Lifshitz Weyl Anomalies in Two and Three Dimensions

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Abstract

Our modern understanding of the forces of nature as described by quantum field theories is fundamentally based on symmetries and their associated conservation laws. Quantum anomalies occur when a symmetry of a classical field theory is violated upon quantization. Gravitational anomalies of one-loop quantum effective actions arise after coupling classical field theories to external background geometry and integrating out all dynamical matter fields in the partition function. A gravitational Weyl anomaly of a relativistic field theory is the statement that the quantum effective action is not invariant under local rescaling of the background geometry.

In this work, we study Weyl anomalies in non-relativistic Lifshitz field theories in (1+1) and (2+1) dimensions. Lifshitz field theories introduce a degree of scaling anisotropy between space and time measured by the dynamical scaling exponent z. In 1+1 dimensions, we analyze and study the physical and mathematical nature of a particular $z = 1$ and $z = 2$ Lifshitz Weyl anomaly. We then use the Fujikawa method to derive the $z = 1$ Lifshitz Weyl anomaly from a two-dimensional massless chiral field theory. We also derive the (1+1)-dimensional $z = 2$ Lifshitz Weyl anomaly from a (2+1)-dimensional non-relativistic Chern-Simons action on a manifold with a boundary. We evaluate the $z = 1$ Lifshitz Weyl anomaly on the Möbius strip and relate it to a topological invariant that counts the parity of its number of half-twists.

In 2+1 dimensions, we extend a background metric optimization procedure for Euclidean path integrals, first introduced for a two-dimensional conformal field theory, to a $z = 2$ anisotropically scale-invariant $(2 + 1)$-dimensional Lifshitz field theory of a free massless scalar field. We find optimal geometries for static and dynamic correlation functions. For the static correlation functions, the optimal background metric is equivalent to an AdS metric on a Poincare patch, while for dynamical correlation functions, we find a Lifshitz-like metric.

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Introduction

1.1 Overview of Anomalies

Our modern understanding of the forces of nature as described by quantum field theories is fundamentally based on symmetries and their associated conservation laws [1]. Quantum anomalies occur when a symmetry of a classical field theory is violated upon quantization. As it later turned out, quantum anomalies are often associated with a deeper understanding of the physics of the underlying field theory.

To be a bit more specific in what an anomaly actually means, consider a quantum theory with a symmetry group G which leaves the classical action invariant [2]. A quantum theory is called anomalous if it breaks G. Hence, anomalies are symmetries of classical theories which are quantum mechanically violated. Depending on the nature of the symmetry group G, anomalies can be classified into: discrete or continuous and global or gauge anomalies. If G is a global symmetry, then anomalies in G do not represent an inconsistency of the full theory and may even have interesting physical consequences. The most important example of this type is the axial anomaly, the non-conservation of the axial current which, as it is famously known, is important for understanding the decay rate of the neutral pion $\pi^0 \to \gamma\gamma$.

Another fundamental pillar that has shaped our understanding of modern field theory is the principle of *gauge* symmetry, or more accurately, gauge redundancy [1]. Contrary to global symmetries, gauge symmetries are essential in guaranteeing the overall consistency of the field theory. It is therefore not surprising that anomalies of *local* gauge symmetries, i.e. those where the gauge transformation parameter depends on spacetime, do indeed render the theory inconsistent. Given an anomaly of a quantum field theory, the most natural question is to ask how does one properly deal with it? Is it always possible or even desirable to cancel the anomaly? For example, the presence of the top quark was initially predicted on the basis of allowing for CP violation in the quark sector of the field theory after it was discovered in kaons a decade earlier. This is an example where the presence of anomalies eventually led to a deeper understanding of the underlying physics.

The study of anomalies span a wide range of areas in field theory and string theory [1–3]. In phenomenology, the calculation of the decay rate for neutral pions in the Standard Model of particle physics is a prime example. Examples of more formal areas include the study of dualities in supersymmetric gauge theories [4], the analysis of black hole thermodynamics and more recently entanglement entropy in anti–de sitter space/conformal field theory (AdS/CFT) correspondence [5].

Quite generally, the understanding of the physics of anomalies and what they actually reveal has passed through multiple phases [1,6]. The first phase started with the discovery of the well-known ABJ or singlet axial anomaly in four dimensions by calculating, using purely *perturbative* methods, the triangle Feynman diagram, which gives the *1-loop* quantum effective action. As we mentioned earlier, an anomaly is associated with a non-conserved current of the quantum effective action. In the axial anomaly, the axial current is broken

$$\mathcal{A} = \partial^{\mu} j^{5}_{\mu} = \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \tag{1.1}$$

which when extended to non-abelian fields $F_{\mu\nu} = F^{a}{}_{\mu\nu} T^{a}$ gives the singlet anomaly

$$\mathcal{A} = \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} \mathrm{tr} F_{\mu\nu} F_{\alpha\beta} \tag{1.2}$$

where tr is the trace over the Lie algebra index a in T^a. It is the regularization of the divergent triangle diagram that actually breaks the axial current and thus induces the anomaly. However, the second phase of understanding the underlying physics of anomalies revealed that considering them a mere perturbation effect lacked a deeper grasp of what they actually mean. This second phase uncovered an exciting connection to the fields of differential geometry, (co)homology theory and topology. For example, the singlet anomaly in (1.2) was cast in the form of *analytical index* of a Dirac operator that satisfies some algebraic properties. The index is simply a number that counts the difference in the chirality or the handedness of the *zero modes* of the Dirac operator

$$\text{index } \widehat{D}_{+} = \frac{1}{2\pi} \int \mathcal{A}(x) = \int dx\, \alpha(x) \sum_{n} \tilde{\varphi}^{\dagger}_{n}(x) \gamma_5 \tilde{\varphi}_{n}(x) = n_{+} - n_{-} \tag{1.3}$$

where the $\tilde{\varphi}_n$ are the eigenfunctions of the $\widehat{D}_{+}$ and $\alpha(x)$ is the local chiral transformation parameter. Using the celebrated Atiyah-Singer (AS) index theorem [7–10] (see also chapter 11 in [1]), the analytical index was found to be equivalent to a *topological index* that assigned the anomaly a characteristic or *cohomology* class, i.e. a certain topological invariant. Running with the example of a singlet anomaly in four dimensions,

$$\text{index } \widehat{D}_{+} = -\frac{1}{8\pi^2} \int \text{tr } FF \tag{1.4}$$

Please note that the AS index theorem simply connects geometry with topology. Intuitively, it tells us that the integral of a *local geometric* invariant on the right hand side, in this case the square of the field strength tensor or curvature, is associated with a global topological property, the Chern number. In the context of the theory of gravity and curved spacetime, the famous Gauss-Bonnet theorem is a special case of the AS index theorem.

The end of the second phase culminated with the work of Fujikawa [11–14] who discovered another connection between anomalies and path integral measures of gauge theories. More specifically, given the following path integral functional of the gauge field A_μ

$$Z[A_\mu] = e^{-W[A_\mu]} = \int (\mathcal{D}\overline{\psi})(\mathcal{D}\psi) e^{S[A_\mu]} \tag{1.5}$$

where the classical action of a massles fermion in the background of A_μ is given by

$$S_\psi = \int d^2x \ \overline{\psi} \ i\gamma^\mu(\partial_\mu + A_\mu)\psi \tag{1.6}$$

The classical action is invariant under the following chiral transformation

$$\psi'(x) = e^{i\alpha(x)\gamma_5}\psi(x), \quad \overline{\psi'}(x) = \overline{\psi'}(x)e^{i\alpha(x)\gamma_5}. \tag{1.7}$$

After quantization, the path integral or quantum effective action $Z[A_\mu] = e^{-W[A_\mu]}$ break the classical chiral symmetry. Fujikawa was able to show that the underlying reason for this is the transformation of the path integral measure

$$(\mathcal{D}\overline{\psi})(\mathcal{D}\psi) \to (\mathcal{D}\overline{\psi})(\mathcal{D}\psi)J[\alpha] \tag{1.8}$$

where $J[\alpha]$ is the Jacobian of the transformation given by

$$J[\alpha] = \exp\left(-2i \int d^2x\, \alpha(x) \sum_n \tilde{\varphi}_n^\dagger(x)\gamma_5\tilde{\varphi}_n(x)\right) \tag{1.9}$$

The *regularized* sum in the exponential is exactly the chiral anomaly

$$2\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\ . \tag{1.10}$$

Fujikawa's method is a simple yet very powerful computational method and has been extended to calculate other types of anomalies such as the gravitational ones.

The third phase of our modern understanding of anomalies started in the eighties with the work of Stora, Wess and Zumino who were able to use a variety of purely mathematical methods to derive anomalies in quantum field theories (see chapters 8, 9 and 10 in [1]). Of special importance to us in our work is the *Wess-Zumino* (WS) consistency conditions [15]. These conditions are relations that must be satisfied by the anomalous currents (or effective actions), the solution of which classify the space of all possible terms that can arise from the variation of the quantum effective action under a specific gauge transformation into two classes: (1) relevant or nontrivial anomalies and (2) irrelevant or trivial anomalies. Terms in the first class are called *cocycles* and represent the actual physical anomalies of the quantum field theory, in the sense that they cannot be *removed* by adding a *counterterm*, a local functional, with appropriate coefficients, to the classical action whose variation exactly cancels the anomaly. Irrelevant anomalies on the other hand, can be removed by adding such terms to the classical action, hence the name trivial.

To be more concrete, following the notations in [16], let us assume that we are given a theory with the classical symmetries:

$$\delta_{\chi^\alpha} S(\{F\},\{\phi\}) = 0, \tag{1.11}$$

where χ^α are the gauge transformation parameters, $\{F\}$ a set of background fields, $\{\phi\}$ the set of dynamical fields in the theory, and S the classical action. In the cohmological description of the WZ consistency conditions, the gauge transformation parameter is replaced by a *Grassmannian* BRST-like ghost, and its action on the fields is defined such that it becomes nilpotent

$$(\delta_\sigma)^2 = 0. \tag{1.12}$$

The WZ consistency conditions are then simply given by

$$\delta_\sigma A_\sigma = 0, \tag{1.13}$$

where σ is now the *Grassmannian BRST ghost*. There are two kinds of solutions to the WS conditions: (1) trivial solutions which take the following form

$$A_\sigma = \delta_\sigma G(\{F\}), \tag{1.14}$$

where $G(\{F\})$ is a local functional of the background fields (of zero ghost number). These solutions are called *exact or coboundary terms* and (2) non-trivial solutions (or 1-cocycles) which also *closed* under δ_σ but may or may not be exact. The problem of finding the *physical or relevant* anomalies now becomes the problem of finding the space of δ_σ-closed terms (or *1-cocycles*) with ghost number 1, modulo the space of δ_χ-exact terms (*or coboundaries*). Those terms then define *cohomology classes* that belong to a *cohomology group.*

Solving the WZ consistency conditions can be compared to the more familiar counterpart in de-Rham cohomology which deals with differential forms. If M is a n-dimensional smooth differential manifold and the set of differential forms of degree k are denoted by $\Omega^k(M)$, then the exterior derivative is a map $d : \Omega^k(M) \mapsto \Omega^{k+1}(M)$

is the counterpart of δ_σ in the WZ condition. The *kernel* of d is

$$Z^k = \ker(d : \Omega^k(M) \mapsto \Omega^{k+1}(M)) \subset \Omega^k \ , \tag{1.15}$$

are coycles of degree k or k-cocycles and they are the counterpart of (1.13). Similarly, the *image* of d are elements of

$$B^k = \mathrm{im}(d : \Omega^{k-1}(M) \mapsto \Omega^k(M)) \subset \Omega^k \ , \tag{1.16}$$

are k-coboundaries or k-exact forms and they correspond to (1.14). Then, space of k-forms closed under d that are not k-exact is the quotient of $H^k = Z^k/B^k$ defines the k-th *cohomology group* whose elements (physical anomalies) are *cohomology classes*. This is one way of seeing how physical anomalies can potentially have a topological characterization and are thus related to topological invariants.

Since we will need it later in this section, we end our formal tour of anomalies by briefly defining the term *descent equations*, which although purely mathematical in nature, their contents carry a simple physical meaning [1,3]. Descent equations simply link together the relevant anomalies in different dimensions by a *chain* of equations (actually polynomials) that start from a symmetric gauge-invariant polynomial in F dimensions *descending* to lower dimensions. Running with our singlet anomaly example, by using the descent equations (sometimes called the Stora-Zumino equations), it was surprising to find the singlet anomaly in (1.2) is linked to the *non-Abelian* anomaly in four dimensions via a Chern-Simons term in three dimensions [1]. Descent equations can either be *non-trivial or trivial.* Non-trivial simply means that the anomaly term in lower dimensions are linked to another in higher dimensions, which has a topological character. Trivial descent equations on the other hand, do not link the chain terms in different dimensions. In the following section, we will see how the

WZ consistency conditions and descent equations have been used to cohomologically classify Weyl anomalies.

1.2 Weyl Anomalies

Gravitational anomalies of one-loop quantum effective actions arise after coupling classical field theories to external background geometry (or gravity) and integrating out all dynamical matter fields in the partition function. In other words, only matter fields are quantized while the gravitational field itself is left classical. Since the quantum energy-momentum tensor, by definition, encodes the response of effective actions to infinitesimal variations in the underlying background metric, they are the central objects in studying gravitational anomalies. There are three types of gravitational anomalies. Lorentz anomalies break the symmetry of local Lorentz transformations and is signaled by the presence of an antisymmetric energy-momentum tensor. The Einstein or diffeomorphism anomaly break the classical diffeomorphism invariance of the underlying field theory and its presence is detected by the fact that the energy-momentum tensor is not covariantly conserved. Lorentz and diffeomorphism anomalies indicate that the underlying quantum field theory cannot be consistently coupled to frame or metric gravity.

A gravitational conformal (or Weyl anomaly) is the statement that the quantum effective action is not invariant under *local rescaling* of the background metric that couples to the dynamical fields in the action. The trace of the expectation value of the energy-momentum tensor is the canonical test of whether the theory is Weyl anomalous or not. If the trace is non-zero, the quantum theory suffers a conformal anomaly. For example, let the classical action of a matter field ψ coupled to background fixed metric $g_{\mu\nu}$ be given by $S[g,\phi]$. Then partition function of the background metric $Z[g]$

is given by

$$Z[g] \;=\int [d\psi]_g \, e^{-S[g,\phi]} = e^{-W[g]} \tag{1.17}$$

where $W[g]$ is the quantum effective action. The variation of $W[g]$ under an infinitesimal local Weyl transformation $g_{\mu\nu} \to (1+\sigma(x))g_{\mu\nu}$ gives

$$\delta^W_\sigma W[g] = -\frac{1}{2}\int d^2x\sqrt{g}\,\sigma(x)\,\left\langle T^\mu_\mu(x)\right\rangle \tag{1.18}$$

where

$$\langle T^{\mu\nu}\rangle = \frac{2}{\sqrt{g}}\frac{\delta W[g]}{\delta g_{\mu\nu(x)}} \tag{1.19}$$

The *cohomological* formulation of the algebraic structure of relativistic *conformal anomalies* [17–21] classified the Weyl anomalies in even dimensions into one of two main classes [19]:

1. *Type-A* anomaly, given by the integrated Euler density of the manifold, the Euler class,

2. *Type-B* anomalies consisting of the integral of the Weyl transformation factor times *Weyl-invariant* scalar densities.

In this classification, type-A anomalies appear in a *scale-independent* effective action where the integrated anomalies vanish and are therefore related to a topological invariant. Type-B anomalies, on the other hand, appear in a scale-dependent action and hence, are not associated to topological invariants [16].

While the authors of [19] emphasized the scale dependence of the effective action, the author of [21] classified the Weyl anomalies using descent-equations. In his classification, he found that all type-B anomalies satisfy a trivial descent of equations, i.e. they are Weyl-invariant densities satisfying $\delta_\sigma\left(\sqrt{g}\phi\right) = 0$ [1]. and therefore, do not

[1] ϕ is a scalar formed from contractions of the Levi-Civita tensor, the Riemann tensor and its covariant derivatives [16]

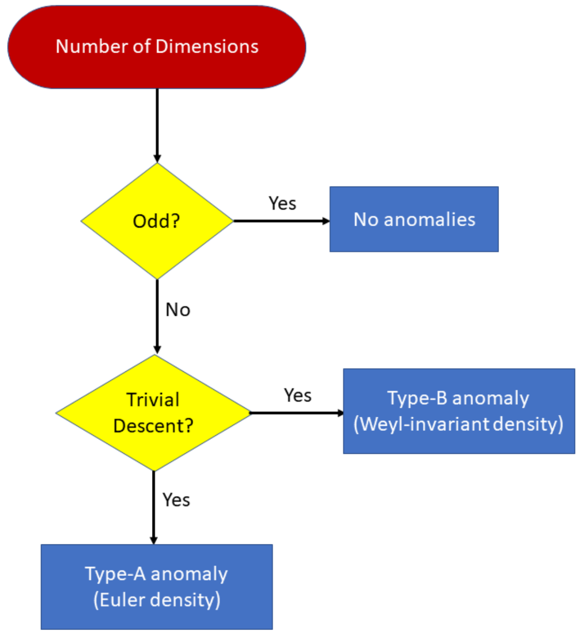


Figure 1.1. Types of Weyl Cocycles. For relativistic CFTs, there are no Weyl anomalies in odd dimensions. In even dimensions, Weyl anomalies can either be Type-A or Type-B. Note that, for non-relativistic CFTs, Weyl anomalies can be found in odd dimensions as we will show in Chapter 5.

have a topological character. He also found that type-A anomalies have non-trivial descent and that the unique anomaly in this class is the Euler density. Fig. 1.1 shows a flowchart of the types of Weyl cocycles.

Another type of classification of anomalies within the framework of the cohomology of a *total* operator, i.e. an operator that represents the variation with respect to the transformation parameters of *all* symmetries of an underlying theory. Let the total operator representing the three gauge symmetries of the theory be given by

$$\delta \equiv \delta^D_\xi + \delta^L_\alpha + \delta^W_\sigma . \tag{1.20}$$

where D denotes diffeomorphism, L Lorentz and W Weyl transformations. According to the classification theory by Bonora [18], the *nontrivial* terms, i.e. nontrivial cocycles (or anomalies), in the cohomology of the total operator are classified into two categories:

1. terms in the relative cohomology with respect to *one* of the symmetries – terms which are closed but not exact under *only one* of the symmetries, when considering only the space of terms invariant under all the rest. Intuitively speaking, if we are, for example, interested in calculating Weyl anomalies of a specific theory with a given set of symmetries, then the set of all possible Weyl-anomalous terms must obey the rest of the symmetries, i.e. Lorentz and diffeomorphisms, of the underlying theory.

2. terms in the cohomology of one of the operators, which admit a *partner* in the cocycle space of another, such that their sum is a cocycle of the total operator.

Terms in the second class are interesting since although they appear as anomalies of some symmetry as a result of anomalies in another symmetry. They are partners of some other type of anomaly in the theory. We will see in the next section that all terms in the relative cohomology of the Lifshitz Weyl operator belong to the first class. [16]. However, the $z = 1$ Lifshitz Weyl anomaly was found to be the Weyl partner of the Lorentz anomaly in the conformal cohomology. In other words, it is in the cocycle space of the *Lorentz* operator that admits a partner in the cocycle space of the *Weyl* operator. Concretely, in a 2d massless *chiral* CFT coupled to background gravity, the anomalous trace of the energy-momentum tensor is given by [1, 22, 23]

$$\langle T^{\mu}_{\mu} \rangle = R - \underbrace{2\epsilon^{ab}\nabla_{\mu}\omega^{\mu}{}_{ab}}_{\text{Weyl partner}} \ . \tag{1.21}$$

where $\omega^{\mu}{}_{ab}$ is the component of the spin connection 1-form ω_{ab}. Whereas the Ricci scalar R is the usual conformal anomaly, the second term (the Weyl partner) only appears as a consequence of the Lorentz anomaly (a Lorentz cocycle), i.e. is a partner of the Lorentz anomaly. In simple words, the term $\epsilon^{ab}\nabla_{\mu}\omega^{\mu}{}_{ab}$ only appears in a chiral CFT which is known to break local Lorentz invariance. We will dwell further into this relationship and explore more of its properties in this thesis in Chapters 2 and 3.

There is another interesting yet subtle aspect of the relationship between the Weyl and Lorentz (or diffeomorphism) anomaly. On the one hand, the subtlety manifests itself in subtracting (or adding) a local counterterm (Bardeen-Zumino term) to the effective action and the effect this term may have on removing one anomaly *and* adding another [24] [17]. Equivalently, the subtlety shows up in the *arbitrariness* of the regularization method used in calculating the anomaly. It was in fact shown in [17] that subtracting a local counterterm from the classical action is equivalent to changing the regularization scheme. In two dimensions, in particular, the Weyl anomaly can be removed by a local counterterm at the expense of a diffeomorphism anomaly and vice versa. With a class of regularizations depending on a continuous parameter α, it was shown that for $\alpha = 0$, diffeomorphism invariance is preserved while a Weyl anomaly appears

$$\delta^{W}_{(\alpha=0)} \propto \int d^2x \sqrt{g}\, \sigma(x)\, R, \qquad \delta^{D}_{(\alpha=0)} = 0 \;, \tag{1.22}$$

while for $\alpha = 1/2$, Weyl symmetry is preserved at the expense of breaking diffeomorphism invariance

$$\delta^{W}_{(\alpha=1/2)} = 0, \qquad \delta^{D}_{(\alpha=0)} \propto \int d^2x \sqrt{g}\, \partial_\rho \xi^\rho \left(-R + g^{\mu\nu} D_\mu \Gamma_\nu\right) \tag{1.23}$$

where ξ_ρ is the diffeomorphism transformation parameter.

1.2.1 Non-relativistic Weyl anomalies

Recently, there has been a considerable level of activity in studying Weyl anomalies in non-relativistic field theories such as Galilean, Schrödinger and Lifshitz field theories [16, 25–32]. Non-relativistic field theories do not place space and time on an equal footing and thus introduce a degree of anisotropy between them. In particular, Lifshitz field theories in D spatial dimensions, are invariant under *global* anisotropic non-relativistic scaling of their temporal t and spatial coordinates x^i

$$t \to \lambda^z t, \quad x^i \to \lambda x^i \tag{1.24}$$

where z is known as *dynamical scaling exponent* and $i = 1 \dots D$ is the spatial index. The dynamical scaling exponent measures the degree of anisotropy between time and space. The symmetry group of a Lifshitz field theory consists of four generators: $H, P^i, L^{ij}, \mathcal{D}$ where H is the generator of time translations, P^i the generator of spatial translations, L^{ij} the generator of rotations and $\mathcal{D}$ is the generator of scaling transformations with dynamical scaling exponent z that sends $t \to \lambda^z t$ and $x^i \to \lambda x^i$ (see section 2 in [33]).

Lifshitz scaling symmetry has naturally appeared in a wide variety of theoretical and experimental setups. In particular, the scaling symmetry with $z = 2$ has emerged near conformal quantum critical points [34] in (2+1)-dimensions, [35–37], with $z = 2$ scaling. More recently, Lifshitz-type scaling has been potentially linked to an emergent symmetry of the continuum limit of gapless quantum spin chains [38]. Lifshitz field theories have applications in high-energy physics [39], hydrodynamics [40–42].

Contrary to *relativistic* field theories which are locally symmetric under fully covariant diffeomorphism transformations, Lifshitz field theories are locally symmetric under foliation-preserving diffeomorphisms (FPD) defined by

$$t \to \tilde{t}(t), \quad x^i \to \tilde{x^i}(x^i, t) \ . \tag{1.25}$$

In FPD-invariant theories, the spacetime is naturally foliated into *equal-time slices or hypersurfaces or leaves* where x^i= const.. On a manifold equipped with by a Riemannian spacetime metric $g_{\mu\nu}$ [43] [16], a foliation is characterized by a smooth timelike 1-form n_μ *normal* to the foliation *leaves*. In Section 2, we will provide a more formal definition of foliated manifolds. Lifshitz field theories are also invariant under local *anisotropic* Weyl scaling transformations characterized by a dynamical scaling exponent $z \geq 1$ as opposed to relativistic conformal symmetry in conformal field theories (CFTs) [44].

Building on the work of [20, 21], in their cohomological classification of Lifshitz anomalies, the authors in [16], found that all Lifshitz Weyl anomalies are type-B and thus belong to the same cohomology class of a trivial descent cocycle. Specifically, if $\mathcal{A}$ is a Lifshitz Weyl anomaly, then it satisfies the following equation

$$\mathcal{A} = \mathcal{H} + \mathcal{F} \tag{1.26}$$

where $\mathcal{H}$ is some trivial descent cocycle and $\mathcal{F}$ is a trivial descent coboundary.

Another very important aspect of studying quantum anomalies in field theories is the type of background geometry to which they can couple. For example, relativistic field theories couple to Lorentzian geometries while on the other hand, non-relativistic field theories typically involves coupling to *non-relativistic* geometries, for example, the Newton-Cartan (NC) or Schrödinger spacetimes. Newton-Cartan theory is a re-formulation and generalization of Newtonian gravity first introduced by Cartan in [45] [46] and Friedrichs [47] and later developed by many others. See [48] for a more comprehensive list. The NC theory places the Newtonian theory of gravity on geometrical grounds in an attempt to mimic the the geometrical formulation of the Einstein's general theory of relativity. However, it wasn't until the appearance of *torsion* in the NC geometry that the interest in NC spacetimes has started to significantly grow with several promising applications in high-energy and condensed-matter systems. NC geometry with *torsion*, or torsional NC geometry (TNC) has recently been the focus of intense study. TNC geometry has appeared in different physical setups and applications, for example, in boundary effective actions of non-relativistic holographic theories [49–54] and in effective field theories of quantum Hall states [55, 56], in Weyl-invariant field theories coupled to flat NC spacetime were constructed in [51, 54]. We will formally define the TNC geometry and its different types of in Section 2.1.

Recently, Weyl anomalies of Lifshitz field theories coupled to NC geometry with *temporal torsion* have been calculated in several spacetime dimensions and for multiple values of the scaling exponent z by solving the WZ consistency condition [16]. It was found in [16] that while the conformal anomalies of (1+1)-dimensional relativistic conformal field theories are *type-A*, those of Lifshitz field theories belong to *type-B*. Specifically, in 1+1 dimensions, which is the main focus of this dissertation, and for *any* value of z, only one *trivial descent* anomaly i.e. a trivial descent cocyle modulo a coboundary term, was found in the *parity-odd*, mixed-derivative sector of the Lifshitz cohomology of the relative Weyl operator [16]. The rest of the cocycles were shown to be trivial descent *coboundaries* and thus, can be removed by local counterterms. It was also found that the (1+1)-dimensional Weyl anomaly breaks time-reversal invariance. More importantly, in [16], it was shown that the $z = 1$ (1+1)-dimensional Lifshitz Weyl anomaly is actually the *Weyl partner* of the Lorentz anomaly in (1+1)-dimensional CFT.

The goal of Chapter 2 is to analyze and investigate the physical as well as the mathematical nature of the (1+1)-dimensional Lifshitz Weyl anomaly within a non-relativistic framework. More specifically, we will see how non-relativistic field theories coupled to *background* Newton-Cartan (NC) geometry with temporal torsion can generate Weyl anomalies. We will use the Arnowitt, Deser and Misner (ADM) parametrization [57] in our study to discuss the geometric nature of the (1+1)-dimensional Lifshitz Weyl anomaly, true for $z \geq 1$.

In addition, we dedicate special attention to understanding the $z = 1$ (1+1)-dimensional Lifshitz Weyl anomaly and how it relates to the Lorentz anomaly in two-dimensional CFTs. In the Appendix to Chapter 2, we will unveil the underlying connection between these anomalies within the framework of the geometry of foliated manifolds [58]. We will attempt to place the connection on firm mathematical grounds using the language of fiber bundles. In particular, we will clearly show that the proper mathematical characterization of the Lifshitz Weyl anomaly described in Section 2.3 is directly related to the *geometry and topology* of *flat* line bundles.

Thus far, the $z = 1$ Lifshitz Weyl has been found formally using a cohomological approach. It would be interesting, however to derive it from a specific field theory. This is the goal of Chapter 3. In Section 3.1, we will use the *Fujikawa* method to derive the anomaly from a 2d massless chiral field theory. We will explain why chirality is an essential requirement to get the correct expression of the 2d $z = 1$ Lifshitz Weyl anomaly. Owing to the true nature of $z = 1$ Weyl anomaly as the Weyl partner of the Lorentz anomaly, we will expand the Dirac operator in the Jacobian of the path integral measure in a *chiral spinor* basis [12] [59] in order to obtain the correct expression of the $z = 1$ Weyl anomaly.

In Appendix 3.A, we present an attempt to derive the respective Lifshitz Weyl anomaly from the simplest $z = 1$ FPD-invariant action of a massless scalar field using heat kernel expansion [60]. Up to first order in perturbation theory, we do not find the *relevant* Weyl anomaly. Concretely, the final expression of the anomaly that we obtain consists only of *irrelevant or coboundary* terms, i.e. terms that can be removed by adding local counterterms to the quantum effective action. Although we did not do to second-order perturbation theory in our heat kernel expansion, we do not expect it would change the parity symmetry $x \to -x$ but this remains to be checked nevertheless.

1.3 Non-relativistic gauge-gravity duality

Gauge/gravity duality [61,62] is a conjectured relationship between a relativistic field theory in flat spacetime and a gravity theory in one higher spatial dimension. The best understood example is the anti de-Sitter/conformal field theory (AdS/CFT) correspondence relating string theory in an anti de-Sitter (a negatively curved spacetime) background to conformal field theories in one less spatial dimension [63, 64]. The AdS/CFT duality is a strong-weak duality in the sense that when the fields of the quantum field theory side are strongly interacting, those in the gravity theory are weakly interacting and thus more mathematically tractable. This feature of the duality has been used to study a wide variety of problems ranging from high-temperature quantum chromodynamics and hydrodynamics [65] to condensed-matter systems. Holographic superconductors [66], cold atoms [67], and non-Fermi liquids [68] are prominent examples to name a few. Figure 1.2 shows a simple depiction of the AdS/CFT correspondence. In the interest of being a bit more concrete, the gauge/gravity duality can be stated in simple mathematical terms as a relationship between the quantum effective action functional $W[\phi]$ on the field theory side, the *boundary*, and the *bulk* on-shell gravitational action with specific asymptotic boundary conditions

$$W[\phi^{(0)}] = -S^{grav}_{\mathrm{on-shell}}[\phi^{(0)}]. \tag{1.27}$$

Non-relativistic geometries such as Lifshitz and Schrödinger spacetimes have been realized holographically using relativistic and non-relativstic (NR) theories of gravity. Denoting the extra holographic dimension by r, the Lifshitz geometry [69] can be expressed as [69]

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{dx^i dx_i}{r^2} \tag{1.28}$$

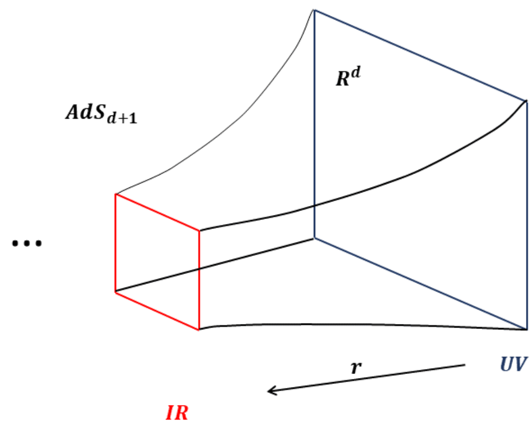


Figure 1.2. The AdS/CFT correspondence relates weakly couples gravity theories in an anti de-Sitter background to conformal field theories in one less spatial dimension.

The Lifshitz geometry has been first obtained as a solution to an Einstein gravity theory coupled to a massive vector field [70]. Lifshitz geometries also arise as solutions to higher derivative gravity theories [71]. Lifshitz black hole solutions to higher derivative theories in various dimensions were given in [72–77]. The Schrödinger geometry

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{2dtd\xi + dr^2 + dx^i dx_i}{r^2}$$

where ξ is an additional null direction resulting from the light cone reduction of a Lorentz-invariant theory in one higher dimension, also arises as a solution to models of Einstein gravity coupled to massive vector fields [67, 78].

On the front of non-relativistic quantum gravity, Horava-Lifshitz (HL) theories of gravity have been introduced as a *power-counting renormalizable* non-relativistic gravitational theories with anisotropic scaling symmetry [79, 80]. The central idea behind Horava-Lifshitz gravity theories is that by introducing terms with higher spatial derivatives, the ultraviolet (UV) behavior of the graviton propagator is improved and the theory eventually becomes power-counting renormalizable. When the number of spatial dimensions equals the dynamical scaling exponent z, Weyl-invariant actions can be found. HL actions break the principle of general covariance by foliating spacetime with space-like surfaces and introducing extra geometric data that affect the number and dynamics of degrees of freedom in the theory. As a result, not only do they describe the dynamics of the helicity-2 modes of the spatial metric but also an extra *helicity-0* scalar mode. Since this foliation mode is an excitation of the global time, it is usually called a *scalar khronon* [81]. The Lifshitz geometry has been realized as solution to a four-dimensional non-projectable HL gravity [82] using the work of [83] which defines the the notion of anisotropic conformal infinity.

Analogous to boundary relativistic field theories which naturally couple to background Lorentzian geometry, non-relativistic field theories couple to background non-relativistic geometries such the NC spacetimes [51]. The connection between dynamical NC geometry, with and without torsion, to HL gravity theories was demonstrated in [52]. More specifically, it was shown that dynamical NC geometries without torsion give rise to the so-called *projectable* HL gravity while those with twistless torsion (TTNC) i.e. those that obey the Frobenius condition and do not allow torsion on the spatial slices, give rise to the non-projectable version of HL gravity. Projectable HL gravity theories are those where the time component of the spacetime metric depends only on time whereas the non-projectable version emerges when it is a function of both space and time, i.e. $N(x,t)$. Weyl-invariant theories of HL gravity can only be non-projectable [84, 85].

Gauging, i.e. making local, a symmetry algebra is closely related to spacetime geometry. For example, gauging the Poincare algebra with some constraints, naturally gives rise to Riemannian geometry that couples to *relativistic* field theories [86] [also see appendix A in [52]]. In non-relativistic systems, it was shown in [87] and [88] that gauging the Bargmann and Schrodinger algebras, both non-relativistic symmetry algebra, leads to NC geometries without and with torsion respectively. More specifically, as noted in [52], adding torsion to the NC geometry amounts to making

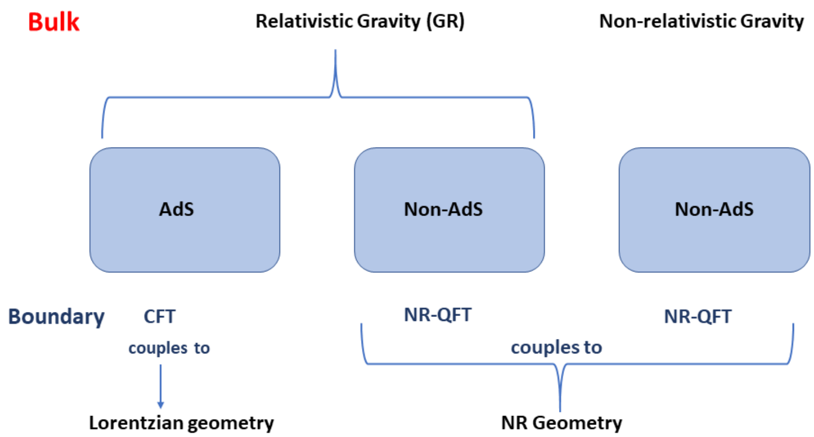


Figure 1.3. The coupling of boundary field theories to background geometries in AdS and non-AdS holography. While relativistic field theories couple to Lorentzian geometries, non-relativistic ones couple to non-relativistic geometries such as the Newton-Cartan.

it locally scale-invariant by gauging the Schrodinger algebra. Therefore, it stands to reason that the 1+1 Lifshitz anomaly is directly linked to the torsion vector of the NC geometry, which as shown in [52], maps directly to the torsion or acceleration vector a_μ in HL gravity theories.

By gauging the non-relativistic Bargmann and centrally-extended Schrodinger symmetry algebras, the authors in [89] constructed a (2+1)-dimensional non-relativistic Bargmann-invariant and Schrödinger-invariant Chern-Simons (NRSCS) actions, respectively. While the former was found to give projectable HL theory of gravity, the latter, was found to be equivalent to $z = 2$ conformal i.e Weyl-invariant non-projectable HL gravity. Chern-Simons CS actions are known to be gauge-invariant up to total derivative terms. On manifolds with boundaries, these total derivative terms can generate anomalies of boundary quantum effective actions. For example, in the context of AdS/CFT, under an infinitesimal diffeomorphism or Lorentz transformation, the boundary term of the gravitational CS (gCS) action added to a three-dimensional on-shell gravitational action generates a diffeomorphism or Lorentz anomaly, respectively, of a two-dimensional *boundary* CFT effective action [90].

In Chapter 4, we will derive the (1+1)-dimensional $z = 2$ Lifshitz Weyl anomaly from the NRSCS action. More specifically, by placing the NRSCS action on a manifold with a boundary, we will show that under a Weyl transformation, the NRSCS action changes by a total derivative term that precisely matches the boundary Weyl anomaly of a $z = 2$ Lifshitz effective action coupled to background TTNC geometry. We will show that the (1+1)-dimensional $z = 2$ Lifshitz Weyl anomaly can be derived holographically from a specific term in the three-dimensional NRSCS action constructed from the gauge fields of the Weyl and special conformal symmetry generators of the Schrodinger algebra. We call this term the torsional CS (tCS) term. We will show that the tCS term added to a three-dimensional Weyl-invariant HL gravity action plays a role similar to what the gCS term plays when the latter is added to a three-dimensional diffeomorphism-invariant action.

In Appendix 4.A, we will show that the differential form $a \wedge da$ is closed and independent of the choices of the foliation and torsion 1-forms n_μ and a_μ. Hence, it defines a cohomology class, known as *Godbillon-Vey class* $GV(\mathcal{F})$ in the third real de-Rham cohomology group $H^3(M; R)$.

1.4 Emergent Geometry and Path Integral Optimization for a Lifshitz Action

An important quest of many body physics is the search for efficient variational characterizations of correlated quantum systems. (for a review see, e.g., [91]). A class of tensor network states, particularly geared towards the description of scale-invariant systems, are called the *multi-scale entanglement renormalization ansatz* (MERA) [92,93]. MERA is used to represent approximate ground states of 1D quantum spin chains at criticality described by 2D conformal field theory (CFT) [94]. The scale-invariance of the MERA network turned out to also play a special role in connecting it to holographic duals in the sense of the AdS/CFT correspondence [95]. Here, the bulk of a MERA network can be understood as a discrete realization of 3D anti-de Sitter space (AdS_3), identifying the extra holographic direction with the renormalization group (RG) flow in the MERA [95].

Motivated by the procedure of tensor network renormalization in [96], where the path integral is first discretized into a lattice and then mapped into a tensor network which turns out to be a MERA, Caputa et. al, in a recent series of works [97, 98], took a step further in studying this relationship from the viewpoint of optimizing Euclidean path integrals that represent the ground state wave functional of two-dimensional CFT. Starting with flat Euclidean metric with a UV cutoff, they argued that their optimization procedure amounts to minimizing the Jacobian of the scale transformation for the path integral measure. In the conformally flat gauge, this translates to solving the equation of motion of the Liouville effective action from which they find that the AdS_3 metric a Poincare patch H_2 naturally emerges. This new approach is very appealing, as it suggests a concrete procedure connecting the AdS/CFT correspondence with numerical approaches to many body systems, such as the MERA tensor network [92, 93, 95, 99].

In Chapter 5, we extend the idea in [97, 98] to a *non-relativistic* field theory, specifically to a $z = 2$ anisotropically scale-invariant $(2+1)$-dimensional Lifshitz field theory of a free massless scalar field and show that the procedure can be successfully applied in systems of interest beyond a CFT. We show how natural geometries arise from the path integral optimization procedure. Our results are illustrated in Fig. 5.2. Concretely, we show the following:

1. Extend the background metric optimization procedure for Euclidean path integrals of two-dimensional conformal field theories, first introduced in [97, 98], to

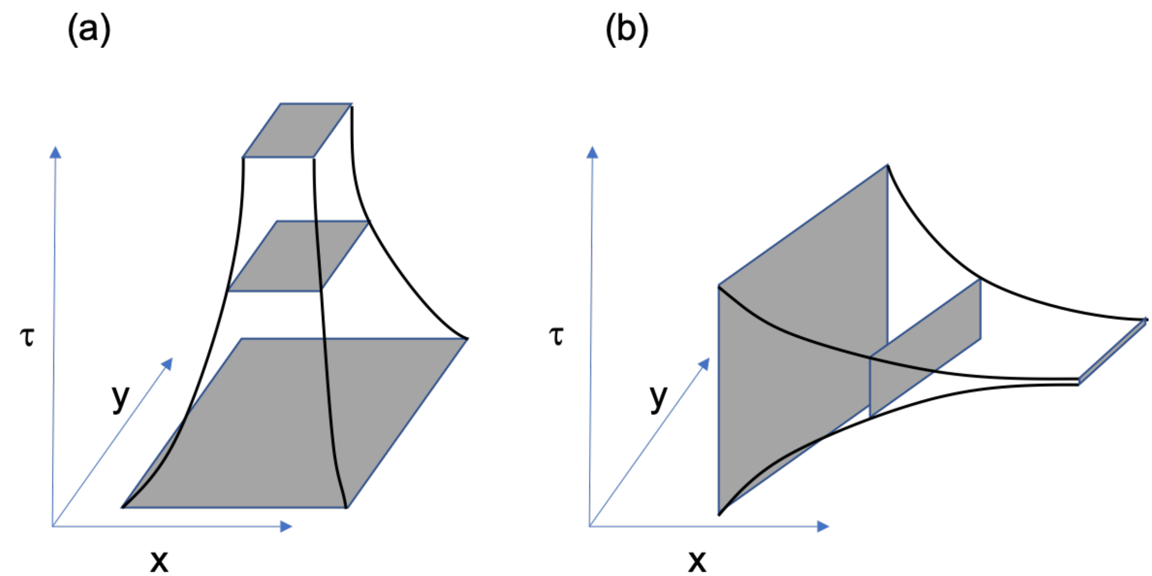


Figure 5.2. The two geometries emerging for the quantum Lifhsitz model. (a) An AdS_3-like geometry arises when considering equal time correlation functions and (b) A Lifshitz metric that is optimal for computing correlation functions with a temporal component.

a $z = 2$ anisotropically scale-invariant $(2 + 1)$-dimensional Lifshitz field theory of a free massless scalar field.

2. Find optimal geometries for static and dynamic correlation functions. For the static correlation functions, the optimal background metric is equivalent to an AdS metric on a Poincare patch while for dynamical correlation functions, we find the Lifshitz geometry.

1.5 The $z = 1$ Lifshitz Weyl Anomaly a Topological Invariant

The Gauss-Bonnet theorem relates the integral of the Ricci scalar R over a smooth compact closed manifold to the Euler characteristic χ of the underlying manifold. Surprisingly, the Gauss-Bonnet theorem links the integral of a *local* geometric quantity, R, to a global topological invariant χ. Intuitively, The Euler characteristic, or the Euler class, is a topological invariant that keeps track or counts of the number of n-dimensional of holes of an n-dimensional *orientable* manifold, those manifolds with orientable tangent spaces.

It is well known that spinor fields do not transform covariantly under the diffeomorphim group [1]. Hence, in a theory where fermions are coupled to background gravity, the orthonormal tangent frame coordinates or vielbeins are used for this purpose. The coupling is achieved using Cartan's formalism [100]. To set up the notation, we introduce vielbeins $e^a{}_\mu$ as

$$g_{\mu\nu} = \delta_{ab} e^a{}_\mu e^b{}_\nu \ , \tag{1.29}$$

where δ_{ab} is the flat Euclidean spacetime metric. The inverse vielbeins $E_a{}^\mu$ is then defined as $g^{\mu\nu} = \delta^{ab} E_a{}^\mu E_b{}^\nu$, and satisfy $E_a{}^\mu e^b{}_\mu = \delta^b_a$. The components of the spin connection 1-form $\omega^{ab}{}_\mu$ are then given by [1]

$$\omega^a{}_{b\mu} = e^a{}_\nu \nabla_\mu E_b{}^\nu = e^a{}_\nu \left(\partial_\mu E_b{}^\nu + \Gamma^\nu_{\mu\lambda} E_b{}^\lambda \right) \ . \tag{1.30}$$

In two spacetime dimensions, the Ricci scalar $\mathcal{R}$ is given by the curvature of the spin connection ω

$$\mathcal{R} = d\omega \, , \tag{1.31}$$

while the curvature U of the *dual* spin connection $\star\omega$ is given by

$$\mathcal{U} = d \star \omega \, , \tag{1.32}$$

where $\star$ is the Hodge dual operator. While the scalar curvature R of $\mathcal{R}$ can be expressed in terms of the spin connection $\tilde{\omega}^\mu = \epsilon^\mu{}_\alpha \omega^\alpha$ as

$$R = 2\nabla_\mu (\tilde{\omega}^\mu) \ , \tag{1.33}$$

the scalar curvature U of $\mathcal{U}$ can be expressed in terms of its dual as

$$U = 2\nabla_\mu(\omega^\mu) \tag{1.34}$$

Two spacetime dimensions are special in that both the spin connection and its dual are 1-forms. It was then noted in [101], if the $\chi = \int \mathcal{R}$ is a topological invariant on smooth, compact and closed manifold, then $\lambda = \int \mathcal{U}$ is also a topological invariant. However, it was argued in [101] that $\lambda = \int \mathcal{U} = 0$ since it is always possible to choose a coordinate frame where the connection has zero divergence. We observe that these coordinates implicitly assume that $\star\omega$ is a connection over an *orientable* bundle, in which case, it can be trivialized. It was also noted in [101] that while the structure group related to ω is $SO(2, R)$, that related to $\star\omega$ is reduced to the group of multiplication by positive real numbers $\mathbb{R}^+ = \{\mathbb{R}^+ - \{0\}\,,\ \times\}$ implying that the fiber bundle associated to $\star\omega$ is a line bundle that can be made trivial.

More importantly, the authors of [102], two years earlier, were able to obtain the U scalar curvature as the chiral anomaly, Weyl anomaly as well as the Lorentz anomaly of a fermionic action constructed from a generalized Dirac operator for which they computed an analytic index. More specifically, they showed that

$$\text{index}\ \widehat{D}_{gen} = \frac{g}{4\pi} \int U\, e\, d^2x\ , \tag{1.35}$$

where $\widehat{D}_{gen}$ is the generalized Dirac operator given by

$$\widehat{D}_{gen} = i\sigma^\mu \left(\nabla_\mu + ig\omega_\mu\sigma_3\right)\ , \tag{1.36}$$

and $\nabla_\mu = \partial_\mu + \omega_\mu\sigma_3$, $e = \sqrt{\det g}$ and σ_3 is the Pauli matrix. By the *Atiyah–Singer* (AS) index theorem, the analytic index equals a *topological* index that describes a purely topological characteristic of the *fiber bundle* considered in the theory. An important consequence of the AS index theorem is the fact that the topological invariant can be expressed as an integral over certain characteristic classes which represent the invariant. In cohomology theory, these characteristic classes are represented by cohomology classes as elements of a cohomology group with coefficients in some field F, typically the field of integers $\mathbb{Z}$. For example, χ is given by an integral over the *Euler class* $e(E)$ of an oriented, real vector bundle $E \to M$. On compact 2-manifolds, the Euler class $e(E)$ is an element of the second integral cohomology group $H^2(M;\mathbb{Z})$. The Euler class exists as an *obstruction*, as with most cohomology classes measuring how *twisted* the vector bundle is [103], [104], and [105].

A special form of the index theorem is the famous *Gauss-Bonnet* theorem. On a 2-dimensional compact manifold with a *boundary*, the Gauss-Bonnet theorem relates the integral of the Ricci scalar R representing the Gaussian curvature, and the integral of the *extrinsic* curvature or the *geodesic* curvature, to the Euler characteristic of the underlying manifold [2]

$$\chi = \frac{1}{4\pi}\left(\int_M R\, e\, d^2x + \int_{\partial M} 2K\sqrt{h}\, d\tau\right), \tag{1.37}$$

where K is the trace of the extrinsic curvature tensor defined in terms of the spatial metric on the boundary in (2.14) and τ is an arbitrary parameter of the boundary ∂M. It is well known that K can be expressed as the covariant divergence of the unit normal to the boundary $\widehat{n}$ as $K = \nabla_\mu \widehat{n}^\mu$. In appendix A of [106], it was shown that in the flat conformal gauge, χ can be written solely in terms the boundary integral of the topological part of K

$$\chi = \frac{1}{4\pi}\int_{\partial M} 2K\sqrt{h}\, d\tau = \frac{1}{2\pi}\int_{\partial M} \partial_\mu \widehat{n}^\mu\,. \tag{1.38}$$

[2]Here M is taken to be diffeomorphic to a subset of $\mathbf{R}^2$.

On a circle with unit normal $\widehat{n}_\mu = (cos\,\theta,\, sin\,\theta)$, we have $\partial_\mu \widehat{n}^\mu = \widehat{n}^\mu \widehat{n}_\mu = 1$ which implies that

$$\chi = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1\ , \tag{1.39}$$

and therefore the Euler characteristic of M, $\chi(M) = 1 - N_H$ where N_H is the number of holes. Hence, on a flat disc with a boundary unit circle, the Euler characteristic $\chi = 1$ and so, indeed, the disc has zero holes.

Owing to the similarities outlined above between R and U, the authors in [102] used the boundary term of the action in (1.40), first derived in [107] to show that the topological invariant λ on a compact 2-manifold with a boundary is given by the following integral

$$\lambda = \frac{1}{4\pi}\left(\int_M U\, e\, d^2x - 2\int_{\partial M}\left[e^{\mu a}\,\epsilon_{ab}\,\nabla_\mu n^b - \epsilon^{\mu\nu}\nabla_\mu n_\nu\right]\sqrt{h}\, d\tau\right)\ , \tag{1.40}$$

The boundary term in (1.40) has first been derived within the framework of Einstein-Cartan theory [108] in the first-order formalism. Looking into λ in (1.40), we observe that (1) λ, as opposed to χ, involves the use of the Levi-Civita antisymmetric symbol ϵ_{ab} and (2) detects the twist in the unit tangent rather than the normal vector.

Analogous to (1.38), the authors [102] used the conformal-Lorentz gauge

$$e^a_\mu = e^\sigma\left(\delta^a_\mu \cos(\theta) + \epsilon^a_\mu \sin(\theta)\right) \tag{1.41}$$

to express λ as the boundary integral of the divergence of a unit *tangent* vector $t_u = \epsilon_{\mu\nu} n^\nu$ (omitting the hat on $\widehat{t}$ and $\widehat{n}$) for a *flat* metric.

$$\lambda = \int_{\partial M} d\tau\, \partial_\mu\, t^\mu\,. \tag{1.42}$$

Although the authors on [102] gave sufficient evidence that λ is a topological invariant, they did not characterize it.

Chapter 6 is a first attempt to study, analyze, and give λ the topological interpretation that was missing in [102] and [101]. More specifically, we show that in a specific *flat* limit on a Möbius strip, λ is indeed a topological invariant and does not vanish as argued in [101] by Myers. Geometrically speaking, the Möbius strip is the simplest two-dimensional non-orientable manifold that has an intrinsic parity. Topologically speaking, it is a *nontrivial* real line bundle.

We will illustrate that the difference between R and U is, in principle, the difference between intrinsic and extrinsic geometry. More specifically, we will show that deformation retracting the Möbius surface by way of taking the *flat limit*, i.e. the limit in which the *thickness* of the Möbius surface goes to zero, of its Ricci scalar simply gives the intrinsic curvature of the core circle of the Möbius strip with no knowledge of the extrinsic curvature induced by its embedding in $\mathbb{R}^3$. On the other hand, deformation retracting the Möbius surface by taking the flat limit of the dual curvature scalar gives the *normal* curvature of the Möbius boundary with no knowledge of the intrinsic curvature of the Möbius core circle. Specifically, we show that

- the $z = 1$ Lifshitz Weyl anomaly is the scalar curvature of U,
- and in the *flat* limit, λ is a topological invariant that detects the parity of the number of twists of the Möbius surface embedded in $\mathbb{R}^3$.

Chapter 2

Analysis of the Lifshitz Weyl Anomaly in (1+1) Dimensions

The key results of this chapter are

1. The Weyl anomaly is directly related to the presence of torsion in the NC geometry

2. The Weyl symmetry is restored by canceling the time dependence of the torsion and restricting it to the foliation leaves.

3. The geometry and topology of the Weyl anomaly pertains to flat line bundles or equivalently, of codimension-1 transverse foliations.

The bulk of ideas and calculations in this chapter appears in [109].

The goal of Chapter 2 is to analyze and investigate the physical as well as mathematical nature of the (1+1)-dimensional Lifshitz Weyl anomaly within a non-relativistic framework. More specifically, we will see how non-relativistic field theories coupled to *background* Newton-Cartan (NC) geometry with temporal torsion can generate Weyl anomalies. We will use the Arnowitt, Deser and Misner (ADM) parametrization [57] in our study to discuss the geometric nature of the (1+1)-dimensional Lifshitz Weyl anomaly, true for $z \geq 1$.

In Section 2.1, we review the basic geometric constructs of the Newton-Cartan geometry with torsion emphasizing the role played by the Frobenius condition in such a context. In Section 2.2, we also briefly review the ADM parametrization of the spacetime before we begin our study of the (1+1)-dimensional Lifshitz Weyl anomaly in Section 2.3. In 2.3, we will present two pictures of the Lifshitz Weyl anomaly in (1+1) dimensions each revealing a different aspect.

Section 2.4 illustrates that by *restricting* the lapse function to be only spatially dependent, the local Weyl symmetry of the effective action is restored. We will show that this amounts to solving a simple equation of motion for a stationary *chiral* boson, one solution of which gives the Rindler metric.

In Section 2.5, we dedicate special attention to understanding the $z = 1$ (1+1)-dimensional Lifshitz Weyl anomaly and how it actually relates to the Lorentz anomaly in two-dimensional CFTs. We explain the interesting connection presented in [16] that shows that the $z = 1$ (1+1) Lifshitz Weyl anomaly is in fact the Weyl partner of the Lorentz anomaly in 2d CFTs.

In the Appendix to this chapter, we use the language of fiber bundles to provide that the proper mathematical characterization of the Lifshitz Weyl anomaly described in Section 2.3. In Appendix 2.A, we will first provide a very brief introduction to the mathematics of foliated manifolds and the associated geometrical and topological structures. More specifically, we will discuss the different types of flat line bundles associated with a foliated manifold and more importantly, using this framework, why it becomes natural to understand that the Lifshitz Weyl anomaly in (1+1) dimensions is the Weyl partner of the Lorentz anomaly. In Appendix 2.B, we then move on to describe the different types of transverse foliations and how they naturally map to NC geometry with and without torsion. We then discuss flat line bundles and how they are related to codimension-1 foliations in Appendix 2.C. We finally, comment on how non-orientable manifolds, flat line bundles, torsion and transverse structures are all related to one another.

2.1 The Newton-Cartan (NC) Geometry

The Newton-Cartan geometry, as opposed to Riemannian geometry in relativstic theories, is what couples naturally to non-relativistic field theories. Anomalies in non-relativistic quantum field theories coupled to NC geometry are prime examples of how the geometrical objects of the NC geometry become manifest. In non-relativistic field theories, the *time* direction plays a major role and spacetime is naturally foliated into *equal-time slices* or surfaces of simultaneity. Assuming the existence of a smooth scalar field globally defined on the spacetime manifold M, a foliated manifold $(M, \mathcal{F})$ with a foliation $\mathcal{F}$ is defined by a smooth non-vanishing foliation 1-form t_μ that is *normal* to the tangent space of the foliation. More concretely, the tangent space to the foliation is defined by the kernel of t_μ i.e. those vector fields which satisfy $t_\mu X^\mu = 0$. If the foliated manifold is equipped with a metric $g_{\mu\nu}$, then instead we use $n_\mu = t_\mu / \sqrt{|g^{\beta\gamma} t_\beta t_\gamma|}$ as the foliation 1-form. A more formal definition of a foliation will be given in Appendix 2.A.

More formally, the basic geometrical structure on a $D = (d{+}1)$-dimensional NC manifold M consists of an everywhere smooth *temporal* metric $n_\mu n_\nu$, a *degenerate* symmetric spatial component $h^{\mu\nu}$ with signature $(0, +, \ldots, +)$ i.e. corank-1 tensor [56] and a notion of a covariant derivative ∇ all satisfying the following constraints

$$h^{\mu\nu} n_\mu = 0, \quad n_\mu n^\nu = -1, \quad \nabla_\mu n_\nu = \nabla_\mu h^{\nu\lambda} = 0\,. \tag{2.1}$$

The covariant derivative is defined with this connection

$$-\Gamma^{\lambda}{}_{\mu\nu} = n^{\lambda}\partial_{\mu}n_{\nu} + \frac{1}{2}h^{\lambda\rho}\left(\partial_{\mu}h_{\nu\rho} + \partial_{\nu}h_{\mu\rho} - \partial_{\rho}h_{\mu\nu}\right) \ . \tag{2.2}$$

While the 1-form n_μ provides a notion of a *clock*, its inverse n^μ denotes the direction of time often called the velocity field. Using the NC geometrical objects defined above, one can construct a non-degenerate symmetric rank-2 tensor $g_{\mu\nu}$ with a Lorentzian signature (-1,1,. . . 1) that has a temporal component $n_\mu n_\nu$ as well as a spatial component $h_{\mu\nu}$, i.e. $g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu$. For a more formal and thorough definition of the NC spacetime, see for example [110]. Following [52], there are three different constraints on the foliation 1-form n_μ that each give a different type of NC geometry:

(i) *Torsionless* NC geometry: $dn = 0$ where the connection $\Gamma^{\lambda}{}_{[\mu\nu]} = 0$

(ii) *Twistless Torsion* Newton-Cartan (TTNC) or *temporal torsion* geometry

$$n_\lambda \Gamma^{\lambda}{}_{[\mu\nu]} = \partial_\mu n_\nu - \partial_\nu n_\mu = a_\mu n_\nu - a_\nu n_\mu \, , \tag{2.3}$$

where the *acceleration or torsion vector* a_μ is a foliation-tangent vector defined as the Lie derivative of the foliation 1-form along n^μ

$$a_\mu = \mathcal{L}_n n_\mu = n^\nu \nabla_\nu n_\mu \, , \tag{2.4}$$

The covariant derivative in (2.4) can be expressed in terms of the Lie derivative using

$$\nabla_\mu n^\nu = K^\nu_\mu - a^\nu n_\mu, \quad h^\rho_\mu \nabla_\rho n^\nu = K^\nu_\mu$$

where $K_{\mu\nu}$ is the *extrinsic* curvature tensor. The TTNC constraint in (2.3) is an expression of the solution of the *Frobenius condition*, an integrability condition that states a 1-form defines a codimension-1 foliation if and only if it satisfies the following constraint:

$$n \wedge dn = 0\,. \tag{2.5}$$

Imposing the Frobenius condition means that it is always possible to find a coordinate system in which the spacetime manifold is foliated into equal-time hypersurfaces or foliation leavs Σ_t to which the time-like 1-form n_μ is normal. The Frobenius condition makes the TTNC spacetime *causal* in the sense that if it does not hold, then each point $p \in M$, has a neighborhood within which all points are spacelike separated. It is also important to mention that TTNC spacetimes, while being causal, still lack the notion of an *absolute* time measured by all observes along their worldlines. The difference between the total coordinate time measured by two observers starting at different points on Σ_{t_1} and traveling to another time slice Σ_{t_2} along their respective wordlines measures the temporal torsion $dn = a \wedge n$ [56]. This point is key to understanding the physical as well the geometrical meaning of the Weyl anomaly.

(iii) *Torsional* NC or TNC geometry where n_μ is not constrained and has therefore arbitrary torsion.

We will later see how the (1+1)-dimensional Lifshitz Weyl anomaly is directly related to the TTNC geometry.

2.2 The ADM Parametrization

In the ADM decomposition, one chooses coordinates (t, x^i) such that the leaves of the foliation are given by constant-time slices $t = const$ and x^i for the coordinates in each leaf. The ADM metric assumes a frame, the *unitary or synchronous* gauge where the time of the *spatial* foliation hypersurfaces coincides with coordinate time t such that the spacetime metric has a well-defined notion of global time. In this gauge, the ADM metric describes the TTNC geometry where the Frobenius condition given in (2.5) is *automatically* satisfied. In these *preferred* coordinates, the metric $g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu$ takes the form [1]

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{ti} \\ g_{jt} & g_{ii} \end{pmatrix} = \begin{pmatrix} -N^2 + N^i N_i & N_i \\ N_j & h_{ij} \end{pmatrix}, \tag{2.6}$$

while the components of the inverse metric are given by

$$g^{\mu\nu} = \begin{pmatrix} g^{tt} & g^{ti} \\ g^{jt} & g^{ii} \end{pmatrix} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^j}{N^2} & h^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix}, \tag{2.7}$$

where h_{ij} is the induced *spatial* metric on the foliation leaves, N^i is the *shift* vector and $N(x,t)$ is the *lapse* function. Intuitively, the lapse function is the ratio between the *coordinate* time t labeling a given time-slice Σ_t and *proper* time measured by an observer. It governs the temporal propagation of points on one time slice to another. In these coordinates, $N(x,t)$ is given by

$$N(x,t) = \left(\vec{\nabla} t \cdot \vec{\nabla} t\right)^{-1/2} \tag{2.8}$$

The covariant volume element in these coordinates is given by:

$$\sqrt{-g}\, d^{(D+1)}x = N\sqrt{h}\, dt\, d^D x. \tag{2.9}$$

The timelike vector fields normal to the foliation are given by

$$\begin{aligned} n_\mu &= N(-1, 0)\,, \\ n^\mu &= \frac{1}{N}(1, -N^i)\,. \end{aligned} \tag{2.10}$$

[1]For details on how the ADM metric is obtained from the fundamental NC objects in the process of gauging the Bargmann algebra, see section 8 in [52].

In D spatial dimensions, the Lie derivative of a foliation-tangent tensor $X_{ijk\ldots}$, i.e. one that satisfies $n^\mu X_{\mu\beta\gamma} = 0$ along n^μ splits into a time derivative normal to the foliation and spatial derivatives inside the foliation leaves and is given by

$$\mathcal{L}_n X_{ijk\ldots} = \frac{1}{N}\partial_t X_{ijk\ldots} - \frac{1}{N}\mathcal{L}^{(D)}_{\vec{N}} X_{ijk\ldots}\,, \tag{2.11}$$

where $\mathcal{L}^{(D)}_{\vec{N}}$ is the Lie derivative inside the foliation leaf.

The torsion or acceleration vector is the spatial gradient of the lapse function. In 1+1 dimensions, the x-component of the acceleration vector is given by

$$a_x = \frac{\partial_x N}{N}\ ,^{2} \tag{2.12}$$

and the temporal component is $a_t = N^x a_x$. In 1+1 dimensions, with a non-zero shift vector N^x, the ADM metric is given by [3]

$$ds^2 = -N^2\,dt^2 + N_x\,dxdt + h_{xx}\,dx^2\,. \tag{2.13}$$

In 1+1 dimensions, the extrinsic curvature tensor is simply given by

$$K_{xx} = \frac{1}{2N}(\partial_t h_{xx})\,. \tag{2.14}$$

[2] To see this, start from (2.4), use that $n_\mu = -N\nabla_\mu t$ and follow the explicit calculation in eq. (3.17) of [43]

$$\begin{aligned}
a_\alpha &= n^\mu\nabla_\mu n_\alpha = -n^\mu\nabla_\mu(N\nabla_\alpha t) = -n^\mu\nabla_\mu N\nabla_\alpha t - Nn^\mu \underbrace{\nabla_\mu\nabla_\alpha t}_{=\nabla_\alpha\nabla_\mu t} \\
&= \frac{1}{N}n_\alpha n^\mu\nabla_\mu N + Nn^\mu\nabla_\alpha\left(-\frac{1}{N}n_\mu\right) = \frac{1}{N}n_\alpha n^\mu\nabla_\mu N + \frac{1}{N}\nabla_\alpha N\,\underbrace{n^\mu n_\mu}_{=-1} - \underbrace{n^\mu\nabla_\alpha n_\mu}_{=0} \\
&= \frac{1}{N}\left(\nabla_\alpha N + n_\alpha n^\mu\nabla_\mu N\right) = \frac{1}{N}h^\mu{}_\alpha\nabla_\mu N \\
&= \frac{1}{N}\partial_\alpha N
\end{aligned}$$

[3] By working in the ADM preferred coordinates, the shift vector N^x can always be removed by an FPD transformation.

In 1+1 spacetime dimensions with a zero shift vector, the spacetime metric has only two degrees of freedom: the lapse function $N(x,t)$ and the spatial metric h_{ij}. The spatial metric h_{ij} is a rank-0 tensor, i.e. a function $h_{xx}(x,t)$.

2.3 The 1+1 Weyl Anomaly And Anomalous Ward Identity

In this section, we attempt to illustrate the geometric nature of the 1+1-dimensional Lifshitz Weyl anomaly and how it is closely related to the NC geometry with temporal torsion. To that effect, we use the ADM coordinates to define some basic TTNC objects required to understand the geometric nature of the 1+1 Weyl anomaly. As mentioned in the introduction, dynamical TTNC gives rises to non-projectable Horava-Lifshitz theory of gravity. Since this approach is useful for our purposes in this section, we use some of the notations and definitions in [54] and [52]. [4]

The antisymmetric part of the torsion tensor $\Gamma^{\lambda}_{\mu\nu}$, is expressed as

$$\Gamma^{\lambda}_{[\mu\nu]} = n^{\lambda}\partial_{[\mu}n_{\nu]} = n^{\lambda}a_{[\mu}n_{\nu]} = n^{\lambda}R_{\mu\nu}(H)\,, \tag{2.15}$$

where $R_{\mu\nu}(H)$[5] is the curvature 2-form of n_ν defined as the *gauge field* of the generator of time translation symmetry, i.e. the Hamiltonian (H)

$$R(H) = (\partial_\mu n_\nu - \partial_\nu n_\mu)\, dx^\mu \wedge dx^\nu\,. \tag{2.16}$$

[4]We would like to point out that adding torsion to the NC geometry by gauging the *Bargamann algebra* is different from adding torsion by gauging the Schrödinger algebra. In the former, the torsion is explicitly given by the antisymmetric part of the *affine* connection which is the curvature of the gauge connection (field) corresponding the generator of time translation symmetry i.e. the Hamiltonian we will see below. By gauging the Schrödinger algebra, torsion is added to the algebra as a *constraint* equation. It is this way of adding torsion that we will use in Chapter 4. Also, we note that torsion in two dimensions is always given by the *trace* of the components of the affine connection.

[5]Note that $R_{\mu\nu}(H)$ does not mean R is a function of H.

In 1+1 spacetime dimensions, imposing the Frobenius condition and using the ADM gauge, the only non-vanishing component of the torsion 2-form Γ as defined by equation (2.27) in [52] is given by

$$\begin{aligned} \Gamma &= \frac{1}{2}\Gamma^t_{[xt]}\, dx \wedge dt = n^t(a_x n_t - a_t n_x)\, dx \wedge dt \\ &= a_x\, dx \wedge dt \\ &= \frac{\partial_x N(x,t)}{N(x,t)}\, dx \wedge dt\,, \end{aligned} \tag{2.17}$$

Equivalently,

$$\begin{aligned} R(H) &= \frac{1}{2} n_t \Gamma^t_{[xt]}\, dx \wedge dt \\ &= (a_x n_t - a_t n_x)\, dt \wedge dx \\ &= (a_x n_t)\, dx \wedge dt \\ &= \partial_x N\;\, dx \wedge dt \\ &= R_{xt}(H)\;\, dx \wedge dt\,. \end{aligned} \tag{2.18}$$

Now we can see that in *torsionless* NC geometry, dn is a closed 1-form, i.e. $dn = 0$ that corresponds to $R_{\mu\nu}(H) = 0$ in (2.16). This, in turn, translates to zero curvature in the gauge field i.e. a flat connection, corresponding to the time translation symmetry generated by the Hamiltonian. On the other hand, the TTNC case corresponds to a non-zero $R_{\mu\nu}(H)$ or $dn \neq 0$. The Frobenius condition then tells us that this curvature is given by the torsion tensor a_μ: $\partial_{[\mu} n_{\nu]} = a_{[\mu} n_{\nu]}$. Lifshitz field theories with classical Weyl invariance couple to TTNC geometry and the Weyl anomaly will be directly related to this torsion or acceleration vector field.

To derive the anomalous Ward identity, we start with a classical action $S[\phi, N, h_{ij}]$ with matter fields ϕ coupled to background TTNC geometry. Throughout this chapter, we set the shift vector N^i to zero which can always be done by an FPD transformation. $S[\phi, N, h_{ij}]$ is assumed to be invariant under infinitesimal anisotropic local Weyl transformation with scaling exponent z

$$\delta N = z\sigma N, \quad h_{ij} = 2\sigma h_{ij}\,, \tag{2.19}$$

where $\sigma(x,t)$ is the infinitesimal Weyl transformation parameter. Quantum mechanically, however, the UV regularization of the partition function $Z = e^{-W[N,h_{ij}]}$ breaks the local Weyl invariance of the quantum effective action $W[N, h_{ij}]$ resulting in a Weyl anomaly. More concretely, the presence of a Weyl anomaly in the effective action necessarily means that the variation $\delta_\sigma W$ is non-zero

$$\delta_\sigma W = \int N\sqrt{h}\, dt\; dx\, \sigma\, \mathcal{A}\,, \tag{2.20}$$

where $\mathcal{A}$ is the non-relativistic counterpart of $\langle T^\mu{}_\mu\rangle$ in a relativistic CFT and is given by the expectation value of the time-projected and spatially-projected components of the trace of the energy-momentum tensor

$$\mathcal{A} = z\left\langle T^t_t\right\rangle + \left\langle T^x_x\right\rangle \neq 0\,. \tag{2.21}$$

where $\langle T^t_t\rangle = \langle T^{\mu\nu}\rangle n_\mu n_\nu$ and $\langle T^x_x\rangle = \langle T^{\mu\nu}\rangle h_{\mu\nu}$ and using the *temporal* metric $n_\mu n_\nu$ and the spatial metric $h_{\mu\nu}$ defined in (2.1). It is important to note that although [16] in their cohomological classification of Lifshitz Weyl anomalies does not explicitly say that the background geometry to which they couple the Lifshitz theory is a TTNC spacetime, it actually implicitly is. In the cohomological classification of Weyl anomalies in FPD-invariant Lifshitz field theories in all spacetime dimensions, the foliation 1-form n_μ satisfies the Frobenius condition which is the key defining property of TTNC geometry. Section 2.4 of [27] contains more information on the relationship between the notations and conventions used in [16] and standard NC geometry.

We now move to demonstrate the geometric and physical nature of the Lifshitz Weyl anomaly after rewriting it in terms of the ADM coordinates defined above. We emphasize that the discussion in this section is valid for *all* values of z. We will present two different yet related pictures. While the first picture emphasizes the fundamental role of n_μ and n^μ in the TTNC geometry, the second one defines a_μ as a *fundamental* foliation 1-form and emphasizes the geometrical picture of the Lifshitz Weyl anomaly. The latter picture will be useful in Chapter 4 when the anomaly is derived from the (2+1)-dimensional NRSCS action.

2.3.1 The vielbein picture

We start from the expression of the (1+1)-dimensional Weyl anomaly as given in [16] and rewrite it terms of the ADM gauge in (2.10) and (2.11). The Weyl anomaly is given by the variation of the one-loop effective action of the (1+1)-dimensional Lifshitz effective action $W[g]$ with respect to the Weyl parameter σ

$$\begin{aligned} \delta_\sigma W &= \int \sqrt{-g}\, \sigma\, \tilde{\epsilon}^\mu\, \mathcal{L}_n a_\mu \\ &= \int \sqrt{-g}\, \sigma\, n_x \epsilon^{xt}\, \mathcal{L}_n a_t + n_t \epsilon^{tx}\, \mathcal{L}_n a_x \\ &= \int \sqrt{-g}\, \sigma\, n_t\, \mathcal{L}_n a_x\,, \end{aligned} \tag{2.22}$$

where $\tilde{\epsilon^\mu} = n_\alpha \epsilon^{\alpha\mu}$ is the foliation-projected Levi-Civita tensor, $\epsilon^{tx} = 1$, $a_t = N^x a_x = 0$, (since $N^x = 0$) and $\sigma(x,t)$ is the Weyl transformation parameter. Using the definition of the Lie derivative in (2.11), the Weyl anomaly is given by

$$\begin{aligned} \delta_\sigma W &= -\int dtdx\; N^2 \sqrt{h}\, \sigma\, \mathcal{L}_n a_x \\ &= -\int dtdx\; N \sqrt{h}\, \sigma \left(\frac{\partial a_x}{\partial t} \right). \end{aligned} \tag{2.23}$$

In terms of the lapse function $N(x,t)$, using (2.18) and (2.17), it takes the following form

$$\begin{aligned}\delta_\sigma W &= -\int dtdx\ N\sqrt{h}\,\sigma\left(\frac{1}{N}\partial_t\partial_x N - \frac{1}{N^2}\partial_t N\partial_x N\right)\\ &= -\int dtdx\ N\sqrt{h}\,\sigma\left(\frac{1}{N}\partial_t R_{xt}(H) - \frac{1}{N}\partial_t N a_x\right)\\ &= -\int dtdx\sqrt{h}\,\sigma\ (\partial_t R_{xt}(H) - (\partial_t N)a_x)\ . \end{aligned} \tag{2.24}$$

Expressing $\tilde{\epsilon}^\mu\ \mathcal{L}_n a_\mu$ in (2.22) in terms of local tangent frame coordinates and using differential forms will better reveal its geometric nature. Using that the vielbeins for the temporal and spatial components of the NC metric $g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu$ can be expressed as

$$n_{\mu\nu} = n_\mu n_\nu\,, \quad h_{\mu\nu} = e_\mu{}^A \delta_{AB} e_\nu{}^B\,, \tag{2.25}$$

the foliation 1-form in terms of ADM ccoordinates (2.13) can be expressed as

$$n = Ndt\ . \tag{2.26}$$

We now use *Cartan's formula* $\mathcal{L}_X d\omega = d\mathcal{L}_X\omega$, which relates the Lie derivative along a vector field X of a k-form $d\omega$ to the exterior derivative of the $(k-1)$-form $\mathcal{L}_X\,\omega$. Acting with the Lie derivative on dn along n^μ, we get

$$\begin{aligned}\mathcal{L}_n\,(dn) = d\mathcal{L}_n\,n &= d\mathcal{L}_n\,(N\,dt) \\ &= d(\frac{1}{N}\partial_t N - N^x\partial_x N)dt \\ &= (\partial_t a_x)dt\wedge dx\,, \end{aligned} \tag{2.27}$$

where we used the definition of the Lie derivative in (2.11) and (2.12) and chose N^x to be zero. Using (2.21), the expectation value of the energy-momentum tensor is therefore given by

$$z\left\langle T_t^t \right\rangle + \langle T_x^x \rangle = \partial_t a_x = \partial_t \left(\frac{R_{xt}(H)}{N(x,t)} \right) = \partial_t \frac{\partial_x N}{N} \,. \tag{2.28}$$

From equations (2.27) and (2.28), we can see that the 1+1 Lifshitz Weyl anomaly, in the 1-form picture, is naturally given by the time derivative of the spatially-dependent lapse function $N(x,t)$ or equivalently the time derivative of a_x, which is the solution of the Frobenius condition (2.18). This makes explicit the relationship between TTNC geometry, the Frobenius condition and the role they both play in the generating the 1+1 Lifshitz Weyl anomaly.

2.3.2 The 1-form Picture

In this 1-form picture, the torsion is expressed as a 1-form

$$a = a_\mu dx^\mu = a_t \, dt + a_x \, dx \,. \tag{2.29}$$

The curvature 2-form $W_{\mu\nu}$ of the torsion 1-form a_μ is then given by

$$W \equiv da = \left(\frac{\partial a_t}{\partial x} - \frac{\partial a_x}{\partial t} \right) dx \wedge dt \,. \tag{2.30}$$

Using $a_t = N^x a_x$, da can be expressed as:

$$da = \left(\frac{\partial a_x}{\partial t} - N^x \frac{\partial a_x}{\partial x} \right) dt \wedge dx \,. \tag{2.31}$$

Setting $a_t = 0$ or equivalently, $N^x = 0$, we get

$$da = \left(\frac{\partial a_x}{\partial t} \right) dt \wedge dx \,. \tag{2.32}$$

which illustrates that the 1+1 Lifshitz Weyl anomaly can be directly interpreted as the curvature of the torsion 1-form a_μ. In this picture, we can thus clearly see that the anomalous *gravitational* degree of freedom is a direct consequence of the *time and* spatial dependence of the lapse function $N(x,t)$.

2.4 Anomaly Cancellation of the $z = 1$ Lifshitz Weyl Anomaly

In this section, we show that by *restricting* the lapse function $N(x,t)$ to one time slice or foliation leaf by making it only x-dependent, the local Weyl symmetry of the effective action is restored. We will show that this amounts to solving an equation of motion for a stationary *chiral* mode which has one particular solution that gives the Rindler metric.

To restore the local Weyl symmetry of the induced effective action, the Weyl anomaly must be canceled. Insisting on the Weyl invariance of the quantum effective action $W[e^a_\mu]$, amounts to satisfying the equation of motion in (2.23) or (2.24). With a zero shift vector $N^x = 0$, and the lapse function parameterized as $N(x,t) = e^{\psi(x,t)}$, the equation of motion is given by

$$\partial_t \partial_x \log N = \partial_t \partial_x \psi(x,t) = 0\,. \tag{2.33}$$

Since, physically, the Weyl anomaly represents a *time-dependent* acceleration, or a non-uniform gravitational field where energy is not conserved, restoring local Weyl invariance in the effective action is tantamount to having observers with proper acceleration in *flat* spacetime or having a uniform gravitational field where energy is conserved. This necessarily means getting rid of the time dependence of the lapse function $N(x,t)$. Mathematically speaking, restoring the local Weyl symmetry requires making a_μ a closed 1-form, i.e. a flat connection $da = 0$ with zero curvature $W_{\mu\nu} = 0$.

The equation of motion in (2.33) has a general solution given by

$$\psi(x,t) = \psi_1(x) + \psi_2(t)\,, \tag{2.34}$$

or equivalently, $N(x,t) = N_1(x)\,N_2(t)$. Imposing the boundary condition $\psi_2(t) = 0$ or $N_2(t) = 1$ necessarily eliminates this spurious degree of freedom thus making (2.33) automatically satisfied. As a result, the Weyl anomaly is canceled and hence a_x becomes a conserved charge of the Weyl gauge symmetry. A spatially-dependent lapse function $N(x)$ gives a family of arbitrary time-independent solutions each of which lives on a hypersurface of constant time t. Choosing $N(x)$ to be linear is a particularly interesting choice of coordinates, since with this choice and $h_{xx} = 1$, the background spacetime metric in ADM coordinates becomes

$$ds^2 = -(\alpha x)^2\,dt^2 + \,dx^2\,. \tag{2.35}$$

which is the *Rindler* metric of a hyperbolically accelerated reference frame with coordinates (x,t) with rapidity $\eta = \alpha t$. If we label the flat Minkowski spacetime coordinates by (X,T) and choose Rindler observer with constant proper acceleration $\alpha = 1$ and proper time τ equal to coordinate time t, then (X,T) are related to Rindler coordinates by the following transformations

$$T = x\,\mathrm{sinh(t)}, \quad X = x\,\mathrm{cosh(t)}\,. \tag{2.36}$$

These linear transformations preserve the hyperbolae $X^2 - T^2 = N^2(x) = x^2$ which describe the worldlines of a family of Rindler observers at rest for *fixed* x. These transformations can be represented by elements of the one-parameter group of Lorentz boosts $SO^+(1,1)$ with boost parameter $\eta = \alpha t$. An element of the indefinite special orthogonal group $SO^+(1,1)$ is represented by a 2×2 real matrix

$$M(\eta) = \begin{bmatrix} \cosh(\eta) & \sinh(\eta) \\ \sinh(\eta) & \cosh(\eta) \end{bmatrix} . \tag{2.37}$$

In light-cone coordinates, $U = X + T,\ V = X - T$, $M(\eta)$ is diagonalized

$$M(\eta) = \begin{bmatrix} e^{\eta} & 0 \\ 0 & e^{-\eta} \end{bmatrix} , \tag{2.38}$$

such that area $U * V = X^2 - T^2$ of the hyperbola is preserved. Therefore, the group $SO^+(1,1)$, in addition to being the group of Lorentz boosts in 1+1 dimensions is also the group of scale (actually squeeze) transformations that preserve the area $U * V$ of the hyperbolic worldline of a Rindler observer at a fixed $x = x_0$.

If we define a frame fields e^0 and e^1 as

$$e^0 = x\, dt, \quad e^1 = dx\,, \tag{2.39}$$

which in terms of the dual basis vector field, is given by

$$n^t = \frac{1}{x}\,\partial_t, \quad n^x = \partial_x\,, \tag{2.40}$$

then the unit timelike vector n^μ defines integral curves consisting of the world lines of a family of Rindler observers each at fixed $x = x_0$. For each such observer, n^t is a *Killing* vector of the Rindler metric. Since the Lie derivative of the torsion vector a_μ along n^μ after canceling the Weyl anomaly is now $\mathcal{L}_n\, a_\mu = 0$, a_μ is conserved and n_μ satisfies the Frobenius condition $n \wedge dn = 0$. It is interesting to note that the *vorticity-free* condition of the worldlines of Rindler observers i.e. the vanishing of the rotation tensor in the Raychaudhuri equation, is the twistless torsion condition in equation (6.8) of [52].

2.5 The $z = 1$ Lifshitz Weyl Anomaly as the Weyl Partner of the Lorentz Anomaly

The authors in [16] revealed that the $z = 1$ (1+1)-dimensional Lifshitz Weyl anomaly is the *Weyl partner* of the Lorentz anomaly in 1+1 CFT as we explained in Section 1.2. Here, we elaborate on this interesting connection and see how it works. The starting point here is a CFT with a local Lorentz anomaly. The idea is to *shift* the Lorentz anomaly of a given $1 + 1$-dimensional CFT to a foliation dependence, i.e. to a *dependence* on n_μ and then rewrite the *anomalous* CFT quantum effective action in terms of n_μ. Concretely, let W_{CFT} be an effective action of a Lorentz-anomalous CFT. Then, the 1+1 Lifshitz effective action W_{Lif} can be *defined* in terms of W_{CFT} as follows

$$W_{\text{Lif}}[e^a{}_\mu, t^a] \equiv W_{\text{CFT}}[-e^a{}_\mu n_a, e^a{}_\mu \widetilde{n}_a] = W_{\text{CFT}}[-n_\mu, \widetilde{n}_\mu],^{6} \tag{2.41}$$

where n_ν and $\widetilde{n}^a \equiv \epsilon^{ab} n_b$ are arbitrary foliation vectors aligned with the frame fields $e^a{}_\mu$ which are defined for a relativistic spacetime as

$$g^{\mu\nu} e^a_\mu e^b_\nu = \eta^{ab} \tag{2.42}$$

$$\eta_{ab} e^a_\mu e^b_\nu = g_{\mu\nu} \,, \tag{2.43}$$

[6] Note that since, classically, the CFT is locally Lorentz-invariant, W_{Lif} is equivalent to W_{CFT} modulo a local term.

where η^{ab} is the flat metric in the two-dimensional tangent frame basis. The component of the Lorentz spin connection 1-form $\omega^a{}_b$, in terms of e^a_μ, is defined as

$$\omega_\mu{}^a{}_b = -e_b{}^\nu \nabla_\mu e^a{}_\nu \,. \tag{2.44}$$

In a 1+1 CFT with local Lorentz anomaly [7], for example, in a chiral CFT, $\langle T^\mu_\mu \rangle$ has an extra term in addition to the Ricci scalar R [1]. This extra term is essentially how the Lorentz anomaly *manifests* itself in $\langle T^\mu_\mu \rangle$ taken as the variation of $W_{\text{Lif}}[e^a{}_\mu]$ with respect to the viebein 1-forms $e^a{}_\mu$. This additional term is the divergence of the Lorentz (spin) connection 1-form $\omega^\mu{}_{ab}$ defined in (2.44)

$$\langle T^\mu_\mu \rangle_{\text{CFT}} = -2\epsilon^{ab}\nabla_\mu \omega^\mu{}_{ab} \,. \tag{2.45}$$

After identifying local tangent frame i.e. the vielbeins with the foliation 1-forms

$$e^0_\mu \equiv n_\mu, \quad e^1_\mu \equiv \widetilde{n}_\mu \,, \tag{2.46}$$

the authors in [16], were able to demonstrate that $\langle T^\mu_\mu \rangle_{\text{CFT}}$ is indeed the Weyl partner of the Lorentz anomaly up to the coboundary terms $(a_\rho K + \widetilde{\nabla}_\rho K)$

$$\begin{aligned} \langle T^\mu_\mu \rangle_{\text{CFT}} = &\; -2\epsilon^{ab}\nabla_\mu \omega^\mu{}_{ab} \\ &= -2\nabla_\mu \left(\epsilon^{ab} e_{a\nu} \nabla^\mu e_b{}^\nu\right) \\ &= 4\tilde{\epsilon}^\rho (\mathcal{L}_n \, a_\rho + a_\rho K + \widetilde{\nabla}_\rho K) \,, \end{aligned} \tag{2.47}$$

where $\widetilde{\nabla}_\mu = h^{\mu'}_\mu \nabla_{\mu'} K$ is the *foliation-projected* covariant derivative of a foliation-tangent tensor and K is the trace of $K_{\mu\nu}$ ([16]). Thus, we see that the anomalous

[7] A diffeomorphism anomaly in 1+1 CFT can be shifted to a local frame anomaly by a local counterterm [1, 111].

local frame rotations of $W_{\text{CFT}}[e^a_\mu]$ is exchanged for anomalous Weyl transformations of the foliation in $W_{\text{CFT}}[-n_\mu, \widetilde{n}_\mu]$.

For a system of left-handed n_L and right-handed n_R chiral fermions, the effective action $W[e^a{}_\mu]$ has a real part $\mathcal{R}[e^a{}_\mu]$ and an imaginary part $\mathcal{I}[e^a{}_\mu]$ given by [112]

$$W[e^a\mu] = (n_R + n_L)\,\mathcal{R}[e^a{}_\mu] + (n_R - n_L)\,i\,\mathcal{I}[e^a{}_\mu]\ , \tag{2.48}$$

The CFT Lorentz anomaly would then be given by

$$\langle T^\mu_\mu \rangle_{\text{CFT}} = \frac{1}{192\pi^2}\,(n_R + n_L)\ R + i\,(n_R - n_L)\,\epsilon^{ab}\nabla_\mu \omega^\mu{}_{ab}\ . \tag{2.49}$$

The above equation clearly shows that while the Ricci scalar R will *not* vanish if $n_R = n_L$, the $z = 1$ Lifshitz Weyl anomaly, $\epsilon^{ab}\nabla_\mu \omega^\mu{}_{ab}$, will disappear if $n_R = n_L$ which makes clear the fact that the $z = 1$ Lifshitz Weyl anomaly is fundamentally induced by the presence of *chirality* in the theory. In two dimensions, chiral matter coupled to external gravity induces a diffeomorphism or local Lorentz anomaly. Thus, in this sense, the Lifshitz Weyl anomaly is indeed the Weyl partner of the Lorentz anomaly. It is important to mention that the $z = 1$ Lifshitz Weyl anomaly has been derived in [22] from a massless chiral fermion action using the BRST cohomological approach. The anomaly appeared in the imaginary part of the quantum effective action as we explained above.

2.6 Discussion and Outlook

In this chapter, we presented an in-depth study of the (1+1)-dimensional Lifhsitz Weyl anomaly within the framework of Newton-Cartan geometry with torsion. We presented two different pictures of the anomaly paying special attention to the $z = 1$

anomaly which as we explained, turns out to be the Weyl partner of the local Lorentz anomaly in a (1+1) conformal field theory.

In Chapter 3, we will use the Fujikawa method to derive the $z = 1$ Weyl anomaly from a two-dimensional massless chiral fermion action. In Chapter 4, we will derive the Lifshitz Weyl anomaly from a specific term, called the torsion Chern-Simons (tCS) term, in a non-relativistic Schrödinger Chern-Simons action. In addition we will show that the differential form $a \wedge da$ is closed and independent of the choices of the foliation 1-form n_μ and a_μ. Hence it defines a cohomology class, known as *Godbillon-Vey class* $GV(\mathcal{F})$ in the third real de Rham cohomology group $H^3(M; R)$.

Appendix

The reader can skip the discussion in this Appendix since it is not required to follow through with the rest of the material in this dissertation.

In this appendix, we discuss the following three points:

1. Illustrate that the proper mathematical characterization of the Lifshitz Weyl anomaly described in Section 2.3 is directly related to the *geometry and topology* of foliated 2-manifolds and *flat* line bundles,

2. show that the torsionless NC geometry and NC geometry with temporal torsion correspond to different types of transverse foliations,

3. and finally, show that non-orientable manifolds, flat line bundles, torsion and transverse structures are all related to one another.

One key purpose is to demonstrate to the reader, in preparation for Chapter 6, that the Möbius strip is the simplest most natural manifold on which the integral of the $z = 1$ Lifshitz Weyl anomaly does not vanish and thus gives a topological invariant. A second purpose is to serve as background for Chapter 4 when we show that $a \wedge da$ is known in the mathematical literature as the Godbillon-Vey foliation invariant.

Here is a summery of the results of this section. Table 2.1 shows which line sub-bundles the different geometrical constructs of the NC geometry belong to. Table 2.2 summarizes how different types of transverse foliation with different diffeomorphism groups of foliation leaves correspond to different types of the NC geometry. Figure 2.1 depicts the mutual relationship between $\mathbb{Z}_2$ torsion in flat line bundles and their non-orientability, on one hand, and between a transverse *(G,M)*-structure on a foliation and flat line bundles on the other hand. Thus, we conclude that the Möbius strip is the topological manifold over which the $z = 1$ Lifshitz Weyl anomaly does not vanish. This will discussed in more detail in Chapter 6.

Geometrical Construct (1-form)	**Line Subbundle**
$t_\mu \in \Gamma(N)$	Normal Subundle N
$n_\mu \in \Omega^1(M)$ of T^*M	Conormal subbundle N^* or T^*M with a metric on M
$a_\mu \in \Omega^1(M)$ of T^*M	Conormal subbundle N^* or T^*M with a metric on M

Table 2.1: The line subbundles associated with a foliation $\mathcal{F}$. The NC geometry constructs t_μ, n_μ, and a_μ naturally belong to these line subbundles.

Group G	**Differential Forms**	**NC Geometry**
Euclidean Translations of $\mathbf{R}$	$d\omega = 0$	Torsionless NC
Affine Translations of $\mathbf{R}$	$d\omega = \theta \wedge \omega,\ d\theta = 0$	TTNC geometry
$PSL(2,R)$ projective transformations	$d\omega = \theta \wedge \omega,\ d\theta = \omega \wedge \eta,\ d\eta = \eta \wedge \theta$	$PSL(2,R)$ geometry

Table 2.2: Different types of transverse foliation with different diffeomorphism groups of foliation leaves correspond to different types of the NC geometry
.

In Appendix 2.A, we will first provide a very brief introduction to the concept of foliated manifolds and the associated geometrical and topological structures. This will allow us to then place the discussion of the Lifshitz Weyl anomaly in the previous sections on a firm mathematical ground. More specifically, we will discuss the different types of flat line bundles associated with a foliated manifold and more importantly, using this framework, why it becomes natural to understand that Lifshitz Weyl anomaly in (1+1) dimensions is the Weyl partner of the Lorentz anomaly. In Appendix 2.B, we then move on to describe the different types of transverse foliations and how they naturally map to NC geometry with and without torsion. We then discuss flat line bundles and how they are related to codimension-1 foliations in Appendix 2.C.

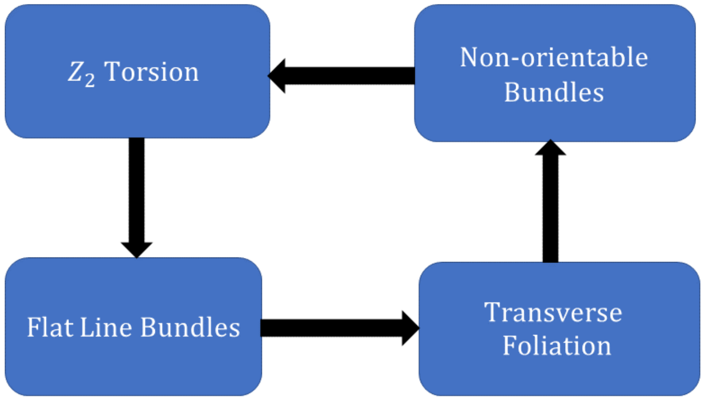


Figure 2.3. This figure depicts the mutual relationship between $\mathbb{Z}_2$ torsion in flat line bundles and their non-orientability, on one hand, and between a transverse *(G,M)*-structure on a foliation and flat line bundles on the other hand.

We note that due to the fact that this section is mathematically-oriented, we will not prove any theorem or conjecture that we use in our discussion below but will instead provide enough references for the interested reader. We will, in particular, assume some familiarity with the notions of fiber bundles, principal bundles and (co)homology classes. In the following subsections, we specialize to a codimension1 foliation although the definitions can be easily generalized to a codimension-1 foliation. Mathematical definitions will clearly be marked as **definitions**.

2.A The geometry of foliated manifolds

Definition 1: A fiber bundle is a structure (E, M, π, F), where E, M, and F are topological spaces and $\pi : E \to M$ is a continuous map (surjective) satisfying a *local* triviality condition. The space M is called the base space of the bundle, E the total space, and F the fiber. The map π is called the projection map (or bundle submersion). The triviality condition is understood as follows. Every point $x \in B$ has an open neighborhood or cover U_α such that the bundle has an open covering

$\{U_\alpha\}_{(\alpha \in A)}$, diffeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$, and transition functions $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \to \text{Diff}(F)$ such that $\varphi_\alpha = g_{\alpha\beta} \circ \varphi_\beta$.

As a concrete and practical *example* of how fiber bundles are used to formally describe anomalies especially in two spacetime dimensions, we closely follow the presentation in [113].

The tangent bundle is an example of a fiber bundle with an associated principle bundle with structure group (or gauge group) $G = SO(2, R)$. At each point on a 2-manifold M equipped with a metric $g_{\mu\nu}$, one can always write the metric using the vielbeins as ${e_\mu}^a$

$$g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \delta_{ab} \ , \tag{2.50}$$

where $a, b = \{1, 2\}$ are tangent (or Lorentz) frame indices. The vielbeins ${e_\mu}^a$ are not uniquely defined by (2.50) since the metric is invariant under a local Lorentz (or frame) transformation with an element ${\Lambda^a}_b \in SO(2, R)$

$$e^a{}_\mu \to (\Lambda^{-1})^a{}_b(x) e^b{}_\mu \ , \tag{2.51}$$

$e^a{}_\mu$ are the components of a Lie-valued 1-form which therefore, in order to ensure covariance under local Lorentz transformations, require us to introduce a spin connection ω_μ with components $\omega^{ab}{}_\mu = -\omega^{ba}{}_\mu$ transforming in the adjoint representation of $SO(2)$

$$\omega_\mu \to \Lambda^{-1}(x)\omega_\mu\Lambda(x) + \Lambda^{-1}(x)\, \partial_\mu(x) \tag{2.52}$$

Now assume the manifold M is covered by open subsets U_α and transitions functions $g_{\alpha\beta}$ between those charts. Depending on whether the manifold is topologically trivial or not, it might not be possible in general to define $e^a{}_\mu$ *globally* on the manifold. However, *locally* on a chart U_α, it is always possible to find one $e^a{}_\mu(\alpha)$. At intersections, $U_\alpha \cap U_\beta$, we can use either the vielbein $e_{(\alpha)}$ or $e_{(\beta)}$. The two are equivalent provided they are related by $\Lambda_{(\alpha\beta)}$

$$e_{(\alpha)}(x) = \Lambda_{(\alpha\beta)}(x)\, e_{(\beta)}(x) \ , \tag{2.53}$$

and similarly for the spin connection

$$\omega_\mu = \Lambda^{-1}{}_{(\alpha\beta)}(x)\omega_{(\beta)}\Lambda_{(\alpha\beta)}(x) + \Lambda^{-1}{}_{(\alpha\beta)}(x)\, \partial\, \Lambda_{(\alpha\beta)}(x)(x) \ . \tag{2.54}$$

In this example, $\Lambda_{(\alpha\beta)}$ are the transition functions of the principal $SO(2)$-bundle and they encode the non-triviality of the manifold. By definition, a manifold is topologically *nontrivial* if it does not admit a *unique globally* defined vielbein $e^a{}_\mu$ such that one cannot choose $\Lambda_{(\alpha\beta)} = 1$ at any intersection $U_\alpha \cap U_\beta$. A manifold is, therefore, topologically trivial if $\Lambda_{(\alpha\beta)} = 1$ at any intersection is admitted [113].

When the principle bundle admits a *flat* connection such that $F = dA = 0$, then it is flat. However, this does not necessarily mean that the bundle is topologically trivial since, as we discussed above, the non-triviality of the fiber bundle is detected or measured by the transition functions $g_{\alpha\beta}$ of the bundle that act on the overlap of the open subsets covering the base manifold M. The Aharonov-Bohm effect is a famous example of a nontrivial flat principle bundle.

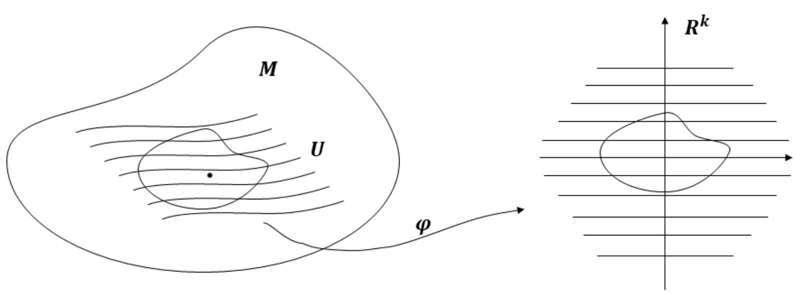


Figure 2.2. The definition of a foliation.

Next we define a codimension-1 foliation and its associated vector bundles and then use and use it to mathematically characterize the $z = 1$ Lifshitz Weyl anomaly.

Definition 2: A codimension-1 foliation [58], [114] of a d-dimensional smooth manifold M is a decomposition of M into a union of disjoint connected subsets $\{\mathcal{L}_\alpha\}_{(\alpha\in A)}$, called the leaves of the foliation $\mathcal{F}$, with the following property: Every point in M has a neighborhood U and a system of local coordinate maps that act by submersions[8] $\varphi_\alpha : U_\alpha \to \mathbb{R}$, such that for each overlap $U_\alpha \cap U_\beta$, there is a local diffeomorphism $g_{\alpha\beta}$ [9] of $\mathbb{R}$ such that

$$\varphi_\alpha = g_{\alpha\beta} \circ \varphi_\beta \tag{2.55}$$

A *local* characterization of a foliation $\mathcal{F}$ is usually given by the Frobenius theorem: a codimension-1 foliation is described by a codimension-1 integrable subbundle $T\mathcal{F}$ of the tangent bundle of M, TM, is locally described by a non-vanishing 1-form φ such that

$$d\varphi = \theta \wedge \varphi.$$

We say that $d\varphi$ *defines* the foliation $\mathcal{F}$. Using the language of fiber bundles, associated to any foliated manifold $(M, \mathcal{F})$ of codimension-1 is a real rank-1 vector bundle $N = TM/T\mathcal{F}$ called the normal bundle. More specifically, a transverse codimension-1 foliated manifold $(M, \mathcal{F})$ amounts to the existence of a nonvanishing section of the top degree line bundle $\Lambda^1 N^*$ of the conormal bundle N^* over M. Any identification of N^* with a subbundle of T^*M, obtained say by equipping M with a Riemannian metric, identifies such a section with a nonvanishing differential form $\varphi \in \Omega^1(M)$ such that

$$\varphi(X) = 0 \tag{2.56}$$

[8]Note that a fiber bundle is a simple case of a foliation where the submersions φ_α are maps into the same fiber. Please also note that this definition of a foliation is implicitly making use of the notion of a foliated atlas.

[9]This is not to be confused with the metric $g_{\mu\nu}$ on M.

for X contained in the space $\Gamma(T\mathcal{F})$ of vector fields tangent to the foliation.

In simple words, a codimension-1 foliation is a partition of a space M into disjoint connected subsets (leaves) which *locally* look like *copies of* $\mathbb{R}^k$, (lines if M is two-dimensional), stacked on top of each other [58], [115], [114] as shown in Fig. 2.2. A natural way to think of a leaf is as an immersed submanifold.

Given a codimension-1 foliation of a 2-dimensional manifold M, we have three different *line subbundles*:

I. The foliation tangent line bundle $T\mathcal{F}$.

II. The normal bundle over M, $N = TM/T\mathcal{F}$, which when a metric g on M is given, is defined by the condition $g(X, Y) = 0$ for all $X \in \Gamma(T\mathcal{F})$ and $Y \in \Gamma(N)$, the space of sections of *normal* vector fields.

III. The conormal line bundle N^*, which when a metric g on M is given, is identified with the cotangent line bundle $T^*\mathcal{F}$.

In light of the above, we can now give the foliation 1-forms t_μ and n_μ both defined in Section 2.1, a more mathematically-precise definition. While n_μ is a section in the normal line bundle i.e. $\nabla t \in \Gamma(N)$, n_μ naturally belongs in the space $\Omega^1(M)$ of the cotangent line bundle T^*F. In the 1-form picture of Section 2.3.2, we can see that the torsion 1-form $a_\mu \in \Omega^1(M)$. In Section 2.3.1 and in [16] however, a_μ is defined as a foliation *tangent* vector field which means that it naturally sits in $T\mathcal{F}$ rather than T^*F.

Given this structure of a codimension-1 foliation, the most important observation is that we are dealing with the geometry and topology of real line bundles. Since the torsion 1-form $a_\mu \in \Omega^1(M)$ is closed when constrained to a foliation leaf, which is a line in this case, it is a flat connection on the conormal bundle with zero curvature, i.e. $da = 0$. This clearly illustrates that the topology of real flat line bundles or of closed 1-forms is what we want to study in order to be able to characterize the topological character of the Lifshitz Weyl anomaly. Indeed, flat line bundles turn out to intimately related to foliating two-dimensional manifolds, i.e. to codimension-1 foliations in many ways. We will carry out this characterization in Chapter 6.2.

With the above background, we now explain the underlying mechanism followed by [16] in Section 2.5 to show that $z = 1$ Lifshitz Weyl anomaly is the Weyl partner of the Lorentz anomaly of conformal effective actions. Suppose we are given an effective action of a conformal field theory defined on a 2-manifold M. In addition, assume the theory, after quantization, breaks local Lorentz symmetry. Since the local Lorentz symmetry group is $SO(2)$, this means the principle G-bundle, associated to the tangent bundle over M, has a structure group (or gauge group) $G = SO(2)$. An effective action which breaks local Lorentz symmetry thus essentially means that the orthonormal frames, ${e^a}_\mu$ can no longer be defined $SO(2)$-valued sections. In other words, they are *not* an irreducible represnetation of the $SO(2)$ group. Instead, the ${e^a}_\mu$ orthonormal tangent frame decomposes into *two* separate *rank-1* vector fields ${e^0}_\mu$ and ${e^1}_\mu$, which become sections of two different line bundles with a reduced structure group $\mathbf{R}^* = \mathbf{R} - \{0\}$.

In terms of foliation geometry, identifying ${e^0}_\mu$ with the foliation 1-form n_μ means it becomes a section in the conormal line bundle N^* or in $\Omega^1(M)$ of T^*M, if M is equipped with a Riemannain metric, while identifying ${e^1}_\mu$ with its dual $\tilde{n}_\mu$ makes it a section in the tangent line bundle $T\mathcal{F}$. Both line bundles are defined over 1-dimensional integral submanifolds, i.e. leaves of the foliation. The torsion 1-form $a_\mu \in \Omega^1(M)$ is a flat connection if restricted to a foliation leaf and the conormal line bundle is therefore flat. This is the underlying mathematical reason of why Lifshitz Weyl anomaly of a theory defined on a foliated two-dimensional manifold can be understood as the Weyl partner of the Lorentz anomaly in a two-dimensional theory. If we can find a foliation of the bundle where a_μ *cannot* be defined globally, then both the tangent and conormal line bundles will be topologically non-trivial and we can thus give the Lifshitz Weyl anomaly a topological characterization.

A natural question therefore is: how does one classify or characterize flat line bundles? How does one distinguish trivial bundles or from nontrivial ones? We will partly answer this question below in Sections 2.B and 2.C To answer this question, we need to first define the notion of a transverse structure on a foliaiton. We do that in the next subsection.

2.B Transverse structures on a foliation

We will now briefly explain what it means to define a transverse (G, X)-structure on a foliation following Goldman and Brooks [116]. We will see that different transverse foliation structures provide the proper mathematical characterization of the three different types of NC geometry presented in Section 2.1. Transverse (G, X)-structures are closely related to the topic of foliated fiber bundles which we discuss in the next subsection and which will ultimately need in Chapter 6 to describe a new two-dimensional topological invariant.

Definition If X is a k-dimensional manifold, and G a group of diffeomorphisms of X, we say that $\mathcal{F}$ has a *transverse* (G, X)-structure if the φ's can be taken to be submersions onto X, and if the transition functions $g_{\alpha\beta}$'s are taken to lie in G.

For our case of interest, X is 1-dimensional manifold and hence G is the group of diffeomorphisms of $\mathbf{R}$, Diff($\mathbf{R}$). Since every foliation is locally a submersion $\varphi_\alpha : U_\alpha \to X$, we have a codimension-1 foliation of M with leaves defined by φ_α^{-1} and where now $g_{\alpha\beta} \in$ Diff($\mathbf{R}$). Let us consider the case when G is a Lie group. According to [115], Table 2.3, shows the types of transverse structures for a codimension-1 foliation, their description in terms of differential forms and the corresponding type of NC geometry they characterize. In Table 2.3, ω is a *global* 1-form that defines the foliation $\mathcal{F}$.

Group G	**Differential Forms**	**NC Geometry**
Euclidean Translations of $\mathbf{R}$	$d\omega = 0$	Torsionless NC
Affine Translations of $\mathbf{R}$	$d\omega = \theta \wedge \omega, \ d\theta = 0$	TTNC geometry
$PSL(2, R)$ projective transformations	$d\omega = \theta \wedge \omega, \ d\theta = \omega \wedge \eta, \ d\eta = \eta \wedge \theta$	$PSL(2, R)$ geometry

Table 2.3: Different types of transverse foliation with different diffeomorphism groups of foliation leaves correspond to different types of the NC geometry

.

It is the clear that ω is identified with the foliation 1-form n and θ is identified the torsion 1-form a in Section 2.3.2 . Notice that when $G = PSL(2, R)$, which is the group of projective transformations of the real projective line $\mathbf{R}P^1$, two things happen: (1) $d\theta$ is *not* closed which is the statement the the Lifshitz Weyl anomaly is non-vanishing and thus not restricted to a foliation leaf and (2) a non-zero 3-form $\theta \wedge d\theta \neq 0$ exists in three dimensions.

In Chapter 4, we will in fact see that $\theta \wedge d\theta$ is indeed a Chern-Simons (CS) term in a centrally-extended Schrödinger-invariant CS theory of gravity that translates within the $SL(2, R)$ group. The differential form $\theta \wedge d\theta$ is closed and independent of the choices of ω and θ. Hence, it defines a cohomology class, known as *Godbillon-Vey* class $GV(\mathcal{F})$ in the third real de-Rham cohomology group $H^3(M; R)$. We will have a lot more to say about the Godbillon-Vey class $GV(\mathcal{F})$ in chapter 4 when we derive the Lifhsitz Weyl anomaly from a Chern-Simons action.

The other important comment we would like to make is that, as we can see from the table above, when G is the group of affine translations, $d\theta = 0$, i.e. closed and therefore, although this type of transverse foliation admits the presence of torsion, it restricts it to a leaf. But this is exactly what happens when the Weyl invariance is restored by making the lapse function only x-dependent. Hence, we conclude that restoring Weyl invariance of the Lifshitz theory or rather canceling the anomaly necessarily means changing the type of transverse structures of the foliation from transversely projective to transversely affine.

2.C Flat line bundles and transverse foliations

In this section, we show how flat line bundles are closely related to transverse (G, M)-structures of a codimension-1 foliation.

Definition 3: A flat *real* line bundle [117] over a manifold M is a local system of one-dimensional $\mathbf{R}$-vector spaces which is determined by the *monodromy homorphsism*

$$Mon_E : \pi_1(M, x) = \mathbf{R}^* \ , \tag{2.57}$$

where π_1 is the fundamental group of the manifold M, $\mathbf{R}^* = \mathbf{R} - \{0\}$ is the group of real numbers under multiplication. This means that the structure group (or the gauge group) of the bundle is $\mathbf{R}^*$. If F_x is the fiber F over the base point x, then the monodromy action defines the F_x as vector space of $\pi_1(M, x)$. The fundamental group [104] of a manifold M is the group of the equivalence classes of loops under *homotopy*. It contains information about the *holes* contained in the manifold. Since $\mathbf{R}^*$ is an Abelian group, the fundamental group can be identified with the first homology group of the manifold $H_1(M)$ [118].

Any real flat bundle, trivial or nontrivial, determines a cohomology class $w_1(E) \in H^1(M; \mathbb{Z}_2)$, the first Stiefel-Whitney class of the bundle E [117]. The important thing is the class $w_1(E)$ itself can be defined in terms of the *monodromy* representation $Mon_E : \pi_1(M, x) = \mathbf{R}^*$ as follows

$$w_1(E) : \pi_1(M, x) \to \mathbb{Z}_2 = \{1, -1\} \,, \quad [\gamma] \mapsto \text{sign } (\text{Mon}([\gamma])) \,. \tag{2.58}$$

In other words, the class $w_1(E)$ assigns an equivalence class of loops in M to an element of $\mathbb{Z}_2$. In this way, the structure group of the bundle has been reduced to $\mathbb{Z}_2$, which tells us that there are *two* equivalence classes of loops in the bundle that the class $w_1(E)$ records. The first Stiefel-Whitney $w_1(E)$ determines the *orientability* of the underlying manifold M. The canonical example of an unorientable manifold is the Möbius strip which, topologically, is a nontrivial line bundle described by $w_1(E)$.

What about $w_1(E)$ for a trivial real line bundle? Trivial flat line bundles E have the first Stiefel-Whitney class $w_1(E) = 0$ and they are in one-to-one correspondence with the first *real*, not integral, cohomology group $H^1(M; \mathbf{R})$. This is the case, for example, of the cylinder which topologically is diffeomorphic to the product space $S^1 \times \mathbf{R}$.

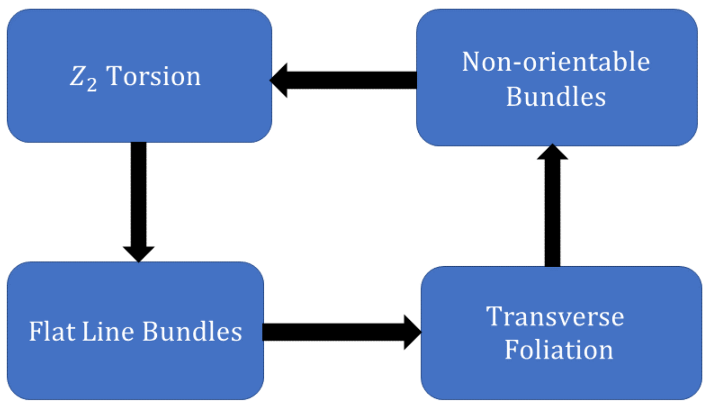


Figure 2.3. This figure depicts the mutual relationship between $\mathbb{Z}_2$ torsion in flat line bundles and their non-orientability, on one hand, and between a transverse *(G,M)*-structure on a foliation and flat line bundles on the other hand.

The connection of flat line bundles to transverse (G, M)-structures of a codimension-1 foliations has been known for more than forty years [119]. Any flat G-bundle over M is *uniquely* determined by the same monodromy (or holonomy) homomorphism for the flat line bundle in (2.57). This essentially means that the *holonomy* of a connection on a flat bundle E is equivalent to the holonomy of a transverse foliation of that of a leaf. For more details and proof of this relationship, please see [119]. The holonomy of a foliated manifold is a very important concept. Intuitively, it tells us how the leaves collectively behave together to give the foliation.

Thus, not only does it contain information about the fundamental groups of the leaves (which may vary from leaf to another) but about the foliation itself [120].

On the other hand, the relationship between torsion and non-orientable manifolds can be explained by the Chern-Weil theory. The Chern-Weil theory [105] says that the Chern classes of a flat bundle over a manifold are often non-trivial in *integral* cohomology, and hence can be used to distinguish between *flat* vector bundles [121]. Concretely, the Chern-Weil theory says that the *integral* Chern classes of a flat bundle over a compact manifold are all *torsion* [121]. In particular, over a non-orientable *2-manifold*, there are only two isomorphism types of flat vector bundles in each dimension: (1) the trivial bundle and (2) the non-trivial bundle, containing torsion, told apart by their first Chern class in the second cohomology group $c_1(E) \in H^2(M; \mathbb{Z}) = \mathbb{Z}_2$. Figure 2.3 illustrate the mutual relationship between the $\mathbb{Z}_2$ torsion in flat line bundles, their non-orientability on one hand and between transverse *(G,M)*-structure on a foliation and line bundles with a flat connection on the other hand.

Chapter 3

The Lifshitz Weyl Anomaly From a Chiral Field Theory

The key result of this chapter is:

1. Use the *Fujikawa* method to derive the $z = 1$ (1+1) Lifshitz Weyl anomaly from a massless chiral fermion action.

The $z = 1$ Lifshitz Weyl has been found formally using a cohomological approach. It would be interesting to derive it from a specific field theory. This is the goal of Chapter 3. In this chapter, we derive the $z = 1$ Lifshitz Weyl anomaly from a 2d massless chiral field theory. We will see why a why chirality is an essential requirement to get the correct expression of the 2d $z = 1$ Lifshitz Weyl anomaly. We will uncover the true nature of the (1+1)d the $z = 1$ Lifshitz Weyl anomaly as the *Weyl partner* of the pure *Lorentz anomaly* of a 2d quantum effective action as first pointed out in [16]. More specifically, in 3.1, we will use the *Fujikawa* method to derive the anomaly. By virtue of the true nature of $z = 1$ Weyl anomaly discussed in Section 2.5, we will expand the Dirac operator in the Jacobian of the path integral measure in a *chiral* *spinor* basis [12] [59] in order to obtain the correct expression of the $z = 1$ Weyl anomaly. We will end this chapter by making a few comments.

In Appendix 3.A, we present an attempt to derive the respective Lifshitz Weyl anomaly from the simplest $z = 1$ FPD-invariant action of a massless scalar field using heat kernel expansion [60]. Up to first order in perturbation theory, we do not find the *relevant* Weyl anomaly. Concretely, the final expression of the anomaly that we obtain consists only of *irrelevant or coboundary* terms, i.e. terms that can be removed by adding local counterterms to the quantum effective action. Although we did not go to second-order perturbation theory in our heat kernel expansion, we do not expect it would change the parity symmetry $x \to -x$ but this remains to be checked nevertheless. The failure to obtain the induced $z = 1$ Lifshitz Weyl anomaly reaffirms the fact this anomaly is only present in a quantum theory with *chiral* matter coupled to gravity which is known to suffer from a local Lorentz anomaly.

3.1 The Anomaly From a 2d Massless Weyl Fermion Action

In this section we use the Fujikawa method [11–14, 59] to derive the $z = 1$ Lifshitz Weyl anomaly from a 2d massles chiral fermion action. In the next two subsections, we set the notations and conventions used throughout this chapter and briefly review the two necessary ingredients that go into the computation of the Lifshitz Weyl anomaly which is actually carried in out in Section 3.1.3. In this section, we closely follow the notations in Appendix A and B of [122].

In Euclidean space, the time coordinate t is Wick-rotated $t \to it$ along with $\gamma_0 \to i\gamma_0$ such that $(\gamma^\mu)^\dagger = \gamma^\mu$ for all spacetime indices. The γ_5 matrix is expressed in four dimensions

$$\gamma_5 = -\gamma_0\gamma_1\gamma_2\gamma_3, \tag{3.1}$$

and it anti-commutes with the Dirac matrices,

$$\{\gamma_5,\ \gamma^\mu\} = 0 \quad \text{and} \quad (\gamma_5)^\dagger = \gamma_5. \tag{3.2}$$

In 2d Euclidean space, the γ matrices are given by

$$\gamma_0 = i\gamma_4, \quad \gamma_5 = -\gamma_4\gamma_1 \tag{3.3}$$

while in Minkowski space, they become Pauli σ-matrices

$$\gamma_0 = \sigma_2, \quad \gamma_1 = i\sigma_1, \quad \gamma_5 = \sigma_3 \tag{3.4}$$

3.1.1 The spin connection

It is well known that spinor fields do not transform covariantly under the diffeomorphim group [1]. Hence, in a theory where fermions are coupled to background gravity, the orthonormal tangent frame coordinates or vielbeins are used for this purpose. The coupling is achieved using Cartan's formalism [100]. To set up the notation, we introduce vielbeins ${e^a}_\mu$ as

$$g_{\mu\nu} = \delta_{ab}{e^a}_\mu {e^b}_\nu \ , \tag{3.5}$$

where δ_{ab} is the flat Euclidean spacetime metric. The inverse vielbeins ${E_a}^\mu$ is then defined as $g^{\mu\nu} = \delta^{ab}{E_a}^\mu {E_b}^\nu$, and satisfy ${E_a}^\mu {e^b}_\mu = \delta^b_a$. The components of the spin connection 1-form ${\omega^{ab}}_\mu$ are then given by [1]

$$\omega^a{}_{b\mu} = {e^a}_\nu \nabla_\mu {E_b}^\nu = {e^a}_\nu \left(\partial_\mu {E_b}^\nu + \Gamma^\nu_{\mu\lambda} {E_b}^\lambda\right) \ . \tag{3.6}$$

Using the inverse vielbeins, the γ-matrices are used to define the coupling of the Dirac operator to the background vilebeins

$$\gamma^\mu = \gamma^a E_a{}^\mu \ , \qquad \{\gamma^a,\ \gamma^b\} = 2\delta_{ab} \ . \tag{3.7}$$

The γ-matrices are expressed in terms of their tangent frame counterparts γ^a and the spinor covariant derivative

$$D_\mu = \partial_\mu + \frac{1}{8}[\gamma_a,\ \gamma_b]\omega_\mu{}^{ab} \ . \tag{3.8}$$

We then have the following important commutation relation between covariant derivatives

$$[D_\mu,\ D_\nu] = \frac{1}{8}[\gamma_a,\ \gamma_b]R^{ab}{}_{\mu\nu} \tag{3.9}$$

.

Using (3.8) and (3.9), we then have the following relation for the Dirac operator

$$(i\gamma^\mu D_\mu)^2 = -D_\mu D^\mu - \frac{1}{32}[\gamma_a,\ \gamma_b]\ [\gamma_c,\ \gamma_d]\ R^{abcd} \ , \tag{3.10}$$

which after contracting with $\gamma_a\gamma_b\gamma_c\gamma_d$ and using the the identity $R^{abcd}+R^{acdb}+R^{adbc} = 0$, we get the well-known *Lichnerowicz formula* for the square of the Dirac operator written expressed only in terms of the Ricci scalar [122, 123]

$$(i\gamma^\mu D_\mu)^2 = -D_\mu D^\mu + \frac{1}{4}R \ . \tag{3.11}$$

3.1.2 The geodesic interval

The second ingredient that we need in ordet to carry out the computation in Section 3.1.3 is the *geodesic interval* [124], which is defined as one half the square of the distance along the geodesic between any two points x and x' [1].

$$\sigma(x, x') = \frac{1}{2}\left(\int_x^{x'} ds\right)^2 . \qquad (3.12)$$

The geodesic interval is basically a symmetric function $\sigma(x, x')$ of x and x' that satisfies the following differential equation

$$\sigma(x, x') = \frac{1}{2}(\nabla_\mu \sigma)(\nabla^\mu \sigma) = \frac{1}{2}(\nabla'_\mu \sigma)(\nabla'^\mu \sigma) \qquad (3.13)$$

with these boundary conditions

$$\sigma(x, x) = 0 \ , \qquad \lim_{x \to x'} \nabla_\mu \sigma(x, x') = 0 = \lim_{x \to x'} \nabla'_\mu \sigma(x, x') \ . \qquad (3.14)$$

In the coincidence limit $x \to x'$, $\sigma(x, x')$ obeys the relation

$$\lim_{x \to x'} \nabla_\mu \nabla_\nu \sigma(x, x') = - \lim_{x \to x'} \nabla_\mu \nabla'_\nu \sigma(x, x') = g_{\mu\nu} \ . \qquad (3.15)$$

and thus, the geodesic interval is considered a curved space generalization of its flat space counterpart $\sigma(x, x') = (x - x')^2/2$. More importantly, $\sigma(x, x')$ can be used to define the delta function on a general Riemannian manifold as follows

$$\frac{1}{\sqrt{g}}\delta(x - x') = \frac{1}{\sqrt{g}} \int \frac{d^2k}{(2\pi)^2} e^{ik_\mu \nabla^\mu \sigma(x,x')} \ . \qquad (3.16)$$

The so-called *Synge–DeWitt tensors* are defined by coincidence limit $x \to x'$ of successive covariant derivatives of the geodesic interval, $\nabla_\mu \nabla_\nu \cdots \nabla_\kappa \sigma(x, x')$. The lowest-rank Synge–DeWitt tensors are expressed as

$$[\sigma] = 0 \ , \qquad [\nabla_\mu \sigma] = 0 \ , \qquad [\nabla_\mu \nabla_\nu \sigma] = g_{\mu\nu} \ . \qquad (3.17)$$

[1] We closely follow the notation and definitions in Appendix B.2 of [122]

where $[\nabla_\mu \nabla_\nu \cdots \nabla_\kappa \sigma] = \lim_{x\to x'} \nabla_\mu \nabla_\nu \cdots \nabla_\kappa \sigma(x, x')$.

3.1.3 Steps of the Derivation

The theory of massless chiral fermions coupled to backgound gravity in $1+1$ space–time dimensions can be written as

$$S_\psi = \int d^2x\, e\, \overline{\psi}\, i\gamma^\mu D_\mu P_- \psi \tag{3.18}$$

where $e = \sqrt{|g|}$, $P_- = \frac{1}{2}(1-\gamma_5)$ is the chirality operator, the conjugate field is defined by $\overline{\psi} = \psi^\dagger \gamma^0$ and the Dirac operator D_μ is used here as defined in (3.8).

Under local Lorentz transformations with infinitesimal parameter α^a_b, the vielbein and spinors transform as follows

$$\delta^L_\alpha\, e^a_\mu = -\alpha^a_b\, e^b_\mu, \quad \delta^L_\alpha\, \psi = -\frac{1}{2}\alpha_{ab}\sigma^{ab}\psi, \quad \overline{\psi} = \frac{1}{2}\alpha_{ab}\overline{\psi}\sigma^{ab}\,, \tag{3.19}$$

where $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$, the action S_ψ changes as

$$\delta^L_\alpha\, S_\psi = \int d^2x\, e\, T^\mu_a\, \delta^L_\alpha e^a_\mu = -\int d^2x\, e\, T^\mu_a\, \alpha^a_b e^b_\mu \quad , \tag{3.20}$$

which implies that a Lorentz-invariant action S_ψ dictates a *symmetric* energy-momentum tensor

$$\delta^L_\alpha\, S_\psi = 0 \Longleftrightarrow T^{ab} = T^{ba} \tag{3.21}$$

Similarly, the action S_ψ is invariant under Weyl transformation of the vielbein and spinor fields which scale finitely as follows

$$e^a{}_\mu{}'(x) = e^{\sigma(x)} e^a{}_\mu(x), \quad \psi'(x) = e^{-\sigma(x)}\psi(x), \quad \overline{\psi'}(x) = e^{-\sigma(x)}\overline{\psi'}(x). \tag{3.22}$$

Then, under a Weyl transformation with infinitesimal parameter $\sigma(x)$, they transform as follows

$$\delta^W_\sigma e^a{}_\mu = \sigma\, e^a{}_\mu, \quad \delta^W_\sigma \psi = -2\sigma\psi, \quad \delta^W_\sigma \overline{\psi} = -2\sigma\overline{\psi} \quad , \tag{3.23}$$

the action S_ψ changes as

$$\delta^W_\sigma S_\psi = \int d^2x\, e\, T^\mu{}_a\, \delta^W_\sigma e^a{}_\mu = -\int d^2x\, e\, \sigma\, T^\mu{}_\mu \qquad , \tag{3.24}$$

which implies that a Weyl-invariant action S_ψ dictates a *traceless* energy-momentum tensor

$$\delta^W_\sigma S_\psi = 0 \Longleftrightarrow T^\mu{}_\mu = 0. \tag{3.25}$$

We now proceed to quantize the classical action S_ψ by computing the path integral *only* over the fermions. The path integral is defined as

$$Z = \int (\mathcal{D}\overline{\psi})(\mathcal{D}\psi) e^{-S_\psi} \tag{3.26}$$

As pointed out in Section 3.A, it is the path integral measure that carries all the information about the coupling to a background metric, in this case, a vielbein. The measure will be expressed as an infinite sum which, to be mathematically well-defined, must be *properly* regularized. By properly here, I mean one has to choose the correct regularization scheme that (1) only breaks the symmetry (or symmetries) for which the anomaly is calculated while respecting the remaining gauge symmetries of the path integral such that the correct anomaly and, more importantly, all anomalies are obtained. Said differently, the Weyl anomaly and in fact, all anomalies, result from the fact that the regularization methods break the symmetries of the classical action.

In this section, we use the Fujikawa *Gaussian regulariztion* method to compute $\langle T^\mu{}_\mu \rangle$. The trick here however, is that despite the fact that this $z = 1$ Lifshitz anomaly appears in $\langle T^\mu{}_\mu \rangle$, as pointed out earlier in Section 2.5, is a direct consequence of the presence of *chirality* in the action as opposed to the conformal anomaly generated from a non-chiral classical action. Therefore, in order to produce the correct anomaly, we will have to use the regularization method used in calculating the local Lorentz anomaly in [59]. More concretely, we will expand the Dirac operator in the chiral basis in the same way the local Lorentz anomaly is calculated. The presence of the γ_5 matrix in the unregulated sum in the Jacobian of the path integral measure turns out to play a very important role in obtaining the $z = 1$ Weyl anomaly.

The following derivation closely resembles that of the gravitational contribution to the axial anomaly in four dimensions given in [122]. Following [12] [59], we first define the spinors $\tilde{\psi}(x) \equiv \sqrt{e}\psi(x)$ and $\tilde{\overline{\psi}}(x) \equiv \sqrt{e}\overline{\psi}(x)$ and consider them as the fundamental fields. We then expand the Hermitian Dirac operator $i\gamma^\mu \tilde{D}_\mu$ in terms of a complete set of eigen-functions $\tilde{\varphi}_n$

$$(i\gamma^\mu \tilde{D}_\mu)\tilde{\varphi}_n(x) = \lambda_n \tilde{\varphi}_n(x) \ , \tag{3.27}$$

which are normalized as

$$\int d^2x \ \tilde{\varphi}_n^\dagger(x)\tilde{\varphi}_m(x) = \delta_{nm} \ . \tag{3.28}$$

The path integral measure is then formally expressed in terms of elements a_n and $\bar{b}_n$ of the Grassmann algebra as

$$(\mathcal{D}\tilde{\psi})(\mathcal{D}\tilde{\overline{\psi}}) = \prod_n a_n \prod_m \bar{b}_m \ , \tag{3.29}$$

after which a general spinor is expanded into a *chiral* eigen-basis as

$$\tilde{\psi}(x) = \sum_n a_n P_- \tilde{\varphi}_n(x), \quad \tilde{\overline{\psi}}(x) = \sum_n \tilde{\varphi}_n^\dagger(x)\bar{b}_n P_+ \ , \tag{3.30}$$

where $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$. Under an infinitesimal Weyl transformation (3.23), the coefficients $\tilde{a}_n$ and $\tilde{\bar{b}}_n$ transform as $\tilde{a}_n \to \sum_m C_{nm}\tilde{a}_m$ and $\tilde{\bar{b}}_n \to \sum_m C_{nm}\tilde{\bar{b}}_m$, where

$$C_{nm} = \delta_{nm} + i \int d^2x\, \sigma(x) \sum_n \tilde{\varphi}_n^\dagger(x)\gamma_5\tilde{\varphi}_n(x) \ . \tag{3.31}$$

As a result, the path integral measure picks up a Jacobian factor and transforms as

$$(\mathcal{D}\tilde{\psi})(\mathcal{D}\tilde{\bar{\psi}}) \to \exp\left(J[\sigma]\right) \tag{3.32}$$

where

$$J[\sigma] = \exp\left(-2i \int d^2x\, \sigma(x) \sum_n \tilde{\varphi}_n^\dagger(x)\gamma_5\tilde{\varphi}_n(x)\right) . \tag{3.33}$$

Since the sum in the exponential is not well defined, the primitive or rather unregulated trace of the energy-momentum density therefore reads

$$\langle T^\mu{}_\mu(x)\rangle = 2\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x) \ . \tag{3.34}$$

To regularize this infinite sum, Fujikawa used a Gaussian cut-off [11, 12, 125]

$$\begin{aligned} A(x) &= \lim_{M\to\infty} \sum_n \varphi_n^\dagger(x)\gamma_5\ e^{-(i\gamma^\mu D_\mu/M)^2}\varphi_n(x) \\ &= \lim_{M\to\infty}\lim_{x\to x'} \mathrm{tr} \int \frac{d^2k}{(2\pi)^2}\ \gamma_5\ e^{-(i\gamma^\mu D_\mu/M)^2} e^{ik_\mu \nabla^\mu \sigma(x,x')} \end{aligned} \tag{3.35}$$

where in the last line, we switched to momentum basis in curved space and used geodesic interval defined in (3.13). The trace tr is taken over the Dirac matrices. The order of limits in the above equation is important. First note that the derivatives of the Dirac operator are taken with respect to x (and *not* x'). It is also important that the action of the operator on $\exp(ik_\mu\nabla^\mu\sigma(x,x'))$ is evaluated *before* the coincidence limit $x \to x'$ is taken. Using Lichnerowicz's formula defined in (3.11), we then obtain the following expression for $A(x)$ [122]

$$\begin{aligned} &\lim_{M\to\infty}\lim_{x\to x'} \mathrm{tr}\int \frac{d^2k}{(2\pi)^2}\; e^{ik_\mu \nabla^\mu \sigma(x,x')}\;\gamma_5 \,\exp\Big[\frac{1}{M^2}\Big((D_\mu + i\Delta_\mu)(D^\mu + i\Delta^\mu) - \frac{1}{4}R\Big)\Big] \\ = &\lim_{M\to\infty} M^2 \;\mathrm{tr}\int \frac{d^2k}{(2\pi)^2}\; e^{-k_\mu k^\mu}\gamma_5 \,\exp\left(\frac{1}{M^2}D_\mu D^\mu + \frac{2i}{M}k_\mu D^\mu - \frac{1}{4M^2}R\right). \end{aligned} \tag{3.36}$$

Acting with the operator $(i\gamma^\mu D_\mu)^2$ in (3.35), replaces D_μ with its counterpart in curved space $D_\mu + i\Delta_\mu$, where $\Delta_\mu(x,x') = k_\nu \nabla_\mu \nabla^\nu \sigma(x,x')$, gives the first line in (3.36). Then, taking the limit $x \to x'$ replaces $\Delta_\mu(x,x')$ by k_μ using the relation in (3.15) which gives the second line in (3.36). Note that k_μ has also been rescaled to Mk_μ.

Next, we expand the exponent in (3.36) in power series of $1/M$. As a result, *only* terms up to order $1/M^2$ will survive in the limit $M \to \infty$. In addition, terms in the expansion with less than *two* Pauli matrices will vanish using their trace identities. Hence, it is clear that the Ricci scalar term in the exponential will *not* contribute since it contains no Pauli matrices and the non-vanishing contribution to the trace will therefore come from the $D_\mu D^\mu$ term [122].

In two Minkowski as well as Euclidean spacetime dimensions, the Dirac-γ matrices become Pauli matrices and in particular the γ_5 matrix is σ_3 for the former and $i\gamma_4\gamma_1$. After dropping all lower order terms, with zero trace, and all higher order terms, which vanish in the limit $M \to \infty$, we get the following expression [122]

$$A(x) = \mathrm{tr}\int \frac{d^2k}{(2\pi)^2}\; e^{-k_\mu k^\mu}\sigma_3 \left(\frac{1}{2}(D^2) - 2k_\mu k_\nu D^\mu D^\nu\right). \tag{3.37}$$

The integrals over k can be carried out by using a unit vector $\widehat{k}_\mu$ $k_\mu = k\widehat{k}_\mu$ with $k_\mu k^\mu = k^2$. Using the following identity [122]

$$\int \frac{d^2k}{(2\pi)^2}\; e^{-k^2}\; k^2\, \widehat{k}_a \widehat{k}_b = \frac{1}{2}\delta_{ab}\frac{2\pi}{(2\pi)^2}\int_0^\infty dk\; k^3 e^{-k^2} = \frac{1}{16\pi}\delta_{ab} \tag{3.38}$$

and the definition of the Dirac operator in (3.8), we finally get

$$\begin{aligned} A(x) &= \frac{1}{16\pi}\mathrm{tr}\,(\sigma_3\, D_\mu D^\mu)\,\mathbb{I} \\ &= \frac{1}{16\pi}\mathrm{tr}\,\sigma_3\left(\partial_\mu + \frac{1}{4}\sigma_3\,\omega_\mu\right)\left(\partial^\mu + \frac{1}{4}\sigma_3\,\omega^\mu\right)\,\mathbb{I} \\ &= \frac{1}{32\pi}\left(\partial_\mu\,\omega^\mu + \omega_\mu\,\partial^\mu\right)\,\mathbb{I} \\ &= \frac{1}{32\pi}\left(\partial_\mu\,\omega^\mu\right) \\ &= \frac{1}{32\pi}\left(\epsilon^{ab}\partial_\mu\,\omega^\mu{}_{ab}\right) \end{aligned} \tag{3.39}$$

where we used the trace identity of Pauli matrices $\mathrm{tr}\,(\sigma_\alpha\sigma_\beta) = 2\delta_{\alpha\beta}$ and the fact that in the definition of the Dirac operator $[\gamma_a, \gamma_b]$ becomes $[\sigma_1, \sigma_2] = 2\sigma_3$. The identity operator $\mathbb{I}$ inserted above has been always implicit when the operator $(i\gamma^\mu D_\mu)^2$ acted on $e^{ik_\mu\nabla^\mu\sigma(x,x')}$ in (3.36) and it is the reason why the term $\omega_\mu\,\partial^\mu$ simply vanishes. Thus, we are left finally with the exact same expression that in Section 2.5, we learned is in fact the $z = 1$ Lifshitz Weyl anomaly.

3.2 Discussion and Outlook

In this chapter, we attempted to derive the $z = 1$ Lifshitz Weyl anomaly from two different 2d field theories. In Section 3.A, we used the heat kernel method to derive the Lifshitz Weyl anomaly from the simplest $z = 1$ action of an FPD-invariant action of a massless scalar field coupled to background non-relativistic gravity. As we have seen, this action actually failed to produce the *relevant* anomaly in the sense that the final expression of the trace of the diagonal matrix element consisted only of *irrelevant or coboundary* terms, i.e. terms that can be removed by adding local counterterms.

In Section 3.1.3, we used the *Fujikawa* method to derive the $z = 1$ 2d Lifshitz Weyl anomaly from the action of a *chiral* massless fermion coupled to relativistic gravity. We showed that only by expanding the spinors in a chiral basis, we were to obtain the correct expression of the Weyl anomaly. This, in turn, confirmed the fact that the $z = 1$ Lifshitz anomaly is indeed the Weyl partner of the Lorentz anomaly of (1+1)-dimensional conformal field theory.

It is important to mention however that this $z = 1$ Weyl anomaly $\left(\epsilon^{ab}\partial_\mu \omega^\mu{}_{ab}\right)$ is a *consistent* anomaly, i.e. obeys the WZ consistency conditions described in Section 1.2 not a *covariant* anomaly, i.e. the anomalous *current*, here the $\langle T^\mu_\mu \rangle = \frac{1}{16\pi}\left(\partial_\mu \omega^{01\mu}\right)$, is not gauge covariant. This is of course obvious since $\langle T^\mu_\mu \rangle$ depends explicitly on the spin connection. In addition, since this $z = 1$ Lifshitz Weyl anomaly, one should not expect it to be covariant in a theory that breaks relativistic invariance to start with. It is well know that by adding a local Bardeen-Zumino polynomial $\mathcal{P}^\mu_\mu$ to $\langle T^\mu_\mu \rangle$, one get obtain a covariant anomaly from a consistent one [1, 23].

Finally, we would like to make an interesting observation. Consider the 2d *Dirac* theory of a free massless theory

$$S_\psi = \int d^2x\, e\, \overline{\psi}\, i\gamma^\mu D_\mu \psi \ . \tag{3.40}$$

Suppose we want to calculate the chiral anomaly of this action. The classical action in (3.40) is invariant under the following chiral transformations

$$\psi \to e^{i\alpha(x)\gamma_5}\psi \ , \overline{\psi} \to \overline{\psi} e^{i\alpha(x)\gamma_5} \ . \tag{3.41}$$

If we follow the exact same procedure of quantizing this classical action using the Fujikawa method with Gaussian regulator, we get the same exact Jacobian factor in (3.33). Carrying out the same steps in Section 3.1.3, we obtain an expression for the 2d chiral anomaly that is exactly the same as that of the Lifshitz Weyl anomaly in (3.39).

Recently, a surprising development in computing trace anomalies of 4d field theories did occur. It was shown in [126] that the *imaginary* part of the quantum effective action of a chiral field theory contains the parity-odd Pontryagin term which meant that the anomalous trace is given by

$$\left\langle T^{\mu}_{\mu} \right\rangle = -\beta_1 C^2 - \beta_2 E_4 - \alpha \Box R - \beta_4 P_4 \tag{3.42}$$

where C^2 is the square of the Weyl tensor, E_4 is the Euler density and P_4 is the parity-odd Pontrayagin term. Here β_4 is a purely imaginary coefficient $\beta_4 = \frac{i}{48\pi^2}$. On dimensional grounds, the P_4 is a possible term but it has been strongly believed that this term does vanish in actual calculations [127]. As emphasized in [127], the relation between parity-odd terms and anomalies in four spacetime dimensions in relation to *diffeomorphism* anomalies has long been been known [128]. The surprise here is the appearance of this term in the trace anomaly.

The appearance of the parity-odd Pontryagin term in the trace anomaly and its relationship to diffeomorphism anomalies bears a striking similarity to our discussion of Weyl partners in two spacetime dimensions. The $z = 1$ Lifshitz Weyl anomaly is parity odd and appears, as discussed in Section 2.4, with an imaginary coefficient in the 2d effective action

$$\left\langle T^{\mu}_{\mu} \right\rangle = \frac{1}{192\pi} \left(n_R + n_L\right) R + i \left(n_R - n_L\right) \epsilon^{ab} \nabla_{\mu} \omega^{\mu}{}_{ab} \ . \tag{3.43}$$

This begs the question of whether one can explain this surprising appearance of the Pontrayagin term in the 4d trace anomaly using the rationale of a *Weyl partner* of the diffeomorphism anomaly.

Appendix

3.A An FPD-invariant Free Massless Scalar Field

In this Appendix, we present an attempt to derive the respective Lifshitz Weyl anomaly from the simplest $z = 1$ FPD-invariant action of a massless scalar field using heat kernel expansion [60].

Consider the following FPD-invariant action of a free massless scalar field

$$\int dx\, dt \sqrt{g} g^{\mu\nu} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) + \sqrt{g} g^{tt} \left(\partial_t \phi\right)^2 \tag{3.44}$$

We can, using an FPD transformation, (5.5) $t \to \tilde{t}(t)$, $\quad x^i \to \tilde{x^i}(x^i, t)$, get to a coordinate system where the metric $g^{\mu\nu}$ is diagonal, and some first derivatives are set to zero. In particular, assume an FPD transformation at a given point p where

$$\begin{aligned}
\tilde{g}^{tt} &= \left(\frac{\partial \tilde{t}}{\partial t}\right)^2 g^{tt} && (3.45)\\
\tilde{g}^{xt} &= \left(\frac{\partial \tilde{x}}{\partial x}\right)\left(\frac{\partial \tilde{t}}{\partial x}\right) g^{xx} + \left(\frac{\partial \tilde{x}}{\partial t}\right)\left(\frac{\partial \tilde{t}}{\partial x}\right) g^{tx} + \left(\frac{\partial \tilde{x}}{\partial x}\right)\left(\frac{\partial \tilde{t}}{\partial t}\right) g^{xt} + \left(\frac{\partial \tilde{x}}{\partial t}\right)\left(\frac{\partial \tilde{t}}{\partial t}\right) g^{tt} \\
&= \left(\frac{\partial \tilde{x}}{\partial x}\right)\left(\frac{\partial \tilde{t}}{\partial t}\right) g^{xt} + \left(\frac{\partial \tilde{x}}{\partial t}\right)\left(\frac{\partial \tilde{t}}{\partial t}\right) g^{tt} \ .
\end{aligned}$$

where in the last step, we use that $\left(\frac{\partial \tilde{t}}{\partial x}\right) = 0$ to kill the first two terms. Given the above transformation, we can now solve for

$$\begin{aligned} \tilde{g}^{xt} = 0 \Longrightarrow 0 &= \left(\frac{\partial \tilde{x}}{\partial x}\right)\left(\frac{\partial \tilde{t}}{\partial t}\right) g^{xt} + \left(\frac{\partial \tilde{x}}{\partial t}\right)\left(\frac{\partial \tilde{t}}{\partial t}\right) g^{\mathrm{tt}} \\ &= \left(\frac{\partial \tilde{t}}{\partial t}\right)\left\{\left(\frac{\partial \tilde{x}}{\partial x}\right) g^{xt} + \left(\frac{\partial \tilde{x}}{\partial t}\right) g^{\mathrm{tt}}\right\} \Longrightarrow \left(\frac{\partial \tilde{x}}{\partial t}\right) = -\frac{g^{xt}}{g^{tt}}\left(\frac{\partial \tilde{x}}{\partial x}\right) . \end{aligned} \tag{3.46}$$

Using the transformation in (3.46), we now have a diagonal form for $g^{\mu\nu}$, which amounts to taking the shift vector $N^i = 0$. From now on, consider following action

$$S_{FPD} = \int dx\, dt \sqrt{g}\left(g^{xx}\left(\partial_x \phi\right)\left(\partial_1 \phi\right) + g^{tt}\left(\partial_t \phi\right)^2\right) . \tag{3.47}$$

The classical action $S_{FPD}[g, \varphi]$ is invariant under infinitesimal local Weyl transformation of the background metric with parameter $\sigma(x)$: $g_{\mu\nu} \rightarrow (1+\sigma(x)) g_{\mu\nu}$, the variation in the classical action $S_{FPD}[g, \varphi]$ is:

$$\delta_\sigma^W S_{FPD}[g, \phi] = -\frac{1}{2}\int d^2x\, T^{\mu\nu}\, \delta g_{\mu\nu} = -\frac{1}{2}\int d^2x\, \sigma(x) T^\mu{}_\mu , \tag{3.48}$$

where

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S[g, \phi]}{\delta g_{\mu\nu}(x)} . \tag{3.49}$$

This invariance of $S[g, \phi]$ under local Weyl transformation implies that *classically*

$$T^\mu{}_\mu = 0 . \tag{3.50}$$

However, the situation is different when we try to quantize the action $S_{FPD}[g, \varphi]$ which we do by computing the path integral $Z[g]$ only over the scalar field ϕ

$$Z_{FPD}[g] = \int [d\phi]_g\, e^{-S[g,\phi]} = e^{-W_{FPD}[g]} , \tag{3.51}$$

where $W_{FPD}[g]$ is the FPD-invariant *quantum effective action*, the variation of which under a local Weyl transformation gives after replacing $g_{\mu\nu} \to (1+\sigma(x))g_{\mu\nu}$ is given by

$$\delta^W_\sigma W_{FPD}[g] = -\frac{1}{2}\int d^2x\sqrt{g}\,\sigma(x)\,\langle T^\mu{}_\mu(x)\rangle \ , \tag{3.52}$$

where the expectation value of the trace of the energy-momentum tensor is defined as

$$\langle T^{\mu\nu}\rangle \ = \frac{2}{\sqrt{g}}\frac{\delta W[g]}{\delta g_{\mu\nu}(x)} \ , \tag{3.53}$$

and therefore the a non-zero $\langle T^{\mu\nu}\rangle$ gives the Weyl or trace anomaly.

3.A.1 Steps of the calculation

In what follows, we follow the steps in Appendix 5A and 5B of [44] to calculate the Weyl anomaly starting from the S_{FPD} classical action in (3.44). The computation of the Weyl anomaly eventually comes down to calculating the following trace

$$\delta W = \frac{1}{2}\mathrm{Tr}\left(\sigma\, e^{-\epsilon\Delta_{FPD}}\right) = -\frac{1}{2}\int dx\,dt\,\delta\sigma(x,t)\,\langle x,t|e^{\epsilon\Delta_{FPD}}|x,t\rangle \tag{3.54}$$

where the operator

$$\Delta_{FPD} = \frac{1}{\sqrt{g}}\partial_x\sqrt{g}g^{xx}\partial_x + \frac{1}{\sqrt{g}}\partial_t\sqrt{g}g^{tt}\partial_t \tag{3.55}$$

is the FPD counterpart of the fully covariant Laplace-Beltrami operator $\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu$ which is used to covariantly couple a free scalar field to a relativistic background metric. Here ϵ is an infinitesimal *proper time* parameter that we take to zero at the end of the calculation. However, before we get to the actual steps of the calculating this trace in Section 3.A.2, we will first make some necessary definitions and expand Δ_{FPD} to second order which we do below.

Let us write, in the vicinity of a point p, the metric in coordinates where

$$\begin{aligned} g^{tt} &= 1, \quad \partial_t g^{tt} = 0, \quad \partial_t^2 g^{tt} = 0 \\ g^{xx} &= 1, \quad \partial_x g^{xx} = 0, \quad \partial_x^2 g^{xx} = 0, \quad g^{\mathrm{xt}} = 0 \end{aligned} \tag{3.56}$$

We then perturbatively expand g^{tt} and g^{xx} in δx and δt taking g^{xt}=0, again the in the vicinity of a point p as follows

$$\begin{aligned} g^{tt} &= 1 + A_p \delta x + \frac{1}{2}\left(\partial_x A\right)_p \delta x^2 + \frac{1}{2}\left(\partial_t A\right)_p \delta x \delta t \\ g^{xx} &= 1 + B_p \delta t + \frac{1}{2}\left(\partial_t B\right)_p \delta t^2 + \frac{1}{2}\left(\partial_x B\right)_p \delta x \delta t \end{aligned} \tag{3.57}$$

where

$$\begin{aligned} A &= \partial_x g^{tt} \\ B &= \partial_t g^{xx} \end{aligned} \tag{3.58}$$

Now let Δ_t be the time component of Δ_{FPD}

$$\begin{aligned} \Delta_t &\equiv \frac{1}{\sqrt{g}} \partial_t \sqrt{g} g^{tt} \partial_t = g^{tt} \partial_t^2 + \left(\partial_t g^{tt}\right) \partial_t + g^{tt} \left(\frac{1}{\sqrt{g}} \partial_t \sqrt{g}\right) \partial_t \\ &= g^{tt} \partial_t^2 + \left(\partial_t g^{tt}\right) \partial_t - g^{tt} \left(\frac{1}{2} \partial_t \log\left(g^{tt} g^{xx}\right)\right) \partial_t \\ &= g^{tt} \partial_t^2 + \frac{1}{2}\left(\partial_t g^{tt}\right) \partial_t - \frac{1}{2} \frac{g^{tt}}{g^{xx}} \left(\partial_t g^{xx}\right) \partial_t \ . \end{aligned} \tag{3.59}$$

The spatial part of Δ_{FPD}, Δ_s, is similarly defined by replacing g^{tt} with g^{xx} and ∂_t with ∂_x in Δ_t . In terms of A and B, $\partial_t g^{tt}$ and $\partial_t g^{xx}$ read

$$\begin{aligned} \partial_t g^{tt} &= \frac{1}{2}\left(\partial_t A\right)_p \delta x; \quad \partial_x g^{xx} \frac{1}{2}\left(\partial_x B\right)_p \delta t \\ \partial_t g^{xx} &= B + \left(\partial_t B\right)_p \delta t + \frac{1}{2}\left(\partial_t B\right)_p \delta x \ . \end{aligned} \tag{3.60}$$

Expanding $\frac{g^{tt}}{g^{xx}}\left(\partial_t g^{xx}\right)$ and ∂_t to second order, Δ_t gives

$$\Delta_t = \left(\partial_t^2 - \frac{B}{2}\partial_t\right) + C_1\partial_t + C_2\partial_t^2 \ , \tag{3.61}$$

where the coefficients C_1 and C_2 are given by

$$\begin{aligned}
C_1(x,t) &= \left(-\frac{A\left(\partial_x B\right)_p}{4} - \frac{B\left(\partial_x A\right)_p}{4}\right)\delta \mathrm{x}^2 \\
&+ \left(\frac{AB^2}{2} - \frac{B\left(\partial_t A\right)_p}{4} - \frac{A\left(\partial_t B\right)_p}{2} + \frac{B\left(\partial_x B\right)_p}{2}\right)\delta t\delta x \\
&+ \left(-\frac{AB}{2} + \frac{\left(\partial_t A\right)_p}{4} - \frac{\left(\partial_x B\right)_p}{4}\right)\delta \mathrm{x} \\
&+ \left(\frac{B^2}{2} - \frac{\left(\partial_t B\right)_p}{2}\right)\delta t + \left(-\frac{B^3}{2} + \frac{3B\left(\partial_t B\right)_p}{4}\right)\delta t^2 \\
C_2(x,t) &= A\delta x + \frac{1}{2}\left(\partial_t A\right)_p \delta t\delta x + \frac{1}{2}\left(\partial_x A\right)_p \delta x^2 \ .
\end{aligned} \tag{3.62}$$

3.A.2 The trace of the heat kernel

With the expansion of Δ_{FPD} in (3.61), we now evaluate the trace in (3.54) using the heat kernel expansion. Concretely, we want to evaluate $\epsilon \to 0$ limit of 3.54 which is dominated by the short distance behavior of the heat kernel trace $\langle x,t|\, e^{\epsilon\Delta_{FPD}}\, |x,t\rangle$.

The heat kernel expansion to first order in ϵ is given by

$$\begin{aligned}
&G_\epsilon\left(t_0,x_0;t,x\right) = G_0\left(t_0,x_0;t,x;\epsilon\right) + \\
&\qquad \int_0^\epsilon ds \int_{-\infty}^{\infty} dx\, dt \int_0^\epsilon ds\ G_{\epsilon-s}\left(t_0,x_0;t,x\right)\left(C_1(x,t)\frac{\partial}{\partial t} + C_2(x,t)\frac{\partial^2}{\partial t^2}\right) \times\ G_s\left(t,x;t_0,x_0\right)
\end{aligned} \tag{3.63}$$

where $G_0\left(t_0, x_0; t, x; \epsilon\right)$ is the *free* propagator. $G_0\left(t_0, x_0; t, x; \epsilon\right)$ can be computed by carrying out the following Gaussian integral in momentum or k-space

$$\begin{aligned} G_0\left(t_0, x_0; t, x; \epsilon\right) &= \int_{-\infty}^{\infty} dk\, e^{\epsilon k^2 - i\,\frac{B}{2}\epsilon\, k}\, e^{i\,k(t-t_0)} \\ &= \frac{\pi e^{-\frac{(B\epsilon+2(t_0-t))^2}{16\epsilon}} e^{-\frac{(A\epsilon+2(x_0-x))^2}{16\epsilon}}}{2\pi\epsilon} . \end{aligned} \tag{3.64}$$

We note that

$$\begin{aligned} G_\epsilon\left(t_0, x_0; t_0, x_0\right) &= \frac{\pi e^{-\frac{(B\epsilon)^2}{16\epsilon}} e^{-\frac{(A\epsilon)^2}{16\epsilon}}}{2\pi\epsilon} \\ &\sim \frac{\pi}{2\pi\epsilon} e^{\frac{-\epsilon}{16}\left(B^2+A^2\right)} \sim \frac{1}{2\epsilon} - \frac{1}{32}\left(B^2 + A^2\right) . \end{aligned} \tag{3.65}$$

Using the transnational invariance of $G_\epsilon\left(t_0, x_0; t, x\right)$, one can take $x_0 = t_0 = 0$. Then, $G_\epsilon\left(t_0, x_0; t, x\right)$ can be simplified to

$$G_\epsilon\left(0, 0; t, x\right) = \frac{e^{-\frac{(B\epsilon+2t)^2}{16\epsilon}} e^{-\frac{(A\epsilon+2x)^2}{16\epsilon}}}{4\pi\epsilon} \tag{3.66}$$

We now proceed to evaluate the next term in the expansion (3.63) which is given by the following integral

$$\begin{aligned} &\int_0^\epsilon ds\, \epsilon \int_{-\infty}^{\infty} dx\, dt \int_0^\epsilon ds\; G_{\epsilon-\overline{\epsilon}}\left(0, 0; t, x\right) \left(C_1(x,t) \frac{\partial}{\partial t} + C_2(x,t) \frac{\partial^2}{\partial t^2} \right) \\ &\times\; G_{\overline{\epsilon}}\left(t, x; 0, 0\right) + \left(A \leftrightarrow B, \frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial x} \right) , \end{aligned}$$

where s represents the proper time integration variable and $\overline{\epsilon} = \epsilon s$. There are three integrals that must be evaluated in (3.67): The x, t, and s integrals. We first carry out the x and t integrals before we expand to first order in ϵ to the obtain the following non-vanishing terms,

$$\left(\frac{\left(A^2-B^2+B-A-A^2(s-1)+A(-1+s)\right)(s-1)}{8\pi}\right) \tag{3.67}$$

Integrating (3.67) over s

$$\int_{s=0}^{s=1} ds\left(\frac{\left(A^2-B^2+B-A-A^2(s-1)+A(-1+s)\right)(s-1)}{8\pi}\right), \tag{3.68}$$

gives after simplification and interchanging A with B

$$-\frac{1}{12}\left(2A^2+2B^2+B+A\right)\pi\,. \tag{3.69}$$

Expanding the zeroth order correction (3.64) to first order in ϵ gives

$$\frac{1}{4\pi\epsilon}+\frac{-A^2-B^2}{64\pi}+\frac{-A^2-B^2+B+A}{24\pi}, \tag{3.70}$$

Adding everything together in (3.69) and (3.70) then gives the final expression

$$\begin{aligned}\langle x,t|e^{\epsilon\Delta_{FPD}}|x,t\rangle &= \frac{1}{4\pi\epsilon}-\frac{11A^2}{192\pi}-\frac{11B^2}{192\pi}+\frac{B}{24\pi}+\frac{A}{24\pi} \\ &= \frac{1}{4\pi\epsilon}-\frac{11}{192\pi}\left((\partial_x g^{tt})^{\,2}+(\partial_t g^{xx})^{\,2}\right)+\frac{1}{24\pi}\left(\partial_t^2 g^{xx}+\partial_x^2 g^{tt}\right) \\ &= \frac{1}{4\pi\epsilon}-\frac{11}{192\pi}\left[\left(\partial_x N^{-2}\right)^2+\left(\partial_t h^{xx}\right)^2\right]+\frac{1}{24\pi}\left(\partial_t^2 h^{xx}+\partial_x^2 N^{-2}\right)\end{aligned} \tag{3.71}$$

where in the last step we used the ADM metric in (2.13) to substitute for A and B. We can directly see that $\langle x,t|e^{\epsilon\Delta_{FPD}}|x,t\rangle$ contains only two spatial and two time derivatives of the matrix elements. As explicitly presented in [16], these terms are coboundary terms that can be removed by adding local counterterms to the effective action, and therefore, we conclude that an FPD-invariant theory of a free massless scalar in two spacetime dimensions does not generate the Lifshitz Weyl anomaly.

As also explained in Section 7.1 of [16], these terms are actually directly related to the decomposition of the Ricci scalar in ADM coordinates. However, since the Ricci scalar is a coboundary in the Lifshitz cohomology, as opposed to being a cocycle in the conformal cohomology, it can removed by adding local counterterms.

As we mentioned at the beginning of this Appendix, this heat kernel calculation is limited to first-order perturbation theory. Going to second order in perturbation theory remains to be checked although we do not expect it to break the parity symmetry required to generate the correct $z = 1$ Weyl anomaly.

Chapter 4

The Lifshitz Weyl Anomaly From A Non-relativistic NRSCS Action

The key result of this chapter is:

1. Derive the $z = 2$ Lifshitz Weyl anomaly from a 2+1-dimensional (3D) non-relativistic Schrödinger-invariant Chern-Simons action (NRSCS) on a manifold with a boundary.

The bulk of ideas and calculations in this chapter appears in [109].
In this chapter, we derive the $z = 2$ Lifshitz Weyl anomaly from a 2+1-dimensional (3d) non-relativistic Schrödinger-invariant Chern-Simons action (NRSCS) on a manifold with a boundary. The boundary theory is a $z = 2$ Lifshitz theory coupled to TTNC geometry. This 3d NRSCS action was recently constructed by gauging the centrally-extended Schrodinger algebra which made dynamical the TTNC geometry. In the metric formalism, it was then shown that the NRSCS action is indeed equivalent to a three-dimensional non-projectable $z = 2$ *conformal* or Weyl-invariant HL theory of gravity which is the counterpart of relativistic conformal gravity [89]. As we will discuss below, this NRSCS action contains two terms which do not contribute to the

solution of the bulk equation of motion. In this section, we place the 3d NRSCS action on a manifold with *boundary* and show that one of these two terms, the tCS term, does in fact induce the $z = 2$ (1+1)-dimensional Lifshitz Weyl anomaly in (2.22). In fact, the authors in [89] wondered if one of these two terms would correspond to a boundary Weyl anomaly. Let us emphasize again that the (1+1)-dimensional Lifshitz Weyl anomaly we are discussing in this paper is *universal* i.e. true for *all* values of z. Therefore, throughout the discussion in this section, the relevant dual boundary theory is a $z = 2$ Lifshitz theory with a background TTNC geometry.

In addition to being an anomaly of a Lifshitz *gravitational* effective action, the 1+1 Lifshitz Weyl anomaly is also *parity-odd* and time-reversal symmetry breaking. This strongly suggests that we should be looking for a bulk 2+1-dimensional Chern-Simons gravity action that, on-shell, and asymptotically, has the same symmetries of a 1+1 Lifshitz effective action living on the boundary. To reproduce the Weyl anomaly of the boundary Lifshitz effective action, we require that the bulk gravity action be conformal or more precisely, Weyl-invariant under anisotropic local Weyl transformations only in the bulk. The solution of this bulk gravity action must also be a Lifshitz metric in 2+1 spacetime dimensions.

One last requirement is the bulk gravity action must be one that contains temporal torsion terms since, as we discussed in Section 2, the (1+1)-dimensional Lifshitz Weyl is generated by coupling the Lifshitz classical action to a background metric with temporal torsion. Thus, a candidate 3d gravity action that satisfies all of these requirements is that of a (2+1)-dimensional Weyl-invariant CS non-projectable (NP) HL gravity action.

The layout of this chapter is as follows. In Section 4.1, we review some background material for this section. After briefly reviewing the Schrödinger group and algebra, we discuss also review how gauging the Poincare algebra naturally produces the geometric constructs of Riemannian geometry. This should serve as background for

Sections 4.2 and 4.3. The key properties of the NRSCS action are reviewed in Section 4.2. In preparation for the derivation of the tCS term in Section 4.3, we very quickly review how the Lorentz anomaly has been derived holographically from a relativistic gravitational CS action in [90]. In Section 4.3, we derive the $z = 2$ Lifshitz Weyl anomaly from the NRSCS action. In Section 4.4, we will discuss several other aspects related to the derivation in 4.3, in particular, its relation to the Abelian Laughlin state in the quantum Hall effect (QHE) and whether the derivation represents a new type of anomaly inflow associated with the Lifshitz Weyl anomaly. Other aspects of the NRSCS action will also be discussed.

In Appendix 4.A, we study some of the mathematical aspects of the tCS term where we will show that the tCS term corresponds to a foliation invariant known as the *Godbillon-Vey* $GV(\mathcal{F})$.

4.1 Background

One of the most well-studied non-relativistic symmetry groups is the Schrödinger group. The Schrödinger group $Sch(D)$ in D spatial dimensions is defined by the following spacetime transformations [129]

$$t \to t' = \frac{at+b}{ct+d}, \quad \mathbf{r} \to \mathbf{r}' = \frac{\mathcal{D}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{ct+d}; \quad ab - cd = 1 \ , \tag{4.1}$$

where $\mathbf{v}$ and $\mathbf{a}$ are the velocity and acceleration vectors respectively while $\mathcal{D}$ is scale factor. In [130,131], it was shown that the Schrödinger group $Sch(D)$ is the symmetry group of the Schrödinger equation. The group $Sch(D)$ acts *projectively* on the space of solutions $\phi(t, \mathbf{r})$ of the Schrödinger equation [129].

Let us now consider the non-relativistic Schrödinger algebra of the Schrödinger group in $D = 2$ spatial dimensions. It contains the Galilean algebra as a subalgebra which has following non-zero commutation relations of the group generators

$$[J_{ab}\,, P_c] = \delta_{ac} P_b - \delta_{bc} P_a\,, \quad [J_{ab}\,, G_c] = \delta_{ac} G_b - \delta_{bc} G_a\,, \quad [H\,, G_a] = P_a \tag{4.2}$$

where in (2+1) dimensions $a = 1, 2$, J_{ab} are the angular momentum generators, P_a the generators of momentum, G_a the generators of Galilean transformations and H is generator of time translations. The Bargmann algebra is another well-studied non-relativistic algebra which is basically the Galilean algebra plus a central extension:

$$[J\,, P_a] = \epsilon_{ab} P_b\,, \quad [J\,, G_a] = \epsilon_{ab} G_b\,, \quad [H\,, G_a] = P_a, \quad [P_a\,, G_b] = N \delta_{ab} \tag{4.3}$$

The *extended* Schrödinger algebra is a combination of the Bargmann and $SL(2, R)$ algebras with central extensions. It has the following non-zero commutations relations:

$$\begin{aligned}
&[W\,, H] = -2H\,, \quad [H\,, K] = W\,, \quad [W\,, K] = 2K \\
&[H\,, Y] = -Z\,, \quad [H\,, S] = -2Y\,, \quad [K\,, Y] = S\,, \\
&[K\,, Z] = 2Y\,, \quad [W\,, S] = 2S\,, \quad [W\,, Z] = -2Z\,,
\end{aligned} \tag{4.4}$$

where W is the generator of scale (Weyl) transformations, K is the generator of special conformal transformations, and Y, Z, S are central extensions of the Schrödinger algebra.

The Schrödinger group has been introduced in [130, 131] as the non-relativistic analogue of the conformal group in D dimensions. Schrödinger symmetry has been used in a wide variety of applications [132], [129], for example, non-relativistic field theory [133–136], gauge-gravity duality [67, 137], higher-spin theories [138], hydrodynamics [139–141] or dynamical scaling [132, 142] and many others.

Gauging a symmetry algebra i.e. by associating a gauge field with each symmetry generator of the algebra, one can obtain a specific geometry described in terms of covariant derivatives, curvature tensors along with consistency conditions. For example, gauging the Poincare algebra gives the familiar Riemannaian geometry from which relativistic theories gravity are constructed [86]. Thus, there is a strong relationship between a symmetry algebra, geometry and gravity theories. This relationship is clearest in three dimensions where a (2+1)-dimensional CS theory becomes equivalent to an Einstein theory of gravity.

Let us demonstrate how the gauging procedure works for the Poincare algebra. We closely follow the example in [86] for this purpose. Take $A = A^a_\mu T_a dx^\mu$ to be a connection 1-form of group G on a 3-manifold M, i.e A is the vector potential of a gauge theory with gauge group G and let the generators of its Lie algebra $\mathcal{G}$ be T_a. The Chern-Simons action for A is then given by

$$S_{CS}[A] = \frac{k}{4\pi}\int_M Tr\left[A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right] \tag{4.5}$$

where k is the CS level and Tr is the non-degenerate invariant bilinear form on the Lie algebra $\mathcal{G}$. The Euler-Lagrange equation of motion of (4.5) is given by

$$F[A] = dA + A \wedge A = 0 \ , \tag{4.6}$$

Hence, A is a flat connection. However, being flat, does not necessarily mean that the principal bundle over which A is defined is topologically trivial. The Aharonov-Bohm effect is a prime example of a such a nontrivial flat connection.

Now let G be the Poincare group $SO^+(2,1)$ and J^a be the generator of Lorentz transformations, P^a the generator of translations with the following commutation relations

$$\begin{aligned} \left[J^a, J^b\right] &= \epsilon^{abc} J_c \\ \left[J^a, P^b\right] &= \epsilon^{abc} P_c \\ \left[P^a, P^b\right] &= 0. \end{aligned} \tag{4.7}$$

If we define the group's invariant bilinear form via

$$Tr(J^a P^b) = \eta^{ab} \tag{4.8}$$

$$Tr(J^a J^b) = Tr(P^a P^b) = 0 \tag{4.9}$$

and write the connection one form as

$$A = e^a P_a + \omega^a J_a \tag{4.10}$$

then up to boundary terms, the Chern-Simons action (4.5) is the same as the Einstein-Hilbert action (4.11) with zero cosmological constant $\Lambda = 0$ and $k = \frac{1}{4\mathrm{G_N}}$

$$S = \frac{1}{8\pi G} \int \left[e^a \wedge (\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c) + \frac{\Lambda}{6}\epsilon_{abc} e^a \wedge e^b \wedge e^c \right]. \tag{4.11}$$

4.2 Key Properties of the NRSCS Action

In a similar fashion, gauging the *extended* Schördinger non-relativistic algebra, i.e. by letting the gauge field A take its value in the centrally-extended Schrödinger algebra and can thus be expanded as a linear combination of the generators of the algebra [89]

$$\begin{aligned} A &= H\tau + P_a e^a + G_a \omega^a + J\omega + Nm + Wa + Kf \\ &\quad + S\zeta + Y\alpha + Z\beta \,, \end{aligned} \tag{4.12}$$

where H, P_a, G_a, J, N, W, K are the generators of time translations (the Hamiltonian), momentum translation, Galilean boosts, rotations, central charge, Weyl transformations and special conformal transformations, respectively.[1] The three central extensions of the Schrodinger group are S, Y and Z respectively. Using the metric on this non semi-simple Lie algebra, the NRSCS action as given in [89] is

$$
\begin{aligned}
S_{NRSCS}[A] &= \int_M Tr\left[A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right] \\
&= \int_M Tr\; c_1 \mathcal{L}_{NRSCS} + c_2 \left[a \wedge da - \tau \wedge df + 2a \wedge \tau \wedge f\right] + 2c_3 \omega \wedge d\omega \qquad (4.13) \\
&= \int_M Tr\; c_1 \mathcal{L}_{NRSCS} + c_2\, \mathcal{L}_{tCS} + 2c_3 \omega \wedge d\omega\,. \qquad (4.14)
\end{aligned}
$$

For $c_2 = c_3 = 0$, $S_{NRSCS}[A]$ is equivalent to a bulk 3d action of non-projectable conformal HL gravity. In $S_{NRSCS}[A]$, the Chern-Simons level $\frac{k}{4\pi}$ has been omitted. The arbitrary constants c_i are defined in terms of the symmetric bilinear form invariant under the Schrodinger algebra, i.e. $B(V_i, V_j) = c_i$ and V_i is a generator of the algebra. For example, $B(W, W) = 2c_2$. A key observation is that a_μ is the gauge field of the dilatation symmetry. The curvature of the torsion gauge connection is given by $R(W) = da - 2df$ where f_μ is the gauge field associated with the generator of special conformal transformations K. (In this section K is not the trace of the extrinsic curvature). Therefore, one should expect that a boundary Weyl anomaly would be generated by the tCS term in the action.

As we pointed out before in Section 2.1, we emphasize again here that the role of adding torsion is to make the gauge connection *dilatation-covariant*. In other words, the role of the gauge field of dilatations (or scaling transformations) is to introduce torsion in the NC geometry. There are two ways to add torsion to the NC geomtry. The first is by gauging the Bargmann algebra which we followed in Section 2.1, where

[1] The generator of scale or dilatation transformation in [89] is denoted by D which we reserve in thesis to denote the number of spatial dimensions.

torsion results directly from imposing the $R(H) = 0$ constraint. The second way of adding torsion to the NC geomtry is by gauging the conformal non-relativistic Schrödinger algebra which gives the (NRSCS) action.

With $c_2 = c_3 = 0$, the NRSCS action (4.2), satisfies a bulk equation of motion $F = dA + A \wedge A = 0$ that gives a $z = 2$ Lifshitz metric $ds^2 = -\frac{dt^2}{r^4} + \frac{dr^2}{r^2} + \frac{dx^2}{r^2}$. Therefore, the bulk theory represented by the NRSCS action is *Weyl-invariant*. However, the tCS term whose coefficient is c_2, transforms under the $SL(2, \mathbb{R})$ subgroup of the Schrodinger algebra and as discussed in [89] cannot be removed by a field redefinition and therefore, as noted there, it may lead to a Weyl anomaly at the boundary. The sole contribution of the tCS term when added to an on-shell bulk HL theory of gravity is to induce a Weyl anomaly of a Weyl-anomalous *dual* boundary theory. This is because, as discussed above, it does *not* contribute to the solution of the bulk $z = 2$ HL gravity theory.

Before we get into the derivation of the boundary Lifhsitz Weyl anomaly, we very briefly highlight the key steps of deriving, holographically, the Lorentz (or diffeomorphism) anomaly of a 2d boundary CFT from the gravitational CS (gCS) term added to a Lorentz-invariant 3d bulk gravity theory [90]. In terms of the spin connection ω, the gCS action defined in terms of the CS 3-form $\Omega_3(\omega)$ is given by

$$S_{gCS}(\omega) = \int_{\mathcal{M}} \Omega_3(\omega) = \int_{\mathcal{M}} Tr(\omega d\omega + \frac{2}{3}\omega^3) \tag{4.15}$$

Under a local Lorentz transformation with parameter α^a_b, $\delta^L_\alpha \, \omega^a_{b\mu} = \partial_\mu \alpha^a_b + \omega^a_{\mu c}\alpha^c_b - \alpha^a_c \omega^c_{b\mu}$, the gCS action changes by a boundary term [90]

$$\delta S_{gCS} = \int_{\partial\mathcal{M}} Tr(\alpha d\omega) \tag{4.16}$$

In a holographic context, this means that while the bulk 3d gravity theory is Lorentz-invariant, the boundary effective action is not. The Lorentz anomaly of the boundary theory $W[e^a_\mu]$ is given by

$$\delta_\alpha W[e^a_\mu] = -\int_{\partial\mathcal{M}} d^2x\, e\, \alpha^{ab} T_{ab} \tag{4.17}$$

If we compare the bulk variation of the gCS action to the anomalous variation of the boundary conformal effective action, we then get

$$\delta_\alpha W[e^a_\mu] = \frac{c_L - c_R}{96\pi} \int_{\partial\mathcal{M}} Tr(\alpha d\omega) \tag{4.18}$$

where the c_L and c_R are left and right central charges of the *chiral* effective active on the boundary. For more details, please see Sections 3 and 4 in [90].

4.3 The Lifshitz Weyl Anomaly from the tCS Term

Denote the (2+1) on-shell HL gravity action by S_{HL}. Using the developed machinery of non-relativistic holography [85, 143], which started when Lifshitz and Schrodinger spacetime solutions to relativistic actions of gravity were found [67, 69, 70, 137], the variation of the on-shell HL action at low energies and to leading order in the metric can be expressed in terms of the TTNC geometry on the boundary

$$\delta S_{HL} = \int d^3x \sqrt{g^{(0)}} T^{ij} g^{(0)}_{ij}\,, \tag{4.19}$$

where $d^3x \equiv dtdxdr$ and $\sqrt{g^{(0)}} = N^{(0)}\sqrt{h^{(0)}}$ is the metric in terms of the boundary lapse and shift vectors and $T^{ij}g_{ij}$ is identified with the trace of the expectation value of the boundary theory effective action $W[N^{(0)}, h^{(0)}_{ij}]$ coupled to a metric that is *anisotropically* conformal to $N^{(0)}h^{(0)}_{ij}$. [2] If the action S_{HL} is Weyl-invariant as for example the one constructed in [89], then $\delta S_{HL} = 0$ under the variations $\delta N = z\sigma N^{(0)}$ and $h^{(0)}_{ij} = 2\sigma h^{(0)}_{ij}$. However, if we were to trust the machinery of non-relativistic holography especially for HL gravity and asymptotically Lifshitz spacetimes, we have to be able to deal with a Weyl-anomalous boundary theory and assume a non Weyl-invariant bulk theory of gravity with a non-vanishing $T^{ij}g^{(0)}_{ij} = z\,\langle T^t_t\rangle + \langle T^x_x\rangle$. Adding the tCS term to the on-shell gravity action S_{HL} is our way out. As we show below, under a Weyl transformation, the tCS term is invariant up to a boundary term. If we assume the coefficient of the tCS term matches that of the boundary Weyl anomaly, then the latter cancels with the variation of the bulk on-shell action. More concretely, the variation of the on-shell bulk gravity action $S = S_{HL} + S_{tCS}$ under a Weyl transformation with parameter $\sigma(x,t)$ should be given by

$$\delta_\sigma S = \delta_\sigma S_{tCS} = c_2 \int_{\partial M} Tr[\delta a\, da]\,. \tag{4.20}$$

Let us see how we do that. We set $c_3 = 0$ in the NRSCS action and start by integrating out the connection β in the NRSCS action. The corresponding equation of motion is $df = -2b \wedge f$. Substituting this solution into the tCS term, we see that $-\tau \wedge df$ cancels with $2b \wedge \tau \wedge f$ such that the tCS term can be written as

$$\mathcal{L}_{tCS}[a] = 2c_2\,(a \wedge da)\ . \tag{4.21}$$

In terms of differential forms, a variation of the torsion field a in $S_{tCS}[a]$ gives

$$\delta S_{tCS}[a] = c_2 \left(\int_M Tr[\delta a \wedge R(W)] + \int_{\partial M} Tr[\delta a\, da] \right)\ , \tag{4.22}$$

[2] To properly define an asymptotically Lifshitz spacetime, we assume the notion of anisotropic conformal infinity of the $D+1$-dimensional Lifshitz geometry at $r \to 0$ where there is an asymptotic codimension-one foliation [85].

where $R(W)$ is the curvature of the gauge connection of dilatations W. In components with coordinates (t, x, r), the $S_{tCS}[a]$ action reads

$$S_{tCS}[a] = c_2 \int d^3x \; \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \,. \tag{4.23}$$

The variation of the tCS action is then given by

$$\begin{aligned} \delta S_{tCS}[a] &= c_2 \int d^3x \; \epsilon^{\mu\nu\rho} \left[\delta a_\mu \partial_\nu a_\rho + a_\mu \partial_\nu \delta a_\rho\right] \\ &= c_2 \int d^3x \; \epsilon^{\mu\nu\rho} \left[\delta a_\mu R_{\nu\rho}(W) + \partial_\mu (a_\nu \delta a_\rho)\right] \;. \end{aligned} \tag{4.24}$$

$R_{\nu\rho}(W) = 0$ is the equation of motion that minimizes the tCS action. The last term must be set to zero on the boundary $r = 0$. One choice is

$$(a_t - N^x a_x)\Big|_{r=0} = 0, \quad N^r = 0 \,, \tag{4.25}$$

since by definition, $a_t = N^x a_x + N^r a_r$. The other sets only a_t to zero

$$a_t = N^x a_x = 0 \,. \tag{4.26}$$

The choice in (4.25) is however more general. Under infinitesimal local Weyl transformation with parameter $\sigma(x, t)$, the gauge connection a_μ transforms as

$$\delta_\sigma a = d\sigma \,, \tag{4.27}$$

and the tCS action varies by a the total derivative term

$$\delta_\sigma S_{tCS}[a] = c_2 \int_{\partial M} Tr(\sigma da) \tag{4.28}$$

In components, this becomes

$$a_\mu \to a_\mu + \partial_\mu \sigma$$

and

$$S_{tCS} \to S_{tCS} + c_2 \int_{r=0} dxdt\ \sigma(\partial_t a_x - \partial_x a_t)\,,$$

which has precisely the same *form* of the (1+1)-dimensional Weyl anomaly in (2.23) and (2.30) of the Lifshitz boundary theory. Thus, the bulk remains Weyl-invariant while the boundary theory does not. We can then conclude that the tCS term added to a 3d Weyl-invariant HL gravity action plays a role similar to what the gCS term plays when added to a 3D diffeomorphism-invariant action. This is the main result of this paper. However, it is important to observe that without knowing the exact value of the coefficient c_2 and matching it with that of the anomaly computed in an example Lifshitz field theory, for example, using heat kernel methods, it would be difficult to claim the derivation is exact.

It stands to reason that we should be able to find the $a \wedge da$ term in the parity-odd sector of the cohomology of the relative Weyl operator in 2+1 dimensions. Indeed, such a term can be found in [16]. The term is given by

$$\begin{aligned} \tilde{\epsilon}^{\alpha\beta} a_\alpha \mathcal{L}_n\, a_\beta &= \tilde{\epsilon}^{xr} a_x \mathcal{L}_n\, a_r + \tilde{\epsilon}^{rx} a_r \mathcal{L}_n\, a_x \\ &= a_x \partial_t a_r - a_r \partial_t a_x\,, \end{aligned} \tag{4.29}$$

where we have used that $\mathcal{L}_n\, a_r = \frac{1}{N}\,(\partial_t a_r - N^r \mathcal{L}_{N^r}\, a_r)$, $\tilde{\epsilon}^{\alpha\beta} = n_\gamma \epsilon^{\gamma\alpha\beta}$ and $a_t = 0$. Now let us show that $a \wedge da$ can be expressed as (4.29). Let us start by expanding the $a \wedge da$ in coordinate bases as $a = a_\mu\, dx^\mu = a_x\, dx + a_r\, dr + a_t\, dt$. The $a \wedge da$ term can be expanded as follows

$$
\begin{aligned}
a \wedge da = \epsilon^{ijk} a_i \partial_j a_k & \\
& = a_t \left(\partial_x a_r - \partial_r a_x\right) dtdxdr - a_x \left(\partial_t a_r - \partial_r a_t\right) dxdrdt \\
& + a_r \left(\partial_t a_x - \partial_x a_t\right) drdxdt \,,
\end{aligned}
$$

where $di\, dj\, dk \equiv di \wedge dj \wedge dk$. The exterior derivative da is given by

$$
\begin{aligned}
a \wedge da &= \epsilon^{xr} a_x \partial_t a_r \\
&= a_x \partial_t a_y - a_r \partial_t a_x
\end{aligned} \tag{4.30}
$$

which matches the one given in [16].

4.4 Discussion and Outlook

4.4.1 Anomaly Cancellation by Anomaly Inflow

It is well known that the Floreannini-Jackiw (FJ) action [144] describes massless chiral self-dual edge bosons for the Abelian Laughlin fractional quantum Hall (FQH) state [37,145,146]. In fact, it is the Wess-Zumino-Witten (WZW) low-energy *boundary* CS action for the Laughlin state. The FJ action is given by

$$
S_{FJ} = \int dt\, dx\ \partial_t \phi \partial_x \phi - v_x (\partial_x \phi)^2 \,, \tag{4.31}
$$

with equation of motion

$$
\partial_t \partial_x \phi - v_x \partial_x^2 \phi = \partial_t \rho(x,t) - v_x \partial_x \rho(x,t) = 0 \,, \tag{4.32}
$$

where $\rho(x,t) \equiv \partial_x \phi(x,t)$ is the chiral boson *density* expressed as spatial derivative of the gauge degree of freedom $\phi(x)$. This equation has solutions of the form $\phi(x+vt)$ which describes a chiral wave propagating with constant velocity v_x. Replacing $\phi(x,t)$ with $N(x,t)$, $\rho(x,t)$ with $R_{xt}(x,t)$ and v_x with a constant N^x, the FJ action becomes

$$S_{FJ} = \int dt\, dx\ \partial_t N \partial_x N - N^x (\partial_x N)^2\,, \tag{4.33}$$

with an equation of motion

$$\partial_t R_{xt}(H) - N^x \partial_x R_{xt}(H) = 0\,. \tag{4.34}$$

We observe that while the first term of (4.34) is the 1+1 Lifshitz Weyl anomaly, a trivial descent cocycle in the parity-odd, mixed-derivative sector of the cohomology of the Lifshitz Weyl operator, the second term $\partial_x R_{xt}(H)$ is a *coboundary* term that belongs to the parity-even two-spatial derivatives sector. It is interesting to note, as pointed out in [147], that in the FJ action, it is as if the chiral boson is propagating in *curved* spacetime with background metric N^x.

Note that in deriving the boundary CS action in (4.31) from the tCS action (4.21), one usually works in Galilean-boosted coordinates where the temporal component of the gauge field a_t is set to zero (see equations 6.7-6.9 in [146]. By doing so, one also sets the velocity of the chiral boson N^x to zero and hence the chiral boson $\rho(x,t)$ is stationary, i.e. with equation of motion $\partial_t \rho(x,t) = 0$. Analogously, in the process of making the TTNC geometry dynamical, there is complete freedom in deciding the value of $a_t = N^x a_x + N^r a_r$ which fixes the special conformal transformation in the $SL(2,R)$ subalgebra of the Schrodinger algebra [88]. Choosing $N^x = N^r = 0$ directly produces the Lifshitz Weyl anomaly in (2.23). On the other hand, setting only $N^r = 0$ with a constant N^x amounts to a boundary condition where $a_t = N^x a_x$ which then adds the *coboundary* term $(\partial_x N)^2$ to (4.34) and gives the FJ action in (4.31). We would like to further understand the relationship, if any, between the Weyl anomaly of the the $z = 1$ Lifshitz theory and the FJ action in the context of the FQHE.

In light of the previous discussion and deriving the 1+1 Lifshitz Weyl anomaly from the tCS action, one is naturally led to wonder if the Weyl anomaly is actually somehow related to chiral edge states of a FQH theory. We discuss this possibility here. According to the classification in [148, 149], four distinct CS terms can appear in the low-energy effective action of the QH state for a microscopic theory with the following symmetries: (1) $U(1)$ gauge transformations, (2) general covariance, and (3) local $SO(2)$ rotations. Written in terms of differential forms, the four CS terms are

$$S_{CS} = \frac{\nu}{4\pi}\int_{\mathcal{M}} A \wedge dA + 2\overline{s} A \wedge d\omega + \overline{s^2}\omega \wedge d\omega + \frac{c}{96\pi}\int_{\mathcal{M}} \Gamma \wedge \Gamma \wedge \Gamma\,, \tag{4.35}$$

where $\Gamma^{\mu}{}_{\nu} \equiv \Gamma^{\mu}{}_{\nu\rho} dx^{\rho}$. The first term is the $U(1)$ electromagnetic Hall conductance term, while the second and third are known as the *Wen-Zee* terms, and the last is the gravitational Chern-Simons (gCS). On a manifold $\mathcal{M}$ with a boundary, the four CS terms appearing in S_{CS} defined above are no longer gauge-invariant because boundary terms spoil gauge-invariance. According to [149], there are then two possibilities for each CS term: (1) it represents a relevant anomaly of the low-energy effective action that cannot be canceled by adding *local boundary terms*, or (2) it is a *trivial* anomaly which can be canceled by adding local boundary terms. The electromagnetic Hall conductance and relativistic gCS terms are of the first kind. The electromagnetic pure CS term can be made invariant by adding a boundary term to cancel the bulk anomaly

$$\delta_{\sigma} S_{edge} = -\delta_{\sigma} S_{bulk} = -k\int_{\partial\mathcal{M}} d^2x\, \sigma\, \epsilon^{\alpha\beta}\partial_{\alpha} A_{\beta}\,, \tag{4.36}$$

where σ is the gauge transformation parameter. This is an example of *anomaly inflow* [2] where there is an influx of charge into the boundary where they are absorbed by the anomalous gapless edge modes and as a result, the anomaly of the boundary theory gets *canceled* by the total derivative term of a CS action. The Lifshitz Weyl is precisely of that nature

$$\delta_\sigma W_{edge} = -\delta_\sigma S_{tCS} = -k \int_{\partial\mathcal{M}} \sqrt{h}\; N\; \sigma\; dtdx\;\; \left(\epsilon^{ij}\; \partial_i a_j\right)\; .$$

Since the (1+1) Lifshitz Weyl anomaly, as we discussed in Section 2.3 is non-trivial, in the sense that it cannot be canceled by a local boundary term, then according the classification in [149], it belongs to the first class. Thus, if the microscopic theory of the Abelian QH state, in addition to having the three symmetries in (4.35), is also symmetric under *anisotropic local Weyl* transformations such that $W_{edge} + S_{tCS}$ is Weyl-invariant, could the boundary tCS term $a \wedge da$ represent a new kind of *torsional anomaly inflow* where torsional (or gravitational) degrees of freedom flow into the Weyl-anomalous boundary Lifshitz effective action? If so, what universal quantity, if any, does the coefficient of the tCS action represent? More importantly, is there physical scale-invariant FQH system? Does the NRSCS action contain the two WZ terms in? (4.35) We leave these questions for future work.

Another related topic where anomaly cancellation by anomaly inflow is relevant is the thermal Hall effect. In [150], it was shown that the thermal Hall current does not vanish in equilibrium and hence, *Luttinger's* idea of coupling the system to a uniform gravitational field such that the gravitational potential gradient exactly balances out the energy flux induced by a thermal gradient cannot be used and thus the thermal Hall conductance can not be determined by its gravitational counterpart as it was argued before in [151]. In other words, it was argued in [150] that a *uniform* gravitational field can not induce a bulk thermal current and thermal energy must therefore be carried entirely by the $(1 + 1)$-dimensional edge modes. The relationship to we discussed in Section (2.4) is to observe that as a result of canceling the Weyl anomaly and restoring Weyl invariance, the system is in *equilibrium*. In fact, if we choose the lapse function $N = e^{\psi} = x$ such that $a_x = \partial_x \psi = \frac{1}{x}$ and the Luttinger potential $\Phi(x) = \psi(x) = \log(x)$, then

$$N(x)\partial_x a_x = -\partial_x \Phi(x)\,, \tag{4.37}$$

$$\frac{1}{T}\frac{\partial T}{\partial x} = -\frac{\partial \Phi}{\partial x}\,. \tag{4.38}$$

if we identify the lapse function with inverse temperature and the torsion with the temperature gradient

$$N(x) = \beta(x) = \frac{1}{T(x)} = x, \quad a_x = T(x) = \frac{1}{x}\,. \tag{4.39}$$

More relevant to the work in this paper is the work in [152] where a non-relativistic analogue of part of the work in [150] was introduced. The authors in [152] coupled a (2+1)-dimensional non-relativistic field theory to a NC geometry with torsion.[3] However, since TTNC geometry only couples to the energy density, they turned on the spatial components of the timelike vector field n_μ and n^μ to couple to the energy current.[4] They proceeded then to construct the most general partition function with time-independent, local space and time translations and gauge symmetries. Using the Euclidean path integral to calculate the partition function, the authors in [152] derived an expression for the thermal current. However, they did not discuss the possibility of Weyl-anomalous effective actions in the context of their work as was done in [150] where it was shown how the gravitational anomaly of the boundary-induced effective action can be canceled by the inflow of the spatial and temporal components of the bulk energy-momentum tensor computed from the three-dimensional gCS term (4.35).

[3] Note that in [152], the torsion tensor is the curvature in the Hamiltonian $R_{xt}(H) = \partial_x \psi e^{\psi}$

[4] As noted in [55], turning on the spatial components of the 1-form n_ν does not violate the Frobenius condition.

Similarly, understanding the nature of this inflow requires calculating the operator conjugate to a_μ in the bulk which we leave for future work.

Anomaly inflow has also been used to cancel the gravitational anomaly of a chiral field theory and obtain the Hawking radiation as was first discussed in [153]. The relationship between Weyl anomalies and the thermal flux of the Hawking radiation as also studied in [154]. Using anomaly inflow, the authors in [153] found that the influx required to cancel the gravitational anomaly at the horizon is proportional to T^2 with $T = \frac{\kappa}{2pi}$ which is blackbody radiation with the Hawking temperature. This is interesting since, if we assume the field theory near the black hole horizon is the action in (6.3), then according to the discussion in Section 2.4, canceling the Lorentz anomaly of this theory (which we recall can always be traded for a diffeomorphism anomaly by a local counterterm), is equivalent to canceling the Weyl anomaly in a $z = 1$ (1+1) Lifshitz theory.

Appendix

4.A The Godbillon-Vey Cohomology Class

In this Appendix, we quickly review some of the mathematical aspects of the tCS term where we will show that the tCS term corresponds to a foliation invariant known as the *Godbillon-Vey* $GV(\mathcal{F})$. The reader can skip the discussion in this Appendix since it is not required to follow through with the rest of the material in this dissertation.

As we have seen in the discussion above, the tCS term $a \wedge da$ is a Chern-Simons (CS) term in a centrally-extended Schrödinger-invariant CS theory that translates within the $SL(2, R)$ group. As with any CS term, there are associated topological considerations. In this section, we study some of these topological aspects of the tCS term. In fact, it turns out that the tCS term is famously known in the mathematics literature as the *Godbillon-Vey* invariant $GV(\mathcal{F})$. We now define and summarize its basic topological as well geometrical aspects.

Given a transverse protective $\mathbb{R}P^1$ foliation, it was observed by Godbillon and Vey [155] that the form $a \wedge da$ is closed and independent of the choices of 1-forms n and a and therefore it is a foliation-invariant. As such, it defines a cohomology class, known as Godbillon-Vey class $GV(\mathcal{F})$ in the third real de Rham cohomology group $H^3(M; R)$. Many aspects of the $GV(\mathcal{F})$ invariant has been extensively studied throughout the last four decades [156]. In this section, we will only focus on those aspects of $GV(\mathcal{F})$ that relate to the Lifshitz Weyl anomaly and the tCS action.

The original construction of the $GV(\mathcal{F})$ invariant is illuminating. It works as follows [156]: consider $\mathcal{F}$ a codimension-1 foliation on a three-dimensional manifold M defined by the equation $n = 0$. The foliation 1-form $n \in \Omega^1(M)$ is then integrable. Therefore one has $dn = a \wedge n$ for some $a \in \Omega^1(M)$ and $da \wedge n = 0$ (since $a \wedge dn = 0$ by $d^2 n = 0$) so there exists $f \in \Omega^1(M)$ such that $da = n \wedge f$. Let $w = a \wedge da$; it satisfies $dw = da \wedge da = n \wedge f \wedge n \wedge f = 0$, and therefore it is closed. This was the original observation of Godbillon-Vey that led them to define their famous GV invariant of the foliation. Thus, we see that since the cohomology class of the tCS form $a \wedge da$ does not depend on the choice of the foliation 1-form n, it is an invariant of the foliation itself. As such, it defines a cohomology class, known as Godbillon-Vey class $GV(\mathcal{F})$ in the third real de Rham cohomology group $H^3(M; R)$.

Finally, we comment briefly on the geometrical and topological characteristic of the $GV(\mathcal{F})$ invariant as presented by Goldman and Brooks in [116]. On a three-dimensional manifold M, the $GV(\mathcal{F})$ is actually a *topological* volume that measures the complexity of representing the fundamental class of M by singular simplices. If the manifold M is hyperbolic, with a transversely protective foliation (see Section 2.B) defined by a monodromy homomorphism $\pi_1(M) \to (PSL(2, \mathbf{R}))$, (monodromy homomorphism was defined in Section 2.C), then the $GV(\mathcal{F})$ invariant gives the actual volume of M. For more details, please see [116].

It is interesting to note that when M is equipped with a metric, $a \wedge da$ has been expressed solely in terms of classical geometric invariants of a family of *normal* curves and immersed subamnaifolds, namely, the second fundamental form, curvature and torsion of the normal curves [157].

Chapter 5

Emergent Geometry and Path Integral Optimization for a Lifshitz Action

The key results of this chapter are:

1. Extend the background metric optimization procedure for Euclidean path integrals of two-dimensional conformal field theories, first introduced in [97, 98], to a $z = 2$ anisotropically scale-invariant $(2 + 1)$-dimensional Lifshitz field theory of a free massless scalar field.

2. Find optimal geometries for static and dynamic correlation functions. For the static correlation functions, the optimal background metric is equivalent to an AdS metric on a Poincare patch while for dynamical correlation functions, we find the Lifshitz geometry.

The entire content of this chapter appears in [158].

An important quest of many body physics is the search for efficient variational characterizations of correlated quantum systems. (for a review see, e.g., [91]). A class of tensor network states, particularly geared towards the description of scale-invariant systems, are called the *multi-scale entanglement renormalization ansatz* (MERA) [92, 93]. MERA is used to represent approximate ground states of 1D quantum spin chains at criticality described by 2D conformal field theory (CFT) [94]. The scale-invariance of the MERA network turned out to also play a special role in connecting it to holographic duals in the sense of the AdS/CFT correspondence [95]. Here, the bulk of a MERA network can be understood as a discrete realization of 3D anti-de Sitter space (AdS_3), identifying the extra holographic direction with the renormalization group (RG) flow in the MERA [95].

Motivated by the procedure of tensor network renormalization in [96], where the path integral is first discretized into a lattice and then mapped into a tensor network which turns out to be a MERA, Caputa et. al, in a recent series of works [97, 98], took a step further in studying this relationship from the viewpoint of optimizing Euclidean path integrals that represent the ground state wave functional of two-dimensional CFT. Starting with flat Euclidean metric with a UV cutoff, they argued that their optimization procedure amounts to minimizing the Jacobian of the scale transformation for the path integral measure. In the conformally flat gauge, this translates to solving the equation of motion of the Liouville effective action from which they find that the AdS_3 metric a Poincare patch H_2 naturally emerges. This new approach is very appealing, as it suggests a concrete procedure connecting the AdS/CFT correspondence with numerical approaches to many body systems, such as the MERA tensor network [92, 93, 95, 99].

In Chapter 5, we extend the idea in [97, 98] to a *non-relativistic* field theory, specifically to a $z = 2$ anisotropically scale-invariant $(2+1)$-dimensional Lifshitz field theory of a free massless scalar field and show that the procedure can be successfully applied in systems of interest beyond a CFT. We show how natural geometries arise

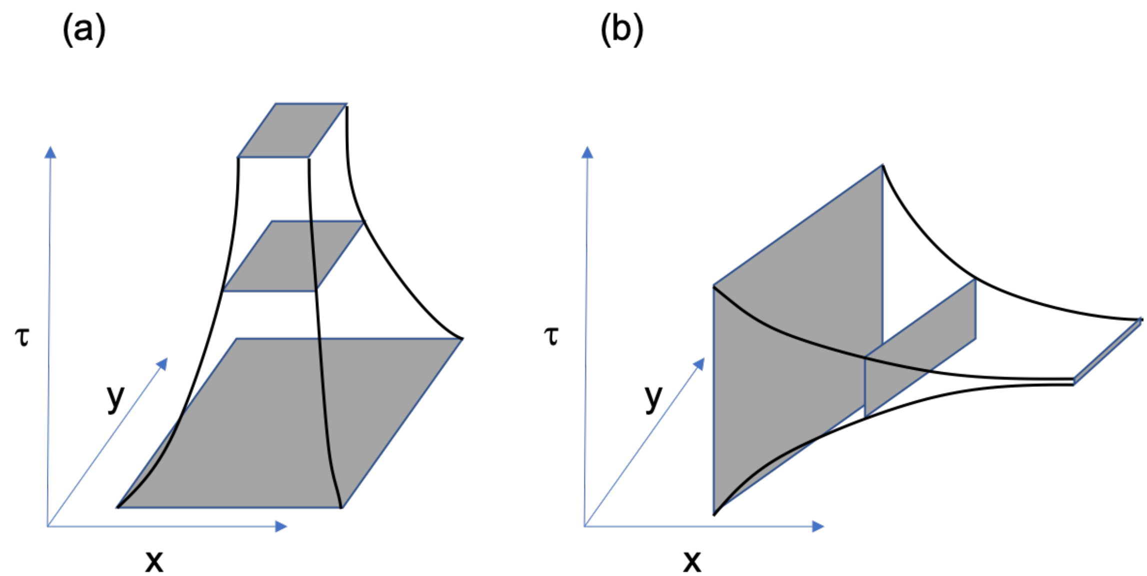


Figure 5.1. The two geometries emerging for the quantum Lifhsitz model. (a) An AdS_3-like geometry arises when considering equal time correlation functions and (b) A Lifshitz metric that is optimal for computing correlation functions with a temporal component.

from the path integral optimization procedure. Our results are illustrated in Fig. 5.1. Concretely, we show the following results:

1. Extend the background metric optimization procedure for Euclidean path integrals of two-dimensional conformal field theories, first introduced in [97,98], to a $z = 2$ anisotropically scale-invariant $(2 + 1)$-dimensional Lifshitz field theory of a free massless scalar field.

2. Find optimal geometries for static and dynamic correlation functions. For the static correlation functions, the optimal background metric is equivalent to an AdS metric on a Poincare patch while for dynamical correlation functions, we find the Lifshitz geometry.

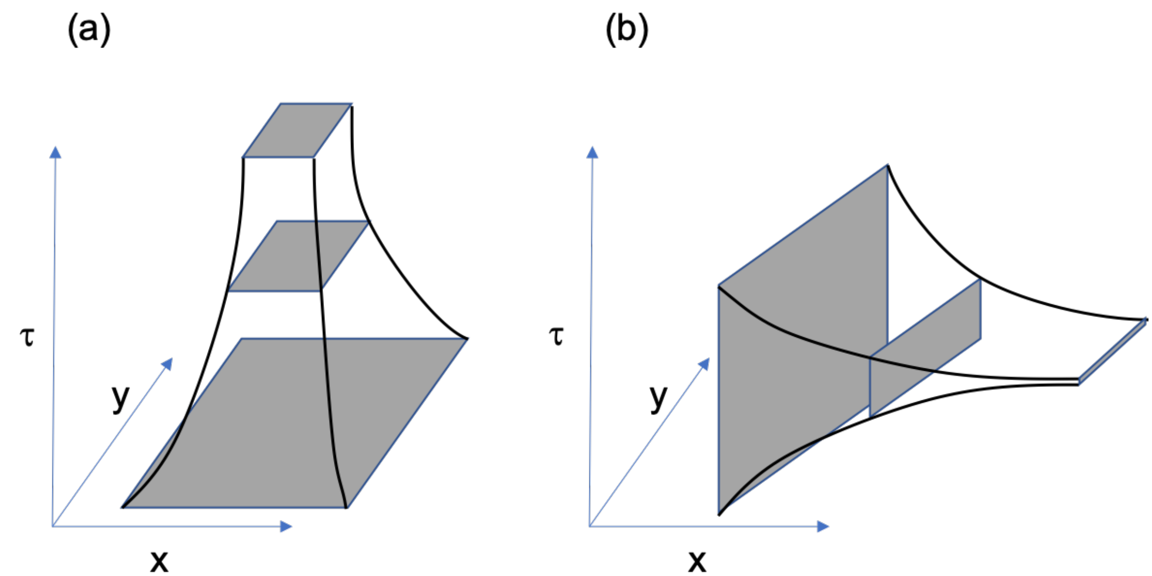
The layout of this chapter is as follows. In Section 5.1, we describe some background of the quantum Lifshitz model and anisotropic Weyl transformation. In Section 5.2, we outline the main steps of the approach used to obtain the optimal geometries. In Section 5.2.1, we explain the optimal geometry for the case of static two-point functions and in Section 5.2.2, for the case of dynamic two-point functions. In Section 5.3, we discuss how this work can potentially be extended and ask some questions pertaining to the approach itself. In Appendix 5.A, we show some details of the calculations.

5.1 Background

The quantum Lifshitz model is a canonical example of a (2+1)-dimensional Lifshitz field theory [35]. This model describes a free massless scalar field with dynamical scaling exponent $z = 2$ and represents an important example of a conformal quantum critical point. Different aspects of this theory have been studied and analyzed in [35,36,159].For example, it emerges as the scaling limit of the square lattice quantum dimer model [36]. Of particular interest to us in this paper, is the Weyl anomaly of this model which has first been computed holographically in [85] and by Baggio et al in [25] using heat kernel expansion and the holographic renormalization methods in [160]. In [16] [29], Lifshitz Weyl anomalies have been computed cohomologically in different dimensions and for different values of the dynamical scaling exponent z. In [161], the heat kernel expansion has been generalized to calculate effective actions and Weyl anomalies for Lifshitz field theories.

The quantum Lifshitz Hamiltonian [35] of a $z = 2$ theory of a massless scalar field $\phi(t, x)$ in $2 + 1$ dimensions is given by

$$H = \int d^2x \; \{{\pi_\phi}^2 + (\Delta_s \phi)^2\} \; . \tag{5.1}$$



The two geometries emerging for the quantum Lifhsitz model. (a) An AdS_3-like geometry arises when considering equal time correlation functions and (b) A Lifshitz metric that is optimal for computing correlation functions with a temporal component.

The Euclidean action of the field $\phi(t,x)$ coupled to a *background* metric g_{ij} is given by

$$S = \int d^2x\mathrm{dt}N\sqrt{h}\left(N^{-2}\left(\partial_t\phi\right)^2 + \lambda\left(\Delta_s\phi\right)^2\right) \ , \tag{5.2}$$

where Δ_s is the *spatial* Laplace-Beltrami operator

$$\Delta_s = \frac{1}{\sqrt{h}}\partial_i h^{\mathrm{ij}}\sqrt{h}\partial_j \ , \tag{5.3}$$

and h_{ij} is the spatial component of the background metric [2].

$$\mathrm{ds}^2 = N^2\mathrm{dt}^2 + h_{\mathrm{ij}}\mathrm{dx}^i\mathrm{dx}^j \ . \tag{5.4}$$

[2]In terms of the ADM metric commonly used in the literature, $g_{tt} = \frac{1}{N^2}$

where N is called the lapse function. The action in (5.2) is invariant under the following foliation-preserving diffeomorphism transformations

$$t \mapsto \tilde{t}(t), \quad x^i \mapsto \tilde{x}^i(\vec{x}) \tag{5.5}$$

and anistoropic Weyl scaling transformations

$$N \to e^{z\sigma}N \ ; \ h_{\rm ij} \to e^{2\sigma}h_{\rm ij} \ , \ i,j \neq t \ \ . \tag{5.6}$$

Starting with the action 5.2 and the background metric in (5.4), the authors in [25] used the heat kernel expansion to the calculate the Weyl anomaly which can expressed as

$$\begin{aligned} \delta W &= \int dt\, d^2x N\sqrt{h}\,\delta\sigma\mathcal{A} \\ &= \int dt\, d^2x N\sqrt{h}\,\delta\sigma\frac{1}{128\pi}\left(K^{ij}K_{ij} - \frac{1}{2}\mathrm{Tr}(K)^2\right) \end{aligned} \tag{5.7}$$

where $K_{ij} = \frac{1}{2N}\partial_t h_{ij}$ and $\mathrm{Tr}(K) = h^{ij}k_{ij}$. In their calculation, they also found a total derivative term that they showed can be removed by a local counterterm.

As stated before, in [98], such a starting point led, via path integral optimization, to an AdS metric. The path integral optimization suggested in [98] looks for the extremal measure over all choices of the gauge σ, due to the Weyl anomaly in the model. Here we use the same structure, though with the anisotropic Weyl scaling appropriate.

5.2 Outline of the Optimization Approach

Here we ask the following question: what is the optimal geometry associated with a path integral computation of correlation functions in the quantum Lifshitz model?

In contrast to the CFT case, due to the non-relativistic nature of the model, equal time correlation functions and dynamical correlation functions should be treated differently. Indeed, we find two separate geometries associated with the optimal calculation, described in Fig. 5.2. For equal-time correlation functions, we consider Weyl transformation which are translationally invariant in space, but not in time, Fig. 5.2(a), covered by case (1) below.

Consider dynamical correlation functions on the other hand. To find the optimized geometry to describe two point functions, such as, say, $\langle\phi(t,r)\phi(t,r')\rangle$, we can choose the spatial axis $r-r'$ to be in the y direction, due to spatial rotational invariance of the model. We concentrate therefore on the computation of the description of the state in the t,y plane, and thus choose a Weyl scaling which is homogeneous in t,y, but can depend on the third coordinate x, Fig. 5.2(b) as explained in case (2) below.

We point out however one difference between our setup and the setup in [98]. Here, We do not start from the quantum effective action and then derive the equation of motion as they do but rather directly compute the variation in the Lifshitz effective action due to an infinitesimal transformation of the Weyl transformation parameter σ which now encodes all the information about the metric. A general framework for computing one loop effective action for Lifshitz theory via heat kernel coefficients has been presented in several places, see e.g. [161, 162]. In our case, our starting point is a flat metric, deformed by a Weyl scaling. We compute the variation of the effective action explicitly utilizing the particular structure of our metric and finally obtain differential equations for the scaling factor σ. Concretely, we compute the variation of the one loop effective action under $\sigma \to \sigma + \delta\sigma$. In this case,

$$\delta W[\sigma] = \frac{1}{2}\int d\boldsymbol{r}\delta\sigma(\boldsymbol{r})\left\langle\boldsymbol{r}\right| e^{\epsilon\rho D}\left|\boldsymbol{r}\right\rangle \ , \tag{5.8}$$

where $\boldsymbol{r} = (\boldsymbol{x}, t)$, $\rho(\boldsymbol{r}) = \frac{1}{\sqrt{g(\boldsymbol{r})}}$ and $D = -\frac{1}{N\sqrt{h}}\partial_\tau N^{-1}\sqrt{h}\partial_\tau + \frac{1}{N}\Delta_s N\Delta_s$ [25]. In our system we fix our gauge so that $N = e^{2\sigma}$, $h_{ij} = \lambda N\delta_{ij}$. In this case we have:

$$D = \left(-\partial_t^2 + \lambda^2\left(\partial_x^2 + \partial_y^2\right)^2\right). \tag{5.9}$$

We note that upon varying σ we have $\delta D = \delta\sigma$. The $\epsilon \to 0$ behavior of (5.8) is dominated by the short distance behavior of the heat kernel $\langle \boldsymbol{r}| e^{\epsilon\rho D} |\boldsymbol{r}\rangle$.

Now, as promised, we specialize to cases where, σ depends either on the time coordinate t alone, or on one of the spacial coordinates, say x. Denoting $\rho = e^{-4\sigma}$, we expand ρ close to a given point $\boldsymbol{r}_0$,

$$\rho\left(\delta\boldsymbol{r} + \boldsymbol{r}_0\right) = \rho_0 + \delta\rho, \tag{5.10}$$

where $\rho_0 = \rho(\boldsymbol{r}_0) = \frac{1}{\sqrt{g(\boldsymbol{r})}}|_{\boldsymbol{r}=\boldsymbol{r}_0}$.

To obtain the variation we carry out a second order perturbation calculation of the heat kernel, using :

$$e^{-\epsilon(\rho_0+\delta\rho)D} = e^{-\tilde{\epsilon}D} - \frac{1}{\rho_0}\int_0^{\tilde{\epsilon}} e^{-(\tilde{\epsilon}-s)\ D}\delta\rho D\ e^{-s\ D}\mathrm{d}s + \tag{5.11}$$
$$\frac{1}{\rho_0^2}\int_0^{\tilde{\epsilon}}\mathrm{d}s\int_0^s \mathrm{d}s_1 e^{-(\tilde{\epsilon}-s)\ D}\delta\rho D\ e^{-(s-s_1)\ D}\delta\rho D\ e^{-s_1 D} + \ldots$$

where $\tilde{\epsilon} = \rho_0\epsilon$. We assume that the operator D is diagonal in momentum, and that $\delta\rho$ depends on a single coordinate such as x or t and has an expansion:

$$\delta\rho = \Sigma_{m=1}c_m(x - x_0)^m \tag{5.12}$$

In [162], the gravitational quantum effective action for a d-dimensional Lifshitz scalar field theory has been calculated using the heat kernel expansion in momentum space. It is important to note, however, that the curved spacetime Lifshitz operator used in [162] is slightly different than the one we use in this paper. We note that the terms above correspond to the heat kernel coefficients b_0, b_2, b_4 in the expansion:

$$\langle \boldsymbol{r}|e^{-\epsilon\delta\rho D}|\boldsymbol{r}\rangle = \sum_{n=0}^{\infty} b_n(\rho D)\epsilon^{(n-d-z)/2z} \tag{5.13}$$

specializing to $d = z = 2$. Note that $b_1, b_3 = 0$ as anticiated e.g. in [161].

5.2.1 Optimal geometry: static correlation functions

Explicitly evaluating the heat kernel through second order perturbation series in $\delta\rho$, we find that the leading (in ϵ) contributions to δW up to two derivatives in the case of $\sigma = \sigma(t)$ are given as

$$\delta \mathrm{W} = \frac{1}{2}\int \mathrm{dtd}^2 x\delta\sigma\left(\frac{e^{4\sigma}}{16\pi\epsilon\lambda} - \frac{1}{24\pi\lambda}\frac{d^2\sigma}{\mathrm{dt}^2}\right) \tag{5.14}$$

Following [98], we search for a profile $\rho(t)$ to minimize the effective action by solving for $\delta W = 0$. Eq. (5.14) implies that the optimal $\sigma(t)$ obeys the Liouville equation:

$$\frac{e^{4\sigma}}{\epsilon} - \frac{2}{3}\frac{d^2\sigma}{\mathrm{dt}^2} = 0 \tag{5.15}$$

Much as in [98], The solution is given by the standard substitution of the form $\sigma(t) = -\frac{1}{2}\log\mu t$, where $\mu = \sqrt{\frac{3}{\epsilon}}$ we find the optimal metric is given by

$$ds^2 = \frac{1}{\mu^2 t^2}\mathrm{dt}^2 + \frac{\lambda}{\mu t}(\mathrm{dx}^2 + \mathrm{dy}\ ^2)\ , \tag{5.16}$$

This surprising result suggests that indeed a some type of a hierarchical tensor network would still be the optimal discrete spacetime configuration even if the field theory we started with is only anistropically scale invariant. It is possible to uniformize the geometry by using a coordinate $u = 2\sqrt{t}$, the optimal metric can also be written as

$$ds^2 = \frac{4}{3u^2}\left(du^2 + \sqrt{3}(dx^2 + dy^2)\right) \ , \tag{5.17}$$

which is the AdS_3 metric of a Poincare patch. Thus, a proper MERA-like description is possible for this non-uniformally rescaled Lifshitz theory. Another possibility, hinted by recent work on exact holographic tensor networks [163], is that a non-unitary MERA-like structure can be chosen that features a scale-invariant tensor network for a non-CFT model.

5.2.2 Optimal geometry: dynamic correlation functions

We now turn to address the optimization in the "lateral" direction. Explicitly evaluating the heat kernel through second order perturbation series in $\delta\rho$, we find that leading contributions to δW in ϵ, read

$$\delta \mathrm{W} = \frac{1}{2}\int \mathrm{dtd}^2 x \delta\sigma \left(\frac{e^{4\sigma}}{16\pi\ \epsilon\lambda} - \frac{e^{2\sigma}((\frac{\mathrm{d}\sigma}{\mathrm{dx}})^2 + \frac{d^2\sigma}{\mathrm{dx}^2})}{12\pi^{3/2}\sqrt{\epsilon}}\right) \ . \tag{5.18}$$

The equation of motion for the case of 5.18 is given by

$$\frac{e^{4\sigma}}{16\pi\ \epsilon\lambda} - \frac{e^{2\sigma}((\frac{\mathrm{d}\sigma}{\mathrm{dx}})^2 + \frac{d^2\sigma}{\mathrm{dx}^2})}{12\pi^{3/2}\sqrt{\epsilon}} = 0 \tag{5.19}$$

To solve this equation, we define: $Y(x) \equiv e^{\sigma(x)}$, and note that (5.19) can be written as:

$$Y" = CY^3 \ \ ; \ \ C = \frac{3\pi^{1/2}}{4\lambda\sqrt{\epsilon}} \tag{5.20}$$

This nonlinear equation is equivalent to the system $Y' = Z \ ; \ Z' = CY^3$, which allows us to find an integral of motion by solving for $\frac{\mathrm{dZ}}{\mathrm{dY}} = \frac{CY^3}{Z}$, from which we obtain the integral of motion:

$$\frac{1}{2}Z^2 = C\log(Y) + \alpha \implies \alpha = \frac{1}{2}Y'^2 - C\log(Y) \tag{5.21}$$

We can solve this equation, getting:

$$Y = \frac{\sqrt{2}}{\left(\sqrt{C}x + \alpha\right)}, \tag{5.22}$$

resulting in the metric, written in terms of Y our metric is

$$\mathrm{ds}^2 = Y^4\mathrm{dt}^2 + \lambda Y^2(\mathrm{dx}^2 + \mathrm{dy}^2) \tag{5.23}$$

and the leading behavior of the metric at large x is thus:

$$\mathrm{ds}^2 \approx 4\frac{\mathrm{dt}^2}{C^2x^4} + 4\lambda\frac{\mathrm{dx}^2 + \mathrm{dy}^2}{Cx^2} \tag{5.24}$$

We emphasize, that as opposed to the usual notion of holographic Lifshitz geometry for this model, where the boundary is (2+1)-dimensional, here we deform one of the original dimensions of the (2+1) spacetime and use it as our holographic direction.

5.3 Discussion and Outlook

The equal-time and dynamical two-point correlation functions for the quantum Lifshitz model that we consider in this work have been studied in [35] and more recently in [164] where they have been compared with the holographic two-point function. The authors find that the correlation functions match quite well with the scaling obtained from a holographic calculation with a Lifshitz geometry, thereby strengthening our expectation that a tensor network description of the system will inherent the features of a Lifshitz geometry.

We also find it quite striking that a semi-classical description of correlation functions is obtained for the system, although there is no manifest small parameter like $\hbar$ or a strong/weak coupling duality to drive us into a semi-classical regime in our original setup. Finally, we remark that although we obtained here an optimal geometry for a specific $z = 2$ (2+1)-dimensional field theory, it is natural to expect that the procedure described here would still work for more general field theories in higher dimensions with arbitrary values of z.

An interesting extension of this approach is to see whether it would work for interacting field theories. The premise so far in [98] and our work too is that field theories are free. Generalizing this work to interacting field theories would be exciting yet very challenging.

Appendix

5.A Some details of the calculation

To obtain our equations we carry out a second order perturbation calculation of the heat kernel, using :

$$\begin{aligned} e^{-\epsilon(\rho_0+\delta\rho)D} = e^{-\tilde{\epsilon}D} - \frac{1}{\rho_0}\int_0^{\tilde{\epsilon}} e^{-(\tilde{\epsilon}-s)\ D}\delta\rho D\ e^{-s\ D}\mathrm{ds} \\ + \frac{1}{\rho_0^2}\int_0^{\tilde{\epsilon}}\mathrm{ds}\int_0^{s}\mathrm{ds}_1 e^{-(\tilde{\epsilon}-s)\ D}\delta\rho D\ e^{-(s-s_1)\ D}\delta\rho D\ e^{-s_1 D} + \dots \end{aligned} \tag{5.25}$$

where $\tilde{\epsilon} = \rho_0\epsilon$. For convenience, set $\mathbf{r}_0 = 0$ throughout the calculation, and reinstate its value in the end. We assume that the operator D is diagonal in momentum, and that $\delta\rho$ depends on a single coordinate x, and has an expansion:

$$\delta\rho = \Sigma_{m=1} c_m x^m \tag{5.26}$$

Taking q to be the momentum in the x direction and K to be the momentum vector in all other directions, the zeroth order contribution to the heat kenrel reads:

$$A_0 = \langle 0|e^{-\tilde{\epsilon}D}|0\rangle = \frac{1}{(2\pi)^{d+1}}\int d^dK\mathrm{dq}\ e^{-\tilde{\epsilon}D(K,q)}\ ; \tag{5.27}$$

The contribution from the first order term in (5.25) is

$$A_1 = -\tfrac{1}{\rho_0}\langle 0|\int_0^{\tilde{\epsilon}} e^{-(\tilde{\epsilon}-s)\ D}\delta\rho D\ e^{-s\ D}|0\rangle \mathrm{ds} \tag{5.28}$$
$$= -\tfrac{1}{\rho_0}\tfrac{2\pi}{(2\pi)^{d+2}}\int_0^{\tilde{\epsilon}} \mathrm{ds}\int d^dK\mathrm{dq}\left(\Sigma c_m\left(i\tfrac{d}{dq}\right)^m e^{-(\tilde{\epsilon}-s)\ D(K,q)}\right)D(K,q)\ e^{-s\ D(K,q)}$$

which can also be expressed in the form:

$$A_1 = -\frac{1}{\rho_0}\frac{1}{(2\pi)^{d+1}}\int_0^{\tilde{\epsilon}} \mathrm{ds}\int d^dK\mathrm{dq}e^{-\tilde{\epsilon}D(K,q)}D(K,q)$$
$$\times\left\{\Sigma_{m=1}i^m\Sigma_{h=1}^m(-1)^h c_m B_{h,m}\left((\tilde{\epsilon}-s)\,D'\left(K,q_1\right),(\tilde{\epsilon}-s)D\text{"}\left(K,q_1\right),\ ...\right)\right\}$$

where $B_{h,m}$ are Bell polynomials. In the case we are interested in, due to the time reversal/space inversion symmetry the first non zero contribution comes from $c_2 = \frac{1}{2}\partial_x{}^2\delta\rho$:

$$A_1 \approx \frac{1}{\rho_0}\frac{c_2}{(2\pi)^{d+1}}\int d^dK\mathrm{dq}e^{-\tilde{\epsilon}\ D(K,q)}\left(-\frac{1}{2}D\text{"}(K,q)\tilde{\epsilon}^2+\frac{1}{3}(D'(K,q))^2\tilde{\epsilon}^3\right) \tag{5.29}$$

The second order contribution is given by

$$A_2 = \frac{1}{\rho_0^2}\langle 0|\int_0^{\tilde{\epsilon}} ds\int_0^s ds_1 e^{-(\tilde{\epsilon}-s)\,D}\delta\rho D\,e^{-(s-s_1)\,D}\delta\rho D\,e^{-s_1 D}|0\rangle$$
$$= \frac{1}{\rho_0^2}\frac{\Sigma_{n,m}c_n c_m}{(2\pi)^{d+1}}\int_0^{\tilde{\epsilon}} ds\int d^dK\,dq\left(\left(i\frac{d}{dq}\right)^m e^{-(\tilde{\epsilon}-s)\ D(K,q)}\right)D(K,q)\,e^{-(s-s_1)\,D(K,q)}$$
$$\times\left(\left(-i\frac{d}{dq}\right)^n D(K,q)\,e^{-s_1\,D(K,q)}\right)$$

Chapter 6

The $z = 1$ Lifshitz Weyl Anomaly as a Topological Invariant

The key result of this chapter is:

1. The integral of the $z = 1$ Lifshitz Weyl anomaly, in a specific *flat* limit, is a topological invariant that detects the parity of the number of twists in the Möbius surface when embedded in $\mathbb{R}^3$.

In this chapter, we make a first attempt to study and analyze a topological invariant in two dimensions that has first been suggested thirty years ago in [102] and [101]. The goal to is attempt to give this new invariant the topological sense missing in [102] as well as [101]. We will show that on a two-dimensional non-orientable manifold with boundary, i.e. Möbius strip, in a certain *flat* limit, to be defined later in this section, the integral of the invariant does not vanish.

In Section 6.1, we summarize the key points made in [102] and [101] pertaining to the relationship between the scalar invariants of the Lorentz connection and its Hodge dual. We show how they both appear as the Lorentz and conformal anomaly of a two-dimensional local theory in which the veilbeins are dynamical variables. We will then demonstrate how the $z = 1$ Lifshitz Weyl anomaly is related to this scalar curvature of the *dual* Lorentz connection. We also contrast the main features of the Ricci scalar and this new scalar curvature.

In Section 6.2, we show that in a specific *flat limit*, λ is a non-vanishing topological invariant on the Möbius manifold, the simplest non-orientable 2-manifold with a boundary. In Section 6.2.1, we evaluate the λ invariant and demonstrate why the Möbius line bundle is the space that makes λ a non-vanishing topological invariant. We demonstrate that λ in this flat limit, is a topological invariant that detects the parity of the number of twists in the Möbius surface as embedded in $\mathbb{R}^3$. In Section 6.3, we discuss various topological, cohomological and geometric aspects of the λ invariant and *speculate* that λ gives the torsion coefficient of the Möbius surface. In Appendix 6.A, we briefly discuss, using certain aspects of integral homology and cohomology theory, how the Möbius flat line bundle contains torsion, and how the latter can be used to detect non-orientable manifolds. This is naturally a continuation of our discussion in Appendix 2.A.

6.1 The Curvature of the Dual Connection

The role of torsion in relativistic gravity is described within the Einstein-Cartan formulation. The vielbeins $e^a{}_\nu$ and spin connection ω_{ab} form a description of gravity equivalent to the metrical one only when, in addition to the local Lorentz symmetry, they are also subject to a constraint. In addition to curvature, there is the geometrical notion of torsion which describes the *internal twist* of the 2d manifold. In the Einstein–Cartan formulation, it is associated to the Lorentz torsion two–form [48,165]

$$T^a = de^a + \epsilon^a{}_b \, \omega \wedge e^b \ . \tag{6.1}$$

The standard connection used in Einstein's theory of gravity is *torsion–free*, i.e. $T^a = 0$. It is this constraint equation together with the connection metricity, that uniquely determines the spin connection ω_{ab} in terms of the vielbein

$$\omega = (*de^a)\, e_a \tag{6.2}$$

In [102], Obukhov and Solodukhin constructed a *local* quadratic action of two-dimensional gravity out of the frame fields e^a_μ

$$\begin{aligned} W[e^a_\mu] &= \frac{1}{4}\int d^2x\, e\, C^a{}_{\mu\nu} C^{\mu\nu}{}_a \\ &= \frac{1}{4}\int d^2x\, e\, (\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu)\, (\partial^\mu e_a{}^\nu - \partial^\nu e_a{}^\mu)\,, \end{aligned} \tag{6.3}$$

where $C^a{}_{\mu\nu}$ is the non-holonomic tensor, regraded as the field strengths of the vielbein fields $e^a{}_\mu$ and $e = \det\, e^a{}_\mu$. According to [102], under a local conformal transformation $\delta e^a{}_\mu = \sigma e^a{}_\mu$ with infinitesimal Weyl parameter σ, the action suffers a conformal anomaly R while under a local Lorentz transformation $\delta e^a{}_\mu = \alpha^a_b e^b{}_\mu$ with infinitesimal Lorentz parameter $\chi\epsilon^a{}_b$, it suffers a Lorentz anomaly U.

Two dimensions are special in the sense that the spin connection ω and its hodge dual $\star\omega$ are *both* 1-forms. Let the curvature of the spin connection ω be defined by

$$\mathcal{R} = d\omega\,, \tag{6.4}$$

and the curvature U of the dual spin connection $\star\omega$ is defined by

$$\mathcal{U} = d\star\omega\,, \tag{6.5}$$

where $\star$ is the Hodge dual operator. Then, the *scalar* curvature R of $\mathcal{R}$ can be expressed in terms of the spin connection as

$$R = 2\nabla\left(e^a{}_\mu C_a{}^{\mu\nu}\right) = \nabla_\mu(\tilde{\omega}^\mu) \ , \tag{6.6}$$

while that of $\mathcal{U}$ can be expressed as

$$U = 2\nabla_\mu\left(e^a{}_\mu \epsilon_{ab} C^{b\mu\nu}\right) = \nabla_\mu(\omega^\mu) \tag{6.7}$$

It was then noted in [101], if the $\chi = \int \mathcal{R}$ is a topological invariant on smooth, compact and closed manifold, then $\lambda = \int \mathcal{U}$ is also a topological invariant. However, it was argued in [101] that $\lambda = \int \mathcal{U} = 0$ since it is always possible to choose a coordinate frame where the connection has zero divergence. We observe that these coordinates implicitly assume that $\star\omega$ is a connection over an *orientable* bundle, in which case, it can be trivialized. It was also noted in [101] that while the structure group related to ω is $SO(2, R)$, that related to $\star\omega$ is reduced to the group of multiplication by positive real numbers $\mathbb{R}^+ = \{\mathbb{R}^+ - \{0\}\,,\ \times\}$ implying that the fiber bundle associated to $\star\omega$ is a line bundle that can be made trivial. Therefore, to properly understand the topological underpinnings of $\lambda = \int \mathcal{U}$, one needs to also understand the topology of line bundles or more concretely of flat line bundles which we discussed in Appendix 2.A of Chapter 2.

More importantly, the authors in [102] were able to obtain the U scalar as the chiral anomaly, Weyl anomaly as well as the Lorentz anomaly of a fermionic action using a generalized Dirac operator with a non-vanishing analytic index. More specifically, they showed that

$$\text{index } \widehat{D}_{gen} = \frac{g}{4\pi} \int U\, e\, d^2x \ , \tag{6.8}$$

where $\widehat{D}_{gen}$ is the generalized Dirac operator given by

$$\widehat{D}_{gen} = i\sigma^{\mu}\left(\nabla_{\mu} + ig\omega_{\mu}\sigma_3\right) \ , \tag{6.9}$$

and $\nabla_{\mu} = \partial_{\mu} + \omega_{\mu}\sigma_3$. It is well known that the analytic index of a positive semi-definite elliptic partial differential operator $\widehat{O}$ acting on sections of vector bundles i.e. over compact manifolds, is a homotopy invariant that accounts for the difference between the left-handed and right-handed zero modes. It is the difference between the dimension of the kernel and co-kernel of $\widehat{O}$.

By the *Atiyah–Singer* (AS) index theorem [7–10], the analytic index equals a *topological* index which describes a purely topological characteristic of the *fiber bundle* considered in the theory. [See for example chapter 5 in [1]]. An important consequence of the AS index theorem is the fact that the topological invariant can be expressed as an integral over certain characteristic classes which represent the invariant. In cohomological algebra, these characteristic classes are represented by cohomology classes as elements of a cohomology group with coefficients in some field F, typically integers $\mathbb{Z}$. For example, χ is given by an integral over the *Euler class* $e(E)$ of an oriented, real vector bundle $E \to M$. On compact 2-manifolds, the Euler class $e(E)$ is an element of the second integral cohomology group $H^2(M; \mathbb{Z})$. The Euler class exists as an *obstruction*, as with most cohomology classes measuring how *twisted* the vector bundle is. For more information about the Euler class in cohomology, please see [103], [104], and [105].

A special form of the index theorem is the famous *Gauss-Bonnet* theorem. On 2-dimensional manifold compact manifold with a boundary, the Gauss-Bonnet theorem relates the integral of the Ricci scalar R representing the Gaussian curvature, and the integral of the *extrinsic* curvature or the second fundamental form representing the *geodesic* curvature, to the Euler characteristic as follows [1]

$$\chi = \frac{1}{4\pi}\left(\int_M R\, e\, d^2x + \int_{\partial M} 2K\sqrt{h}\, d\tau\right) , \tag{6.10}$$

where K is the trace of the extrinsic curvature tensor defined in terms of the spatial metric on the boundary in (2.14) and τ is an arbitrary parameter of the ∂M. It is well known that K can be expressed as the covariant divergence of the unit normal to the boundary $\widehat{n}$ as $K = \nabla_\mu \widehat{n}^\mu$. In Appendix A of [106], it was shown that in the flat conformal gauge, χ can be written solely in terms the boundary integral of the topological part of K as follows

$$\chi = \frac{1}{4\pi}\int_{\partial M} 2K\sqrt{h}\, d\tau = \frac{1}{2\pi}\int_{\partial M} \partial_\mu \widehat{n}^\mu \, . \tag{6.11}$$

On a circle with unit normal $\widehat{n}_\mu = (cos\,\theta,\, sin\,\theta)$, we have $\partial_\mu \widehat{n}^\mu = \widehat{n}^\mu \widehat{n}_\mu = 1$ which implies that

$$\chi = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1 \ , \tag{6.12}$$

and therefore the Euler characteristic of M, $\chi(M) = 1 - N_H$ where N_H is the number of holes. Hence, on a flat disc with a boundary unit circle, the Euler characteristic $\chi = 1$ and so, indeed, the disc has zero holes. In terms of the Gauss-Bonnet theorem, the trace of the second fundamental form measures the geodesic curvature κ_g of a 1d boundary curve $\gamma(v)$ on a 2d surface embedded in $\mathbb{R}^3$. The geodesic curvature measures the magnitude of the *surface-tangential* component of the acceleration vector [166,167]. This projection onto the tangent plane extracts the *intrinsic curvature* of $\gamma(v)$ as it moves on the 2d surface *regardless* of how it bends in the *normal* direction of $\mathbb{R}^3$, i.e. regardless of its extrinsic curvature in $\mathbb{R}^3$.

[1] Here M is taken to be diffeomorphic to a subset of $\mathbb{R}^2$.

	Connection	Structure Group	Fiber Bundle	Boundary Term
Ricci Scalar (R)	$d\omega$	$SO(2,R)$	Tangent frame	$\partial_\mu \widehat{n}^\mu$
Lorentz Scalar (U)	$d\star\omega$	$\mathbb{R}$	Orthonormal frame	$\partial_\mu \widehat{t}^\mu$

Table 6.1: Comparison between the Ricci and Lorentz Scalars

Owing to the similarities outlined above between R and U, the authors in [102] used the boundary term of the action in (6.3), first derived in [107], to show that the topological invariant λ on a compact 2-manifold with a boundary is given by the following integral

$$\lambda = \frac{1}{4\pi}\left(\int_M U\, e\, d^2x - 2\int_{\partial M}\left[e^{\mu a}\,\epsilon_{ab}\,\nabla_\mu n^b - \epsilon^{\mu\nu}\nabla_\mu n_\nu\right]\sqrt{h}\, d\tau\right)\ , \tag{6.13}$$

where for simplicity I dropped the hat on $\widehat{n}$. The boundary term in (6.13) has been derived for the action in (6.3) within the framework of Einstein-Cartan theory [107] in the first-order formalism .

By using the conformal-Lorentz gauge (1.41), λ was expressed as the boundary integral of the divergence of a unit *tangent* vector $t_u = \epsilon_{\mu\nu} n^\nu$ (omitting the hat on $\widehat{t}$ and $\widehat{n}$) for a *flat* metric.

$$\lambda = \int_{\partial M} d\tau\, \partial_\mu\, t^\mu\,. \tag{6.14}$$

Table 6.1 compares the key features of the R and U invariants.

Chapter 6 represents a first attempt to study, analyze, and give λ the topological interpretation that was missing in [102] and [101]. More specifically, we show that in the *flat* limit on a Möbius strip, λ is indeed a topological invariant and does not vanish as argued in [101] by Myers.

How is the (1+1)-dimensional Lifshitz Weyl anomaly related to $U = \nabla_\mu \omega^\mu$? In terms of ADM coordinates, if the Lorentz connection 1-form is expressed in terms of a_x and K as

$$\omega = a_x dt + K_{xx} dx\ , \tag{6.15}$$

then, its dual which is also a 1-form, is defined as

$$\star\,\omega = a_x dx - K_{xx} dt\,. \tag{6.16}$$

The curvature $d\star\omega$ is then

$$d\star\omega = \left(\partial_t a_x + \left(\partial_x K - a_x K\right)\right) dt \wedge dx\ . \tag{6.17}$$

The first term in (6.17) is the $z = 1$ Lifshitz Weyl anomaly as we demonstrated in (2.47) while the last two terms are the Lifshitz coboundary terms found in [16]. Thus, the scalar curvature U of $d\star\omega$ can be written as

$$U \equiv \left\langle T^{\mu}_{\mu}\right\rangle = 2\nabla_{\mu}(\omega^{\mu}) = 2\nabla_{\mu}\left(\epsilon_{ab} e^{a}{}_{\nu} C^{b\mu\nu}\right)\ . \tag{6.18}$$

By comparing this equation with (2.47), we directly see that the U scalar is indeed the Weyl partner of the local Lorentz anomaly, which, in turn is the $z = 1$ Lifshitz Weyl anomaly.

6.2 Topological Characterization Of the Anomaly

In this section, we show that, similar to χ, λ, in a specific *flat* limit, is a non-vanishing topological invariant on a Möbius strip, the simplest non-orientable 2-manifold with a boundary. As a 2d topological manifold, the Möbius strip is a non-trivial flat line bundle over the circle $\mathbb{S}^1$ with a $\mathbb{Z}_2$ structure group.

The Möbius bundle is described by the first *Stiefel-Whitney* (SW) cohomology class that detects the obstruction to defining an orientable manifold [118]. The first SW class detects the orientability of a real line bundle $E \to M$, i.e, $w_1(E) = 0$ if and only if E is orientable. The Möbius strip is a non-orientable line bundle

$\pi : E \to \mathbb{S}^1$ and thus $w_1(E) \in H^1(\mathbb{S}^1, \mathbb{Z}_2)$, the first cohomology group with $\mathbb{Z}_2$ coefficients. Generally speaking, a manifold M is orientable if its tangent bundle is orientable and thus $w_1(TM) = 0$.

As a parametric surface, the Möbius strip has an embedding in three-dimensional Euclidean space $\mathbb{R}^3$. The unit normal to the surface reverses orientation after a 2π rotation and is thus discontinuous. It is this discontinuity or rather *ambiguity* in the unit normal that renders Möbius surface unorientable and thus captures the non-triviality of the Möbius line bundle. Without this additional cycle, the Möbius strip is the just a trivial line bundle $S^1 \times \mathbb{R}$ that describes the cylinder as a product manifold.

Next, we evaluate the λ invariant on the Möbius band.

6.2.1 Evaluating the λ invariant

6.2.1.1 Problem setup

The Möbius strip is, topologically speaking, defined as a rectangle with its top and bottom sides identified as shown in Fig. 6.1. As a 2d surface, the Möbius surface can be embedded in three-dimensional Euclidean space with coordinates $\mathbf{X}(t, v)$

$$\begin{aligned} x(t,v) &= \left(R - t\sin\frac{v}{2}\right)\cos(v) \\ y(t,v) &= \left(R - t\sin\frac{v}{2}\right)\sin(v) \\ z(t,v) &= t\cos\frac{v}{2}\ , \end{aligned} \tag{6.19}$$

where $0 \leq v < 2\pi$ and $-W \leq t \leq W$. This embedding describes a straight line segment with width $2W$ which as it rotates about the z-axis with angle v also rotates counterclockwise about its center at half speed $\frac{v}{2}$ and thus it returns with a reversed orientation to its starting point. Note that we assume that the half-width $W < R$ to ensure that the resulting surface has no self-intersections. This Möbius strip has a core circle with radius R which lies in the xy-plane. Contrast this to the cylinder, which is described *only* by a line that rotates around the z-axis in a circle with coordinates $(\cos(v), \sin(v))$. The cylinder is thus topologically homeomorphic to the trivial product space $S^1 \times \mathbf{R}$. Fig. 6.2 shows a Möbius band with the above parametrization.

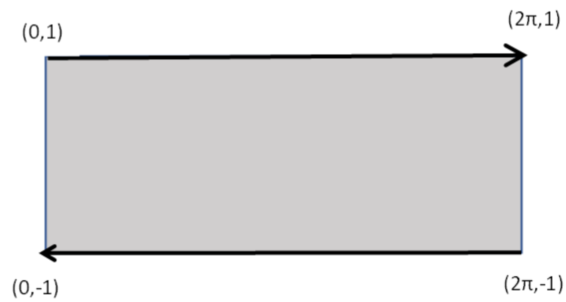


Figure 6.1. Topologically, the Möbius strip is a rectangle with its top and bottom sides identified

We now identify the *boundary* curve γ to this surface shown in 6.3. The *boundary curve* of the strip can be described by taking $t = W$ in the surface parametrization

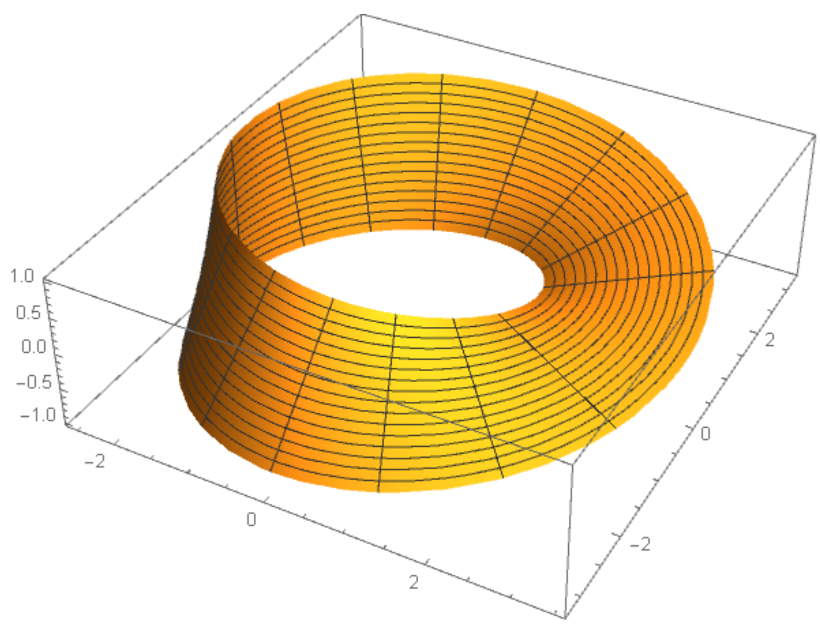


Figure 6.2. The Möbius band embedded in $\mathbb{R}^3$

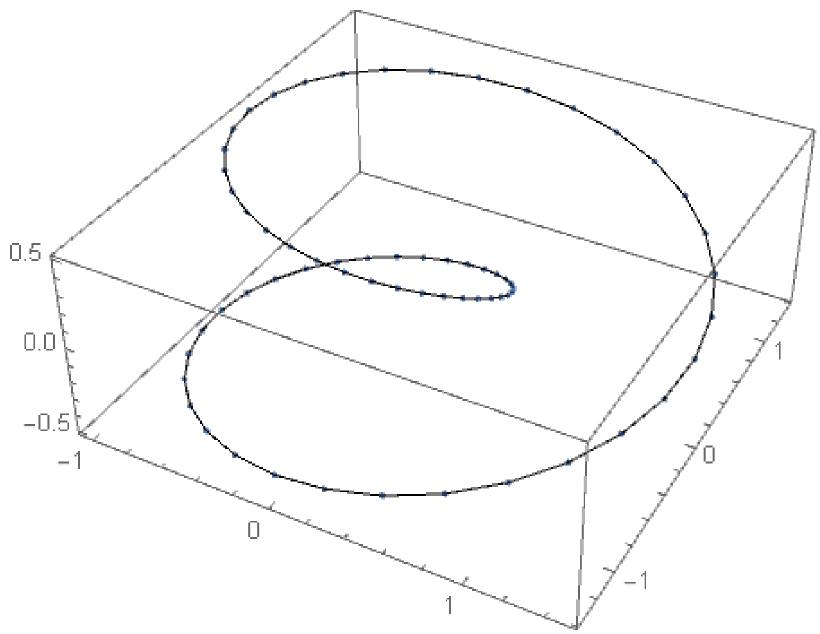


Figure 6.3. The Möbius Boundary

(6.19) with $0 \leq v \leq 4\pi$ as follows

$$\begin{aligned} x(t,v) &= \left(R - W\sin\frac{v}{2}\right)\cos(v) \\ y(t,v) &= \left(R - W\sin\frac{v}{2}\right)\sin(v) \\ z(t,v) &= W\cos\frac{v}{2}\ . \end{aligned} \tag{6.20}$$

If we denote $\mathbf{X}(t,v) = \langle x(t,v), y(t,v), z(t,v)\rangle$, then the tangent plane at each point $\mathbf{x} \in M \subset \mathbb{R}^3$ is $T_x M$ is spanned by two orthonormal basis vectors $\left(\widehat{\mathbf{X}}_t, \widehat{\mathbf{X}}_v\right)$ whose components are given by

$$\widehat{\mathbf{X}}_t = \left\{-\sin\left(\frac{v}{2}\right)\cos(v), -\sin\left(\frac{v}{2}\right)\sin(v), \cos\left(\frac{v}{2}\right)\right\} \tag{6.21}$$

and $\widehat{\mathbf{X}}_v$

$$\begin{aligned}
x'(t,v) &= \frac{\cos\left(\frac{v}{2}\right)\left(-4R\sin\left(\frac{v}{2}\right)-3t\cos(v)+2t\right)}{\sqrt{4R^2-2t\left(4R\sin\left(\frac{v}{2}\right)+t\cos(v)\right)+3t^2}} \\
y'(t,v) &= \frac{4R\cos(v)+t\left(\sin\left(\frac{v}{2}\right)-3\sin\left(\frac{3v}{2}\right)\right)}{2\sqrt{4R^2-2t\left(4R\sin\left(\frac{v}{2}\right)+t\cos(v)\right)+3t^2}} \\
z'(t,v) &= -\frac{t\sin\left(\frac{v}{2}\right)}{\sqrt{4R^2-2t\left(4R\sin\left(\frac{v}{2}\right)+t\cos(v)\right)+3t^2}}
\end{aligned} \tag{6.22}$$

where $\widehat{\mathbf{X}}_t$ and $\widehat{\mathbf{X}}_v$ denote derivatives with respect to ∂_t and ∂_v respectively. Moving forward, we will drop the *hats* over $\mathbf{X}_t$ and $\mathbf{X}_v$ since it will always be assumed we are working unit vectors throughout this section. The components of the unit normal $\widehat{\mathbf{N}}(t,v)$ to the *surface* or equivalently to the tangent plane, defined by $\widehat{\mathbf{N}} = \mathbf{X}_t \times \mathbf{X}_v$, the cross product of $\mathbf{X}_t$ and $\mathbf{X}_v$, are

$$\left\{\begin{array}{l}
\frac{\cos\left(\frac{v}{2}\right)\left(t\left(\sin\left(\frac{v}{2}\right)+\sin\left(\frac{3v}{2}\right)\right)-2R\cos(v)\right)}{\sqrt{4R^2-2t\left(4R\sin\left(\frac{v}{2}\right)+t\cos(v)\right)+3t^2}}, \frac{\sin^2(v)\left(t-R\csc\left(\frac{v}{2}\right)\right)-t\cos(v)}{\sqrt{4R^2-2t\left(4R\sin\left(\frac{v}{2}\right)+t\cos(v)\right)+3t^2}}, \\
\frac{-2R\sin\left(\frac{v}{2}\right)+t(-\cos(v))+t}{\sqrt{4R^2-2t\left(4R\sin\left(\frac{v}{2}\right)+t\cos(v)\right)+3t^2}}
\end{array}\right\} \tag{6.23}$$

In the $\left(\widehat{\mathbf{X}}_t, \widehat{\mathbf{X}}_v, \widehat{\mathbf{N}}\right)$ basis, the first fundamental form or equivalently the line element is given as follows

$$\begin{aligned}
\mathrm{ds}^2 &= g_{tt}(t,v)\ \mathrm{dt}^2 + g_{vv}(t,v)\ \mathrm{dv}^2 \\
&= E\ \mathrm{dt}^2 + G\ \mathrm{dv}^2 \\
&= ||\mathbf{X}_t||^2\,\mathrm{dt}^2 + ||\mathbf{X}_v||^2\,\mathrm{dv}^2 \\
&= \mathrm{dt}^2 + R^2 + \frac{3t^2}{4} - \frac{1}{2}t\left(4R\sin\left(\frac{v}{2}\right)+t\cos(v)\right)
\end{aligned} \tag{6.24}$$

The Christoffel symbols of the second kind can be directly computed from the metric in (6.24). For a general holonomic coordinate system with basis tangent vectors $\mathbf{e}_i$, the Christoffel symbols are generally defined as

$$\nabla_\mu \mathbf{e}_\nu = \Gamma^\rho_{\mu\nu} \mathbf{e}_\rho. \tag{6.25}$$

In this coordinate basis, $\nabla_\mu \mathbf{e}_\nu = \nabla_\nu \mathbf{e}_\mu$ and hence $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$. In an arbitrary non-holonomic basis of tangent vectors denoted by $\mathbf{u}_\alpha$, the covariant derivatives may not necessarily commute and therefore, the Christoffel symbols (or the affine connection) may contain torsion.

Let us denote the basis vectors $\mathbf{X}_t$ as $\mathbf{u}_t = \partial_t \equiv \partial_1$ and $\mathbf{X}_v$ as $\mathbf{u}_v = \partial_v \equiv \partial_2$. The Christoffel symbols can then be explicitly computed in terms if the metric (6.24) as follows

$$\begin{aligned}
\Gamma^t{}_{vv} &= -\frac{\partial_t g_{vv}}{2g_{tt}} = R\sin\left(\frac{v}{2}\right) + \frac{1}{4}t(2\cos(v) - 3) \\
\Gamma^v{}_{tv} &= \frac{\partial_t g_{vv}}{2g_{vv}} = \frac{1}{t - \frac{4R\left(R - t\sin\left(\frac{v}{2}\right)\right)}{4R\sin\left(\frac{v}{2}\right) + 2t\cos(v) - 3t}} \\
\Gamma^v{}_{vv} &= \frac{\partial_v g_{vv}}{2g_{vv}} \frac{t\left(t\sin(v) - 2R\cos\left(\frac{v}{2}\right)\right)}{4R^2 - 2t\left(4R\sin\left(\frac{v}{2}\right) + t\cos(v)\right) + 3t^2}
\end{aligned} \tag{6.26}$$

With this embedding in (6.19) and the metric in (6.24), the second fundamental form is given explicitly by [166, 168]

$$K_{tt}\ \mathrm{dt}^2 + K_{vt}\ \mathrm{dt\ dv}\ + K_{vv}\ \mathrm{dv}^2 \tag{6.27}$$

where the coefficients are calculated as follows

$$\begin{aligned} e = K_{tt} &= \widehat{\mathbf{n}} \cdot \dot{\mathbf{X}}_t = 0 \\ f = K_{tv} &= \widehat{\mathbf{n}} \cdot \mathbf{X}_t' = \frac{R}{\sqrt{4R^2 - 2t\left(4R\sin\left(\frac{v}{2}\right) + t\cos(v)\right) + 3t^2}} \\ g = K_{vv} &= \widehat{\mathbf{n}} \cdot \mathbf{X}_v' = \cos\left(\frac{v}{2}\right)\left(1 + \frac{t^2}{4R^2 - 2t\left(4R\sin\left(\frac{v}{2}\right) + t\cos(v)\right) + 3t^2}\right) \end{aligned} \tag{6.28}$$

One way to demonstrate the presence of the torsion in the Möbius band is to see the behavior of the unit tangent vector $\mathbf{X}_v$ as it completes a rotation by 2π. It simply fails to close. Figure 6.4 shows a plot of $\mathbf{X}_v$ as a discrete curve, i.e. as a polygon. This is consistent with the geometrical notion of torsion which describes the internal twist of a two-dimensional manifold. In two dimensions, torsion is special in several ways. For example, it does not preserve tangent vectors in the sense that a vector field does not maintain the same angle with the tangent vector as it is parallel transported [169].

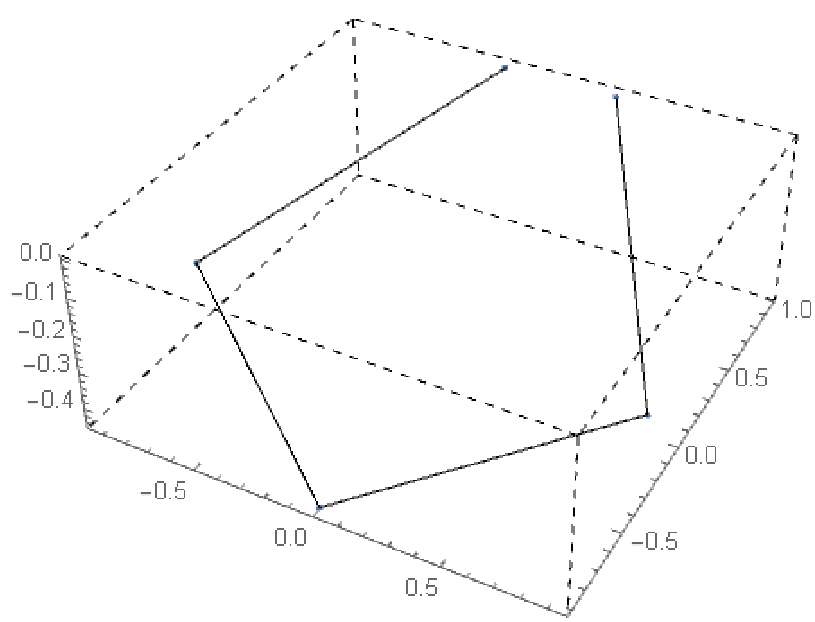


Figure 6.4. An illustration of the torsion in the Möbius band. The unit tangent vector $\mathbf{X}_v$ does not close after a rotation by 2π.

Let $\mathbf{X}_v'$ be the *unit* acceleration vector. Then the projection of $\mathbf{X}_v'(t, v)$ onto the unit tangent vector $\mathbf{X}_t(t, v)$ gives the geodesic curvature κ_g

$$\begin{aligned}\kappa_g &= \mathbf{X}_v' \cdot \mathbf{X}_t \\ &= \frac{12t\left(2R^2+t^2\right)\cos(v)+4R\sin\left(\frac{v}{2}\right)\left(4R^2-6t^2\cos(v)+9t^2\right)-28R^2t-2t^3\cos(2v)-11t^3}{2\left(4R^2-2t\left(4R\sin\left(\frac{v}{2}\right)+t\cos(v)\right)+3t^2\right)^{3/2}}\ ,\end{aligned} \tag{6.29}$$

and its projection of onto the normal $\mathbf{N}(v, t)$ gives the normal curvature κ_n

$$\kappa_n = \mathbf{X}_v' \cdot \mathbf{N} = \cos\left(\frac{v}{2}\right)\left(\frac{t^2}{4R^2-2t\left(4R\sin\left(\frac{v}{2}\right)+t\cos(v)\right)+3t^2}+1\right)\ , \tag{6.30}$$

such that the curvature κ is given

$$\kappa = \sqrt{\kappa_g^2 + \kappa_n^2}\ . \tag{6.31}$$

Fig. 6.5 shows that the curvature of a 1d curve on a 2d surface embedded in $\mathbb{R}^3$ decomposes into a linear combination of the geodesic curvature, κ_g, the component along the unit surface tangent $\mathbf{X}_t$ and the normal curvature, the component along the unit normal to the surface $\mathbf{N}$.

The normal curvature, κ_n, as defined above, is given by magnitude of the component the unit acceleration vector $\mathbf{X}_v'$ along the *normal* to the surface $\mathbf{N}(v)$. Thus, κ_n detects the curvature of a 1d curve on a 2d surface as seen in the *ambient* space $\mathbb{R}^3$. In other words, κ_n depends on the curvature of the 2d surface in $\mathbb{R}^3$, and, therefore, measures the *extrinsic* curvature of the curve, as opposed to κ_g which measures its intrinsic curvature in the 2d surface.

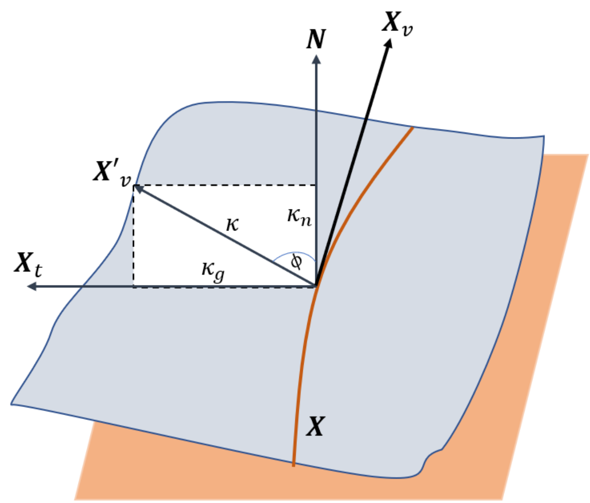


Figure 6.5. The curvature of a 1d curve on a 2d surface embedded in $\mathbb{R}^3$ is a linear combination of the geodesic curvature, the component along the unit surface tangent $\mathbf{X}_t$ and the normal curvature, the component along the unit normal to the surface $\mathbf{N}$.

6.2.1.2 The Calculation

With the above setup, we can now proceed to carry out the following two calculations for the Möbius strip: (1) the Euler number χ using Gauss-Bonnet theorem followed by (2) the λ invariant.. This will illustrate the key difference between the Gaussian curvature scalar $K = R/2$ where R the Ricci scalar and the dual or Lorentz scalar scalar U. We note that the general Gauss-Bonnet theorem can be directly applied to non-orinetable surfaces [170]. One direct and simple proof of this fact uses the orientable double cover of the Möbius strip [170]. It is more common to see statements that the Euler class is only defined for orientable manifolds with orientable tangent bundles which is of course true given the integral is over a volume form not a volume density.

The Gaussian curvature K is given by the following formula [166, 167] (in our above setup, $G = g_{vv}$, $E = g_{tt} = 1$)

$$\begin{aligned} K &= \frac{1}{\sqrt{G}} \left[\partial_v \left(\frac{\sqrt{E}}{E} \Gamma^v_{tt} \right) - \partial_t \left(\frac{\sqrt{G}}{G} \Gamma^v_{tv} \right) \right] \\ &= -\frac{1}{\sqrt{G}} \partial_t \left(\frac{\sqrt{G}}{G} \Gamma^v_{tv} \right) \\ &= -\frac{1}{2\sqrt{G}} \partial_t \left(\frac{G_t}{\sqrt{G}} \right) \\ &= -\frac{1}{2\sqrt{G}} \partial_t^2 \left(\frac{G}{\sqrt{G}} \right) . \end{aligned} \tag{6.32}$$

The geodesic curvature is given by [171]

$$\kappa_g = \frac{1}{2\sqrt{EG}} \big(-G_t t'(s) + E_v v'(s) \big) . \tag{6.33}$$

where s is the arc length parameter. Since $ds = \frac{ds}{dv} dv$, then $\kappa_g \, ds = \big(-\frac{1}{2} G_t/\sqrt{G}\big) v'(s) \, ds = \big(-\frac{1}{2} \frac{G_t}{\sqrt{G}}\big) \, dv$. The Gaussian curvature has been directly evaluated in [171] as follows

$$\int_M K \, dA = \iint_{[0,2\pi]\times[-1/2,1/2]} -\frac{1}{2\sqrt{G}} \left(\partial_t^2 \frac{G}{\sqrt{G}} \right) \sqrt{G} \, dt \, dv \tag{6.34}$$

$$= -\frac{1}{2} \int_0^{2\pi} \int_{-1/2}^{1} /2 \Big(\partial_t^2 \frac{G}{\sqrt{G}} \Big) \, dt \, dv \tag{6.35}$$

$$= -\frac{1}{2} \int_0^{2\pi} \left(\partial_t \frac{G}{\sqrt{G}} \right) \bigg|_{t=1/2} - \left(\partial_t \frac{G}{\sqrt{G}} \right) \bigg|_{t=-1/2} dv \tag{6.36}$$

$$\approx -1.97. \tag{6.37}$$

where the in last step, Mathematica was used to numerically evaluate the integral. Evaluating the integral of the geodesic curvature on the top gives [171]

$$-\frac{1}{2} \int_0^{2\pi} \partial_t \frac{G}{\sqrt{G}} \bigg|_{t=1} dt \approx 4.53,$$

and on the bottom boundary gives

$$-\frac{1}{2}\int_0^{2\pi} \partial_t \frac{G}{\sqrt{G}}\Big|_{t=-1} dt \approx -2.56,$$

Therefore, we finally get

$$\int_M K dA = -1.97 \qquad \text{and} \qquad \int_{\partial M} \kappa_g \, ds = +4.53 - 2.56 = +1.97. \tag{6.38}$$

and hence, as expected, the sum is $2\pi\chi(M) = 0$ which confirms the fact that the Euler number of the Möbius strip is zero [171].

We now calculate the dual curvature scalar, U, for the Möbius strip. The calculation proceeds in parallel to the calculation of the Gaussian curvature. The dual curvature U can be calculated in two complementary ways depending on whether we choose t or v coordinates to be the temporal coordinate, or in other words, how we choose to foliate our Möbius strip. If we take v to be the spatial direction, then the leaves of the Möbius strip are the *spatial* circles with the rotating lines pointing in the temporal t direction. Alternatively, we can choose the spatial leaves of the Möbius strip to be the rotating lines along t and consider the circles to be moving in the temporal v direction. Fig. 6.6 illustrates the rotating lines of the Möbius strip. [2]

In the former case, we use Γ^v_{vv} and then U is given explicitly by

$$\begin{aligned} U &= -\frac{1}{\sqrt{G}}\partial_t\left(\frac{\sqrt{G}}{G}\Gamma^v_{vv}\right) \qquad (6.39)\\ &= -\frac{1}{2\sqrt{G}}\partial_t\left(\frac{G_v}{\sqrt{G}}\right)\\ &= -\frac{1}{2\sqrt{G}}\partial_t\left(\partial_v\frac{G}{\sqrt{G}}\right)\\ &= \frac{t\left(3\left(4R^2+t^2\right)\sin(v)+6Rt\cos\left(\frac{3v}{2}\right)+t^2(-\sin(2v))\right)-2R\left(R^2+3t^2\right)\cos\left(\frac{v}{2}\right)}{\left(4R^2-2t\left(4R\sin\left(\frac{v}{2}\right)+t\cos(v)\right)+3t^2\right)^{3/2}}. \end{aligned}$$

[2]Figure adapted from http://webmath2.unito.it/paginepersonali/sergio.console/CurveSuperfici/Notebooks/.

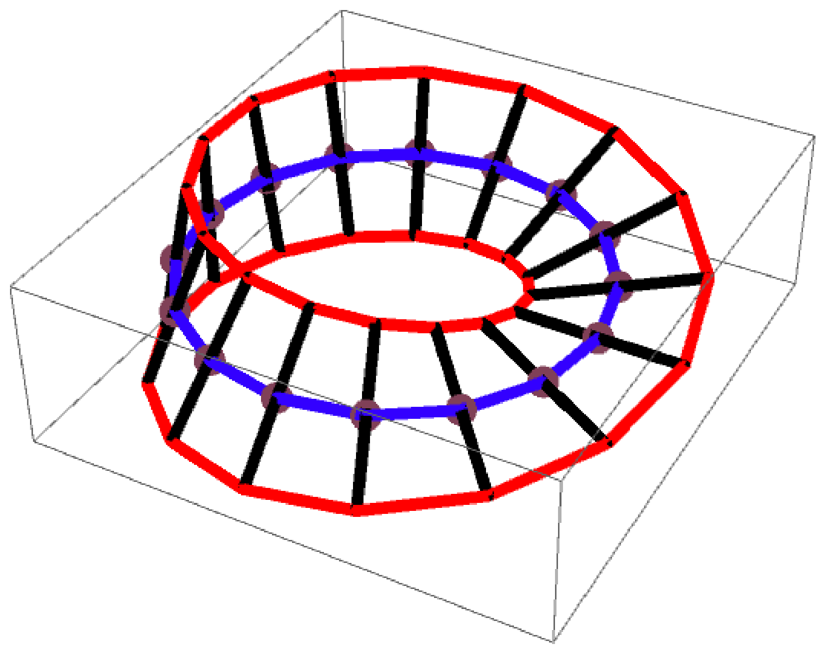


Figure 6.6. The Möbius boundary is shown in red, the rotating lines in black and the core circle in blue.

while in the latter case we instead use Γ^v_{tv} as in the Gaussian curvature case and U is given explicitly by

$$\begin{aligned} U &= -\frac{1}{\sqrt{G}}\partial_v\left(\frac{\sqrt{G}}{G}\Gamma^v_{tv}\right) \\ &= -\frac{1}{2\sqrt{G}}\partial_v\left(\frac{G_t}{\sqrt{G}}\right) \\ &= -\frac{1}{2\sqrt{G}}\partial_v\left(\partial_t\frac{G}{\sqrt{G}}\right) \\ &= \frac{t\left(3\left(4R^2+t^2\right)\sin(v)+6Rt\cos\left(\frac{3v}{2}\right)+t^2(-\sin(2v))\right)-2R\left(R^2+3t^2\right)\cos\left(\frac{v}{2}\right)}{\left(4R^2-2t\left(4R\sin\left(\frac{v}{2}\right)+t\cos(v)\right)+3t^2\right)^{3/2}}. \end{aligned} \tag{6.40}$$

Since the difference between $R = 2K$ and U becomes strikingly clear in the limit where the width t of the strip goes to zero, which we define as the *flat limit*, we present it here and defer the case of a finite t to Section 6.3. It is well known that the Möbius strip, although not homeomorphic to the circle, is the *deformation retract* of its core circle. Taking the limit $t \to 0$ of R amounts to deformation retracting the Möbius surface onto the core circle such that $G = 1$ in metric (6.24). This is why we call this deformation retraction process the *flat Möbius* limit. Taking the radius R of the circle to be 1, the $t \to 0$ flat limit of the Gaussian curvature K gives the constant *intrinsic* curvature of the core circle

$$K\Big|_{t=0} = \frac{e}{G}\Big|_{t=0} = -\frac{1}{4} \ . \tag{6.41}$$

On the other hand, taking the $t \to 0$ flat limit of U in (6.40) instead gives

$$U\Big|_{t=0} = \frac{1}{4}\cos(v/2) \ . \tag{6.42}$$

The flat limit clearly illustrates that the difference between K and U is, in principle, the difference between intrinsic and extrinsic geometry. In other words, deforming the Möbius surface by way of taking the flat limit of $K = 2R$ simply gives the intrinsic curvature of the core circle which has no knowledge of the extrinsic curvature induced by its embedding in $\mathbb{R}^3$. On the other hand, deforming the Möbius surface by taking the same $t \to 0$ of the dual curvature scalar U extracts the extrinsic curvature of the Möbius boundary with absolutely no knowledge of the intrinsic curvature of the core circle. This is another way of saying that $U\Big|_{t=0}$ measures the normal curvature along $\mathbf{N}(v)$ which detects how the surface bends in the normal direction $\mathbf{N}(v)$. Thus, in a sense, $U\Big|_{t=0}$ measures the deviation from *planarity*, which is the most basic definition of torsion in space curves.

To better understand this $t \to 0$ of U, we will derive it directly from the (6.40) by taking the flat limit of Γ^v_{tv} to $t = 0$ as follows

$$\begin{aligned} U\Big|_{t=0} &= -\frac{1}{\sqrt{G}}\partial_v\left(\frac{\sqrt{G}}{G}\frac{\partial_t G}{2G}\Big|_{t=0}\right) \\ &= -\frac{1}{2\sqrt{G}}\partial_v\left(\frac{1}{\sqrt{G}}(\sin(-v/2))\right) \\ &= \frac{1}{2\sqrt{G}}\partial_v\left(\sin(\frac{v}{2})\right) \\ &= \frac{1}{4}\cos(v/2)\ , \end{aligned} \tag{6.43}$$

where in the last step we used that in the flat limit $G = 1$. Integrating $U\Big|_{t=0}$ gives

$$\int_{-\pi}^{\pi} \frac{1}{4}\cos(v/2)\, dv = 1\ . \tag{6.44}$$

We now move to demonstrate how in this flat limit, the integral of U detects the *parity* of the number of twists of the Möbius band. It is important to note that the number of twists in the Möbius band is an *embedding* invariant or more accurately, an *isotopy* invariant. An isotopy invariant can be used to classify different embeddings of the Möbius surface in $\mathbb{R}^3$. The twist number is *not* a topological invariant since all strips with an *even* number of half-twists are homeomorphic to $[0, 1] \times S^1$, which is a trivial line bundle. Since, as we explained in Appendix 2.C, the cylinder and the Möbius strip are two topologically *distinct* isomorphism classes of the real line bundles, we can directly conclude that the parity of the number of twists is indeed a topological invariant.

The Möbius strip with a number of half-twists w and unit radius has the following embedding in $\mathbb{R}^3$

$$\begin{aligned}
x(t,v) &= \Big(1 - t\cos(\frac{wv}{2})\Big)\cos(v) \\
y(t,v) &= \Big(1 - t\cos(\frac{wv}{2})\Big)\sin(v) \\
z(t,v) &= t\sin(\frac{wv}{2}) \ .
\end{aligned} \tag{6.45}$$

Let $w = 2$. Calculating the dual curvature $U^{w=2}$ using (6.39) gives

$$U^{w=2} = \frac{\cos(v)\left(6t^2 + t(t(t(\sin(3v) - 7\sin(v)) - 6\cos(2v)) - 12\sin(v)) + 4\right)}{2\sqrt{2}\left(3t^2 - t(t\cos(2v) + 4\sin(v)) + 2\right)^{3/2}} \ , \tag{6.46}$$

which in the flat limit gives

$$U^{w=2}\big|_{t=0} = \frac{\cos(v)}{2} \ . \tag{6.47}$$

Upon integration, it yields

$$\lambda = \int_{-\pi}^{\pi} U^{w=2}\big|_{t=0}\, dv = \int_{-\pi}^{\pi} \frac{\cos(v)}{2}\, dv = 0 \ . \tag{6.48}$$

Now, let $w = 3$. Calculating the dual curvature $U^{w=3}$ gives

$$\frac{3\left(8R^3\cos\left(\frac{3v}{2}\right) + t\sin(3v)\left(-12R^2 + 2t\left(6R\sin\left(\frac{3v}{2}\right) + t\cos(3v)\right) - 11t^2\right)\right)}{4\left(4R^2 - 2t\left(4R\sin\left(\frac{3v}{2}\right) + t\cos(3v)\right) + 11t^2\right)^{3/2}} \ , \tag{6.49}$$

which in the flat limit gives

$$U^{w=3}\big|_{t=0} = \frac{3}{4}\cos\left(\frac{3v}{2}\right) \ , \tag{6.50}$$

and upon integration yields

$$\lambda = \int_{\pi}^{-\pi} U^{w=3}\Big|_{t=0} dv = \int_{\pi}^{-\pi} \frac{3}{4}\cos\left(\frac{3v}{2}\right) dv = 1 \ . \tag{6.51}$$

Therefore, we see that the integral of U in this flat limit distinguishes between an even number of twists and an odd number of twists of the Möbius strip and hence it represents a topological invariant. We discuss and further elaborate on the topological and geometrical underpinnings of this result in Section 6.3.

6.2.1.3 The flat limit of the dual spin connection

In this section, we work out an explicit example using vielbeins and the dual spin connection to demonstrate how $U\big|_{t=0} = \frac{1}{4}\cos(v/4)$ can be obtained in the flat limit. Here, we follow the example in Section 12.2.3 of [1]. The frame fields are given by

$$e^1 = -2\cos(v/4)\, dt, \quad e^2 = -2\cos(v/4)^{-1}\, dv \ , \tag{6.52}$$

with components

$$e^1_t = -2\cos(v/4), \quad e^2_v = -2\cos(v/4)^{-1} \ . \tag{6.53}$$

The inverse vielbeins are then

$$E_1 = -\frac{1}{2}\cos(v/4)^{-1}\frac{\partial}{\partial t}, \quad E_2 = -\frac{1}{2}\cos(v/4)\frac{\partial}{\partial v} \ , \tag{6.54}$$

with components

$$E^t_1 = -\frac{1}{2}\cos(v/4)^{-1}, \quad E^v_2 = -\frac{1}{2}\cos(v/4) \ , \tag{6.55}$$

such that

$$e^a_t\, E^t_b + e^a_v\, E^v_b = \delta^a_b \ . \tag{6.56}$$

Acting with the exterior derivative

$$d = \frac{\partial}{\partial t}\, dt + \frac{\partial}{\partial v}\, dv \ ,$$

we get

$$de^1 = \frac{1}{2}\sin(v/4)\, dv \wedge dt, \quad de^2 = 0 \ . \tag{6.57}$$

Using $dt\, dv(E_1, E_2) = \frac{1}{4}$, the coefficients of the spin connection 1-form are given by

$$\xi^1_{21} = \frac{1}{8}\sin(v/4)$$

and therefore

$$\omega^1_{21} = \frac{1}{8}\sin(v/4)$$

such that the spin connection 1-form is

$$\omega^1_{21} e^1 = -\frac{1}{4}\sin(v/4)\ \cos(v/4) dt \wedge dv \tag{6.58}$$

and the curvature 2-form $d\omega^1_2$ is given by

$$\begin{aligned} U^1_2 &= -\frac{1}{16}\big(\cos^2(v/4) - \sin(v/4)\big)\, dv\, dt \\ &= -\frac{1}{16}\cos(v/2) \\ &= \frac{1}{4}\cos(v/2) e^2\, e^1 \ , \end{aligned} \tag{6.59}$$

where in the last step we used (6.52). Thus, from this example, we see that a black hole metric is one way of obtaining the flat limit of the dual curvature U.

6.3 Discussion and Outlook

On topological grounds, we *speculate* that this invariant counts the dimension of the $\mathbb{Z}_2$ torsion subgroup in the same way that the Euler class $e(E) \in H^n(M;\mathbb{Z})$ of an orientable tangent bundle counts the number of n-dimensional holes of a compact n-dimensional manifold M. We note that for orientable tangent bundles (equivalently manifolds), $e([M]) = \chi(M)$, where $\chi(M) = \sum_{i=0}^{n}(-1)^i \mathrm{dim}\ H^i(M,\mathbb{Z}) = \sum_{i=0}^{n}(-1)^i\, b_n(M)$ where $b_n(M)$ is the n-th Betti number of the base manifold M. For a proof of this statement, see [118]. However, since the rational Betti numbers $b_n(M)$, and hence, the Euler characteristic, do not take into account any torsion in the homology groups present in non-orientable bundles, another number called the *torsion coefficient*, analogous to $\chi(M)$ is used to classify manifolds with *torsional cycles*. We *speculate* that the λ invariant computes the torsion coefficient of the Möbius bundle.

In non-orientable manifolds (or bundles), for example, the Möbius surface, there is a class of *cycles* that only close after going around twice. This phenomenon is called *torsion* in homology theory. To classify these type of cycles that must be followed around twice before they close, another number, other than the Euler number, known as the *torsion coefficient* is used. Hence, torsion, as defined in homology theory, tells us about the non-orientability of surfaces. As we will further discuss in appendix 6.A, for surfaces, non-orientability can be detected by the presence of a torsion subgroup in the first homology group.

It is important to note that unlike $\chi(M)$ which classifies different types of holes in the *base* manifold M of a vector bundle $E \xrightarrow{\pi} M$, $\lambda(E)$ detects torsion in the bundle itself. This is an important distinction since it underlies another reason why the Ricci scalar R is fundamentally different from the dual Lorentz scalar U. In the flat limit, U captures the extrinsic curvature of the Möbius bundle which principally detects the twist in the frame of the Möbius line bundle as seen in $\mathbb{R}^3$ rather than the curvature of its base circle that the Ricci scalar detects. This, in a sense, can be considered the reason why the Euler characteristic $\chi(M)$ does not take into account the torsion in the Möbius bundle.

We also would like to point out that, in hindsight, this result is not very surprising. Based on our discussion in Section 2, this result is compatible with the fact that the $z = 1$ Weyl anomaly is *parity-odd*. Therefore, the fact that λ measures the parity of the number of half-twists is quite natural.

Geometrically speaking, another way to understand the difference between K (or R) and U is to try to find a statement of $\int U\, dA$ analogous to that of the Gauss-Bonnet theorem in terms of the curvature of a geodesic triangles. Specifically, for a geodesic triangle T, the integral $\int_T K$ is the deviation of the sum of its turning angles $\theta_i = (\pi - \alpha_i)$ from π

$$\begin{aligned} \int_T K &= 2\pi - \sum_i \alpha_i - \int_{\partial T} \kappa_g \\ &= \sum_i (\pi - \alpha_i) - \pi - \int_{\partial T} \kappa_g \ , \end{aligned} \tag{6.60}$$

where α_i is the interior angle of T. The turning/winding number theorem [166, 168] states that for a smooth closed curve, the total (signed) curvature is an integer multiple of 2π. For a Möbius strip, therefore, one can see, given the calculation in (6.34), that $\sum_i(\pi - \alpha) - \pi = -\int_{\partial T} \kappa_g$ gives an Euler characteristic $\chi(\text{Möbius}) = 0$. It is important to observe that the geodesic triangle T is a closed *discrete* piecewise curve, which implies that the discrete turning angle θ_i at each vertex v_i of T is given by an angle θ_i such that, in the flat limit, $\sum_i \theta_i = 2\pi n$ where $n \in \mathbb{Z}$ [172].

On the contrary, trying to construct a discrete version of the normal curvature vector defined as the gradient of arc length of the curve, i.e. $\mathbf{X}'_v(v) = \kappa(v)\mathbf{N}(v)$ actually fails (see equation 3 in [172]) in a way that *violates* the turning number theorem. This is because the discrete curvature κ_i at each vertex v_i of the closed discrete curve is $2\sin(v/2)$ as opposed to a mere angle θ_i. We therefore *speculate* that this may be one way of explaining why the unit tangent vector in Fig. 6.4, which is a discrete curve, does not close: at each vertex of $\mathbf{X}_v$, the curvature κ_i is actually given by $2\sin(\theta_i/2)$ such that in the limit of large n of an n-gon with turning angle $\theta_i = \frac{2\pi}{n}$, $\sum_i = 2n\sin(\frac{2\pi}{2n}) < 2\pi$. To verify the above argument in the case of the Möbius strip, one can directly compute the gradient of the arc length in the flat limit as $\frac{\partial_t G}{2G}$ which indeed gives $2\sin(v/2)$ exactly. Hence, at least in the case of the Möbius strip, one can speculate that in the *flat* limit, $\int U\Big|_{t=0}$ is given by the integral over a discrete curve C (we are using θ for the angle since v_i denotes a vertex of a a discrete polygon)

$$\int_C U \;=\; \int_C \sum_i 2\sin(\theta_i) - \int_C \kappa_n \ . \tag{6.61}$$

In the case of a *finite* thickness t, one can calculate $\int_M U\,dA$ as follows

$$\begin{aligned}
\int_M U\,dA &= \iint_{[0,2\pi]\times[-1/2,1/2]} \frac{1}{2\sqrt{G}}\partial_t\left(\frac{\partial_v G}{\sqrt{G}}\right)\sqrt{G}\,dt\,dv \\
&= \frac{1}{2}\int_{-\pi}^{\pi}\int_{-1/2}^{1/2}\left(\partial_t\frac{\partial_v G}{\sqrt{G}}\right)dt\,dv \\
&= \frac{1}{2}\int_{-\pi}^{\pi}\left(\partial_v\frac{G}{\sqrt{G}}\right)\Big|_{t=1/2} - \left(\partial_v\frac{G}{\sqrt{G}}\right)\Big|_{t=-1/2} dv \\
&\approx -0.96\ ,
\end{aligned}$$

where again Mathematica was used to numerically compute the last integral. Observe that $\int U$ has an extra minus sign which $\int K$ does not have. This extra minus sign cancels with the minus sign in the definition of K. The reason for this extra minus is simply because, as we explained at the end of Section 6.39, U is actually the exterior derivative of the Hodge dual of the spin connection 1-form $d\star\omega$ as shown in (6.17).

We also note that the integral is taken from π to $-\pi$ rather than from 0 to 2π as in the case of the Gaussian curvature. Another difference between K and U is while K does not change sign at 0 and 2π, U actually does.

The *dual* geodesic curvature, $\tilde{\kappa}_g$, is now given by

$$\begin{aligned} \tilde{\kappa}_g &= \left(\frac{1}{2}\frac{G_v}{\sqrt{G}}\right) dv \\ &= -\frac{t\left(2R\cos\left(\frac{v}{2}\right) - t\sin(v)\right)}{2\sqrt{R^2 - \frac{1}{2}t\left(4R\sin\left(\frac{v}{2}\right) + t\cos(v)\right) + \frac{3t^2}{4}}} \ . \end{aligned} \tag{6.62}$$

Evaluating the integral of the dual geodesic curvature on the top boundary gives

$$\frac{1}{2}\int_{\pi}^{-\pi} \frac{G_v}{\sqrt{G}}\Big|_{t=1/2} dv \approx 0.48$$

and on the bottom boundary gives

$$\frac{1}{2}\int_{-\pi}^{\pi} \frac{G_v}{\sqrt{G}}\Big|_{t=-1/2} dv \approx 0.48$$

such that one finally gets

$$\int_M U\, dv\, dt + \int_{\partial M} \tilde{\kappa}_g\, dv \approx 0.96 + 0.48 + 0.48 \approx 1.92 \ . \tag{6.63}$$

So we can directly see that the boundary terms actually add up and do not cancel out as they do in the case of the Gaussian curvature K.

Although we did not give a rigorous proof of the topological invariance of λ, we gave sufficient evidence that it actually is. A logical next step is to provide a proof that λ actually counts the torsion coefficient of 2-dimensional non-orientable manifolds. A useful formula for the Gaussian curvature is Liouville's equation written in terms of the isothermal coordinates. It would be nice if an analogous equation can be found for the dual curvature U.

Another natural extension of this work is to find a physical application of this λ invariant in condensed-matter theory especially in regards to the $\mathbb{Z}_2$ topological invariant in the quantum spin Hall Effect [173, 174]. Also, a torsional topological invariant in four dimensions has been found in [175–177]. It would be very exciting if a connection between this four-dimensional invariant can be found.

Another very interesting direction is the time reversal anomaly in 2+1d topological phases [178,179]. Time reversal invariance of a (2+1) quantum field theory is a global symmetry of the theory. Coupling the QFT to a background gauge field or, in other words, gauging the time reversal symmetry, is, interestingly enough, equivalent to placing the theory on a Möbius strip. We have seen that the Lifshitz Weyl anomaly is parity-odd and breaks time-reversal invariance. In light of our discussion of the $a \wedge da$ term in Chapter 4 and how we derived the Lifshitz Weyl anomaly as a boundary term of the NRSCS action, it would indeed very interesting to find any connection of the work in this thesis to that in [178].

Appendix

6.A Torsion class and flat vector bundles

In this Appendix, we briefly discuss, using certain aspects of integral homology and cohomology theory, how the Möbius flat line bundle contains torsion, and how the latter can be used to detect non-orientable manifolds. This is a continuation of our discussion in Appendix 2.A.

In orientable manifolds, it is well known that the Euler number only captures the *free* part of the second integral homology group $H_2(M; Z)$. For non-orientable manifolds, e.g. the Möbius surface, on the other hand, there is a class of *cycles* that only close after going around twice. The phenomenon is called *torsion*. To classify these types of cycles that must be followed around *twice* before they close, another number, other than the Euler number, known as the *torsion coefficient* is used. Hence, we see that torsion can tell us about non-orientability in 2-manifolds. Geometrically speaking, the Euler number only counts handles and holes in surfaces that can be triangulated. The Möbius surface fails to be triangulated exactly because of the cycle that only closes after going around twice.

More concretely, a closed n-manifold M is orientable iff $H_n(M; \mathbb{Z}) = \mathbb{Z}$, and non-orientable iff $H_n(M; \mathbb{Z}) = 0$. Simply put, $H_{n-1}(M; \mathbb{Z})$ is torsion-free if and only if M is orientable and has torsion subgroup $\mathbb{Z}_2$ if M is non-orientable. Thus, for 2-manifolds, orientability can be identified directly from H_1 [121].

Given these facts about the relationship between homology, torsion and non-orinetability, let us apply them to the case of a Möbius strip which is described topologically by a real real line bundle over the circle $\mathbb{S}^1$. The Möbius strip is known to have the following homology groups with integer coefficients [105]

$$\begin{aligned} H_2(\mathbb{S}^1,\mathbb{Z}) &\cong \{0\} \\ H_1(\mathbb{S}^1,\mathbb{Z}) &\cong \mathbb{Z} \\ H_0(\mathbb{S}^1,\mathbb{Z}) &\cong \mathbb{Z}. \end{aligned} \tag{6.64}$$

and with $\mathbb{Z}_2$ coeffiecients

$$\begin{aligned} H_0(S^1,\mathbb{Z}_2) &\cong \mathbb{Z}_2 \\ H_1(S^1,\mathbb{Z}_2) &\cong \mathbb{Z}_2 \\ H_2(S^1,\mathbb{Z}_2) &\cong 0. \end{aligned} \tag{6.65}$$

Therefore, we directly conclude that the Möbius flat line bundle is non-orientable and has torsion which implies that the structure group of the bundle i.e. the gauge group can be reduced to $\mathbb{Z}_2$.

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