**Visualizing Complex-Valued Functions with GPU Computing**

(Technical Paper)

**Infinite Controversy: Reactions to Cantor's Theory**

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On my honor as a University Student, I have neither given nor received unauthorized aid on this assignment as defined by the Honor Guidelines for Thesis-Related Assignments

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# Visualizing Complex-Valued Functions with GPU Computing

**Abstract**

Complex-valued functions play an important role in physics, electrical engineering, and many branches of mathematics; however, to visualize such a function in the same way we do real-valued functions would require four spatial dimensions — a luxury we unfortunately lack. To address this need, mathematicians have devised alternative methods of visualizing complex-valued functions, such as *domain coloring*, *conformal mapping*, and more. Here, we explore how to efficiently implement these algorithms on graphics processing units (GPUs), which allow for greater concurrency than traditional CPUs.

**Introduction**

The Mandelbrot set consists of the points $c$ in the complex plane for which the sequence defined iteratively by $z_{n+1} = z_n^2 + c$ remains bounded, with $z_0 = 0$. Domain coloring this fractal produces images shown in Figure 1.

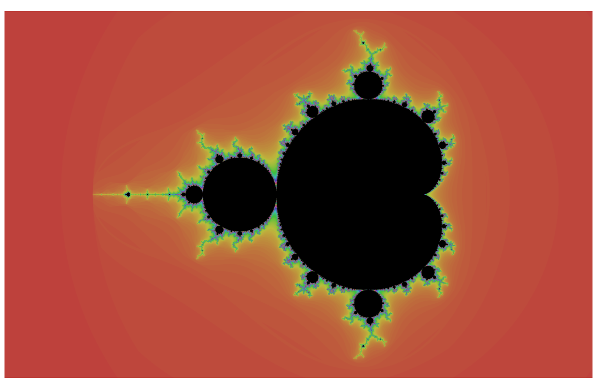

**Figure 1: Mandelbrot Set**

The Mandelbrot set itself consists of the black points; the other points are colored according to "how quickly" the sequence of $z_n$ diverges. Visualizations like the one above are expensive to compute. For example, the above image has a resolution of 2560x1600, meaning it contains over 4 million pixels. For each pixel, on the order of 100 floating point operations are required to determine that pixel's color. In an unparalleled program, each of these floating point operations would be performed sequentially. However, if we parallelize our program, such that many operations can be performed concurrently, the time to generate the image can be greatly reduced. On CPUs, one can accomplish this via multithreading or libraries like AVX-512, which enable arithmetic operations to be vectorized, so that they act on chunks of memory concurrently, rather than on a single value.

The problem is that on a modern consumer-grade CPU, the ability to multithread is rather limited. It is unlikely to be able to use more than 20 threads. Likewise, AVX-512 acts on 512 bits of memory at a time. In C++, 8 double precision floating point numbers can fit in this amount of memory. Thus, when working with variables of this type, AVX-512 intrinsics are unlikely to speed up computations by more than 8x. To speed up our computations even further, we must look to GPUs. In contrast to our CPU's hypothetical 20 threads, GPUs have on the order of thousands, which is very promising for applications requiring a high degree of parallelism, such as domain coloring.

**Related Works**

The techniques we discuss for visualizing complex-valued functions are not new; Wegert discusses domain coloring and phase portraiture extensively in (Wegert, 2016). Similarly, Frederick and Schwartz give an overview of conformal mapping (Frederick & Schwartz, 1990).

However, our work is novel in that we focus not on the mathematics of how these techniques work, but instead on how to implement them for execution on a GPU. NVIDIA, a leading GPU manufacturer, maintains a tool kit called CUDA, which enables one to write C++

style code to be executed on GPUs. The syntax of CUDA is very similar to C++, with some added decorators on functions one wishes to parallelize. NVIDIA maintains a very informative introduction to CUDA on their blog for those interested in learning (Harris, 2017).

**Project Design**

We have so far implemented two flavors of visualization: domain coloring and conformal mapping. These techniques are similar in that if we wish to visualize a complex-valued function *f*, both involve coloring a point *z* in the complex plane according to the value of *f(z)*. Color is used because *z* and *f(z)* are complex numbers, and therefore each are 2-dimensional, hence plotting *f* as we do real functions would require four spatial dimensions.

The distinction between domain coloring and conformal mapping is in how *f(z)* is used to determine the color of *z*. For domain coloring, we use the angle of *f(z)* to determine the hue of *z*, and the magnitude of *f(z)* to determine the lightness of *z*. In particular, we use the functions below to determine an HSL representation of the color of *z* (Saravanan et al., 2016).

$$H = arg(z) + \frac{2\pi}{3}$$

$$S = 1$$

$$L = \frac{2}{\pi} arctan(z)$$

However, the pixels in our image are colored using an RGB representation. Therefore, it is necessary that we convert our HSL values to RGB values. To accomplish this, we use an algorithm detailed in [3]. Using the above, we can produce domain colorings like that of the function *exp(1 / z)*, which has an essential singularity at 0 pictured in Figure 2 below.

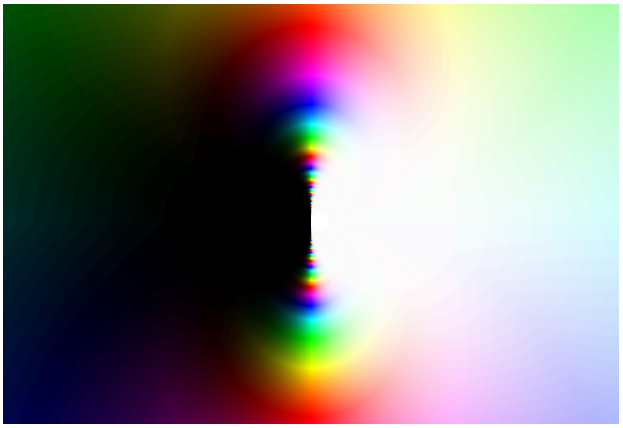

**Figure 2: Essential Singularity**

Intuitively, we can observe how the function behaves by inspecting the image. At the white portions, the function diverges off to infinity, while the black portions are where it converges to zero; where the black and white meet is the function's singularity. How the angles change is also represented by the changes in hue.

On the other hand, conformal mapping involves tiling the complex plane with a "pattern" image, such as the one pictured in the left of Figure 3 below, then applying $f$ to distort the image in the plane.

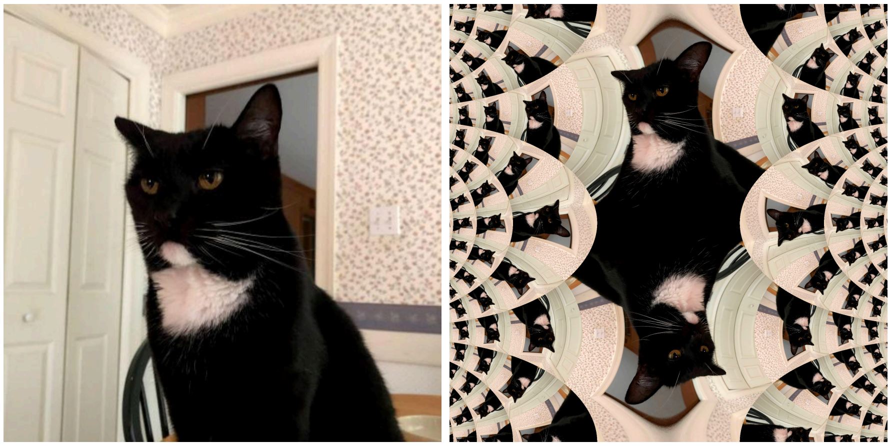

**Figure 3: Conformal Map of 2cosh($z$)**

The general algorithm for implementing domain coloring and conformal mapping on a GPU proceeds as follows: for each pixel in the output image, we determine which point in the complex plane it represents. To this point, we then apply the function we wish to produce a domain coloring of, which produces another complex number. To this number, we then apply a function for coloring the pixel depending on whether we are producing a domain coloring or a conformal mapping. Since for each pixel this process depends on no other pixels, this algorithm can be parallelized. That is, supposing we had an eight core CPU, we assign every eighth pixel to each core, so each core does ⅛ of the total work. For a CPU, eight cores is a respectable number. The code in this project was written to be run on a GTX 3060, which has 3584 CUDA cores.

After having determined the color for every pixel, we write the results to a .ppm file for output. This image format is simple enough to implement a reader and writer from scratch, and can easily be converted to more conventional image formats such as .jpg or .png should the user wish.

**Results**

The project is still ongoing, however so far we have seen significant speed-ups in the GPU implementation of the algorithm, especially on large images. The current benchmarking suite includes visualizations of 61 fractals, each with an image size of 2048x2048 pixels, and amounting to 778MB of raw RGB values. Our implementations are able to produce these visualizations in 12 seconds, or about 0.2 seconds per image. We have also started work on a website (https://elijahkin.github.io/) for documenting each of the visualizations produced.

**Conclusion**

Students of complex analysis should find the domain colorings and conformal mappings produced with our implementations pedagogically useful. Moreover, being implemented on a

GPU, it may eventually be possible to interact with the visualizations live; instead of saving them as image files, we envision the visualization opening a new window and allowing the user to zoom and pan as they wish. The fact that our visualizations are generated on a GPU makes this dream closer to reality.

## Future Work

We plan to implement other techniques of visualization going forward. For example, phase portraiture is a technique similar to domain coloring except caring only about the hue of *f(z)*, ignoring its magnitude. This is useful for functions, such as the Riemann zeta function, where ordinary domain coloring is not especially illuminating such as that shown in Figure 4.

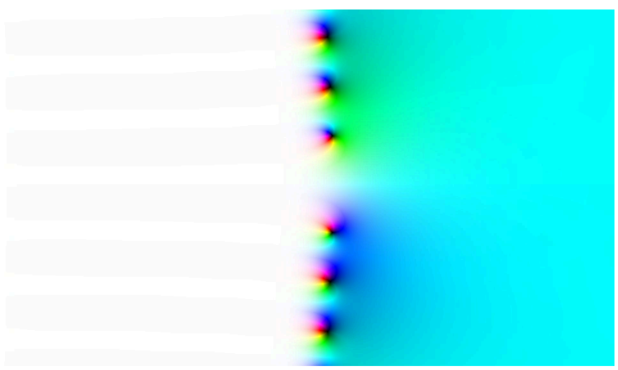

**Figure 4: Riemann Zeta Domain Coloring**

Further, we intend to implement visualizations for escape-time fractals like the Mandelbrot set and burning-ship fractals. Moreover, we will continue work on the website to more thoroughly document existing visualizations, as well as uploading new ones.

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# Infinite Controversy: Reactions to Cantor's Theory

## Introduction

As mathematics has evolved, different schools of mathematicians have evolved along with it. In the late nineteenth century, the predominant schools were formalism and pre-intuitionism, the latter of which influenced or was closely related to schools such as intuitionism, constructivism, and finitism.

In 1874, a mathematician named Georg Cantor published a proof suggesting that two sets, each containing infinitely many elements, can be of different "sizes." Today, this sentiment is often stated informally as "some infinities are bigger than others" and almost all mathematicians working today are in agreement with this. However, this was not always the case; Leopold Kronecker, Cantor's former teacher, labeled him a "scientific charlatan" and "corrupter of youth" and Henri Poincaré called his work "a grave mathematical malady, a perverse pathological illness that would one day be cured." Conversely, Bertrand Russell described him as "one of the greatest intellects of the nineteenth century" (Dauben, 1990). In this paper, we will analyze the responses to Cantor's theory from several schools of mathematicians through the lens of the sociology of scientific knowledge (SSK) framework.

## Sociology of Scientific Knowledge

The SSK framework emphasizes understanding science as a social activity, and in particular, that the scientists doing research are humans with their own beliefs and convictions. For example, the field of cosmology deals heavily with the dating of the universe. In analyzing a cosmological result through the SSK framework, one might consider whether the scientists involved, either spiritual or secular, are biased towards finding results supporting their faith or lack thereof (Porta et al., 2016).

In the case of Cantor's theory, we will analyze its detractors and proponents, whose disagreement stemmed largely from differing views on infinity, God, and the nature of mathematics.

**Cantor's Theory**

To understand the disagreement, it is necessary to first familiarize oneself with Cantor's theory that "infinite sets can have different cardinalities." To understand this statement, it is necessary to define "set" and "cardinality"; a set is simply a collection of objects. For example, $A = \{a, b, c\}$ and $B = \{\alpha, \beta, \gamma, \delta\}$ are sets containing three Latin letters and four Greek letters respectively. Intuitively, the "size" of B is in some sense larger than that of A. Mathematicians formalize this notion of size as a set's *cardinality*. For a finite set, its cardinality is simply the number of elements it contains. However, many of the most important sets are infinite, such as the set of non-negative integers, commonly denoted as $\mathbb{N}$. Therefore, we would like to generalize our definition of cardinality to any (possibly infinite) set, being careful to do so in such a way that this generalization is consistent with our definition of cardinality for finite sets.

This is accomplished via surjections; a surjection from a set X to a set Y is a function f such that for every element y of Y, there exists an element x of X such that $f(x) = y$. Colloquially, f is a surjection if given an arbitrary element of Y, we can find an element of X that maps to it. If there exists a surjection from X to Y, we say that the cardinality of X is at least as large as the cardinality of Y, since the surjection suggests we can "cover" Y with elements of X. To illustrate this concept of surjection, recall the sets A and B discussed previously. We concluded that the cardinality of B is at least that of A, since B contains more elements. Hence, for this new definition of cardinality in terms of surjections to be consistent with our previous one, we should be able to demonstrate a surjection from B to A, and indeed,

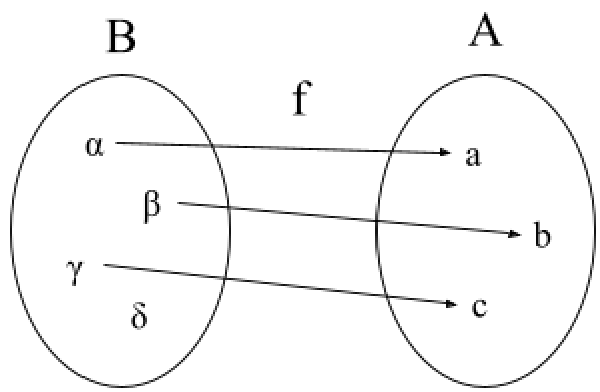


**Figure 1: Surjection from B to A**

Figure 1 is a surjection because for every element of A, there exists an element of B that maps to it. Thus, there exists a surjection from B to A, so we may conclude that the cardinality of B must be at least the cardinality of A.[1] For finite sets, it turns out that this definition is exactly the same as a set having more elements than another, however, it also allows us to measure the cardinality of infinite sets, whose elements we cannot count directly.

Cantor's argument is essentially this: the set of natural numbers, $\mathbb{N}$, is an infinite set. Were there only one "size" of infinity, then for any other infinite set A, we should be able to find a surjection from $\mathbb{N}$ to A, that is, conclude that the cardinality of $\mathbb{N}$ is at least that of A. If we were to demonstrate that for some set A this cannot happen, then it must be that the cardinality of $\mathbb{N}$ is not at least the cardinality of A, and thus the cardinality of A is "larger" infinity. Cantor's proof that there exists no such surjection relies on a line of argumentation known as *proof by contradiction*; if one assumes the negation of a statement and is able to derive a contradiction, then it must be that the statement is true.

[1]A mathematical idea analogous to *Hume's principle* in philosophy.

In 1891, Cantor published his *diagonal argument*, which is the version of the proof most commonly taught to mathematics students today. The argument proceeds by showing there exist no surjection from $\mathbb{N}$ to the closed interval [0, 1], that is, the set of all real numbers between 0 and 1, including both endpoints, and is constructive – by assuming there exists a surjection from $\mathbb{N}$ to [0, 1], one can construct an element of [0, 1] mapped to by no element of $\mathbb{N}$. Hence, there must be no such surjection, so the cardinality of $\mathbb{N}$ is *not* at least the cardinality of [0, 1], and so the cardinality of [0, 1] must be a "larger" infinity than the cardinality of $\mathbb{N}$.[2]

**Detractors**

The term *pre-intuitionism* was coined retroactively in 1951 by L. E. J. Brouwer, in order to distinguish its members from its philosophical successor, *intuitionism* (Brouwer, 1981). A primary tenet of pre-intuitionism and its derivatives is a dissatisfaction with proof by contradiction. To most of the mathematical community, it is acceptable to define a term, for example *group*, and prove by contradiction that it cannot be that no objects satisfying the definition of a group exist. Therefore, one may conclude that *group* is a useful definition – it refers to a non-empty set of mathematical objects. However, the pre-intuitionist would find this argument insufficient; unless one is able to give a concrete example of a group, they would argue the definition is useless.

This explains why some pre-intuitionist mathematicians may have opposed Cantor's theory. His argument relied fundamentally on deriving a contradiction from the assumption that a surjection exists. While his 1891 proof is constructive, in that he demonstrates a specific element of [0, 1] that is not mapped to by any element of $\mathbb{N}$, some of his earlier proofs were not, and therefore unsatisfying to mathematicians with beliefs of a more constructive nature.

[2] Today, we refer to the cardinality of $\mathbb{N}$ as *countable infinity*, and use *uncountable infinity* for larger varieties.

The finitists were a related school, whose central tenet was that they accepted only finite mathematical objects to exist. Their issue with Cantor's theory is clear – not only did they reject that the cardinalities of $\mathbb{N}$ and [0, 1] were not equal; they rejected the existence of $\mathbb{N}$ and [0,1] at all. For many, the motivation for finitist beliefs was religious – infinitude was an inherently godly quality such that it would be seen as sacrilege to ascribe it to anything other than the divine. Cantor himself was a Lutheran and believed his mathematical work to have revolutionary theological value, saying "From me, Christian philosophy will be offered for the first time the true theory of the infinite." (Dauben, 1990). However, not all theists were as eager to explore the religious consequences of Cantor's theory. Such was the case for prominent finitist and former teacher of Cantor's, Leopold Kronecker, who famously said, "God created the integers; all else is the work of man." Moreover, on his student, he wrote, "I don't know what predominates in Cantor's theory – philosophy or theology, but I am sure that there is not any mathematics here" (Zenkin, 2004).

Secular mathematicians had reason to be finitist as well. Since the time of Aristotle, there has been a distinction between *potential infinity* and *actual infinity*; potential infinity refers to an interminable process, for example, one can never finish enumerating the natural numbers because each one has a successor. *Actual infinity*, in contrast, would be to speak about the set of natural numbers, as if it had been completed. In rejecting actual infinity, Aristotle was able to refute Zeno's paradox[3], and thereby set a precedent for others doing so, which would continue until quite recently. In this way, one could acknowledge the infinitude of natural numbers, but reject the existence of $\mathbb{N}$ and therefore certainly Cantor's theory.

[3] The paradox states that to walk 1 meter, one must first walk ½ meter, which in turn necessitates one to walk ¼ meter, and so on. Repeating this division *ad infinitum*, Zeno concludes that to walk 1 meter requires one to complete infinitely many tasks, and therefore, motion is impossible, contradicting common sensibility.

**Proponents**

Among Cantor's proponents were the formalist school of mathematicians, who were guided by the idea that mathematics need not be representative of reality, but instead a kind of game. As Weir writes, "This idea has some intuitive plausibility: consider the tyro toiling at multiplication tables or the student using a standard algorithm for differentiating or integrating a function." These processes are not unlike a chess player searching for the optimal move. In the formalist view, mathematics is about successively inventing definitions, applying rules, and exploring the consequences. For example, that every square is a rectangle is a simple consequence of the standard definitions of square and rectangle. It is unequivocally true from our definitions, even if no squares or rectangles exist in reality (Weir, 2019).

It is not surprising then that formalists generally embraced Cantor's theory. David Hilbert, a prominent formalist, wrote, "No one shall expel us from the paradise which Cantor has created for us!" (Zenkin, 2004). To them, it must have seemed like a curious, amusing consequence of his definitions of surjection and cardinality, and would not have concerned themselves with the philosophical implications of the existence of different infinities.

**Conclusion**

Most mathematicians working today accept Cantor's theory. Perhaps this is due to an increase in secularism, or a shift towards the view that mathematics need not be representative of reality; perhaps it is due to his ideas no longer seeming as radical, scary, or disruptive after enduring nearly 150 years of criticism; perhaps it is due to set theory, the natural setting for cardinality and surjections, becoming the standard foundation of mathematics done today. More than likely, it is several of the above in combination with other factors I have not realized.

There is much room for further analysis. In particular, Bertrand Russell was another prominent figure who supported Cantor's theory, and one could explore how Cantor's ideas influenced later conceptions of infinity, for example Hilbert's analogy of the infinite hotel. In contrast, Ludwig Wittgenstein opposed Cantor's theory, sharing views with the finitists, while also trying to distance himself from that school. Further, one could explore the development of more modern schools, such as ultrafinitism, which rejects not only infinite quantities but even large finite ones. The sociology of scientific knowledge STS framework provides the appropriate lens through which to analyze the reactions to Cantor's theory; principally, it emphasizes the beliefs and convictions of Cantor's contemporaries to understand why some reacted so vitriolically while others praised him as a visionary.

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