Contact surgery and LOSS invariant

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Abstract

Let L and S be two disjoint Legendrian knots in a contact manifold (Y, ξ). Ozsváth and Stipsicz [24] showed that the LOSS invariant of L is natural under $+1$ contact surgery on Legendrian knot S. This thesis extend their result and prove the naturality of the LOSS invariant of L under any positive integer contact surgery along S.

In addition, when S is rationally null-homologous, we also entirely characterize the $Spin^c$ structure in the surgery cobordism that makes the naturality of contact invariant or LOSS invariant work (without conjugation ambiguity). In particular this implies that the contact invariant of the $+n$ contact surgery along a rationally null-homologous Legendrian S depends only on the classical invariants of S.

The additional generality provided by those results allows us to prove that if two Legendrian knots have different LOSS invariants then, after adding the same positive twists to each in a suitable sense, the two new Legendrian knots will also have different LOSS invariants. This leads to new infinite families of examples of Legendrian (or transverse) non-simple knots that are distinguished by their LOSS invariants.

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Contents

Chapter 1

Introduction

1.1 Contact 3-manifold, Heegaard Floer, and LOSS invariant.

A contact 3-manifold (Y, ξ) is a smooth 3-manifold Y together with a special 2-plane field distribution ξ. Though the idea of contact topology was born over two centuries ago, only in recent decades it has been developed and moved to the front of mathematics. There are surprisingly subtle relationships arose between contact manifolds and 3- (and 4-) dimensional geometry and topology. For example contact 3-manifolds have essential and deep connection to a revolutionary package of invariants of 3- and 4- dimensional manifolds developed by Ozsváth-Szabó [27] called Heegaard Floer homology.

As one of the most important concept in contact topology, Legendrian knots also have strong connection to the knot Floer homology introduced by Jacob Rasmussen [31] and Ozsváth-Szabó [25] independently. Given a Legendrian knot L in (Y, ξ), Lisca-Ozsváth-Stipsicz-Szabó associated L to elements "LOSS invariant" $\mathfrak{L}(L)$ and "LOSS-hat invariant" $\hat{\mathfrak{L}}$ which lives in the knot Fleor homology $HFK^{-}(-Y, L)$ and $\widehat{HFK}$ respectively [20]. Moreover Ozsváth-Stipsicz prove the naturality of this LOSS invariant under $+1$ contact surgery.

Theorem 1.1.1 (Ozsváth-Stipsicz [24]). *Let $L, S \in (Y, \xi)$ be two disjoint oriented*

Legendrian knots in the contact 3-manifold (Y, ξ) *with* L *null-homologous. Let* $(Y_1(S), \xi_1(S))$ *denote the contact 3-manifold we get by performing contact* $(+1)$*-surgery along* S*, and we denote* L_S *the oriented Legendrian knot corresponding to* L *in* $(Y_1(S), \xi_1^-(S))$*. Moreover suppose that* L_S *is null-homologous in* $Y_1(S)$*. Let* W *be the 2-handle cobordism from* Y *to* $Y_1(S)$ *induced by the surgery, and let*

$$F_{S,\mathfrak{s}} : HFK^-(-Y, L) \to HFK^-(-Y_1(S), L_S) \tag{1.1.2}$$

be the homomorphism in knot Floer homology induced by $-W$*, the cobordism with reversed orientation, for* $\mathfrak{s}$ *a* $Spin^c$ *structure on* $-W$*. If* Y *is a rational homology sphere then there is a unique choice of* $\mathfrak{s}$ *for which*

$$F_{S,\mathfrak{s}}(\mathfrak{L}(Y, \xi, L)) = \mathfrak{L}(Y_1(S), \xi_1(S), L_S) \tag{1.1.3}$$

holds. A similar identity holds for the Legendrian invariant $\hat{\mathfrak{L}}$ *in* $\widehat{HFK}$*.*

As a consequence of this theorem, they also shows the following.

Theorem 1.1.4 (Ozsváth-Stipsicz [24])**.** *The twist knot which is the mirror of* 7_2 *in Rolfsen's table is not transversely simple. (For the definition of transversely simple see section 2.2)*

The goal of this thesis is to generalized the Theorem 1.1.1 to contact $+n$ surgery, and using the generalization to give more example of non-simple knot.

1.2 Summary of results

In [33], we first extend Theorem 1.1.1 to general contact $+n$ surgery.

Theorem 1.2.1. *Let* $L, S \in (Y, \xi)$ *be two disjoint oriented Legendrian knots in the contact 3-manifold* (Y, ξ) *with* L *null-homologous. Let* $(Y_n(S), \xi_n^-(S))$ *denote the contact 3-manifold we get by performing contact* $(+n)$*-surgery along* S*, and we denote* L_S *the oriented Legendrian knot corresponding to* L *in* $(Y_n(S), \xi_n^-(S))$*. Moreover suppose* *that L_S is null-homologous in $Y_n(S)$. Let W be the 2-handle cobordism from Y to $Y_n(S)$ induced by the surgery, and let*

$$F_{S,\mathfrak{s}} : HFK^-(-Y, L) \to HFK^-(-Y_n(S), L_S) \tag{1.2.2}$$

be the homomorphism in knot Floer homology induced by $-W$, the cobordism with reversed orientation, for $\mathfrak{s}$ a Spin^c structure on $-W$. If Y is a rational homology sphere then there is a choice of $\mathfrak{s}$ for which

$$F_{S,\mathfrak{s}}(\mathfrak{L}(Y, \xi, L)) = \mathfrak{L}(Y_n(S), \xi_n^-(S), L_S) \tag{1.2.3}$$

holds. A similar identity holds for the Legendrian invariant $\hat{\mathfrak{L}}$ in $\widehat{HFK}$.

Remark 1.2.4. doing contact surgery on a Legendrian knot in a contact 3-manifold (Y, ξ) gives new contact 3-manifold, but if we are doing contact $+n$ surgery for $n > 1$ the resulting contact structure is not unique and we need to make a choice of stabilization [9]. In this thesis we are choosing the contact structure corresponding to all stabilization being negative, and denote by ξ_n^- the resulting contact structure (see section 2.3 for more detail).

Since the LOSS invariant stays unchanged under negative stabilization [20], it gives rise to an invariant of transverse knots. If we have a transverse knot T in (Y, ξ) the transverse invariants $\mathfrak{T}$ and $\hat{\mathfrak{T}}$ are defined to be the LOSS invariants of a Legendrian approximation of T [13]. Thus we obtain a parallel naturality statement for transverse invariants $\mathfrak{T}$.

Corollary 1.2.5. *Let T be a null-homologous transverse knot and S an oriented Legendrian knot in (Y, ξ) which is disjoint from T. Let $(Y_n(S), \xi_n^-(S))$ denote the contact 3-manifold we get by performing contact $(+n)$-surgery along S, and we denote T_S the oriented Legendrian knot corresponding to T in $(Y_n(S), \xi_n^-(S))$. Moreover suppose that T_S is null-homologous in $Y_n(S)$. Let W be the 2-handle cobordism from*

Y to $Y_n(S)$ induced by the surgery, and let

$$F_{S,\mathfrak{s}} : HFK^-(-Y, T) \to HFK^-(-Y_n(S), T_S) \qquad (1.2.6)$$

be the homomorphism in knot Floer homology induced by $-W$, the cobordism with reversed orientation, for $\mathfrak{s}$ a spin^c structure on $-W$. If Y is a rational homology sphere then there is a choice of for which

$$F_{S,\mathfrak{s}}(\mathfrak{L}(Y, \xi, T)) = \mathfrak{L}(Y_n(S), \xi_n^-(S), T_S) \qquad (1.2.7)$$

holds. A similar identity holds for the Legendrian invariant $\hat{\mathfrak{L}}$ in $\widehat{HFK}$.

The way of proving Theorem 1.2.1 combines the ideas of [24], [4], [22], and [21], and can be briefly described as follows. We first interpret the contact $+n$ surgery cobordism as a capping off cobordism by viewing it upside down. Then we construct a doubly pointed Heegaard triple describing the capping off cobordism and the induced map $F_{B,\mathfrak{s}}$ in knot Floer homology where B is the binding component being capped off, and finally we show this map carries the LOSS invariant of L to the LOSS invariant of L_S.

In particular, Theorem 1.2.1 follows from a naturality property for the LOSS invariant under capping off cobordisms. Recall that an (abstract) open book consists of a pair (S, ϕ) where S is a compact oriented 2-manifold with boundary and ϕ is a diffeomorphism of S fixing ∂S(For more detailed description of open book see section 2.4). If a boundary component B of S is chosen, then the capped-off open book (S', ϕ') is obtained by attaching a disk to S along B and extending ϕ by the identity.

Theorem 1.2.8. *Let $(S_{g,r}, \phi)$ be an abstract open book with genus g and $r > 1$ binding components. Suppose T and B are distinct binding components; then capping off B we get a new open book $(S_{g,r-1}, \phi')$ which has a binding component T' correspond to T.*

Denote by (M, ξ), (M', ξ') the contact 3 manifolds corresponding to those two open books, so that T, T' naturally become transverse knots. The capping off cobordism *gives rise to a map*

$$F_{B,\mathfrak{s}} : HFK^-(-M', T') \to HFK^-(-M, T) \tag{1.2.9}$$

where $\mathfrak{s}$ *is a spin*^{c} *structure on the cobordism* W *from* $-M'$ *to* $-M$*. If* M' *is a rational homology sphere, and both* T*,* T' *are null-homologous, then then there is a choice of* $\mathfrak{s}$ *for which*

$$F_{B,\mathfrak{s}}(\mathfrak{T}(M', \xi', T')) = \mathfrak{T}(M, \xi, T) \tag{1.2.10}$$

holds. A similar identity holds for the transverse invariant $\hat{\mathfrak{T}}$ *in* $\widehat{HFK}$*.*

Combining the proof of Theorem 1.2.1 in this paper and the proof of Theorem 1.1 in [22], it's easy to see that $Spin^c$ structures on the cobordism that makes the naturality of contact invariant and LOSS invariant work are the same $Spin^c$ structure, and we can say more about this $Spin^c$ structure $\mathfrak{s}$.

Proposition 1.2.11. *Assume in the situation of Theorem 1.2.1 that* S *is null-homologous and both* Y *and* $Y_n(S)$ *are rational homology sphere. Then the* $\mathfrak{s}$ *in Theorem 1.2.1, Corollary 1.2.5, as well as in [22, Theorem 1.1 (for integer surgery)] has the property that*

$$\langle c_1(\mathfrak{s}), [\tilde{F}] \rangle = rot(L) + n - 1$$

where F *is a Seifert surface for* S *and* $\tilde{F}$ *is obtained by attaching the core of the* 2*-handle to* F *in* W*.*

The above proposition is a direct consequence of Theorem 6.1.4 which is an analogous statement of the above proposition but for contact $+n$ surgery on a rationally null-homologous Legendrian S.

Remark 1.2.12. In [22] Mark-Tosun prove the above proposition (for the contact invariant) but with a **sign** ambiguity (such sign ambiguity also shows up in [10]). Here we resolve this ambiguity. In other words we can characterize this $Spin^c$ structure not just up to conjugation, and the proof is entirely different.

As a direct consequence of Theorem 6.1.4 we have the following corollary which extends a result of Golla [16, Proposition 6.10] from (S^3, ξ_{std}) to all contact rational homology sphere (Golla does not require the smooth coefficient to be non-zero but we do).

Corollary 1.2.13. *Let (Y, ξ) be a contact rational homology sphere, S be a rationally null-homologous Legendrian knot, and L be a null-homologous Legendrian knot. If $Y_n(S)$ is again a rational homology sphere, then the contact invariant $c(\xi_n^-(S))$ as well as the LOSS invariant $\mathfrak{L}(Y_n(S), \xi_n^-(S), L_S)$ are independent of the Legendrian isotopy class of S, when the classical invariants are fixed.*

Remark 1.2.14. The LOSS invariants are actually only well defined up to sign, and up to the action of the mapping class group on (Y, L) [24] (that is, the group of isotopy classes of diffeomorphisms of Y fixing L). We denote by $[\mathfrak{L}] \in HFK^-(-Y, L)/ \pm MCG(Y, L)$ the image of $\mathfrak{L}$ when we quotient out these actions, and similarly for the other types of LOSS invariants.

Using the main theorem one can produce infinite families of smooth knots that have distinct Legendrian (resp. transverse) representatives with same Thurston-Bennequin and rotation numbers (resp. self linking number). More specifically starting with two Legendrian representatives of knot K with different $[\mathfrak{L}]$ or $[\hat{\mathfrak{L}}]$, it is always possible to produce two Legendrian representatives of a new knot K' that also have different $[\mathfrak{L}]$ or $[\hat{\mathfrak{L}}]$, essentially by adding positive twists to parallel strands in K. More specifically the procedure can be described as follows.

Let (Y, ξ) be a contact 3-manifold, and consider a triple (L, σ_n, B) where L is a Legendrian knot in (Y, ξ), $\sigma_n = \{e_i | e_i$ is an oriented Legendrian arc of L for $i = 1, 2, ..., n\}$, and B is a Darboux ball (a ball with standard contact structure). We say this is a **compatible triple** if the following hold:

1. B only intersects L at σ_n.

2. Inside the Darboux ball the front projections of the arcs e_i are horizontal, parallel and have the same orientation for all $i = 1, 2, ..., n$. In other words σ_n is a collection of n Legendrian push-offs of one oriented horizontal arc.

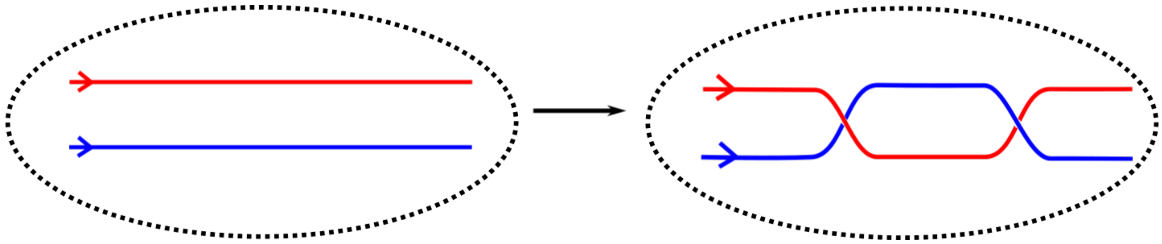


Figure 1-1: Example when there are two parallel arcs (the blue and red arcs are e_1 and e_2, and the dotted circle represents a Darboux ball). On the left is part of L inside a standard Darboux ball. After doing the twist we get the right diagram which is still inside the Darboux ball and is part of the new knot L_σ

Given a compatible triple (L, σ_n, B) we can construct a new oriented Legendrian knot $\boldsymbol{L_\sigma}$ by adding a full non-zigzagged positive twist to the front projection of σ_n in B (See figure 1-1 for example when $n = 2$).

Note that since all arcs e_i are horizontal, parallel and oriented in the same direction there is no ambiguity of the new Legendrian knot L_σ once given a compatible (L, σ_n, B). Now we are able to state the theorem.

Theorem 1.2.15. *In the above setting let (L, σ_n, B) and (L', σ'_n, B') be two compatible triples in (Y, ξ). Assume*

- *L and L' are smoothly ambiently isotopic*
- *The isotopy sends B contactmorphically to B', and e_i to e'_i.*

Then L_σ, $L'_{\sigma'}$ are smoothly isotopic. Moreover if Y is a rational homology sphere and L is null-homologous, and if L and L' have different $[\mathfrak{L}]$ or $[\hat{\mathfrak{L}}]$, then so do L_σ and $L'_{\sigma'}$.

As an example (application) of Theorem 1.6 we will see the following corollary.

Corollary 1.2.16. *In standard tight S^3, the mirror of the knot 9_7 in Rolfsen's table is neither Legendrian simple nor transversely simple.*

In section 5.2 we will see more non-simple knot examples that we can derive from Theorem 1.2.15 (see Theorem 5.2.1 and 5.2.2, and the discussion about figure 5-10).

1.3 Outline

The thesis is outlined as follows. In Chapter 2 we will review relevant background about contact 3-manifold. We will talk about Legendrian and transverse knot, classical invariants of Legendrian and transverse knot, contact surgery, and open book decomposition. In Chapter 3 we will recall the basic definition of knot Floer homology, how knot Floer behaves under surgery, and the definition of the LOSS invariant. Then we will prove Theorem 1.2.1 in Chapter 4, and give application of the Naturality in Chapter 5. Last, in Chapter 6 we will prove the $Spin^c$ structure formula, (Theorem 6.1.4) that implies Proposition 1.2.11.

Chapter 2

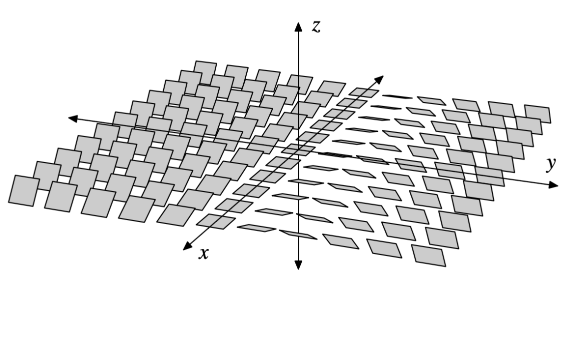
Background on contact 3-manifold

2.1 Basic contact structures

Definition 2.1.1. A contact 3-manifold (Y, ξ) is a smooth 3-manifold Y together with a 2-plane field distribution ξ such that for any one form α with $\ker(\alpha) = \xi$, we have $\alpha \wedge d\alpha > 0$.

Definition 2.1.2. Two contact manifold Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are said to be contactomorphic (or contact isotopy) if there is a diffeomorphism (or isotopy) $f : M_1 \to M_2$ with $df(\xi_1) = \xi_2$, where $df : TM_1 \to TM_2$ denotes the differential of f.

Example 2.1.3. The standard contact structure ξ_{std} on the 3 space $\mathbb{R}^3$ is defined as the $ker(dz - ydx)$.



Example 2.1.4. The standard contact structure ξ_{std} on the 3-sphere $S^3 \subset \mathbf{R}^4$ (with cartesian coordinates x, y, z, t) is defined as the kernel of $ker(xdy - ydx + zdt - tdz)$.

We remark that if we remove the north pole of the S^3 the induced contact structure will be contactmorphic to $(\mathbb{R}^3, \xi_{std})$.

There are two basic types of contact structures we are considering. To make it clear we first give a definition of overtwisted disk.

Definition 2.1.5. An embedded disk $D \in (Y, \xi)$ is called an overtwisted disk if on the boundary of the disk the contact plane coincide with the tangent plane.

The overtwisted and tight contact manifolds are distinguished by the fact if it contains any overtwisted disk or not.

Definition 2.1.6. The contact 3-manifold (Y, ξ) is called overtwisted if it contains a overtwisted disk, and called tight otherwise.

Examples given above are all tight contact structures. Eliashberg had classified the overtwisted contact structures for any 3-manifold [11]. However the classification for the tight contact structures are much subtle, and since it's not the major topics here we are not going to explore in this thesis.

2.2 Legendrian and transverse knots

As there is always a special contact plane ξ associate to a contact 3-manifold, there are two important types of knot we can consider relative to this plane.

Definitions 2.2.1. Let (Y, ξ) be a contact 3-manifold.

· A Legendrian knot L in (Y, ξ) is an embedded S^1 that is always tangent to ξ. (We will assume it always come with a choice of orientation)

· A transverse knot T in (Y, ξ) is an embedded oriented S^1 that is always positively transverse to ξ.

We say two Legendrian (resp transverse) knots are Legendrian (resp transverse) isotopic if they are isotopic through Legendrian (resp transverse) knots.

Given a Legendrian knot L in $(\mathbb{R}^3, \xi_{std})$ we can look at the **Front Project** of L, which is defined to be the image of L under the map:

$$\Pi : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \to (x, z).$$

For more detail see [13, Etnyre].

There are two important facts about the front project of L.

1. Front projections $\Pi(L)$ have no vertical tangencies.

2. Front projections can be parameterized by maps that are immersions except at finite many points, at which there is still a well defined tangent line in $\mathbb{R}^3$. Such points are called generalized cusps.

In terms of knot diagram those projections $\Pi(L)$ have the following properties:

1. There is no vertical tangency in $\Pi(L)$.

2. The vertical tangencies change to generalized cusps, and the generalized cusps are the only non-smooth points.

3. At each crossing the slope of the overcrossing is smaller than the undercrossing.

Example 2.2.2. Figure 2-1 is an example of Legendrian trefoil, and when we give it an orientation the cusps are divided into two categories, either upward cusp or downward cusps.

By manipulating a general knot diagram it's not hard to show that every knot has a Legendrian representative in $(\mathbb{R}^3, \xi_{std})$, we can then use Darboux's theorem which says that every contact 3-manifold locally looks like $(\mathbb{R}^3, \xi_{std})$, and obtain result for general contact 3-manifolds.

There are smooth Reidemeister moves for smooth knot diagram. In the contact world, there are Legendrian Reidemeister moves for the front projection of a Legendrian knot.

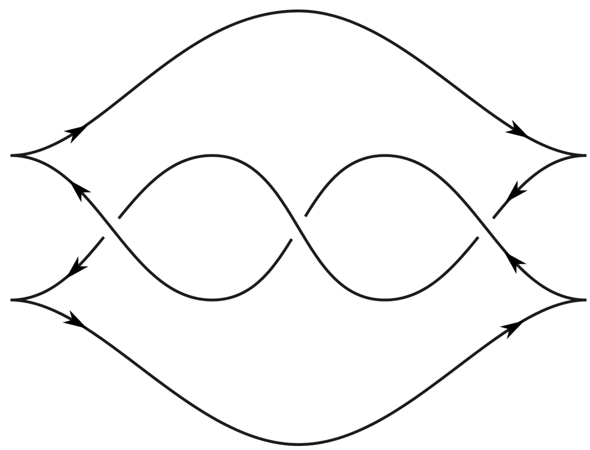


Figure 2-1: A front projection of right handed trefoil.

Definition 2.2.3. There are 3 types of Legendrian Reidemesiter moves for the front projection of Legendrian knot showing in Figure 2-2

We also have the Reidemeister's theorem for front projection of Legendrian knot in $(\mathbb{R}^3, \xi_{std})$.

Theorem 2.2.4 ([32])**.** *Two front diagrams represents the Legendrian isotopic Legendrian knots if and only if they are related by regular homotopy and sequences of Legendrian Reidemeister moves.*

2.2.1 Invariants of Legendrian and transverse knots

Except the smooth knot type there are two classical invariants associated to an null-homologous Legendrian knot in a contact 3-manifold (Y, ξ). To define the invariants we first recall that for a smooth knot K in a 3-manifold Y, a Seifert surface Σ of K is a smooth surface in Y such that $\partial\Sigma$, the boundary of Σ, is K.

The first invariant is called the **Thurston-Bennequin number**, and is denoted by $tb(L)$, roughly speaking it measures the "twisting of ξ around L". More precisely let ν be the normal bundle of L, then the intersection of ν_x and ξ_x gives a line bundle over L which induce a framing (trivialization of normal bundle) on L called the **Thurston-Bennequin framing** or **contact framing**, and the difference between

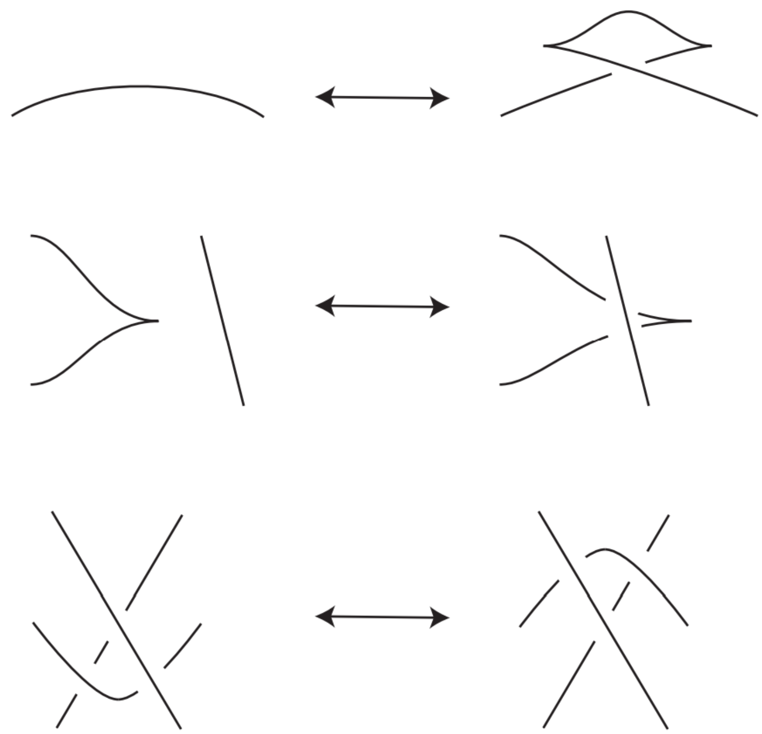


Figure 2-2: Three types of Legendrian Reidemeister moves. (To obtain all moves we also need the corresponding figures rotated 180 degrees about all axes) Picture used from [13]

the Thurston-Bennequin framing and the smooth framing is the $tb(L)$ (where the smooth framing is determined by the Seifert surface of the knot).

Second one is called the **rotation number**, and is denoted by $rot(L)$. We start with any Seifert surface Σ of L, then the rotation number is defined to be the relative Euler class $e(\xi|_\Sigma) \in \mathbb{Z}$ with the trivialization along $\partial\Sigma$ given by the tangents of L, and it can be shown it does not depend on the choice of Seifert surface.

Even though the definition of $tb(L)$ and $rot(L)$ seems a bit subtle and hard to calculate for Legendrian knots in general contact 3-manifold, for a Legendrian knot L in $(\mathbb{R}^3, \xi_s td)$ it's easy to derive $tb(L)$ and $rot(L)$ from the front projection of L, and they can be calculated as follow.

$$tb(L) = writhe(\Pi(L)) - \frac{1}{2}(\text{ number of cusps in } \Pi(L)),$$

where writhe is the number of positive crossing minus the number of negative crossing.

$$rot(L) = \frac{1}{2}((\text{number of downward cusps}) - (\text{number of upward cusps}))$$

Example 2.2.5. The Legendrian trefoil in Figure 2-1 has $tb = 1$ and $rot = 0$.

Next we will define a way of **stabilizing** and **destabilizing** a Legendrian knot in $(\mathbb{R}^s, \xi_{sdt})$ using the front projection, and we can naturally extend the stabilization and destabilization to other contact 3-manifold Darboux's theorem.

Definition 2.2.6. If a strand of L in the front projection is shown on the left side of Figure 2-3, then the stabilization of L is obtained by removing the original strand and replacing it with one of the zig-zags on the right side of Figure 2-3. If down cusps are added then the stabilization is call positive stabilization of L, and denoted $S^+(L)$. If up cusps are added then the stabilization is called negative stabilization, and denoted $S^-(L)$. The reverse of the above procedure is called destabilization.

Remark 2.2.7. To shorthand the notation we often use L^{+n} and L^{-n} to denote n positive or negative stabilization of L, and it will be the most common used notation for the rest of the thesis. However the notation mentioned in the definition is still useful and necessary because positive and negative stabilization is not canceling each other.

Remark 2.2.8. It's easy to see how classical Legendrian invariants change under the stabilization. That is: $tb(L^-) = tb(L^+) = tb(L) - 1$; $rot(L^-) = rot(L) - 1$, and $rot(L^+) = rot(L) + 1$

There is also an invariant for transverse knot T called **self linking number**, and is denoted by $sl(T)$. The reason why we did not talk the transverse knot as detailed as Legendrian knot is that we can always obtained transverse knot T by taking a transverse push-off of the corresponding Legendrian knot L, the self linking number of the push-off is equal to $tb(L) - rot(L)$, moreover if two Legendrian knots differ by a sequence of negative stabilizations, then they will have the same transverse push-offs.

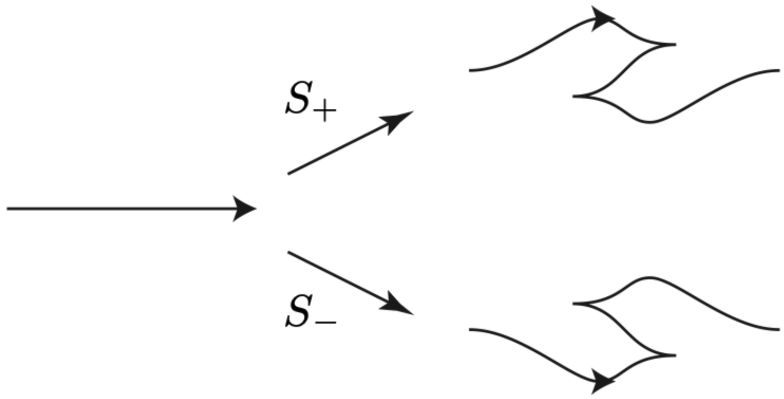


Figure 2-3: Top one is a positive stabilization and the bottom one is a negative stabilization. Picture used from [13]

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On the other hand, if we have a transverse knot we can take it's Legendrian push-off to obtain Legendrian knot. However the Legendrian push-offs of a transverse knot is only well defined up to negative stabilizations, that is the two Legendrian push-offs of a transverse knot might differ by sequences of negative stabilizations.

In general for Legendrian and transverse knots the classical invariants are not so difficult to compute and it is a very effective way to distinguish Legendrian and transverse non-isotopic knots However it's very hard, but also interesting, to distinguish Legendrian and transverse knot if they have the same classical invariants. To make the statements more clear we introduce an important definition in next subsection.

2.2.2 Non-simple knot

Definitions 2.2.9. Let K be a smooth knot in a contact 3-manifold (Y, ξ). We say

- K is Legendrian non-simple if K has two Legendrian representatives L_1 and L_2 with the same tb and rot but not Legendrian isotopic.

- K is transversely non-simple if K has two transverse representatives T_1 and T_2 with the same self link number but not transverse isotopic.

Example 2.2.10. The most famous and important example of Legendrian non-simple knots are Eliashberg–Chekanov twist knot E_n. The Legendrian representatives of them are $E(k,l)$ (see figure 2-4). Note that $E(k,l)$ and $E(k',l')$ are topologically isotopic if and only if $k+l=k'+l'$, and all of those Legendrian representatives have $tb(k,l)=1$ and $rot(k,l)=0$. In [12] Epstein-Fuchs-Meyer shows that $E(k,l)$ and $E(k',l')$ are Legendrian isotopic if and only if the unordered pair $\{k,l\}$ is equal to $\{k',l'\}$. Later in [24] Ozsváth-Stipsicz shows that E_n is transversely non-simple for $n>3$ and odd.

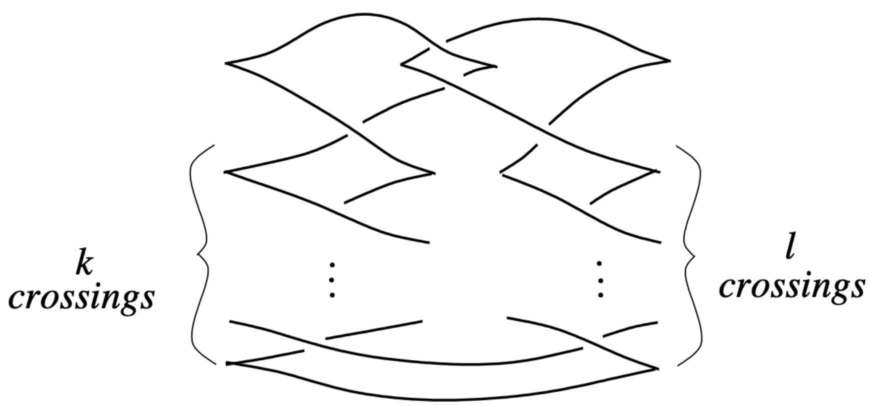

Figure 2-4: The Eliashberg–Chekanov Legendrian knots $E(k,l)$

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2.3 Contact surgery and contact surgery diagram

Given a Legendrian knot L in contact 3-manifold (Y,ξ) it always has a canonical contact framing, defined by a vector field along K that is transverse to ξ. A notion of contact r-surgery along a Legendrian knot L in (Y,ξ) is described as follows ([7]): This amounts to a topological surgery, with surgery coefficient $r\in\mathbb{Q}\cup\infty$ measured relative to the contact framing. We can further assign a contact structure on the surgered manifold $(Y-\nu L)\cup(S^1\times D^2)$, (where νL denotes a tubular neighbourhood of L) as follows: for $r\neq 0$ we require this contact structure to coincide with ξ on $Y-\nu L$ and its extension over $S^1\times D^2$ to be tight (we only need it's tight on $S^1\times D^2$, not necessarily on the whole surgered manifold).

According to [17], such an extension always exists and is unique (up to isotopy) for $r = 1/k$ with $k \in \mathbb{Z}$. (For $r = 0$, that extension is always overtwisted and thus requires a different treatment. For that reason we shall not discuss the case of contact 0-surgery any further).

Therefore, if $r = 1/k$ with $k \in \mathbb{Z}$, there is a canonical procedure for this surgery, in other words the resulting contact manifold is completely determined by the initial manifold (Y, ξ), the Legendrian knot L in (Y, ξ), and the surgery coefficient $r = 1/k$.

For general r there are non-canonical choices involve in order to give a contact structure on the surgered manifold. To better describe the choices we first introduce the following algorithms that change the contact r surgery to a sequence of contact ± 1 surgery on a sequence of Legendrian knots, and it's divided into two cases when $r > 0$ and $r < 0$ respectively.

Remark 2.3.1. Since for a Legendrian knot L, $tb(L)$ is the difference between contact framing and smooth framing, the contact r-surgery is smooth $(r + tb(L))$-surgery.

Theorem 2.3.2 (DGS algorithm for $r > 0$ [9])**.** *Given a Legendrian knot L in (Y, ξ). Let $0 < \frac{x}{y} = r \in \mathbb{Q}$ be a contact surgery coefficient. Let $c \in \mathbb{Z}$ be the minimal positive integer such that $\frac{x}{y - cx} < 0$, with the continued fraction*

$$\frac{x}{y - cx} = [a_1, a_2, ...a_m] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{... - \cfrac{1}{a_m}}} \tag{2.3.3}$$

where each $a_i \leq -2$. Then any contact $\frac{x}{y}$ surgery on L can be described as contact surgery along a link $(L_0^1 \cup L_0^2 \cup ... \cup L_0^c) \cup L_1 \cup ... \cup L_m$, where

- *$L_0^1, ..., L_0^c$ are Legendrian push-offs of L.*
- *L_1 is obtained from a Legendrian push-off of L_0^c by stabilizing $|a_1 + 1|$ times.*
- *L_i is obtained from a Legendrian push-off of L_{i-1} by stabilizing $|a_i + 2|$ times, for $i \geq 2$.*

- *The contact surgery coefficients are* $+1$ *on each* L_0^j *and* -1 *on each* L_i.

Theorem 2.3.4 (DGS algorithm for $r < 0$[9]). *Given a Legendrian knot* L *in* (Y, ξ). *Let* $0 > -x/y = r \in \mathbb{Q}$ *be a contact surgery coefficient with the continued fraction*

$$-\frac{x}{y} = [a_1 + 1, a_2, ..., a_m] = a_1 + 1 - \cfrac{1}{a_2 - \cfrac{1}{... - \cfrac{1}{a_m}}} \tag{2.3.5}$$

where each $a_i \leq -2$. *Then any contact* (x/y)*-surgery on* L *can be described as contact surgery along a link* $L_1 \cup ... \cup L_m$, *where*

- L_1 *is obtained from a Legendrian push-off of* L *by stabilizing* $|a_1 + 2|$ *times.*
- L_j *is obtained from a Legendrian push-off of* L_{j-1} *by stabilizing* $|a_j + 2|$ *times, for* $j \geq 2$.
- *The contact surgery coefficients* -1 *on each* L_j.

The choices we mentioned above correspond to the choices of stabilizations for each L_i, each of which can be either positive or negative. The case we are interested in is positive integer contact $+n$ surgery, and if we follow the algorithm carefully we can see that $+n$ contact surgery on a Legendrian knot L is the same as doing contact surgery along the link $(L_0^1) \cup L_1 \cup ... \cup L_{n-1}$ where L_0^1 is the Legendrian push-off of L, L_1 is one stabilization of a Legendrian push-off of L_0^1, and L_i is a Legendrian push-off of L_{i-1} for $i \in \{2, 3, ..., n-1\}$. In particular there only one choice of stabilization involve. The contact structure $\xi_n^-(L)$ we consider corresponds to choosing the negative stabilization for L_1. Below is one reason why we want to consider the one with negative stabilizations.

Proposition 2.3.6 ([21] Proposition 2.4). *Let* L *be an oriented Legendrian knot and* L^- *be negative stabilization of* L, *and let* $n > 0$ *be a positive integer. Then* $\xi_n^-(L) = \xi_{n+1}^-(L^-)$.

Just like we can use smooth surgery diagram to describe the smooth surgery, we can use contact surgery diagram (Legendrian knots with contact framing) to describe a contact surgery.

Example 2.3.7. The Figure 2-5 describes a contact $(-5/3)$-surgery on the Legendrian unknot K with $tb = -1$ and $rot = 0$. By the Theorem 2.3.4 we have $a_1 = a_2 = -3$, and thus it's equivalent to the contact (-1)-surgery on K_1 and K_2. Since it's smoothly $(-8/3)$ surgery on the unknot, this contact surgery diagram describes a contact structure on the lens space $l(8, 3)$.

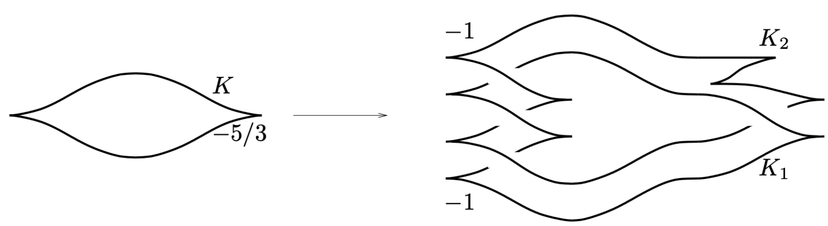

Figure 2-5: Example of contact surgery and DGS algorithm for $r = -5/3$

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2.4 Open book decomposition

Even though the contact structure on a 3-manifold Y seems very geometric, Giroux showed that it can be represent by a topological object called open book decomposition.

Definition 2.4.1. An abstract open book is a pair (P, ϕ) where

(1) P is an oriented compact surface with boundary.

(2) $\phi : P \to P$ is a diffeomorphism such that ϕ is equal to identity near the boundary of P.

P and ϕ are called the "page" and "monodromy" of the open book, respectively, and the boundary ∂P is called the binding of P.

Given an abstract open book (P, ϕ) we can form a 3-manifold $M(\phi)$ correspond to (P, ϕ) as follows:

$$M(\phi) = P_\phi \cup_\psi (\bigsqcup_{|\partial P|} S^1 \times D^2)$$

where $|\partial P|$ denotes the number of boundary components of P and

$$P_\phi = P \times [0,1]/(x,0) \sim (\phi(x),1)$$

is the mapping torus of ϕ, and $\cup_\psi$ means that the diffeomorphism ψ is used to identify the boundaries of the two sides. For each boundary component B of P the map $\psi : \partial(S^1 \times D^2) \to B \times D^2$ is defined to be the unique diffeomorphism that takes $S^1 \times \{x\}$ to B where $x \in \partial D^2$ and $\{y\} \times \partial D^2$ to $(\{y'\} \times [0,1])/\sim) = S^1$, where $y \in S^1$ and $y' \in \partial P$. If $M = M(\phi)$ for some (P, ϕ) we say (P, ϕ) is an open book decomposition of M.

Example 2.4.2. The easiest example would be P is a disk and ϕ is the identity, then $M(\phi) = S^3$. A slightly harder example would be P' is an annulus, and ϕ' is the right hand Dehn twist along the homologically essential circle, and in this case $M(\phi') = S^3$ also.

The open book (P', ϕ') above is an example of positive stabilization of (P, ϕ) which is an common operation we can do on open book.

Definition 2.4.3. A positive (negative) stabilization of an abstract open book (P, ϕ) is the open book

(1) with page $P' = P \cup$ 1-handle, and

(2) monodromy $\phi' = \tau_k \circ \phi$ where τ_k is a right- (left-) handed Dehn twist along a curve k in P' that intersects the co-core of the 1-handle exactly once.

It's natural to ask the question that if every 3-manifold admit an open book decomposition, and it turn out to be true.

Theorem 2.4.4 ([1]). *Every closed oriented 3-manifold has an open book decomposition.*

Next we are going to state one of the most important theorems in contact topology.

Theorem 2.4.5 (Giroux correspondence [15]). *Let Y be a closed oriented 3-manifold. Then there is a one-to-one correspondence between*

$$\{\textit{Oriented contact structures on } Y \textit{ up to contact isotopy}\}$$

and

$$\{\textit{Open book decompositions of } Y \textit{ up to positive stabilization}\}.$$

We say (P, ϕ) and (Y, ξ) are compatible if the above correspondence send one to another.

We will frequently use both the open book and surgery diagram as ways of describing a contact manifold, and we will see how they are related in the next subsection.

2.4.1 Contact surgery and Capping off cobordism

If $(S_{g,r}, \phi)$ is an abstract open book with genus g and $r > 1$ binding components. We can cap off one of the binding components B of $S_{g,r}$ with a disk, we obtain a new open book $(S_{g,r-1}, \phi')$ where ϕ' is the extension of ϕ to $S_{g,r-1}$ by the identity on the disk. If we denote $M(\phi)$ and $M(\phi')$ be the corresponding 3-manifold with open book decomposition $(S_{g,r}, \phi)$ and $(S_{g,r-1}, \phi')$ respectively. Then there is a natural cobordism W from $M(\phi)$ to $M(\phi')$ obtained by attaching a page 0-framed 2-handle along the binding component B in $M(\phi)$ corresponding. Alternatively, we can also think X as a cobordism from $-M(\phi')$ to $-M(\phi)$. This cobordism X is called the capping off cobordism of (P, ϕ) long B.

Next we will see how to represent the contact surgery on the open book, and how surgery cobordism related to the capping off cobordism.

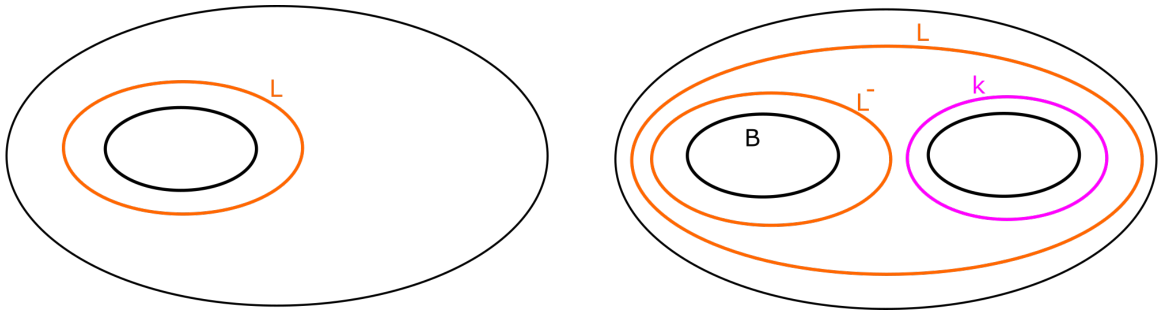

Figure 2-6: The left diagram describes the open book with Legendrian L lie on it and parallel to some binding, and the right one is the stabilization of the left one (we do right hand twist along k), where L^- now is parallel to some binding B. Then after we do $n-1$ right hand twists along L^- and 1 left hand twist along L we obtain an open book (P', ϕ') for $(Y_n(L), \xi_n^-(L))$.

Let L be a Legendrian knot in (Y, ξ), then we can always find an open book (P, ϕ) that is compatible with (Y, ξ) and contains L as a homologically nontrivial curve on the page with page framing equal to contact framing [20, Proposition 2.4]. We can first get an open book that is compatible with $(Y_n(L), \xi_n^-(L))$ in the following way. First observe that by replacing L and (P, ϕ) by a stabilization if necessary (as in [3, Lemma 6.5] or in [22, Section 3]), we can assume that L is parallel to a boundary component of P (by Proposition 2.3.6 this does not lose generality).

We then stabilize the open book again such that L^- also lies on the page, and again by the observation in [3, Lemma 6.5] we may further assume L^- is parallel to a binding component B (though L is no longer boundary parallel; see figure 2-6 for the description of the stabilization).

We denote the stabilized open book $(P', k \circ \phi)$, where here and below we will use the same symbol for a simple closed curve on the page and the right-handed Dehn twist along that curve. By Theorem 2.3.2 and the correspondence between surgery and Dehn twists, we conclude that $(P', \phi') = (P', (L^-)^{n-1} \circ (L)^{-1} \circ k \circ \phi)$ is an open book compatible with $(Y_n(L), \xi_n^-(L))$.

Now we are able to describe the important theorem that relates contact $+n$ surgery cobordism and capping off cobordism.

Theorem 2.4.6 ([21] Proposition 4.1)**.** *In the above setting let B_L be the binding of (P', ϕ') that corresponds to B in $(P', k \circ \phi)$. Then*

- *Capping off* B_L *gives us back* (P, ϕ)

- *Let* $X : Y_n(L) \to Y$ *be the cobordism corresponding to capping off* B_L*, and let* $W_{L,n} : Y \to Y_n(L)$ *be the topological cobordism obtained by attaching a* 4*-dimensional* $2-$*handle along* L *with framing* $tb(L)+n$*. Then,* $X = -W_{L,n}$*, i.e.* X *is obtained from* $W_{L,n}$ *by viewing it upside–down and reversing its orientation.*

Chapter 3

Background on knot Floer homology and the LOSS invariant

We will first review some necessary background on knot Floer homology (see [25] [27] [28] for detail)

3.1 Knot Floer Homology

We use the same notation and construction as in [25]. A doubly pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ consists of the following information. Σ is an oriented genus g surface, $\alpha = \{\alpha_1, ..., \alpha_g\}$ is a g-tuple of disjoint homologically linearly independent circles on Σ, $\beta = \{\beta_1, ...\beta_g\}$ is another g-tuple of circles on Σ similar to α, and z, w are two points on the complement of the α and β curves. Such a diagram gives rise to a 3-manifold Y in a standard way, by thinking of the α and β circles as determining (the compressing disks in) handlebodies H_α and H_β with $\partial H_\alpha = -\partial H_\beta = \Sigma$, and setting $Y = H_\alpha \cup_\Sigma H_\beta$. We call (Σ, α, β) the Heegaard diagram representing Y, and it can be shown any closed oriented 3-manifold admits a Heegaard diagram.

Given an oriented null-homologous knot K in some three manifold Y, one can construct a doubly pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ describing (Y, K) in the following sense. (Σ, α, β) is a Heegaard diagram representing Y. Moreover if we connect z to w by an embedded arc missing the α circles and pushed a little bit into the α handlebody, and connect w to z by another embedded arc missing the β circles and pushed into the β handlebody, then the oriented closed curve given by the union of those two arcs is exactly the knot K in Y. We say such a doubly pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ is compatible with (Y, K), and it can be shown that every pair (Y, K) admits a compatible doubly pointed Heegaard diagram [25].

We can further associate a chain complex CFK^- to a doubly pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ as follows. Assuming the α and β curves intersect transversely we consider the two tori

$$\mathbb{T}_\alpha = \alpha_1 \times \alpha_2 \times ... \times \alpha_g, \quad \mathbb{T}_\beta = \beta_1 \times \beta_2 \times ... \times \beta_g$$

in the g^{th} symmetric power $Sym^g(\Sigma)$. The chain complex CFK^- is the free $\mathbb{F}[U]$-module generated by the intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The differential ∂^- is defined as following:

$$\partial^- \mathbf{x} = \sum_{\{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta\}} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \mu(\phi)=1, n_z(\phi)=0\}} \#\widehat{\mathfrak{M}}(\phi) \cdot U^{n_w(\phi)} \cdot \mathbf{y}$$

where $\pi_2(\mathbf{x}, \mathbf{y})$ is the set of homotopy class of disk connecting $\mathbf{x}$ to $\mathbf{y}$; $\mu(\phi)$ is the expected dimension of the moduli space $\mathfrak{M}(\phi)$ of holomorphic disks in the homotopy class ϕ, and $\widehat{\mathfrak{M}(\phi)}$ is the expected dimension of the moduli space $\mathfrak{M}(\phi)$ modulo out the $\mathbb{R}$ action; $n_z(\phi)$ and $n_w(\phi)$ are the algebraic intersection numbers between ϕ and $\{z\} \times Sym^{g-1}(\Sigma)$ and $\{w\} \times Sym^{g-1}(\Sigma)$, respectively. The knot Floer homology groups HFK^- and $\widehat{HFK}$ are the homology of the complexes CFK^- and $\widehat{CFK}$ respectively, where $\widehat{CFK}$ is the same as CFK^- except by specializing $U = 0$. Under suitable admissible conditions for the doubly pointed Heegaard diagram [27] [25], these homology groups are invariants of the smooth knot type K in Y.

When the knot is null-homologous these groups are bi-graded and can be decomposed as follows:

$$HFK^-(Y, K) = \bigoplus_{d \in \mathbb{Q}, \mathfrak{s} \in Spin^c(Y,K)} HFK_d^-(Y, K, \mathfrak{s}),$$

where d is the Maslov grading and $\mathfrak{s}$, which run through relative $Spin^c$ structures, is called the Alexander grading, and the Euler characteristic class of the $\widehat{HFK}$ is the Alexander polynomial of K.

3.2 Maps induced by surgery

Let Y be a 3-manifold and K be a framed knot in Y with framing f (i.e. a trivialization of normal bundle), and denote $Y_f(K)$ to be the 3 manifold obtained from Y by surgery along K with framing f. Then there exists a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$ that is "compatible" with (or "subordinate" to) the cobordism induced by surgery, in particular (Σ, α, β) describes Y, (Σ, α, γ) describes Y_f, and (Σ, β, γ) is a Heegaard diagram for a connected sum of copies of $S^1 \times S^2$, and furthermore we can explicitly relate the framed knot (K, f) to such Heegaard triple (see [28] Section 4 for details). This Heegaard triple induces a well defined map from $HF^-(Y)$ to $HF^-(Y_f(k))$, and a similar construction works for knot Floer homology [25], as we now outline. Assume L is a homologically trivial knot and assume the induced knot L' in $Y_f(K)$ is also homologically trivial. Then there exists a doubly pointed Heegaard triple $(\Sigma, \alpha, \beta, \gamma, w, z)$ describing the surgery cobordism and giving rise to a map

$$F_{K(f),\mathfrak{s}} : HFK^-(Y, L) \to HFK^-(Y_f(K), L') \tag{3.2.1}$$

which is induced by a chain map

$$f_{K(f),\mathfrak{s}} : CFK^-(Y, L) \to CFK^-(Y_f(K), L'). \tag{3.2.2}$$

The latter is defined for a compatible doubly pointed Heegaard triple $(\Sigma, \alpha, \beta, \gamma, w, z)$ by the formula

$$f_{K(f),\mathfrak{s}}(\mathbf{x}) = \sum_{\{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma\}} \sum_{\psi} \#\mathfrak{M}(\psi) \cdot U^{n_w(\phi)} \cdot \mathbf{y}$$

where $\mathfrak{s}$ is a $spin^c$ structure on the cobordism. The inner sum is over homotopy classes $\psi \in \pi_2(\mathbf{x}, \Theta_{\beta,\gamma}, \mathbf{y})$ of Whitney triangles connecting $\mathbf{x}$, $\mathbf{y}$, and a representative

$\Theta_{\beta,\gamma}$ of the top dimensional class in $HFK^-(\Sigma, \beta, \gamma, w, z)$, and satisfying $s_w(\psi) = \mathfrak{s}$, $n_z(\psi) = 0$, and $\mu(\psi) = 0$, where the latter is the expected dimension of the moduli space $\mathfrak{M}(\psi)$ of holomorphic triangles in homotopy class ψ.

We recall that a Whitney triangle connecting $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{r} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ is a map

$$u : \Delta \to Sym^n(\Sigma)$$

where Δ is an oriented 2-simplex with vertices v_α, v_β, and v_γ labeled clockwise, and e_α, e_β, and e_γ are the edges opposite to v_α, v_β, and v_γ respectively. Moreover we want the boundary conditions satisfy that $u(v_\alpha) = \mathbf{r}$, $u(v_\beta) = \mathbf{y}$ and $u(v_\gamma) = \mathbf{x}$, and $u(e_\alpha) \subset \mathbb{T}_\alpha$, $u(e_\beta) \subset \mathbb{T}_\beta$ and $u(e_\gamma) \subset \mathbb{T}_\gamma$. In particular if we start at any vertex of Δ and go clockwise we should travel along the α, β, and γ curves in cyclic order. See Figure 3-1 for a schematic picture.

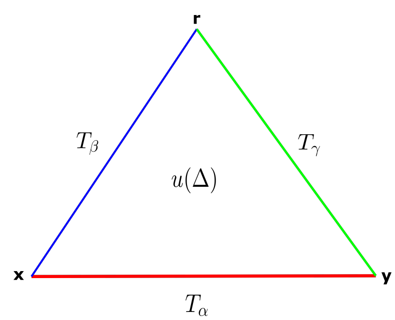

Figure 3-1: Schematic Whitney triangle for $(\Sigma, \alpha, \beta, \gamma)$

3.3 LOSS invariant for Legendrian knots

In this section we review the construction for LOSS invariant [20]. As we describe below, given an oriented null-homologous Legendrian knot L in some contact three manifold (Y, ξ), one can find a doubly pointed Heegaard diagram $(\Sigma, \beta, \alpha, w, z)$ that is compatible with $(-Y, L)$ and associate a cycle in CFK^-, giving rise to an element in $HFK^-(-Y, L)$.

To begin, we first find an open book (P, ϕ) that is compatible with (Y, ξ) and contains L as a homologically nontrivial curve on the page [20, Proposition 2.4] such that the page framing is equal to the contact framing.

Then we choose a family of properly embedded arcs $\{\mathbf{a_i}\}$ as basis for P, meaning that if we cut P along $\{\mathbf{a_i}\}$ we get a disk. Moreover we can choose the basis $\{\mathbf{a_i}\}$ such that L, considered as lying on $P = P_{+1}$, only intersects a_1 transversely at one point and does not intersect with other a_i for $i \neq 1$. We now construct a doubly pointed Heegaard diagram $(\Sigma, \beta, \alpha, w, z)$ that is compatible with $(-Y, L)$ using $(P, \phi, \{\mathbf{a_i}\})$.

We first form the Heegaard surface Σ as the union of two pages, $P_{+1} \cup -P_{-1}$. For the α and β curves we start with the basis $\{\mathbf{a_i}\}$ lying on P_{+1}, and let b_i be a push off of a_i for all i on P_{+1}, such that a_i and b_i intersect transversely at one point on P_{+1} for each i. In particular the boundary points of b_i are obtained from those of a_i by pushing along ∂P_{+1} in the direction determined by the orientation. Now we let $\alpha_i = a_i \cup \overline{a_i}$ and $\beta_i = b_i \cup \overline{\phi(b_i)}$ for all i, where $\overline{a_i}$ is the image of a_i under the identity map on the opposite page $-P_{-1}$ and $\overline{\phi(b_i)}$ is the image of b_i under the monodromy map ϕ on $-P_{-1}$. Finally we place the base points w, z on P_{+1} such that z is "outside" the thin strips between a_i and b_i for all i, and w is in between a_1 and b_1. Note that there are two possibilities for the placement of w; we choose the one compatible with the orientation of L. Let $\mathbf{x} = (x_1, x_2, ..., x_g) \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ where $x_i = a_i \cap b_i$. Now we change the orientation of Y and consider the diagram $(\Sigma, \beta, \alpha, w, z)$, which is compatible with $(-Y, L)$. We view $\mathbf{x}$ as an element in $CFK^-(-Y, L)$. It was shown in [20] that $\mathbf{x}$ is a cycle, and the homology class of $\mathbf{x}$, written $\mathfrak{L}(L) \in HFK^-(-Y, L)$, is an invariant of the oriented Legendrian knot L with values in the graded module $HFK^-(-Y, L)$, modulo its graded automorphisms. The construction for $\hat{\mathfrak{L}}(L) \in \widehat{HFK}(-Y, L)$ is the same, simply considering $\mathbf{x}$ as a cycle in $\widehat{CFK}(-Y, L)$.

Here is an important property of the LOSS invariant that we will use.

Theorem 3.3.1 ([20])**.** *Suppose that L is an oriented Legendrian knot and denote the negative and positive stabilizations of L as L^- and L^+. Then, $\mathfrak{L}(L^-) = \mathfrak{L}(L)$ and $\mathfrak{L}(L^+) = U \cdot \mathfrak{L}(L)$. Similarly $\hat{\mathfrak{L}}(L^-) = \hat{\mathfrak{L}}(L)$ and $\hat{\mathfrak{L}}(L^+) = U \cdot \hat{\mathfrak{L}}(L) = 0$.*

In particular, since both $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ are unchanged under negative stabilization they are also invariants for transverse knots. Thus for a transverse knot T we define the

LOSS invariant $\mathfrak{T}, \hat{\mathfrak{T}}$ of T to be the LOSS invariant $\mathfrak{L}, \hat{\mathfrak{L}}$ of a Legendrian approximation of T.

Chapter 4

Naturality of LOSS invariant

In this chapter we will prove Theorem 1.2.1 and Theorem 1.2.8.

4.1 Naturality of LOSS invariant under capping off

As we have seen in Section 2.4.1, the contact $+n$ surgery is actually a special case of capping off, so instead of directly proving the $+n$ situation we will first prove the naturality result for the LOSS invariant under capping off cobordisms. In order to precisely state the proposition we need to first introduce a new definition.

Definition 4.1.1. Let (M, ξ) be a contact 3-manifold, and (P, ϕ) an open book decomposition with at least 2 binding components supporting (M, ξ). Consider L a null-homologous oriented Legendrian knot, B a binding component of (P, ϕ), and $\{\mathbf{a_i}\}$ a basis for P. We say a triple $(L, B, \{\mathbf{a_i}\})$ is adapted to (P, ϕ) if the following hold.

1. L is sitting on the page P and is parallel to some binding component T other than B
2. Up to reordering $\{\mathbf{a_i}\}$, L intersects a_1 transversely at one point and does not intersect other a_i for $i \neq 1$
3. Up to reordering $\{\mathbf{a_i}\}$, B intersects a_2 at exactly one point and does not intersect other a_i for $i \neq 2$.

We say (L, B) is adapted to (P, ϕ) if condition 1 holds. See Figure 4-1 for an example of an adapted triple. It's easy to see that if (L, B) is adapted to (P, ϕ) then (maybe after further stabilization of the open book) we can always find basis $\{\mathbf{a_i}\}$ such that $(L, B, \{\mathbf{a_i}\})$ is adapted to (P, ϕ)

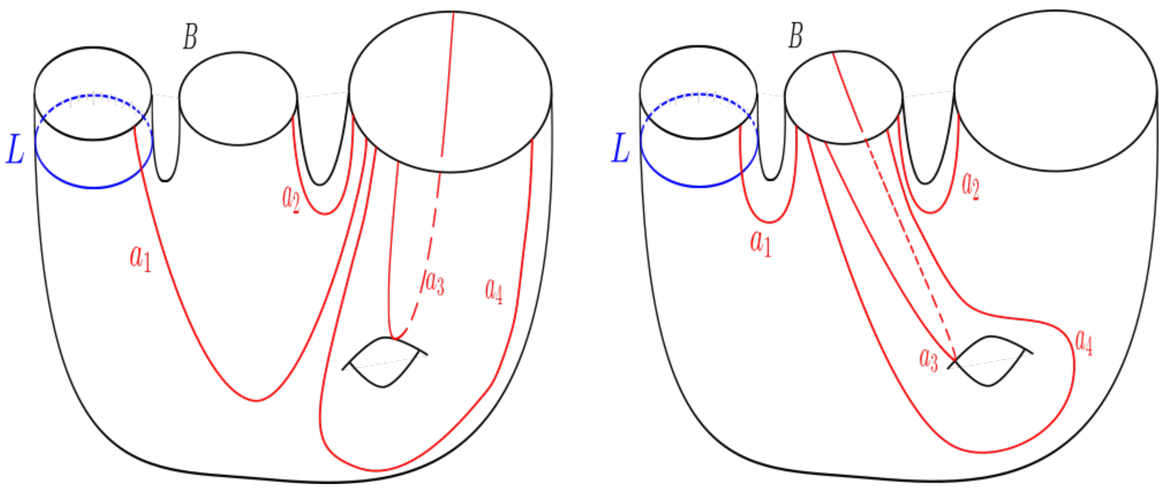

Figure 4-1: The left diagram is an adapted triple $(L, B, \{\mathbf{a_i}\})$ and the right one is not; one can transform the $\{a_i\}$ from one to the other by arcslides.

Similar to what we saw in section 3.3, given an adapted $(L, B, \{\mathbf{a_i}\})$ we can associate a doubly pointed Heegaard triple $(\Sigma, \alpha, \gamma, \beta, z, w)$ as follows.

Let Σ be the Heegaard surface that is the union of two pages $P_{+1} \cup -P_{-1}$, and consider the basis $\{\mathbf{a_i}\}$ as lying on P_{+1}. Let c_i be a push off of a_i for all i, and for all $i \neq 2$ let b_i be a further push off of c_i. When $i = 2$ we let b_2 be a parallel push of the binding component B on the page P_{+1}. We require that the push offs satisfy that each a_i, b_i, and c_i intersect transversely at one point for all i (as before, we arrange that the endpoints of the pushoff slide in the direction of the induced orientation of the boundary of P_{+1}). In particular, for all i, there is a "small triangle" formed by the arcs a_i, c_i, b_i, see Figure 4-2.

For the α and γ curves in the Heegaard diagram we let $\alpha_i = a_i \cup \overline{a_i}$ and $\gamma_i = c_i \cup \overline{\phi(c_i)}$ for all i. For the β curves let $\beta_i = b_i \cup \overline{\phi(b_i)}$ for $i \neq 2$, and $\beta_2 = b_2$. Finally we place the base points w, z such that they specify the Legendrian knot L the same way as we define for LOSS invariant.

Thus $(\Sigma, \alpha, \gamma, z, w)$ is a diagram for (M, L), while $(\Sigma, \alpha, \beta, z, w)$ describes the induced knot L' lying in the contact manifold (M', ξ') obtained by capping off the binding component B (this is clear after destabilizing the diagram using the single intersection between α_2 and β_2). Furthermore, one can see (as in [4]) that $(\Sigma, \alpha, \gamma, \beta, z, w)$ describes the capping off cobordism map from M to M' and send L to L'. When we turn the cobordism upside down $(\Sigma, \gamma, \beta, \alpha)$ describes the cobordism map from $-M'$ to $-M$ (As in [4],[22], $(\Sigma, \gamma, \beta, \alpha)$ is left-subordinate to this cobordism; see [28] sections 4 and 5). After verifying admissibility conditions, this means that the doubly pointed Heegaard triple $(\Sigma, \gamma, \beta, \alpha, w, z)$ (Figure 4-2) can be used to calculate the map

$$F_{B,\mathfrak{s}} : HFK^{-}(-M', L') \to HFK^{-}(-M, L) \tag{4.1.2}$$

Let $\mathbf{x} = \{x_1, x_2, ...x_g\}$, $\mathbf{\Theta} = \{\theta_1, \theta_2, ...\theta_g\}$ and $\mathbf{y} = \{y_1, y_2, ...y_g\}$ where $x_i = a_i \cap b_i$, $\theta_i = b_i \cap c_i$, and $y_i = a_i \cap c_i$ on P_{+1}. If we denote by Δ_i the small triangle connecting x_i, θ_i, y_i then the $spin^c$ structure $\mathfrak{s}$ we wish to use in equation (4.1) is described by the Whitney triangle $\psi \in \pi_2(\mathbf{x}, \mathbf{\Theta}, \mathbf{y})$ (i.e. $\mathfrak{s}_z(\psi) = \mathfrak{s}$) where the domain $\mathbb{D}(\psi)$ is the sum of all Δ_i ([27] Proposition 8.4). Then we have the following key proposition.

Proposition 4.1.3. *Given adapted $(L, B, \{\mathbf{a_i}\})$ and map $F_{B,\mathfrak{s}}$ in the above setting. If we further assume M' is a rational homology sphere and L' is null-homologous in M' then we have*

$$F_{B,\mathfrak{s}}(\mathfrak{L}(M', \xi', L')) = \mathfrak{L}(M, \xi, L) \tag{4.1.4}$$

We remark that by the choices we made in the pushoffs of a_i, the intersections x_i, y_i, and θ_i appear in clockwise order around Δ_i for each i.

To prove the above Proposition we divide it into several lemmas.

Lemma 4.1.5. *(cf. [4, Lemma 2.2]) The doubly pointed diagram $(\Sigma, \gamma, \beta, \alpha, w, z)$ is weakly admissible, in the sense that any non-trivial triply-periodic domain has both positive and negative multiplicities.*

Observe that if we ignore the w base point then the local picture near Δ_i are all the same except for $i = 2$. (See figure 4-3, for local description of Δ_i and Δ_2.)

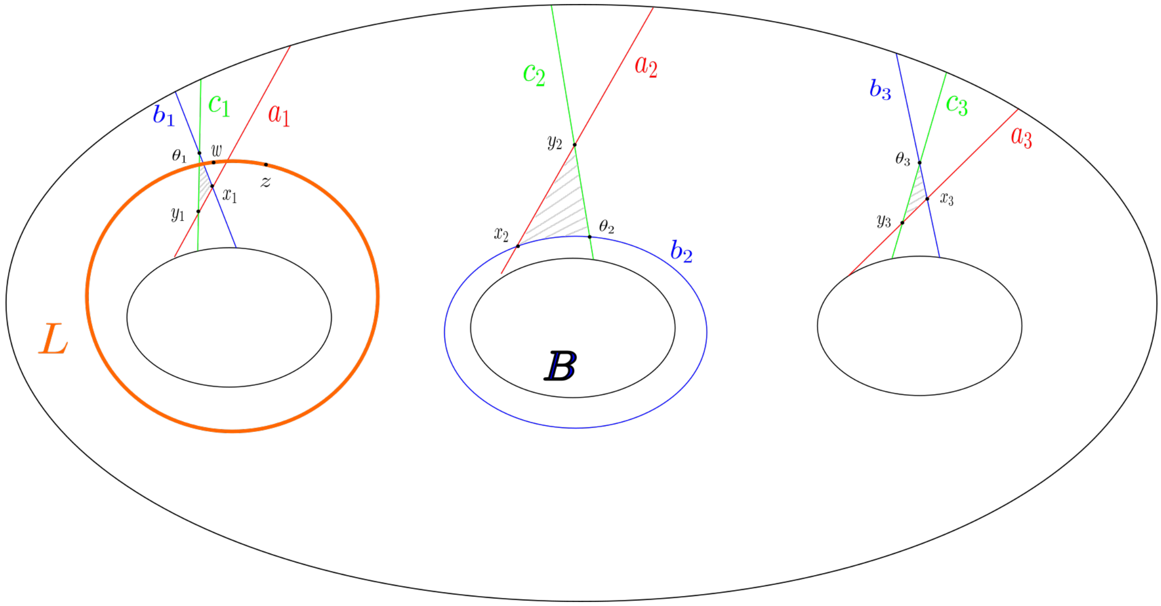

Figure 4-2: Since all of what we care are on the P_{+1} page, we can just draw things on P_{+1} to capture all the information instead of drawing the whole doubly pointed Heegaard triple. (The black circles are binding, red curves are a_i (parts of the α_i), blue curves are b_i (parts of the β_i), green curves are c_i (parts of the γ_i). The *spin*^{c} structure $\mathfrak{s}$ is represented by the small shaded triangle. There might be genus but it's not shown on the picture.)

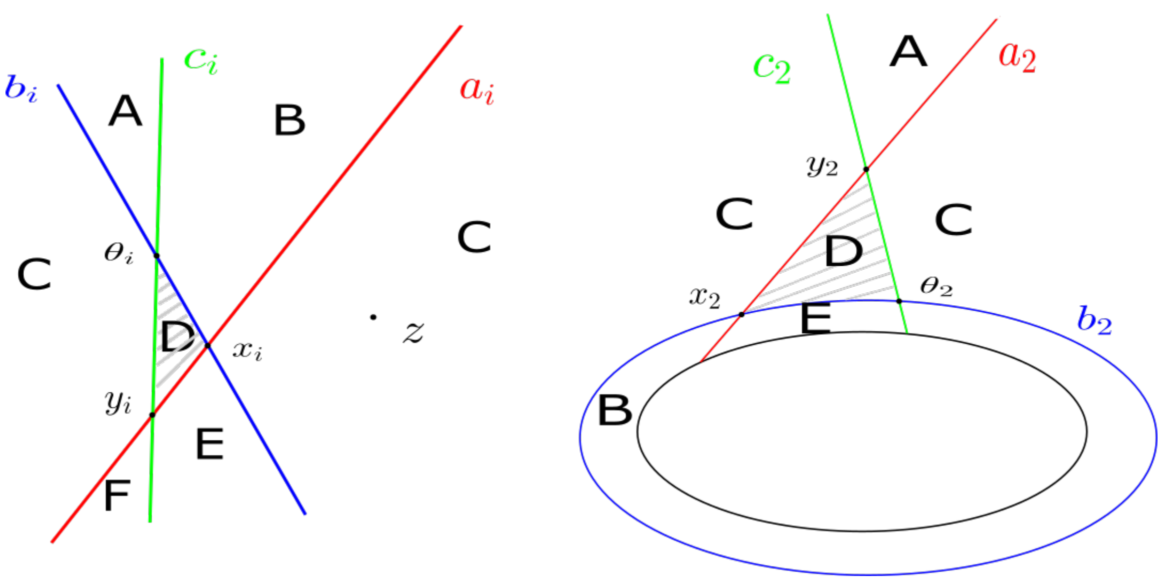

Figure 4-3: On the left is the local picture for Δ_i for $i \neq 2$, and on the right is the local picture for Δ_2. A,B,C,D,E,F are the letters used to label the regions in local picture.

Proof. We first analyze the local picture for Δ_i where $i \neq 2$. Let Q be a triply-periodic domain whose multiplicities in the regions A, B, C, D, E and F are given by the integers a, b, c, d, e and f, respectively. Since ∂Q (rather, the portion of ∂Q lying on the α circles) consists of full α arcs, we must have

$$b - c = d - e = c - f.$$

Similarly ∂Q also contains only full β arcs so we have

$$c - a = d - b = e - c.$$

Note that the region C contains base point z so $c = 0$, which implies Q has both positive and negative multiplicity unless

$$a = b = c = d = e = f = 0.$$

Since for each Δ_i $i \neq 2$ the local pictures are the same, this tells us that if Q has only positive or only negative multiplicities then Q contains no $\alpha_i, \beta_i, \gamma_i$ as boundary for $i \neq 2$. So the only possibility for Q to be nonzero is near Δ_2.

For the region around Δ_2 we again label the regions A, B, C, D, and E as in Figure 4-3, and the multiplicities are given by the integers a, b, c, d, and e respectively. Again since ∂Q contains only full α curves we must have

$$a - c = c - d = b - e,$$

and since C is the region containing base point z we have $c = 0$. Hence if Q has only positive or only negative multiplicities then $a = d = 0$, and $b = e$. If $b = e \neq 0$ we infer ∂Q contains only the curve β_2, however since β_2 is not null-homologous in Σ it can't bound a 2 chain by itself. So $b = e = 0$ which shows the diagram is weakly admissible.

□

Lemma 4.1.6. *(cf. [4, Proposition 2.3]) In the above setting, let* $\mathbf{y}' = (\mathbf{y_1'}, \mathbf{y_2'}, ..., \mathbf{y_g'}) \in \mathbb{T}_\gamma \cap \mathbb{T}_\alpha$ *be an intersection point and* $\psi' \in \pi_2(\mathbf{x}, \boldsymbol{\Theta}, \mathbf{y}')$ *a Whitney triangle with only nonnegative local multiplicities. If* $n_z(\psi') = 0$ *and* $\mathfrak{s}_z(\psi') = \mathfrak{s}$*, then* $\mathbf{y}' = \mathbf{y}$*, and* $\psi' = \psi$*.*

Proof. We want to show that the domain of ψ', $\mathbb{D}(\psi')$ is the same as that of ψ, i.e. that $\mathbb{D}(\psi') = \mathbb{D}(\psi) = \Delta_1 + \Delta_2 + ... + \Delta_g$. As what we did above, we again look at what happens locally near Δ_i $(i \neq 2)$ and Δ_2.

First we look around Δ_i $(i \neq 2)$. Let a, b, c, d, e and f be the multiplicities of $\mathbb{D}(\psi')$ at A, B, C, D, E and F. Since $\mathbb{D}(\psi')$ has corners at x_i and θ_i, we have

$$a + d = b + c + 1, \; d + c = b + e + 1 \tag{1}$$

Since $c = 0$, equations (1) imply $a = -e$, and because the domain only contains nonnegative multiplicities $a = e = 0$. Therefore (1) becomes

$$d = b + 1 \tag{2}$$

Now if $y_i' \neq y_i$ for some $i \neq 2$ it implies $d + f = 0$, so $d = f = 0$, but when we put $d = 0$ in (2) we have $b = -1$ which is a contradiction. So $y_i = y_i'$, which implies $d + f = 1$ (since $c = e = 0$). If $d = 0$ then combining with (2) again we get $b = -1$, a contradiction. So we must have $d = 1$ and $f = 0$, which means that altogether $d = 1$ and $a = b = c = e = f = 0$.

Because the above argument works for all $i \neq 2$, we conclude that $\mathbb{D}(\psi')$ is locally just Δ_i for $i \neq 2$, in other words $\mathbb{D}(\psi') = \Delta_1 + \Delta_2' + \Delta_3 + ... + \Delta_g$, where Δ_2' is a region missing base point z and whose oriented boundary consists of arcs along β_2 from θ_2 to x_2; along α_2 from x_2 to y_2'; and along γ_2 from y_2' to θ_2. So we are left to show $y_2' = y_2$, and $\Delta_2' = \Delta_2$.

Since $\mathfrak{s}_z(\psi') = \mathfrak{s}(\psi) = \mathfrak{s}$, by [27, Proposition 8.5] we have

$$\mathbb{D}(\psi') - \mathbb{D}(\psi) = \mathbb{D}(\phi_1) + \mathbb{D}(\phi_2) + \mathbb{D}(\phi_3)$$

where ϕ_1, ϕ_2, and ϕ_3 are in $\pi_2(\mathbf{x},\mathbf{x})$, $\pi_2(\mathbf{\Theta},\mathbf{\Theta})$, and $\pi_2(\mathbf{y},\mathbf{y}')$ respectively. We want to show $\mathbb{D}(\psi') - \mathbb{D}(\psi) = 0$.

First since $-M'$, which has Heegaard description (Σ, β, α), is a rational homology sphere, we have $\pi_2(\mathbf{x},\mathbf{x}) = 0$ and therefore we can suppose $\mathbb{D}(\phi_1) = 0$ (strictly, $\pi_2(\mathbf{x},\mathbf{x})$ always contains a copy of $\mathbb{Z}$ corresponding to multiples of the Heegaard surface, but these are not relevant here because of positivity of multiplicities and the condition that $n_z(\psi) = n_z(\psi') = 0$). Moreover since $\mathbb{D}(\psi') - \mathbb{D}(\psi) = \Delta_2' - \Delta_2$ and $\pi_2(\mathbf{x},\mathbf{x}) = 0$, if $\mathbb{D}(\phi_2) \neq 0$ the (β_2 portion of the) boundary of $\mathbb{D}(\phi_2)$ can contain only multiples of β_2. However β_2 is homologically independent from all linear combinations of β_i for $i \neq 2$ and γ_j for all j, and we conclude $\mathbb{D}(\phi_2) = 0$.

Therefore

$$\mathbb{D}(\psi') - \mathbb{D}(\psi) = \Delta_2' - \Delta_2 = \mathbb{D}(\phi_3)$$

where $\mathbb{D}(\phi_3)$ is a domain not containing z, whose oriented boundary consists of arcs along α_2 from y_2 to y_2' and arcs along γ_2 from y_2' to y_2. Since Δ_2 has multiplicities 1 and 0 in the regions D and E respectively, the multiplicities of Δ_2' at D and E must satisfy

$$d - 1 = e. \tag{3}$$

At the same time, the boundary conditions at x_2 and θ_2 tell us

$$d + b - 1 = c + e.$$

Combining these two we get $b = c = 0$.

Now suppose $y_2 \neq y_2'$. The boundary conditions then say $a + d = c + c = 0$, so that $a = d = 0$. Then when we return to (3) we get $e = -1$, which is a contradiction—so $y_2 = y_2'$. In this case the boundary constraint tells us

$$a + d = 1.$$

If we combine this with (3) it follows $a = 0 = c = e = b$ and $d = 1$. Hence $\Delta_2' = \Delta_2$, which implies $\mathbb{D}(\psi') - \mathbb{D}(\psi) = 0$.

□

Now we are ready to show Proposition 4.1.3.

Proof of Proposition 4.1.3. Lemma 4.1.5 says the map $F_{B,\mathfrak{s}}$ can be computed from the Heegaard diagram now under consideration. (Strictly, weak admissibility suffices to compute the homomorphism in the hat theory, while the minus theory requires strong admissibility for the *spin*^{c} structure under consideration. Weak and strong admissibility coincide if the *spin*^{c} structure is torsion on each boundary component. Alternatively, weak admissibility is also sufficient to define maps in the minus theory if we work over the power series ring $\mathbb{F}[[U]]$, so we work in that setting in the most general case.) Since the small triangle ψ has a unique holomorphic representative, and Lemma 4.1.6 implies that the small triangle is the only one contributing to the map $F_{B,\mathfrak{s}}$, we have $F_{B,\mathfrak{s}}(\mathbf{x}) = \mathbf{y}$. We only left to show $\mathbf{x} = \mathfrak{L}(M', \xi', L')$, and $\mathbf{y} = \mathfrak{L}(M, \xi, L)$. The latter is clear by the definition of LOSS invariant .

For $\mathfrak{L}(M', \xi', L')$, denote (P_B, ϕ_B) the corresponding open book after capping off binding B, and by abuse of notation consider $\{a_i\}$ (for $i \neq 2$) as a basis for P_B. By definition of LOSS again we can see that $\mathfrak{L}(M', \xi', L')$ is represented by $\mathbf{x}' = (x_1, x_3, ..., x_g)$. The diagram $(\Sigma, \beta, \alpha, z, w)$ then differs from the one obtained from P_B by a stabilization of the Heegaard diagram. Then by [27, Section 10], we see under the isomorphism induced by the stabilization we map $\mathbf{x}'$ to $\mathbf{x}$. So $\mathbf{x} = \mathfrak{L}(M', \xi', L')$. □

Remark 4.1.7. Since the Legendrian knot L is a Legendrian approximation of T (L is parallel to T in the open book), the Theorem 1.2.8 follows.

4.2 Naturality of LOSS invariant under $+n$ contact surgery

Now we are ready to prove Theorem 1.2.1.

Proof of Theorem 1.1. We first choose an arbitrary open book (P, ϕ) supporting (Y, ξ) and having L and S on the page P, where as we saw in section 2.4.1 we may further assume S is parallel to some binding of P. Again by [3, Lemma 6.5] we can stabilize the open book such that S^-, the negative stabilization of S, is parallel to a binding component B, and L^- is also parallel to some other binding component T. We again call the stabilized open book (P, ϕ).

Now by Theorem 2.4.6 the smooth cobordism from $-Y$ to $-Y_n(S)$ induced by contact n surgery (smooth $tb(S) + n$) is the same as a capping off cobordism from $Y_n(S)$ to Y viewed upside down, where we cap off a binding component $B_S \subset Y_n(S)$ as we saw in section 2.4.1. (Note here that S is playing the role of L in section 2.4.1 and the above parts of this section.)

In other words we have an open book (P_S, ϕ_S) for $(Y_n(S), \xi_n^-(S))$ such that capping off B_S gives us back (P, ϕ), and such that the knot induced by $(L^-)_S$ is L^-. Then (possibly after further stabilization of (P_S, ϕ_S)) we choose a basis $\{\mathbf{a_i}\}$ such that $((L^-)_S, B_S, \{\mathbf{a_i}\})$ is adapted to (P_S, ϕ_S), and thus by Proposition 4.1.3

$$F_{B_S,\mathfrak{s}}(\mathfrak{L}(-Y, \xi, L^-)) = \mathfrak{L}(-Y_n(S), \xi_n^-(S), (L^-)_S).$$

By equivalence of contact surgery and capping off we have

$$F_{S,\mathfrak{s}}(\mathfrak{L}(-Y, \xi, L^-)) = \mathfrak{L}(-Y_n(S), \xi_n^-(S), (L^-)_S)$$

Finally because $(L^-)_S = (L_S)^-$, and the LOSS invariant is invariant under negative stabilization (Theorem 3.3.1) we conclude that

$$F_{S,\mathfrak{s}}(\mathfrak{L}(-Y, \xi, L)) = \mathfrak{L}(-Y_n(S), \xi_n^-(S), L_S)$$

□

Chapter 5

Application to Legendrian and transverse non-simple knot

One interesting application of the main theorem is to give many more examples of Legendrian and transversely non-simple knots which are distinguished by their LOSS invariants.

5.1 Adding twist preserve the distinction of LOSS invariant

In this section we will prove the Theorem 1.2.15 which can be summarised as follows: if we start with two Legendrian knots with different LOSS invariants, then, after adding twist, the new Legendrian knots we get also have different LOSS invariants.

To prove Theorem 1.2.15 we need the following lemma.

Lemma 5.1.1. *Let L be an oriented Legendrian knot in (Y, ξ), and for $i = 1, 2, ...n$ let e_i be arcs of L such that they are horizontal parallel with the same orientation inside some Darboux ball B. Moreover let S be an oriented max tb unknot in B that links each e_i positively once (so the linking number between S and L is $+n$). Then after doing $+2$ contact surgery on S, with the choice of stabilization being negative, the resulting contact manifold is contactomorphic to (Y, ξ), but the resulting e_i's are* *parallel Legendrian pushoffs of a negative stabilization of* e_1 *(thus, smoothly, the new strands have a full negative twist).*

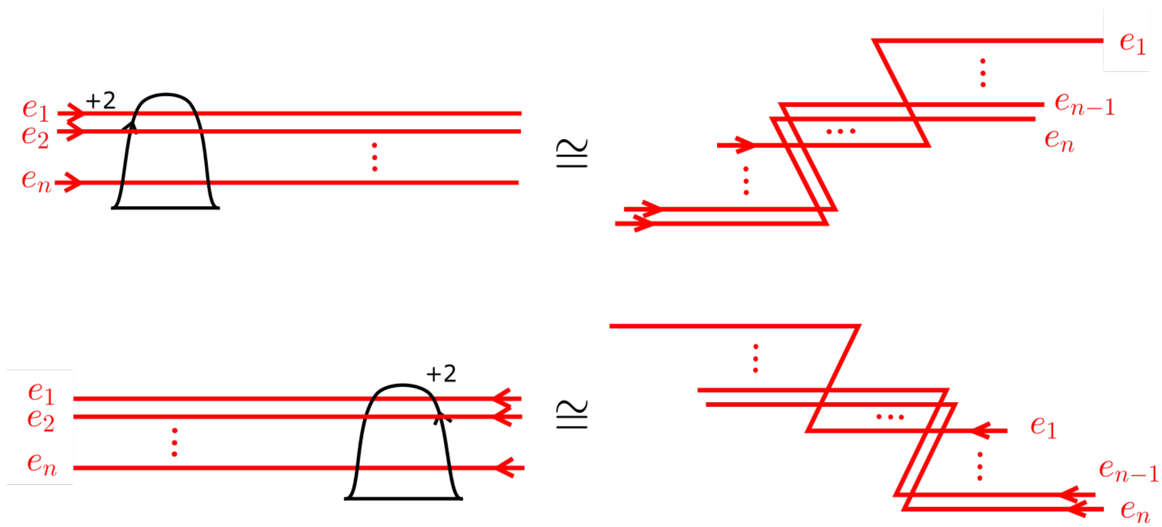

Figure 5-1: there are n red arcs e_1 to e_n

Proof. There are two possibilities of how those e_i are oriented, either from left to right or from right to left. So to prove the Lemma it's the same to show the equivalence of the pair of contact surgery diagrams in Figure 5-1.

Since the proof is symmetric with one and the other we will only show top case of the figure (strand orientation from left to right). Note that since the e_i are Legendrian push-offs of each other, it's enough to consider the case when $n = 1$. In Figure 5-2 we exhibit a sequence of Legendrian isotopies, contact surgery and contact handle moves to show the equivalence of the two diagrams when $n = 1$.

□

To prove Theorem 1.2.15 notice that given a compatible triple (L, σ_n, B) the new Legendrian knot L_σ we form only differs from L by a positive twist. We intend to use the above Lemma on L_σ, so we can cancel out the positive twist with a negative twist and give back L. With this idea in mind we are ready to start the proof.

Proof of Theorem 1.2.15. Since there exist a smoothly isotopy from L to L' that takes the B to B', and inside the balls we are doing the same operation to the arcs, we infer the resulting knots L_σ and $L'_{\sigma'}$ are smoothly isotopic, proving the first part.

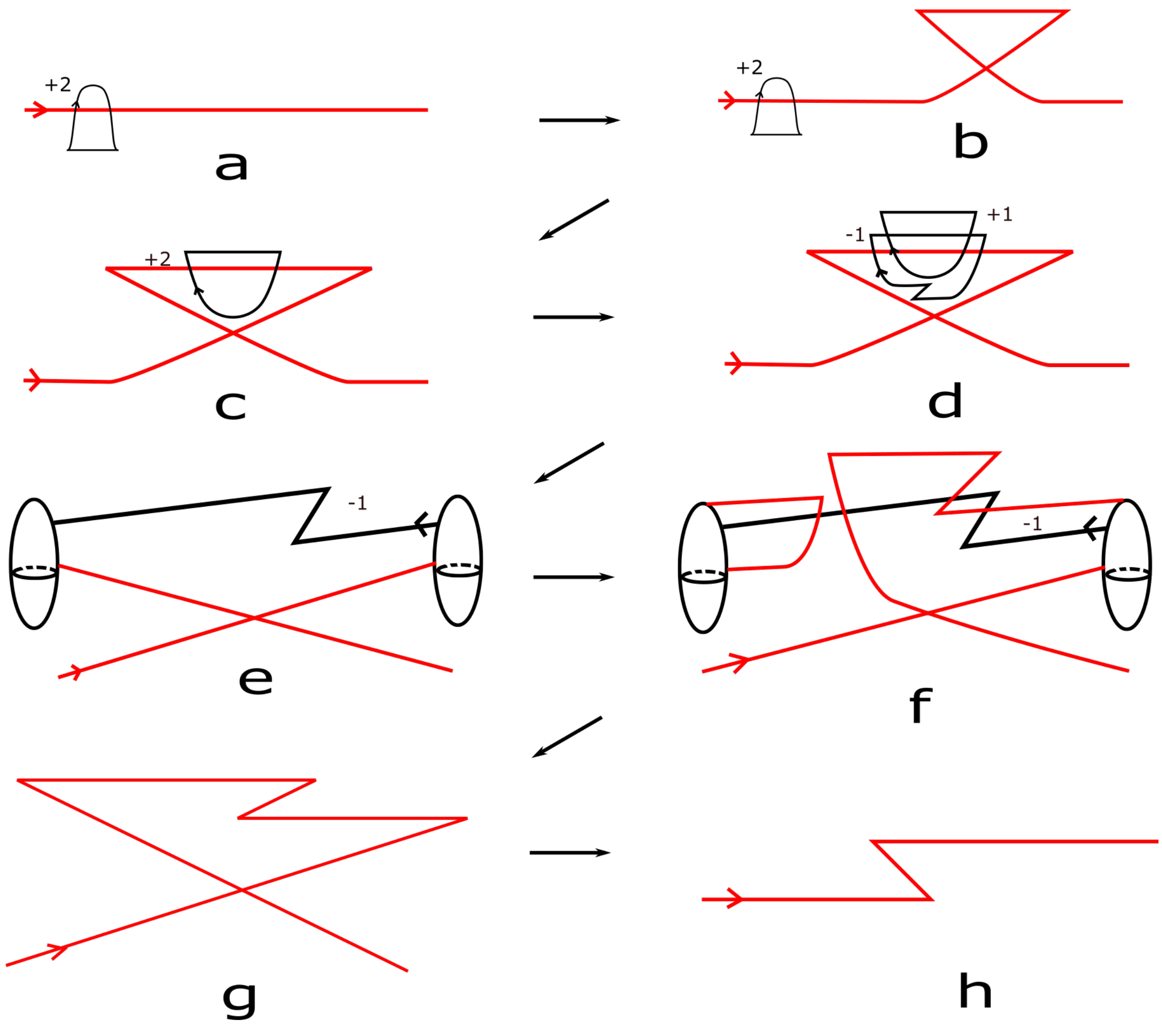

Figure 5-2: From **a** to **b** we use Legendrian Reidemeister 1 moves; from **b** to **c** we isotopy Legendrian meridian from bottom to top using [8, Figure 13-15]; from **c** to **d** we use the DGS algorithm [9] to change +2 contact surgery to +1 and −1 contact surgeries, and we use negative stabilization as the assumption; from **d** to **e** we use [8, Theorem 4] to identify surgery diagram with handle diagram; from **e** to **f** we handle slide the red curve over the −1 framed handle using [8, Proposition 1]; from **f** to **g** we cancel out the −1 framed 2 handle with the 1 handle; and last we perform a Legendrian Reidemeister move to get rid of the extra crossing and attain **h**.

Then let's consider the Darboux ball B and the new Legendrian knot L_σ. Inside the ball the arcs e_i have the same orientation and can be considered to be initially horizontal and parallel (near the left side of the ball), then they start doing a positive twist as we move from left to right. Now as in Lemma 5.1 let S be an oriented max tb unknot that links both e_i positively one time, and perform $+2$ contact surgery on S. We can think of this as happening near the horizontal parallel part of the e_i, so it looks like the top left diagram of figure 5-3. By Lemma 5.1.1 this is equivalent to the top right of figure 5-3, then after sequence of Legendrian Reidemeister moves it's not hard to see we obtain the bottom right.

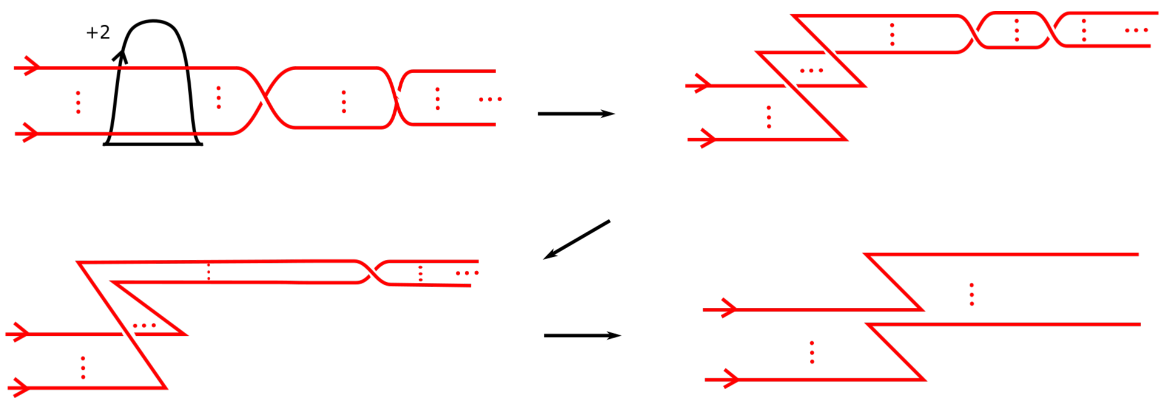

Figure 5-3: We apply the Lemma 5.1.1 on the arcs in L_σ, then we undo the twist which gives back L with n additional negative stabilization.

From the picture we can easily see that doing this $+2$ contact surgery on S transforms L_σ to the n-fold negative stabilization L^{-n} of L. Now we want to apply Theorem 1.2.1. Since L is null-homologous, by construction L_σ is also null-homologous (we can add bands to the Seifert surface of L), and since we do not change the ambient contact 3 manifold (Y, ξ) by doing $+2$ contact surgery, the map in Theorem 1.2.1 is of the form

$$F_{S,\mathfrak{s}} : HFK^-(-Y, L_\sigma) \to HFK^-(-Y, L). \tag{5.1.2}$$

Since Y is a rational homology sphere, and LOSS invariant is unchanged under negative stabilization, by Theorem 1.2.1 there exist some *spin*^{c} structure $\mathfrak{s}$ such that

$$F_{S,\mathfrak{s}}(\mathfrak{L}(L_\sigma)) = \mathfrak{L}(L^{-n}) = \mathfrak{L}(L) \tag{5.1.3}$$

We do the same thing, +2 contact surgery on S' for arcs e'_i of $L'_{\sigma'}$ inside $B_{\sigma'}$. So we also get a map

$$F_{S',\mathfrak{s}'} : HFK^-(-Y, L'_{\sigma'}) \to HFK^-(-Y, L') \tag{5.1.4}$$

such that

$$F_{S',\mathfrak{s}'}(\mathfrak{L}(L'_{\sigma'})) = \mathfrak{L}(L'^{-n}) = \mathfrak{L}(L') \tag{5.1.5}$$

By the construction of L_σ and $L'_{\sigma'}$ and the assumption about the isotopy from L to L' it's easy to see that there also exist a smooth isotopy from L_σ to $L'_{\sigma'}$ that sends S to S'. Moreover since S and S' are null-homologous (they are inside the ball) and ξ has torsion first Chern class (Y is a rational homology sphere), by Proposition 1.2.11

$$\pm\langle c_1(\mathfrak{s}), [\tilde{Z}]\rangle = rot(S) + 1$$

and

$$\pm\langle c_1(\mathfrak{s}'), [\tilde{Z}]\rangle = rot(S') + 1$$

Note that both $\mathfrak{s}$ and $\mathfrak{s}'$ restrict to the *spin*^{c} structure corresponding to ξ on both boundaries of the cobordism. This condition together with the value of $\langle c_1(\mathfrak{s}), [\tilde{Z}]\rangle$ determines a *spin*^{c} structure uniquely on the surgery cobordism. Thus, $\mathfrak{s}$ and $\mathfrak{s}'$ are either equal or conjugate to each other. If $\mathfrak{s}$ and $\mathfrak{s}'$ are conjugate to each other we have the following commutative diagram.

because $rot(S) = rot(S')$, we infer $\mathfrak{s}$ is equal to $\mathfrak{s}'$.

Let us write K and K_σ for the smooth knot types underlying L, L' and L_σ, $L'_{\sigma'}$, respectively. By the results of [19], we can consider the LOSS invariants of L and L' to lie in the same group $HFK^-(-Y, K)$, and similarly those of L_σ and $L'_{\sigma'}$ lie in

$HFK^-(-Y,K_\sigma)$. More precisely, this means that there are canonical isomorphisms

$$HFK^-(-Y,L_\sigma) \to HFK^-(-Y,K_\sigma), \qquad HFK^-(-Y,L'_{\sigma'}) \to HFK^-(-Y,K_\sigma)$$
$$HFK^-(-Y,L) \to HFK^-(-Y,K), \qquad HFK^-(-Y,L') \to HFK^-(-Y,K).$$

With these identifications in mind, we will drop the distinction between the circles S and S', as they are ambiently smoothly isotopic.

Now for any $d \in MCG(Y,K_\sigma)$, let $d(S)$ denote the induced knot and $d_S \in MCG(Y,K)$ the induced diffeomorphism after the surgery on S; moreover let d^*,d^*_S be the induced maps on knot Floer homology and $d(\mathfrak{s})$ be the induced $spin^c$ structure. Then by Theorem 8.9 and Corollary 11.17 in [18], and Theorem 1.8 in [19] we have the following commutative diagram.

$$\begin{array}{ccc} HFK^-(-Y,K_\sigma) & \xrightarrow{F_{S,\mathfrak{s}}} & HFK^-(-Y,K) \\ \downarrow d^* & & \downarrow d^*_S \\ HFK^-(-Y,K_\sigma) & \xrightarrow[F_{d(S),d(\mathfrak{s})}]{} & HFK^-(-Y,K) \end{array} \tag{5.1.6}$$

which implies

$$d^*_S(F_{S,\mathfrak{s}}(\mathfrak{L}(L_\sigma))) = F_{d(S),d(\mathfrak{s})}(d^*(\mathfrak{L}(L_\sigma))).$$

Since $F_{S,\mathfrak{s}}(\mathfrak{L}(L_\sigma)) = \mathfrak{L}(L)$, we have

$$d^*_S(\mathfrak{L}(L)) = F_{d(S),d(\mathfrak{s})}(d^*(\mathfrak{L}(L_\sigma))).$$

Our assumption is that $[\mathfrak{L}(L)] \neq [\mathfrak{L}(L')]$ (strictly, that these MCG orbits are different under the canonical isomorphisms above). Now suppose $[\mathfrak{L}(L_\sigma)] = [\mathfrak{L}(L'_{\sigma'})]$, so that there exists $d \in MCG(Y,L_\sigma)$ such that $d^*(\mathfrak{L}(L_\sigma)) = \mathfrak{L}(L'_{\sigma'})$. Combined with the above, we infer

$$d^*_S(\mathfrak{L}(L)) = F_{d(S),d(\mathfrak{s})}(\mathfrak{L}(L'_{\sigma'})).$$

Now we claim that this class is the same as $F_{S,\mathfrak{s}}(\mathfrak{L}(L'_{\sigma'}))$. To see this, note first that implicit in the condition that $d^*(\mathfrak{L}_\sigma) = \mathfrak{L}'_{\sigma'}$ is the requirement that $d_*(\xi) = \xi$.

Moreover, since we are free to modify d by an isotopy (fixing K_σ), we can suppose that d is the identity on the ball containing S. Since d preserves the contact structures it must fix the induced *spin*^{c} structures on the boundary $-Y \sqcup -Y$. As the Chern number evaluation on the cobordism is also preserved, we infer $(d(S), d(\mathfrak{s})) = (S, \mathfrak{s})$. By the naturality theorem, it then follows that $F_{d(S),d(\mathfrak{s})}(\mathfrak{L}'_{\sigma'}) = F_{S',\mathfrak{s}'}(\mathfrak{L}'_{\sigma'}) = \mathfrak{L}(L')$. From the equation above, we obtain $d^*_S(\mathfrak{L}(L)) = \mathfrak{L}(L')$, contrary to assumption.

Exactly same arguments work for $\hat{\mathfrak{L}}$. □

5.2 Non-simplicity of Legendrian and transverse knot

Let's now construct an example of non-simple knot using Theorem 1.2.15.

Proof of Corollary 1.2.16. It's easy to see the two Legendrian knots in figure 5-4 are smoothly isotopic to $m(9_7)$ and have same *tb* and *rot*. We claim that they have different $[\hat{\mathfrak{L}}]$, which will imply the two are not Legendrian isotopic, and also that their transverse push-offs are not transverse isotopic. The Legendrians in Figure 5-4 were obtained by an application of (the construction leading to) Theorem 1.2.15 to the two knots in figure 5-5. According to [24, Theorem 1.3] the two knots in Figure 5-5 have different $[\hat{\mathfrak{L}}]$; moreover we can smoothly isotop the left side of Figure 5-5 to the right while fixing everything in the green circle. This verifies the assumptions of Theorem 1.2.15, thus after adding a full twist to the arcs in green circle, two new Legendrian knots in figure 5-4 have different $[\hat{\mathfrak{L}}]$ invariant.

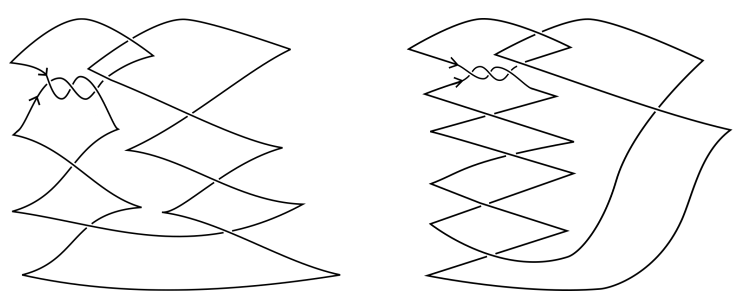

Figure 5-4: Both of these are smoothly isotopic to $m(9_7)$

□

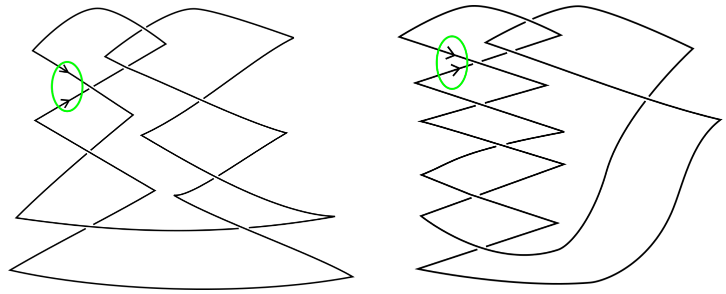

Figure 5-5: Two different Legendrian representatives of the Eliashberg–Chekanov twist knot $E_5 = m(7_2)$. The green circle indicate where we apply Theorem 1.2.15

Notice that the knot $m(9_7)$ is a rational knot. In Conway's notation [6] this is the [-3,-5,2] knot. Using similar ideas as the above, we can get infinite families of knots that are Legendrian and transversely non-simple.

Theorem 5.2.1. *Let m, n be positive integers with $n > 3$ and odd. In Conway's notation the knot $[-2m-1, -n, 2]$ (Figure 5-6) has at least $\lceil \frac{n}{4} \rceil$ Legendrian (transverse) representatives that have $tb = 2m+1$ and $rot = 0$ (self-linking number $2m+1$) that are pairwise not Legendrian (transverse) isotopic.*

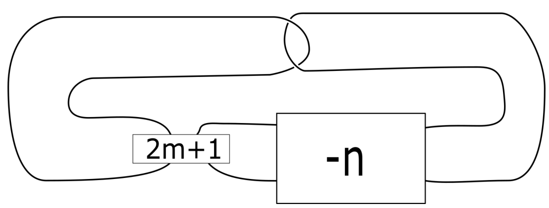

Figure 5-6: $+1$ means one right handed half twist and -1 means one left handed half twist. For $m = 1$ $n = 5$ the result is $m(9_7)$; if $m = 2$ $n = 5$ we have $m(11a_{242})$; if $m = 1$ $n = 7$ we have $m(11a_{246})$

Proof. Again in [24, Theorem 1.3] Ozsváth and Stipsicz prove that the Eliashberg–Chekanov twist knot E_n shown in figure 5-7 has $\lceil \frac{n}{4} \rceil$ many Legendrian representatives $E(k, l)$ (see figure 5-8), moreover $E(k, l)$ and $E(k'.l')$ have different $[\hat{\mathfrak{L}}]$ provided that k, l, k', l' are odd, $k + l - 1 = k' + l' - 1 = n$, $k \geq l$, $k' \geq l'$, and $k \neq k'$.

So similar to the proof of Corollary 1.2.16, to construct Legendrian $[-2m-1, -n, 2]$ we just apply Theorem 1.2.15 m times to all pairs of $E(k, l)$, in the Darboux ball represented by green circle in figure 5-8. Notice that each time we apply the Theorem

1.2.15 we add one full right handed twist to the green circle in figure 5-7. So we still have $\lceil \frac{n}{4} \rceil$ many representative of the new knot $[-2m-1, -n, 2]$ that have pairwise distinct $[\hat{\mathfrak{L}}]$. Moreover, $E(k,l)$ all have $rot = 0$ and $tb = 1$, and adding a positive twist to $E(k,l)$ does not change the rotation number and adds two to the Thurston–Bennequin number. So all those new representative have the same $tb = 2m+1$ and $rot = 0$. Since these representatives are distinguished by $[\hat{\mathfrak{L}}]$, their transverse pushoffs are also not transversely isotopic and have self-linking number $2m+1$. □

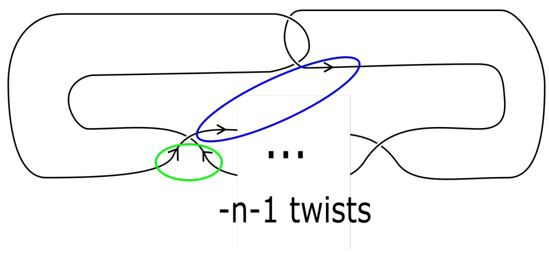

Figure 5-7: The Eliashberg–Chekanov twist knot E_n. The green (blue) circle smoothly corresponds to the green (blue) circle on the different Legendrian realizations of $E(k,l)$ of $E(n)$ in Figure 5-8.

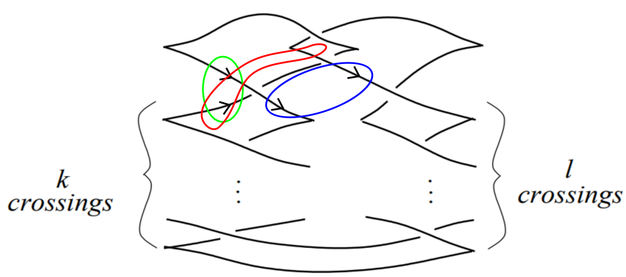

Figure 5-8: The Legendrian knots $E(k,l)$, $k,l \geq 1$ odd. These knots are smoothly isotopic to E_n, with $k+l-1 = n$. The green, blue or red circle indicates possible place where we can apply the Theorem 1.2.15. (Diagram from Figure 8 in [24])

Instead of applying Theorem 1.2.15 to the green circle on $E(k,l)$, we can also apply it to the blue circle. The exact same proof applies, and it will give us a family of double twist knot that are non-simple.

Theorem 5.2.2. *Let m, n be positive integers with $n > 3$ and odd. The double twist knot $K(2m+2, -n)$ (Figure 5-9, the knot $[-n, -2m-2]$ in Conway notation) has at* *least* $\lceil \frac{n}{4} \rceil$ *Legendrian (transverse) representatives that have* $tb = 2m+1$ *and* $rot = 0$ *(self-linking number* $2m+1$*) and are pairwise not Legendrian (transverse) isotopic.*

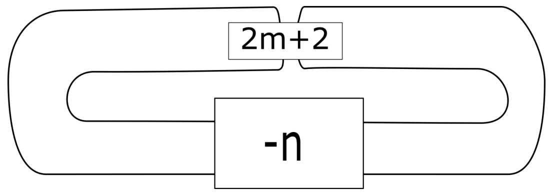

Figure 5-9: The double twist knot $K(2m+2, -n)$, or in Conway notation $[-n, -2m-2]$. For $m = 1$ $n = 5$ this is $m(9,4)$; if $m = 2$ $n = 5$ we have $m(11a_{358})$; if $m = 1$ $n = 7$ it is $m(11a_{342})$

We remark that in [24, Theorem 5.8] and [14, Theorem 1.2] there are transverse non-simplicity statements about certain families of rational knots, and all the double twist in Theorem 5.2.2 are included there. However, the knots $[-2m-1, -n, 2]$ in Theorem 5.2.1 are not. Moreover, instead of applying Theorem 1.2.15 to individual green or blue circles, we can apply it to them simultaneously, or to the red circle on 3 arcs, or to some green, some blue and some red. Each of those gives different families of non-simple knots.

We can also apply our theorem to more complicated (non 2-bridge) non-simple knot. In [23] Ng, Ozsváth, and Thurston found many examples of non-simple knots. Those knots have pair of representatives T_1 and T_2 with same tb and rot but $\hat{\theta}(T_1) = 0$ and $\hat{\theta}(T_2) \neq 0$, where $\hat{\theta}$ is the Legendrian invariant living in grid homology [29]. It has been shown in [5] that $\hat{\theta}$ is the same as $\hat{\mathfrak{L}}$, so we can apply Theorem 1.2.15 to those knots. For example the two knots in figure 5-10 are two Legendrian representatives of $(2,3)$ cable of $(2,3)$ torus knot with same tb and rot but different $\hat{\mathfrak{L}}$ invariant, so if we apply Theorem 1.2.15 to add twists in the circled region we will produce more non-simple knots. As we have seen, it's easy to produce a lot of infinite families of non-simple knots as long as we start with a non-simple ones that distinguished by $\hat{\mathfrak{L}}$ invariants.

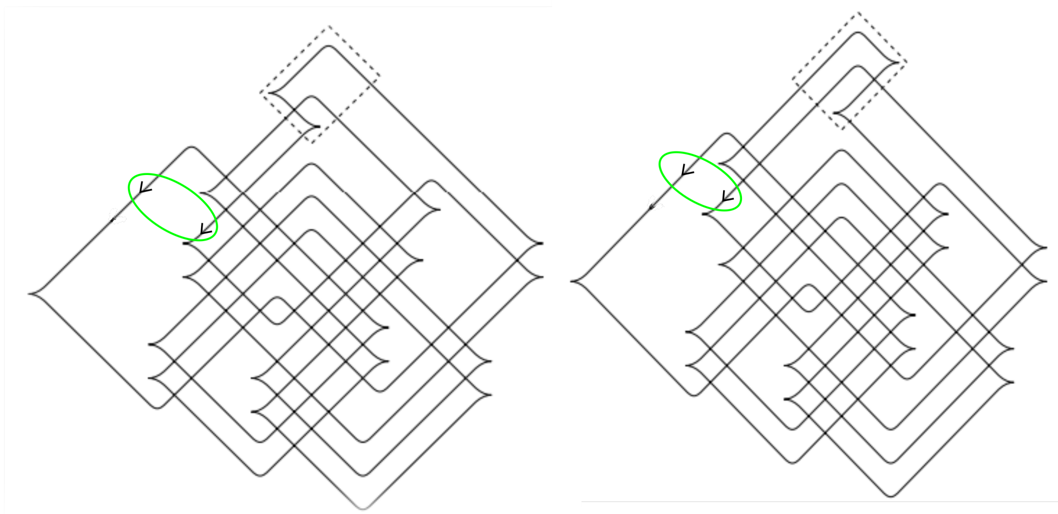

Figure 5-10: Legendrian fronts for L_1 (left) and L_2 (right), which are both $(2,3)$ cables of the $(2,3)$ torus knot. They have same tb and rot ,but $\hat{\mathfrak{L}}(L_1) = 0$, $\hat{\mathfrak{L}}(L_2) \neq 0$ (this diagram is from Figure 6 in [29]; the dotted circle indicates the only region in which the diagrams differ).

Chapter 6

$Spin^c$ structure in contact surgery cobordism

In this chapter, we will prove the $Spin^c$ formula in Proposition 1.2.11 but in a more general setting where we allow the surgery Legendrian to be rational null-homologous.

6.1 Rational null-homologous Legendrian knot and surgery cobordism

We start with the setting of Y being a rational homology sphere, K an oriented rationally null-homologous knot with order p which means $[K]$ in $H_1(Y, \mathbb{Z})$ has order p. Following the definition and convention in [10], a **rational Seifert surface** for smooth K is a smooth map $i : F \to Y$ from a connected compact oriented surface F to Y that is an embedding from the interior of F into the exterior of L, together with a p-fold cover from ∂F to K. Let $N(K)$ be the closed neighborhood of K, and $\mu \subset \partial N(K)$ a meridian. We can assume that $i(F) \cap \partial N(K)$ consists of c parallel oriented simple closed curves such that each represents the same homology class ν in $H_1(\partial N(K), \mathbb{Z})$. Define the canonical longitude λ_{can} be the longitude satisfying $\nu = t\lambda_{can} + r\mu$, where for simplicity the homology classes are also denoted by λ_{can} and ν respectively, t and r are coprime integers with $0 \leq r < t$. In other words,

$[\partial F] = c(t\lambda_{can} + r\mu)$, and λ_{can} is the choice of longitude for which r/t is the unique representative of the rational self-linking number of K in $[0, 1)$ (see also [22] and [30, Section 2.6]).

If we perform smooth integer n surgery along an order p rationally null-homologous knot K, our convention is that the surgery coefficient is measured with respect to the canonical longitude λ_{can}, and we denote by $Y_n(K)$ the resulting manifold. Let $X_n(K)$ be the 4-manifold obtained by attaching a 4-dimensional 2-handle H to $Y \times I$ along $K \times \{1\}$ where the coefficient n is with respect to λ_{can}, in other words we have $\partial X_n(K) = (-Y) \cup Y_n(K)$. Let C be the core of the 2-handle in $X_n(K)$, with $\partial C = K \times \{1\}$. For a rational Seifert surface $i : F \to Y$ of K we let $\tilde{F}$ be the 2-cycle in $X_n(K)$ given by $\tilde{F} = (i(F) \times \{1\}) \cup (-pC)$. We think of $\tilde{F}$ as a "capped off Seifert surface", which is precisely true if $p = 1$, i.e., the knot is null-homologous. Here $\tilde{F}$ is merely a 2-chain; with more care one can construct a smooth representative of the class $[\tilde{F}] \in H_2(X_n(K); \mathbb{Z})$ if desired.

Now we move on to an order p Legendrian rationally null-homologous knot L in some contact rational homology sphere (Y, ξ). In [2], using rational Seifert surfaces Baker-Etnyre defined the rational Thurston-Bennequin number $tb_{\mathbb{Q}}(L)$ and rational rotation number $rot_{\mathbb{Q}}(L)$ for such a Legendrian knot L, and we refer the reader to [2] for detailed definitions. Now, the Legendrian L has a canonical framing λ_c induced by the contact planes, which is given by $\lambda_c = \lambda_{can} + k\mu$ for some k. In the null-homologous case k is precisely the Thurston-Bennequin number, but in general we have the following. The definition in [2] is that

$$tb_{\mathbb{Q}}(L) := \frac{1}{p}([F] \cdot \lambda_c) = -\frac{1}{p}(p\lambda_{can} + cr\mu) \cdot (\lambda_{can} + k\mu).$$

(The sign appears because the intersection on the left happens in $\partial(Y - N(K))$ while on the right we work in $\partial N(K)$.) If we write $tb_{\mathbb{Q}}(L) = \frac{q}{p}$, then the above yields

$$k = \frac{q + cr}{p} = tb_{\mathbb{Q}}(L) + \frac{cr}{p}. \tag{6.1.1}$$

Observe that the number k is uniquely determined by L, and we denote the corresponding k as $k(L)$. Finally, note that performing $+n$-contact surgery on L is given smoothly by $+n$ surgery with respect to contact framing λ_c, which is equivalent to doing $k+n$ surgery on L with respect to the canonical framing λ_{can}.

We have the following naturality theorem of the contact invariant.

Theorem 6.1.2. *[22, Theorem 1.1] Let L be an oriented rationally null-homologous Legendrian knot in a contact rational homology sphere (Y, ξ) with non-vanishing contact invariant $c(\xi)$. Let $0 < n \in \mathbb{Z}$ be the contact surgery coefficient, and K be the smooth knot type of L. Let $W : Y \to Y_{k(L)+n}(K)$ be the corresponding rational surgery cobordism where $W = X_{k(L)+n}(K)$, and consider $\xi_n^-(L)$ on $Y_{k(L)+n}(K)$. There exist a Spin^c structure $\mathfrak{s}$ on $-W$ such that the homomorphism*

$$F_{-W,\mathfrak{s}} : \widehat{\mathrm{HF}}(-Y) \to \widehat{\mathrm{HF}}(-Y_{k(L)+n}(K))$$

induced by W with its orientation reversed satisfies

$$F_{-W,\mathfrak{s}}(c(\xi)) = c(\xi_n^-(L)).$$

Remark 6.1.3. In [22] the knot is assumed to be null-homologous, but the version stated above follows by the same arguments. In fact, even though we cite the above theorem from [22], naturality under $+n$ contact surgery actually comes from the combination of [4, Theorem 1.2] (naturality of the contact invariant under capping off cobordism) and Theorem 2.4.6 (Equivalence of capping off and $+n$-contact surgery). Since we don't require the knot to be null-homologous in those theorems we have the rationally null-homologous version for the contact surgery.

What we want to characterize here is the *Spin*^{c} structure mentioned in the above theorem without conjugation ambiguity. Note that in the following statement we use y to denote the order of L in homology instead of p, since elsewhere p refers to the order of the induced knot in $Y_n(L) = Y_{k(L)+n}(K)$.

Theorem 6.1.4. *In the above setting, assume $Y_n(L)$ is also a rational homology* *sphere. Then the* $\mathfrak{s}$ *in Theorem 6.1.2 has the property that*

$$\langle c_1(\mathfrak{s}), [\tilde{F}]\rangle = y(rot_{\mathbb{Q}}(L) + n - 1)$$

where y *is the order of* $[L]$*,* F *is a rational Seifert surface for* L *and* $\tilde{F}$ *is the "capped off" surface of* F*.*

We will prove the above theorem at the end of this section. Then Proposition 1.2.11 follows directly from the above theorem where L is null-homologous in Y.

6.2 $Spin^c$ for capping off cobordism

To prove Theorem 6.1.4 we again need to first prove the analogue theorem for the capping off cobordism.

Theorem 6.2.1. *Let* $(P_{g,r}, \phi)$ *be an abstract open book with genus* g *and* $r > 1$ *binding components with a chosen transverse binding component* B*.Then capping off* B *we get a new open book* $(P_{g,r-1}, \phi')$*. Denote by* (M, ξ)*,* (M', ξ') *the contact 3 manifolds corresponding to those two open books. The capping off cobordism gives rise to a map*

$$F_{B,\mathfrak{s}} : \widehat{\mathrm{HF}}(-M') \to \widehat{\mathrm{HF}}(-M) \tag{6.2.2}$$

where $\mathfrak{s}$ *is a* $Spin^c$ *structure on the cobordism* W *from* $-M'$ *to* $-M$*. Then*

(i) [4, Theorem 1.2] if M' *is a rational homology sphere, there is a choice of* $\mathfrak{s}$ *for which*

$$F_{B,\mathfrak{s}}(c(\xi')) = c(\xi) \tag{6.2.3}$$

holds.

(ii) Assume also that M *is a rational homology sphere, and let* L *be the Legendrian push off of* B*, realised as a curve on the page of the open book* $(S_{g,r}, \phi)$ *that is parallel to the binding* B*. Then the* $Spin^c$ *structure* $\mathfrak{s}$*mentioned in* (i) *satisfies:*

$$\langle c_1(\mathfrak{s}), [\tilde{V}]\rangle = -p(rot_{\mathbb{Q}}(L) + 1),$$

where p is the order of $[B]$, and $\tilde{V}$ is homology class in W represented by the "capped off" rational Seifert surface V of B.

To prove the second part of the above theorem we first need the following (rationally null-homologous version of) Lemmas from [24, section 4].

Lemma 6.2.4. *Let (P, ϕ) be an abstract open book and $M(\phi)$ the corresponding 3-manifold. Let L be a homologically non-trivial closed curve on P, then $p[L] \in H_1(P)$ is in the kernel of $H_1(P) \to H_1(M(\phi))$ if and only if it can be written as $p[L] = \phi_*(Z) - Z$ for some $Z \in H_1(P, \partial P)$.*

Proof. The case for the connected binding is proved in [24, Lemma 4.2], where in the connected binding case Z is actually an absolute class in $H_1(P)$. Now we suppose (P, ϕ) has two binding components. To make the proof clear we first pick useful basis representatives for $H_1(P)$ as follows. We let l_2 be a curve that is parallel to one of the boundary components of P, then we pick curves l_i for $3 \leq i \leq k$ such that they are disjoint from l_2 and form a standard symplectic basis for $H_1(P/l_2)$. Then it is clear that $\{[l_2], [l_3], ..., [l_k]\}$ form the basis of $H_1(P)$.

Now we positively stabilize the open book by attaching a 1-handle between the two binding components to obtain a new page P^+. Then $(P^+, \phi^+) = (P^+, \phi \circ \tau_c)$ is an open book with connected binding where τ_c is a right handed Dehn twist along a curve c in P^+ that intersects both the co-core of the 1-handle and l_2 exactly one time but misses all the other l_i. Then from the one-boundary case we know $p[L] = \phi^+_*(Z^+) - Z^+$ for some $Z^+ \in H_1(P^+)$. We view P as a subsurface of P^+, and let $l_1 = c$ where the orientation is chosen so that $l_1 \cdot l_2 = 1$; then $\{[l_1], [l_2], ..., [l_k]\}$ is a basis for $H_1(P^+)$. First we have the following observations regarding this basis:

1. If we let $\overline{l_1} = l_1 \cap P$ then $\overline{l_1} \in H_1(P, \partial P)$, and moreover $\phi^+_*(l_1) - l_1 = \phi_*(\overline{l_1}) - \overline{l_1}$ is an absolute class on $H_1(P)$.

2. $\phi^+_*(l_2) - l_2 = c = l_1$

3. $\phi^+_*(l_i) - l_i = \phi_*(l_i) - l_i$ which is also an absolute class on $H_1(P)$, for $i \neq 1, 2$.

4. For any absolute class $[O] \in H_1(P)$, $O \cdot l_2 = 0$.

Using the above basis we write $Z^+ = \sum_{i=1} u_i l_i$ and $\phi^+_*(Z^+) - Z^+ = \sum_{i=1} v_i l_i$. We can also express $\phi^+_*(Z^+) - Z^+$ as $\sum_{i=1} u_i(\phi^+_*(l_i) - l_i)$, and we claim $v_1 = u_2$. This is because the coefficient v_1 of l_1 is the same as the intersection number between $\phi^+_*(Z^+) - Z^+ = \sum_{i=1} u_i(\phi^+_*(l_i) - l_i)$ and l_2. The observations above imply $(\phi^+_*(l_2) - l_2) \cdot l_2 = 1$, and $(\phi^+_*(l_i) - l_i) \cdot l_2 = 0$ for $i \neq 2$. Thus $(\phi^+_*(Z^+) - Z^+) \cdot l_2 = u_2$ which proves the claim.

Our assumptions tell us $L \subset P$ which implies $[L] \cdot l_2 = 0$, and we also know that $p[L] = \phi^+_*(Z^+) - Z^+$, so $(\phi^+_*(Z^+) - Z^+) \cdot l_2 = 0$ thus we conclude $u_2 = 0$. Using the observation 1 and 3, we can express $p[L] = \phi_*(Z) - Z$, where $Z = u_1\overline{l_1} + \sum_{i=3} u_i l_i \in H_1(P, \partial P)$.

For open book with more than two binding components the above argument easily extends inductively. □

Lemma 6.2.5. *[24, Lemma 4.4] Let $L \in M(\phi)$ be an order p rationally null-homologous Legendrian knot supported in the page P of the open book (page framing equals to contact framing). Let $\mathfrak{p}$ be a two-chain with $\partial\mathfrak{p} = p[L] + (Z - \phi_*(Z))$ for some one–cycle $Z \in H_1(P, \partial P)$. Then, $p \cdot rot_{\mathbb{Q}}(L)$ is equal to $e(\mathfrak{p})$, the Euler measure of $\mathfrak{p}$ (see [26, section 7.1] for details about Euler measure).*

Proof. This is more straight forward than the previous lemma. We start with the same set up as above let (P, ϕ) be the open book with multiple binding components and (P', ϕ') be the positive stabilization of (P, ϕ) with connected binding (note positive stabilization does not change the contact structure, so when we view P as subsurface of P' the Legendrian knot L sitting on P' is Legendrian isotopic to the one sitting on P). [24, Lemma 4.4] says any two chain $\mathfrak{p}'$ with $\partial\mathfrak{p}' = p[L] + (Z' - \phi_*(Z'))$, where $Z' \in H_1(P')$ is the cycle in the proof of the above lemma, satisfies $e(\mathfrak{p}') = p \cdot rot(L)$.

Then according to the proof of the above lemma it's clear that the corresponding two chain $\mathfrak{p} = p[L] + (Z - \phi_*(Z))$, where $Z \in H_1(P, \partial P)$ as described above, gives rise to such a $\mathfrak{p}'$ when we include $\mathfrak{p}$ into P'. Thus the lemma follows. □

For a rationally null-homologous Legendrian L on page P with order p, we Let $\{a_1, ..., a_k\}$ be a basis for $H_1(P, \partial P)$ with the property that a_i are embedded **arcs** in P, $a_2, ..., a_k$ are disjoint from L and a_1 meets L in a single transverse intersection point with the convention $L \cdot a_1 = +1$. For $Z \in H_1(P, \partial P)$ and two-chain $\mathfrak{p}$ we found in the above two lemmas, using above basis we can rewrite them as

$$Z = \sum_{i=1}^{k} n_i \cdot a_i, \text{ and } \partial\mathfrak{p} = p[L] + \sum_{i=1}^{k} n_i \cdot (a_i - \phi_*(a_i)).$$

Now we are ready to proceed the proof of Theorem 6.2.1. We first recall the setting of the Heegaard triple diagram $(\Sigma, \alpha, \gamma, \beta, z)$ that describe the capping off cobordism from M to M', for simplicity we let $P = P_{g,r}$, and $P' = P_{g,r-1}$.

Σ is the Heegard surface of the union of two pages $P_{+1} \cup -P_{-1}$. Let arcs $\{a_1, ..., a_k\}$ be a basis for P_{+1} with the property that $a_2, ..., a_k$ are disjoint from L and a_1 meets L in a single transverse intersection point (note this is the exactly same basis we used for Z and $\mathfrak{p}$), where L is the Legendrian push off of the binding B on P_{+1}. We let c_i be a push off of a_i for all i, b_i be a further push off of c_i for $i \neq 1$, and b_1 be the parallel push of of binding B on P_{+1}. Then in particular we will have a triangle Δ_i formed by a_i, c_i and b_i for all i (see Figure 6-1).

Now we let $\alpha_i = a_i \cup \overline{a_i}$ and $\gamma_i = c_i \cup \overline{\phi(c_i)}$ for all i. For the β curves let $\beta_i = b_i \cup \overline{\phi(b_i)}$ for $i \neq 1$, and $\beta_1 = b_1$. For base point z, we put it outside the thin strips region between the arcs. Then the manifolds M and M' are represented by (Σ, α, γ) and (Σ, α, β) respectively. The pointed Heegaard triple $(\Sigma, \alpha, \gamma, \beta, z)$ describes the capping off cobordism from M to M', and $(\Sigma, \gamma, \beta, \alpha, z)$ is the opposite cobordism from $-M'$ to $-M$. This is exactly the same setting in section 4 but ignoring the extra Legendrian knot.

According to [4], the small triangle we formed by a_i, c_i, b_i is representing the $Spin^c$ structure in Theorem 6.2.1 (i), since $(\Sigma, \alpha, \gamma, \beta, z)$ and $(\Sigma, \gamma, \beta, \alpha, z)$ represent the same 4-manifold W we will calculate this $Spin^c$ structures using $(\Sigma, \alpha, \gamma, \beta, z)$.

To calculate the first Chern class we will use the formula in [28, Proposition 6.3]. If we let $\psi : \Delta \to Sym^k(\Sigma)$ be the Whitney triangle correspond to the domain consists of little triangles Δ we are interested in, D a triply periodic domain representing the two-dimensional homology class $H(D) \in H_2(W, \mathbb{Q})$, then

$$\langle c_1(\mathfrak{s}_z(\psi), H(D)\rangle = e(D) + \#(\partial D) - 2n_z(D) + 2\sigma(\psi, D), \tag{6.2.6}$$

where $e(D)$ is the Euler measure of D, $\#(\partial D)$ is the coefficient sum of all terms in ∂D, and $\sigma(\psi, D)$ is the dual spider number. The dual spider number $\sigma(\psi, D)$ can be calculated as follows:

We first choose an orientation on α, γ and β we let α', γ' and β' be the leftward push offs of the corresponding curve. Let $\partial_{\alpha'}(D)$, $\partial_{\gamma'}(D)$ and $\partial_{\beta'}(D)$ be the 1-chains obtained by translating the corresponding components of ∂D. Let u be an interior point of Δ so that $\psi(u)$ misses α, γ, β curves, then choose three oriented paths r, t and s, from u to the α, γ, β boundaries respectively such that r, t and s are in the 2-simplex Δ that is the domain of $\psi : \Delta \to Sym^k(\Sigma)$. Identifying these arcs with their image 1-chain in Σ, the dual spider number is given by

$$\sigma(\psi, D) = n_{\psi(u)}(D) + \partial_{\alpha'}(D) \cdot r + \partial_{\gamma'}(D) \cdot t + \partial_{\beta'}(D) \cdot s \tag{6.2.7}$$

Now we are ready to prove the Theorem 6.2.1 *(ii)*

Proof of Theorem 6.2.1 (ii). We first need to identify a triply periodic domain in the triple diagram.

Recall that we have a domain $\mathfrak{p}$ in the page P with $\partial\mathfrak{p} = p[L] + \sum_{i=1}^{k} n_i \cdot (a_i - \phi_*(a_i))$ and $e(\mathfrak{p}) = p \cdot rot_{\mathbb{Q}}(L)$. Consider $\mathfrak{p}$ as lying on P_{-1}, and P_{-1} lying in $\Sigma = P_{+1} \cup -P_{-1}$. With this point of view we write $\overline{\mathfrak{p}}$ instead as the domain $\mathfrak{p}$ on Σ with

$$\partial\overline{\mathfrak{p}} = p[\overline{L}] + \sum_{i=1}^{k} n_i \cdot (\overline{a_i} - \phi_*(\overline{a_i})),$$

where we are using $\overline{L}$ and $\overline{a_i}$ to mean the images of L and a_i (considered on P_{+1}) under identity map on the opposite page $-P_{-1}$. Then $\overline{\mathfrak{p}}$ satisfies $e(\overline{\mathfrak{p}}) = -p \cdot rot_{\mathbb{Q}}(L)$.

We then observe that the cycle $(\alpha_i - \gamma_i)$ is exactly $(\overline{a_i} - \phi_*(\overline{a_i}))$, and thus when

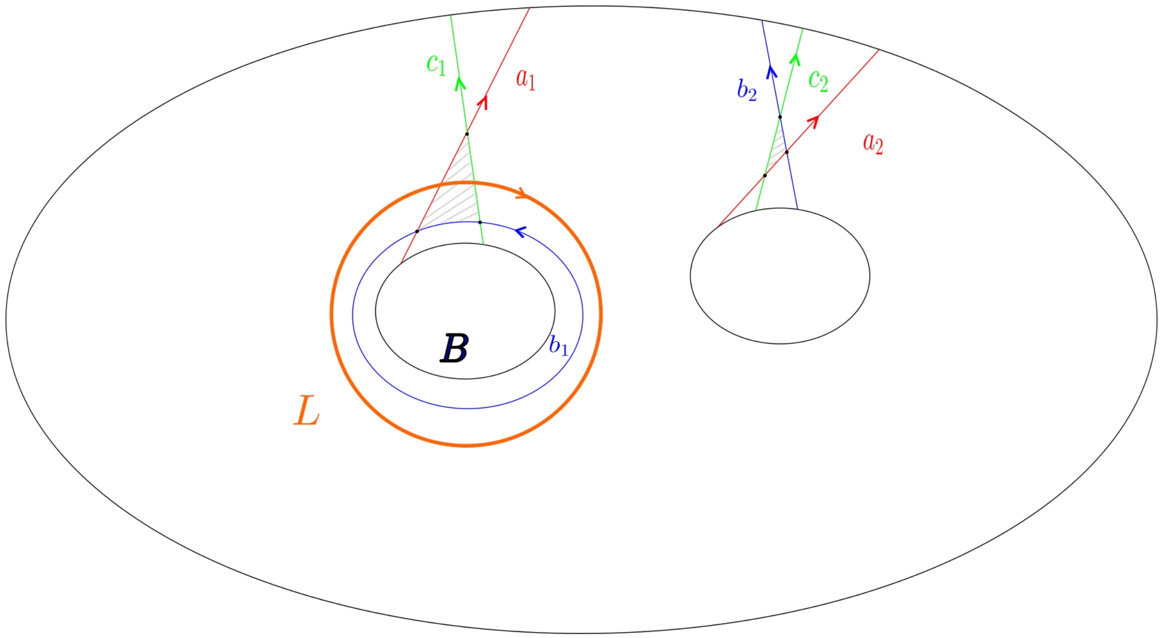

Figure 6-1: We oriented the curves as it shown, and the $Spin^c$ structure $\mathfrak{s}$ corresponds to the shaded small triangle.

we push $\overline{L}$ across B to L (from $-P_{-1}$ to P_{+1}) we obtain a corresponding domain (still denoted $\overline{\mathfrak{p}}$) on Σ with

$$\partial\overline{\mathfrak{p}} = p[L] + \sum_{i=1}^{k} n_i \cdot (\alpha_i - \gamma_i)$$

such that $e(\overline{\mathfrak{p}}) = -p \cdot rot_{\mathbb{Q}}(L)$.

Now since L is the Legendrian push off of the binding B its orientation coincides with that of B, which is compatible with the orientation of the page (the page P_{+1} is oriented counter clockwise). Thus the arc a_1 is oriented by the requirement $L.a_1 = +1$, which induces natural orientations on α_1 and γ_1. We manually give an orientation to β_1 that is opposite to the orientation of L, and for the orientation of the rest curves we make choice such that near each little triangle Δ_i for $i \neq 1$, it looks like Δ_2 as shown in Figure 6-1. We will use those orientation for calculation.

Notice that $L \simeq -\beta_1$, thus we can take our $\overline{\mathfrak{p}}$ to be a triply periodic domain on $(\Sigma, \alpha, \gamma, \beta, z)$ with boundary $-p\beta_1 + \sum_{i=1}^{k} n_i \cdot (\alpha_i - \gamma_i)$. Moreover we can add or subtract a multiple of the whole Σ to make $n_z(\overline{\mathfrak{p}}) = 0$. Thus by formula 6.2.6

$$\begin{aligned}\langle c_1(\mathfrak{s}_z(\psi), H(\bar{\mathfrak{p}})\rangle &= e(\bar{\mathfrak{p}}) + \#(\partial\bar{\mathfrak{p}}) - 2n_z(\bar{\mathfrak{p}}) + 2\sigma(\psi,\bar{\mathfrak{p}}) \\ &= -p\cdot rot_{\mathbb{Q}}(L) + (-p + \sum n_i - n_i) - 0 + 2\sigma(\psi,\bar{\mathfrak{p}}) \\ &= -p\cdot rot_{\mathbb{Q}}(L) - p + 2\sigma(\psi,\bar{\mathfrak{p}})\end{aligned}$$

We claim that $2\sigma(\psi,\bar{\mathfrak{p}}) = 0$. We draw the "dual spider" α', γ', β' and r, s, t based on the orientation, recall $\Delta = \Delta_1 + \Delta_2 + ... + \Delta_k$, and except Δ_1 the neighborhood of the rest triangles look the same, so the contribute for the dual spider can be divided into two case for Δ_1 and Δ_i where $i \neq 1$.

We first look at the contribution of dual spider number $\sigma(\psi,\bar{\mathfrak{p}}|)_{\Delta_1}$from Δ_1, see Figure 6-2. Since we assume $n_z(\bar{\mathfrak{p}}) = 0$, and the multiplicity of α_1 and γ_1 are n_1 and $-n_1$ respectively, this force $n_{\psi(u)}(\bar{\mathfrak{p}})|_{\Delta_1} = -n_1$. It is easy to see $\partial_{\alpha'_1}(\bar{\mathfrak{p}})\cdot r = 0$ and $\partial_{\beta'_1}(\bar{\mathfrak{p}})\cdot s = 0$ because both α'_1 and β'_1 are outside of the triangle Δ_1. The last quantity $\partial_{\gamma'_1}(\bar{\mathfrak{p}})\cdot t = n_1$ because $\gamma'_1 \cdot t = -1$ and γ_1 has multiplicity $-n_1$. Thus

$$\begin{aligned}\sigma(\psi,\bar{\mathfrak{p}})|_{\Delta_1} &= n_{\psi(u)}(\bar{\mathfrak{p}})|_{\Delta_1} + \partial_{\alpha'_1}(\bar{\mathfrak{p}})\cdot r + \partial_{\gamma'_1}(\bar{\mathfrak{p}})\cdot t + \partial_{\beta'_1}(\bar{\mathfrak{p}})\cdot s \\ &= -n_1 + 0 + n_1 + 0. \\ &= 0\end{aligned}$$

We then look at the contribution of dual spider number $\sigma(\psi,\bar{\mathfrak{p}}|)_{\Delta_i}$ from Δ_i for $i \neq 1$, see Figure 6-2. Again the assumption $n_z(\bar{\mathfrak{p}}) = 0$ with the fact that the multiplicity of α_i and γ_i are n_i and $-n_i$ respectively force $n_{\psi(u)}(\bar{\mathfrak{p}})|_{\Delta_i} = n_i$. This time we see $\partial_{\gamma'_i}(\bar{\mathfrak{p}})\cdot t = 0$ because γ'_i are outside of the triangle Δ_i, and $\partial_{\beta'_1}(\bar{\mathfrak{p}})\cdot s = 0$ because β_i has multiplicity 0. The last quantity $\partial_{\alpha'_1}(\bar{\mathfrak{p}})\cdot r = -n_1$ because $\alpha'_i \cdot t = -1$ and α_i has multiplicity n_1. Thus

$$\begin{aligned}\sigma(\psi,\bar{\mathfrak{p}})|_{\Delta_i} &= n_{\psi(u)}(\bar{\mathfrak{p}})|_{\Delta_i} + \partial_{\alpha'_i}(\bar{\mathfrak{p}})\cdot r + \partial_{\gamma'_i}(\bar{\mathfrak{p}})\cdot t + \partial_{\beta'_i}(\bar{\mathfrak{p}})\cdot s \\ &= n_i - n_i + 0 + 0 \\ &= 0\end{aligned}$$

Thus each triangles contribute 0, we have $\sigma(\psi, \bar{\mathfrak{p}}) = 0$, which proves the claim. Hence when we plug in back to the Chern class evaluation we get

$$\langle c_1(\mathfrak{s}_z(\psi), H(\bar{\mathfrak{p}})\rangle = -p \cdot rot_{\mathbb{Q}}(L) - p$$

To complete the proof we claim that $H(\bar{\mathfrak{p}})$ is the same as the class of the capped off rational Seifert surface. This is clear by the fact that the β-boundary of $\bar{\mathfrak{p}}$ is just pL, together with the construction of the identification between periodic domains in $(\Sigma, \alpha, \gamma, \beta)$ and homology classes in the cobordism between M and M' [27, Proposition 8.2]. □

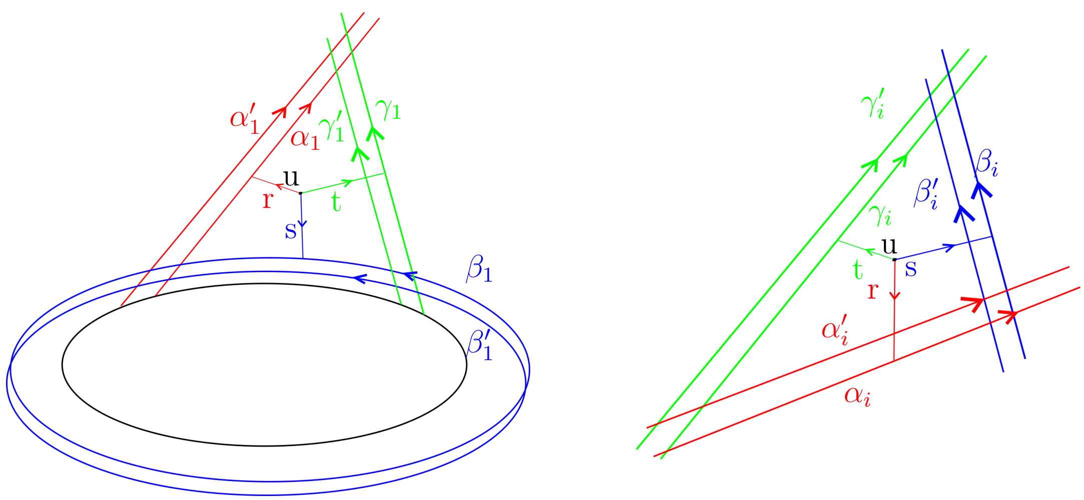

Figure 6-2: The picture on the left describes the dual spider near Δ_1, and the picture on the right describes the dual spider near Δ_i for $i \neq 1$.

6.3 *Spin*^{c} characterization from capping off cobordism to surgery cobordism

Before going in to the proof of Theorem 6.1.4, we first recall some the general settings about the $+n$ contact surgery, capping off and rationally null-homologous knot.

We will use the same notation as is described in section 2.4.1. Namely, $(P', k\circ\phi)$ is a stabilized open book compatible with (Y,ξ) with L on its page, and the stabilization L^- parallel to some binding B; (P',ϕ') is an open book compatible with $(Y_n(L),\xi_n^-(L))$ (where ϕ' is obtained by composing $k\circ\phi$ with some twists along L and L^-); and B_L is the binding in (P',ϕ') corresponding to B in $(P', k\circ\phi)$. Thus the cobordism corresponding to $+n$ contact surgery on L in Y is the same 4-manifold (with opposite orientation) as the cobordism corresponding to capping off B_L in $Y_n(L)$. Observe that the capped-off rational Seifert surfaces $\tilde{F}$ (for L) and $\tilde{V}$ (for B_L) are both generators of the second rational homology of this cobordism, but need not be identical classes.

We also recall that $[\partial F] = c(t\lambda_{can} + r\mu)$, where λ_{can} is the canonical framing and $ct = y$ is the order of L. Moreover if $tb_{\mathbb{Q}}(L) = \frac{x}{y}$, then contact $+n$ surgery on L is the same as smooth $k(L)+n$ surgery on L which is equivalent to smooth $\frac{x+cr+ny}{y}$ (according to 6.1.1) surgery on L.

Under the above setting we are able to state the most essential Lemma we need to prove the Theorem 6.1.4.

Lemma 6.3.1. *As classes in $H_2(W;\mathbb{Z})/Tors$, we have*

$$[\tilde{F}] = \begin{cases} -[\tilde{V}] & \text{if } x+ny>0 \\ [\tilde{V}] & \text{if } x+ny<0 \end{cases}.$$

Proof. First notice the classes $[\tilde{F}]$ and $[\tilde{V}]$ are represented by 2-chains obtained by adding y (the order of L in homology) copies of the core disk C of the surgery handle to F in the first case, or p (the order of B_L in homology) copies of the cocore disk C_c to V in the second case. Here the disk C is oriented so that $\partial C = -L$, and we take the cocore disk to be oriented so that its signed intersection with the core disk is $+1$. It follows that $[\tilde{F}]\cdot[\tilde{V}] = \pm yp$, where the sign depends on the relative orientation between ∂V (or equivalently L') and the boundary of the cocore. We make the following 2 claims.

Claim 1: $[\tilde{F}]\cdot[\tilde{V}] = -yp$

Claim 2: $[\tilde{F}]\cdot[\tilde{F}] = y(x+ny)$

Once we have achieved the above two claims then by [22, commment after Lemma 5.1] or [10, Lemma 5.2] the order of L' is $p = |c(k(L)+n)t - cr| = |c(\frac{x+cr+ny}{y})t - cr| = |x+ny|$. Thus Claim 1 implies $[\tilde{F}] \cdot [\tilde{V}] = -y|x+ny|$. But $[\tilde{F}]$ and $[\tilde{V}]$ rational generators of $H_2(W)$, hence (modulo torsion) one is a multiple of the other; on the other hand we now have that $[\tilde{F}] \cdot [\tilde{F}] = \pm[\tilde{F}] \cdot [\tilde{V}]$ where the sign depends on the positivity of $x + ny$. The lemma follows.

Now we will first prove Claim 1. We denote $C_c(L')$ to be the oriented cocore of the handle correspond to the orientation of L', then the Claim 1 is equivalent to $C \cdot C_c(L') = -1$.

To figure out the intersection number we recall that the DGS alogrithm (Theorem 2.3.2, c.f. [9]) says that $+n$ surgery on L is the same as surgery along a link $L_1 = L, L_2, \dots, L_n$, so that L_2 is the negativly stabilized Legendrian push off of L_1, and L_i is the Legendrian push off of L_{i-1} for $i = 3, \dots, n$. Moreover, if we denote L_0 be a further push off of L_n then L' corresponds to L_0 after performing contact $+1$ surgery on L_1 and contact -1 surgery on $L_2, \dots, L_n$. (Remark here the notation is a bit different from the notation in Theorem 2.3.2.)

We will first slide every L_i for $i = 2, \dots, n$ with smooth framing $k(L) - 2$ over L_1 in order and further slide L_0 over L_1. If we denote $s(L_i)$ to be the new knots in the surgery diagram corresponding to L_i after the slide, then it's not hard to see that for $i = 2, \dots, n$ $s(L_i)$ are all isotopic to the meridian of L with smooth framing -1, and $s(L_0)$ is also isotopic to the meridian of L. Thus after we blow down all the $s(L_i)$ for $i = 2, \dots, n$, we are back to smooth $k(L) + n$ surgery on $L_1 = L$, and $s(L_0)$ is still the meridian of L_1 that corresponds to the cocore of the handle.

Now $[C] \cdot [C_c(L')]$ is the same as $lk_{\mathbb{Q}}(L_1, s(L_0))$. Equipping L_1 and L_0 with the same orientation at the beginning, the slide of L_0 over L_1 becomes a handle subtraction, so that $lk_{\mathbb{Q}}(L_1, s(L_0)) = lk_{\mathbb{Q}}(L_1, L_0) - lk_{\mathbb{Q}}(L_1, L_1) = tb_{\mathbb{Q}}(L_1) - (tb_{\mathbb{Q}}(L_1) + 1) = -1 = [C] \cdot [C_c(L')]$, which finishes the proof of Claim 1.

Claim 2 is more straight forward following [22, Lemma 5.1] that tells us the self-intersection $[\tilde{F}] \cdot [\tilde{F}] = y((k(L) + n)y - cr) = y(\frac{x+cr+ny}{y} y - cr) = y(x + ny)$, which completes the proof. □

Proof of Theorem 6.1.4. First from Theorem 2.4.6 and 6.2.1 we have that the *Spin*^{c} structure we are interested in satisfies

$$\langle c_1(\mathfrak{s}), [\tilde{V}]\rangle = -p(rot_{\mathbb{Q}}(L') + 1).$$

Combined with Lemma 6.3.1 this means

$$\langle c_1(\mathfrak{s}), [\tilde{F}]\rangle = \begin{cases} p(rot_{\mathbb{Q}}(L') + 1) & \text{if } x + ny > 0 \\ -p(rot_{\mathbb{Q}}(L') + 1) & \text{if } x + ny < 0 \end{cases}. \tag{6.3.2}$$

The last step is to represent $rot_{\mathbb{Q}}(L')$ using $rot_{\mathbb{Q}}(L)$. To simplify the notation in calculation we let $rot_{\mathbb{Q}}(L) = r$ and $tb_{\mathbb{Q}}(L) = a = \frac{x}{y}$ where y is the order of L; then the order of L' is $p = |x + ny|$ as we discussed above. Again we express contact $+n$ surgery on L as surgery along a link $L_1 = L, L_2, \ldots, L_n$ following the DGS algorithm (Theorem 2.3.2), and L_0 is a further push off of L_n that correspond to L' before performing contact surgery. Then [10, Lemma 4.1] indicates that the rational rotation number of L' is given by

$$rot_{\mathbb{Q}}(L') = rot_{\mathbb{Q}}(L_0) - \left\langle \begin{pmatrix} rot_{\mathbb{Q}}(L_1) \\ \vdots \\ rot_{\mathbb{Q}}(L_n) \end{pmatrix}, M^{-1} \begin{pmatrix} lk_{\mathbb{Q}}(L_0, L_1) \\ \vdots \\ lk_{\mathbb{Q}}(L_0, L_n) \end{pmatrix} \right\rangle \tag{6.3.3}$$

, where the $<\ ,\ >$ in the above equation is just the usual inner product (dot product) of two vectors.

It's easy to find those rotation numbers. In (Y, ξ) we have $L_1 = L$ and $L_i = L^-$ for $i = 0, 2, ..., n$, so $rot_{\mathbb{Q}}(L_1) = r$ and $rot_{\mathbb{Q}}(L_i) = r - 1$ for $i = 0, 2, ..., n$. In (6.3.3),

M is the $n \times n$ rational linking matrix

$$M = \begin{bmatrix} a+1 & a & \dots & \dots & \dots & a \\ a & a-2 & a-1 & \dots & \dots & a-1 \\ \vdots & a-1 & a-2 & a-1 & \dots & \vdots \\ \vdots & \vdots & a-1 & a-2 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & a-1 \\ a & a-1 & \dots & \dots & a-1 & a-2 \end{bmatrix}$$

whose inverse is

$$M^{-1} = \frac{1}{a+n} \begin{bmatrix} n-(n-1)a & a & \dots & \dots & \dots & a \\ a & 1-a-n & 1 & \dots & \dots & 1 \\ \vdots & 1 & 1-a-n & 1 & \dots & \vdots \\ \vdots & \vdots & 1 & 1-a-n & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \ddots & 1 \\ a & 1 & \dots & \dots & 1 & 1-a-n \end{bmatrix}$$

Moreover since $lk_{\mathbb{Q}}(L_0, L_1) = a$, and $lk_{\mathbb{Q}}(L_0, L_i) = a-1$ for all $i = 2, ...n$ we want to calculate

$$\begin{bmatrix} n-(n-1)a & a & \dots & \dots & \dots & a \\ a & 1-a-n & 1 & \dots & \dots & 1 \\ \vdots & 1 & 1-a-n & 1 & \dots & \vdots \\ \vdots & \vdots & 1 & 1-a-n & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \ddots & 1 \\ a & 1 & \dots & \dots & 1 & 1-a-n \end{bmatrix} \begin{pmatrix} a \\ a-1 \\ \vdots \\ \vdots \\ \vdots \\ a-1 \end{pmatrix} = \begin{pmatrix} a \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

Thus equation 6.3.3 becomes

$$
\begin{aligned}
rot_{\mathbb{Q}}(L') &= (r-1) - \left\langle \begin{pmatrix} r \\ r-1 \\ \vdots \\ r-1 \end{pmatrix}, \frac{1}{a+n} \begin{pmatrix} a \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle \\
&= (r-1) - \frac{ra + (n-1)(r-1)}{a+n} \\
&= \frac{r-a-1}{a+n} \text{ (substitute } a = \frac{x}{y}) \\
&= \frac{yr - x - y}{x+ny} \\
&= \frac{yr + ny - y - (x+ny)}{x+ny} \\
&= \frac{y(r+n-1)}{x+ny} - 1
\end{aligned}
$$

Now when we plug in back to equation 6.3.2 and $p = |x + ny|$ we obtain

$$
\begin{aligned}
\langle c_1(\mathfrak{s}), [\tilde{V}]\rangle &= \begin{cases} p(rot_{\mathbb{Q}}(L') + 1) & \text{if } x + ny > 0 \\ -p(rot_{\mathbb{Q}}(L') + 1) & \text{if } x + ny < 0 \end{cases} \\
&= \begin{cases} |x + ny|(\frac{y(r+n-1)}{x+ny} - 1 + 1) & \text{if } x + ny > 0 \\ -|x + ny|(\frac{y(r+n-1)}{x+ny} - 1 + 1) & \text{if } x + ny < 0 \end{cases} \\
&= \begin{cases} y(r + n - 1) & \text{if } x + ny > 0 \\ y(r + n - 1) & \text{if } x + ny < 0 \end{cases} \\
&= y(r + n - 1) \quad \text{if } x + ny \neq 0.
\end{aligned}
$$

Last the condition both Y and $Y_n(L)$ are rational homology sphere implies the order of L', namely $p = |x + ny|$, is nonzero, which conclude the proof. □

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