Well-posedness and Stability for Nonlinear Schrödinger Equations with Dynamic/Wentzell Boundary Conditions

Christopher Grant Lefler
Reston, Virginia

Bachelor of Science, College of Creative Studies,
University of California - Santa Barbara, 2007

A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia

May, 2014

Acknowledgements

I would like to thank my research collaborators Marcelo Cavalcanti, Wellington Corrêa, and most especially my advisor Irena Lasiecka, without whom this thesis could not have been written. I would also like to thank Roberto Triggiani for in many ways serving as a secondary advisor and for being the one to point me in the direction of the study of Wentzell boundary conditions. I would also like to thank Zoran Grujic for always having the patience to spend his time serving as a sounding board for my ideas. I cannot overstate my gratitude for all the things the many ways these individuals have helped me personally and professionally.

I wish also to thank my undergraduate advisor Charles Akemann who played a crtical role in steering me towards mathematics and analysis in particular, and for always keeping an open door for whenever I came in to chat. I would like to thank Paul Gendron for believing in me enough to take me to sea on his scientific experiments and for his support and understanding of my atheltic commitments.

Finally, I would like to thank my father and mother, Mike and Joan Lefler, and my brother, Jonathan Lefler, for always believing in me even during times when I didn't believe in myself. Thank you for your love and support. I dedicate this thesis to you.

Contents

Chapter 1

Introduction

1.1 Defining the Objectives of this Thesis

In this thesis the issues of well-posedness and stability for semilinear Schrödinger equations with time dependent boundary conditions of the form $\frac{\partial y}{\partial n} = -y_t$ are studied. Here n represents the unit normal vector on the boundary of a connected, bounded domain in $\mathbb{R}^N$ with smooth boundary for dimensions $N = 2, 3$. Before discussing the details of the model, a brief overview of the objectives of this research is given in the most general terms possible.

The issue of well-posedness is the most essential question in the theoretical study of differential equations. This issue consists of three parts:

1. Does a solution exist?
2. Is the solution uniquely determined?
3. Does the solution depend continuously on the initial data?

The question of existence of solutions is the most fundamental. Mathematically speaking, it does not make sense to address and question about behavior of solutions (e.g. uniqueness or stability) without existence first being established. This is one area in which the theoretical study of mathematics differs from applied studies in which solutions to physically observable problems are studied and therefore must exist as long as the problem is correctly modeled. The study of well-posedness is therefore very abstract in nature. However, well-posedness remains essential to applications. Without uniqueness a solution may split into two or more separate solution paths. Without continuous dependence on the initial data, a solution that is not continuous in time may "skip" similar to a song being played by a broken record player. In the study of numerical modeling schemes, well-posedness is essential (though neither necessary nor sufficient) for the stability of computer algorithms. Ill-posed problems must be reformulated with additional assumptions.

An important functional analysis technique for establishing well-posedness is to show that the differential operator generates a semigroup. Thus, a solution $y(t)$ can be written

$$y(t) = [S(t)](y_0)$$

where $S(t)$ denotes the evolution operator for the differential equation.

Finally, we are interested in the long time behavior of solutions. A solution is said to be stable if it remains within a neighborhood of a given orbit. The solution is asymptotically stable if it converges to that orbit. If a solution is asymptotically stable, how fast does it converge? If it is unstable, then does the solution simply never settle around a particular orbit, or does the energy of the system increase ad infinitum?

1.2 Introducing the Model

The goal of this thesis is to establish well-posedness and exponential stability of the following non-linear Schrodinger equation with dynamic boundary conditions:

$$\begin{cases} y_t = i\Delta y - i|y|^2 y & \text{in } \Omega \times (0,\infty) \\ y = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial y}{\partial n} = -y_t & \text{on } \Gamma_1 \times (0,\infty) \end{cases} \tag{1.2.1}$$

where $\Omega \subset \mathbb{R}^N$ is bounded in dimension $N = 2, 3$. The boundary of Ω is assumed to be comprised of two smooth, closed, disjoint pieces Γ_0 and Γ_1, both of which have non-empty interiors.

To study the nonlinear model, we first establish well-posedness of the following linear model:

$$\begin{cases} y_t = i\Delta y & \text{in } \Omega \times (0,\infty) \\ y = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial y}{\partial n} = -y_t & \text{on } \Gamma_1 \times (0,\infty) \end{cases} \tag{1.2.2}$$

This is done by treating the above problem as a Wentzell problem, i.e. by substituting $i\Delta y$ for y_t on the boundary. We note that the argument for well-posedness of the linear model is independent of the dimension of the space.

A fixed point method will be used to ultimately show well-posedness of the non-linear model. Here, global existence of solutions is achieved in dimension $N = 2$; however, in dimension $N = 3$ only local existence of solutions can be achieved. One of the many delicacies of this problem becomes apparent here: the fixed point method is done by treating (1.2.1) as a dynamic problem, rather than using the Wentzell formulation used for the linear problem (1.2.2).

The existence of weak solutions to (1.2.1) are also established using a Galerkin method. The virtue of these weak solutions is that they are global in time in both dimensions $N = 2$ and $N = 3$.

And finally, the long time behavior of (1.2.1) is studied via classical methods that are used to demonstrate exponential stability of solutions: that is to say that solutions are asymptotically stable with exponential decay rates.

More specifics about the results of this thesis will be elaborated on in Chapter 2.

1.3 Physical Interpretation

Semilinear Schrödinger equations have been studied extensively due to their applications to plasma physics and laser optics. The cubic nonlinear term is of particular interest to the physics community as a naturally occuring potential energy term. To make sense of the boundary condition being imposed, we must look at how the Schrodinger equation is derived.

As opposed to other well known differential equations arising in physics, the Schrodinger equation cannot be derived from first physical principles. To each elementary particle we ascribe a de Broglie wave function

$$\Psi(x,t) = Ae^{(p\cdot x - Et)/\hbar} \tag{1.3.1}$$

where p represents momentum and E represents energy. Physically, the wave function is not observable and must be interpreted through a philosophical framework, however, the square of the amplitude of the wave function for a particular state gives rise to a probability distribution that the particle will be observed in that state:

$$\rho(x,t) = \frac{|\Psi(x,t)|^2}{\displaystyle\int_\Omega |\Psi(x,t)|^2 \, d\Omega}.$$

More generally, it can be stated that this wave function contains all the information that can be known about the particle. And furthermore, it is a fundamental postulate of quantum mechanics that all the variables of the wave function can be represented as linear Hermitian operators and that any measurement of a variable will be an eigenvalue of the corresponding operator. Thus, we may solve for energy by differentiating in the time variable

$$-\frac{iE}{\hbar}\frac{d}{dt}\Psi = E\Psi.$$

We can likewise solve for the kinetic energy of the system by observing that

$$-i\hbar\nabla\Psi = p\Psi$$

and

$$KE = \frac{p \cdot p}{2m} = -\frac{\hbar^2}{2m}\Delta\Psi$$

where m is the mass of the particle. The Schrodinger equation is then derived by observing that

$$\text{Total Energy } = (\text{Kinetic Energy}) + (\text{Potential Energy})\,.$$

Our insight into the boundary condition is as follows: the normal derivative of the wave function Ψ on the boundary is proportional to the momentum, while the time derivative of Ψ is proportional to the energy. Thus, rather than interpreting y_t on the boundary as a velocity feedback (as in, for examle, the wave equation), it should be interpreted as a dissipative energy feedback relation.

1.4 Wentzell Boundary Conditions

Important to the study of the Schrödinger problem (1.2.1) is the treatment of the linear problem (1.2.2) as a Wentzell problem (see also Venttsel). Wentzell was interested in studying the problem of the most general boundary conditions which restrict a second order diffusive elliptic operator to the infinitestimal generator of a positive contraction semigroup on the space of continuous functions over the domain. Let Ω be a bounded region in $\mathbb{R}^N$ with smooth boundary Γ. The result of this work was the discovery of the generalized Wentzell boundary condition

$$\alpha\Delta y + \beta\frac{\partial y}{\partial n} + \gamma y = 0 \text{ on } \Gamma \tag{1.4.1}$$

carried the desired property (1959) for $\alpha > 0, \beta \geq 0, \gamma \geq 0$ [45]. Here, Δ on the boundary should be interpreted as the Laplacian coming from the interior:

$$\frac{\partial^2 y}{\partial n^2} + \frac{\partial y}{\partial n}(\text{div } n) + \sum_{i=1}^{N-1}\frac{\partial^2 y}{\partial \tau_i^2} = \Delta y \text{ on } \Gamma. \tag{1.4.2}$$

Physically, this boundary condition can be interpreted as a (damped) harmonic oscillator acting at each point on the boundary. In the case of the heat equation, this means that the boundary can act as a heat source or sink depending on physical conditions. These boundary conditions also arise naturally in the study of the wave equation. In particular, generalized Wentzell boundary conditions can be thought of as a closed subclass of acoustic boundary conditions. This thesis marks the introduction of the use of Wentzell boundary conditions in the study of the Schrödinger equation.

A key aspect of Wentzell boundary conditions is their behavior at the resolvent level. Consider for example the heat operator $B = \Delta$ on Ω equipped with Wentzell boundary conditions as above. At the resolvent level,

$$\begin{cases} \lambda u - Bu = h \\ Bu + \beta \frac{\partial u}{\partial n} + \gamma u = 0 \end{cases} \tag{1.4.3}$$

may be rewritten

$$\begin{cases} \lambda u - Bu = h \\ \beta \frac{\partial u}{\partial n} + (\gamma + \lambda) u = h, \end{cases} \tag{1.4.4}$$

which is an elliptic problem with inhomogeneous Robin boundary conditions that can be solved, as seen in [17]. A notable difference with the Schrödinger equation is that we will take $\alpha = i$. This follows naturally from considering the operator framework

$$Ay = i\Delta y = y_t \tag{1.4.5}$$

and hence at the resolvent level we again have

$$\begin{cases} \lambda u - Au = h \\ \\ \beta \frac{\partial u}{\partial n} + (\gamma + \lambda) u = h. \end{cases} \tag{1.4.6}$$

Chapter 2

Summary of Results

2.1 Linear Theory

Stability of the linear model

$$\begin{cases} y_t = i\Delta y & \text{in } \Omega \times (0,\infty) \\ y = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial y}{\partial n} = -y_t & \text{on } \Gamma_1 \times (0,\infty) \end{cases} \tag{2.1.1}$$

for Ω bounded in $\mathbb{R}^N$ (where $N = 2,3$) has been known for some time; however, there is no known proof of well-posedness elsewhere in the literature. To develop a well-posedness theory of the nonlinear model (1.2.1), a theory must first be developed for the linear model. We introduce the operator A given by

$$A = i\Delta$$

with domain

$$D(A) = \left\{ y \in V, \Delta y \in V, \frac{\partial y}{\partial n} = -i\Delta|_{\Gamma_1} y \text{ on } \Gamma_1 \right\}.$$

We prove the following result:

Theorem 2.1.1. *The operator $(A, D(A))$ generates a C_0 semigroup of contractions on the space $V = H^1_{\Gamma_0}$.*

As noted, the introduced of this operator recasts the above linear problem with dynamic boundary condition as a Wentzell problem with Wentzell boundary condition given by $i\Delta y + \frac{\partial y}{\partial n} = 0$ on Γ_1. It is shown that this operative is dissipative on $H^1(\Omega)$, but not on $L^2(\Omega)$. Maximality is a nontrivial issue that does not follow directly from classical results. The Banach space

$$Z = \left\{ y \in V, \Delta y \in L^2(\Omega), \frac{\partial y}{\partial n} \in L^2(\Gamma_1) \right\}$$

which we equip with the norm

$$\|u\|_Z = \|u\|_V + \|\Delta u\|_{L^2(\Omega)} + \|\frac{\partial y}{\partial n}\|_{L^2(\Gamma_1)}$$

is introduced, on which it is shown that the operator A is continuous and coercive, thus allowing for semigroup generation through classical results.

Two critical details are needed to pass to the nonlinear model. First, the well-posedness theory must be extended to inhomogeneous problems. Second, additional regularity will be required to obtain a priori estimates needed to produce a fixed point argument. By Duhamel's formula, we can assert well-posedness of

$$\begin{cases} y_t - i\Delta y = f & \text{in } \Omega \times (0, \infty) \\ y = 0 & \text{on } \Gamma_0 \times (0, \infty) \\ \dfrac{\partial y}{\partial n} = -i\Delta y & \text{on } \Gamma_1 \times (0, \infty). \end{cases} \tag{2.1.2}$$

on V. Using this result, we can generalize to the problem

$$\begin{cases} y_t - i\Delta y = f & \text{in } \Omega \times (0,\infty) \\ y = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial y}{\partial n} + i\Delta y = g & \text{on } \Gamma_1 \times (0,\infty) \end{cases} \tag{2.1.3}$$

by defining an auxillary function $\tilde{y} = y - Ng$ where N is a Neumann map. We are then able to prove

Theorem 2.1.2. *Let $f \in L^1(0,\infty;V)$ and $g \in L^2(0,\infty;L^2(\Gamma_1))$. Then for each $y_0 \in V$ there exists a unique solution $y \in C(0,\infty;V)$ to (2.1.3).*

Furthermore, by taking $g = f|_{\Gamma_1}$, we may make the identification of the above Wentzell problem with the dynamic problem

$$\begin{cases} y_t - i\Delta y = f & \text{in } \Omega \\ y = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial y}{\partial n} + y_t = 0 & \text{on } \Gamma_1. \end{cases} \tag{2.1.4}$$

Then as a corollary to the above theorem,

Corollary 2.1.3. *Let $f \in L^2(0,\infty;V)$. Then for each $y_0 \in V$ there exists a unique solution $y \in C(0,\infty;V)$ to (2.1.4).*

The fixed point argument used to acquire well-posedness of the nonlinear model requires additional regularity, thus we seek the following result:

Theorem 2.1.4. *Let $y_0 \in D(A)$ and $f \in H^1(0,\infty,V)$. Then there exists a unique solution*

$$y \in C(0,\infty,H^2(\Omega)) \cap C^1(0,\infty,V)$$

to (2.1.4).

This result is obtained by differentiating (2.1.4) in time. Defining $z = y_t$, we study the equation

$$\begin{cases} z_t - i\Delta z = f_t & \text{in } \Omega \\ z = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial z}{\partial n} + z_t = 0 & \text{on } \Gamma_1 \end{cases} \tag{2.1.5}$$

which is well-posed on V by the previous corollary. Using elliptic regularity results we are able to obtain well-posedness of (2.1.4) on $H^2(\Omega)$.

We can in fact extend the linear theory to include Lipschitz perturbations (both on the interior and the boundary) with nonlinear boundary dissipation:

$$\begin{cases} y_t = i\Delta y + f(y) & \text{in } \Omega \times (0,\infty) \\ y = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial y}{\partial n} = -g(y_t) \text{ on } \Gamma_0 \times (0,\infty) & \\ y_0 \in V = H^1_{\Gamma_0} & \end{cases} \tag{2.1.6}$$

Here, we assume that $f(y) : H^1_{\Gamma_0}(\Omega) \longmapsto H^1_{\Gamma_0}(\Omega)$ and $h(y) : H^1_{\Gamma_0}(\Omega) \longmapsto L^2(\Gamma_1)$ are Lipschitz and we make the following assumptions on the boundary dissipation:

Assumption 2.1.1. *Assume that $g(z)$ is a continuous function on $\mathbb{C}$ such that both $g(z)$ and its inverse $g^{-1}(z)$ satisfy:*

(i) $Re(g(z) - g(v))(\bar{z} - \bar{v}) \geq m|z - v|^2$

(ii) $Re(g(z)) \geq m|z|^2$

(iii) $Im(g(z)\bar{z}) = 0$

(iv) $|g(z)| \leq M|z|$

for some constants $m, M \in \mathbb{R}_+$.

Nonlinear boundary feedback of this form appears in literature for wave and Schrödinger equations e.g. [23] and [24] respectively. In particular, assumptions (i) and (iii) form a complex analog to the assumption of monotonicity that appears in the study of wave equations.

This problem is solved using the same approach as the linear model. We define an operator A_f by

$$A_f y = i\Delta y + f(y) \tag{2.1.7}$$

with accompanying domain

$$D(A_f) = \left\{ y \in V, \Delta y \in V, \frac{\partial y}{\partial n} = -g(i\Delta|_{\Gamma_1} y + h(y)) \text{ on } \Gamma_1 \right\} \tag{2.1.8}$$

to which we apply the same method as before to obtain the result

Theorem 2.1.5. *The operator* $(A, D(A_f))$ *generates a strongly continuous semigroup.*

Unlike in the linear model, ω-maximal dissipativity is obtained for some value of ω that is sufficiently large. It can no longer be said that the semigroup is a contraction semigroup.

2.2 Nonlinear Theory

We return to the model of interest:

$$\begin{cases} y_t - i\Delta y = F(y) = -i|y|^2 y & \text{in } \Omega \\ y = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial y}{\partial n} + y_t = 0 & \text{on } \Gamma_1. \end{cases} \tag{2.2.1}$$

Define the spaces

$$X_0 = \left\{(y, z) \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) H^1_{\Gamma_0}(\Omega) : z = y_t\right\}$$

with norm

$$\|(y, z)\|_{X_0} = \|y\|_{H^2(\Omega)} + \|z\|_{H^1_{\Gamma_0}}(\Omega)$$

and

$$X_T = \left\{(y, z) : y \in C[0, T; H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)], z \in C(0, T; H^1_{\Gamma_0}(\Omega)), y_t = z\right\}$$

with norm

$$\|(y,z)\|_{X_T} = \sup_{t\in[0,T]} \|y\|_{H^2(\Omega)} + \sup_{t\in[0,T]} \|z\|_{H^1_{\Gamma_0}(\Omega)}.$$

Then we have the following well-posedness result:

Theorem 2.2.1. *For every bounded subset $B \subset X_0$, there exists $T > 0$ such that for all $(y_0, z_0) \in B$, there exists a unique solution y of (2.2.1) with time derivative $y_t = z$ such that the pair $(y, z) \in X_T$.*

Given the association $z = y_t$, we can rewrite the result $(y, z) \in X_T$ as

$$y(x,t) \in C[0,T; H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)] \cap C^1(0,T; H^1_{\Gamma_0}(\Omega)). \tag{2.2.2}$$

This result follows by way of fixed point argument. Fixed point arguments are commonly used in the study of semilinear Schrödinger equations frequently in the accompaniment of Strichartz estimates. Due to the inhomogeneous nature of the boundary condition on Γ_1, these types of estimates cannot be applied. However, we are able to use variational estimates. In particular, these estimates are performed both on $\|y\|_{H^2(\Omega)}$ and $\|z\|_V$. Use of estimates on $\|z\|_V$ are unique to this problem and highlights one of the challenges of this research. Whereas the linear problem was treated as a Wentzell problem, the nonlinear problem must be treated as a dynamic problem for the fixed point method to work.

The fixed point argument is done in space and time and thus is only able to provide a local existence theory. Global existence of solutions in two dimensions follows from the use of the Brezis-Gallouet inequality. The idea is that the cubic nonlinearity $|y|^2 y$ is almost Lipschitz on $H^2(\Omega)$. This allows us to find a bound on the growth rate of the form $Me^{\alpha e^{\beta t}}$, which in turn allows for the following theorem:

Theorem 2.2.2. *Suppose $N = 2$. For all $(y_0, z_0) \in X_0$ and for all $T > 0$, there exists a unique solution y of (2.2.1) with time derivative $y_t = z$ such that the pair $(y, z) \in X_T$.*

The problem of being unable to obtain a global well-posedness theory when $N = 3$ is typical in the literature. In particular, there is no global well-posedness theory in the literature for semilinear Schrödinger equations even for homogeneous Dirichlet and Neumann boundary conditions on bounded domains. Currently, the literature is focused on global existence results for weak solutions. We are able to provide a similar global existence result by the Galerkin approach. We define a weak solution of (2.2.1) as a solution to

$$i(y', v)_{L^2(\Omega)} - (\nabla y, \nabla v)_{L^2(\Omega)} + (y', v)_{L^2(\Gamma_1)} - (|y|^2 y, v)_{L^2(\Omega)} = 0, \; \forall t \in [0, \infty) \quad (2.2.3)$$

Note that since we obtain this result by solving a finite dimensional approximate problem for $y_m \in V_m \subset V = H^1_{\Gamma_0}(\Omega)$, the boundary condition $\frac{\partial y}{\partial n} + y_t = 0$ is not preserved. This prevents us from seeking strong solutions as was done by fixed point argument. Instead, we obtain one final existence result:

Theorem 2.2.3. *Let $y_0 \in V$. Then for all $v \in V$ there exists a solution $y \in C^1(0, \infty; V)$ to (2.2.3).*

2.3 Stability

As mentioned earlier, stability of the linear model was proved in dimensions $N = 2, 3$ by Machtyngier using the method of integrating against the multiplier $q(x) \cdot \nabla \bar{y}$. We are able to prove a similar result by the same method; however, currently existence of global regular solutions to (2.2.1) has only been proved in dimension $N = 2$ as seen above. We prove the following stability result:

Theorem 2.3.1 (Stabilization)**.** *Assume that Ω is star-shaped and let y be a regular solution of the problem (1.2.1). Then, there exist positive constants γ and C such that the H^1-energy associated to problem (1.2.1) decays exponentially, that is,*

$$E(t) \leq Ce^{-\gamma t}E(0), \quad \textit{for all } t > T_0,$$

$T_0 > 0$ *large enough.*

Chapter 3

Overview of the Literature

3.1 General Overview

Due to the dispersive nature of the Schrödinger equation, the research naturally separates into two distinct categories: results for the Schrödinger equation on $\mathbb{R}^N$ and results for bounded domains. The former has been well-studied. On $\mathbb{R}^N$the Schrödinger equation is self-regularizing. Indeed, it is well known that for Schrödinger equations with nonlinear component $k|y|^p y$, with $p > 0$, are globally well-posed on $\mathbb{R}^N$ in the defocusing case as long as $p < \frac{4}{N-2}$ and in the focusing case as long as $p < \frac{4}{N}$ ([13]). Recent studies have extended well-posedness to $L^r(\mathbb{R}^N)$ functions. Much of the theory for unbounded problems relies on the use of Strichartz estimates (1977) [40], which are of the general form

$$\|y(t)\|_{L_t^p L_x^q} \leq c\|y_0\|_{L^r}. \tag{3.1.1}$$

These results have since been generalized to inhomogeneous problems by Yajima (1987) [47] and by Cazenave and Weissler (1988) [12].

Few results exist that bridge the gap between Schrödinger problems in unbounded domains and Schrödinger problems in bounded domains. Strichartz estimates have only recently found application to bounded domains within the past decade. To the author's knowledge, the first result proved by Burq, Gerard, and Tzvetkov (2004) [8] was for compact boundaryless manifolds and came with some loss of derivatives, e.g. using bounds of the form $c\|y_0\|_{H^s}$. It has been proved that in some geometries this loss is unavoidable. Strichartz inequalities have been extended to domains with boundary by Anton (2008) [2] and more recently by Ozsari [35]. Providing a further complication, the time dynamic nature of the boundary condition in (1.2.1) prevents the consideration of classical Strichartz estimates although some similar variational estimates to the inhomogeneous case will be applied.

Several additional results are discussed below in greater detail. These following results have played essential roles in shaping the course of this thesis research.

3.2 Nonlinear Schrödinger Equations in 2D

The first known result for nonlinear Schrödinger equations on a bounded domain is due to Brezis and Gallouet (1980) [14]. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ. Then for initial condition $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique solution to the equation

$$\begin{cases} i\dfrac{dy}{dt} - \Delta y + k|y|^2 y = 0 & \text{in } \Omega \times [0,\infty) \\ \\ y(x,t) = 0 & \text{on } \Gamma \times [0,\infty) \end{cases} \tag{3.2.1}$$

such that $y \in C[0,\infty, H^2(\Omega)) \cap C^1[0,\infty, L^2(\Omega))$ provided either:

a) $k \geq 0$

b) $k < 0$ and $|k| \int |y_0|^2\, dx < 4$

Several key lemmas are used to prove this result, which will in turn be critical in acquiring global existence of strong solutions to this thesis problem in dimension $N = 2$. The first follows from what are now considered standard Sobolev space inequalities:

Lemma 3.2.1. *For every $y \in H^2(\Omega)$,*

$$\||y|^2 y\|_{H^2(\Omega)} \leq C\|y\|^2_{L^\infty(\Omega)}\|y\|_{H^2(\Omega)}.$$

More critical to this thesis is to Brezis-Gallouet inequality:

Lemma 3.2.2. *Let $\Omega \subset \mathbb{R}^2$. For every $y \in H^2(\Omega)$ such that $\|y\|_{H^1(\Omega)} \leq 1$,*

$$\|y\|_{L^\infty(\Omega)} \leq C(1 + \sqrt{\log(1 + \|y\|_{H^2(\Omega)})}). \tag{3.2.2}$$

3.3 Complications Arising in Higher Dimensions

More generally, suppose $\Omega \subset \mathbb{R}^n$ is an open set, bounded or unbounded, such that the boundary Γ (if indeed there is one) is C^∞ smooth. Then we may consider the inhomogeneous problem:

$$\begin{cases} i\dfrac{dy}{dt} = \Delta y - m|y|^{p-1}y & \text{in } \Omega \\ \\ y(t,x) = Q(t,x) & \text{on } \Gamma \end{cases} \tag{3.3.1}$$

Strauss and Bu [39] attempted to prove existence and uniqueness of solutions to this problem for $m > 0$ with initial data $y_0 \in H^1(\Omega)$ and the inhomogenous boundary condition $Q \in C^3(-\infty, \infty, \Omega)$. In the course of this thesis, a critical error in the existence portion of the proof was discovered. Central to the argument is the use of truncating functions on the nonlinear term $k|y|^{p-1}y$. Truncations of the form

$$f_k(y) = \begin{cases} m|y|^{p-1}y & |y| < k \\ \\ mk^{p-1}y & |y| \geq k \end{cases}$$

are utilized; however, while these truncations are Lipschitz on $L^2(\Omega)$, they are not Lipschitz on $H^1(\Omega)$. Taking the gradient reveals

$$\nabla f_k(y) = \begin{cases} (p-1)m|y|^{p-2}y\nabla|y| + m|y|^{p-1}\nabla y & |y| < k \\ \\ mk^{p-1}\nabla y & |y| \geq k \end{cases}$$

and since the term $(p-1)m|y|^{p-2}y\nabla|y|$ contributes nontrivially to the derivative, there is a jump discontinuity in the derivative along the spherical shell $|y| = k$. For a real valued problem one might consider the truncation

$$f_k(y) = \begin{cases} m|y|^{p-1}y & |y| < k \\ \\ mk^{p-1}\left[py + \dfrac{y}{|y|}(1-p)k\right] & |y| \geq k \end{cases}$$

however, the term $\dfrac{y}{|y|}$ again contributes nontrivially to the derivative for complex valued functions y. If we view the truncation $f_k(y)$ as a composition between a truncating function ϕ_k and the nonlinear term $f(y) = m|y|^{p-1}y$, the only way to avoid this jump discontinuity is for for the truncating function ϕ_k to be differentiable. For real-valued functions this is not a strong condition; for complex valued functions it is since differentiability implies analyticity.

This difficulty with truncationing functions played a critical role in shaping the course of this thesis work. Our original intention was to adapt the techniques pioneered by Lasiecka and Tataru [23] for the wave equation (1993) to the Schrödinger equation. But, as will be seen in the following chapter, the natural space to consider for Schrödinger equations with Wentzell boundary conditions is $H^1(\Omega)$. This forced us to consider different methods for studying the problem.

The proof of uniqueness of solutions is, however, correct. Let $e^{it\Delta_D}$ denote the evolution operator for the Schrödinger equation with homogeneous Dirichlet boundary conditions on Γ. Then if the following dispersive estimate:

$$\|e^{it\Delta_D}\|_{\mathcal{L}(L^1(\Omega),L^\infty(\Omega))} \leq \frac{C}{t^{n/2}} \tag{3.3.2}$$

holds and if $1 < p < 1 + \frac{4}{N-2}$, then solutions to (3.3.1) – if they exist – are unique. This result highlights the difficulty of establishing a general well-posedness for nonlinear Schrödinger equations in bounded domains. Specifically, this dispersive estimate is domain dependent and generally satisfied by unbounded domains such as $\mathbb{R}^N$, the half-plane, or in exterior domains of regular bounded sets.

3.4 Known Results on Bounded Domains in 3D

Well-posedness of Schrödinger equations in one and two dimensions has been well studied; however, there is no general well-posedness theory on bounded domains in three dimensions. The earliest result the author is aware of is for homogeneous Dirichlet boundary conditions due to Vladimirov (1986) [46]. Existence and uniqueness of solutions is proved under assumptions on the boundedness of the dissipation.

The study of existence of global solutions to nonlinear Schrödinger models in dimension $N \geq 3$ on bounded domains with inhomogeneous boundary conditions is more recent. Most of the literature on such models has centered around inhomogeneous Dirichlet boundary conditions. Currently, global existence of weak $H^1(\Omega)$ solutions in any dimension has been proved for defocusing Schrödinger equations with inhomogeneous Dirichlet boundary conditions by Ozsari (2011) [36]. Existence of global solutions to the focusing model was achieved by Ozsari in the following year [33] using hidden trace regularity for nonlinearities $|y|^p y$ in the case where $p \in (0, 4/(n+2))$. The author is only aware of results for inhomogeneous Neumann problems dating from within the past two years. Existence of solutions with Neumann boundary conditions has been obtained by Ozsari (2013) in the focusing case for $p > 0$ and in the focusing case for $p \in (0, 4/(n+2))$ [34]. Uniqueness and continuous dependence on the data are not well understood on bounded domains in dimension $N = 3$ or higher.

3.5 Stability of the Linear Model

Stability of nonlinear Schrödinger equations is much less delicate as techniques developed for the wave equation carry over more naturally. It has already been shown by Machtyngier (1990) ([37], [38]) that the linear model

$$\begin{cases} y_t = i\Delta y & \text{in } \Omega \times (0,\infty) \\ y = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial y}{\partial n} = -(m(x)\cdot n(x))y_t & \text{on } \Gamma_1 \times (0,\infty) \end{cases} \tag{3.5.1}$$

is exponentially stable, i.e., for every $C > 1$, there exists $\gamma > 0$ such that the energy decays exponentially:

$$E(t) \leq CE(0)e^{-\gamma t}. \tag{3.5.2}$$

This proof follows a well known method of multiplying by $q(x)\cdot\nabla\bar{y}$ under the integral. This requires the additional assumption that Ω is a star-shaped domain. This result of Machtyngier is extended to nonlinear Schrödinger equations in Chapter 7 of this thesis.

Chapter 4

Well-Posedness of the Linear Model

4.1 Recasting the Linear Problem as a Wentzell Problem

Consider the model

$$\begin{cases} y_t = i\Delta y & \text{in } \Omega \times (0,\infty) \\ y = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial y}{\partial n} = -y_t & \text{on } \Gamma_1 \times (0,\infty) \end{cases} \tag{4.1.1}$$

with Ω, Γ_0, and Γ_1 as above. Well-posedness of the linear problem requires careful consideration. The appearance of the principal part of the equation on the boundary prevents classical semigroup considerations. Instead, we define an operator A by

$$A = i\Delta$$

with domain

$$D(A) = \left\{ y \in V, \Delta y \in V, \frac{\partial y}{\partial n} = -i\Delta|_{\Gamma_1} y \text{ on } \Gamma_1 \right\}$$

where $V = H^1_{\Gamma_0}(\Omega)$. $\Delta|_{\Gamma_1}$ should be interpreted as the restriction of the Laplacian from the interior to the boundary.

The above operator formulation recasts (1.2.2) as a Wentzell problem:

$$\begin{cases} y_t = i\Delta y & \text{in } \Omega \times (0,\infty) \\ y = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial y}{\partial n} = -i\Delta y & \text{on } \Gamma_1 \times (0,\infty) \end{cases} \tag{4.1.2}$$

which we solve on the space V.

Several key points must be made. Classically, well-posedness of Wentzell problems for the heat equation is achieved on spaces of the form $X_p = L^p(\Omega) \cup L^p(\Gamma)$ [17]. This treats the problem as a coupled system of two PDE's: one acting on the interior and one acting on the boundary. We skirt this issue by incorporating the boundary condition into the domain of the operator. However, semigroup generation of the operator A is not obvious. On the space $L^2(\Omega)$, the operator A is not dissipative:

$$(Ay, y)_{L^2(\Omega)} = (i\Delta y, y)_{L^2(\Omega)} = -i(\nabla y, \nabla y)_{L^2(\Omega)} + i\left(\frac{\partial y}{\partial n}, y\right)_{L^2(\Gamma_1)} \tag{4.1.3}$$

hence,

$$\mathrm{Re}(Ay, y)_{L^2(\Omega)} = \mathrm{Re}(-\Delta y, y)_{L^2(\Gamma_1)}. \tag{4.1.4}$$

This means we cannot use $L^2(\Omega)$ energy estimates.

4.2 Dissipativity on $H^1_{\Gamma_0}$

To the space V, we apply the gradient norm via Poincaré. On V dissipativity holds:

$$(\nabla Ay, \nabla y)_{L^2(\Omega)} = (i\nabla \Delta y, \nabla y)_{L^2(\Omega)} = -i(\Delta y, \Delta y)_{L^2(\Omega)} + i\left(\Delta y, \frac{\partial y}{\partial n}\right)_{L^2(\Gamma_1)} \tag{4.2.1}$$

whereby substituting $\dfrac{\partial y}{\partial n} = -i\Delta y$ on the boundary, we achieve:

$$(\nabla Ay, \nabla y)_{L^2(\Omega)} = i\|\Delta y\|_{L^2(\Omega)} - \left\|\frac{\partial y}{\partial n}\right\|_{L^2(\Gamma_1)} \tag{4.2.2}$$

hence,

$$\mathrm{Re}(\nabla Ay, \nabla y)_{L^2(\Omega)} \leq 0.$$

Maximality remains an issue. If we define a bilinear form

$$a(y, v) = (-Ay + \lambda y, v)_V \tag{4.2.3}$$

we discover that it is not continuous on V. Moreover, there is no space of the form $H^s(\Omega)$ on which it is both continuous and coercive.

4.3 Maximality: Choosing the Correct Space

We introduce the space

$$Z = \left\{ y \in V, \Delta y \in L^2(\Omega), \frac{\partial y}{\partial n} \in L^2(\Gamma_1) \right\}$$

which we equip with the norm

$$\|u\|_Z = \|u\|_V + \|\Delta u\|_{L^2(\Omega)} + \left\|\frac{\partial y}{\partial n}\right\|_{L^2(\Gamma_1)}.$$

Lemma 4.3.1. *The space Z is Banach.*

Proof. It needs to be shown that Z is complete. Let

$$\begin{cases} z_n \to z \text{ in } H^1_{\Gamma_0}(\Omega) \\ \Delta z_n \to y \text{ in } L^2(\Omega) \\ \frac{\partial}{\partial n} z_n \to w \text{ in } L^2(\Gamma_1) \end{cases} \tag{4.3.1}$$

It needs to be shown that $v = \Delta z$ and $w = \frac{\partial z}{\partial n}$. The first follows since the operator $(\Delta, D(\Delta) = H^1_{\Gamma_0})$ is densely defined on $H^{-1}(\Omega)$, hence by closeability

$$\Delta z_n = \Delta z \text{ in } H^{-1}(\Omega) \tag{4.3.2}$$

For the latter, observe that if $z \in V$ is a solution of the elliptic problem then $w = \frac{\partial z}{\partial n} \in H^{-1/2}(\Gamma_1)$ follows from trace theory. However, $\frac{\partial z_n}{\partial n} \to w$ in $L^2(\Gamma_1)$ thus $\frac{\partial z}{\partial n} = w$ in $L^2(\Gamma_1)$, thus the desired result. □

We wish to invoke the Browder-Minty theorem ([4], Ch. 5) to show that for any fixed $f \in V$, there exists a unique weak solution $y \in V$ satisfying

$$a(y, v) = (-f, v)_V$$

for all $v \in V$. This is done by showing that $a(y, v)$ is continuous and coercive on Z. Observe that

$$\begin{aligned} a(y, v) =& -i(\Delta y, v)_V + (y, v)_V \\ =& i(\Delta y, \Delta v)_{L^2(\Omega)} - i\left(\Delta y, \frac{\partial v}{\partial n}\right)_{L^2(\Gamma_1)} + (y, v)_V \end{aligned} \tag{4.3.3}$$

whereby the triangle inequality,

$$|a(y,v)| \leq |(\Delta y, \Delta v)_{L^2(\Omega)}| + |(\Delta y, \frac{\partial v}{\partial n}_{L^2(\Gamma_1)} + |\lambda||(y,v)_V|. \tag{4.3.4}$$

Applying Cauchy-Schwarz to each of the respective inner products yields

$$|a(y,v)| \leq C(\lambda)\|y\|_Z\|v\|_Z \tag{4.3.5}$$

which proves continuity. For coercivity,

$$|a(y,y)| = \left|\lambda\|y\|_V + i\|\Delta y\|_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial n}\right\|_{L^2(\Gamma_1)}\right| \geq C(\lambda)\|y\|_Z^2. \tag{4.3.6}$$

We conclude from the Browder-Minty theorem that for all $f \in Z'$, where Z' denotes the dual spacce of Z, that there is a solution $y \in Z$ to $a(y,v) = (-f,v)_V$. Moreover, we observe that $D(A) \subset Z \subset V \subset Z'$, hence for all $f \in V$ there is a solution $y \in Z \subset V$. Furthermore, if

$$i\Delta y - \lambda y = f \in V$$

then $\Delta y \in V$, hence $y \in D(A)$. And moreover, $\Delta y \in V$ implies that $\Delta|_{\Gamma_1} y \in H^{1/2}(\Gamma_1)$ and thus $\frac{\partial v}{\partial n} \in H^{1/2}(\Gamma_1)$ as well. Trace theory tells us that $y \in H^2(\Omega)$, thus we know that the regularity of $D(A)$ is at least $H^2(\Omega)$. We are now in a position to apply Lumer-Phillips to get the following result:

Theorem 4.3.2. *The operator $(A, D(A))$ generates a C_0 semigroup of contractions on the space $V = H^1_{\Gamma_0}$.*

Thus, for any $y_0 \in V$ we can write

$$y(t) = e^{tA}y_0$$

where e^{tA} represents the evolution operator for the Linear Schrödinger (4.1.2) equation with Wentzell boundary conditions.

4.4 Inhomogeneous Linear Problems

Suppose now that $f(x,s) \in L^1(0,\infty,V)$. Then by Duhamel's formula,

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A} f(s)\, ds \tag{4.4.1}$$

is a solution to the problem

$$\begin{cases} y_t - i\Delta y = f & \text{in } \Omega \times (0,\infty) \\ y = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial y}{\partial n} = -i\Delta y & \text{on } \Gamma_1 \times (0,\infty). \end{cases} \tag{4.4.2}$$

Since $f(x,s) \in L^1(0,\infty,V)$, but the fundamental theorem of calculus we establish that $y \in C(0,\infty,V)$. We wish to extend well-posedness to the inhomogenous Wentzell problem

$$\begin{cases} y_t - i\Delta y = f & \text{in } \Omega \times (0,\infty) \\ y = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial y}{\partial n} + i\Delta y = g & \text{on } \Gamma_1 \times (0,\infty). \end{cases} \tag{4.4.3}$$

It suffices to solve this problem for $f = 0$ and use superposition to ontain well-posedness of the above problem. Define a Neumann map as follows:

$$Ng = \begin{cases} \Delta Ng = 0 \\ \frac{\partial}{\partial n} Ng = g. \end{cases}$$

For any $s \in \mathbb{R}$, $N : H^s(\Gamma_1) \longmapsto H^{s+3/2}(\Omega)$.

Define $\tilde{y} = y - Ng$. Then $\frac{\partial}{\partial n}\tilde{y} = -i\Delta y$ and since $\Delta Ng = 0$,

$$\frac{\partial}{\partial n}\tilde{y} = -i\Delta\tilde{y}. \tag{4.4.4}$$

Moreover,

$$\tilde{y}_t = y_t - Ng_t = i\Delta y - Ng_t$$

and again using $\Delta Ng = 0$,

$$\tilde{y}_t = i\Delta(y - Ng) - Ng_t = i\Delta\tilde{y} - Ng_t. \tag{4.4.5}$$

Combining (4.4.4) and (4.4.5), the $\tilde{y}$ problem becomes

$$\begin{cases} \tilde{y}_t = i\Delta(y - Ng) - Ng_t = i\Delta\tilde{y} - Ng_t & \text{in } \Omega \times (0,\infty) \\ \tilde{y} = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \frac{\partial}{\partial n}\tilde{y} = -i\Delta\tilde{y} & \text{on } \Gamma_1 \times (0,\infty) \\ \tilde{y}(0) = y_0 - Ng(0). \end{cases} \tag{4.4.6}$$

Lemma 4.4.1. *If $g \in W^{1,1}(0,\infty; H^{-1/2}(\gamma_1))$, then there exists a unique solution $\tilde{y} \in C(0,\infty;V)$ to (4.4.6).*

Proof. If $g \in H^{-1}(0,\infty; H^{-1/2}(\gamma_1))$, then $g(0) \in H^{-1/2}(\Gamma_1)$ and $g_t \in L^1(0,\infty; H^{-1/2}(\gamma_1))$ and since $N : H^s(\Gamma_1) \longmapsto H^{s+3/2}(\Omega)$,

$$\begin{cases} Ng_t \in L^1(0,\infty;V) \\ Ng(0). \in V \end{cases} \tag{4.4.7}$$

Thus, (4.4.6) reduces to (4.4.2), which was solved above. □

We are now prepared to show well-posedness of the inhomogeneous model. Recalling that

$$y = \tilde{y} + Ng$$

we can now say that since $Ng \in C(0,\infty;V)$ and since by the above lemma $\tilde{y} \in C(0,\infty;V)$, we conclude that there exists a unique solution $y \in C(0,\infty;V)$ to (4.4.3) for all $y_0 \in V$. We are not through. Ultimately we wish to identify this Wentzell problem with the dynamic problem that arises when taking $g = f|_{\Gamma_1}$.

Theorem 4.4.2. *Let $f \in L^1(0,\infty;V)$ and $g \in L^2(0,\infty;L^2(\Gamma_1))$. Then for each $y_0 \in V$ there exists a unique solution $y \in C(0,\infty;V)$ to (4.4.3).*

A lemma is needed.

Lemma 4.4.3. *For any $g \in L^2(0,\infty;L^2(\Gamma_1))$ and for any constant $c > 0$,*

$$\left|(g, \partial_n y)_{L^2(\Gamma_1)}\right| \leq c\|\partial_n y\|^2_{L^2(\Gamma_1)} + \frac{1}{c}\|g\|^2_{L^2(\Gamma_1)} \tag{4.4.8}$$

Proof. By the Cauchy-Schwarz inequality,

$$\left|(g, \partial_n y)_{L^2(\Gamma_1)}\right| \leq \|g\|_{L^2(\Gamma_1)} \|\partial_n y\|_{L^2(\Gamma_1)} \tag{4.4.9}$$

to which we apply the following well known inequality: if a and b are nonnegative real numbers and $\varepsilon > 0$, then

$$ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$$

to achieve the desired result. □

Again by superposition, we take $f = 0$. Taking the H^1 inner product with $\bar{y}$ and integrating in time,

$$\|y(t)\|_V^2 = \int_0^t \frac{d}{dt} \|y(s)\|_V^2 \, ds = \frac{1}{2} \int_0^t (y_t, y)_V \tag{4.4.10}$$

where $y_t = i\Delta y$ in Ω. Integrating by parts,

$$\int_0^t (i\Delta y, y)_V \, ds = \int_0^t -i\|\Delta y\|_{L^2(\Omega)}^2 + i(\Delta y, \partial_n y)_{L^2(\Gamma_1)} \, ds \tag{4.4.11}$$

into which we can substitute the boundary condition to obtain

$$\int_0^t (i\Delta y, y)_V \, ds = \int_0^t -i\|\Delta y\|_{L^2(\Omega)}^2 - \|\partial_n y\|_{L^2(\Gamma_1)}^2 + (g, \partial_n y)_{L^2(\Gamma_1)} \, ds. \tag{4.4.12}$$

Taking real parts,

$$\text{Re}\left[\int_0^t (i\Delta y, y)_V \, ds\right] \leq \int_0^t -\frac{1}{2}\|\partial_n y\|_{L^2(\Gamma_1)}^2 + 2\|g\|_{L^2(\Gamma_1)}^2 \, ds \tag{4.4.13}$$

hence $\sup_t \|y(t)\|_V^2$ remains bounded as long as $g \in L^2(0, \infty; L^2(\Gamma_1))$, proving the theorem.

By making the identification $g = f|_{\Gamma_1}$, we can identify (4.4.3) with the dynamic boundary condition problem

$$\begin{cases} y_t - i\Delta y = f & \text{in } \Omega \\ y = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial y}{\partial n} + y_t = 0 & \text{on } \Gamma_1 \end{cases} \tag{4.4.14}$$

Note that if $f \in L^2(0,\infty;V)$, by trace theory $g \in L^2(0,\infty;H^{1/2}(\Gamma_1))$, hence the following result:

Corollary 4.4.4. *Let $f \in L^2(0,\infty;V)$. Then for each $y_0 \in V$ there exists a unique solution $y \in C(0,\infty;V)$ to (4.4.14).*

We now have a continuous map

$$K_1 : (f, y_0) \longmapsto y(t) \tag{4.4.15}$$

which is bounded from $L^2(0,\infty,V) \times V$ to $C(0,\infty,H^1(\Omega))$.

4.5 Regularity of Solutions

It needs to be shown that this map K_1 is continuous on $H^2(\Omega)$. That is,

$$K_1 : H^1(0,\infty,V) \times D(A) \longrightarrow C(0,\infty,H^2(\Omega)) \cap C^1(0,\infty,V). \tag{4.5.1}$$

Let $z = y_t$. By differentiating (4.4.14) in time, we get

$$\begin{cases} z_t - i\Delta z = f_t & \text{in } \Omega \\ z = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial z}{\partial n} + z_t = 0 & \text{on } \Gamma_1 \end{cases} \tag{4.5.2}$$

to which we wish to apply the map K_1. If $z_0 \in V$ and $f_t \in L^2(0, \infty, V)$, then

$$K_1 : (f_t, z_0) \longmapsto z(t) \in C(0, \infty, V). \tag{4.5.3}$$

However, if $y_0 \in D(A)$ then $z_0 \in V$ and if $f_t \in L^2(0, \infty, V)$ then $f \in H^1(0, \infty, V)$. We have shown that

$$K_1 : H^1(0, \infty, V) \times D(A) \longrightarrow C^1(0, \infty, V).$$

We wish to show that

$$K_1 : H^1(0, \infty, V) \times D(A) \longrightarrow C(0, \infty, H^2(\Omega))$$

as well. Consider that if $z \in C(0, \infty, V)$ and $f \in H^1(0, \infty, V)$, then $z - f \in C(0, \infty, V)$. Furthermore, if $z \in C(0, \infty, V)$ then $z|_{\Gamma_0} \in C(0, \infty, H^{1/2}(\Omega))$. However, substituting z for y_t in (4.4.14) shows that

$$\Delta y \in C(0, \infty, V) \text{ and } \frac{\partial y}{\partial n} \in C(0, \infty, H^{1/2}(\Omega)). \tag{4.5.4}$$

Elliptic regularity estimates provide us with $y \in C(0, \infty, H^2(\Omega))$. We arrive at the following result:

Theorem 4.5.1. *Let* $y_0 \in D(A)$ *and* $f \in H^1(0, \infty, V)$. *Then there exists a unique solution*

$$y \in C(0, \infty, H^2(\Omega)) \cap C^1(0, \infty, V)$$

to (4.4.14).

Chapter 5

Lipschitz Perturbations of the Linear Model

5.1 Generalizing the Linear Theory

As stated in Chapter 2, the initial strategy of this thesis work was to solve the issue of well-posedness for a suitable collection of approximating problems that converge to the nonlinear problem (1.2.1). In particular, this strategy involved proving that Lipschitz perturbations of the linear model remain well-posed and then choosing a series of Lipschitz approximations to the nonlinear term $|y|^2y$. While we do not presently believe that such a construction of a series of approximations is possible, the following result remains interesting for its own sake:

Assumption 5.1.1. *Assume that $g(z)$ is a continuous function on $\mathbb{C}$ such that both $g(z)$ and its inverse $g^{-1}(z)$ satisfy:*

(i) $Re(g(z) - g(v))(\bar{z} - \bar{v}) \geq m|z - v|^2$

(ii) $Re(g(z)) \geq m|z|^2$

(iii) $Im(g(z)\bar{z}) = 0$

(iv) $|g(z)| \leq M|z|$

for some constants $m, M \in \mathbb{R}_+$.

It is worth noting that the above assumptions are satisfied by any the identity function. Moreover, we note that the condition (i) together with the condition (iii) form a complex counterpart to the assumption of monotonicity.

Consider now the model

$$\begin{cases} y_t = i\Delta y + f(y) & \text{in } \Omega \times (0,\infty) \\ y = 0 & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial y}{\partial n} = -g(y_t) \text{ on } \Gamma_0 \times (0,\infty) & \\ y_0 \in V = H^1_{\Gamma_0} & \end{cases} \tag{5.1.1}$$

where $V = H^1_{\Gamma_0}$, and Ω, Γ_0, and Γ_1 as are in the previous chapter and $f(y) : H^1_{\Gamma_0}(\Omega) \to H^1_{\Gamma_0}(\Omega)$ is Lipschitz continuous. That is, for every pair $y, v \in H^1_{\Gamma_0}(\Omega)$,

$$\|f(y) - f(v)\|_{H^1_{\Gamma_0}(\Omega)} \leq L\|y - v\|_{H^1_{\Gamma_0}(\Omega)} \tag{5.1.2}$$

for some fixed constant L.

As was the case for the linear theory, well-posedness is achieved by converting this dynamic problem into a Wentzell problem. Namely, we replace $g(y_t)$ on the boundary by $g(i\Delta + h(y))$. Here we assume that $h : H^1(\Omega) \to L^2(\Gamma_1)$ is Lipschitz, i.e.

$$\|h(y) - h(v)\|_{H^{1/2}(\Gamma_1)} \leq K\|y - v\|_V. \tag{5.1.3}$$

Since the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ is continuous and linear this formulation actually generalizes the above problem, which can be reduced to the special case where $h(y) = \gamma_0(f(y))$. With that in mind, define the operator A_f by

$$A_f y = i\Delta y + f(y) \tag{5.1.4}$$

with accompanying domain

$$D(A_f) = \left\{ y \in V, \Delta y \in V, \frac{\partial y}{\partial n} = -g(i\Delta|_{\Gamma_1} y + h(y)) \text{ on } \Gamma_1 \right\}. \tag{5.1.5}$$

The presence of f itself no bearing on the domain. Under the assumptions that $g : H^{1/2}(\Gamma_1) \to H^{1/2}(\Gamma_1)$ and that the range of h is also $H^{1/2}(\Gamma)$, then by the same argument applied in the previous chapter it is apparent that $D(A_f)$ contains $H^2(\Omega)$ elements. We note that since the trace operator γ_0 has range $H^{1/2}(\Gamma)$ this assumption does not impose any restrictions on f.

Following the same strategy as used in the linear theory, the following theorem will be proved:

Theorem 5.1.1. *The operator $(A, D(A_f))$ generates a strongly continuous semi-group.*

Unlike in the previous chapter where the linear Schrödinger model was discussed, it can no longer be stated arbitrarily that this is a contraction semigroup. We will instead prove ω-maximal dissipativity of the operator A_f, hence the bound on the evolution operator becomes:

$$\|e^{tA_f}\|_{\mathcal{L}(V)} \leq Ce^{\omega t}. \tag{5.1.6}$$

5.2 Dissipativity

Since (5.1.1) is nonlinear we will have to take the difference of two solutions. Before this we observe that by Green's theorem,

$$\begin{aligned}(Ay, v)_V &= i(\nabla \cdot \Delta y, \nabla v)_{L^2(\Omega)} + (f(y), v)_V \\ &= -\, i(\Delta y, \Delta v)_{L^2(\Omega)} + i\left(\Delta y, \frac{\partial v}{\partial n}\right)_{L^2(\Gamma_1)} + (f(y), v)_V \\ &= -\, i(\Delta y, \Delta v)_{L^2(\Omega)} + \left(g^{-1}(-\frac{\partial y}{\partial n}), \frac{\partial v}{\partial n}\right)_{L^2(\Gamma_1)} - \left(h(y), \frac{\partial v}{\partial n}\right)_{L^2(\Gamma_1)} + (f(y), v)_V\end{aligned} \tag{5.2.1}$$

Hence if we consider the difference between two solutions $y, v \in V$ and recall assumption (ii) on g^{-1}:

$$\begin{aligned}(A_f y - A_f v, y - v)_V = -i\|\Delta y - \Delta v\|^2_{L^2(\Omega)} - m\left\|\frac{\partial y}{\partial n} - \frac{\partial v}{\partial n}\right\|^2_{L^2(\Gamma_1)} - \left(h(y) - h(v), \frac{\partial y}{\partial n} - \frac{\partial v}{\partial n}\right)_{L^2(\Gamma_1)} \\ +(f(y) - f(v), y - v)_V.\end{aligned} \tag{5.2.2}$$

By the Cauchy-Schwarz inequality,

$$\left(h(y) - h(v), \frac{\partial y}{\partial n} - \frac{\partial v}{\partial n}\right)_{L^2(\Gamma_1)} \leq \|h(y) - h(v)\|_{L^2(\Gamma_1)}\left\|\frac{\partial y}{\partial n} - \frac{\partial v}{\partial n}\right\|_{L^2(\Gamma_1)}$$

and

$$(f(y) - f(v), y - v)_V \leq \|f(y) - f(v)\|_V \|y - v\|_V.$$

Lipschitz continuity of h and f now plays an essential role. Since $\|h(y) - h(v)\|_{H^{1/2}(\Gamma_1)} \leq K\|y - v\|_V$,

$$\|h(y) - h(v)\|_{L^2(\Gamma_1)} \left\| \frac{\partial y}{\partial n} - \frac{\partial v}{\partial n} \right\|_{L^2(\Gamma_1)} \leq K\|y - v\|_V \left\| \frac{\partial y}{\partial n} - \frac{\partial v}{\partial n} \right\|_{L^2(\Gamma_1)} \tag{5.2.3}$$

and since $\|f(y) - f(v)\|_V \leq L\|y - v\|_V$,

$$\|f(y) - f(v)\|_V \|y - v\|_V \leq L\|y - v\|_V^2. \tag{5.2.4}$$

To (5.2.3) we apply the following well known result:

$$ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2. \tag{5.2.5}$$

Using $\varepsilon = \dfrac{2}{m}$,

$$K\|y - v\|_V \left\| \frac{\partial y}{\partial n} - \frac{\partial v}{\partial n} \right\|_{L^2(\Gamma_1)} \leq \frac{2}{m} K^2 \|y - v\|_V^2 + \frac{m}{2} \left\| \frac{\partial y}{\partial n} - \frac{\partial v}{\partial n} \right\|_{L^2(\Gamma_1)}^2. \tag{5.2.6}$$

Combining (5.2.4) and (5.2.6) with (5.2.2),

$$\begin{aligned} \mathrm{Re}(A_f y - A_f v, y - v)_V \leq -m & \left\| \frac{\partial y}{\partial n} - \frac{\partial v}{\partial n} \right\|_{L^2(\Gamma_1)}^2 \\ & +2mK^2 \|y - v\|_V^2 + \frac{m}{2} \left\| \frac{\partial y}{\partial n} - \frac{\partial v}{\partial n} \right\|_{L^2(\Gamma_1)}^2 + L\|y - v\|_V^2 \end{aligned} \tag{5.2.7}$$

where by taking $\omega > 2mK^2 + L$ we may conclude that

$$\mathrm{Re}(A_f y - A_f v - \omega I(y - v), y - v)_V < 0. \tag{5.2.8}$$

5.3 Maximality

As with the linear problem (4.1.2), maximality will be proved on the Banach space Z:

$$Z = \left\{ y \in V, \Delta y \in L^2(\Omega), \frac{\partial y}{\partial n} \in L^2(\Gamma) \right\}$$

which is equipped with the norm

$$\|y\|_Z = \|y\|_V + \|y\|_{L^2(\Omega)} + \left\| \frac{\partial y}{\partial n} \right\|_{L^2(\Gamma)}.$$

As before, define

$$a(y, v) = (\lambda y - A_f y, v)_V.$$

Although $a(y, v)$ is no longer a bilinear form, the same theory applies. Namely, if it can be shown that this form is continuous and coercive then the Browder-Minty theorem can still be applied. Hence, for every $j \in V \subset Z'$ (where Z' represents the dual space of Z), there exists a unique $y \in Z$ satisfying

$$a(y, v) = (-j, v)_V \text{ for all } v \in Z$$

for some value of λ such that $\mathrm{Re}(\lambda)$ is sufficiently large.

To see that $a(y, v)$ is continuous on Z,

$$\begin{aligned} a(y, v) = \lambda(y, v)_V + i(\Delta y, \Delta v)_{L^2(\Omega)} + \left(g^{-1}\left(\frac{\partial y}{\partial n}\right), \frac{\partial v}{\partial n} \right)_{L^2(\Gamma_1)} \\ -(f(y), v)_V + \left(h(y), \frac{\partial v}{\partial n} \right)_{L^2(\Gamma_1)} \end{aligned} \tag{5.3.1}$$

whereby the triangle inequality and using the bounds on f, g, and h,

$$\begin{aligned} |a(y,v)| \le |\lambda(y,v)_V| + |(\Delta y, \Delta v)_{L^2(\Omega)}| + M \left| \left(\frac{\partial y}{\partial n}, \frac{\partial v}{\partial n} \right)_{L^2(\Gamma_1)} \right| \\ + L\|y\|_V \|v\|_V + K\|y\|_V \left\| \frac{\partial v}{\partial n} \right\|_{L^2(\Gamma_1)} \end{aligned} \tag{5.3.2}$$

for which there exists a bound $C(\lambda, M, L, K)$ such that

$$|a(y,v)| \le C(\lambda, M, L, K)\|y\|_Z \|v\|_Z. \tag{5.3.3}$$

For coercivity, observe that

$$\begin{aligned} a(y,y) = \lambda\|y\|_V^2 + \|\Delta y\|_{L^2(\Omega)}^2 + \left(g^{-1}\left(\frac{\partial y}{\partial n} \right), \frac{\partial y}{\partial n} \right)_{L^2(\Gamma_1)} \\ -(f(y),y)_V + \left(h(y), \frac{\partial y}{\partial n} \right)_{L^2(\Gamma_1)}. \end{aligned} \tag{5.3.4}$$

For any complex number $z = x + iy$, the bound $|z| \ge \frac{1}{2}|x| + \frac{1}{2}|y|$ can be applied. Furthermore, the bound $g^{-1}(z) \ge m|z|$ from assumption (ii) can be applied, hence by taking $\mathrm{Im}(\lambda) \ge 0$ so as to prevent cancellation of components, we arrive at the crude estimate

$$\begin{aligned} |a(y,y)| \ge \frac{1}{4}\mathrm{Re}(\lambda)\|y\|_V^2 + \frac{1}{4}\|\Delta y\|_{L^2(\Omega)}^2 + \frac{m}{4} \left\| \frac{\partial y}{\partial n} \right\|_{L^2(\Gamma_1)} \\ -|(f(y),y)_V| - \left| \left(h(y), \frac{\partial y}{\partial n} \right)_{L^2(\Gamma_1)} \right|. \end{aligned} \tag{5.3.5}$$

Recycling the estimates (5.2.4) and (5.2.6) stemming from the Lipschitz bounds on f and h with the modification made to (5.2.6) that we take $\varepsilon = \frac{m}{8}$ instead of $\frac{m}{2}$ from the calculation (5.2.5), we arrive at the estimate

$$\begin{aligned}|a(y,y)| \geq& \frac{1}{4}\mathrm{Re}(\lambda)\|y\|_V^2 + \frac{1}{4}\|\Delta y\|_{L^2(\Omega)}^2 + \frac{m}{8}\left\|\frac{\partial y}{\partial n}\right\|_{L^2(\Gamma_1)} - L\|y\|_V^2 - \frac{8}{m}K^2\|y\|_V^2 \\ \geq& C\|y\|_Z^2\end{aligned} \tag{5.3.6}$$

for some constant $C > 0$ as long as $\mathrm{Re}(\lambda) > 4L + \frac{32}{m}K^2$.

We are now in a position to apply the Lumer-Philips theorem. If $\omega > 4L + \frac{32}{m}K^2$, then by the calculations above, the operator $A_f - \omega I$ is maximally dissipative, thus the operator A_f generates a strongly continuous semigroup.

Chapter 6

Well-Posedness of the Nonlinear Model

6.1 Strategy

We have shown that for linear functions $f : H^1_{\Gamma_0}(\Omega) \to H^1_{\Gamma_0}(\Omega)$,

$$\begin{cases} y_t - i\Delta y = f & \text{in } \Omega \\ y = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial y}{\partial n} + y_t = 0 & \text{on } \Gamma_1 \end{cases} \tag{6.1.1}$$

is well-posed. We wish now to show that the nonlinear model

$$\begin{cases} y_t - i\Delta y = F(y) = -i|y|^2 y & \text{in } \Omega \\ y = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial y}{\partial n} + y_t = 0 & \text{on } \Gamma_1 \end{cases} \tag{6.1.2}$$

is well-posed globally in dimension $N = 2$ and locally in dimension $N = 3$. To that end, we wish to apply the linear semigroup theory to the non-linear problem that arises when we take $f(y) = -i|y|^2 y$. A fixed point method will be used. Apriori estimates are needed. These variational estimates are distinct from the Strichartz estimates commonly found in the literature, but will serve a similar role in the analysis of the problem.

Let $z = y_t$. As seen in Chapter 4, the appearance of y_t on the boundary means that if we wish to show that solutions exist for data $y \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$, we need also show that $z \in H^1_{\Gamma_0}(\Omega)$ over the same time period. This follows from trace theory: if $y \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$, then by trace theory, $\dfrac{\partial y}{\partial n} \in H^{1/2}(\Gamma_1)$; however, from equation (6.1.2) we have the relation $\dfrac{\partial y}{\partial n} = z$ on Γ_1, hence we require $z|_{\Gamma_1} \in H^{1/2}(\Gamma_1)$, which means that we must have $z \in H^1_{\Gamma_0}(\Omega)$.

To acquire estimates on z, we differentiate equation (6.1.2) in time,

$$\begin{cases} z_t - i\Delta z = F_t & \text{in } \Omega \\ z = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial z}{\partial n} + z_t = 0 & \text{on } \Gamma_1 \end{cases} \tag{6.1.3}$$

where

$$\begin{aligned} F_t =& - i\frac{d}{dt}y^2\bar{y} \\ =& - 2iyy_t\bar{y} - iy^2\bar{y}_t \\ =& - 2i|y|^2 z - iy^2\bar{z} \end{aligned} \tag{6.1.4}$$

6.2 A Priori Estimates

Lemma 6.2.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded in dimension $N = 2, 3$. Let $F(y) = -i|y|^2 y$ where $y \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$. Also let $z = y_t \in H^1_{\Gamma_0}(\Omega)$. Then the following estimates* *hold:*

A1)

$$\|F(y)\|_{H^2(\Omega)} \leq C\|y\|^2_{L^\infty(\Omega)}\|y\|_{H^2(\Omega)} \tag{6.2.1}$$

A2)

$$\|F(y)\|_{H^2(\Omega)} \leq C\|y\|^3_{H^2(\Omega)} \tag{6.2.2}$$

B1)

$$\|F_t(y)\|_{H^1_{\Gamma_0}(\Omega)} \leq C\|\nabla z\|_{L^2(\Omega)}\|y\|^2_{H^2(\Omega)} \tag{6.2.3}$$

B2)

$$\|F_t(y)\|_{H^1_{\Gamma_0}(\Omega)} \leq C\|y\|_{L^\infty(\Omega)}\|\nabla z\|_{L^2(\Omega)}\|y\|_{H^2(\Omega)} \tag{6.2.4}$$

(A1) was proved by Brezis and Gallouet. (A2) follows directly since $H^2(\Omega)$ embeds continuously into $L^\infty(\Omega)$ and thus

$$\|F(y)\|_{H^2(\Omega)} \leq C\|y\|^3_{H^2(\Omega)}. \tag{6.2.5}$$

To estimate $\|F_t(y)\|_{H^1_{\Gamma_0}(\Omega)}$ we first calculate $\nabla F_t(y)$:

$$\begin{aligned}\nabla(-2i|y|^2 z - iy^2\bar{z}) =& \nabla(-2iy\bar{y}z - iy^2\bar{z})\\ =& -i(2y\bar{y}\nabla z + 2y\nabla\bar{y}z + 2\nabla y\bar{y}z + 2y\nabla y\bar{z} + y^2\nabla\bar{z}).\end{aligned}$$

By the triangle inequality, we have

$$\|F_t(y)\|_{H^1_{\Gamma_0}(\Omega)} \leq 2\|y\bar{y}\nabla z\|_{L^2(\Omega)}+2\|y\nabla\bar{y}z\|_{L^2(\Omega)}+2\|\nabla y\bar{y}z\|_{L^2(\Omega)}+2\|y\nabla y\bar{z}\|_{L^2(\Omega)}+\|y^2\nabla\bar{z}\|_{L^2(\Omega)}.$$

We approximate each term independently. Using Holder's inequality,

$$2\|y\bar{y}\nabla z\|_{L^2(\Omega)} \leq 2\|y\|^2_{L^\infty(\Omega)}\|\nabla z\|_{L^2(\Omega)}$$

and likewise

$$\|y^2\nabla\bar{z}\|_{L^2(\Omega)} \leq \|y\|^2_{L^\infty(\Omega)}\|\nabla z\|_{L^2(\Omega)}.$$

The estimate for $2\|y\nabla\bar{y}z\|_{L^2(\Omega)}$ must be more carefully constructed. Again using Holder's inequality,

$$\begin{aligned}\|y\nabla\bar{y}z\|_{L^2(\Omega)} \leq & \|y\|_{L^\infty(\Omega)}\|\nabla\bar{y}z\|_{L^2(\Omega)} \\ \leq & \|y\|_{L^\infty(\Omega)}\left(\||z|^2\|_{L^3(\Omega)}\||\nabla\bar{y}|^2\|_{L^{3/2}(\Omega)}\right)^{1/2} \\ = & \|y\|_{L^\infty(\Omega)}\|z\|_{L^6(\Omega)}\|\nabla y\|_{L^3(\Omega)}.\end{aligned}$$

The choices for spaces in the use of Holder's inequality on $\|\nabla\bar{y}z\|_{L^2(\Omega)}$ is particularly essential. In dimensions $n = 2, 3$, the Sobolev imbeddings $H^1(\Omega) \subset L^6(\Omega)$, $H^2(\Omega) \subset W^{1,3}(\Omega)$ and $H^2(\Omega) \subset L^\infty(\Omega)$ hold, thus

$$\|y\nabla\bar{y}z\|_{L^2(\Omega)} \leq C\|y\|_{L^\infty(\Omega)}\|\nabla z\|_{L^2(\Omega)}\|y\|_{H^2(\Omega)}$$

and

$$\|y\nabla\bar{y}z\|_{L^2(\Omega)} \leq C\|\nabla z\|_{L^2(\Omega)}\|y\|^2_{H^2(\Omega)}.$$

The same approach can be used to bound the remaining two terms, $\|\nabla y\bar{y}z\|_{L^2(\Omega)}$ and

$\|y\nabla y\bar{z}\|_{L^2(\Omega)}$. This leaves us with the following a priori estimates:

$$\begin{aligned} \|F_t(y)\|_{H^1_{\Gamma_0}(\Omega)} \leq & C\|\nabla z\|_{L^2(\Omega)}\|y\|^2_{H^2(\Omega)} \\ \|F_t(y)\|_{H^1_{\Gamma_0}(\Omega)} \leq & C\|y\|_{L^\infty(\Omega)}\|\nabla z\|_{L^2(\Omega)}\|y\|_{H^2(\Omega)}. \end{aligned} \tag{6.2.6}$$

6.3 Energy Estimates

Multiplying the equation

$$y_t = i\Delta y - i|y|^2 y$$

by $\overline{y_t}$, integrating by parts and taking real parts gives rise to the energy relations

$$E(0) = \frac{1}{2}\int_\Omega |\nabla y_0|^2 \, d\Omega + \frac{1}{4}\int_\Omega |y_0|^4 \, d\Omega \tag{6.3.1}$$

and

$$E(t_2) = E(t_1) - \int_{t_1}^{t_2}\int_{\Gamma_1} \left|\frac{\partial y}{\partial n}\right|^2 d\Gamma_1. \tag{6.3.2}$$

In dimension $N = 2$, the Sobolev space $H^1(\Omega)$ imbeds continuously into $L^4(\Omega)$, thus from the former energy relation,

$$E(0) \leq \frac{1}{2}\|y_0\|^2_{H^1(\Omega)} + C\|y_0\|^4_{H^1(\Omega)} \tag{6.3.3}$$

and from the latter we observe that the energy is decreasing in time, hence

$$\|y(t)\|_{H^1(\Omega)} \leq C \tag{6.3.4}$$

for all $t \in [0, \infty)$.

6.4 Fixed Point Argument

To prove existence and uniqueness of solutions, we first prove local existence and uniqueness. In dimension $N = 2$, existence of global solutions will be shown. However, in dimension $N = 3$, existence of global solutions remains an open question.

We begin by defining the following spaces

$$X_0 = \left\{(y, z) \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) H^1_{\Gamma_0}(\Omega) : z = y_t\right\}$$

which we equip with the following norm:

$$\|(y, z)\|_{X_0} = \|y\|_{H^2(\Omega)} + \|z\|_{H^1_{\Gamma_0}(\Omega)}$$

and we define also the Banach space

$$X_T = \left\{(y, z) : y \in C^1[0, T; H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)], z \in C(0, T; H^1_{\Gamma_0}(\Omega)), y_t = z\right\}$$

to which we equip the norm:

$$\|(y, z)\|_{X_T} = \sup_{t \in [0,T]} \|y\|_{H^2(\Omega)} + \sup_{t \in [0,T]} \|z\|_{H^1_{\Gamma_0}(\Omega)}.$$

Theorem 6.4.1. *For every bounded subset $B \subset X_0$, there exists $T > 0$ such that for all $(y_0, z_0) \in B$, there exists a unique solution y of (6.1.2) with time derivative $y_t = z$ such that the pair $(y, z) \in X_T$.*

The classical time derivative of y is not defined at time $T = 0$. Here, z_0 is taken to mean $\lim_{t \to 0^+} y_t$.

Proof. For $y_0 \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$ and $y(t) \in C^1[0, T; H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)]$, denote by $\Phi(u)$ the functional

$$[\Phi(y)](t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} F(y)\, ds \tag{6.4.1}$$

with the defined operator $A = i\Delta$ with associated boundary conditions as was used earlier to prove well-posedness of the linear model. Likewise, for $z_0 \in H^1_{\Gamma_0}(\Omega)$ and $z(t) \in C(0, T; H^1_{\Gamma_0}(\Omega))$, denote by $\Psi(z)$ the functional

$$[\Psi(z)](t) = e^{tA} z_0 + \int_0^t e^{(t-s)A} F_t(z)\, ds. \tag{6.4.2}$$

We note that these functionals are returning variational formulas for $y(t)$ and $z(t)$ respectively as given by Duhamel's formula. We will show that there is a time $T > 0$ such that the map $T(y, z) = (\Phi(y), \Psi(z))$ is a contraction on the space X_T.

We first need to verify that $T(y, z)$ maps $B_R(X_T)$ into $B_R(X_T)$, where B_R denotes a ball of radius R, for suitable choices of R and T. Using conservation of the flow e^{tA} and (A2) from Lemma 3,

$$\begin{aligned} \|[\Phi(y)](t)\|_{H^2(\Omega)} \leq & \|y_0\|_{H^2(\Omega)} + \int_0^T \|F(y)\|_{H^2(\Omega)}\, ds \\ \leq & \|y_0\|_{H^2(\Omega)} + CT\|y\|^3_{H^2(\Omega)}. \end{aligned}$$

Likewise, by combining the conservation law with (B2),

$$\begin{aligned} \|[\Psi(z)](t)\|_{H^1_{\Gamma_0}} \leq & \|z_0\|_{H^1_{\Gamma_0}} + \int_0^T \|F_t(z)\|_{H^1_{\Gamma_0}}\, ds \\ \leq & \|z_0\|_{H^1_{\Gamma_0}} + CT\|\nabla z\|_{L^2(\Omega)}\|y\|^2_{H^2(\Omega)}. \end{aligned}$$

Since $(y_0, z_0) \in B$, a bounded subset of X_0, we can take $\|(y_0, z_0)\|_{X_0} \leq \frac{R}{2}$, thus $\|y_0\|_{H^2(\Omega)} \leq \frac{R}{2}$ and $\|z\|_{H^1_{\Gamma_0}(\Omega)} \leq \frac{R}{2}$. Similarly, if $(y, z) \in B_R(X_T)$, then $\|y\|_{H^2(\Omega)} \leq R$ and $\|z_0\|_{H^1_{\Gamma_0}(\Omega)} \leq R$. Hence for $(y, z) \in B_R(X_T)$

$$\|[\Phi(y)](t)\|_{H^2(\Omega)} \leq \frac{R}{2} + CTR^3$$

and

$$\|[\Psi(z)](t)\|_{H^1_{\Gamma_0}} \leq \frac{R}{2} + CTR^3$$

as well. Taking $T < \frac{1}{2CR^2}$ ensures that $T(y, z)$ does not leave the ball $B_R(X_T)$.

To apply a contraction mapping argument, contractive estimates are now needed. Let $(y_1, z_1), (y_2, z_2) \in X_T$. Then by similar arguments as above,

$$\begin{cases} \|[\Phi(y_1)](t) - \Phi(y_2)](t)\|_{H^2(\Omega)} \leq & \int_0^T \|F(y_1) - F(y_2)\|_{H^2(\Omega)}\, ds \\ \|[\Psi(z_1)](t) - [\Psi(z_2)](t)\|_{H^1_{\Gamma_0}} \leq & \int_0^T \|F_t(z_1) - F_t(z_2)\|_{H^1_{\Gamma_0}}\, ds \end{cases}$$

and by using the crude estimate $\|a - b\| \leq \|a\| + \|b\|$, we find that $\|[\Phi(y_1)](t) - \Phi(y_2)](t)\|_{H^2(\Omega)} < R$ and $\|[\Psi(z_1)](t) - [\Psi(z_2)](t)\|_{H^1_{\Gamma_0}} < R$ for $T < \frac{1}{2CR^2}$. Hence by the Banach Contraction Mapping theorem, there exists a fixed point $(y, z) \in B_R(X_T)$ such that $y(t)$ is a strong solution of (1.2.1) and $y_t(t) = z(t)$.

$\square$

6.5 Global Solutions in 2D

We wish to show that $\|y(t)\|_{H^2(\Omega)}$ and $\|z(t)\|_{H^1_{\Gamma_0}}$ remain bounded for all $t \in [0, \infty)$ as well. The latter does not immediately follow from the former due to the appearance of $z = y_t$ on the boundary. To verify that $\|y(t)\|_{H^2(\Omega)}$ and $\|z(t)\|_{H^1_{\Gamma_0}}$ remain bounded, we use the Brezis-Gallouet inequality (3.2.2) on the variational inequalities used in the fixed point argument. For the former we follow the strategy used by Brezis and Gallouet:

$$\|y(t)\|_{H^2(\Omega)} = \|[\Phi(y)](t)\|_{H^2(\Omega)} \leq \|y_0\|_{H^2(\Omega)} + \int_0^T \|F(y)\|_{H^2(\Omega)}\, ds \tag{6.5.1}$$

whereby

$$\||y|^2 y\|_{H^2(\Omega)} \leq C\|y\|^2_{L^\infty(\Omega)}\|y\|_{H^2(\Omega)}$$

and

$$\|y\|_{L^\infty(\Omega)} \leq C(1 + \sqrt{\log(1 + \|y\|_{H^2(\Omega)})})$$

leading to the inequality

$$\|y(t)\|_{H^2(\Omega)} \leq C + C\int_0^t \|y(s)\|_{H^2(\Omega)}[1 + \log(1 + \|y(s)\|_{H^2(\Omega)})]\, ds. \tag{6.5.2}$$

As in the argument by Brezis and Gallouet, we denote the right hand side by $G(t)$. Then,

$$G'(t) = C\|y(t)\|_{H^2(\Omega)}[1 + \log(1 + \|y(t)\|_{H^2(\Omega)})] \leq CG(t)[1 + \log(1 + G(t))] \tag{6.5.3}$$

and hence by separation of variables,

$$\frac{d}{dt}\log[1 + \log(1 + G(t))] \leq C. \tag{6.5.4}$$

Exponentiating the above inequality provides the following estimate:

$$\|y(t)\|_{H^2(\Omega)} \leq Me^{\alpha e^{\beta t}} \tag{6.5.5}$$

for some constants M, α and β. It needs also be verified that $\|z(t)\|_{H^1_{\Gamma_0}}$ remains bounded for all time. Again using the variational form,

$$\|z(t)\|_{H^1_{\Gamma_0}} = \|[\Psi(z)](t)\|_{H^1_{\Gamma_0}} \leq \|z_0\|_{H^1_{\Gamma_0}} + C\int_0^t \|\nabla z\|_{L^2(\Omega)}\|y\|^2_{H^2(\Omega)}\,ds. \tag{6.5.6}$$

However, we can make use of the bound on $\|y(t)\|_{H^2(\Omega)}$ to get

$$\|z(t)\|_{H^1_{\Gamma_0}} \leq C + C\int_0^t \|z(s)\|_{H^1_{\Gamma_0}}\,ds.$$

Taking the time derivative of both sides,

$$\frac{d}{dt}\|z(t)\|_{H^1_{\Gamma_0}} \leq C\|z(t)\|_{H^1_{\Gamma_0}}$$

and therefore we achieve an estimate of the form

$$\|z(t)\|_{H^1_{\Gamma_0}} \leq M_2 e^{\gamma t} \tag{6.5.7}$$

where γ is a constant depending on $\sup\limits_{t\in[0,T]} \|y(t)\|_{H^2(\Omega)}$. We have proved the following result:

Theorem 6.5.1. *For dimension $N = 2$, for all $(y_0, z_0) \in X_0$ and for all $T > 0$, there exists a unique solution y of (6.1.2) with time derivative $y_t = z$ such that the pair $(y, z) \in X_T$.*

Chapter 7

Weak Solutions by the Galerkin Method

7.1 Defining Weak Solutions

In the previous chapter, global regular solutions to (1.2.1) were obtained in dimension $N = 2$, but the result in dimension $N = 3$ is only partial. In this chapter we solve (1.2.1) using the method of Faedo - Galerkin on $H^1(\Omega)$ to achieve weak solutions. The advantage of this approach is that weak solutions are global in both dimension $N = 2$ and $N = 3$. However, weak solutions come with the disadvantage that uniqueness cannot be assertained. This is the trade-off that must be made when $N = 3$: either we achieve well-posedness on a finite time interval which cannot be extended arbitrarily or we achieve global existence of solutions but not well-posedness.

We multiply the equation

$$iy_t + \Delta y - |y|^2 y = 0 \tag{7.1.1}$$

by an admissible function v and integrate in Ω. Incorporating the boundary conditions of (1.2.1) and using Green's theorem, we arrive at

$$i(y', v)_{L^2(\Omega)} - (\nabla y, \nabla v)_{L^2(\Omega)} + (y', v)_{L^2(\Gamma_1)} - (|y|^2 y, v)_{L^2(\Omega)} = 0, \forall t \in [0, \infty).$$

We define $y \in H^1_{\Gamma_0}(\Omega)$ as a weak solution to (1.2.1) if it satisfies the above equality for all $v \in H^1_{\Gamma_0}(\Omega)$. We will prove the following result:

Theorem 7.1.1. *For any dimension $N \leq 3$, given $y_0 \in H^1_{\Gamma_0}(\Omega)$ there exists a global weak solution $y(t) \in H^1_{\Gamma_0}(\Omega)$ to (1.2.1).*

7.2 Constructing a Convergent Subsequence

Let $\{\omega_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $H^1_{\Gamma_0}(\Omega)$. Although an explicit basis cannot be computed, we know a priori that one exists because $H^1_{\Gamma_0}(\Omega)$ is a separable Hilbert space.

$$\begin{cases} y = 0 & \text{on } \Gamma_0 \\ \frac{\partial y}{\partial n} = 0 & \text{on } \Gamma_1. \end{cases}$$

We note that $\{\omega_j\}$ is dense in $H^1_{\Gamma_0}(\Omega)$. Define $V_m = [\omega_1, \ldots, \omega_m]$ and let $v \in V_m$. Then (1.2.1) reduces to the following approximate problem on V_m:

$$\begin{cases} y_m(t) \in V_m \Leftrightarrow y_m(t) = \sum_{j=1}^{m} h_{jm}(t)\, \omega_j(t) \\ (i\, y_m', v)_{L^2(\Omega)} - (\nabla y_m, \nabla v)_{L^2(\Omega)} + (y_m', v)_{L^2(\Gamma_1)} - (|y_m|^2 y_m, v)_{L^2(\Omega)} = 0 \\ y_m(0) = y_m^0 \quad \text{in } \Omega \\ y_m(0) = y_m^0 \to y_0 \quad \text{in } V = H^1_{\Gamma_0}(\Omega) \end{cases} \tag{7.2.1}$$

It must be shown that the approximate system (7.2.1) gives rise to an ordinary differential equation which we may solve. Substituting $y_m(t) = \sum_{j=1}^{m} h_{jm}(t)\,\omega_j(t)$ back into the second equation in (7.2.1), we may write

$$i\sum_{j,k\leq m}(h'_{jm}\omega_j,\omega_k)_{L^2(\Omega)}+\sum_{j,k\leq m}(h'_{jm}\omega_j,\omega_k)_{L^2(\Gamma_1)} = \tag{7.2.2}$$

$$+\sum_{j,k\leq m}(\nabla\, h_{jm}\omega_j,\nabla\,\omega_k)_{L^2(\Omega)}+\sum_{j,k\leq m}(|h_{jm}|^2\,h_{jm}\omega_j,\omega_k)_{L^2(\Omega)}. \tag{7.2.3}$$

Define h_m to be the vector given by $h_m = \langle h_{1m},\, h_{2m},\, ...,\, h_{mm}\rangle$. Then the left hand side (LHS) of the above may be written

$$M_{jk}h'_m = i\sum_{j,k\leq m}(h'_{jm}\omega_j,\omega_k)_{L^2(\Omega)}+\sum_{j,k\leq m}(h'_{jm}\omega_j,\omega_k)_{L^2(\Gamma_1)} \tag{7.2.4}$$

where M_{jk} is a matrix with elements given by

$$M_{jk} = i(\omega_j,\omega_k)_{L^2(\Omega)}+(\omega_j,\omega_k)_{L^2(\Gamma_1)}. \tag{7.2.5}$$

It will be shown that this matrix is invertible. Observe that

$$(M_{jk}h_m,\, h_m)_{L^2(\Omega)} = i(h_m,h_m)_{L^2(\Omega)}+(h_m,h_m)_{L^2(\Gamma_1)},$$

which has real part $\|h_m\|^2_{L^2(\Gamma_1)} > 0$ and imaginary part $\|h_m\|^2_{L^2(\Omega)} > 0$. Hence, M_{jk} can be written as a sum of a real-valued matrix that is positive definite and an imaginary-valued matrix that can be written as iI, where I is the identity on V_m.

Hence, M_{jk} is invertible and we may write for any $v \in V_m$,

$$\sum_{j\le m}(h'_{jm}\omega_j, v)_{L^2(\Omega)} = M_{jk}^{-1}\left(\sum_{j\le m}(\nabla\, h_{jm}\omega_j, \nabla\, v)_{L^2(\Omega)} + \sum_{j\le m}(|h_{jm}|^2\, h_{jm}\omega_j, v)_{L^2(\Omega)}\right). \tag{7.2.6}$$

Since (7.2.1) can be rewritten as an ordinary differential equation, the approximate system has a local solution on $[0, t_m)$ guaranteed by the Caratheodory's theorem with $y_m(t)$ absolutely continuous and $y'_m(t)$ existing a. e. in Dini's sense. This solution can be extended to the interval $[0, T]$. Since $y_m \in V_m$, we can write

$$y_m(t) = \sum_{j=1}^{m} h_{jm}(t)\,\omega_j \tag{7.2.7}$$

and by (7.2.1) above we have that for all $t \in\ (0, t_m)$

$$y'_m \in\ L^2(0, t; [H^1_{\Gamma_0}(\Omega)]') \tag{7.2.8}$$

We note that the derivative (7.2.8) is in Dini's sense (i. e., a. e.). We take $\frac{d}{dt}$to be the time derivative in the distributional sense of $\mathcal{D}'(0, t; L^2(\Omega))$. By integrating against test functions $\theta \in\ C_0^\infty(0, t)$ and making sense of the L^2 inner product as a duality pairing we get

$$\frac{d}{dt}\,(y_m(t), v)_{L^2(\Omega)} \quad = \quad (y'_m(t), v)_{L^2(\Omega)} \tag{7.2.9}$$

for all $v \in\ V_m$ and all $t \in\ (0, t_m)$.

7.3 A Priori Estimates

We observe that if we consider $v = \omega_j, j = 1, \ldots, m$ and multiply the second equation of (7.2.1) by $\overline{h'_{jm}}(t)$ and then sum in j, then by taking into account the boundary conditions and considering only the real part, we obtain

$$\frac{d}{dt}\Big[\|\nabla y_m\|^2_{L^2(\Omega)} + \frac{1}{4}\|y_m\|^4_{L^4(\Omega)}\Big] + \|y'_m\|^2_{L^2(\Gamma_1)} = 0\,.$$

Integrating this expression in time over $t \in [0, T]$, and having in mind that the Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$, $q \leq 6$, for $N \leq 3$, we obtain for all $t \in [0, T]$,

$$\|\nabla y_m\|^2_{L^2(\Omega)} + \frac{1}{4}\|y_m\|^4_{L^4(\Omega)} + \int_0^t \|y'_m\|^2_{L^2(\Gamma_1)}\,dt \leq C\|\nabla y^0_m\|^2_{L^2(\Omega)}. \tag{7.3.1}$$

We note in particular that since the sequence $\{\nabla y^0_m\}$ converges in $H^1(\Omega)$ to ∇y_0, $\sup\{\nabla y^0_m\}$ must be finite and therefore the left hand side (LHS) of the above estimate is bounded independently of m and hence there is a convergent subsequence of $\{y_m\}$. The nature of this convergence will be discussed in the following section. We can also infer from the estimate (7.3.1) that

$$\{y_m\} \quad \text{is bounded in} \quad L^\infty(0, T; H^1_{\Gamma_0}(\Omega)) \tag{7.3.2}$$

$$\{y_m\} \quad \text{is bounded in} \quad L^\infty(0, T; L^4(\Omega)) \tag{7.3.3}$$

$$\{y'_m\} \quad \text{is bounded in} \quad L^2(0, T; L^2(\Gamma_1)) \tag{7.3.4}$$

$$\{|y_m|^2\, y_m\} \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)) \tag{7.3.5}$$

where the last assertion again comes from use of the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$.

7.4 Passage Through the Limit

From the a priori estimates, there exists a subsequence of $\{y_m\}_{m\in\mathbb{N}}$, which is still denoted in the same way, such that

$$y_m \overset{\star}{\rightharpoonup} y \text{ weak star in } L^\infty(0,T;H^1_{\Gamma_0}(\Omega))\,. \tag{7.4.1}$$

$$y_m \rightharpoonup y \text{ weak in } L^\infty(0,T;L^4(\Omega))\,. \tag{7.4.2}$$

$$y'_m \overset{\star}{\rightharpoonup} y' \text{ weak in } L^\infty(0,T;[H^1_{\Gamma_0}(\Omega)]')\,. \tag{7.4.3}$$

$$y'_m \overset{\star}{\rightharpoonup} y' \text{ weak star in } L^\infty(0,T;L^2(\Gamma_1))\,. \tag{7.4.4}$$

Using the chain of Sobolev embeddings

$$H^1_{\Gamma_0}(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \hookrightarrow Ł^\infty(\Omega),$$

it follows from the boundness of (7.3.2) that by the Aubin-Lions theorem, that there exists a subsequence of $\{y_m\}$, which we again denote the same way, such that,

$$y_m \longrightarrow y \;\text{ strongly in }\; L^2(0,T,L^2(\Omega)), \tag{7.4.5}$$

that is,

$$y_m \longrightarrow y \;\text{ a. e. in }\; \Omega \times (0,T). \tag{7.4.6}$$

By continuity of map $z \mapsto |z|^2\, z$ from (7.4.6), we have

$$|y_m|^2\, y_m \longrightarrow |y|^2\, y \;\text{ a. e. in }\; \Omega \times (0,T). \tag{7.4.7}$$

So, combining (7.3.5) and (7.4.7) jointly with Lions' Lemma, we obtain,

$$|y_m|^2\, y_m \rightharpoonup |y|^2\, y \text{ weak star in } L^\infty(0,T;L^2(\Omega))\,. \tag{7.4.8}$$

Moreover, as $\omega_j\,\theta \in L^1(0,T;L^2(\Omega))$ and $\omega_j\,\theta \in L^1(0,T;L^2(\Gamma_1)$, from (7.4.1) – (7.4.4), we can assert the following convergences:

$$\int_0^T (i\,y'_m,\omega_j)_{L^2(\Omega)}\,\theta(t)\,dt \longrightarrow \int_0^T (i\,y',\omega_j)_{H^1_{\Gamma_0}(\Omega)}\,\theta(t)\,dt\,. \tag{7.4.9}$$

$$\int_0^T (\nabla\,y_m,\nabla\,\omega_j)_{L^2(\Omega)}\,\theta(t)\,dt \longrightarrow \int_0^T (\nabla\,y,\nabla\,\omega_j)_{L^2(\Omega)}\,\theta(t)\,dt\,. \tag{7.4.10}$$

$$\int_0^T (|y_m|^2\,y_m,\omega_j)_{L^2(\Omega)}\,\theta(t)\,dt \longrightarrow \int_0^T (|y|^2\,y,\omega_j)_{L^2(\Omega)}\,\theta(t)\,dt\,. \tag{7.4.11}$$

$$\int_0^T (y'_m,\omega_j)_{L^2(\Gamma_1)}\,\theta(t)\,dt \longrightarrow \int_0^T (y',\omega_j)_{L^2(\Gamma_1)}\,\theta(t)\,dt\,. \tag{7.4.12}$$

Let $j \in \mathbb{N}$ and consider $m > j$. Multiplying the second equation of (7.2.1) by $\theta \in \mathcal{D}(0,T)$, taking $v = \omega_j$ and integrating from 0 to T,

$$\begin{aligned} 0 \;=\; & \int_0^T (i\,y'_m,\omega_j)_{L^2(\Omega)}\,\theta(t)\,dt - \int_0^T (\nabla\,y_m,\nabla\,\omega_j)_{L^2(\Omega)}\,\theta(t)\,dt \\ + & \int_0^T (y'_m,\omega_j)_{L^2(\Gamma_1)}\,\theta(t)\,dt - \int_0^T (|y_m|^2\,y_m,\omega_j)_{L^2(\Omega)}\,\theta(t)\,dt\,. \end{aligned} \tag{7.4.13}$$

From convergences (7.4.9) – (7.4.12), we can pass through the limit as $m \to +\infty$ in (7.4.13) to obtain

$$\begin{aligned} 0 \;=\; & \int_0^T (i\,y'_m,\omega_j)_{L^2(\Omega)}\,\theta(t)\,dt - \int_0^T (\nabla\,y,\nabla\,\omega_j)_{L^2(\Omega)}\,\theta(t)\,dt \\ + & \int_0^T (y',\omega_j)_{L^2(\Gamma_1)}\,\theta(t)\,dt - \int_0^T (|y|^2\,y,\omega_j)_{L^2(\Omega)}\,\theta(t)\,dt\,. \end{aligned} \tag{7.4.14}$$

By the totality of the $\omega_j's$ in $H^1_{\Gamma_0}(\Omega)$, we have

$$\begin{aligned} 0 \;=\; & \int_0^T (i\,y'_m,v)_{L^2(\Omega)}\,\theta(t)\,dt - \int_0^T (\nabla\,y,\nabla\,v)_{L^2(\Omega)}\,\theta(t)\,dt \\ + & \int_0^T (y',v)_{L^2(\Gamma_1)}\,\theta(t)\,dt - \int_0^T (|y|^2\,y,v)_{L^2(\Omega)}\,\theta(t)\,dt, \forall\,v \in H^1_{\Gamma_0}(\Omega), \forall\,\theta \in \mathcal{D}(0,T)\,. \end{aligned}$$

Hence, for all $v \in H^1_{\Gamma_0}(\Omega)$,

$$i(y'(t), v)_{L^2(\Omega)} - (\nabla y(t), \nabla v)_{L^2(\Omega)} + (y'(t), v)_{L^2(\Gamma_1)} - (|y(t)|^2 y(t), v)_{L^2(\Omega)} = 0$$

holds for all $t \in [0, T]$, where by (7.4.1) – (7.4.4) and (7.4.8), T can be taken arbitrarily large.

Chapter 8

Exponential Stability

8.1 Introduction of a Multiplier

Since well-posedness of regular solutions in dimension $N = 2$ has been established, we may now prove the following result:

Theorem 8.1.1 (Stabilization). *Assume that Ω is a star-shaped domain and let y be a regular solution of the problem (1.2.1). Then, there exist positive constants γ and C such that the H^1-energy associated to problem (1.2.1) decays exponentially, that is,*

$$E(t) \leq Ce^{-\gamma t}E(0), \quad \textit{for all } t > T_0,$$

$T_0 > 0$ large enough.

The method used for achieving this stability result is classical. A multiplier is used to construct an integral identity. By choosing a particular vector field for the multiplier it is shown that the energy contracts in time. Use of the multiplier $h(x) \cdot \nabla w$ was introduced by Lasiecka, Lions and Triggiani (1986) in the study of regularity of the wave equation [22]. Triggiani exported the use of this multiplier (1989) in a result that pioneered an operator approach to stability of the wave equation [42]. This multiplier method was first translated to the Schrödinger equation by Machtyngier (1990) [37] for the linear version of the problem we are interested in studying. This complex multiplier becomes $(q \cdot \nabla \overline{y})$.

Lemma 8.1.2. *Let Ω be a bounded domain of $\mathbb{R}^2$, with smooth boundary Γ. Let $q \in \left[C^2(\overline{\Omega})\right]^2$ be a vector field. Then, for all regular solutions (e.g. solutions in the sense of Theorem 6.5.1) of the problem (1.2.1) the following identity holds*

$$\begin{aligned}
&\mathrm{Re}\left(2\int_0^T\int_\Omega \frac{\partial y}{\partial x_j}\frac{\partial q_k}{\partial x_j}\frac{\partial \overline{y}}{\partial x_k}\,dx\,dt\right) + \frac{1}{2}\int_0^T\int_\Omega (divq)|y|^4\,dx\,dt \qquad (8.1.1)\\
&= \mathrm{Re}\left[i\int_\Omega y(q\cdot\nabla\overline{y})\,dx\right]_0^T + \mathrm{Re}\left(2\int_0^T\int_\Gamma \partial_n y(q\cdot\nabla\overline{y})\,d\gamma\,dt\right)\\
&-\int_0^T\int_\Gamma (q\cdot n)|\nabla y|^2\,d\gamma\,dt - \mathrm{Re}\left(\int_0^T\int_\Omega (\nabla\overline{y}\cdot\nabla(divq))y\,dx\,dt\right)\\
&-\frac{1}{2}\int_0^T\int_{\Gamma_1}(q\cdot n)|y|^4\,d\gamma\,dt + \int_0^T\int_{\Gamma_1}\partial_n\overline{y}(divq)y\,d\gamma\,dt.
\end{aligned}$$

Proof. Multiplying equation (1.2.1) by $(q \cdot \nabla \overline{y})$ and integrating over $\Omega \times (0,T)$, we obtain

$$0 = \int_0^T (iy' + \Delta y - |y|^2 y)(q\cdot\nabla\overline{y})\,dx\,dt. \qquad (8.1.2)$$

Next, we shall analyze the first term on the RHS of (8.1.2).

Estimate for $I_1 := \int_0^T\int_\Omega iy'(q\cdot\nabla\overline{y})\,dx\,dt$.

Integrating by parts, we deduce that

$$I_1 = \left[i \int_\Omega y(q \cdot \nabla \overline{y})\, dx \right]_0^T - i \int_0^T \int_\Omega y(q \cdot \nabla \overline{y}')\, dx\, dt. \tag{8.1.3}$$

On the other hand, by making use of Gauss' formula, we infer

$$\begin{aligned}
& i \int_0^T \int_\Omega (q \cdot \nabla y)\overline{y}'\, dx\, dt = i \int_0^T \int_\Omega (q_k \overline{y}') \frac{\partial y}{\partial x_k}\, dx\, dt \\
& = -i \int_0^T \int_\Omega \frac{\partial}{\partial x_k}(q_k \overline{y}') u\, dx\, dt + \underbrace{i \int_0^T \int_{\Gamma_1} (q \cdot n) y \overline{y}'\, d\gamma\, dt}_{\text{since } y=0 \text{ on } \Gamma_0} \\
& = -i \int_0^T \int_\Omega (divq) y \overline{y}'\, dx\, dt - i \int_0^T \int_\Omega (q \cdot \nabla \overline{y}') y\, dx\, dt \\
& + i \int_0^T \int_{\Gamma_1} (q \cdot n) y \overline{y}'\, d\gamma\, dt.
\end{aligned}$$

which implies that

$$\begin{aligned}
& -i \int_0^T \int_\Omega (q \cdot \nabla \overline{y}') y\, dx\, dt \\
= \; & i \int_0^T \int_\Omega (q \cdot \nabla y) \overline{y}'\, dx\, dt + i \int_0^T \int_\Omega (divq) y \overline{y}'\, dx\, dt \\
- \; & i \int_0^T \int_{\Gamma_1} (q \cdot n) y \overline{y}'\, d\gamma\, dt.
\end{aligned} \tag{8.1.4}$$

Substituting (8.1.4) in (8.1.3), we arrive at

$$\begin{aligned}
I_1 \; = \; & \left[i \int_\Omega y(q \cdot \nabla \overline{y})\, dx \right]_0^T + i \int_0^T \int_\Omega (q \cdot \nabla u) \overline{y}'\, dx\, dt \\
+ \; & i \int_0^T \int_\Omega (divq) y \overline{y}'\, dx\, dt - i \int_0^T \int_{\Gamma_1} (q \cdot n) y \overline{y}'\, d\gamma\, dt,
\end{aligned} \tag{8.1.5}$$

and since

$$iy' = -\Delta y + |y|^2 y \;\text{ in } \Omega \;\Leftrightarrow\; \overline{y}' = -i\Delta \overline{y} + i|y|^2 \overline{y} \;\text{ in } \Omega,$$

from (8.1.5) we can write

$$
\begin{aligned}
I_1 &= \left[i \int_\Omega y(q\cdot\nabla\overline{y})\,dx \right]_0^T + \int_0^T \int_\Omega (q\cdot\nabla y)\Delta\overline{y}\,dx\,dt \\
&- \int_0^T \int_\Omega (q\cdot\nabla y)|u|^2\overline{y}\,dx\,dt + \int_0^T \int_\Omega (divq)\Delta\overline{y}y\,dx\,dt \\
&- \int_0^T \int_\Omega (divq)|y|^4\,dx\,dt - i\int_0^T \int_{\Gamma_1} (q\cdot n)y\overline{y}'\,d\gamma\,dt.
\end{aligned} \tag{8.1.6}
$$

Taking the real part of (8.1.1), having in mind (8.1.6), and observing that $\mathrm{Re}(z) = \mathrm{Re}(\overline{z})$, for all $z\in\mathbb{C}$, we deduce that

$$
\begin{aligned}
0 &= \mathrm{Re}\left[i \int_\Omega y(q\cdot\nabla\overline{y})\,dx \right]_0^T + 2\,\mathrm{Re}\int_0^T \int_\Omega \Delta y(q\cdot\nabla\overline{y})\,dx\,dt \\
&- 2\,\mathrm{Re}\int_0^T \int_\Omega (q\cdot\nabla\overline{y})|y|^2 y\,dx\,dt + \mathrm{Re}\int_0^T \int_\Omega (divq)\Delta\overline{y}y\,dx\,dt \\
&- \int_0^T \int_\Omega (divq)|y|^4\,dx\,dt - \mathrm{Re}\,i\int_0^T \int_{\Gamma_1} (q\cdot n)y\overline{y}'\,d\gamma\,dt.
\end{aligned} \tag{8.1.7}
$$

In what follows, we analyze the terms on the RHS of (8.1.7).

Estimate for $I_2 := 2\int_0^T \int_\Omega \Delta y(q\cdot\nabla\overline{y})\,dx\,dt$.

Employing Green formula, we have

$$
\begin{aligned}
I_2 &= -2\int_0^T \int_\Omega \nabla y\cdot\nabla(q\cdot\nabla\overline{y})\,dx\,dt + 2\int_0^T \int_\Gamma \partial_n y(q\cdot\nabla\overline{y})\,d\gamma\,dt \\
&= -2\int_0^T \int_\Omega \frac{\partial y}{\partial x_j}\frac{\partial q_k}{\partial x_j}\frac{\partial\overline{y}}{\partial x_k}\,dx\,dt - 2\int_0^T \int_\Omega q_k \frac{\partial y}{\partial x_j}\frac{\partial^2\overline{y}}{\partial x_k \partial x_j}\,dx\,dt \\
&+ 2\int_0^T \int_\Gamma \partial_n y(q\cdot\nabla\overline{y})\,d\gamma\,dt.
\end{aligned}
$$

Taking the real part of I_2, yields,

$$\begin{aligned}\mathrm{Re}(I_2) &= \mathrm{Re}\left(-2\int_0^T\int_\Omega \frac{\partial y}{\partial x_j}\frac{\partial q_k}{\partial x_j}\frac{\partial \overline{y}}{\partial x_k}\,dx\,dt\right) \\ &- 2\int_0^T\int_\Omega q_k\,\mathrm{Re}\left(\frac{\partial y}{\partial x_j}\frac{\partial^2 \overline{y}}{\partial x_k\partial x_j}\right)dx\,dt \\ &+ \mathrm{Re}\left(2\int_0^T\int_\Gamma \partial_n y(q\cdot\nabla\overline{y})\,d\gamma\,dt\right).\end{aligned} \tag{8.1.8}$$

Having in mind that

$$2\,\mathrm{Re}\left[\frac{\partial y}{\partial x_j}\frac{\partial^2\overline{y}}{\partial x_k\partial x_j}\right] = \frac{\partial}{\partial x_k}\left[\left|\frac{\partial y}{\partial x_j}\right|^2\right],$$

using (4.8) and applying Green's formula, we find that

$$\begin{aligned}\mathrm{Re}(I_2) &= \mathrm{Re}\left(-2\int_0^T\int_\Omega \frac{\partial y}{\partial x_j}\frac{\partial q_k}{\partial x_j}\frac{\partial \overline{y}}{\partial x_k}\,dx\,dt\right) + \int_0^T\int_\Omega (divq)\,|\nabla y|^2\,dx\,dt \\ &- \int_0^T\int_\Gamma (q\cdot n)|\nabla y|^2\,d\gamma\,dt + \mathrm{Re}\left(2\int_0^T\int_\Gamma \partial_n y(q\cdot\nabla\overline{y})\,d\gamma\,dt\right).\end{aligned} \tag{8.1.9}$$

Estimate for $I_3 := -2\int_0^T\int_\Omega |y|^2 y(q\cdot\nabla\overline{y})\,dx\,dt$.

We have,

$$I_3 = -2\int_0^T\int_\Omega |y|^2 y\,q_k\frac{\partial\overline{y}}{\partial x_k}\,dx\,dt,$$

and since

$$4\,\mathrm{Re}\left(y\frac{\partial\overline{y}}{\partial x_k}\right)|y|^2 = \frac{\partial}{\partial x_k}\left[|y|^4\right].$$

By employing Green's formula we deduce that

$$\mathrm{Re}(I_3) = \frac{1}{2}\int_0^T\int_\Omega (divq)|y|^4\,dx\,dt - \frac{1}{2}\underbrace{\int_0^T\int_{\Gamma_1}(q\cdot n)|y|^4\,d\gamma\,dt}_{\text{since } y=0 \text{ on } \Gamma_0} \tag{8.1.10}$$

Estimate for $I_4 := \int_0^T \int_\Omega (divq)\Delta\overline{y}y\,dx\,dt.$

Again applying Green's formula,

$$I_4 = -\int_0^T \int_\Omega \nabla\overline{y}\cdot\nabla\left((divq)\,y\right)\,dx\,dt + \underbrace{\int_0^T \int_{\Gamma_1} \partial_n\overline{y}\,(divq)y\,d\gamma\,dt}_{\text{since } y=0 \text{ on } \Gamma_0} \quad (8.1.11)$$
$$= -\int_0^T \int_\Omega (\nabla\overline{y}\cdot\nabla(divq))\,y\,dx\,dt - \int_0^T \int_\Omega (divq)\,|\nabla y|^2\,dx\,dt$$
$$+ \int_0^T \int_{\Gamma_1} \partial_n\overline{y}\,(divq)y\,d\gamma\,dt.$$

Through combining the results we have obtained, namely (8.1.7), (8.1.9), (8.1.10) and (8.1.11), we may now conclude that

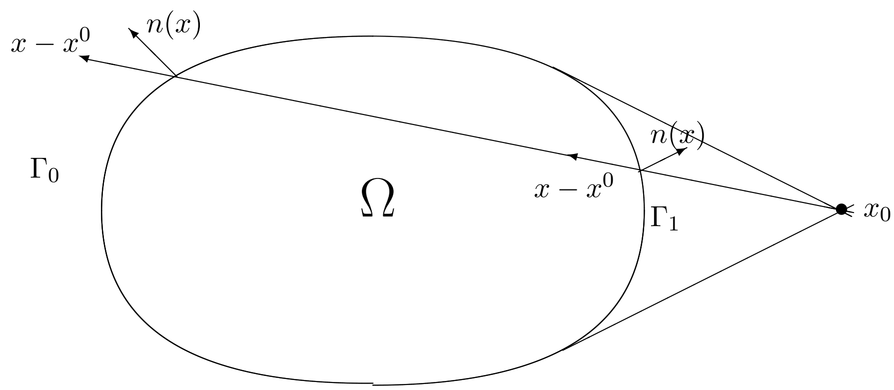
$$0 = \mathrm{Re}\left[i\int_\Omega y(q\cdot\nabla\overline{y})\,dx\right]_0^T - \frac{1}{2}\int_0^T \int_\Omega (divq)|y|^4\,dx\,dt$$
$$- 2\,\mathrm{Re}\left(\int_0^T \int_\Omega \frac{\partial y}{\partial x_j}\frac{\partial q_k}{\partial x_j}\frac{\partial\overline{y}}{\partial x_k}\,dx\,dt\right)$$
$$- \int_0^T \int_\Gamma (q\cdot n)|\nabla y|^2\,d\gamma\,dt + \mathrm{Re}\left(2\int_0^T \int_\Gamma \partial_n y(q\cdot\nabla\overline{y})\,d\gamma\,dt\right)$$
$$- \mathrm{Re}\int_0^T \int_\Omega (\nabla\overline{y}\cdot\nabla(divq))\,y\,dx\,dt - \frac{1}{2}\int_0^T \int_{\Gamma_1} (q\cdot n)|y|^4\,d\gamma\,dt$$
$$+ \int_0^T \int_{\Gamma_1} \partial_n\overline{y}\,(divq)y\,d\gamma\,dt - \mathrm{Re}\,i\int_0^T \int_{\Gamma_1} (q\cdot n)y\overline{y}'\,d\gamma\,dt.$$

which finishes the proof. □

8.2 Contraction of Energy

Until now we have only required that Ω be a connected, bounded domain with smooth boundary. We now require the additional assumption that Ω be star-shaped, namely, that for a fixed $x_0 \in \mathbb{R}^n$ we have,

$$(x - x_0) \cdot n(x) \leq 0 \ \text{ on } \Gamma_0 \ \text{ and } \ (x - x_0) \cdot n(x) > 0 \ \text{ on } \Gamma_1. \tag{8.2.1}$$



Substitute the vector field $m(x) = x - x^0$ for the vector field $q(x)$ and taking $x^0 \in \mathbb{R}^n$ to be fixed. Then from Lemma 8.1.2 we obtain

$$\begin{aligned}
&2\int_0^T \int_\Omega |\nabla y|^2 \, dx \, dt + \int_0^T \int_\Omega |y|^4 \, dx \, dt \\
&= \operatorname{Re}\left[i \int_\Omega u(m \cdot \nabla \overline{y}) \, dx \right]_0^T \\
&- \int_0^T \int_\Gamma (m \cdot n)|\nabla y|^2 \, d\gamma \, dt + \operatorname{Re}\left(2 \int_0^T \int_\Gamma \partial_n y (m \cdot \nabla \overline{y}) \, d\gamma \, dt \right) \\
&- \underbrace{\int_0^T \int_\Omega (\nabla \overline{y} \cdot \nabla (div\, m)) \, y \, dx \, dt}_{=0 \text{ since } div m = n} - \frac{1}{2} \int_0^T \int_{\Gamma_1} (m \cdot n)|y|^4 \, d\gamma \, dt \\
&+ n \int_0^T \int_{\Gamma_1} \partial_n \overline{y} \, u \, d\gamma \, dt - \operatorname{Re} i \int_0^T \int_{\Gamma_1} (m \cdot n) y \overline{y}' \, d\gamma \, dt.
\end{aligned}$$

and since $m \cdot n > 0$ on Γ_1, we deduce,

$$\begin{aligned} 4\int_0^T E(t)\,dt &\leq \operatorname{Re}\left[i\int_\Omega y(m\cdot\nabla\overline{y})\,dx\right]_0^T \\ &- \int_0^T\int_\Gamma (m\cdot n)|\nabla y|^2\,d\gamma\,dt + \operatorname{Re}\left(2\int_0^T\int_\Gamma \partial_n y(m\cdot\nabla\overline{y})\,d\gamma\,dt\right) \\ &+ n\int_0^T\int_{\Gamma_1}\partial_n\overline{y}\,y\,d\gamma\,dt - \operatorname{Re} i\int_0^T\int_{\Gamma_1}(m\cdot n)y\overline{y}'\,d\gamma\,dt. \end{aligned} \tag{8.2.2}$$

Since $y = 0$ on Γ_0 it follows that $\nabla\overline{y} = \partial_n\overline{y}\,n$ on Γ_0, and consequently,

$$\begin{cases} |\nabla y|^2 = |\partial_n y|^2 \quad \text{on } \Gamma_0, \\ m\cdot\nabla\overline{y} = (m\cdot n)\partial_n\overline{y} \Rightarrow \partial_n y(m\cdot\nabla\overline{y}) = (m\cdot n)\,|\partial_n y|^2 \quad \text{on } \Gamma_0. \end{cases} \tag{8.2.3}$$

By combining (8.2.2) and (8.2.3) we obtain

$$\begin{aligned} 4\int_0^T E(t)\,dt &\leq \operatorname{Re}\left[i\int_\Omega y(m\cdot\nabla\overline{y})\,dx\right]_0^T + \int_0^T\int_{\Gamma_0}(m\cdot n)|\partial_n y|^2\,d\gamma\,dt \\ &- \int_0^T\int_{\Gamma_1} (m\cdot n)|\nabla y|^2\,d\gamma\,dt + \operatorname{Re}\left(2\int_0^T\int_{\Gamma_1} \partial_n y(m\cdot\nabla\overline{y})\,d\gamma\,dt\right) \\ &+ n\int_0^T\int_{\Gamma_1}\partial_n\overline{y}\,y\,d\gamma\,dt - \operatorname{Re} i\int_0^T\int_{\Gamma_1}(m\cdot n)y\overline{y}'\,d\gamma\,dt. \end{aligned} \tag{8.2.4}$$

Having in mind that $m(x)\cdot n(x) \leq 0$ for all $x \in \Gamma_0$, $m(x)\cdot n(x) \geq \delta > 0$ for all $x \in \Gamma_1$, $\partial_n y = -y'$ on Γ_1 and recalling that the trace map $\gamma_0 : H^1_{\Gamma_0}(\Omega) \to L^2(\Gamma_1)$ is continuous, we see that

$$\begin{aligned} 4\int_0^T E(t)\,dt &\leq \operatorname{Re}\left[i\int_\Omega y(m\cdot\nabla\overline{y})\,dx\right]_0^T - \delta\int_0^T\int_{\Gamma_1}|\nabla y|^2\,d\gamma\,dt \\ &+ \frac{2R^2}{4\eta}\int_0^T\int_{\Gamma_1}|y'|^2\,d\gamma\,dt + 2\eta\int_0^T\int_{\Gamma_1}|\nabla y|^2\,d\gamma\,dt \\ &+ \frac{n^2\lambda_1}{4\eta}\int_0^T\int_{\Gamma_1}|y'|^2\,d\gamma\,dt + 2\eta\int_0^T E(t)\,dt \\ &+ \frac{R}{4\eta}\int_0^T\int_{\Gamma_1}|y'|^2\,d\gamma\,dt + \eta\int_0^T E(t)\,dt, \end{aligned} \tag{8.2.5}$$

where

$$R := \max_{x \in \overline{\Omega}} ||x - x^0||_{\mathbb{R}^n},$$

$\lambda_1 > 0$ comes from the Poincaré inequality and η is an arbitrary positive constant.

Choosing η sufficiently small, from (8.2.5) it holds that

$$\int_0^T E(t)\,dt \quad \leq \quad C\,\mathrm{Re}\left[i\int_\Omega u(m\cdot\nabla\overline{y})\,dx\right]_0^T + C\int_0^T\int_{\Gamma_1}|y'|^2\,d\gamma\,dt, \qquad (8.2.6)$$

where $C = C(\lambda_1, |\Omega|, n)$ is a positive constant.

Combining (8.2.6) with the energy identity we obtain

$$E(T) \leq \gamma E(0), \quad \text{for } T > T_0,$$

with T_0 sufficiently large and $0 < \gamma < 1$, which gives us the exponential stability and we conclude the proof of the theorem (8.1.1) .

Bibliography

[1] E. Anderson. *Modern Physics and Quantum Mechanics*. W. B. Saunders Company, 1971.

[2] R. Anton. Strichartz inequalities for Lipschitz metrics on manifolds and nonlinear Schrödinger equation on domains. *Bull. Soc. Math. France 136* (2008), no. 1, 27-65.

[3] H. Brézis. *Operateurs Maximaux Monotones et Semigroups de Contractions dans les Spaces de Hilbert*. Amsterdam: North Holland Publishing Co., 1973.

[4] H. Brézis. *Functional Analysis, Sobolev Spaces and Parcial Differential Equations*. New York: Springer, 2010.

[5] V. Barbu. *Nonlinear Semigroups and Diferential Equations in Banach Spaces*. New York: Springer Monographs in Mathematics, 2010.

[6] V. Barbu. *Nonlinear Differential Equations of Monotone Types in Banach Spaces*. 1976. Romania, Bucuresti: Noordhoff International Publishing,

[7] C. A. Bortot, M. M. Cavalcanti, W. J. Corrêa, V. N. Domingos Cavalcanti. Uniform decay rate estimates for Schrödinger and plate equations with nonlinear locally distributed damping. *Journal of Differential Equations 254* (2013), no. 9, 3729-3764.

[8] N. Burq, P. Gerard, N. Tzvetkov. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. *Amer. J. Math. 126* (2004) 569-605.

[9] M. M. Cavalcanti, V. N. Domingos Cavalcanti, R. Fukuoka, F. Natali, Exponential stability for the 2-D defocusing Schrödinger equation with locally distributed damping. *Differential Integral Equations 22* (2009), no. 7-8, 617-636.

[10] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano, F. Natali, Qualitative aspects for the cubic nonlinear Schrödinger equations with localized damping: exponential and polynomial stabilization. *Journal of Differential Equations 248* (2010), no. 12, 2955-2971.

[11] M. M. Cavalcanti, V. N. Domingos Cavalcanti, I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping - source. *Journal of Differential Equations 236* (2007), no. 2, 407-459.

[12] T. Cazenave, F. B. Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. *Nonlin. Anal. T.M.A. 14* (1990) 807-836.

[13] T. Cazenave, A. Haraux. *An Introduction to Semilinear Evolution Equations.* Oxford University Press, 1998.

[14] H. Brézis, T. Gallouet, Nonlinear Schrödinger evolution equations. *Nonlinear Analysis 4* (1980), 677-681.

[15] R. Cipollati; E. Matchtyngier; E. San Pedro Siqueira. Nonlinear Boundary Feedback Stabilization for Schrödinger Equations. *Differential and Integral Equations*, 1996.

[16] I. Chueshov, M. Eller, I. Lasiecka, Finite dimensionality of the attractor for a semilinear wave equation with nonlinear boundary dissipation. *Comm. Partial Differential Equations 29* (2004), no. 11, 1847-1876.

[17] A. Favini, G.R. Goldstein, J.A. Goldstein and S. Romanelli, The heat equation with generalized Wentzell boundary conditions, *Journal of Evolution Equations* (2002), 1-19.

[18] C. G. Gal, G. R. Goldstein, J. A. Goldstein. Oscillatory boundary conditions for acoustic wave equations. *J. Evol. Equ. 3* (2003), 623-635.

[19] G. R. Goldstein. Derivation and physical interpretation of general boundary conditions. *Advances in Differential Equations 11* (2006) no. 4, 457-480.

[20] R. Hyakuna, M. Tsutsumi. On the global wellposedness for the nonlinear Schrödinger equations with Lp-large initial data. *Nonlinear Differ. Equ. Appl. 18* (2011), 309-327.

[21] S. Kesavan. *Topics in Functional Analysis and Applications.* New Age International, 1989.

[22] I. Lasiecka, J. L. Lions, R. Triggiani, Nonhomogeneous boundary value problems for second order hyperbolic operators. *J. Math. Pures Appl. (9) 65* (1986), no.2, 149-192.

[23] I. Lasiecka, D. Tataru, Uniform boundary stabilization of semilinear wave equation with nonlinear boundary dissipation, *Differential Integral Equations 6* (1993) 507-533.

[24] I. Lasiecka, R. Triggiani, Optimal regularity, exact controllability and uniform stabilization of Schrödinger equations with Dirichlet control. *Differential Integral Equations 5* (1992), no. 3, 521-535.

[25] I. Lasiecka, R. Triggiani, X. Zhang. Nonconservative Schrödinger equations with unobserved Neumann B.C.: global uniqueness and observability in one shot. *Analysis and optimization of differential systems (Constanta, 2002)*, 235-246, *Kluwer Acad. Publ.*, Boston, MA, 2003.

[26] I. Lasiecka, R. Triggiani, X. Zhang. Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates. I. $H^1(\Omega)$-estimates. *J. Inverse Ill-Posed Probl. 12* (2004), no. 1, 43-123.

[27] I. Lasiecka, R. Triggiani, X. Zhang. Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates. II. $L2(\Omega)$-estimates. *J. Inverse Ill-Posed Probl. 12* (2004), no. 2, 183-231.

[28] I. Lasiecka, R. Triggiani. Well-posedness and sharp uniform decay rates at the $L^2(\Omega)$ -level of the Schrödinger equation with nonlinear boundary dissipation. *J. Evol. Equ. 6* (2006), no. 3, 485-537.

[29] I. Lasiecka, R. Triggiani. *Control Theory for Partial Differential Equations: Volume 1, Abstract Parabolic Systems: Continuous and Approximation Theories.* Cambridge University Press (2010).

[30] I. Lasiecka, R. Triggiani. *Control Theory for Partial Differential Equations I.* Cambridge University Press, 2000.

[31] J.L. Lions. *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*. Dunod, Paris, 1969.

[32] J.L. Lions; E. Magenes. *Problémes aux Limites non Homogénes, Aplications.* Dunod, Paris, 1968.

[33] T. Özsari. Weakly-damped focusing nonlinear Schrödinger equations with Dirichlet control. *J. Math. Anal. Appl. 389* (2012), no. 1, 84-97.

[34] T. Özsari. Global existence and open loop exponential stabilization of weak solutions for nonlinear Schrödinger equations with localized external Neumann manipulation. *Nonlinear Anal. 80* (2013), 179-193.

[35] T. Özsari. Well-posedness for NLS with boundary forces in low dimensions by Strichartz estimates. Pre-print.

[36] T. Özsari, V. K. Kalantarov, I. Lasiecka, Uniform decay rates for the energy of weakly damped defocusing semilinear Schrödinger equations with inhomogeneous Dirichlet boundary control. *J. Differential Equations 251* (2011), no. 7, 1841-1863.

[37] E. Machtyngier. Controlabilite exacte et stabilisation frontiere de l'equation de Schrödinger. *C. R. Acad. Sci. Paris 310* (1990), 806-811.

[38] Machtyngier, E. , Zuazua, E. Stabilization of the Schrödinger equation, *Portugaliae Mathematica 51* (1994), 243-256.

[39] Strauss, W. , Bu, C. An inhomogeneous boundary value problem for nonlinear Schrödinger equations, *Journal of Differential Equations 173* (2001) 79-91.

[40] R. S. Strichartz. Restriction of Fourier transforms to quadratic surfaces and decay of solutions of wave equation. *Duke Math. J. 44* (1977) 705-714.

[41] R.E. Showalter. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equation.* AMS, Providence, 1997.

[42] R. Triggiani. Wave equation on a bounded domain with boundary dissipation: an operator approach, *Journal of Mathematical Analysis and Applications. 137* (1989) 438-461.

[43] M. Tsutsumi. On Smooth Solutions to the Initial-Boundary Value Problem for the Nonlinear Schrödinger Equation in Two Space Dimension. *Nonlinear Analysis*, 1989.

[44] M. Tsutsumi. Global solutions of the nonlinear Schrödinger equations in exterior domains. *Commun. Partial. Diff. Eq. 8* (1983), 1337-1374.

[45] A. D. Venttsel (translated by Ralph DeMarr). On boundary conditions for multidimensional diffusion processes. *Theory of Probability and its Applications, Vol IV* (1959), no. 2, 164-177.

[46] M. V. Vladimirov, Solvability of a mixed problem for the nonlinear Schrödinger equation. *Mat. Sb. (N.S.) 130(172)* (1986), no. 4, 520-536, 576.

[47] K. Yajima. Existence of solutions for Schrödinger evolution equation. *Comm. Math. Phys. 110* (1987), 415-426.