A METHOD OF REPRESENTING POINTS IN SPACE AND THE APPLICATION OF THIS METHOD TO THE INVESTIGATION OF SOME PROBLEMS IN SPACE GEOMETRY

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TO THE INVESTIGATION OF

SOME PROBLEMS IN SPACE GEOMETRY

By

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THE STRAIGHT LINE

The position of a point in space is usually determined by three numbers, which give the distances of the point from three mutually perpendicular coordinate planes. However, it is possible to determine the position of a point by two numbers, one real, and one complex. For instance, the combination of any real and any complex number, as $(a, b+ic)$, determines a unique point in space, which can be found as follows:

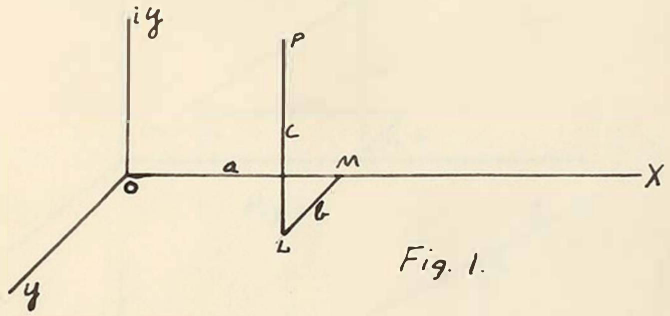


Fig. 1.

Draw three mutually perpendicular coordinate axes, and call them, for convenience, the x, y, and iy axes, as in Fig. 1. Now from O measure off a distance (a) along the x-axis. This distance is represented in the figure by OM. From M measure off a distance (b) in direction parallel to the y-axis. This puts us at point L. From L measure off a distance (c) in a direction parallel to the iy-axis. This traces out the line segment LP. Now P is the point $(a, b+ic)$. In what follows we shall call (a) the abscissa of the point P, and $(b+ic)$ the ordinate of the point P. An equation in which the abscissas and ordinates, just defined, are the variables, will represent a locus of points in this system. In all such equations we shall represent

the variable abscissas by the small letter x, and the variable ordinates by the capital letter Y. In these equations, x, of course, must be real, but Y may be complex.

If we have, given, two points in space, $(a, b+ic)$ and (a_1, b_1+ic_1), the distance between the points is at once, from an examination of Fig. 2, seen to be

$$\sqrt{(a_1-a)^2+(b_1-b)^2+(c_1-c)^2}\,.$$

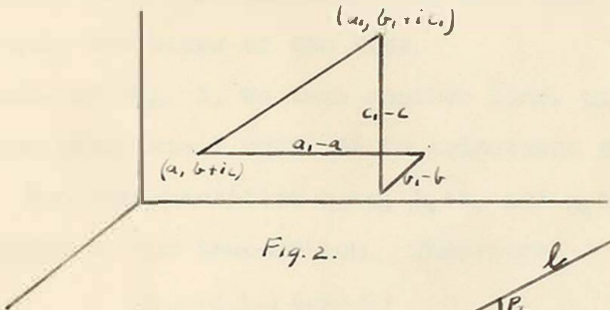


Fig. 2.

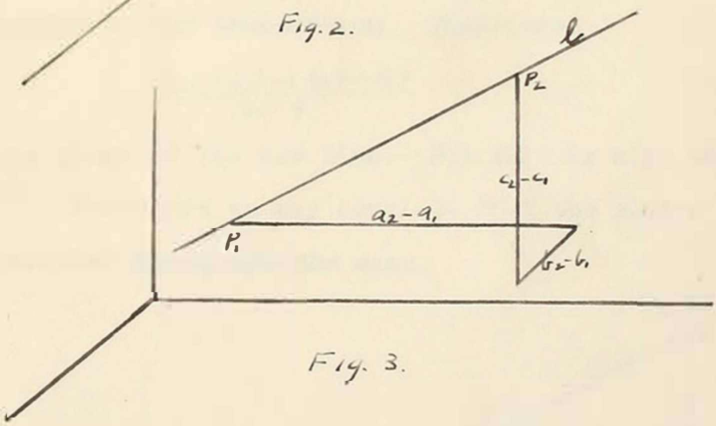


Fig. 3.

If we let l be any line in space, and if we take any two points on l, P_1 and P_2, and let the coordinates of P_1 be (a_1, b_1+ic_1), and the coordinates of P_2 be (a_2, b_2+ic_2), then from Fig. 3, it is seen that the three quantities c_2-c_1, b_2-b_1, a_2-a_1, are in constant ratio, no matter what points on the line are chosen as P_1 and P_2. But if c_2-c_1, b_2-b_1, and a_2-a_1 are in constant ratio, we can say that the variable abscissas by the small letter x, and the variable ordinates by the capital letter Y. In these equations, x, of course, must be real, but Y may be complex.

If we have, given, two points in space, $(a,b+ic)$ and $(a',b'+ic')$, the distance between the points is at once, from an examination of Fig. 2, seen to be

$$\sqrt{(a-a')^2+(b-b')^2+(c-c')^2}$$

Fig. 2

If we let L be any line in space, and if we take any two points on L, P_1 and P_2, and let the coordinates of P_1 be $(a,b+ic)$, and the coordinates of P_2 be $(a',b'+ic')$, then from Fig. 3, it is seen that the three quantities $a-a'$, $b-b'$, $c-c'$, are the direction ratios, no matter what points on the line are chosen as P_1 and P_2. Now if $a-a'$, $b-b'$, and $c-c'$, are in direction ratio, we can say that

$$\frac{b_2 - b_1 + i(c_2 - c_1)}{a_2 - a_1} = K ,$$

where K is some, constant, complex number. Therefore

$$\frac{(b_2 + ic_2) - (b_1 + ic_1)}{a_2 - a_1} = K .$$

Therefore the ratio of the difference between the ordinates, and the difference between the abscissas, is a constant for any two points on a line whatever. We will call this constant ratio the slope of the line.

If now, in Fig. 3, we take another line, parallel to l, we can translate l until it is coincident with the new line. But the quantities $c_2 - c_1$, $b_2 - b_1$, and $a_2 - a_1$, will not be changed in the translation. Therefore

$$\frac{(b_2 + ic_2) - (b_1 + ic_1)}{a_2 - a_1}$$

becomes the slope of the new line. But this is also the slope of l. Therefore we may conclude that the slopes of any two parallel lines are the same.

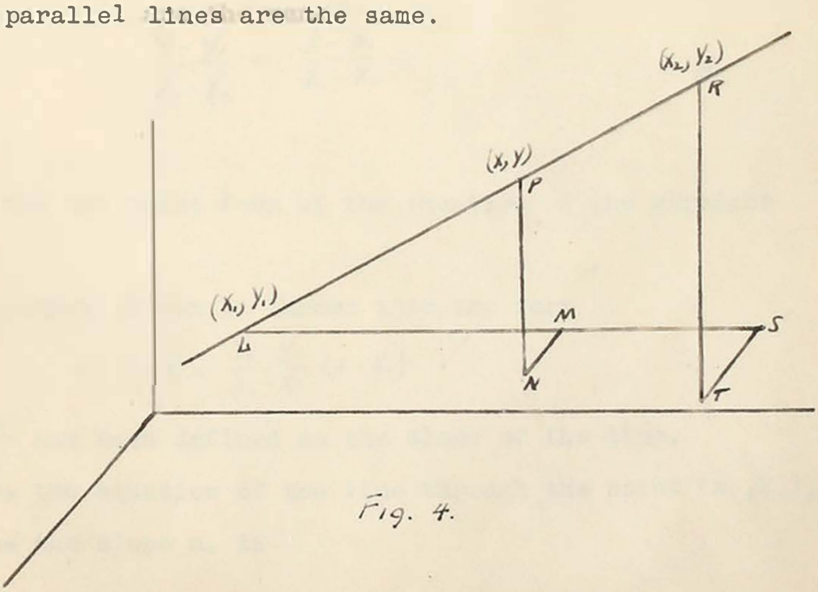


Fig. 4.

$$[illegible] = k$$

where k is some, constant, complex number. Therefore

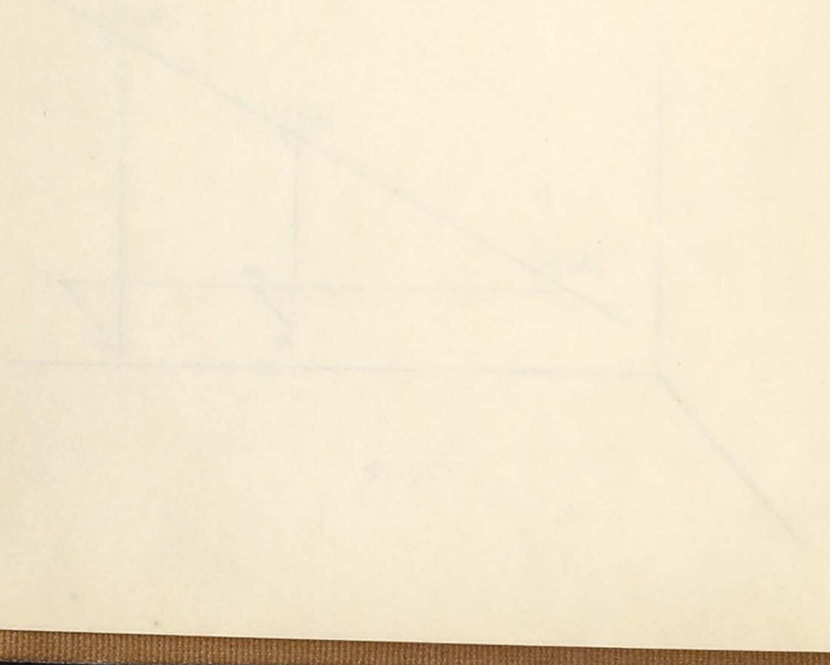
$$[illegible] = k$$

Therefore the ratio of the difference between the ordinates, and the difference between the abscissas, is a constant for any two points on a line whatever. We will call this constant ratio the slope of the line.

If now, in Fig. 3, we take another line, parallel to l, we can translate l until l' is coincident with the new line. But the quantities [illegible], [illegible], and [illegible] will not be changed in the translation. Therefore

$$[illegible]$$

becomes the slope of the new line. But this is also the slope of l. Therefore we may conclude that the slopes of any two parallel lines are the same.



Given two points in space, the equation of the line joining these two points may be derived as follows. Let (x_1, Y_1) and (x_2, Y_2) be the given points, and let (x, Y) be a running point on the line. In Fig. 4, from a consideration of similar triangles, it is evident that

$$\frac{ST}{MN} = \frac{LS}{LM} = \frac{X_2 - X_1}{X - X_1}$$

also $$\frac{TR}{NP} = \frac{LS}{LM} = \frac{X_2 - X_1}{X - X_1}$$

Therefore $$ST = MN \frac{X_2 - X_1}{X - X_1}$$

and $$TRi = NP \frac{X_2 - X_1}{X - X_1} i$$

Adding $$ST + TRi = (MN + NPi) \frac{X_2 - X_1}{X - X_1}$$

But $ST + TRi = Y_2 - Y_1$, and $MN + NPi = Y - Y_1$

Therefore $$Y_2 - Y_1 = (Y - Y_1) \frac{X_2 - X_1}{X - X_1}$$

$$\frac{Y - Y_1}{Y_1 - Y_2} = \frac{X - X_1}{X_1 - X_2} \quad . \qquad (1)$$

This is the two point form of the equation of the straight line.

Equation (1) can be thrown into the form

$$Y - Y_1 = \frac{Y_2 - Y_1}{X_2 - X_1} (X - X_1)$$

But $\frac{Y_2 - Y_1}{X_2 - X_1}$ has been defined as the slope of the line. Therefore the equation of the line through the point (x_1, Y_1), which has the slope m, is

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[illegible]

Also [illegible]

Therefore [illegible]

and [illegible]

Adding [illegible]

But [illegible]

Therefore $y_1 - y_2 = (y - y_1)\frac{x_1 - x_2}{x - x_1}$ [illegible]

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$ [illegible]

This is the two point form of the equation of the straight line.

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$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$$

But $\frac{y_1 - y_2}{x_1 - x_2}$ has been defined as the slope of the line. Therefore the equation of the line through the point (x_1, y_1), which has the slope m, is

$$y - y_1 = m(x - x_1) \quad (2)$$

This is the point-slope form of the equation of the straight line.

If the line (2) happens to go through the origin, the point (0,0) can be taken as the reference point, and the equation becomes

$$y = mx \quad (3)$$

If (2) does not go through the origin, it willcut the coordinate plane, $x = 0$, at some point, (0,b). If we use this as the reference point, (2) takes the form

$$y = mx + b \quad (4)$$

If we define the Y-intercept of a line to be the ordinate of the point in which the line cuts the plane, $x = 0$, then b is the Y-intercept of line (4), and (4) is the equation of the line in the slope-intercept form.

To find the angle between two lines whose slopes are $(a + ib)$ and $(c + id)$.

For convenience let the two lines go through the origin. Then they will cut the plane, $x = 1$, in the points $(1, a + ib)$ and $(1, c + id)$ respectively. These are the points Pand P_1 of Fig. 5.

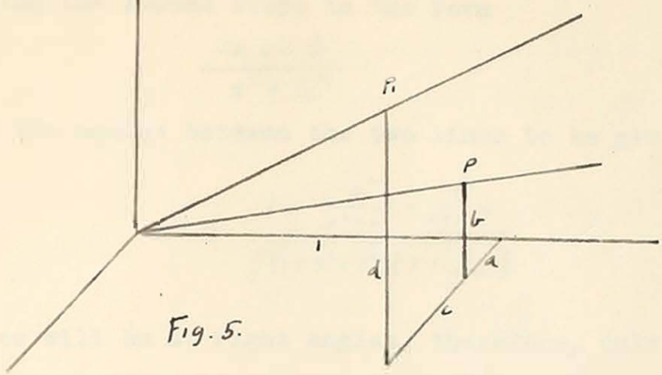

Fig. 5.

The cosines of the angles between OP_1 and the three coordinate axes are

$$\frac{1}{OP_1}, \quad \frac{c}{OP_1}, \quad \frac{d}{OP_1} \qquad \text{respectively.}$$

Let θ be the angle between the two lines. Now the projection of OP on OP_1 is equal to the sum of the projections of the lines marked 1, a, and b, in the figure. Therefore

$$OP \cos\theta = \frac{1}{OP_1} + \frac{ac}{OP_1} + \frac{bd}{OP_1}$$

$$\cos\theta = \frac{1+ac+bd}{OP_1 \, . \, OP}$$

But $OP = \sqrt{1+a^2+b^2}$, and $OP_1 = \sqrt{1+c^2+d^2}$

Therefore $$\cos\theta = \frac{1+ac+bd}{\sqrt{(1+a^2+b^2)(1+c^2+d^2)}}$$

The lines will be at right angles to each other if $ac+bd = -1$. If the slopes of the two lines be negative reciprocals, that is, if the slopes be

$$a+ib \qquad \text{and} \qquad \frac{-1}{a+ib}$$

Fig. 4

The cosines of the angles between OP, and the coordinate axes are

$\frac{l}{OP_1}$, [illegible] respectively.

Let θ be the angle between the two lines. Now the projection of OP on OP_1 is equal to the sum of the projections of the lines marked l, m, and n, in the figure. Therefore

$OP \cos\theta = \frac{l}{OP_1} + [illegible]$

$\cos\theta = [illegible]$

But $OP = \sqrt{1+k^2+k'^2}$ and $OP_1 = \sqrt{l^2+m^2+n^2}$

Therefore $\cos\theta = \frac{l + mk + nk'}{\sqrt{1+k^2+k'^2}\sqrt{l^2+m^2+n^2}}$

The lines will be at right angles to each other if $mm' = -1$, if the slopes of the two lines be negative reciprocals, that is, if the slopes be

[illegible] and [illegible]

on writing the second slope in the form

$$\frac{-a+ib}{a^2+b^2}$$

we find the angles between the two lines to be given by

$$\cos\theta = \frac{1 - \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}}{\sqrt{(1+a^2+b^2)(1+\frac{1}{a^2+b^2})}} .$$

The lines will be at right angles, therefore, only in case $b = 0$; that is, in case the slopes are real. The test for perpendicularity, used in plane geometry, falls down in the general case, when the slopes are complex, and is valid in this system only in the special case, when the slopes are real.

To find the condition that two straight lines may intersect.

Given the two straight lines

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

Unless $m_1 = m_2$ these equations can always be solved for x. Solving, we get

$$x = -\frac{b_1 - b_2}{m_1 - m_2}$$

Since b_1, b_2, m_1, and m_2 are all complex,

$$-\frac{b_1 - b_2}{m_1 - m_2}.$$

will, in general, be complex. But if the two lines are to intersect, they must intersect at some point, the abscissa of which is real. Therefore if $Y = m_1 x + b_1$ and $Y = m_2 x + b_2$ are to intersect,

$$-\frac{b_1 - b_2}{m_1 - m_2} \quad \text{must be real.}$$

We shall now determine whether this condition corresponds to the condition for intersection given in ordinary solid geometry.

Consider the two straight lines, whose equations, when written in the ordinary form, are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad\qquad \frac{x-a}{\lambda} = \frac{y-b}{\mu} = \frac{z-c}{\nu}$$

$$y = \beta + \frac{m}{l}(x-\alpha) \qquad\qquad y = b + \frac{\mu}{\lambda}(x-a)$$

$$z = \gamma + \frac{n}{l}(x-\alpha) \qquad\qquad z = c + \frac{\nu}{\lambda}(x-a)$$

In the complex form, $(Y = y + iz)$, these equations become

$$Y = \frac{m+in}{l}x + \beta - \frac{\alpha m}{l} + i\left(\gamma - \frac{\alpha n}{l}\right)$$

$$Y = \frac{\mu-i\nu}{\lambda}x + b - \frac{a\mu}{\lambda} - i\left(c - \frac{a\nu}{\lambda}\right) .$$

Solving these two equations simultaneously for x,

$$x = \frac{\begin{vmatrix} \beta - \frac{\alpha m}{l} + i\left(\gamma - \frac{\alpha n}{l}\right) & 1 \\ b - \frac{a\mu}{\lambda} + i\left(c - \frac{a\nu}{\lambda}\right) & 1 \end{vmatrix}}{\begin{vmatrix} -\frac{m+in}{l} & 1 \\ -\frac{\mu+i\nu}{\lambda} & 1 \end{vmatrix}}$$

$$x = \frac{\begin{vmatrix} \beta - \frac{\alpha m}{l} & 1 \\ b - \frac{a\mu}{\lambda} & 1 \end{vmatrix} + i\begin{vmatrix} \gamma - \frac{\alpha n}{l} & 1 \\ c - \frac{a\nu}{\lambda} & 1 \end{vmatrix}}{\begin{vmatrix} -\frac{m}{l} & 1 \\ -\frac{\mu}{\lambda} & 1 \end{vmatrix} + i\begin{vmatrix} -\frac{n}{l} & 1 \\ -\frac{\nu}{\lambda} & 1 \end{vmatrix}}$$

Multiplying both numerator and denominator by

$$\begin{vmatrix} -\frac{m}{l} & 1 \\ -\frac{\mu}{\lambda} & 1 \end{vmatrix} - i \begin{vmatrix} -\frac{n}{l} & 1 \\ -\frac{\nu}{\lambda} & 1 \end{vmatrix},$$

we get

$$x = \frac{S_1 + i\begin{vmatrix} \frac{n}{l} & 1 \\ \frac{\nu}{\lambda} & 1 \end{vmatrix} \times \begin{vmatrix} \beta - \frac{\alpha m}{l} & 1 \\ b - \frac{a\mu}{\lambda} & 1 \end{vmatrix} + i\begin{vmatrix} -\frac{m}{l} & 1 \\ -\frac{\mu}{\lambda} & 1 \end{vmatrix} \times \begin{vmatrix} \gamma - \frac{\alpha n}{l} & 1 \\ c - \frac{a\nu}{\lambda} & 1 \end{vmatrix}}{S_2}$$

where S_1 and S_2 are some, real numbers. If x is to be real, the coefficient of i in the above expression must be equal to zero. Hence

$$\begin{vmatrix} \frac{n}{l} & 1 \\ \frac{\nu}{\lambda} & 1 \end{vmatrix} \times \begin{vmatrix} \beta l - \alpha m & l \\ b\lambda - a\mu & \lambda \end{vmatrix} + \begin{vmatrix} -\frac{m}{l} & 1 \\ -\frac{\mu}{\lambda} & 1 \end{vmatrix} \times \begin{vmatrix} l\gamma - \alpha n & l \\ \lambda c - a\nu & \lambda \end{vmatrix} = 0$$

Expanding,

$$\left(\frac{n}{l} - \frac{\nu}{\lambda}\right)(\beta\lambda l - \alpha m\lambda - b\lambda l + a\mu l) + \left(\frac{\mu}{\lambda} - \frac{m}{l}\right)(l\gamma\lambda - \alpha n\lambda - \lambda c l + a\nu l) = 0$$

$$n\beta\lambda - \frac{\alpha m\lambda n}{l} - nb\lambda + a\mu n - \nu\beta l + \alpha m\nu + bl\nu - \frac{a\mu l\nu}{\lambda} + l\gamma\mu - \alpha n\mu$$
$$- cl\mu - \frac{a\nu l\mu}{\lambda} - \gamma\lambda m + \frac{\alpha m\lambda n}{l} + \lambda cm - ma\nu = 0$$

$$(\alpha - a)(m\nu - n\mu) + (\beta - b)(n\lambda - l\nu) + (\gamma - c)(l\mu - m\lambda) = 0$$

$$\begin{vmatrix} \alpha - a & \beta - b & \gamma - c \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix} = 0$$

This is the ordinary condition of solid geometry that two straight lines may intersect.

THE PLANE

The equation $x = k$ represents a plane perpendicular to the x-axis and at a distance k from the origin. An equation in the form $Y = f(x)$, where x is the only variable present in the function $f(x)$, cannot represent a plane, or a surface of any kind, for this equation gives only a finite number of values of Y for every assigned value of x. That is, geometrically, a plane $x = k$ cuts the locus $Y = f(x)$ in only a finite number of points, and since a plane would, in general, cut a surface in an infinite number of points, $Y = f(x)$ cannot be a surface. If however we have an equation involving a variable parameter t, which shall always be real, the equation of our locus takes the form $Y = g(x,t)$, and we have an infinite number of values of Y for every assigned value of x. The locus in this case will, in general, be a surface.

The plane can best be studied by considering the family of straight lines $Y = m(t)x + b(t)$, where $m(t)$ and $b(t)$ are such functions of t that all the lines of the family lie in a plane.

To find the equation of the plane determined by three points, (x_1, Y_1), (x_2, Y_2), (x_3, Y_3).

The line joining (x_1, Y_1), and (x_2, Y_2) is

$$\frac{Y - Y_1}{Y_1 - Y_2} = \frac{X - X_1}{X_1 - X_2} \qquad (1)$$

This line has for its slope,

$$\frac{Y_1 - Y_2}{X_1 - X_2}.$$

The plane to be determined can be thought of as a family of lines each of which is parallel to ① and cuts the line joining (x_1, Y_1), and (x_3, Y_3). That is

$$y - y_t = \frac{y_1 - y_2}{x_1 - x_2}(x - x_t), \qquad ②$$

where

$$\frac{y_t - y_1}{y_1 - y_3} = \frac{x_t - x_1}{x_1 - x_3}. \qquad y_t = y_1 + m_{13}(x_t - x_1) \qquad ③$$

Now we can let x_t be our variable parameter; then eliminating y_t between ② and ③, we get

$$y = \frac{y_1 - y_2}{x_1 - x_2}(x - x_t) + \frac{y_1 - y_3}{x_1 - x_3}(x_t - x) + y_1$$

or putting $t = x_t$,

$$y = \frac{y_1 - y_2}{x_1 - x_2}(x - t) + \frac{y_1 - y_3}{x_1 - x_3}(t - x) + y_1 \qquad ④$$

We can check our result in this case, for if this be the desired plane ④ should be satisfied for each of the three points, (x_1, Y_1), (x_2, Y_2), (x_3, Y_3). That is if we put in these values of x and Y, there should be some real value of t for which ④ is identically satisfied.

Solving ④ for t,

$$t = \frac{y_1 - y + x\dfrac{y_1 - y_2}{x_1 - x_2} - x_1\dfrac{y_1 - y_3}{x_1 - x_3}}{\dfrac{y_1 - y_2}{x_1 - x_2} - \dfrac{y_1 - y_3}{x_1 - x_3}} \qquad ⑤$$

The plane to be determined can be thought of as a family of lines each of which is parallel to ⑦ and cuts the line joining (x_1, Y_1), and (x_2, Y_2). That is

$$y - y_1 = \frac{Y - Y_1}{x_1 - x_2}(x - x_1), \quad ②$$

where

$$\frac{Y - Y_1}{Y - Y_2} = \frac{x_1 - x}{x - x_2}. \quad ③$$

Now we can let x_1 be our variable parameter, then eliminating Y between ② and ③ we get

$$y = \frac{Y_2 - Y_1}{x_2 - y_1}(x - x_1) + \frac{Y_2 - Y_1}{x_2 - x_1}(x_1 - x) + Y_1$$

or putting $Y = Y_1$

$$y = \frac{Y_2 - Y_1}{x_2 - x_1}(x - x_1) + \frac{Y_2 - Y_1}{x_2 - x_1}(x - x_1) + Y_1 \quad ④$$

We can check our result in this case, for it must be that desired plane ④ should be satisfied for each of the three points, $(x_1, Y, 1)$, $(x_2, Y, 1)$, (x_3, Y_3). Then as if we put in these values of x and Y, there should be some real value of x for which ⑤ is identically satisfied.
Solving ④ for x,

$$x = \frac{y - Y + x\frac{Y_2 - Y_1}{x_2 - x_1} - x_1\frac{Y_2 - Y_1}{x_2 - x_1}}{\frac{Y_2 - Y_1}{x_2 - x_1} - \frac{Y_2 - Y_1}{x_2 - x_1}} \quad ⑥$$

for $x = x_1$, $y = y_1$,

$$t = \frac{y_1 - y + x_1\left(\frac{y_1-y_2}{x_1-x_2} - \frac{y_1-y_3}{x_1-x_3}\right)}{\frac{y_1-y_2}{x_1-x_2} - \frac{y_1-y_3}{x_1-x_3}} = x_1 ,$$

which is real. For $x = x_2$, $y = y_2$,

$$t = \frac{y_1 - y_2 + x_2\frac{y_1-y_2}{x_1-x_2} - x_1\frac{y_1-y_3}{x_1-x_3}}{\frac{y_1-y_2}{x_1-x_2} - \frac{y_1-y_3}{x_1-x_3}}$$

$$= \frac{\left(1 - \frac{x_2}{x_1-x_2}\right)(y_1-y_2) - x_1\left(\frac{y_1-y_3}{x_1-x_3}\right)}{\frac{y_1-y_2}{x_1-x_2} - \frac{y_1-y_3}{x_1-x_3}}$$

$$= \frac{x_1\frac{y_1-y_2}{x_1-x_2} - x_1\frac{y_1-y_3}{x_1-x_3}}{\frac{y_1-y_2}{x_1-x_2} - \frac{y_1-y_3}{x_1-x_3}} = x_1 ,$$

which is real. Observe that t has the same value for (x_2, Y_2) that it had for (x_1, Y_1), which is what we might expect since the same line of the family goes through both (x_2, Y_2) and (x_1, Y_1). For $x = x_3$, $y = y_3$,

$$t = \frac{(y_1-y_3)\left(1 - \frac{x_1}{x_1-x_3}\right) + x_3\frac{y_1-y_2}{x_1-x_2}}{\frac{y_1-y_2}{x_1-x_2} - \frac{y_1-y_3}{x_1-x_3}}$$

$$= \frac{x_3\frac{y_1-y_2}{x_1-x_2} - x_3\frac{y_1-y_3}{x_1-x_3}}{\frac{y_1-y_2}{x_1-x_2} - \frac{y_1-y_3}{x_1-x_3}} = x_3 ,$$

which is real. Therefore 4 is satisfied for each of the three points.

To find the plane determined by two intersecting straight lines.

Given the two straight lines in space

$$y = m_1 x + b_1 \,, \qquad (6)$$

$$y = m_2 x + b_2 \,, \qquad (7)$$

such that

$$\frac{b_1 - b_2}{m_1 - m_2} \text{ is real.} \qquad (8)$$

Consider the family of lines that go through some point of (6), say $(0, b_1)$ and always intersects (7).

$$y = m(t)\, x + b_1 \,. \qquad (9)$$

This is satisfied for $(0, b_1)$. Now it is left for us to choose $m(t)$ such a function of t that (9) and (7) will intersect for all values of t.

$$y = m(t)\, x + b_1 \,,$$

$$y = m_2 x + b_2 \,.$$

These will cut each other if

$$\frac{b_1 - b_2}{m(t) - m_2} \text{ is real.}$$

where This condition is satisfied for $m(t) = (b_1 - b_2)t + m_2$. Hence the plane determined by (6) and (7) is

$$y = (b_1 - b_2)\, t x + m_2 x + b_1 \,. \qquad (10)$$

Again we can check our result, for if (10) be the desired plane, the equation should be satisfied for every point on (6) and (7). Therefore putting in $Y = m_1 x + b_1$, we get

$$m_1 x + b_1 = (b_1 - b_2)\, t x + m_2 x + b_1$$

$$t = \frac{m_1 - m_2}{b_1 - b_2} \,,$$

To find the plane determined by two intersecting straight lines.

Given the two straight lines in space

$y = m_1 x + t_1$, [illegible]

$y_1 = m_2 x + t_2$, [illegible]

such that

[illegible] is real. [illegible]

Consider the family of lines that go through each point of [illegible] , say $(0, y)$ and always intersects [illegible] .

$y = [illegible]$ [illegible]

This is satisfied for $(0, y)$. Now it is left for us to choose $m(t)$ such a function of t that [illegible] will intersect for all values of t.

$y = [illegible]$,

$y_1 = [illegible]$.

These will cut each other if

[illegible] is real.

This condition is satisfied for $m(t) = [illegible]$.

Hence the plane determined by [illegible] and [illegible] is

$y_1 = [illegible]$ [illegible]

Again we can check our result, for if [illegible] is the desired plane, the equation should be satisfied for every point on [illegible] and [illegible] . Therefore putting in [illegible] , we get

[illegible]

which is real because of condition (8). Here we notice that all points on (6) are satisfied for the same valueof t, which is what we might expect since (6) is a member of the family. For $Y = m_2 x + b_2$, we get,

$$m_2 x + b_2 = (b_1 - b_2) t x + m_2 x + b_1$$

$$t = -\frac{1}{x} \,,$$

which is real. In this case there is a different value of t for each point on the line since (7) is not a member of the family.

To find the equation of the straight line which is the intersection of two given planes.

Let $Y = H(x,t)$ and $Y = G(x,t)$ be the equations of two given planes, t being the same parameter in both cases. Cutting the two planes by a third plane $x = k$, we get in the plane $x = k$ two straight lines, $Y = H(k,t)$ and $Y = G(k,t)$. The elimination of t between these two equations gives a value of Y, which together with $x = k$, determines the common point of the two lines. If now we put in for k the variable, x, we get the locus of the common points of $Y = H(x,t)$ and $Y = G(x,t)$, or the line common to the two planes. Therefore the line of intersection of two given planes, where the same parameter is used for each plane, is determined by eliminating the parameter between the equations of the planes. See Fig. 8. page 34.

It will be of interest to apply this method in two cases where the results are already known. Consider the two planes which is real because of condition C. Here we notice that all points on C are satisfied for the same values of t, which is what we might expect since C is a member of the family. For $Y = mX + b$, we get,

[illegible]

$t = \frac{1}{2}$ [illegible]

which is real. In this case there is a different value of t for each point on the line since C is not a member of the family.

To find the equation of the straight line which is the intersection of two given planes.

Let $Y = f(X,t)$ and $Y = \phi(X,t)$ be the equations of two given planes, t being the same parameter in both cases. Cutting the two planes by a third plane $X = k$, we get in the plane $x = k$ two straight lines, $Y = f(k,t)$ and $Y = \phi(k,t)$. The elimination of t between these two equations gives a value of Y, which together with $x = k$, determines the common point of the two lines. If now we put in for k the variable, x, we get the locus of the common points of $Y = f(x,t)$ and $Y = \phi(x,t)$, or the line common to the two planes. Therefore the line of intersection of two given planes, where the same parameter is used for each plane, is determined by eliminating the parameter between the equations of the planes. See Fig. 8, page 34.

It will be of interest to apply this method to two cases where the results are already known. Consider the two planes

$$y = \frac{y_1 - y_2}{x_1 - x_2}(x - t) + \frac{y_1 - y_3}{x_1 - x_3}(t - x) + y_1, \qquad (1)$$

$$y = \frac{y_1 - y_4}{x_1 - x_4}(x - t) + \frac{y_1 - y_3}{x_1 - x_3}(t - x) + y_1. \qquad (2)$$

Eliminating t,

$$\frac{y_1 - y + x\frac{y_1 - y_2}{x_1 - x_2} - x_1\frac{y_1 - y_3}{x_1 - x_3}}{\frac{y_1 - y_2}{x_1 - x_2} - \frac{y_1 - y_3}{x_1 - x_3}} = \frac{y_1 - y + x\frac{y_1 - y_4}{x_1 - x_4} - x_1\frac{y_1 - y_3}{x_1 - x_3}}{\frac{y_1 - y_4}{x_1 - x_4} - \frac{y_1 - y_3}{x_1 - x_3}}.$$

Simplifying,

$$(y_1 - y)\left(\frac{y_1 - y_4}{x_1 - x_4} - \frac{y_1 - y_2}{x_1 - x_2}\right) + x\frac{y_1 - y_2}{x_1 - x_2}\left(\frac{y_1 - y_4}{x_1 - x_4} - \frac{y_1 - y_3}{x_1 - x_3}\right) - x_1\frac{y_1 - y_3}{x_1 - x_3}\left(\frac{y_1 - y_4}{x_1 - x_4} - \frac{y_1 - y_3}{x_1 - x_3}\right)$$
$$- x\frac{y_1 - y_4}{x_1 - x_4}\left(\frac{y_1 - y_2}{x_1 - x_2} - \frac{y_1 - y_3}{x_1 - x_3}\right) + x_1\frac{y_1 - y_3}{x_1 - x_3}\left(\frac{y_1 - y_2}{x_1 - x_2} - \frac{y_1 - y_3}{x_1 - x_3}\right) = 0$$

$$(y_1 - y)\left(\frac{y_1 - y_4}{x_1 - x_4} - \frac{y_1 - y_2}{x_1 - x_2}\right) + (x - x_1)\frac{y_1 - y_3}{x_1 - x_3}\left(\frac{y_1 - y_4}{x_1 - x_4} - \frac{y_1 - y_2}{x_1 - x_2}\right) = 0$$

$$\frac{y - y_1}{y_1 - y_3} = \frac{x - x_1}{x_1 - x_3}. \qquad (3)$$

But since (1) and (2) are the planes determined by (x_1, Y_1), (x_3, Y_3), (x_2, Y_2), and (x_1, Y_1), (x_3, Y_3), (x_4, Y_4) we know that (3) is the line of intersection.

Again, consider the two planes

$$y = (b_1 - b_2)tx + m_2x + b_1, \qquad (4)$$

$$y = (b_3 - b_2)tx + m_2x + b_3. \qquad (5)$$

Eliminating t,

$$\frac{y - m_2 x - b_1}{b_1 - b_2} = \frac{y - m_2 x - b_3}{b_3 - b_2} ,$$

Clearing of fractions,

$$(y - m_2 x)(b_3 - b_1) + b_1 b_2 - b_1 b_3 + b_1 b_3 - b_2 b_3 = 0$$

$$(y - m_2 x - b_2)(b_3 - b_1) = 0$$

$$y = m_2 x + b_2 ,$$

which we know to be the line determined by 4 and 5 .

CURVES IN SPACE

The equation $Y = f(x)$, where $f(x)$ is a complex function, and x a real variable, picks out a point in space for each value of x. Therefore if $f(x)$ is a continuous function, $Y = f(x)$ represents a continuous curve in space.

To find the equation in the form $Y = f(x)$ of the curve whose equations in ordinary coordinates are

$$x = \varphi(t), \quad y = \chi(t), \quad z = \psi(t). \qquad (1)$$

We can from the first of these equations express t as a function of x, $t = \varphi^{-1}(x)$. Then the equation of the curve can be put in the form

$$y = \chi(\varphi^{-1}(x)), \quad z = \psi(\varphi^{-1}(x)).$$

But these two equations, for any value of x, say $x = x_1$, simply pick out a point in the plane $x = x_1$. This point can be determined equally well in complex coordinates by

$$Y = \chi(\varphi^{-1}(x_1)) + i\,\psi(\varphi^{-1}(x_1)) .$$

If now we replace x_1 by x, we pick out a point for every value of x, and get the required curve in space. Therefore the equations

$$Y = \chi(\varphi^{-1}(x)) + i\,\psi(\varphi^{-1}(x)).$$

and $\quad x = \varphi(t), \quad y = \chi(t), \quad z = \psi(t),$

represent the same curve.

Given the equation of the helix in the ordinary form

$$x = rt\tan\alpha, \quad y = r\cos t, \quad z = r\sin t.$$

To find its equation in the form Y = f(x).

$$t = \frac{x}{r\tan\alpha}$$

$$y = r\cos\frac{x}{r\tan\alpha}$$

$$z = r\sin\frac{x}{r\tan\alpha}$$

$$Y = r\left(\cos\frac{x}{r\tan\alpha} + i\sin\frac{x}{r\tan\alpha}\right)$$

$$Y = r\,e^{\frac{ix}{r\tan\alpha}}$$

Putting $\frac{1}{r\tan\alpha} = m$ we find the general equation of the helix to be

$$Y = r\,e^{imx}.$$

It can readily be shown, that in ordinary geometry, if we have a plane curve $\rho = f(\theta)$, and if we roll the plane of the curve into the cone $y^2 + z^2 + x^2\tan^2\alpha = 0$, the equation of the resulting space curve will be

$$x = \rho\cos\alpha$$

$$y = \rho\sin\alpha\cos(\theta\csc\alpha)$$

$$z = \rho\sin\alpha\sin(\theta\csc\alpha).$$

If we consider the plane curve $\rho = k\theta$, we get

$$x = k\theta\cos\alpha$$

$$y = k\theta\sin\alpha\cos(\theta\csc\alpha)$$

$$z = k\theta\sin\alpha\sin(\theta\csc\alpha).$$

Throwing this into the complex form by the method outlined above,

$$\theta = \frac{x}{K \cos \alpha}$$

$$y = x \tan \alpha \cos \frac{x}{K \sin \alpha \cos \alpha}$$

$$z = x \tan \alpha \sin \frac{x}{K \sin \alpha \cos \alpha}$$

$$Y = \left(\cos \frac{x}{K \sin \alpha \cos \alpha} + i \sin \frac{x}{K \sin \alpha \cos \alpha} \right) x \tan \alpha$$

$$Y = e^{\frac{ix}{K \sin \alpha \cos \alpha}} \, x \tan \alpha . \qquad (a)$$

Compare this with equation (a'), page 28.

Likewise, in the general case, where the plane curve takes the form $\theta = f(\rho)$,

$$x = \rho \cos \alpha$$
$$y = \rho \sin \alpha \cos (f(\rho) \csc \alpha)$$
$$z = \rho \sin \alpha \sin (f(\rho) \csc \alpha)$$

$$\rho = \frac{x}{\cos \alpha}$$

$$y = x \tan \alpha \cos \left(\frac{f\left(\frac{x}{\cos \alpha}\right)}{\sin \alpha} \right)$$

$$= x \tan \alpha \sin \left(\frac{f\left(\frac{x}{\cos \alpha}\right)}{\sin \alpha} \right) .$$

Hence, in the complex form,

$$Y = \left(\cos \frac{f\left(\frac{x}{\cos \alpha}\right)}{\sin \alpha} + i \sin \frac{f\left(\frac{x}{\cos \alpha}\right)}{\sin \alpha} \right) x \tan \alpha$$

$$Y = e^{\frac{i f\left(\frac{x}{\cos \alpha}\right)}{\sin \alpha}} \, x \tan \alpha . \qquad (b)$$

Compare this with equation (b'), page 29.

[illegible]

Compare this with equation [illegible], page [illegible].

Likewise, in the general case, where the plate curve takes the form [illegible],

[illegible]

Hence, in the complex form

[illegible]

Compare this with equation [illegible], page [illegible].

Let us consider the space curve that when written in the ordinary form is

$$x = \varphi(t), \quad y = \chi(t), \quad z = \psi(t) \; . \qquad (1)$$

The tangent to this curve at any point, (x_o, y_o, z_o), is

$$\frac{x - x_o}{\varphi'(t_o)} = \frac{y - y_o}{\chi'(t_o)} = \frac{z - z_o}{\psi'(t_o)}$$

or
$$y = y_o + \frac{\chi'(t_o)}{\varphi'(t_o)}(x - x_o)$$

$$z = z_o + \frac{\psi'(t_o)}{\varphi'(t_o)}(x - x_o) \, .$$

In the complex form

$$Y = \left(\frac{\chi'(t_o)}{\varphi'(t_o)} + i \frac{\psi'(t_o)}{\varphi'(t_o)} \right)(x - x_o) + y_o + i\, x_o \qquad (2)$$

But $y_o + iz_o = Y_o$. Therefore (2) becomes

$$Y - Y_o = \left(\frac{\chi'(t_o) + i\psi'(t_o)}{\varphi'(t_o)} \right)(x - x_o) \, . \qquad (3)$$

And (3) is the complex form of the tangent to (1) at (x_o, y_o, z_o).

If we throw (1) immediately into the complex form,

$$\left. \begin{array}{l} Y = \chi_t + i\,\psi(t) \\ x = \varphi(t) \end{array} \right\} \qquad (4)$$

From (4),

$$\frac{dy}{dx} = \left(X'(t) + i\,\psi'(t) \right) \frac{dt}{dx}$$

But $$\frac{dt}{dx} = \frac{1}{\varphi'(t)}$$

Therefore $$\frac{dy}{dx_0} = \frac{X'(t_0) + i\,\psi'(t_0)}{\varphi'(t_0)} . \qquad (5)$$

But the right hand member of (5) is the coefficient of $(x-x_0)$ in (3). Therefore equation (3) can be written in the form

$$y - y_0 = \frac{dy}{dx_0}(x - x_0) .$$

Hence we can say the equation of the tangent to any curve in space, $Y = f(x)$, at (x_0, Y_0) is

$$y - y_0 = f'(x_0)(x - x_0) .$$

We have seen that the equation of a plane must contain a variable parameter. It is equally true, and for the same reasons, that the equation of any surface must contain a variable parameter. It is also true that, since the parameter is an arbitrary one, the equation of the same surface may take an infinite number of forms.

To find the equation of the surface of revolution about the x-axis.

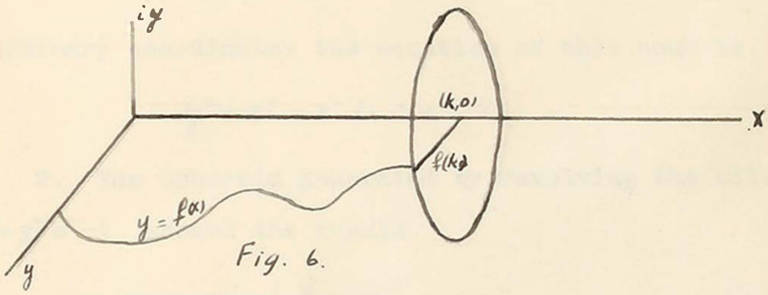

Fig. 6.

If we have a curve, $y = f(x)$, in the xoy plane, and we revolve that curve around the x-axis, we get e surface, which, if cut by the plane $x = k$, will give a circular section. Moreover the radius of this circle is $f(k)$. Now the equation of this circle in the $x = k$ plane is

$$y = (\cos t + i \sin t)\, f(k)\,.$$

But the locus of all these circles is the required surface, hence, putting in for k the running coordinate x we get

$$y = (\cos t + i \sin t)\, f(x)$$

as the equation of the surface of revolution of the plane curve, $y = f(x)$, about the x-axis. This equation can be thrown into the simplified form

We have seen that the equation of a plane must contain a variable parameter. It is equally true, and for the same reasons, that the equation of any surface must contain a variable parameter. It is also true that, since the parameter is an arbitrary one, the equation of the same surface may take an infinite number of forms.

To find the equation of the surface of revolution about the x-axis.

Fig. 4

If we have a curve, $y = f(x)$, in the xy plane, and we revolve that curve around the x-axis, we get a surface, which, if cut by the plane $x = k$, will give a circular section. Moreover the radius of this circle is $f(k)$. Now the equation of this circle in the $x = k$ plane is

$$y^2 + z^2 = [f(k)]^2 .$$

But the locus of all these circles is the required surface, hence, putting in for k the running coordinate x we get

$$y^2 + z^2 = [f(x)]^2$$

as the equation of the surface of revolution of the plane curve, $y = f(x)$, about the x-axis. This equation can be thrown into the simplified form

$$Y = e^{it} f(x) . \qquad (1)$$

The equation of this surface in the ordinary system is

$$\sqrt{y^2 + z^2} = f(x) ,$$

By making use of equation (1) we can write down the equations of several well known surfaces of revolution.

1. The cone whose axis is the x-axis and whose generator is $y = x\tan\alpha$ is

$$Y = e^{it} x \tan\alpha . \qquad (2)$$

In ordinary coordinates the equation of this cone is

$$y^2 + z^2 - x^2 \tan^2\alpha = 0$$

2. The spheroid generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$ around the x-axis.

$$y = b\sqrt{1 - x^2/a^2}$$

Hence from (1) ,

$$Y = be^{it}\sqrt{1 - \frac{x^2}{a^2}} ,$$

In ordinary coordinates the equation of this surface is $x^2/a^2 + y^2/b^2 + z^2/b^2 = 1$. See Fig. 9. page 34.

3. The sphere of radius a, center at the origin.

$$x^2 + y^2 = a^2$$

$$y = \sqrt{a^2 - x^2}$$

Hence from (1) ,

$$Y = e^{it}\sqrt{a^2 - x^2} .$$

In ordinary coordinates the equation of the sphere is $x^2 + y^2 + z^2 = a$.

We can get the above equations in the complex form from the equation in the ordinary form in the following manner.

1. $y^2 + z^2 + x^2 \tan^2\alpha = 0.$ ④

Introduce an arbitrary convenient parameter, say

$$t = \tan^{-1}\frac{z}{y}$$

From ⑤

$$z^2 = y^2\tan^2 t, \qquad y^2 = z^2/\tan^2 t.$$

Introducing these values in ④

$$y^2(1+\tan^2 t) = x^2\tan^2\alpha$$
$$z^2(1+1/\tan^2 t) = x^2\tan^2\alpha$$

$$y^2 = \frac{x^2\tan^2\alpha}{\sec^2 t}, \qquad y = x\tan\alpha . \cos t .$$

$$z^2 = \frac{x^2\tan^2\alpha}{\csc^2 t}, \qquad z = x\tan\alpha . \sin t .$$

$$Y = y + iz = (\cos t + i\sin t)\, x\tan\alpha.$$

$$Y = e^{it} x\tan\alpha.$$

Compare this equation with ② above.

2. Consider the spheroid $x^2/a^2 + y^2/b^2 + z^2/b^2 = 1.$

$$y^2 + z^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

Introduce $\quad t = \tan^{-1}\frac{z}{y}$

Eliminating first y and then z,

$$z = \sin t . b\sqrt{1 - x^2/a^2}$$
$$y = \cos t . b\sqrt{1 - x^2/a^2}$$
$$Y = (\cos t + i\sin t)\, b\sqrt{1 - x^2/a^2}$$

$$Y = e^{it} b\sqrt{1 - \frac{x^2}{a^2}} .$$ Compare with ③ above.

We can now give the following general rule.

Given the equation of a surface in the ordinary form, $F(x,y,z) = 0$, to write the equation of the same surface in the complex form, $G(x,Y,t) = 0$.

Impose some arbitrary convenient relation between z, y, and t, say $P(z,y,t) = 0$. Now eliminate first y and then z between F and P, and we get the equation in the parametric form,

$$\left.\begin{array}{l} y = \varphi(x,t) \\ z = \psi(x,t) \end{array}\right\}$$

Then the equation of the surface in the complex form is

$$Y = \varphi(x,t) + i\,\psi(x,t) . \qquad (6)$$

It can readily be shown that $F(x,y,z) = 0$ and $Y = \varphi(x,t) + i\psi(x,t)$ represent the same surface, for

$$\begin{array}{l} y = \varphi(x,t) \\ z = \psi(x,t) \end{array} \qquad (7)$$

are simply the parametric equations of the surface $F(x,y,z) = 0$. Now consider the curve of section of (7) by the plane $x = k$,

$$\begin{array}{l} y = \varphi(k,t) \\ z = \psi(k,t) . \end{array}$$

This curve can be represented in complex coordinates by $Y = \varphi(k,t) + i\psi(k,t)$. By putting x in place of k we get the locus of the intersection of the surface with all the planes $x = k$. Therefore (6) and (7) represent the same surface.

We have seen that the tangent to the curve $Y = f(x)$, at the point (x_0, Y_0), is

$$y = f'(x_0)(x - x_0) + f(x_0) .$$

If in the above equation we let x_0 be a variable parameter, we get all the tangents to the curve $Y = f(x)$. In other words, if we let $x = t$, we get

$$y = f'(t)(x - t) + f(t),$$

which is the surface of tangents to $Y = f(x)$.

We know that $Y = mx + b$ represents a straight line. If we replace m and b by functions of t, the resulting equation,

$$y = m(t)\,x + b(t),$$

represents a surface of straight lines, or a ruled surface.

The condition that a ruled surface shall also be a developable surface is that consecutive lines of the surface shall intersect. Let

$$y = m(t)\,x + b(t)$$

and

$$y = m(t+\Delta t)\,x + b(t+\Delta t)$$

be two lines which in the limit shall be consecutive lines. The condition that these two lines shall intersect is that

$$\frac{b(t+\Delta t) - b(t)}{m(t+\Delta t) - m(t)} \quad \text{shall be real.}$$

$$\frac{\dfrac{b(t+\Delta t) - b(t)}{\Delta t}}{\dfrac{m(t+\Delta t) - m(t)}{\Delta t}} \quad \text{shall be real.}$$

In the limit, $\dfrac{D_t\ b(t)}{D_t\ m(t)}$ shall be real.

Therefore the condition that $Y = m(t)x + b(t)$ shall represent a developable surface is that

$$\frac{D_t\ b(t)}{D_t\ m(t)} \quad \text{shall be real.}$$

The surface of tangents is

$$Y = f'(t)\,x - f'(t)\,t + f(t).$$

The condition that this shall be a developable surface is, from above, that

$$\frac{f'(t) - f'(t) - t f''(t)}{f''(t)} \quad \text{shall be real.}$$

But

$$\frac{f'(t) - f'(t) - t f''(t)}{f''(t)} = -t$$

Therefore the condition is satisfied and the surface of tangents is a developable surface.

CURVES ON SURFACES

We have seen that, in general, an equation in the form $F(x,Y,t)=0$ represents a surface in space. If now for any value of x we assign a particular value of t, we pick out a particular point on the surface. But this can be accomplished for every value of x by letting t be a function of x. Thus we pick out a locus of points on the surface. Therefore $F(x,Y,f(x))=0$ is the equation of acurve that lies wholly on the surface $F(x,Y,t)=0$.

Given the curve $\rho = K\theta$ in a plane. To find the equation of the space curve that $\rho = k\theta$ goes into when the plane of the curve is rolled into the cone $y = e^{it} x \tan\alpha$.

$$t = \frac{\theta}{\sin\alpha}$$

$$\theta = \frac{\rho}{K}$$

$$\rho = \frac{x}{\cos\alpha}$$

$$t = \frac{x}{K\sin\alpha\cos\alpha}$$

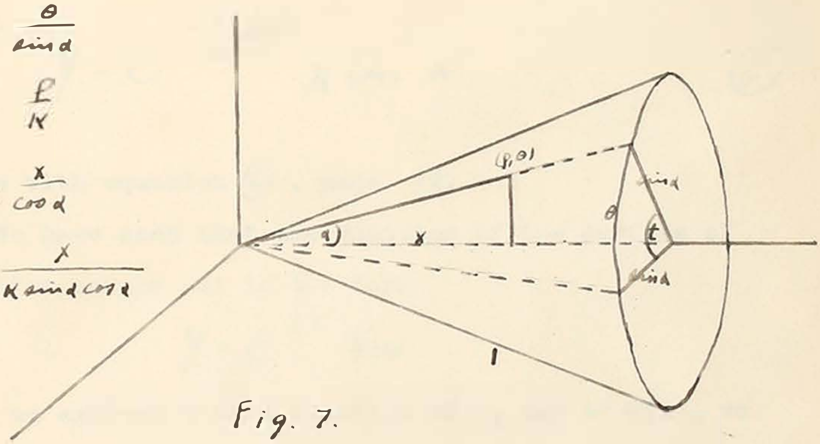

Fig. 7.

Hence the equation of the space curve becomes

$$y = e^{\frac{ix}{K\sin\alpha\cos\alpha}} x \tan\alpha. \qquad (a')$$

Compare with equation (a), page 19.

If now we consider any plane curve, $\theta = f(\rho)$, we can immediately yet the equation of the space curve that this goes into when the plane is rolled into the cone

$$y = e^{it} x \tan\alpha .$$

For as above

$$t = \frac{\theta}{\sin\alpha}$$

$$\theta = f(\rho)$$

$$\rho = \frac{x}{\cos\alpha}$$

$$t = \frac{f\left(\frac{x}{\cos\alpha}\right)}{\sin\alpha} .$$

And the equation of the curve is

$$y = e^{\frac{i f\left(\frac{x}{\cos\alpha}\right)}{\sin\alpha}} x \tan\alpha . \qquad (6')$$

Compare with equation (6), page 19.

We have seen that the equation of the surface of revolution can be put in the form

$$y = e^{it} f(x)$$

If now we express t as a function of x, say $t = \varphi(x)$, we get the equation of a curve on the surface. Therefore the equation of any curve which lies wholly on any surface of revolution can be put in the form

$$y = e^{i\varphi(x)} f(x) .$$

We have seen that the surface of tangents to the curve $Y = f(x)$ is

$$y = f'(t)\,x - f'(t)\,t + f(t)$$

If in the above equation, t is expressed as a function of x, say $t = g(x)$, the resulting equation,

$$y = f'(g(x))\,x + f'(g(x))\,g(x) + f(g(x)),$$

is the equation of a curve that lies wholly on the surface of tangents.

SUMMARY

A point in space may be determined by two numbers, one real, one complex.

The equation of the straight line determined by the two points (x_1, Y_1) and (x_2, Y_2) is

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}.$$

The equation of the straight line which has the slope m, and whose Y-intercept is b, is

$$y = mx + b.$$

Two straight lines in space,

$$y = m_1 x + b_1,$$
$$y = m_2 x + b_2,$$

will intersect if

$$\frac{b_1 - b_2}{m_1 - m_2} \quad \text{is real.}$$

The equation of a plane, or a surface, must contain a variable parameter.

The plane which is determined by the three points (x_1, Y_1), (x_2, Y_2), (x_3, Y_3), can be represented by the equation

$$y = \frac{y_1 - y_2}{x_1 - x_2}(x - t) + \frac{y_1 - y_3}{x_1 - x_3}(t - x) + y_1$$

where t is a variable parameter.

The plane determined by the intersecting straight lines,

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2,$$

where $\frac{b_1 - b_2}{m_1 - m_2}$ is real, can be represented by the equation

$$y = (b_1 - b_2)tx + m_2 x + b_1,$$

SUMMARY

A point in space may be determined by two numbers, one real, one complex.

The equation of the straight line determined by the two points (x_1, Y_1) and (x_2, Y_2) is

$$\frac{Y - Y_1}{Y_1 - Y_2} = \frac{x - x_1}{x_1 - x_2}$$

The equation of the straight line which has the slope m, and whose Y-intercept is b, is

$$Y = mx + b$$

Two straight lines in space,

$$Y = m_1 x + b_1$$
$$Y = m_2 x + b_2$$

will intersect if

$$\frac{b_1 - b_2}{m_2 - m_1} \quad \text{(real).}$$

The equation of a plane, or a surface, must contain a variable parameter.

The plane which is determined by the three points (x_1, Y_1), (x_2, Y_2), (x_3, Y_3), can be represented by the equation

$$Y = \frac{Y_1 - Y_2}{x_1 - x_2}(x - x_1) + t\frac{Y_1 - Y_3}{x_1 - x_3}(x - x_1) + Y_1$$

where t is a variable parameter.

The plane determined by the intersecting straight lines,

$$Y = m_1 x + b_1$$
$$Y = m_2 x + b_2$$

where $\frac{b_1 - b_2}{m_2 - m_1}$ is real, can be represented by the equation

$$Y = [illegible]$$

If the equations of two planes are given, with the same parameter used for both planes, the equation of the line of intersection can be gotten by eliminating the parameter.

The equation $Y = f(x)$, where $f(x)$ is a complex function and x a real variable, represents a curve in space.

The equation of the tangent at (x_0, Y_0) to the curve $Y = f(x)$ is

$$Y = f'(x_0)(x - x_0) + f(x_0)$$

Any surface of revolution can be represented by the equation

$$Y = e^{it} f(x)$$

The surface of tangents to the curve $Y = f(x)$ can be represented by the equation

$$Y = f'(t)(x - t) - f(t).$$

The equation $Y = m(t)x + b(t)$ represents a ruled surface. This surface will be a developable surface if

$$\frac{D_t\ b(t)}{D_t\ m(t)} \text{ is real.}$$

Applying the above condition to the surface of tangents, it is found that the surface of tangents is a developable surface.

The equation $F(x, Y, f(x)) = 0$ represents a curve that lies wholly on the surface $F(x, Y, t) = 0$.

The equation of any curve, which lies wholly on any surface of revolution, can be put in the form

$$Y = e^{i\varphi(x)} f(x)$$

The equation

$$Y = x f'(g(x)) - f'(g(x))\, g(x) + f(g(x))$$

is the equation of a curve which lies wholly on the surface of tangents to the curve $Y = f(x)$.

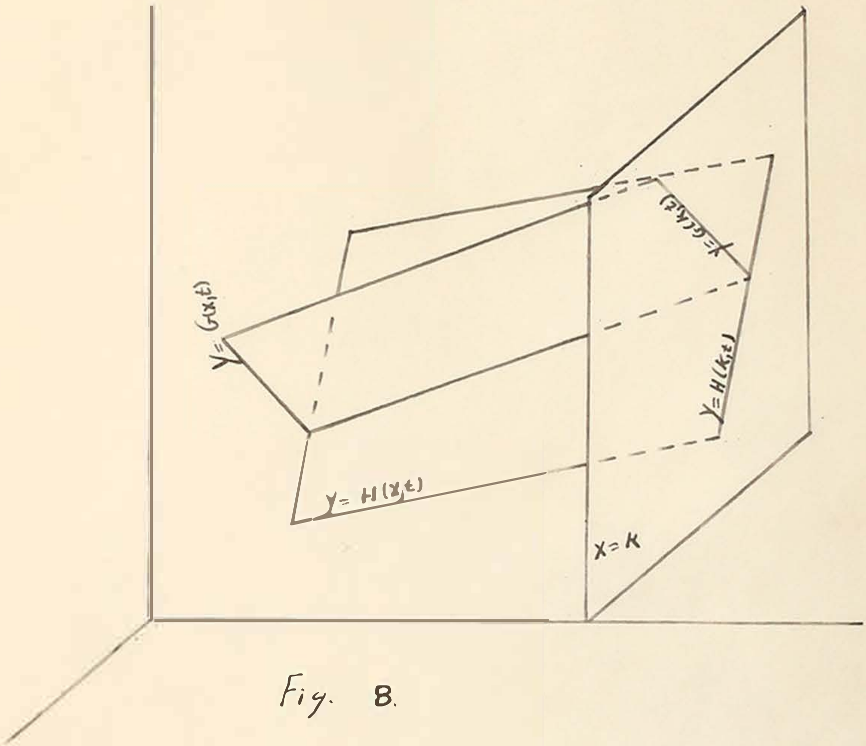

Fig. 8.

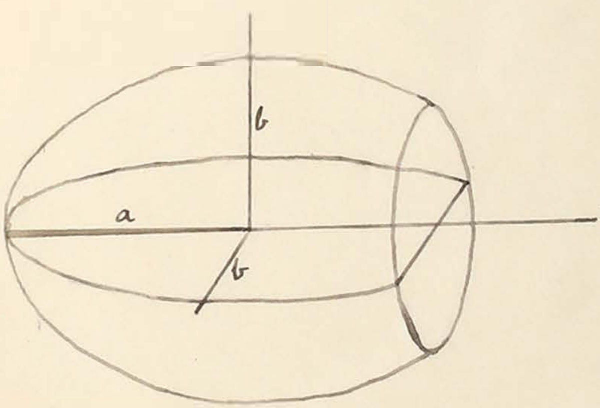

Fig. 9.