Productivity, Model Uncertainty, Home Production, and the New Home Sale Price in the U.S.

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Abstract

This dissertation investigates the role of productivity, model uncertainty, and home production in macroeconomics and the housing market, focusing on the resource allocation among different production sectors and the new home sales price.

In the first chapter, I construct sector-specific total factor productivity shocks in a two-sector DSGE model and study to what extent that the TFP shock in housing sector accounts for the rise and fall during the most recent housing market crisis. I solve the model using the methodology proposed by Hansen and Sargent (1995). The discounted Linear Exponential Quadratic Gaussian control method successfully avoids computational difficulties such as high-dimensional state space, explosive value function, etc., and facilitates the simulation of the worst-case model to implement the likelihood ratio tests. I calibrate the parameters in the benchmark model by the simulated method of moments. The benchmark model, with TFP shocks alone, does a decent job in fitting the first and second moments of real-world data; it accounts for 32 percent of the increase and 40 percent of the decrease in the new home sales price. The implied changes in the resource allocation between the two sectors are also in line with data. Then I introduce model uncertainty into the benchmark framework and dedicate to answer the question that whether the fear of model misspecification help account for the boom-bust in the new home sales price, and I examine the first and second moments of the worst-case distribution of the TFP shocks. I allow a one-time temporary change in the model uncertainty level during the time span with enhanced dispersions of professional forecasts. The model uncertainty parameter is calibrated by the detection error probability as in Hansen and Sargent (2007). When households hold a concern for model misspecification, they are seeking robust decision rules which perform well over a set of alternative models that are statistically indistinguishable from the not-fully-

trusted approximating model. When model uncertainty level rises, the worst-case model distorts the Gaussian mean of TFP shocks negatively in households' minds but keeps the variance-covariance matrix of the TFP shocks almost unchanged. Since the TFP shock in housing production is more volatile, it receives a larger negative mean-shift effect than the TFP shock in the consumption goods sector from the robust state transition law. As a result, the new home sales price is pushed up even further. Thus with the one-time change in model uncertainty, the model is capable of accounting for 40 percent of the surge in new home sales price. The inclusion of the fear for model misspecification improves model's fitting with data.

The second chapter studies the implications of introducing home production into an otherwise standard two-sector production economy from the theoretical perspective. While the benchmark model shows reasonable impulse responses of the key economic variables to different productivity shocks in the two market production sectors, the existence of home production highlights more interesting mechanisms due to the substitution incentive between market-produced and home-produced goods. When there is a positive productivity shock in the nonhousing goods production sector, households are inclined to substitute market consumption for home consumption as it is relatively more efficient to work in the market. As a result, resources flow to the two market sectors as if the relative productivity change in the nonhousing sector to the housing sector were amplified. In the case with a positive productivity shock in the housing sector, although it is more efficient to produce housing goods, the increment of housing stock requires more home hours to pair with it in home production. Thus the effects on resource allocations in the market sectors driven by this relative productivity change are tempered. This substitution mechanism can help improve the two-sector model's performance in fitting of the correlations between factor inputs that observed in the data.

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Contents

Chapter 1

Productivity, Model Uncertainty, and the New Home Sales Price in the U.S.

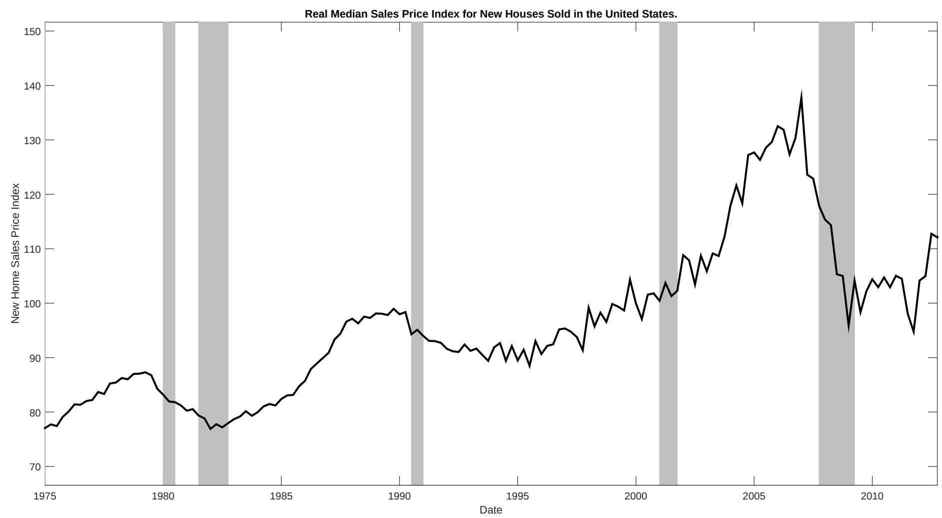
1.1 Introduction

The U.S. housing market experienced the unprecedented fluctuations from 2003 to 2009. Empirical data documented the most prominent boom and bust of the inflation-adjusted median sales price for new houses sold in the U.S. as shown in Figure 1.1. The real new home sales price surged since the second quarter of 2003, peaked in the fourth quarter of 2006 with a sharp increase of 30 percent, and plummeted back to the pre-boom level in the following three years.

Unlike the price index for the existing homes, the new home sales price index measures the market value for new houses constructed.[1] Davis and Heathcote (2007) decompose the

[1] The Case-Shiller Home Price Index (HPI) measures the repeated sales value of the existing homes. The historical data series of the Case-Shiller HPI is shown in Appendix 1.G. The Case-Shiller HPI exhibits a larger increase and decrease during 2003 - 2009. But this paper focuses on the new home sales price, which co-moves very closely with the Case-Shiller HPI.

Figure 1.1: Real Median Sales Price Index for New Houses Sold in U.S. (1975 - 2012)



Data source: Federal Reserve Economic Data, St. Louis Fed.

aggregate value of the housing stock into structures and land components, and they argue that the value of *new* homes relies heavily on the production of new structures while the price of land plays a more important role in the value of *existing* homes. This paper dedicates to study the dynamics of new home sales prices with an emphasis on the productivity shocks in both construction and non-construction sectors. The business cycle facts show that construction output is about 8.5 times as volatile as the non-construction output. If the variance of productivity shocks is assumed to be equal across the two sectors, the resulting relative volatility will be counterfactual. Davis and Heathcote (2005) calibrate a multi-sector model in which the price of new structures is driven by changes in relative productivity across sectors. Their model successfully matches the U.S. housing market data over the post-war period. I adopt the idea of sector-specific productivity shocks and sector-specific technology (i.e. distinct capital shares, labor shares, etc.) for production. However, I only consider two production sectors to keep the problem tractable while still seize the core that relative productivity changes lead to resource reallocations and drive the dynamics of the new home sales price.[2] In addition to the two-sector setup, my model features indivisible labor in the preference which allows greater movements in labor supply along the extensive margin. Moreover, change capital or labor input is costly. I use a quadratic function to characterize the adjustment costs associated with any resource reallocation. The purpose of embedding these features is to fine-tune the model to best match the data series of factor inputs so that it can be used to investigate the behavior of new home prices driven by the productivity shocks. The model parameters are calibrated by matching the first and the second business cycle moments using the pre-boom macroeconomic data.

My benchmark model with productivity shocks alone matches relatively well the scale of new home sales price fluctuations before the housing boom. However, the sector-specific productivity shocks only account for about 25 percent of the skyrocketing new home prices from the year of 2003 to 2007. The main driving force underlying the boom of the new home sales price is a decrease in the relative productivity of construction sector to non-construction sector. This mechanism is consistent with the empirical findings shown in Sveikauskas et al. (2014) and Galesi (2014). Productivity variations also lead to changes in the marginal products of capital and labor input, which gives an incentive for the economy to specialize in the sector with highest production efficiency with the resources. A relatively low productivity in the construction sector discourages building more new homes and pushes up the new home sales price.

Recently, a growing body of literature works on the profound economic decline during 2008-2009 from the perspective of model uncertainty (uncertainty in the Knightian sense). According to the seminal work by Knight (1921), there are two kinds of uncer-

[2]The multi-sector business cycle literature studying the housing market and residential investment includes, but is not limited to, Long and Plosser (1983), Baxter (1996), Iacoviello and Neri (2010), Dorofeenko et al. (2014), etc.

tainty: the first, often called *risk*, corresponds to a situation in which a uniquely (either objectively or subjectively) determined probability is assigned to any unrealized event; the second, often called *Knightian uncertainty*, corresponds to a situation in which the probability distribution of a future event is unknown.[3] Ellsberg (1961) implemented experimental analysis to formally discuss the distinction between risk and uncertainty. Such a distinction is meaningful since no probability distribution is actually given or easily computable in most economic contexts where agents are facing uncertainties. As argued in Caballero and Krishnamurthy (2008), the Knightian uncertainty is triggered by unusual events and untested financial innovations that lead the agents to question their understanding of the economic conditions and challenge their trust in the data-generating process. Survey data on households' expectations about future macroeconomic outcomes also reveal significant pessimistic biases prior to the Great Recession.[4] Moreover, a number of research papers studying the connection between the model uncertainty and the business cycles find that an increase in the uncertainty level acts as a negative impact on economic productivity. Ilut and Schneider (2014) use confidence shocks to describe model uncertainty and claim that if model uncertainty increases, households behave as if future productivity is expected to fall and firms set prices as if productivity is lower although the actual productivity is higher than what they fear. Bhandari et al. (2016) conclude that an increase in model uncertainty makes the households' worst-case expectations much more pessimistic and negatively distorts households' subjective perception of productivity shocks. Productivity shock is the driving force behind the housing price dynamics in my model, and, to the best of my knowledge, there is no previous research that studies housing prices from the perspective of model uncertainty. With this motivation, I introduce model uncertainty

[3] Knightian uncertainty is now more commonly known as *model uncertainty*, *ambiguity*, *robust control*, or *risk sensitivity*, etc.

[4] See Bhandari et al. (2016).

into my benchmark model and dedicate to answer the question that whether a one-time temporary increase in model uncertainty level, along with the sector-specific TFP shocks, can account for a larger proportion of the changes in new home sales prices.

In the existing literature, there are three commonly used methods to characterize the model uncertainty: multiplier (or max-min) preference, multiple-prior preference, and smooth recursive preference. In this paper, I adopt the multiplier preference framework introduced by Hansen and Sargent (2001). In the environment with model uncertainty, agents do not know the true data-generating process (denoted the true model) and are concerned about the possibility that the model used for making decisions (denoted the approximating model) is misspecified. They also know that the approximating model is close enough to the true model in that the "distance" between these two models – measured by the entropy between the two probability distributions – has an upper bound. The agents cannot distinguish the approximating models within the entropy ball (denoted the alternative model set) surrounding the true model from each other statistically and consequently, they choose optimal decisions as if the worst-case model that delivers the lowest expected lifetime utility (denoted the worst-case model) were the data-generating process. Formally, the fear of model misspecification is depicted by a max-min optimal control problem where a malevolent agent (nature) minimizes the household's lifetime utility by choosing the worst-case model and simultaneously a benevolent agent maximizes his intertemporal utility by choosing the optimal consumption and investment plans.[5] Solving the max-min control problem yields a log-exponential utility recursion and the value function in the Bellman equation is modified accordingly.

I solve the model numerically using the discounted Linear Exponential Quadratic Gaus-

[5]Also see the max-min expected utility theory of Gilboa and Schmeidler (1989) and the applications of robust control theory proposed by Anderson et al. (2000).

sian (LEQG) method introduced by Hansen and Sargent (1995). The LEQG method has many advantages over the possible alternatives. It is computationally efficient and does not suffer from the overflow problem as the value function iteration algorithm does. More importantly, it is very convenient to simulate the worst-case state transition processes and compute the likelihood of a sample using the Kalman filter, which is a nice property that facilitates the calibration of model uncertainty parameter. To study the effects of model uncertainty on the new home sales price and other macroeconomic variables, I consider a one-time temporary change in the fear for model misspecification of the productivity shocks. I determine the time span of the model uncertainty variations based on the dispersion measurements in the Survey of Professional Forecasters. The size of the model uncertainty level change is governed by the uncertainty parameter in households' preference, which is calibrated by implementing the likelihood ratio test and model selection procedure to calculate the detection error probabilities as proposed by Hansen and Sargent (2007).

With the one-time temporary change in model uncertainty and feeding in the actual TFP shocks, the simulated dynamic paths of capital, labor input, and other aggregate variables are consistent with the data. In particular, the presence of model uncertainty gives rise to the precautionary savings motive. As a result, an increase in model uncertainty level leads to an immediate reduction in consumption; in the meanwhile, capital stock accumulates in all sectors and households work for more hours. Now the model can capture about 35 percent of the new home sales price boom. However, the model does not match the behavior of housing rental rates very well. I compare the impulse responses of the key variables to the TFP shocks under different model uncertainty levels. The result confirms that enhanced model uncertainty does not amplify the effects of the TFP shocks. Hence the one-time change in model uncertainty acts more like a shift in levels of the model variables.

Finally, I simulate for the TFP shocks under the worst-case model to investigate the mechanism of model uncertainty for the results. Intuitively, the fear of model misspecification skews the households' subjective probabilities towards the bad outcomes when they are making expectations about future utility and returns. The simulated worst-case model displays a negative mean distortion in both TFP shocks. Moreover, since the TFP shock in the housing sector is much more volatile, it also receives a larger negative distortion in its mean. Therefore, the relative productivity of the construction sector to the non-construction sector lowers further. Consequently, the economy builds even less new houses, which raises the new home sales price further. I also compute the covariance matrix of the joint distribution of the TFP shocks. However, there is no significant change in the variance nor in the correlation between the two TFP shocks, which stands in line with the conclusion in Hansen and Sargent (2007).

The rest of the paper proceeds as follows. Section 2 presents the model and section 3 defines the market equilibrium. In section 4, I elaborate the solution method. Section 5 describes the data used in the quantitative analysis and the calibration procedures of model parameters. Section 6 shows the quantitative results. Section 7 discusses the worst-case shocks under the environment with model uncertainty. Section 8 concludes. All supplementary materials are included in the Appendix.

1.2 Model

I construct a dynamic stochastic model to study a perfectly competitive economy, which consists of a continuum of households, a representative firm producing the consumption (or investment) goods, and a representative firm that builds new home units for sale. There is no govenment in this economy.

1.2.1 Households and Preferences

Households are identical, infinitely lived, and of measure one. The preference of the households is defined over consumption goods c_t, housing services s_t, and leisure ℓ_t. Each household is endowed with one unit of time in every period.

Labor supply is indivisible. The households cannot choose the number of hours worked; rather, once being employed, they have to work a full amount of time, $\bar{N}$. As discussed in Hansen (1985), the households' decision problem will be discontinuous and the consumption possibility set will be non-convex due to the discrete work-or-not-work choice. To circumvent these technical difficulties, I adopt the "lottery" fashion over employment and consumption that depicted by Hansen (1985) and Rogerson (1988). In detail, households have opportunities to work in two sectors every period, a consumption sector and a housing sector. The probability of being selected to work in either sector is $n_{c,t}$ or $n_{h,t}$ and thus the probability of being unemployed is $1 - n_{c,t} - n_{h,t}$. If employed, households enjoy consumption level c_t^E; if not, they consume c_t^U. There is complete unemployment insurance provided by the firms so that every household gets paid whether it works or not.[6] Thus the consumption-leisure choice set only contains three points:

$$(c_t, \ell_t) = \begin{cases} (c_t^E, 1 - \bar{N}), & \text{with probability } n_{c,t}; \quad \text{(employed in consumption sector)} \\ (c_t^E, 1 - \bar{N}), & \text{with probability } n_{h,t}; \quad \text{(employed in housing sector)} \\ (c_t^U, 1), & \text{with probability } 1 - n_{c,t} - n_{h,t}. \quad \text{(unemployed)} \end{cases}$$

[6]Hansen (1985) elaborates the work contract traded in the economy, which commits the household to work a certain amount of time with certain probability. Since all households are identical, all will choose the same contract with the same probability. And the households differ ex post depending on the outcome of the lottery.

Therefore, the expected utility in period t is given by

$$u(c_t, s_t, \ell_t) = (n_{c,t} + n_{h,t})u(c_t^E, s_t, 1 - \bar{N}) + (1 - n_{c,t} - n_{h,t})u(c_t^U, s_t, 1). \tag{1.1}$$

The households own h_t units of housing stock. In each period, one unit of housing stock delivers one unit of housing services, i.e. $s_t = h_t$, and it depreciates at rate δ_h. The housing stock is perfectly divisible and can be accumulated overtime through residential investment $I_{h,t}$. However, changing housing stock is costly and bears a quadratic adjustment cost.

In addition, the households own the representative firms and receive all of the profits Π_t. There is no money in this economy. The price of consumption goods is normalized to 1; the wage rate is w_t per hour; the unit price of new homes is $P_{I_h,t}$; the price paid to housing services can be measured by the implicit home rent to the owners, $R_{h,t}$.[7]

The households face the following budget constraint in period t:[8]

$$\begin{aligned} &(n_{c,t} + n_{h,t})c_t^E + (1 - n_{c,t} - n_{h,t})c_t^U + R_{h,t}s_t + P_{Ih,t}I_{h,t} \\ &\leq w_t(n_{c,t} + n_{h,t})\bar{N} + R_{h,t}h_t + \Pi_t - \frac{\omega}{2}\left(\frac{h_{t+1} - h_t}{h_t}\right)^2 h_t, \end{aligned} \tag{1.2}$$

with the law of motion for housing stock:

$$h_{t+1} = (1 - \delta_h)h_t + I_{h,t}. \tag{1.3}$$

The key feature of households' preference is that it reflects a fear of model uncertainty.[9]

[7] I exempt the discussion of own-or-rent a home in this paper. Every household is a home owner.

[8] The adjustment cost of housing stock is ascribed to transaction costs like preparing relevant documents, signing contracts, etc. Thus it is evaluated by units of consumption goods.

[9] Similar terminologies used in the literature are "ambiguity", "model misspecification", "robustness", "risk sensitivity".

Specifically, when households use some stochastic model to make expectations about future states, they do not completely trust the information about the randomness in the economy and fear that the probability distributions of the underlying shocks might be misspecified. The households regard the model in hand as an approximation to the unknown data generating process. They believe that data will come from an unknown member of a set of alternative models that closely surrounds the approximating model.[10] However, there is no way to distinguish the approximating model from the data generating process given the available information.[11] In order to induce a robust decision rule that performs well over the set of alternative models, the households plan against the worst-case process. The worst-case model consists of probability distributions of the shocks under which households' discounted lifetime utility reaches the lowest level among all the other alternative models. In a sense, all probability distributions in households' minds are *subjective*; only the probability distribution of the true data generating process is *objective*, which is unknown.

Anderson et al. (2003) and Hansen and Sargent (2007) formally introduce model uncertainty into household preference using a max-min multiplier optimization problem:[12]

$$V(\boldsymbol{s_t}) = \max_{\boldsymbol{x_t}} \min_{\{p(\epsilon_{t+1}|\epsilon_t)\}\in[0,1]^{\#\epsilon}} \left\{ u(\boldsymbol{x_t}) + \beta \sum_{\epsilon_{t+1}} p(\epsilon_{t+1}|\epsilon_t) V(\boldsymbol{s_{t+1}}) \right\}, \tag{1.4}$$

$$\text{s.t.} \quad \sum_{\epsilon_{t+1}} p(\epsilon_{t+1}|\epsilon_t) = 1, \tag{1.5}$$

$$\sum_{\epsilon_{t+1}} p(\epsilon_{t+1}|\epsilon_t) \log\left(\frac{p(\epsilon_{t+1}|\epsilon_t)}{\pi(\epsilon_{t+1}|\epsilon_t)}\right) \leq \eta_0, \tag{1.6}$$

where $\boldsymbol{s_t}$ stands for all state variables including the shock(s) ϵ_t; $\boldsymbol{x_t}$ stands for all con-

[10]The "closeness" is measured by the entropy between two probability distributions. Refer to Appendix 1.A for more details.

[11]This is very likely to be the case when there are many conflicting views in information and only a moderate size of data sample can be trusted.

[12]Strzalecki (2011) provides the axiomatic foundations for multiplier preferences.

trol variables; $p(\epsilon_{t+1}|\epsilon_t)$ characterizes the subjective probability of the realization of the shock(s) being ϵ_{t+1} at period $t+1$ given that the period-t realization being ϵ_t; the counterpart objective probability is $\pi(\epsilon_{t+1}|\epsilon_t)$. The minimization part reflects the idea that the households guard themselves against the worst-case scenario, and the maximization part shows households' desire for robustness. The two constraints (1.5) and (1.6) indicate that any subjective probability distribution must satisfy two requirements. First, it must be a valid probability measure and sum up to one. Second, it needs to be "close enough" to the objective probabilities so that the households cannot differentiate them. The discrepancy between two probability specifications is measured by "entropy", and it cannot be greater than some positive value.[13]

Solving the max-min problem (1.4) - (1.6) yields the utility recursion in a log-exponential form:

$$V(\boldsymbol{s_t}) = \max_{\boldsymbol{x_t}} \left\{ u(\boldsymbol{x_t}) + \beta \frac{2}{\sigma} \log \left(\mathbb{E}_{\pi(\epsilon_{t+1}|\epsilon_t)} \left[\exp \left(\frac{\sigma V(\boldsymbol{s_{t+1}})}{2} \right) \right] \right) \right\}, \tag{1.7}$$

where $\sigma \leq 0$ regulates households' fear of model misspecification.[14]

According to Hansen et al. (1999), a smaller absolute value of σ means that the households care less about model uncertainty.[15] In an extreme case when $\sigma = 0$, the utility recursion converges to the conventional Von Neumann-Morgenstern utility under rational expectation.[16]

[13] See Backus et al. (2005) and Hansen and Sargent (2007) for the definition of entropy.

[14] The parameter σ relates to the Lagrangian multiplier in the minimization problem. Appendix 1.A shows how this functional form is derived following the steps suggested by Young (2012).

[15] The parameter σ also acts as a form of risk aversion. Tallarini (2000) shows that this log-exponential specification can be regarded as a special case of the preferences depicted in Epstein and Zin (1989) with the intertemporal elasticity of substitution being unity. See more details in Appendix 1.A.

[16] Note that

$$\lim_{\sigma \to 0} \log \left(\mathbb{E}_{\pi} \left[\exp \left(\frac{\sigma V(\boldsymbol{s_{t+1}})}{2} \right) \middle| \epsilon_t \right] \right) = 0.$$

By L'Hôpital's rule, $\lim_{\sigma \to 0} \frac{2}{\sigma} \log \left(\mathbb{E}_{\pi} \left[\exp \left(\frac{\sigma V(\boldsymbol{s_{t+1}})}{2} \right) \middle| \epsilon_t \right] \right) = \mathbb{E}_{\pi}[V(\boldsymbol{s_{t+1}})|\epsilon_t]$.

The households' problem can be characterized by the following Bellman equation:

$$V(h_t, z_t, \xi_t) = \max_{\substack{c_t^E, c_t^U, n_{c,t}, \\ n_{h,t}, h_{t+1}, I_{h,t}}} \Bigg\{ (n_{c,t} + n_{h,t}) u(c_t^E, s_t, 1 - \bar{N}) + (1 - n_{c,t} - n_{h,t}) u(c_t^U, s_t, 1)$$

$$+ \beta \frac{2}{\sigma} \log \left(\mathbb{E}_{\pi,t} \left[\exp \left(\frac{\sigma V(h_{t+1}, z_{t+1}, \xi_{t+1})}{2} \right) \Big| z_t, \xi_t \right] \right) \Bigg\}, \tag{1.8}$$

$$\text{s.t. } (n_{c,t} + n_{h,t}) c_t^E + (1 - n_{c,t} - n_{h,t}) c_t^U + R_{h,t} s_t + P_{Ih,t} I_{h,t}$$

$$\leq w_t (n_{c,t} + n_{h,t}) \bar{N} + R_{h,t} h_t + \Pi_t - \frac{\omega}{2} \left(\frac{h_{t+1} - h_t}{h_t} \right)^2 h_t, \tag{1.9}$$

$$h_{t+1} = (1 - \delta_h) h_t + I_{h,t}, \tag{1.10}$$

where z_t and ξ_t represent sector-specific total factor productivity (TFP) shocks. The expectation operator $\mathbb{E}_{\pi,t}[\cdot]$ is based on the information set given the objective probabilities $\pi(\epsilon_{t+1}|\epsilon_t)$.

In addition, I use the following parametric form for the single-period utility function:

$$u(c_t^i, s_t, n_t^i) = \mu_c \log(c_t^i) + (1 - \mu_c) \log(s_t) + \phi \log(1 - n_t^i), \quad , i = E, U. \tag{1.11}$$

1.2.2 Firms and Production Technologies

There are two representative firms in this economy. One firm produces consumption and investment goods using capital and labor; the other firm builds new homes using capital, labor, and land. Both firms own capital; the firm in the housing construction sector also owns land. They employ labor force, make investment decisions for physical capital, and pay the entire profits to the households.

The Consumption Sector

The representative firm in the consumption goods sector produces output $Y_{c,t}$ using a constant-returns-to-scale technology: [17]

$$Y_{c,t} = \exp(z_t) K_{c,t}^{\alpha} N_{c,t}^{1-\alpha}, \tag{1.12}$$

where $K_{c,t}$ and $N_{c,t}$ stand for capital and labor input; z_t represents the TFP shock in the consumption goods sector.

The output $Y_{c,t}$ is a numeraire good, whose price is normalized to 1. It can be consumed by households, invested and installed as new capital stock, and utilized to compensate any costs caused by economic activities.

The Housing Sector

New homes $Y_{h,t}$ are produced in the housing sector using capital $K_{h,t}$, labor $N_{h,t}$, and land L_t:[18]

$$Y_{h,t} = P_{Ih,t} \exp(\xi_t) K_{h,t}^{\theta} N_{h,t}^{\nu} L_t^{1-\theta-\nu}, \tag{1.13}$$

where ξ_t represents the TFP shock in the housing sector.

Housing construction also uses a third input, the land. Land supply is fixed and normalized to one unit in every period, i.e. $L_t = 1, \forall t$. In line with Davis and Heathcote (2005), fixed land supply acts as an adjustment cost in residential investment. New homes produced are sold to the households at price $P_{I_h,t}$ per unit.

[17] With appropriate parameterization, the production function is increasing and concave in both arguments $K_{c,t}$ and $N_{c,t}$, and it satisfies the Inada conditions.

[18] The production function is increasing and concave in all arguments and satisfies the Inada conditions.

Adjustment Costs

Firms' decisions about how much capital to be invested and how much labor force to be employed give rise to adjustment costs. I assume that all factor input changes are subject to quadratic adjustment costs. Therefore, the law of motions for capital inputs are stated as follows:

$$K_{c,t+1} = (1-\delta_{kc})K_{c,t} + I_{kc,t} - \frac{\kappa_c}{2}\left(\frac{K_{c,t+1}-K_{c,t}}{K_{c,t}}\right)^2 K_{c,t}, \tag{1.14}$$

$$K_{h,t+1} = (1-\delta_{kh})K_{h,t} + I_{kh,t} - \frac{\kappa_h}{2}\left(\frac{K_{h,t+1}-K_{h,t}}{K_{h,t}}\right)^2 K_{h,t}. \tag{1.15}$$

In each period, capital stocks depreciate at rates δ_{kc} and δ_{kh} in the consumption goods sector and housing sector respectively. The parameters κ_c and κ_h control the size of adjustment costs proportionate to the change rates in capital stocks.

The adjustment costs for altering labor inputs are given by

$$\tau(N_{c,t-1}, N_{c,t}) = \frac{\tau_c}{2}\left(\frac{N_{c,t}}{\bar{N}} - \frac{N_{c,t-1}}{\bar{N}}\right)^2 \tag{1.16}$$

and

$$\tau(N_{h,t-1}, N_{h,t}) = \frac{\tau_h}{2}\left(\frac{N_{h,t}}{\bar{N}} - \frac{N_{h,t-1}}{\bar{N}}\right)^2, \tag{1.17}$$

where τ_c and τ_h are the adjustment cost parameters; $N_{c,t}/\bar{N}$ and $N_{h,t}/\bar{N}$ are the number of workers hired in the two sectors. Note that the subscripts in the labor adjustment cost functions are different from that in the capital adjustment cost functions. This is because of the difference in timing: capital adjustment costs arise from the decisions in capital investment which is to choose the future capital stock level given the current state; however, labor cannot be stored and the relevant adjustment costs come from the gap between current and previous labor usage.

Sector-Specific TFP Shocks

A large portion of literature studying housing market in a multi-sector model assumes that the housing sector shares the same TFP shock as the non-housing sectors. In contrast, I model the TFP shocks in the consumption sector and housing sector separately. Both TFP shocks, z_t and ξ_t follow an mean-zero AR(1) process:

$$z_{t+1} = \rho_z z_t + \sigma_z \epsilon_{z,t+1}, \quad \epsilon_{z,t+1} \sim \text{i.i.d. } \mathcal{N}(0,1); \tag{1.18}$$

$$\xi_{t+1} = \rho_\xi \xi_t + \sigma_\xi \epsilon_{\xi,t+1}, \quad \epsilon_{\xi,t+1} \sim \text{i.i.d. } \mathcal{N}(0,1). \tag{1.19}$$

I assume that innovations $\epsilon_{z,t}$ and $\epsilon_{\xi,t}$ are uncorrelated for simplicity and tractability concerns. Thus $cov(\epsilon_{z,j}, \epsilon_{\xi,k}) = 0, \forall j, k$. This assumption is consistent with the findings in Davis and Heathcote (2005) that productivity shocks are only weakly correlated across sectors, and in particular shocks to the construction sector are essentially uncorrelated with those in the non-construction sectors.

The sector-specific TFP shocks is important to generate the key mechanism of this model. First, this setup is in line with data showing that productivity shocks in housing and non-housing sectors are very different in their volatilities. Second, certainty equivalence fails when households have concerns about model misspecification. Hansen and Sargent (2007) demonstrate that the variance of the shocks have a significant effect on the robust decision rules. Third, the determination of new home sales price $P_{I_h,t}$ will also depend on the relative changes between z_t and ξ_t. The equilibrium conditions show that

$$P_{I_h,t} = \frac{\exp(z_t)(1-\alpha)K_{c,t}^{\alpha} N_{c,t}^{-\alpha}}{\exp(\xi_t)\nu K_{h,t}^{\theta} N_{h,t}^{\nu-1}}. \tag{1.20}$$

We can see that not only does the new home sales price depend on resource allocations across the two sectors, it also depends on the relative productivity level $\exp(z_t)/\exp(\xi_t)$. The changes in this relative productivity level will be amplified in an environment with model uncertainty.

Firms' Optimization Problems

The representative firm in the consumption sector solves the following Bellman equation:

$$J^C(K_{c,t}, N_{c,t-1}, z_t, \xi_t) = \max_{K_{c,t+1}, N_{c,t}, I_{kc,t}} Y_{c,t} - w_t N_{c,t} - I_{kc,t} - \frac{\tau_c}{2}\left(\frac{N_{c,t}}{\bar{N}} - \frac{N_{c,t-1}}{\bar{N}}\right)^2 + \mathbb{E}_t\left[\Lambda_{t+1} \cdot J^C(K_{c,t+1}, N_{c,t}, z_{t+1}, \xi_{t+1}) | z_t, \xi_t\right], \tag{1.21}$$

$$\text{s.t.} \quad K_{c,t+1} = (1-\delta_{kc})K_{c,t} + I_{kc,t} - \frac{\kappa_c}{2}\left(\frac{K_{c,t+1} - K_{c,t}}{K_{c,t}}\right)^2 K_{c,t}. \tag{1.22}$$

Similarly, the Bellman equation for the representative firm in the housing sector is

$$J^H(K_{h,t}, N_{h,t-1}, z_t, \xi_t) = \max_{K_{h,t+1}, N_{h,t}, I_{kh,t}} Y_{h,t} - w_t N_{h,t} - I_{kh,t} - \frac{\tau_h}{2}\left(\frac{N_{h,t}}{\bar{N}} - \frac{N_{h,t-1}}{\bar{N}}\right)^2 + \mathbb{E}_t\left[\Lambda_{t+1} \cdot J^H(K_{h,t+1}, N_{h,t}, z_{t+1}, \xi_{t+1}) | z_t, \xi_t\right], \tag{1.23}$$

$$\text{s.t.} \quad K_{h,t+1} = (1-\delta_{kh})K_{h,t} + I_{kh,t} - \frac{\kappa_h}{2}\left(\frac{K_{h,t+1} - K_{h,t}}{K_{h,t}}\right)^2 K_{h,t}. \tag{1.24}$$

The firms discount the future profit flows in the same way as the households. It is easy to prove that the discount factor Λ_t also has a log-exponential form as households' discount factor.

1.3 Equilibrium

There are four markets in the economy: the housing market, the labor market, the consumption market, and the implicit home rental market.

I use the uppercase letters to represent the aggregate counterparts of the individual variables. Since the households are homogeneous and of measure 1, the aggregate variables simply equal to the individual variables, e.g. $H_t = h_t$, $C_t^E = c_t^E$, $C_t^U = c_t^U$, etc.

Definition 1. A *recursive competitive equilibrium* for this economy is given by the value functions of the households and the firms $\{V, J^C, J^H\}$, households' optimal choices for $\{c_t^E, c_t^U, n_{c,t},$
$n_{h,t}, h_{t+1}, I_{h,t}\}$, firms' decision rules for $\{K_{c,t+1}, K_{h,t+1}, N_{c,t}, N_{h,t}, I_{kc,t}, I_{kh,t}\}$, price functions $\{P_{I_h,t}, w_t, R_{h,t}\}$, and law of motions for housing stock and physical capital stocks (2.2.6), (2.2.13) (2.2.14), such that:

(1) Given the prices, households' decision rules derived by solving the problem (2.2.7) - (2.2.9) maximize the lifetime utility;

(2) Given the prices, firms' choices solve the profit maximization probelms described in (2.2.19) - (2.2.20) and (2.2.21) - (2.2.22);

(3) All markets clear;

(i) Housing market clears,

$$Y_{h,t} = P_{Ih,t} I_{h,t}, \tag{1.25}$$

(ii) Labor market clears,

$$N_{c,t} = n_{c,t}\bar{N}, \quad N_{h,t} = n_{h,t}\bar{N}, \tag{1.26}$$

(iii) Implicit home rental market clears,

$$s_t = h_t = H_t, \tag{1.27}$$

(iv) Consumption market clears,

$$\begin{aligned}
&(n_{c,t} + n_{h,t})C_t^E + (1 - n_{c,t} - n_{h,t})C_t^U + I_{kc,t} + I_{kh,t} \\
&= Y_{c,t} - \frac{\omega}{2}\left(\frac{H_{t+1} - H_t}{H_t}\right)^2 H_t - \frac{\tau_c}{2}\left(\frac{N_{c,t}}{\bar{N}} - \frac{N_{c,t-1}}{\bar{N}}\right)^2 - \frac{\tau_h}{2}\left(\frac{N_{h,t}}{\bar{N}} - \frac{N_{h,t-1}}{\bar{N}}\right)^2
\end{aligned} \tag{1.28}$$

(4) Law of motions for the stock variables hold.

$$H_{t+1} = (1 - \delta_h)H_t + I_{h,t}, \tag{1.29}$$

$$K_{c,t+1} = (1 - \delta_{kc})K_{c,t} + I_{kc,t} - \frac{\kappa_c}{2}\left(\frac{K_{c,t+1} - K_{c,t}}{K_{c,t}}\right)^2 K_{c,t}, \tag{1.30}$$

$$K_{h,t+1} = (1 - \delta_{kh})K_{h,t} + I_{kh,t} - \frac{\kappa_h}{2}\left(\frac{K_{h,t+1} - K_{h,t}}{K_{h,t}}\right)^2 K_{h,t}. \tag{1.31}$$

1.4 Solution

1.4.1 Solution Method

The economy is under perfect competition with complete markets and perfect information. The First Fundamental Theorem of Welfare Economics implies that the decentralized equilibrium is equivalent to the Pareto optimum of a benevolent social planner's problem. I derive the optimality conditions of the planner's problem and the deterministic steady state in Appendix 1.B.

The optimality conditions describe the robust decision rules under the approximating model which performs well over the set of alternative models. However, there are several difficulties in solving the model by commonly-used numerical methods . First, the model has so many state variables and thus suffers from the curse of dimensionality. Second, the exponential operator in the value function will frequently cause the overflow problem in computation procedures such as the value function iteration algorithm. Third, I need to form the worst-case shocks in the likelihood ratio tests to calibrate the model uncertainty parameter. It is very hard to characterize the worst-case shocks base on the decision rules obtained by the accurate solution methods.

To circumvent these obstacles, I adopt the Linear Exponential Quadratic Gaussian (LEQG) control method proposed in Hansen and Sargent (1995) and Hansen and Sargent (2007). This method requires taking a second-order Taylor expansion of the objective function and linearizing the state transition laws with respect to all of the state and control variables. The LEQG control problem can be solved analytically by iterating on a Riccati equation. This procedure is computationally efficient. Moreover, the solution shows that certainty equivalence no longer holds, for the covariance matrix of the TFP innovation shocks will affect the robust decision rules.[19]

1.4.2 Computation Algorithm

I solve for the robust decision rules numerically via the following steps:

[19] In Appendix 1.C, I show the technical details of how to solve for a discounted LEQG control problem.

Step 1: Define the objective function as the Lagrangian of the planner's problem,

$$\begin{aligned}
\mathcal{L} &= \mu_c \log(C_t) + (1-\mu_c)\log(H_t) + (n_{c,t}+n_{h,t})\phi \log(1-\bar{N}) \\
&+ \beta \left(\frac{2}{\sigma}\right) \log\left(\mathbb{E}_t\left[\exp\left(\frac{\sigma}{2} V(K_{c,t+1}, K_{h,t+1}, n_{c,t}, n_{h,t}, H_{t+1}, z_{t+1}, \xi_{t+1})\right) \Big| z_t, \xi_t\right]\right) \\
&+ \lambda_{1,t}\Big[\exp(z_t) K_{c,t}^{\alpha}(n_{c,t}\bar{N})^{1-\alpha} - C_t - K_{c,t+1} + (1-\delta_{kc})K_{c,t} - K_{h,t+1} \\
&+ (1-\delta_{kh})K_{h,t} - \frac{\kappa_c}{2}\left(\frac{K_{c,t+1}-K_{c,t}}{K_{c,t}}\right)^2 K_{c,t} - \frac{\kappa_h}{2}\left(\frac{K_{h,t+1}-K_{h,t}}{K_{h,t}}\right)^2 K_{h,t} \\
&- \frac{\omega}{2}\left(\frac{H_{t+1}-H_t}{H_t}\right)^2 H_t - \frac{\tau_c}{2}(n_{c,t}-n_{c,t-1})^2 - \frac{\tau_h}{2}(n_{h,t}-n_{h,t-1})^2\Big] \\
&+ \lambda_{2,t}\Big[\exp(\xi_t) K_{h,t}^{\theta}(n_{h,t}\bar{N})^{\nu} - H_{t+1} + (1-\delta_h)H_t\Big];
\end{aligned} \tag{1.32}$$

Step 2: Define the vector of state variables, $\boldsymbol{x}_t = [1, z_t, \xi_t, K_{c,t}, K_{h,t}, H_t, n_{c,t-1}, n_{h,t-1}]^T$; the vector of control variables, $\boldsymbol{a}_t = [K_{c,t+1}, K_{h,t+1}, H_{t+1}, n_{c,t}, n_{h,t}, C_t, \lambda_{1,t}, \lambda_{2,t}]^T$; and the vector of Gaussian shocks $\boldsymbol{\epsilon}_t = [\epsilon_{z,t}, \epsilon_{\xi,t}]^T$;

Step 3: Write out the linear state transition law and determine the coefficient matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$:

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{a}_t + \boldsymbol{C}\boldsymbol{\epsilon}_{t+1}, \quad \boldsymbol{\epsilon}_{t+1} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}), \tag{1.33}$$

where

$$
\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_\xi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \boldsymbol{C} = \begin{pmatrix} 0 & 0 \\ \sigma_z & 0 \\ 0 & \sigma_\xi \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.34}
$$

Step 4: Solve for the deterministic steady state values of $\{\bar{K}_c, \bar{K}_h, \bar{H}, \bar{n}_c, \bar{n}_h, \bar{C}\}$ from the equation system:

$$
1 = \beta \left[\alpha K_c^{\alpha-1} (n_c \bar{N})^{1-\alpha} + 1 - \delta_{kc} \right], \tag{1.35}
$$

$$
(1-\alpha) K_c^{\alpha} (n_c \bar{N})^{-\alpha} = -\frac{\phi \log(1-\bar{N}) C}{\mu_c \bar{N}}, \tag{1.36}
$$

$$
\left(\frac{1}{\beta} - 1 + \delta_{kh} \right) \frac{\nu K_h}{\theta n_h \bar{N}} = -\frac{\phi \log(1-\bar{N}) C}{\mu_c \bar{N}}, \tag{1.37}
$$

$$
\frac{(1-\alpha) K_c^{\alpha} (n_c \bar{N})^{-\alpha}}{\nu K_h^{\theta} (n_h \bar{N})^{\nu-1}} = \frac{1}{\frac{1}{\beta} - 1 + \delta_h} \frac{(1-\mu_c) C}{\mu_c H}, \tag{1.38}
$$

$$
C + \delta_{kc} K_c + \delta_{kh} K_h = K_c^{\alpha} (n_c \bar{N})^{1-\alpha}, \tag{1.39}
$$

$$
\delta_h H = K_h^{\theta} (n_h \bar{N})^{\nu}; \tag{1.40}
$$

Step 5: Around the given steady state $[\bar{\boldsymbol{x}}, \bar{\boldsymbol{a}}]$, compute the first and second numerical derivatives of the Lagrangian $\mathcal{L}$ with respect to all states and controls; obtain matrices $\boldsymbol{f}_n$,

$\boldsymbol{f}_k$, $\boldsymbol{S}$, $\boldsymbol{T}$, and $\boldsymbol{L}$; then construct matrices $\boldsymbol{Q}$, $\boldsymbol{W}$, and $\boldsymbol{R}$;

Step 6: Iterate on the Riccati equation until convergence,

$$\begin{aligned}\boldsymbol{P} &= \boldsymbol{Q} + \beta \boldsymbol{A}^T D(\boldsymbol{P})\boldsymbol{A} \\ &\quad - \left(\boldsymbol{W} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{A}\right)^T \left(\boldsymbol{R} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{B}\right)^{-1} \left(\boldsymbol{W} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{A}\right),\end{aligned} \tag{1.41}$$

where

$$D(\boldsymbol{P}) = \boldsymbol{P} + \sigma \boldsymbol{P}\boldsymbol{C}(\boldsymbol{I} - \sigma \boldsymbol{C}^T \boldsymbol{P}\boldsymbol{C})^{-1}\boldsymbol{C}^T\boldsymbol{P}; \tag{1.42}$$

obtain the fixed point solution for $\boldsymbol{P}$ and compute $D(\boldsymbol{P})$;

Step 7: Compute the robust decision rule

$$\boldsymbol{a}_t = -\left(\boldsymbol{R} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{B}\right)^{-1} \left(\boldsymbol{W} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{A}\right)\boldsymbol{x}_t \equiv \boldsymbol{F}\boldsymbol{x}_t, \tag{1.43}$$

and the state transition law is given by

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{a}_t + \boldsymbol{C}\boldsymbol{\epsilon}_{t+1} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})\boldsymbol{x}_t + \boldsymbol{C}\boldsymbol{\epsilon}_{t+1}; \tag{1.44}$$

Step 8: Update the given steady state by $\bar{\boldsymbol{x}}' = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})\bar{\boldsymbol{x}}$; if $\bar{\boldsymbol{x}}' = \bar{\boldsymbol{x}}$, then we have obtained the stochastic steady state; if $\bar{\boldsymbol{x}}' \neq \bar{\boldsymbol{x}}$, repeat Step 5 to Step 7 until the state variables converge to the stochastic steady state; the associated matrices $\boldsymbol{P}$, $D(\boldsymbol{P})$ and $\boldsymbol{F}$ are what we need to construct the robust decision rules.

1.5 Calibration

In this section, I calibrate the parameter values by matching model implications with key moments observed in the macro economy at a quarterly frequency.[20]

The postwar data suggest that per-capita leisure has been approximately constant over time in spite of the steady increase of real wages.[21] In addition, the fraction of household income spent on residential services remained roughly constant according to the Consumer Expenditure Survey. In line with these empirical findings, I choose the following functional form for the households' preference:

$$u(c_t, s_t, n_t) = \mu_c \log(c_t) + (1 - \mu_c) \log(s_t) + \phi \log(1 - n_t), \tag{1.45}$$

where μ_c and ϕ determine the weights on consumption and leisure in utility.

There are 20 parameters in my model: β, μ_c, ϕ, $\bar{N}$, α, θ, ν, δ_{kc}, δ_{kh}, δ_h, κ_c, κ_h, ω, τ_c, τ_h, ρ_z, ρ_ξ, σ_z, σ_ξ, and σ. I divide the parameters into 4 groups and determine their values separately. The first group of parameters, $\{\beta, \mu_c, \phi, \bar{N}, \alpha, \theta, \nu, \delta_{kc}, \delta_{kh}, \delta_h\}$, relates to preferences and production technologies; these parameters only depends on the long-run relationships among the macroeconomic variables and I calibrate their values by matching the first moments of model and data in the steady state. The second group of parameters, $\{\rho_z, \rho_\xi, \sigma_z, \sigma_\xi\}$, are pined down by the autocorrelation and volatility statistics computed from the TFP shock series directly. The third group of parameters, $\{\kappa_c, \kappa_h, \omega, \tau_c, \tau_h\}$, governs the adjustment costs of factor inputs in the consumption sector and housing sector; I

[20] I follow the calibration methodologies advocated by Kydland and Prescott (1982), Prescott (1986), Greenwood and Hercowitz (1991), Benhabib et al. (1991), Cooley and Prescott (1995), Cooley (1997), Favero (2001) and Gomme and Rupert (2007).

[21] This observation implies that the income effect and the substitution effect of increases in real wages on leisure exactly offset each other. Therefore, the elasticity of substitution between consumption and leisure should be near unity.

set their values to match the volatilities of the related variables. Lastly, I calibrate the model uncertainty parameter, σ, by implementing likelihood ratio tests between the approximating model and the worst-case model; then I compute the detection error probability from these tests and aim it to a commonly accepted value in literature.

1.5.1 Data

I use quarterly U.S. data from 1973Q1 to 2014Q4. The measurements of key macroeconomic variables are constructed using data sets from Bureau of Economic Analysis (BEA) and Federal Reserve Economic Data (FRED).[22] The sample data series are segmented into two subsets: (1) 1973Q1 - 2002Q4, the pre-boom period; (2) 2003Q1 - 2014Q4, the boom-bust period. I assume that there is no model uncertainty before the housing market boom and bust, i.e. $\sigma = 0$. I use the pre-boom data to decide model parameter values except for σ by simulated method of moments. Then I calibrate the model uncertainty parameter, σ, by computing the detection error probabilities from likelihood ratio tests. Data sample from 2003Q1 to 2014Q4 is a test set for evaluating model performance with calibrated model uncertainty parameter value.

The key variable measurements are summarized in Table 1.1, Table 1.2, and Table 1.3. Nominal quantities and prices are deflated by the Consumer Price Index. I use the Civilian Noninstitutional Population to convert the quantities to per-capita terms. I then take the natural logarithm of the data series and use the Hodrick-Prescott (HP) filter to extract the trend components from the series at the quarterly frequency.

[22] I show the detailed data source for each variable in Appendix 1.D

Table 1.1: Data Measurements – Output, Investment, and Consumption

Variable	Description	Measurement in Data
Y_T	Total Output	Gross Domestic Product (GDP)
Y_h	Value of New Housing	Private Residential Fixed Investment
$R_h H$	Expenditure on Housing Services	Consumption on Housing and Utilities
Y_c	Non-Housing Sector Output	$Y_T - Y_h - R_h H$
I_T	Total Investment	Gross Private Domestic Investment
		+ Federal Nondefense Gross Investment
		+ State and Local Investment
		+ Consumer Durables
I_k	Non-Residential Investment	$I_T - Y_h$
C	Non-Housing Consumption	$Y_c - I_k$
I_h	New Home Units for Sale	Units of New Houses for Sale
H	Housing Stock	$(R_h H)/R_h$, R_h is described in Table 1.3

Table 1.2: Data Measurements – Capital and Labor

Variable	Description	Measurement in Data
K	Total Capital Stock	Gross Fixed Assets + Consumer Durables
		- Gross Residential Structures
K_h	Capital in Housing Sector	Private Fixed Assets in Construction Industry
K_c	Capital in Consumption Sector	$K - K_h$
N	Total Labor Input	Total Hours Worked
N_h	Labor in Housing Sector	Total Hours Worked in Construction Industry
N_c	Labor in Consumption Sector	$N - N_h$
$1 - n_c - n_h$	Unemployment Rate	Civilian Unemployment Rate

Table 1.3: Data Measurements – Prices

Variable	Description	Measurement in Data
P_{Ih}	New Home Sales Price	Median Sales Price for New Houses Sold in U.S.
R_h	Implicit Home Rent	Consumer Price Index: Housing
r_f	Risk-Free Rate	3-Month Effective Federal Funds Rate

1.5.2 First Moments

I use the data measurements to compute the ratios among several variables in long-run equilibrium. The value for landshare is set to 0.3 according to an estimate given by BEA. Thus $\theta + \nu = 0.7$. There is no sufficient and accurate data for capital depreciations by industry. So I simply assume that the capital depreciation rates are the same across sectors, i.e. $\delta_{kc} = \delta_{kh} = \delta_k$.

Table 1.4 reports the selected first moments that are computed from the data sample and the model steady state. The model does a good job in matching the first moments with data.

Table 1.4: First Moments

Target Moment	Data	Model
3-month effective federal funds rate, $r_f = \frac{1}{\beta} - 1$	1.26%	1.26%
Share of non-housing consumption in total consumption, $\frac{C}{C+R_h H}$	0.8386	0.8386
Residential investment to housing stock ratio, $\frac{I_h}{H}$	0.0151	0.0151
Non-residential investment to capital stock ratio, $\frac{I_k}{K}$	0.0284	0.0284
Capital input in construction sector, $\frac{K_h}{K}$	0.0075	0.0075
Labor input in construction sector, $\frac{N_h}{N}$	0.054	0.055
Residential investment to total output ratio, $\frac{Y_h}{Y_T}$	0.045	0.044
Total investment to total output ratio, $\frac{I_T}{Y_T}$	0.296	0.295
Total consumption to total output ratio, $\frac{C+R_h H}{Y_T}$	0.704	0.705
Unemployment rate, $1 - n_c - n_h$	0.0635	0.0635
Total labor input, $N = (n_c + n_h)\bar{N}$	0.3	0.3

Table 1.5 lists the estimated parameter values. These parameter values stand in line with macro-housing literature. For example, in Davis and Heathcote (2005), they estimate a housing depreciation rate of 0.0141; in Iacoviello and Pavan (2013), they estimate a capital depreciation rate of 0.0233. Capital share is much higher in the consumption sector than in the housing sector, which is consistent with the fact that non-housing production is capital intensive and housing production is labor intensive.

Table 1.5: Parameter Values Determined by Matching First Moments

Parameter	Interpretation	Estimated Value
β	discount factor	$\beta = 0.987$
μ_c	weight of consumption in utility	$\mu_c = 0.8386$
ϕ	weight of leisure in utility	$\phi = 2$
$\bar{N}$	fixed labor input if employed	$\bar{N} = 0.318$
α	capital share in consumption sector	$\alpha = 0.433$
θ	capital share in housing sector	$\theta = 0.063$
ν	labor share in housing sector	$\nu = 0.637$
δ_k	depreciation rate of physical capital stock	$\delta_k = 0.0284$
δ_h	depreciation rate of housing stock	$\delta_h = 0.0151$

1.5.3 TFP Shocks

Based on the Cobb-Douglas production technologies for consumption goods and housing,

$$Y_{c,t} = \exp(z_t) K_{c,t}^{\alpha} N_{c,t}^{1-\alpha}, \tag{1.46}$$

$$I_{h,t} = \exp(\xi_t) K_{h,t}^{\theta} N_{h,t}^{\nu} L_t^{1-\theta-\nu}, \tag{1.47}$$

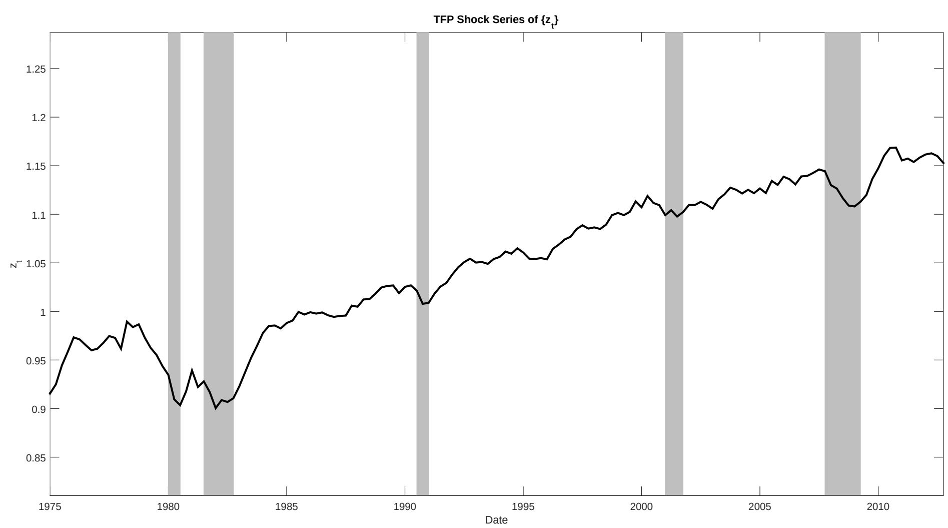
and that land supply is fixed $L_t \equiv 1$, I use Solow residuals to construct the TFP shocks in the two sectors, [23]

$$z_t = \ln(Y_{c,t}) - \alpha \ln(K_{c,t}) - (1-\alpha)\ln(N_{c,t}), \tag{1.48}$$

$$\xi_t = \ln(I_{h,t}) - \theta \ln(K_{h,t}) - \nu \ln(N_{h,t}). \tag{1.49}$$

Figure 1.2 and Figure 1.3 show the logarithmic values of the TFP shocks z_t and ξ_t. We can see that there is an upward trend in z_t and a downward trend in ξ_t. These trends are consistent with the empirical findings in Galesi (2014), which also illustrates an increasing path for non-construction TFP and a decreasing path for construction TFP using productivity data from the EU KLEMS database.

Figure 1.2: TFP Shock in the Consumption Sector $\{z_t\}$, 1975Q1 - 2012Q4.



[23]Note that $I_{h,t}$ is the *units*, rather than the *value*, of new residential construction completed.

Figure 1.3: TFP Shock in the Housing Sector $\{\xi_t\}$, 1975Q1 - 2012Q4.

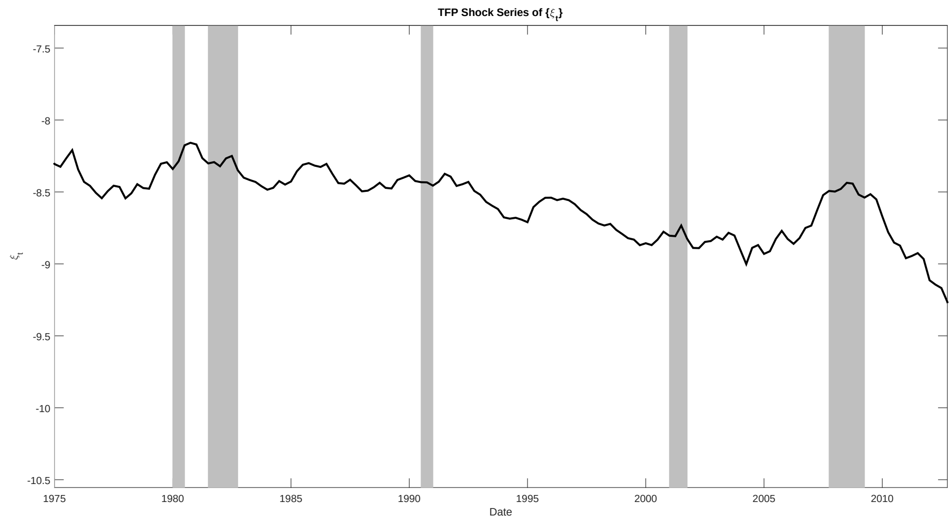


Figure 1.4 and Figure 1.5 plot the detrended TFP shocks obtained by applying the HP filter. The volatility of the TFP shock in housing sector is much higher than that in the consumption sector.

Figure 1.4: Cyclical Component of Shock z_t, 1975Q1 - 2012Q4

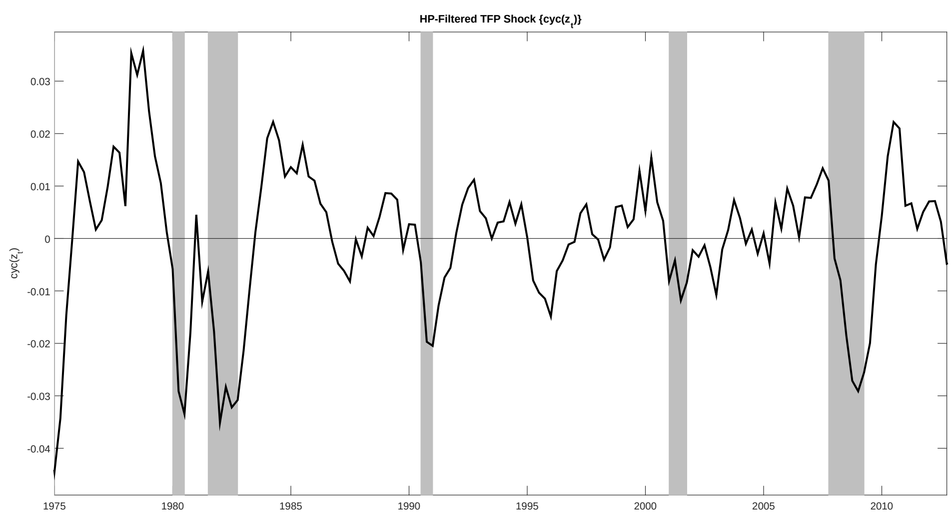
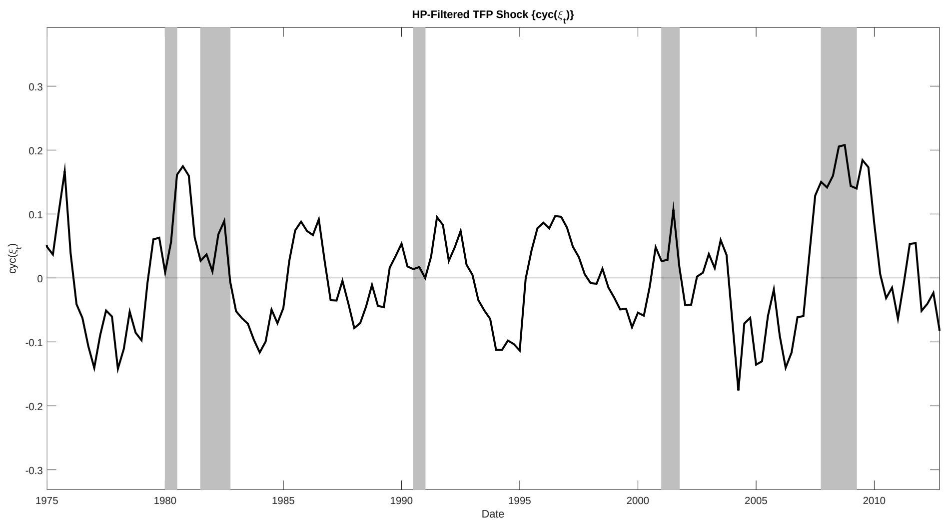


Figure 1.5: Cyclical Component of Shock ξ_t, 1975Q1 - 2012Q4



I adopt the parametric form of an AR(1) process for both detrended shock series,

$$z_{t+1} = \rho_z z_t + \sigma_z \epsilon_{z,t+1}, \quad \epsilon_{z,t+1} \sim \text{i.i.d. } \mathcal{N}(0,1) \tag{1.50}$$

$$\xi_{t+1} = \rho_\xi \xi_t + \sigma_\xi \epsilon_{\xi,t+1}, \quad \epsilon_{\xi,t+1} \sim \text{i.i.d. } \mathcal{N}(0,1) \tag{1.51}$$

then I estimate the autocorrelation coefficients and the standard deviations of the innovations by linear regression. I examine the residuals from the regression by Ljung-Box test, which confirms that there is no significant autocorrelations among the residuals and thus the AR(1) model is appropriate for the TFP shocks. The estimated parameter values are reported in Table 1.6. The productivity in the housing sector is more persistent and much more volatile than the productivity in the consumption sector. This result stands in line with the estimation in Davis and Heathcote (2005).

Table 1.6: Parameters Relates to TFP Shocks

Parameter	Interpretation	Estimated Value
ρ_z	autocorrelation of z_t	$\rho_z = 0.8245$
σ_z	standard deviation of innovation $\epsilon_{z,t}$	$\sigma_z = 0.0079$
ρ_ξ	autocorrelation of ξ_t	$\rho_\xi = 0.8169$
σ_ξ	standard deviation of innovation $\epsilon_{\xi,t}$	$\sigma_\xi = 0.0512$

1.5.4 Second Moments

I choose the values for the third set of parameters $\{\kappa_c, \kappa_h, \tau_c, \tau_h, \omega\}$ by matching the second moments of data. I use the relative volatilities of housing stock and capital and labor inputs in two sectors to exactly identify the five parameters. Table 1.7 and Table 1.8 report the results.

Table 1.7: Second Moments

Target Moments (Relative Volatilities)	Data	Model
Capital input in non-housing sector relative to total output, $\frac{\sigma(K_c)}{\sigma(Y_T)}$	0.590705	0.591033
Capital input in housing sector relative to total output, $\frac{\sigma(K_h)}{\sigma(Y_T)}$	1.354407	1.354326
Labor input in non-housing sector relative to total output, $\frac{\sigma(N_c)}{\sigma(Y_T)}$	0.745078	0.744951
Labor input in housing sector relative to total output, $\frac{\sigma(N_h)}{\sigma(Y_T)}$	2.683119	2.683273
Total housing stock relative to total output, $\frac{\sigma(H)}{\sigma(Y_T)}$	0.675463	0.675415

Table 1.8: Parameter Values Determined by Matching Second Moments

Parameter	Interpretation	Estimated Value
κ_c	adjustment cost parameter of K_c	$\kappa_c = 0.2631$
κ_h	adjustment cost parameter of K_h	$\kappa_h = 1.1605$
τ_c	adjustment cost parameter of N_c	$\tau_c = 3.0130$
τ_h	adjustment cost parameter of N_h	$\tau_h = 0.1134$
ω	adjustment cost parameter of H	$\omega = 2.2343$

Table 1.9: Second Moments (continued)

Second Moments (Correlations)	Data	Model
Consumption and non-housing output, $corr(C, Y_c)$	0.5880	0.6915
Non-residential investment and non-housing output, $corr(I_k, Y_c)$	0.9293	0.9843
Residential investment and non-housing output, $corr(I_h, Y_c)$	0.0403	-0.0801
Total labor input and non-housing output, $corr(N, Y_c)$	0.5363	0.6041
Labor input in non-housing sector and total labor input, $corr(N_c, N)$	0.9984	0.9967
Labor input in housing sector and total labor input, $corr(N_h, N)$	0.7716	0.7322
Labor input in non-housing sector and housing sector, $corr(N_c, N_h)$	0.5328	0.0696
Non-residential investment and residential investment, $corr(I_k, I_h)$	0.1193	0.0775

1.5.5 Model Uncertainty

Finally, I calibrate the model uncertainty parameter σ by likelihood ratio tests and detection error probabilities.Hansen and Sargent (2007) propose the basic idea of this procedure, which is to use a statistical theory of model selection to define a mapping from the model uncertainty parameter σ to a detection error probability for discriminating between the approximating model and an endogenous worst-case model associated with that σ.

According to Hansen and Sargent (2007), the approximating model (i.e. an estimate of the true model) under the linear quadratic setup takes the form of a time-invariant linear state transition law

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{a}_t + \boldsymbol{C}\breve{\boldsymbol{\epsilon}}_{t+1}, \quad \breve{\boldsymbol{\epsilon}}_{t+1} \sim \text{i.i.d. } \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}). \tag{1.52}$$

where $\boldsymbol{x}_t$ and $\boldsymbol{a}_t$ are the state and control variables in the model that defined in section 1.4.2.

The distorted model is a member of the set of alternative models that surround the approximating model, which takes the form of

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{a}_t + \boldsymbol{C}(\boldsymbol{\epsilon}_{t+1} + \boldsymbol{w}_{t+1}), \quad \boldsymbol{\epsilon}_{t+1} \sim \text{i.i.d. } \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}), \tag{1.53}$$

where $\boldsymbol{\epsilon}_{t+1}$ is a different white Gaussian noise from $\breve{\boldsymbol{\epsilon}}_{t+1}$. The component $\boldsymbol{w}_{t+1}$ reflects the distortion on the innovations in the approximating model, and it depends on the history of the states. To make the approximating model a good estimate of the true model, there is a upper bound that controls the approximating error

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{t+1} \boldsymbol{w}_{t+1}^T \boldsymbol{w}_{t+1} \leq \eta_0. \tag{1.54}$$

The robust decision rules is computed by solving a two-player zero-sum games: a maximizing decision maker chooses controls $\{\boldsymbol{a}_t\}$ and a evil agent chooses model distortions $\{\boldsymbol{w}_{t+1}\}$.

The worst-case model is endogenous. Given the robust decision rules $\boldsymbol{a}_t = \boldsymbol{F}\boldsymbol{x}_t$ as obtained in section 1.4.2, the worst-case model is a distorted model in which the distortion component assumes a specific form

$$\begin{aligned} \boldsymbol{x}_{t+1} &= (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})\boldsymbol{x}_t + \boldsymbol{C}(\boldsymbol{\epsilon}_{t+1} + \boldsymbol{w}_{t+1}) \\ &= (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F} + \boldsymbol{C}\boldsymbol{\kappa})\boldsymbol{x}_t + \boldsymbol{C}\boldsymbol{\epsilon}_{t+1}, \end{aligned} \tag{1.55}$$

where $\boldsymbol{w}_{t+1} = \boldsymbol{\kappa}\boldsymbol{x}_t$ with $\boldsymbol{\kappa} = \sigma(\boldsymbol{I} - \sigma\boldsymbol{C}^T\boldsymbol{P}\boldsymbol{C})^{-1}\boldsymbol{C}^T\boldsymbol{P}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})$.[24]

Detection error probabilities can be calculated using likelihood ratio tests. Consider two models. Model A is the approximating model described by (1.52), and model B is the worst-case (distorted) model (1.55).[25] The full calibration procedure is as follows:

Step 1: Generate a random sample of TFP shocks z_t and ξ_t by model A. Compute the likelihood of this sample being generated by model A, L_A, and the likelihood of this sample being generated by model B, L_B. The likelihood ratio test selects model A when $\log(L_A) > \log(L_B)$; it selects model B when $\log(L_B) > \log(L_A)$. The probability of detection error is essentially the probability of choosing the wrong model by mistake. By running the likelihood ratio test for thousands of times over thousands of different samples, I compute the detection error probability when model A generates the data,

$$p_A = \text{Prob}\left(\log\frac{L_A}{L_B} < 0 \,\middle|\, A\right). \tag{1.56}$$

[24]The worst-case model is *endogenous* because the distortion to the shocks, $\boldsymbol{w}_{t+1} = \boldsymbol{\kappa}\boldsymbol{x}_t$, feed back on the history of endogenous states $\boldsymbol{x}_t$ nonlinearly. Refer to Hansen and Sargent (2007) chapter 2 for more details about the worst-case model.

[25]The transition densities associated with model A and model B are absolutely continuous with respect to each other, i.e. they put positive probabilities on the same events. Hence it is difficult to distinguish these two models empirically.

Step 2: Similarly, use model B to generate random samples of $\{z_t, \xi_t\}_{t=1}^{T}$ and compute the corresponding detection error probability ,

$$p_B = \text{Prob}\left(\log \frac{L_A}{L_B} > 0 \,\middle|\, B\right). \tag{1.57}$$

Step 3: Compute the average of these two propabilities of detection error and denote

$$p(\sigma) = \frac{1}{2}(p_A + p_B). \tag{1.58}$$

Step 4: Adjust the value of σ and repeat the Step 1 to Step 3 until $p(\sigma) \approx 0.1$.[26]

The sample size T cannot be too large. Hansen and Sargent (2007) show that when data series is long enough, then it is very easy to distinguish between the approximating model and the worst-case model, thus $\sigma = 0$. I choose a sample size of $T = 28$ (that is, seven years of quarterly data), and set the model uncertainty parameter to a value of -17.5. In Appendix 1.E, I show how to use Kalman filter to compute the likelihood of a given sample under each of the models.

1.6 Quantitative Analysis

1.6.1 Impulse Response Functions

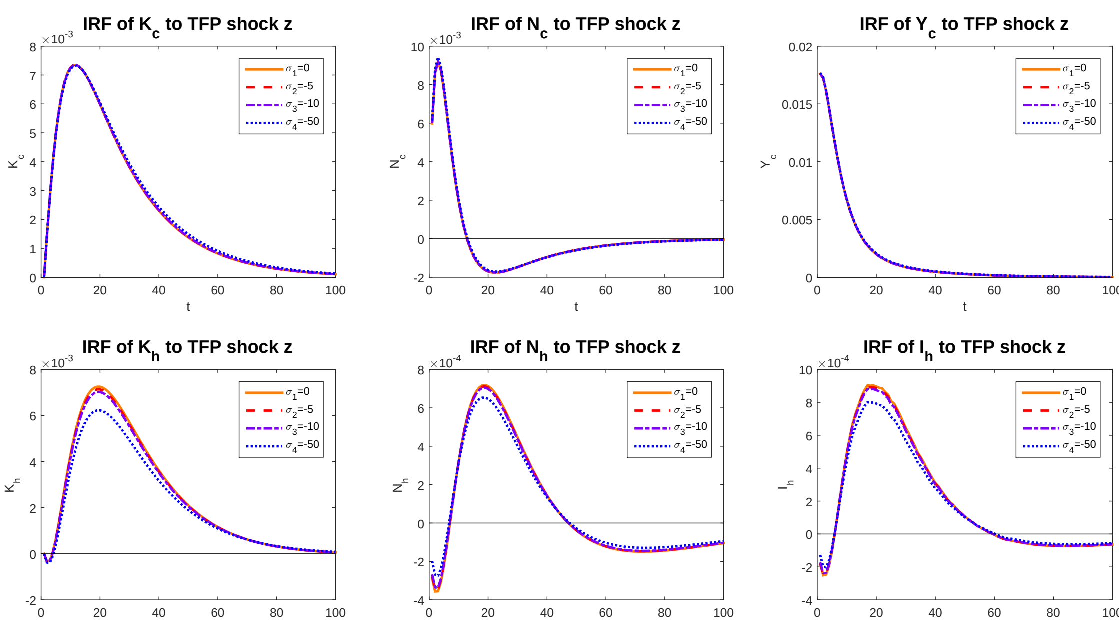
The source of uncertainty in this model is from the TFP shocks in the two production sectors, z and ξ. The fear of model misspecification in the households' minds is also about these two shocks' probability distributions. In order to understand the model mechanism,

[26]This value is suggested by Hansen and Sargent (2007). It is the commonly accepted value in the literature concerning robust control problems.

it is important to examine how the variables of interest respond to an increase (or decrease) of the TFP shocks. In this section, I study the effects of the two TFP shocks on the key model variables separately. Further, I explore if different model uncertainty levels would change the pattern of these impulse responses or not.

Figure 1.6 and Figure 1.7 show the impulse responses of resource allocation, output, consumption, labor choice, investment, new home sales price, and home rents to a one-standard-deviation increment in the TFP shock z_t in the consumption sector. Each figure plots the impulse responses of a specific variable under four model uncertainty levels, $\sigma = 0, -5, -10, -50$. Recall that the more negative the value of σ, the less confident the households are about the model specification.

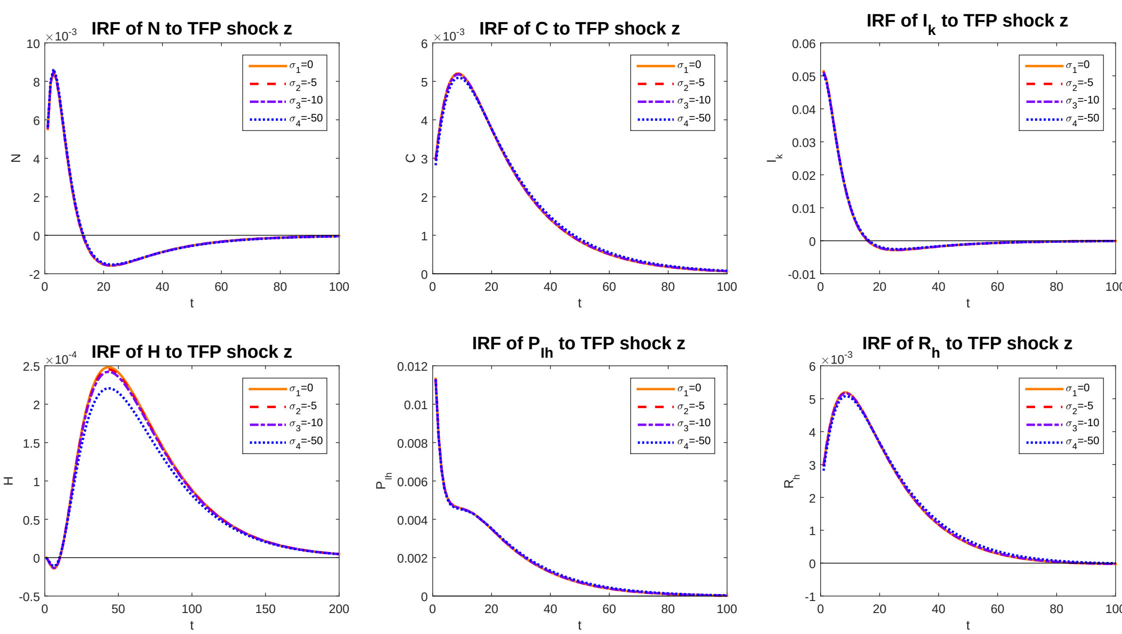
Figure 1.6: IRF of $\{K_c, N_c, Y_c, K_h, N_h, I_h\}$ to a Positive Shock of z_t



Note: z_t increases by one standard deviation; no change in ξ_t.

On one hand, an increase in productivity in the consumption sector attracts higher capital and labor inputs thus leads to greater output of consumption (or/and investment) goods. On the other hand, capital input in the housing sector also accumulates, though there is a tiny decrease at the very beginning. This is because of more investment goods produced and hence more new capital installed, which increases capital usage in both sectors. Labor input in the housing sector jump down initially, as the marginal return of labor is lower than that in the consumption sector due to the productivity change. However, it increases later to catch up with the higher level of capital input. As a result, the number of new home units built declines first then rises and eventually returns to the original level.

Figure 1.7: IRF of $\{N, C, I_k, H, P_{Ih}, R_h\}$ to a Positive Shock of z_t



Note: z_t increases by one standard deviation; no change in ξ_t.

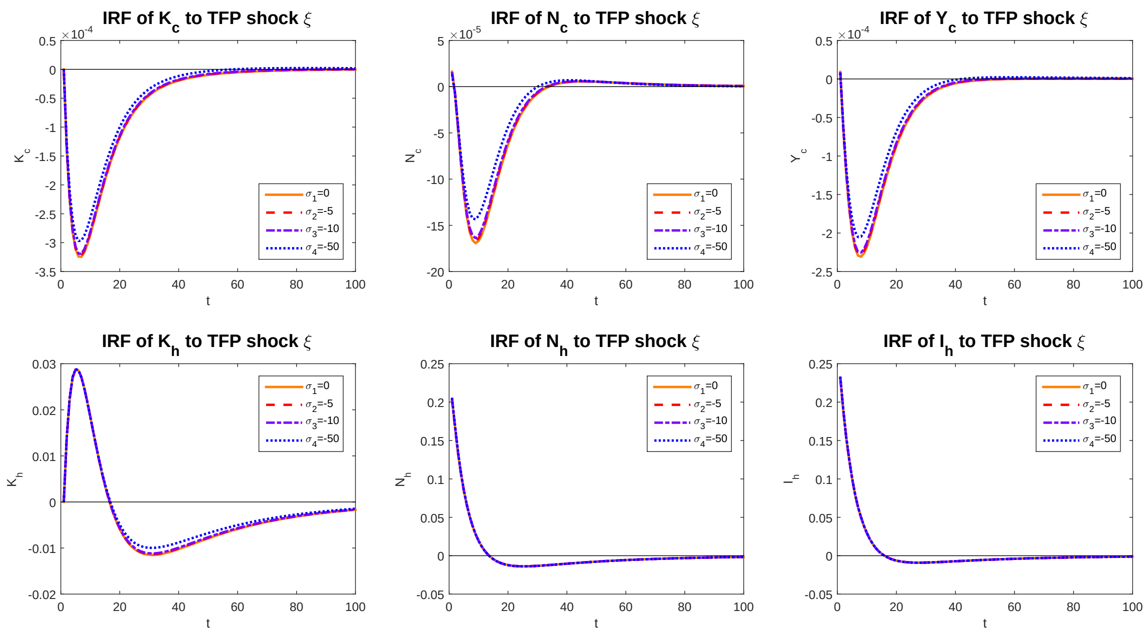
Additionally, with a positive productivity shock in the consumption sector, households work more hours and consume more at first. But then households choose to work less and enjoy more leisure due to the wealth effect. There is a higher investment in the physical capital and the total housing stock also increases.

The dynamic response of the new home sales price mainly come from two driven forces. First, an increase of TFP shock in the consumption sector will directly raise the new home sales price according to the relationship shown in Equation (1.20); second, the productivity change also leads to adjustments in production factors in both sectors, which alter the marginal products of the capital and labor and in turn affect the price of new homes indirectly. It is hard to determine the net effect without numerical solutions. In Figure 1.7, the impulse response of P_{Ih} shows that a positive shock in TFP z will increase the new home sales price.

The home rent, R_h, is determined by the ratio of consumption to housing services enjoyed by the households. Since both consumption and available housing units available in the economy, the net effect depends on the numerical results and is shown in Figure 1.7.

Figure 1.8 and Figure 1.9 show the impulse responses of the variables of interest to a one-standard-deviation increment in the TFP shock ξ_t in the housing sector. Again, each figure plots the specific impulse response function under the four different model uncertainty levels, $\sigma = 0, -5, -10, -50$.

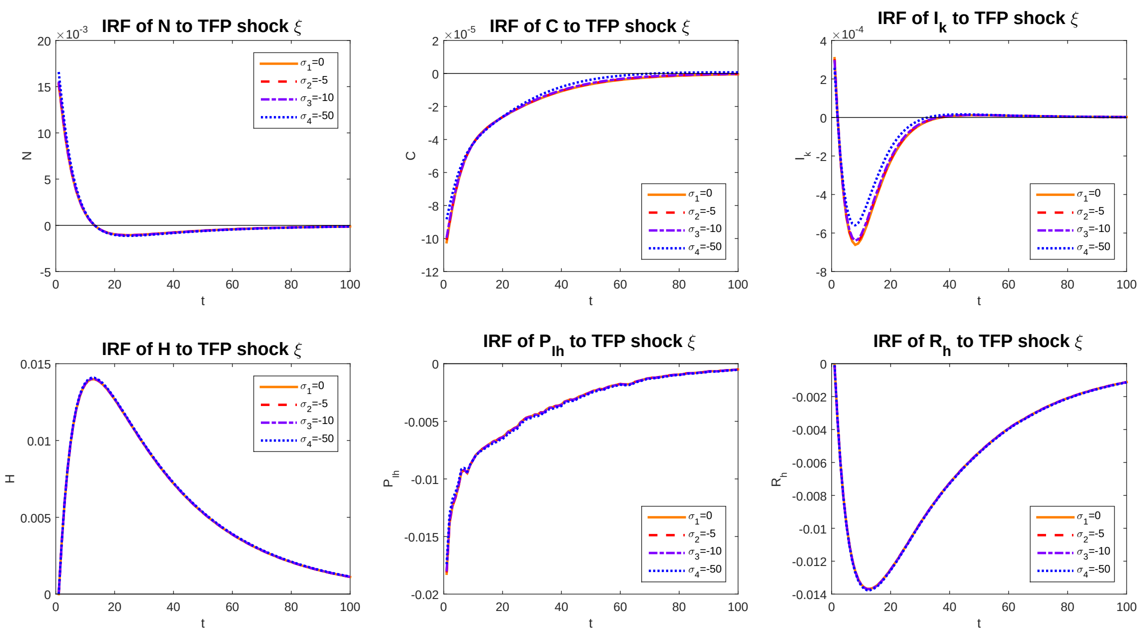
Figure 1.8: IRF of $\{K_c, N_c, Y_c, K_h, N_h, I_h\}$ to a Positive Shock of ξ_t



Note: ξ_t increases by one standard deviation; no change in z_t.

We can see that both capital and labor inputs increase in the housing sector to take the advantage of higher total factor productivity. On the contrary, there are less capital and labor inputs in the consumption/investment goods production. Capital input in the housing sector later falls below the level of its starting point because of insufficient investment in physical capital, I_k. This positive productivity shock leads to a higher output of new housing units and a lower output of consumption/investment goods.

Figure 1.9: IRF of $\{N, C, I_k, H, P_{Ih}, R_h\}$ to a Positive Shock of ξ_t



Note: ξ_t increases by one standard deviation; no change in z_t.

The impulse response of total hours worked is similar to that shown in Figure 1.7. However, both consumption and investment in physical capital decrease because of lower output Y_c. Housing stock accumulates and home rent becomes lower. New home sales price, P_{Ih}, also decreases as a consequence of higher productivity in the housing sector.

Finally, I notice that the impulse response curves of each variable under different model uncertainty levels almost stack on top of each other. It indicates that whether the households fear more or less about model misspecification has nearly no effects on the aftermaths of a positive shock of a given size in either z_t or ξ_t. It confirms that model uncertainty does not generate an amplification mechanism for the TFP shocks in my model.[27]

[27] I also investigated the impulse response functions of these variables to negative shocks in z_t and ξ_t. They are just the mirror images of the ones responding to the positive shocks. Further, changing the model uncertainty level does not affect this symmetry at all.

1.6.2 Implications of Model Uncertainty Changes

In this section, I study the implications of my model compared to the observed data series. First, I feed the TFP shocks derived in section 1.5.3 into a benchmark model where there is no model uncertainty at all, i.e. $\sigma \equiv 0$, and simulate the dynamic paths for variables of interest. The results from the benchmark model show how the TFP shocks alone account for housing market dynamics and movements in related variables. Then I study the contribution of a one-time change to the model uncertainty level in matching the data paths. Time-varying model uncertainty is not computationally tractable and thus beyond my discussion. Instead, I increase model uncertainty to a constant level only within a specific time period. The model paths with this one-time model uncertainty shock differ from the paths under the benchmark model, which demonstrates that households' fear of model misspecification improves the performance of my model in matching real world data.

The sample period of consideration is from 1974Q1 to 2014Q4. I increase the model uncertainty level to the calibrated value, -17.5, between 2004Q2 and 2010Q1.[28] The model uncertainty level is set to 0 during other time periods. I choose the second quarter of the year 2004 as the start point of model uncertainty change because the dispersion of housing start forecasts jumped up around that time, indicating that the information available is less trustable than before thus the model used for decision making can be misspecified. The end point of this increase in model uncertainty is the first quarter of the year 2010 as the dispersion of the professional forecasts for main economic activity indicators, such as GDP, housing starts, industrial production, etc., all drop to a relatively low level since then. The data evidence can be found in Appendix 1.F.

[28] Recall that σ always takes on a negative value. The more negative the value, the higher the model uncertainty level. This choice of the time span with elevated model uncertainty stands in line with Bloom et al. (2014) and Bloom (2014), which use dispersion of industry-level TFP shocks.

In the rest part of this section, I discuss the implied variable paths with and without model uncertainty and compare them to the observed data series. In Figure 1.10 to Figure 1.22, the black solid line represents the data series; the blue dotted line plots the variable path under the benchmark model (i.e. $\sigma = 0$, the no-model-uncertainty case); the red dashed line shows the variable path with a one-time model uncertainty shock (i.e. $\sigma = 0$ from 1974Q1 to 2004Q1, $\sigma = -17.5$ from 2004Q2 to 2010Q1, and $\sigma = 0$ from 2010Q2 to 2014Q4). The green dashed lines mark the start point and the end point of the time span of increased model uncertainty level. The grey bars depict the recessions in the economy.

Resource Allocations

Figure 1.10 and Figure 1.11 show the cyclical dynamics of total physical capital stock and total employment in the economy. In general, the TFP shocks alone capture a significant portion of the variations in line with the data series, especially for the years around the recent business cycle.

When model uncertainty increases, or in other words, when households become less confident about the model specifications, they save more and accumulate more capital stock. This result is consistent with Hansen et al. (1999) and Hansen and Sargent (2007), which argue that concern about model uncertainty introduces precautionary savings because the risk-sensitive households want to protect themselves against model specification errors.[29] In addition, households will work more hours if they are more uncertain towards model specification. After the model uncertainty level decreases to 0 again, both capital input and labor input fall. Capital stock remains at a higher level than the benchmark model, but the total workforce drops below the benchmark level.

[29]"Risk-sensitive households" refers to the households who fear of model uncertainty and seek robust decisions.

Figure 1.10: Cyclical Dynamics of Total Capital Stock, K

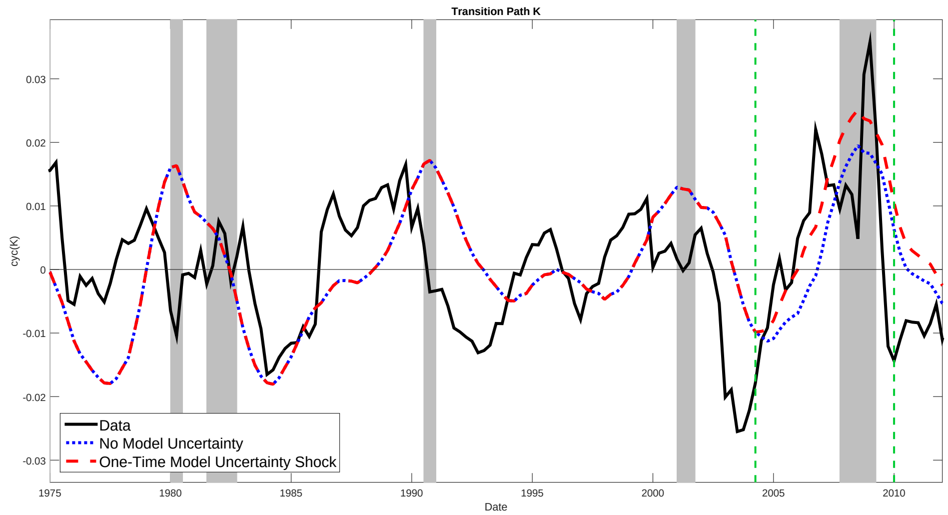
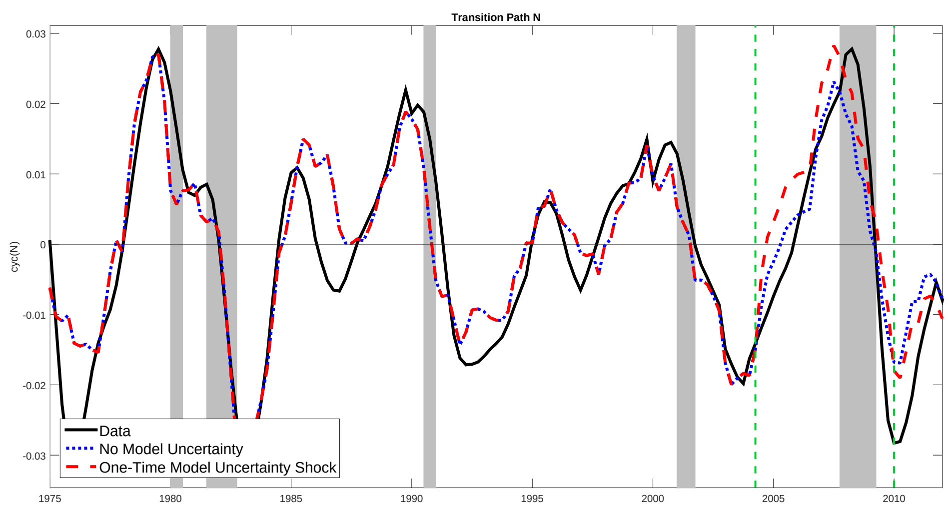


Figure 1.11: Cyclical Dynamics of Total Labor Input, N



The plots of capital and labor in the consumption goods sector are very similar to the plots of total capital stock and total labor input, which attributes to that the housing construction sector is relatively small – it only possesses 5.5 percent of total labor and less than

1 percent of total physical capital according to data.

From Figure 1.12 to Figure 1.15 we can see that the excess capital stock and labor input due to the model uncertainty increase mostly flow into the consumption goods sector.

Figure 1.12: Cyclical Dynamics of Capital Input in Consumption Sector, K_c

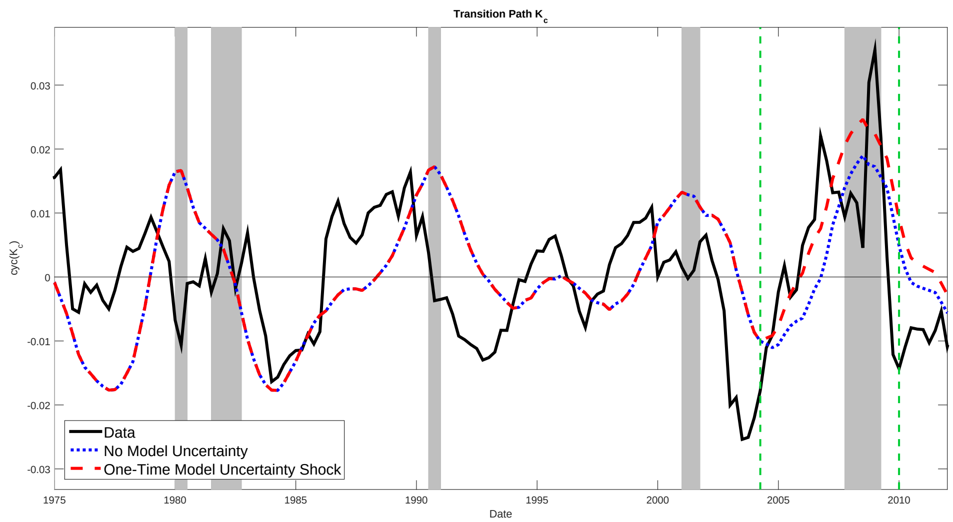


Figure 1.13: Cyclical Dynamics of Labor Input in Consumption Sector, N_c

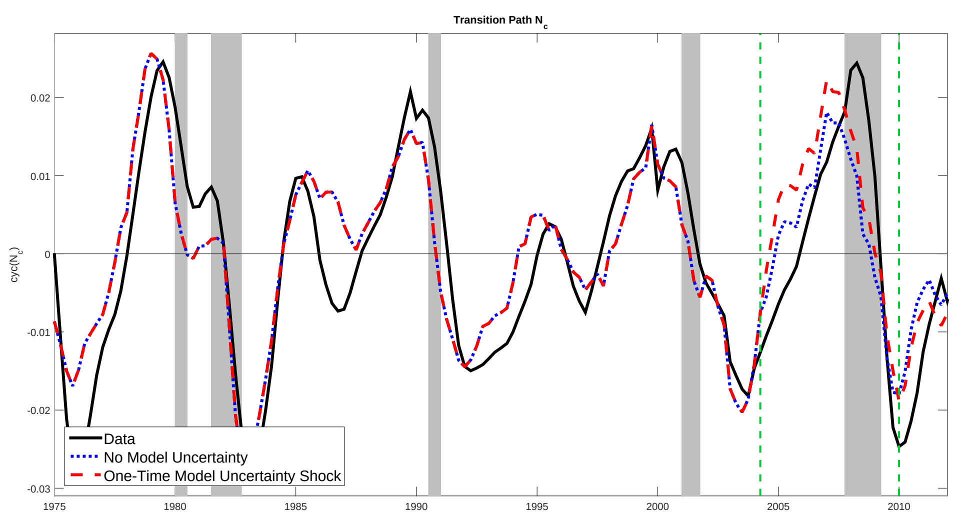


Figure 1.14: Cyclical Dynamics of Capital Input in Housing Sector, K_h

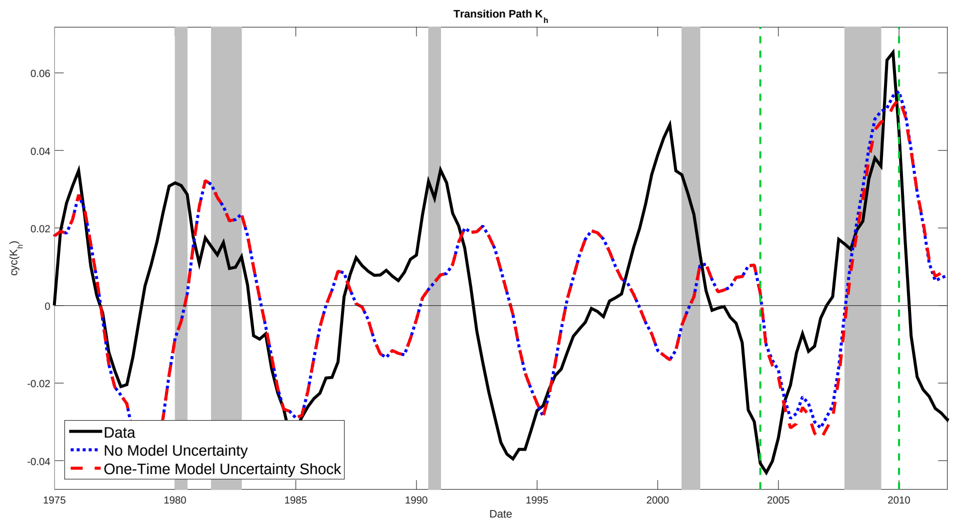
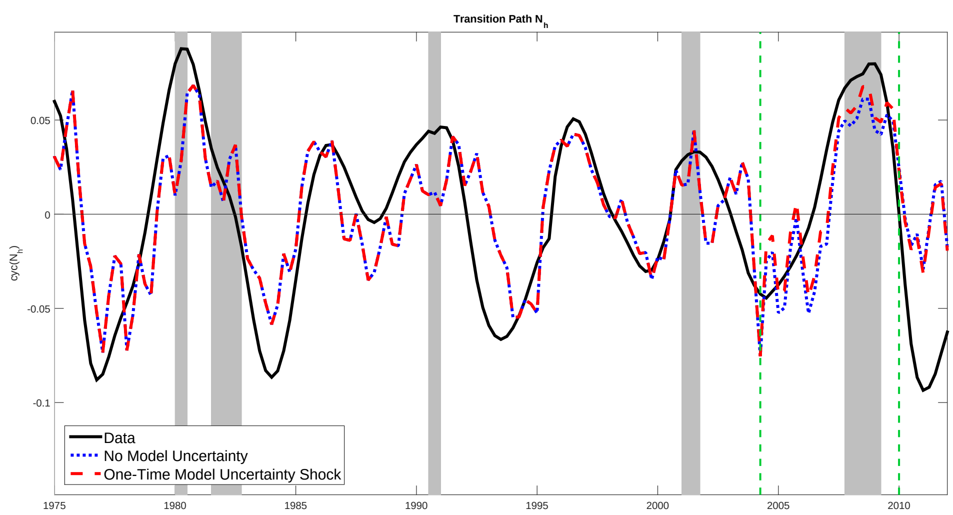


Figure 1.15: Cyclical Dynamics of Labor Input in Housing Sector, N_h



The model paths mimic the data series roughly well, except for the cyclical behavior of capital input in the housing sector. The model series assume comparative volatilities as their data counterparts. In particular, featured indivisible labor and adjustment costs, the model can be fine tuned to match the actual fluctuations of labor input in business cycles as affirmed in Hansen (1985).

Output, Investment and Consumption

The model does a good job in matching with data of output and investment, even with the TFP shocks alone. The output in the housing sector is about 10 times as volatile as the output in the consumption sector, which results from the fact that there are much more fluctuations in factor inputs and productivity of the housing sector.

An increase in model uncertainty level leads to higher non-housing output and more new homes built. Moreover, the model uncertainty change favors the consumption goods sector as we can see a greater percentage increment of consumption output in Figure 1.16 compared to Figure 1.17. This is not a surprising result since we have already seen in previous figures that capital and labor input in the consumption sector incline to increase by a larger amount.

Figure 1.16: Cyclical Dynamics of Output in Consumption Sector, Y_c

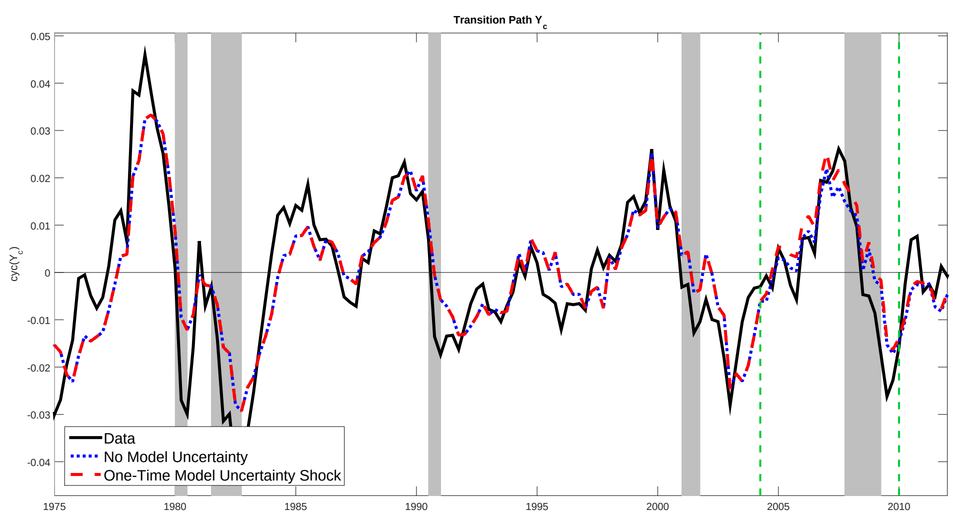
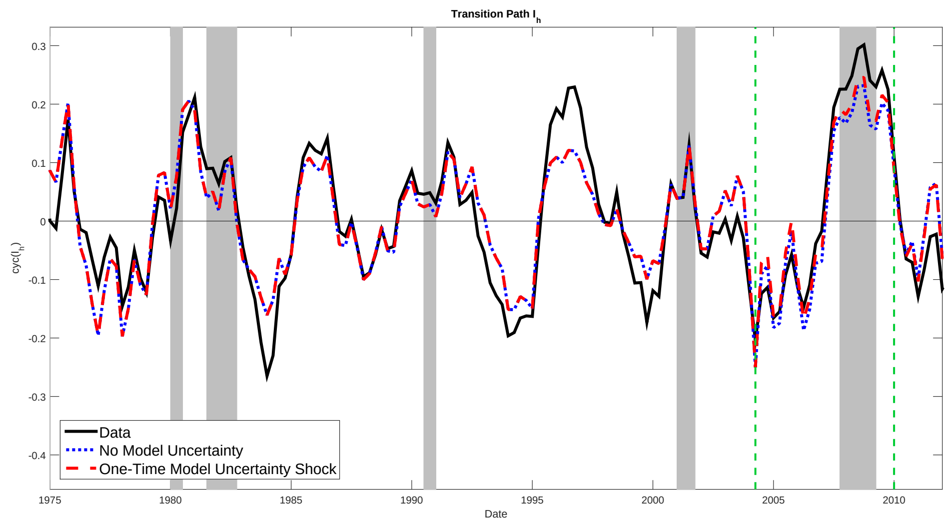
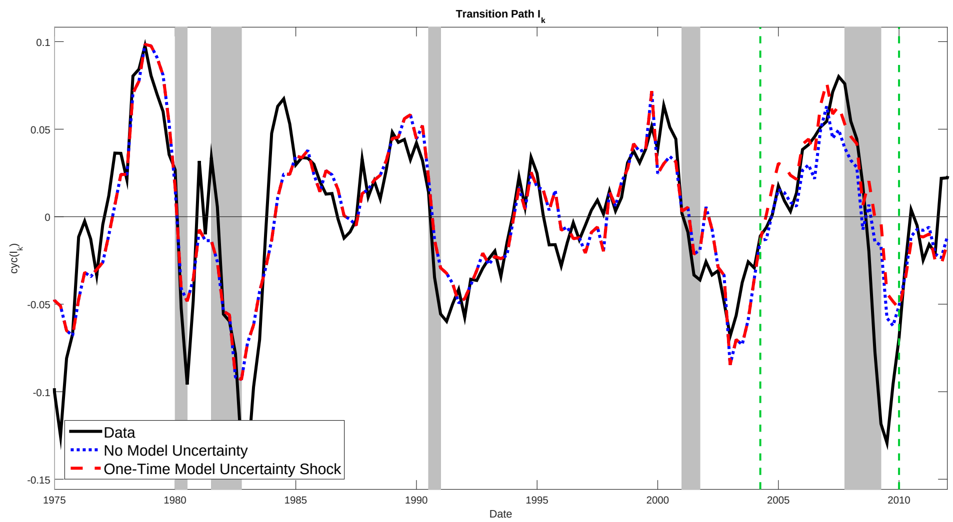


Figure 1.17: Cyclical Dynamics of Output in Housing Sector, I_h



The output from the consumption goods sector can be used either as capital investment or as personal consumption. During the periods of high model uncertainty, households consume less and save more even though there are more consumption/investment goods produced as shown by Figure 1.18 and Figure 1.19.

Figure 1.18: Cyclical Dynamics of Capital Investment, I_k



In Figure 1.19, the cutback in consumption immediately after the model uncertainty increase partially capture the downturn in personal expenditure observed from 2004Q1 to 2006Q4.

Figure 1.19: Cyclical Dynamics of Non-Housing Consumption, C

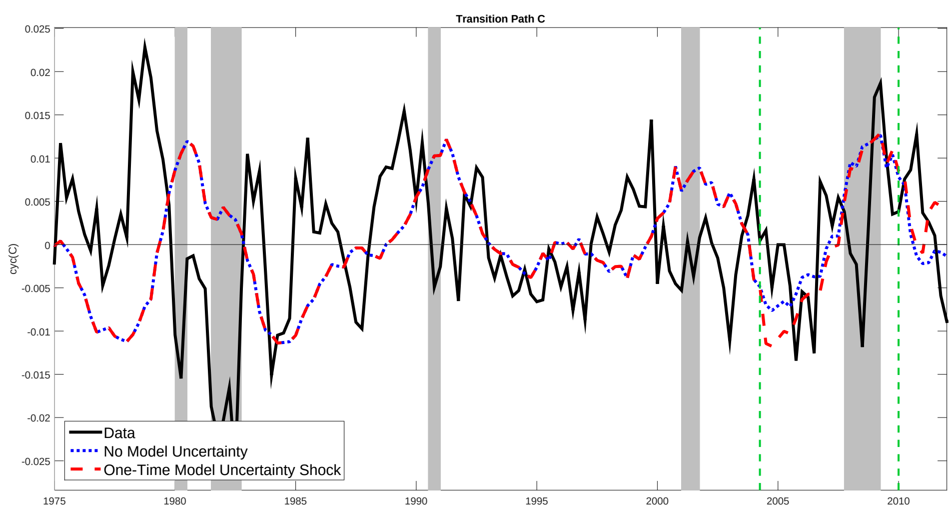
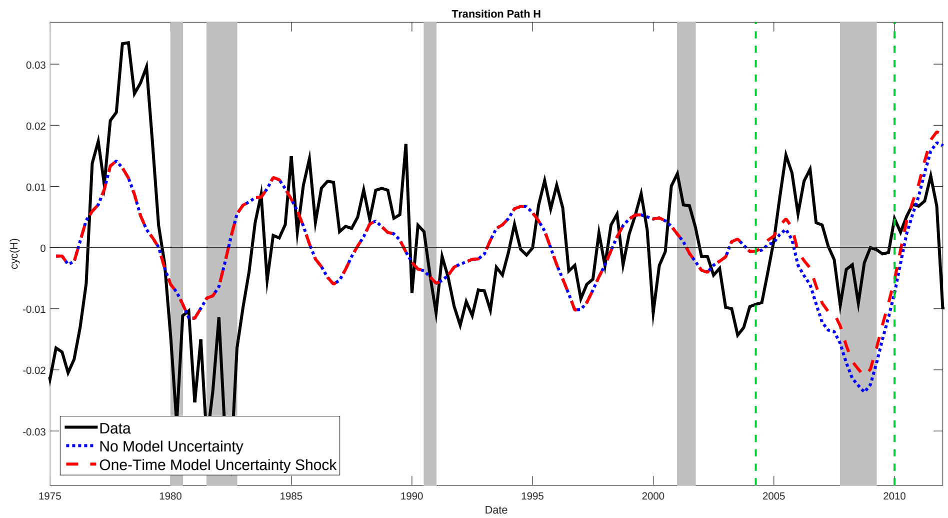


Figure 1.20: Cyclical Dynamics of Total Housing Stock, H

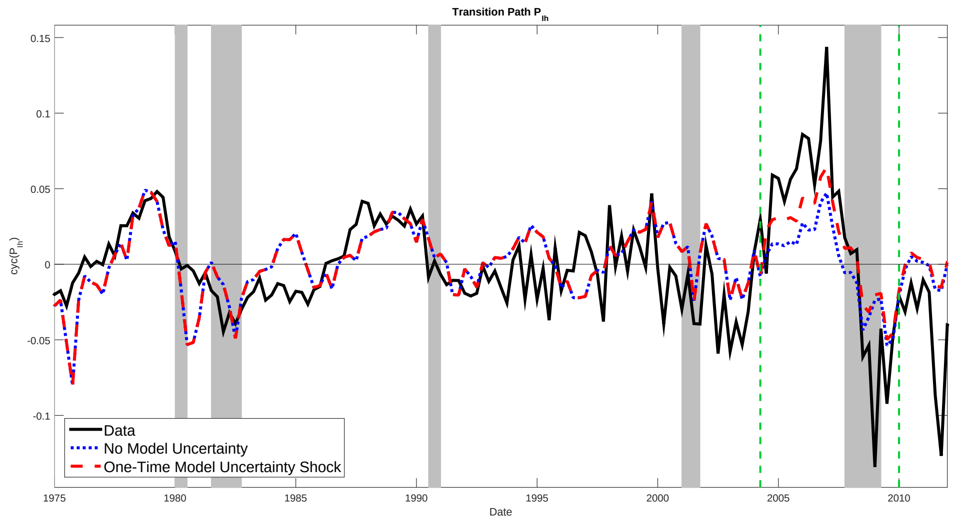


The model does not match the dynamic behavior of total housing stock very well. There is no measurement directly available for existing housing stock, so the data series per se could be imprecise. The model series is able to replicate the decrease in housing stock from 2006 to 2008 and the recovery afterward, which is consistent with the pattern shown by data.

New Home Sales Price and Home Rents

Figure 1.21 answers the key question asked in this paper – does an increase in model uncertainty level help account for the boom and bust of the new home sales price? The benchmark model with TFP shocks alone exhibits similar variations as the data, but it can only account for about 32 percent of the price boom that peaked at 2007Q1. However, with elevated model uncertainty, the model is able to capture 40 percent of the price surge.

Figure 1.21: Cyclical Dynamics of New Home Sales Price, P_{Ih}



As shown by Hansen and Sargent (2007), the robust decision rules derived from the linear quadratic regulator problem with model uncertainty is equivalent to the solution to an ordinary linear quadratic regulator problem under rational expectation with a distorted transition laws of the state variables. The distorted transition law negatively distorts the mean of the innovation. That been said, the households maximize their lifetime utility as if they were facing a negative impact on TFP shocks at all times. From the impulse response functions displayed in section 1.6.1, a negative shock of z_t (TFP in the consumption sector) leads to an increase in the new home sales price $P_{I_h,t}$, while a negative shock of ξ_t (TFP in the housing sector) gives rise to a decrease in $P_{I_h,t}$. This result is as expected from the pricing function

$$P_{I_h,t} = \frac{\exp(z_t)(1-\alpha)K_{c,t}^{\alpha}N_{c,t}^{-\alpha}}{\exp(\xi_t)\nu K_{h,t}^{\theta}N_{h,t}^{\nu-1}}. \tag{1.59}$$

Although the change in $P_{I_h,t}$ also depends on the resource allocation, the net effect is dominated by the change in the TFP shocks.

Moreover, the TFP shock in the housing sector is about four times as volatile as the TFP shock in the consumption sector. The quantitative result in section 1.7 shows that the more volatile the TFP shock is, the larger room there is for model specification error, thus the greater the "imaginary" negative distortion in households' minds.

In summary, an increase in the fear for model misspecification imposes a positive effect on the new home sales price.

Figure 1.22: Cyclical Dynamics of Home Rents, R_h

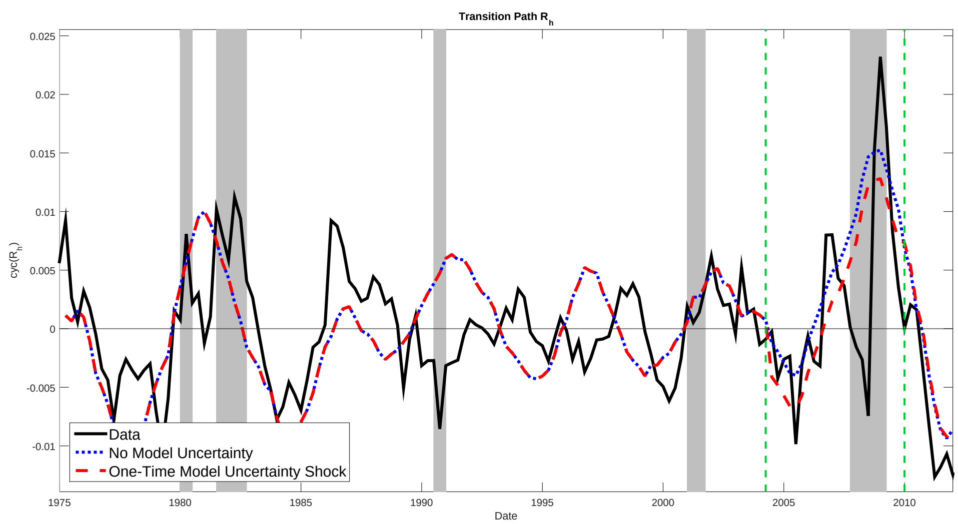


Figure 1.22 plots the data series and the model paths of home rental rate R_h. The home rental rate is determined by the ratio of marginal utility from housing services to marginal utility from consumption goods. Specifically,

$$R_{h,t} = \frac{(1-\mu_c)C_t}{\mu_c H_t}. \tag{1.60}$$

Holding the fear for model specification errors, the households consume less and the economy produces more housing units, as shown by Figure 1.19 and Figure 1.20, which pushes up the marginal utility of consumption and drags down the marginal utility of housing services. As a result, the home rents are lower compared to the benchmark model.

1.7 The Worst-Case Shocks

In this section, I study the moments of the distributions of innovation shocks under the worst-case model.

As discussed in section 1.5.5, the approximating model and the endogenous worst-case model can be described by the time-invariant state transition laws (1.52) and (1.55). In the approximating model, the TFP innovations in two sectors, $\epsilon_{z,t}$ and $\epsilon_{\xi,t}$, are assumed to be independent, i.e. $corr(\epsilon_{z,t}, \epsilon_{\xi,t}) = 0$.

Hansen and Sargent (2007) show that the transition density for the approximating model is

$$f_o(\boldsymbol{x}_{t+1}|\boldsymbol{x}_t) \sim \mathcal{N}((\boldsymbol{A} + \boldsymbol{BF})\boldsymbol{x}_t, \boldsymbol{CC}^T), \tag{1.61}$$

and the transition density for the worst-case model is

$$f(\boldsymbol{x}_{t+1}|\boldsymbol{x}_t) \sim \mathcal{N}((\boldsymbol{A} + \boldsymbol{BF} + \boldsymbol{C\kappa})\boldsymbol{x}_t, \hat{\boldsymbol{C}}\hat{\boldsymbol{C}}^T), \tag{1.62}$$

where $\boldsymbol{\kappa} = \sigma(\boldsymbol{I} - \sigma\boldsymbol{C}^T\boldsymbol{PC})^{-1}\boldsymbol{C}^T\boldsymbol{P}(\boldsymbol{A} + \boldsymbol{BF})$, and $\hat{\boldsymbol{C}}\hat{\boldsymbol{C}}^T = \boldsymbol{C}(\boldsymbol{I} - \sigma\boldsymbol{C}^T\boldsymbol{PC})\boldsymbol{C}^T$. The matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{F}, \boldsymbol{P}$ are derived as in section 1.5.5. We can see that not only does the model misspecification distort the Gaussian mean by $\boldsymbol{C\kappa x}_t$, the implied worst-case model also distorts the covariance matrix of the innovations, as $\hat{\boldsymbol{C}}\hat{\boldsymbol{C}}^T \neq \boldsymbol{CC}^T$ when $\sigma < 0$.

First, I examine the mean distortion effect of the worst-case model. The distortion term, $\boldsymbol{C\kappa x}_t$, is endogenous and feed back on current state $\boldsymbol{x}_t$. Thus the distortion is changing over time. But I can compute the coefficient matrix $\boldsymbol{C\kappa}$ to quantify the one-period-ahead distortion on the state variables,

$$\boldsymbol{C\kappa x}_t = \sigma \boldsymbol{C}(\boldsymbol{I} - \sigma \boldsymbol{C}^T \boldsymbol{P} \boldsymbol{C})^{-1} \boldsymbol{C}^T \boldsymbol{P}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})\boldsymbol{x}_t$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0044 & -0.0019 & 0.0000 & 0.0000 & 0.0001 & 0.0000 & -0.0002 & 0.0000 \\ -0.0204 & 0.0000 & -0.0102 & -0.0000 & -0.0037 & 0.0021 & -0.0000 & 0.0000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ z_t \\ \xi_t \\ K_{c,t} \\ K_{h,t} \\ H_t \\ n_{c,t} \\ n_{h,t} \end{pmatrix}. \tag{1.63}$$

It shows that there are no direct distortion effects on the endogenous states $K_{c,t}, K_{h,t}, H_t, n_{c,t}$ and $n_{h,t}$, as all elements in the coefficient matrix $\boldsymbol{C\kappa}$ corresponding to these states are zero. The worst-case model directly twists the TFP shocks z_t and ξ_t and slant the dynamic transitions of the endogenous states via the TFP shocks. The second and the third row of matrix $\boldsymbol{C\kappa}$ demonstrate the distortion effects on z_{t+1} and ξ_{t+1}. The coefficients on the constant terms, -0.0044 and -0.0204, are most prominent among all elements in these two rows, which indicate the negative mean shift effects on the TFP shocks. The second most significant coefficients are -0.0019 and -0.0102, which associate with the past state of the shock per se. These negative values reduce the autocorrelations of the shock series and make them less persistent. The distortion also allow future values of the TFP shocks to depend on the endogenous states, but most of these effects are minimal.

The examine the long-run distortion effects, I compare the stochastic steady states of the approximating model versus the worst-case model. The transition laws (1.52) and (1.55) specify a vector autoregressive process of order one for each of the two models respectively. Define

$$\boldsymbol{D}_A = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}, \quad \boldsymbol{D}_B = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{F} + \boldsymbol{C}\boldsymbol{\kappa} \tag{1.64}$$

The two VAR(1) models are stationary, as all eigenvalues of $\boldsymbol{D}_A$ and $\boldsymbol{D}_B$ lie within the unit circle (except for the eigenvalue that corresponds to the constant term). The steady states of the approximating (or worst-case) model can be computed by multiplying matrix $\boldsymbol{D}_A$ (or $\boldsymbol{D}_B$) by itself until convergence. Table 1.10 reports the result.

Table 1.10: Stochastic Steady States

Variable	Approximating Model	Worst-Case Model
z	0	-0.02024
ξ	0	-0.05552
K_c	16.90	17.97
K_h	0.1769	0.1819
N_c	0.2817	0.2821
N_h	0.0227	0.0235
H	5.0555	5.4874
P_{Ih}	1.53195	1.53374
$V(\boldsymbol{x})$	-25.29	-26.71

The worst-model represents an economy where the households hold a pessimistic view about the productivity shocks and they behave as if the worst-case shocks were taking place. Due to precautionary savings motive, the households save more and also work harder. Consequently, the economy converges to a steady state with higher capital and labor input in both sectors, higher housing stock, and higher new home sales price.

Then I turn to the distortion effects on volatility and correlation in the worst-case model. From the numerical solution of the model, with the parameter value of model uncertainty set to -17.5, I have

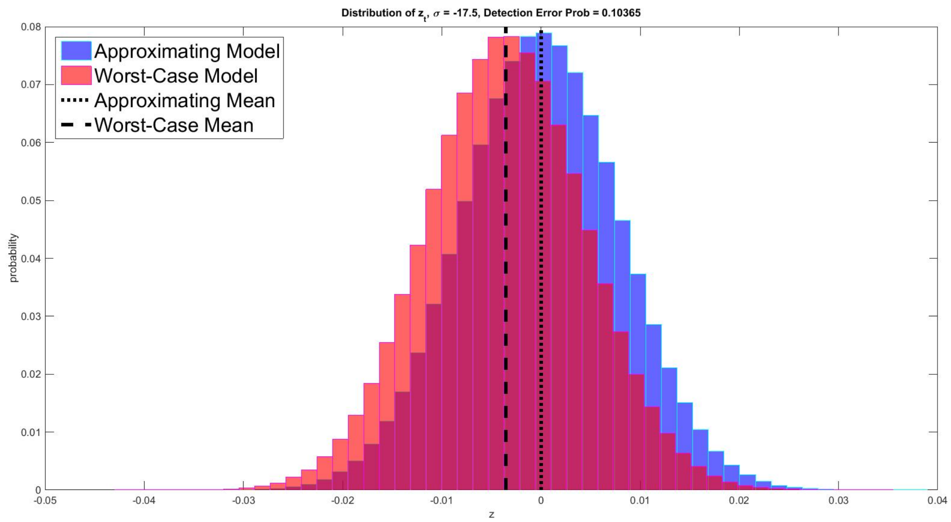
$$\boldsymbol{C}\boldsymbol{C}^T = \begin{pmatrix} 0.000063198 & 0 \\ 0 & 0.002624658 \end{pmatrix}, \quad \hat{\boldsymbol{C}}\hat{\boldsymbol{C}}^T = \begin{pmatrix} 0.000063341 & -0.000000005 \\ -0.000000005 & 0.002658685 \end{pmatrix}. \tag{1.65}$$

From the result above, we can see that the magnitude of $cov(\epsilon_{z,t}, \epsilon_{\xi,t})$ is negligible and $corr(\epsilon_{z,t}, \epsilon_{\xi,t}) \approx 0$ under the worst-case model, which means the distortion generates minimal correlation between the two TFP innovations. Moreover, concern for model misspecification slightly increases the volatility of the TFP shocks. The standard deviation of the innovation in z_t is increased from 0.00795 to 0.00796 (by 0.11%); the standard deviation of the innovation in ξ_t has a greater increment from 0.05123 to 0.05156 (by 0.65%).[30] This result is consistent with Hansen-Sargent's argument that "the volatility covariance matrix is slightly altered" under the worst-case model.

Next I use simulated shock series to varify the theoretical results. I simulate a very long sample of z_t and ξ_t under the approximating model and the worst-case model respectively. Since the distortion depends on current level of the states. I assume the distortion starts from the steady state when there is no model uncertainty. This experiment mimics the responses of variables when there is a sudden increase in ambiguity level. Figure 1.23 and Figure 1.24 show the one-period-ahead distortion to the TFP shocks. I compute $corr(z_t, \xi_t) \approx 0$, which indicates that the correlation between the productivities of two sectors is still insignificant after the distortion under the worst-case model.

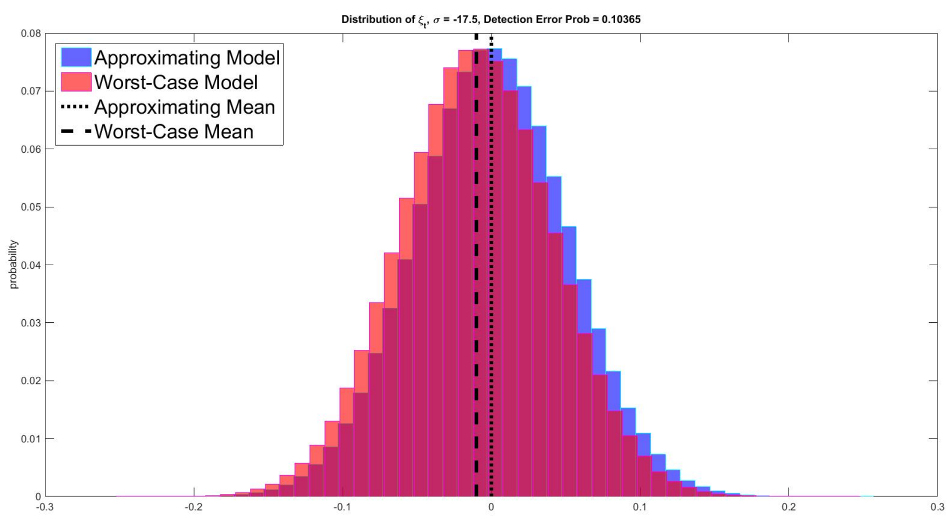
[30] The standard deviations of TFP innovations are computed by taking the square roots of the diagonal elements of the variance-covariance matrices $\boldsymbol{C}\boldsymbol{C}^T$ and $\hat{\boldsymbol{C}}\hat{\boldsymbol{C}}^T$.

Figure 1.23: Simulated Distribution of $\{z_t\}$



Note: $\sigma = -17.5$. Detection error probability $p(\sigma) = 0.10015$.

Figure 1.24: Simulated Distribution of $\{\xi_t\}$

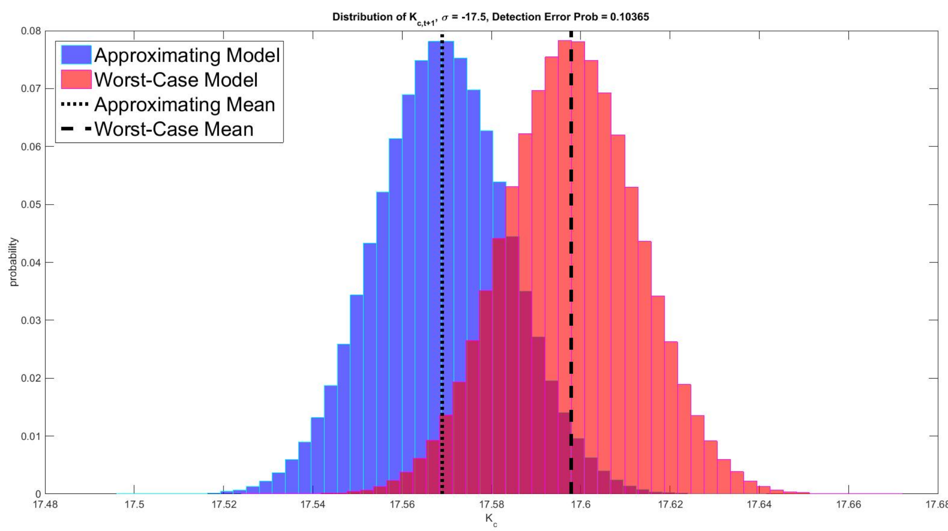


Note: $\sigma = -17.5$. Detection error probability $p(\sigma) = 0.10015$.

Figure 1.25 - Figure 1.29 plot the simulation results for resource allocations and the

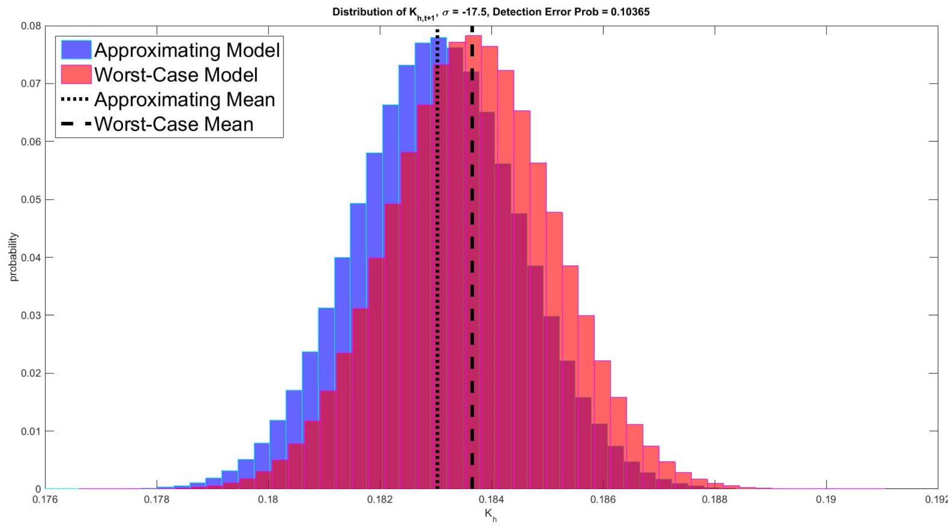
new home sales price. It is clear that the worst-case shocks, starting from the steady state under the approximating model, slant factor inputs upward in both sectors except for capital input in the housing sector, hence result in higher new home sales price.

Figure 1.25: Simulated Distribution of $\{K_{c,t+1}\}$



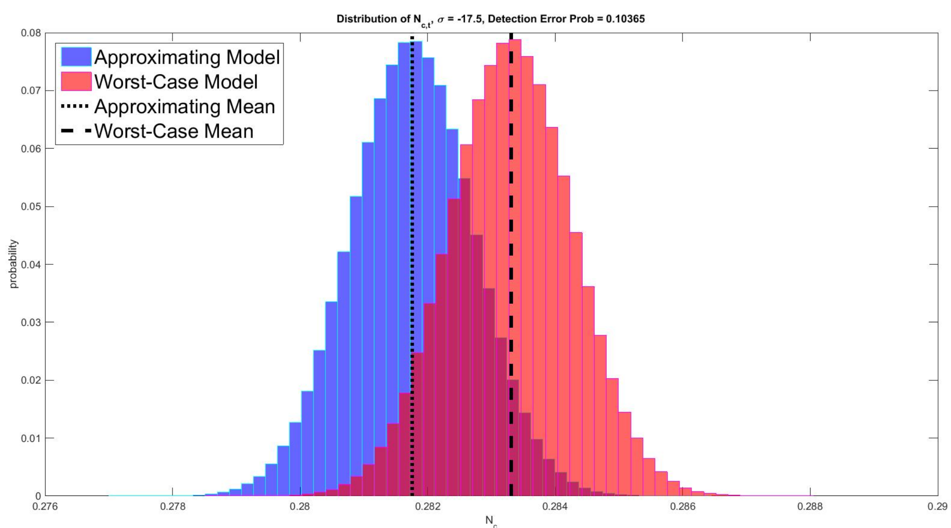
Note: $\sigma = -17.5$. Detection error probability $p(\sigma) = 0.10015$.

Figure 1.26: Simulated Distribution of $\{K_{h,t+1}\}$



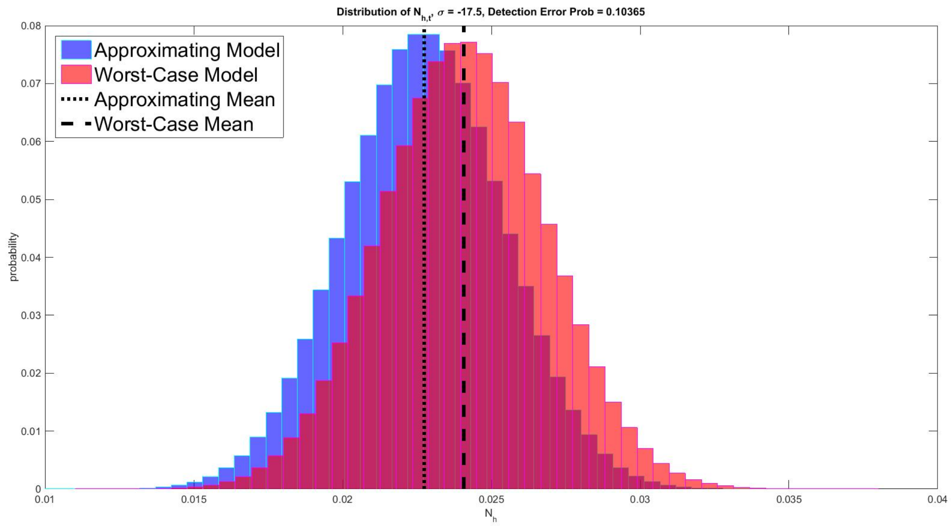
Note: $\sigma = -17.5$. Detection error probability $p(\sigma) = 0.10015$.

Figure 1.27: Simulated Distribution of $\{N_{c,t}\}$



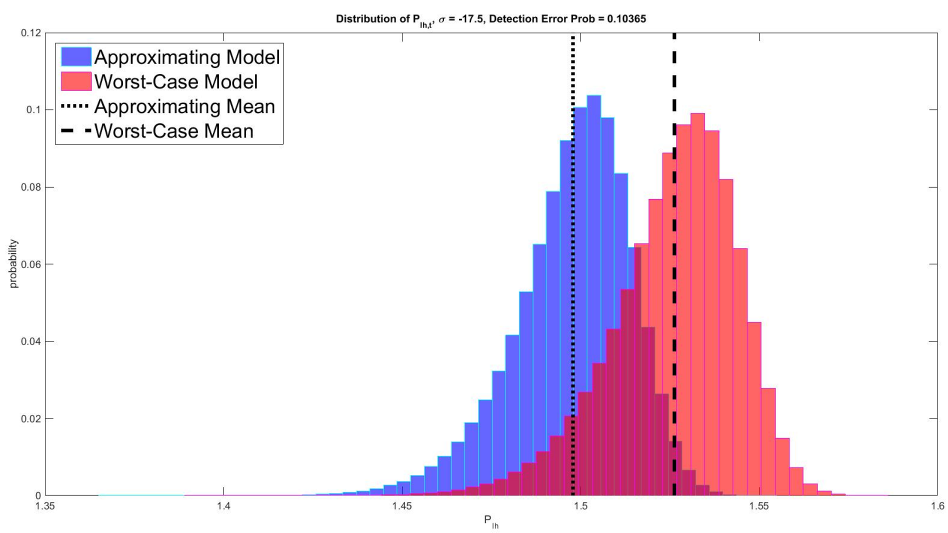
Note: $\sigma = -17.5$. Detection error probability $p(\sigma) = 0.10015$.

Figure 1.28: Simulated Distribution of $\{N_{h,t}\}$



Note: $\sigma = -17.5$. Detection error probability $p(\sigma) = 0.10015$.

Figure 1.29: Simulated Distribution of $\{P_{I_h,t}\}$



Note: $\sigma = -17.5$. Detection error probability $p(\sigma) = 0.10015$.

The bottom line is that the worst-case model essentially distorts the mean rather than the covariance matrix of the TFP shocks. Thus my result confirms that it is sufficient to keep track of the mean distortion when solving the linear quadratic control problem, which also reflects a form of certainty equivalence discussed in Anderson et al. (2003).

1.8 Conclusion

The U.S. housing market experienced the unprecedented fluctuation from 2002 to 2010. The real new home sales price sky-rocketed by about 30% within only five years and plummeted soon after the year of 2006, leaving its net increase virtually nil. Observing that the volatility of new housing construction is much larger than that of non-housing production, I construct sector-specific total factor productivity shocks in a two-sector DSGE model and study to what extent that the TFP shock in housing sector accounts for the rise and fall during the most recent housing market crisis. Then I introduce model uncertainty into the benchmark framework and dedicate to answer the question that whether the fear of model misspecification help account for the boom-bust in the new home sales price. Finally, I examine the first and second moments of the worst-case distribution of the TFP shocks.

I solve the model using the methodology proposed by Hansen and Sargent (1995). The discounted Linear Exponential Quadratic Gaussian control method successfully avoids computational difficulties such as high-dimensional state space, explosive value function, etc., and facilitates the simulation of the worst-case model to implement the likelihood ratio tests. I calibrate the parameters in the benchmark model by the simulated method of moments. The benchmark model, with TFP shocks alone, does a decent job in fitting the first and second moments of real-world data; it accounts for 32 percent of the increase and 40 percent of the decrease in the new home sales price. The implied changes in the resource allocation between the two sectors are also in line with data.

Further, I allow a one-time temporary change in the model uncertainty level during the time span with enhanced dispersions of professional forecasts. The model uncertainty parameter is calibrated by the detection error probability as in Hansen and Sargent (2007). When households hold a concern for model misspecification, they are seeking robust decision rules which perform well over a set of alternative models that are statistically indistinguishable from the not-fully-trusted approximating model. When model uncertainty level rises, the worst-case model distorts the Gaussian mean of TFP shocks negatively in households' minds but keeps the variance-covariance matrix of the TFP shocks almost unchanged. Since the TFP shock in housing production is more volatile, it receives a larger negative mean-shift effect than the TFP shock in the consumption goods sector from the robust state transition law. As a result, the new home sales price is pushed up even further. Thus with the one-time change in model uncertainty, the model is capable of accounting for 40 percent of the surge in new home sales price. The inclusion of the fear for model misspecification improves model's fitting with data.

Appendix

1.A Utility Recursion and Epstein-Zin Preferences

First, I prove that the continuation value of the representative household who fears model uncertainty takes a log-exponential form.[31]

As mentioned in section 2.2.1, the household optimizes under the worst-case scenario and solves for the max-min problem (1.4) - (1.6).

I rewrite the objective function of the max-min problem as

$$V(\boldsymbol{s_t}) = \max_{\boldsymbol{x_t}} \left\{ u(\boldsymbol{x_t}) + \beta \min_{\{p(\epsilon_{t+1}|\epsilon_t)\} \in [0,1]^{\#\epsilon}} \sum_{\epsilon_{t+1}} p(\epsilon_{t+1}|\epsilon_t) V(\boldsymbol{s_{t+1}}) \right\}, \tag{1.A.1}$$

and define the continuation value given the information set at time t as

$$\mathcal{R}_t(\boldsymbol{s_{t+1}}) = \min_{\{p(\epsilon_{t+1}|\epsilon_t)\} \in [0,1]^{\#\epsilon}} \sum_{\epsilon_{t+1}} p(\epsilon_{t+1}|\epsilon_t) V(\boldsymbol{s_{t+1}}), \tag{1.A.2}$$

$$\text{s.t.} \quad \sum_{\epsilon_{t+1}} p(\epsilon_{t+1}|\epsilon_t) = 1, \quad \text{(probability constraint)} \tag{1.A.3}$$

$$\sum_{\epsilon_{t+1}} p(\epsilon_{t+1}|\epsilon_t) \log\left(\frac{p(\epsilon_{t+1}|\epsilon_t)}{\pi(\epsilon_{t+1}|\epsilon_t)}\right) \leq \eta_0, \quad \text{(entropy constraint)} \tag{1.A.4}$$

where $p(\epsilon_{t+1}|\epsilon_t)$ is the *subjective* probability and $\pi(\epsilon_{t+1}|\epsilon_t)$ is the *objective* probability.

[31] The proof follows Backus et al. (2005), Hansen and Sargent (2007) and Young (2012).

Entropy measures the distance between these two probability distributions and is defined as in Backus et al. (2005).

Proposition 1. The continuation value $\mathcal{R}_t(\boldsymbol{s_{t+1}})$ has a log-exponential functional form

$$\mathcal{R}_t(\boldsymbol{s_{t+1}}) = \frac{2}{\sigma} \log \left(\mathbb{E}_{\pi,t} \left[\exp \left(\frac{\sigma V(\boldsymbol{s_{t+1}})}{2} \right) \right] \right). \tag{1.A.5}$$

Proof. I set up the Lagrangian as follows to solve for the minimization problem,

$$\begin{aligned}\mathcal{R}_t(\boldsymbol{s_{t+1}}) = \min_{\{p(\epsilon_{t+1}|\epsilon_t)\}\in[0,1]^{\#\epsilon}} & \Bigg\{ \sum_{\epsilon_{t+1}} p(\epsilon_{t+1}|\epsilon_t) V(\boldsymbol{s_{t+1}}) + \lambda \left(\sum_{\epsilon_{t+1}} p(\epsilon_{t+1}|\epsilon_t) - 1 \right) \\ & - \frac{2}{\sigma} \sum_{\epsilon_{t+1}} p(\epsilon_{t+1}|\epsilon_t) \log \left(\frac{p(\epsilon_{t+1}|\epsilon_t)}{\pi(\epsilon_{t+1}|\epsilon_t)} \right) \Bigg\}. \end{aligned} \tag{1.A.6}$$

The parameters $\lambda > 0$ and $-\frac{2}{\sigma} > 0$ are the Lagrangian multipliers corresponding to the probability constraint and the entropy constraint, which shows the value of loosening respective constraint to the household.[32] The more negative the parameter σ is, the less desirable it is to relax the entropy constraint, which means that the household prefers a tighter model specification.

Since the objective function is convex and the constraint set is convex and compact in $p(\epsilon_{t+1}|\epsilon_t)$, the minimization problem is a convex programming problem. Therefore, the Kuhn-Tucker first-order condition with respect to $p(\epsilon_{t+1}|\epsilon_t)$ is both sufficient and necessary,

$$V(\boldsymbol{s_{t+1}}) + \lambda - \frac{2}{\sigma} \left[\log \left(\frac{p(\epsilon_{t+1}|\epsilon_t)}{\pi(\epsilon_{t+1}|\epsilon_t)} \right) + 1 \right] = 0. \tag{1.A.7}$$

[32] A constant term, $\frac{2\eta_0}{\sigma}$, is dropped from the Lagrangian. This trick does not change the optimal solution, but it enables me to get a closed-form expression for $\mathcal{R}_t(\boldsymbol{s_{t+1}})$.

Multiplying both sides by $p(\epsilon_{t+1}|\epsilon_t)$ and summing over ϵ_{t+1} yields

$$\sum_{\epsilon_{t+1}} p(\epsilon_{t+1}|\epsilon_t)V(\boldsymbol{s_{t+1}}) + \lambda - \frac{2}{\sigma}\sum_{\epsilon_{t+1}} p(\epsilon_{t+1}|\epsilon_t)\log\left(\frac{p(\epsilon_{t+1}|\epsilon_t)}{\pi(\epsilon_{t+1}|\epsilon_t)}\right) - \frac{2}{\sigma} = 0. \tag{1.A.8}$$

Substituting (1.A.3) and (1.A.8) into the Lagranian (1.A.6) returns

$$\mathcal{R}_t(\boldsymbol{s_{t+1}}) = -\lambda + \frac{2}{\sigma}. \tag{1.A.9}$$

Then combine (1.A.7) and (1.A.9),

$$V(\boldsymbol{s_{t+1}}) - \mathcal{R}_t(\boldsymbol{s_{t+1}}) = \frac{2}{\sigma}\log\left(\frac{p(\epsilon_{t+1}|\epsilon_t)}{\pi(\epsilon_{t+1}|\epsilon_t)}\right). \tag{1.A.10}$$

Rearranging the terms in (1.A.10), we have

$$p(\epsilon_{t+1}|\epsilon_t) = \pi(\epsilon_{t+1}|\epsilon_t)\exp\left(\frac{\sigma(V(\boldsymbol{s_{t+1}}) - \mathcal{R}_t(\boldsymbol{s_{t+1}}))}{2}\right). \tag{1.A.11}$$

Summing (1.A.11) over ϵ_{t+1} gives

$$1 = \exp\left(-\frac{\sigma\mathcal{R}_t(\boldsymbol{s_{t+1}})}{2}\right)\left(\sum_{\epsilon_{t+1}}\pi(\epsilon_{t+1}|\epsilon_t)\exp\left(\frac{\sigma V(\boldsymbol{s_{t+1}})}{2}\right)\right). \tag{1.A.12}$$

Note that $\mathcal{R}_t(\boldsymbol{s_{t+1}})$ is the continuation value given the information set at time t and thus is independent of the probability $\pi(\epsilon_{t+1}|\epsilon_t)$.

Finally, take log on both sides of (1.A.12) and rearrange terms again, we have

$$\mathcal{R}_t(\boldsymbol{s_{t+1}}) = \frac{2}{\sigma}\log\left(\sum_{\epsilon_{t+1}}\pi(\epsilon_{t+1}|\epsilon_t)\exp\left(\frac{\sigma V(\boldsymbol{s_{t+1}})}{2}\right)\right) \tag{1.A.13}$$

$$= \frac{2}{\sigma}\log\left(\mathbb{E}_{\pi,t}\left[\exp\left(\frac{\sigma V(\boldsymbol{s_{t+1}})}{2}\right)\right]\right). \tag{1.A.14}$$

□

This is how we derive the utility recursion for the households

$$V(\boldsymbol{s_t}) = \max_{\boldsymbol{x_t}}\left\{u(\boldsymbol{x_t}) + \beta\frac{2}{\sigma}\log\left(\mathbb{E}_{\pi(\epsilon_{t+1}|\epsilon_t)}\left[\exp\left(\frac{\sigma V(\boldsymbol{s_{t+1}})}{2}\right)\right]\right)\right\}. \tag{1.A.15}$$

Proposition 2. The log-exponential utility recursion (1.A.15) is a special case of the Epstein-Zin preferences with the intertemporal elasticity of substitution being one.

Proof. The proof is based on Epstein and Zin (1989) and Tallarini (2000).

The Epstein-Zin preference is depicted by a constant-elasticity-of-substitution (CES) utility function defined over the current consumption, c_t, and the certainty equivalent of future utility, $\mu_t(U_{t+1})$,

$$U_t(c_t, \mu(U_{t+1})) = \left[(1-\beta)c_t^{1-\rho} + \beta(\mu_t(U_{t+1}))^{1-\rho}\right]^{\frac{1}{1-\rho}}, \quad 0 < \rho \neq 1, \tag{1.A.16}$$

where the marginal rate of time preference is $\frac{1}{\beta} - 1$; the intertemporal elasticity of substitution between c_t and $\mu_t(U_{t+1})$ is $\frac{1}{\rho}$.

In the special case where the intertemporal elasticity of substitution equals to one, i.e. $\rho = 1$, the CES function degenerates to the Cobb-Douglas form,

$$U_t = c_t^{1-\beta}\left[\mu_t(U_{t+1})\right]^{\beta}. \tag{1.A.17}$$

Take log on both sides of (1.A.17), we have

$$\log(U_t) = (1-\beta)\log(c_t) + \beta\log(\mu_t(U_{t+1})).$$

Assume that the certainty equivalent takes a power functional form

$$\mu_t(U_{t+1}) = \left[\mathbb{E}_t(U_{t+1}^{1-\gamma})\right]^{\frac{1}{1-\gamma}}, \tag{1.A.18}$$

where γ is the relative risk aversion coefficient. Then

$$\begin{aligned}\frac{\log(U_t)}{1-\beta} &= \log(c_t) + \frac{\beta}{1-\beta}\log\left(\left[\mathbb{E}_t(U_{t+1}^{1-\gamma})\right]^{\frac{1}{1-\gamma}}\right) \\ &= \log(c_t) + \frac{\beta}{(1-\beta)(1-\gamma)}\log\left(\mathbb{E}_t[U_{t+1}^{1-\gamma}]\right).\end{aligned} \tag{1.A.19}$$

Define $V_t \equiv \dfrac{\log(U_t)}{1-\beta}$, hence $U_t = \exp((1-\beta)V_t)$. Equation (1.A.19) becomes

$$V_t = \log(c_t) + \frac{\beta}{(1-\beta)(1-\gamma)}\log\left(\mathbb{E}_t[\exp((1-\beta)(1-\gamma)V_{t+1})]\right).$$

Define $\sigma \equiv 2(1-\beta)(1-\gamma)$. Therefore, we obatin the log-exponential form of the utility recursion:

$$V_t = \log(c_t) + \beta\frac{2}{\sigma}\log\left(\mathbb{E}_t\left[\exp\left(\frac{\sigma V_{t+1}}{2}\right)\right]\right).$$

$\square$

1.B Planner's Problem and Optimality Conditions

The social planner maximizes households' lifetime utility subject to the resources constraints for production.

The utility function of households has the following form,

$$u(C_t, H_t, N_t) = (n_{c,t} + n_{h,t})u(C_t^E, H_t, 1-\bar{N}) + (1 - n_{c,t} - n_{h,t})u(C_t^U, H_t, 1). \quad (1.B.1)$$

This utility function has a nice property that we will have $C^E = C^U$ if (C, H) and N are separable.

Proposition 3. If the utility function can be written as $u(C, H, N) = f(C, H) + g(N)$ where function $f(C, H)$ is differentiable in C and its first partial derivative $f_c(C, H)$ is monotone, then the optimal consumption choice guarantees that $C^E = C^U$.

Proof. Seeing that $u(C, H, N) = f(C, H) + g(N)$, we have

$$u(C_t^E, H_t, 1-\bar{N}) = f(C_t^E, H_t) + g(1-\bar{N}), \quad (1.B.2)$$

$$u(C_t^U, H_t, 1) = f(C_t^U, H_t) + g(1), \quad (1.B.3)$$

and thus

$$u(C_t, H_t, N_t) = (n_{c,t}+n_{h,t})[f(C_t^E, H_t)+g(1-\bar{N})]+(1-n_{c,t}-n_{h,t})[f(C_t^U, H_t)+g(1)]. \quad (1.B.4)$$

The optimal consumption choices, C_t^E and C_t^U, come from the solution to the following utility maximization problem,

$$V(\boldsymbol{s_t}) = \max_{C_t^E, C_t^U, \boldsymbol{x_t}} \Big\{ (n_{c,t} + n_{h,t})[f(C_t^E, H_t) + g(1-\bar{N})] + (1 - n_{c,t} - n_{h,t})[f(C_t^U, H_t) + g(1)] \\ + \beta \mathbb{E}_t \left[V(\boldsymbol{s_{t+1}}) \right] + \lambda \Big(\Pi_t - (n_{c,t} + n_{h,t}) C_t^E - (1 - n_{c,t} - n_{h,t}) C_t^U \Big) \Big\}, \tag{1.B.5}$$

where $\boldsymbol{s_t}$ represents all of the state variables; $\boldsymbol{x_t}$ contains control variables other than C_t^E and C_t^U; Π_t is the total disposable income; $\lambda > 0$ is the Lagrangian multiplier associated with the budget constraint. The first order condisions with respect to C_t^E and C_t^U give that

$$(n_{c,t} + n_{h,t}) f_c(C_t^E, H) = \lambda (n_{c,t} + n_{h,t}), \tag{1.B.6}$$

$$(1 - n_{c,t} - n_{h,t}) f_c(C_t^U, H) = \lambda (1 - n_{c,t} - n_{h,t}). \tag{1.B.7}$$

Compare (1.B.6) and (1.B.7), we have

$$f_c(C_t^E, H) = f_c(C_t^U, H). \tag{1.B.8}$$

Since the partial derivative $f_c(C_t, H_t)$ is monotone, hence we have $C_t^E = C_t^U, \forall t$.

□

Given the specific functional form of the utility function,

$$u(C_t^i, H_t, N_t^i) = \mu_c \log(C_t^i) + (1 - \mu_c) \log(H_t) + \phi \log(1 - N_t^i), \quad i = E, U, \tag{1.B.9}$$

and use the property that $C_t^E = C_t^U = C_t$, we can write down the social planner's utility function as

$$
\begin{aligned}
u(C_t, H_t, N_t) &= (n_{c,t} + n_{h,t})u(C_t^E, H_t, 1 - \bar{N}) + (1 - n_{c,t} - n_{h,t})u(C_t^U, H_t, 1) \\
&= (n_{c,t} + n_{h,t}) \Big[\mu_c \log(C_t^E) + (1 - \mu_c)\log(H_t) + \phi \log(1 - \bar{N})\Big] \\
&+ (1 - n_{c,t} - n_{h,t}) \Big[\mu_c \log(C_t^U) + (1 - \mu_c)\log(H_t) + \phi \log(1)\Big] \\
&= \mu_c \log(C_t) + (1 - \mu_c)\log(H_t) + (n_{c,t} + n_{h,t})\phi \log(1 - \bar{N}). \qquad (1.B.10)
\end{aligned}
$$

1.B.1 Planner's Problem

The Bellman equation of the planner's optimization problem can be written as follows:

$$
\begin{aligned}
&V(z_t, \xi_t, K_{c,t}, K_{h,t}, H_t, n_{c,t-1}, n_{h,t-1}) \\
=& \max_{\substack{K_{c,t+1}, K_{h,t+1}, H_{t+1}, n_{c,t}, n_{h,t}, \\ C_t, I_{kc,t}, I_{kh,t}, I_{h,t}}} \Bigg\{ \mu_c \log(C_t) + (1-\mu_c)\log(H_t) + (n_{c,t} + n_{h,t})\phi \log(1-\bar{N}) \\
&+ \beta \left(\frac{2}{\sigma}\right) \log\left(\mathbb{E}_t \left[\exp\left(\frac{\sigma}{2} V(z_{t+1}, \xi_{t+1}, K_{c,t+1}, K_{h,t+1}, H_{t+1}, n_{c,t}, n_{h,t})\right) \Big| z_t, \xi_t\right]\right) \Bigg\},
\end{aligned}
\tag{1.B.11}
$$

$$
\text{s.t.} \quad C_t + I_{kc,t} + I_{kh,t} \leq Y_{c,t} - \frac{\omega}{2}\left(\frac{H_{t+1} - H_t}{H_t}\right)^2 H_t - \frac{\tau_c}{2}\left(n_{c,t} - n_{c,t-1}\right)^2 - \frac{\tau_h}{2}\left(n_{h,t} - n_{h,t-1}\right)^2, \tag{1.B.12}
$$

$$
I_{h,t} \leq Y_{h,t}, \tag{1.B.13}
$$

$$
H_{t+1} \leq (1-\delta_h)H_t + I_{h,t}, \tag{1.B.14}
$$

$$
K_{c,t+1} = (1-\delta_{kc})K_{c,t} + I_{kc,t} - \frac{\kappa_c}{2}\left(\frac{K_{c,t+1} - K_{c,t}}{K_{c,t}}\right)^2 K_{c,t}, \tag{1.B.15}
$$

$$
K_{h,t+1} = (1-\delta_{kh})K_{h,t} + I_{kh,t} - \frac{\kappa_h}{2}\left(\frac{K_{h,t+1} - K_{h,t}}{K_{h,t}}\right)^2 K_{h,t}, \tag{1.B.16}
$$

$$
Y_{c,t} = \exp(z_t) K_{c,t}^{\alpha} (n_{c,t}\bar{N})^{1-\alpha}, \tag{1.B.17}
$$

$$
Y_{h,t} = \exp(\xi_t) K_{h,t}^{\theta} (n_{h,t}\bar{N})^{\nu} L_t^{1-\theta-\nu}, \quad L_t = 1, \forall t. \tag{1.B.18}
$$

Simplify the constraints and construct the Lagrangian:

$$
\begin{aligned}
\mathcal{L} = {} & \mu_c \log(C_t) + (1-\mu_c)\log(H_t) + (n_{c,t}+n_{h,t})\phi\log(1-\bar{N}) \\
& + \beta\left(\frac{2}{\sigma}\right)\log\left(\mathbb{E}_t\left[\exp\left(\frac{\sigma}{2}V(K_{c,t+1},K_{h,t+1},n_{c,t},n_{h,t},H_{t+1},z_{t+1},\xi_{t+1})\right)\Big|z_t,\xi_t\right]\right) \\
& + \lambda_{1,t}\Bigg[\exp(z_t)K_{c,t}^{\alpha}(n_{c,t}\bar{N})^{1-\alpha} - C_t - K_{c,t+1} + (1-\delta_{kc})K_{c,t} - K_{h,t+1} + (1-\delta_{kh})K_{h,t} \\
& - \frac{\kappa_c}{2}\left(\frac{K_{c,t+1}-K_{c,t}}{K_{c,t}}\right)^2 K_{c,t} - \frac{\kappa_h}{2}\left(\frac{K_{h,t+1}-K_{h,t}}{K_{h,t}}\right)^2 K_{h,t} - \frac{\omega}{2}\left(\frac{H_{t+1}-H_t}{H_t}\right)^2 H_t \\
& - \frac{\tau_c}{2}(n_{c,t}-n_{c,t-1})^2 - \frac{\tau_h}{2}(n_{h,t}-n_{h,t-1})^2\Bigg] \\
& + \lambda_{2,t}\Big[\exp(\xi_t)K_{h,t}^{\theta}(n_{h,t}\bar{N})^{\nu} - H_{t+1} + (1-\delta_h)H_t\Big]
\end{aligned}
\tag{1.B.19}
$$

To make notations clearer, denote $V(t) \equiv V(z_t,\xi_t,K_{c,t},K_{h,t},H_t,n_{c,t-1},n_{h,t-1})$ and $V(t+1) \equiv V(z_{t+1},\xi_{t+1},K_{c,t+1},K_{h,t+1},H_{t+1},n_{c,t},n_{h,t})$. Take the first order conditions with respect to the choice variables:

$$K_{c,t+1}: \quad \beta \mathbb{E}_t \left[\frac{\exp\left(\frac{\sigma}{2} V(t+1)\right)}{\mathbb{E}_t\left[\exp\left(\frac{\sigma}{2} V(t+1)\right)\right]} V'_{Kc}(t+1) \right] = \lambda_{1,t} \left[1 + \kappa_c \left(\frac{K_{c,t+1} - K_{c,t}}{K_{c,t}} \right) \right], \tag{1.B.20}$$

$$K_{h,t+1}: \quad \beta \mathbb{E}_t \left[\frac{\exp\left(\frac{\sigma}{2} V(t+1)\right)}{\mathbb{E}_t\left[\exp\left(\frac{\sigma}{2} V(t+1)\right)\right]} V'_{Kh}(t+1) \right] = \lambda_{1,t} \left[1 + \kappa_h \left(\frac{K_{h,t+1} - K_{h,t}}{K_{h,t}} \right) \right], \tag{1.B.21}$$

$$H_{t+1}: \quad \beta \mathbb{E}_t \left[\frac{\exp\left(\frac{\sigma}{2} V(t+1)\right)}{\mathbb{E}_t\left[\exp\left(\frac{\sigma}{2} V(t+1)\right)\right]} V'_{H}(t+1) \right] = \lambda_{1,t}\, \omega \left(\frac{H_{t+1} - H_t}{H_t} \right) + \lambda_{2,t}, \tag{1.B.22}$$

$$\begin{aligned} n_{c,t}: \quad & \phi \log(1 - \bar{N}) + \beta \mathbb{E}_t \left[\frac{\exp\left(\frac{\sigma}{2} V(t+1)\right)}{\mathbb{E}_t\left[\exp\left(\frac{\sigma}{2} V(t+1)\right)\right]} V'_{nc}(t+1) \right] \\ & + \lambda_{1,t} \left[\exp(z_t)(1-\alpha) K_{c,t}^{\alpha} (n_{c,t}\bar{N})^{-\alpha} \bar{N} - \tau_c (n_{c,t} - n_{c,t-1}) \right] = 0, \end{aligned} \tag{1.B.23}$$

$$\begin{aligned} n_{h,t}: \quad & \phi \log(1 - \bar{N}) + \beta \mathbb{E}_t \left[\frac{\exp\left(\frac{\sigma}{2} V(t+1)\right)}{\mathbb{E}_t\left[\exp\left(\frac{\sigma}{2} V(t+1)\right)\right]} V'_{nh}(t+1) \right] \\ & - \lambda_{1,t}\, \tau_h (n_{h,t} - n_{h,t-1}) + \lambda_{2,t} \exp(\xi_t) \nu K_{h,t}^{\theta} (n_{h,t}\bar{N})^{\nu-1} \bar{N} = 0, \end{aligned} \tag{1.B.24}$$

$$C_t: \quad \frac{\mu_c}{C_t} = \lambda_{1,t}, \tag{1.B.25}$$

$$\begin{aligned} \lambda_{1,t}: \quad & C_t + K_{c,t+1} - (1-\delta_{kc}) K_{c,t} + K_{h,t+1} - (1-\delta_{kh}) K_{h,t} \\ = & \exp(z_t) K_{c,t}^{\alpha} (n_{c,t}\bar{N})^{1-\alpha} - \frac{\kappa_c}{2} \left(\frac{K_{c,t+1} - K_{c,t}}{K_{c,t}} \right)^2 K_{c,t} - \frac{\kappa_h}{2} \left(\frac{K_{h,t+1} - K_{h,t}}{K_{h,t}} \right)^2 K_{h,t} \\ - & \frac{\omega}{2} \left(\frac{H_{t+1} - H_t}{H_t} \right)^2 H_t - \frac{\tau_c}{2} (n_{c,t} - n_{c,t-1})^2 - \frac{\tau_h}{2} (n_{h,t} - n_{h,t-1})^2, \end{aligned} \tag{1.B.26}$$

$$\lambda_{2,t}: \quad H_{t+1} - (1-\delta_h) H_t = \exp(\xi_t) K_{h,t}^{\theta} (n_{h,t}\bar{N})^{\nu}. \tag{1.B.27}$$

Applying the Envelope theorem to the value function yields

$$V'_{Kc}(t) = \lambda_{1,t} \left[\exp(z_t)\alpha K_{c,t}^{\alpha-1}(n_{c,t}\bar{N})^{1-\alpha} + 1 - \delta_{kc} + \frac{\kappa_c}{2}\left(\left(\frac{K_{c,t+1}}{K_{c,t}} \right)^2 - 1 \right) \right], \tag{1.B.28}$$

$$V'_{Kh}(t) = \lambda_{1,t} \left[1 - \delta_{kh} + \frac{\kappa_h}{2}\left(\left(\frac{K_{h,t+1}}{K_{h,t}} \right)^2 - 1 \right) \right] + \lambda_{2,t} \exp(\xi_t)\theta K_{h,t}^{\theta-1}(n_{h,t}\bar{N})^{\nu}, \tag{1.B.29}$$

$$V'_{H}(t) = \frac{1-\mu_c}{H_t} + \lambda_{1,t}\ \frac{\omega}{2}\left(\left(\frac{H_{t+1}}{H_t} \right)^2 - 1 \right) + \lambda_{2,t}(1-\delta_h), \tag{1.B.30}$$

$$V'_{nc}(t) = \lambda_{1,t}\ \tau_c(n_{c,t} - n_{c,t-1}), \tag{1.B.31}$$

$$V'_{nh}(t) = \lambda_{1,t}\ \tau_h(n_{h,t} - n_{h,t-1}). \tag{1.B.32}$$

1.B.2 Optimality Conditions

Combining the first order conditions (1.B.20) - (1.B.27) and the Envelope theorem conditions (1.B.28) - (1.B.32), we obtain the following optimality conditions from which the robust decision rules can be solved.

$$(1)\quad \lambda_{1,t}\left[1+\kappa_c\left(\frac{K_{c,t+1}-K_{c,t}}{K_{c,t}}\right)\right]=\beta\mathbb{E}_t\left[\frac{\exp\left(\frac{\sigma}{2}V(t+1)\right)}{\mathbb{E}_t\Big[\exp\left(\frac{\sigma}{2}V(t+1)\right)\Big]}\lambda_{1,t+1}\bigg\{(1-\delta_{kc}) + \frac{\kappa_c}{2}\left(\left(\frac{K_{c,t+2}}{K_{c,t+1}}\right)^2-1\right)+\exp(z_{t+1})\alpha K_{c,t+1}^{\alpha-1}(n_{c,t+1}\bar{N})^{1-\alpha}\bigg\}\right], \tag{1.B.33}$$

$$(2)\quad \lambda_{1,t}\left[1+\kappa_h\left(\frac{K_{h,t+1}-K_{h,t}}{K_{h,t}}\right)\right]=\beta\mathbb{E}_t\left[\frac{\exp\left(\frac{\sigma}{2}V(t+1)\right)}{\mathbb{E}_t\Big[\exp\left(\frac{\sigma}{2}V(t+1)\right)\Big]}\bigg\{\lambda_{1,t+1}\Big[(1-\delta_{kh}) + \frac{\kappa_h}{2}\left(\left(\frac{K_{h,t+2}}{K_{h,t+1}}\right)^2-1\right)\Big]+\lambda_{2,t+1}\exp(\xi_{t+1})\theta K_{h,t+1}^{\theta-1}(n_{h,t+1}\bar{N})^{\nu}\bigg\}\right], \tag{1.B.34}$$

$$(3)\quad \lambda_{1,t}\,\omega\left(\frac{H_{t+1}-H_t}{H_t}\right)+\lambda_{2,t} = \beta\mathbb{E}_t\left[\frac{\exp\left(\frac{\sigma}{2}V(t+1)\right)}{\mathbb{E}_t\Big[\exp\left(\frac{\sigma}{2}V(t+1)\right)\Big]}\left\{\frac{1-\mu_c}{H_{t+1}}+\lambda_{1,t+1}\,\frac{\omega}{2}\left(\left(\frac{H_{t+1}}{H_t}\right)^2-1\right)+\lambda_{2,t+1}(1-\delta_h)\right\}\right], \tag{1.B.35}$$

$$(4)\quad -\phi\log(1-\bar{N})+\lambda_{1,t}\Big[\tau_c(n_{c,t}-n_{c,t-1})-\exp(z_t)(1-\alpha)K_{c,t}^{\alpha}(n_{c,t}\bar{N})^{-\alpha}\bar{N}\Big] = \beta\mathbb{E}_t\left[\frac{\exp\left(\frac{\sigma}{2}V(t+1)\right)}{\mathbb{E}_t\Big[\exp\left(\frac{\sigma}{2}V(t+1)\right)\Big]}\cdot\lambda_{1,t+1}\,\tau_c(n_{c,t+1}-n_{c,t})\right], \tag{1.B.36}$$

$$(5)\quad -\phi\log(1-\bar{N})+\lambda_{1,t}\,\tau_h(n_{h,t}-n_{h,t-1})-\lambda_{2,t}\exp(\xi_t)\nu K_{h,t}^{\theta}(n_{h,t}\bar{N})^{\nu-1}\bar{N} = \beta\mathbb{E}_t\left[\frac{\exp\left(\frac{\sigma}{2}V(t+1)\right)}{\mathbb{E}_t\Big[\exp\left(\frac{\sigma}{2}V(t+1)\right)\Big]}\cdot\lambda_{1,t+1}\,\tau_h(n_{h,t+1}-n_{h,t})\right], \tag{1.B.37}$$

$$(6)\quad \frac{\mu_c}{C_t}=\lambda_{1,t}, \tag{1.B.38}$$

$$(7)\quad C_t+K_{c,t+1}-(1-\delta_{kc})K_{c,t}+K_{h,t+1}-(1-\delta_{kh})K_{h,t} = \exp(z_t)K_{c,t}^{\alpha}(n_{c,t}\bar{N})^{1-\alpha}-\frac{\kappa_c}{2}\left(\frac{K_{c,t+1}-K_{c,t}}{K_{c,t}}\right)^2K_{c,t}-\frac{\kappa_h}{2}\left(\frac{K_{h,t+1}-K_{h,t}}{K_{h,t}}\right)^2K_{h,t} - \frac{\omega}{2}\left(\frac{H_{t+1}-H_t}{H_t}\right)^2H_t-\frac{\tau_c}{2}(n_{c,t}-n_{c,t-1})^2-\frac{\tau_h}{2}(n_{h,t}-n_{h,t-1})^2, \tag{1.B.39}$$

$$(8)\quad H_{t+1}-(1-\delta_h)H_t=\exp(\xi_t)K_{h,t}^{\theta}(n_{h,t}\bar{N})^{\nu}. \tag{1.B.40}$$

These optimality conditions characterize the equilibrium in the economy.

1.B.3 Steady State

In steady state, all variables do not evolve over time. In particular,

$$\frac{\exp\left(\frac{\sigma}{2}V(t+1)\right)}{\mathbb{E}_t\left[\exp\left(\frac{\sigma}{2}V(t+1)\right)\right]} = 1. \tag{1.B.41}$$

Therefore, we can compute the deterministic steady state values of $\{K_c, K_h, H, n_c, n_h, C\}$ in the economy from the equation system:

$$1 = \beta\left[\alpha K_c^{\alpha-1}(n_c\bar{N})^{1-\alpha} + 1 - \delta_{kc}\right], \tag{1.B.42}$$

$$(1-\alpha)K_c^{\alpha}(n_c\bar{N})^{-\alpha} = -\frac{\phi\log(1-\bar{N})C}{\mu_c\bar{N}}, \tag{1.B.43}$$

$$\left(\frac{1}{\beta} - 1 + \delta_{kh}\right)\frac{\nu K_h}{\theta n_h\bar{N}} = -\frac{\phi\log(1-\bar{N})C}{\mu_c\bar{N}}, \tag{1.B.44}$$

$$\frac{(1-\alpha)K_c^{\alpha}(n_c\bar{N})^{-\alpha}}{\nu K_h^{\theta}(n_h\bar{N})^{\nu-1}} = \frac{1}{\frac{1}{\beta} - 1 + \delta_h}\frac{(1-\mu_c)C}{\mu_c H}, \tag{1.B.45}$$

$$C + \delta_{kc}K_c + \delta_{kh}K_h = K_c^{\alpha}(n_c\bar{N})^{1-\alpha}, \tag{1.B.46}$$

$$\delta_h H = K_h^{\theta}(n_h\bar{N})^{\nu}. \tag{1.B.47}$$

1.B.4 Price Functions

Although prices do not exist in a centralized economy, we can back up the prices in the competitive equilibrium from the planner's policy function.

The optimal resource allocation equalizes the marginal product of labor (or capital) across the two sectors. Based on this relationship, the new home sales price must satisfy

$$P_{Ih,t} = \frac{\exp(z_t)(1-\alpha)K_{c,t}^{\alpha}(n_{c,t}\bar{N})^{-\alpha}}{\exp(\xi_t)\nu K_{h,t}^{\theta}(n_{h,t}\bar{N})^{\nu-1}}. \tag{1.B.48}$$

The home rents must equal to the ratio of marginal utility of housing services to consumption goods, thus

$$R_{h,t} = \frac{(1-\mu_c)C_t}{\mu_c H_t}. \tag{1.B.49}$$

Finally, the wage rate equals to the marginal product of labor in either sector.

$$w_t = \exp(z_t)(1-\alpha)K_{c,t}^{\alpha}(n_{c,t}\bar{N})^{-\alpha}\bar{N}. \tag{1.B.50}$$

1.C Linear Exponential Quadratic Gaussian Control Problem with Model Uncertainty

A Linear Quadratic Gaussian (LQG) control problem has a format in which the utility function is quadratic in the state vector $\boldsymbol{x}_t$ and the control vector $\boldsymbol{a}_t$,

$$\begin{aligned} V(\boldsymbol{x}_t) &= \max_{\boldsymbol{a}_t, \boldsymbol{x}_{t+1}} \left\{ u(\boldsymbol{x}_t, \boldsymbol{a}_t) + \beta \mathbb{E}_t \left[V(\boldsymbol{x}_{t+1}) \right] \right\}, \\ u(\boldsymbol{x}_t, \boldsymbol{a}_t) &= \boldsymbol{x}_t^T \boldsymbol{Q} \boldsymbol{x}_t + \boldsymbol{a}_t^T \boldsymbol{R} \boldsymbol{a}_t + 2\boldsymbol{a}_t^T \boldsymbol{W} \boldsymbol{x}_t; \end{aligned} \tag{1.C.1}$$

the state transition function is linear,

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{a}_t + \boldsymbol{C}\boldsymbol{\epsilon}_{t+1}; \tag{1.C.2}$$

and the uncertainty comes from white Gaussian noises,

$$\boldsymbol{\epsilon}_{t+1} \sim \text{i.i.d. } \mathcal{N}(\mathbf{0}, \boldsymbol{I}). \tag{1.C.3}$$

The matrix $\boldsymbol{Q}$ is negative semidefinite, and the matrix $\boldsymbol{R}$ is negative definite. The solution to this problem consists of a quadratic value function and a time-invariant optimal linear control law. This type of problems have a property called *Certainty Equivalence* – the optimal control law is independent of the uncertainty and we will obtain exactly the same control law as in a deterministic environment.

Hansen and Sargent (1995) introduce the discounted Linear Exponential Quadratic Gaussian (LEQG) method to solve for a generalized LQG control problem embedded with model uncertainty. The most intriguing result is that the certainty equivalence no longer holds when the agent fears about model misspecification. In this section, I elaborate this methodology which is the footstone of the solution algorithm of my model.

Consider the following problem:

$$V(\boldsymbol{x}_t) = \max_{\boldsymbol{a}_t, \boldsymbol{x}_{t+1}} \left\{ \boldsymbol{x}_t^T \boldsymbol{Q} \boldsymbol{x}_t + \boldsymbol{a}_t^T \boldsymbol{R} \boldsymbol{a}_t + 2\boldsymbol{a}_t^T \boldsymbol{W} \boldsymbol{x}_t + \beta\left(\frac{2}{\sigma}\right) \log\left(\mathbb{E}_t\left[\exp\left(\frac{\sigma}{2} V(\boldsymbol{x}_{t+1})\right) | \boldsymbol{x}_t\right]\right)\right\}, \tag{1.C.4}$$

$$\text{s.t. } \boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{a}_t + \boldsymbol{C}\boldsymbol{\epsilon}_{t+1}, \quad \boldsymbol{\epsilon}_{t+1} \sim \text{i.i.d. } \mathcal{N}(\mathbf{0}, \boldsymbol{I}). \tag{1.C.5}$$

It can be solved via guess and verify. Guess that value function is quadratic and concave, i.e. $V(\boldsymbol{x}_t) = \boldsymbol{x}_t^T \boldsymbol{P} \boldsymbol{x}_t + d$ and $\boldsymbol{P}$ is negative definite, then

$$\begin{aligned} V(\boldsymbol{x}_{t+1}) =& \boldsymbol{x}_{t+1}^T \boldsymbol{P} \boldsymbol{x}_{t+1} + d \\ =& (\boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{a}_t + \boldsymbol{C}\boldsymbol{\epsilon}_{t+1})^T \boldsymbol{P} (\boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{a}_t + \boldsymbol{C}\boldsymbol{\epsilon}_{t+1}) + d. \end{aligned} \tag{1.C.6}$$

Substituting (1.C.6) into the Bellman equation (1.C.4) and using the moment generating function of the Gaussian random variable, $\boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I})$, we have

$$\boldsymbol{x}_t^T \boldsymbol{P} \boldsymbol{x}_t + d = \max_{\boldsymbol{a}_t} \Big\{ \boldsymbol{x}_t^T \boldsymbol{Q} \boldsymbol{x}_t + \boldsymbol{a}_t^T \boldsymbol{R} \boldsymbol{a}_t + 2\boldsymbol{a}_t^T \boldsymbol{W} \boldsymbol{x}_t + \beta(\boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{a}_t)^T D(\boldsymbol{P})(\boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{a}_t) + U(\boldsymbol{P}, d) \Big\}, \tag{1.C.7}$$

where $D(\boldsymbol{P}) = \boldsymbol{P} + \sigma \boldsymbol{P}\boldsymbol{C}(\boldsymbol{I} - \sigma \boldsymbol{C}^T \boldsymbol{P}\boldsymbol{C})^{-1} \boldsymbol{C}^T \boldsymbol{P}$, matrix $(\boldsymbol{I} - \sigma \boldsymbol{C}^T \boldsymbol{P}\boldsymbol{C})$ is positive definite, and $U(\boldsymbol{P}, d)$ is some constant term.[33]

Maximizing with respect to the control vector $\boldsymbol{a}_t$ yields the decision rule

$$\boldsymbol{a}_t = -\left(\boldsymbol{R} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{B}\right)^{-1} \left(\boldsymbol{W} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{A}\right) \boldsymbol{x}_t. \tag{1.C.8}$$

Substituting (1.C.8) back into (1.C.7) and matching coefficients yields the *matrix Riccati equation*:

$$\begin{aligned} &\boldsymbol{x}_t^T \boldsymbol{P} \boldsymbol{x}_t + d \\ &= \boldsymbol{x}_t \left[\boldsymbol{Q} + \beta \boldsymbol{A}^T D(\boldsymbol{P})\boldsymbol{A} - \left(\boldsymbol{W} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{A}\right)^T \left(\boldsymbol{R} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{B}\right)^{-1} \left(\boldsymbol{W} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{A}\right) \right] \boldsymbol{x}_t \\ &+ U(\boldsymbol{P}, d). \end{aligned} \tag{1.C.9}$$

Therefore, the solution matrix $\boldsymbol{P}$ can be obtained by iterating on the recursive equation

$$\boldsymbol{P} = \boldsymbol{Q} + \beta \boldsymbol{A}^T D(\boldsymbol{P})\boldsymbol{A} - \left(\boldsymbol{W} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{A}\right)^T \left(\boldsymbol{R} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{B}\right)^{-1} \left(\boldsymbol{W} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{A}\right), \tag{1.C.10}$$

[33] I use the results from Jacobson (1973). The detailed algebraic derivation is ommited here.

with that

$$D(\boldsymbol{P}) = \boldsymbol{P} + \sigma \boldsymbol{P}\boldsymbol{C}(\boldsymbol{I} - \sigma \boldsymbol{C}^T \boldsymbol{P}\boldsymbol{C})^{-1}\boldsymbol{C}^T \boldsymbol{P}. \tag{1.C.11}$$

In addition, $d = U(\boldsymbol{P}, d)$.

After solving for $\boldsymbol{P}$, the decision rule for $\boldsymbol{a}_t$ is determined by (1.C.8). Denote

$$\begin{aligned} \boldsymbol{a}_t =& \boldsymbol{F}\boldsymbol{x}_t, \\ \boldsymbol{F} :=& -\left(\boldsymbol{R} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{B}\right)^{-1}\left(\boldsymbol{W} + \beta \boldsymbol{B}^T D(\boldsymbol{P})\boldsymbol{A}\right). \end{aligned} \tag{1.C.12}$$

The system also needs to be saddle-stable, which requires that the matrix $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}$ have eigenvalues within the unit circle.

From (1.C.10) - (1.C.12) we can see that $D(\boldsymbol{P})$ depends on the volatility matrix $\boldsymbol{C}$ and the model uncertainty parameter σ. Hence certainty equivalence property fails in this model, and model uncertainty plays a role in the control law.

Notice that the solution of the LEQG problem is a function of the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{W}$. In order to compute the numerical solution, we must compute the matrices $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{W}$ first. That is, we need to derive the quadratic form for the utility function

$$u(\boldsymbol{x}_t, \boldsymbol{a}_t) = \boldsymbol{x}_t^T \boldsymbol{Q}\boldsymbol{x}_t + \boldsymbol{a}_t^T \boldsymbol{R}\boldsymbol{a}_t + 2\boldsymbol{a}_t^T \boldsymbol{W}\boldsymbol{x}_t = \begin{pmatrix} \boldsymbol{x}_t^T & \boldsymbol{a}_t^T \end{pmatrix} \underbrace{\begin{pmatrix} \boldsymbol{Q} & \boldsymbol{W}^T \\ \boldsymbol{W} & \boldsymbol{R} \end{pmatrix}}_{\boldsymbol{\Theta}} \begin{pmatrix} \boldsymbol{x}_t \\ \boldsymbol{a}_t \end{pmatrix}. \tag{1.C.13}$$

In a problem where the utility function is not quadratic, we can use the second order multivariate Taylor expansion of $u(\cdot)$ around the stochastic steady state as the approximation. I omit the subscript t henceforth.

Suppose that $\boldsymbol{y}$ is a vector containing all of the states and all of the controls as well as the constant 1:

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{a} \end{pmatrix} = \begin{pmatrix} 1 \\ \underset{n\times 1}{\hat{\boldsymbol{x}}} \\ \underset{k\times 1}{\boldsymbol{a}} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ \hat{\boldsymbol{x}} \end{pmatrix}. \tag{1.C.14}$$

Partitioning the matrix $\boldsymbol{\Theta}$ according to the dimensions of $\boldsymbol{x}$ and $\boldsymbol{a}$, we can write

$$\begin{aligned} u(\boldsymbol{x},\boldsymbol{a}) =& \begin{pmatrix} 1 & \underset{1\times n}{\hat{\boldsymbol{x}}^T} & \underset{1\times k}{\boldsymbol{a}^T} \end{pmatrix} \begin{pmatrix} \underset{1\times 1}{\Theta_{11}} & \underset{1\times n}{\Theta_{12}} & \underset{1\times k}{\Theta_{13}} \\ \underset{n\times 1}{\Theta_{21}} & \underset{n\times n}{\Theta_{22}} & \underset{n\times k}{\Theta_{23}} \\ \underset{k\times 1}{\Theta_{31}} & \underset{k\times n}{\Theta_{32}} & \underset{k\times k}{\Theta_{33}} \end{pmatrix} \begin{pmatrix} 1 \\ \underset{n\times 1}{\hat{\boldsymbol{x}}^T} \\ \underset{k\times 1}{\boldsymbol{a}} \end{pmatrix} \\ =& \Theta_{11} + \hat{\boldsymbol{x}}^T\Theta_{21} + \boldsymbol{a}^T\Theta_{31} + \Theta_{12}\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}^T\Theta_{22}\hat{\boldsymbol{x}} + \boldsymbol{a}^T\Theta_{32}\hat{\boldsymbol{x}} + \Theta_{13}\boldsymbol{a} + \hat{\boldsymbol{x}}^T\Theta_{23}\boldsymbol{a} + \boldsymbol{a}^T\Theta_{33}\boldsymbol{a}. \end{aligned} \tag{1.C.15}$$

Denote $\bar{\boldsymbol{y}} = (1, \bar{\hat{\boldsymbol{x}}}, \bar{\boldsymbol{a}})^T$ as the point around which the utility function is expanded. Then the second order Taylor approximation gives

$$u(\boldsymbol{y}) \approx u(\bar{\boldsymbol{y}}) + D_u(\bar{\boldsymbol{y}})^T(\boldsymbol{y} - \bar{\boldsymbol{y}}) + \frac{1}{2}(\boldsymbol{y} - \bar{\boldsymbol{y}})^T D_u^2(\bar{\boldsymbol{y}})(\boldsymbol{y} - \bar{\boldsymbol{y}}), \tag{1.C.16}$$

where $D_u(\bar{\boldsymbol{y}})$ and $D_u^2(\bar{\boldsymbol{y}})$ are the gradient vector and the Hessian matrix of function $u(\cdot)$ respectively. Denote[34]

$$D_u(\bar{\boldsymbol{y}}) := \boldsymbol{f} = \begin{pmatrix} 0 \\ \underset{n\times 1}{\boldsymbol{f}_n} \\ \underset{k\times 1}{\boldsymbol{f}_k} \end{pmatrix}, \quad D_u^2(\bar{\boldsymbol{y}}) := \boldsymbol{V}, \quad \frac{1}{2}\boldsymbol{V} := \begin{pmatrix} \underset{1\times 1}{0} & \underset{1\times n}{\boldsymbol{0}} & \underset{1\times k}{\boldsymbol{0}} \\ \underset{n\times 1}{\boldsymbol{0}} & \underset{n\times n}{\boldsymbol{S}} & \underset{n\times k}{\boldsymbol{T}^T} \\ \underset{k\times 1}{\boldsymbol{0}} & \underset{k\times n}{\boldsymbol{T}} & \underset{k\times k}{\boldsymbol{L}} \end{pmatrix}. \tag{1.C.17}$$

[34]Note that the first and second partial derivative of $u(\cdot)$ with respect to the constant state 1 is always 0.

Since $D_u^2(\cdot)$ is symmetric, so $\boldsymbol{S}$ and $\boldsymbol{L}$ are also symmetric. The Taylor expansion becomes

$$\begin{aligned} u(\boldsymbol{y}) =& u(\bar{\boldsymbol{y}}) + \boldsymbol{f}^T(\boldsymbol{y}-\bar{\boldsymbol{y}}) + (\boldsymbol{y}-\bar{\boldsymbol{y}})^T\left(\frac{1}{2}\boldsymbol{V}\right)(\boldsymbol{y}-\bar{\boldsymbol{y}}) \\ =& u(\bar{\boldsymbol{y}}) - \boldsymbol{f}^T\bar{\boldsymbol{y}} + \bar{\boldsymbol{y}}^T\left(\frac{1}{2}\boldsymbol{V}\right)\bar{\boldsymbol{y}} + \boldsymbol{f}^T\boldsymbol{y} + \boldsymbol{y}^T\left(\frac{1}{2}\boldsymbol{V}\right)\boldsymbol{y} - \bar{\boldsymbol{y}}^T\left(\frac{1}{2}\boldsymbol{V}\right)\boldsymbol{y} - \boldsymbol{y}^T\left(\frac{1}{2}\boldsymbol{V}\right)\bar{\boldsymbol{y}}. \end{aligned} \tag{1.C.18}$$

Let $\boldsymbol{G} \equiv u(\bar{\boldsymbol{y}}) - \boldsymbol{f}^T\bar{\boldsymbol{y}} + \bar{\boldsymbol{y}}^T\left(\frac{1}{2}\boldsymbol{V}\right)\bar{\boldsymbol{y}}$ be the constant term, substituting (1.C.14) and (1.C.17) into (1.C.18) yields

$$\begin{aligned} u(\boldsymbol{y}) =& \boldsymbol{G} + \begin{pmatrix}\boldsymbol{f}_n^T & \boldsymbol{f}_k^T\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{x}} \\ \boldsymbol{a}\end{pmatrix} + \begin{pmatrix}\hat{\boldsymbol{x}}^T & \boldsymbol{a}^T\end{pmatrix}\begin{pmatrix}\boldsymbol{S} & \boldsymbol{T}^T \\ \boldsymbol{T} & \boldsymbol{L}\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{x}} \\ \boldsymbol{a}\end{pmatrix} \\ & - \begin{pmatrix}\bar{\hat{\boldsymbol{x}}}^T & \bar{\boldsymbol{a}}^T\end{pmatrix}\begin{pmatrix}\boldsymbol{S} & \boldsymbol{T}^T \\ \boldsymbol{T} & \boldsymbol{L}\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{x}} \\ \boldsymbol{a}\end{pmatrix} - \begin{pmatrix}\hat{\boldsymbol{x}}^T & \boldsymbol{a}^T\end{pmatrix}\begin{pmatrix}\boldsymbol{S} & \boldsymbol{T}^T \\ \boldsymbol{T} & \boldsymbol{L}\end{pmatrix}\begin{pmatrix}\bar{\hat{\boldsymbol{x}}} \\ \bar{\boldsymbol{a}}\end{pmatrix} \\ =& \boldsymbol{G} + \hat{\boldsymbol{x}}^T\boldsymbol{S}\hat{\boldsymbol{x}} + \boldsymbol{a}^T\boldsymbol{L}\boldsymbol{a} + \boldsymbol{a}^T\boldsymbol{T}\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}^T\boldsymbol{T}^T\boldsymbol{a} + \boldsymbol{f}_n^T\hat{\boldsymbol{x}} + \boldsymbol{f}_k^T\boldsymbol{a} + \left(\bar{\hat{\boldsymbol{x}}}^T\boldsymbol{S} + \bar{\boldsymbol{a}}^T\boldsymbol{T}\right)\hat{\boldsymbol{x}} \\ & - \hat{\boldsymbol{x}}^T\left(\boldsymbol{S}\bar{\hat{\boldsymbol{x}}} + \boldsymbol{T}^T\bar{\boldsymbol{a}}\right) - \left(\bar{\hat{\boldsymbol{x}}}^T\boldsymbol{T}^T + \bar{\boldsymbol{a}}^T\boldsymbol{L}\right)\boldsymbol{a} - \boldsymbol{a}^T\left(\boldsymbol{T}\bar{\hat{\boldsymbol{x}}} + \boldsymbol{L}\bar{\boldsymbol{a}}\right) \end{aligned} \tag{1.C.19}$$

Since $\boldsymbol{\Theta}$ is symmetric, so is Θ_{12}, Θ_{13}, and Θ_{23}.

Comparing (1.C.15) with (1.C.19) and matching the coefficients, we have

$$\Theta_{11} = \boldsymbol{G}, \tag{1.C.20}$$

$$\Theta_{22} = \boldsymbol{S}, \tag{1.C.21}$$

$$\Theta_{33} = \boldsymbol{L}, \tag{1.C.22}$$

$$\Theta_{32} = \boldsymbol{T}, \quad \Theta_{23} = \boldsymbol{T}^T, \tag{1.C.23}$$

$$\Theta_{12} = \frac{1}{2}\boldsymbol{f}_n^T - \bar{\hat{\boldsymbol{x}}}^T\boldsymbol{S} - \bar{\boldsymbol{a}}^T\boldsymbol{T}, \quad \Theta_{21} = \Theta_{12}^T, \tag{1.C.24}$$

$$\Theta_{13} = \frac{1}{2}\boldsymbol{f}_k^T - \bar{\hat{\boldsymbol{x}}}^T\boldsymbol{T}^T - \bar{\boldsymbol{a}}^T\boldsymbol{L}, \quad \Theta_{31} = \Theta_{13}^T. \tag{1.C.25}$$

Finally,

$$\boldsymbol{Q} = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}, \quad \boldsymbol{R} = \Theta_{33}, \quad \boldsymbol{W} = \begin{pmatrix} \Theta_{31} & \Theta_{32} \end{pmatrix}. \tag{1.C.26}$$

Hence we have obtained the quadratic approximation of utility function $u(\cdot)$ using the numerical first and second order partial derivative matrices $\boldsymbol{f}_n$, $\boldsymbol{f}_k$, $\boldsymbol{S}$, $\boldsymbol{T}$, and $\boldsymbol{L}$.

1.D Data Source

Table 1.D.1: Data Source Description

Data	Source
Gross Domestic Product	BEA NIPA Table 1.1.5
Private Residential Fixed Investment	FRED series *PRFI*
Consumption on Housing and Utilities	BEA NIPA Table 2.3.5
Gross Private Domestic Investment	BEA NIPA Table 1.1.5
Federal Nondefense Gross Investment	BEA NIPA Table 3.9.5
State and Local Investment	BEA NIPA Table 3.9.5
Consumer Durables	BEA FA Table 1.1.5
Units of New Houses for Sale	FRED series *NHFSEPC*
Gross Fixed Assets	BEA FA Table 1.1
Gross Residential Structures	BEA FA Table 1.1
Private Fixed Assets in Construction Industry	BEA FA Table 3.1ESI
Total Hours Worked	BEA NIPA Table 6.9B - 6.9D
Total Hours Worked in Construction Industry	BEA NIPA Table 6.9B - 6.9D
Median Sales Price for New Houses Sold in U.S.	FRED series *MSPNHSUS*
Consumer Price Index: Housing	FRED series *CPIHOSSL*, 2000Q1 = 100
Consumer Price Index	FRED series *CPIAUCSL*, 2000Q1 = 100
Civilian Noninstitutional Population	FRED series *CNP16OV*
3-Month Effective Federal Funds Rate	FRED series *FEDFUNDS*
Civilian Unemployment Rate	FRED series *UNRATE*

BEA: Bureau of Economic Analysis; NIPA: National Income and Product Accounts; FA: Fixed Assets;

FRED: Federal Reserve Economic Data

1.E Kalman Filter

A nice property of the Linear Quadratic Gaussian problem is that it is very easy to apply the Kalman filter to compute the likelihood of a sample generated by the Linear Quadratic model.

Consider the state space representation of a model, which consists of

$$\text{State equation:} \quad \boldsymbol{u}_{t+1} = \boldsymbol{D}\boldsymbol{u}_t + \boldsymbol{v}_{t+1}, \quad \boldsymbol{v}_t \sim \text{i.i.d. } (\boldsymbol{0}, \boldsymbol{\Sigma}_v), \tag{1.E.1}$$

$$\text{Observation equation:} \quad \boldsymbol{y}_t = \boldsymbol{G}^T\boldsymbol{s}_t + \boldsymbol{H}^T\boldsymbol{u}_t + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim \text{i.i.d. } (\boldsymbol{0}, \boldsymbol{\Sigma}_\eta), \tag{1.E.2}$$

where $\boldsymbol{u}_t$ is the vector of *unobserved* states; $\boldsymbol{y}_t$ is the $n \times 1$ vector of *observed* variables and $\boldsymbol{s}_t$ is the $k \times 1$ vector of *exogenous/predetermined* variables; the error terms $\boldsymbol{v}_t$ and $\boldsymbol{\eta}_t$ are both mean-zero and uncorrelated with each other.

The Kalman filter produces an forecast of the unobseved state of the system as an weighted average of the predicted state and the new measurement. The weights are calculated from the mean squared error (MSE) of the forecast so that higher weights are given to the more confident forecasts. Specifically, the recursive formula of the forecast given by Kalman filter is

$$\hat{\boldsymbol{u}}_{t+1|t} = \boldsymbol{D}\hat{\boldsymbol{u}}_{t|t-1} + \boldsymbol{\mathcal{K}}_t(\boldsymbol{y}_t - \boldsymbol{G}^T\boldsymbol{s}_t - \boldsymbol{H}^T\hat{\boldsymbol{u}}_{t|t-1}), \tag{1.E.3}$$

where $\boldsymbol{\mathcal{K}}_t \equiv \boldsymbol{D}\boldsymbol{P}_{t|t-1}\boldsymbol{H}(\boldsymbol{H}^T\boldsymbol{P}_{t|t-1}\boldsymbol{H} + \Sigma_\eta)^{-1}$ is the Kalman gain, and the associated MSE of the forecast $\hat{\boldsymbol{u}}_{t+1|t}$ is

$$\boldsymbol{P}_{t+1|t} = \boldsymbol{D}\left[\boldsymbol{P}_{t|t-1} - \boldsymbol{P}_{t|t-1}\boldsymbol{H}(\boldsymbol{H}\boldsymbol{P}_{t|t-1}\boldsymbol{H} + \Sigma_\eta)^{-1}\boldsymbol{H}^T\boldsymbol{P}_{t|t-1}\right]\boldsymbol{D}^T + \Sigma_v. \tag{1.E.4}$$

The following proposition in Hamilton (1994) shows how to compute the likelihood of a sample of the observed variables, $\{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_T\}$, being generated by this model.

Proposition 4. If $\boldsymbol{u}_1$ and $\{\boldsymbol{v}_t, \boldsymbol{\eta}_t\}_{t=1}^T$ are multivariate Gaussian, and with the given information set $\mathcal{I}_t := \{\boldsymbol{s}_t, \boldsymbol{s}_{t-1}, ..., \boldsymbol{s}_1; \boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2}, ..., \boldsymbol{y}_1\}$, then

(i) Kalman filter gives the optimal forecast of $\hat{\boldsymbol{u}}_{t|t-1}$ and $\hat{\boldsymbol{y}}_{t|t-1}$;

(ii) The conditional distribution of $\boldsymbol{y}_t$ is Gaussian:

$$\boldsymbol{y}_t|\mathcal{I}_t \sim \mathcal{N}(\boldsymbol{G}^T\boldsymbol{s}_t + \boldsymbol{H}^T\hat{\boldsymbol{u}}_{t|t-1}, \boldsymbol{H}^T\boldsymbol{P}_{t|t-1}\boldsymbol{H} + \boldsymbol{\Sigma}_\eta); \tag{1.E.5}$$

(iii) The likelihood of a sample point $\boldsymbol{y}_t$ is given by

$$\begin{aligned}\text{Prob}(\boldsymbol{y}_t|\mathcal{I}_t) =& f_{\boldsymbol{Y}_t|\mathcal{I}_t}(\boldsymbol{y}_t|\mathcal{I}_t) \\ =& (2\pi)^{-\frac{n}{2}}|\boldsymbol{H}^T\boldsymbol{P}_{t|t-1}\boldsymbol{H} + \boldsymbol{\Sigma}_\eta|^{-\frac{1}{2}} \exp\Big\{ -\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{G}^T\boldsymbol{s}_t - \boldsymbol{H}^T\hat{\boldsymbol{u}}_{t|t-1})^T \\ & (\boldsymbol{H}^T\boldsymbol{P}_{t|t-1}\boldsymbol{H} + \boldsymbol{\Sigma}_\eta)^{-1}(\boldsymbol{y}_t - \boldsymbol{G}^T\boldsymbol{s}_t - \boldsymbol{H}^T\hat{\boldsymbol{u}}_{t|t-1})\Big\},\end{aligned} \tag{1.E.6}$$

thus the log-likelihood of the whole sample $\{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_T\}$ is given by

$$\sum_{t=1}^T \log f_{\boldsymbol{Y}_t|\mathcal{I}_t}(\boldsymbol{y}_t|\mathcal{I}_t). \tag{1.E.7}$$

In order to apply the proposition to compute the likelihood, first, I need to write down the approximating model and the worst-case model in the state space representation. For the approximating model, the state equation is simply the transition law:

$$\boldsymbol{x}_{t+1} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})\boldsymbol{x}_t + \tilde{\boldsymbol{\epsilon}}_{t+1}, \quad \tilde{\boldsymbol{\epsilon}}_{t+1} = \boldsymbol{C}\boldsymbol{\epsilon}_{t+1}, \tag{1.E.8}$$

and since all states are observable, the observation equation is

$$\boldsymbol{y}_t = \boldsymbol{x}_t. \tag{1.E.9}$$

So $\boldsymbol{D} = \boldsymbol{A} + \boldsymbol{BF}$, $\boldsymbol{G} = \boldsymbol{0}$, $\boldsymbol{H} = \boldsymbol{I}$, $\boldsymbol{\Sigma}_\eta = \boldsymbol{0}$, and $\boldsymbol{\Sigma}_v = \boldsymbol{\Sigma}_{\tilde{\epsilon}}$.

However, to avoid a singularity problem in the matrix computation, I should only include the stochastic components of the system in the state vector. Therefore, I modify the state equation and the observation equation as follows:

$$\begin{pmatrix} z_{t+1} \\ \xi_{t+1} \end{pmatrix} = \begin{pmatrix} \rho_z & 0 \\ 0 & \rho_\xi \end{pmatrix} \begin{pmatrix} z_t \\ \xi_t \end{pmatrix} + \begin{pmatrix} \epsilon_{z,\tilde{t}+1} \\ \epsilon_{\xi,\tilde{t}+1} \end{pmatrix}, \quad \begin{pmatrix} \epsilon_{z,\tilde{t}+1} \\ \epsilon_{\xi,\tilde{t}+1} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_z^2 & 0 \\ 0 & \sigma_\xi^2 \end{pmatrix} \right), \tag{1.E.10}$$

$$\boldsymbol{y}_t = \begin{pmatrix} z_t \\ \xi_t \end{pmatrix}. \tag{1.E.11}$$

Similarly, for the worst-case model, I use the transition law

$$\boldsymbol{x}_{t+1} = (\boldsymbol{A} + \boldsymbol{BF} + \boldsymbol{C\kappa})\boldsymbol{x}_t + \boldsymbol{C}\boldsymbol{\epsilon}_{t+1}, \tag{1.E.12}$$

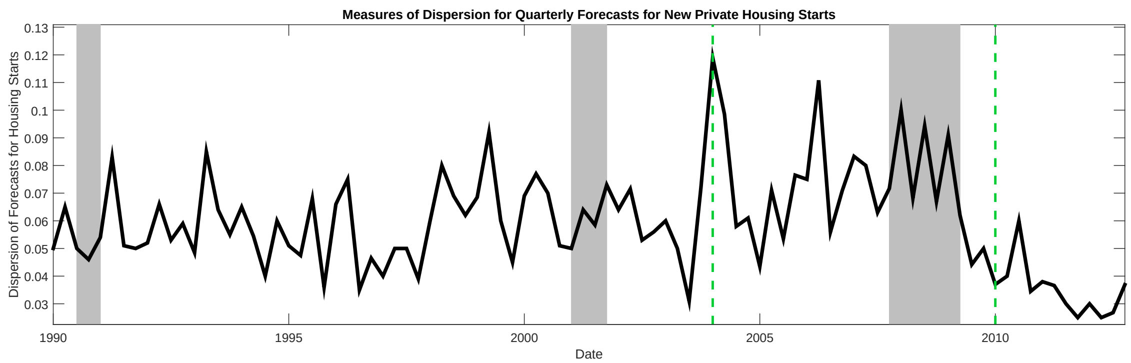
where $\boldsymbol{\kappa} = \sigma(\boldsymbol{I} - \sigma\boldsymbol{C}^T\boldsymbol{PC})^{-1}\boldsymbol{C}^T\boldsymbol{P}(\boldsymbol{A} + \boldsymbol{BF})$. Then I select the rows and columns in the coefficient matrix $(\boldsymbol{A} + \boldsymbol{BF} + \boldsymbol{C\kappa})$ that correspond to z_t and ξ_t to construct the matrix $\boldsymbol{D}$ in the state equation appropriately.

The rest part of the computation is straight forward. I use the formula of Kalman filter to compute the forecast $\hat{\boldsymbol{\epsilon}}_{t|t-1} = \{\hat{z}_{t|t-1}, \hat{\xi}_{t|t-1}\}$ and its associated MSE $\boldsymbol{P}_{t|t-1}$, for $t = 1, 2, ..., T$. Then I use the Proposition 4-(iii) to compute the likelihood of sample

$\{z_1, \xi_1, z_2, \xi_2, ..., z_T, \xi_T\}$ being generated by model A and model B respectively.[35]

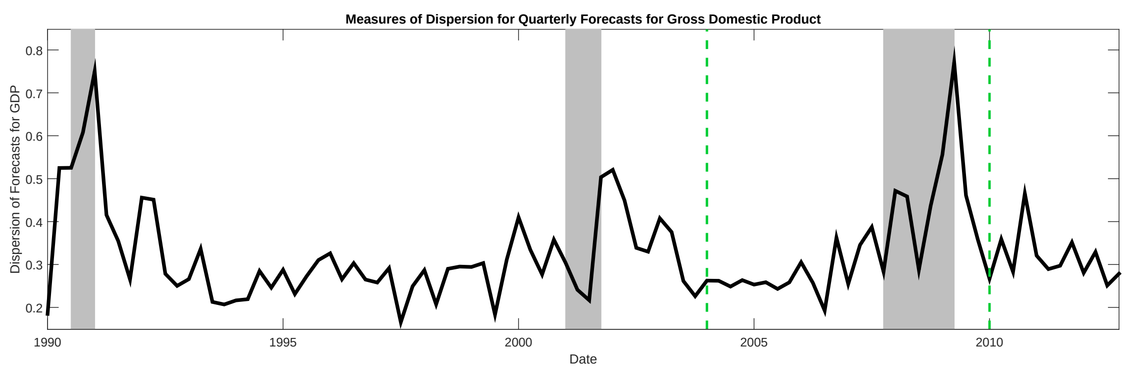
1.F Dispersion of Professional Forecasts

Figure 1.F.1: Measures of Dispersion for Quarterly Forecasts for New Private Housing Starts



Data source: Survey of Professional Forecasters database provided by Philadelphia FED.

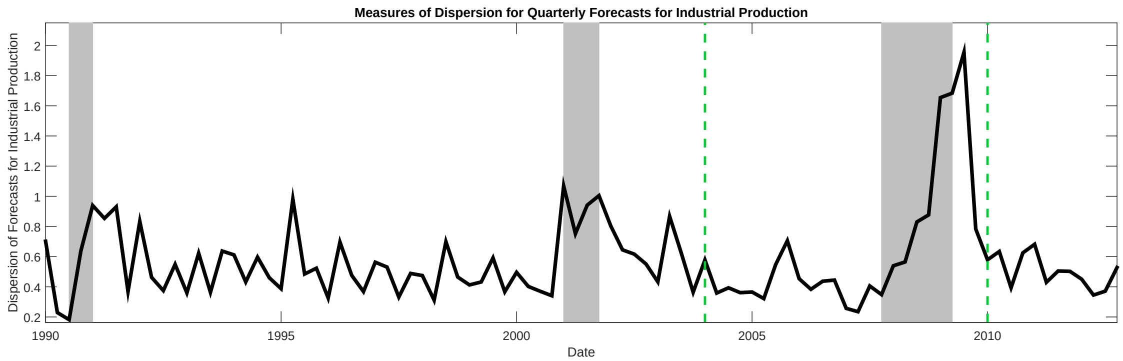
Figure 1.F.2: Measures of Dispersion for Quarterly Forecasts for Gross Domestic Product



Data source: Survey of Professional Forecasters database provided by Philadelphia FED.

[35]The choice of initial value of $\hat{\epsilon}_{1|0}$ is trivial. So I choose the steady state of (z, ξ).

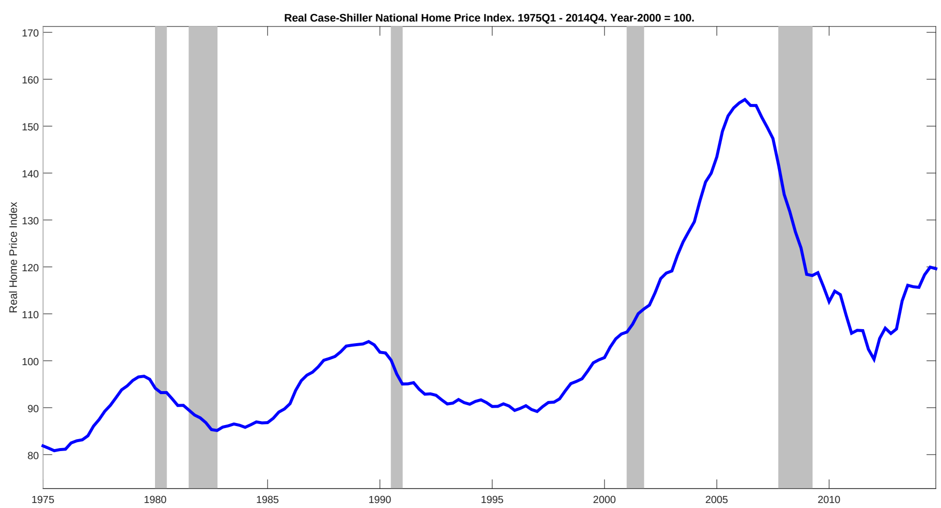
Figure 1.F.3: Measures of Dispersion for Quarterly Forecasts for Industrial Production



Data source: Survey of Professional Forecasters database provided by Philadelphia FED.

1.G Other Figures

Figure 1.G.1: Real Case-Shiller National Home Price Index in U.S. (1975 - 2014)



Chapter 2

Productivity, Home Production, and the New Home Sales Price in the U.S.

2.1 Introduction

In the macroeconomic literature, the private economic activities take place in two sectors, the business sector and the private sector. There is no need to reemphasize the importance of the business sectors. However, much less effort has gone into modeling the private sector (i.e. the home production sector), especially in the literature related to housing production and the new home sales price.

The home production sector is large. The Michigan Time Use Survey indicates that a typical marries couple spend 25 percent of their discretionary time on unpaid work in the home, such as houseworks and child care, compared to 33 percent on paid jobs in the market.[1] The postwar U.S. national income and product accounts indicate that home investment (purchase of consumer durables and residential structures, e.g. housing) exceeds

[1] See Hill (1984) and Juster and Stafford (1991).

market investment (purchase of producer durables and nonresidential structures, e.g. equipment) by about 15 percent. Additionally, Greenwood and Hercowitz (1991) report that the household capital stock actually exceeds market capital stock, and Benhabib et al. (1991) estimate that the size of the household sector's output is almost one half as large as that of the market output. Further, Benhabib et al. (1990) and Rios-Rull (1993) show the empirical evidence that individuals employed in the market sector spend much less time working in the home than unemployed individuals and also that employed individuals with higher wages substitute out of home and into market production. These findings suggest that not only is the home sector large, but that there is a significant amount of substitutability between it and the market, and thus lead us believe that accounting for home production and its interaction with market production may be important for understanding many macroeconomic phenomena.

In terms of literature, Greenwood and Hercowitz (1991) and Benhabib et al. (1991) show that real business cycle models with explicit household production sectors perform better than the standard business cycle model along several dimensions, especially in the aspects of better matching the fluctuations of output, labor hours, consumption and investment, and the comovements between productivity and output or labor hours observed in the data. However, to the best of my knowledge, all of the studies regarding home production only consider one production sector in the market. This paper explores the implications of introducing the home sector into a two-sector production economy as in Chapter 1 from a theoretical perspective.

In the standard two-sector economy, the relative productivity changes lead to resource reallocation and drive the dynamics of the new home sales price. Including home production enriches the choice set facing the households: in the standard model, households can only allocate their time between leisure and work in the market; with home production, they can divide their time among leisure, business work, and home work. The investment in housing goods now becomes the investment in the input of home production. More importantly, the relative productivity changes also give incentive for the households to substitute between market-produced goods and home-produced goods, which affects the demand for the housing goods and in turn influences the resource allocation between the two market sectors. This substitution effect generates very different responses of the economic variables to productivity shocks than that in an economy without home production. Compared with the benchmark model in which there is no home production, variables in the model with home production have amplified impulse responses to productivity shocks in the non-housing sector; and they exhibit mitigated impulse responses to productivity shocks in the housing sector.

The rest of the paper is organized as follows: Section 2 presents the model, Section 3 describes the market equilibrium, Section 4 discusses the quantitative results of the model. Finally, Section 5 concludes and provides comments.

2.2 Model

I build a dynamic stochastic model to study a perfectly competitive economy, which consists of a continuum of households that work both in the market and at home, a representative firm producing the consumption and investment goods, and a representative firm that builds new home units for sale. There is no government in this economy.

2.2.1 Households and Home Production

There are measure one of perpetual and identical households. The household's preference is defined over leisure ℓ_t, and the composite of market-produced consumption goods c_t and home-produced consumption goods s_t:

$$u(c_t, s_t, \ell_t) = \ln\left(C(c_t, s_t)\right) + \phi \ln(\ell_t), \tag{2.2.1}$$

where the consumption composite takes a CES functional form

$$C(c_t, s_t) = \left[\mu_c c_t^{\eta} + (1-\mu_c) s_t^{\eta}\right]^{\frac{1}{\eta}}. \tag{2.2.2}$$

The elasticity of substitution between market- and home-consumption is $\frac{1}{1-\eta}$. According to Benhabib et al. (1991), the CES form with $\eta \neq 0$ is necessary to assure the substantive effects from introducing home production. Otherwise, if $\eta = 0$ (i.e. Coubb-Douglas function), the model will generate the same outcome for market quantities as the case without home production.

The total time endowment for each household is one unit in every period, and it is distributed among market production N_t, home production $N_{s,t}$, and leisure time ℓ_t.

Market labor supply is indivisible. The households cannot choose the number of hours worked; rather, once being employed, they have to work a full amount of time, $\bar{N}$. I adopt the "lottery" fashion over employment and consumption that depicted by Hansen (1985) and Rogerson (1988). In each time period, the households have a probability of $n_{c,t}$ to work in the consumption goods sector and a probability of $n_{h,t}$ to work in the housing goods sector. Thus the probability of being unemployed is $1-n_{c,t}-n_{h,t}$. If employed, households enjoy consumption level c_t^E; if not, they consume c_t^U. There is complete unemployment insurance provided by the firms so that every household gets paid whether it works or not.

The households own h_t units of housing stock and combine it with home labor $N_{s,t}$ (which equals to $N^E_{s,t}$ if employed, or $N^U_{s,t}$ if unemployed) to produce home consumptions s_t. The home production function assumes the Cobb-Douglas form:

$$s_t = h_t^\gamma N_{s,t}^{1-\gamma}. \tag{2.2.3}$$

The productivity in home production is normalized to 1. This assumption is legitimate because Benhabib et al. (1991) showed that it is the relative productivity variation that matters for the implications on market variables.

Thus the consumption-leisure choice set only contains three points,

$$(C(c_t, s_t), \ell_t) = \begin{cases} (C(c_t^E, s_t^E), 1 - \bar{N} - N_{s,t}^E), & \text{with prob. } n_{c,t}; \quad \text{(employed in C sector)} \\ (C(c_t^E, s_t^E), 1 - \bar{N} - N_{s,t}^E), & \text{with prob. } n_{h,t}; \quad \text{(employed in H sector)} \\ (C(c_t^U, s_t^U), 1 - N_{s,t}^U), & \text{with prob. } 1 - n_{c,t} - n_{h,t}. \quad \text{(unemployed)} \end{cases}$$

and the households' expected utility in period t is given by

$$u(c_t, s_t, \ell_t) = (n_{c,t} + n_{h,t}) u(c_t^E, s_t^E, 1 - \bar{N} - N_{s,t}^E) + (1 - n_{c,t} - n_{h,t}) u(c_t^U, s_t^U, 1 - N_{s,t}^U) \tag{2.2.4}$$

combined with the detailed specifications (2.2.1), (2.2.2), and (2.2.3).

The housing stock is perfectly divisible and can be accumulated overtime through residential investment $I_{h,t}$. However, changing housing stock is costly and bears a quadratic adjustment cost. The housing stock depreciates at a constant rate of δ_h per period. In addition, the households own the representative firms and receive all of the profits Π_t. There is no money in this economy. The price of consumption goods is normalized to 1; the wage rate is w_t per hour; the unit price of new homes is $P_{I_h,t}$; the price paid to housing services can be measured by the implicit home rent to the owners, $R_{h,t}$.[2]

The households face the following budget constraint in period t:[3]

$$\begin{aligned}&(n_{c,t}+n_{h,t})c_t^E+(1-n_{c,t}-n_{h,t})c_t^U+R_{h,t}s_t+P_{Ih,t}I_{h,t}\\ \leq\ & w_t(n_{c,t}+n_{h,t})\bar{N}+R_{h,t}h_t+\Pi_t-\frac{\omega}{2}\left(\frac{h_{t+1}-h_t}{h_t}\right)^2h_t,\end{aligned} \tag{2.2.5}$$

with the law of motion for housing stock accumulation,

$$h_{t+1}=(1-\delta_h)h_t+I_{h,t}. \tag{2.2.6}$$

The households' problem can be characterized by the following Bellman equation:

$$\begin{aligned}V(h_t,z_t,\xi_t)=\max_{\substack{c_t^E,c_t^U,N_{s,t}^E,N_{s,t}^U,\\ n_{c,t},n_{h,t},h_{t+1},I_{h,t}}}&\Big\{(n_{c,t}+n_{h,t})u(c_t^E,s_t^E,1-\bar{N}-N_{s,t}^E)\\ &+(1-n_{c,t}-n_{h,t})u(c_t^U,s_t^U,1-N_{s,t}^U)+\beta\mathbb{E}_t\left[V(h_{t+1},z_{t+1},\xi_{t+1}\Big|z_t,\xi_t\right]\Big\},\end{aligned} \tag{2.2.7}$$

$$\begin{aligned}\text{s.t. }&(n_{c,t}+n_{h,t})c_t^E+(1-n_{c,t}-n_{h,t})c_t^U+R_{h,t}s_t+P_{Ih,t}I_{h,t}\\ &\leq w_t(n_{c,t}+n_{h,t})\bar{N}+R_{h,t}h_t+\Pi_t-\frac{\omega}{2}\left(\frac{h_{t+1}-h_t}{h_t}\right)^2h_t,\end{aligned} \tag{2.2.8}$$

$$h_{t+1}=(1-\delta_h)h_t+I_{h,t}, \tag{2.2.9}$$

$$s_t^i=h_t^\gamma(N_{s,t}^i)^{1-\gamma},\quad i\in\{E,U\} \tag{2.2.10}$$

[2] I exempt the discussion of own-or-rent a home in this paper. Every household is a home owner.

[3] The adjustment cost of housing stock is ascribed to transaction costs like preparing relevant documents, signing contracts, etc. Thus it is evaluated by units of consumption goods.

where z_t and ξ_t represent sector-specific total factor productivity (TFP) shocks.

2.2.2 Firms and Production Technologies

The characterizations of representative firms are the same as in Chapter 1. I just briefly restate the key functions here.

The Consumption Sector

The representative firm in the consumption goods sector produces output $Y_{c,t}$ using a constant-returns-to-scale technology:

$$Y_{c,t} = \exp(z_t) K_{c,t}^{\alpha} N_{c,t}^{1-\alpha}. \tag{2.2.11}$$

The output $Y_{c,t}$ is a numeraire good, whose price is normalized to 1. It can be consumed by households, invested and installed as new capital stock, and utilized to compensate any costs caused by economic activities.

The Housing Sector

New homes $Y_{h,t}$ are produced in the housing sector using capital $K_{h,t}$, labor $N_{h,t}$, and land L_t:

$$Y_{h,t} = P_{Ih,t} \exp(\xi_t) K_{h,t}^{\theta} N_{h,t}^{\nu} L_t^{1-\theta-\nu}. \tag{2.2.12}$$

The housing construction also uses land as an input, which has fixed supply of one unit in every period. New homes produced are sold to the households at price $P_{I_h,t}$ per unit.

Adjustment Costs

I assume quadratic adjustment costs as before. The law of motions for capital inputs are:

$$K_{c,t+1} = (1-\delta_{kc})K_{c,t} + I_{kc,t} - \frac{\kappa_c}{2}\left(\frac{K_{c,t+1}}{K_{c,t}} - 1\right)^2 K_{c,t}, \tag{2.2.13}$$

$$K_{h,t+1} = (1-\delta_{kh})K_{h,t} + I_{kh,t} - \frac{\kappa_h}{2}\left(\frac{K_{h,t+1}}{K_{h,t}} - 1\right)^2 K_{h,t}. \tag{2.2.14}$$

where δ_{kc} and δ_{kh} stand for capital depreciation rates; κ_c and κ_h are the parameters in the capital adjustment cost functions.

The adjustment costs for altering labor inputs are given by

$$\tau(N_{c,t-1}, N_{c,t}) = \frac{\tau_c}{2}\left(\frac{N_{c,t}}{\bar{N}} - \frac{N_{c,t-1}}{\bar{N}}\right)^2 \tag{2.2.15}$$

and

$$\tau(N_{h,t-1}, N_{h,t}) = \frac{\tau_h}{2}\left(\frac{N_{h,t}}{\bar{N}} - \frac{N_{h,t-1}}{\bar{N}}\right)^2, \tag{2.2.16}$$

where τ_c and τ_h are the adjustment cost parameters; $N_{c,t}/\bar{N}$ and $N_{h,t}/\bar{N}$ are the number of workers hired in the two sectors.

Sector-Specific TFP Shocks

I model the TFP shocks in the consumption sector and housing sector separately. Both TFP shocks, z_t and ξ_t follow an mean-zero AR(1) process:

$$z_{t+1} = \rho_z z_t + \sigma_z \epsilon_{z,t+1}, \quad \epsilon_{z,t+1} \sim \text{i.i.d. } \mathcal{N}(0,1); \tag{2.2.17}$$

$$\xi_{t+1} = \rho_\xi \xi_t + \sigma_\xi \epsilon_{\xi,t+1}, \quad \epsilon_{\xi,t+1} \sim \text{i.i.d. } \mathcal{N}(0,1). \tag{2.2.18}$$

I assume that innovations $\epsilon_{z,t}$ and $\epsilon_{\xi,t}$ are uncorrelated for simplicity and tractability concerns. Thus $cov(\epsilon_{z,j}, \epsilon_{\xi,k}) = 0, \forall j, k$.[4]

Firms' Optimization Problems

The representative firm in the consumption sector solves the following Bellman equation:

$$J^C(K_{c,t}, N_{c,t-1}, z_t, \xi_t) = \max_{K_{c,t+1}, N_{c,t}, I_{kc,t}} Y_{c,t} - w_t N_{c,t} - I_{kc,t} - \frac{\tau_c}{2}\left(\frac{N_{c,t}}{\bar{N}} - \frac{N_{c,t-1}}{\bar{N}}\right)^2 + \beta \mathbb{E}_t \left[J^C(K_{c,t+1}, N_{c,t}, z_{t+1}, \xi_{t+1}) | z_t, \xi_t\right], \tag{2.2.19}$$

$$\text{s.t.} \quad K_{c,t+1} = (1-\delta_{kc})K_{c,t} + I_{kc,t} - \frac{\kappa_c}{2}\left(\frac{K_{c,t+1} - K_{c,t}}{K_{c,t}}\right)^2 K_{c,t}. \tag{2.2.20}$$

Similarly, the Bellman equation for the representative firm in the housing sector is

$$J^H(K_{h,t}, N_{h,t-1}, z_t, \xi_t) = \max_{K_{h,t+1}, N_{h,t}, I_{kh,t}} Y_{h,t} - w_t N_{h,t} - I_{kh,t} - \frac{\tau_h}{2}\left(\frac{N_{h,t}}{\bar{N}} - \frac{N_{h,t-1}}{\bar{N}}\right)^2 + \beta \mathbb{E}_t \left[J^H(K_{h,t+1}, N_{h,t}, z_{t+1}, \xi_{t+1}) | z_t, \xi_t\right], \tag{2.2.21}$$

$$\text{s.t.} \quad K_{h,t+1} = (1-\delta_{kh})K_{h,t} + I_{kh,t} - \frac{\kappa_h}{2}\left(\frac{K_{h,t+1} - K_{h,t}}{K_{h,t}}\right)^2 K_{h,t}. \tag{2.2.22}$$

2.3 Equilibrium

There are four markets in the economy: the housing market, the labor market, the consumption market, and the implicit home rental market.

[4]This assumption is consistent with the findings in Davis and Heathcote (2005) that productivity shocks are only weakly correlated across sectors, and in particular shocks to the construction sector are essentially uncorrelated with those in the non-construction sectors.

I use the uppercase letters to represent the aggregate counterparts of the individual variables. Since the households are homogeneous and of measure 1, the aggregate variables simply equal to the individual variables, e.g. $H_t = h_t$, $C_t^E = c_t^E$, $C_t^U = c_t^U$, etc.

Definition 2. A *recursive competitive equilibrium* for this economy is given by the value functions of the households and the firms $\{V, J^C, J^H\}$, households' optimal choices $\{c_t^E, c_t^U, N_{s,t}^E, N_{s,t}^U, n_{c,t}, n_{h,t}, h_{t+1}, I_{h,t}\}$, firms' decision rules $\{K_{c,t+1}, K_{h,t+1}, N_{c,t}, N_{h,t}, I_{kc,t}, I_{kh,t}\}$, price functions $\{P_{I_h,t}, w_t, R_{h,t}\}$, and law of motions for housing stock and physical capital stocks (2.2.6), (2.2.13) (2.2.14), such that:

(1) Given the prices, households' decision rules derived by solving the problem (2.2.7) - (2.2.10) maximize the lifetime utility;

(2) Given the prices, firms' choices solve the profit maximization probelms described in (2.2.19) - (2.2.20) and (2.2.21) - (2.2.22);

(3) All markets clear;

(i) Housing market clears,

$$Y_{h,t} = P_{Ih,t} I_{h,t}, \tag{2.3.1}$$

(ii) Labor market clears,

$$N_{c,t} = n_{c,t}\bar{N}, \quad N_{h,t} = n_{h,t}\bar{N}, \tag{2.3.2}$$

(iii) Implicit home rental market clears,

$$h_t = H_t, \tag{2.3.3}$$

(iv) Consumption market clears,

$$
\begin{aligned}
&(n_{c,t} + n_{h,t})C_t^E + (1 - n_{c,t} - n_{h,t})C_t^U + I_{kc,t} + I_{kh,t} \\
&= Y_{c,t} - \frac{\omega}{2}\left(\frac{H_{t+1} - H_t}{H_t}\right)^2 H_t - \frac{\tau_c}{2}\left(\frac{N_{c,t}}{\bar{N}} - \frac{N_{c,t-1}}{\bar{N}}\right)^2 - \frac{\tau_h}{2}\left(\frac{N_{h,t}}{\bar{N}} - \frac{N_{h,t-1}}{\bar{N}}\right)^2
\end{aligned}
\tag{2.3.4}
$$

(4) Law of motions for the stock variables hold.

$$H_{t+1} = (1 - \delta_h)H_t + I_{h,t}, \tag{2.3.5}$$

$$K_{c,t+1} = (1 - \delta_{kc})K_{c,t} + I_{kc,t} - \frac{\kappa_c}{2}\left(\frac{K_{c,t+1} - K_{c,t}}{K_{c,t}}\right)^2 K_{c,t}, \tag{2.3.6}$$

$$K_{h,t+1} = (1 - \delta_{kh})K_{h,t} + I_{kh,t} - \frac{\kappa_h}{2}\left(\frac{K_{h,t+1} - K_{h,t}}{K_{h,t}}\right)^2 K_{h,t}. \tag{2.3.7}$$

2.4 Quantitative Analysis

2.4.1 Solution Method and Computation Algorithm

The economy is under perfect competition with complete markets and perfect information. According to the First Fundamental Theorem of Welfare Economics, the decentralized market equilibrium is equivalent to the Pareto optimum of the corresponding social planner's problem. I derive the optimality conditions characterizing the decision rules and the steady states in Appendix 2.A and 2.B. The benchmark model is the situation without home production, which can be regarded as a special case that $\gamma = 1$ in the home production function and that $N_{s,t}^E = N_{s,t}^U = 0$.

The computation method is the same as that described in Section 1.4.2 in Chapter 1.

I adopt the Linear Exponential Quadratic Gaussian (LEQG) control method proposed in Hansen and Sargent (1995) and Hansen and Sargent (2007). This method requires taking a second-order Taylor expansion of the objective function and linearizing the state transition laws with respect to all of the state and control variables. The LEQG control problem can be solved analytically by iterating on a Riccati equation. This procedure is computationally efficient.

2.4.2 Calibration

The parameters to be calibrated in this model are β, ϕ, μ_c, η, γ, $\bar{N}$, α, θ, ν, δ_{kc}, δ_{kh}, δ_h, κ_c, κ_h, ω, τ_c, τ_h, ρ_z, ρ_ξ, σ_z and σ_ξ. I divide these 21 parameters into 3 groups and calibrate their values separately. The first group of parameters, $\{\beta, \phi, \mu_c, \eta, \gamma, \bar{N}, \alpha, \theta, \nu, \delta_{kc}, \delta_{kh}, \delta_h\}$, relates to preferences and production technologies; these parameters only depends on the long-run relationships among the macroeconomic variables and I calibrate their values by matching the first moments of model and data in the steady state. The second group of parameters, $\{\rho_z, \rho_\xi, \sigma_z, \sigma_\xi\}$, are pined down by the autocorrelation and volatility statistics computed from the TFP shock series directly. The third group of parameters, $\{\kappa_c, \kappa_h, \omega, \tau_c, \tau_h\}$, governs the adjustment costs of factor inputs in the consumption sector and housing sector; I set their values to match the volatilities of the related variables.

I use quarterly U.S. data from 1973Q1 to 2014Q4. The measurements of key macroeconomic variables are constructed using data sets from Bureau of Economic Analysis (BEA) and Federal Reserve Economic Data (FRED).[5]. The key variable measurements are summarized in Table 1.1, Table 1.2, and Table 1.3. Nominal quantities and prices are deflated by the Consumer Price Index. I use the Civilian Noninstitutional Population to convert the

[5] I show the detailed data source for each variable in Appendix 1.D

quantities to per-capita terms. I then take the natural logarithm of the data series and use the Hodrick-Prescott (HP) filter to extract the trend components from the series at the quarterly frequency.

First, I use the data measurements to compute the ratios among several variables in long-run equilibrium. I matched the same first moment targets as reported in Table 1.4 to calibrate the parameters $\{\beta, \phi, \mu_c, \bar{N}, \alpha, \theta, \nu, \delta_{kc}, \delta_{kh}, \delta_h\}$. The additional parameter γ in the home production function is calibrated the time allocation between market jobs and working at home. According to the data from the Michigan Time Use Survey, it indicate that an average household spends 33 percent of time working for the paid jobs in the market and 28 percent working in the home. I calibrate the parameter $\gamma = 0.4$ in the home production function to match this fact. As for the parameter η, which relates to the elasticity of substitution between market-produced consumption goods and home-produced consumption goods,[6] is set to 0.8. This follows the guidelines suggested in Benhabib et al. (1991).[7]

The calibrated parameter values are listed in Table 2.4.1. These parameter values stand in line with macro-housing literature.

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[6]The elasticity of substitution is $\frac{1}{1-\eta}$, based on the CES form of the consumption composite function.

[7]In Benhabib et al. (1991), they discussed two sources for this parameter value: one is based on Eichenbaum and Hansen (1990), using aggregate data to estimate a model in which individuals value both the services of market consumption goods and the flow of services from consumer durables such as the output of a home production, and set $\eta = 1$; the other method is to estimate a reduced-form regression depicted in Benhabib et al. (1991) using the pooled data from the Panel Study of Income Dynamics described in Rios-Rull (1993), which derives a value of $\eta = 0.6$. In the paper Benhabib et al. (1991), they suggest use $\eta = 0.8$, the mean value of the two estimates obtained from the macro-data based model and the micro-data based model.

Table 2.4.1: Parameter Values Determined by Matching First Moments

Parameter	Interpretation	Estimated Value
β	discount factor	$\beta = 0.987$
ϕ	weight of leisure in utility	$\phi = 1.5$
μ_c	weight of market consumption goods in consumption composite	$\mu_c = 0.6231$
η	elasticity between market consumption and home consumption	$\eta = 0.8$
γ	housing stock share in home production	$\gamma = 0.4$
$\bar{N}$	fixed labor input if employed	$\bar{N} = 0.318$
α	capital share in consumption sector	$\alpha = 0.433$
θ	capital share in housing sector	$\theta = 0.063$
ν	labor share in housing sector	$\nu = 0.637$
δ_k	depreciation rate of physical capital stock	$\delta_k = 0.0284$
δ_h	depreciation rate of housing stock	$\delta_h = 0.0151$

The parameters that govern the TFP shocks are estimated based on the calculation of Solow residuals based on the Cobb-Douglas production functions and some of the parameter values in the first group. The estimated parameter values are the same as reported in Table 1.6.

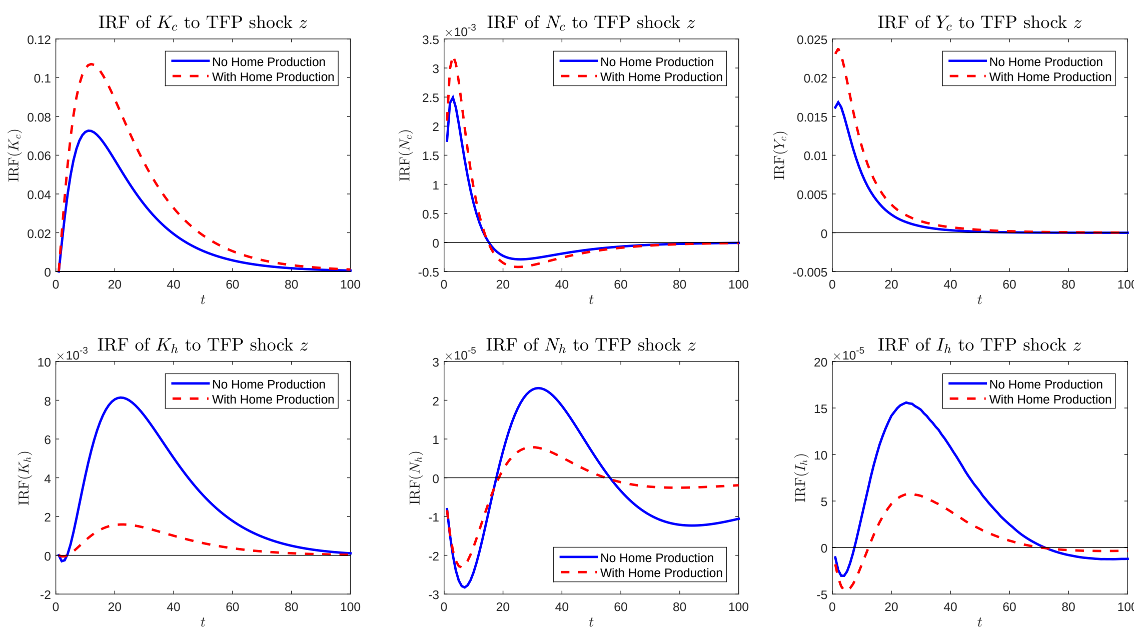
Finally, I choose the values for the third set of parameters $\{\kappa_c, \kappa_h, \tau_c, \tau_h, \omega\}$ by matching the second moments of data. I use the relative volatilities of housing stock and capital and labor inputs in two sectors to exactly identify the five parameters and get $\kappa_c = 0.2368$, $\kappa_h = 2.16$, $\omega = 2.2343$, $\tau_c = 2.591$, $\tau_h = 10.115$.

2.4.3 Impulse Response Functions

In order to understand the model mechanism, I study the effects of the two TFP shocks on the key model variables separately.

Figure 2.1, Figure 2.2 and Figure 2.3 show the impulse responses of resource allocation, output, consumption, labor choice, investment, new home sales price, and home rents to a one-standard-deviation increment in the TFP shock z_t in the non-housing production sector. Each figure plots the impulse responses of a specific variable under two cases: with home production and without the home production (the benchmark model).

Figure 2.1: IRF of $\{K_c, N_c, Y_c, K_h, N_h, I_h\}$ to a Positive Shock of z_t

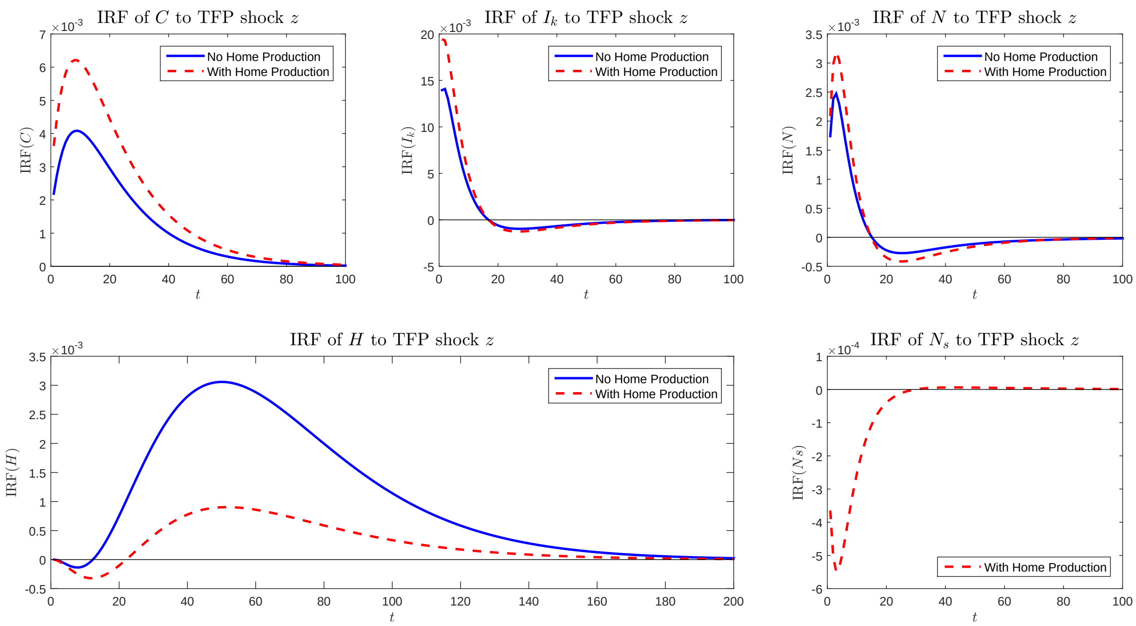


Note: z_t increases by one standard deviation; no change in ξ_t.

As shown in Figure 2.1, an increase in productivity in the consumption goods sector attracts more capital and labor inputs thus leads to greater output of consumption (or/and investment) goods. Capital input in the housing sector also accumulates because of more investment goods produced and hence more new capital installed in both sectors. Labor input in the housing sector jump down initially, as the marginal return of labor is lower than that in the consumption sector due to the productivity change. However, it increases later to catch up with the higher level of capital input.

With home production, these impulse responses exhibit similar patterns but with very different magnitudes. When there is a productivity increase in the consumption goods sector, it is relatively more efficient to work in the market and it gives incentive to the households for substituting market-produced consumption goods for home-produced consumption goods. As a result, capital input, labor input and output all increase in the consumption goods market. On the other hand, since the demand for home-produced consumption is lower, the incentive to work in the housing sector and produce housing goods (which is an input for home-produced consumption) is also lower, which leads to less capital and labor input and lower output in the housing sector. It is intriguing to notice that this substitution effect between market-produced and home-produced goods, due to the existence of home production, affects the resource allocations as if the relative productivity change in the consumption goods sector to the housing sector were amplified.

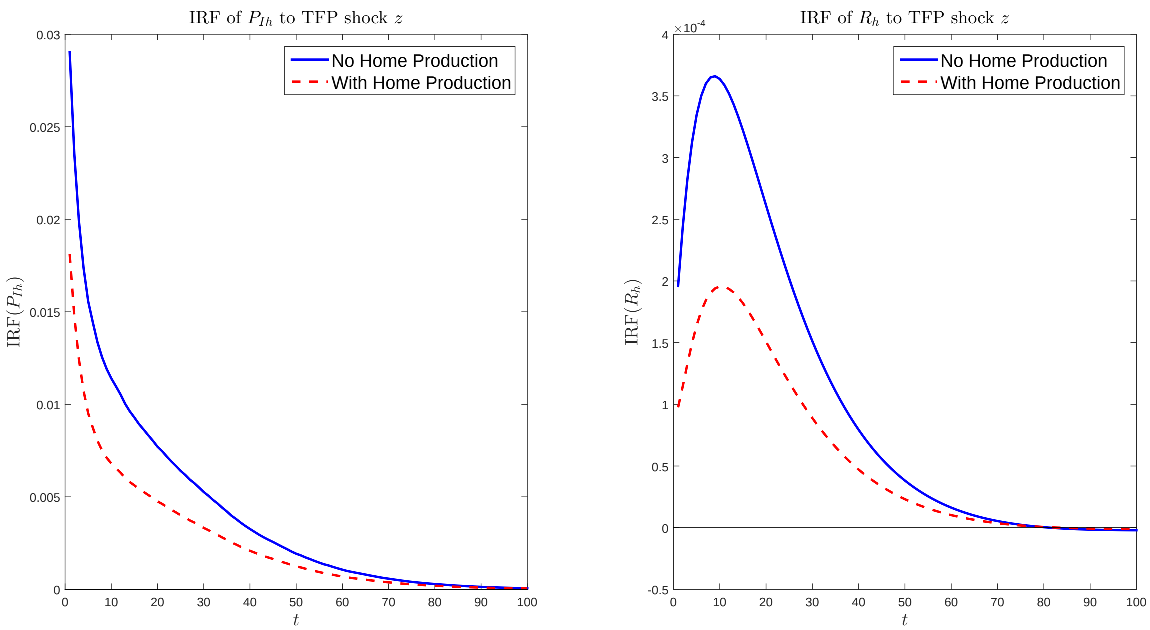
Figure 2.2: IRF of $\{C, I_k, H, N, N_s\}$ to a Positive Shock of z_t



Note: z_t increases by one standard deviation; no change in ξ_t.

Additionally, as shown in Figure 2.2, with a positive productivity shock in the consumption sector, households work more hours and consume more at first. But then households choose to work less and enjoy more leisure due to the wealth effect. There is a higher investment in the physical capital and the total housing stock also increases. In the case with home production, as discussed above, the households intend to consume more market-produced consumption goods and produce less housing goods. Investment in physical capital is also higher as total non-housing output increases. Total hours worked in the market rises and home working time declines.

Figure 2.3: IRF of $\{P_{Ih}, R_h\}$ to a Positive Shock of z_t



Note: z_t increases by one standard deviation; no change in ξ_t.

As derived in Appendix 2.B.4, the pricing function of new home units and implied home rents are

$$P_{Ih,t} = \frac{\exp(z_t)(1-\alpha)K_{c,t}^{\alpha}(n_{c,t}\bar{N})^{-\alpha}}{\exp(\xi_t)\nu K_{h,t}^{\theta}(n_{h,t}\bar{N})^{\nu-1}}. \tag{2.4.1}$$

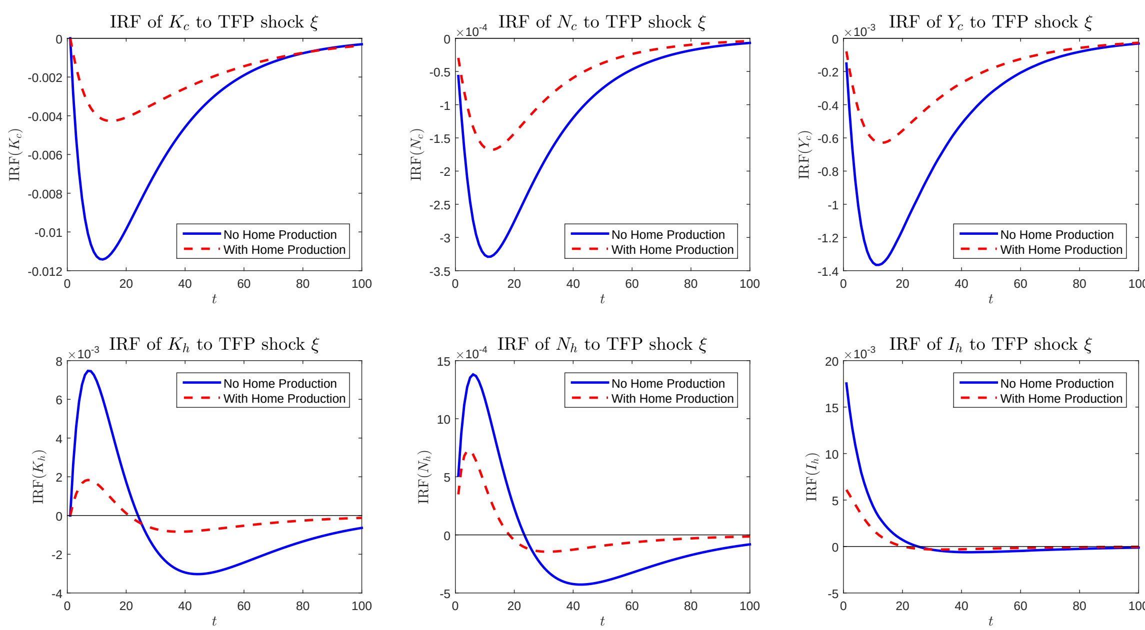
and

$$R_{h,t} = \frac{\frac{\phi\gamma}{(1-\gamma)H_t}\left[(n_{c,t}+n_{h,t})\frac{N_{s,t}^{E}}{1-\bar{N}-N_{s,t}^{E}} + (1-n_{c,t}-n_{h,t})\frac{N_{s,t}^{U}}{1-N_{s,t}^{U}}\right]}{\frac{\mu_c(C_t^E)^{\eta-1}}{\mu_c(C_t^E)^{\eta}+(1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^{E})^{(1-\gamma)\eta}}}. \tag{2.4.2}$$

The dynamic response of the new home sales price is driven by two forces: (1) the relative productivity between the consumption goods sector and the housing sector; (2) the marginal products of capital and labor implied by the resource allocation. The home rent, R_h, is determined by the ratio of marginal utility gained from housing services to the market consumption goods. Figure 2.3 shows the effects of a positive shock in TFP z on the new home sales price and home rents. When home production is in place, the same positive productivity shock in the consumption sector generate smaller increment in the new home sales price compared with the case without home production. This is consistent with the fact that the substitution incentive towards more market-produced consumption goods suppresses the demand for housing goods. Due to the similar reason, home rent also has dampened response to the positive productivity shock.

Figure 2.4, Figure 2.5 and Figure 2.6 show the impulse responses of the variables of interest to a one-standard-deviation increment in the TFP shock ξ_t in the housing sector.

Figure 2.4: IRF of $\{K_c, N_c, Y_c, K_h, N_h, I_h\}$ to a Positive Shock of ξ_t



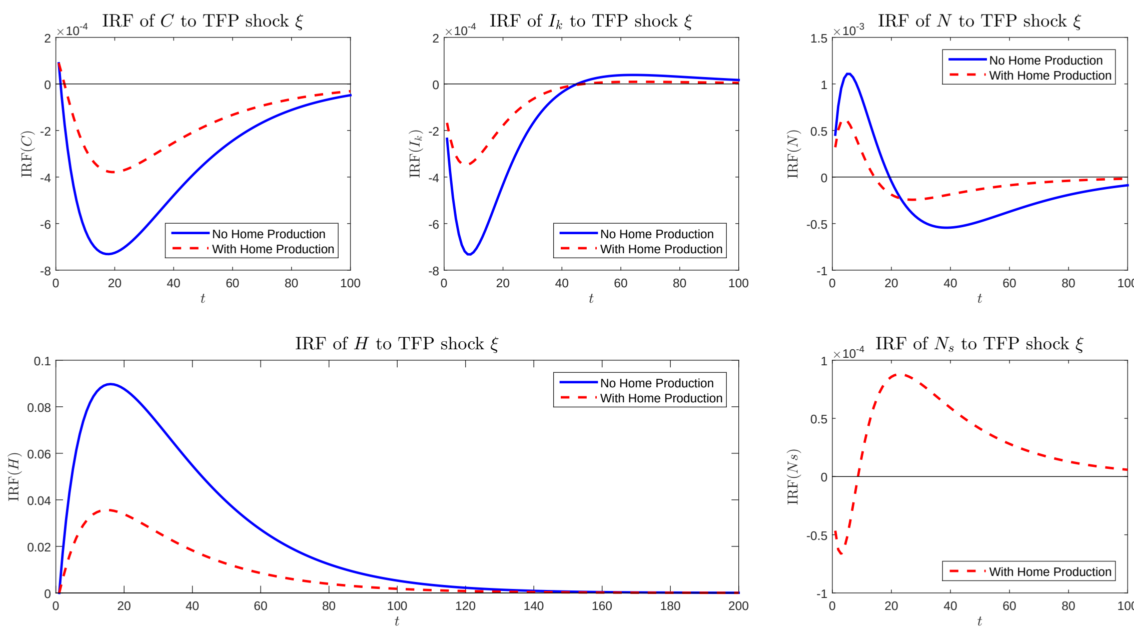
Note: ξ_t increases by one standard deviation; no change in z_t.

In Figure 2.4, we can see that both capital and labor inputs increase in the housing sector to take the advantage of higher efficiency of new home production. On the contrary, there are less capital and labor inputs in the consumption/investment goods production.

Capital input in the housing sector later falls below the level of its starting point because of insufficient investment in physical capital, I_k. This positive productivity shock leads to a higher output of new housing units and a lower output of consumption/investment goods.

In the case with home production, higher output of new housing units needs more home working hours to pair with this increment of asset to produce home goods. Consequently, total market labor input declines, which leads to decreases in the factor inputs both market sectors. In this case, as opposed to the case with positive TFP shock in the consumption sector, the home production mechanism affects the resource allocations as if the relative productivity change in the housing goods sector to the consumption goods sector were tempered.

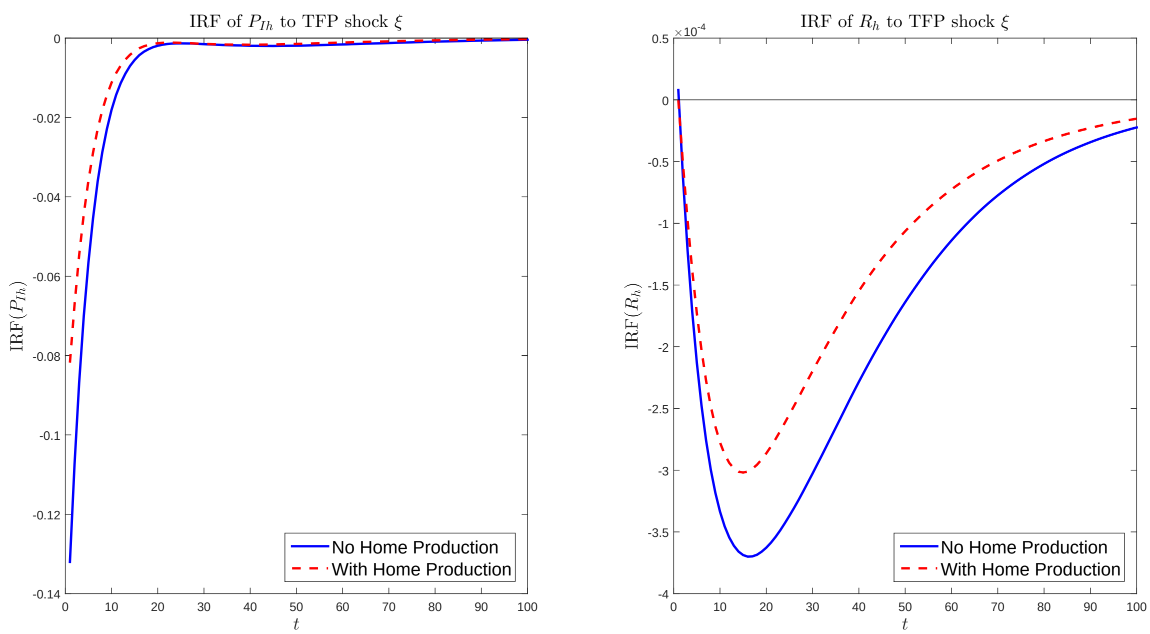
Figure 2.5: IRF of $\{C, I_k, H, N, N_s\}$ to a Positive Shock of ξ_t



Note: ξ_t increases by one standard deviation; no change in z_t.

Additionally, as shown in Figure 2.5, with a positive productivity shock in the housing sector, both consumption and investment in physical capital decrease because of lower output Y_c. Total labor input in the market increases because higher productivity leads to higher labor input in the housing sector. Housing stock also accumulates. When there is opportunity to produce home goods, higher housing stock means higher marginal product of one unit of housing in the home goods production. Thus the households would work more at home and reduce working hours in the market sectors. As a result, the market produces less consumption/investment goods and less new home units as well.

Figure 2.6: IRF of $\{P_{Ih}, R_h\}$ to a Positive Shock of ξ_t



Note: ξ_t increases by one standard deviation; no change in z_t.

Finally, Figure 2.6 shows the effects of a positive shock in the housing sector on the new home sales price and home rents. On the one hand, abundant supply of new home units lowers the sales price, which also lowers the home rent; on the other hand, the need for home production increases the demand for housing stock, so we can see that the negative effects on the new home sales price and home rent diminish in the case with home production.

2.5 Conclusion

Home production is not new to the macroeconomic literature. A number of existing papers studies the interaction between home production and market production and show their success in accounting for business cycle phenomena such as volatilities and correlations of key economic variables. But very few dedicated in investigating the relationship between home production and the housing market, as the housing sector has not been explicitly modeled in the home production literature. This paper studies the implications of introducing home production into an otherwise standard two-sector production economy from the theoretical perspective.

I solve the model using the methodology proposed by Hansen and Sargent (1995). The discounted Linear Exponential Quadratic Gaussian control method effectively avoids computational difficulties such as high-dimensional state space, explosive value function, etc.. Then I calibrate the parameters in the benchmark model (without home production) by the simulated method of moments.

While the benchmark model shows reasonable impulse responses of the key economic variables to different productivity shocks in the two market production sectors, the existence of home production highlights more interesting mechanisms due to the substitution incentive between market-produced and home-produced goods. When there is a positive productivity shock in the nonhousing goods production sector, households are inclined to substitute market consumption for home consumption as it is relatively more efficient to work in the market. As a result, resources flow to the two market sectors as if the relative productivity change in the nonhousing sector to the housing sector were amplified. In the case with a positive productivity shock in the housing sector, although it is more efficient to produce housing goods, the increment of housing stock requires more home hours to pair with it in home production. Thus the effects on resource allocations in the market sectors driven by this relative productivity change are tempered. This substitution mechanism can help improve the two-sector model's performance in fitting of the correlations between factor inputs that observed in the data.

Appendix

2.A Solve the Benchmark Model: No Home Production

2.A.1 Planner's Problem

The social planner maximizes households' lifetime utility subject to the resources constraints for production.

The utility function of all households aggregates the utility of the employed and unemployed individuals,

$$U(C_t, H_t, N_t) = (n_{c,t} + n_{h,t})u(C_t^E, H_t, N_t^E) + (1 - n_{c,t} - n_{h,t})u(C_t^U, H_t, N_t^U), \quad (2.A.1)$$

in which the momentary utility function has the following form

$$u(C_t^i, H_t, N_t^i) = \frac{1}{\eta} \ln \Big[\mu_c (C_t^i)^\eta + (1 - \mu_c) H_t^\eta\Big] + \phi \ln(1 - N_t^i - N_{s,t}^i), \quad i \in \{E, U\}. \quad (2.A.2)$$

According to the model assumptions, we have

$$N_t^i = \begin{cases} \bar{N}, & i = E, \\ 0, & i = U. \end{cases} \tag{2.A.3}$$

Therefore,

$$\begin{aligned} & U(C_t, H_t, N_t) = (n_{c,t} + n_{h,t})u(C_t^E, H_t, \bar{N}) + (1 - n_{c,t} - n_{h,t})u(C_t^U, H_t, 0) \\ & = (n_{c,t} + n_{h,t})\frac{1}{\eta} \ln\left[\mu_c (C_t^E)^\eta + (1 - \mu_c) H_t^\eta\right] + (n_{c,t} + n_{h,t})\phi \ln(1 - \bar{N}) \\ & + (1 - n_{c,t} - n_{h,t})\frac{1}{\eta} \ln\left[\mu_c (C_t^U)^\eta + (1 - \mu_c) H_t^\eta\right]. \end{aligned} \tag{2.A.4}$$

The Bellman equation of the planner's optimization problem can be written as follows:

$$
\begin{aligned}
& V(z_t, \xi_t, K_{c,t}, K_{h,t}, H_t, n_{c,t-1}, n_{h,t-1}) \\
= & \max_{\substack{K_{c,t+1}, K_{h,t+1}, H_{t+1}, n_{c,t}, n_{h,t}, \\ C_t^E, C_t^U, I_{kc,t}, I_{kh,t}, I_{h,t}}} \Big\{ (n_{c,t} + n_{h,t}) u(C_t^E, H_t, \bar{N}) + (1 - n_{c,t} - n_{h,t}) u(C_t^U, H_t, 0) \\
& + \beta \mathbb{E}_t \Big[V(z_{t+1}, \xi_{t+1}, K_{c,t+1}, K_{h,t+1}, H_{t+1}, n_{c,t}, n_{h,t}) \Big| z_t, \xi_t \Big] \Big\}, \qquad (2.A.5)
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.} \quad & (n_{c,t} + n_{h,t}) C_t^E + (1 - n_{c,t} - n_{h,t}) C_t^U + I_{kc,t} + I_{kh,t} \\
& \leq Y_{c,t} - \frac{\omega}{2} \left(\frac{H_{t+1}}{H_t} - 1 \right)^2 H_t - \frac{\tau_c}{2} (n_{c,t} - n_{c,t-1})^2 - \frac{\tau_h}{2} (n_{h,t} - n_{h,t-1})^2, \qquad (2.A.6)
\end{aligned}
$$

$$I_{h,t} \leq Y_{h,t}, \qquad (2.A.7)$$

$$H_{t+1} = (1 - \delta_h) H_t + I_{h,t}, \qquad (2.A.8)$$

$$K_{c,t+1} = (1 - \delta_{kc}) K_{c,t} + I_{kc,t} - \frac{\kappa_c}{2} \left(\frac{K_{c,t+1}}{K_{c,t}} - 1 \right)^2 K_{c,t}, \qquad (2.A.9)$$

$$K_{h,t+1} = (1 - \delta_{kh}) K_{h,t} + I_{kh,t} - \frac{\kappa_h}{2} \left(\frac{K_{h,t+1}}{K_{h,t}} - 1 \right)^2 K_{h,t}, \qquad (2.A.10)$$

$$Y_{c,t} = \exp(z_t) K_{c,t}^{\alpha} (n_{c,t} \bar{N})^{1-\alpha}, \qquad (2.A.11)$$

$$Y_{h,t} = \exp(\xi_t) K_{h,t}^{\theta} (n_{h,t} \bar{N})^{\nu} L_t^{1-\theta-\nu}, \qquad (2.A.12)$$

$$L_t = 1, \quad \forall t. \qquad (2.A.13)$$

Simplify the constraints and construct the Lagrangian:

$$
\begin{aligned}
\mathcal{L} =& (n_{c,t} + n_{h,t})\frac{1}{\eta}\ln\left[\mu_c(C_t^E)^\eta + (1-\mu_c)H_t^\eta\right] + (n_{c,t} + n_{h,t})\phi\ln(1-\bar{N}) \\
&+ (1 - n_{c,t} - n_{h,t})\frac{1}{\eta}\ln\left[\mu_c(C_t^U)^\eta + (1-\mu_c)H_t^\eta\right] \\
&+ \beta\mathbb{E}_t\left[V(z_{t+1}, \xi_{t+1}, K_{c,t+1}, K_{h,t+1}, H_{t+1}, n_{c,t}, n_{h,t})\Big| z_t, \xi_t\right] \\
&+ \lambda_{1,t}\Bigg[\exp(z_t)K_{c,t}^\alpha(n_{c,t}\bar{N})^{1-\alpha} - (n_{c,t}+n_{h,t})C_t^E - (1-n_{c,t}-n_{h,t})C_t^U \\
&- K_{c,t+1} + (1-\delta_{kc})K_{c,t} - K_{h,t+1} + (1-\delta_{kh})K_{h,t} - \frac{\kappa_c}{2}\left(\frac{K_{c,t+1}}{K_{c,t}} - 1\right)^2 K_{c,t} \\
&- \frac{\kappa_h}{2}\left(\frac{K_{h,t+1}}{K_{h,t}} - 1\right)^2 K_{h,t} - \frac{\omega}{2}\left(\frac{H_{t+1}}{H_t} - 1\right)^2 H_t - \frac{\tau_c}{2}(n_{c,t} - n_{c,t-1})^2 \\
&- \frac{\tau_h}{2}(n_{h,t} - n_{h,t-1})^2\Bigg] + \lambda_{2,t}\Bigg[\exp(\xi_t)K_{h,t}^\theta(n_{h,t}\bar{N})^\nu - H_{t+1} + (1-\delta_h)H_t\Bigg]. \quad (2.A.14)
\end{aligned}
$$

2.A.2 Optimality Conditions

To simplify notations, I denote $V(t) \equiv V(z_t, \xi_t, K_{c,t}, K_{h,t}, H_t, n_{c,t-1}, n_{h,t-1})$.

Take the first order conditions with respect to the choice variables:

$$
K_{c,t+1}: \quad \beta\mathbb{E}_t\left[V'_{Kc}(t+1)\right] = \lambda_{1,t}\left[1 + \kappa_c\left(\frac{K_{c,t+1}}{K_{c,t}} - 1\right)\right], \quad (2.A.15)
$$

$$
K_{h,t+1}: \quad \beta\mathbb{E}_t\left[V'_{Kh}(t+1)\right] = \lambda_{1,t}\left[1 + \kappa_h\left(\frac{K_{h,t+1}}{K_{h,t}} - 1\right)\right], \quad (2.A.16)
$$

$$
H_{t+1}: \quad \beta\mathbb{E}_t\left[V'_H(t+1)\right] = \lambda_{1,t}\,\omega\left(\frac{H_{t+1}}{H_t} - 1\right) + \lambda_{2,t}, \quad (2.A.17)
$$

$$
\begin{aligned}
n_{c,t}: \quad \beta\mathbb{E}_t\left[V'_{nc}(t+1)\right] =& \frac{1}{\eta}\ln\left[\frac{\mu_c(C_t^U)^\eta + (1-\mu_c)H_t^\eta}{\mu_c(C_t^E)^\eta + (1-\mu_c)H_t^\eta}\right] - \phi\ln(1-\bar{N}) \\
&+ \lambda_{1,t}\Big[C_t^E - C_t^U + \tau_c(n_{c,t} - n_{c,t-1}) \\
&- \exp(z_t)(1-\alpha)K_{c,t}^\alpha(n_{c,t}\bar{N})^{-\alpha}\bar{N}\Big], \quad (2.A.18)
\end{aligned}
$$

$$
\begin{aligned}
n_{h,t}: \quad \beta \mathbb{E}_t \left[V'_{nh}(t+1) \right] =& \frac{1}{\eta} \ln \left[\frac{\mu_c (C_t^U)^\eta + (1-\mu_c) H_t^\eta}{\mu_c (C_t^E)^\eta + (1-\mu_c) H_t^\eta} \right] - \phi \ln(1-\bar{N}) \\
&+ \lambda_{1,t} \left[C_t^E - C_t^U + \tau_h (n_{h,t} - n_{h,t-1}) \right] \\
&- \lambda_{2,t} \exp(\xi_t) \nu K_{h,t}^\theta (n_{h,t} \bar{N})^{\nu-1} \bar{N}, \qquad (2.A.19)
\end{aligned}
$$

$$
C_t^E: \quad \frac{\mu_c (C_t^E)^{\eta-1}}{\mu_c (C_t^E)^\eta + (1-\mu_c) H_t^\eta} = \lambda_{1,t}, \qquad (2.A.20)
$$

$$
C_t^U: \quad \frac{\mu_c (C_t^U)^{\eta-1}}{\mu_c (C_t^U)^\eta + (1-\mu_c) H_t^\eta} = \lambda_{1,t}, \qquad (2.A.21)
$$

$$
\begin{aligned}
\lambda_{1,t}: \quad & (n_{c,t} + n_{h,t}) C_t^E + (1 - n_{c,t} - n_{h,t}) C_t^U + K_{c,t+1} - (1-\delta_{kc}) K_{c,t} + K_{h,t+1} - (1-\delta_{kh}) K_{h,t} \\
=& \exp(z_t) K_{c,t}^\alpha (n_{c,t} \bar{N})^{1-\alpha} - \frac{\kappa_c}{2} \left(\frac{K_{c,t+1}}{K_{c,t}} - 1 \right)^2 K_{c,t} - \frac{\kappa_h}{2} \left(\frac{K_{h,t+1}}{K_{h,t}} - 1 \right)^2 K_{h,t} \\
&- \frac{\omega}{2} \left(\frac{H_{t+1}}{H_t} - 1 \right)^2 H_t - \frac{\tau_c}{2} (n_{c,t} - n_{c,t-1})^2 - \frac{\tau_h}{2} (n_{h,t} - n_{h,t-1})^2, \qquad (2.A.22)
\end{aligned}
$$

$$
\lambda_{2,t}: \quad H_{t+1} - (1-\delta_h) H_t = \exp(\xi_t) K_{h,t}^\theta (n_{h,t} \bar{N})^\nu. \qquad (2.A.23)
$$

Applying the Envelope theorem to the value function yields

$$
V'_{Kc}(t) = \lambda_{1,t} \left[1 - \delta_{kc} + \frac{\kappa_c}{2} \left(\left(\frac{K_{c,t+1}}{K_{c,t}} \right)^2 - 1 \right) + \exp(z_t) \alpha K_{c,t}^{\alpha-1} (n_{c,t} \bar{N})^{1-\alpha} \right], \qquad (2.A.24)
$$

$$
V'_{Kh}(t) = \lambda_{1,t} \left[1 - \delta_{kh} + \frac{\kappa_h}{2} \left(\left(\frac{K_{h,t+1}}{K_{h,t}} \right)^2 - 1 \right) \right] + \lambda_{2,t} \exp(\xi_t) \theta K_{h,t}^{\theta-1} (n_{h,t} \bar{N})^\nu, \qquad (2.A.25)
$$

$$V'_H(t) = \frac{(n_{c,t}+n_{h,t})(1-\mu_c)H_t^{\eta-1}}{\mu_c(C_t^E)^\eta+(1-\mu_c)H_t^\eta} + \frac{(1-n_{c,t}-n_{h,t})(1-\mu_c)H_t^{\eta-1}}{\mu_c(C_t^U)^\eta+(1-\mu_c)H_t^\eta}$$
$$+ \lambda_{1,t}\frac{\omega}{2}\left(\left(\frac{H_{t+1}}{H_t}\right)^2-1\right) + \lambda_{2,t}(1-\delta_h), \tag{2.A.26}$$

$$V'_{nc}(t) = \lambda_{1,t}\tau_c(n_{c,t}-n_{c,t-1}), \tag{2.A.27}$$

$$V'_{nh}(t) = \lambda_{1,t}\tau_h(n_{h,t}-n_{h,t-1}). \tag{2.A.28}$$

Combining the first order conditions (2.A.15) - (2.A.23) and the Envelope theorem conditions (2.A.24) - (2.A.28), I construct the following conditions to solve for optimal decision rules.

$$\lambda_{1,t}\left[1+\kappa_c\left(\frac{K_{c,t+1}}{K_{c,t}}-1\right)\right] = \beta\mathbb{E}_t\Bigg[\lambda_{1,t+1}\Bigg((1-\delta_{kc})+\frac{\kappa_c}{2}\left(\left(\frac{K_{c,t+2}}{K_{c,t+1}}\right)^2-1\right)$$
$$+\ \exp(z_{t+1})\alpha K_{c,t+1}^{\alpha-1}(n_{c,t+1}\bar{N})^{1-\alpha}\Bigg)\Bigg], \tag{2.A.29}$$

$$\lambda_{1,t}\left[1+\kappa_h\left(\frac{K_{h,t+1}}{K_{h,t}}-1\right)\right] = \beta\mathbb{E}_t\Bigg[\lambda_{1,t+1}\left((1-\delta_{kh})+\frac{\kappa_h}{2}\left(\left(\frac{K_{h,t+2}}{K_{h,t+1}}\right)^2-1\right)\right)$$
$$+\ \lambda_{2,t+1}\exp(\xi_{t+1})\theta K_{h,t+1}^{\theta-1}(n_{h,t+1}\bar{N})^{\nu}\Bigg], \tag{2.A.30}$$

$$\lambda_{1,t}\omega\left(\frac{H_{t+1}}{H_t}-1\right)+\lambda_{2,t}$$
$$= \beta\mathbb{E}_t\Bigg[\frac{(n_{c,t+1}+n_{h,t+1})(1-\mu_c)H_{t+1}^{\eta-1}}{\mu_c(C_{t+1}^E)^\eta+(1-\mu_c)H_{t+1}^\eta} + \frac{(1-n_{c,t+1}-n_{h,t+1})(1-\mu_c)H_{t+1}^{\eta-1}}{\mu_c(C_{t+1}^U)^\eta+(1-\mu_c)H_{t+1}^\eta}$$
$$+\lambda_{1,t+1}\frac{\omega}{2}\left(\left(\frac{H_{t+2}}{H_{t+1}}\right)^2-1\right)+\lambda_{2,t+1}(1-\delta_h)\Bigg], \tag{2.A.31}$$

$$\beta\mathbb{E}_t\left[\lambda_{1,t+1}\tau_c(n_{c,t+1}-n_{c,t})\right]=\frac{1}{\eta}\ln\left[\frac{\mu_c(C_t^U)^\eta+(1-\mu_c)H_t^\eta}{\mu_c(C_t^E)^\eta+(1-\mu_c)H_t^\eta}\right]-\phi\ln(1-\bar{N})$$
$$+\lambda_{1,t}\left[C_t^E-C_t^U+\tau_c(n_{c,t}-n_{c,t-1})-\exp(z_t)(1-\alpha)K_{c,t}^\alpha(n_{c,t}\bar{N})^{-\alpha}\bar{N}\right], \tag{2.A.32}$$

$$\beta\mathbb{E}_t\left[\lambda_{1,t+1}\,\tau_h(n_{h,t+1}-n_{h,t})\right]=\frac{1}{\eta}\ln\left[\frac{\mu_c(C_t^U)^\eta+(1-\mu_c)H_t^\eta}{\mu_c(C_t^E)^\eta+(1-\mu_c)H_t^\eta}\right]-\phi\ln(1-\bar{N})$$
$$+\lambda_{1,t}\left[C_t^E-C_t^U+\tau_h(n_{h,t}-n_{h,t-1})\right]-\lambda_{2,t}\exp(\xi_t)\nu K_{h,t}^\theta(n_{h,t}\bar{N})^{\nu-1}\bar{N}, \tag{2.A.33}$$

$$\lambda_{1,t}=\frac{\mu_c(C_t^E)^{\eta-1}}{\mu_c(C_t^E)^\eta+(1-\mu_c)H_t^\eta}, \tag{2.A.34}$$

$$\lambda_{1,t}=\frac{\mu_c(C_t^U)^{\eta-1}}{\mu_c(C_t^U)^\eta+(1-\mu_c)H_t^\eta}, \tag{2.A.35}$$

$$(n_{c,t}+n_{h,t})C_t^E+(1-n_{c,t}-n_{h,t})C_t^U+K_{c,t+1}-(1-\delta_{kc})K_{c,t}+K_{h,t+1}-(1-\delta_{kh})K_{h,t}$$
$$=\exp(z_t)K_{c,t}^\alpha(n_{c,t}\bar{N})^{1-\alpha}-\frac{\kappa_c}{2}\left(\frac{K_{c,t+1}}{K_{c,t}}-1\right)^2K_{c,t}-\frac{\kappa_h}{2}\left(\frac{K_{h,t+1}}{K_{h,t}}-1\right)^2K_{h,t}$$
$$-\frac{\omega}{2}\left(\frac{H_{t+1}}{H_t}-1\right)^2H_t-\frac{\tau_c}{2}(n_{c,t}-n_{c,t-1})^2-\frac{\tau_h}{2}(n_{h,t}-n_{h,t-1})^2, \tag{2.A.36}$$

$$H_{t+1}-(1-\delta_h)H_t=\exp(\xi_t)K_{h,t}^\theta(n_{h,t}\bar{N})^\nu. \tag{2.A.37}$$

These optimality conditions characterize the equilibrium in the economy.

2.A.3 Steady State

In steady state, all variables do not evolve over time. In addition, the stochastic shock variables are in their mean values, $z_t = z_{t+1} = 0$, $\xi_t = \xi_{t+1} = 0$. Therefore the optimal conditions (2.A.29) - (2.A.37) can be simplified to the following equation system:

$$1 = \beta \left[\alpha K_c^{\alpha-1} (n_c \bar{N})^{1-\alpha} + 1 - \delta_{kc} \right], \tag{2.A.38}$$

$$\lambda_1 = \beta \left[\lambda_1 (1 - \delta_{kh}) + \lambda_2 \theta K_h^{\theta-1} (n_h \bar{N})^{\nu} \right], \tag{2.A.39}$$

$$\lambda_2 = \beta \left[\frac{(n_c + n_h)(1-\mu_c) H^{\eta-1}}{\mu_c (C^E)^\eta + (1-\mu_c) H^\eta} + \frac{(1 - n_c - n_h)(1-\mu_c) H^{\eta-1}}{\mu_c (C^U)^\eta + (1-\mu_c) H^\eta} + \lambda_2 (1 - \delta_h) \right]$$

$$\lambda_1 \left[(1-\alpha) K_c^{\alpha} (n_c \bar{N})^{-\alpha} \bar{N} - C^E + C^U \right] = \frac{1}{\eta} \ln \left[\frac{\mu_c (C^U)^\eta + (1-\mu_c) H^\eta}{\mu_c (C^E)^\eta + (1-\mu_c) H^\eta} \right] - \phi \ln(1 - \bar{N}), \tag{2.A.40}$$

$$\lambda_2 \nu K_h^{\theta} (n_h \bar{N})^{\nu-1} \bar{N} - \lambda_1 (C^E - C^U) = \frac{1}{\eta} \ln \left[\frac{\mu_c (C^U)^\eta + (1-\mu_c) H^\eta}{\mu_c (C^E)^\eta + (1-\mu_c) H^\eta} \right] - \phi \ln(1 - \bar{N}), \tag{2.A.41}$$

$$\lambda_1 = \frac{\mu_c (C^E)^{\eta-1}}{\mu_c (C^E)^\eta + (1-\mu_c) H^\eta}, \tag{2.A.42}$$

$$\lambda_1 = \frac{\mu_c (C^U)^{\eta-1}}{\mu_c (C^U)^\eta + (1-\mu_c) H^\eta}, \tag{2.A.43}$$

$$(n_c + n_h) C^E + (1 - n_c - n_h) C^U + \delta_{kc} K_c + \delta_{kh} K_h = K_c^{\alpha} (n_c \bar{N})^{1-\alpha} \tag{2.A.44}$$

$$\delta_h H = K_h^{\theta} (n_h \bar{N})^{\nu}. \tag{2.A.45}$$

2.A.4 Price Functions

Although prices do not exist in a centralized economy, we can back up the prices from the competitive equilibrium of the corresponding decentralized economy.

The optimal resource allocation equalizes the marginal product of labor (or capital) across the two sectors. Based on this relationship, the new home sales price must satisfy the following condition:

$$P_{Ih,t} = \frac{\exp(z_t)(1-\alpha) K_{c,t}^{\alpha} (n_{c,t} \bar{N})^{-\alpha}}{\exp(\xi_t) \nu K_{h,t}^{\theta} (n_{h,t} \bar{N})^{\nu-1}}. \tag{2.A.46}$$

The home rents must solve the following optimal consumption choice problem in the decentralized economy:

$$\max_{C_t^E, C_t^U, H_t} \Bigg\{ (n_{c,t}+n_{h,t})\frac{1}{\eta}\ln\left[\mu_c (C_t^E)^\eta + (1-\mu_c)H_t^\eta\right] + (n_{c,t}+n_{h,t})\phi\ln(1-\bar{N})$$
$$+ (1-n_{c,t}-n_{h,t})\frac{1}{\eta}\ln\left[\mu_c (C_t^U)^\eta + (1-\mu_c)H_t^\eta\right]$$
$$+ \beta\mathbb{E}_t\left[V(z_{t+1},\xi_{t+1},K_{c,t+1},K_{h,t+1},H_{t+1},n_{c,t},n_{h,t})\Big| z_t,\xi_t\right]\Bigg\} \tag{2.A.47}$$

$$\text{s.t.} \quad (n_{c,t}+n_{h,t})C_t^E + (1-n_{c,t}-n_{h,t})C_t^U + R_{h,t}H_t \leq (n_{c,t}+n_{h,t})w_t + \text{Non-Labor Income}. \tag{2.A.48}$$

The first-order conditions with respect to C_t^E, C_t^U, and H_t are:

$$C_t^E: \quad \frac{\mu_c (C_t^E)^{\eta-1}}{\mu_c (C_t^E)^\eta + (1-\mu_c)H_t^\eta} = \lambda_t, \tag{2.A.49}$$

$$C_t^U: \quad \frac{\mu_c (C_t^U)^{\eta-1}}{\mu_c (C_t^U)^\eta + (1-\mu_c)H_t^\eta} = \lambda_t, \tag{2.A.50}$$

$$H_t: \quad \frac{(n_{c,t}+n_{h,t})(1-\mu_c)H_t^{\eta-1}}{\mu_c (C_t^E)^\eta + (1-\mu_c)H_t^\eta} + \frac{(1-n_{c,t}-n_{h,t})(1-\mu_c)H_t^{\eta-1}}{\mu_c (C_t^U)^\eta + (1-\mu_c)H_t^\eta} = \lambda_t R_{h,t}. \tag{2.A.51}$$

It's easy to derive that

$$R_{h,t} = \left[(n_{c,t}+n_{h,t}) + (1-n_{c,t}-n_{h,t})\left(\frac{C_t^E}{C_t^U}\right)^{\eta-1}\right]\left(\frac{1-\mu_c}{\mu_c}\right)\left(\frac{H_t}{C_t^E}\right)^{\eta-1}. \tag{2.A.52}$$

2.B Solve the Model with Home Production

2.B.1 Planner's Problem

The utility function of all households aggregates the utility of the employed and unemployed individuals. With home production,

$$U(C_t, S_t, N_t, N_{s,t}) = (n_{c,t}+n_{h,t})u(C_t^E, S_t^E, N_t^E, N_{s,t}^E)+(1-n_{c,t}-n_{h,t})u(C_t^U, S_t^U, N_t^U, N_{s,t}^U). \tag{2.B.1}$$

in which the momentary utility function has the following form

$$u(C_t^i, S_t^i, N_t^i, N_{s,t}^i) = \frac{1}{\eta}\ln\left[\mu_c(C_t^i)^\eta + (1-\mu_c)(S_t^i)^\eta\right] + \phi\ln(1-N_t^i-N_{s,t}^i), \quad i \in \{E, U\}, \tag{2.B.2}$$

and the market labor is indivisible in that

$$N_t^i = \begin{cases} \bar{N}, & i = E, \\ 0, & i = U. \end{cases} \tag{2.B.3}$$

The home production assumes the Cobb-Douglas form,

$$S_t^i = H_t^\gamma (N_{s,t}^i)^{1-\gamma}, \quad i \in \{E, U\}. \tag{2.B.4}$$

Therefore I can write the social planner's utility function as

$$
\begin{aligned}
&U(C_t, S_t, N_t, N_{s,t}) \\
&= (n_{c,t} + n_{h,t}) u(C_t^E, H_t^\gamma (N_{s,t}^E)^{1-\gamma}, \bar{N}, N_{s,t}^E) + (1 - n_{c,t} - n_{h,t}) u(C_t^U, H_t^\gamma (N_{s,t}^U)^{1-\gamma}, 0, N_{s,t}^U) \\
&= (n_{c,t} + n_{h,t}) \frac{1}{\eta} \ln \left[\mu_c (C_t^E)^\eta + (1-\mu_c)(H_t^\gamma (N_{s,t}^E)^{1-\gamma})^\eta \right] + (n_{c,t} + n_{h,t}) \phi \ln(1 - \bar{N} - N_{s,t}^E) \\
&+ (1 - n_{c,t} - n_{h,t}) \frac{1}{\eta} \ln \left[\mu_c (C_t^U)^\eta + (1-\mu_c)(H_t^\gamma (N_{s,t}^U)^{1-\gamma})^\eta \right] + (1 - n_{c,t} - n_{h,t}) \phi \ln(1 - N_{s,t}^U).
\end{aligned}
\tag{2.B.5}
$$

The social planner maximizes households' lifetime utility subject to the resources constraints for production.

$$
\begin{aligned}
& V(z_t, \xi_t, K_{c,t}, K_{h,t}, H_t, n_{c,t-1}, n_{h,t-1}) \\
= & \max_{\substack{K_{c,t+1}, K_{h,t+1}, H_{t+1}, n_{c,t}, n_{h,t}, \\ C_t^E, C_t^U, I_{kc,t}, I_{kh,t}, I_{h,t}, N_{s,t}^E, N_{s,t}^U}} \Big\{ (n_{c,t} + n_{h,t}) u(C_t^E, H_t^\gamma (N_{s,t}^E)^{1-\gamma}, \bar{N}, N_{s,t}^E) \\
& + (1 - n_{c,t} - n_{h,t}) u(C_t^U, H_t^\gamma (N_{s,t}^U)^{1-\gamma}, 0, N_{s,t}^U) \\
& + \beta \mathbb{E}_t \Big[V(z_{t+1}, \xi_{t+1}, K_{c,t+1}, K_{h,t+1}, H_{t+1}, n_{c,t}, n_{h,t}) \Big| z_t, \xi_t \Big] \Big\}, \qquad (2.B.6)
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.} \quad & (n_{c,t} + n_{h,t}) C_t^E + (1 - n_{c,t} - n_{h,t}) C_t^U + I_{kc,t} + I_{kh,t} \\
& \leq Y_{c,t} - \frac{\omega}{2} \left(\frac{H_{t+1}}{H_t} - 1 \right)^2 H_t - \frac{\tau_c}{2} (n_{c,t} - n_{c,t-1})^2 - \frac{\tau_h}{2} (n_{h,t} - n_{h,t-1})^2, \qquad (2.B.7) \\
& I_{h,t} \leq Y_{h,t}, \qquad (2.B.8) \\
& H_{t+1} = (1 - \delta_h) H_t + I_{h,t}, \qquad (2.B.9) \\
& K_{c,t+1} = (1 - \delta_{kc}) K_{c,t} + I_{kc,t} - \frac{\kappa_c}{2} \left(\frac{K_{c,t+1}}{K_{c,t}} - 1 \right)^2 K_{c,t}, \qquad (2.B.10) \\
& K_{h,t+1} = (1 - \delta_{kh}) K_{h,t} + I_{kh,t} - \frac{\kappa_h}{2} \left(\frac{K_{h,t+1}}{K_{h,t}} - 1 \right)^2 K_{h,t}, \qquad (2.B.11) \\
& Y_{c,t} = \exp(z_t) K_{c,t}^\alpha (n_{c,t} \bar{N})^{1-\alpha}, \qquad (2.B.12) \\
& Y_{h,t} = \exp(\xi_t) K_{h,t}^\theta (n_{h,t} \bar{N})^\nu L_t^{1-\theta-\nu}, \qquad (2.B.13) \\
& L_t = 1, \quad \forall t. \qquad (2.B.14)
\end{aligned}
$$

Simplify the constraints and construct the Lagrangian:

$$
\begin{aligned}
\mathcal{L} = {} & (n_{c,t}+n_{h,t})\frac{1}{\eta}\ln\Big[\mu_c(C_t^E)^\eta + (1-\mu_c)(H^\gamma(N_{s,t}^E)^{1-\gamma})^\eta\Big] + (n_{c,t}+n_{h,t})\phi\ln(1-\bar{N}-N_{s,t}^E) \\
& + (1-n_{c,t}-n_{h,t})\frac{1}{\eta}\ln\Big[\mu_c(C_t^U)^\eta + (1-\mu_c)(H^\gamma(N_{s,t}^U)^{1-\gamma})^\eta\Big] + (1-n_{c,t}-n_{h,t})\phi\ln(1-N_{s,t}^U) \\
& + \beta\mathbb{E}_t\Big[V(z_{t+1},\xi_{t+1},K_{c,t+1},K_{h,t+1},H_{t+1},n_{c,t},n_{h,t})\Big|z_t,\xi_t\Big] \\
& + \lambda_{1,t}\Big[\exp(z_t)K_{c,t}^\alpha(n_{c,t}\bar{N})^{1-\alpha} - (n_{c,t}+n_{h,t})C_t^E - (1-n_{c,t}-n_{h,t})C_t^U \\
& - K_{c,t+1} + (1-\delta_{kc})K_{c,t} - K_{h,t+1} + (1-\delta_{kh})K_{h,t} - \frac{\kappa_c}{2}\left(\frac{K_{c,t+1}}{K_{c,t}}-1\right)^2 K_{c,t} \\
& - \frac{\kappa_h}{2}\left(\frac{K_{h,t+1}}{K_{h,t}}-1\right)^2 K_{h,t} - \frac{\omega}{2}\left(\frac{H_{t+1}}{H_t}-1\right)^2 H_t - \frac{\tau_c}{2}(n_{c,t}-n_{c,t-1})^2 \\
& - \frac{\tau_h}{2}(n_{h,t}-n_{h,t-1})^2\Big] + \lambda_{2,t}\Big[\exp(\xi_t)K_{h,t}^\theta(n_{h,t}\bar{N})^\nu - H_{t+1} + (1-\delta_h)H_t\Big]. \qquad (2.B.15)
\end{aligned}
$$

2.B.2 Optimality Conditions

I denote $V(t) \equiv V(z_t,\xi_t,K_{c,t},K_{h,t},H_t,n_{c,t-1},n_{h,t-1})$ as before.

The first-order conditions with respect to the choice variables are:

$$
K_{c,t+1}: \quad \beta\mathbb{E}_t\left[V'_{Kc}(t+1)\right] = \lambda_{1,t}\left[1+\kappa_c\left(\frac{K_{c,t+1}}{K_{c,t}}-1\right)\right], \qquad (2.B.16)
$$

$$
K_{h,t+1}: \quad \beta\mathbb{E}_t\left[V'_{Kh}(t+1)\right] = \lambda_{1,t}\left[1+\kappa_h\left(\frac{K_{h,t+1}}{K_{h,t}}-1\right)\right], \qquad (2.B.17)
$$

$$
H_{t+1}: \quad \beta\mathbb{E}_t\left[V'_H(t+1)\right] = \lambda_{1,t}\,\omega\left(\frac{H_{t+1}}{H_t}-1\right) + \lambda_{2,t}, \qquad (2.B.18)
$$

$$
\begin{aligned}
n_{c,t}: \quad \beta\mathbb{E}_t\left[V'_{nc}(t+1)\right] = {} & \frac{1}{\eta}\ln\left[\frac{\mu_c(C_t^U)^\eta + (1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^U)^{(1-\gamma)\eta}}{\mu_c(C_t^E)^\eta + (1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^E)^{(1-\gamma)\eta}}\right] + \phi\ln\left(\frac{1-N_{s,t}^U}{1-\bar{N}-N_{s,t}^E}\right) \\
& + \lambda_{1,t}\left[C_t^E - C_t^U + \tau_c(n_{c,t}-n_{c,t-1}) - \exp(z_t)(1-\alpha)K_{c,t}^\alpha(n_{c,t}\bar{N})^{-\alpha}\bar{N}\right], \qquad (2.B.19)
\end{aligned}
$$

$$n_{h,t}: \quad \beta \mathbb{E}_t \left[V'_{nh}(t+1) \right] = \frac{1}{\eta} \ln \left[\frac{\mu_c (C_t^U)^\eta + (1-\mu_c) H_t^{\gamma\eta} (N_{s,t}^U)^{(1-\gamma)\eta}}{\mu_c (C_t^E)^\eta + (1-\mu_c) H_t^{\gamma\eta} (N_{s,t}^E)^{(1-\gamma)\eta}} \right] + \phi \ln \left(\frac{1 - N_{s,t}^U}{1 - \bar{N} - N_{s,t}^E} \right)$$

$$+ \lambda_{1,t} \left[C_t^E - C_t^U + \tau_h (n_{h,t} - n_{h,t-1}) \right] - \lambda_{2,t} \exp(\xi_t) \nu K_{h,t}^\theta (n_{h,t} \bar{N})^{\nu - 1} \bar{N}, \tag{2.B.20}$$

$$C_t^E: \quad \frac{\mu_c (C_t^E)^{\eta-1}}{\mu_c (C_t^E)^\eta + (1-\mu_c) H_t^{\gamma\eta} (N_{s,t}^E)^{(1-\gamma)\eta}} = \lambda_{1,t}, \tag{2.B.21}$$

$$C_t^U: \quad \frac{\mu_c (C_t^U)^{\eta-1}}{\mu_c (C_t^U)^\eta + (1-\mu_c) H_t^{\gamma\eta} (N_{s,t}^U)^{(1-\gamma)\eta}} = \lambda_{1,t}, \tag{2.B.22}$$

$$N_{s,t}^E: \quad \frac{(1-\mu_c)(1-\gamma) H_t^{\gamma\eta} (N_{s,t}^E)^{(1-\gamma)\eta - 1}}{\mu_c (C_t^E)^\eta + (1-\mu_c) H_t^{\gamma\eta} (N_{s,t}^E)^{(1-\gamma)\eta}} = \frac{\phi}{1 - \bar{N} - N_{s,t}^E}, \tag{2.B.23}$$

$$N_{s,t}^U: \quad \frac{(1-\mu_c)(1-\gamma) H_t^{\gamma\eta} (N_{s,t}^U)^{(1-\gamma)\eta - 1}}{\mu_c (C_t^U)^\eta + (1-\mu_c) H_t^{\gamma\eta} (N_{s,t}^U)^{(1-\gamma)\eta}} = \frac{\phi}{1 - N_{s,t}^U}, \tag{2.B.24}$$

$$\lambda_{1,t}: \quad (n_{c,t} + n_{h,t}) C_t^E + (1 - n_{c,t} - n_{h,t}) C_t^U + K_{c,t+1} - (1-\delta_{kc}) K_{c,t} + K_{h,t+1} - (1-\delta_{kh}) K_{h,t}$$

$$= \exp(z_t) K_{c,t}^\alpha (n_{c,t} \bar{N})^{1-\alpha} - \frac{\kappa_c}{2} \left(\frac{K_{c,t+1}}{K_{c,t}} - 1 \right)^2 K_{c,t} - \frac{\kappa_h}{2} \left(\frac{K_{h,t+1}}{K_{h,t}} - 1 \right)^2 K_{h,t}$$

$$- \frac{\omega}{2} \left(\frac{H_{t+1}}{H_t} - 1 \right)^2 H_t - \frac{\tau_c}{2} (n_{c,t} - n_{c,t-1})^2 - \frac{\tau_h}{2} (n_{h,t} - n_{h,t-1})^2, \tag{2.B.25}$$

$$\lambda_{2,t}: \quad H_{t+1} - (1-\delta_h) H_t = \exp(\xi_t) K_{h,t}^\theta (n_{h,t} \bar{N})^\nu. \tag{2.B.26}$$

Applying the Envelope theorem to the value function yields

$$V'_{Kc}(t) = \lambda_{1,t} \left[1 - \delta_{kc} + \frac{\kappa_c}{2} \left(\left(\frac{K_{c,t+1}}{K_{c,t}} \right)^2 - 1 \right) + \exp(z_t) \alpha K_{c,t}^{\alpha-1} (n_{c,t} \bar{N})^{1-\alpha} \right], \tag{2.B.27}$$

$$V'_{Kh}(t) = \lambda_{1,t} \left[1 - \delta_{kh} + \frac{\kappa_h}{2} \left(\left(\frac{K_{h,t+1}}{K_{h,t}} \right)^2 - 1 \right) \right] + \lambda_{2,t} \exp(\xi_t) \theta K_{h,t}^{\theta-1} (n_{h,t} \bar{N})^\nu, \tag{2.B.28}$$

$$V'_H(t) = \frac{(n_{c,t}+n_{h,t})(1-\mu_c)\gamma H_t^{\gamma\eta-1}(N_{s,t}^E)^{(1-\gamma)\eta}}{\mu_c(C_t^E)^\eta + (1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^E)^{(1-\gamma)\eta}} + \frac{(1-n_{c,t}-n_{h,t})(1-\mu_c)\gamma H_t^{\gamma\eta-1}(N_{s,t}^U)^{(1-\gamma)\eta}}{\mu_c(C_t^U)^\eta + (1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^U)^{(1-\gamma)\eta}} + \lambda_{1,t}\frac{\omega}{2}\left(\left(\frac{H_{t+1}}{H_t}\right)^2 - 1\right) + \lambda_{2,t}(1-\delta_h), \tag{2.B.29}$$

$$V'_{nc}(t) = \lambda_{1,t}\ \tau_c(n_{c,t} - n_{c,t-1}), \tag{2.B.30}$$

$$V'_{nh}(t) = \lambda_{1,t}\ \tau_h(n_{h,t} - n_{h,t-1}). \tag{2.B.31}$$

Combining the first order conditions (2.B.16) - (2.B.26) and the Envelope theorem conditions (2.B.27) - (2.B.31), we obtain the following conditions from which the optimal decision rules can be solved.

$$\lambda_{1,t}\left[1+\kappa_c\left(\frac{K_{c,t+1}}{K_{c,t}}-1\right)\right] = \beta\mathbb{E}_t\left[\lambda_{1,t+1}\left((1-\delta_{kc}) + \frac{\kappa_c}{2}\left(\left(\frac{K_{c,t+2}}{K_{c,t+1}}\right)^2 - 1\right) + \exp(z_{t+1})\alpha K_{c,t+1}^{\alpha-1}(n_{c,t+1}\bar{N})^{1-\alpha}\right)\right], \tag{2.B.32}$$

$$\lambda_{1,t}\left[1+\kappa_h\left(\frac{K_{h,t+1}}{K_{h,t}}-1\right)\right] = \beta\mathbb{E}_t\left[\lambda_{1,t+1}\left((1-\delta_{kh}) + \frac{\kappa_h}{2}\left(\left(\frac{K_{h,t+2}}{K_{h,t+1}}\right)^2 - 1\right)\right) + \lambda_{2,t+1}\exp(\xi_{t+1})\theta K_{h,t+1}^{\theta-1}(n_{h,t+1}\bar{N})^{\nu}\right], \tag{2.B.33}$$

$$\lambda_{1,t}\omega\left(\frac{H_{t+1}}{H_t}-1\right)+\lambda_{2,t}$$

$$=\beta\mathbb{E}_t\Bigg[\frac{(n_{c,t+1}+n_{h,t+1})(1-\mu_c)\gamma H_{t+1}^{\gamma\eta-1}(N_{s,t+1}^E)^{(1-\gamma)\eta}}{\mu_c(C_{t+1}^E)^\eta+(1-\mu_c)H_{t+1}^{\gamma\eta}(N_{s,t+1}^E)^{(1-\gamma)\eta}}$$

$$+\frac{(1-n_{c,t+1}-n_{h,t+1})(1-\mu_c)\gamma H_{t+1}^{\gamma\eta-1}(N_{s,t+1}^U)^{(1-\gamma)\eta}}{\mu_c(C_{t+1}^U)^\eta+(1-\mu_c)H_{t+1}^{\gamma\eta}(N_{s,t+1}^U)^{(1-\gamma)\eta}}$$

$$+\lambda_{1,t+1}\frac{\omega}{2}\left(\left(\frac{H_{t+2}}{H_{t+1}}\right)^2-1\right)+\lambda_{2,t+1}(1-\delta_h)\Bigg], \tag{2.B.34}$$

$$\beta\mathbb{E}_t\left[\lambda_{1,t+1}\tau_c(n_{c,t+1}-n_{c,t})\right]$$

$$=\frac{1}{\eta}\ln\left[\frac{\mu_c(C_t^U)^\eta+(1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^U)^{(1-\gamma)\eta}}{\mu_c(C_t^E)^\eta+(1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^E)^{(1-\gamma)\eta}}\right]+\phi\ln\left(\frac{1-N_{s,t}^U}{1-\bar{N}-N_{s,t}^E}\right)$$

$$+\lambda_{1,t}\left[C_t^E-C_t^U+\tau_c(n_{c,t}-n_{c,t-1})-\exp(z_t)(1-\alpha)K_{c,t}^\alpha(n_{c,t}\bar{N})^{-\alpha}\bar{N}\right], \tag{2.B.35}$$

$$\beta\mathbb{E}_t\left[\lambda_{1,t+1}\,\tau_h(n_{h,t+1}-n_{h,t})\right]$$

$$=\frac{1}{\eta}\ln\left[\frac{\mu_c(C_t^U)^\eta+(1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^U)^{(1-\gamma)\eta}}{\mu_c(C_t^E)^\eta+(1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^E)^{(1-\gamma)\eta}}\right]+\phi\ln\left(\frac{1-N_{s,t}^U}{1-\bar{N}-N_{s,t}^E}\right)$$

$$+\lambda_{1,t}\left[C_t^E-C_t^U+\tau_h(n_{h,t}-n_{h,t-1})\right]-\lambda_{2,t}\exp(\xi_t)\nu K_{h,t}^\theta(n_{h,t}\bar{N})^{\nu-1}\bar{N}, \tag{2.B.36}$$

$$\lambda_{1,t}=\frac{\mu_c(C_t^E)^{\eta-1}}{\mu_c(C_t^E)^\eta+(1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^E)^{(1-\gamma)\eta}}, \tag{2.B.37}$$

$$\lambda_{1,t}=\frac{\mu_c(C_t^U)^{\eta-1}}{\mu_c(C_t^U)^\eta+(1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^U)^{(1-\gamma)\eta}}, \tag{2.B.38}$$

$$\frac{\phi}{1-\bar{N}-N_{s,t}^E}=\frac{(1-\mu_c)(1-\gamma)H_t^{\gamma\eta}(N_{s,t}^E)^{(1-\gamma)\eta-1}}{\mu_c(C_t^E)^\eta+(1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^E)^{(1-\gamma)\eta}}, \tag{2.B.39}$$

$$\frac{\phi}{1-N_{s,t}^U}=\frac{(1-\mu_c)(1-\gamma)H_t^{\gamma\eta}(N_{s,t}^U)^{(1-\gamma)\eta-1}}{\mu_c(C_t^U)^\eta+(1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^U)^{(1-\gamma)\eta}}, \tag{2.B.40}$$

$$(n_{c,t}+n_{h,t})C_t^E+(1-n_{c,t}-n_{h,t})C_t^U+K_{c,t+1}-(1-\delta_{kc})K_{c,t}+K_{h,t+1}-(1-\delta_{kh})K_{h,t}$$

$$=\exp(z_t)K_{c,t}^\alpha(n_{c,t}\bar{N})^{1-\alpha}-\frac{\kappa_c}{2}\left(\frac{K_{c,t+1}}{K_{c,t}}-1\right)^2K_{c,t}-\frac{\kappa_h}{2}\left(\frac{K_{h,t+1}}{K_{h,t}}-1\right)^2K_{h,t}$$

$$-\frac{\omega}{2}\left(\frac{H_{t+1}}{H_t}-1\right)^2H_t-\frac{\tau_c}{2}(n_{c,t}-n_{c,t-1})^2-\frac{\tau_h}{2}(n_{h,t}-n_{h,t-1})^2, \tag{2.B.41}$$

$$H_{t+1}-(1-\delta_h)H_t=\exp(\xi_t)K_{h,t}^\theta(n_{h,t}\bar{N})^\nu. \tag{2.B.42}$$

These optimality conditions characterize the equilibrium in the economy.

2.B.3 Steady State

In steady state, all variables do not evolve over time. In addition, the stochastic shock variables are in their mean values, $z_t = z_{t+1} = 0$, $\xi_t = \xi_{t+1} = 0$. Therefore the optimal conditions (2.B.32) - (2.B.42) can be simplified to the following equation system:

$$1 = \beta \left[\alpha K_c^{\alpha-1}(n_c\bar{N})^{1-\alpha} + 1 - \delta_{kc} \right], \tag{2.B.43}$$

$$\lambda_1 = \beta \left[\lambda_1(1-\delta_{kh}) + \lambda_2 \theta K_h^{\theta-1}(n_h\bar{N})^{\nu} \right], \tag{2.B.44}$$

$$\begin{aligned} \lambda_2 = \beta \Bigg[& \frac{(n_c+n_h)(1-\mu_c)\gamma H^{\gamma\eta-1}(N_s^E)^{(1-\gamma)\eta}}{\mu_c(C^E)^\eta + (1-\mu_c)H^{\gamma\eta}(N_s^E)^{(1-\gamma)\eta}} \\ & + \frac{(1-n_c-n_h)(1-\mu_c)\gamma H^{\gamma\eta-1}(N_s^U)^{(1-\gamma)\eta}}{\mu_c(C^U)^\eta + (1-\mu_c)H^{\gamma\eta}(N_s^U)^{(1-\gamma)\eta}} + \lambda_2(1-\delta_h) \Bigg], \end{aligned} \tag{2.B.45}$$

$$\begin{aligned} \lambda_1 \left[(1-\alpha)K_c^{\alpha}(n_c\bar{N})^{-\alpha}\bar{N} - C^E + C^U \right] = & \frac{1}{\eta} \ln \left[\frac{\mu_c(C^U)^\eta + (1-\mu_c)H^{\gamma\eta}(N_s^U)^{(1-\gamma)\eta}}{\mu_c(C^E)^\eta + (1-\mu_c)H^{\gamma\eta}(N_s^E)^{(1-\gamma)\eta}} \right] \\ & + \phi \ln \left(\frac{1-N_s^U}{1-\bar{N}-N_s^E} \right), \end{aligned} \tag{2.B.46}$$

$$\begin{aligned} \lambda_2 \nu K_h^{\theta}(n_h\bar{N})^{\nu-1}\bar{N} - \lambda_1(C^E - C^U) = & \frac{1}{\eta} \ln \left[\frac{\mu_c(C^U)^\eta + (1-\mu_c)H^{\gamma\eta}(N_s^U)^{(1-\gamma)\eta}}{\mu_c(C^E)^\eta + (1-\mu_c)H^{\gamma\eta}(N_s^E)^{(1-\gamma)\eta}} \right] \\ & + \phi \ln \left(\frac{1-N_s^U}{1-\bar{N}-N_s^E} \right), \end{aligned} \tag{2.B.47}$$

$$\lambda_1 = \frac{\mu_c(C^E)^{\eta-1}}{\mu_c(C^E)^\eta + (1-\mu_c)H^{\gamma\eta}(N_s^E)^{(1-\gamma)\eta}}, \tag{2.B.48}$$

$$\lambda_1 = \frac{\mu_c(C^U)^{\eta-1}}{\mu_c(C^U)^\eta + (1-\mu_c)H^{\gamma\eta}(N_s^U)^{(1-\gamma)\eta}}, \tag{2.B.49}$$

$$\frac{\phi}{1-\bar{N}-N_s^E} = \frac{(1-\mu_c)(1-\gamma)H^{\gamma\eta}(N_s^E)^{(1-\gamma)\eta-1}}{\mu_c(C^E)^\eta + (1-\mu_c)H^{\gamma\eta}(N_s^E)^{(1-\gamma)\eta}}, \tag{2.B.50}$$

$$\frac{\phi}{1-N_s^U} = \frac{(1-\mu_c)(1-\gamma)H^{\gamma\eta}(N_s^U)^{(1-\gamma)\eta-1}}{\mu_c(C^U)^\eta + (1-\mu_c)H^{\gamma\eta}(N_s^U)^{(1-\gamma)\eta}}, \tag{2.B.51}$$

$$(n_c+n_h)C^E + (1-n_c-n_h)C^U + \delta_{kc}K_c + \delta_{kh}K_h = K_c^\alpha(n_c\bar{N})^{1-\alpha} \tag{2.B.52}$$

$$\delta_h H = K_h^\theta(n_h\bar{N})^\nu. \tag{2.B.53}$$

2.B.4 Price Functions

The optimal resource allocation equalizes the marginal product of labor (or capital) across the two sectors. Based on this relationship, the new home sales price must satisfy

$$P_{Ih,t} = \frac{\exp(z_t)(1-\alpha)K_{c,t}^\alpha(n_{c,t}\bar{N})^{-\alpha}}{\exp(\xi_t)\nu K_{h,t}^\theta(n_{h,t}\bar{N})^{\nu-1}}. \tag{2.B.54}$$

The home rents must solve the following optimal consumption choice problem in the decentralized economy:

$$\begin{aligned}\max_{C_t^E, C_t^U, H_t} \Bigg\{ & (n_{c,t}+n_{h,t})\frac{1}{\eta}\ln\left[\mu_c(C_t^E)^\eta + (1-\mu_c)(H^\gamma(N_{s,t}^E)^{1-\gamma})^\eta\right] + (n_{c,t}+n_{h,t})\phi\ln(1-\bar{N}-N_{s,t}^E) \\ & + (1-n_{c,t}-n_{h,t})\frac{1}{\eta}\ln\left[\mu_c(C_t^U)^\eta + (1-\mu_c)(H^\gamma(N_{s,t}^U)^{1-\gamma})^\eta\right] + (1-n_{c,t}-n_{h,t})\phi\ln(1-N_{s,t}^U) \\ & + \beta\mathbb{E}_t\left[V(z_{t+1},\xi_{t+1},K_{c,t+1},K_{h,t+1},H_{t+1},n_{c,t},n_{h,t})\Big|z_t,\xi_t\right]\Bigg\} \end{aligned} \tag{2.B.55}$$

$$\text{s.t.} \quad (n_{c,t}+n_{h,t})C_t^E + (1-n_{c,t}-n_{h,t})C_t^U + R_{h,t}H_t \le (n_{c,t}+n_{h,t})w_t + \text{Non-Labor Income} \tag{2.B.56}$$

According to the first-order conditions, it's easy to compute the home rents as

$$R_{h,t} = \frac{\frac{\phi\gamma}{(1-\gamma)H_t}\left[(n_{c,t}+n_{h,t})\frac{N_{s,t}^E}{1-\bar{N}-N_{s,t}^E} + (1-n_{c,t}-n_{h,t})\frac{N_{s,t}^U}{1-N_{s,t}^U}\right]}{\frac{\mu_c(C_t^E)^{\eta-1}}{\mu_c(C_t^E)^\eta+(1-\mu_c)H_t^{\gamma\eta}(N_{s,t}^E)^{(1-\gamma)\eta}}}. \tag{2.B.57}$$

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