Estimates on Functional Integrals of Non-Relativistic Quantum Field Theory, with Applications to the Nelson and Polaron Models

Gonzalo Alfredo Bley Delgado
Santiago, Chile

B.S. in Physics, Pontificia Universidad Católica de Chile, 2011
B.S. in Mathematics, Pontificia Universidad Católica de Chile, 2012
M.S. in Mathematics, University of Virginia, 2014

A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia
May, 2016

Lawrence E. Thomas (Advisor, Mathematics) ____________________

John Z. Imbrie (Mathematics) ____________________

Ira W. Herbst (Mathematics) ____________________

Harry B. Thacker (Physics) ____________________

Abstract

In this dissertation we present a computational tool that allows one to provide lower bounds for the ground state energy of several quantum-mechanical Hamiltonians. Given a Hamiltonian H, its ground state energy can be expressed as a Feynman-Kac formula

$$\inf \text{spec } H = -\lim_{T\to\infty} \frac{1}{T} \log \left[E\left(e^{\mathcal{A}_T}\right)\right],$$

where $\mathcal{A}_T$ is negative the time integral of the effective potential of the Schrödinger operator H, evaluated at a Brownian path on the time interval $[0, T]$. It is shown how a Clark-Ocone expansion for the action $\mathcal{A}_T$, namely an expansion as its deterministic expectation plus a random oscillatory part, represented as a stochastic integral, can lead to an upper bound for the exponential moment of $\mathcal{A}_T$, $E\left(e^{\mathcal{A}_T}\right)$, of the form $e^{BT+o(T)}$, for some constant B. This leads, in particular, to a lower bound for H given by $-B$. The method is illustrated in two canonical models in non-relativistic quantum field theory: the Nelson and polaron models.

Contents

Chapter 1

Introduction

In this dissertation a new tool to derive lower bounds for many-body quantum mechanical Hamiltonians is presented. Part of this thesis is based on joint work with L.E. Thomas [5]. After providing enough background and introducing the bounds, they are applied to two models of chief interest in non-relativistic quantum field theory, the Nelson and polaron models. Both models have as Hamiltonian

$$H = -\frac{1}{2}\sum_{n=1}^{N}\Delta_n + \int_{\mathbb{R}^3}\omega(k)a(k)^\dagger a(k)\,dk + \int_{\mathbb{R}^3}\left[F_k(x)a(k) + \overline{F_k(x)}a(k)^\dagger\right]\,dk, \tag{1.1}$$

for some complex valued functions ω and F on $\mathbb{R}^3$ and $\mathbb{R}^3\times\mathbb{R}^{3N}$, respectively. H acts on $L^2(\mathbb{R}^{3N})\otimes\mathcal{F}$, where $\mathcal{F}$ is the Fock space over $L^2(\mathbb{R}^3)$, which may be interpreted as a space accomodating one quantum mechanical harmonic oscillator for each mode $k\in\mathbb{R}^3$. $a(k)$ and $a(k)^\dagger$ are simply the lowering and raising operators for the mode k, which satisfy the canonical commutation relations $[a(k), a(l)^\dagger] = \delta(k-l)$, $[a(k), a(l)] = [a(k)^\dagger, a(l)^\dagger] = 0$. By explicitly integrating out the quantum field variables, R.P. Feynman found in 1950 [11] a rather explicit formula for the matrix element $(f\otimes\Omega, e^{-itH}g\otimes\Omega)$, where Ω is the ground state of the field, in which each one of the oscillators is in the ground state, and f and g are functions that depend on the spacial variables only. Around that time, M. Kac [24], in perhaps an attempt to understand the Ph.D. thesis of Feynman [12], worked with e^{-tH} instead of e^{-itH}, which allowed him to make sense of the matrix element through a well-defined description in terms of Brownian motion. From Kac's work it follows that for a Hamiltonian of the form $-\Delta/2+V$ acting on $L^2(\mathbb{R}^d)$, where V involves spacial coordinates alone and is subject to certain regularity restrictions,

$$(f, e^{-tH}g) = \int E^x\left[f(x)g(X_T)\exp\left(-\int_0^T V(X_t)\,dt\right)\right]\,dx, \tag{1.2}$$

where X denotes d-dimensional Brownian motion and E^x is expectation with respect to Brownian motion starting at $x\in\mathbb{R}^d$. (E^0 will be denoted by E.) (1.2) is known as a "Feynman-Kac formula." By combining the works of Feynman [11] and Kac [24], one finds that, if H is as given above, equation (1.1),

$$(f\otimes\Omega, e^{-tH}g\otimes\Omega) = \int E^x\left[f(x)g(X_T)\exp\left(\int_0^T\!\!\int_0^t h(X_t - X_s, t-s)\,ds\,dt\right)\right]\,dx, \tag{1.3}$$

for a certain real-valued function h, uniquely determined from ω and F, appearing in (1.1).

It was noticed by Kac [25] that the long-time behavior of the matrix element $(\delta_0, e^{-t(-\Delta/2+V)}1)$ determines the ground-state energy of $-\Delta/2+V$ when V is purely spacial, an idea that was then adapted by Feynman to the case of the polaron [13]. In particular, Feynman realized that, again by integrating the field coordinates,

$$-\lim_{T\to\infty}\frac{1}{T}\log\left\{E\left[\exp\left(\int_0^T\!\!\int_0^t h(X_t-X_s,t-s)\,ds\,dt\right)\right]\right\} \tag{1.4}$$

is the ground-state energy of H, as given above, equation (1.1). Chapter 2 will be devoted to deriving and explaining this formula. What motivates this work is that an upper bound on the functional integral present in (1.4), namely

$$E\left[\exp\left(\int_0^T\!\!\int_0^t h(X_t-X_s,t-s)\,ds\,dt\right)\right], \tag{1.5}$$

will yield a lower bound for the ground state energy of H.

Let us now concentrate on the object $\mathcal{A}=\int_0^T\!\int_0^t h(X_t-X_s,t-s)\,ds\,dt$, the "effective action" of the Hamiltonian H, equation (1.1). $\mathcal{A}$ will happen to be a Brownian functional: an L^2 functional of Brownian paths. Then, $\mathcal{A}$ may be expanded by means of the so-called Clark-Ocone formula

$$\mathcal{A}=E(\mathcal{A})+\int_0^T \rho_t\,dX_t, \tag{1.6}$$

for some unique $\mathbb{R}^3$-valued stochastic process ρ, where the second term is an Itô integral. By using this expansion we are able to find lower bounds for the Nelson and polaron models. This will be described further in the next chapters. Let us now briefly give an overview of these two models.

The polaron model was conceived by H. Fröhlich [15], describing the interaction of an electron with a polar crystal. Its Hamiltonian is of the form (1.1) but its exact shape is irrelevant at this time, and will be studied in some detail below. Many results have been published about the model since Fröhlich's paper, including estimates on the ground state energy [34, 13, 18, 28, 29, 16], the effective mass problem [13, 42, 43, 44, 31], asymptotics for large coupling [39, 9, 33], stability and absence of binding for multiparticle polarons [14], and many others. In Chapter 4 we give an explicit lower bound on the N-polaron (N electrons interacting with a polar crystal) without interelectronic repulsion, $E\geq-\alpha N-\alpha^2N^3/4$, where α is the polaron coupling constant. Although it is known that N^3 is the correct behavior of the ground state energy for large N [14], we do not know of the existence of a previous lower bound that is explicit, valid for all N and α, and has the correct large-N behavior, and so we believe that this lower bound is a noteworthy contribution. For $N=1$ the best known lower bound valid for all coupling constants α is $-\alpha-\alpha^2/3$ [34], and this is an improvement over that bound, especially for large α. We also prove absence of binding of bipolarons for strong enough Coulomb repulsion between the electrons. This is an improvement over previous work in [14, 4], of more than 50% on the relevant estimate. We remark in passing that one can show, by using the main result in [4] and the lower bound above for $N=1$, that there is always binding for every α, small enough repulsion parameter, and large enough number of electrons, so the question of no-binding is a non-trivial one. See Chapter 5 for relevant definitions and a precise statement of the result.

The Nelson model was formulated by E. Nelson in [37, 38], which considers a system of nucleons interacting with a meson field. As said before, its Hamiltonian has the form (1.1), but we will again leave a more precise description of it for later. The literature of the Nelson model is extensive, and contributions include a proof of the analyticity of the ground state energy in the coupling paramater [1], studies into the effective dynamics of the nucleons [46, 45], research into atoms with "Nelson interaction" [19, 21], the finding of effective weak coupling interactions [8, 22, 23], asymptotics for large number of particles [10], non-existence of ground state in the massless case [35, 20, 2], among several others. Here, we would like to single out one recent work by Gubinelli, Hiroshima and Lörinczi from 2014 [17]. In this article the authors take up Nelson's first paper on his model [38] and, by following what amounts essentially to the same ideas as Nelson's, involving stochastic calculus, prove that there is a renormalized Hamiltonian when an ultraviolet cutoff is removed. This was first proved by Nelson a couple of years after his first work on the model, in [37], using operator techniques. Their work can be seen as a continuation of previous work by Nelson [38] that was left, in some ways, unfinished. The main idea that Nelson uses, which is the same that Gubinelli et al. pursue, consists of performing an expansion on the action $\mathcal{A}$ that does not correspond to Clark-Ocone, but to something one could call an "Itô expansion," for lack of a better name for it. To understand this terminology, let us consider the simpler example of the action $\mathcal{A} = -\int_0^T V(X_t)\,dt$. Recall Itô's formula, namely

$$W(X_T) = W(0) + \int_0^T \nabla W(X_t)\,dX_t + \frac{1}{2}\int_0^T \Delta W(X_t)\,dt, \tag{1.7}$$

for functions W in C^2. Since the last term in (1.7) looks just like $-\int_0^T V(X_t)\,dt$, we let W be a function such that $-(\Delta/2)W = V$. We then find

$$\mathcal{A} = -W(0) + W(X_T) - \int_0^T \nabla W(X_t)\,dX_t. \tag{1.8}$$

Assuming sufficient regularity for V (by, for instance, dictating that it be smooth and of compact support), one can write W as $C\int V(y)/|x-y|\,dy$ for some constant C, and so $\mathcal{A}$ will have an explicit expansion. The reader is warned however that this is not the Clark-Ocone expansion, but something definitely different. The only deterministic term here is $-W(0)$, which is independent of T, whereas $E(\mathcal{A})$ certainly depends on time. One could even compute both terms and convince oneself they are different.

The Itô expansion has the drawback, in the particular case of the Nelson model, that many more terms are obtained than with the Clark-Ocone expansion (Gubinelli et al. obtain a total of 4, and Nelson gets 16), some of which require an extranous infrared cutoff, that both Nelson and Gubinelli et al. apply. Indeed, Nelson recognized that the infrared cutoff he puts in the terms he finds is an artifact of his calculations that is not really required (see the first paragraph of Section 6 in [38]). The Clark-Ocone expansion we perform instead has the advantage of being simpler, since only two terms are obtained, but far more importantly, both of the terms we get behave well for small values of k, meaning that our calculations do not require any kind of infrared cutoff, which confirms that it is indeed an artifact. This is not to say that it is completely unnecessary, since it is known that no ground state exists without it in the massless case [35, 20, 2], but we show one can make do without it, as far as our calculations are concerned. What we compute in Chapter 4 is a very explicit lower bound for the renormalized Nelson model, in terms of the coupling constant appearing and the number of nucleons, uniform in the mass paramater of the mesons $\mu \geq 0$, which is valid even in the massless case $\mu = 0$ without an infrared cutoff. Even though there is an implicit lower bound for the massless case but with infrared cutoff in the work of Gubinelli et al. [17, Corollary 2.18], it is unclear to us if it is indeed at all possible to disassemble their calculations and derive from their computations alone an explicit lower bound on the ground state energy such as we provide. Another result we prove for the Nelson model is the analogue of the conclusion for the polaron model concerning absence of binding. See Chapter 5 for an explicit statement.

We now give a brief overview of the structure of the thesis. The dissertation has been split into five chapters. Chapter 1 is the current introduction. Chapter 2 presents a derivation of a Feynman-Kac formula for the action of the semigroup e^{-TH} against a vector ψ in the case of a non-relativistic particle interacting with a quantum field. The formula is useful in particular since it unveils the effective potential of H. In Chapter 3, a summary of an article written by the author and L.E. Thomas [5] is given, in which upper bounds were provided for the exponential moments of several effective actions $\mathcal{A}_T$ appearing in quantum mechanics and non-relativistic quantum field theory. The Clark-Ocone formula and further estimates used are presented here. Chapter 4 applies the ideas of Chapter 3 to the multiparticle Nelson and polaron models, finding explicit lower bounds for them. Finally, Chapter 5 uses the ideas presented in previous chapters to find new estimates, in the cases of the Nelson and polaron models, on the critical repulsion parameter needed for a Coulomb interaction between two particles to prevent them from binding.

Chapter 2

Feynman-Kac Formula For Non-Relativistic Quantum Field Theories

Here we will derive a Feynman-Kac formula for a rather general, but simple model of a non-relativistic particle interacting with a quantum field. By this we mean a functional integral describing the evolution of any state under the semigroup e^{-TH}, where H is the Hamiltonian of the model. The formula will be quite explicit, as we will see. Despite that this is not the true quantum mechanical propagator, given by e^{-iTH}, a formula for its imaginary time version can lead to useful computations, such as the ground state energy, which is determined by the long-time behavior of certain matrix elements of e^{-TH}. We start by considering the following Hamiltonian

$$-\frac{\Delta}{2} + \omega a^\dagger a + u(x)a + \overline{u(x)}a^\dagger, \tag{2.1}$$

acting on $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R})$. The Laplacian here is d-dimensional and $x \in \mathbb{R}^d$. a and $a^\dagger$ are the lowering and raising operators for a simple, one dimensional harmonic oscillator. This Hamiltonian is a simple model for a non-relativistic particle with d spatial degrees of freedom interacting with a quantum mechanical harmonic oscillator. ω is simply the frequency of the oscillator. Later on, we will consider a particle interacting with several harmonic oscillators. The collection of oscillators will be described by a Fock space. For the moment, however, the discussion will continue with just one oscillator. Since d is any dimension, the model accepts many particles interacting with this one oscillator. For simplicity, however, the reader may want to keep in mind for now the case of a single three-dimensional particle, in which case $d = 3$.

Since $\omega a^\dagger a + u(x)a + \overline{u(x)}a^\dagger$ is a potential mixing the spatial coordinates with the coordinates of the oscillator, it is natural to try to obtain a Feynman-Kac formula for e^{-TH} using the Trotter product formula, which is the classical way to prove that

$$e^{-TH}\psi(x) = E^x \left[\exp\left(-\int_0^T V(X_t)\, dt \right) \psi(X_t) \right], \tag{2.2}$$

when $\omega a^\dagger a + u(x)a + \overline{u(x)}a^\dagger$ is replaced by a potential $V(x)$ involving the spatial coordinates alone. Here X_t is d-dimensional Brownian motion, and E^x denotes expectation with respect to Brownian motion starting at x. (E^0 will be typically denoted simply by E.) This line of thought works well, and is the one we will pursue. We assume from now on that u is a continuous, complex valued function on $\mathbb{R}^d$.

Let us quickly review how the formula (2.2) is obtained. The discussion in this paragraph will proceed formally, and technical points of rigor will be omitted. One simply splits $H = -\Delta/2 + V$ using the Trotter product formula,

$$(e^{-TH}\psi)(x) = \lim_{n\to\infty} \left[e^{T\Delta/(2n)}e^{-TV(x)/n}\right]^n \psi(x), \tag{2.3}$$

and recalling that $e^{\alpha\Delta/2}f = f * p_\alpha$, where $p_\alpha(x)$ is the heat kernel $(2\pi\alpha)^{-d/2}e^{-x^2/(2\alpha)}$, one obtains, after repeated applications of $e^{T\Delta/(2n)}$,

$$\begin{aligned}(e^{-TH}\psi)(x) &= \lim_{n\to\infty} E^x \left\{\exp\left[-\frac{T}{n}\sum_{m=1}^{n} V(X_{mT/n})\right]\psi(X_T)\right\} \\ &= E^x\left\{\exp\left[-\int_0^T V(X_t)\,dt\right]\psi(X_T)\right\}.\end{aligned} \tag{2.4}$$

The proof we just gave is standard. See, for instance, [40]. Equation (2.2) is useful not only because it is the solution of the heat equation with potential V – if $\Psi(x,t) = (e^{-tH}\psi)(x)$,

$$\frac{\partial\Psi}{\partial t} = \frac{\Delta}{2}\Psi - V(x)\Psi, \tag{2.5}$$

when $\Psi(x,0) = \psi(x)$ – but also because it provides information into the corresponding stationary Schrödinger eigenvalue equation

$$H\psi = -\frac{\Delta}{2}\psi + V\psi = \lambda\psi; \tag{2.6}$$

in particular, since Ψ may also be written as

$$\Psi(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(x) \tag{2.7}$$

for some $\Psi(\cdot,0)$-dependent constants a_n, where $H\phi_n = \lambda_n\phi_n$, with $\lambda_1 \le \lambda_2 \le \ldots$, we have

$$\lambda_1 = -\lim_{T\to\infty}\frac{1}{T}\log\Psi(x,T). \tag{2.8}$$

By setting $\Psi(\cdot,0)$ equal to 1, one gets the Feynman-Kac formula

$$\text{inf spec } H = -\lim_{T\to\infty}\frac{1}{T}\log\left\{E^x\left[\exp\left(-\int_0^T V(X_t)\,dt\right)\right]\right\}. \tag{2.9}$$

The reader should note that the functional integral inside the logarithm on the right side of (2.9) can be conveniently written as the inner product $(\delta_x, e^{-TH}1)$. The upshot then is that the long-time behavior of the matrix element $(\delta_x, e^{-TH}1)$ determines the ground state energy of the Hamiltonian H. This result will be used extensively in the next chapters. They will be devoted to finding upper bounds for this matrix element in particular examples. The upper bounds in turn will produce lower bounds for the ground-state energy of H.

2.1 Particle interacting with Harmonic Oscillators

We now go back to the case where the potential has a certain prescribed form involving raising and lowering operators, equation (2.1). We will assume that initially the particle is in a certain configuration described by a state $\psi \in L^2$ and that the oscillator is in an unnormalized coherent state $\widehat{\xi}$. By this we mean

$$\widehat{\xi} = |\xi\rangle = \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{n!}} |n\rangle, \tag{2.10}$$

where $|n\rangle$ is the n-th eigenstate of $a^\dagger a$: $a^\dagger a|n\rangle = n|n\rangle$. The oscillator and the particle are then decoupled. This is not a restriction, however, since any state can be expressed as an L^2-limit of linear combinations of product states like the one we are considering. By means of the Trotter product formula, we then obtain that

$$e^{-TH}\psi \otimes |\xi\rangle = \lim_{n\to\infty} \left[\exp\left(\frac{T\Delta}{2n}\right) \exp\left(-\frac{T}{n} \left[\omega a^\dagger a + u(x)a + \overline{u(x)}a^\dagger \right] \right) \right]^n \psi \otimes |\xi\rangle. \tag{2.11}$$

This involves the repeated application of operators of the type $\exp[-t(\omega a^\dagger a + ua + \overline{u}a^\dagger)]$ for some constant u. The key is that an application of this operator over a coherent state delivers again a coherent state with some additional constants in front. We will prove this in a lemma.

Lemma 2.1. *Let $|\xi\rangle$ be a coherent state, and u be a complex constant. Then, for any real numbers t and ω,*

$$\begin{aligned} &\exp\left[-t\left(\omega a^\dagger a + ua + \overline{u}a^\dagger\right)\right]|\xi\rangle \\ &= \exp\left[-\frac{(1 - t\omega - e^{-t\omega})}{\omega^2}|u|^2 - \frac{(1-e^{-t\omega})}{\omega}\xi u \right] \left| e^{-t\omega}\xi - (1 - e^{-t\omega})\frac{\overline{u}}{\omega} \right\rangle. \end{aligned} \tag{2.12}$$

Proof. By completing the square, we see it is enough to consider the study of $f(t) \equiv \exp\left(-tA^\dagger A\right)|\xi\rangle$, where $A = a + \eta$, for some complex number η. Differentiating f, we find

$$f'(t) = -(\xi + \eta)\exp\left(-tA^\dagger A\right)A^\dagger|\xi\rangle. \tag{2.13}$$

Computation of $g(t) \equiv \left[\exp(-tA^\dagger A), A^\dagger\right]$ will lead to a differential equation for f. Since $[A, A^\dagger] = 1$, by using the fact that $[UV, W] = U[V, W] + [U, W]V$ we find, by differentiating g,

$$g'(t) = -g(t)(I + A^\dagger A) - A^\dagger \exp(-tA^\dagger A), \tag{2.14}$$

and therefore,

$$g(t) = -(1 - e^{-t})A^\dagger \exp\left(-tA^\dagger A\right), \tag{2.15}$$

which implies that

$$f'(t) = -(\xi + \eta)e^{-t}A^\dagger f(t), \tag{2.16}$$

and so

$$f(t) = \exp\left[-(1 - e^{-t})(\xi + \eta)A^\dagger\right]|\xi\rangle. \tag{2.17}$$

The result then follows by picking $A = a + \overline{u}/\omega$ at time $t\omega$, and using the fact that, for $\mu \in \mathbb{C}$, the coherent state $|\mu\rangle$ is equal to $\exp(\mu a^\dagger)|0\rangle$. □

By repeatedly applying Lemma 2.1, one easily gets

Lemma 2.2. *For complex constants $u_1, \dots, u_n$ and real numbers t and ω,*

$$\prod_{i=1}^{n} \exp\left[-t\left(\omega a^\dagger a + u_{n-i+1} a + \overline{u_{n-i+1}} a^\dagger\right)\right] |\xi\rangle, \tag{2.18}$$

where a product $\prod_{i=1}^{m} a_i$ is to be ordered as $a_1 \dots a_m$, is equal to

$$\begin{aligned} &\exp\left[\frac{1}{\omega^2}(e^{-t\omega} - 1 + \omega t)\sum_{i=1}^{n}|u_i|^2\right] \exp\left[\frac{1}{\omega^2}(1-e^{-t\omega})^2 \sum_{i=2}^{n}\sum_{j=1}^{i-1} u_i \overline{u_j} e^{-t(i-j-1)\omega}\right] \\ &\times \exp\left[-(1-e^{-t\omega})\frac{\xi}{\omega}\sum_{i=1}^{n} u_i e^{-t(i-1)\omega}\right] \left| \frac{(e^{-t\omega}-1)}{\omega} \sum_{i=1}^{n} \overline{u_i} e^{-(n-i)t\omega} + \xi e^{-nt\omega} \right\rangle. \end{aligned} \tag{2.19}$$

Repeated application of $\exp[T\Delta/(2n)]$ then yields the result that

$$\begin{aligned} e^{-TH}[\psi \otimes |\xi\rangle](x) &= \lim_{n\to\infty} E^x \left[\prod_{i=1}^{n} \exp\left(-\frac{T}{n}\left[\omega a^\dagger a + u(X_{iT/n}) a + \overline{u(X_{iT/n})} a^\dagger\right]\right) \psi(X_T)|\xi\rangle\right] \\ &= E^x \left\{ \exp\left[\int_0^T\int_0^t \overline{u(X_t)} u(X_s) e^{-(t-s)\omega}\, ds\, dt - \xi \int_0^T e^{-\omega(T-t)} u(X_t)\, dt\right] \right. \\ &\qquad \left. \times\, \psi(X_T) \left| -\int_0^T \overline{u(X_t)} e^{-t\omega}\, dt + \xi e^{-T\omega} \right\rangle \right\}. \end{aligned} \tag{2.20}$$

Interesting to note in the formula above is that the functional integral is sensitive to the initial coherent state of the oscillator. This persists even after computing transition probabilities. For instance, the matrix element $(\delta_0 \otimes \widehat{0}, e^{-TH} 1 \otimes \widehat{\xi})$ is equal to

$$E\left\{\exp\left[\int_0^T\int_0^t \overline{u(X_t)} u(X_s) e^{-\omega(t-s)}\, ds\, dt - \xi \int_0^T e^{-\omega(T-t)} u(X_t)\, dt\right]\right\}, \tag{2.21}$$

a formula first derived by Feynman [11] in the particular case $\xi = 0$. (He followed a route different than the Trotter product formula above, however.) We now consider the case of a particle interacting with a collection of oscillators, described by the Fock space on $L^2(\mathbb{R}^m)$. The Hamiltonian in question will be

$$H = -\frac{\Delta}{2} + \int \omega(k) a_k^\dagger a_k\, dk + \int \left[f_k(x) a_k + \overline{f_k(x)} a_k^\dagger\right] dk, \tag{2.22}$$

where now the operators a_k satisfy $[a_k, a_{k'}^\dagger] = \delta(k-k')$. In applications, ω and f will have compact support in k, uniformly in x, and the reader may want to keep that case in mind. (A function $h_k(x)$ has this property if there is a ball B such that $\operatorname{supp} h_\cdot(x) \subset B$ for all x.) Since there is one harmonic oscillator for each mode k, one simply uses the previous result for each individual mode, obtaining finally

$$\begin{aligned} e^{-TH}[\psi \otimes |\xi\rangle](x) = E^x &\left\{ \exp\left[\int\int_0^T\int_0^t \overline{f_k(X_t)} f_k(X_s) e^{-\omega(k)(t-s)}\, ds\, dt\, dk \right.\right. \\ &\left.\left. - \int\int_0^T e^{-\omega(k)(T-t)} \xi_k f_k(X_t)\, dt\, dk\right] \psi(X_T) \otimes \left| -\int_0^T \overline{f(X_t)} e^{-t\omega}\, dt + \xi e^{-T\omega} \right\rangle \right\}, \end{aligned} \tag{2.23}$$

where what is on the right hand side of the tensor product is a coherent state for the Fock space defined through the relation $a_k|\zeta\rangle = \zeta_k|\zeta\rangle$. Equation (2.23) is then a Feynman-Kac formula for a single, non-relativistic d-dimensional particle interacting with a quantum field, described by a Fock space over $L^2(\mathbb{R}^m)$. We will now proceed to give two examples of utmost importance, where this formula will prove useful.

2.2 The Multipolaron Model

The polaron model was devised by H. Fröhlich [15] as a description of a single non-relativistic electron interacting with a polar crystal. The movement of the electron distorts the lattice surrounding it, and this distortion in turn affects the movement of the electron itself. The extension to many electrons follows easily from the model of Fröhlich, except that an interelectronic repulsion term appears. The full Hamiltonian for N electrons interacting with a polar crystal is given by

$$H = -\sum_{n=1}^{N} \frac{\Delta_n}{2} + \int \chi_\Lambda(k) a_k^\dagger a_k \, dk + \frac{\sqrt{\alpha}}{2^{3/4}\pi} \sum_{n=1}^{N} \int \frac{\chi_\Lambda(k)}{|k|} (a_k e^{ikx_n} + a_k^\dagger e^{-ikx_n}) \, dk + \sum_{m<n} \frac{U}{|x_n - x_m|}, \tag{2.24}$$

where α and U are non-negative constants. Here all the spacial coordinates and Laplacians are three-dimensional, and so are the k-integrals appearing. $\chi_\Lambda(k)$ is just the indicator function of the ball of radius Λ in $\mathbb{R}^3$. This ultraviolet cutoff is, in a sense, unnecessary. The only reason why we included it is that without it the operator only makes sense as a quadratic form. Keeping the cutoff ensures that it can be made sense of as a self-adjoint operator, with a definite domain. See [16] for more details. The Feynman-Kac formula above (2.23) immediately applies. Here we will study only the matrix element $(\delta_0 \otimes \widehat{0}, e^{-TH} 1 \otimes \widehat{0})$. After using the formula with

$$\omega(k) = \chi_\Lambda(k), \tag{2.25}$$

$$f_k(x) = \frac{\chi_\Lambda(k)\sqrt{\alpha}}{2^{3/4}\pi|k|} \sum_{n=1}^{N} e^{ikx_n}, \tag{2.26}$$

and computing explicitly the k-integral and taking the limit $\Lambda \to \infty$, which amounts to computing the Fourier transform of $1/|k|^2$ (equal to $1/|x|$, up to constants), one obtains

$$(\delta_0 \otimes \widehat{0}, e^{-TH} 1 \otimes \widehat{0}) = E\left[\exp\left(\sum_{m,n} \frac{\alpha}{\sqrt{2}} \int_0^T \int_0^t \frac{e^{-(t-s)}}{|X_t^m - X_s^n|} \, ds \, dt - \sum_{m<n} \int_0^T \frac{U}{|X_t^m - X_t^n|} \, dt\right)\right], \tag{2.27}$$

where $(X^1, X^2, \ldots, X^N)$ is a $3N$-dimensional Brownian motion. The long time behavior of this functional integral will then determine the ground state energy of the model. We will explore this energy and its dependence on U later below.

2.3 The Nelson Model

The Nelson model refers to a model conceived by E. Nelson [37, 38], describing the interaction of N non-relativistic nucleons with a meson field, the latter represented by a Boson field, in analogy with the polaron model. The Hamiltonian in this case reads as

$$H = -\sum_{n=1}^{N} \frac{\Delta_n}{2} + \int \chi_\Lambda(k)\omega(k) a_k^\dagger a_k \, dk + \sqrt{\alpha} \sum_{n=1}^{N} \int \frac{\chi_\Lambda(k)}{\sqrt{\omega(k)}} (a_k e^{ikx_n} + a_k^\dagger e^{-ikx_n}) \, dk, \tag{2.28}$$

acting on $L^2(\mathbb{R}^3) \otimes \mathcal{F}$, as before, where $\mathcal{F}$ is the Fock space on $L^2(\mathbb{R}^3)$. α is again a non-negative constant. Here the frequency of the oscillators satisfies the dispersion relation $\omega(k) = \sqrt{k^2 + \mu^2}$, where μ is any non-negative number, representing the mass of the mesons. The formula (2.23) again applies immediately – the matrix element $(\delta_0 \otimes \widehat{0}, e^{-TH} 1 \otimes \widehat{0})$ reads as

$$(\delta_0 \otimes \widehat{0}, e^{-TH} 1 \otimes \widehat{0}) = E\left[\exp\left(\alpha \sum_{m,n} \iint_0^T \int_0^t \frac{\chi_\Lambda(k) e^{-\omega(k)(t-s)}}{\omega(k)} e^{-ik(X_t^m - X_s^n)}\, ds\, dt\, dk\right)\right]. \tag{2.29}$$

In contrast with the polaron model, this time the ultraviolet cutoff is a requirement to make sense of both the Hamiltonian and the functional integral above. One can easily verify that the expectation of the argument of the exponential in (2.29) is infinite if one takes $\Lambda = \infty$, and so the functional integral is meaningless in this case. How one deals with the case $\Lambda = \infty$ will be described in the chapters to follow. Essentially, a divergent function of Λ, behaving as a constant times $\log \Lambda$ for large Λ, has to be added to the Hamiltonian. The renormalized Hamiltonian $H + \infty$ is then well defined [37]. We will explore this in more detail in later chapters.

Chapter 3

Estimates on Functional Integrals of Non-Relativistic Quantum Field Theory

In this chapter we will provide upper bounds for exponential moments that appear when estimating from above certain matrix elements of the form $(\delta_0, e^{-TH}1)$ or $(\delta_0 \otimes \widehat{0}, e^{-TH}1 \otimes \widehat{0})$ in the case of interaction with a quantum field; this in turn will provide lower bounds on the Hamiltonians H by the Feynman-Kac formula, as explained in the previous chapter. We will base the discussion upon an article written by the author and L.E. Thomas [5]. We will refer to that article extensively during the present chapter. The reader may refer to it for more information on certain aspects that will be omitted in this discussion. Our main theorem will be the bounding of three different types of functional integrals; we will group here Theorems 2.1, 2.2, and 2.3 from [5] into a single one, in simplified form. Even though some applications of the theorem will be given in the present chapter, it will be used more extensively in the next one.

Theorem 3.1. *Let $f : [0,\infty) \to [0,\infty)$ be a non-increasing function and d an integer greater than 1. For any $1 \leq \theta < 2$,*

$$E\left[\exp\left(\int_0^T \frac{f(t)}{|X_t|^\theta}\,dt\right)\right] \leq \exp\left(A_\theta \|f^{2/(2-\theta)}\|_1 + B_\theta \|f(t)/t^{\theta/2}\|_1\right), \tag{3.1}$$

$$E\left[\exp\left(\int_0^T\int_0^t \frac{f(t-s)}{|X_t - X_s|^\theta}\,ds\,dt\right)\right] \leq \exp\left(A_\theta \|f\|_1^{2/(2-\theta)} T + B_\theta \|f(t)/t^{\theta/2}\|_1 T\right), \tag{3.2}$$

$$\begin{aligned} &E\left[\exp\left(\int_0^T\int_0^t \frac{f(t-s)}{|X_t - Y_s|^\theta}\,ds\,dt\right)\right] \\ &\leq \exp\left(2^{-\theta/(2-\theta)} A_\theta \|f\|_1^{2/(2-\theta)} T + 2^{-\theta/2}(1-\theta/2)^{-1} B_\theta \|f\|_1 T^{1-\theta/2}\right), \end{aligned} \tag{3.3}$$

where X and Y denote independent d-dimensional Brownian motions, and

$$A_\theta = \frac{2^{(3\theta-2)/(2-\theta)}\theta^{\theta/(2-\theta)}(2-\theta)}{(d-\theta)^{2\theta/(2-\theta)}}, \quad B_\theta = \frac{\theta\Gamma[(d-\theta)/2]}{2^{\theta/2}\Gamma(d/2)}. \tag{3.4}$$

In the theorems mentioned in [5] a slightly sharper and more general version of the bounds was provided. In particular, arbitrary non-negative functions f, Brownian motions starting at points other than zero, and the case $0 \leq \theta < 1$ were covered. For purposes of our discussion here, however, the more general version will not be needed, and so what is provided here will suffice.

Based on the functional integral given in the previous chapter for the polaron model, we hope the reader has already noticed that inequalities (3.2) and (3.3) will be of use for bounding the ground-state energy of the N-polaron from below. In the case $N = 1$, for instance, $f(t) = \alpha e^{-t}$ and $\theta = 1$, and inequality (3.2) yields the lower bound $-\alpha - \alpha^2/4$, sharp for small α [13] but off by a large factor of about 2.5 for large α [9, 33]. In any case, this is an explicit lower bound valid for all $\alpha \geq 0$, and is an improvement over Lieb and Yamazaki's bound [34], $-\alpha - \alpha^2/3$.

Inequality (3.1) will yield, for instance, a lower bound for the hydrogen-atom Hamiltonian

$$H = -\frac{\Delta}{2} - \frac{\alpha}{|x|}, \tag{3.5}$$

in three dimensions, where $\alpha \geq 0$. This inequality turns out to be sharp, for the ground state energy is $-\alpha^2/2$ and

$$E\left[\exp\left(\alpha \int_0^T \frac{dt}{|X_t|}\right)\right] \leq e^{\alpha^2 T/2 + o(T)}. \tag{3.6}$$

By means of the same inequality (3.1), lower bounds for the Schrödinger operators with potential $-\alpha/|x|^\theta$, with $1 \leq \theta < 2$, follow. The cases $0 \leq \theta < 1$ are also covered, by using the analogue of (3.1) in that range of values of θ, found in [5]. When $\theta = 2$ there is a phase transition in α: there is a critical behavior at $\alpha = (d-2)^2/8$, in the sense that the ground state energy is 0, 0, and $-\infty$ to the left, at, and to the right of this point, respectively. The lower bounds for these operators lead to a generalized form of Hardy's inequality [32]. To see this, by using (3.1), we get that, for any function $f \in C_c^\infty(\mathbb{R}^d)$ and $1 \leq \theta < 2$,

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla f(x)|^2\,dx - \alpha \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^\theta}\,dx \geq -(2-\theta)\left[\frac{(8\theta)^\theta \alpha^2}{4(d-\theta)^{2\theta}}\right]^{1/(2-\theta)}. \tag{3.7}$$

The right-hand side provides a lower bound for the operator $-\Delta/2 - \alpha/|x|^\theta$ in d dimensions. Hardy's inequality with the sharp constant $(d-2)^2/4$ is in particular recovered by selecting α equal to $(d-\theta)^\theta/8$ and letting θ go to 2:

$$\int_{\mathbb{R}^d} |\nabla f(x)|^2\,dx \geq \frac{(d-2)^2}{4}\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2}\,dx. \tag{3.8}$$

The inequalities are of use for the Nelson model as well, as will be developed in the next chapter. See also [5] for more information on how they relate to recent work by Gubinelli, Hiroshima, and Lörinczi [17] on this model.

3.1 The Idea Behind The Proof

The idea behind the proof of the theorem is a simple one, consisting of the use of the Clark-Ocone formula plus the estimates given in Lemmas 3.2 and 3.3 below. We start with the canonical finite-time horizon Wiener space $(\Omega, \mathcal{F}_T, P)$, where $\Omega = C_d[0,T]$, the set of continuous $\mathbb{R}^d$-valued functions on $[0,T]$; $\mathcal{F}_t$ is the standard filtration for Brownian motion $\sigma(X_s : 0 \leq s \leq t)$, with X d-dimensional Brownian motion on Ω, namely $X_t(\omega) = \omega(t)$; and P is Wiener measure. Real-valued functions F in $L^2(\Omega, \mathcal{F}_T)$ will be called "Brownian functionals." The Clark-Ocone formula then says that all Brownian functionals may be expanded as

$$F = E(F) + \int_0^T \rho_t\,dX_t, \tag{3.9}$$

for a unique $\mathbb{R}^d$-valued, adapted L^2-process ρ, the stochastic derivative of $\mathcal{A}$, and where the stochastic integral $\int_0^T \rho_t \, dX_t = \sum_{n=1}^d \int_0^T \rho_t^n \, dX_t^n$ is to be understood in the sense of Itô. This formula appears in [26, Section 3.4]; see also [36, 27, 41]. From now on the fixed time T will be implicit; for example, whenever we speak of a Brownian functional, the time T will be tacitly assumed. Two lemmas are central to the proof of the theorem above. Their proof will appear in a separate section below.

Lemma 3.2. *(Martingale Estimate Lemma.) Let $\mathcal{A}$ be a Brownian functional. Then, if ρ is the stochastic derivative of $\mathcal{A}$, for any $p > 1$,*

$$E(e^{\mathcal{A}}) \leq e^{E(\mathcal{A})} E\left[\exp\left(\frac{p^2}{2(p-1)} \int_0^T \rho_t^2 \, dt\right)\right]^{1-1/p}. \tag{3.10}$$

Lemma 3.3. *(Convolution Lemma.) If p_t denotes the heat kernel $p_t(x) = (2\pi t)^{-d/2} e^{-x^2/(2t)}$ on $\mathbb{R}^d$, for any bounded measurable function $h : [0, \infty) \to \mathbb{C}$ and $0 < \theta < 2$,*

$$\int_0^\infty h(t) \int_{\mathbb{R}^d} \frac{(x-y)}{|x-y|^{\theta+2}} p_t(y) \, dy \, dt = a(\theta, |x|, h) \frac{x}{|x|^\theta}, \tag{3.11}$$

for some complex-valued function a satisfying

$$|a(\theta, |x|, h)| \leq \frac{2\|h\|_\infty}{\theta(d-\theta)}. \tag{3.12}$$

Lemma 3.2 is of no real use unless one can compute, or at least give an estimate on, ρ. There is a certain class of Brownian functionals for which the stochastic derivative can be computed explicitly. For all the actions we encounter in this text, namely of the form prescribed by equations (3.1), (3.2), (3.3), the stochastic derivative may be computed explicitly through something called the Malliavin derivative [36]. We will not give a full explanation of how this is done, and the interested reader may refer to [5] for more details – we will simply provide here enough background to make the ensuing discussion understandable. Let H be the Hilbert space of square integrable functions from $[0, T]$ to $\mathbb{R}^d$ and define, for any g in H, $W(g) \equiv \int_0^T g_t \, dX_t$. Let $\mathcal{S}$ be the space of $L^2(\Omega, \mathcal{F}_T)$-functions of the form $F = f[W(g_1), \ldots, W(g_m)]$ for some smooth function $f : \mathbb{R}^m \to \mathbb{R}$ with polynomial growth and $m \in \mathbb{N}$. Then, we define the Malliavin derivative of F as

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \left[W(g_1), \ldots, W(g_m)\right] g_i(t), \tag{3.13}$$

a function in $L^2(\Omega \times [0, T])$. The formula can be extended to a wider class of functions by approximating them by limits of functions in $\mathcal{S}$; see [5] for more details. It turns out that for functions $\mathcal{A}$ in this wider class, the stochastic derivative of $\mathcal{A}$ may be calculated as $\rho_t = E(D_t\mathcal{A}|\mathcal{F}_t)$. This is how we will compute stochastic derivatives in this and the next chapter. For instance, for

$$\mathcal{A} = \int_0^T \frac{dt}{|X_t|^\theta}, \tag{3.14}$$

with $1 \leq \theta < 2$, we find, formally, by means of equation (3.13),

$$D_t \mathcal{A} = -\theta \int_t^T \frac{X_s}{|X_s|^{\theta+2}} \, ds, \tag{3.15}$$

and then,

$$\rho_t = E(D_t\mathcal{A}|\mathcal{F}_t) = -\theta \int_{\mathbb{R}^d} \int_t^T \frac{y + X_t}{|y + X_t|^{\theta+2}} p_{s-t}(y) \, ds \, dy. \tag{3.16}$$

The final answer is not just formal, but actually correct. The idea is to mollify integrands and approximate integrals with Riemann sums; see [5]. The attentive reader will notice that ρ in this case has the form that appears in Lemma 3.3 – this is where that lemma comes in handy: it provides a further estimate on ρ that shows that it is bounded above by something that is less singular than the original action. Indeed, if one follows the calculations carefully, one will notice that one passes from an exponent θ to an exponent $2\theta - 2$. This "improvement" in the action is the key step in the proof of the bounds. By a repeated application of Lemmas 3.2 and 3.3, it becomes clear then how one can get an upper bound for the exponential moment of the action in question: one simply repeats the argument until one gets an action that is bounded by a number times T. This is not what we will do at the end, however, but will follow a simpler, shorter route. The philosophy remains the same in any case, and is that one application of the two lemmas above exchanges the original action for something better, at the price of an upper bound and some constants.

3.2 Proof of Lemmas 3.2 and 3.3

The Martingale Estimate Lemma is a simple consequence of the Clark-Ocone expansion and Hölder's inequality with parameter p:

$$
\begin{aligned}
E\left[\exp\left(\mathcal{A}\right)\right] &= E\left[\exp\left(E\left(\mathcal{A}\right) + \int_0^T \rho_t\, dX_t\right)\right] \\
&= E\left(e^{\mathcal{A}}\right) E\left\{\exp\left[\left(\int_0^T \rho_t\, dX_t - \frac{p}{2}\int_0^T \rho_t^2\, dt\right) + \frac{p}{2}\int_0^T \rho_t^2\, dt\right]\right\} \\
&\leq E\left(e^{\mathcal{A}}\right) E\left[\exp\left(\int_0^T (p\rho_t)\, dX_t - \frac{1}{2}\int_0^T (p\rho_t)^2\, dt\right)\right]^{1/p} \\
&\quad \times E\left[\exp\left(\frac{p^2}{2(p-1)}\int_0^T \rho_t^2\, dt\right)\right]^{1-1/p}.
\end{aligned}
\tag{3.17}
$$

The key observation now is that the process

$$
\Omega_t = \exp\left[\int_0^t (p\rho_s)\, dX_s - \frac{1}{2}\int_0^t (p\rho_s)^2\, ds\right] \tag{3.18}
$$

is a supermartingale, since ρ is an L^2 process. (See [26, Subsection 3.5 D] for a proof, involving an easy stopping time argument.) It then follows that $E(\Omega_t) \leq E(\Omega_0) = 1$ for all $t \geq 0$, and the assertion of the lemma is proved.

As regards the Convolution Lemma, one first rewrites $(x-y)/|x-y|^{\theta+2}$ as a gradient, namely

$$
\frac{x-y}{|x-y|^{\theta+2}} = -\frac{1}{\theta}\nabla_x \frac{1}{|x-y|^{\theta}}, \tag{3.19}
$$

and then one expresses $|x-y|^{-\theta}$ as a Laplace transform:

$$
\frac{1}{|x-y|^{\theta}} = \frac{(2\pi)^{d/2}}{2^{\theta/2}\Gamma(\theta/2)}\int_0^\infty s^{(d-\theta-2)/2} p_s(x-y)\, ds, \tag{3.20}
$$

where p is the d-dimensional heat kernel as defined in Chapter 2, namely $p_s(x) = (2\pi s)^{-d/2} e^{-x^2/(2s)}$. Using these two computations and the fact that $p_t * p_s = p_{t+s}$, where $(f*g)(x)$ denotes the convolution $\int f(x-y)g(y)\, dy$, one finds

$$
\int_0^\infty h(t)\int_{\mathbb{R}^d} \frac{p_t(y)(x-y)}{|x-y|^{\theta+2}}\, dy\, dt = \frac{(2\pi)^{d/2}}{2^{\theta/2}\Gamma(\theta/2)}\int_0^\infty\int_0^\infty h(t|x|^2)\frac{s^{(d-\theta-2)/2}}{t+s} p_{t+s}(1)\, ds\, dt \frac{x}{|x|^{\theta}}. \tag{3.21}
$$

The expression to the left of $x/|x|^\theta$, on the right side of (3.21), is just the function a in the statement of the lemma. Bounding h by its L^∞ norm and computing the remaining integral leads to the desired result. See [5] for more details on the proof of this lemma. □

3.3 Proof of Theorem 3.1

We will only prove here inequality (3.2). At the end of the proof we will describe briefly how the proofs of the other inequalities go about. We will be using the Martingale Estimate Lemma first, and for that reason one first has to check that

$$\mathcal{A} \equiv \int_0^T \int_0^t \frac{f(t-s)}{|X_t - X_s|^\theta}\, ds\, dt \tag{3.22}$$

is a Brownian functional; the only step that really has to be verified is that it is in L^2. It is indeed in this space, and its proof will be omitted – one can use the Convolution Lemma for this. The expectation of $\mathcal{A}$ can be calculated in a straightforward way; omitting details, one finds

$$E\left(\int_0^T \int_0^t \frac{f(t-s)}{|X_t - X_s|^\theta}\, ds\, dt\right) = \frac{\Gamma\left[(d-\theta)/2\right]}{2^{\theta/2}\Gamma(d/2)} \int_0^T \int_0^t \frac{f(s)}{s^{\theta/2}}\, ds\, dt. \tag{3.23}$$

Next comes the conditional expectation of the Malliavin derivative of $\mathcal{A}$, which gives, after some computations

$$E\left(D_u \mathcal{A} | \mathcal{F}_u\right) = \theta \int_0^u \int_0^{T-u} f(t+u-s) \int_{\mathbb{R}^d} \frac{(X_s - X_u) - y}{|(X_s - X_u) - y|^{\theta+2}} p_t(y)\, dy\, dt\, ds. \tag{3.24}$$

This is in the form prescribed by the convolution lemma. One application of the lemma and the realization that

$$\|f(\cdot + u - s)\|_{\infty, T-u} = f(u-s), \tag{3.25}$$

leads to the conclusion that

$$|E\left(D_u \mathcal{A} | \mathcal{F}_u\right)| \le \frac{2}{d-\theta} \|f\|_1^{1/2} \left(\int_0^u \frac{f(u-s)}{|X_u - X_s|^{2\theta-2}}\, ds\right)^{1/2}, \tag{3.26}$$

where the Cauchy-Schwarz inequality was used. From now on we may assume that f is integrable, for otherwise the inequality is trivial. From the Martingale Estimate Lemma one then finds that

$$\begin{aligned} &E\left[\exp\left(\int_0^T \int_0^t \frac{f(t-s)}{|X_t - X_s|^\theta}\, ds\, dt\right)\right] \\ &\le \exp\left(\frac{\Gamma\left[(d-\theta)/2\right]}{2^{\theta/2}\Gamma(d/2)} \|f(s)/s^{\theta/2}\|_1 T\right) E\left[\exp\left(\frac{2p^2 \|f\|_1}{(d-\theta)^2 (p-1)} \int_0^T \int_0^t \frac{f(t-s)}{|X_t - X_s|^{2\theta-2}}\, ds\, dt\right)\right]^{1-1/p}, \end{aligned} \tag{3.27}$$

which proves in particular that the left side of (3.2) is finite, the reason being that an effective reduction of exponent from θ to $2\theta - 2$ has been accomplished (see the longer proof of this same theorem in [5] for more details on this finiteness statement). An application of Young's inequality

$ab \le a^x/x + b^y/y$, $1/x + 1/y = 1$, then leads to

$$\begin{aligned}\frac{2p^2\|f\|_1}{(d-\theta)^2(p-1)}\frac{f(t-s)}{|X_t-X_s|^{2\theta-2}} &= \frac{2(2\theta-2)^{(2\theta-2)/\theta}p^2\|f\|_1 f(t-s)^{(2-\theta)/\theta}}{(d-\theta)^2(p-1)\theta^{(2\theta-2)/\theta}} \\ &\quad\times \frac{\theta^{(2\theta-2)/\theta}f(t-s)^{(2\theta-2)/\theta}}{(2\theta-2)^{(2\theta-2)/\theta}|X_t-X_s|^{2\theta-2}} \\ &\le \frac{2^{\theta/(2-\theta)}(2-\theta)(2\theta-2)^{(2\theta-2)/(2-\theta)}p^{2\theta/(2-\theta)}\|f\|_1^{\theta/(2-\theta)}f(t-s)}{(d-\theta)^{2\theta/(2-\theta)}(p-1)^{\theta/(2-\theta)}\theta^{\theta/(2-\theta)}} \\ &\quad+ \frac{f(t-s)}{|X_t-X_s|^{\theta}},\end{aligned} \tag{3.28}$$

and this is how we get that, by choosing $p = \theta$,

$$\begin{aligned}&E\left[\exp\left(\int_0^T\int_0^t \frac{f(t-s)}{|X_t-X_s|^{\theta}}\,ds\,dt\right)\right] \\ &\le \exp\left(\frac{\theta\Gamma\left[(d-\theta)/2\right]}{2^{\theta/2}\Gamma(d/2)}\|f(s)/s^{\theta/2}\|_1 T + \frac{2^{(3\theta-2)/(2-\theta)}(2-\theta)\theta^{\theta/(2-\theta)}\|f\|_1^{2/(2-\theta)}}{(d-\theta)^{2\theta/(2-\theta)}}T\right).\end{aligned} \tag{3.29}$$

(This choice of θ minimizes the second summand in expression (3.29).) The proof of (3.1) is similar, and will be omitted (see [5] for a complete proof, however). The proof of (3.3) is somewhat different, involving the reduction to a single Brownian motion through Jensen's inequality; see [5] for more information. □

Chapter 4

Lower Bound on The Renormalized Nelson Model

In a 1964 article, E. Nelson [37] presented a model describing the interaction of N non-relativistic nucleons with a meson field with mass $\mu \geq 0$. Its Hamiltonian was written as

$$H_{\alpha,\mu}^{N,\Lambda} = -\sum_{n=1}^{N} \frac{\Delta_n}{2} + \int \chi_\Lambda \omega(k) a_k^\dagger a_k \, dk + \sqrt{\alpha} \sum_{n=1}^{N} \int \frac{\chi_\Lambda}{\sqrt{\omega(k)}} (e^{ikx_n} a_k + e^{-ikx_n} a_k^\dagger) \, dk, \tag{4.1}$$

where the Laplacian and integrals are three-dimensional, $\alpha, \Lambda \geq 0$, $\omega(k) = \sqrt{\mu^2 + k^2}$, and χ_Λ is the indicator function of the three-dimensional ball of radius Λ centered at the origin. This operator acts on $L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}$, where $\mathcal{F}$ is the standard Fock space on $L^2(\mathbb{R}^3)$. For finite Λ, H is self-adjoint and bounded below [37]. As $\Lambda \to \infty$, however, the ground state energy of $H_{\alpha,\mu}^{N,\Lambda}$ goes to $-\infty$, and therefore the ultraviolet cutoff is necessary as a means of controlling the ground-state energy of the Hamiltonian. In his 1964 paper, Nelson discovered that by adding a logarithmically divergent constant, which we will denote by $Q_{\alpha,\mu}^{N,\Lambda}$, the Hamiltonian can be stabilized in the limit $\Lambda \to \infty$, yielding a well-defined self-adjoint operator $\widehat{H}$ that is bounded below. The precise way the operator is renormalized is in the strong limit sense at the unitary group level: for every real t, $e^{it(H_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda})} \to e^{it\hat{H}}$, strongly. The purpose of this article is to provide an explicit lower bound on $\widehat{H}$. More precisely, we will provide a lower bound on $H_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda}$ that is uniform in Λ and μ – it will be written in terms of α and N, and will be valid for any value of these two quantities. The $\Lambda \to \infty$ limit then gives a lower bound for $\widehat{H}$. (See, for instance, [16] for details on this limit passage.) The actual result is that, if $E_{\alpha,\mu}^{N,\Lambda}$ is the ground-state energy of $H_{\alpha,\mu}^{N,\Lambda}$,

$$E_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda} \geq -C_\delta \alpha^{3+\delta} N^{4+\delta}, \tag{4.2}$$

where C_δ is an explicit positive constant that depends only on $\delta > 0$, and $C_\delta \to \infty$ as $\delta \to 0$.

The main tool will be the unified approach to find lower bounds for several Hamiltonians appearing in non-relativistic quantum mechanics and field theory that was presented in the previous chapter. As a preliminary computation, we will work with the N-polaron model of H. Fröhlich [15] without interelectronic repulsion, which describes a system of N electrons interacting with the phonons of a crystal lattice, as was already discussed in previous chapters. Its Hamiltonian is akin to that of the Nelson model,

$$H_\alpha^N = -\sum_{n=1}^{N} \frac{\Delta_n}{2} + \int a_k^\dagger a_k \, dk + \sum_{n=1}^{N} \frac{\sqrt{\alpha}}{2^{3/4}\pi} \int \frac{1}{|k|} (a_k e^{ikx_n} + a_k^\dagger e^{-ikx_n}) \, dk, \tag{4.3}$$

and its analysis will serve as a stepping stone toward finding a lower bound for the Nelson model. The reason why we will follow this path is that the Hamiltonian of the N-polaron model does not require any kind of renormalization, and thus a faster route toward finding a lower bound to its Hamiltonian can be found, using the methods mentioned.

4.1 The Main Strategy

The main idea behind the lower bound is, as was said in the introduction to this chapter, the use of the strategy presented in the previous chapter. In our case here, it will consist of the following: if $\mathcal{A}$ is the action for the Nelson model, derived in the second chapter, and reproduced here for completeness,

$$\mathcal{A} = \alpha \sum_{m,n} \iint_0^T \int_0^t \frac{\chi_\Lambda(k) e^{-\omega(k)(t-s)}}{\omega(k)} e^{-ik(X_t^m - X_s^n)}\, ds\, dt\, dk, \tag{4.4}$$

the mission is to find an upper bound for $E(e^{\mathcal{A}})$ that is log-linear in T. We first expand $\mathcal{A}$ using the Clark-Ocone formula, equation (3.9), reproduced here for the convenience of the reader,

$$\mathcal{A} = E(\mathcal{A}) + \int_0^T \rho_t\, dX_t, \tag{4.5}$$

for some unique L^2, adapted process ρ. An effective replacement of $\mathcal{A}$ by another action is then done by means of the martingale estimate lemma, Lemma 3.2 from the previous chapter,

$$E(e^{\mathcal{A}}) \le e^{E(\mathcal{A})} E\left[\exp\left(\frac{p^2}{2(p-1)} \int_0^T \rho_t^2\, dt\right)\right]. \tag{4.6}$$

One application of this estimate for the Nelson model then leads to functional integrals for which estimates were given in Theorem 3.1. We will make use here of the following inequalities, which are special cases of that theorem: for any $1 \le \theta < 2$,

$$E\left[\exp\left(\beta \int_0^T \int_0^t \frac{1_{[0,\varepsilon]}(t-s)}{|X_t - X_s|^\theta}\, ds\, dt\right)\right] \le \exp\left[\left(A_\theta \beta^{2/(2-\theta)} \varepsilon^{2/(2-\theta)} + \frac{B_\theta \varepsilon^{1-\theta/2} \beta}{1-\theta/2}\right) T\right], \tag{4.7}$$

$$E\left[\exp\left(\beta \int_0^T \int_0^t \frac{1_{[0,\varepsilon]}(t-s)}{|X_t - Y_s|^\theta}\, ds\, dt\right)\right] \le \exp\left[2^{-\theta/(2-\theta)} A_\theta \beta^{2/(2-\theta)} \varepsilon^{2/(2-\theta)} T + o(T)\right], \tag{4.8}$$

$$E\left[\exp\left(\beta \int_0^T \int_0^t \frac{e^{-(t-s)}}{|X_t - X_s|}\, ds\, dt\right)\right] \le \exp\left[\left(\frac{\beta^2}{2} + \sqrt{2}\beta\right) T\right], \tag{4.9}$$

$$E\left[\exp\left(\beta \int_0^T \int_0^t \frac{e^{-(t-s)}}{|X_t - Y_s|}\, ds\, dt\right)\right] \le \exp\left[\frac{\beta^2 T}{4} + o(T)\right]. \tag{4.10}$$

These bounds will be used extensively in the next sections. Before starting with the actual proofs, we will give a brief outline of what is to come. In the next section, Section 4.2, a combinatorial argument will be given that will allow us to substantially improve the exponent in the number of particles in the final lower bound. The idea is to perform an optimal block design for all the pairs of particles involved. Instead of constructing the partition directly for the Nelson model, we will do it first for the polaron. The same block design will work for the Nelson model as well. With the combinatorial argument ready, in Section 4.3 we will use all the tools introduced to construct the lower bound we are seeking.

4.2 Optimal Block Design through the Polaron Model

As was said in the previous section, we will construct a partition for the pairs of particles that will turn out to be optimal in a sense that will be made precise later. The construction will be performed on the multi-particle polaron model, and then we will use the same partition for the Nelson model. From the form of the Hamiltonian for the multi-particle polaron model, the functional integral leading to the ground-state energy is given by

$$\begin{aligned} & E\left[\exp\left(\frac{\alpha}{\sqrt{2}}\sum_{m=1}^{N}\sum_{n=1}^{N}\int_0^T\int_0^t \frac{e^{-(t-s)}}{|X_t^m - X_s^n|}\,ds\,dt\right)\right] \\ &= E\left[\exp\left(\frac{\alpha}{\sqrt{2}}\sum_{m=1}^{N}\int_0^T\int_0^t \frac{e^{-(t-s)}}{|X_t^m - X_s^m|}\,ds\,dt + \frac{\alpha}{\sqrt{2}}\sum_{m<n}\int_0^T\int_0^T \frac{e^{-|t-s|}}{|X_t^m - X_s^n|}\,ds\,dt\right)\right] \\ &= E\left[\prod_{m=1}^{N}\exp\left(\frac{\alpha}{\sqrt{2}}\int_0^T\int_0^t \frac{e^{-(t-s)}}{|X_t^m - X_s^m|}\,ds\,dt\right)\prod_{m<n}\exp\left(\frac{\alpha}{\sqrt{2}}\int_0^T\int_0^T \frac{e^{-|t-s|}}{|X_t^m - X_s^n|}\,ds\,dt\right)\right] \\ &\equiv E\left(\prod_{m=1}^{N}\Omega_{m,m}\prod_{m<n}\Omega_{m,n}\right) = E\left(\prod_{m\le n}\Omega_{m,n}\right). \end{aligned} \tag{4.11}$$

If all the $\Omega_{m,n}$ were independent, it would be an easy matter to split the above expectation. The problem is that this is not true: assuming so leads, for instance, to a lower bound that is quadratic in N, which is untenable for large N. (See, for instance, [3] for an upper bound that is cubic in N.) What is certainly true, however, is that the objects of any subset S of $R \equiv \{\Omega_{m,n} : 1 \le m \le n \le N\}$ in which every $1 \le j \le N$ appears as an index of at most one of its elements are independent. For example, if $N = 17$, the elements of $S = \{\Omega_{1,1}, \Omega_{2,3}, \Omega_{5,7}\}$ are independent. 1 appears only as an index of $\Omega_{1,1}$, and 5 appears only in $\Omega_{5,7}$. 16 is not an index of any of the elements of S. The question now arises, what is the optimal way to decompose R into disjoint subsets $S_1, \ldots, S_M$, $M \le N(N+1)/2$, so that the elements of S_i are independent? As it stands, the question is vague, since we have not specified what "optimal" means. Our desire is to take advantage of the independence of the elements of certain special subsets of R in a way that gives the best upper bound possible for the functional integral above, (4.11). To illustrate this, suppose that we have picked a partition $S_1, \ldots, S_M$, as above. Let T_i be the index set of S_i – the set of ordered pairs of elements of S. (For example, if $S_i = \{\Omega_{1,1}, \Omega_{2,2}\}$, then $T_i = \{(1,1),(2,2)\}$.) Then, if $p_1, \ldots, p_M > 1$ are such that $1/p_1 + \ldots + 1/p_M = 1$,

$$\begin{aligned} E\left(\prod_{m\le n}\Omega_{m,n}\right) &= E\left(\prod_{k=1}^{M}\prod_{(m,n)\in T_k}\Omega_{m,n}\right) \le \prod_{k=1}^{M} E\left(\prod_{(m,n)\in T_k}\Omega_{m,n}^{p_k}\right)^{1/p_k} \\ &= \prod_{k=1}^{M}\prod_{(m,n)\in T_k} E\left(\Omega_{m,n}^{p_k}\right)^{1/p_k}. \end{aligned} \tag{4.12}$$

Here we will content ourselves with picking all the p_i equal to one another. In this case, the above product reduces to

$$\prod_{m\le n} E(\Omega_{m,n}^M)^{1/M}. \tag{4.13}$$

The expression (4.13) may be bounded above, as explained in the introduction, by a product of terms of the form e^{aM^q}, where $a > 0, q > 0$. This will be optimized if the number of blocks M is optimized. It turns out that $M = N$. Before proving this as a lemma, let us make some definitions.

Let $N \geq 1$, and consider the set $I_N \equiv \{(m,n) : 1 \leq m \leq n \leq N\}$. Call a partition $\{G_1, \ldots, G_M\}$ of I_N "non-repeating" if $I_N = G_1 \cup \ldots \cup G_M$, the G_i's are non-empty and pairwise disjoint, and for each $1 \leq i \leq M$, every $1 \leq j \leq N$ appears in at most one ordered pair of G_i. (For example, some possible elements of a non-repeating partition of I_{17} are $\{(1,1),(2,3)\}$ and $\{(1,2),(3,4),(5,6)\}$. $\{(1,2),(2,3)\}$, in contrast, cannot possibly belong to any non-repeating partition of I_{17}.) We will say that a non-repeating partition of I_N is "optimal" if its cardinality is minimal, meaning that there is no non-repeating partition of I_N with fewer elements. As a final definition, recall that an $N \times N$ commutative latin square is a symmetric $N \times N$ matrix with exactly N objects appearing in it, such that all N objects appear in each row and each column. The existence of such a matrix is guaranteed by the existence of abelian groups. The addition table of any abelian group of N elements is a commutative latin square. For more information on latin squares and combinatorial designs in general, see [6].

Lemma 4.1. *The minimum number of elements of a non-repeating partition of I_N is N. Therefore, optimal non-repeating partitions of I_N have N elements. Furthermore, the collection of all optimal non-repeating partitions of I_N is in one-to-one correspondence with the set of $N \times N$ commutative latin squares.*

Proof. A good way of visualizing partitions of I_N is assigning to each ordered pair (m,n) a label, representing the block, or partition element, it corresponds to. This is illustrated in Figure 4.1, in the case $N = 5$, for a partition that is repeating. We call this way of representing a partition "block representation."

A	A	A	C	B
	B	A	C	A
		B	A	A
			B	C
				B

Figure 4.1: Block representation of some partition $\{A, B, C\}$ of I_5. Each square represents an ordered pair, whose first component must be read from top to bottom and the second component from left to right, as in a matrix. For instance, the rightmost block in the second row, from top to bottom, corresponds to the pair $(2,5)$, with label A. A in this case has 7 elements and $(2,3)$ is in it but $(5,5)$ is not.

Suppose there is a non-repeating partition of I_N made out of $N-1$ subsets. The pair $(1,1)$ must be in exactly one element of the partition; let us call it G_1. G_1 does not contain any other element of the first row. Likewise, the pair $(1,2)$ must be in exactly one element, G_2. The only element of the first row G_2 contains is $(1,2)$. Continuing this way up until the pair $(1, N-1)$ will produce $N-1$ different objects of the partition, and so the partition has been exhausted. The pair $(1,N)$ does not belong to any element of the partition. The proof is illustrated in the case $N = 5$ in Figure 4.2.

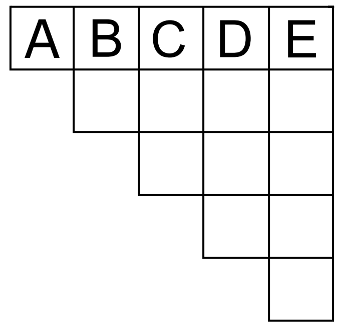

Figure 4.2: Proof of the non-existence of a partition with fewer than N components, in the case $N = 5$. Simply quering which square belongs to which block along the first row reveals 5 different components, which we call here A, B, C, D, and E.

We will now construct explicitly a non-repeating partition with N elements. In fact, the existence of such a partition is equivalent to the existence of an $N \times N$ commutative latin square, as we will now prove. Given a non-repeating partition with N elements, one could first draw its corresponding block representation. One may then take a copy of the resulting triangle, flip it about the diagonal, and attach it to the diagonal of the original triangle. The resulting symmetric matrix is a commutative latin square. (To see this last point, if there is a component of the partition repeated in a row, then the component contains two pairs that share a common index, which is not possible. The argument works for columns as well.) If now one takes a commutative $N \times N$ latin square and cuts off the bottom diagonal half, one gets the block representation of a non-repeating partition with components labeled by the objects of the triangle, if one puts the pair (m, n) into the component that is written in that position. The proof is illustrated in Figure 4.3.

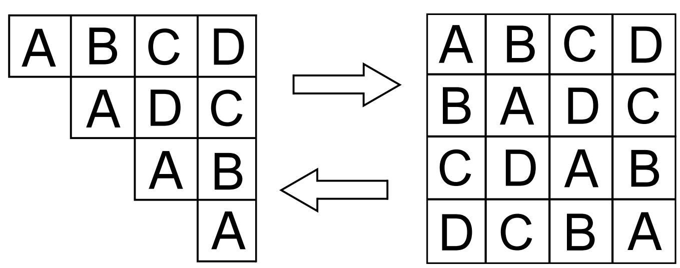

Figure 4.3: Proof of the equivalence between a 4×4 commutative latin square and a non-repeating partition of I_4. Copying the original half and pasting it onto the bottom creates a commutative latin square, and viceversa, taking a commutative latin square and removing its bottom diagonal half creates a non-repeating partition when each pair is assigned to the letter lying on it.

For concreteness of exposition, from now on one may always take the top diagonal half of the additive table of $\mathbb{Z}_N$ as a partition for N particles. For $N = 4$ this is illustrated in Figure 4.4. □

Let us now return to the lower bound for the N-polaron ground-state energy. What we have

A	B	C	D
	C	D	A
		A	B
			C

Figure 4.4: The upper diagonal half of the additive table of $\mathbb{Z}_4$. Here, for instance, $A = \{(1,1),(2,4),(3,3)\}$.

accomplished so far is that

$$\begin{aligned}
&E\left[\exp\left(\sum_{m,n}\frac{\alpha}{\sqrt{2}}\int_0^T\!\!\int_0^t \frac{e^{-(t-s)}}{|X_t^m - X_s^n|}\,ds\,dt\right)\right] \\
&\leq \prod_{m=1}^{N} E\left[\exp\left(\frac{N\alpha}{\sqrt{2}}\int_0^T\!\!\int_0^t \frac{e^{-(t-s)}}{|X_t^m - X_s^m|}\,ds\,dt\right)\right]^{1/N} \\
&\quad\times \prod_{m<n} E\left[\exp\left(\frac{N\alpha}{\sqrt{2}}\int_0^T\!\!\int_0^T \frac{e^{-|t-s|}}{|X_t^m - X_s^n|}\,ds\,dt\right)\right]^{1/N} \\
&= E\left[\exp\left(\frac{N\alpha}{\sqrt{2}}\int_0^T\!\!\int_0^t \frac{e^{-(t-s)}}{|X_t - X_s|}\,ds\,dt\right)\right] E\left[\exp\left(\frac{N\alpha}{\sqrt{2}}\int_0^T\!\!\int_0^T \frac{e^{-|t-s|}}{|X_t - Y_s|}\,ds\,dt\right)\right]^{(N-1)/2}.
\end{aligned} \tag{4.14}$$

Furthermore, by means of the Cauchy-Schwarz inequality,

$$\begin{aligned}
&E\left[\exp\left(\frac{N\alpha}{\sqrt{2}}\int_0^T\!\!\int_0^T \frac{e^{-|t-s|}}{|X_t - Y_s|}\,ds\,dt\right)\right] \\
&= E\left[\exp\left(\frac{N\alpha}{\sqrt{2}}\int_0^T\!\!\int_0^t \frac{e^{-(t-s)}}{|X_t - Y_s|}\,ds\,dt + \frac{N\alpha}{\sqrt{2}}\int_0^T\!\!\int_0^t \frac{e^{-(t-s)}}{|Y_t - X_s|}\,ds\,dt\right)\right] \\
&\leq E\left[\exp\left(\sqrt{2}N\alpha\int_0^T\!\!\int_0^t \frac{e^{-(t-s)}}{|X_t - Y_s|}\,ds\,dt\right)\right]^{1/2} E\left[\exp\left(\sqrt{2}N\alpha\int_0^T\!\!\int_0^t \frac{e^{-(t-s)}}{|Y_t - X_s|}\,ds\,dt\right)\right]^{1/2} \\
&= E\left[\exp\left(\sqrt{2}N\alpha\int_0^T\!\!\int_0^t \frac{e^{-(t-s)}}{|X_t - Y_s|}\,ds\,dt\right)\right],
\end{aligned} \tag{4.15}$$

and therefore, from inequalities (4.9) and (4.10), we obtain

$$E\left[\exp\left(\sum_{m,n}\frac{\alpha}{\sqrt{2}}\int_0^T\!\!\int_0^t \frac{e^{-(t-s)}}{|X_t^m - X_s^n|}\,ds\,dt\right)\right] \leq \exp\left\{\left[\frac{N^2\alpha^2}{4} + N\alpha + \frac{N^2(N-1)\alpha^2}{4}\right]T + o(T)\right\}. \tag{4.16}$$

Finally, by using the Feynman-Kac formula for the ground-state energy, we find that E_α^N, the ground state energy of the N-polaron, satisfies the bound

$$E_\alpha^N \geq -\frac{N^3\alpha^2}{4} - N\alpha. \tag{4.17}$$

A nice feature of this lower bound, that we hope the reader has already noticed, is that it has the exact same shape as the lower bound we already gave for only one particle in the previous chapter, $-\alpha - \alpha^2/4$. One may compare this with the upper bound obtained from the Pekar-Tomasevich functional

$$E_\alpha^N \le -0.108\alpha^2 N^3. \tag{4.18}$$

(See, for instance, [3].) This shows that the N^3 factor in the first summand of the lower bound is correct. It is a curious fact that after performing an optimal partition, applying Hölder's inequality and Cauchy-Schwarz, and using inequality (4.10), one arrives at an answer that could have been guessed with a previous knowledge of our bound for one particle and the N^3 dependence present in (4.18).

4.3 Proof of The Lower Bound for The Nelson Model

4.3.1 Overview

In the previous section we saw how a well-chosen block design for pairs of particles can substantially improve lower bounds for many-body quantum systems, by means of the bounds introduced in the previous chapter. We also obtained a new lower bound for the N-polaron model, which agrees with a known upper bound, as far as the N-dependence is concerned. In this section we will pursue a similar strategy for the renormalized N-particle Nelson model Hamiltonian. The idea, as was said in Section 4.1, is to bound from above the functional integral corresponding to the matrix element $(\delta_0 \otimes \widehat{0}, e^{-tH_{\alpha,\mu}^{N,\Lambda}} 1 \otimes \widehat{0})$, given by $E(e^{\mathcal{A}})$, with $\mathcal{A}$ defined by equation (4.4), that is

$$\mathcal{A} = \alpha \sum_{m,n} \iint_0^T \int_0^t \frac{\chi_\Lambda(k) e^{-\omega(k)(t-s)}}{\omega(k)} e^{-ik(X_t^m - X_s^n)}\, ds\, dt\, dk, \tag{4.19}$$

by means of the Martingale Estimate Lemma, Lemma 3.2. All we need to check in order to use the lemma is that $\mathcal{A}$ is L^2 and real-valued. Square integrability is immediate, as can be seen from the bound

$$\left| \int \frac{e^{-ik(X_t^m - X_s^n)}}{\omega(k)} e^{-\omega(k)(t-s)} \chi_\Lambda(k)\, dk \right| \le 2\pi\Lambda^2, \tag{4.20}$$

and so is real-valuedness, which follows from noting that

$$\int \frac{e^{-ik(X_t^m - X_s^n)}}{\omega(k)} e^{-\omega(k)(t-s)} \chi_\Lambda(k)\, dk = \int \frac{\cos[k(X_t^m - X_s^n)]}{\omega(k)} e^{-\omega(k)(t-s)} \chi_\Lambda(k)\, dk. \tag{4.21}$$

After applying the Martingale Estimate Lemma, further estimates will be required, as well as more results from the previous chapter, which will be introduced here as needed. The first term we will study is the expectation $E(\mathcal{A})$. After that, we will proceed to estimate $\int_0^T \rho_t^2\, dt$, with ρ the stochastic derivative of $\mathcal{A}$.

4.3.2 The Expectation Term $E(\mathcal{A})$

The expectation reads as, using the fact that $E(e^{ikZ}) = e^{-k^2\sigma^2/2}$ for a Gaussian variable Z with mean zero and variance σ^2, and the independence of X_m and X_n for n different than m,

$$\alpha N T \int \frac{\chi_\Lambda(k)}{\omega(k)(k^2/2 + \omega(k))}\, dk + \alpha N(N-1) \int_0^T \int_0^t \int \frac{e^{-k^2 t^2/2} e^{-k^2 s^2/2}}{\omega(k)} e^{-\omega(k)(t-s)} \chi_\Lambda(k)\, dk\, ds\, dt. \tag{4.22}$$

The term on the left in (4.22), divided by T,

$$\alpha N \int \frac{\chi_\Lambda(k)}{\omega(k)[k^2/2+\omega(k)]}\, dk \equiv Q^{N,\Lambda}_{\alpha,\mu}, \tag{4.23}$$

is especially important for us for a simple reason: it is precisely the term Nelson used to renormalize the Hamiltonian (4.1) [37, Equation (8)]. It is easily seen to be logarithmically divergent in Λ. Nelson found this term in [37] after performing a Gross transformation on the Hamiltonian. The same term is reappearing now as one of the summands in the expectation of the action. It turns out that this is the only term that contributes at the end: the term on the right in (4.22) is sublinear, uniformly in Λ and μ. To see this, by using the fact that $|k| \le \omega(k)$ and $\chi_\Lambda \le 1$, and computing the Fourier transform of $e^{-|k|}/|k|$, we can bound the second term from above as

$$\begin{aligned} &\alpha N(N-1)\int_0^T\int_0^t\int \frac{e^{-k^2t^2/2}e^{-k^2s^2/2}}{|k|}e^{-|k|(t-s)}\, dk\, ds\, dt \\ &= \alpha N(N-1)E\left(\int_0^T\int_0^t\int \frac{e^{-ik(X_t^m-X_s^n)}}{|k|}e^{-|k|(t-s)}\, dk\, ds\, dt\right) \\ &\le 2\pi\alpha N(N-1)\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\chi_{[0,T]}(t)\chi_{[0,T]}(s)p_t(x)p_s(y)}{|x-y|^2+(t-s)^2}\, dx\, dy\, ds\, dt, \end{aligned} \tag{4.24}$$

where p_t is the heat kernel, as defined in the previous chapter. The four-dimensional Hardy-Littlewood-Sobolev inequality [30] then immediately applies, yielding the upper bound $CT^{3/4}$ for some constant C.

4.3.3 The Integral Term $\int_0^T \rho_t^2\, dt$

We will first introduce some notation. X will represent a vector in $\mathbb{R}^{3N}$, accomodating $3N$ independent Brownian motions, one for each spatial coordinate of the system,

$$X = (X_1^1, X_2^1, X_3^1, X_1^2, X_2^2, X_3^2, \ldots, X_1^N, X_2^N, X_3^N), \tag{4.25}$$

and k^l will refer to the vector

$$(0, 0, \ldots, \underbrace{k_1}_{(3l-2)\text{nd slot}}, \underbrace{k_2}_{(3l-1)\text{st slot}}, \underbrace{k_3}_{3l\text{th slot}}, \ldots, 0), \tag{4.26}$$

that is, given a vector k in $\mathbb{R}^3$, k^l is the embedding of k into the corresponding slots of a vector in $\mathbb{R}^{3N}$. Now, in this particular case, the stochastic derivative ρ_t may be computed as $E(D_t\mathcal{A}|\mathcal{F}_t)$, where D is the Malliavin derivative operator. Recall that, as was said in the previous chapter, for a smooth function f with polynomial growth from $\mathbb{R}^n$ to $\mathbb{R}$ and $\mathbb{R}^d$-valued functions $g_1, \ldots, g_n$ in $L^2([0,T])$, the Malliavin derivative D_t of the function $f(W(g_1), \ldots, W(g_n))$, with $W(g_i)$ equal to the Itô integral $\int_0^T g_i(t)\, dX_t$, is defined as

$$D_t f(W(g_1), \ldots, W(g_n)) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(g_1), \ldots, W(g_m))g_i(t). \tag{4.27}$$

In applying this definition to the particular $\mathcal{A}$ in question here, there are some technicalities, such as justifying the passage of D under integration. These issues are addressed and resolved in [5], and the interested reader may refer to that to see how this is taken care of; the idea is to essentially discretize integrals and mollify integrands. The Malliavin derivative $D_u\mathcal{A}$ then reads as

$$-\alpha i\sum_{m,n}\int_0^T\int_0^t\int \frac{e^{-\omega(k)(t-s)}}{\omega(k)}e^{-ik(X_t^m-X_s^n)}\chi_\Lambda(k)\left\{k^m 1_{[0,t]}(u) - k^n 1_{[0,s]}(u)\right\}\, dk\, ds\, dt. \tag{4.28}$$

By writing e^{ikY_r} as $e^{ik(Y_r-Y_u)}e^{ikY_u}$ for a three dimensional Brownian motion Y and $r \geq u$, and using the independence of X^n and X^m for different n and m and their Markov property, one sees that $E(D_u\mathcal{A}|\mathcal{F}_u)$ is then equal to

$$2\alpha \sum_{m,n} \int_u^T \int_0^t \int \frac{e^{-\omega(k)(t-s)}}{\omega(k)} \chi_\Lambda(k) k^m e^{-k^2(t-u)/2} e^{-k^2(s-u)_+/2} \sin[k(X^n_{s\wedge u} - X^m_u)]\, dk\, ds\, dt, \tag{4.29}$$

and therefore $E(D_u\mathcal{A}|\mathcal{F}_u)^2$ reads as

$$\begin{aligned} &4\alpha^2 \sum_{m=1}^N \left(\sum_{n=1}^N \int_u^T \int_0^t \int \frac{e^{-\omega(k)(t-s)}}{\omega(k)} \chi_\Lambda(k) k e^{-k^2(t-u)/2} e^{-k^2(s-u)_+/2} \sin[k(X^m_u - X^n_{s\wedge u})]\, dk\, ds\, dt \right)^2 \\ &\equiv 4\alpha^2 \sum_{m=1}^N \left(\sum_{n=1}^N \mathcal{C}_{m,n} \right)^2 \leq 4\alpha^2 N \sum_{m=1}^N \sum_{n=1}^N \mathcal{C}^2_{m,n}. \end{aligned} \tag{4.30}$$

The idea is to now bound (4.30) from above. We first perform the angular integrals involved:

$$\mathcal{C}_{m,n} = 4\pi \int_u^T \int_0^t \frac{X^m_u - X^n_{s\wedge u}}{|X^m_u - X^n_{s\wedge u}|} \int_0^\Lambda \frac{e^{-\nu(r)(t-s)}}{\nu(r)} e^{-r^2(t-u)/2} e^{-r^2(s-u)_+/2} \varphi(r|X^m_u - X^n_{s\wedge u}|) r^3\, dr\, ds\, dt, \tag{4.31}$$

where $\varphi(x) = (\sin x - x\cos x)/x^2$, and $\nu(r) = \sqrt{r^2+\mu^2}$. Then, bounding by the massless case, $\nu(r) \geq r$,

$$\begin{aligned} |\mathcal{C}_{m,n}| &\leq 4\pi \int_0^\Lambda \int_0^u \frac{1-e^{-(r+r^2/2)(T-u)}}{1+r/2} r e^{-r(u-s)} |\varphi(r|X^m_u - X^n_s|)|\, ds\, dr \\ &\quad + 4\pi \int_0^\Lambda \frac{r}{r/2-1} \int_u^T \left[e^{-(r+r^2/2)(t-u)} - e^{-r^2(t-u)} \right] |\varphi(r|X^m_u - X^n_u|)|\, dt\, dr \\ &\equiv \mathcal{D}_{m,n} + \mathcal{E}_{m,n}. \end{aligned} \tag{4.32}$$

It turns out that $\mathcal{E}_{m,n}$ is bounded uniformly in all the relevant variables Λ, u, and T; and that $\mathcal{D}_{m,n}$ is bounded uniformly in Λ and T. We state and prove this in the form of two lemmas, starting with $\mathcal{E}_{m,n}$.

Lemma 4.2.

$$\mathcal{E}_{m,n} \leq 4\pi(\|\varphi(r)/r\|_1 + \sqrt{2\|\varphi(r)/r\|_1\|\varphi\|_\infty} + \|\varphi\|_\infty). \tag{4.33}$$

Proof. We first split the integral involved into two fragments. We assume here that Λ is greater than 2, which does not cause any loss of generality since the integrand is positive. By denoting this integrand $f(r)$, we then split $\int_0^\Lambda f(r)\, dr$ as

$$\int_0^\varepsilon f(r)\, dr + \int_\varepsilon^2 f(r)\, dr + \int_2^\Lambda f(r)\, dr, \tag{4.34}$$

where ε is a number in $(0,2)$, to be fixed later. The first integral may be written as

$$\begin{aligned} &4\pi \int_0^\varepsilon \frac{r}{1-r/2} |\varphi(r|X^m_u - X^n_u|)| \int_u^T e^{-r^2(t-u)} \left[1 - e^{-(r-r^2/2)(t-u)} \right] dt\, dr \\ &\leq 4\pi \int_0^\varepsilon \frac{r}{1-r/2} |\varphi(r|X^m_u - X^n_u|)| \int_u^T e^{-r^2(t-u)}\, dt\, dr \leq 4\pi \int_0^\varepsilon \frac{|\varphi(r|X^m_u - X^n_u|)|}{r(1-r/2)}\, dr \\ &\leq \frac{4\pi}{1-\varepsilon/2} \int_0^\infty \frac{|\varphi(r|X^m_u - X^n_u|)|}{r}\, dr = \frac{4\pi}{1-\varepsilon/2} \|\varphi(r)/r\|_1. \end{aligned} \tag{4.35}$$

As for the second integral, it can be bounded as

$$\int_{\varepsilon}^{2} f(r)\,dr \leq 4\pi\|\varphi\|_\infty \int_{\varepsilon}^{2} r^2 \int_u^T e^{-r^2(t-u)}(t-u)\,dt\,dr \leq 4\pi\|\varphi\|_\infty \int_{\varepsilon}^{2} \frac{1}{r^2}\,dr = 4\pi\|\varphi\|_\infty\left(\frac{1}{\varepsilon}-\frac{1}{2}\right), \tag{4.36}$$

and, finally, for the third integral we get

$$\begin{aligned}\int_2^{\Lambda} f(r)\,dr &\leq 4\pi\|\varphi\|_\infty \int_2^{\Lambda} \frac{r}{r/2-1}\int_u^T e^{-(r+r^2/2)(t-u)}\left[1-e^{-(r^2/2-r)(t-u)}\right]dt\,dr \\ &\leq 4\pi\|\varphi\|_\infty \int_2^{\Lambda} \frac{r^2}{(r^2/2+r)^2}\,dr \leq 4\pi\|\varphi\|_\infty.\end{aligned} \tag{4.37}$$

With all this, we obtain

$$\begin{aligned}\int_0^{\Lambda} f(r)\,dr &\leq \frac{4\pi}{1-\varepsilon/2}\|\varphi(r)/r\|_1 + \frac{4\pi\|\varphi\|_\infty}{\varepsilon} + 2\pi\|\varphi\|_\infty \\ &= 4\pi(\|\varphi(r)/r\|_1 + \sqrt{2\|\varphi(r)/r\|_1\|\varphi\|_\infty} + \|\varphi\|_\infty),\end{aligned} \tag{4.38}$$

where an optimization in ε was performed. □

Lemma 4.3. *For any $\varepsilon > 0$ and $0 \leq \theta < 1/4$,*

$$\mathcal{D}_{m,n} \leq \frac{4\pi\|\varphi\|_\infty}{\varepsilon} + F_\theta\left(\int_{(u-\varepsilon)_+}^{u} \frac{ds}{|X_u^m - X_s^n|^{2-4\theta}}\right)^{1/2}, \tag{4.39}$$

where

$$F_\theta \equiv \frac{8\pi}{\sqrt{2}}\|x\varphi(x)\|_\infty^{1-\theta}\|\varphi(x)/x\|_\infty^{\theta}\int_0^\infty \frac{r^{2\theta-1/2}}{2+r}\,dr. \tag{4.40}$$

Proof. Let $\varepsilon > 0$. Then,

$$\begin{aligned}D_{m,n} &\leq 4\pi\int_0^u\int_0^\infty \frac{r}{1+r/2}e^{-r(u-s)}|\varphi(r|X_u^m - X_s^n|)|\,dr\,ds \\ &= 8\pi\int_0^u\int_0^\infty \frac{re^{-r(u-s)/|X_u^m-X_s^n|}}{|X_u^m-X_s^n|(2|X_u^m-X_s^n|+r)}|\varphi(r)|\,dr\,ds \\ &= 8\pi\int_0^{(u-\varepsilon)_+}\int_0^\infty \frac{re^{-r(u-s)/|X_u^m-X_s^n|}}{|X_u^m-X_s^n|(2|X_u^m-X_s^n|+r)}|\varphi(r)|\,dr\,ds \\ &\quad + 8\pi\int_{(u-\varepsilon)_+}^u\int_0^\infty \frac{re^{-r(u-s)/|X_u^m-X_s^n|}}{|X_u^m-X_s^n|(2|X_u^m-X_s^n|+r)}|\varphi(r)|\,dr\,ds.\end{aligned} \tag{4.41}$$

The first term can be estimated from above as

$$\begin{aligned}4\pi\|\varphi\|_\infty\int_0^{(u-\varepsilon)_+}\int_0^\infty \frac{re^{-r(u-s)/|X_u^m-X_s^n|}}{|X_u^m-X_s^n|^2}\,dr\,ds &= 4\pi\|\varphi\|_\infty\int_0^{(u-\varepsilon)_+}\int_0^\infty re^{-r(u-s)}\,dr\,ds \\ &= 4\pi\|\varphi\|_\infty\left(\frac{1}{u-(u-\varepsilon)_+}-\frac{1}{u}\right) \leq \frac{4\pi\|\varphi\|_\infty}{\varepsilon}.\end{aligned} \tag{4.42}$$

As for the second term, we first note that $r|\varphi(r)|$ is both bounded by $\|x\varphi(x)\|_\infty$ and $\|\varphi(x)/x\|_\infty r^2$. Then, for $0 \le \theta \le 1$, $r|\varphi(r)| \le \|x\varphi(x)\|_\infty^{1-\theta}\|\varphi(x)/x\|_\infty^{\theta} r^{2\theta} \equiv E_\theta r^{2\theta}$. Applying this we find that it can be bounded above as

$$\begin{aligned}
&8\pi \int_{(u-\varepsilon)_+}^{u} \frac{1}{|X_u^m - X_s^n|} \int_0^\infty \frac{e^{-r(u-s)}}{2+r} r|X_u^m - X_s^n||\varphi(r|X_u^m - X_s^n|)|\,dr\,ds \\
\le{}& 8\pi E_\theta \int_0^\infty \frac{r^{2\theta}}{2+r} \int_{(u-\varepsilon)_+}^{u} \frac{e^{-r(u-s)}}{|X_u^m - X_s^n|^{1-2\theta}}\,ds\,dr \\
\le{}& 8\pi E_\theta \int_0^\infty \frac{r^{2\theta}}{2+r} \left(\int_{(u-\varepsilon)_+}^{u} e^{-2r(u-s)}\,ds\right)^{1/2} \left(\int_{(u-\varepsilon)_+}^{u} \frac{ds}{|X_u^m - X_s^n|^{2-4\theta}}\right)^{1/2} dr \qquad (4.43)\\
\le{}& \frac{8\pi E_\theta}{2^{1/2}} \int_0^\infty \frac{r^{2\theta-1/2}}{2+r}\,dr \left(\int_{(u-\varepsilon)_+}^{u} \frac{ds}{|X_u^m - X_s^n|^{2-4\theta}}\right)^{1/2} \\
={}& F_\theta \left(\int_{(u-\varepsilon)_+}^{u} \frac{ds}{|X_u^m - X_s^n|^{2-4\theta}}\right)^{1/2}.
\end{aligned}$$

We further restrict θ to $[0, 1/4)$ to ensure finiteness of F_θ. □

We now put all together, using Lemmas 4.2, 4.3, to find

$$\begin{aligned}
|\mathcal{C}_{m,n}| &\le 4\pi(\|\varphi(r)/r\|_1 + \sqrt{2\|\varphi(r)/r\|_1\|\varphi\|_\infty} + \|\varphi\|_\infty) + \frac{4\pi\|\varphi\|_\infty}{\varepsilon} \\
&\quad + F_\theta \left(\int_{(u-\varepsilon)_+}^{u} \frac{ds}{|X_u^m - X_s^n|^{2-4\theta}}\right)^{1/2} \qquad (4.44)\\
&\equiv C + D_\varepsilon + F_\theta \left(\int_0^u \frac{1_{[0,\varepsilon]}(u-s)}{|X_u^m - X_s^n|^{2-4\theta}}\,ds\right)^{1/2}
\end{aligned}$$

if $m \ne n$, and

$$|\mathcal{C}_{m,m}| \le D_\varepsilon + F_\theta \left(\int_0^u \frac{1_{[0,\varepsilon]}(u-s)}{|X_u^m - X_s^m|^{2-4\theta}}\,ds\right)^{1/2}. \qquad (4.45)$$

It then follows that, by using the inequality $(a+b)^2 \le 2(a^2+b^2)$,

$$\sum_{m,n} \mathcal{C}_{m,n}^2 \le 2N(N-1)(C+D_\varepsilon)^2 + 2ND_\varepsilon^2 + 2F_\theta^2 \sum_{m,n} \int_0^u \frac{1_{[0,\varepsilon]}(u-s)}{|X_u^m - X_s^n|^{2-4\theta}}\,ds, \qquad (4.46)$$

and therefore,

$$\begin{aligned}
E\left[\exp\left(4\alpha^2 N \sum_{m,n} \int_0^T \mathcal{C}_{m,n}^2\,du\right)\right] &\le \exp\left[\left(8\alpha^2 N^2 (N-1)(C+D_\varepsilon)^2 + 8\alpha^2 N^2 D_\varepsilon^2\right) T\right] \\
&\quad \times E\left[\exp\left(\sum_{m,n} 8\alpha^2 N F_\theta^2 \int_0^T\!\!\int_0^u \frac{1_{[0,\varepsilon]}(u-s)}{|X_u^m - X_s^n|^{2-4\theta}}\,ds\,du\right)\right] \\
&\equiv e^{\gamma T} E\left[\exp\left(\sum_{m,n} \beta \int_0^T\!\!\int_0^u \frac{1_{[0,\varepsilon]}(u-s)}{|X_u^m - X_s^n|^{2-4\theta}}\,ds\,du\right)\right].
\end{aligned} \qquad (4.47)$$

At this point we are in the same situation as that in the N-polaron model. The main result from that section was that there is a bound that allows one to take the sum outside of the expectation, at the price of an upper bound and constants and exponents. By using that result here, we get that the term (4.47) is bounded above by

$$e^{\gamma T} E\left[\exp\left(N\beta\int_0^T\int_0^t \frac{1_{[0,\varepsilon]}(t-s)}{|X_t-X_s|^{2-4\theta}}\,ds\,dt\right)\right] E\left[\exp\left(2\beta N\int_0^T\int_0^t \frac{1_{[0,\varepsilon]}(t-s)}{|X_t-Y_s|^{2-4\theta}}\,ds\,dt\right)\right]^{(N-1)/2}, \tag{4.48}$$

and, by means of equations (4.7) and (4.8), we get

$$\begin{aligned} &E\left[\exp\left(4\alpha^2 N\sum_{m,n}\int_0^T \mathcal{C}_{m,n}^2\,du\right)\right] \\ &\quad\le e^{\gamma T}\exp\left[\left(A_{2-4\theta}2^{3/(2\theta)}\alpha^{1/\theta}N^{1+1/\theta}\varepsilon^{1/(2\theta)}F_\theta^{1/\theta}+4\alpha^2F_\theta^2B_{2-4\theta}\varepsilon^{2\theta}N^2\theta^{-1}\right)T+o(T)\right]. \end{aligned} \tag{4.49}$$

4.3.4 The Lower Bound

By the results found in the previous subsections, if $E_{\alpha,\mu}^{N,\Lambda}$ is the ground-state energy of $H_{\alpha,\mu}^{N,\Lambda}$,

$$\begin{aligned} E_{\alpha,\mu}^{N,\Lambda}+Q_{\alpha,\mu}^{N,\Lambda}\ge &-A_{2-4\theta}2^{3/(2\theta)}\alpha^{1/\theta}N^{1+1/\theta}\varepsilon^{1/(2\theta)}F_\theta^{1/\theta}-4\alpha^2F_\theta^2B_{2-4\theta}\varepsilon^{2\theta}N^2\theta^{-1} \\ &-8\alpha^2N^2(N-1)(C^2+2CD_\varepsilon)-8\alpha^2N^3D_\varepsilon^2. \end{aligned} \tag{4.50}$$

We optimize $-A_{2-4\theta}2^{3/(2\theta)}\alpha^{1/\theta}N^{1+1/\theta}\varepsilon^{1/(2\theta)}F_\theta^{1/\theta}-8\alpha^2N^3D_\varepsilon^2$ in ε. This yields the following lower bound for $E_{\alpha,\mu}^{N,\Lambda}+Q_{\alpha,\mu}^{N,\Lambda}$,

$$\begin{aligned} &-\Phi_\theta\alpha^{6/(1+4\theta)}N^{(7+4\theta)/(1+4\theta)}-8C^2\alpha^2(N-1)N^2-\Psi_\theta\alpha^{4(1+\theta)/(1+4\theta)}(N-1)N^{4(1+\theta)/(1+4\theta)} \\ &-\Upsilon_\theta\alpha^{2(1+2\theta+4\theta^2)/(1+4\theta)}N^{2(1+2\theta+4\theta^2)/(1+4\theta)}, \end{aligned} \tag{4.51}$$

where

$$\begin{aligned} \Phi_\theta &=2^{13/(1+4\theta)}A_{2-4\theta}^{4\theta/(1+4\theta)}F_\theta^{4/(1+4\theta)}\pi^{2/(1+4\theta)}\|\varphi\|_\infty^{2/(1+4\theta)}(4\theta)^{-4\theta/(1+4\theta)}(1+4\theta), \\ \Psi_\theta &=2^{3(3+2\theta)/(1+4\theta)}\pi^{1/(1+4\theta)}C\|\varphi\|_\infty^{1/(1+4\theta)}\theta^{-2\theta/(1+4\theta)}A_{2-4\theta}^{2\theta/(1+4\theta)}F_\theta^{2/(1+4\theta)}, \\ \Upsilon_\theta &=2^{2(1+\theta+18\theta^2)/(1+4\theta)}F_\theta^{2(1+2\theta)/(1+4\theta)}B_{2-4\theta}\|\varphi\|_\infty^{8\theta^2/(1+4\theta)}\pi^{8\theta^2/(1+4\theta)}\theta^{-1/(1+4\theta^2)}A_{2-4\theta}^{-4\theta^2/(1+4\theta)}. \end{aligned} \tag{4.52}$$

This concludes the finding of the lower bound for the renormalized Nelson model. The upshot is that the energy is bounded below by $-C_\delta\alpha^{3+\delta}N^{4+\delta}$, for a diverging positive constant C_δ, when $\delta\to 0^+$. It is not possible for us for the moment to give an upper bound with a reasonable N or α dependence that could be of use to corroborate the behavior of the lower bound we have just found. There is, however, a rather trivial upper bound for $E_{\alpha,\mu}^{N,\Lambda}+Q_{\alpha,\mu}^{N,\Lambda}$ that is easy to obtain, and is provided by Jensen's inequality,

$$\begin{aligned} &E\left[\exp\left(\alpha\sum_{m,n}\int_0^T\int_0^t\int \frac{e^{-ik(X_t^m-X_s^n)}}{\omega(k)}e^{-\omega(k)(t-s)}\chi_\Lambda(k)\,dk\,ds\,dt\right)\right] \\ \ge&\exp\left\{\alpha\sum_{m,n}E\left[\int_0^T\int_0^t\int \frac{e^{-ik(X_t^m-X_s^n)}}{\omega(k)}e^{-\omega(k)(t-s)}\chi_\Lambda(k)\,dk\,ds\,dt\right]\right\} \\ =&\exp\left[Q_{\alpha,\mu}^{N,\Lambda}T+o(T)\right], \end{aligned} \tag{4.53}$$

as was computed before, in Subsection 4.3.2. By means of the Feynman-Kac formula, we obtain then that $E_{\alpha,\mu}^{N,\Lambda} + Q_{\alpha,\mu}^{N,\Lambda} \leq 0$.

Chapter 5

No-Binding Theorem

In this section we will prove that for a sufficiently strong Coulomb repulsion, two particles subject to the massless Nelson model interaction will not bind. This was first proven by Frank, Lieb, Seiringer and Thomas in [14] for the bipolaron. The result was then slightly sharpened for all coupling constants, with separate, better results for small and large coupling, by Benguria and Bley in [4]. The idea is to take a model of non-relativistic quantum field theory, involving two non-relativistic particles interacting with a quantum field, and then include a repulsive Coulomb interaction term. The Hamiltonian will look like

$$H_U = -\frac{\Delta_1}{2} - \frac{\Delta_2}{2} + \int \omega_k a_k^\dagger a_k \, dk + \sqrt{\alpha} \int \left[f_k(x_1, x_2) a_k + \overline{f_k(x_1, x_2)} a_k^\dagger \right] dk + \frac{U}{|x_2 - x_1|}, \tag{5.1}$$

for a complex function f, non-negative function ω, and non-negative constants α and U. H_U acts on $L^2(\mathbb{R}^6) \otimes \mathcal{F}$, where $\mathcal{F}$ is the Fock space on $L^2(\mathbb{R}^3)$. In applications f and ω are continuous and have compact support in k (they include an ultraviolet cutoff). Here α is a coupling constant, and U is a repulsion parameter. A Feynman-Kac formula for this Hamiltonian, describing the evolution of a given vector in the domain of H_U, would involve the term

$$\exp\left[\alpha \iint_0^T \int_0^t e^{-\omega(k)(t-s)} \overline{f_k(X_t, Y_t)} f_k(X_s, Y_s) \, ds \, dt \, dk - \int_0^T \frac{U}{|Y_t - X_t|} \, dt \right], \tag{5.2}$$

as discussed in the first chapter. Now, for a fixed value of k, what is inside the k-integral reads as

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \overline{g(t)} h(t-s) g(s) \, ds \, dt, \tag{5.3}$$

where $g(t) = f_k(X_t, Y_t)\chi_{[0,T]}(t)$ and $h(t) = e^{-\omega(k)t}\chi_{[0,\infty)}(t)$. Even though h is not positive definite, its even extension to all real numbers, namely $h(|t|) = e^{-\omega(k)|t|}$, is positive definite, as can be seen by computing its Fourier transform – a positive function. The key observation now is that

$$\mathrm{Re}\left[\int_{\mathbb{R}} \int_{\mathbb{R}} \overline{g(t)} h(t-s) g(s) \, ds \, dt \right] = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{g(t)} h(|t-s|) g(s) \, ds \, dt \geq 0. \tag{5.4}$$

In applications, the integral over all k of the imaginary part of (5.3) will be equal to zero. This leads to the conclusion that the interaction of the particles with the field leads to an effective attractive potential between the particles and themselves. The question now arises whether a sufficiently strong U can overcome the effective attraction between the particles. If this is the case, only two residual self-interaction terms will be left. We would then expect that for large enough U the ground state energy of H would be greater than or equal to two times the ground-state energy of the corresponding Hamiltonian for just one particle. Indeed, we have the following theorem, that will be proven in the next two sections.

Theorem 5.1. *In the bipolaron case, if $U \geq 17.8\alpha$, then no binding occurs, meaning that $E_U^2 \geq 2E^1$, where E_U^2 and E^1 are the bipolaron and polaron ground-state energies, respectively. For the 2-particle Nelson model, no binding occurs for $U \geq D_3\alpha^3 + D_2\alpha^2 + D_1\alpha + D_{3/4}\alpha^{3/4}$, for some explicit positive constants D_i.*

Theorem 5.1 was proven in [14] for the bipolaron, yielding $U \geq 37.7\alpha$ for all α. In [4], the result was $U \geq 36.9\alpha$, with better estimates for small and large coupling α, to wit, $U \geq 28.6\alpha + o(\alpha)$ and $U \geq 27.4\alpha + o(\alpha)$, respectively. (The reader will find the stated estimates in those two papers to be substantially bigger than these numbers, which is due to a rescaling that has to be made; see the end of Section 5.1 below.)

We note in passing that one may change $E_U^2 \geq 2E^1$ in the theorem to $E_U^2 = 2E^1$, since $E_U^2 \leq 2E^1$ is always true for translation invariant two-body quantum systems, such as the two-particle Nelson and polaron models; this result is known as the subadditivity of energy. It follows by considering two bumps as the wave function, one for each particle, and separating them as much as possible. This is true for many-body systems as well, meaning here, with the same notation, $E_U^N \leq NE^1$. See, for instance, [32] for more details on this. The main point here is that $E_U^2 \geq 2E^1$, since this means that the two particles being separated is as favorable energetically as the two repelling each other through the Coulomb potential. The binding energy $E_U^2 - 2E^1$ is then non-negative.

To prove the theorem, we will proceed as in [14], partitioning the interparticle distance $|x_1 - x_2|$ into several intervals in the positive real line. This is the most natural object to gain control on, since it is the one governing the strength of the Coulomb repulsion term. After partitioning the positive real line, we will focus on each element of the partition, thereby localizing the interparticle distance. On each localization region, several bounds will be performed, which will allow us at the end to determine a value of U for which there will be no binding for the particles in the system. Toward the end of the calculation we will depart from what was done in [14]: no further localizations will be performed – the interparticle partition is all that will be made. The main drawback of the proof in [14] is that particles are relocalized into disjoint balls, after partitioning the interparticle distance, which causes an unnecesary increase in the final estimate for U. Here we will show how one can simply solve the problem by partitioning just the interelectronic distance. The proof in [4] mimics that of [14], only that the parameters of the partition were fully optimized. The further ball relocalization was still performed, however, which did not allow for much improvement on the no-binding condition for U. Careful estimates on polaron ground state energies for large and small coupling permitted then the obtention of sharper results in these two regimes.

Let us start by partitioning the positive real line. We cover the real line with infinitely many bumps that overlap. Each bump will be represented by a certain function, φ_n. Perhaps Figure 5.1 will be useful. Each bump φ_n is defined piecewise with a quarter of a sine or cosine function for

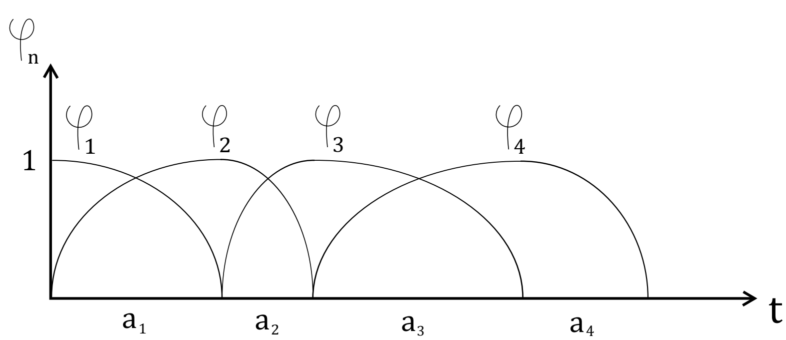

Figure 5.1: The functions φ_n.

each length a_m. For instance, $\varphi_3(t)$ is equal to $\sin[(t - a_1)\pi/(2a_2)]$ when $a_1 \leq t \leq a_1 + a_2$, it is $\cos[(t - a_1 - a_2)\pi/(2a_3)]$ for $a_1 + a_2 \leq t \leq a_1 + a_2 + a_3$, and is 0 otherwise. The functions φ_n satisfy the condition $\sum_{n=1}^{\infty} \varphi_n^2 = 1$. They then form a good IMS partition of unity [7]. We now use the functions φ_n to partition the inter-particle distance by defining $\phi_n(x_1, x_2) = \varphi_n(|x_2 - x_1|)$. Then, certainly, $\sum_{n=1}^{\infty} \phi_n^2 = 1$, and then the IMS formula holds:

$$H_U = \sum_{n=1}^{\infty} \phi_n H_U \phi_n - \sum_{n=1}^{\infty} |\nabla \phi_n|^2, \tag{5.5}$$

which tells us that localizing comes with a kinetic energy cost, the second term in (5.5). The gradient there is 6-dimensional, involving both particles. Let ψ be any state in the quadratic form domain of the operator H_U. Let ψ_n be defined as $\phi_n \psi$. Then, if $c_n = \sum_{m=1}^{n} a_m$,

$$\begin{aligned}(\psi, H_U \psi) &= \sum_{n=1}^{\infty} (\psi_n, H_U \psi_n) - \sum_{n=1}^{\infty} (\psi, |\nabla \phi_n|^2 \psi) \\ &= \sum_{n=1}^{\infty} (\psi_n, H_0 \psi_n) + U \sum_{n=1}^{\infty} \left(\psi_n, \frac{1}{|x_2 - x_1|} \psi_n \right) - \sum_{n=1}^{\infty} (\psi, |\nabla \phi_n|^2 \psi) \\ &\geq (\psi_1, H_0 \psi_1) + (\psi_2, H_0 \psi_2) + \sum_{n=3}^{\infty} (\psi_n, H_0 \psi_n) + \sum_{n=1}^{\infty} \frac{U}{c_n} \|\psi_n\|^2 - \sum_{n=1}^{\infty} (\psi, |\nabla \phi_n|^2 \psi).\end{aligned} \tag{5.6}$$

We now proceed to find lower bounds for each of the terms $(\psi_n, H_0 \psi_n)$. ψ_1 and ψ_2 correspond to the regions where the particles are close to each other. At this stage the bounds will be sensitive to the model in question. We will divide the estimates into two subsections, covering the polaron and Nelson model cases separately.

5.1 The Polaron Case

In this situation we may simply bound $(\psi_n, H_0 \psi_n)$ as follows

$$(\psi_1, H_0 \psi_1) \geq \text{inf spec } H_0 \|\psi_1\|^2 \geq (-2\alpha - 2\alpha^2) \|\psi_1\|^2 \geq (2E_1 - 2\alpha^2) \|\psi_1\|^2. \tag{5.7}$$

The second inequality was proved in Chapter 4. The third inequality follows from a result of Feynman [13] (which amounts to an application of Jensen's inequality to the functional integral defining the ground-state energy of the polaron). Here E_1 denotes the ground state energy of the single polaron Hamiltonian

$$-\frac{\Delta}{2} + \int a_k^\dagger a_k \, dk + \frac{\sqrt{\alpha}}{2^{3/4} \pi} \int \frac{1}{|k|} (a_k e^{ikx} + a_k^\dagger e^{-ikx}) \, dk. \tag{5.8}$$

For $(\psi_2, H_0 \psi_2)$, the same bound may be made, obtaining $(\psi_2, H_0 \psi_2) \geq (2E_1 - 2\alpha^2) \|\psi_2\|^2$. For $(\psi_n, H_0 \psi_n)$, $n \geq 3$, we let V_n be the separating potential between the particles equal to 0 if $|x_1 - x_2| \geq d_n$ and equal to ∞ otherwise. Here d_n will be equal to $a_1 + \ldots + a_{n-2}$. The region $\Omega_n \equiv \{(x_1, x_2) : |x_1 - x_2| \geq d_n\}$ can easily be seen to be a rotation by 45° about three different axes of the region $\left\{(x, y) \in \mathbb{R}^6 : |x| \geq d_n / \sqrt{2}\right\}$. Ω_n is then just the complement of a solid cylinder in $\mathbb{R}^6$. The function of V_n is to confine the six coordinates to the complement of this cylinder. We now find that

$$(\psi_n, H_0 \psi_n) = (\psi_n, (H_0 + V_n) \psi_n) \geq \text{inf spec } (H_0 + V_n) \|\psi_n\|^2. \tag{5.9}$$

We can write a Feynman-Kac formula for inf spec $(H_0 + V_n)$. V_n does not involve Fock space variables, and therefore commutes with the other potentials in H_0. The following formula is then obtained

$$\text{inf spec } (H_0 + V_n) = -\lim_{T \to \infty} \frac{1}{T} \log E^{(x_1, x_2)} \left\{ \exp\left(\mathcal{A}_{1,1}^T + \mathcal{A}_{2,2}^T + \mathcal{A}_{1,2}^T + \mathcal{A}_{2,1}^T \right) \chi_{\Omega_n} [X([0, T])] \right\}, \tag{5.10}$$

where $\chi_{\Omega_n}[X([0,T])]$ is 1 if the Brownian path X is completely contained in Ω_n for all times in $[0,T]$ and is zero otherwise. The expectation is thus taken only on those paths. Here (x_1, x_2) is any conveniently located point in the interior of Ω_n, and $\mathcal{A}_{i,j}^T$ is defined as

$$\frac{\alpha}{\sqrt{2}} \int_0^T \int_0^t \frac{e^{-(t-s)}}{|X_t^i - X_s^j|} \, ds \, dt, \tag{5.11}$$

with X^1 denoting the first three components of X and X^2 the second three. We then clearly have that

$$\begin{aligned} &E^{(x_1,x_2)} \left\{ \exp\left(\mathcal{A}_{1,1}^T + \mathcal{A}_{2,2}^T + \mathcal{A}_{1,2}^T + \mathcal{A}_{2,1}^T\right) \chi_{\Omega_n}[X([0,T])] \right\} \\ &\leq E^{(x_1,x_2)} \left[\exp(\mathcal{A}_{1,1}^T) \exp(\mathcal{A}_{2,2}^T)\right] \exp\left(\frac{\sqrt{2}\alpha T}{d_n}\right) \\ &= E^{x_1} \left[\exp(\mathcal{A}_{1,1}^T)\right] E^{x_2} \left[\exp(\mathcal{A}_{2,2}^T)\right] \exp\left(\frac{\sqrt{2}\alpha T}{d_n}\right). \end{aligned} \tag{5.12}$$

From here it follows that

$$(\psi_n, H_0 \psi_n) \geq \left(2E_1 - \frac{\sqrt{2}\alpha}{d_n}\right) \|\psi_n\|^2. \tag{5.13}$$

After the estimates performed for $(\psi_n, H_0\psi_n)$, we find, by means of inequality (5.6),

$$\begin{aligned} (\psi, H_U \psi) \geq &\left(2E_1 - 2\alpha^2 + \frac{U}{a_1}\right) \|\psi_1\|^2 + \left(2E_1 - 2\alpha^2 + \frac{U}{a_1 + a_2}\right) \|\psi_2\|^2 \\ &+ \sum_{n=3}^{\infty} \left(2E_1 - \frac{\sqrt{2}\alpha}{d_n} + \frac{U}{c_n}\right) \|\psi_n\|^2 - \sum_{n=1}^{\infty} (\psi, |\nabla\phi_n|^2 \psi). \end{aligned} \tag{5.14}$$

By further noticing that

$$|\nabla\phi_n(x_1, x_2)|^2 = 2|\varphi_n'(|x_2 - x_1|)|^2, \tag{5.15}$$

we see that

$$\begin{aligned} \sum_{m=1}^{\infty} (\psi, |\nabla\phi_m|^2 \psi) &= \sum_{n=1}^{\infty} \left(\psi_n, \sum_{m=1}^{\infty} |\nabla\phi_m|^2 \psi_n\right) = 2 \sum_{n=1}^{\infty} \left(\psi_n, \sum_{m=1}^{\infty} |\varphi_n'(|x_2 - x_1|)|^2 \psi_n\right) \\ &\leq \frac{\pi^2}{a_1^2} \|\psi_1\|^2 + \pi^2 \sum_{n=2}^{\infty} \frac{\|\psi_n\|^2}{\min(a_n, a_{n-1})^2}. \end{aligned} \tag{5.16}$$

Hence,

$$\begin{aligned} (\psi, H_U \psi) \geq &\left(2E_1 - 2\alpha^2 + \frac{U}{a_1} - \frac{\pi^2}{a_1^2}\right) \|\psi_1\|^2 + \left(2E_1 - 2\alpha^2 + \frac{U}{a_1 + a_2} - \frac{\pi^2}{\min(a_2, a_1)^2}\right) \|\psi_2\|^2 \\ &+ \sum_{n=3}^{\infty} \left(2E_1 - \frac{\sqrt{2}\alpha}{d_n} + \frac{U}{c_n} - \frac{\pi^2}{\min(a_n, a_{n-1})^2}\right) \|\psi_n\|^2. \end{aligned} \tag{5.17}$$

We notice now that if

$$U \geq 2\alpha^2 a_1 + \frac{\pi^2}{a_1}, \tag{5.18}$$

$$U \geq 2\alpha^2(a_1 + a_2) + \frac{\pi^2(a_1 + a_2)}{\min(a_1, a_2)^2}, \tag{5.19}$$

$$U \geq \frac{\sqrt{2}\alpha c_n}{d_n} + \frac{\pi^2 c_n}{\min(a_n, a_{n-1})^2} = \sqrt{2}\alpha + \frac{\sqrt{2}\alpha(a_{n-1} + a_n)}{a_1 + \ldots + a_{n-2}} + \frac{\pi^2(a_1 + \ldots + a_n)}{\min(a_n, a_{n-1})^2} \quad \text{for all } n \geq 3, \tag{5.20}$$

then $(\psi, H_U\psi) \geq 2E_1\|\psi\|^2$. By setting $a_i = b_i/\alpha$, the constant α may be scaled out. The task now is to minimize the function

$$\begin{aligned} F(b_1, b_2, b_3, \ldots) \equiv & \left(2b_1 + \frac{\pi^2}{b_1}\right) \vee \left[2(b_1 + b_2) + \frac{\pi^2(b_1 + b_2)}{\min(b_1, b_2)^2}\right] \\ & \vee \max_{n \geq 3} \left[\sqrt{2} + \frac{\sqrt{2}(b_{n-1} + b_n)}{b_1 + \ldots + b_{n-2}} + \frac{\pi^2(b_1 + \ldots + b_n)}{\min(b_n, b_{n-1})^2}\right]. \end{aligned} \tag{5.21}$$

Even though we will not prove that a minimum exists nor will we find it exactly, we will provide parameters b_i that will give an answer that may be construed as the minimum by all practical means.

Let us explain what we mean by this. We first define F_n as the n-th term of F, in the obvious order, that is

$$\begin{aligned} F_1 &\equiv 2b_1 + \frac{\pi^2}{b_1}, \\ F_2 &\equiv 2(b_1 + b_2) + \frac{\pi^2(b_1 + b_2)}{\min(b_1, b_2)^2}, \\ F_n &\equiv \sqrt{2} + \frac{\sqrt{2}(b_{n-1} + b_n)}{b_1 + \ldots + b_{n-2}} + \frac{\pi^2(b_1 + \ldots + b_n)}{\min(b_n, b_{n-1})^2}, \quad n \geq 3. \end{aligned} \tag{5.22}$$

Then we notice that the second term in F is always bigger than the first one, since

$$2(b_1 + b_2) + \frac{\pi^2(b_1 + b_2)}{\min(b_1, b_2)^2} > 2b_1 + \frac{\pi^2 b_1}{\min(b_1, b_2)^2} \geq 2b_1 + \frac{\pi^2}{b_1}. \tag{5.23}$$

The second term may then be minimized exactly, yielding the answer $4\sqrt{2}\pi$ for $b_1 = b_2 = \pi/\sqrt{2}$. To three digits of precision, $4\sqrt{2}\pi = 17.8$. It turns out that the entire function F can be squashed to yield a number that is smaller than 17.8. We will state our result then by saying that F can be made smaller than 17.8 when appropriate parameters b_i are taken. In fact, if one takes $b_1 = 2.3436, b_2 = 2.3436, b_3 = 2.7273, b_4 = 3.5106$, and $b_n = b_4(n-3)$ for $n \geq 5$, one can verify directly that F_2, F_3, F_4, and F_5 are all (by a very small margin) less than 17.8. One can then provide an easy proof of the fact that $F_n > F_{n+1}$ for $n \geq 5$.

We conclude that if $U \geq 17.8\alpha$, then no binding occurs. This is an improvement of more than 50% over the previous result in [4], $U \geq 36.9\alpha$. That particular result, $U \geq 36.9\alpha$, was reported in [4] as $U \geq 52.1\alpha$. The reason is that the Laplacian in [4] (and also in [14]) is written without a factor of 1/2, whereas here it is. By scaling, $U \geq 52.1\alpha$ becomes $U \geq (52.1/\sqrt{2})\alpha$ when the factor of 1/2 is introduced, and that explains the discrepancy between the way we state the results of [4, 14] and theirs.

5.2 The Nelson Model Case

In this subsection we will repeat the calculations performed to find the lower bounds for each term $(\psi_n, H_0\psi_n)$, this time with H_U as the Nelson Hamiltonian for two particles. We first work with

$(\psi_n, H_0\psi_n)$ when $n \geq 3$. We are again led to study the functional integral

$$E^{(x_1,x_2)}\left\{\exp(\mathcal{A}^T_{1,1}+\mathcal{A}^T_{2,2}+\mathcal{A}^T_{1,2}+\mathcal{A}^T_{2,1})\chi_{\Omega_n}[X([0,T])]\right\}, \tag{5.24}$$

where this time

$$\mathcal{A}^T_{i,j} = \alpha\int_0^T\int_0^t\int \frac{1}{\omega(k)}e^{-ik(X^i_t-X^j_s)}e^{-\omega(k)(t-s)}\chi_\Lambda(k)\,dk\,ds\,dt. \tag{5.25}$$

Even though we believe the following results are true for the massive case $\mu > 0$, we presently only have a proof when the mesons have no mass, $\mu = 0$, and this will be the situation we will discuss. Here the action $\mathcal{A}^T_{i,j}$ may be computed explicitly as

$$\begin{aligned}4\pi\alpha\int_0^T\int_0^t &\frac{1}{|X^i_t-X^j_s|^2+(t-s)^2}\\ &\times\left\{1-e^{-(t-s)\Lambda}\left[\frac{(t-s)}{|X^i_t-X^j_s|}\sin(|X^i_t-X^j_s|\Lambda)+\cos(|X^i_t-X^j_s|\Lambda)\right]\right\}ds\,dt.\end{aligned} \tag{5.26}$$

It may be easily checked that the expression in braces in the action (5.26) is non-negative and that it is bounded above by 2. Now, to estimate $(\psi_n, H_0\psi_n)$, with H_0 the Nelson Hamiltonian, we proceed as before, using the Feynman-Kac formula found above, equation (5.10). By using that formula, the terms $\mathcal{A}^T_{1,2}$ and $\mathcal{A}^T_{2,1}$ inside the exponential may be bounded above as

$$8\pi\alpha\int_0^T\int_0^t\frac{1}{d_n^2+(t-s)^2}\,ds\,dt \leq 8\pi\alpha\int_0^T\int_0^\infty\frac{1}{d_n^2+s^2}\,ds\,dt = \frac{4\pi^2\alpha}{d_n}T. \tag{5.27}$$

This leads to the estimate

$$(\psi_n, H_0\psi_n) \geq \left(2E_1 - \frac{8\pi^2\alpha}{d_n}\right)\|\psi_n\|^2. \tag{5.28}$$

We now move on to study $(\psi_n, H_0\psi_n)$ for $1 \leq n \leq 2$. This will require a lower bound on the Nelson model Hamiltonian for two nucleons. That is precisely what was provided in the last chapter, in equation (4.51). After setting $N = 2$ in that lower bound, there is still a free parameter that should be optimized, θ. The fact that α is still arbitrary makes the optimization calculation impossible to be done analytically. That is why we will simply pick the average between the two endpoints of the set of viable values of θ, 1/8. All this leads to the lower bound

$$\begin{aligned}E^{2,\Lambda}_{\alpha,\mu}+Q^{2,\Lambda}_{\alpha,\mu} &\geq -2^5\Phi_{1/8}\alpha^4-2^3\Psi_{1/8}\alpha^3-2^5C^2\alpha^2-2^{7/4}\Upsilon_{1/8}\alpha^{7/4}\\ &\equiv -C_4\alpha^4-C_3\alpha^3-C_2\alpha^2-C_{7/4}\alpha^{7/4},\end{aligned} \tag{5.29}$$

and therefore, by using the fact that $-Q^{2,\Lambda}_{\alpha,\mu} = -2Q^{1,\Lambda}_{\alpha,\mu} \geq 2E^{1,\Lambda}_{\alpha,\mu}$,

$$\begin{aligned}(\psi_n, H_0\psi_n) \geq \text{inf spec } H_0\|\psi_n\|^2 &= E^{2,\Lambda}_{\alpha,\mu}\|\psi_n\|^2\\ &\geq \left[-(C_4\alpha^4+C_3\alpha^3+C_2\alpha^2+C_{7/4}\alpha^{7/4})+2E^{1,\Lambda}_{\alpha,\mu}\right]\|\psi_n\|^2.\end{aligned} \tag{5.30}$$

By the same calculations as before, there will be no binding if U satisfies the inequalities

$$U \geq (C_4\alpha^4+C_3\alpha^3+C_2\alpha^2+C_{7/4}\alpha^{7/4})a_1+\frac{\pi^2}{a_1}, \tag{5.31}$$

$$U \geq (C_4\alpha^4+C_3\alpha^3+C_2\alpha^2+C_{7/4}\alpha^{7/4})(a_1+a_2)+\frac{\pi^2(a_1+a_2)}{\min(a_1,a_2)^2}, \tag{5.32}$$

$$U \geq 8\pi^2\alpha+\frac{8\pi^2\alpha(a_{n-1}+a_n)}{a_1+\ldots+a_{n-2}}+\frac{\pi^2(a_1+\ldots+a_n)}{\min(a_n,a_{n-1})^2}\quad\text{for all } n \geq 3. \tag{5.33}$$

Things clarify a bit if one rescales $a_i = b_i/\alpha$, as before. Since the right side of (5.32) is greater than or equal to the right side of (5.31) (as explained in the previous subsection), one finds that it is enough that U satisfy

$$U \geq (C_4\alpha^3 + C_3\alpha^2 + C_2\alpha + C_{7/4}\alpha^{3/4})(b_1 + b_2) + \frac{\pi^2(b_1+b_2)}{\min(b_1,b_2)^2}\alpha, \tag{5.34}$$

$$U \geq 8\pi^2\alpha + \frac{8\pi^2(b_{n-1}+b_n)}{b_1+\ldots+b_{n-2}}\alpha + \frac{\pi^2(b_1+\ldots+b_n)}{\min(b_n,b_{n-1})^2}\alpha \quad \text{for all } n \geq 3. \tag{5.35}$$

Even though it is not possible to fully optimize this collection of inequalities, due to the fact that one cannot take the factor α out completely, there are several things one can do that will provide a reasonable answer. One could for instance study only the coeffiecients of the first powers of α, leading to the minimization of

$$\left[C_2(b_1+b_2) + \frac{\pi(b_1+b_2)}{\min(b_1,b_2)^2}\right] \vee \max_{n\geq 3}\left[8\pi^2 + \frac{8\pi^2(b_{n-1}+b_n)}{b_1+\ldots+b_{n-2}} + \frac{\pi^2(b_1+\ldots+b_n)}{\min(b_n,b_{n-1})^2}\right]. \tag{5.36}$$

After finding parameters leading to a reasonable number for (5.36), then no binding occurs if $U \geq D_3\alpha^3 + D_2\alpha^2 + D_1\alpha + D_{3/4}\alpha^{3/4}$, where D_i are explicit. (For instance, D_1 is just (5.36) and $D_3 = C_4(b_1+b_2)$.) The attentive reader will have noticed that we have proved no-binding for both the unrenormalized and the renormalized two-particle Nelson models. The unrenormalized case is what we have proved in principle, since, for U sufficiently large, $E^{2,\Lambda}_{\alpha,\mu} \geq 2E^{2,\Lambda}_{\alpha,\mu}$. But the same result for the renormalized model follows immediately by recalling that $Q^{2,\Lambda}_{\alpha,\mu} = 2Q^{1,\Lambda}_{\alpha,\mu}$, since $E^{2,\Lambda}_{\alpha,\mu} + Q^{2,\Lambda}_{\alpha,\mu} \geq 2(E^{2,\Lambda}_{\alpha,\mu} + Q^{1,\Lambda}_{\alpha,\mu})$ implies the assertion by taking $\Lambda \to \infty$.

Bibliography

[1] A. Abdesselam, D. Hasler, Analyticity of the Ground State Energy for Massless Nelson Models, Comm. Math. Phys. 310, 511-536 (2012).

[2] A. Panati, Existence and nonexistence of a ground state for the massless Nelson model under binding condition, Rep. Math. Phys. 63, 3, 305-330 (2009).

[3] R.D. Benguria, G.A. Bley, Exact asymptotic behavior of the Pekar-Tomasevich functional, J. Math. Phys. 52, 052110 (2011).

[4] R.D. Benguria, G.A. Bley, Improved results on the no-binding of bipolarons, J. Phys. A: Math. Theor. 45 045205 (2012).

[5] G.A. Bley, L.E. Thomas, Estimates on Functional Integrals of Quantum Mechanics and Non-Relativistic Quantum Field Theory, arXiv:1512.00356 (2015).

[6] C.J. Coulborn, J.H. Dinitz, Handbook of Combinatorial Designs, 2nd edition, Chapman & Hall/CRC (2006).

[7] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, Schrödinger Operators, Springer (2008).

[8] E.B. Davies, Particle-boson interactions and weak coupling limit, J. Math. Phys. 20, 345-351 (1979).

[9] M.D. Donsker, S.R.S. Varadhan, Asymptotics for the polaron, Comm. Pure Appl. Math., 36, 505-528 (1983).

[10] M. Falconi, Classical limit of the Nelson model with cutoff, J. Math. Phys. 54, 012303 (2013).

[11] R.P. Feynman, Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction, Phys. Rev. 80, 440 (1950).

[12] R.P. Feynman, Space-Time Approach to Non-Relativistic Quantum Mechanics, Rev. Mod. Phys. 20, 2, 367 (1948).

[13] R.P. Feynman, Slow Electrons in a Polar Crystal, Phys. Rev., 97, 3 (1955).

[14] R.L. Frank, E.H. Lieb, R. Seiringer, L.E. Thomas, Stability and absence of binding for multi-polaron systems, Publications mathématiques de l'IHÉS, 113, 1, 39–67 (2011).

[15] H. Fröhlich, Electrons in lattice fields, Adv. in Phys. 3, 325–361 (1954).

[16] M. Griesemer, J.S. Møller, Bounds on the Minimal Energy of Translation Invariant N-Polaron Systems, Comm. Math. Phys. 297, 283-297 (2010).

[17] M. Gubinelli, F. Hiroshima, J. Lörinczi, Ultraviolet renormalization of the Nelson Hamiltonian through functional integration, J. Funct. Anal., 267, 3125-3153 (2014).

[18] M. Gurari, Self energy of slow electrons in polar materials, Phil. Mag. Ser. 7, 44, 350, 329-336 (1953).

[19] C. Hainzl, M. Hirokawa, H. Spohn, Binding energy for hydrogen-like atoms in the Nelson model without cutoffs, J. Funct. Anal., 220, 424-459 (2005).

[20] M. Hirokawa, Infrared catastrophe for Nelson's model – non existence of ground state and soft-boson divergence, Publ. RIMS, Kyoto Univ. 42, 897-922 (2006).

[21] M. Hirokawa, F. Hiroshima, H. Spohn, Ground state for point particles interacting through a massless scalar Bose field, Adv. Math. 191, 339-392 (2005).

[22] F. Hiroshima, Asymptotic behaviors of the interaction Hamiltonians of quantum fields and particles, Nonlin. Anal., 30, 8, 4863-4874 (1997).

[23] F. Hiroshima, Weak coupling limit and removing an ultraviolet cutoff for a Hamiltonian of particles interacting with a quantized scalar field, J. Math. Phys. 40, 3, 1215 (1999).

[24] M. Kac, On distributions of certain Wiener functionals, Trans. Amer. Math. Soc. 65 (1949).

[25] M. Kac, On Some Connections between Probability Theory and Differential and Integral Equations, Proc. Second Berkeley Symp. on Math. Statist. and Prob., Univ. of Calif. Press, 189-215 (1951).

[26] I. Karatzas, S. Shreve, Brownian Motion and Stochastic Calculus, 2nd ed., Springer, New York, (1991).

[27] I. Karatzas, S. Shreve, Methods of Mathematical Finance, Springer, (2001).

[28] T.-D. Lee, F.E. Low, D. Pines, The Motion of Slow Electrons in a Polar Crystal, Phys. Rev. 90, 297 (1953)

[29] T.-D. Lee, D. Pines, The Motion of Slow Electrons in Polar Crystals, Phys. Rev. 88, 960 (1952).

[30] E.H. Lieb, M. Loss, Analysis, 2nd ed., AMS (2011).

[31] E.H. Lieb, R. Seiringer, Equivalence of Two Definitions of the Effective Mass of a Polaron, J. Stat. Phys. 154, 51-57 (2014).

[32] E.H. Lieb, R. Seiringer, The Stability of Matter in Quantum Mechanics, Cambridge (2009).

[33] E.H. Lieb, L.E. Thomas, Exact ground state energy of the strong-coupling polaron, Comm. Math. Phys. 183, 511-519 (1997). Erratum 188, 499-500 (1997).

[34] E.H. Lieb, K. Yamazaki, Ground-State Energy and Effective Mass of the Polaron, Phys. Rev. 111, 3, 728 (1958).

[35] J. Lörinczi, R.A. Minlos, H. Spohn, The infrared behavior in Nelson's model of a quantum particle coupled to a massless scalar field, Ann. Henri Poincaré 3, 2, 269-295 (2002).

[36] D. Nualart, The Malliavin Calculus and Related Topics, 2nd. ed., Springer, Berlin (2006).

[37] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, J. Math. Phys. 5, 1190-1197 (1964).

[38] E. Nelson, Schrödinger particles interacting with a quantized scalar field, in Analysis in Function Space: Proceedings of a conference on the theory and application of analysis in function space held at Endicott House in Dedham, Mass. June 9-13, 1963, W.T. Martin and I. Segal, eds., 87-121, MIT Press, Cambridge, Mass, (1964).

[39] S.I. Pekar, Untersuchung über die Elektronentheorie der Kristalle, Akademie Verlag, Berlin (1954).

[40] M. Reed, B. Simon, Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness, Academic Press, San Diego, (1975).

[41] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, 3rd ed., Springer, Berlin, (2010).

[42] H. Spohn, Effective mass of the polaron: A functional integral approach, Ann. Phys. 175, 2, 278-318 (1987).

[43] H. Spohn, Roughening and pinning transitions for the polaron, J. Phys. A: Math. Gen. 19, 533-545 (1986).

[44] H. Spohn, The polaron at large total momentum, J. Phys. A: Math. Gen. 21, 1199-1211 (1988).

[45] L. Tenuta, S. Teufel, Effective Dynamics for Particles Coupled to a Quantized Scalar Field, Comm. Math. Phys. 280, 751-805 (2008).

[46] S. Teufel, H. Spohn, Semiclassical Motion of Dressed Electrons, Rev. Math. Phys., 14, 1, 1-28 (2002).