, giving her an intimate understanding of the task, as well as her own modeling strategies. To mitigate how this may have influenced data analysis, both authors recorded field notes separately and they also triangulated their findings across multiple data sources. The second potential limitation is that students' problem solving strategies may have been influenced by the decision to have students work in groups of three. Having students work on the task independently or in groups of two may have uncovered different problem solving strategies. The third limitation is related to the generalizability of the findings. The students who participated in the activity had all taken an engineering course in seventh grade and had prior experience working with solenoids. The students who were the focus of this paper were also advanced in their mathematics coursework, having completed algebra. While these students are not representative of typical middle school students, their selection was purposeful. The primary goal of this study was to see if students could develop a realistic model and secondarily, to see how students utilized their conceptual and procedural knowledge of algebra. Given the nature of the research questions for this study, selecting students who had already taken algebra was beneficial.

Conclusion

Brent, Caitlin, and Erin were able to develop a mathematical model involving three independent variables, despite having not done this type of modeling prior to participating in the model-eliciting *Deriving Ampere's Law* activity. This activity emphasized the modeling process itself rather than asking students to replicate known procedures. These students needed to apply their prior conceptual and procedural knowledge and utilize productive problem-solving strategies in order to develop their mathematical models. These students were excited by the challenge of the *Deriving Ampere's Law* activity and remained engaged throughout the entire session. Despite the limitations noted above, the findings from this study are incredibly encouraging. The findings from this study and from other iterations of the *Deriving Ampere's Law* activity with different student groups show that challenging model-eliciting activities such as this are accessible and enjoyable for middle school students.

For this paper, the *Deriving Ampere's Law* activity was separated into four investigations. However, depending on the depth of participants' prior knowledge, the task can be approached holistically. The immediate plan for future research is to have pre-service mathematics teachers and pre-service science teachers complete the *Deriving Ampere's Law* activity as a single task rather than as separate investigations. Additional plans for future research include: (1) further unpacking students' affective reactions and (2) reworking some design aspects of the *Deriving Ampere's Law* activity. The data collected for this paper, particularly the data from the pre-algebra group, indicated that there were aspects of the *Deriving Ampere's Law* activity that influenced students' persistence and perseverance with the task. This data will be further analyzed to explore students' affective responses. If possible, the students who participated in the activity will be interviewed again to further reflect on their experience and to what extent, if any, the experience influenced their thinking about mathematics and science during the school year.

In addition to unpacking students' affective responses, plans for future research also include reworking some design aspects of the activity itself. The original set of solenoids developed for the activity were calibrated using 3.16 A because of the limitations of the variable power supply used. A new set of solenoids, using a more reliable power supply, is under development to provide students with the ability to interact with current as a variable in a more meaningful way. With more reliable materials, a holistic version of the *Deriving Ampere's Law* activity could be implemented with middle school students to get a better understanding of how students might approach the activity with limited scaffolding. Finally, opportunities for commercialization of the activity are underway to be able to develop and mass-produce sets of pre-wound, calibrated solenoids. These efforts would allow the *Deriving Ampere's Law* activity to be implemented in a classroom setting, such as a middle school mathematics or science class.

CHAPTER 4

Pre-Service Mathematics and Science Teachers' Modeling Strategies: The Derivation of Ampere's Law

Modeling is a key component of authentic science and mathematics practices. Scientific progress depends on the ability of scientists to develop models to "represent, replicate, observe, and test their ideas, hypotheses, and theories" (Atkan, 2016, p.7). Mathematical models are essential tools for finding solutions to real-world problems that exist outside of the domain of mathematics (Daher & Shahbari, 2015). Both the Next Generation Science Standards (National Research Council, 2012) and the Common Core State Standards (National Governors Association Center for Best Practices, 2010) emphasize modeling as an essential practice in their respective domains. The National Council for Teachers of Mathematics (2000) emphasizes the importance of students experiencing mathematics in context and encourages that students be given the opportunity to apply their understanding of mathematics to solve problems that exist outside of the realm of mathematics.

Despite the emphasis placed on modeling in curricular standards, national and international assessments consistently document that students struggle with applying their understanding of science and mathematics to novel problems and contexts. The results from the 2011 Trends in International Mathematics and Science Study (TIMSS) indicated that 60% of eighth-grade students in the United States did not meet the high benchmark in science and 70% did not meet the high benchmark in mathematics (Martin, Mullis, Foy, & Stanco, 2012; Mullis, Martin, Foy, & Arora, 2012). At the high level in science, “students demonstrate understanding of concepts related to science cycles, systems, and principles” (Martin, Mullis, Foy, & Stanco, 2012, p. 111). At the high level in mathematics, “students can apply their understanding and knowledge in a variety of relatively complex situations” (Mullis, Martin, Foy, & Arora, 2012, p. 113). Being able to communicate their understanding of science or model situations using mathematics is indicative of students performing at the advanced level, the highest achievement benchmark on the TIMSS scale. Transitioning between real-world problems and models is particularly challenging for students, especially since formal education emphasizes abstraction (Crouch & Hanes, 2004). In order to better support students’ ability to develop and apply scientific and mathematical models, it is critical that the modeling process be incorporated into classroom instruction.

Relevant Literature

A model can be broadly defined as “a representation of an idea, an object, an event, a process or a system” (Gilbert & Bouter, 1998, p. 53). Models can be realized through a variety of different mediums, such as written descriptions, diagrams, verbal explanations, or physical object and they provide us with the opportunity to study concepts that might otherwise be inaccessible or invisible (Atkan, 2016). When modeling real-world situations, students must interpret and make sense of complex and imperfect information in order to create a meaningful representation of the given situation (Daher & Shahbari, 2015). Students are also engaged in several cognitive processes during modeling activities, including “interpreting, discussing, translating, [and] validating”

(Daher & Shahbari, 2015, p. 27). The translation between the real-world problem and the developed model is a defining characteristic of science and mathematics (Crouch & Hanes, 2004). Within each discipline, however, there is some variation in the way models are defined and used.

Modeling in Science and Mathematics

In science, models act as a bridge between theory and reality, making models an integral part of the study and advancement of science (Gilbert, 2004). A scientific model can be defined as "an abstraction and simplification of a system that make its central features explicit and visible" (Kenyon, Davis, & Hug, 2011, p. 2). There are two basic types of scientific models, conceptual models and expressed models. A conceptual model is one's internal representation, while an expressed model is an external representation of one's conceptual model (Kenyon, Davis, & Hug, 2011).

In mathematics, modeling is the process of representing real-world situations using mathematics as a way to understand and solve a specified problem (Daher & Shahbari, 2015) and the model itself is the mathematical description of the real-world situation (Lesh & Lehrer, 2003). Mathematical modeling requires one to be able to move fluidly between the real world and the mathematized world. Mathematical modeling is rooted in the "assumption that humans interpret their experiences using internal conceptual systems (or constructs) whose functions are to select, filter, organize, and transform information, or to infer patterns and regularities beneath the surface of things" (Lesh & Lehrer, 2003).

Both scientific modeling and mathematical modeling are cyclical processes (see Figures 1 and 2). The scientific modeling process includes four phases: constructing the model, using the model, evaluating the model, and revising the model. The construction phase involves analyzing the system or phenomenon to be modeled, identifying the key features, and determining how these features are best represented. The next phase is using the model as a way to describe, analyze, and make predictions about the system or phenomenon. Based on the outcomes from the "using" phase, the model is then evaluated and revised (if necessary) to better accomplish the model's intended purpose.

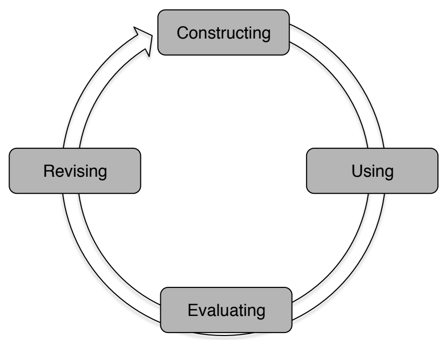

Figure 1. Scientific modeling process (Kenyon, Davis, & Hug, 2011).

When creating a mathematical model, one needs to first decide how a real-world problem should be mathematized and then interpret what information given in the real-world problem is relevant and which mathematical techniques are appropriate in developing the model (Crouch & Hanes, 2004). As mathematical models are developed, they are tested and revised and the initial real-world problem itself is revisited and reinterpreted as the model is amended (Delice & Kertil, 2015).

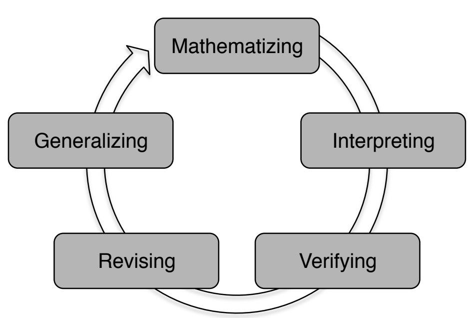

Figure 2. Mathematical modeling cycle (Delice & Kertil, 2015).

Incorporating Modeling into the Classroom

Modeling allows students to engage in authentic science and mathematics practices (Gilbert, 2004; Crouch & Hanes, 2004). There are several characteristics of authentic curricula, including replicating the work of professionals in the field, encouraging creative thought, exploring a small network of key ideas in depth, and working across disciplines to solve human problems (Gilbert, 2004). When engaged with authentic science and mathematics curriculum, students must be afforded the opportunity to develop and test their own models. However, there is evidence that this practice is not commonplace in schools. The empirical literature documents that students struggle with the modeling process. That being said, modeling can be taught. According to Gilbert (2004), there are four discrete steps to learning the modeling process: (1) using existing models, (2) revising existing models, (3) reconstructing existing models, and (4) constructing new models. In order for students to learn how to engage in scientific and mathematical modeling practices, teachers themselves need to be proficient in the modeling process.

The way in which one engages with models and the modeling process can be categorized based on the depth of use (Grosslight, Unger, & Jay, 1991; Harrison & Treagust, 2000). At the highest level, one recognizes that models are meant to explore concepts and that models can be manipulated based on what is needed to solve a given problem. This level of use describes the way in which modeling experts engage with the process. At the lowest level, one assumes that there is a one-to-one correspondence between the model and reality. At this level, one does not explore the model beyond its surface appearances. This level of use generally describes the way in which those with limited modeling experience engage in the process. Some of the difficulties with modeling can be attributed to having a limited knowledge base and one's ability to translate between different contexts.

Pre-Service Teachers' Proficiency with Modeling

Facilitating modeling activities requires flexibility with pedagogical content knowledge, comfort with reform-teaching methods, and the ability to understand and interpret students' thinking. Teachers often develop these skills through experience, making it particularly difficult for pre-service and early career teachers to incorporate modeling activities into their classroom instruction.

Pre-Service Science Teachers. The challenges that new science teachers face can be summarized into five major themes: understanding the content and the disciplines of science, understanding learners, understanding instruction, understanding the learning environment, and understanding professionalism (Davis, Petish, & Smithey, 2006). The first four of these themes can influence teachers' ability to incorporate modeling activities into their own classrooms. Pre-service science teachers (PSSTs) recognize that active learning is more effective for promoting student learning, but are still hesitant towards incorporating this type of instruction in their own teaching (Doster, Jackson, & Smith, 1997). Two main areas of concern for pre-service teachers are their depth of content and pedagogical knowledge and their unfamiliarity with the modeling process.

Insufficient content knowledge and pedagogical knowledge are sources of anxiety for PSSTs (Doster, Jackson, & Smith, 1997). Many new science teachers lack adequate content knowledge (Davis, Petish, & Smithey, 2006). Science teachers, especially at the elementary and middle grades, are expected to posses a broad range of content knowledge (e.g., earth science, life science, physical science) and this content knowledge needs to be of sufficient depth in order to be able to teach effectively. However, accumulating this level of content knowledge is challenging at the university level since courses are often taught modularly (Gilbert, 2004). Pedagogically, science instruction relies heavily on teacher-dominated methods, such as lecture and demonstrations, and many PSSTs have experienced some level of success in these types of classroom environments during their own K-12 careers (Doster, Jackson, & Smith, 1997). Given their anxiety regarding their own content knowledge, PSSTs may revert to teacher-dominated pedagogical methods when given the opportunity to plan and implement their own instructional activities rather than incorporating modeling activities.

Science curriculum is rooted in understanding the nature of models and being able to develop and use models, but engaging students in authentic modeling experiences is difficult for experienced teachers. It is even more challenging for PSSTs who may have limited modeling experiences themselves. In a case study of seven pre-service teachers, Atkan (2016) found that all of his participants recognized the importance of models in science teaching and learning, but were less enthusiastic about providing instructional opportunities for students to develop their own models, citing concerns of students' prior knowledge and limited classroom time as being hindrances to the modeling process. Engaging students in modeling activities requires teachers to not only understand the scientific modeling process themselves, but to also be able to interpret students' thinking and understanding (Davis, Petish, & Smithey, 2006). Atkan (2016) concluded that it was the pre-service teachers' own limited experience with and understanding of scientific modeling that made it difficult to incorporate modeling activities into their own instruction.

Pre-Service Mathematics Teachers. Lesh and Lehrer (2003) argue that the use of modeling perspectives in classroom instruction emphasizes the idea that “expertise in teaching is reflected not only in what teachers can ‘do,’ but also what they ‘see’ in teaching, learning, and problem-solving situations” (p. 111). Previous success with routine mathematics tasks does not imply proficiency with mathematical modeling. Garofalo and Trinter (2013) found that pre-service mathematics teachers (PSMTs) who were able to successfully complete textbook trigonometry exercises struggled with generating mathematical models to represent the projectile motion of a softball and the periodic motion of a pendulum. Mathematical modeling requires flexibility with multiple representations, making the modeling process difficult for both students and teachers. Engaging in mathematical modeling requires more than computational proficiency; it requires being able to interpret situations through a mathematical lens. Because of this, teachers have a tendency to avoid using modeling tasks in their own classrooms (Delice & Kertil, 2015).

Being able to represent a problem in multiple ways is an important skill for successful modeling. One representation that can be particularly helpful early in the modeling process is creating a pictorial representation of the situation. Delice and Kertil (2015) studied pre-service teachers’ ability to develop a mathematical model that described the change in radii of two rolls of cassette tape as the tape from one roll is transferred to the other roll. They found that most of the participants began the task by drawing a diagram, but noted that very few of the diagrams were correct representations of the problem. They also observed that the drawn diagram played a significant role in scaffolding the next steps of the modeling process, which could be problematic considering the inaccuracies in many of the pre-service teachers' diagrams. The pre-service teachers' inability to generate a correct pictorial representation hindered their ability to successfully develop a mathematical model.

Successful mathematical modeling also requires the ability to critically analyze mathematical models. Zbiek (1998) found that PSMTs who relied too heavily on technological tools to generate mathematical models for different sets of data struggled with explaining the appropriateness of models in mathematical terms. Of the PSTMs who participated in a four-week unit on mathematical modeling, those who exclusively used curve-fitting software to generate a variety of different functions to model given sets of data tended to choose their models based on "goodness of fit" (i.e., r^2 values) regardless of whether or not the model reflected the relationship visible in the data's scatterplot.

For some pre-service teachers, connecting mathematics with real-world situations is difficult when solving tasks that are even simpler than developing mathematical models. Verschaffel, de Corte, and Gorghart (1997) found that pre-service teachers tended to disregard the value of real-world knowledge when solving word problems. A group of pre-service teachers were tested on a series of word problems, with some of the problems worded in such a way that a correct answer required a realistic analysis of the given constraints of the problem (e.g., If a school bus seats 36 students, then how many school buses are needed to transport 450 students?). These same pre-service teachers were also tasked with grading students' responses to the same word problems. Overall, the pre-service teachers tended to disregard the realistic context of each word problem when generating their own solutions and they had a tendency to preference non-realistic student solutions as well. A lack of representational fluency and a tendency to disregard real-world knowledge are two major issues that pre-service teachers face that would make incorporating modeling activities into classroom instruction particularly difficult.

Providing pre-service teachers with authentic modeling experiences during their teacher preparation program can help uncover their proficiency with and attitudes towards incorporating modeling activities into their own classroom instruction. The modeling activity on which this study is based was designed to provide students with the opportunity to develop a mathematical model to describe a scientific phenomenon. Pre-service teachers are asked to relate magnetic field strength to the different attributes of a solenoid (further explained below). The research question we explore in this paper is: What strategies do pre-service mathematics teachers and pre-service science teachers use when developing mathematical models and to what extent (if any) does their academic track influence their modeling strategies?

Methodology

Ampere's Law ($B = \mu \frac{N}{L} I$) relates the strength of the magnetic field produced by a solenoid (B) to the number of coils of wire (N), the length of the solenoid (L), and the current passing through the wire (I). A solenoid is a coil of conductive wire; when electric current flows through the wire, the coil generates a magnetic field (see Figure 3). Ampere's Law can be derived experimentally by systematically varying the different attributes of a solenoid. The number of wraps of wire and the current are directly related to magnetic field strength, while the length of the solenoid is inversely related to magnetic field strength.

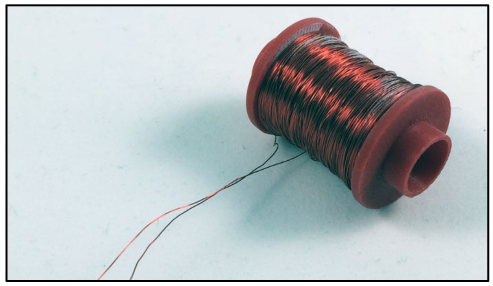

Figure 3. Example of a solenoid.

Setting

The setting for this study was a major public university located in central Virginia. Within this university is a self-contained college of education that offers both graduate and undergraduate programs of study, including a dual-degree teacher preparation program and a post-graduate teacher preparation program. The *Deriving Ampere's Law* activity took place at this university during the fall semester of the 2016-2017 school year.

Participants

Four pre-service teachers participated in the *Deriving Ampere's Law* activity. These pre-service teachers were purposefully selected based on their teacher certification program and their prior coursework. Of the four pre-service teachers, Anna and Emily were pre-service mathematics teachers (PSMTs) and Michael and Reid were pre-service science teachers (PSSTs). At the time of data collection, Anna, Emily, and Michael were in their fourth year of the five-year dual-degree program (bachelor's degree in mathematics or science and master's degree in teaching). Reid was in his first year of a two-year post-graduate teacher preparation program (master's degree in teaching). Anna and Emily were enrolled in a yearlong secondary mathematics pedagogy course and were completing their undergraduate degree in mathematics. Both Michael and Reid were enrolled in a yearlong secondary science pedagogy course and Michael was completing his undergraduate degree in biology, while Reid had already completed his undergraduate degree in physics.

The *Deriving Ampere's Law* activity was scheduled for two separate sessions, ranging from two hours to two-and-half hours in length, which included a post-activity debriefing. The pre-service teachers were grouped based on their academic track. The two PSMTs completed the activity in November 2016 and spent approximately two hours on the activity. The two PSSTs completed the activity in December 2016 and spent approximately two-and-a-half hours on the activity.

Task Description

The impetus for the *Deriving Ampere's Law* activity was to explore how solenoids could be utilized as a hands-on manipulative in the teaching and learning of mathematics. The two authors[3] met to research how different parameters affected the magnetic field strength of a solenoid, which inspired the question of whether or not Ampere's Law could be derived experimentally. To test this theory, the authors decided to create a calibrated set of solenoids. Over the course of several months in early 2016, the first author developed multiple sets of solenoids and used a variety of different methods to measure magnetic field strength before the activity was finalized.

To complete the *Deriving Ampere's Law* activity, pre-service teachers were provided with a set of pre-made solenoids that vary in both the number of wraps of wire and solenoid length. Pre-service teachers were also provided with a variable DC power supply and a magnetic field sensor. In order to generate magnetic field measurements that

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[3] J. Garofalo (second author)

would allow a wide range of participants to be successful with this activity, the pre-wrapped solenoids were calibrated during task development (Table 1). The power supply used for this activity consistently and reliably held 3.16 A, which is why the solenoids were calibrated using this current value.

Table 1

Solenoid Data Collected under Laboratory Conditions

Number of Wraps (N)	**Solenoid Length (L)**	**Electric Current (I)**	**Field Strength (B)**	**Constant (μ)**
50	2 in	3.16 A	35.97 G	0.455
100	2 in	3.16 A	71.87 G	0.455
150	2 in	3.16 A	107.8 G	0.455
50	1 in	3.16 A	71.87 G	0.455
50	2 in	3.16 A	35.97 G	0.455
50	3 in	3.16 A	24.20 G	0.459
50	4 in	3.16 A	18.10 G	0.458
50	2 in	0.79 A	9.0 G	0.456
50	2 in	1.58 A	17.97 G	0.455
50	2 in	3.16 A	35.97 G	0.455

In previous iterations of the *Deriving Ampere's Law* activity, the activity was divided into four separate investigations for use with several groups of middle school students (see Corum & Garofalo, 2017). For the first three investigations, students were asked to develop separate models for each of the independent variables and for the fourth investigation, the students were asked to look at their three separate models and use the structure of these models to help them develop a final model relating the three independent variables to a single dependent variable. All of the middle school students who completed this four-part version of the *Deriving Ampere's Law* activity were able to derive the final model with varying degrees of scaffolding. Given the nature of the research questions for this project, the *Deriving Ampere's Law* activity was presented to the pre-service teachers more holistically. The pre-service teachers were given all of the solenoids upfront and were asked to develop a model that related number of wraps of wire, solenoid length, and electric current to magnetic field strength.

Data Collection

Pre-service teachers were video recorded while working on the *Deriving Ampere's Law* activity and the audio was transcribed. Pre-service teachers' written work was also collected for analysis. While pre-service teachers worked on the activity, they engaged in discussions with their partner regarding their modeling strategies. Both authors recorded field notes throughout the activity session to further capture pre-service teachers' conversations and their written work. Upon completing the activity, the pre-service teachers participated in a debriefing interview to further explore their opinions about the activity and how this activity compared to their classroom experiences.

Data Analysis

The primary goals of this project were to understand how pre-service teachers' prior experiences influenced how they approached a modeling activity and which strategies they utilized when developing their models. Data analysis began after both groups of pre-service teachers completed the *Deriving Ampere's Law* activity. The first author analyzed the transcript and the written work for the PSMT group and prepared narrative descriptions of the PSMTs solution strategies. The first author then went through the same initial data analysis procedures for the PSST group. After completing the initial round of data analysis, the second author reviewed the narrative descriptions separately from the first author.

The two authors then met to confirm the first author's interpretation. The two authors reread parts of the transcript, reanalyzed pre-service teachers' written work, and reviewed their separately collected field notes. During this meeting, both authors regularly revisited their multiple data sources to ensure that their analysis and interpretations were warranted. The two authors triangulated the narrative descriptions of both groups with the observational field notes, audio transcripts, and pre-service teachers' written work and both authors came to a consensus that the narrative accurately captured what the pre-service teachers had done to complete the *Deriving Ampere's Law* activity.

Findings

Both groups of pre-service teachers were able to experimentally derive Ampere's Law. The PSMTs and the PSSTs had little difficulty with recognizing the structure of their final model based on the nature of the relationships between the independent variables and the dependent variable (i.e., direct or inverse variation). Both groups established a strategy prior to collecting data and were systematic in their data collection and data analysis. Both groups also regularly tested and revised their model against their collected data. However, the groups differed in approach to determining the constant of proportionality and the extent to which they focused on the units of measurement.

Pre-Service Mathematics Teachers

Prior to starting the activity, the first author asked Anna and Emily if they were familiar with solenoids. After seeing the pre-wrapped solenoids, Emily recalled an activity from elementary school where she wrapped wire around a nail to create an electromagnet. This familiarity with solenoids was sufficient for completing the task, so the first author proceeded to demonstrate how to use the equipment to measure the magnetic field strengths of the different solenoids. When demonstrating the equipment, the first author asked Anna and Emily to select one of the solenoids. After selecting a solenoid, the first author told them the length of the solenoid (two-inches) and the number of wraps (150). Both Anna and Emily wanted to record that information; Anna set up a data table, while Emily listed the information.

After the equipment demonstration, Anna and Emily decided how they were going to proceed with the data collection. Anna asked if they should measure the "tiny one" (one-inch, 50-wraps) next, but Emily suggested that they establish a plan. She asked Anna, "We don't want to vary a bunch of things at once, so do you want to look at same wraps, different length? Same length, different wraps? Same solenoid, different current?" Anna wanted to focus on the same length with different wraps of wire. Emily asked Anna if she wanted to change the current, but Anna explained that since they have already measured the two-inch, 150-wrap solenoid at 3.16 A, they should collect the rest of their data for the two-inch solenoids at 3.16 A.

Data collection. They then measured the two-inch, 50-wrap solenoid (36 G) and the two-inch, 100-wrap solenoid (72 G). Emily described the relationship as, "Alright, so more wraps, more magnets." Anna asked if Emily wanted to change the current next and Emily agreed. Because the maximum electrical current with the power supply was 3.16 A, Anna suggested lowering the current to 3.0 A. They measured the two-inch, 100-wrap solenoid at 3.0 A (67 G) and the two-inch, 50-wrap solenoid at 3.0 A (33 G). Anna commented that there was a difference between the field strengths, but explained that she had not yet noticed much of a pattern. Emily predicted that the relationship would be a ratio because "addition is too easy."

Next, they measured the two-inch, 150-wrap solenoid at 3.0 A (102 G). Emily and Anna noticed that the magnetic field strength generated by the two-inch, 100-wrap solenoid and the two-inch, 150-wrap solenoid at 3.0 A was 5 G less than the field strength generated at 3.16 A, but that the difference in the two-inch, 50-wrap solenoid was 3 G. They measured the two-inch, 50-wrap solenoid again and saw that their second reading was still 33 G. With the data collected, Emily checked to see if the ratio was the same between the change in number of wraps and the change in magnetic field strength. She noticed that 50 wraps multiplied by two equals 100 wraps and 36 G multiplied by two equals 72 G. Similarly, 100 wraps multiplied by 1.5 equals 150 wraps and 72 G multiplied by 1.5 approximately equals 108 G (Figure 4).

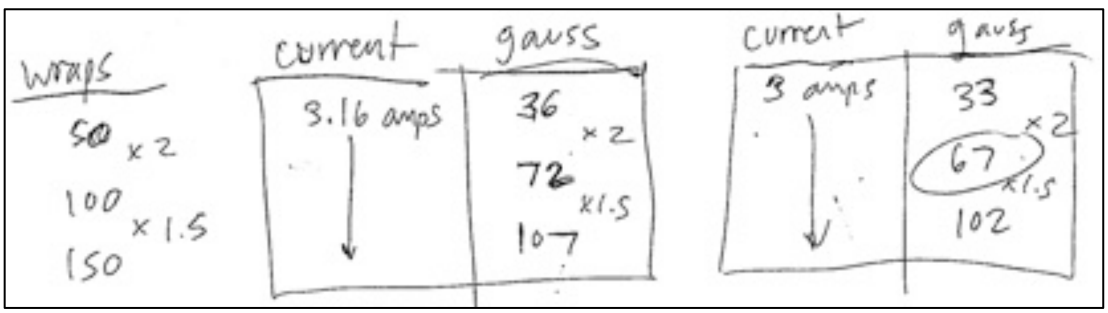


Figure 4. Emily's data table for solenoids of varying wraps of wire.

Emily noticed that relationship she observed at 3.16 A also held true at 3.0 A. Anna asked if there was a relationship between the 50-wrap solenoid and the 150-wrap solenoid. Emily checked and confirmed that the magnetic field strength produced by the 150-wrap solenoid (107 G) is approximately three times the field strength produced by the 50-wrap solenoid (36 G). She summarized this relationship as, "As the number of wraps increases by a factor of *x*, so does the magnet strength," to which Anna responds, "We have a direct variation." Both Anna and Emily shared that direct variation was a topic they had recently observed during their practicum placements. Having described the relationship qualitatively, Anna recalled the ultimate goal of the task and asked how they would write the relationship as a formula. Emily noted that because the relationship between number of wraps of wire and magnetic field strength was direct variation, the formula should be in the form, “wraps over magnet is some factor” and that they needed to determine the factor. Emily divided the number of wraps of wire by magnetic field strength and found that this resulted in a different factor for readings taken at 3.16 A versus 3.0 A.

Emily suggested that the relationship might be the number of wraps of wire divided by magnetic field strength equals one-half current ($\frac{wraps}{gauss} = \frac{1}{2}amps$) and that the difference they observed at 3.16 A could be attributed to measurement error. Anna suggested that they measure the two-inch solenoids at 2.0 A to see if the relationship held true. As they set up the equipment to collect more data, Emily reminded Anna that they still have not considered solenoid length, but realized that variable should not affect the relationship between number of wraps of wire and magnetic field strength because it was being held constant. Anna cautioned against varying too many things at once:

Anna: *…we have too many variables.*
Emily: *We have too much going on.*
Anna: *And the amps.*
Emily: *That’s why we’re trying to vary just one thing.*
Anna: *Exactly.*
Emily: *Okay, do you want to do two amps and see what would happen for these three?*
Anna: *Mmm hmm.*

After collecting data at 2.0 A, Anna again questioned the decision to vary the current while analyzing the relationship between number of wraps of wire and magnetic field strength. Anna commented, “We’re kind of varying two things at the same time…We’re keeping length constant, but we’re changing number of amps and wraps.” Emily reassured Anna that they were "being systematic" and that they are limited in the extent to which they can change the number of wraps of wire. Emily wanted to confirm that the relationship between number of wraps of wire and magnetic field strength held at different current values, while Anna interpreted this as varying two independent variables at the same time. With the data collected at 2.0 A, Emily divided number of wraps of wire by magnetic field strength, which equaled approximately 2.17. While this did not fit Emily's earlier hypothesized relationship ($\frac{wraps}{gauss} = \frac{1}{2} amps$), both Emily and Anna noticed that the direct variation they observed at 3.0 A also held true at 2.0 A.

At this point, Emily abandoned her initial relationship of $\frac{wraps}{gauss} = \frac{1}{2} amps$, but confirmed that there was a direct variation between number of wraps of wire and magnetic field strength. She then suggested, "If we find out all the different ways they [the independent variables] vary, we can just kind of put it together." Anna then suggested that they collect data at 1.0 A and calculate the constant (see Figure 5). Anna recommended discarding the data they collected at 3.16 A because the data collected in trials two, three, and four "increment at a nicer rate." She then commented that they could take measurements at 0 A to be "really thorough," but Emily objected, explaining "It's all gonna be zero, it's not gonna make a magnet!"

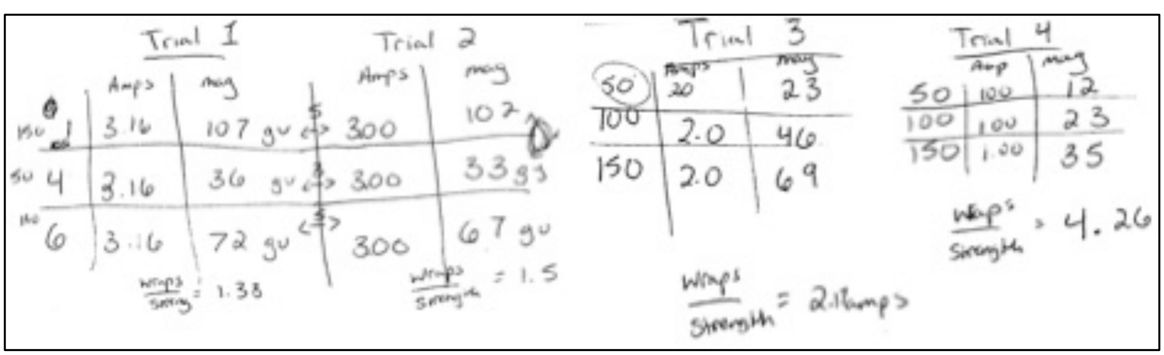


Figure 5. Anna's data table for solenoids of varying wraps of wire.

Anna then asked if Emily wanted to collect data for different lengths. Emily confirmed that they will be keeping the number of wraps constant and only varying the length. Both Anna and Emily set up tables before collecting data (Figure 6). Emily suggested collecting data at the same current values that they used previously, but disregard 3.16 A. Both Anna and Emily agreed that they should first collect their data at 1.0 A because that was the current setting for the power supply.

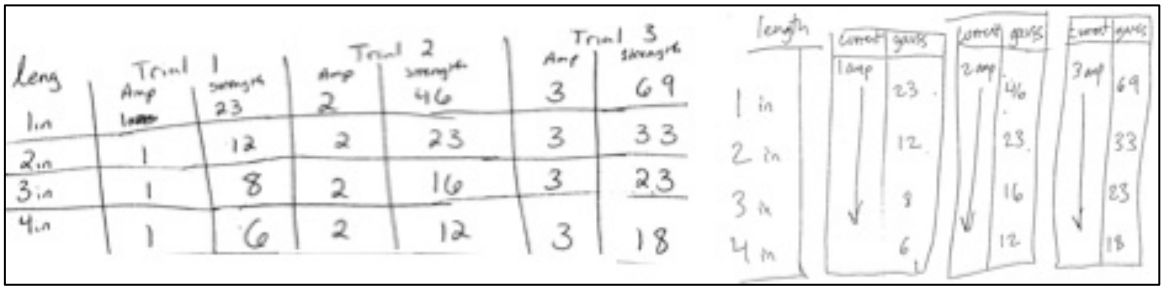


Figure 6. Anna's (left) and Emily's (right) data for solenoids of varying length.

They first measured the one-inch, 50-wrap solenoid at 1.0 A, which generated a magnetic field strength of 23 G. Anna commented that she had seen 23 G previously. Looking back at her data tables, she saw that the two-inch, 100-wrap solenoid also generated a magnetic field strength of 23 G at 1.0 A. Emily then looked over her data and noticed that the two-inch, 50-wrap solenoid generated a field strength of 23 G at 2.0 A. They then set the power supply to 2.0 A and 3.0 A and saw that the one-inch, 50-wrap solenoid generated magnetic field strengths of 46 G and 69 G, respectively. Emily observed that the field strength at 2.0 A was double the field strength at 1.0 A and Anna observed that the set of data was identical to the data they collected for the two-inch solenoid of varying wraps at 2.0 A. Anna reminded Emily that they had already collected data for the two-inch, 50-wrap solenoid, so she suggested that they refer to their first data table. Looking at their previously collected data, Anna commented that the magnetic field strength was approximately increasing by 10 G as the current increased by 1.0 A. Emily noticed a different relationship. Multiplying 12 G (the magnetic field strength generated at 1.0 A) by two approximately equaled the magnetic field strength generated at 2.0 A. Multiplying 12 by three approximately equaled the magnetic field strength generated at 3.0 A. This relationship was the same relationship she observed with the one-inch, 50-wrap solenoid.

It is important to note that while they were varying the length of the solenoids, they were analyzing how the magnetic field strength changed for each solenoid as the current changed. Emily described the relationship between current and magnetic field strength as another direct variation. They measured the three-inch, 50-wrap solenoid at 1.0 A (8 G), 2.0 A (16 G), and 3.0 A (23 G) next and Anna noticed that one of their magnetic field strengths was again 23 G. They then measured the four-inch, 50-wrap solenoid at 1.0 A (6 G) and 2.0 A (12 G). Before changing the current to 3.0 A, Anna prompted Emily to make a prediction. Emily predicted that the field strength at 3.0 A would be 18 G, which they confirmed experimentally. Emily analyzed the data that they had collected so far and summarized the relationship between current and magnetic field strength. She explained, "There's direct variation again…Current over Gauss equals *x*, so both the number of wraps and the current vary directly with the strength." Emily also clarified that they still had not considered how length affects magnetic field strength, which seemed to confuse Anna, who commented, "But we were measuring length. Aren't we supposed to be doing lengths?"

Emily acknowledged that when they started this round of data collection, the intent was to explore the relationship between the length of the solenoid and the strength of the magnetic field. However, they ended up identifying the relationship between electric current and magnetic field strength. Anna commented that because they varied electric current when looking at the relationship between number of wraps of wire and magnetic field strength, they had already identified the relationship between electric current and magnetic field strength. As Anna explained,

> The same thing happened, amp-wise, for these [solenoids] when we looked at wraps and strength…If we're looking at amps and strength here [with solenoids of varying lengths], then this should be the same…We're not looking at the length and the strength. We kind of lost what we were trying to measure.

Because of the way they collected their second round of data, Emily thought that the most obvious relationship was the relationship between electric current and magnetic field strength. But with the data they had, they could also analyze the relationship between solenoid length and magnetic field strength. Emily observed that, "The length increases, the strength decreases. That makes sense because it's [wraps of wire] not as close together." Looking at the data they collected at 1.0 A, Anna observed that as you double the electric current, the magnetic field strength "decreases by two." Emily initially disagreed with Anna's observation, but then realized that Anna did not mean to subtract 2 G when she said "decreases by two." Emily restated Anna's observation as, "As the length is times two, the strength is divided by two."

Structuring the model. Once Emily articulated the relationship between electric current and magnetic field strength in this way, she immediately recognized the relationship as inverse variation. She then confirmed this relationship with the data collected at 2.0 A and 3.0 A. Emily recalled that the an inverse relationship is represented by the equation, $xy = k$, and Anna recalled that in an inverse relationship, "as one goes up, the other goes down." Anna then suggested another possibility, "It could technically be direct almost, where that’s [the constant] is something smaller than one.” Anna quickly realized that this would not work, but this caused Emily to question whether or not the relationship they observed was inverse variation. They then decided to check the relationship using their collected data.

Looking at their data, Emily and Anna multiplied the length of the solenoid (x) by the magnetic field strength (y) to determine whether or not this resulted in a constant ($xy = k$). They did this for the data they collected at each of the different current values. They found that at 1.0 A, $k \approx 24$, at 2.0 A, $k \approx 46$, and at 3.0 A, $k \approx 66$ (see Figure 7).

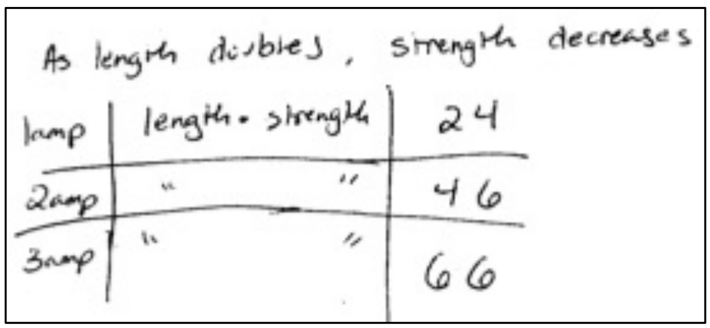


Figure 7. Anna’s verification that *xy* = *k* for data collected at different current values.

Emily then questioned how they were verifying whether or not the relationship between electric current and magnetic field strength was inverse variation. Emily asked, “Wait. Is that how we want to find it? I’m confusing myself.” She then suggested that they review all of the relationships they had identified with the three independent variables.

The relationships they observed were: (1) length increases, strength decreases, (2) amps increase, strength increases, and (3) wraps increase, strength increases. Emily assigned letters to represent their different variables and then recommended that they think about the relationships in terms of strength increasing and drew the following diagram on her paper (Figure 8).

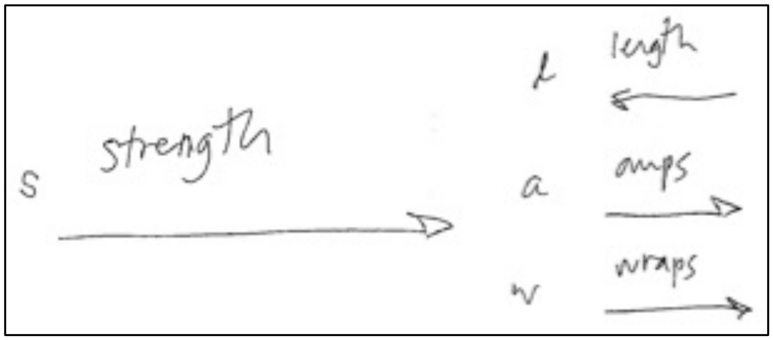


Figure 8. Emily's summary of the relationships between the variables.

Anna then suggested that, using what they identified in terms of direct and inverse variations, they could set up the structure for their final equation. Anna explained, "Alright, so *a* and *w*, in some way or form, have to be on top because they increase when strength increases."

Anna wondered if the two variables in the numerator would be added or multiplied together. Emily hypothesized that the two variables would be multiplied because every relationship they have seen thus far has involved a ratio. Using that line of reasoning, they then decide that they would need to divide by the length of the solenoid. This resulted in the following structure for their final model: $s = \frac{aw}{l}$.

Recognizing the need for a constant. From this model, Anna noticed that the magnetic field strength divided by the number of wraps ($\frac{s}{w}$) of wire should equal the current divided by the length ($\frac{a}{l}$) and that this could be used to help them determine the constant for the equation. Emily then posed the question of whether the constant would be added or multiplied to their equation. Returning to the observation that all of the relationships involved ratios, Emily predicted that the constant would be multiplied. Anna agreed that the constant should be multiplied, but for a different reason. Anna explained, "[In] science, every single time you add something [to an equation], it's a variable of some kind. All of our variables are already taken up, so it can't be that."

To determine the constant, both Anna and Emily agreed that they should use their collected data. Anna suggested using the data they collected for the two-inch, 50-wrap solenoid at 2.0 A. She used $\frac{s}{w} = \frac{a}{l}$ and set up the equation below (Figure 9).

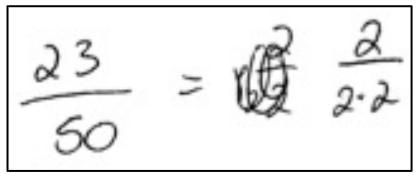


Figure 9. Anna's initial attempt at determining the constant for their equation.

Emily also calculated $\frac{s}{w}$ and $\frac{a}{l}$ and found that those two were not equal using their collected data. However, she noticed that the two expressions were approximately equal if they multiplied the denominator by two. Emily explained, "We need to find a relationship to make this true. Our strength needs to be multiplied by two, which means it's this over $2L$. She then proposed the equation $s = \frac{aw}{2l}$ and used their collected data to see if that equation held true (Figure 10).

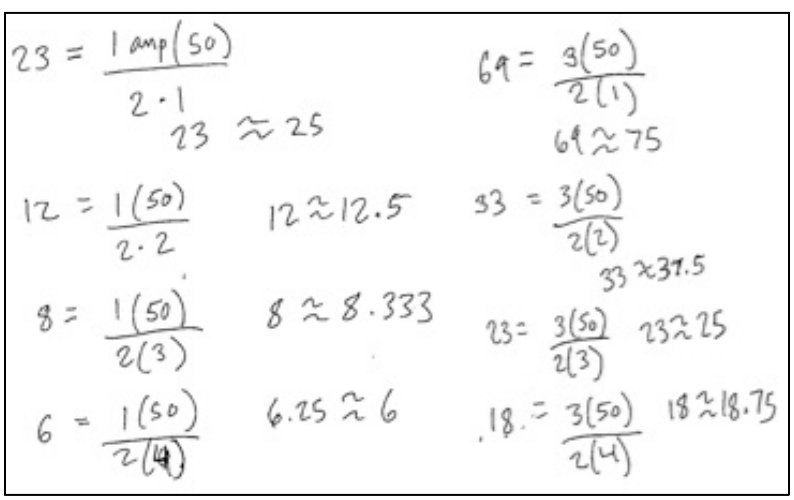


Figure 10. Emily's verification of their initial equation.

Emily saw that $s = \frac{aw}{2l}$ held true for some of their collected data, but not all of it, which led her to question whether or not their constant was correct. Emily said to Anna, "Our constant's not quite two, I don't think. Unless there's just a lot of error." Anna then recalled that during data collection, they had rounded their measurements for magnetic field strength. Anna and Emily continued to use their collected data to verify whether or not $s = \frac{aw}{2l}$ is the correct equation. After testing the equation using all of their collected data, both Anna and Emily acknowledged that the model is not entirely correct, but were unsure whether or not the discrepancy between the predicted values and their collected data could be attributed to measurement error. They discussed other possible factors that could have caused the discrepancy, including not collecting enough trials of data and rounding the magnetic field strengths during data collection.

Testing and revising the model. Anna and Emily were confident that their model's structure was correct, but questioned the accuracy of their constant, as demonstrated in the following exchange:

Emily: *Maybe it's [the constant] not exactly two…We found the direct and inverse variations for all the different factors, so we know that our variables are in the right places, so the only thing we're not 100% sure about is this constant.*
Anna: *I'm wondering if it's a little over two.*
Emily: *Or what if it's a decimal? Oh god, what are we gonna do? How are we gonna tell?*

Thinking that the constant might be a value greater than two, Anna recalled that the metric system is most commonly used in science and suggested that the constant in their equation might be the conversion factor between inches and centimeters. Emily acknowledged that their constant was not exactly correct, but then wondered if they were "reading into it too much" by considering the conversion between inches to centimeters.

Anna looked up the conversion factor (2.54) and Emily agreed that they could try to see if that worked better as the constant in their model. Using $s = \frac{aw}{2.54l}$ as their new model, Emily used their data for the one-inch, 50-wrap solenoid at 1.0 A and saw that this newly proposed constant also did not work.

Anna no longer believed that the constant was related to a conversion factor, but she still wanted to find a value for their constant that was more accurate. Emily disagreed, as evidenced in the exchange below:

Anna:	*Well, it was a nice hunch. That's how scientists did it in the old days, right?*
Emily:	*Yeah, they would just try stuff and see if it worked.*
Anna:	*So, let's see. What else?*
Emily:	*I feel like this is pretty good, but I don't know. I'm 90% confident in this response.*

Emily liked that the constant in the denominator was a "nice number" (an integer) because "that's how a lot of science things look." Anna, however, suggested that they rearrange their equation so that the constant was isolated, giving the "gravitational constant" as an analogy. Emily then solved $s = \frac{aw}{cl}$ for c, which resulted in $c = \frac{aw}{sl}$.

With the equation $c = \frac{aw}{sl}$, Emily suggested that they calculate values for their constant using their collected data point. Both Anna and Emily worked together and found that the constant ranged from 2.08 to 2.27. Given their range of values, Emily asked Anna if she wanted to use $c = 2.2$ instead of $c = 2$ in their final equation. Emily suggested that if the constant was an irrational number, then they would never be able to calculate the constant exactly. Anna then suggested that the constant could be either pi ($\pi \approx 3.14$) or e ($e \approx 2.72$), but quickly realized that neither of those would work.

Anna and Emily now had $s = \frac{aw}{2.2l}$ as their equation. Emily suggested that the only way that they could get a more accurate constant would be if they collected more data. Yet, she was still hesitant to claim that they had their final model because she was "worried that we're gonna say we're done and then we're going to be wrong." Anna confirmed that they had accounted for all of their variables. Emily realized that they determined the value of their constant using the data they collected at 1.0 A, 2.0 A, and 3.0 A, but that they still had the data they collected at 3.16 A. Using $s = \frac{aw}{2.2l}$, Emily confirmed that the magnetic field strength predicted by this equation matched the data they collected at 3.16 A. At this point, both Anna and Emily agreed that their final equation was $s = \frac{aw}{2.2l}$.

Pre-Service Science Teachers

Prior to demonstrating the materials, the first author asked Michael and Reid if they were familiar with solenoids. Michael said that a solenoid is "sort of an electromagnet." The first author then demonstrated how to use the materials and identified the parameters of the different sets of solenoids. At this point, Reid suggested that they write down this information in a data table. The second author then summarized the goal of the task, which was to find a formula that related magnetic field strength to the three independent variables. Michael confirmed that the three independent variables were electric current, solenoid length, and number of wraps of wire. The variable power supply that Michael and Reid used had a safety feature that would hold either electric current or voltage constant. Since there was a possibility that the electric current would need to be manipulated using the knobs labeled "voltage," Michael asked about the relationship between electric current and voltage. He acknowledged that his understanding of electromagnetism was limited and wanted to affirm that voltage was not a variable they were considering, but that it was related to current. Reid clarified that while they may need to use the voltage knobs to adjust the current, they are only concerned with current as one of the independent variables.

Initial data collection. Once the authors finished demonstrating the materials, Michael and Reid began their data collection process. Reid first suggested that they label their variables. He labeled current as *I*, magnetic field strength as *B*, solenoid length as *L*, and number of wraps of wire as *N*. Reid then recalled that he had memorized this relationship once before, but could no longer remember it. Michael asked if Reid wanted to measure the solenoids of varying lengths first. Reid agreed, but clarified that their strategy was to "figure out direct or indirect relationships." Reid then outlined a very detailed plan for data collection and organization. He recommended that they first name the different solenoids using letters *A* through *F*. Then, he said that they should organize their data table based on the variables they will manipulate (Figure 11). He suggested that they represent length as multiples of one (the shortest solenoid length) and number of wraps of wire as multiples of 50 (the least number of wraps).

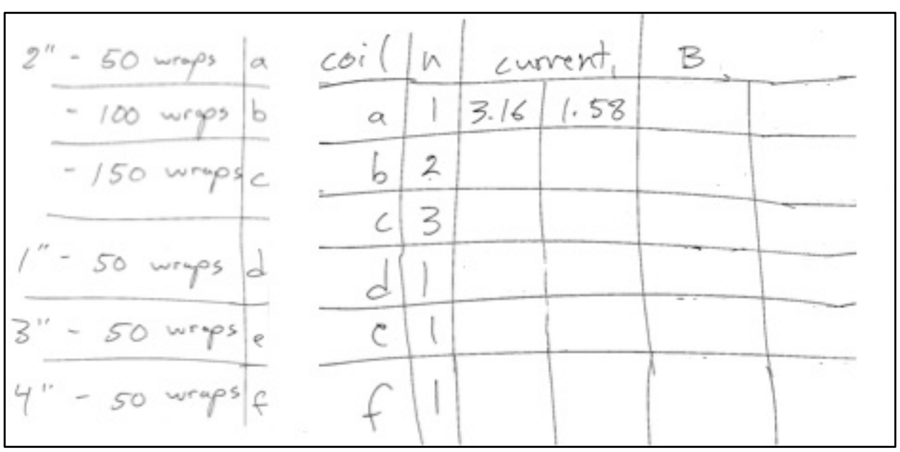


Figure 11. Reid's data table structure.

Reid also recommended that they measure each solenoid at two current values to confirm that they "should see a stronger magnetic field with a higher current." As Reid outlined their strategy, he mentioned that he had forgotten the definition of a Gauss, which was the unit they were using to measure magnetic field strength.

Michael and Reid measured the two-inch, 50-wrap solenoid first. As Michael inserted the probe into the solenoid, Reid explained why the end of the solenoid where the probe was inserted was important. Reid referenced the "right hand rule" and explained that the direction of the magnetic field is based on the direction of the electric current. At 3.16 A, the two-inch, 50-wrap solenoid generated a magnetic field strength of approximately 35.2 G. Reid then suggested that they should take their next measurement at 1.58 A because that was half of the electric current from their first reading. At 1.58 A, the solenoid produced a magnetic field strength of 17.3 G, which Reid recognized was approximately half their previous reading. Reid recommended that they measure another solenoid at 1.58 A and 3.16 A to confirm the relationship between magnetic field strength and current. They measure the two-inch, 100-wrap solenoid next and see that they solenoid generated a magnetic field strength of 37.2 G and 73.7 G at 1.58 A and 3.16 A, respectively. Michael noticed that once again the magnetic field strength doubled as the current doubled. At this point, Reid said to Michael that they did not need to measure all of the solenoids and that they could just measure two solenoids of varying lengths.

Reid clarified how they are representing the number of wraps of wire. Since the number of wraps of wire varied by multiples of 50, Reid felt that representing *N* as multiples of 50 would better "reveal the effect of [the wraps of wire] doubling or tripling" (Figure 12).

Figure 12. Redefining the number of wraps of wire.

Returning to their collected data, Reid summarized what they had observed thus far. Looking at the data collected at 3.16 A, Reid explained that when $N = 1$ (i.e., 50 wraps of wire), the magnetic field strength was 35.2 G and when $N = 2$ (i.e., 100 wraps of wire), the magnetic field strength was 73.7 G, meaning that as the number of wraps of wire doubled, the magnetic field strength doubled. Michael described the relationship as, "it would be the proportion of N," which Reid restated as, "B is directly proportionate with N."

Reid then looked at how electric current related to magnetic field strength. He explained, "We halved [electric current] and we essentially halved [magnetic field strength]…so, the current is directly proportional to the magnetic field too." Knowing that both number of wraps of wire and electric current were directly proportional to magnetic field strength, Reid suggested that in their final equation, the number of wraps of wire (N) and electric current (I) would be in the numerator. At this point, Reid commented, "This is actually the first time I've ever done something like this…In physics class, all the time, we are just given these expressions. Somebody sat down and had to figure out the relationship, you know?"

Michael suggested that they think about another relationship they already knew to help them think about how to better understand the relationship they were trying to model in this task. He referred to this as a "relatable expression" to help him "picture a type of expression that is similar to what we think this is." Michael gave the example that density was equal to mass over volume ($D = \frac{M}{V}$). Using this example, Reid explained that if you doubled the volume, in order for the equation to still hold true, then you would also need to double the mass. Relating this back to magnetic field strength, Reid further explained that if they are observing that the magnetic field strength has doubled, then that means either the number of wraps of wire or the current has also doubled. Reid acknowledged that using density as a proxy to help them make sense of the relationship they were trying to figure out was a good suggestion. He then recommended that they look at the relationship between length and magnetic field strength.

Data collection using a new sensor and structuring the model. At this point, Michael and Reid had left the power supply turned on for an extended period of time and the plastic tip of the magnetic field sensor had melted inside of the two-inch, 100-wrap solenoid. Neither of the authors knew whether or not the melted probe tip would significantly affect the sensor readings, so Michael and Reid continued to use this probe.

Michael and Reid measured the one-inch, 50-wrap probe at 3.16 A and got a magnetic field strength of 81.7 G. They then lowered the electric current to 1.58 A and saw that the magnetic field strength was now 45.3 G. Reid suggested that they add another column to their data table that represents the length of the solenoid divided by one, which was the length of the shortest solenoid (see Figure 14). To see the relationship between solenoid length and magnetic field strength, Michael and Reid compared the magnetic field strengths generated by the two-inch, 50-wrap solenoid and the one-inch, 50-wrap solenoid at 3.16 A, which were 35.2 G and 81.7 G, respectively. Reid said that the relationship was approximately double, but Michael said that based on their data, that was not exactly correctly. Reid suggested that they should measure the other solenoids of different length to confirm the relationship. Michael predicted that, “as the [solenoid] length increases, the magnetic field decreases about proportionally.” Reid recalled from his prior knowledge of solenoids that, “the length of the solenoid, or the density of the wraps, is a factor…if you have the same number of wraps over a shorter length,” and added this to their list of observations (Figure 13), which Reid referred to as their “Things We Think” chart.

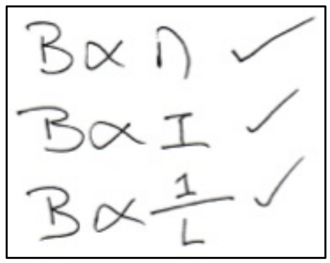
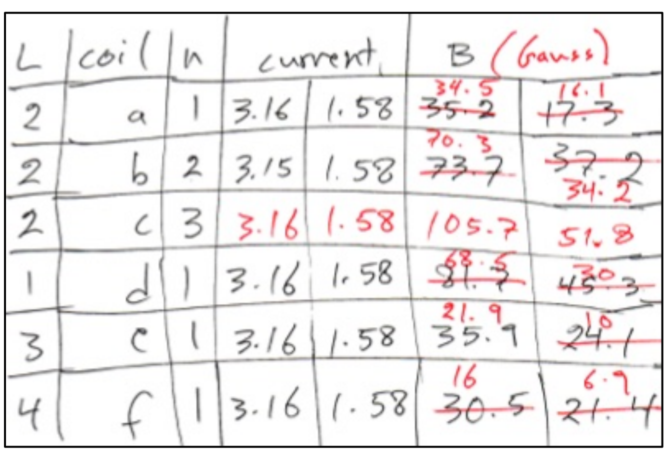


Figure 13. Michael’s “Things We Think” chart.

Michael and Reid then measured the three-inch, 50-wrap solenoid. Based on their belief that magnetic field strength was inversely proportion to solenoid length, the magnetic field strength of this solenoid should be one-third of the magnetic field strength generated by the one-inch, 50-wrap solenoid. The magnetic field strength of the three-inch, 50-wrap solenoid was 24.1 G at 1.58 A and 35.9 G at 3.16 A, which did not match their predictions. At this point, Reid questioned whether or not the magnetic field sensor was producing accurate readings. They then measured the four-inch, 50-wrap solenoid at 3.16 A and 1.58 A and got 30.5 G and 21.4 G, respectively. Reid asked the authors if they were getting “friendly readings” with the damaged probe and the first author questioned the accuracy of their data. She gave Michael and Reid a new probe to use and Reid suggested that they measure one of the solenoids that they had measured prior to the probe getting damaged. Michael clarified that they should measure the solenoid that was already attached to the power supply (four-inch, 50-wrap solenoid) and compare whether or not the new probe gave the same reading as the first probe.

Using the new probe, Michael and Reid measured the four-inch, 50-wrap solenoid at 1.58 A and saw that the magnetic field strength was 9.5 G. Reid asked the two authors which magnetic field strength (21.4 G with the original probe or 9.5 G with the new probe) was more aligned with the data other groups had collected. The first author confirmed that the original probe was not producing accurate readings. Reid noticed that the magnetic field strength for the four-inch, 50-wrap solenoid was changing the longer the solenoid remained connected to the power supply and attributed this to increased resistance as the solenoid was heating up. Michael noticed that the new probe was more sensitive to detecting an ambient magnetic field in the room compared with the original probe. Michael and Reid recognized that they needed to recollect their data because the original probe had been compromised during data collection. Both agreed that they should measure all of the solenoids again to standardize their data. Reid used the same table to record the data collected with the new probe, but wrote the new data in red ink (Figure 14).



L	coil	n	current		B (Gauss)	
2	a	1	3.16	1.58	34.5 ~~35.2~~	16.1 ~~17.3~~
2	b	2	3.15	1.58	70.3 ~~73.7~~	~~37.2~~ 34.2
2	c	3	3.16	1.58	105.7	51.8
1	d	1	3.16	1.58	68.5 ~~81.7~~	30 ~~45.3~~
3	e	1	3.16	1.58	21.9 35.9	10 ~~24.1~~
4	f	1	3.16	1.58	16 ~~30.5~~	6.9 ~~21.4~~

Figure 14. Reid's data table with the data collected using the new probe.

As Michael and Reid collected their data using the new probe, Michael again commented that the probe was detecting a magnetic field even when power supply was not turned off and that this field ranged from -6.0 G to -3.0 G. Reid questioned whether or not that could be attributed to an ambient field in the room since it was so high. He then questioned how that might affect their ability to generate their final relationship. As Reid explained,

> That means we're seeing that there's an ambient [field]. I don't know how to account for that. What do you do? I guess it's all relative, except no, it's not, because we're trying to figure out a relationship, but this [probe] is five to ten [Gauss] off and it's giving us an improper reading of how things are related. We'll deal with it later.

Reid's solution to this issue was to just "keep in mind that we could be five or ten off" when they analyze their data. After Michael and Reid finished measuring all of the solenoids using the new probe, Reid suggested that they return to their "Things We Think" chart (see Figure 13) and confirm that the relationships they previously observed still hold true. Using their newly collected data, Michael and Reid both came to the conclusion that magnetic field strength was directly related to the number of wraps of wire and electric current and inversely related to the length of the solenoid.

Testing the model and grappling with units. Reid summarized their model as $B = \frac{NI}{L}$, but Michael explained to Reid that substituting values for N, I, and L did not result in a value for B that matched their collected data. Using the two-inch, 50-wrap solenoid at 3.16 A, Michael found that the magnetic field strength predicted by their equation was $\frac{1 \times 3.16}{2} = 1.58$, whereas the magnetic field strength they measured was 34.5 G. Reid reminded Michael that "N has a 50 tied up in it" because the values for N in their data table were actually the number of wraps of wire divided by 50. However, using

$N = 50$ in their equation still did not result in the correct value for magnetic field strength ($\frac{50 \times 3.16}{2} = 79$). Reid then suggested that there might be an issue with their units. He tried to recall the definition of a Gauss and how the unit related to magnetic field strength. Looking at their equation ($B = \frac{NI}{L}$), Reid restated the units involved (i.e., number of wraps of wire was "unit-less", electric current was measured in amps, solenoid length was measured in inches). Reid knew that Gauss was a metric unit, which led him to believe that the length of the solenoid needed to be converted from inches to meters.

Reid recalled that one inch is equal to 0.0254 meters, which led him to ask himself if the definition of a Gauss is an amp per meter. He then asked himself the definition of an amp, which he thought was a "Joule per second or something." Michael asked Reid to clarify how they had redefined length in their data table and Reid explained that they divided the length by one-inch to "cancel" the units. At this point, Reid asked if he would be able to look up the definition of a Gauss. The second author asked him why he thought he needed that information. Reid explained,

> We're trying to check an expression that we've come up with…A Gauss is a metric unit and we're measuring [length] in inches…There's no way it [a Gauss] could be anything other than an amp per meter, according to what we think it is, right? I hope I'm doing horrible on this because I'm supposed to be a physics person. Let's just check the first one.

Reid then asked Michael to use their equation to calculate the strength of the magnetic field generated by a one-inch (0.0254 meters), 50-wrap solenoid at 3.16 A. Michael set up the equation (Figure 15) and found that the equation predicted a magnetic field strength of 6,220.5 G. Reid then asked Michael to divide that by 68.5 (the magnetic field strength they had measured for this solenoid). This resulted in 90.8, which Reid classified as "not very friendly."

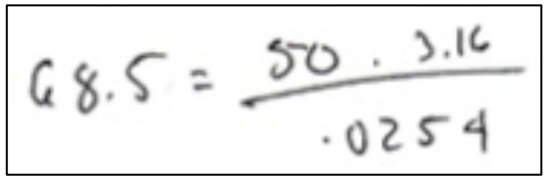


Figure 15. Michael's work to verify their initial equation.

Michael revisited their earlier decision to redefine *N* as the number of wraps of wire divided by 50 (see Figure 12). Michael recommended defining the variables in their table so that the numbers reflected how they would be used in their equation (e.g., since they were using $N = 50$ in their equation, then the value for *N* in their table should also be 50). Reid explained that redefining *N* and *L* in their data table was to help them quickly identify the types of relationships (i.e., direct or inverse), but that was not a necessary step. Michael decided to revise his data table so that the values for *N* reflected the actual number of wraps and the values for *L* were converted to meters (Figure 16).

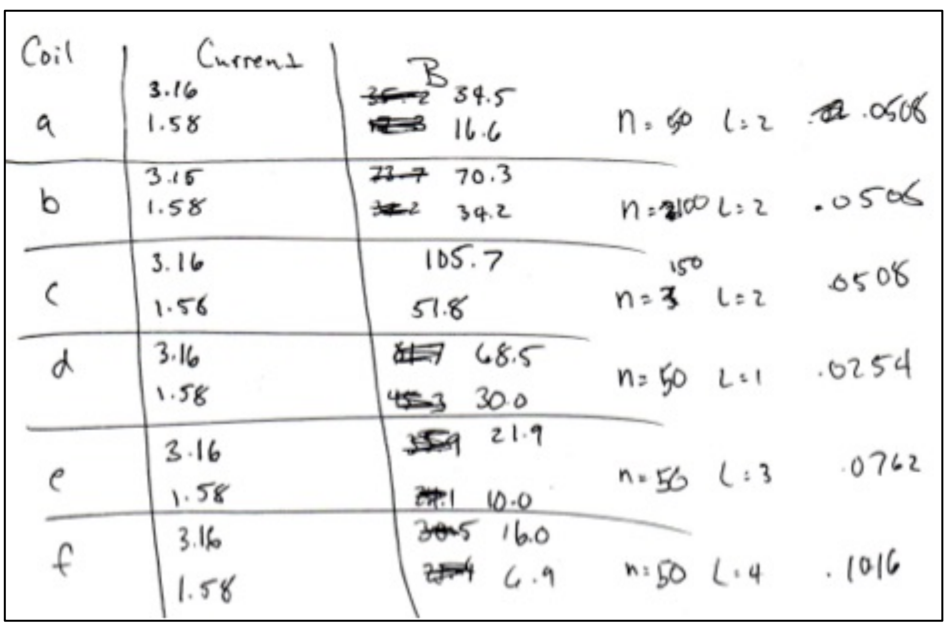


Figure 16. Michael's revised data table.

Recognizing the need for a constant. After revising his data table, Michael said to Reid, "I'm wondering if there's just a constant that'll help us out." Reid agreed and suggested that their constant would be 90, which was based on the predicted magnetic field strength of the one-inch, 50-wrap solenoid at 3.16 A (6,220.5 G) divided by the actual magnetic field strength (68.5 G). Michael questioned whether or not that would be the constant, so the two began analyzing their data separately.

After working separately for a few minutes, Reid asked Michael to explain his thinking about the constant in their equation. Michael told Reid that the constant was not 90, but was actually closer to 1/100. Michael explained, "If your constant is some value *x*, you solve this [$68.5 = x\left(\frac{50\times3.16}{0.0254}\right)$] and get 6,220. We did it the other way." Instead of dividing the predicted value by the measured value, Michael explained that they needed to solve for *x*, which meant dividing the measured value by the predicted value ($68.5 \div 6220 \approx 0.011$). Using 1/100 as his constant, Michael used the revised equation ($B = \frac{NI}{100L}$) to calculate the magnetic field strength for three different solenoids and found that the predicted magnetic field strength was within 3 G of the measured magnetic field strength. Reid immediately thought of 1/100 as having to do with their unit conversions. Initially, Michael and Reid had converted the solenoid lengths from inches to meters. Reid explained,

> If I take it [solenoid length] back to centimeters, I multiply by 100 and our constant goes away. Is it possible that a Gauss is an amp per centimeter? What the hell does that even mean? Centimeters of what? Length of coil? … But no one defines anything in terms of centimeters. I guess it's possible that there's just a 100 in the denominator.

Michael reiterated that when calculating the constant, the exact value was not 1/100, but that it ranged from 0.0111 to 0.0113. Michael used three sets of solenoid data; Reid recommended that they calculate the constant of proportionality for each of the remaining sets of solenoid data to confirm. The two worked together and found that the constant ranged from 0.0103 to 0.0113, which they averaged to 0.011. Michael and Reid agreed that the constant could be approximated as 1/100, which caused Reid to again question whether or not the constant was related to unit conversions.

Michael reviewed the units they had been using for their independent variables thus far in their data analysis. They were measuring length in meters and electric current in amps. Reid inspected the power supply to confirm that the unit for electric current was amps. He asked Michael to connect one of the solenoids to the power supply. Once they turned on the power supply, Reid verified that the units were amps. He then explored the different menu options in the SparkVue software to see if the program's settings could help him better understand the units used for measuring magnetic field strength. As Reid examined the software, he again tried to recall the definition of a Gauss. Reid explained to Michael, "If you want to know the number of [magnetic] field lines, it's called a Gaussian surface…It's a surface that you can figure out how many field lines penetrate it. But magnetic field is not measured in Gauss. It's measured in Tesla." Reid then asked the two authors if they designed the activity so that magnetic field was purposefully not measured in Tesla. The first author explained that there were two options for units when measuring magnetic field strength, one of those options was Tesla and the other was Gauss.

Determining the constant. The second author interjected to ask what was bothering Michael and Reid about their current equation ($B = \frac{NI}{100L}$), where L had been converted to meters. Michael explained that, "100 is too much of a good number." Reid questioned the decision to measure magnetic field strength in Gauss. The second author reiterated that the only reason magnetic field strength was measured in Gauss was because that was one of the options in the software. The two authors asked what values they were using for their variables and they explained that they were using the actual number of wraps of wire (i.e., 50, 100, 150) for N and meters for L. Reid further explained, "I'm thinking that to go from centimeters to meters, we had a one over one-hundred, which if Gauss is in units of centimeters, then that's where our one over one-hundred constant comes from and we don't need to have it. But I don't remember what a Gauss is." The second author then asked, "Why does that [units] even matter if you have a constant of proportionality?" To which Reid answered, "Because we want to know why that one over one-hundred is so friendly over there." During this exchange, Michael continued to calculate the predicted magnetic field strengths using their model and realized that the model did not accurately predict what they had measured. The second author asked what would happen to their equation if they did not convert the length of the solenoid from inches to meters and explained that, "The units don't matter because they'll all be accounted for with a different constant of proportionality." Reid disagreed and explained, "You can't equate a Gauss to something that's in terms of amperes and inches…If you're talking about amperes, you're talking about the metric system…I'm just assuming that if there's going to be a unit of length, it's going to be metric." Reid then wondered if Gauss was the standard unit for magnetic field strength and Tesla was the metric unit. Michael responded that the unit of length should not matter and that if they used inches instead of meters, then their constant would be approximately 1/2 instead of 1/100. The second author again asked, "What would you get if you kept it in inches?" Reid answered that they would need a different constant, which Michael had already calculated to be 0.43.

Reid suggested that both he and Michael solve for the new constant, using inches for their solenoid length, and compare their answers. The second author reminded them that Michael had already done that, but Michael explained that because of the variability they observed when calculating their first constant, he was not confident that 0.43 was correct for all of their solenoid data. Michael further explained his reasoning behind calculating a new constant using the two-inch, 50-wrap solenoid as an example:

> Fifty times 3.16 divided by two inches is 79…34.5 divided by 79 is 0.436. This is the same way that I got one over one-hundred, right? … If we convert it [solenoid length] from inches to meters, it's not getting rid of whatever constant if we think there's a constant. It's just the constant no longer incorporates that conversion.

Reid and Michael worked together to calculate the constant for each of their sets of solenoid data (see Figure 17). They found that the average of their constants was approximately equal to 0.44, but Reid again questioned the constant because "you're never going to have an equation that's 0.44, though."

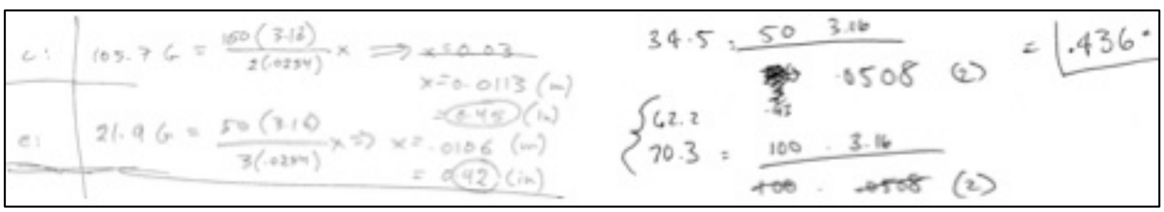


Figure 17. Reid's (L) and Michael's (R) calculations to find the constant.

Michael suggested that they revisit the relationship between magnetic field strength and solenoid length. Michael thought that if the relationship was not "perfectly inverse," then that might explain why their constant was 0.44. Reid was confident that the magnetic field strength was inversely related to solenoid length, but agreed to revisit their data to affirm their observation. At this point, the first author paused both Reid and Michael and asked them to consider other possible variables that could affect the strength of the magnetic field generated by the solenoid. Reid suggested that the radius of the coil, which was held constant, and the properties of the materials could also affect the magnetic field strength. The first author shared her observation that Reid and Michael were fixating on the numeric value of their constant and whether or not having a 0.44 in their equation made sense. Reid explained that 0.44 "makes sense based on our observations, but we're just trying to say what it [the relationship] is really." The first author clarified that the goal was to find a relationship based on their observations and their collected data, to which Reid responded that they were confident in their final model "as long as you'll accept an approximation symbol instead of an equals sign." With this confirmation, Reid and Michael agree that their final equation was $B = 0.44\left(\frac{NI}{L}\right)$.

Discussion

While the *Deriving Ampere's Law* activity incorporates scientific concepts (i.e., electromagnetism), the activity itself asks for a mathematical model and hence is much more aligned with the process of mathematical modeling (see Figure 2) rather than the process of scientific modeling (see Figure 1). The mathematical modeling cycle involves five phases: mathematizing, interpreting, verifying, revising, and generalizing. To better understand the pre-service teachers' modeling strategies, both productive and non-productive, the mathematical modeling cycle was utilized to interpret decisions they made over the course of the activity. Both groups of pre-service teachers were able to successfully derive Ampere's Law and both groups demonstrated all five phases of the mathematical modeling cycle.

There were several similarities and differences in the way the PSMTs and the PSSTs approached the *Deriving Ampere's Law* activity. Both groups outlined a plan for data collection prior to starting the activity and when analyzing their collected data, both groups identified the types of relationships between the independent variables and the dependent variable. Both groups also regularly tested and revised their model against their collected data. The groups differed in the way that they structured their model, the amount of consideration they placed on units and unit conversions, and their recognition of the need for a constant in their final equation. Their work on the *Deriving Ampere's Law* activity also revealed beliefs both groups held about the nature of traditional "school" mathematics and science formulas and activities.

Similarities between PSMTs and PSSTs

The PSMTs and the PSSTs shared several similarities in their approach to the *Deriving Ampere's Law* activity. These similarities included planning a systematic approach to data collection, identifying types of variation, and routinely verifying and revising their developed models. After the equipment demonstrations, both the PSMTs and the PSSTs agreed upon a plan for data collection. In the PSMT group, Emily said to Anna, "We don't want to vary a bunch of things at once, so do you want to look at the same wraps, different length? Same length, different wraps? Same solenoid, different current?" This implied that Emily wanted to isolate the independent variables as they collected their data to better understand the relationships between the independent and dependent variables. In the PSST group, Reid told Michael that their goal was to "figure out direct and indirect relationships." Similar to the PSMT group, the PSST group wanted to identify how each of the independent variables related to the dependent variable.

While both members of the PSST group seemed to be in agreement with their data collection plan throughout the *Deriving Ampere's Law* activity, there were moments of confusion between the members of the PSMT group. For example, when measuring the solenoids of varying wraps of wire, Emily wanted to collect data at different current outputs. For Emily, this was a way to confirm that the relationship they hypothesized between number of wraps of wire and magnetic field strength held true at different electric currents. Anna, however, interpreted this as varying two independent variables simultaneously, which she found to be problematic. Emily attempted to explain her reasoning to Anna, but struggled to clearly articulate her rationale for measuring the solenoids at different electric currents.

Both the PSMTs and the PSSTs analyzed their collected data by looking for relationships between the independent variables and the dependent variable. The PSST group articulated this approach to data analysis early on when Reid told Michael that their goal was to identify types of relationships (i.e., direct or inverse). This approach came about more organically for the PSMT group. After having measured the two-inch solenoids of varying wraps at 3.16 A, Emily described the relationship she observed qualitatively ("more wraps, more magnets") and then checked to see if the change in magnetic field strength was proportional to the change in number of wraps of wire. Once she confirmed this, Emily then described the relationship, "As the number of wraps increases by a factor of x, so does the magnet strength," which Anna recognized as an example of direct variation. The PSMT group approached their data analysis for the remaining solenoids in a similar fashion.

The PSMTs and the PSSTs also regularly verified and revised their model. The PSMTs initial model was $s = \frac{aw}{l}$ and the PSSTs initial model was $B = \frac{NI}{L}$. With an initial structure in mind, both the PSMTs and the PSSTs used their collected data to confirm whether or not their model was accurate. They selected one of the solenoids they had measured (e.g., two-inch, 50-wraps) and used their values for number of wraps of wire, solenoid length, and electric current to see whether or not their model predicted the same magnetic field strength as their collected data. This then informed their next steps for model revision. The PSMTs and the PSSTs continued this cycle of testing and revising until they settled upon a generalizable model that they felt accurately predicted their collected data.

Differences between PSMTs and PSSTs

There were several differences between the PSMTs and the PSSTs as they worked through the *Deriving Ampere's Law* activity. These differences included structuring their model, considering units and unit conversions, and recognizing the need for a constant. These differences may have been influenced by the pre-service teachers' beliefs about the nature of school science and mathematics, which is further explored in the forthcoming subsection. Recall that both the PSMTs and the PSSTs derived their initial model structure by relying on the types of variation (i.e., direct or inverse) they observed for each of the independent variables. The PSSTs did not attempt to generate intermediate models over the course of the activity. Instead, they translated the relationships they had described qualitatively (see Figure 13) into their initial model structure ($B = \frac{NI}{L}$).

The PSMT group, however, considered informal models for each of the independent variables separately. When analyzing the relationship between number of wraps of wire and magnetic field strength, Emily described the relationship, "As the number of wraps increase by a factor of x, so does the magnet strength." When analyzing the relationship between electric current and magnetic field strength, Emily described the relationship as, "current over Gauss equals x." When Anna and Emily recognized the relationship between solenoid length and magnetic field strength was inverse variation, Emily described the relationship as $xy = k$.

It is worth noting that both Anna and Emily mentioned that direct and inverse variations were topics they had recently seen in their middle school practicum placements. Their recent experiences with direct and inverse variations might have influenced the way that they approached these relationships in their data analysis. Consider how inverse variation is defined in a popular Algebra I textbook – "y varies inversely as x if there is some nonzero constant such that $xy = k$" (Holliday, Cuevas, Moore-Harris, & Carter, 2005, p. 624). This definition of inverse variation is identical to the way that Emily described the relationship between solenoid length and magnetic field strength. When developing their model, the PSMT group first set up the equation $\frac{s}{w} = \frac{a}{l}$, which was structured differently than the PSST group's equation. However, given the PSTM group's recent classroom experiences with direct and inverse variation, their formulation of the relationship is not surprising. Another interesting difference between the PSST group and the PSMT group was the letters they chose to represent their variables. The PSST group used B (magnetic field strength), N (number of wraps), I (intensity), and L (length) and the PSMT group used s (magnetic field strength), w (number of wraps), a (Amps), and l (length). The PSST group named their variables as they would appear in physics textbooks.

While both the PSMTs and the PSSTs considered the role of units when revising their models, the extent to which units and unit conversions affected their progress with the task differed significantly. After deciding upon the structure of the model, the PSMTs realized that they needed a constant in their denominator ($s = \frac{aw}{2l}$), but both Anna and

Emily acknowledged that using $k = 2$ was not particularly precise. Unsure of whether or not they could attribute the discrepancy between the model's predicted magnetic field strength and their collected data to measurement error, Anna suggested that their constant should be greater than two and that it might be the conversion factor between inches and centimeters (1 inch = 2.54 centimeters). After testing $s = \frac{aw}{2.54l}$, both Anna and Emily saw that this revised model was not more accurate and they abandoned the idea that the constant was related to unit conversions.

The issue with units played a much greater role in the PSSTs' approach to the *Deriving Ampere's Law* activity. The issue of units first appeared when the PSSTs prepared for their initial round of data collection and Reid mentioned that he had forgotten the definition of a Gauss. After Reid and Michael had identified the relationships between the independent variables and the dependent variable (see Figure 13), units became an even greater concern. When Michael explained to Reid that the magnetic field strengths predicted by their model ($B = \frac{NI}{L}$) did match their collected data, Reid's initial thought was that the discrepancy was because of their units. Two of their variables were measured in metric units (i.e., amps, Gauss) and one of their variables was measured in standard units (i.e., inches), which introduced the issue of dimensional analysis. They then converted inches to meters, which resulted in a constant that Michael approximated as 1/100. The nature of this number, which resembled a metric conversion, led both Michael and Reid to question again whether or not the constant was related to unit conversions. Reid again asked about the definition of a Gauss and noted that traditionally, magnetic field strength was measured in Tesla. For the PSST group, the units in the equation were incredibly important and both Reid and Michael lost sight of the fact that a constant of proportionality would take into account any need for unit conversions.

Both the PSMTs' final model ($s = \frac{aw}{2.2l}$) and the PSSTs' final model ($B = 0.44\frac{NI}{L}$) included a constant. However, the two groups recognized the need for a constant at different points in the activity. This could have been due in part to how the two groups mathematized and interpreted the *Deriving Ampere's Law* activity. The PSMT group interpreted the goal of the task was to develop a mathematical model that could describe their collected data. As soon as the PSMT group recognized the structure of their model was $s = \frac{aw}{l}$, Anna set up the equation $\frac{s}{w} = \frac{a}{l}$ to verify the accuracy of their model using their collected data. This happened approximately 50 minutes into the activity. Emily recognized fairly quickly that if they multiplied their length by two, then the relationship was approximately equal and both Anna and Emily described multiplying by two as their constant. They then spent approximately 15 minutes calculating the constant using their collected data.

The PSST group, however, did not articulate the need for a constant until later. While the *Deriving Ampere's Law* activity was posed the same way to the PSST group as the PSMT group, Reid interpreted the goal of the task was to identify the direct and inverse variations. Once Reid confirmed that his group had identified the types of variations, he was satisfied with their final answer. It was Michael who suggested that they use their collected data to verify their model. Recall that when their initial model ($B = \frac{NI}{L}$) did not fit their collected data, both Reid and Michael attributed the error to unit conversions. It was not until approximately 75 minutes into the activity that Michael suggested they needed a constant in their equation and it was approximately 30 minutes after that when Michael and Reid arrived upon a constant based on their collected data.

Beliefs about "School" Formulas and Numbers

The *Deriving Ampere's Law* activity also revealed several beliefs that the PSMTs and the PSSTs had about the difference between math and science in school and real-world applications of math and science. Both the PSMTs and the PSSTs spoke about "nice" numbers versus "not nice" numbers. For Emily (PSMT), she preferred the equation $s = \frac{aw}{2l}$ because the constant was a "nice number" and "that's how a lot of science things look." Even though she recognized their model did not accurately predict magnetic field strength when compared to their collected data, there was something about the nature of the model that she preferred. Reid and Michael engaged in a similar conversation when developing their final model. Because of their unit conversions, Michael approximated their first constant to be 1/100, but that "the 100 was too nice to be an observational value." Similarly, Reid described the 100 as being "so friendly." However, both Reid and Michael shared that having 0.44 as their constant also did not feel correct. Reid explained that he was "used to seeing cleaner relationships" and Michael explained that they "wanted it [the model] to be too clean." Reid and Michael were describing the difference between theoretical relationships they had seen in textbooks compared to relationships that were derived experimentally.

The PSSTs, and to a lesser extent, the PSMTs, introduced the idea of metric conversions, which seemed to derail their progress. The PSST group, in particular, did not consider the fact that a constant of proportionality would account for differences in units. While acknowledging the role of units is important, both the PSMTs and the PSSTs seemed to lose sight of how a constant of proportionality affects an equation. The PSMTs momentarily considered mathematical constants (i.e., π, e) as a possible constant in their final equation. While it did not take much time for them to recognize these values did not work in their equation, the fact that these values had no relation whatsoever to the problem at hand did not deter them from testing these possible constants in their final model.

Both the PSMTs and the PSSTs spoke about how the *Deriving Ampere's Law* activity compared to their previous classroom experiences. Reid shared how this activity differed from physics labs when he said, "I just remember lab activities can be so structured that you just move through them and you follow one direction and then the next and you have no idea." Similarly, Emily explained, "Science experiments were never really something that I enjoyed. I can look up this relationship in a book." Emily found traditional science experiments to be over-scripted and she found filling in pre-made data tables to be tedious and not meaningful. Anna shared a similar sentiment. She explained,

> You're given the equation, you write it down…you solve a bunch of problems and stuff, so that's what takes the fun out of science because you are just memorizing formulas…But here, you're bringing in the scientific process and they have to think like scientists.

Both Michael and Reid connected their experiences from the *Deriving Ampere's Law* activity to the concept of inquiry learning, which was a topic they had recently discussed in their secondary science methods course. Michael shared that he found the open-ended nature of the *Deriving Ampere's Law* activity to be much more beneficial than labs that are more structured. Reid, who had already completed his undergraduate degree in physics, shared, "I've never been given things to make observations and then come up with the expression as a result of the observation." He spoke about how he recalled classroom experiences where he would have to derive equations from existing models or lab experiences where he would have to verify known relationships, but the process of deriving a model from collected data was something he did not recall ever doing before. Both Anna and Emily shared that while the *Deriving Ampere's Law* activity was related to electromagnetism, they interpreted the activity as being relevant in a mathematics classroom.

Limitations

There are three potential limitations to this research study. The first limitation is that researcher bias may have influenced data analysis. The first author developed the materials for the *Deriving Ampere's Law* activity, giving her an intimate understanding of the task, as well as her own modeling strategies. To mitigate how this may have influenced data analysis, both authors recorded field notes separately and they also triangulated their findings across multiple data sources. The second limitation is that the magnetic field sensor probe tip was damaged while the PSSTs were working on the activity. The authors were able to immediately replace the damaged probe with an identical probe, so this minor equipment malfunction did not seem to hinder the PSSTs' progress. The third limitation is related to the generalizability of the findings. There was a lot of variation in the pre-service teachers' prior knowledge. The two PSMTs shared that direct and inverse variation was a topic they had recently seen in their practicum placements. As a result, their understanding of direct and inverse variations was influenced by their recent experiences in a middle school classroom. One of the PSSTs had significant prior experiences in electromagnetism, having majored in physics as an undergraduate student and having worked as an engineer prior to beginning his teacher preparation program. However, given the nature of the research questions for this study, the differences in the pre-service teachers' prior knowledge were expected.

Conclusion

Both the pre-service mathematics teachers (Anna and Emily) and the pre-service science teachers (Michael and Reid) were able to experimentally derive Ampere's Law, which relates three independent variables (number of wraps of wire, electric current, and solenoid length) to the strength of the magnetic field produced by a solenoid. While it is possible to complete the *Deriving Ampere's Law* activity individually, as evidenced during early pilot testing of the activity itself, it was useful for both the PSMTs and the PSSTs to have a partner to discuss their ideas and strategies for completing the activity. The PSMTs' and the PSSTs' approach to completing the *Deriving Ampere's Law* activity revealed similarities and differences between the groups' modeling process based on their prior knowledge and also highlighted how their beliefs about the nature of traditional school activities influenced their strategies.

Both groups were excited by the *Deriving Ampere's Law* activity and made connections between the activity and their experiences in their methods courses and their practicum placements. Despite the limitations noted above, the findings from this study indicate the importance of providing pre-service teachers with the opportunity to engage in modeling activities. By engaging in the modeling process themselves, the pre-service teachers who participated in this study became more aware of their own understanding of modeling and considered how they might incorporate modeling activities such as the *Deriving Ampere's Law* activity into their own classroom instruction. Furthermore, based on these findings, it appears that incorporating activities that involve this type of data collection and analysis into teacher preparation for mathematics and science teachers would be beneficial. Not only is there value in providing pre-service teachers with the opportunity to engage in the kind of thinking associated with modeling activities, but activities such as the *Deriving Ampere's Law* activity also provide pre-service teachers with examples of inquiry-driven and collaborative instructional activities.

For this paper, the *Deriving Ampere's Law* activity was approached holistically. The activity has also been successfully implemented as four separate investigations with middle school students (see Corum & Garofalo, 2017). Plans for future research include: (1) further unpacking affective reactions and (2) reworking some design aspects of the *Deriving Ampere's Law* activity. The data collected for this paper, as well as the data collected when the activity was implemented with middle school students, indicated that there were aspects of the *Deriving Ampere's Law* activity that influenced students' persistence and perseverance with the task. This data will be further analyzed to explore students' affective responses. If possible, the middle school students who participated in the activity will be interviewed again to further reflect on their experience and to what extent, if any, the experience influenced their thinking about mathematics and science during the school year.

In addition to unpacking students' affective responses, plans for future research include reworking some design aspects of the activity itself. The original set of solenoids developed for the activity were calibrated using 3.16 A because of the limitations of the variable power supply used. A new set of solenoids, using a more reliable power supply, is under development to provide students with the ability to interact with current as a variable in a more meaningful way. With more reliable materials, the holistic version of the *Deriving Ampere's Law* activity that was used with the pre-service teachers' can be implemented with middle school students to get a better understanding of how younger students might approach the activity with limited scaffolding. Finally, opportunities for commercialization of the activity are underway to be able to develop and mass-produce sets of pre-wound, calibrated solenoids. These efforts would allow the *Deriving Ampere's Law* activity to be implemented in a classroom setting, such as a secondary methods course or a middle school mathematics or science class.

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APPENDICES

Appendix A

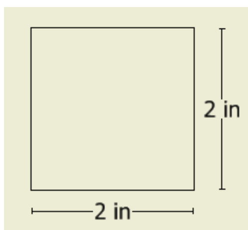
Surface Area and Volume Unit Pretest/Posttest

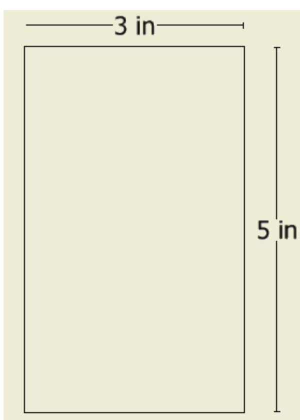
Name:_______________________________

Please us the spaces next to, or below, the questions and figures to show your work or explain your thinking.

1. Area is measured in what kind of units?

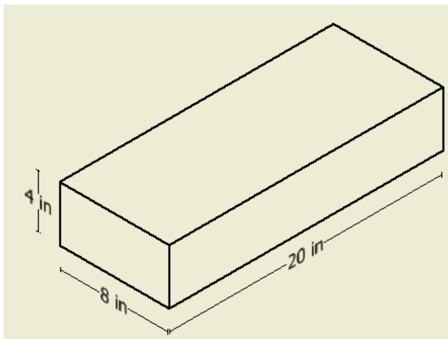
2. Find the area of the shapes below:



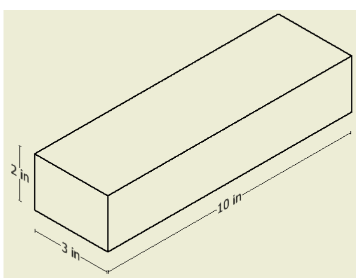


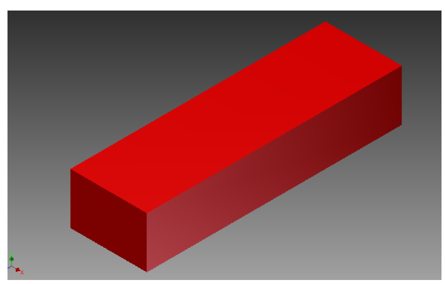
3. Write a formula for the area of any rectangle, or tell what information about a rectangle is needed to find the area.

4. Find the surface area of the following solid:

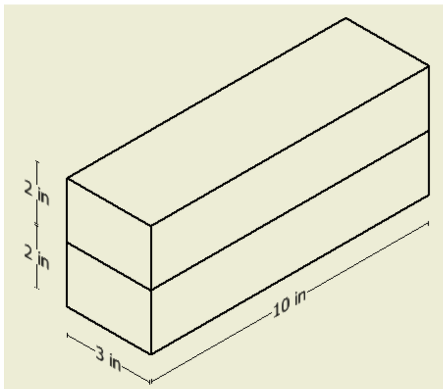


5. What is the minimum amount of wrapping paper you would need to wrap the long pencil box shown below with the dimensions: 2 inches, by 3 inches, by 10 inches?





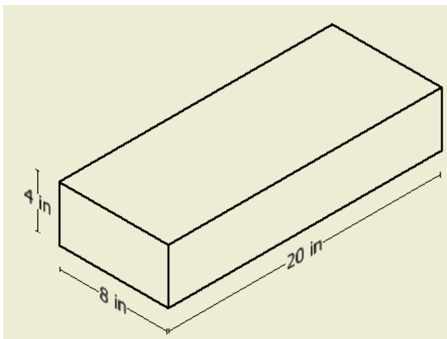
6. What is the minimum amount of wrapping paper you would need to wrap 2 long pencil boxes, if one is stacked on top of the other, like below, and then wrapped together?



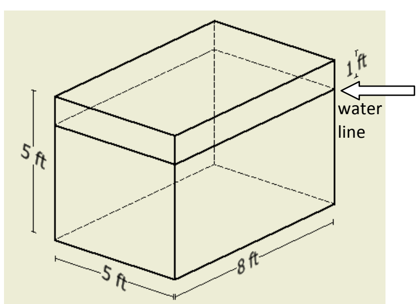
7. Volume is a measure of____

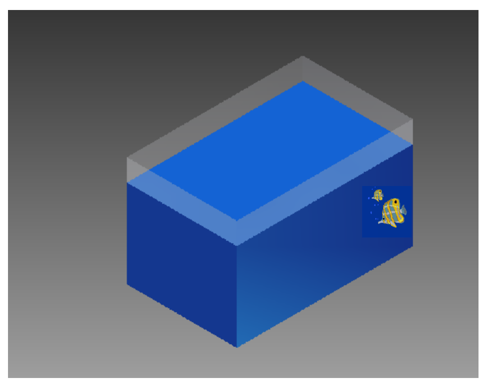
8. Volume is measured in what kind of units?

9. Find the volume of the solid shown below:

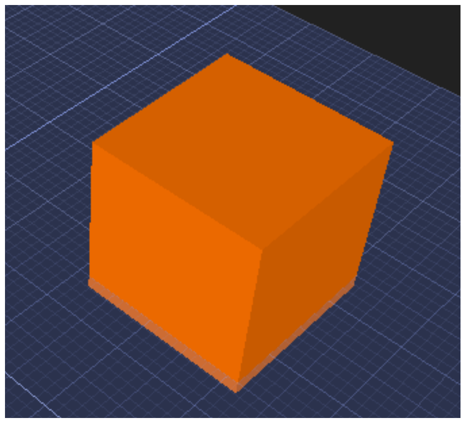


10. What is the volume of water in the fish tank shown below:

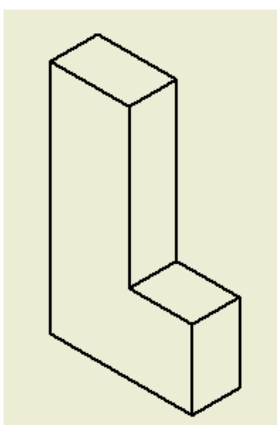




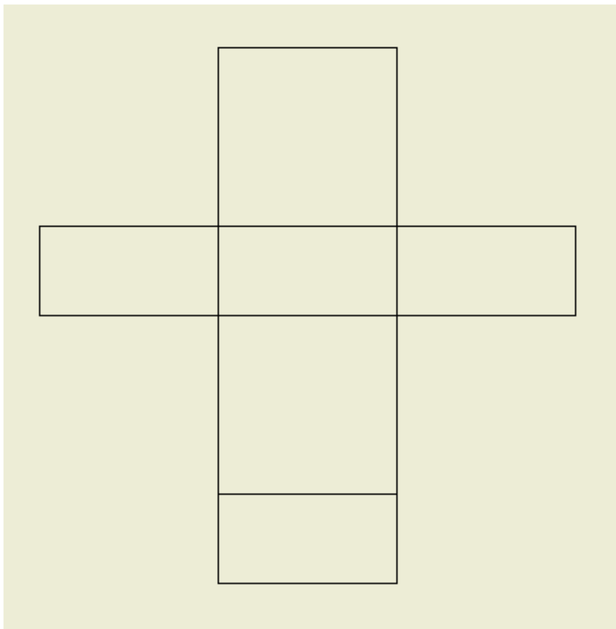
11. The solid below is a cube. How many faces, edges, and vertices does it have?



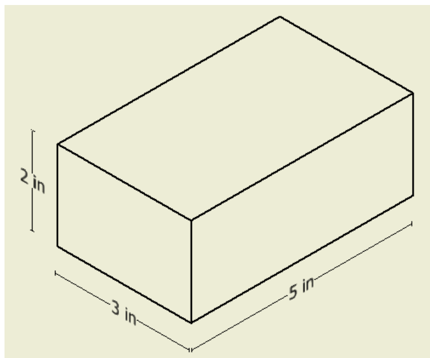
12. How many faces, edges, and vertices does the solid below have?



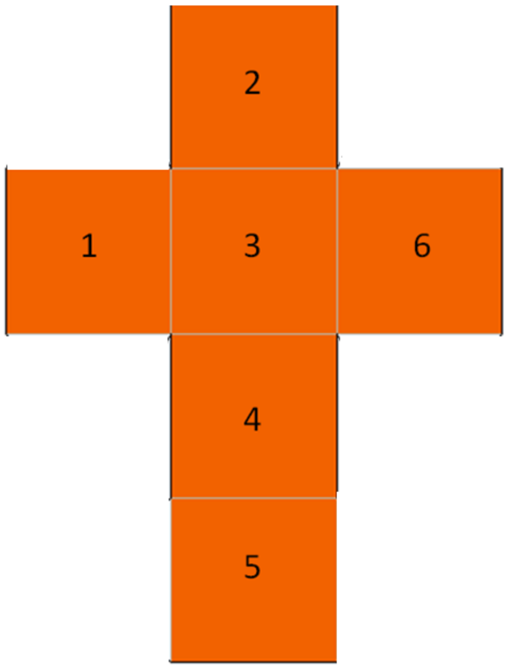
13. When the figure below is cut out and folded into a 3-dimensional solid, how many vertices, faces, and edges will the resulting solid figure have? Explain your thinking.

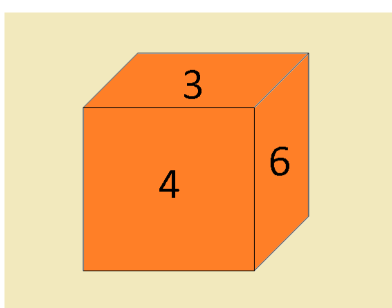


14. Suzie folded a flat shape and taped the sides together to form the prism below. Can you draw a shape that could have been folded into this prism?

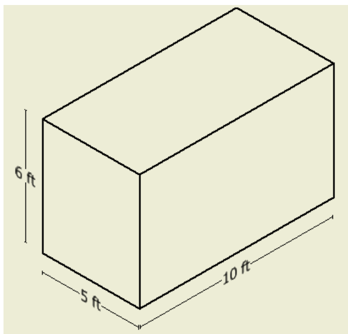


15. The shape below has faces numbered 1 through 6. If Billy folds the 2-dimensional shape below to form the cube below it, what number face would be on the bottom of the cube?





16. You have a moving truck with a bed with dimensions 6 ft, by 5 ft, by 10 ft, like in the figure below.

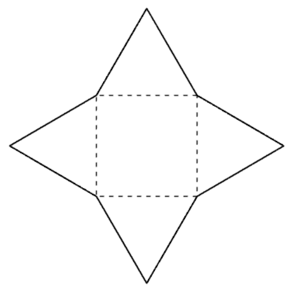


If you load 6 boxes, each with dimensions 1 ft by 2 ft by 3 ft, and two mattress boxes each with dimension 2 ft by 5 ft by 7 ft into the truck, how much space would be left over for other stuff?

Appendix B

Surface Area and Volume Unit Posttest SOL Items

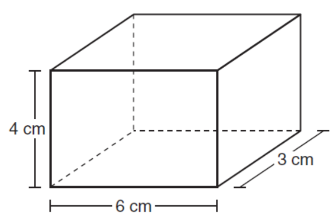
29 **Sarah folded the following figure along the dotted lines to make a three-dimensional shape.**



Which *best* describes the shape Sarah made?

A Triangular pyramid
B Triangular prism
C Rectangular prism
D Square pyramid

22 **Chelsea wants to cover a rectangular prism-shaped box with paper. Which is closest to the minimum amount of paper Chelsea needs?**



F 26 cm^2
G 54 cm^2
H 72 cm^2
J 108 cm^2

Appendix C

IRB-SBS Documentation

UNIVERSITY of VIRGINIA
OFFICE OF THE VICE PRESIDENT FOR RESEARCH
INSTITUTIONAL REVIEW BOARD FOR THE SOCIAL AND BEHAVIORAL SCIENCES

In reply, please refer to: Project # 2016-0391-00

October 17, 2016

Kimberly Corum
Joe Garofalo
CISE (Curriculum, Instruction & Special Ed)
[redacted]
Charlottesville, VA 22903

Dear Kimberly Corum and Joe Garofalo:

The Institutional Review Board for the Social and Behavioral Sciences has approved your research project entitled "Pre-Service Teachersâ€™ Modeling Strategies." You may proceed with this study. Please use the enclosed Consent Forms as the masters for copying forms for participants.

This project # 2016-0391-00 has been approved for the period October 14, 2016 to October 13, 2017. If the study continues beyond the approval period, you will need to submit a continuation request to the Review Board. If you make changes in the study, you will need to notify the Board of the changes.

Sincerely,

Tonya R. Moon, Ph.D.
Chair, Institutional Review Board for the Social and Behavioral Sciences

One Morton Drive, Suite 500 • Charlottesville, VA 22903
P.O. Box 800392 • Charlottesville, VA 22908-0392
Phone: 434-924-5999 • Fax: 434-924-1992
www.virginia.edu/vpr/irb/sbs.html

Informed Consent Agreement

Please read this consent agreement carefully before you decide to participate in the study.

Purpose of the research study: The purpose of this research study is to explore secondary mathematics and science pre-service teachers' understanding of the integration of STEM disciplines through the process of mathematical and scientific modeling.

What you will do in the study: You will work with two other students in your program to complete a task that asks you to investigate the different parameters that affect the strength of the magnetic field generated by a solenoid. You will also be asked to participate in a task-based group interview as you work through the activity. The session will be video taped for transcription and data analysis purposes. The participants can stop the session at any time.

Time required: The study will require approximately two hours of your time.

Risks: There are no anticipated risks in this study.

Benefits: There are no direct benefits to you for participating in this research study. The study may help us understand how to better prepare teachers to incorporate authentic STEM modeling tasks into their own classroom instruction.

Confidentiality: The information that you give in the study will be handled confidentially. Your information will be assigned a code name. The list connecting your name to this code will be kept in a locked file. When the study is completed and the data have been analyzed, this list will be destroyed. Your name will not be used in any report.

Voluntary participation: Your participation in the study is completely voluntary.

Right to withdraw from the study: You have the right to withdraw from the study at any time without penalty. All audio recordings will be destroyed should you decide to withdraw.

How to withdraw from the study: If you want to withdraw from the study, please inform me of your intent via email (kc3v@virginia.edu). There is no penalty for withdrawing.

Payment: You will receive no payment for participating in the study.

IRB-SBS Office Use Only		
Protocol #	2016-0391	
Approved	from: 10/14/16	to: 10/13/17
SBS Staff	[signature]	

If you have questions about the study, contact:

Kimberly Corum
Curry School of Education, Department of Curriculum, Instruction and Special Education
University of Virginia, Charlottesville, VA 22903
Telephone: [redacted]
Email address: kc3v@virginia.edu

Joe Garofalo
Curry School of Education, Department of Curriculum, Instruction, and Special Education
P.O. Box 400273
University of Virginia, Charlottesville, VA 22903
Telephone: [redacted]
Email address: jg2e@virginia.edu

If you have questions about your rights in the study, contact:

Tonya R. Moon, Ph.D.
Chair, Institutional Review Board for the Social and Behavioral Sciences
One Morton Dr Suite 500
University of Virginia, P.O. Box 800392
Charlottesville, VA 22908-0392
Telephone: (434) 924-5999
Email: irbsbshelp@virginia.edu
Website: www.virginia.edu/vpr/irb/sbs

Agreement:

I agree to participate in the research study described above.

Signature: ______________________________ **Date:** __________

You will receive a copy of this form for your records.

IRB-SBS Office Use Only	
Protocol #	2016-0391
Approved	from: 10/14/16 to: 10/13/17
SBS Staff	

Materials Release Form for Audio/Video Recordings

Project Title: Pre-Service Teachers' Modeling Strategies

During the interview, you were recorded on videotape so that your information may be preserved as an historical record. Upon completion of the interview, the interviewer compiled the recording into a written transcript. Having read the transcript of the interview, you have three choices regarding the videotape and transcript of the interview. The materials may be designated either **"public,"** **"for research only,"** or **"private."**

If you designate the materials **"public,"** the videotape and transcript will be accessible to members of the community through Kimberly Corum. Kimberly Corum may use the materials from the interview for future exhibits and your materials will remain part of its permanent collection.

If you designate the materials **"for research only,"** your videotape and transcript will be analyzed by the researcher and your information will be used to complete the research study. Your information will be reported in a way that does not identify you and your materials will be destroyed after the study is complete.

If you designate the materials **"private,"** the videotape and transcript will be given to you and never released to Kimberly Corum. The only records of the interview will belong solely to you.

If in the future you wish to change the status of your videotape and transcript, you may contact Kimberly Corum:

Kimberly Corum
Curry School of Education, Department of Curriculum, Instruction and Special Education
University of Virginia, Charlottesville, VA 22903
Telephone: [redacted]
Email address: kc3v@virginia.edu

IRB-SBS Office Use Only		
Protocol #	2016-0391	
Approved	from: 10/14/16	to: 10/13/17
SBS Staff	[signature]	

___ I hereby designate the materials as **public** and give permission for my videotape and transcript to be used by Kimberly Corum.

___ I hereby designate the videotape and transcript **for research only** and give my permission for the researcher to use my materials as part of the research study. I want my materials to be reported so that they will not identify me and destroyed when the study is complete.

___ I hereby designate these materials as **private** and do NOT give my permission for my videotape and transcript to be used by Kimberly Corum. The materials will be given to you for your own private use.

Signature: ____

Date:________________

You will receive a copy of this form for your records.

IRB-SBS Office Use Only		
Protocol #	2016-0391	
Approved	from: 10/14/16	to: 10/13/17
SBS Staff	[signature]	

Materials Release Form for Future Data Analysis

Project Title: Pre-Service Teachers' Modeling Strategies

During the experiment, you worked with other participants to complete the *Deriving Ampere's Law* activity, which challenged you to derive Ampere's Law by measuring a series of pre-wrapped solenoids. We would like to ask permission to use the videotape, transcript, and collected works samples from the session for future research studies. This data will be used to prepare journal manuscripts to be considered for publication and shared during conference presentations. If you agree to have your data used in subsequent research, your name will not be linked to these materials and your data will be reported using pseudonyms. All video files and interview transcripts will be stored electronically on a password-protected cloud server (www.dropbox.com). Hand-written data will be stored in a secure filing cabinet that is only accessible by the researcher. If you choose not to give us permission to use your data, there is no penalty. You will still receive full credit for your participation in the experiment.

In the future, if you wish to change the status of your data, you may contact:

Kimberly Corum
Curry School of Education, Department of Curriculum, Instruction and Special Education
University of Virginia, Charlottesville, VA 22903
Telephone: [redacted]
Email address: kc3v@virginia.edu

____ I give permission for my data to be used for future research.

____ I do NOT give permission for my data to be used for future research. Please destroy it once this study is complete.

Signature: ____

Date: ________________

You will receive a copy of this form for your records.

IRB-SBS Office Use Only		
Protocol #	2016-0391	
Approved	from: 10/14/16	to: 10/13/17
SBS Staff	[signature]	