A New Framework of Model Reference Adaptive Control

Partial-State Feedback Designs and Applications

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Abstract

Model reference adaptive control (MRAC) is an important methodology to accommodate various system uncertainties. Traditionally, for output tracking MRAC, either a state feedback controller or an output feedback controller is used. As an effort to provide additional feedback capacity and design flexibility to the existing MRAC family, this dissertation focuses on the development of partial-state feedback MRAC framework and the application of such novel MRAC designs. For partial-state feedback MRAC, plant-model matching is achievable as with full-state feedback control, while the controller structure enjoys less complexity as compared with an output feedback MRAC design. In this study, adaptive partial-state feedback control designs are developed for single-input-single-output systems and multi-input-multi-output systems, respectively. Both adaptive control schemes ensure closed-loop system stability and asymptotic output tracking. Related issues such as plant-model matching, error model, adaptive law, and stability analysis are investigated in this dissertation.

Applications of the new partial-state feedback MRAC designs are also explored. Based on the enhanced robustness brought by the partial-state feedback MRAC designs, new sensor failure compensation control schemes for single-input-single-output systems and multi-input-multi-output systems are developed and investigated in this dissertation. The new adaptive sensor failure compensation schemes have the capability of ensuring asymptotic output tracking while compensating all possible uncertain sensor failures in the presence of the system parametric uncertainties. The new sensorredundancy-free compensation schemes relax some requirements of traditional faulttolerant control techniques. This dissertation also extends the partial-state feedback MRAC application to the multi-agent system control area. A new adaptive output consensus control scheme via partial-state feedback MRAC is developed which can increase control design flexibility and make full use of all possible system measurements for multi-agent consensus control. The new consensus control scheme is able to achieve closed-loop signal boundedness and asymptotic output consensus in the presence of system parameter uncertainties.

The effectiveness of the developed adaptive partial-state feedback control designs and sensor fault compensation designs have been assessed on some high-fidelity aircraft systems or quadrotor systems by MATLAB. The simulation results have demonstrated the desired performance of our designs. This dissertation is dedicated to my parents, my husband and my daughter

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Chapter 1 Introduction

Adaptive control attracted tremendous attention in the past decades due to its capacity of accommodating system uncertainties [2,17,33,37,39,40,51,53,82]. Recently, more results have been developed such as adaptive backstepping control [11,86,98], adaptive posicast control [18], adaptive sliding mode control [10], robust adaptive control [31,58,61] and other adaptive control designs [5,60,87,92]. Among different design methods, model reference adaptive control (MRAC) is one of the most important methods [14], [28]. However, although systematic research work has been conducted for MRAC, some technical improvements and new designs of MRAC schemes are still needed for completing the MRAC theory.

1.1 Background and Research Motivations

Model reference adaptive control is an essential adaptive control approach. It provides feedback controller structures and stable adaptive laws for plants to guarantee asymptotic output or state tracking of a given reference model system and closedloop signal boundedness, in the presence of system uncertainties [3, 19, 24, 33, 65]. Most existing MRAC frameworks can be classified into three different types: (i) state feedback MRAC for state tracking [33, 53, 85]; (ii) state feedback MRAC for output tracking [7,27]; and (iii) output feedback MRAC for output tracking [24,88,89]. The controller structure of state feedback MRAC for state tracking is simple, but the plant-model state matching condition is restrictive, which can only be satisfied for system matrices in certain canonical forms. It turns out that state feedback MRAC for output tracking is suitable for broader applications because of its simple structure and unrestrictive matching condition [27]. However, in applications, the requirement for a full-state vector may confine the use of state feedback MRAC. On the other hand, output feedback MRAC for output tracking attracts interest, as its implementation does not need the information of state variables, although its controller structure is more complex.

From the above observations, we conclude that (i) the feedback signal of MRAC schemes is either state vector or output signal; (ii) for the existing MRAC designs, either the requirement for the feedback signal is hard to be satisfied or the controller implementation complexity is high. In practice, the sensors in closed-loop systems can be selected by users at desired locations, in other words, the number of measurable signals in a closed-loop system could be larger than one but less than the number of state variables, which shows the potential to improve the existing MRAC frameworks.

Thus, it is desirable to develop a new MRAC framework for output tracking which can provides

• new feedback capacity to make full use of available system measurements; and

• new design flexibility between the required system measurements for feedback and controller implementation complexity.

In this study, we develop such a new MRAC framework by using partial-state vectors. The new adaptive control framework, with a general feedback signal, provides an additional and complete control approach to guarantee asymptotic output of a given reference model system and closed-loop signal boundedness, in the presence of system uncertainties, and brings a new manageable trade-off between the required system measurements and implementation complexity into the existing MRAC family.

One application of partial-state feedback MRAC is sensor failure compensation. For some applications, although the full-state can be measured, the measurements may not be accurate due to sensor failures and the control system may break down when sensor failures happen. Inspired by the robustness of partial-state feedback MRAC brings in, we develop sensor compensation schemes to improve the reliability of control systems in this research. The new sensor failure compensation schemes have the capability to achieve asymptotic output tracking and closed-loop boundedness in the presence of parametric uncertainties and possible uncertain sensor failures.

Multi-agent system cooperative control has received tremendous attention during the past decade due to its broad applications. To provide additional design flexibility and feedback capacity to multi-agent MRAC systems, the newly developed partial-state feedback MRAC is applied for multi-agent consensus control. Our works [73,74] have verified the effectiveness of state feedback MRAC designs for state consensus control for multi-agent systems. In this research, we incorporate the developed partial-state feedback MRAC schemes to the multi-agent systems for output consensus control.

1.2 Literature Review

In this section, we will present a brief overview of research on model reference adaptive control, partial-state feedback control, fault-tolerant control, and multi-agent consensus control, which provides solid technical foundations for the studies in this dissertation.

1.2.1 Model Reference Adaptive Control

Considerable effort has been devoted to the development of MRAC [3, 19, 24, 33, 39, 51, 53, 59, 65, 82, 88], which plays an important role in the adaptive control family. It provides feedback controller structures for plants to guarantee asymptotic output or state tracking of a given reference model system and closed-loop signal boundedness for the objective model reference control, and it adopts adaptive control method to provide stable adaptive laws for the plants to deal with system uncertainties. The desired closed-loop performance of a MRAC system, such as time constant, damping ratio, natural frequency, is determined by the reference model [80]. The block diagram of a general MRAC system is given in Fig. 1.1.



Figure 1.1: Block diagram of a general MRAC system structure.

According to the different implementation scheme, there are two approaches of MRAC designs: direct MRAC designs [27, 46, 53, 78, 85], and indirect MRAC designs [12, 36, 62]. For direct MRAC designs, the adaptive controller parameters are updated by an adaptive law directly, while for in direct MRAC design, the adaptive controller parameters are solved from some algebraic equations related to the system parameters which are estimated from an adaptive law.

Direct MRAC designs, further, can be classified into three types: (i) state feed-

back MRAC for state tracking [33,53,85]; (ii) state feedback MRAC for output tracking [7,27]; and (iii) output feedback MRAC for output tracking [24, 88, 89]. The controller structure of state feedback MRAC for state tracking is simple, but the plant-model state matching condition is restrictive, which can only be satisfied for system matrices in certain canonical forms. It turns out that state feedback MRAC for output tracking is suitable for a wider range of applications because of its simple structure and unrestrictive matching condition [27]. However, in applications, the requirement for a full-state vector may confine the use of state feedback MRAC. On the other hand, output feedback MRAC for output tracking attracts interest, as its implementation does not need the information of state variables, although its controller structure is more complex. To learn more details, please refer the related textbooks [3,24,33,37,53,82].

1.2.2 Partial-State Feedback Control

Research in partial-state feedback control has been reported. In [37], a partial-state feedback design is developed for nonlinear systems in a canonical form to achieve asymptotic output tracking by using a vector with a subset of state variables. In [40], by using a full-order Luenberger-based state observer, an adaptive model reference controller using system measurements of dimension greater than the number of inputs is developed for bounded output tracking of multi-input-multi-output systems with (A, B, C, C_z) known whose dynamics may have high relative degree and are not necessarily minimum-phase. In [52], a partial-state feedback adaptive controller for a discrete-time system model is proposed using linear quadratic control design, which needs to solve the computation of a matrix factorization. In [22], a backstepping technique is utilized to construct a controller to achieve global convergence, whose design procedure may become complex when the plant order is high. Moreover, some partial-state feedback control designs without adaptation for certain classes of plants have been developed. In [4], a partial-state feedback controller is developed for an induction motor system. In [23], the authors develop a discrete-time partial-state feedback controller for a fourth-order wind power system for achieving closed-loop signal boundedness. Control designs in [4] and [23] have the capability of solving some practical problems without full-state measurements. However, a rigorous and systematic study for partial-state feedback MRAC which has the capability of handling system parametric uncertainties for general linear time-invariant (LTI) systems is still not available in the literature.

From the above observations, a rigorous and systematic study for partial-state feedback MRAC which has the capability of handling system parametric uncertainties for general linear time-invariant (LTI) systems is still not available in the literature. Thus, one of the aims of this research is to propose and study a new model reference adaptive control framework using partial-state feedback for output tracking, which can achieve closed-loop system stability as well as asymptotic output tracking, in the presence of parameter uncertainties.

1.2.3 Fault-Tolerant Control

Fault-tolerant control algorithms are capable of eliminating severe consequences of various faults and maintaining desirable system stability and performance [6, 54, 84, 94]. For systems with system sensor failures, one typical type of methods, based on neural networks, is to detect the failed sensors and recover the correct measurement by collecting and comparing the measurements from both faulty sensors and correct sensors [25, 67, 96]. Such designs rely on the redundancy of sensors. Some recent results may avoid such a requirement. In [42], an adaptive sensor failure inverse (compensator) is developed and added into the closed-loop system to adaptively can-

cel the effects of sensor uncertainties for output tracking. In [34], a practical case is considered, wherein sensor bias is only present on the rate measurements but not the position measurements. For such a special case, an observer can be constructed based on the bias-free position measurements to estimate the actual values of the rate, in order to achieve asymptotic state tracking. In [26], a new error model is developed for updating the unknown parameters of a compensator and a feedback controller, for a multivariable control system being subject to sensor failures. For those results, fault estimates are needed, although they do not rely on sensor redundancy.

The development of fault-tolerant control methods for emerging technology applications, especially, for safety-critical applications, is essential. Quadrotor control attracts considerable attention in the past decades, one of a main challenges for quadrotor control for is sensor fault. Most of the control methods proposed in the literature are full-state feedback control method. However, the accurate measurement of the full state vector may not be available during the whole flight since the sensors are sensitive to temperature and vibration. Both additive faults (bias, drift, loss of accuracy) and one multiplicative fault (loss of effectiveness) may be existed in quadrotor control systems. In response to the sensor faults problem, sensor fusion techniques are widely utilized for quadrotor control applications. To make this technique function well, the quadrotor system has to equipment at least double sets of sensors, or even triple sets of sensors, which makes a large sensor redundancy, brings additional time delay, and increases technical difficulty for restrict stability analysis. Hence, researchers try to find direct control schemes to guarantee tracking control in the presence of sensor failures [1, 32, 50]. In [32], two adaptive fuzzy controllers are developed to compensate four different types of sensor faults for quadrotor position tracking and altitude tracking respectively. In [1], a sensor fault detection algorithm is proposed by building a neural network adaptive structure, with an assumption that all the system states can be correctly measured. In [50], a nonlinear compensation scheme is presented through feedback linearization technique for compensating a constant bias sensor measurement.

From the above observations, we conclude that failure detection and identification or failure estimation are required procedures for most of the sensor failure tolerant control designs. In this dissertation, we will propose new compensation schemes based on MRAC so as to improve the art of sensor failure compensation techniques.

1.2.4 Multi-Agent Consensus Control

Cooperative control of multi-agent systems has attracted considerable attention in recent years with the development of sensor networks [16, 20, 30, 38, 56, 57, 63, 64, 90]. Its applications, for instance, unmanned air vehicles (UAV) formation, robot position synchronization, and satellite altitude alignment, cover broad areas. The fundamental problem of cooperative control is to enable a group of agents to reach consensus (to converge to a common value on their states or outputs), which can be classified as leaderless consensus and leader-following consensus. A large number of control algorithms are proposed for consensus of multi-agent systems to deal with measurement noise [43, 48], time-delay [13, 81] and switching topology [45, 56, 90].

In the past years, leader-following consensus attracts more attention. Most of the pioneering results on leader-following consensus control are for some simple dynamics [29,69,91,97]. Recently, researchers started to investigate adaptive leader-following consensus problems for multi-agent systems where the agent dynamics are given in general linear systems [35,41,44,46,49,93] or given in nonlinear systems [15,64,95]. For linear time-invariant systems, efforts are dedicated to adaptive state feedback state consensus problem. In [46], leader-following state consensus of a multi-agent system

is achieved by a distributed adaptive state feedback disturbance compensation protocol. In [44, 49], state feedback adaptive control protocols are developed for identical followers to achieve state consensus of a multi-agent system with a graph having a spanning tree. Since state consensus requires strict matching conditions among the leader and all the followers that is not suitable for many applications, adaptive output consensus has been actively studied in the literature. In [35], without a stability analysis, an output feedback control protocol is presented with a distributed adaptive law to achieve leader-following output consensus of a multi-agent system with relative-degree-one followers where the graph has a spanning tree. In [41], a distributed adaptive protocol is presented for multi-agent systems with identical follower agents, for leader-following consensus with a constant reference signal. In [93], a distributed adaptive output feedback control scheme is developed for output consensus of a multi-agent system where the graph has a spanning tree.

1.3 Dissertation Outline

In this research, there are in total six MRAC problems, divided into two parts (the structure of this dissertation is illustrated in Fig. 1.2). The first part is to build up the technical foundation of partial-state feedback MRAC which includes

- a partial-state feedback model reference adaptive control scheme for single-input single-output (SISO) systems;
- a partial-state feedback model reference adaptive control scheme for multi-input multi-output (MIMO) systems; and
- an analysis on higher-order convergence properties of multivariable model reference adaptive control systems.

The second part is to apply the design ideas of partial-state feedback MRAC to develop

- a sensor failure compensation scheme for SISO systems;
- a sensor failure compensation scheme for MIMO systems; and
- an output consensus control scheme for multi-agent systems.



Figure 1.2: A schematic diagram for the structure of this dissertation.

The dissertation is organized as follows, where the major results have been documented and published in the journal and conference papers [71–73, 75–79].

• In Chapter 2, the fundamental MRAC problem is formulated which is the main control problem in this research, and different MRAC problems, incorporated with partial-state feedback control, fault-tolerant control, and consensus control, are formulated respectively.

- In Chapter 3, the adaptive partial-state feedback MRAC scheme is developed for SISO systems, which has the capability to make the closed-loop signals bounded and the plant output track the reference signal asymptotically in the presence of parameter uncertainties.
- In Chapter 4, the knowledge of the adaptive partial-state feedback control for SISO systems is expanded to MIMO systems, where some advanced control techniques are used to deal with the unique features of MIMO systems.
- In Chapter 5, a stronger higher-order convergence property of the multivariable MRAC systems is shown. It is proved that under the same MRAC design conditions, not only a tracking error component but its up to higher-order time-derivatives converge to zero.
- In Chapter 6, a sensor failure compensation scheme is designed for SISO systems based on the robustness of the partial-state feedback MRAC scheme that developed in Chapter 3, which can achieve asymptotic output tracking and closed-loop signal boundedness, in the presence of uncertain sensor failures and uncertain system parameters.
- In Chapter 7, a multivariable sensor failure compensation scheme is designed for quadrotor systems, which can achieve asymptotic output tracking and closedloop signal boundedness, in the presence of uncertain sensor failures for multivariable plants with an non-equilibrium off-set term; and
- In Chapter 8, a distributed output consensus control scheme is developed, by applying the partial-state feedback MRAC scheme developed in Chapter 3 to the multi-agent systems, which guarantees output consensus and closed-loop

signal boundedness in the multi-agent systems, and provides new additional design flexibilities to multi-agent control systems.

• In Chapter 9, conclusions of this research are given and some future research topics in this area are presented for discussion.

Chapter 2 Background and Problem Statement

As the control method that to be studied and explored in this dissertation, model reference adaptive control, with an introduction to some typical MRAC designs, is briefly reviewed in this chapter. Then two control applications are briefly formulated, where some essential background knowledge for the following control designs is introduced.

2.1 Model Reference Adaptive Control Problem

The basic feature of MRAC is to use control adaptation to make the output or state of an unknown plant to track that of a chosen reference model system. A brief introduction of the basic model reference adaptive control systems is given in this section. For completeness, a brief review of the existing MRAC designs for multiinput multi-output (MIMO) systems is given as follows.

Plant description. Consider an *M*-input and *M*-output linear time-invariant plant described by

$$\dot{x}(t) = Ax(t) + Bu(t), \ y(t) = Cx(t),$$
(2.1.1)

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times M}$ and $C \in \mathbb{R}^{M \times n}$ are unknown constant parameter matrices, and $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^M$ and $y(t) \in \mathbb{R}^M$ are the state, input and output vectors, respectively. The input-output description of the plant (2.1.1) is

$$y(t) = G(s)[u](t), \ G(s) = C(sI - A)^{-1}B.$$
 (2.1.2)

The notation, y(t) = G(s)[u](t), is used to denote the output y(t) of a system represented by a transfer function matrix G(s) with a control input signal u(t). It is a simple notation to combine both the time domain and the frequency domain signal operations, suitable for adaptive control system presentation.

For better understanding the control objective and the plant assumptions, we first introduce a crucial concept for multivariable MRAC designs, as defined in the following lemma.

Lemma 2.1.1. [82] For any $M \times M$ strictly proper and full rank rational matrix G(s), there exists a lower triangular polynomial matrix $\xi_m(s)$, defined as the modified left interactor (MLI) matrix of G(s), of the form

$$\xi_m(s) = \begin{bmatrix} d_1(s) & 0 & \dots & \dots & 0 \\ h_{21}^m(s) & d_2(s) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{M1}^m(s) & \dots & \dots & h_{MM-1}^m(s) & d_M(s) \end{bmatrix},$$
 (2.1.3)

where $h_{ij}^m(s)$, j = 1, ..., M - 1, i = 2, ..., M are polynomials, and $d_i(s)$, i = 1, ..., Mare monic stable polynomials of degrees $l_i > 0$, such that the high-frequency gain matrix of G(s), defined as $K_p = \lim_{s \to \infty} \xi_m(s)G(s)$ is finite and nonsingular.

This interactor matrix $\xi_m(s)$ characterizes the plant infinity zero structure of G(s), whose property of having a stable inverse is essential for MRAC designs. To better understand the plant infinite zero structure, we consider SISO systems here. For a SISO system, the plant infinite zero structure is characterized by the term s^{n-m} such that the system high-frequency gain $k_p = \lim_{s \to \infty} s^{n-m}G(s) \neq 0$. **Plant assumptions.** The assumptions for MRAC designs are given as follows.

- (A2.1) All zeros of $G(s) = C(sI A)^{-1}B$ are stable, and (A, B, C) is stabilizable and detectable.
- (A2.2) G(s) has full rank and its modified left interactor matrix $\xi_m(s)$ is known.

Assumption (A2.1) is for a stable plant-model output matching, and Assumption (A2.2) is for choosing a reference model system $W_m(s) = \xi_m^{-1}(s)$ suitable for plant-model output matching. Note that the zeros of G(s) are defined as the system transmission zeros (the values of s making G(s) nonsingular).

Control objective. The objective of multivariable MRAC is to construct a feedback control law for generating the control input signal u(t) in (2.1.1) such that all signals in the closed-loop system are bounded and the output vector y(t) asymptotically tracks a given reference output $y_m(t)$ generated from a reference model system

$$y_m(t) = W_m(s)[r](t), \ W_m(s) = \xi_m^{-1}(s),$$
 (2.1.4)

where $r(t) \in \mathbb{R}^M$ is a bounded reference input signal, and $\xi_m(s)$, defined in Lemma 2.1.1, is a modified left interactor matrix of the system transfer matrix $G(s) = C(sI - A)^{-1}B$, whose inverse matrix is stable, i.e., $W_m(s)$ is stable. The block diagram of an MRAC system is given in Fig. 2.1.

Typical multivariable MRAC designs. In the literature, there are two types of multivariable MRAC designs, namely, state feedback output tracking design and output feedback output tracking design.

(a) State feedback for output tracking. When the full state vector x(t) is available for measurement, the following simple adaptive controller structure can be used:

$$u(t) = K_1^{\mathrm{T}}(t)x(t) + K_2(t)r(t), \qquad (2.1.5)$$



Figure 2.1: Block diagram of a general MRAC system structure.

where $K_1(t) \in \mathbb{R}^{n \times M}$ and $K_2(t) \in \mathbb{R}^{M \times M}$ are controller parameters to be adaptively updated by stable adaptive laws. Such controller parameters $K_1(t)$ and $K_2(t)$ are the adaptive estimates of the nominal controller parameters K_1^* and K_2^* satisfying the matching condition

$$C(sI - A - BK_1^{*T})^{-1}BK_2^* = W_m(s), \ K_2^{*-1} = K_p,$$
(2.1.6)

with K_p being the system high-frequency gain matrix of G(s) for plant-model output matching: $y(t) = W_m(s)[r](t) = y_m(t)$. The existence of the nominal controller parameters K_1^* and K_2^* is guaranteed as long as the plant interactor matrix $\xi_m(s)$ is used for $W_m(s) = \xi_m^{-1}(s)$. In addition, to ensure the output tracking as well as the system internal signal boundedness, (A, B, C) needs to be stabilizable and detectable and all zeros of G(s) need to be stable [27].

(b) Output feedback for output tracking. In applications, when the full state vector x(t) is not accessible, the standard output feedback adaptive controller

$$u(t) = \Theta_1^{\mathrm{T}}(t)\omega_1(t) + \Theta_2^{\mathrm{T}}(t)\omega_2(t) + \Theta_{20}(t)y(t) + \Theta_3(t)r(t)$$
(2.1.7)

needs to be used, where

$$\omega_1(t) = \frac{A_0(s)}{\Lambda(s)} [u](t), \ \omega_2(t) = \frac{A_0(s)}{\Lambda(s)} [y](t)$$
(2.1.8)

with $A_0(s) = [I_M, sI_M, \dots, s^{\bar{\nu}-2}I_M]^{\mathrm{T}}$, $\Theta_1(t) \in \mathbb{R}^{(\bar{\nu}-1)M \times M}$, $\Theta_2(t) \in \mathbb{R}^{(\bar{\nu}-1)M \times M}$, $\Theta_{20}(t) \in \mathbb{R}^{M \times M}$, $\Theta_3(t) \in \mathbb{R}^{M \times M}$, $\bar{\nu}$ being the upper bound of the observability index of the plant, and $\Lambda(s)$ being a monic stable polynomial of degree $\bar{\nu} - 1$. To ensure the internal signal boundedness while achieving output tracking, it is needed that all zeros of G(s) are stable and (A, B, C) needs to be stabilizable and detectable.

Remark 2.1.1. Another MRAC system is the one which makes the plant-model state matching achievable using state feedback. The controller structure for state feedback state tracking is the same with the one for state feedback output tracking. The control objective is to make x(t) track $x_m(t)$ from a chosen stable reference model system $\dot{x}_m(t) = A_m x_m(t) + B_m r(t)$. However, the plant-model matching condition: $A + BK_1^{*T} = A_m, BK_2^* = B_m$, is restrictive for many applications, since the reference model parameters (A_m, B_m) needs to be chosen in advance. In this dissertation, we do not consider the state tracking problem. Please refer to [82] for details.

2.2 Partial-State Feedback MRAC Problem

In this section, the concept of partial-state signal is first clarified, and a basic partialstate feedback MRAC problem is then formulated.

Partial-state signal for feedback control. The central idea of feedback control is that a systems output can be measured and fed back to a controller of some kind and used to effect the control. It has been shown that signal feedback can be used to control a vast array of dynamic systems [21]. For a general feedback control system shown as Fig. 2.2, a partial-state signal can be represented as

$$y_0(t) = C_0 x(t). (2.2.1)$$

<u>Multiple possibilities of $y_0(t)$ </u>. For systems with single-input single-output, the partialstate signal $y_0(t)$ could be

- (i) a vector which contains the output y(t);
- (ii) a vector which does not contain the output y(t); and
- (iii) a scalar which is not equal to the output y(t).

For systems with multi-inputs multi-outputs, the partial-state signal $y_0(t)$ could be

- (i) $y_0(t)$ is a vector containing some or all elements of y(t);
- (ii) $y_0(t)$ is vector which does not contain any element of y(t);
- (iii) $y_0(t)$ is a scalar as one element of y(t); and
- (iv) $y_0(t)$ is a scalar not being any element of y(t).

In the studies of partial-state feedback MRAC, we assume a partial-state vector $y_0(t) = C_0 x(t)$ is available for measurement with the constant matrix C_0 unknown. The multiple possibilities of $y_0(t)$ shown above confirm that the partial-state feedback MRAC framework provides additional design flexibilities and feedback complicity.



Figure 2.2: A control system with partial-state feedback signal.

Partial-state feedback MRAC. The basic partial-state feedback control problem is to find a control law for generating the control signal u(t) in (2.1.1) by using the partial-state signal $y_0(t)$ to ensure closed-loop signal boundedness and output tracking, in the presence of parameter uncertainties.

The introduction of partial-state feedback signals allows the MRAC systems to take advantage of all possible systems measurements, which makes partial-state feedback MRAC provide additional possibilities for the choice of feedback signals. Also, from control implementation points of view, the partial-state MRAC framework can provide controller structures with

- less state information requirement than state feedback design; and
- less complex controller structure than output feedback design

From the in-depth study of the new MRAC framework to be shown next, there exists a manageable inverse relationship between the number of system measurements for feedback control and controller implementation complexity. In other words, besides the basic function of MRAC designs, the partial-state feedback MRAC framework also provides more choices, with the respective of the system measurements and controller implementation complexity, for control applications.

2.3 Sensor Failure Compensation Problem

There are two types of failures commonly seen in control systems. The first type is actuator failures such that the actuators will not respond to control signals. The other is sensor failures such that the sensors will not read the actual system state variables. In this dissertation, we will briefly investigate the compensation schemes for sensor failures.

Sensor failures. To formulate the sensor failure, first consider a linear timeinvariant plant (2.1.1): $\dot{x}(t) = Ax(t) + Bu(t), y(t) = Cx(t)$. We consider the case when a set of sensors $S_i, i = 1, 2, ..., n$, is used to measure the *n* state variables $x_i(t)$. In the presence of a fault at the *j*th sensor, the sensor output may be described as

$$z_j = S_j(x_j) = \begin{cases} x_j & \text{with the healthy sensor } S_j \\ \bar{s}_j & \text{with the failed sensor } S_j \end{cases}$$
(2.3.1)

for some unknown bounded values \bar{s}_j with unknown indices $j \in \{1, 2, ..., n\}$. Thus,

for the state vector x(t) constructed by the *n* state variables x_i , the sensor output vector with possible uncertain state sensor failures is $z(t) = [z_1, z_2, \ldots, z_n]^{\mathrm{T}}$.

This failure model characterizes the most typical classes of sensor failures that may occur, that is, some unknown sensor outputs are stuck at some unknown fixed or varying values. For example, the humidity-sensitive pressure sensor may be stuck at some unknown values due to water or moisture. In addition, the state sensor failures investigated in this dissertation are uncertain, which means we do not know which sensors are failed, how much the failures are and when the failures occur. Such uncertain state sensor failures require effective adaptive compensation in the design of a control scheme to guarantee desired system performance.

Failure patterns. For the *n* sensors corresponding to the *n* state variables x_i , i = 1, 2, ..., n, there are different sensor failure patterns which can be represented by a generic failure pattern matrix

$$\sigma = \operatorname{diag}\{\sigma_1, \sigma_2, \dots, \sigma_n\}$$
(2.3.2)

where $\sigma_i = 1$ if the *i*th sensor fails and $\sigma_i = 0$ if the *i*th sensor is healthy. With such a matrix, we can express the sensor output vector as

$$z(t) = x(t) - \sigma(x(t) - \bar{s})$$
(2.3.3)

where $\bar{s} = [\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n]^{\mathrm{T}}$ is defined as a failure vector. For each individual failure pattern, we denote σ as $\sigma = \sigma_{(k)}$, for $k = 0, 1, 2, \ldots, N-1$, where N is the total number of different sensor failure patterns including the no failure pattern, and correspondingly, denote \bar{s} as $\bar{s} = \bar{s}_{(k)}$. We use $\sigma_{(0)} = \text{diag}\{0, 0, \ldots, 0\}$ and $\bar{s}_{(0)} = [0, 0, \ldots, 0]^{\mathrm{T}}$ to represent the no failure case (for the k = 0 case).

Sensor failure compensation problem. The basic sensor failure compensation control problem is to find a control law for generating the control signal u(t) in (2.1.1) to ensure closed-loop signal boundedness and output tracking, in the presence of sensor failure (2.3.3).

2.4 Leader-Following Output Consensus Control Problem

In this section, a leader-follower system is introduced, and some preliminary topology knowledge is given for describing the information flow among the agents.

Leader-follower multi-agent systems. Consider a multi-agent system including N followers and one virtual leader. The *i*th followers' dynamic equation is

$$\dot{x}_i(t) = A_i x_i(t) + b_i u_i(t), \quad y_i(t) = b_i x_i(t), \ i = 1, \dots, N,$$
(2.4.1)

for the unknown parameter matrices $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times 1}$ and $C_i \in \mathbb{R}^{1 \times n}$, where $x_i(t) \in \mathbb{R}^n$ is the state vector of the *i*th follower, $u_i(t) \in \mathbb{R}$ is the control input of the *i*th follower, and $y_i(t) \in \mathbb{R}$ is the output of the *i*th follower. The input-output description of each follower is

$$y_i(t) = G_i(s)[u_i](t), G_i(s) = k_{pi} \frac{Z_i(s)}{P_i(s)}, i = 1, \dots, N$$
 (2.4.2)

where $k_{pi} \neq 0$, $P_i(s) = \det(sI - A_i) = s^n + p_{(n-1)i}s^{n-1}\cdots + p_{1i}s + p_{0i}$, and $Z(s) = s^m + \cdots + z_{1i}s + z_{0i}$ for some $m \geq 0$. The notation: y(t) = G(s)[u](t), is used to denote the output y(t) of a LTI system represented by a transfer function G(s) with input signal u(t).

The dynamic model of the virtual leader is given by

$$y_l(t) = W_l(s)[r](t), W_l(s) = \frac{1}{P_l(s)},$$
(2.4.3)

where $P_l(s)$ is a desired stable polynomial of degree $n^* = n - m$ (the followers' relative degree n^* is assumed to be known), and r(t) is a bounded piecewise continuous reference input signal. It is natural to model the information exchange between agents by algebraic graph theory which is introduced next.

Graph theory. The information exchange among the N follower agents in this dissertation is denoted by a undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with a set of nodes \mathcal{V} , a set of undirected edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the adjacency matrix of the graph $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$. The node v_i represents the *i*th follower agent. An unordered edge $(v_i, v_j) \in \mathcal{E}$ (or equivalently $(v_j, v_i) \in \mathcal{E}$) represents that the information exchanges between the agents v_i and v_j , and v_i and v_j are neighbors. In addition, $(v_i, v_j) \in \mathcal{E}$ follows that the adjacency element $a_{ij} = a_{ji} = 1$. A path is a sequence of unordered edge edges of the form $(v_{i1}, v_{i2}), (v_{i2}, v_{i3}), \ldots$, in a graph, where $v_{ij} \in \mathcal{V}$. If, for any two nodes $v_i, v_j \in \mathcal{V}$, there is a path between them, then \mathcal{G} is called a connected graph.

To describe the information exchange from the leader to the followers, we denote the leader as v_0 . Let $\mathcal{V}_{\Sigma} = \{\mathcal{V}, v_0\}$ be the node set consisting of all the follower agents and the leader. Since the leader v_0 can not be affected by the followers v_i , the connection edges (v_i, v_0) between the leader v_0 and the *i*th agent v_i are directed which means that the follower v_i can obtain the information from v_0 , but not vice versa. Let \mathcal{E}_l be the edges set consisting of all edges (v_i, v_0) . Define $\mathcal{N}_i = \{v_j \in \mathcal{V}_{\Sigma} : (v_j, v_i) \in$ $\{\mathcal{E} \bigcup \mathcal{E}_l\}\}$ as the neighborhood of the *i*th follower and $\mathcal{N}_0 = \{v_j \in \mathcal{V} : (v_j, v_0) \in \mathcal{E}_l\}$ as the set of follower agents that are directly connected to the leader.

Output consensus problem. For the multi-agent system consisting of (2.4.1) and (2.4.3), the basic output consensus control objective is to find a control protocol for generating the control signal $u_i(t)$ in (2.4.1) for each follower such that all the signals in the multi-agent system are bounded and the output of all followers track the output of the given leader asymptotically.
Chapter 3

Partial-State Feedback MRAC for SISO Systems

This chapter presents the novel MRAC scheme using *partial-state feedback signal* for *achieving output tracking* for uncertain single-input single-output systems, where we

- develop a nominal partial-state feedback model reference control scheme;
- develop two partial-state feedback MRAC designs for relative-degree-one plants and general plants, respectively;
- analyze plant-model matching, stability and tracking performance; and
- verify the effectiveness of the designs by simulation studies.

The developed partial-state feedback MRAC scheme has less restrictive matching conditions, less state information requirement, and less complex controller structure compared to an output feedback MRAC design, which can achieve closed-loop system stability as well as asymptotic output tracking.

3.1 Review of MRAC and Problem Statement

In this section, first, MRAC designs using state feedback or output feedback are reviewed and their features are discussed. In order to make a full use of all possibly available system signals, which can reduce the controller complexity and improve system response performance, we propose to study a new MRAC scheme with partialstate feedback for output tracking. The problem statement is given, following the review of the existing MRAC designs.

3.1.1 Review of MRAC

The basic feature of MRAC is to use control adaptation to make the output or state of an unknown plant to track that of a chosen reference model system. There are two typical MRAC designs for output tracking: design using state feedback and design using output feedback. A review of the existing MRAC designs is given as follows.

Plant description. Consider a linear time-invariant plant:

$$\dot{x}(t) = Ax(t) + bu(t), \ x(t) \in \mathbb{R}^{n}, u(t) \in \mathbb{R}$$
$$y(t) = cx(t), \ y(t) \in \mathbb{R}$$
(3.1.1)

with $x(0) = x_0$, where $A \in \mathbb{R}^{n \times n}$ is an unknown matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^{1 \times n}$ are unknown vectors. The input-output description of the plant (3.1.1) is

$$P(s)[y](t) = k_p Z(s)[u](t), \qquad (3.1.2)$$

where $k_p \neq 0$, $P(s) = \det(sI - A) = s^n + p_{n-1}s^{n-1}\cdots + p_1s + p_0$, and $Z(s) = s^m + \cdots + z_1s + z_0$ for some $m \geq 0$. We will use the notation: y(t) = G(s)[u](t), to denote the output y(t) of a LTI system represented by a transfer function G(s) with input signal u(t). The symbol s is a differentiation operator: $s[x](t) = \dot{x}(t)$, or the

Laplace transform variable as the case may be. It is a simple notation to combine both time domain and frequency domain signal operations, suitable for adaptive control system presentation.

Goal of MRAC. The goal of output tracking MRAC is to construct a feedback control law for u(t) in (3.1.1) such that all signals in the closed-loop system are bounded and the system output y(t) asymptotically tracks a given reference output signal $y_m(t)$ generated from a reference model system

$$y_m(t) = W_m(s)[r](t), \ W_m(s) = \frac{1}{P_m(s)},$$
(3.1.3)

where $P_m(s)$ is a desired stable polynomial of degree n - m (which is assumed to be known), and r(t) is a bounded piecewise continuous reference input signal.

State feedback for output tracking. In applications when the full-state vector x(t) is available, a simple adaptive controller structure is

$$u(t) = k_1^{\mathrm{T}}(t)x(t) + k_2(t)r(t), \qquad (3.1.4)$$

where $k_1(t) \in \mathbb{R}^n$ and $k_2(t) \in \mathbb{R}$ are to be adaptively updated. Such an adaptive controller with stable adaptive laws to update the controller parameter $k_1(t)$ and $k_2(t)$ has the capability of driving the control plant output y(t) to track $y_m(t)$ generated from the reference model (3.1.3) asymptotically. The controller parameters $k_1(t)$ and $k_2(t)$ are the adaptive estimates of the nominal controller parameters k_1^* and k_2^* depending on the unknown system parameters (A, b, c) and satisfying

$$\det(sI - A - bk_1^{*\mathrm{T}}) = P_m(s)Z(s)\frac{1}{k_p}, \ k_2^* = \frac{1}{k_p},$$
(3.1.5)

to achieve stable plant-model output matching: $y(t) = W_m(s)[r](t) = y_m(t)$, if $k_1(t) = k_1^*$ and $k_2(t) = k_2^*$ in (3.1.4). The existence of k_1^* and k_2^* is guaranteed by the degree of $P_m(s)$ being n - m and by Z(s) being a stable polynomial (a basic assumption of

MRAC design for output tracking for internal stability) [53]. Note that Z(s) is stable also implies that (A, b, c) is stabilizable and detectable.

Output feedback for output tracking. In applications when x(t) is not accessible for measurement, we can use an output feedback adaptive controller structure:

$$u(t) = \theta_1^{\mathrm{T}}(t)\omega_1(t) + \theta_2^{\mathrm{T}}(t)\omega_2(t) + \theta_{20}(t)y(t) + \theta_3(t)r(t), \qquad (3.1.6)$$

where $\omega_1(t) = \frac{a(s)}{\Lambda(s)}[u](t)$, $\omega_2(t) = \frac{a(s)}{\Lambda(s)}[y](t)$, with $a(s) = [1, s, \dots, s^{n-2}]^T$, $\theta_1(t) \in \mathbb{R}^{n-1}$, $\theta_2(t) \in \mathbb{R}^{n-1}$, $\theta_{20}(t) \in \mathbb{R}$, $\theta_3(t) \in \mathbb{R}$, and $\Lambda(s)$ is a monic stable polynomial of degree n-1. Adaptive controller parameters $\theta_1(t)$, $\theta_2(t)$, $\theta_{20}(t)$ and $\theta_3(t)$ are the estimates of the constant nominal controller parameters $\theta_1^*, \theta_2^*, \theta_{20}^*$ and θ_3^* depending on system parameters and satisfying the desired matching equation:

$$\theta_1^{*\mathrm{T}}a(s)P(s) + (\theta_2^{*\mathrm{T}}a(s) + \theta_{20}^{*}\Lambda(s))k_pZ(s) = \Lambda(s)(P(s) - k_p\theta_3^{*}Z(s)P_m(s)), \quad (3.1.7)$$

for plant-model output matching: $y(t) = W_m(s)[r](t) = y_m(t)$. To ensure the internal signal boundedness and achieving output tracking, Z(s) needs to be a stable polynomial.

Research motivation. In summary, stable output matching can always be achieved with Z(s) being a stable polynomial and the relative degree of the system n - m known. However, to implement a state feedback controller, the full-state vector x(t) is needed which may not be practical in many applications, while for an output feedback controller, its complexity with the filters $\frac{a(s)}{\Lambda(s)}$ for generating $\omega_1(t)$ and $\omega_2(t)$ in (3.1.6) may be an issue of control implementation for some applications (see Remark 3.4.1 for details).

This motivates our new research on developing partial-state feedback MRAC designs whose controller structures are simpler than an output feedback control design and whose implementations do not need full-state measurements, to provide additional design flexibility and feedback capacity, as formulated next.

3.2 Problem Statement

For the plant (3.1.1): $\dot{x}(t) = Ax(t) + bu(t)$, y(t) = cx(t), with (A, b, c) unknown and $c(sI - A)^{-1}b = k_p \frac{Z(s)}{P(s)}$, the goal of this chapter is to design a partial-state feedback MRAC scheme. We make the following basic assumptions:

- (A3.1) A vector signal $y_0(t) = C_0 x(t) \in \mathbb{R}^{n_0}$ is available for measurement, with (A, C_0) observable for $C_0 \in \mathbb{R}^{n_0 \times n}$ and $\operatorname{rank}[C_0] = n_0$; and
- (A3.2) all zeros of Z(s) are stable.

For Assumption (A3.1), the vector signal $y_0(t)$ can be a subset of components of x(t) or a linear combination of them. Assumption (A3.2) is a basic assumption of MRAC designs for guaranteeing internal stability while achieving output tracking.

Partial-state feedback MRAC. The objective of partial-state feedback model reference adaptive control is to design a control law using the partial-state vector $y_0(t)$ to generate the control signal u(t) to make all the closed-loop signals bounded and the plant output y(t) asymptotically track a reference output signal $y_m(t)$ generated from (3.1.3): $y_m(t) = W_m(s)[r](t)$.

Technical issues. We will solve three new technical issues for partial-state feedback MRAC which are described in details as follows:

Issue I: Output matching using partial-state feedback. To achieve the stated control objective, a new controller structure, capable of ensuring plant-model output matching: $y(t) = W_m(s)[r](t)$ for (A, b, c) known, and suitable for controller adaptation for $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$ for (A, b, c) unknown, is needed. In other words, for partial-state feedback MRAC, a controller structure, which is able to ensure plantmodel output matching, needs to be developed.

<u>Issue II: Adaptation of partial-state feedback controller.</u> To achieve desired output tracking in the presence of parameter uncertainties, a partial-state adaptive controller with stable adaptive laws for parameter adaptation is needed. To derive the stable adaptive laws, it is crucial to obtain a tracking error equation based on the partial-state vector $y_0(t)$.

<u>Issue III: Stability analysis and performance evaluation.</u> To prove the effectiveness of the proposed adaptive partial-state feedback control scheme, closed-loop system stability and tracking performance analysis is to be conducted.

Different from the state feedback and output feedback cases, the partial-state vector $y_0(t)$ is a subset of components of x(t) or a linear combination of them so that the formation of $y_0(t)$ has three possibilities:

(i) $y_0(t)$ being a vector which contains the output y(t);

(ii) $y_0(t)$ being a vector which does not contain the output y(t); and

(iii) $y_0(t)$ being a scalar which is not equal to the output y(t).

Remark 3.2.1. The proposed method in our work solves an adaptive output tracking problem using the partial-state signal $y_0(t) = C_0 x(t)$ for general LTI systems with parameter uncertainties. The application of such a proposed control design is broader due to the absence of the need of an explicit state transformation, compared to the partial-state feedback design in Section 7.4.2 of [37]. In particular, comparing the proposed control scheme to the work shown in Section 7.4.2 of [37], it is worth noting that the partial-state $y_0(t)$ in our work could be either a subset of the state variables or a linear combination of them or even not contain the output y(t).

Remark 3.2.2. The partial-state feedback adaptive control problem can also be

illustrated for nonlinear systems, which helps to understand the problem novelty in reducing the complexity of control laws. It is known that for a nonlinear system

$$\dot{z} = f(z) + g(z)u, \ z \in \mathbb{R}^n, \tag{3.2.1}$$

under certain conditions [37], there exists a state transformation x = T(z) such that the system can be transformed into an output-feedback form

$$\dot{x}_{i} = x_{i+1} + \varphi_{i0}(y) + \sum_{j=1}^{q_{i}} a_{ij}\varphi_{ij}(y), \ i = 1, \dots, \rho - 1
\dot{x}_{\rho} = x_{\rho+1} + \varphi_{\rho0}(y) + \sum_{j=1}^{q_{\rho}} a_{\rho j}\varphi_{\rho j}(y) + b_{n-\rho}\sigma(y)u,
\dot{x}_{i} = x_{i+1} + \varphi_{i0}(y) + \sum_{j=1}^{q_{i}} a_{ij}\varphi_{ij}(y) + b_{n-i}\sigma(y)u,
i = \rho + 1, \dots, n
y = x_{1}$$
(3.2.2)

with $x_{n+1} = 0$, where a_{ik} and b_j are unknown parameters, and φ_{ij} and σ are known nonlinear functions.

Adaptive state-feedback control of the system (3.2.1) (or its canonical forms such as a parametric-feedback form) and adaptive output-feedback control of the system (3.2.2), for output tracking, have been extensively studied in the literature. In particular, a full-order state observer has been used for an output feedback control design for (3.2.2). Furthermore, a partial-state feedback control scheme is proposed in [37] for systems of a fixed canonical form (3.2.1) of (3.2.2).

Being different with the partial-state feedback problem solved in [37] where the feedback signal is a subset of state variables in x(t) or z(t), our partial-state feedback control problem of the system (3.2.1) may be stated as: how to effectively solve an adaptive output tracking problem without the full knowledge of z(t) but with some partial-state knowledge of $y_0(t) = h_0(z(t))$, in addition to the knowledge of the system output y(t) (for systems not in a canonical form)? Some technical issues are: what are the new conditions under which the system (3.2.1) can be transformed to one similar but more general system than (3.2.2), with the additional knowledge of $y_0(t)$ explicitly contained (for example, those functions $\varphi_{ik}(y)$ are replaced by $\varphi_{ik}(y_0)$)? how can the adaptive output tracking control problem be solved using a simpler controller structure based on a partial-state observer, less complex than that used for the system model (3.2.2) with a full-order observer.

In this thesis, we will only solve some related technical issues for adaptive partialstate feedback model reference control of linear systems, which can be helpful for understanding the nonlinear system case. \Box

3.3 Nominal Partial-State Feedback Control

This section solves the first technical issue: plant-model output matching using partial-state feedback, by developing a partial-state observer and deriving a parametrized nominal partial-state feedback controller. With such a nominal partial-state feedback controller, several desired plant-model output matching properties are established.

3.3.1 Partial-State Observer

When the full-state vector x(t) is not available for measurement, the control law u(t) is constructed with an estimate $\hat{x}(t)$ (generated from a state estimator or observer) of the state vector x(t): $u(t) = k_1^{*T} \hat{x}(t) + k_2^* r(t)$. In partial-state feedback control problem, a partial-state vector $y_0(t)$ is measurable, we start the partial-state feedback control controller derivation from developing a partial-state observer using $y_0(t)$, in order to obtain an estimate $\hat{x}(t)$ of the state vector x(t).

State transformation. For the state equation: $\dot{x}(t) = Ax(t) + bu(t)$, as the design in [9], introducing a transformation matrix $P \in \mathbb{R}^{n \times n}$ such that $C_0 P^{-1} = [I_{n_0}, 0]$ with $n_0 = \operatorname{rank}[C_0]$, we can transfer it to

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} u(t),$$
(3.3.1)

where $\bar{x}(t) = Px(t) = [\bar{x}_1^{\mathrm{T}}(t), \bar{x}_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ with $\bar{x}_1(t) \in \mathbb{R}^{n_0}, \bar{x}_2(t) \in \mathbb{R}^{n-n_0}, \bar{A}_{11} \in \mathbb{R}^{n_0 \times n_0},$ $\bar{A}_{12} \in \mathbb{R}^{n_0 \times (n-n_0)}, \bar{A}_{21} \in \mathbb{R}^{(n-n_0) \times n_0}, \bar{A}_{22} \in \mathbb{R}^{(n-n_0) \times (n-n_0)}, \bar{b}_1 \in \mathbb{R}^{n_0} \text{ and } \bar{b}_2 \in \mathbb{R}^{n-n_0}.$ With such a transformation, the available vector signal $y_0(t) = C_0 x(t) = \bar{x}_1(t)$, and only $\bar{x}_2(t)$ is to be estimated.

Estimation of $\bar{x}(t)$. We generate an estimate $\hat{x}(t)$ for $\bar{x}(t)$ with a reduced-order dynamic system generating an estimate $\hat{x}_2(t)$ for $\bar{x}_2(t)$, in the form:

$$\hat{\bar{x}}(t) = \left[\bar{x}_1^{\mathrm{T}}, \hat{\bar{x}}_2^{\mathrm{T}}\right]^{\mathrm{T}} = \left[y_0^{\mathrm{T}}(t), (w(t) + L_r y_0(t))^{\mathrm{T}}\right]^{\mathrm{T}},$$
(3.3.2)

where $L_r \in \mathbb{R}^{(n-n_0) \times n_0}$ is a constant gain matrix such that the eigenvalues of the $(n-n_0) \times (n-n_0)$ matrix $\bar{A}_{22} - L_r \bar{A}_{12}$ are stable and prespecified, and $w(t) \in \mathbb{R}^{n-n_0}$ is generated from the observer equation

$$\dot{w}(t) = (\bar{A}_{22} - L_r \bar{A}_{12})w(t) + (b_2 - L_r b_1)u(t) + ((\bar{A}_{22} - L_r \bar{A}_{12})L_r + \bar{A}_{21} - L_r \bar{A}_{11})y_0(t),$$

$$w(0) = w_0.$$
(3.3.3)

From (3.3.1)–(3.3.3), the estimated error $\tilde{\bar{x}}_2(t) = \bar{x}_2(t) - \hat{\bar{x}}_2(t)$ satisfies $\tilde{\bar{x}}_2(t) = (\bar{A}_{22} - L_r \bar{A}_{12})\tilde{\bar{x}}_2(t)$, which decays to zero exponentially since $\bar{A}_{22} - L_r \bar{A}_{12}$ is stable.

Estimation of x(t). Together with $\hat{x}_1 = \bar{x}_1 = y_0$, we have $\lim_{t\to\infty}(\bar{x}(t) - \hat{x}(t)) = 0$. Finally, with $\hat{x}(t) = P^{-1}\hat{x}(t)$, it is ensured that $\lim_{t\to\infty}(x(t) - \hat{x}(t)) = \lim_{t\to\infty} P^{-1}(\bar{x}(t) - \hat{x}(t)) = 0$ exponentially, at a prespecified rate, as desired.

3.3.2 Parametrized Partial-State Feedback Controller

Based on the technique in [9], the estimate $\hat{x}(t)$ converges to x(t) exponentially by a partial-state observer designed in the above subsection. Hence plant-model output matching can be achievable by the observer-based control law $u(t) = k_1^{*T} \hat{x}(t) + k_2^{*} r(t)$ as a nominal control law $u(t) = k_1^{*T} x(t) + k_2^* r(t)$ does it. However, since $\hat{x}(t)$ also depends on the unknown plant parameters, such an observer-based control law needs to be reparameterized, for the design of an adaptive control scheme for the unknown plant case.

We are now developing a parameterized partial-state feedback controller based on the partial-state observer.

Reparameterization of $u(t) = k_1^{*T} \hat{x}(t) + k_2^* r(t)$. We first express the partialstate estimate w(t) in (3.3.3) as

$$w(t) = (sI - \bar{A}_{22} + L_r \bar{A}_{12})^{-1} (\bar{b}_2 - L_r \bar{b}_1) [u](t) + (sI - \bar{A}_{22} + L_r \bar{A}_{12})^{-1} \times ((\bar{A}_{22} - L_r \bar{A}_{12}) L_r + \bar{A}_{21} - L_r \bar{A}_{11}) [y_0](t) + e^{(\bar{A}_{22} - L_r \bar{A}_{12})t} w(0) = \frac{n_1(s)}{\Lambda(s)} [u](t) + \frac{n_2(s)}{\Lambda(s)} [y_0](t) + e^{(\bar{A}_{22} - L_r \bar{A}_{12})t} w(0),$$
(3.3.4)

where $w(0) \in \mathbb{R}^{n-n_0}$ is an estimate of $L_r y_0(0) - \bar{x}_2(0)$, $\Lambda(s) = \det(sI - \bar{A}_{22} + L_r \bar{A}_{12})$ whose degree is $n - n_0$ and stability properties can be prespecified and by assigning the eigenvalues of $\bar{A}_{22} - L_r \bar{A}_{12}$ as a set of given (known) values, and $n_1(s)$ is an $(n-n_0) \times 1$ polynomial vector and $n_2(s)$ is an $(n-n_0) \times n_0$ polynomial matrix, whose maximum degrees are $n - n_0 - 1$ or less.

Using (3.3.2) and (3.3.4), we can express $k_1^{*T}\hat{x}(t)$ as

$$k_1^{*\mathrm{T}}\hat{x}(t) = \theta_1^{*\mathrm{T}}\frac{a_1(s)}{\Lambda(s)}[u](t) + \theta_2^{*\mathrm{T}}\frac{A_2(s)}{\Lambda(s)}[y_0](t) + \varepsilon_0(t), \qquad (3.3.5)$$

for some exponentially decaying signal $\varepsilon_0(t)$ representing the effect of the initial condition w(0), where $\theta_1^* \in \mathbb{R}^{n-n_0}$, $\theta_2^* \in \mathbb{R}^{n_0(n-n_0)}$ and $\theta_{20}^* \in \mathbb{R}^{n_0}$, such that $\theta_{20}^{*T} = k_{p1}^{*T} + k_{p2}^{*T}L_r$, $k_{p2}^{*T}n_1(s) = \theta_1^{*T}a_1(s)$ and $k_{p2}^{*T}n_2(s) = \theta_2^{*T}A_2(s)$, for $k_1^{*T}P^{-1} = [k_{p1}^{*T}, k_{p2}^{*T}]$ with $k_{p1}^* \in \mathbb{R}^{n_0}$ and $k_{p2}^* \in \mathbb{R}^{n-n_0}$, and $a_1(s) = [1, s, \dots, s^{n-n_0-1}]^T$, $A_2(s) = [I_{n_0}, sI_{n_0}, \dots, s^{n-n_0-1}I_{n_0}]^T$. Then, we express the observer-based control law $u(t) = k_1^{*\mathrm{T}} \hat{x}(t) + k_2^{*} r(t)$ as

$$u(t) = \theta_1^{*\mathrm{T}} \frac{a_1(s)}{\Lambda(s)} [u](t) + \theta_2^{*\mathrm{T}} \frac{A_2(s)}{\Lambda(s)} [y_0](t) + \theta_{20}^{*\mathrm{T}} y_0(t) + \theta_3^{*} r(t) + \varepsilon_0(t), \qquad (3.3.6)$$

with $\theta_3^* = k_2^* \in \mathbb{R}$.

Ignoring the decaying term $\varepsilon_0(t)$ in (3.3.6), we obtain the parametrized nominal partial-state feedback controller:

$$u(t) = \theta_1^{*T} \omega_1(t) + \theta_2^{*T} \omega_2(t) + \theta_{20}^{*T} y_0(t) + \theta_3^{*} r(t), \qquad (3.3.7)$$

where $\omega_1(t) = \frac{a_1(s)}{\Lambda(s)}[u](t), \ \omega_2(t) = \frac{A_2(s)}{\Lambda(s)}[y_0](t).$

3.3.3 Partial-State Feedback Based Output Matching

Recall that the first control problem is to achieve plant-model output matching by partial-state feedback control. In this section, several output matching properties of the partial-state feedback controller (3.3.7) are presented.

Theorem 3.3.1. There exist constant parameters θ_1^* , θ_2^* , θ_{20}^* satisfying (3.3.5) and $\theta_3^* = k_p^{-1}$ such that the controller (3.3.7) ensures that all signals in the closed-loop system are bounded and partial-state feedback based output matching: $y(t) - y_m(t) = \varepsilon(t)$, for some initial condition-related exponentially decaying $\varepsilon(t)$, is achieved, where $y_m(t)$ is the output of the reference model (3.1.3).

Proof: The proof is divided into four steps.

<u>Step 1: Output matching by state feedback control.</u> The transfer function of the closed-loop system consisting of (3.1.1) and the nominal control law $u(t) = k_1^{*T}x(t) + k_2^*r(t)$ becomes

$$G_c(s) = c(sI - A - bk_1^{*T})^{-1}bk_2^* = \frac{k_2^*Z(s)}{\det(sI - A - bk_1^{*T})},$$
(3.3.8)

such that the system output $y(t) = G_c(s)[r](t)$. The desired output matching requires that $G_c(s) = W_m(s)$ where $W_m(s)$ is the transfer function of the reference model (3.1.3), so we need to establish

$$c(sI - A - Bk_1^{*T})^{-1}bk_2^* = \frac{k_2^*Z(s)}{\det(sI - A - bk_1^{*T})} = \frac{1}{P_m(s)} = W_m(s).$$
(3.3.9)

Let $k_2^* = \frac{1}{k_p}$, it follows that

$$\det(sI - A - bk_1^{*T}) = Z(s)P_m(s)\frac{1}{k_p}.$$
(3.3.10)

If (A, b) is controllable, according to the linear system theory, there exists a vector k_1^{*T} such that the closed-loop system eigenvalues can be placed in arbitrary locations, which means that (3.3.10) is achievable.

If (A, b) is only stabilizable, zero-pole cancellation exists between det(sI - A) and Z(s) for the stable uncontrollable modes of A. Letting $Z(s) = Z_c(s)Z_{\bar{c}}(s)$ where the zeros of $Z_c(s)$ are the controllable modes of A and the zeros of $Z_{\bar{c}}(s)$ are the uncontrollable modes of A, with the degree of n_c and $n_{\bar{c}} = m - n_c$, respectively $(n_c + n_{\bar{c}} = m)$, we have

$$c(sI - A - bk_1^{*\mathrm{T}})^{-1}bk_2^* = \frac{k_2^* Z_c(s) Z_{\bar{c}}(s)}{\det(sI - A - bk_1^{*\mathrm{T}})} = \frac{1}{P_m(s)}.$$
(3.3.11)

Due to the incapability of k_1^* of moving the uncontrollable modes of A, there exists zero-pole cancellation between det $(sI - A - bk_1^{*T})$ and $Z_c(s)Z_{\bar{c}}(s)$ to cancel out the uncontrollable modes $Z_{\bar{c}}(s)$. We can express the result after zero-pole cancellation as

$$c(sI - A - bk_1^{*\mathrm{T}})^{-1}bk_2^* = \frac{k_2^* Z_c(s)}{(s - \alpha_1) \cdots (s - \alpha_{n-n_c})} = \frac{1}{P_m(s)},$$
(3.3.12)

with α_i , $i = 1, \ldots, n - n_{\bar{c}}$ representing some closed-loop poles which can be arbitrarily placed. In other words, there exists a vector k_1^* to move the poles $\alpha_1, \ldots, \alpha_{n_c}$ to match the zeros of $Z_c(s)P_m(s)\frac{1}{k_p}$. Therefore, (3.3.10) is achievable as well even if (A, b) is only stabilizable. In summary, there exists a $k_1^* \in \mathbb{R}^n$ and a $k_2^* \in \mathbb{R}$ satisfying the matching equation (3.3.9) to ensure the desired matching $G_c(s) = W_m(s)$.

<u>Step 2: Output matching by observer-based feedback control.</u> According to the properties of the partial-state observer, we express $\hat{x}(t)$ as $x(t) + \varepsilon_1(t)$ with some exponentially decaying $\varepsilon_1(t)$ such that the observer-based control law can be expressed as $u(t) = k_1^{*T} x(t) + k_1^{*T} \varepsilon_1(t) + k_2^{*T} r(t)$. Substituting such a control law into the plant (3.1.1), the output becomes

$$y(t) = \frac{c \operatorname{adj}(sI - A - bk_1^{*\mathrm{T}})bk_1^{*\mathrm{T}}}{\det(sI - A - bk_1^{*})} [\varepsilon_1](t) + \frac{c \operatorname{adj}(sI - A - bk_1^{*\mathrm{T}})bk_2^{*}}{\det(sI - A - bk_1^{*})} [r](t). \quad (3.3.13)$$

Ignoring the exponentially decaying term, we conclude that the transfer function of the closed-loop system consisting of the plant (3.1.1) and the observer-based control law $u(t) = k_1^{*T} \hat{x}(t) + k_2^* r(t)$ is the same as the one of the closed-loop system consisting of (3.1.1) and the state feedback control law $u(t) = k_1^{*T} x(t) + k_2^* r(t)$, which is $Y(s)/R(s) = c(sI - A - bk_1^{*T})^{-1}bk_2^*$. Therefore, the nominal parameters k_1^* and k_2^* which make plant-model output matching achievable by state feedback control law can also ensure output matching when used for an observer-based control law.

<u>Step 3: Output matching by partial-state feedback control.</u> From (3.3.6), we obtain the nominal partial-state feedback control law (3.3.7) by ignoring the exponentially decaying $\varepsilon_0(t)$:

$$u(t) = \theta_1^{*T} \omega_1(t) + \theta_2^{*T} \omega_2(t) + \theta_{20}^{*T} y_0(t) + \theta_3^{*} r(t)$$
(3.3.14)

(to get this, we have expressed the term $k_1^{*T}\hat{x}(t)$ as shown in (3.3.5):

$$k_1^{*\mathrm{T}}\hat{x}(t) = \theta_1^{*\mathrm{T}}\omega_1(t) + \theta_2^{*\mathrm{T}}\omega_2(t) + \theta_{20}^{*\mathrm{T}}y_0(t) + \varepsilon_0(t),$$

which is a new parametrization of $k_1^{*T}\hat{x}(t)$).

Hence the existence of the constant parameters θ_1^* , θ_2^* , θ_{20}^* and θ_3^* for output matching by the above nominal partial-state feedback control law is ensured if they satisfy (3.3.5).

The existence of $\theta_1^* \in \mathbb{R}^{n-n_0}$ is guaranteed for $\theta_1^{*T}a_1(s) = k_{p2}^{*T}n_1(s)$, since the polynomial $n_1(s)$ in (3.3.4) for has degree $n - n_0$, with $a_1(s) = [1, s, \dots, s^{n-n_0-1}]^T$. Similarly, the existence of θ_2^* is guaranteed for $\theta_2^{*T}A_2(s) = k_{p2}^{*T}n_2(s)$ in (3.3.6) and (3.3.7). With $k_1^{*T}P^{-1} = [k_{p1}^{*T}, k_{p2}^{*T}]$ and the prespecified $L_r \in \mathbb{R}^{(n-n_0) \times n_0}$, the existence of $\theta_{20}^* = k_{p1}^{*T} + k_{p2}^{*T}L_r$ in (3.3.7) is guaranteed.

In summary, based on the partial-state observer, there always exist constant parameters θ_1^* , θ_2^* , θ_{20}^* and θ_3^* to make (3.3.5) satisfied so that the nominal partial state feedback control law (3.3.14) with θ_1^* , θ_2^* , θ_{20}^* and θ_3^* makes plant-model output matching: $y(t) - y_m(t) = \varepsilon(t)$ achievable, for an exponentially decaying term $\varepsilon(t)$.

<u>Step 4: Closed-loop signal boundedness.</u> From $y(t) - y_m(t) = \varepsilon(t)$, we have the *i*th derivative as

$$y^{(i)}(t) = \varepsilon^{(i)}(t) + y_m^{(i)}(t), \ i = 1, 2, \dots, n^*.$$
 (3.3.15)

Using the reference model: $y_m(t) = \frac{1}{P_m(s)}[r](t)$, we have

$$y_m^{(i)}(t) = s^i [y_m](t) = \frac{s^i}{P_m(s)} [r](t), \qquad (3.3.16)$$

which is bounded for $i = 1, ..., n^*$, because $\frac{s^i}{P_m(s)}$ is stable and proper and $r(t) \in L^{\infty}$. This implies that $y^{(i)}(t) \in L^{\infty}$ for $i = 1, ..., n^*$ as $\varepsilon^{(i)}(t) \in L^{\infty}$.

For the plant (3.1.1), the input-output relationship is $P(s)[y](t) = k_p Z(s)[u](t)$. A relationship between $y_0(t)$ and u(t) can also be obtained: $P(s)[y_0](t) = Z_0(s)[u](t)$, for a polynomial vector $Z_0(s)$. Therefore, a useful relationship between y(t) and $y_0(t)$ can be found as

$$y_0(t) = \frac{1}{k_p} Z^{-1}(s) Z_0(s)[y](t) = \frac{Z_0(s)}{k_p P_m(s) Z(s)} P_m(s)[y](t).$$
(3.3.17)

Since $P_m(s)[y](t)$ is bounded as from (3.1.3) and $\frac{Z_0(s)}{k_p P_m(s)Z(s)}$ is stable and proper or strictly proper, we have $y_0(t)$ is bounded, and so is $\omega_2(t)$.

Finally, using the plant, $P(s)[y](t) = k_p Z(s)[u](t)$, and ignoring the exponentially decaying effect of the initial conditions, we have

$$u(t) = \frac{P(s)}{k_p P_m(s) Z(s)} P_m(s)[y](t), \qquad (3.3.18)$$

which is bounded because $\frac{P(s)}{k_p P_m(s)Z(s)}$ is stable and proper and $P_m(s)[y](t)$ is bounded, and so is $\omega_1(t)$. ∇

Theorem 3.3.1 shows that when the plant parameters are known, the partialstate feedback control law (3.3.7) with the nominal parameters θ_1^* , θ_2^* , θ_{20}^* and θ_3^* is the solution to the model reference control problem. The new features of such a solution include: (a) the nominal partial-state feedback control law can achieve output matching even with $y_0(t) \in \mathbb{R}$ for $y_0(t) \neq y(t)$ (see Corollary 3.3.2 next); and (b) the boundedness of $y^{(i)}(t)$ is guaranteed for $i = 1, \ldots, n^*$.

With $\theta_3^* = k_p^{-1}$, the nominal partial-state feedback controller parameters θ_1^* , θ_2^* , θ_{20}^* and θ_3^* whose existence are guaranteed by Theorem 3.3.1 also satisfy the matching polynomial equation described as follows:

Corollary 3.3.1. Under Assumption (A3.1), constant parameters $\theta_1^* \in \mathbb{R}^{n-n_0}$, $\theta_2^* \in \mathbb{R}^{n_0(n-n_0)}$, $\theta_{20}^* \in \mathbb{R}^{n_0}$ and $\theta_3^* \in \mathbb{R}$ exist such that the output matching equation holds:

$$\theta_1^{*\mathrm{T}}a_1(s)P(s) + (\theta_2^{*\mathrm{T}}A_2(s) + \theta_{20}^{*\mathrm{T}}\Lambda(s))Z_0(s) = \Lambda(s)(P(s) - k_p\theta_3^*Z(s)P_m(s)). \quad (3.3.19)$$

Proof: With the output matching parameters $\theta_1^*, \theta_2^*, \theta_{20}^*$ and θ_3^* , we rewrite (3.3.7) as

$$u(t) = \theta_1^{*\mathrm{T}} \frac{a_1(s)}{\Lambda(s)} [u](t) + \theta_2^{*\mathrm{T}} \frac{A_2(s)}{\Lambda(s)} G_0(s)[u](t) + \theta_{20}^{*\mathrm{T}} G_0(s)[u](t) + \theta_3^{*} r(t), \quad (3.3.20)$$

with $y_0(t) = G_0(s)[u](t)$. This leads the plant y(t) = G(s)[u](t) to the closed-loop system with transfer function

$$G_{c}(s) = G(s) \left(1 - \theta_{1}^{*\mathrm{T}} \frac{a_{1}(s)}{\Lambda(s)} - (\theta_{2}^{*\mathrm{T}} \frac{A_{2}(s)}{\Lambda(s)} + \theta_{20}^{*\mathrm{T}}) G_{0}(s)\right)^{-1} \theta_{3}^{*},$$

which has been made to match $W_m(s)$. From $G_c(s) = W_m(s)$, we obtain

$$1 - \theta_1^{*\mathrm{T}} \frac{a_1(s)}{\Lambda(s)} - (\theta_2^{*\mathrm{T}} \frac{A_2(s)}{\Lambda(s)} + \theta_{20}^{*\mathrm{T}}) G_0(s) = \theta_3^* W_m^{-1}(s) G(s), \qquad (3.3.21)$$

which, for $G(s) = k_p \frac{Z(s)}{P(s)}$ and $G_0(s) = \frac{Z_0(s)}{P(s)}$, can be expressed as (3.3.19). Hence, there always exist θ_1^* , θ_2^* , θ_{20}^* and θ_3^* satisfying the matching equation (3.3.19). ∇

According to Theorem 3.3.1, desired plant-model output matching and closed-loop stability are guaranteed no matter what components of x(t) are included in $y_0(t)$, as long as Assumption (A3.1) is met. A particular case of interest is when $y_0(t) \in \mathbb{R}$ and $y_0(t) \neq y(t)$. For such a special case, we have the following result.

Corollary 3.3.2. There exist parameters θ_1^* , θ_2^* , θ_{20}^* and θ_3^* for nominal partial-state feedback controller (3.3.7) to achieve the desired plant-model matching and closed-loop signal boundedness for $y_0(t) = C_0 x(t) \in \mathbb{R}$ and $y_0(t) \neq y(t)$, if (A, C_0) is observable.

Corollary 3.3.2 shows that model reference control can be designed to make the plant output y(t) track the reference output $y_m(t)$, using a scalar $y_0(t) \neq y(t)$.

Remark 3.3.1. According to the derivation of the partial-state feedback MRC scheme, it is obviously that output feedback model reference control (MRC) is a special case of partial-state feedback MRC when $C_0 = c$. For output feedback MRC, the matching equation (3.3.19) becomes

$$\theta_1^{*\mathrm{T}}a(s)P(s) + (\theta_2^{*\mathrm{T}}a(s) + \theta_{20}^{*}\Lambda(s))k_pZ(s) = \Lambda(s)(P(s) - k_p\theta_3^{*}Z(s)P_m(s)), \quad (3.3.22)$$

whose existence was an interesting topic of the early development of MRAC [53]. \Box

3.4 Adaptive Partial-State Feedback Control

For the plant (3.1.1) with unknown (A, b, c), nominal controller parameters θ_1^* , θ_2^* , θ_{20}^* and θ_3^* in (3.3.7) depending on system parameters (A, b, c) can not be calculated so that the nominal partial-state feedback control design cannot be applied to the plant (3.1.1). In this section, we will develop a general adaptive partial-state feedback control design for general systems and a simpler adaptive design for relative-degreeone systems, respectively, to solve the adaptive control problem.

Recall the LTI plant (3.1.1): $\dot{x}(t) = Ax(t) + bu(t)$, y(t) = cx(t) with (A, b, c)unknown. Assume that assumptions (A3.1) and (A3.2) are satisfied, that is, the partial-state vector $y_0(t) = C_0x(t)$ with (A, C_0) observable is measurable and all zeros of Z(s) are stable.

The control objective is to design an adaptive partial-state feedback controller to generate a control signal u(t) for (A, b, c) unknown such that all closed-loop signals are bounded and the plant output y(t) asymptotically tracks the given reference signal $y_m(t) = W_m(s)[r](t).$

For adaptive control, we need the following assumption:

(A3.3) the sign of the high frequency gain k_p is known.

3.4.1 Controller Structure and Tracking Error Equation

In this subsection, we will propose an adaptive partial-state feedback controller structure, and derive a tracking error equation which is crucial for develop an adaptive law for control adaptation.

Adaptive controller structure. To handle the plant (3.1.1) with (A, b, c) unknown, we design an adaptive version of the controller (3.3.7) as

$$u(t) = \theta_1^{\mathrm{T}}(t)\omega_1(t) + \theta_2^{\mathrm{T}}(t)\omega_2(t) + \theta_{20}^{\mathrm{T}}(t)y_0(t) + \theta_3(t)r(t), \qquad (3.4.1)$$

where $\theta_1(t) \in \mathbb{R}^{n-n_0}$, $\theta_2(t) \in \mathbb{R}^{n_0(n-n_0)}$, $\theta_{20}(t) \in \mathbb{R}^{n_0}$, $\theta_3(t) \in \mathbb{R}$ are the adaptive estimates of the unknown nominal parameters θ_1^* , θ_2^* , θ_{20}^* , θ_3^* respectively, and

$$\omega_1(t) = \frac{a_1(s)}{\Lambda(s)} [u](t), \ \omega_2(t) = \frac{A_2(s)}{\Lambda(s)} [y_0](t)$$
(3.4.2)

with $a_1(s) = [1, s, \dots, s^{n-n_0-1}]^{\mathrm{T}}$, $A_2(s) = [I_{n_0}, sI_{n_0}, \dots, s^{n-n_0-1}I_{n_0}]^{\mathrm{T}}$ and $\Lambda(s)$ being a monic stable polynomial of degree $n - n_0$.

Remark 3.4.1. The order of the filter $\frac{1}{\Lambda(s)}$ in the partial-state feedback controller (3.4.1) is $n - n_0$ which is less than the order n - 1 in an output feedback controller when $n_0 > 1$. This feature can make the signals $\omega_1(t)$ and $\omega_2(t)$ more responsive and less oscillating, and can reduce the controller implementation complexity that caused by high-order filters.

Tracking error equation. To obtain the tracking error equation, we first operate both sides of (3.3.19) on y(t) so that

$$\theta_1^{*T} a_1(s) P(s)[y](t) + (\theta_2^{*T} A_2(s) + \theta_{20}^{*T} \Lambda(s)) Z_0(s)[y](t)$$

= $\Lambda(s) (P(s) - k_p \theta_3^* Z(s) P_m(s))[y](t).$ (3.4.3)

Recall the relationship between $y_0(t)$ and y(t) obtained in Section 3.2.3:

$$Z_0(s)[y](t) = k_p Z(s)[y_0](t).$$
(3.4.4)

Substituting (3.4.4) and the plant: $P(s)[y](t) = k_p Z(s)[u](t)$, into (3.4.3), we have

$$\theta_1^{*T} a_1(s) k_p Z(s)[u](t) + (\theta_2^{*T} A_2(s) + \theta_{20}^{*T} \Lambda(s)) k_p Z(s)[y_0](t)$$

= $\Lambda(s) k_p Z(s)[u](t) - \Lambda(s) k_p \theta_3^* Z(s) P_m(s))[y](t).$ (3.4.5)

Because $\Lambda(s)$ and Z(s) are stable, (3.4.5) can be expressed as

$$u(t) = \theta_1^{*T} \frac{a_1(s)}{\Lambda(s)} [u](t) + \theta_2^{*T} \frac{A_2(s)}{\Lambda(s)} [y_0](t) + \theta_{20}^{*T} y_0(t) + \theta_3^{*} P_m[y](t) + \varepsilon_1(t)$$
(3.4.6)

for some initial condition-related exponentially decaying $\varepsilon_1(t)$. Substituting (3.4.6) in (3.4.1), we have the tracking error equation

$$e(t) = y(t) - y_m(t) = \frac{k_p}{P_m(s)} [\tilde{\theta}^{\mathrm{T}} \omega](t)$$

= $-k_p (\theta^{*\mathrm{T}} \frac{1}{P_m(s)} [\omega](t) - \frac{1}{P_m(s)} [\theta^{\mathrm{T}} \omega](t)),$ (3.4.7)

where $\theta^* = [\theta_1^{*T}, \theta_2^{*T}, \theta_{20}^{*T}, \theta_3^{*}]^T$, $\theta(t) = [\theta_1^T(t), \theta_2^T(t), \theta_{20}^T(t), \theta_3(t)]^T$, $\omega(t) = [\omega_1^T(t), \omega_2^T(t), \psi_2^T(t), \psi_3^T(t), \psi_3(t)]^T$, $\tilde{\theta}(t) = \theta(t) - \theta^*$. The tracking error expression (3.4.7) is the basis for the adaptive designs in the following subsections.

3.4.2 Adaptive Design for General Systems

In this subsection, for the plant (3.1.1) with unknown parameters and relative degree $n^* \ge 1$, we will develop an adaptive law for updating the adaptive controller parameters in (3.4.1), based on a gradient method. System stability and tracking performance analysis is given following the control design.

Estimation error. From the tracking error equation (3.4.7), we define the estimate error as

$$\epsilon(t) = e(t) + \rho(t)\xi(t) \tag{3.4.8}$$

for the estimates $\theta(t)$ and $\rho(t)$ of θ^* and $\rho^* = k_p$, where

$$\xi(t) = \theta^{\mathrm{T}}(t)\zeta(t) - \frac{1}{P_m(s)} [\theta^{\mathrm{T}}\omega](t), \zeta(t) = \frac{1}{P_m(s)} [\omega](t).$$
(3.4.9)

From (3.4.7) and (3.4.8), it follows that

$$\epsilon(t) = \rho^* \tilde{\theta}^{\mathrm{T}}(t) \zeta(t) + \tilde{\rho}(t) \xi(t)$$
(3.4.10)

with $\tilde{\rho}(t) = \rho(t) - \rho^*$, a desired linear form.

3.4.2.1 Adaptive Laws

Based on the desired estimation error form (3.4.10), we choose the gradient-type adaptive update laws for $\theta(t)$ and $\rho(t)$ as

$$\dot{\theta}(t) = -\frac{\operatorname{sign}(k_p)\Gamma\epsilon(t)\zeta(t)}{m_0^2(t)}, \quad \dot{\rho}(t) = -\frac{\gamma\epsilon(t)\xi(t)}{m_0^2(t)}, \quad (3.4.11)$$

with an adaptation gain matrix $\Gamma = \Gamma^{T} > 0$, an adaptation gain $\gamma > 0$, initial estimates $\theta(0)$ and $\rho(0)$ of θ^{*} and ρ^{*} , and $m_{0}(t) = \sqrt{1 + \zeta^{T}(t)\zeta(t) + \xi^{2}(t)}$.

3.4.2.2 Stability and Tracking Performance Analysis

For a general system, the adaptive law (3.4.11) and the control system have the following desired properties.

Lemma 3.4.1. The adaptive law (3.4.11) guarantees that $\theta(t) \in L^{\infty}$, $\rho(t) \in L^{\infty}$, and $\frac{\epsilon(t)}{m_0(t)} \in L^2 \bigcap L^{\infty}$, $\dot{\theta}(t) \in L^2 \bigcap L^{\infty}$ and $\dot{\rho}(t) \in L^2 \bigcap L^{\infty}$.

Proof: With (3.4.10), the time-derivative of the positive definite function $V(\tilde{\theta}, \tilde{\rho}) = |\rho^*|\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} + \gamma^{-1}\tilde{\rho}^2$ along the trajectories of (3.4.11), satisfies

$$\dot{V} = \frac{-2\epsilon^2(t)}{m_0^2(t)} \le 0.$$
 (3.4.12)

Hence, $\theta(t) \in L^{\infty}$, $\rho(t) \in L^{\infty}$ and $\frac{\epsilon(t)}{m_0(t)} \in L^2$, which, with (3.4.10) and (3.4.11), in turn, implies $\frac{\epsilon(t)}{m_0(t)} \in L^{\infty}$, $\dot{\theta}(t) \in L^2 \bigcap L^{\infty}$ and $\dot{\rho}(t) \in L^2 \bigcap L^{\infty}$. ∇

Theorem 3.4.1. The adaptive controller (3.4.1) with the adaptive law (3.4.11), applied to the plant (3.1.1) with relative degree $n^* \ge 1$, guarantees the closed-loop signal bounded and $\lim_{t\to\infty} e(t) = 0$.

Proof: The stability proof of the developed partial-state feedback MRAC scheme is based on a feedback structure for the closed-loop system and on a small loop gain property of such a feedback structure.

<u>Step 1: introducing filtered signals for u(t) and y(t).</u> To obtain the desired feedback structure, we first two fictitious signals

$$z_0(t) = \frac{1}{s+a_0}[u](t), \ z(t) = \frac{1}{s+a_0}[y](t),$$
(3.4.13)

and two fictitious filters $K_1(s)$ and K(s) as

$$sK_1(s) = 1 - K(s), \ K(s) = \frac{a^{n^*}}{(s+a)^{n^*}},$$
 (3.4.14)

where $a_0 > 0$ is arbitrary and a > 0 is to be specified. From the introduced fictitious signals and filters, we can obtain the following equality: $-a_0K_1(s) + (s+a_0)K_1(s) = 1 - K(s)$.

From the equality, with $G(s) = k_p \frac{Z(s)}{P(s)}$, we obtain

$$z_0(t) + a_0 K_1(s)[z_0](t) - K_1(s)[u](t) = K(s)G^{-1}(s)[z](t).$$
(3.4.15)

Hence, with the substitution of (3.4.13) and $K_1(s)$ operated on both sides the controller structure (3.4.1), the following identity is obtained:

$$K_{1}(s)[u](t) = K_{1}(s)\theta_{1}^{\mathrm{T}}(\cdot)\frac{a_{1}(s)}{\Lambda(s)}(s+a_{0})[z_{0}](t) + K_{1}(s)\theta_{2}^{\mathrm{T}}(\cdot)\frac{A_{2}(s)}{\Lambda(s)}[y_{0}](t) + K_{1}(s)\theta_{20}^{\mathrm{T}}(\cdot)[y_{0}](t) + K_{1}(s)[\theta_{3}r](t).$$
(3.4.16)

<u>Step 2: expressing $y_0(t)$ by the filtered signals.</u> To cope with the new partial-state feedback control scheme, we need the following signal transformation. According to the state observer theory, for (A, c) detectable, we can express the system state x(t) as

$$x(t) = (sI - A + Lc)^{-1}b[u](t) + (sI - A + Lc)^{-1}L[y](t)$$

= $\frac{G_1(s)}{\Lambda_0(s)}[u](t) + \frac{G_2(s)}{\Lambda_0(s)}[y](t),$ (3.4.17)

where the eigenvalues of the $n \times n$ matrix A - Lc are stable for some constant gain vector $L \in \mathbb{R}^{n \times 1}$, $\Lambda_0(s) = \det(sI - A + Lc)$ whose degree is n, L is a matrix such that and $G_1(s) = \operatorname{adj}(sI - A + Lc)b$ and $G_2(s) = \operatorname{adj}(sI - A + Lc)L$ are polynomial vectors whose maximum degrees are n - 1. (If (A, c) is observable, the system state vector x(t) can be expressed as (3.4.17) for some constant gain vector L for all t > 0such that the eigenvalues of the matrix A - Lc are stable and prespecified; If (A, c) is only detectable, the system states vector x(t) can be expressed as (3.4.17) for some constant gain vector L such that the stable unobservable modes of A are included in the eigenvalue set of the matrix A - Lc.)

With (3.4.17) and (3.4.13), $y_0(t) = C_0 x(t)$ can be expressed as

$$y_{0}(t) = C_{0} \frac{G_{1}(s)}{\Lambda_{0}(s)} [u](t) + C_{0} \frac{G_{2}(s)}{\Lambda_{0}(s)} [y](t)$$

$$= C_{0} \frac{G_{1}(s)}{\Lambda_{0}(s)} (s + a_{0}) [z_{0}](t) + C_{0} \frac{G_{2}(s)}{\Lambda_{0}(s)} (s + a_{0}) [z](t).$$
(3.4.18)

Step 3: establishing a relationship between the filtered u(t) and the filtered y(t). Using (3.4.16) and (3.4.18), we can express $K_1(s)[u](t)$ in (3.4.16) as

$$\begin{aligned} K_{1}(s)[u](t) &= K_{1}(s)\theta_{1}^{\mathrm{T}}(\cdot)\frac{a_{1}(s)}{\Lambda(s)}(s+a_{0})[z_{0}](t) + K_{1}(s)\theta_{21}^{\mathrm{T}}(\cdot)\frac{N_{1}(s)}{\Lambda_{1}(s)}(s+a_{0})[z_{0}](t) \\ &+ K_{1}(s)\theta_{22}^{\mathrm{T}}(\cdot)\frac{N_{1}(s)}{\Lambda_{1}(s)}(s+a_{0})[z](t) + K_{1}(s)\theta_{201}^{\mathrm{T}}(\cdot)\frac{N_{2}(s)}{\Lambda_{0}(s)}(s+a_{0})[z_{0}](t) \\ &+ K_{1}(s)\theta_{202}^{\mathrm{T}}(\cdot)\frac{N_{2}(s)}{\Lambda_{0}(s)}(s+a_{0})[z](t) + K_{1}(s)[\theta_{3}r](t), \end{aligned}$$

where $\theta_{21}(\cdot) \in \mathbb{R}^{2n-n_0-1}$, $\theta_{22}(\cdot) \in \mathbb{R}^{2n-n_0-1}$, $\theta_{201}(\cdot) \in \mathbb{R}^n$ and $\theta_{202}(\cdot) \in \mathbb{R}^n$, such that $\theta_{21}^{\mathrm{T}}(\cdot)N_1(s) = \theta_2^{\mathrm{T}}(\cdot)A_2(s)C_0G_1(s)$, $\theta_{22}^{\mathrm{T}}(\cdot)N_1(s) = \theta_2^{\mathrm{T}}(\cdot)A_2(s)C_0G_2(s)$, $\theta_{201}^{\mathrm{T}}(\cdot)N_2(s) = \theta_{20}^{\mathrm{T}}(\cdot)C_0G_1(s)$ and $\theta_{202}^{\mathrm{T}}(\cdot)N_2(s) = \theta_{20}^{\mathrm{T}}(\cdot)C_0G_2(s)$, for $N_1(s) = [1, s, \dots, s^{2n-n_0-2}]^{\mathrm{T}}, N_2(s) = [1, s, \dots, s^{n-1}]^{\mathrm{T}}$, and $\Lambda_1(s) = \Lambda(s)\Lambda_0(s)$ whose degree is $2n - n_0$. Substituting (3.4.19) into (3.4.15) and defining

$$P_0(s,\cdot) = 1 + K_1(s)a_0 - \theta_1^{\mathrm{T}}(\cdot)\frac{a(s)}{\Lambda(s)}(s+a_0) - \theta_{21}^{\mathrm{T}}(\cdot)\frac{N_1(s)}{\Lambda_1(s)}(s+a_0)$$

$$-\theta_{201}^{\mathrm{T}}(\cdot)\frac{N_2(s)}{\Lambda_0(s)}(s+a_0), \qquad (3.4.19)$$

we then obtain the desired signal expression

$$P_{0}(s)[z_{0}](t)$$

$$= \left(K(s)G^{-1}(s) + K_{1}(s)\theta_{22}^{\mathrm{T}}(\cdot)\frac{N_{1}(s)}{\Lambda_{1}(s)} + K_{1}(s)\theta_{202}(\cdot)\frac{N_{2}(s)}{\Lambda_{0}(s)}\right)[z](t) + K_{1}(s)[\theta_{3}r](t).$$
(3.4.20)

To further deal with the signal expression (3.4.20), we first find that the impulse response $k_1(t)$ of $K_1(s)$ is

$$k_1(t) = \mathcal{L}^{-1}[K_1(s)] = e^{-at} \sum_{i=1}^{n^*} \frac{a^{n^*-i}}{(n^*-i)!} t^{n^*-i}, \qquad (3.4.21)$$

where $\mathcal{L}^{-1}[\cdot]$ is the inverse Laplace transform operator and whose L^1 signal norm satisfies

$$|k_1(\cdot)||_1 = \int_0^\infty |k_1(t)| dt = \frac{n^*}{a}.$$
(3.4.22)

Hence, it is concluded that there exists $a^0 > 0$ such that for any fixed $a > a^0$, the operator $T_0(s, \cdot) = (P_0(s, \cdot))^{-1}$ is stable and proper¹ [53] (similar to the case: if the H^{∞} gain of $G_0(s)$ is small enough, $(1 + G_0(s))^{-1}$ is stable). Letting $a > a^0$ be finite in K(s) and $K_1(s)$, from (3.4.20), we have

$$z_0(t) = T_1(s, \cdot)[z](t) + b_0(t), \qquad (3.4.23)$$

where $T_1(s, \cdot)$ is a time-varying stable and strictly proper operator, and $b_0(t) \in L^{\infty}$.

<u>Step 4: formulating a closed-loop inequality of the filtered y(t).</u> For $P_m(s) = s^{n^*} + a_{n^*-1}s^{n^*-1} + \dots + a_1s + a_0$, we express $\xi(t)$ in (3.4.9) as

$$\xi(t) = \frac{s^{n^*-1} + a_{n^*-1}s^{n^*-2} + \dots + a_2s + a_1}{P_m(s)} [\dot{\theta}^{\mathrm{T}} \frac{1}{P_m(s)} [\omega]](t)$$

¹A linear operator T(s,t) is stable and proper if $|T(s,\cdot)[x](t)| \leq \beta \int_0^T e^{-\alpha(t-\tau)} |x(\tau)| d\tau + \gamma |x(t)|$ for some constants $\beta \geq 0$, $\alpha > 0$ and $\gamma > 0$, for all $t \geq 0$, where $T(s,\cdot)[x](t)$ denotes the convolution of the impulse response of $T(s,\cdot)$ with $x(\cdot)$ at t. A linear operator T(s,t) is stable and strictly proper if it is stable with $\gamma = 0$.

$$+\frac{s^{n^{*}-2}+a_{n^{*}-1}s^{n^{*}-3}+\dots+a_{2}}{P_{m}(s)}[\dot{\theta}^{\mathrm{T}}\frac{s}{P_{m}(s)}[\omega]](t)$$

+\dots+\frac{s+a_{n^{*}-1}}{P_{m}(s)}[\dot{\theta}^{\mathrm{T}}\frac{s^{n^{*}-2}}{P_{m}(s)}[\omega]](t)+\frac{1}{P_{m}(s)}[\dot{\theta}^{\mathrm{T}}\frac{s^{n^{*}-1}}{P_{m}(s)}[\omega]](t), \qquad (3.4.24)

with the regression signal $\omega = [\omega_1^{\mathrm{T}}(t), \omega_2^{\mathrm{T}}(t), y_0^{\mathrm{T}}(t), r(t)]^{\mathrm{T}}$.

Filtering both sides of (3.4.8) by $\frac{1}{s+a_0}$, we obtain

$$z(t) = \frac{1}{s+a_0} [y_m](t) + \frac{1}{s+a_0} [\epsilon - \rho \xi](t).$$
(3.4.25)

Then, from the inequality

$$|\epsilon(t)| \le \frac{|\epsilon(t)|}{m_0(t)} (1 + ||\zeta(t)||_1 + |\xi(t)|), \qquad (3.4.26)$$

the signal boundedness property shown in Lemma 4.1, and the expressions (3.4.9) and (3.4.23)-(3.4.25), we obtain the desired feedback structure

$$|z(t)| \le x_0(t) + T_2(s, \cdot)[x_1 T_3(s, \cdot)[|z|]](t)$$
(3.4.27)

for some $x_0(t) \in L^{\infty}$, $x_1(t) \in L^{\infty} \cap L^2$ with $x_1(t) \ge 0$, some stable and strictly proper operator $T_2(s,t)$, and some stable and proper operator $T_3(s,t)$ with a non-negative impulse response function.

<u>Step 5: applying Gronwall-Bellman Lemma for signal boundedness.</u> Introducing $z_1(t) = T_3(s, \cdot)[|z|](t)$, operating $T_3(s, t)$ on both sides of (3.4.27), noting that $T_3(s, t)$ has a non-negative impulse response, we have

$$z_1(t) \le b_1 + b_2 \int_0^t e^{-\alpha(t-\tau)} x_1(\tau) z_1(\tau) d\tau$$
(3.4.28)

for some α , b_1 , $b_2 > 0$. Applying a corollary of the Bellman-Gronwall Lemma (see p.64 of [82] for a proof) to (3.4.28) with $x_1(t) \in L^2 \cap L^\infty$, we conclude that $z_1(t) \in L^\infty$, and so $z(t) \in L^\infty$. Hence, $z_0(t) \in L^\infty$ in (3.4.23), $\xi(t) \in L^\infty$ in (3.4.9), $\zeta(t) \in L^\infty$ in (3.4.9), $\epsilon(t) \in L^\infty$ in (3.4.10), $y(t) \in L^\infty$ in (3.4.8), and $u(t) \in L^\infty$ in (3.4.1). From (3.4.10) we have $\dot{\epsilon}(t) \in L^{\infty}$, which, with $\epsilon(t) \in L^2 \cap L^{\infty}$, implies $\lim_{t\to\infty} \epsilon(t) = 0$ in (3.4.10), and $\lim_{t\to\infty} \dot{\theta}(t) = 0$ in (3.4.11). From this property and $\dot{\theta}(t) \in L^2$, it follows that $\xi(t) \in L^2$ and $\lim_{t\to\infty} \xi(t) = 0$ in (3.4.24). Finally, from (3.4.8), we have $e(t) = y(t) - y_m(t) \in L^2$ and $\dot{e}(t) \in L^{\infty}$, so that $\lim_{t\to\infty} e(t) = 0$. ∇

In summary, the main ideas of the developed adaptive control design for general plants are: the controller (3.4.1) with bounded parameters leads to the closed-loop inequality (3.4.27), the adaptive law (3.4.11), through the L^2 property of $\dot{\theta}(t)$ and $\frac{\epsilon(t)}{m_0(t)}$, ensures that the loop gain of (3.4.27) is small so that the signal boundedness is guaranteed, and the L^2 property and signal boundedness ensure that $\lim_{t\to\infty} e(t) = 0$. This proof unifies the stability and tracking analysis for a general MRAC framework: the partial-state feedback design for SISO systems, the commonly seen output feedback design with $y_0(t) = y(t)$, and the state feedback design with $y_0(t) = x(t)$ [82,83].

3.4.3 Adaptive Design for Relative-Degree-One Systems

In this subsection, we will, in particular, develop a simpler adaptive law to update the controller parameters in the adaptive controller (3.4.1) for the unknown plant (3.1.1) with relative degree $n^* = 1$.

Tracking error equation and adaptation law. When the system transfer function $G(s) = k_p \frac{Z(s)}{P(s)}$ has relative degree $n^* = 1$, we choose the characteristic polynomial of the reference model as $P_m(s) = s + a_m, a_m > 0$. The error equation (3.4.7) becomes

$$\dot{e}(t) = -a_m e(t) + k_p \tilde{\theta}^{\mathrm{T}}(t) \omega(t). \qquad (3.4.29)$$

With (3.4.29), we choose the adaptive law for $\theta(t)$ as

$$\tilde{\dot{\theta}}(t) = \dot{\theta} = -\operatorname{sign}(k_p)\Gamma\omega(t)e(t)$$
(3.4.30)

with an adaptation gain matrix $\Gamma = \Gamma^{T} > 0$, and an initial estimate $\theta(0)$ of θ^{*} , chosen as close as possible to θ^{*} .

System stability and tracking performance analysis. For showing the closed-loop stability, we consider the positive definite function $V(e_c) = e^2 + |k_p|\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}$ as a measure of the closed-loop system error $e_c(t) = [e(t), \tilde{\theta}^{\mathrm{T}}(t)]^{\mathrm{T}}$. The time-derivative of $V(e_c)$ is

$$\dot{V} = -2a_m e^2(t) \le 0. \tag{3.4.31}$$

From $V(e_c) > 0$ and $\dot{V}(e_c) \leq 0$, we can conclude that $V(e_c)$ is bounded, which in turn, implies that $\theta(t) \in L^{\infty}$, $e(t) \in L^{\infty}$ and so $y(t) \in L^{\infty}$. From (3.3.17), we have $y_0(t) \in L^{\infty}$, so that $\omega_2(t) = \frac{A_2(s)}{\Lambda(s)}[y_0](t) \in L^{\infty}$. Using (3.1.2): $P(s)[y](t) = k_p Z(s)[u](t)$, ignoring the exponentially decaying initial conditions, we have

$$\frac{s^i}{\Lambda(s)}[u](t) = \frac{P(s)}{k_p Z(s)} \frac{s^i}{\Lambda(s)}[y](t), \qquad (3.4.32)$$

which is bounded for $i = 0, 1, \dots, n - n_0 - 1$, because $\frac{P(s)}{k_p Z(s)} \frac{s^i}{\Lambda(s)}$ is stable and proper and $y(t) \in L^{\infty}$. This implies $\omega_1(t) \in L^{\infty}$, and so does u(t). From (3.4.29), we have $\dot{e}(t) \in L^{\infty}$, and from (3.4.31), we have $e(t) \in L^2$. Therefore, using Barhalat Lemma, we can conclude that $\lim_{t\to\infty} e(t) = 0$. ∇

In summary, we have the following result.

Theorem 3.4.2. The adaptive controller (3.4.1) with the adaptive law (3.4.30), applied to the plant (3.1.1) with relative degree $n^* = 1$, guarantees that all closed-loop signals are bounded and $\lim_{t\to\infty} e(t) = 0$.

Discussion. So far, two adaptive partial-state feedback control designs are developed and the corresponding stability and tracking performances have been proved. It is worth noting that the use of the partial state $y_0(t) = C_0 x(t)$ in the proposed partial-state feedback MRAC scheme, as compared with y(t) = Cx(t) in a traditional output feedback MRAC design, provides new flexibilities in designing MRAC schemes for three typical cases of $y_0(t) = C_0 x(t)$:

- (i) $y_0(t)$ being a vector which contains the output y(t);
- (ii) $y_0(t)$ being a vector which does not contain y(t); and
- (iii) $y_0(t)$ being a scalar which is not equal to y(t).

In all three cases the output y(t) is ensured to track $y_m(t)$ asymptotically, by the partial-state feedback MRAC scheme which has reduced controller complexity as compared with a traditional output feedback MRAC scheme, and has a new MRAC framework linking a state feedback MRAC design and an output feedback MRAC design, as a new addition to the MRAC theory.

Remark 3.4.2. In this thesis we consider direct model reference adaptive control with partial-state feedback: the adaptive controller parameters are updated directly by adaptive laws. For an indirect MRAC design, system parameters are estimated first and the controller parameters are calculated from the estimated system parameters.

For output tracking, one typical indirect MRAC design is based on an inputoutput model: $P(s)[y](t) = k_p Z(s)[u](t)$. The adaptive law for estimating the system parameters in P(s) and Z(s) is developed based on an output estimation error (see p.303 of [5] for details). This method does not involve state nor partial-state information for control design, so that the partial-state signal is not applicable for this input-output model based indirect adaptive control design.

Another possible design of indirect model reference adaptive control is to first obtain the estimates (\hat{A}, \hat{b}) of the parameters (A, b) of a system model $\dot{x}(t) = Ax(t) + bu(t)$ by using x(t) and u(t), and then to solve the matching equation $c(\lambda I - \hat{A}\hat{b}K_1^T)^{-1} \times \hat{b}K_2 = W_m(s)$ to obtain the parameters K_1 and K_2 , for the adaptive control law: $u(t) = K_1^{\mathrm{T}}x(t) + K_2r(t)$, assuming c is known in the output equation: y(t) = cx(t)(which is often the case in applications) [83]. There is a potential for this design to be developed using a partial-state vector $y_0(t) = x_0(t)$ for $x(t) = [x_0^{\mathrm{T}}(t), x_1^{\mathrm{T}}(t)]^{\mathrm{T}}$ such that $\dot{x}_0(t) = A_{11}x_0(t) + A_{12}x_1(t) + b_1u(t)$, where $x_1(t)$ is not available but can be parametrized in terms of filtered versions of y(t) and u(t). In this formulation, the system parameters (A_{11}, A_{12}, b_1) and some additional parameters in parametrization of $A_{12}x_1(t)$ (which are the total equivalent system parameters in terms of the system signals $x_0(t)$ plus u(t) and y(t)) can be estimated. An indirect partial-state feedback adaptive controller (of the same structure as that presented in this chapter) may be constructed using parameters calculated from a design equation (different from that for a state feedback or output feedback design) using the estimates of the equivalent system parameters, for output tracking control. Development of such a new indirect adaptive control scheme is beyond the scope of this thesis.

3.5 Illustrative Examples

In this section, we present simulation studies on a relative-degree-two plant and a relative-degree-one plant to evaluate the effectiveness of the proposed partial-state feedback adaptive control designs, respectively.

3.5.1 Simulation Study of A Relative-Degree-Two Plant

In this subsection, we use the longitudinal dynamic model of a Boeing 737 airplane [84] as the plant whose relative degree is two, for the simulation study. The proposed partial-state feedback adaptive control design for general plants developed in Section 3.3.2 is applied.

Simulation system (longitudinal aircraft system model). For pitch angle control, with the system state vector chosen as $x = [U_b, W_b, Q_b, \theta_0]^{\mathrm{T}}$, the linearized

longitudinal motion equation with the elevator angle $\delta_e(\text{deg})$ as the input is

$$\begin{bmatrix} U_b \\ \dot{W}_b \\ \dot{Q}_b \\ \dot{\theta}_0 \end{bmatrix} = \begin{bmatrix} -0.0264 & 0.1269 & -12.9260 & -32.1690 \\ -0.2501 & -0.8017 & 220.5500 & -0.1631 \\ 0.0002 & -0.0075 & -0.5510 & -0.0003 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} U_b \\ W_b \\ Q_b \\ \theta_0 \end{bmatrix} + \begin{bmatrix} 0.0109 \\ -0.1858 \\ -0.0230 \\ 0 \end{bmatrix} \delta_e,$$
$$y(t) = \theta_0(t). \tag{3.5.1}$$

The four state variables are the axial forward velocity $U_b(\text{ft/s})$, vertical forward velocity $W_b(\text{ft/s})$, vertical pitch velocity $Q_b(\text{rad/s})$ and axial Euler pitch angle $\theta_0(\text{rad})$ (the notation θ_0 is used to avoid possible confusion with $\theta(t)$ in the adaptive controller), and the plant output y(t) is the pitch angle θ_0 . This is a relative degree 2 system model.

Simulation results. Three cases have been studied, with the reference input being $r(t) = 0.08 \sin(0.1t)$, to show the new features of the partial-state feedback MRAC scheme:

Case I: the partial-state y_0 is a vector which contains $y = \theta_0$: $y_0 = [Q_b, \theta_0]^T$;

Case III: the partial-state y_0 is a vector which does not contain $y = \theta_0$: $y_0 = [U_b, W_b]^T$; **Case III:** the partial-state y_0 is a scalar and not equal to $y = \theta_0$: $y_0 = U_b$.

In addition, tracking performance and the control input signal are given, when an adaptive output feedback controller (3.1.6) is applied to (3.5.1), for showing the capability of improving the transient response of the partial-state feedback design.

For all simulation studies, $\Gamma = 2I$, $W_m(s) = \frac{1}{(s+1)^2}$, $\Lambda(s) = (s+1)^{4-n_0}$ $(n_0 = 2, 2, 1, 1$ for the above cases, respectively), y(0) = 0.01, $y_m(0) = 0$, $r(t) = 0.5 \sin(0.08t)$, and initial controller parameters are chosen as 90% of the nominal controller parameters calculated by (3.3.19).

Simulation results for Case I–Case III are shown in Fig. 3.1–Fig. 3.3, respectively,



Figure 3.1: System response for $y_0(t) = [Q_b(t), \theta(t)]^T, y(t) = \theta(t) \ (n^* = 2).$



Figure 3.2: System response for $y_0(t) = [U_b(t), W_b(t)], y(t) = \theta_0(t)$ $(n^* = 2)$.



Figure 3.3: System response for $y_0(t) = Q_b(t), y(t) = \theta_0(t)$ $(n^* = 2)$.



Figure 3.4: System response for output feedback $(n^* = 2)$.

and the simulation results for output feedback design is shown in Fig. 3.4. In Fig. 3.1(a)-Fig. 3.3(a), the dashed lines represent the reference pitch angle and the solid lines represent the aircraft outputs. The tracking performance plots in Fig.3.1(a)-Fig. 3.3(a) show that the asymptotic tracking are achieved in all three cases. Fig. 3.1(b)-Fig. 3.3(b) and Fig.3.1(c)-Fig. 3.3(c) confirm that all control signals and their rates stay in acceptable ranges. Also, all signals in closed-loop systems are bounded whose plots are not shown due to the space limit. In addition, under the same conditions, less oscillating at the beginning of the simulation can be observed from Fig. 3.1-Fig.3.2 produced by the developed partial-state feedback control design, compared to the simulation result of the output feedback design shown in Fig. 3.4, which verifies that the new MRAC scheme improves system transient response.

3.5.2 Simulation Study of A Relative-Degree-One Plant

As an example, we use the lateral dynamic model of a Boeing 747 airplane [21] as the plant to which the proposed partial-state feedback adaptive control design for relative-degree-one systems (developed in Section 3.3.3) is applied.

Simulation system (latitudinal aircraft system model). For yaw rate control, with the system state vector chosen as $x = [\beta, r_0, p, \phi]^T$, the linearized lateralperturbation motion equation with the rudder angle δ_r as the input is

$$\begin{bmatrix} \beta \\ \dot{r}_0 \\ \dot{p} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.598 & -0.115 & -0.0318 & 0 \\ -3.05 & 0.388 & -0.4650 & 0 \\ 0 & 0.0805 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r_0 \\ p \\ \phi \end{bmatrix} + \begin{bmatrix} 0.00729 \\ -0.475 \\ 0.153 \\ 0 \end{bmatrix} \delta_r,$$

$$y(t) = r_0(t).$$

$$(3.5.2)$$

The four state variables are the side-slip angel $\beta(\text{rad})$, yaw rate $r_0(\text{rad/s})$ (the notation r_0 is used to avoid possible confusion with r(t) in $y_m(s) = W_m(s)[r](t)$), roll rate p(rad/s) and roll angle $\phi(\text{rad})$, and the plant output y(t) is the yaw rate. It can be

verified that this is a relative degree 1 system model.

Simulation results. The following Case I–Case III are selected to show the unique features of the developed MRAC scheme.

Case I: the partial-state y_0 is a vector which contains $y = r_0$: $y_0 = [r_0, p]^T$;

Case II: the partial-state y_0 is a vector which does not contain $y = r_0$: $y_0 = [p, \phi]^T$;

Case III: the partial-state y_0 is a scalar which is not equal to $y = r_0$: $y_0 = \phi$.

Similar to the simulation study of relative-degree-two systems, tracking performance and the control input signal are given, when an adaptive output feedback controller (3.1.6) is applied to (3.5.2) for a comparative study.

For all simulation cases, $\Gamma = 5I$, $W_m(s) = \frac{1}{s+3}$, $\Lambda(s) = (s+1)^{4-n_0}$ $(n_0 = 2, 2, 1, 1)$ for the above cases, respectively), y(0) = -0.05, $y_m(0) = 0$, and $r(t) = 0.5 \sin(0.08t)$. Initial conditions in all cases are chosen as 90% of the nominal controller parameters respectively.

Simulation results for Case I, Case II and Case III are shown in Fig. 3.5, Fig. 3.6 and Fig. 3.7, respectively, and the simulation result for the output feedback design is shown in Fig. 3.8. All simulation results verify the desired system performance: the asymptotic tracking is achieved, and the control signals and their rates stay in acceptable ranges respectively. Also, all signals in closed-loop systems are bounded whose plots are not shown due to the space limit. In particular, simulation results for Case II and Case III show that the asymptotic tracking can be achieved without explicitly using the output information in the feedback controller structure, a new feature of the developed partial-state feedback control scheme. In addition, compared to Fig. 3.5–Fig. 3.6, larger oscillation and longer response time are observed from the tracking performance when an output feedback controller is applied (see Fig. 3.8), which supports that the new partial-state feedback MRAC scheme makes system



Figure 3.5: System response for $y_0(t) = [r_0(t), p(t)]^T$, $y(t) = r_0(t)$ $(n^* = 1)$.



Figure 3.6: System response for $y_0(t) = [p(t), \phi(t)]^T$, $y(t) = r_0(t)$ $(n^* = 1)$.



Figure 3.7: System response for $y_0(t) = \phi(t), y(t) = r_0(t)$ $(n^* = 1)$.



Figure 3.8: System response for output feedback $(n^* = 1)$.

response more responsive and less oscillating, as discussed in Remark 4.1.

Summary

In this chapter, we have developed a new framework of MRAC using partial-state feedback for output tracking, with new solutions to three technical issues: plantmodel output matching, parameterized error model, and stable adaptive law design and analysis, for ensuring closed-loop system stability and asymptotic tracking in the presence of plant uncertainties. We developed two adaptive control designs: one Lyapunov type design for relative-degree-one plants, and one gradient type design for general plants. This work has shown that partial-state feedback MRAC provides additional design flexibilities in utilizing system signals, while using less complex controller structures than output feedback. We presented a complete analysis of the closed-loop system stability and tracking performance of partial-state feedback MRAC. It has been shown that such a new MRAC framework builds a natural transition from full-state feedback MRAC to output feedback MRAC, adding new members to the family of MRAC. We presented simulation results for different adaptive control designs, which verify the desired adaptive control system performance, indicating that such a new partial-state feedback MRAC technique has potential for applications.
Chapter 4

Partial-State Feedback MRAC for MIMO Systems

This chapter develops a new model reference adaptive control (MRAC) scheme by partial-state feedback for solving a multivariable adaptive output tracking problem. The new proposed MRAC scheme has full capability to deal with plant uncertainties for output tracking and has desired design flexibility to combine the advantages of full-state feedback MRAC and output feedback MRAC. The new multivariable MRAC scheme provides new features to the control system including additional design flexibility and feedback capacity. Based on the additional design flexibility it provides, a minimum-order MRAC scheme is also proposed in this chapter, which reduces the control adaptation complexity and relaxes the feedback information requirement, compared to the existing MRAC schemes.

This main contributions of this chapter are:

- developing an adaptive multivariable MRAC scheme by using partial-state feedback signal which can guarantee asymptotic output tracking and closed-loop signal boundedness in the presence of plant parameter uncertainties;
- conducting a complete analysis of plant-model output matching for the nominal control design, and a complete analysis of stability and tracking performance

for the adaptive control design;

- presenting a complete system computation complexity analysis of the partialstate feedback reduced-order multivariable MRAC scheme; and
- providing a minimal-order MRAC scheme which enjoys minimum feedback signal requirement and reduces the system adaptation complexity, compared to the other observer-based MRAC designs.

4.1 Problem Statement and Research Motivations

In this section, a brief review of the existing multivariable MRAC schemes is first given in Section 4.1.1. Then, the multivariable MRAC problems: (a) partial-state feedback reduced-order multivariable MRAC; and (b) minimal-order multivariable MRAC, are formulated in Section 4.1.2.

4.1.1 Review of Multivariable MRAC Schemes

Before we formulate the new multivariable MRAC problems, it is necessary to review the existing multivariable MRAC schemes first in this section.

Plant description. Consider an *M*-input and *M*-output linear time-invariant plant described by

$$\dot{x}(t) = Ax(t) + Bu(t), \ y(t) = Cx(t),$$
(4.1.1)

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times M}$ and $C \in \mathbb{R}^{M \times n}$ are unknown constant parameter matrices, and $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^M$ and $y(t) \in \mathbb{R}^M$ are the state, input and output vectors, respectively. The input-output description of the plant (4.1.1) is

$$y(t) = G(s)[u](t), \ G(s) = C(sI - A)^{-1}B.$$
 (4.1.2)

The notation, y(t) = G(s)[u](t), is used to denote the output y(t) of a system represented by a transfer function matrix G(s) with a control input signal u(t). It is a simple notation to combine both the time domain and the frequency domain signal operations, suitable for adaptive control system presentation.

Control goal and plant assumptions. The control goal of multivariable MRAC is to construct a feedback control law by using the state vector x(t) or the output signal y(t) for generating the control input signal u(t) in (4.1.1) such that all signals in the closed-loop system are bounded and the output vector y(t) asymptotically tracks a given reference output vector $y_m(t)$ generated from a reference model system

$$y_m(t) = W_m(s)[r](t), \ W_m(s) = \xi_m^{-1}(s),$$
(4.1.3)

where $r(t) \in \mathbb{R}^M$ is a bounded reference input signal, and $\xi_m(s)$ is a modified left interactor matrix of the system transfer matrix $G(s) = C(sI - A)^{-1}B$, whose inverse matrix is stable, i.e., $W_m(s)$ is stable.

The basic assumptions are made for achieving the control objective for multivariable MRAC systems:

(A4.1) All zeros of $G(s) = C(sI - A)^{-1}B$ are stable, and (A, B, C) is stabilizable and detectable.

(A4.2) G(s) has full rank and its modified left interactor matrix $\xi_m(s)$ is known.

Assumption (A4.1) is for a stable plant-model output matching, and Assumption (A4.2) is for choosing a reference model system $W_m(s) = \xi_m^{-1}(s)$ suitable for plantmodel output matching. Note that the zeros of G(s) are defined as the system transmission zeros (the values of s making G(s) nonsingular). In addition, the interactor matrix $\xi_m(s)$ does not explicitly depends on the parameters of G(s) in this case. **Review of the existing MRAC designs.** According to different types of the feedback signal used to construct the controller, there are two multivariable MRAC designs for output tracking in the literature.

(i) State feedback for output tracking. When the full state vector x(t) is available for measurement, the following simple adaptive controller structure can be used:

$$u(t) = K_1^{\mathrm{T}}(t)x(t) + K_2(t)r(t), \qquad (4.1.4)$$

where $K_1(t) \in \mathbb{R}^{n \times M}$ and $K_2(t) \in \mathbb{R}^{M \times M}$ are controller parameters to be adaptively updated by stable adaptive laws. Such controller parameters $K_1(t)$ and $K_2(t)$ are the adaptive estimates of the nominal controller parameters K_1^* and K_2^* satisfying the matching condition

$$C(sI - A - BK_1^{*T})^{-1}BK_2^* = W_m(s), \ K_2^{*-1} = K_p,$$
(4.1.5)

with K_p being the system high-frequency gain matrix of G(s), for plant-model output matching: $y(t) = W_m(s)[r](t) = y_m(t)$. The existence of the nominal controller parameters K_1^* and K_2^* is guaranteed as long as the plant interactor matrix $\xi_m(s)$ is used for $W_m(s) = \xi_m^{-1}(s)$. In addition, to ensure the output tracking as well as the system internal signal boundedness, (A, B, C) needs to be stabilizable and detectable and all zeros of G(s) need to be stable [27].

(*ii*) Output feedback for output tracking. In applications, when the full state vector x(t) is not accessible for measurement, the standard output feedback adaptive controller

$$u(t) = \Theta_1^{\mathrm{T}}(t)\omega_1(t) + \Theta_2^{\mathrm{T}}(t)\omega_2(t) + \Theta_{20}(t)y(t) + \Theta_3(t)r(t)$$
(4.1.6)

needs to be used, where $\omega_1(t) = \frac{A_0(s)}{\Lambda(s)}[u](t), \ \omega_2(t) = \frac{A_0(s)}{\Lambda(s)}[y](t)$ with $A_0(s) = [I_M, sI_M, \cdots, s^{\bar{\nu}-2}I_M]^{\mathrm{T}}, \ \Theta_1(t) \in \mathbb{R}^{(\bar{\nu}-1)M \times M}, \ \Theta_2(t) \in \mathbb{R}^{(\bar{\nu}-1)M \times M}, \ \Theta_{20}(t) \in \mathbb{R}^{M \times M}, \ \Theta_3(t) \in \mathbb{R}^{M \times M}$

 $\mathbb{R}^{M \times M}$, $\bar{\nu}$ being the upper bound of the observability index ν of the plant, and $\Lambda(s)$ being a monic stable polynomial of degree $\bar{\nu} - 1$. To ensure the internal signal boundedness while achieving output tracking, it is needed that all zeros of G(s) are stable and (A, B, C) needs to be stabilizable and detectable.

Research motivations. In summary, stable output matching can always be achieved with all zeros of G(s) being stable and (A, B, C) being stabilizable and detectable, and the modified left interactor matrix $\xi_m(s)$ being known as well. However, the requirement of the full state vector x(t) for a state feedback controller may not be practical in applications, and the implementation complexity of an output feedback controller may also be an issue. Therefore, we develop the partial-state feedback reduced-order multivariable MRAC design, which

- increases design flexibility of multivariable MRAC systems;
- introduces a unification of multivariable MRAC schemes; and
- provides a manageable trade-off between feedback capacity and system complexity;

In addition, we develop the minimal-order multivariable MRAC scheme, which

- minimizes the number of feedback signal; and
- minimizes the controller implementation complexity.

4.1.2 Partial-State Reduced-Order MRAC

In this chapter, we will investigate the partial-state feedback reduced-order multivariable MRAC problem. Thus, besides the assumptions (A4.1) and (A4.2), we assume (A4.3) a partial-state vector signal $y_0(t) = C_0 x(t) \in \mathbb{R}^{n_0}$, which is a subset of the components of x(t) or a linear combination of them, is available for measurement, with (A, C_0) observable for $C_0 \in \mathbb{R}^{n_0 \times n}$ and $\operatorname{rank}[C_0] = n_0$.

Problem 1: Partial-state multivariable feedback MRAC: The control objective of this problem is to construct an adaptive control law u(t) in (4.1.1) by using the *partial-state vector* $y_0(t)$ such that

- (i) all signals in the closed-loop system are bounded;
- (ii) the output vector y(t) asymptotically tracks the given reference output vector $y_m(t)$, i.e., $\lim_{t\to\infty}(y(t) y_m(t)) = 0$.

Problem 2: Minimal-order multivariable MRAC: The control objective of this problem is to construct an adaptive control law u(t) by using the partial-state vector $y_0(t) \in \mathbb{R}^{n_0}$ with a minimum n_0 such that

- (i) all signals in the closed-loop system are bounded;
- (ii) the asymptotic *M*-output: $\lim_{t\to\infty}(y(t) y_m(t)) = 0$, is achieved;
- (iii) the number of parameters to be adaptively updated are reduced.

4.2 New Multivariable MRAC Using Partial-State Feedback

In this section, we will first solve the partial-state feedback plant-model output matching problem by developing a new controller structure with the signal $y_0(t)$ in Section 4.2.1. Such a nominal controller gives the solution to the plant-model matching problem when the system parameters are known and provides a priori knowledge to the counterpart adaptive control problem which will be solved in Section 4.2.2.

4.2.1 Nominal Partial-State Feedback Control

In this section, the nominal partial-state feedback controller is developed for the plant (4.1.1) with known parameters, which provides the foundation for an adaptive control design with unknown parameters.

4.2.1.1 Partial-State Feedback Controller Structure

In this section, we will develop a parametrized partial-state feedback controller by the partial-state $y_0(t)$ through a virtual observer.

Partial-state observer. When the state x(t) is not accessible, an observer-based state feedback control law:

$$u(t) = K_1^{*T} \hat{x}(t) + K_2^{*T} r(t)$$
(4.2.1)

can be used for plant-model matching, with a suitable state estimate $\hat{x}(t)$. For deriving a parameterized partial-state feedback control law for plant-model output matching, we first obtain a partial-state observer with the available partial-state vector $y_0(t)$ to obtain an estimated state $\hat{x}(t)$.

For the system state equation: $\dot{x}(t) = Ax(t) + Bu(t)$, as the techniques shown in [9], we introduce a transformation matrix $P \in \mathbb{R}^{n \times n}$ such that $C_0 P^{-1} = [I_{n_0}, 0]$ with $n_0 = \operatorname{rank}[C_0]$, and transfer the system state equation as

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u(t),$$
(4.2.2)

where $\bar{x}(t) = Px(t) = [\bar{x}_1^{\mathrm{T}}(t), \ \bar{x}_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ with $\bar{x}_1(t) \in \mathbb{R}^{n_0}, \ \bar{x}_2(t) \in \mathbb{R}^{n-n_0}, \ \bar{A}_{11} \in \mathbb{R}^{n_0 \times n_0}, \ \bar{A}_{12} \in \mathbb{R}^{n_0 \times (n-n_0)}, \ \bar{A}_{21} \in \mathbb{R}^{(n-n_0) \times n_0}, \ \bar{A}_{22} \in \mathbb{R}^{(n-n_0) \times (n-n_0)}, \ \bar{B}_1 \in \mathbb{R}^{n_0 \times M}$ and $\bar{B}_2 \in \mathbb{R}^{(n-n_0) \times M}.$

By the techniques shown in [9], an estimate $\hat{\bar{x}}(t)$ for $\bar{x}(t)$ can be generated as

$$\hat{\bar{x}}(t) = \begin{bmatrix} \bar{x}_1 \\ \hat{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} y_0(t) \\ w(t) + L_r y_0(t) \end{bmatrix}, \qquad (4.2.3)$$

where $\hat{x}_2(t)$ is an estimate for $\bar{x}_2(t)$, $L_r \in \mathbb{R}^{(n-n_0) \times n_0}$ is a constant gain matrix such that the eigenvalues of the $(n - n_0) \times (n - n_0)$ matrix $\bar{A}_{22} - L_r \bar{A}_{12}$ are stable and prespecified, and $w(t) \in \mathbb{R}^{n-n_0}$ is generated from the dynamic equation

$$\dot{w}(t) = (\bar{A}_{22} - L_r \bar{A}_{12})w(t) + (\bar{B}_2 - L_r \bar{B}_1)u(t)$$

$$+ ((\bar{A}_{22} - L_r \bar{A}_{12})L_r + \bar{A}_{21} - L_r \bar{A}_{11})y_0(t).$$
(4.2.4)

Based on the observer-based theory, we have $\lim_{t\to\infty} (x(t) - \hat{x}(t)) = \lim_{t\to\infty} P^{-1}(\bar{x}(t) - \hat{x}(t)) = 0$ exponentially, with the above partial-state observer.

Partial-state feedback controller. The above result shows the estimate $\hat{x}(t)$ converges to x(t) exponentially. Therefore, plant-model output matching should also be achievable by the observer-based control law $u(t) = K_1^{*T}\hat{x}(t) + K_2^*r(t)$, as the nominal control law $u(t) = K_1^{*T}x(t) + K_2^*r(t)$ does it. Since $\hat{x}(t)$ is still parameters-depending, further reparameterization of the observer-based control law is conducted for the purpose of adaptive control design for the unknown plant.

First, we solve the partial-state estimate w(t) in (4.2.4) and express it as

$$w(t) = \varepsilon_0(t) + (sI - \bar{A}_{22} + L_r \bar{A}_{12})^{-1} (\bar{B}_2 - L_r \bar{B}_1)[u](t) + (sI - \bar{A}_{22} + L_r \bar{A}_{12})^{-1} ((\bar{A}_{22} - L_r \bar{A}_{12})L_r + \bar{A}_{21} - L_r \bar{A}_{11})[y_0](t) = \frac{N_1(s)}{\Lambda(s)}[u](t) + \frac{N_2(s)}{\Lambda(s)}[y_0](t) + \varepsilon_0(t),$$
(4.2.5)

where $\varepsilon_0(t) = e^{(\bar{A}_{22}-L_r\bar{A}_{12})t}w(0)$ with w(0) being an estimate of $L_ry_0(0) - \bar{x}_2(0)$, $\Lambda(s) = det(sI - \bar{A}_{22} + L_r\bar{A}_{12})$ whose degree is $n - n_0$ and stability properties can be prespecified by assigning the eigenvalues of $\bar{A}_{22} - L_r\bar{A}_{12}$ as a set of given (known) values, $N_1(s)$ and $N_2(s)$ are some $(n - n_0) \times M$ and $(n - n_0) \times n_0$ polynomial matrices whose maximum degrees are $n - n_0 - 1$ or less.

Using (4.2.3) and (4.2.5), we can express the term $K_1^{*T}\hat{x}(t)$ as

$$K_1^{*\mathrm{T}}\hat{x}(t) = \Theta_1^{*\mathrm{T}}\frac{A_1(s)}{\Lambda(s)}[u](t) + \Theta_2^{*\mathrm{T}}\frac{A_2(s)}{\Lambda(s)}[y_0](t) + \Theta_{20}^{*\mathrm{T}}y_0(t) + \varepsilon_1(t)$$
(4.2.6)

for $\varepsilon_1(t) = K_{p2}^* e^{(\bar{A}_{22} - L_r \bar{A}_{12})t} w(0)$ representing the effect of the initial condition, where $\Theta_1^* \in \mathbb{R}^{M(n-n_0) \times M}, \ \Theta_2^* \in \mathbb{R}^{n_0(n-n_0) \times M}, \ \Theta_{20}^* \in \mathbb{R}^{n_0 \times M}$ and $\Theta_3^* \in \mathbb{R}^{M \times M}$, such that $\Theta_{20}^{*\mathrm{T}} = K_{p1}^* + K_{p2}^* L_r, \ K_{p2}^* N_1(s) = \Theta_1^{*\mathrm{T}} A_1(s)$ and $K_{p2}^* N_2(s) = \Theta_2^{*\mathrm{T}} A_2(s)$, for $K_1^{*\mathrm{T}} P^{-1} = [K_{p1}^*, K_{p2}^*]$ with $K_{p1}^* \in \mathbb{R}^{M \times n_0}$ and $K_{p2}^* \in \mathbb{R}^{M \times (n-n_0)}$, and $A_1(s) = [I_M, sI_M, \dots, s^{n-n_0-1}I_M]^{\mathrm{T}}, \ A_2(s) = [I_{n_0}, sI_{n_0}, \dots, s^{n-n_0-1}I_{n_0}]^{\mathrm{T}}.$

Substituting (4.2.6) into the observer-based control law $u(t) = K_1^{*T} \hat{x}(t) + K_2^* r(t)$ with $\Theta_3^* = K_2^*$ and ignoring the exponentially decaying term $\varepsilon_1(t)$, we obtain the parametrized nominal partial-state feedback controller:

$$u(t) = \Theta_1^{*T} \omega_1(t) + \Theta_2^{*T} \omega_2(t) + \Theta_{20}^{*T} y_0(t) + \Theta_3^{*T} r(t), \qquad (4.2.7)$$

where $\omega_1(t) = \frac{A_1(s)}{\Lambda(s)}[u](t), \ \omega_2(t) = \frac{A_2(s)}{\Lambda(s)}[y_0](t).$

The above controller structure is the desired parameterized controller structure with the partial-state vector $y_0(t)$. Next, the desired plant-model output matching properties based on this controller structure are to be established.

4.2.1.2 Plant-Model Output Matching

The above derivation shows the partial-state feedback control law (4.2.7) is derived from the observer-based control law $u(t) = K_1^{*T} \hat{x}(t) + K_2^* r(t)$ which is a substitution of the state feedback control law $u(t) = K_1^{*T} x(t) + K_2^* r(t)$ when the state x(t) is not available. This fact indicates that by the partial-state feedback control law (4.2.7), desired plant-model matching can be achieved, as the other two control laws do it.

Matching by observer-based control. It has been shown that when K_1^* and K_2^* satisfy the matching condition (4.1.5), plant-model matching can be achieved by the nominal state feedback control law: $u(t) = K_1^{*T}x(t) + K_2^*r(t)$ [27]. For the same plant (4.1.1) and the same reference model (4.1.3), plant-model matching can be achieved by the nominal observer-based state feedback control law: $u(t) = K_1^{*T}\hat{x}(t) + K_2^*r(t)$

 $K_2^*r(t)$, with the same nominal parameters K_1^* and K_2^* . The result is shown as follows.

Lemma 4.2.1. The observer-based state feedback controller $u(t) = K_1^{*T} \hat{x}(t) + K_2^{*} r(t)$, with the nominal controller parameters K_1^{*} and K_2^{*} satisfying the matching condition (4.1.5):

$$C(sI - A - BK_1^{*T})^{-1}BK_2^* = W_m(s), \ K_2^{*-1} = K_p,$$

ensures plant-model output matching: $y(t) - y_m(t) = \varepsilon(t)$, for some initial conditionrelated exponentially decaying $\varepsilon(t)$, where $y_m(t)$ is the output of the reference model (4.1.3).

Proof: Representing $\hat{x}(t) = x(t) + \varepsilon_2(t)$ with $\varepsilon_2(t)$ being an exponential decaying term, the observer-based control law can be expressed as $u(t) = K_1^{*T}x(t) + K_1^{*T}\varepsilon_2(t) + K_2^{*r}(t)$. Substituting this u(t) into the plant (4.1.1), the output y(t) becomes $y(t) = C(sI - A - BK_1^{*T})^{-1}BK_1^{*T}[\varepsilon_2](t) + C(sI - A - BK_1^{*T})^{-1}BK_2^{*}[r](t)$. From the output matching condition: $C(sI - A - BK_1^{*T})^{-1}BK_2^{*} = W_m(s)$, we have $y(t) - y_m(t) = \varepsilon(t)$, for some $\varepsilon(t) = C(sI - A - BK_1^{*T})^{-1}BK_1^{*T}[\varepsilon_2](t)$. ∇

Lemma 4.2.1 confirms the existence of the nominal controller parameters K_1^* and K_2^* of the observer-based control law $u(t) = K_1^{*T} \hat{x}(t) + K_2^* r(t)$, for ensuring plant-model matching.

Matching by partial-state feedback control. We now present the desired output matching properties by the nominal partial-state feedback controller (4.2.7).

Theorem 4.2.1. Constant parameters Θ_1^* , Θ_2^* , Θ_{20}^* and Θ_3^* exist such that the controller (4.2.7) guarantees closed-loop signal boundedness and partial-state feedback based output matching: $y(t) - y_m(t) = \varepsilon(t)$, for some exponentially decaying $\varepsilon(t)$.

Proof: The proof can be divided into two parts. The first part is for plant-model output matching by the controller (4.2.7) and the controller parameters Θ_1^* , Θ_2^* , Θ_{20}^*

and Θ_3^* , which is guaranteed based on the derivation of the partial-state feedback controller shown in Section 4.2.1.

The second part is for closed-loop signal boundedness. From the plant-model output matching property: $y(t) = y_m(t) + \varepsilon(t) \in L^\infty$, we have $\xi_m(s)[y](t) \in L^\infty$ since $\xi_m(s)[y_m](t) = r(t)$ and $\xi_m(s)[\varepsilon](t)$ are bounded (as $\varepsilon(t)$ is exponentially decaying).

From y(t) = G(s)[u](t) with $G(s) = C(sI - A)^{-1}B$ having full rank, ignoring the exponentially decaying effect of the initial conditions, we have $u(t) = G^{-1}(s)\xi_m^{-1}(s) \times$ $\xi_m(s)[y](t)$, which is bounded, because $G^{-1}(s)\xi_m^{-1}(s)$ is stable and proper and $\xi_m(s)[y](t)$ is bounded.

According to the full-state observer theory, for (A, C) detectable, we can express the system state x(t) as

$$\begin{aligned} x(t) &= (sI - A + LC)^{-1}B[u](t) + (sI - A + LC)^{-1}L[y](t) \\ &= \frac{N_{01}(s)}{\Lambda_0(s)}[u](t) + \frac{N_{02}(s)}{\Lambda_0(s)}[y](t), \end{aligned}$$
(4.2.8)

where the eigenvalues of the $n \times n$ matrix A - LC are stable for some constant gain vector $L \in \mathbb{R}^{n \times M}$, $\Lambda_0(s) = \det(sI - A + LC)$ whose degree is n, L is a matrix such that $N_{01}(s) = \operatorname{adj}(sI - A + LC)B$ and $N_{02}(s) = \operatorname{adj}(sI - A + LC)L$ are $n \times M$ polynomial matrices whose maximum degrees are n - 1. Hence, the internal state x(t) is bounded as u(t) and y(t) are bounded, and so is $y_0(t) = C_0 x(t)$. Also, it turns out the boundedness of $\omega_1(t) = \frac{A_1(s)}{\Lambda(s)}[u](t), \, \omega_2(t) = \frac{A_2(t)}{\Lambda(t)}[y_0](t)$. ∇

Theorem 3.1 shows that when the system parameter (A, B, C) are known, the partial-state feedback control law (4.2.7) with the nominal controller parameters given in (4.2.6) and $\Theta_3^* = K_2^*$ solves the nonadpative partial-state feedback model reference adaptive control problem.

In addition, the nominal controller parameters $\Theta_1^*, \Theta_2^*, \Theta_{20}^*$ and Θ_3^* for output matching can also be found through a matching polynomial equation.

Corollary 4.2.1. For partial-state feedback multivariable model reference control, constant parameter matrices $\Theta_1^* \in \mathbb{R}^{M(n-n_0) \times M}$, $\Theta_2^* \in \mathbb{R}^{n_0(n-n_0) \times M}$, $\Theta_{20}^* \in \mathbb{R}^{n_0 \times M}$ and $\Theta_3^* \in \mathbb{R}^{M \times M}$ exist such that the output matching equation holds:

$$\Theta_1^{*T} A_1(s) P(s) + (\Theta_2^{*T} A_2(s) + \Theta_{20}^{*T} \Lambda(s)) Z_0(s)$$

= $\Lambda(s) (P(s) - \Theta_3^{*} K_p \xi_m(s) Z(s)).$ (4.2.9)

Proof: With $y_0(t) = G_0(s)[u](t)$ and $G_0(s) = C_0(sI - A)^{-1}B$, the transfer function matrix of the closed-loop system is

$$G_{c}(s) = G(s) \left(I_{M} - \Theta_{1}^{*T} \frac{A_{1}(s)}{\Lambda(s)} - \left(\Theta_{2}^{*T} \frac{A_{2}(s)}{\Lambda(s)} + \Theta_{20}^{*T} \right) G_{0}(s) \right)^{-1} \Theta_{3}^{*}, \qquad (4.2.10)$$

which has been made to match $W_m(s) = \xi_m^{-1}(s)$. From $G_c(s) = W_m(s)$, we obtain

$$I_M - \Theta_1^{*\mathrm{T}} \frac{A_1(s)}{\Lambda(s)} - (\Theta_2^{*\mathrm{T}} \frac{A_2(s)}{\Lambda(s)} + \Theta_{20}^{*\mathrm{T}}) G_0(s) = \Theta_3^* W_m^{-1}(s) G(s), \qquad (4.2.11)$$

which, for $G(s) = Z(s)P^{-1}(s)$ and $G_0(s) = Z_0(s)P^{-1}(s)$, can be expressed as (4.2.9). Hence, there exist Θ_1^* , Θ_2^* , Θ_{20}^* and Θ_3^* satisfying the matching equation (4.2.9). ∇

Such a matching equation is also crucial for deriving the tracking error model for the adaptive control design in the next section.

4.2.2 Adaptive Partial-State Feedback Control

For the plant (4.1.1) with unknown (A, B, C), nominal controller parameters Θ_1^* , Θ_2^* , Θ_{20}^* and Θ_3^* in (4.2.7) depending on system parameters (A, B, C) can not be calculated so that the nominal partial-state feedback control law cannot be applied to the plant (4.1.1). Thus, an adaptive partial-state feedback controller is needed to deal with the parameter uncertainties. For adaptive control, we need the following assumption:

(A4.4) all leading principle minors Δ_i , i = 1, 2, ..., M, of the high frequency matrix K_p of G(s) are nonzero and their signs are known.

4.2.2.1 Adaptive Controller and Error Model

In this subsection, we propose an adaptive partial-state feedback controller structure, and derive a tracking error equation.

Controller structure. To handle the plant (4.1.1) with (A, B, C) unknown, we design the adaptive version of the controller (4.2.7) as

$$u(t) = \Theta_1^{\mathrm{T}}(t)\omega_1(t) + \Theta_2^{\mathrm{T}}(t)\omega_2(t) + \Theta_{20}^{\mathrm{T}}(t)y_0(t) + \Theta_3(t)r(t), \qquad (4.2.12)$$

where $\Theta_1(t) \in \mathbb{R}^{M(n-n_0) \times M}$, $\Theta_2(t) \in \mathbb{R}^{n_0(n-n_0) \times M}$, $\Theta_{20}(t) \in \mathbb{R}^{n_0 \times M}$, $\Theta_3(t) \in \mathbb{R}^{M \times M}$ are the adaptive estimates of the unknown nominal parameters Θ_1^* , Θ_2^* , Θ_{20}^* , Θ_3^* (defined from (4.2.6) or (4.2.9)), respectively, and

$$\omega_1(t) = \frac{A_1(s)}{\Lambda(s)} [u](t), \ \omega_2(t) = \frac{A_2(s)}{\Lambda(s)} [y_0](t)$$
(4.2.13)

with $A_1(s) = [I_M, sI_M, ..., s^{n-n_0-1}I_M]^T$, $A_2(s) = [I_{n_0}, sI_{n_0}, ..., s^{n-n_0-1}I_{n_0}]^T$, and $\Lambda(s)$ being a monic stable polynomial of degree $n - n_0$.

Tracking error equation. Recall the equation (4.2.11):

$$I_M - \Theta_1^{*\mathrm{T}} \frac{A_1(s)}{\Lambda(s)} - (\Theta_2^{*\mathrm{T}} \frac{A_2(s)}{\Lambda(s)} + \Theta_{20}^{*\mathrm{T}}) G_0(s) = \Theta_3^* W_m^{-1}(s) G(s).$$

For y(t) = G(s)[u](t) and $y_0(t) = G_0(s)[u](t)$, we operate u(t) on both sides of (4.2.11), and have the signal identity:

$$u(t) - \Theta_1^{*\mathrm{T}} \frac{A_1(s)}{\Lambda(s)} [u](t) - (\Theta_2^{*\mathrm{T}} \frac{A_2(s)}{\Lambda(s)} + \Theta_{20}^{*\mathrm{T}}) [y_0](t) = \Theta_3^{*} W_m^{-1}(s) [y](t), \qquad (4.2.14)$$

Such an equation leads to

$$u(t) = \Theta_1^{*T} \frac{A_1(s)}{\Lambda(s)} [u](t) + (\Theta_2^{*T} \frac{A_2(s)}{\Lambda(s)} [y_0](t) + \Theta_{20}^{*T} y_0(t) + \Theta_3^* \xi_m(s) [y](t).$$
(4.2.15)

Substituting (4.2.12) from (4.2.15) with $r(t) = \xi_m(s)[y_m](t)$, we obtain the tracking error equation as

$$e(t) = y(t) - y_m(t) = W_m(s)K_p[u - \Theta^{*T}\omega](t), \qquad (4.2.16)$$

where $\Theta^* = \left[\Theta_1^{*T}, \Theta_2^{*T}, \Theta_{20}^{*T}, \Theta_3^{*}\right]^T$, $\omega(t) = \left[\omega_1^T(t), \omega_2^T(t), y_0^T(t), r^T(t)\right]^T$. Such an equation can be used to develop different parameterizations for adaptive control designs, using different decompositions of K_p .

LDS decomposition of K_p . Given that all principle minors of the high-frequency gain matrix K_p are non-zero, the LDS decomposition of K_p exists and can be employed for dealing with the uncertainty of the unknown matrix K_p .

Lemma 4.2.2. [82] The high-frequency gain matrix $K_p \in \mathbb{R}^{M \times M}$ with all leading principle minors nonzero has a non-unique decomposition: $K_p = L_s D_s S$, where $S \in \mathbb{R}^{M \times M}$ is such that $S = S^T > 0$, $L_s \in \mathbb{R}^{M \times M}$ is a unit upper triangle matrix, and $D_s = \text{diag}\{s_1^*, s_2^*, \ldots, s_M^*\} = \text{diag}\{\text{sign}[d_1^*]\gamma_1, \ldots, \text{sign}[d_M^*]\gamma_M\}$ with arbitrary and chosen constant $\gamma_i > 0$, $i = 1, 2, \ldots, M$.

To employ this LDS decomposition of K_p for adaptive control, substituting $K_p = L_s D_s S$ into the tracking error equation (4.2.16), with u(t) from (4.2.12), we have $L_s^{-1} \xi_m(s)[e](t) = D_s S \tilde{\Theta}^{\mathrm{T}}(t) \omega(t)$, where $\tilde{\Theta}(t) = \Theta(t) - \Theta^*(t)$ with $\Theta(t) = \left[\Theta_1^{\mathrm{T}}(t), \Theta_2^{\mathrm{T}}(t), \Theta_2^{\mathrm{T}}(t), \Theta_2^{\mathrm{T}}(t), \Theta_3^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ being the estimate of $\Theta^* = \left[\Theta_1^{*\mathrm{T}}, \Theta_2^{*\mathrm{T}}, \Theta_{20}^{*\mathrm{T}}, \Theta_3^{*}\right]^{\mathrm{T}}$.

To parametrize the unknown matrix L_s , introducing a constant matrix $\Theta_0^* = L_s^{-1} - I = \{\theta_{ij}^*\}$ with $\theta_{ij}^* = 0$ for i = 1, 2, ..., M and $j \ge i$, we have

$$\xi_m(s)[e](t) + \Theta_0^* \xi_m(s)[e](t) = D_s S \Theta^{\mathrm{T}}(t) \omega(t).$$
(4.2.17)

To parametrize this tracking error equation, choosing a filter $h(s) = \frac{1}{f(s)}$, where f(s)is a stable and monic polynomial whose degree is equal to the maximum degree of the modified interactor matrix $\xi_m(s)$ and operating $h(s)I_M$ on both sides of (4.2.17), we have

$$\bar{e}(t) + \left[0, \,\theta_2^{*^{\mathrm{T}}} \eta_2(t), \,\theta_3^{*^{\mathrm{T}}} \eta_3(t), \dots, \theta_M^{*^{\mathrm{T}}} \eta_M(t)\right]^{\mathrm{T}} = D_s Sh(s) [\tilde{\Theta}^{\mathrm{T}} \omega](t), \qquad (4.2.18)$$

where

$$\bar{e}(t) = \xi_m(s)h(s)[e](t) = [\bar{e}_1(t), \dots, \bar{e}_M(t)]^{\mathrm{T}},$$

$$\eta_i(t) = [\bar{e}_1(t), \dots, \bar{e}_{i-1}(t)]^{\mathrm{T}} \in \mathbb{R}^{i-1}, i = 2, \dots, M,$$

$$\theta_i^* = [\theta_{i1}^*, \dots, \theta_{ii-1}]^{\mathrm{T}}, i = 2, \dots, M.$$
(4.2.19)

Estimation error model based on LDS decomposition of K_p . Based on the tracking error equation (4.2.18), we introduce the estimation error signal:

$$\epsilon(t) = \left[0, \,\theta_2^{\mathrm{T}}\eta_2(t), \,\theta_3^{\mathrm{T}}\eta_3(t), \dots, \theta_M^{\mathrm{T}}\eta_M(t)\right]^{\mathrm{T}} + \Psi(t)\xi(t) + \bar{e}(t), \qquad (4.2.20)$$

with $\Psi(t)$ being the estimate of $\Psi^* = D_s S$, and

$$\xi(t) = \Theta^{\mathrm{T}}(t)\zeta(t) - h(s)[\Theta^{\mathrm{T}}\omega](t), \zeta(t) = h(s)[\omega](t).$$
(4.2.21)

It follows from (4.2.18)-(4.2.21) that

$$\epsilon(t) = \left[0, \,\tilde{\theta}_2^{\mathrm{T}}\eta_2(t), \,\tilde{\theta}_3^{\mathrm{T}}\eta_3(t), \dots, \tilde{\theta}_M^{\mathrm{T}}\eta_M(t)\right]^{\mathrm{T}} + D_s S \tilde{\Theta}^{\mathrm{T}}(t)\zeta(t) + \tilde{\Psi}(t)\xi(t), \quad (4.2.22)$$

where $\tilde{\theta}_i(t) = \theta(t) - \theta_i^*$, i = 2, ..., M, and $\tilde{\Psi}(t) = \Psi(t) - \Psi^*(t)$ are parameter errors. Such an error equation is linear in parameter errors, which is crucial for choosing the adaptive laws for updating the controller parameters.

4.2.2.2 Adaptive Parameter Update Law

Based on the error model (4.2.22), the adaptive laws for updating parameter estimates are chosen as

$$\dot{\theta}_i(t) = -\frac{\Gamma_{\theta i}\epsilon_i(t)\eta_i(t)}{m^2(t)}, i = 2, 3, \dots, M$$
(4.2.23)

$$\dot{\Theta}^{\mathrm{T}}(t) = -\frac{D_s \epsilon(t) \zeta^{\mathrm{T}}(t)}{m^2(t)}, \ \dot{\Psi}(t) = -\frac{\Gamma \epsilon(t) \xi^{\mathrm{T}}(t)}{m^2(t)},$$
(4.2.24)

where $\epsilon(t) = [\epsilon_1(t), \epsilon_2(t), \dots, \epsilon_M(t)]^{\mathrm{T}}$ is computed from (4.2.20), $\Gamma_{\theta i} = \Gamma_{\theta i}^{\mathrm{T}} > 0$, $i = 2, 3, \dots, M$ and $\Gamma = \Gamma^{\mathrm{T}} > 0$ are adaption gain matrices, and

$$m^{2}(t) = 1 + \zeta^{\mathrm{T}}(t)\zeta(t) + \xi^{\mathrm{T}}(t)\xi(t) + \sum_{i=2}^{M} \eta_{i}^{\mathrm{T}}(t)\eta_{i}(t). \qquad (4.2.25)$$

With the positive definition function

$$V = \frac{1}{2} \left(\sum_{i=2}^{M} \tilde{\theta}_{i}^{\mathrm{T}}(t) \Gamma_{\theta i}^{-1} \tilde{\theta}_{i} + \mathrm{tr}[\tilde{\Psi}^{\mathrm{T}} \Gamma^{-1} \tilde{\Psi}] + \mathrm{tr}[\tilde{\Theta} S \tilde{\Theta}^{\mathrm{T}}] \right), \qquad (4.2.26)$$

and its time-derivative $\dot{V} = -\frac{\epsilon^{\mathrm{T}(t)}\epsilon(t)}{m^{2}(t)} \leq 0$, we conclude that (i) $\theta_{i}(t) \in L^{\infty}$, $i = 2, 3, \ldots, M$, $\Theta(t) \in L^{\infty}$, $\Psi(t) \in L^{\infty}$, $\frac{\epsilon(t)}{m(t)} \in L^{2} \cap L^{\infty}$, and (2) $\dot{\theta}_{i}(t) \in L^{2} \cap L^{\infty}$, $i = 2, 3, \ldots, M$, $\dot{\Theta}(t) \in L^{2} \cap L^{\infty}$ and $\dot{\Psi}_{i}(t) \in L^{2} \cap L^{\infty}$. The L^{∞} and L^{2} properties of these signals are crucial for closed-loop stability, as shown next.

4.2.2.3 System Stability and Tracking Properties

Based on the above desired properties of the adaptive law (4.2.23)-(4.2.24), the following desired closed-loop system properties are established.

Theorem 4.2.2. The adaptive partial-state feedback controller (4.2.12) with the adaptive law (4.2.23)–(4.2.24), when applied to the plant (4.1.1), guarantees the closed-loop signal boundedness and asymptotic output tracking: $\lim_{t\to\infty} (y(t) - y_m(t)) = 0$.

The proof of Theorem 4.2.2 can be obtained in a similar way to that described in [82] for output feedback design. The proof is based on a well-defined feedback structure for the closed-loop system which has a small loop gain, leading to closedloop stability. A key step in such an analysis procedure is to express a filtered version of the plant output y(t) in a feedback framework which has a small gain due to the L^2 properties of the adaptive laws. The asymptotic tracking property follows from the complete parametrization of the error equation (4.2.20), the L^2 properties, and the signal boundedness of the closed-loop system. To cope with the partial-state signal $y_0(t)$ in the new partial-state feedback control law, we need to express $y_0(t)$ in terms of the output y(t), for which a new proof derivation is necessary. A detailed proof is shown as follows.

Proof of Theorem 4.2.2.

<u>Step 1: introducing filtered signals for signal processing.</u> Introduce some fictitious filters $H_i(s)$ and $K_i(s)$ as

$$sH_i(s) = 1 - K_i(s), \ K_i(s) = \frac{a_i^{d_m}}{(s+a_i)^{d_m}}, \ i = 1, 2, 3,$$
 (4.2.27)

where $a_i > 0$ is chosen to be sufficiently large and finite for i = 1, 2, 3, and d_m is the maximum degree of the modified interactor matrix $\xi_m(s)$ of G(s).

Denote $h_i(t)$ as the impulse response functions of the transfer function $H_i(s)$, i = 1, 2, 3. From Proposition 2.10 in [82], we have the L^1 operator norms

$$||h_i(\cdot)|| = \frac{d_m}{a_i}, \ a_i > 0, \ i = 1, 2, 3.$$
 (4.2.28)

From y(t) = G(s)[u](t) with G(s) being full rank, $\omega_1(t) = F_1(s)[u](t) = \frac{A_1(s)}{\Lambda(s)}[u](t)$ in (4.2.13) and $H_1(s)$, $K_1(s)$ in (4.2.27), we obtain

$$F_1(s)G^{-1}(s)[y](t) = K_1^{-1}(s)[\omega_1 - H_1(s)s[\omega_1]](t).$$
(4.2.29)

<u>Step 2: expressing $y_0(t)$ by u(t) and y(t).</u> To handle the new partial-state feedback control scheme, we need the transformation for the partial-state signal $y_0(t)$. First, recall the expression of internal state x(t) in (4.2.8): $x(t) = \frac{N_{01}(s)}{\Lambda_0(s)}[u](t) + \frac{N_{02}(s)}{\Lambda_0(s)}[y](t)$ where $\frac{N_{01}(s)}{\Lambda_0(s)}$ and $\frac{N_{02}(s)}{\Lambda_0(s)}$ are stable and proper.

It follows that the partial-state signal $y_0(t) = C_0 x(t)$ can be expressed as

$$y_{0}(t) = C_{0} \frac{N_{01}(s)}{\Lambda_{0}(s)} [u](t) + C_{0} \frac{N_{02}(s)}{\Lambda_{0}(s)} [y](t)$$

= $Q_{1}(s)[u](t) + Q_{2}(s)[y](t)$ (4.2.30)

with $Q_1(s) = C_0 \frac{N_{01}(s)}{\Lambda_0(s)}$ and $Q_2(s) = C_0 \frac{N_{02}(s)}{\Lambda_0(s)}$ being stable and proper.

<u>Step 3: establishing a relationship between u(t) and a filtered y(t).</u> Let $\omega_1(s) = F_1(s)[u](t)$ have a controllable realization (A_c, B_c) :

$$\dot{\omega}_1(t) = A_c \omega_1(t) + B_c u(t), \qquad (4.2.31)$$

where A_c is a stable matrix. From (4.2.12), (4.2.29), (4.2.30), (4.2.31) and $\omega_2(t) = F_2(s)[y_0](t) = \frac{A_2(s)}{\Lambda(s)}[y_0](t)$ in (4.2.13) we obtain

$$\omega_{1}(t) = K_{1}(s)F_{1}(s)G^{-1}(s)[y](t) + H_{1}(s)[\dot{\omega}_{1}](t)$$

= $K_{1}(s)F_{1}(s)G^{-1}(s)[y](t) + H_{1}(s)[A_{c}\omega_{1}](t) + H_{1}(s)B_{c}[\Theta_{1}^{\mathrm{T}}\omega_{1} + \Theta_{2}^{\mathrm{T}}F_{2}(s)Q_{1}(s)[u]$
+ $\Theta_{2}^{\mathrm{T}}F_{2}(s)Q_{2}(s)[y] + \Theta_{20}^{\mathrm{T}}Q_{1}(s)[u] + \Theta_{20}^{\mathrm{T}}Q_{2}(s)[y] + \Theta_{3}r](t).$ (4.2.32)

Since $H_1(s)$ satisfies (4.2.28) and $\Theta_1(t)$ is bounded, there exists $a_1^0 > 0$ such that $(I - H_1(s)(A_c + B_c\Theta_1^{\mathrm{T}}(t)))^{-1}$ is a stable and proper operator with a finite gain for any finite $a_1 > a_1^0$. The above fact can be proved similarly to the proof of Lemma 2.5 in [82]. For $0 < a_1^0 < a_1$, it follows from (4.2.32) that

$$\omega_1(t) = G_1(s, \cdot)[u](t) + G_2(s, \cdot)[y](t) + G_3(s, \cdot)[r](t), \qquad (4.2.33)$$

where for $T_1(s,t) \triangleq (I - H_1(s)(A_c + B_c \Theta_1^{\mathrm{T}}(t)))^{-1},$

$$G_{1}(s,t) = T_{1}(s,t)(H_{1}(s)B_{c}\Theta_{2}^{\mathrm{T}}(t)F_{2}(s)Q_{1}(s) + H_{1}(s)B_{c}\Theta_{20}^{\mathrm{T}}(t)Q_{1}(s))$$

$$G_{2}(s,t) = T_{1}(s,t)(K_{1}(s)F_{1}(s)G^{-1}(s) + H_{1}(s)B_{c}\Theta_{2}^{\mathrm{T}}(t)F_{2}(s)Q_{2}(s) + H_{1}(s)B_{c}\Theta_{20}^{\mathrm{T}}(t)Q_{2}(s))$$

$$G_{3}(s,t) = T_{1}(s,t)H_{1}(s)B_{c}\Theta_{3}^{\mathrm{T}}(t)$$

$$(4.2.34)$$

are stable and proper operators with finite gains¹. It follows from (4.2.33) with $\omega_1(t) = F_1(s)[u](t), \ \omega_2(t) = F_2(s)[y_0](t)$ and $\omega(t) = \left[\omega_1^{\mathrm{T}}(t), \ \omega_2^{\mathrm{T}}(t), \ y_0^{\mathrm{T}}(t), r^{\mathrm{T}}(t)\right]^{\mathrm{T}}$,

¹A linear operator T(s,t) is stable and proper if $|T(s,\cdot)[x](t)| \leq \beta \int_0^T e^{-\alpha(t-\tau)} |x(\tau)| d\tau + \gamma |x(t)|$ for some constants $\beta \geq 0$, $\alpha > 0$ and $\gamma > 0$, for all $t \geq 0$, where $T(s,\cdot)[x](t)$ denotes the convolution of the impulse response of $T(s,\cdot)$ with $x(\cdot)$ at t. A linear operator T(s,t) is stable and strictly proper if it is stable with $\gamma = 0$.

that

$$\omega(t) = G_4(s, \cdot)[u](t) + G_5(s, \cdot)[y](t) + G_6(s, \cdot)[r](t), \qquad (4.2.35)$$

where

$$G_4(s,t) = [G_1(s,t), F_1(s)Q_1(s), Q_1(s), 0]^{\mathrm{T}},$$

$$G_5(s,t) = [G_2(s,t), F_2(s)Q_2(s), Q_2(s), 0]^{\mathrm{T}},$$

$$G_6(s,t) = [G_3(s,t), 0, 0, I]^{\mathrm{T}}.$$
(4.2.36)

From (4.2.16), we have

$$\dot{y}(t) = \dot{y}_m(t) + sW_m(s)\Theta_3^{*-1}[\tilde{\Theta}^{\mathrm{T}}\omega](t).$$
 (4.2.37)

Operating $H_2(s)$ on both sides of (4.2.37) and noting that $sH_2(s) = 1 - K_2(s)$, we have

$$y(t) = K_2(s)h^{-1}(s)[\bar{y}](t) + H_2(s)sW_m(s)[r](t) + H_2(s)sW_m(s)\Theta_3^{*-1}\tilde{\Theta}^{\mathrm{T}}[G_4(s,\cdot)[u] + G_5(s,\cdot)[y] + G_6(s,\cdot)[r]](t)$$
(4.2.38)

with $\bar{y}(t) \triangleq h(s)[y](t)$. Similar to the operator $T_1(s,t)$, $(I-H_2(s)sW_m(s)\Theta_3^{*-1}\tilde{\Theta}^{\mathrm{T}}G_5(s,t))^{-1}$ can be proved to be a stable and proper operator with a finite gain for any finite $a_2 > a_2^0$ and some $a_2^0 > 0$. For $0 < a_2^0 < a_2$, it follows from (4.2.38) that

$$y(t) = G_7(s, \cdot)[u](t) + G_8(s, \cdot)[\bar{y}](t) + G_9(s, \cdot)[r](t), \qquad (4.2.39)$$

where for $T_2(s,t) \triangleq (I - H_2(s)sW_m(s)\Theta_3^{*-1}\tilde{\Theta}^{\mathrm{T}}G_5(s,t))^{-1}$,

$$G_{7}(s,t) = T_{2}(s,t)H_{2}(s)sW_{m}(s)\Theta_{3}^{*-1}\tilde{\Theta}^{T}G_{4}(s,\cdot)$$

$$G_{8}(s,t) = T_{2}(s,t)K_{2}(s)h^{-1}(s)$$

$$G_{9}(s,t) = T_{2}(s,t)H_{2}(s)sW_{m}(s)(I + \Theta_{3}^{*-1}\tilde{\Theta}^{T}G_{6}(s,\cdot))$$
(4.2.40)

are stable and proper operators with finite gains. It follows from (4.2.35) and (4.2.39) that

$$\omega(t) = (G_4(s, \cdot) + G_5(s, \cdot)G_7(s, \cdot))[u](t) + G_5(s, \cdot)G_8(s, \cdot)[\bar{y}](t) + (G_5(s, \cdot)G_9(s, \cdot) + G_6(s, \cdot))[r](t).$$
(4.2.41)

From (4.2.20), we express

$$\bar{y}(t) = \bar{y}_m(t) + W_m(s)[\epsilon - \Psi\xi - \chi](t)$$
(4.2.42)

with $\bar{y}_m(t) = h(s)[y_m](t)$ and $\chi = \left[0, \theta_2^{\mathrm{T}}\eta_2(t), \theta_3^{\mathrm{T}}\eta_3(t), \ldots, \theta_M^{\mathrm{T}}\eta_M(t)\right]^{\mathrm{T}}$. From (4.2.12) and (4.2.41), we obtain

$$u(t) = \Theta^{\mathrm{T}}(t)(G_{4}(s,\cdot) + G_{5}(s,\cdot)G_{7}(s,\cdot))[u](t) + \Theta^{\mathrm{T}}(t)G_{5}(s,\cdot)G_{8}(s,\cdot)[\bar{y}](t) + \Theta^{\mathrm{T}}(t)(G_{6}(s,\cdot) + G_{5}(s,\cdot)G_{9}(s,\cdot))[r](t).$$
(4.2.43)

From (4.2.43), it follows that

$$u(t) = G_{10}(s, \cdot)\Theta^{\mathrm{T}}(t)G_{5}(s, \cdot)G_{8}(s, \cdot)[\bar{y}](t)$$

$$+ G_{10}(s, \cdot)\Theta^{\mathrm{T}}(t)(G_{6}(s, \cdot) + G_{5}(s, \cdot)G_{9}(s, \cdot))[r](t)$$
(4.2.44)

where $G_{10}(s,t) = (I - \Theta^{\mathrm{T}}(t)(G_4(s,\cdot) + G_5(s,\cdot)G_7(s,\cdot)))^{-1}$ is stable and proper operators with finite gains.

<u>Step 4: formulating a closed-loop inequality of the filtered y(t).</u> From (4.2.21), we denote $\xi(t) = [\xi_1(t), \dots, \xi_M(t)]^{\mathrm{T}}$, $\Theta(t) = [\bar{\theta}_1^{\mathrm{T}}(t), \dots, \bar{\theta}_M^{\mathrm{T}}(t)]^{\mathrm{T}}$ with $\bar{\theta}_i(t) \in \mathbb{R}^{(n_0+M)(n-n_0+1)}$, $i = 1, \dots, M$ and $f(s) = s^{d_m} + \hat{a}_{d_m}s^{d_m-1} + \dots + \hat{a}_1s + \hat{a}_0$. Then $\xi_i(t) = \bar{\theta}_i^{\mathrm{T}}(t)\zeta(t) - \frac{1}{f(s)}[\bar{\theta}_i^{\mathrm{T}}\omega](t)$, $i = 1, \dots, M$ can be expressed as

$$\xi_{i}(t) = \frac{s^{d_{m}-1} + \hat{a}_{d_{m}-1}s^{d_{m}-2} + \dots + \hat{a}_{2}s + \hat{a}_{1}}{f(s)} [\dot{\theta}_{i}^{\mathrm{T}} \frac{1}{f(s)}[\omega]](t) + \frac{s^{d_{m}-2} + \hat{a}_{d_{m}-1}s^{d_{m}-3} + \dots + \hat{a}_{2}}{f(s)} [\dot{\theta}_{i}^{\mathrm{T}} \frac{s}{f(s)}[\omega]](t)$$

$$+\dots + \frac{s + \hat{a}_{d_m-1}}{f(s)} [\dot{\theta}_i^{\mathrm{T}} \frac{s^{d_m-2}}{f(s)} [\omega]](t) + \frac{1}{f(s)} [\dot{\theta}_i^{\mathrm{T}} \frac{s^{d_m-1}}{f(s)} [\omega]](t).$$
(4.2.45)

Since $r(t) \in L^{\infty}$, from (4.2.39), (4.2.42), (4.2.44) and (4.2.45), we have

$$\|\bar{y}(t)\| \le x_0 + T_3(s, \cdot) [x_1 T_4(s, \cdot)] \|\bar{y}(t)\|](t)$$
(4.2.46)

for some $x_0(t) \in L^{\infty}$, $x_1(t) \in L^{\infty} \cap L^2$ with $x_1(t) \ge 0$, some stable and strictly proper operator $T_3(s,t)$, and some stable and proper operator $T_4(s,t)$ with a non-negative impulse response function. It follows that

$$\|\bar{y}(t)\| \le x_0(t) + \beta_1 \int_0^T e^{-\alpha_1(t-\tau)} x_1(\tau) (\int_0^\tau e^{-\alpha_2(\tau-\omega)} \|\bar{y}(\omega)\| \, d\omega) d\tau \qquad (4.2.47)$$

for some β_1 , α_1 , $\alpha_2 > 0$.

<u>Step 5: applying Gronwall-Bellman Lemma for signal boundedness.</u> Applying the Small Gain Lemma (Lemma 2.3 in [82]) on (4.2.47), we conclude that $\bar{y}(t)$ is bounded, so are u(t) in (4.2.44) and y(t) in (4.2.39). We can also obtain that $\omega(t) \in L^{\infty}$ in (4.2.41), $x(t) \in L^{\infty}$ in (4.2.8), $y_0(t) \in L^{\infty}$ in (4.2.30), $\zeta(t) \in L^{\infty}$ in (4.2.21), $\xi(t) \in L^{\infty}$ in (4.2.19), $\eta_i(t) \in L^{\infty}$ in (4.2.19), $m(t) \in L^{\infty}$ in (4.2.25) and $\epsilon(t) \in L^{\infty}$ in (4.2.20).

For $\bar{e}(t) = \xi_m(s)h(s)[e](t)$, we have

$$e(t) = W_m(s)\Theta_3^{*-1}[\tilde{\Theta}^{\mathrm{T}}\omega](t)$$

$$= H_3(s)sW_m(s)\Theta_3^{*-1}[\tilde{\Theta}^{\mathrm{T}}\omega](t) + W_m(s)K_3(s)h^{-1}(s)[\bar{e}](t)$$
(4.2.48)

where

$$\lim_{t \to \infty} W_m(s) K_3(s) h^{-1}(s) [\bar{e}](t) = 0$$
(4.2.49)

for a finite $a_3 > 0$ in $K_3(s)$, and $sW_m(s)\Theta_3^{*-1}[\tilde{\Theta}^{\mathrm{T}}\omega](t) \in L^{\infty}$. From (4.2.48) and (4.2.49), we get

$$\|e(t)\| \le c_3 \|h_3(t)\|_1 + z_1(t) \le \frac{c_4}{a_3} + z_1(t)$$
(4.2.50)

where c_3 , $c_4 > 0$ and $\lim_{t\to\infty} z_1(t) = 0$. Since $a_3 > 0$ in $H_3(s)$ can be set arbitrarily large, from (4.2.50), we can conclude that $\lim_{t\to\infty} e(t) = 0$. ∇

This new partial-state feedback multivariable MRAC scheme has never been reported in the literature, it has several unique features that will be discussed in the next section.

4.3 Features of Partial-State Feedback Multivariable MRAC Framework

In this section, we discuss some advantages and unique features of the newly developed partial-state feedback adaptive control framework.

4.3.1 Unification of Multivariable MRAC

As we have mentioned before, the use of the partial-state $y_0(t) = C_0 x(t)$ provides new flexibilities in designing MRAC schemes for four typical cases of $y_0(t) = C_0 x(t)$:

- (1) $y_0(t)$ is a vector containing some or all elements of y(t);
- (2) $y_0(t)$ is vector which does not contain any element of y(t);
- (3) $y_0(t)$ is a scalar as one element of y(t); and
- (4) $y_0(t)$ is a scalar not being any element of y(t).

Among the four cases listed above, case (3) is a special case of the case (1) when $y_0(t) \in \mathbb{R}$, and the case (4) is a special case of the case (2) when $y_0(t) \in \mathbb{R}$. In addition, two more extreme cases are considered:

(5) $y_0(t)$ is the output y(t), and

(6) $y_0(t)$ is the state x(t).

Case (5) makes the partial-state feedback scheme become to a pure output feedback MRAC scheme, and case (6) makes the partial-state feedback MRAC scheme become to a pure state feedback MRAC scheme.

The above six cases cover all kinds of possible feedback control signals one may have for output tracking, which shows the design flexibility and application significance of partial-state feedback MRAC. In other words, the partial-state feedback MRAC scheme provides a unified control scheme that bridges the state feedback control and output feedback control together. It is the unified solution to all multivariable MRAC problems for output tracking, and adds new design possibilities to the MRAC family.

With the help of using the partial-state signal $y_0(t)$, this partial-state feedback multivariable MRAC schemes combines the advantages of the state feedback control design and the output feedback control design. It provides a manageable trade-off between the two existing schemes.

4.3.2 Reduction of Adaptive System Complexity

When n_0 satisfies some certain conditions, the developed partial-state feedback MRAC scheme reduces the adaptation complexity, compared to an output feedback MRAC scheme. In this chapter, we use the number of updated parameters and the number of first-order integrator to measure the system adaptation complexity.

Number of updated parameters. According to the adaptive law (4.2.23)–(4.2.24), the total number of parameters to be updated in the partial-state feedback adaptive law (4.2.23)–(4.2.24) is

$$N_{ps} = \frac{M^2 - M}{2} + (n - n_0)M^2 + (n - n_0)Mn_0 + Mn_0 + M^2 + M^2.$$
(4.3.1)

On the other hand, the total number of parameters to be updated in an output feedback for output tracking adaptive law is

$$N_o = \frac{M^2 - M}{2} + 2(\bar{\nu} - 1)M^2 + 2M^2 + M^2$$
(4.3.2)

with $\bar{\nu}$ being the upper bound of the observability index. According to [82], the range of the observability index ν is $\frac{n}{M} \leq \nu \leq n - M + 1$. Thus, we have $N_o = \frac{M^2 - M}{2} + 2(n - M)M^2 + 2M^2 + M^2 = \frac{M^2 - M}{2} - 2M^3 + (2n + 2)M^2 + M^2$. Therefore, whenever the following inequality:

$$N_{ps} - N_o = -n_0^2 + (n+1-M)n_0 - nM - M + 2M^2 < 0, \qquad (4.3.3)$$

is satisfied, the number of parameters to be updated is reduced by the new control scheme, compared to the output feedback control scheme. By solving the inequality (4.3.3), we conclude that for the systems with n > 3M - 1, when $n_0 < M$ or $n_0 >$ n - 2M + 1, the number of parameters to be updated is reduced by the developed partial-state feedback scheme, and for the systems with n < 3M - 1, when $n_0 > M$, the number of parameters to be updated is reduced by the developed partial-state feedback MRAC scheme.

Number of first-order integrator. For the partial-state feedback multivariable MRAC scheme, the number of first-order integrators for constructing the filtered signals $\zeta(t)$ and $\xi(t)$ is $n_h^*((M + n_0)(n - n_0 + 1) + M)$ with n_h^* being the degree of the polynomial f(s), and the number of first-order integrators for constructing $\bar{e}(t)$ is n_e^* with n_e^* being related to the filter $\xi_m(s)h(s)$. Therefore, the total first-order integrators used for partial-state feedback control adaptation is $N'_{ps} = n_h^*((M + n_0)(n - n_0 + 1) + M) + n_e^*$. Similarly, the number of first-order integrators used for output feedback control adaptation is $N'_o = n_h^*(2\bar{\nu}M + M) + n_e^*$ with $\bar{\nu} = n - M + 1$.

Therefore, whenever the following inequality:

$$N'_{ps} - N'_{o} = n_{h}^{*}(-n_{0}^{2} + (n+1-M)n_{0} - nM - M + 2M^{2}) < 0, \qquad (4.3.4)$$

is satisfied, the number of first-order integrators used for control adaptation is reduced by the partial-state control scheme. By solving the inequality (4.3.4), we conclude that for the systems with n > 3M - 1, when $n_0 < M$ or $n_0 > n - 2M + 1$, the number of first-order filters is reduced by the developed partial-state feedback scheme, and for the systems with n < 3M - 1, when $n_0 > M$, the number of first-order filters is reduced by the developed partial-state feedback scheme.

Summarizing the above results, we could make the following conclusion.

Proposition 4.3.1. For a plant in the form of (4.1.1) with n > 3M - 1, the adaptation complexity is reduced by the partial-state feedback multivariable MRAC scheme using the partial-state $y_0(t) \in \mathbb{R}^{n_0}$ with the condition $n_0 < M$ or $n_0 > n - 2M + 1$; For a plant in the form of (4.1.1) with n < 3M - 1, the adaptation complexity is reduced by the partial-state feedback multivariable MRAC scheme using the partial-state $y_0(t) \in \mathbb{R}^{n_0}$ with the condition $n_0 > M$.

4.4 Observer-Based Minimal-Order Multivariable MRAC

In this section, we will present an observer-based minimal-order multivariable MRAC scheme, which allows the least number of feedback signals for multivariable feedback control and significantly reduces the system complexity compared to an output feedback control scheme.

4.4.1 Minimization of the Number of Feedback Signals

Recall the six feedback possibilities we have mentioned in Section 4.3.1. When apply case (3) and (4) to the developed partial-state feedback MRAC scheme, we can obtain a controller structure as follows.

$$u(t) = \Theta_1^{\mathrm{T}}(t)\omega_1(t) + \Theta_2^{\mathrm{T}}(t)\omega_2(t) + \Theta_{20}^{\mathrm{T}}(t)y_0(t) + \Theta_3(t)r(t), \qquad (4.4.1)$$

where $\Theta_1(t) \in \mathbb{R}^{M(n-1)\times M}$, $\Theta_2(t) \in \mathbb{R}^{(n-1)\times M}$, $\Theta_{20}(t) \in \mathbb{R}^{1\times M}$, $\Theta_3(t) \in \mathbb{R}^{M\times M}$ are the adaptive estimates of the unknown nominal parameters Θ_1^* , Θ_2^* , Θ_{20}^* , Θ_3^* (defined from (4.2.6)), respectively, and $\omega_1(t)$, $\omega_2(t)$ are in the form of (4.2.13) with $A_1(s) = [I_M, sI_M, \dots, s^{n-2}I_M]^{\mathrm{T}}$, $A_2(s) = [1, s, \dots, s^{n-2}]^{\mathrm{T}}$, and $\Lambda(s)$ being a monic stable polynomial of degree n-1.

From Theorem 4.2.2, the controller structure (4.4.1), with the feedback signal $y_0(t)$ being a scalar, guarantees *M*-output tracking for a multivariable plant. Such an adaptive control scheme, when applied to the case (3), shows that it is sufficient for the controller to only use one component of y(t) for feedback control to achieve *M*-output tracking; and when applied to the case (4), it shows that the controller can only use a scalar signal $y_0(t)$ (which is not even from the components of y(t)) for feedback control to achieve *M*-output tracking. Such a result has never been seen in the literature and is believed to be a novel concept in adaptive control and can be summarized as follows.

Corollary 4.4.1. For partial-state feedback multivariable MRAC, it is sufficient to use a partial-state signal $y_0(t) = C_0x(t)$ with (A, C_0) observable to construct the adaptive feedback controller (4.4.1) to achieve the desired performance: closed-loop signal boundedness and asymptotic output tracking: $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$ for $y(t) \in \mathbb{R}^M$. In particular, $y_0(t)$ can be a scalar and not be a component of y(t). The controller (4.4.1) only uses a scalar signal $(y_0(t) \in \mathbb{R})$ from the controlled system for feedback to guarantee an *M*-output tracking, which minimizes the amount of feedback signals for constructing the adaptive controller.

4.4.2 Reduction of Adaptation Complexity

In this section, we will show that the minimal-order controller (4.4.1), without the requirement of the full-state vector, reduces the order of control to cover more control applications.

Substituting the condition $n_0 = 1$ into the inequality (4.3.3) and (4.3.4), we obtain an equivalent inequality: $n - M - nM - M + 2M^2 < 0$, for finding the condition that makes the adaptation complexity of the minimal-order multivariable control less than the one of the output feedback multivariable control. Solving this inequality, we can readily conclude that the inequalities hold when $M < \frac{n}{2}$. Such a result means that for MIMO systems ($M \ge 2$), the system adaptation complexity (i.e., the number of updated control parameters and the number of first-order integrator) are reduced by the controller structure (4.4.1), when the output dimension M is less than the half of the state dimension n. Such an adaptation complexity reduction condition is often the case of real multivariable control systems, such as the aircraft control system shown in Section 4.5.

In addition, we could also conclude that when $M = \frac{n+2}{4}$, the function $f(M) = n - M - nM - M + 2M^2$ has the minimal value: $f(M) = -\frac{1}{8}(n-2)^2$. In other words, compared to the output feedback output tracking scheme, the number of parameters to be adaptively updated and the number of first-order integrators used in the adaptive control system can be reduced by $-\frac{1}{8}(n-2)^2$, when the output dimension M is chosen as $\frac{n+2}{4}$, by using the minimal-order multivariable control scheme. Such a result is also helpful for the choice of system output.

Proposition 4.4.1. For multivariable model reference adaptive control systems, as long as the dimension M of the plant output y(t) is less than $\frac{n}{2}$, the system adaptation complexity is reduced by the minimal-order controller (4.4.1). In particular, for the control system with $y_0(t) \in \mathbb{R}$, when $M = \frac{n+2}{4}$, the system adaptation complexity is minimized, which is $-\frac{1}{8}(n-2)^2$ less than the output feedback multivariable MRAC system.

So far, we have confirmed the two features of the minimal-order multivariable controller (4.4.1): (a) the amount of feedback signal used for constructing the feedback controller is minimum; and (b) the system adaptation complexity can be reduced.

4.5 Simulation Study

In this section, we present a simulation study to evaluate the effectiveness of the proposed partial-state feedback adaptive control designs.

4.5.1 Simulation System

The NASA GTM model [47] is chosen as the plant, which the proposed partial-state feedback adaptive control design is applied on.

Plant dynamics. The linearized NASA GTM model is in the form of (4.1.1): $\dot{x} = Ax + Bu, y = Cx$. The system state vector is $x = [u_b, w_b, q_b, \theta, v_b, r_b, p_b, \phi]^T$ with u_b, v_b, w_b being the body-axis velocity components of origin of body-axis frame, p_b, q_b and r_b being the body-axis components of angular velocity and θ , ϕ being the pitch and roll angle. The control inputs are the elevator angular δ_e and the aileron angular δ_a , and the plant outputs are chosen as the pitch angle θ and the roll angle ϕ . The system parameter matrices are shown in (4.5.1).

$$A = \begin{bmatrix} -0.019 & 0.1364 & -9.7778 & -32.0829 & -0.0018 & -0.0004 & 0 & 0 \\ -0.2804 & -2.7567 & 120.1968 & -2.42 & -0.0001 & 0 & 0.0004 & -0.0061 \\ 0.0205 & -0.3106 & -3.5393 & 0 & 0.007 & 0.0328 & -0.0014 & 0 \\ 0 & 0 & 1 & 0 & 0 & -0.0002 & 0 & 0.0002 \\ 0 & -0.0027 & 0 & -0.0005 & -0.5765 & -125.9974 & 10.4690 & 32.0829 \\ 0 & 0 & -0.0255 & 0 & 0.2245 & -1.4053 & -0.2794 & 0 \\ 0 & 0 & 0.0018 & 0 & -0.629 & 1.9689 & -5.4759 & 0 \\ 0 & 0 & 0 & 0 & -0.0002 & 0 & 0.0754 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0056 & -0.0423 \\ -0.6119 & 0.1579 \\ -0.7486 & 0.0859 \\ 0 & 0 & 0 \\ 0 & -0.0223 \\ 0 & -0.7657 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (4.5.1)$$

Verification of design conditions. For the aircraft model (A, B, C) in (4.5.1), it can be verified that the transfer function $G(s) = C(sI - A)^{-1}B$ has stable zeros: $s_{1,2} = -1.0059 \pm 5.5340i$, $s_3 = -2.4867$ and $s_4 = -0.035$, and G(s) is strictly proper and full rank. The modified interactor matrix $\xi_m(s)$ can be chosen as $\xi_m(s) =$ diag $\{(s+2)^2, (s+2)^2\}$ so that

$$K_p = \lim_{s \to \infty} \xi_m(s) G(s) = \begin{bmatrix} -0.7486 & 0.0859 \\ -0.00001 & -0.7675 \end{bmatrix}$$
(4.5.2)

is finite and non-singular. From (4.5.2), the design condition that the signs of leading principle minors of K_p are positive can also be verified.

Reference model. Since the modified interactor matrix $\xi_m(s)$ is chosen as diag $\{(s+2)^2, (s+2)^2\}$, the transfer function of the chosen reference model (4.1.3) is

$$W_m(s) = \xi_m^{-1}(s) = \text{diag}\left\{\frac{1}{(s+2)^2}, \frac{1}{(s+2)^2}\right\},\tag{4.5.3}$$

which is proper and stable. The reference inputs are chosen as $r(t) = [-40\pi/180\sin(0.1t) - 15\pi/180\sin(0.1t)]^{\mathrm{T}}$.

,



Figure 4.1: Plant Output: pitch angle θ and roll angle ϕ in Case I.



Figure 4.2: Control input signal: elevator angle δ_e and aileron angle δ_a in Case I.



Figure 4.3: Plant output: pitch angle θ and roll angle ϕ in Case II.



Figure 4.4: Control input signal: elevator angle δ_e and aileron angle δ_a in Case II.



Figure 4.5: Plant output: pitch angle θ and roll angle ϕ in Case III.



Figure 4.6: Control input signal: elevator angle δ_e and aileron angle δ_a in Case III.



Figure 4.7: Plant output: pitch angle θ and roll angle ϕ in Case IV.



Figure 4.8: Control input signal: elevator angle δ_e and aileron angle δ_a in Case IV.

4.5.2 Simulation Results

Four cases have been systematically studied to show the new features of the partialstate feedback MRAC scheme.

Case I: $y_0(t) = [q_b(t), \theta(t), p_b(t)]^{\mathrm{T}}$ is a vector containing one element of $y = [\theta(t), \phi(t)]^{\mathrm{T}}$;

Case II: $y_0(t) = [q_b(t), r_b(t), p_b(t)]^{\mathrm{T}}$ is vector which does not contain any element of $y = [\theta(t), \phi(t)]^{\mathrm{T}};$

Case III: $y_0(t) = \phi(t)$ is a scalar as one element of $y(t) = [\theta(t), \phi(t)]^{\mathrm{T}}$; and

Case IV: $y_0(t) = r_b(t)$ is a scalar not being any element of $y(t) = [\theta(t), \phi(t)]^{\mathrm{T}}$.

For all simulation cases, the adaptation gains are chosen as $\Gamma = 5I$, $\Gamma_{\theta} = 5$, and the initial condition are chosen as $y(0) = [-0.01, -0.01]^{\mathrm{T}}$, $y_m(0) = [0, 0]^{\mathrm{T}}$. Case I and II tests the plant output tracking performance when the partial-state feedback signal $y_0(t)$ are vectors, and Case III and IV tests the tracking performance when the partialstate feedback signal $y_0(t)$ are scalars. The plant output tracking performances of Case I – Case IV are shown in Fig. 4.1, Fig. 4.3, Fig. 4.5 and Fig. 4.7, respectively, in which the dotted lines represent the reference pitch angle and roll angle and the solid lines represent the aircraft outputs. The tracking performance plots show that the asymptotic tracking are achieved in all four cases, in particular, the one for Case III and IV confirms the result in Corollary 4.4.1 that, for partial-state feedback MRAC, a scalar feedback signal is sufficient for constructing an adaptive controller to make the M output tracking achievable.

Also, for Case III and IV, adaptive controllers are constructed based on (4.4.1) whose parameter order is 48. While for the same plant, if a standard output feedback controller (4.1.6) is constructed, the controller parameter order would be 56, since

the upper bound of the plant observability index $\bar{\nu}$ is 7, which supports that results in Proposition 4.4.1.

Moreover, Fig. 4.2, Fig. 4.4, Fig. 4.6 and Fig. 4.8 show the control input signals of Case I – Case IV, respectively, which confirm that all control signals stay in acceptable ranges. In addition, signals in closed-loop systems for all four cases are bounded whose plots are not shown due to the space limit.

Summary

In this chapter, we have developed a new framework of multivariable MRAC using partial-state feedback for output tracking, with new solutions to three technical issues: plant-model output matching, parameterized error model based on LDS decomposition, and stable adaptive law design and analysis, for ensuring closed-loop system stability and asymptotic tracking in the presence of plant uncertainties. This work has shown that partial-state feedback MRAC provides additional design flexibilities in utilizing system signals, while using less complex controller structures than output feedback. We presented a complete analysis of the closed-loop system stability and tracking performance of partial-state feedback MRAC. It has been shown that such a new MRAC framework builds a natural transition from full state feedback MRAC to output feedback MRAC, adding new members to the family of MRAC. Moreover, we conclude that for the partial-state feedback MRAC scheme, asymptotic tracking for M ($M \ge 1$) output is achievable by the adaptive controller constructed by some scalar feedback signals, and provide a observer-based minimal-order MRAC scheme based on which. We presented simulation results for different adaptive control designs, which verify the desired adaptive control system performance.

Chapter 5 Higher-Order Convergence Properties of MRAC Systems

For a general multi-input multi-output linear time-invariant system with unknown parameters, a multivariable model reference adaptive control (MRAC) scheme guarantees asymptotic output tracking, under some design conditions. This chapter further shows a stronger higher-order convergence property for the signal components of the tracking error $e(t) = [e_1(t), e_2(t), \ldots, e_M(t)]^T$.

It is proved that under the same MRAC design conditions, not only a tracking error component $e_i(t)$ but its up to q_i th-order time-derivatives converge to zero, where q_i is related to system's infinite zero structure characterized by the system interactor matrix $\xi_m(s)$.

Both cases of a diagonal $\xi_m(s)$ and a non-diagonal $\xi_m(s)$ are studied in this chapter, and the new MRAC tracking property is proved for different forms of the modified interactor matrix. Simulation study is conducted on a transport aircraft model, whose results verify the higher-order error convergence property.
5.1 Review of Multivariable MRAC and Research Motivations

In this section, we first give a brief review of MRAC systems to make the chapter self-contained, and then discuss the research motivations of this chapter.

5.1.1 Multivariable MRAC System

Consider a MIMO linear time-invariant plant

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$
(5.1.1)

for some unknown system parameter matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times M}$ and $C \in \mathbb{R}^{M \times n}$, with $x(t) \in \mathbb{R}^n$ bing the state, $u(t) \in \mathbb{R}^M$ being the input vector, and $y(t) \in \mathbb{R}^M$ being the output vector, respectively. The input-output description of the plant (5.1.1) is

$$y(t) = G(s)[u](t), \ G(s) = C(sI - A)^{-1}B.$$
 (5.1.2)

The notation, y(t) = G(s)[u](t), is to denote the output y(t) of a system represented by G(s) with a control input signal u(t). The plant infinite zero structure is characterized by a modified left interactor (MLI) matrix $\xi_m(s)$ such that the system high-frequency gain matrix $K_p = \lim_{s\to\infty} \xi_m(s)G(s)$ is finite and nonsingular.

Control objective. The control objective of MRAC is to design a feedback control law for u(t) in (5.1.1) to ensure that all the closed-loop signal are bounded and the output y(t) asymptotically tracks the reference signal

$$y_m(t) = W_m(s)[r](t), \ W_m(s) = \xi_m^{-1}(s),$$
(5.1.3)

with $r(t) \in \mathbb{R}^M$ being a bounded reference input signal and $\xi_m(s)$ being a MLI matrix of the G(s) and having a stable inverse (i.e., $W_m(s)$ is stable).

Design conditions. The following standard assumptions are made as design conditions for the control objective to be achieved:

- (A5.1) All zeros of $G(s) = C(sI A)^{-1}B$ are stable, and (A, B, C) is stabilizable and detectable.
- (A5.2) G(s) has full rank and its MLI matrix $\xi_m(s)$ is known.
- (A5.3) An upper bound $\bar{\nu}$ of the observability index ν of G(s) is known.

Assumptions (A5.1)–(A5.3) are standard assumptions for MRAC output tracking. Assumption (A5.1) is for stable plant-model output matching, Assumption (A5.2) is for choosing a reference model system $W_m(s) = \xi_m^{-1}(s)$ suitable for plant-model output matching, Assumption (A5.3) is for constructing an output feedback controller.

Controller structure. The output feedback model reference adaptive controller structure is

$$u(t) = \Theta_1^{\mathrm{T}}(t)\omega_1(t) + \Theta_2^{\mathrm{T}}(t)\omega_2(t) + \Theta_{20}(t)y(t) + \Theta_3(t)r(t), \qquad (5.1.4)$$

where $\omega_1(t) = \frac{A(s)}{\Lambda(s)}[u](t)$, $\omega_2(t) = \frac{A(s)}{\Lambda(s)}[y](t)$ with $\Lambda(s)$ being a monic stable polynomial of degree $\bar{\nu} - 1$ and $A(s) = [I_M, sI_M, \cdots, s^{\bar{\nu}-2}I_M]^{\mathrm{T}}$, and $\Theta_1(t) \in \mathbb{R}^{(\bar{\nu}-1)M \times M}$, $\Theta_2(t) \in \mathbb{R}^{(\bar{\nu}-1)M \times M}$, $\Theta_{20}(t) \in \mathbb{R}^{M \times M}$, $\Theta_3(t) \in \mathbb{R}^{M \times M}$ are controller parameters to be adaptively updated by stable adaptive laws. Such controller parameters are the adaptive estimates of the nominal controller parameters Θ_1^* , Θ_2^* , Θ_{20}^* and Θ_3^* . It has been shown in [19, 53, 65] that for $G(s) = Z(s)P^{-1}(s)$ and $\Theta_3^* = K_p^{-1}$, there exist constant nominal controller parameters such that

$$\Theta_1^{*T} A(s) P(s) + (\Theta_2^{*T} A(s) + \Theta_{20}^* \Lambda(s)) Z(s) = \Lambda(s) (P(s) - \Theta_3^* \xi_m(s) Z(s))$$
(5.1.5)

for output matching: $y(t) = \xi_m^{-1}(s)[r](t) = y_m(t)$.

Nominal control performance. When the system parameters are known, the nominal controller parameters Θ_1^* , Θ_2^* , Θ_{20}^* and Θ_3^* can be calculated according to

output matching polynomial (5.1.5). From (5.1.5), we can obtain

$$\Theta_1^{*T} A(s) Z(s)[u](t) + (\Theta_2^{*T} A(s) + \Theta_{20}^* \Lambda(s)) Z(s)[y](t)$$

= $\Lambda(s) Z(s)[u](t) - \Lambda(s) \Theta_3^* \xi_m(s) Z(s)[y](t),$ (5.1.6)

by using the plant equation: P(s)[y](t) = Z(s)[u](t).

Since $\Lambda(s)$ is a stable filter and Z(s) is stable polynomial matrix, the signal matching equation (5.1.6) can be derived as

$$u(t) = \Theta_1^{*\mathrm{T}} \frac{A(s)}{\Lambda(s)} [u](t) + \Theta_2^{*\mathrm{T}} \frac{A(s)}{\Lambda(s)} [y](t) + \Theta_{20}^{*} y(t) + \Theta_3^{*} \xi_m(s) [y](t) + \epsilon_1(t)$$
(5.1.7)

with the initial-condition related term $\epsilon_1(t)$ being exponentially decaying.

Substituting (5.1.7) into the nominal controller $u(t) = \Theta_1^{*T}\omega_1(t) + \Theta_2^{*T}\omega_2(t) + \Theta_{20}^*y(t) + \Theta_3^*r(t)$ with $r(t) = \xi_m^{-1}(s)[y_m](t)$, we obtain

$$\Theta_3^* \xi_m(s) [y - y_m](t) + \epsilon_1(t) = 0.$$
(5.1.8)

which implies that y(t) is bounded, and $\lim_{t\to\infty} e(t) = 0$ for $e(t) = y(t) - y_m(t)$.

Adaptive control performance. For plants with unknown parameters, an adaptive law for updating the adaptive controller parameters $\Theta_1(t)$, $\Theta_2(t)$, $\Theta_{20}(t)$ and $\Theta_3(t)$ is needed. Ignoring the exponentially decaying term $\epsilon_1(t)$, we derive the tracking error equation as

$$e(t) = y(t) - y_m(t) = W_m(s)K_p[\Theta^{\mathrm{T}}\omega](t),$$
 (5.1.9)

using (5.1.4) and (5.1.7), for the purpose of choosing the stable adaptive law, where $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$ with $\Theta^* = \left[\Theta_1^{*T}, \Theta_2^{*T}, \Theta_{20}^*, \Theta_3^*\right]^T$, $\Theta(t) = \left[\Theta_1^{T}(t), \Theta_2^{T}(t), \Theta_{20}(t), \Theta_3(t)\right]^T$, and the regressor $\omega(t) = \left[\omega_1^{T}(t), \omega_2^{T}(t), y^{T}(t), r^{T}(t)\right]^T$.

Let f(s) be a stable polynomial and $h(s) = \frac{1}{f(s)}$ such that $h(s)\xi_m(s)$ is proper, filter both sides of (5.1.9) with h(s), and define $\bar{e}(t) = h(s)\xi_m(s)[e](t)$. Then, we obtain the following filtered error equation: $\bar{e}(t) = K_p h(s)[\tilde{\Theta}^{\mathrm{T}}\omega](t)$. To update the controller parameters adaptively, based on the filter error equation, the estimation error is defined as $\epsilon(t) = K_p \tilde{\Theta}^T \zeta(t) + \tilde{\Psi}(t)\xi(t)$, where $\xi(t) = \Theta^T(t)\zeta(t) - h(s)[\Theta^T\omega](t), \zeta(t) = h(s)[\omega](t)$, and $\tilde{\Theta} = \Theta(t) - \Theta^*$ and $\tilde{\Psi}(t) = \Psi(t) - \Psi^*$ are parameter errors, with $\Psi(t)$ being the estimation of $\Psi^* = K_p$.

Then, based on the gradient algorithm, the controller parameter adaptation laws are chosen as:

$$\dot{\Theta}^{\mathrm{T}}(t) = -\frac{S_p \epsilon(t) \zeta^{\mathrm{T}}(t)}{m^2(t)}, \dot{\Psi}(t) = -\frac{\Gamma \epsilon(t) \xi^{\mathrm{T}}(t)}{m^2(t)}, \qquad (5.1.10)$$

where S_p is known such that $K_p S_p = (K_p S_p)^T > 0$, $\Gamma = \Gamma^T > 0$ is an adaption gain matrix, and $m^2(t) = 1 + \zeta^T(t)\zeta(t) + \xi^T(t)\xi(t)$.

Such a closed-loop adaptive control system has the desired stability and tracking properties, shown as follows.

Theorem 5.1.1. The adaptive output feedback controller (5.1.4) with the adaptive law (5.1.10), when applied to the plant (5.1.1), guarantees the closed-loop signal boundedness and asymptotic output tracking: $\lim_{t\to\infty} (y(t) - y_m(t)) = 0.$

The proof of Theorem 5.1.1 can be found in [82]. Such a tracking performance result of multivariable MRAC systems has been established for decades. For improving the understanding of multivariable MRAC systems and for guaranteeing extra performance of MRAC applications, in this chapter, we will further investigate the higher-order tracking performance of multivariable MRAC systems.

5.1.2 Research Motivations

The convergence property derivation in this chapter provides new insights into multivariable MRAC and benefits MRAC system applications as well.

Theory development. Although the established results of MRAC systems guarantee asymptotic output tracking, there are open questions to be answered for fully understanding the tracking performance of multivariable MRAC systems. In this chapter, we will derive the higher-order tracking properties of multivariable MRAC systems which has not been reported in the literature, to further mature the theoretical systems of multivariable MRAC.

Application significance. Consider a linearized NASA GTM model [47] in the form of (5.1.1): $\dot{x} = Ax + Bu, y = Cx$, with

$A = \frac{1}{2}$	-0.019	0.1364	-9.7778	-32.0829	-0.0018	-0.0004	0	0]
	-0.2804	-2.7567	120.1968	-2.42	-0.0001	0	0.0004	-0.0061
	0.0205	-0.3106	-3.5393	0	0.007	0.0328	-0.0014	0
	0	0	1	0	0	-0.0002	0	0.0002
	0	-0.0027	0	-0.0005	-0.5765	-125.9974	10.4690	32.0829
	0	0	-0.0255	0	0.2245	-1.4053	-0.2794	0
	0	0	0.0018	0	-0.629	1.9689	-5.4759	0
	0	0	0	-0.0002	0	0.0754	1	0

The system state vector is $x = [u_b, w_b, q_b, \theta, v_b, r_b, p_b, \phi]^T$ with u_b, v_b, w_b being the body-axis velocity components of origin of body-axis frame, p_b, q_b and r_b being the body-axis components of angular velocity and θ , ϕ being the pitch and roll angles. The plant outputs are chosen as the pitch angle θ_b and the roll angle ϕ_b .

For the aircraft model, all the design conditions are satisfied. The MLI matrix $\xi_m(s)$ can be chosen as $\xi_m(s) = \text{diag} \{(s+2)^2, (s+2)^2\}$ so that K_p is finite and nonsingular. Based on the existing MRAC results, asymptotic tracking is guaranteed when a standard output model reference adaptive controller (5.1.4) is applied with the stable adaptive law (5.1.10), that is, $\lim_{t\to\infty}(\theta(t) - \theta_m(t)) = 0$ and $\lim_{t\to\infty}(\phi(t) - \theta_m(t)) = 0$ $\phi_m(t) = 0$, with $\theta_m(t)$ being the desired pitch angle and $\phi_m(t)$ being the desired roll angle generated from the reference model (5.1.3).

However, from the existing multivariable MRAC results, the tracking performances of pitch rate $\frac{d\theta(t)}{dt}$ and roll rate $\frac{d\phi(t)}{dt}$ are not clear yet. The properties: $\lim_{t\to\infty} \left(\frac{d\phi(t)}{dt} - \frac{d\phi_m(t)}{dt}\right) = 0$ and $\lim_{t\to\infty} \left(\frac{d\phi(t)}{dt} - \frac{d\phi_m(t)}{dt}\right) = 0$, are desired, for avoiding some extreme (nonsmooth) oscillations which are harmful to the vehicle. Such desired properties imply that the pitch angle and the roll angle converge to the desired values in a smooth manner. In fact, besides having a desired smooth position, having a smooth velocity and a smooth acceleration are also crucial to extend service life of a vehicle and to improve the passengers experiences. Thus, it is also desired to investigate the higher-order tracking performance of MRAC systems.

5.2 Main Results

Nominal control design is used when the system parameters (A, B, C) are known. It provides priori knowledge for adaptive control design. Thus, in this chapter, we first clarify the high-order convergence properties of the nominal model reference control (MRC) systems, and then show the desired higher-order tracking properties for adaptive control systems.

5.2.1 Nominal Control System Performance

When the system parameters are known, recall the tracking error $e(t) = y(t) - y_m(t)$ shown in (5.1.8):

$$\Theta_3^* \xi_m(s)[e](t) + \epsilon_1(t) = 0. \tag{5.2.1}$$

Next, we will first analyze the nominal higher-order convergence performance of the MRC system when the MLI matrix $\xi_m(s)$ is diagonal, and then extend the discussion to general cases as $\xi_m(s)$ being non-diagonal.

5.2.1.1 Case I: The MLI Matrix $\xi_m(s)$ Is Diagonal

From (5.2.1) and $\Theta_3^* = K_p^{-1}$, we have

$$\xi_m(s)[e](t) = \bar{\epsilon}_1(t), \qquad (5.2.2)$$

with $\bar{\epsilon}_1(t) = -K_p \epsilon_1(t)$ being an exponentially decaying term. With this equation and a diagonal MLI matrix:

$$\xi_m(s) = \text{diag} \{ d_1(s), d_2(s), \dots, d_M(s) \}, \qquad (5.2.3)$$

for each tracking error component $e_i(t) = y_i(t) - y_{mi}(t)$ with $y_i(t)$ being the *i*th element of the output vector signal y(t) and $y_{mi}(t)$ being the *i*th element of the reference signal $y_m(t)$, we obtain

$$d_i(s)[e_i](t) = \bar{\epsilon}_{1i}(t), \quad i = 1, 2, \dots, M,$$
(5.2.4)

where $\bar{\epsilon}_{1i}(t)$ is the *i*th element of $\bar{\epsilon}_1(t)$. Since $d_i(s)$ is a stable polynomial of degree $l_i \geq 1$, (5.2.4) indicates the following new nominal tracking properties for each $e_i(t)$, $i = 1, 2, \ldots, M$:

$$\lim_{t \to \infty} \frac{\mathrm{d}^{q_i} e_i(t)}{\mathrm{d} t^{q_i}} = 0 \quad \text{exponentially}, \quad q_i = 0, \dots, l_i - 1. \tag{5.2.5}$$

The above equation shows that, for multivariable MRC systems, up to $(l_i - 1)$ th derivatives of the tracking error component $e_i(t)$ converge to zero exponentially.

5.2.1.2 Case II: The MLI Matrix $\xi_m(s)$ Is Non-Diagonal

Ignoring the exponentially decaying term related to the initial conditions e(0), from (5.2.1), we have

$$e(t) = \xi_m^{-1}(s) K_p[\epsilon_1](t) = W_m(s) K_p[\epsilon_1](t).$$
(5.2.6)

When the MLI matrix $\xi_m(s)$ is non-diagonal, the inverse of the MLI matrix: $W_m(s) = \xi_m^{-1}(s)$, is expressed as

$$W_m(s) = \begin{bmatrix} \frac{1}{d_1(s)} & 0 & \dots & 0 \\ w_{21}(s) & \frac{1}{d_2(s)} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{M1}(s) & \dots & \dots & w_{MM-1}(s) & \frac{1}{d_M(s)} \end{bmatrix},$$
 (5.2.7)

with $w_{ik}(s)$, i = 2, ..., M, k = 1, ..., i - 1 being some stable polynomials of relative degree $l_{ik} > 0$.

Based on (5.2.7), the tracking error component between the plant output and the desired output in each channel is

$$e_i(t) = -\sum_{k=1}^{i-1} \sum_{j=1}^{M} w_{ik}(s) k_{pkj}[\epsilon_{1j}](t) - \sum_{j=1}^{M} \frac{k_{pij}}{d_i(s)}[\epsilon_{1j}](t), \ i = 1, 2, \dots, M$$
(5.2.8)

with k_{pkj} being the (k, j)th element of the high-frequency gain matrix K_p and $\epsilon_{1j}(t)$ is the *j*th element of the exponentially decaying term $\epsilon_1(t)$.

From (5.2.8), we conclude that the higher-order performance of each $e_i(t)$, $i = 1, 2, \ldots, M$, is

$$\lim_{t \to \infty} \frac{\mathrm{d}^{q_i} e_i(t)}{\mathrm{d} t^{q_i}} = 0 \quad \text{exponentially}, \quad q_i = 1, \dots, \bar{l}_i - 1, \tag{5.2.9}$$

for $\bar{l}_i = \min\{l_i, l_{i1}, \dots, l_{i(i-1)}\}.$

The above equation shows that, for multivariable MRC systems, up to $(\bar{l}_i - 1)$ th derivatives, for $\bar{l}_i = \min\{l_i, l_{i1}, \ldots, l_{i(i-1)}\}$, of tracking error component $e_i(t)$ converge to zero exponentially.

In summary, for nonadaptive model reference control, in addition to the basic tracking error convergence property: $\lim_{t\to\infty} e_i(t) = 0$, certain higher-oder derivatives of $e_i(t)$ are also go to zero exponentially, i = 1, 2, ..., M. The results are summarized as follows.

Theorem 5.2.1. For model reference control systems with a trigonal MLI matrix $\xi_m(s)$ whose inverse matrix is shown in (5.2.7), the ith element of the tracking error e(t) satisfies

$$\lim_{t \to \infty} \frac{\mathrm{d}^{q_i} e_i(t)}{\mathrm{d}t^{q_i}} = 0 \quad exponentailly, \quad i = 1, \dots, M, \tag{5.2.10}$$

for $q_i = 0, 1, ..., \overline{l_i} - 1$ with $\overline{l_i} = \min\{l_i, l_{i1}, ..., l_{i(i-1)}\}$. In particular, when the MLI matrix $\xi_m(s)$ is diagonal as shown in (5.2.3), the ith element of the tracking error e(t) satisfies (5.2.10) for $q_i = 0, 1, ..., l_i - 1$.

The above theorem provides new tracking properties for model reference control systems, that is, the properties for the case when the system parameters (A, B, C) are known. However, for practical applications, accurate system parameters are usually unobtainable. Although the current established MRAC results guarantee asymptotic output tracking, the higher-order tracking performance is not clear yet. So a natural question is: does a MRAC system, under the case of parametric uncertainties, also enjoy a similar high-order tracking performance? A firm and rigorous answer to such a question will be derived in the next section.

5.2.2 Adaptive Control System Performance

When the system parameters (A, B, C) are unknown, control adaptation is needed. For model reference adaptive control systems, we first recall the tracking error equation shown in (5.1.9):

$$e(t) = W_m(s)K_p[\tilde{\Theta}^T\omega](t), \qquad (5.2.11)$$

where $W_m(s) = \xi_m^{-1}(s)$ is a stable polynomial matrix, $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$ is the parameter error matrix, and $\omega(t)$ is the regressor vector. Next, we will analyze the high-order convergence property based on different forms of $\xi_m(s)$. Next, to analyze the higher-order property, we recall the following definition for signal convergence.

Definition 5.2.1. For a function f(t) defined on $[0, \infty]$, $\lim_{t\to\infty} f(t) = 0$ if for every $\eta > 0$, there exists a $T = T(\eta) > 0$ such that $|f(t)| < \eta$, for $\forall t > T$.

This definition can also be used as a necessary and sufficient condition to prove signal convergence: $\lim_{t\to\infty} f(t) = 0$. In this definition, η can be arbitrarily small but cannot equal to zero so that $1/\eta$ is finite.

5.2.2.1 Case I: The MLI Matrix $\xi_m(s)$ Is Diagonal

To demonstrate the idea of investigating high-order tracking error performance, next, we will first give a brief higher-order convergence analysis for the NASA GTM control system shown in Section 5.1.2, and then extend the proof to general MRAC systems.

Example 5.2.1. Convergence study for the NASA GTM MRAC system. Recall the NASA GTM model we have given in Section 5.1.2. We have shown that the MLI matrix $\xi_m(s)$ of the linearized NASA GTM model is diagonal, and we choose it as $\xi_m(s) = \text{diag} \{(s+2)^2, (s+2)^2\}$. Thus, when an output feedback controller (5.1.4) is applied with the adaptive laws (5.1.10), we have

$$e_{1}(t) = \frac{k_{p11}}{d_{1}(s)} [\tilde{\theta}_{1}^{T}\omega](t) + \frac{k_{p12}}{d_{1}(s)} [\tilde{\theta}_{2}^{T}\omega](t),$$

$$e_{2}(t) = \frac{k_{p21}}{d_{2}(s)} [\tilde{\theta}_{1}^{T}\omega](t) + \frac{k_{p22}}{d_{2}(s)} [\tilde{\theta}_{2}^{T}\omega](t),$$
(5.2.12)

for $e_1(t) = \theta(t) - \theta_m(t)$ and $e_2(t) = \phi(t) - \phi_m(t)$, where $d_1(s) = d_2(s) = (s+2)^2$, $\tilde{\theta}_1(t)$ and $\tilde{\theta}_2(t)$ are the first and second column of the parameter error matrix $\tilde{\Theta}(t)$, and k_{pij} , for i = 1, 2 and j = 1, 2, is the (i, j)th element of the high-frequency gain K_p .

For the tracking error component $e_1(t)$, we introduce two fictitious filters: $K_1(s) = \frac{a_1^2}{(s+a_1)^2}$ and $sH_1(s) = 1 - K_1(s)$. In the term of $K_1(s)$, we express the virtual filter

 $H_1(s)$ as

$$H_1(s) = \frac{1}{s}(1 - K_1(s)) = \frac{1}{s}\frac{(s + a_1)^2 - a_1^2}{(s + a_1)^2},$$
(5.2.13)

which is strictly proper and stable and whose impulse response function is $h_1(t) = \mathcal{L}^{-1}[H_1(s)] = a_1 t e^{-a_1 t} + e^{-a_1 t}$, with its L^1 signal norm being $||h_1(\cdot)|| = \int_0^\infty |h_1(t)| dt = \frac{2}{a_1}$.

By using the equality: $1 = sH_1(s) + K_1(s)$, we decompose the first-order derivative signal $\dot{e}_1(t)$ as

$$\dot{e}_{1}(t) = \frac{k_{p11}s}{d_{1}(s)} [\tilde{\theta}_{1}^{T}\omega](t) + \frac{k_{p12}s}{d_{1}(s)} [\tilde{\theta}_{2}^{T}\omega](t)$$

$$= H_{1}(s) \frac{k_{p11}s^{2}}{d_{1}(s)} [\tilde{\theta}_{1}^{T}\omega](t) + H_{1}(s) \frac{k_{p12}s^{2}}{d_{1}(s)} [\tilde{\theta}_{2}^{T}\omega](t) + sK_{1}(s)[e_{1}](t), \qquad (5.2.14)$$

where the terms $H_1(s) \frac{k_{p1j}s^2}{d_1(s)} [\tilde{\theta}_j^T \omega](t)$, j = 1, 2 are to be shown small enough and the term $sK_1(s)[e_1](t)$ is to be shown converging to zero asymptotically when time goes to infinity.

On one hand, for each term of $H_1(s)\frac{k_{p1j}s^2}{d_1(s)}[\tilde{\theta}_j^T\omega](t)$, j = 1, 2, since $\tilde{\theta}_j^T(t)\omega(t)$ is bounded and $\frac{k_{p1j}s^2}{d_1(s)}$ is stable and proper, we have $\frac{k_{p1j}s^2}{d_1(s)}[\tilde{\theta}_j^T\omega](t)$ being bounded. Thus, we can further obtain

$$\left| H_1(s) \frac{k_{p1j} s^2}{d_1(s)} [\tilde{\theta}_j^T \omega](t) \right| \le \frac{c_{1j}}{a_1}, \ j = 1, 2,$$
(5.2.15)

for any $t \ge 0$ and some constant $c_{1j} > 0$ independent of $a_1 > 0$, which follows from the above L^1 signal norm expression of $H_1(s)$: $||h_1(\cdot)||_1 = \frac{2}{a_1}$.

On the other hand, due to the established convergence property: $\lim_{t\to\infty} e_1(t) = 0$, and the property of $sK_1(s)$ being stable and strictly proper, we have $\lim_{t\to\infty} sK_1(s)[e_1](t) = 0$, for any finite $a_1 > 0$ in $K_1(s)$.

Next, to show that the signal convergence of $\dot{e}_1(t)$, it is desired to show that, for every $\eta_1 > 0$, there exists a $T_1 > 0$ such that $|\dot{e}_1(t)| < \eta_1$ for $\forall t > T_1$, according to Definition 5.2.1. On one hand, we set $a_1 = a_1(\eta_1) \ge \max\{\frac{4c_{11}}{\eta_1}, \frac{4c_{12}}{\eta_1}\}$ for the fictitious (virtual) filter $H_1(s)$ in (5.2.13) so that $\frac{c_{11}}{a_1} \le \frac{\eta_1}{4}$ and $\frac{c_{12}}{a_1} \le \frac{\eta_1}{4}$ in (5.2.15), respectively. On the other hand, we let $T_1 = T_{a1}(a_1(\eta_1), \eta_1) \stackrel{\triangle}{=} T_1(\eta_1) > 0$ such that $|sK_1(s)[e_1](t)| < \frac{\eta_1}{2}$ for all $t \ge T_1$. Since the time response of $|sK_1(s)[e_1](t)|$ depends on the parameter $a_1 = a_1(\eta_1)$, the above time instant $T_1 = T_{a_1}(a_1(\eta_1), \eta_1)$ also depends on $a_1 = a_1(\eta_1)$. Then, following (5.2.14) and (5.2.15), we obtain

$$|\dot{e}_1(t)| < 2 \cdot \frac{\eta_1}{4} + \frac{\eta_1}{2} = \eta_1, \ \forall t > T_1,$$
 (5.2.16)

which implies that $\lim_{t\to\infty} \dot{e}_1(t) = 0.$

By using the same decomposition technique for the time derivative of tracking error component $e_2(t)$ with $K_2(s) = \frac{a_2^2}{(s+a_2)^2}$ and $sH_2(s) = 1 - K_2(s)$, $a_2 > 0$, the desired first-order derivative convergence performance: $\lim_{t\to\infty} \dot{e}_2(t) = 0$, can also be guaranteed. From the above analysis, we confirm that $\frac{d\theta(t)}{dt}$ and $\frac{d\phi(t)}{dt}$ track the desired pitch rate and roll rate asymptotically, as we expected.

General converge properties. Next, we will generalize the above analysis to give a rigorous proof to show the higher-order convergence property of multivariable MRAC systems when $\xi_m(s)$ is diagonal.

In general, for the q_i th-order time derivative $\frac{d^{q_i}e_i(t)}{dt^{q_i}}$ of $e_i(t)$, i = 1, 2, ..., M, we introduce two virtual filters:

$$K_i(s) = \frac{a_i^{l_i}}{(s+a_i)^{l_i}}, sH_i(s) = 1 - K_i(s).$$
(5.2.17)

for decomposing the time derivative signal $\frac{d^{q_i}e_i(t)}{dt^{q_i}}$, where $a_i > 0$ is a parameter to be specified, and the filter $H_i(s)$ is given as

$$H_i(s) = \frac{1}{s}(1 - K_i(s)) = \frac{1}{s}\frac{(s + a_i)^{l_i} - a^{l_i}}{(s + a_i)^{l_i}}.$$
(5.2.18)

The stable filter $H_i(s)$ is strictly proper of relative degree one and its impulse response function is

$$h_i(t) = \mathcal{L}^{-1}[H_i(s)] = e^{-a_i t} \sum_{i=1}^{l_i} \frac{a^{l_i - i}}{(l_i - i)!} t^{l_i - i}.$$
 (5.2.19)

It can be verified [65] that the L^1 signal norm of $h_i(t)$ is

$$||h_i(\cdot)||_1 = \int_0^\infty |h(t)| \, dt = \frac{l^i}{a_i}.$$
(5.2.20)

From $e(t) = \xi_m^{-1}(s) K_p[\tilde{\Theta}^T \omega](t)$ and a diagonal MLI matrix $\xi_m(s) = \text{diag}\{d_1(s), d_2(s), \ldots, d_M(s)\}$, we express the tracking error component in each channel as

$$e_i(t) = y_i(t) - y_{mi}(t) = \sum_{j=1}^M \frac{k_{pij}}{d_i(s)} [\tilde{\theta}_j^T \omega](t), \qquad (5.2.21)$$

where $\tilde{\theta}_j(t)$ is the *j*th column of the parameter error matrix $\tilde{\Theta}(t)$, and k_{pij} is the (i, j)th element of K_p .

By operating the equality: $1 = sH_i(s) + K_i(s)$, on $\frac{d^{q_i}e_i(t)}{dt^{q_i}}$, we express the q_i th-order time derivative $\frac{d^{q_i}e_i(t)}{dt^{q_i}}$ of $e_i(t)$ as

$$\frac{\mathrm{d}^{q_i}e_i(t)}{\mathrm{d}t^{q_i}} = \sum_{j=1}^M H_i(s) \frac{k_{pij}s^{q_i+1}}{d_i(s)} [\tilde{\theta}_j^T \omega](t) + s^{q_i} K_i(s)[e_i](t).$$
(5.2.22)

Since $\tilde{\theta}_j^T(t)\omega(t)$ for each j = 1, 2..., M is bounded, and $\frac{k_{pij}s^{q_i+1}}{d_i(s)}$ is stable and proper for each $q_i = 1, 2, ..., l_i - 1$ (strictly proper for $q_i = 1, 2, ..., l_i - 2$), we have

$$|H_i(s)\frac{k_{pij}s^{q_i+1}}{d_i(s)}[\tilde{\theta}_j^T\omega](t)| \le \frac{c_{ij}}{a_i}, \quad j = 1, 2, \dots, M,$$
(5.2.23)

with some $c_{ij} > 0$ being independent of the parameter a_i .

In addition, since $s^{q_i}K_i(s)$ is stable and strictly proper for each $q_i = 1, 2, ..., l_i - 1$, with the established property: $\lim_{t\to\infty} e_i(t) = 0$, we have $\lim_{t\to\infty} s^{q_i}K_i(s)[e_i](t) = 0$.

Hence, with the use of the parameter $a_i > 0$ in $H_i(s)$ and $K_i(s)$, it can be shown that (a) $\frac{c_{ij}}{a_i} \leq \frac{\eta_i}{2M}$ in (5.2.23) for all i, j = 1, 2, ..., M, similar to the analysis before (5.2.16); and (b) $|s^{q_i}K_i(s)[e_i](t)| < \frac{\eta_i}{2}$ for all $t > T_i$ with $T_i = T_{a_i}(a_i(\eta_i), \eta_i)$. Thus, it follows that

$$\left|\frac{\mathrm{d}^{q_i}e_i(t)}{\mathrm{d}t^{q_i}}\right| < \frac{\eta_i}{2} + \frac{\eta_i}{2} = \eta_i, \text{ for all } t > T_i, \tag{5.2.24}$$

which shows $\lim_{\to\infty} \frac{\mathrm{d}^{q_i} e_i(t)}{\mathrm{d} t^{q_i}} = 0$, for $q_i = 1, \dots, l_i - 1$. ∇

Theorem 5.2.2. In multivariable MRAC systems with a diagonal MLI matrix $\xi_m(s) =$ diag $\{d_1(s), d_2(s), \ldots, d_M(s)\}$, each tracking error signal $e_i(t) = y_i(t) - y_{mi}(t)$ for $i = 1, 2, \ldots, M$, satisfies:

$$\lim_{t \to \infty} \frac{\mathrm{d}^{q_i} e_i(t)}{\mathrm{d}t^{q_i}} = 0, \quad q_i = 0, 1, \dots, l_i - 1, \tag{5.2.25}$$

with l_i being the degree of $d_i(s)$.

Thus far, we have derived a stronger tracking property of multivariable MRAC systems with the MLI matrix being diagonal, compared to the existing results in the literature.

5.2.2.2 Case II: The MLI Matrix $\xi_m(s)$ Is Non-Diagonal

Consider a system (5.1.1) with the inverse of the MLI matrix: $W_m(s)$, given in (5.2.7). From $e(t) = \xi_m^{-1}(s) K_p[\tilde{\Theta}^T \omega](t)$, we express the *i*th element of the output tracking error e(t) as

$$e_i(t) = \sum_{k=1}^{i-1} \sum_{j=1}^{M} w_{ik}(s) k_{pkj} [\tilde{\theta}_j^T \omega](t) + \sum_{j=1}^{M} \frac{k_{pij}}{d_i(s)} [\tilde{\theta}_j^T \omega](t), \quad i = 1, 2, \dots, M$$
(5.2.26)

General convergence properties. Based on (5.2.26), we first obtain the q_i -th order derivative of $e_i(t)$, i = 1, 2, ..., M as

$$\frac{\mathrm{d}^{q_i} e_i(t)}{\mathrm{d}t^{q_i}} = \sum_{k=1}^{i-1} \sum_{j=1}^M s^{q_i} w_{ik}(s) k_{pkj} [\tilde{\theta}_j^T \omega](t) + \sum_{j=1}^M \frac{s^{q_i} k_{pij}}{d_i(s)} [\tilde{\theta}_j^T \omega](t).$$
(5.2.27)

For the q_i th-order time derivative $\frac{d^{q_i}e_i(t)}{dt^{q_i}}$ given in (5.2.27), we introduce the virtual filters $K_i(s)$ and $H_i(s)$ as shown in (5.2.17), with a parameter $a_i > 0$.

Operating the equality: $1 = sH_i(s) + K_i(s)$, on (5.2.27), we decompose the q_i thorder time derivative $\frac{d^{q_i}e_i(t)}{dt^{q_i}}$ of $e_i(t)$ as

$$\frac{\mathrm{d}^{q_i}e_i(t)}{\mathrm{d}t^{q_i}} = \sum_{k=1}^{i-1} \sum_{j=1}^M H_i(s) s^{q_i+1} w_{ik}(s) k_{pkj} [\tilde{\theta}_j^T \omega](t) + \sum_{j=1}^M H_i(s) \frac{k_{pij} s^{q_i+1}}{d_i(s)} [\tilde{\theta}_j^T \omega](t) + s^{q_i} K_i(s) [e_i](t).$$
(5.2.28)

In (5.2.28), for each term of $H_i(s)s^{q_i+1}w_{ik}(s)k_{pkj}[\tilde{\theta}_j^T\omega](t)$, $k = 1, 2, \ldots, i-1$, $j = 1, 2, \ldots, M$, since $\tilde{\theta}_j^T(t)\omega(t)$ is bounded and $s^{q_i+1}w_{ik}(s)$ is stable and proper for each $q_i = 1, 2, \ldots, \bar{l}_i - 1$ with $\bar{l}_i = \min\{l_i, l_{i1}, \ldots, l_{i(i-1)}\}$, (strictly proper for $q_i = 1, 2, \ldots, \bar{l}_i - 2$), we have

$$|H_i(s)s^{q_i+1}w_{ik}(s)k_{pkj}[\tilde{\theta}_j^T\omega](t)| \le \frac{b_{ikj}}{a_i}$$
(5.2.29)

with some $b_{ikj} > 0$ being independent of the parameter a_i .

Similarly, in (5.2.28), for each of $H_i(s) \frac{k_{pij}s^{q_i+1}}{d_i(s)} [\tilde{\theta}_j^T \omega](t), j = 1, \dots, M$ we have

$$|H_i(s)\frac{k_{pij}s^{q_i+1}}{d_i(s)}[\tilde{\theta}_j^T\omega](t)| \le \frac{c_{ij}}{a_i}.$$
(5.2.30)

with some $c_{ij} > 0$ being independent of the parameter a_i .

Also, since $s^{q_i}K_i(s)$ is stable and strictly proper for each $q_i = 1, 2, ..., \bar{l}_i - 1$, and $\lim_{t\to\infty} e_i(t) = 0$, we have $\lim_{t\to\infty} s^{q_i}K_i(s)[e_i](t) = 0$.

Hence, with the use of the parameter $a_i > 0$ in $H_i(s)$ and $K_i(s)$, it is shown that for every $\eta_i > 0$, there exists a $T_i = T_i(\eta_i) > 0$ such that $\left| \frac{\mathrm{d}^{q_i} e_i(t)}{\mathrm{d}t^{q_i}} \right| < \eta_i$ for all $t > T_i$, so that $\lim_{\to \infty} \frac{\mathrm{d}^{q_i} e_i(t)}{\mathrm{d}t^{q_i}} = 0$, for $q_i = 1, \ldots, \bar{l}_i - 1$ with $\bar{l}_i = \min\{l_i, l_{i1}, \ldots, l_{i(i-1)}\}$. ∇

In summary, a new tracking error property is obtained based on the above analysis, which is summarized as follows. **Theorem 5.2.3.** In multivariable MRAC systems with a non-diagonal MLI matrix whose inverse is shown as in (5.2.7), each tracking error component $e_i(t) = y_i(t) - y_{mi}(t)$ for i = 1, 2, ..., M, satisfies:

$$\lim_{t \to \infty} \frac{\mathrm{d}^{q_i} e_i(t)}{\mathrm{d}t^{q_i}} = 0, \quad q_i = 0, 1, \dots, \bar{l}_i - 1, \tag{5.2.31}$$

with $\bar{l}_i = \min\{l_i, l_{i1}, \dots, l_{i(i-1)}\}.$

Theorem 5.2.3 guarantees a stronger tracking property of multivariable model reference adaptive control systems with an non-diagonal MLI matrix, which has not been reported in the literature yet.

Remark 5.2.1. It is worth noting here that the higher-order tracking convergence properties given in Theorems 5.2.1, 5.2.2 and 5.2.3 are *inherent* properties of MRAC systems. Although the MLI matrix of the plant is not unique, the structure of MLI matrix is determined by the unique interactor matrix. As the theorems show, the higher-order convergence performance of a system depends on the structure of the MLI matrix rather than its parameters. Thus, the higher-order convergence properties are inherent properties of the MRAC systems.

Next, we will use an quadrotor MRAC system to briefly demonstrate the technique used above.

Example 5.2.2. Higher-order convergence study for a quadrotor MRAC system. Consider a twelve-th order linearized quadrotor system in the form of (5.1.1): $\dot{x} = Ax + Bu, y = Cx$. The system state vector is $x = [x_E, y_E, z_E, \dot{x}_E, \dot{y}_E, \dot{z}_E, \phi, \theta, \psi, p, q, r]^T$ with x_E, y_E, x_E being the quadrotor positions in the earth frame, ϕ, θ, ψ being the attitude angles in the earth frame, and p, q, r being the angular velocities in the body frame. The control inputs are the lifting force F_z and the three torques T_x, T_y and T_z , and the plant outputs are chosen as the positions x_E, y_E, z_E

and the yaw angle ψ . With the modified left interactor matrix of the quadrotor system chosen in [68] (under the cruise condition with $\theta \approx \pi/60 \text{ rad}$), we obtain the inverse of modified left interactor matrix as

$$\xi_m^{-1}(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & 0 & 0 & 0\\ 0 & \frac{1}{(s+1)^4} & 0 & 0\\ \frac{0.05}{(s+1)^2} & 0 & \frac{1}{(s+1)^4} & 0\\ 0 & 0 & 0 & \frac{1}{(s+1)^2} \end{bmatrix}.$$
 (5.2.32)

According to the results shown in Theorem 5.2.3, in addition to $\lim_{t\to\infty} e_i(t) = 0$ for i = 1, 2, 3, 4, we also have $\lim_{t\to\infty} \dot{e}_1(t) = 0$, $\lim_{t\to\infty} \dot{e}_2(t) = \lim_{t\to\infty} \ddot{e}_2(t) = \lim_{t\to\infty} \frac{\mathrm{d}^3 e_2(t)}{\mathrm{d}t^3} = 0$, $\lim_{t\to\infty} \dot{e}_3(t) = 0$, and $\lim_{t\to\infty} \dot{e}_4(t) = 0$. The analysis of the time derivatives of $e_1(t)$, $e_2(t)$ and $e_4(t)$ is straightforward, so we omit it and focus on the convergence analysis of $\dot{e}_3(t)$ next.

For $e_3(t)$, from the inverse modified left interactor matrix (5.2.32), we have

$$e_{3}(t) = \frac{0.05k_{p11}}{(s+1)^{2}} [\tilde{\theta}_{1}^{T}\omega](t) + \frac{0.05k_{p12}}{(s+1)^{2}} [\tilde{\theta}_{2}^{T}\omega](t) + \frac{0.05k_{p13}}{(s+1)^{2}} [\tilde{\theta}_{3}^{T}\omega](t) + \frac{0.05k_{p14}}{(s+1)^{2}} [\tilde{\theta}_{4}^{T}\omega](t) + \frac{k_{p31}}{(s+1)^{4}} [\tilde{\theta}_{1}^{T}\omega](t) + \frac{k_{p32}}{(s+1)^{4}} [\tilde{\theta}_{2}^{T}\omega](t) + \frac{k_{p33}}{(s+1)^{4}} [\tilde{\theta}_{3}^{T}\omega](t) + \frac{k_{p34}}{(s+1)^{4}} [\tilde{\theta}_{4}^{T}\omega](t).$$

$$(5.2.33)$$

Operating $1 = sH_3(s) + K_3(s)$ to $\dot{e}_3(t)$, with the introduced virtual filters: $K_3(s) = \frac{a_3^4}{(s+a_3)^4}$, $H_3(s) = \frac{1}{s} \frac{(s+a_3)^4 - a_3^4}{(s+a_3)^4}$, we have $\dot{e}_3(t) = H_3(s) \frac{0.05k_{p11}s^2}{(s+1)^2} [\tilde{\theta}_1^T \omega](t) + H_3(s) \frac{0.05k_{p12}s^2}{(s+1)^2} [\tilde{\theta}_2^T \omega](t) + H_3(s) \frac{0.05k_{p13}s^2}{(s+1)^2} [\tilde{\theta}_3^T \omega](t) + H_3(s) \frac{k_{p31}s^2}{(s+1)^2} [\tilde{\theta}_1^T \omega](t) + H_3(s) \frac{k_{p32}s^2}{(s+1)^4} [\tilde{\theta}_2^T \omega](t) + H_3(s) \frac{k_{p33}s^2}{(s+1)^4} [\tilde{\theta}_1^T \omega](t) + H_3(s) \frac{k_{p33}s^2}{(s+1)^4} [\tilde{\theta}_1^T \omega](t) + H_3(s) \frac{k_{p34}s^2}{(s+1)^4} [\tilde{\theta}_1^T \omega](t) + sK(s)[e_3](t).$ (5.2.34)

From $\frac{0.05k_{p1j}s^2}{(s+1)^2}$ being stable and proper, $\frac{0.05k_{p3j}s^2}{(s+1)^4}$ being stable and strictly proper, and $||h_3(\cdot)|| = \frac{4}{a_3}$, we have

$$\left| H_3(s) \frac{0.05k_{p1j}s^2}{(s+1)^2} [\tilde{\theta}_j \omega](t) \right| \le \frac{b_{31j}}{a_3}, j = 1, 2, 3, 4,$$
(5.2.35)

for some $b_{31j} > 0$ being independent of a_3 and

$$\left| H_3(s) \frac{0.05k_{p3j}s^2}{(s+1)^4} [\tilde{\theta}_j \omega](t) \right| \le \frac{c_{3j}}{a_3}, j = 1, 2, 3, 4,$$
(5.2.36)

for some $c_{3j} > 0$ being independent of a_3 .

Hence, with the parameter $a_3 \ge \max\{\frac{16b_{31j}}{\eta_3}, \frac{16c_{3j}}{\eta_3}\}$, so that $\frac{b_{31j}}{a_1} \le \frac{\eta_3}{16}$ and $\frac{c_{3j}}{a_3} \le \frac{\eta_3}{16}$ in (5.2.15), for all j = 1, 2, 3, 4. In addition, we let $T_3 = T_3(\eta_3) > 0$ such that $|sK_3(s)[e_3](t)| < \frac{\eta_3}{2}$ for all $t \ge T_3$, since $\lim_{t\to\infty} sK_3[e_3](t) = 0$. Then, it follows that

$$|\dot{e}_3(t)| < 8 \cdot \frac{\eta_3}{16} + \frac{\eta_3}{2} = \eta_3, \ \forall t > T_3.$$
 (5.2.37)

Hence, we verify that $\lim_{t\to\infty} \dot{e}_3(t) = 0$.

In summary, for this quadrotor MRAC system, we confirm that the standard output feedback MRAC scheme guarantees asymptotic tracking for

- the position of x_E , y_E and z_E axes;
- the velocity on x_E , y_E and z_E axes;
- the acceleration and the jerk on y_E axes; and
- the yaw angle and the yaw rate.

Such a stronger result helps the researchers to choose the reference signals more suitable in the future.

5.3 Simulation Study

In this subsection, we study the response of the NASA GTM MRAC system shown in Section 5.1.2 and 5.2.2, to verify the desired tracking error convergence properties we derived in Theorem 5.2.2.

$$\dot{x} = Ax + Bu, y = Cx$$
, with

A =	-0.019	0.1364	-9.7778	-32.0829	-0.0018	-0.0004	0	0]
	-0.2804	-2.7567	120.1968	-2.42	-0.0001	0	0.0004	-0.0061
	0.0205	-0.3106	-3.5393	0	0.007	0.0328	-0.0014	0
	0	0	1	0	0	-0.0002	0	0.0002
	0	-0.0027	0	-0.0005	-0.5765	-125.9974	10.4690	32.0829
	0	0	-0.0255	0	0.2245	-1.4053	-0.2794	0
	0	0	0.0018	0	-0.629	1.9689	-5.4759	0
	0	0	0	-0.0002	0	0.0754	1	0

The modified interactor matrix $\xi_m(s)$ is chosen as diag $\{(s+2)^2, (s+2)^2\}$, the transfer function of the chosen reference model (5.1.3) is $W_m(s) = \text{diag}\left\{\frac{1}{(s+2)^2}, \frac{1}{(s+2)^2}\right\}$, which is proper and stable. The reference inputs are chosen as $r(t) = \left[\frac{-40\pi}{180}\sin(0.1t), \frac{-15\pi}{180}\sin(0.1t)\right]^{\text{T}}$.

Simulation results. For simulation, the adaptation gains are chosen as $\Gamma = 5I$, $\Gamma_{\theta} = 5$, and the initial condition are chosen as $y(0) = [-0.01, -0.01]^{\mathrm{T}}$, $y_m(0) = [0, 0]^{\mathrm{T}}$. The performances of the pitch angle $\theta(t)$ and the pitch rate $\dot{\theta}(t)$ are shown in Fig.5.1(a) and Fig. 5.1(b), respectively, in which the dotted lines represent the reference pitch angle/rate and the solid lines represent the actual pitch angle/rate. The performances of the roll angle $\phi(t)$ and the roll rate $\dot{\phi}(t)$ are shown in Fig. 5.2(a) and Fig.5.2(b), respectively, in which the dotted lines represent the reference roll angle/rate and the solid lines represent the actual roll angle/rate. The simulation results verify the desired system tracking performance.



Figure 5.2: Roll angle $\phi(t)$ and its derivative.

Summary

In this chapter, compared to the existing MRAC result in the literature, we have derived a stronger higher-order tracking property of multivariable MRAC systems: not only the tracking error component $e_i(t)$, i = 1, 2, ..., M but its up to q_i th derivatives converge to zero asymptotically, with q_i related to the system MLI matrix. Such a new convergence property avoids certain undesirable oscillations, brings the MRAC system performance closer to that of a nominal control system, and benefits for practical applications. The convergence analysis on the NASA GTM system illustrates the derivation techniques and show the engineering signification of the derived results.

Chapter 6

Sensor Failure Compensation for SISO Systems

This chapter addresses a new adaptive output tracking problem in the presence of uncertain plant dynamics and uncertain sensor failures. A new unified nominal state feedback control law is developed to deal with various sensor failures, which is directly constructed by state sensor outputs. Such a new state feedback compensation control law is able to ensure the desired plant-model matching properties under different failure patterns. Based on the nominal compensation control design, a new adaptive compensation control scheme is proposed, which guarantees closed-loop signal boundedness and asymptotic output tracking. The new adaptive compensation scheme not only expands the sensor failures types that the system could tolerate, but avoids some signal processing procedures that the traditional fault-tolerant control techniques are forced to encounter. A complete stability analysis and a representative simulation study are conducted to evaluate the effectiveness of the proposed adaptive compensation control scheme.

6.1 Problem Statement

In this section, a state sensor failure model is first established, and a new MRAC problem, namely, adaptive tracking control in the presence of uncertain state sensor failures, is formulated.

6.1.1 Plant Model and State Sensor Failures

Before we establish the state sensor failure model in this subsection, we will first introduce the plant model, and discuss the main advantages of state feedback control and issues with possible state sensor failures.

Plant model. Consider a linear time-invariant plant:

$$\dot{x}(t) = Ax(t) + bu(t), \ x(t) \in \mathbb{R}^{n}, u(t) \in \mathbb{R}$$
$$y(t) = cx(t), \ y(t) \in \mathbb{R}$$
(6.1.1)

where $A \in \mathbb{R}^{n \times n}$ is an unknown matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^{1 \times n}$ are unknown vectors. The input-output description of the plant (6.1.1) is

$$y(t) = G(s)[u](t) = k_p \frac{Z(s)}{P(s)}[u](t), \qquad (6.1.2)$$

where $k_p \neq 0$, $P(s) = \det(sI - A) = s^n + p_{n-1}s^{n-1}\cdots + p_1s + p_0$ and $Z(s) = s^m + \cdots + z_1s + z_0$ for some $m \ge 0$. The notation: y(t) = G(s)[u](t), is used to denote the output y(t) of a LTI system represented by a transfer function G(s) with input signal u(t). The symbol s is a differentiation operator: $s[x](t) = \dot{x}(t)$, or the Laplace transform variable as the case may be. It is a simple notation to combine both the time domain and the frequency domain signal operations, suitable for adaptive control system presentation.

State feedback control. A state feedback controller has the form:

$$u(t) = k_1^T x(t) + k_2 r(t), (6.1.3)$$

where r(t) is a reference signal. For feedback control applications where the state x(t) is accessible, compared to an output feedback controller structure, such a controller structure is a more preferred choice, because

- the state feedback controller structure is much simpler, which leads to easier controller implementation with reduced computation especially in the adaptive control case;
- the state feedback controller structure carries more state information, which results in a better transient performance; and
- the state feedback controller structure has certain redundant capacity for achieving desired control system performance, which is desirable for tolerating uncertain sensor faults.

In particular for MRAC (model reference adaptive control), the state feedback controller structure (6.1.3) is a natural choice when the full state vector x(t) is available, because it enjoys the least dimension regressor in parameter adaptation laws which are used to update adaptive controller parameters. Compared to an output feedback MRAC design where the dimension of the regressor is 2n, the dimension of the regressor in the state feedback MRAC design is n + 1 (in both designs, such a regressor will be processed by a filter whose order is n^* , the plant relative degree, further indicating the advantage of a state feedback adaptive control design). This also confirms that the state feedback control design significantly reduces the controller implementation and computing complexities, compared to an output feedback control design, as we have mentioned above.

It is worthwhile to note that for implementing the state feedback controller structure (6.1.3), a crucial precondition is to obtain the precisely measured state vector x(t). However, in practice, state sensors may be subject to some uncertain failures so that the desired system performance may not be ensured unless effective sensor failure tolerance is provided, which is possible with the state feedback design.

State sensor failure model. In this study, we consider the case when a set of sensors S_i , i = 1, ..., n, is used to measure the *n* state variables $x_i(t)$. In the presence of a unrecoverable fault at the *j*th sensor, the sensor output may be described as

$$z_j = S_j(x_j) = \begin{cases} x_j & \text{with the healthy sensor } S_j \\ \bar{s}_j & \text{with the failed sensor } S_j \end{cases}$$
(6.1.4)

for some unknown bounded values \bar{s}_j with unknown indices $j \in \{1, 2, ..., n\}$. Thus, for the state vector x(t) constructed by the *n* state variables x_i , the sensor output vector with possible uncertain state sensor failures is $z(t) = [z_1, ..., z_n]^T$. Such a sensor output vector z(t) is the actual signal feeding for control implementation: $u = k_1^T z(t) + k_2 r(t)$ (see Fig. 6.1), which results in destruction of the feedback control system.

This failure model characterizes the most typical classes of sensor failures that may occur, that is, some unknown sensor outputs are stuck at some unknown fixed or varying values. For example, the humidity-sensitive pressure sensor may be stuck at some unknown values due to water or moisture.

Discussion on issues and problems with failures. In general, state sensor failures result in invalidation of control laws. For example, without the precisely measured state variables, a pole placement control design cannot position the closed-loop poles; a state matching control design cannot guarantee plant-model state matching; and an adaptive control design with state feedback cannot achieve asymptotic tracking. Thus, effective compensation of such state sensor failures is necessary.

In addition, the state sensor failures investigated in this chapter are uncertain, which means we do not know which sensors are failed, how much the failures are and



Figure 6.1: Control system under possible uncertain sensor failures.

when the failures occur. Such uncertain state sensor failures require effective adaptive compensation control scheme to guarantee desired system performance.

6.1.2 State Sensor Failure Compensation Problem

In this subsection, we formulate the new adaptive control problem for state sensor failure compensation to control the unknown linear time-invariant (LTI) systems (6.1.1) with uncertain sensor failures (6.1.4), for achieving asymptotic output tracking.

6.1.2.1 Control Objective

The control objective of the state sensor failure compensation problem is to construct a feedback control law u(t) in the unknown plant (6.1.1) with the sensor output vector z(t) being subject to the uncertain state sensor failures (6.1.4) such that all signals in the closed-loop system are bounded and the system output y(t) asymptotically tracks a given reference output signal $y_m(t)$ generated from a reference model system

$$y_m(t) = W_m(s)[r](t), \ W_m(s) = \frac{1}{P_m(s)}$$
(6.1.5)

where $P_m(s)$ is a monic stable polynomial with the plant relative degree $n^* = n - m$ and r(t) is a bounded piecewise continuous reference input signal.

Desired features of a sensor failure compensation controller. A new state feedback control design, which can guarantee desired system performance for the case

of no sensor failures (that is, z(t) = x(t) in Fig. 6.1) and for the case of uncertain state sensor failures (that is, $z(t) \neq x(t)$ in Fig. 6.1), is expected. In this chapter, we propose to develop an adaptive state sensor failure compensation scheme, whose controller structure has a form *as close as possible to* the state feedback controller structure for dealing with the *possible uncertain state sensor failures*. For our new state sensor failure compensation problem, we directly use the sensor output vector z(t) without additional signal processing often used in some literature, to maintain a similar controller structure close to a state feedback control design using the full state vector x(t).

6.1.2.2 Failure Pattern and Failure Pattern Set

In this subsection, we specify the class of sensor failures to be compensated by our adaptive control scheme, and define a failure pattern set for which an adaptive control design is developed to handle all its elements (failures).

Failure pattern. For the *n* sensors corresponding to the *n* state variables x_i , i = 1, 2, ..., n, there are different sensor failure patterns which can be represented by a generic failure pattern matrix

$$\sigma = \operatorname{diag}\{\sigma_1, \sigma_2, \dots, \sigma_n\}$$
(6.1.6)

where $\sigma_i = 1$ if the *i*th sensor fails and $\sigma_i = 0$ if the *i*th sensor is healthy. With such a matrix, we can express the sensor output vector as

$$z(t) = x(t) - \sigma(x(t) - \bar{s})$$
(6.1.7)

where $\bar{s} = [\bar{s}_1, \bar{s}_2, \dots, \bar{s}_n]^T$ is defined as a failure vector which can either be a constant vector or a scalar vector. For each individual failure pattern, we denote σ as $\sigma = \sigma_{(k)}$, for $k = 0, 1, 2, \dots, N - 1$, where N is the total number of different sensor failure patterns including the no failure pattern, and correspondingly, denote \bar{s} as $\bar{s} = \bar{s}_{(k)}$. We use $\sigma_{(0)} = \text{diag}\{0, 0, \dots, 0\}$ and $\bar{s}_{(0)} = [0, 0, \dots, 0]^T$ to represent the no failure case (for the k = 0 case).

Failure pattern set. To specify all possible sensor failures to be dealt with by an adaptive control design, we define the sensor failure pattern set as

$$\Sigma = \{ \sigma \mid \sigma = \sigma_{(k)}, \ k = 0, 1, 2, \dots, N - 1 \}.$$
(6.1.8)

Such a failure pattern set includes N failure patterns caused by up to $n - n_0$ failed sensors for some $n_0 \in \{1, 2, ..., n\}$ with n_0 representing the number of healthy sensors. The total number N of failure patterns depends on the problem of interest, and in terms of n_0 , N may be between 1 and $N_0 = \sum_{i=0}^{n-n_0} {n \choose i}$. The maximum number N_0 represents the problem wherein all the possible failure patterns that can be caused by up to $n - n_0$ failed state sensors are to be considered. The reason that the total number N of failure patterns in the failure pattern set Σ may be less than N_0 is because, in practice, some sensors may be more vulnerable to failure than some other sensors, thus there may be no need to consider all possible cases caused by up to $n - n_0$ failed state sensors but just some particular cases.

The case of no failure is always be considered. In other words, N = 2 represents the problem wherein there are two failure patterns: either all sensors are healthy (when z(t) = x(t), with the failure pattern $\sigma_{(0)}$ and the failure vector $\bar{s}_{(0)}$), or a particular group of $n - n_0$ sensors are failed (when $z(t) = x(t) - \sigma_{(1)}(x(t) - \bar{s}_{(1)})$), with a specific failure pattern matrix $\sigma_{(1)}$ and a specific failure vector $\bar{s}_{(1)}$). For a particular application, all failure patterns of interest belong to a specific failure pattern set Σ .

It is valuable to notice that in this problem, it is uncertain what is the value of σ during the system operation, and a desired control design should be able to handle such an uncertainty, which is a nontrivial control problem.

From the number-of-failed-sensors point of view, we classify all the sensor failure patterns in a failure pattern set of interest that we mentioned before into two groups:

- (1) failure patterns caused by exactly $n n_0$ failed sensors which are the worst cases; and
- (2) failure patterns caused by $n \bar{n}_0$ failed sensors with some $n \bar{n}_0 < n n_0$, which represents all the other non-worst cases.

While our adaptive failure compensation scheme to be developed is able to deal with all the sensor failure patterns in a failure pattern set of interest either in case (1) or case (2), the information of n_0 will be used to construct a unified controller structure.

6.1.2.3 Technical Goals

In the procedure of solving this state sensor compensation problem, we will

- build a nominal controller structure with the direct state sensor output z(t), which is capable of achieving plant-model output matching: y(t) = W_m(s)[r](t), for (A, b, c) known for all possible failure patterns in the failure pattern set of interest in Section 6.2 (such a controller structure is necessary and suitable for control adaptation to achieve lim_{t→∞}(y(t) y_m(t)) = 0 in Section 6.3, for (A, b, c) unknown);
- develop an adaptive compensation scheme with stable parameter update laws, which is capable of achieving output tracking in the presence of all possible uncertain sensor failures and system parameter uncertainties; and
- conduct a closed-loop system stability analysis to verify the effectiveness of the proposed adaptive state sensor failure compensation scheme.

6.2 Nominal Sensor Failure Compensation Scheme

In this section, for the case of the system parameters (A, b, c) and the failure pattern $\sigma_{(k)}$ being known, we propose a nominal controller structure to compensate all possible sensor failures in a failure pattern set Σ in Section 6.2.1, which provides a key *a priori* knowledge for the adaptive control design to be developed, and we show that for all possible failure patterns in a failure pattern set Σ the desired plant-model output matching properties can be achieved by the proposed compensation controller in Section 6.2.2.

A basic assumption. To meet the control objective, we first give a basic assumption before the control design. For the state sensor failure compensation problem, we assume that for each sensor failure pattern $\sigma_{(k)}$ of all N possible sensor failure patterns in a chosen failure pattern set Σ of interest, the vector $z_{(k)}(t)$, consisting of all precisely measured state variables $z_i = x_i(t)$ from at least n_0 healthy sensors, exist and is state-observable. Such an assumption is described mathematically as

(A6.1) For each failure pattern $\sigma_{(k)}$ in a chosen failure pattern set Σ , the corresponding healthy sensor vector $z_{(k)}(t) = C_{(k)}x(t) \in \mathbb{R}^{n_{(k)}}$ exists for some matrix $C_{(k)} \in \mathbb{R}^{n_{(k)} \times n}$ and $n_{(k)} \ge n_0$, such that $(A, C_{(k)})$ is observable, $k = 0, 1, 2, \ldots, N - 1$.

Based on Assumption (A6.1), the healthy sensor vectors $z_{(k)}(t)$ collecting n_0 precisely measured state variables can be uniformly denoted as $z_{n_0}(t) = C_{n_0}x(t) \in \mathbb{R}^{n_0}$ with (A, C_{n_0}) being observable for a generic $C_{n_0} \in \mathbb{R}^{n_0 \times n}$ which may take different forms for different cases. With the above definition, the sensor output measurements z(t) can be expressed as

$$z(t) = P^{-1}[z_{n_0}^T(t), \bar{z}_{n_0}^T(t)]^T$$
(6.2.1)

with $P \in \mathbb{R}^{n \times n}$ being an nonsingular matrix such that $C_{n_0}P^{-1} = [I_{n_0}, 0]$ and $\bar{z}_{n_0}(t)$

being the corresponding failed sensor output vector (the transformation matrix P will be discussed in details in Section 6.2.1).

Remark 6.2.1. It is worth noting that $z_{n_0}(t)$ or C_{n_0} may not be unique as there may be different cases with n_0 healthy sensors, and that although $z_{n_0}(t)$ may have effect in an applied control law, it is actually uncertain to the control law what components of x(t) are contained in such a healthy sensor output vector $z_{n_0}(t)$. It is also the case for the vectors $z_{(k)}(t)$: it is unknown what components of x(t) are in such a vector $z_{(k)}(t)$ which influences a feedback control law.

General idea. Besides the properties of the state feedback controller (6.1.3) that we have discussed before, the state feedback controller (6.1.3) is also able to ensure plant-model output matching: $y(t) = y_m(t) = W_m(s)[r](t)$, as long as the matching equation

$$\det(sI - A - bk_1^{*T}) = P_m(s)Z(s)\frac{1}{k_p}, \ k_2^* = \frac{1}{k_p},$$
(6.2.2)

is satisfied [82]. In practice when the actual system state x(t) cannot be obtained, an alternative controller:

$$u(t) = k_1^{*T} \hat{x}(t) + k_2^{*} r(t), \qquad (6.2.3)$$

could be used to achieve plant-model matching, where the state estimation $\hat{x}(t)$ is produced by an observer/state estimator. The controller (6.2.3) is called the observerbased state feedback controller which inherits the desired plant-model output matching properties from the state feedback controller as long as the state estimate $\hat{x}(t)$ converges to the state x(t) exponentially.

Although the observer-based state feedback controller becomes ineffectual either for the problem with uncertain system parameters and uncertain state sensor failures in this chapter, we could bridge the gap between the state feedback controller (6.1.3) and the sensor failure compensation controller to be proposed through the observerbased state feedback controller.

In the rest of this section, we will first derive a parameterizable controller structure for the worst cases (the failure patterns caused by $n - n_0$ failed sensors) in Section 6.2.1 through (6.2.3), and then establish the desired plant-model output matching properties of the new compensation controller structure under worst and non-worst cases in Section 6.2.2 by utilizing the plant-model output matching properties of the observer-based state feedback controller structure.

6.2.1 Nominal Controller Structure

In this subsection, a unified controller structure, with the capability of handling all the possible failures in the failure set Σ is to be proposed. To derive the state sensor failure compensation controller structure, the following three steps are performed.

Step 1: Worst case observer with the healthy sensor vector $z_{n_0}(t)$. For the state equation: $\dot{x}(t) = Ax(t) + bu(t)$, and the transformation matrix $P \in \mathbb{R}^{n \times n}$ such that $C_{n_0}P^{-1} = [I_{n_0}, 0]$ with $n_0 = \operatorname{rank}[C_0]$ (*P* is also used in (6.2.1)), we can first transfer it to

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} u(t),$$
(6.2.4)

where $\bar{x}(t) = Px(t) = [\bar{x}_1^T(t), \bar{x}_2^T(t)]^T$ with $\bar{x}_1(t) \in \mathbb{R}^{n_0}, \bar{x}_2(t) \in \mathbb{R}^{n-n_0}, \bar{A}_{11} \in \mathbb{R}^{n_0 \times n_0},$ $\bar{A}_{12} \in \mathbb{R}^{n_0 \times (n-n_0)}, \bar{A}_{21} \in \mathbb{R}^{(n-n_0) \times n_0}, \bar{A}_{22} \in \mathbb{R}^{(n-n_0) \times (n-n_0)}, \bar{b}_1 \in \mathbb{R}^{n_0}$ and $\bar{b}_2 \in \mathbb{R}^{n-n_0}.$ For such a transformed state equation with the transformation matrix P, the n_0 state variables in $\bar{x}_1(t)$ are the n_0 precisely measured state variables from the healthy sensors, that is, $z_{n_0}(t) = \bar{x}_1(t)$. Thus, $\bar{x}_2(t)$ is to be estimated to obtain the state estimate $\hat{x}(t) = [\bar{x}_1^T(t), \hat{x}_2^T(t)] = [z_{n_0}^T(t), \hat{x}_2^T(t)]^T.$ Then, we generate an estimate $\hat{x}(t)$ for $\bar{x}(t)$ with a reduced-order dynamic system generating an estimate $\hat{x}_2(t)$ for $\bar{x}_2(t)$, in the form:

$$\hat{\bar{x}}(t) = \begin{bmatrix} \bar{x}_1 & \hat{x}_2 \end{bmatrix}^T = \begin{bmatrix} z_{n_0}(t) & w(t) + L_r z_{n_0}(t) \end{bmatrix}^T,$$
 (6.2.5)

where $L_r \in \mathbb{R}^{(n-n_0) \times n_0}$ is a constant gain matrix such that the eigenvalues of the $(n-n_0) \times (n-n_0)$ matrix $\bar{A}_{22} - L_r \bar{A}_{12}$ are stable and prespecified, and $w(t) \in \mathbb{R}^{n-n_0}$ is generated from the dynamic equation

$$\dot{w}(t) = (\bar{A}_{22} - L_r \bar{A}_{12})w(t) + (\bar{b}_2 - L_r \bar{b}_1)u(t) + ((\bar{A}_{22} - L_r \bar{A}_{12})L_r + \bar{A}_{21} - L_r \bar{A}_{11})z_{n_0}(t), \ w(0) = w_0.$$
(6.2.6)

with $w(0) \in \mathbb{R}^{n-n_0}$ being an estimate of $L_r z_{n_0}(0) - \bar{x}_2(0)$.

The solution of (6.2.6) can be expressed as

$$w(t) = e^{(\bar{A}_{22} - L_r \bar{A}_{12})t} w(0) + (sI - \bar{A}_{22} + L_r \bar{A}_{12})^{-1} (\bar{b}_2 - L_r \bar{b}_1)[u](t) + (sI - \bar{A}_{22} + L_r \bar{A}_{12})^{-1} ((\bar{A}_{22} - L_r \bar{A}_{12})L_r + \bar{A}_{21} - L_r \bar{A}_{11})[z_{n_0}](t) = \frac{n_1(s)}{\Lambda(s)} [u](t) + \frac{n_2(s)}{\Lambda(s)} [z_{n_0}](t) + e^{(\bar{A}_{22} - L_r \bar{A}_{12})t} w(0),$$
(6.2.7)

where $\Lambda(s) = \det(sI - \bar{A}_{22} + L_r \bar{A}_{12})$ whose degree is $n - n_0$ and stability properties can be prespecified and by assigning the eigenvalues of $\bar{A}_{22} - L_r \bar{A}_{12}$ as a set of given (known) values, and $n_1(s)$ is an $(n - n_0) \times 1$ polynomial vector and $n_2(s)$ is an $(n - n_0) \times n_0$ polynomial matrix, whose maximum degrees are $n - n_0 - 1$ or less.

From (6.2.4)–(6.2.6), it can be verified that the estimated error $\tilde{x}_2(t) = \bar{x}_2(t) - \hat{x}_2(t)$ satisfies $\dot{\bar{x}}_2(t) = (\bar{A}_{22} - L_r \bar{A}_{12})\tilde{\bar{x}}_2(t)$, which decays to zero exponentially since the matrix $\bar{A}_{22} - L_r \bar{A}_{12}$ is stable.

Step 2: Controller structure with the healthy sensor output $z_{n_0}(t)$. From the observer shown in Step 1, we are able to obtain a state estimate $\hat{x}(t)$ with $\lim_{t\to\infty}(\hat{x}(t) - x(t)) = 0$. To obtain a parameterized controller structure which does not depend on the system parameters, we reparameterize the observer-based controller structure $u(t) = k_1^{*T} \hat{x}(t) + k_2^{*T}(t)$ by using (6.2.5), (6.2.7) and the relationship $\hat{x}(t) = P^{-1}\hat{x}(t)$, and have a controller structure with the healthy sensor output $z_{n_0}(t)$, in the form of

$$u(t) = \theta_1^{*T} \frac{a(s)}{\Lambda(s)} [u](t) + \bar{\theta}_{21}^{*T} \frac{A_{21}(s)}{\Lambda(s)} [z_{n_0}](t) + \bar{\theta}_{201}^{*T} z_{n_0}(t) + \theta_3^* r(t) + \varepsilon_0(t), \qquad (6.2.8)$$

for an exponentially decaying signal $\varepsilon_0(t) = k_{p2}^{*T} e^{(\bar{A}_{22} - L_r \bar{A}_{12})t} w(0)$ representing the effect of the initial condition w(0), where $\theta_1^* \in \mathbb{R}^{n-n_0}$, $\bar{\theta}_{21}^* \in \mathbb{R}^{n_0(n-n_0)}$, $\bar{\theta}_{201}^* \in \mathbb{R}^{n_0}$ and $\theta_3^* \in \mathbb{R}$, such that $\bar{\theta}_{201}^{*T} = k_{p1}^{*T} + k_{p2}^{*T} L_r$, $k_{p2}^{*T} n_1(s) = \theta_1^{*T} a(s)$ and $k_{p2}^{*T} n_2(s) = \bar{\theta}_{21}^{*T} A_{21}(s)$, for $k_1^{*T} P^{-1} = [k_{p1}^{*T}, k_{p2}^{*T}]$ with $k_{p1}^* \in \mathbb{R}^{n_0}$ and $k_{p2}^* \in \mathbb{R}^{n-n_0}$, and

$$a(s) = [1, s, \dots, s^{n-n_0-1}]^T, A_{21}(s) = [I_{n_0}, sI_{n_0}, \dots, s^{n-n_0-1}I_{n_0}]^T.$$
(6.2.9)

Step 3: Compensation controller structure with the sensor output z(t).

To achieve the first technical goal of this chapter which is to develop a state feedback controller structure using the sensor output z(t) directly for dealing with the *uncertain* state sensor failures, we first express the relationship between the healthy sensor output vector $z_{n_0}(t)$ and the sensor output measurement z(t) as

$$z_{n_0}(t) = C_{n_0} z(t), (6.2.10)$$

with $z(t) = P^{-1}[z_{n_0}^T(t), \bar{z}_{n_0}^T(t)]^T$. Then, we parameterize the controller structure (6.2.8) constructed with the healthy sensor output $z_{n_0}(t)$ in terms of z(t), and obtain the nominal state sensor failure compensation controller structure in the form:

$$u(t) = \theta_1^{*T} \omega_1(t) + \theta_2^{*T} \omega_2(t) + \theta_{20}^{*T} z(t) + \theta_3^{*} r(t), \qquad (6.2.11)$$

for some controller parameters $\theta_1^* \in \mathbb{R}^{n-n_0}$, $\theta_2^* \in \mathbb{R}^{n(n-n_0)}$, $\theta_{20}^* \in \mathbb{R}^n$, $\theta_3^* \in \mathbb{R}$, and the

regressors

$$\omega_1(t) = \frac{a(s)}{\Lambda(s)}[u](t), \quad \omega_2(t) = \frac{A(s)}{\Lambda(s)}[z](t)$$
(6.2.12)

with $a(s) = [1, s, ..., s^{n-n_0-1}]^T$, $A(s) = [I_n, sI_n, ..., s^{n-n_0-1}I_n]^T$, and $\Lambda(s)$ is a stable and monic polynomial of degree $n - n_0$. As a MRAC problem, to guarantee the stability of the internal signals, we assume: (A6.2) All zeros of Z(s) are stable.

Remark 6.2.2. In fact, based on the relationship: $z(t) = P^{-1}[z_{n_0}^T(t), \bar{z}_{n_0}^T(t)]^T$, the nominal controller structure (6.2.11) can also be expressed in terms of $z_{n_0}(t)$ and $\bar{z}_{n_0}(t)$ as

$$u(t) = \theta_1^{*T} \frac{a(s)}{\Lambda(s)} [u](t) + \bar{\theta}_{21}^{*T} \frac{A_{21}(s)}{\Lambda(s)} [z_{n_0}](t) + \bar{\theta}_{201}^{*T} z_{n_0}(t) + \theta_3^* r(t) + \bar{\theta}_{22}^{*T} \frac{A_{22}(s)}{\Lambda(s)} [\bar{z}_{n_0}](t) + \bar{\theta}_{202}^{*T} \bar{z}_{n_0}(t)$$
(6.2.13)

with $\bar{\theta}_{22}^* = 0_{(n-n_0)(n-n_0)\times 1}, \ \bar{\theta}_{202}^* = 0_{(n-n_0)\times 1}, \ \text{and} \ A_{22}(s) = [I_{n-n_0}, \dots, s^{n-n_0-1}I_{n-n_0}]^T.$

Such an extended form of the controller (6.2.11) shows the essence of the nominal controller. The nominal controller parameters $\theta_1^*, \bar{\theta}_{21}^*, \bar{\theta}_{201}^*$ and θ_3^* associating with the healthy sensor output $z_{n_0}(t)$ are for plant-model matching , and the nominal parameters $\bar{\theta}_{22}^*$ and $\bar{\theta}_{202}^*$ associating with the failed sensor output $\bar{z}_{n_0}(t)$ are for failure compensation. This extended structure will be used for the proof of plant-model output matching in Section 6.2.2, for the derivation of tracking error equation and for the analysis of closed-loop stability in Section 6.3. The relationship $\theta_2^{*T}A(s)[z](t) = \bar{\theta}_{21}^{*T}A_{21}(s)[z_{n_0}](t) + \bar{\theta}_{22}^{*T}A_{22}(s)[\bar{z}_{n_0}](t)$ for $\theta_2^{*T}A(s)P^{-1} = [\bar{\theta}_{21}^{*T}A_{21}(s), \bar{\theta}_{22}^{*T}A_{22}(s)]$ and the relationship $\theta_{20}^{*T}z(t) = \bar{\theta}_{201}^{*T}z_{n_0}(t) + \bar{\theta}_{202}^{*T}\bar{z}_{n_0}(t)$ for $\theta_{20}^{*T}P^{-1} = [\bar{\theta}_{201}^{*T}, \bar{\theta}_{202}^{*T}]$ can be verified by the definition of $A_{21}(s)$ and $A_{22}(s)$ and $z(t) = P^{-1}[z_{n_0}^{T}(t), \bar{z}_{n_0}^{T}(t)]^T$.

So far, we have obtained the parameterized nominal state sensor failure compensation controller structure (6.2.11) which is constructed by the sensor output z(t)



Figure 6.2: Design procedure of the compensation controller structure (6.2.11).

directly. The design procedure is summarized as Fig. 6.2 shows. Corresponding to the first technical goal, there are two design logic chains: (a) keeping finding parameterized controller structures which make control adaptation realizable (as shown in Section 6.2.1); and (b) keeping finding controllers that can guarantee the plant-model output matching properties inherited from the state feedback controller structure (to be shown in Section 6.2.2), until the technical goal is achieved.

Remark 6.2.3. It is worth noting that although the nominal controller structure (6.2.11) is able to achieve plant-model matching for all failure patterns, including the worst and the general cases, in a chosen failure pattern set Σ of interest (to be proved in the next subsection), for each different failure pattern $\sigma_{(k)}$ the values of the corresponding constant nominal controller parameters θ_1^* , θ_2^* , θ_{20}^* are different. In other words, each special failure pattern $\sigma_{(k)}$ corresponds to a special set of controller parameters $\theta_{1(k)}^*$, $\theta_{2(k)}^*$, $\theta_{20(k)}^*$. The actual controller parameters $\theta_{1(k)}^*$, $\theta_{2(k)}^*$, $\theta_{20(k)}^*$ for each different failure pattern $\sigma_{(k)}$ are denoted as θ_1^* , θ_2^* , θ_{20}^* , due to the use of the vector $z_{n_0}(t)$.
6.2.2 Plant-Model Output Matching

As we discussed in Section 6.1.2.2, all N possible sensor failures in a chosen failure pattern set Σ of interest can be classified into two groups: (1) failure patterns caused by exactly $n - n_0$ failed sensors which are the worst cases; and (2) failure patterns caused by $n - \bar{n}_0$ failed sensors with some $n - \bar{n}_0 < n - n_0$. Although the compensation controller structure (6.2.11) is designed for the worst case of $n - n_0$ sensor failures, it should also be able to make the output of the plant match the one of the reference model for less than $n - n_0$ failures. In other words, the compensation controller structure (6.2.11) should be capable of ensuring output matching for all possible failure patterns under a failure pattern set Σ . In this subsection, we will establish the desired plant-model output matching properties of the nominal compensation controller structure (6.2.11) under the worst cases and the non-worst cases, respectively.

6.2.2.1 Output Matching for $n - n_0$ Failures

Under the situation of $n - n_0$ sensor failures, plant-model output matching properties of the nominal controller structure (6.2.11) can be summarized as below.

Theorem 6.2.1. There exist constant parameters $\theta_1^* \in \mathbb{R}^{n-n_0}, \theta_2^* \in \mathbb{R}^{n(n-n_0)}, \theta_{20}^* \in \mathbb{R}^n, \theta_3^* \in \mathbb{R}$ such that the controller structure (6.2.11) ensures closed-loop signal boundedness and plant-model output matching: $y(t) - y_m(t) = \varepsilon(t)$, for some exponentially decaying $\varepsilon(t)$ and $y_m(t)$ being the output of reference model (6.1.5), in the presence of $n - n_0$ failed sensors.

Proof: In the proof, we will first show the plant-model matching properties, and then show closed-loop signal boundedness. The proof can be divided into four steps.

<u>Step 1: Output matching by observer-based state feedback control.</u> Since the state estimate error $\hat{x}(t) - x(t)$ can be expressed as $\hat{x}(t) - x(t) = [0, \varepsilon_1^T(t)]^T$ with $\varepsilon_1(t) =$ $P_{12}e^{(\bar{A}_{22}-L_r\bar{A}_{12})t}(\bar{x}_2(0)-\hat{x}_2(0))$ for $P_{12}=P^{-1}[0_{(n-n_0)\times n_0},I_{n-n_0}]^T \in \mathbb{R}^{n\times(n-n_0)}$, we express the observer-based control law as $u(t)=k_1^{*T}x(t)+k_1^{*T}\varepsilon_1(t)+k_2^{*r}(t)$. Substituting such a control law into the plant (6.1.1) and ignoring the exponentially decaying term, we have the plant output expression:

$$y(t) = c(sI - A - bk_1^{*T})^{-1}bk_2^*[r](t) + c(sI - A - bk_1^{*T})^{-1}bk_1^*[\varepsilon_1](t).$$
(6.2.14)

Choosing the same controller parameters k_1^* and k_2^* to make the matching equation (6.2.2) of state feedback control be satisfied and ignoring the exponentially decaying term, we obtain the output matching equation

$$y(t) = c(sI - A - bk_1^{*T})^{-1}bk_2^*[r](t) = G_c(s)[r](t) = W_m(s)[r](t), \qquad (6.2.15)$$

with $G_c(s)$ representing the closed-loop transfer function. It is verified that with the same controller parameters k_1^* and k_2^* , the observer-based state feedback controller structure $u(t) = k_1^{*T} \hat{x}(t) + k_2^* r(t)$ has the equivalent capability of ensuring plant-model matching, as the state feedback controller $u(t) = k_1^{*T} x(t) + k_2^* r(t)$ does.

Step 2: Output matching by control with the healthy sensor output $z_{n_0(t)}$. from the derivation of (6.2.8), we have obtained $\theta_3^* = k_2^*$ and

$$k_1^{*T}\hat{x}(t) = \theta_1^{*T} \frac{a(s)}{\Lambda(s)} [u](t) + \bar{\theta}_{21}^{*T} \frac{A_{21}(s)}{\Lambda(s)} [z_{n_0}](t) + \bar{\theta}_{201}^{*T} z_{n_0}(t) + \varepsilon_0(t).$$
(6.2.16)

Thus, to prove plant-model output matching by (6.2.8), we only need to prove the existence of the constant parameters $\theta_1^*, \bar{\theta}_{21}^*, \bar{\theta}_{201}^*$ which satisfy (6.2.16), since the existence of k_1^* and k_2^* to guarantee plant-model output matching have been ensured in the last step.

The existence of $\theta_1^* \in \mathbb{R}^{n-n_0}$ is guaranteed for $\theta_1^{*T}a(s) = k_{p2}^{*T}n_1(s)$, since the polynomial $n_1(s)$ in (6.2.7) for has degree $n - n_0$ with $a(s) = [1, s, \dots, s^{n-n_0-1}]^T$. Similarly, the existence of $\bar{\theta}_{21}^*$ is guaranteed for $\bar{\theta}_{21}^{*T}A_{21}(s) = k_{p2}^{*T}n_2(s)$. With $k_1^{*T}P^{-1} =$ $[k_{p1}^{*T}, k_{p2}^{*T}]$ and the prespecified $L_r \in \mathbb{R}^{(n-n_0) \times n_0}$, the existence of $\theta_{20}^* = k_{p1}^{*T} + k_{p2}^{*T} L_r$ in (6.2.8) is guaranteed.

<u>Step 3: Output matching by the nominal failure compensation control.</u> As we have discussed before, we will use the the equation (6.2.13), the extended version of the nominal controller structure (6.2.11), to prove plant-model output matching. By utilizing the transformation matrix P in (6.2.1), the relationship between the constant controller parameters θ_2^* in (6.2.11) and the constant parameters $\bar{\theta}_{21}^*$, $\bar{\theta}_{22}^*$ in (6.2.13) can shown as

$$\theta_2^* = \mathcal{P}[\bar{\theta}_{21}^{*T}, \, \bar{\theta}_{22}^{*T}]^T \tag{6.2.17}$$

with $\bar{\theta}_{22}^* = 0_{(n-n_0)(n-n_0) \times 1}$, and

$$\mathcal{P} = \begin{bmatrix} P_{11} & 0 & 0 & 0 & | & P_{12} & 0 & 0 & 0 \\ 0 & P_{11} & 0 & 0 & | & 0 & P_{12} & 0 & 0 \\ 0 & 0 & \ddots & 0 & | & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & P_{11} & | & 0 & 0 & 0 & P_{12} \end{bmatrix} \in \mathbb{R}^{n(n-n_0) \times n(n-n_0)}, \quad (6.2.18)$$

for $P_{11} = P^{-1}[I_{n_0}, 0_{n_0 \times (n-n_0)}]^T \in \mathbb{R}^{n \times n_0}$ and $P_{12} = P^{-1}[0_{(n-n_0) \times n_0}, I_{n-n_0}]^T \in \mathbb{R}^{n \times (n-n_0)}$. Similarly, the relationship between the constant controller parameters θ_{20}^* in (6.2.11) and the parameters $\bar{\theta}_{201}^*$, $\bar{\theta}_{202}^*$ in (6.2.13) is shown as

$$\theta_{20}^* = P^{-1} [\bar{\theta}_{201}^{*T}, \, \bar{\theta}_{202}^{*T}]^T \tag{6.2.19}$$

with $\bar{\theta}_{202}^* = 0_{(n-n_0) \times 1}$.

Based on (6.2.17) and (6.2.19), the existence of the constant parameters θ_2^* and θ_{20}^* in (6.2.11) can be verified. Furthermore, thanks to the zero value of $\bar{\theta}_{22}^*$ and $\bar{\theta}_{202}^*$, the plant-model matching properties have not been changed by the additional two terms $\bar{\theta}_{22}^{*T} \frac{A_{22}(s)}{\Lambda(s)} [\bar{z}_{n_0}](t)$ and $\bar{\theta}_{202}^{*T} \bar{z}_{n_0}(t)$. In other words, output matching by the nominal compensation controller (6.2.11) is also guaranteed by some existing constant controller parameters θ_1^* , θ_2^* , θ_{20}^* and θ_3^* .

So far, we have proved that despite the imprecisely measurement state variables in z(t), the capability of guaranteeing plant-model output matching: $y(t) - y_m(t) = \varepsilon(t)$ is retained by the nominal compensation controller (6.2.11) which has the closet form of the state feedback controller, with the constant controller parameters $\theta_1^*, \theta_2^*, \theta_{20}^*, \theta_3^*$.

<u>Step 4: Closed-loop signal boundedness.</u> From $y(t) - y_m(t) = \varepsilon(t)$, we have the *i*th derivative as $y^{(i)}(t) = \varepsilon^{(i)}(t) + y_m^{(i)}(t)$, $i = 1, 2, ..., n^*$. Using the reference model: $y_m(t) = \frac{1}{P_m(s)}[r](t)$, we have

$$y_m^{(i)}(t) = s^i [y_m](t) = \frac{s^i}{P_m(s)} [r](t).$$
(6.2.20)

which is bounded for $i = 1, ..., n^*$, because $\frac{s^i}{P_m(s)}$ is stable and proper and $r(t) \in L^{\infty}$. This implies that $y^{(i)}(t) \in L^{\infty}$ for $i = 1, ..., n^*$ as $\varepsilon^{(i)}(t) \in L^{\infty}$.

For the plant (6.1.1), the input-output relationship is $P(s)[y](t) = k_p Z(s)[u](t)$. A relationship between $z_{n_0}(t)$ and u(t) can also be obtained: $P(s)[z_{n_0}](t) = Z_0(s)[u](t)$, for a polynomial vector $Z_0(s)$. Therefore, a useful relationship between y(t) and $z_{n_0}(t)$ can be found as

$$z_{n_0}(t) = \frac{1}{k_p} Z^{-1}(s) Z_0(s)[y](t) = \frac{Z_0(s)}{k_p P_m(s) Z(s)} P_m(s)[y](t).$$
(6.2.21)

Since $P_m(s)[y](t)$ is bounded as from (6.1.5) and $\frac{Z_0(s)}{k_p P_m(s)Z(s)}$ is stable and proper or strictly proper, we have $z_{n_0}(t)$ is bounded. Then, the sensor output vector z(t) is bounded due to the boundedness of $\bar{z}_{n_0}(t)$, and so is $\omega_2(t)$.

Finally, using the plant, $P(s)[y](t) = k_p Z(s)[u](t)$, and ignoring the exponentially decaying effect of the initial conditions, we have

$$u(t) = \frac{P(s)}{k_p P_m(s) Z(s)} P_m(s)[y](t), \qquad (6.2.22)$$

which is bounded because $\frac{P(s)}{k_p P_m(s)Z(s)}$ is stable and proper and $P_m(s)[y](t)$ is bounded, and so is $\omega_1(t)$. ∇ So far, desired output matching for the worst case scenarios have been established. Next, we will show that the same desired matching properties can be obtained by the same nominal compensation controller (6.2.11) when there are less than $n - n_0$ failed sensors, i.e., the general failure cases.

Plant-model matching equation. With the output matching parameters $\theta_1^*, \theta_2^*, \theta_{20}^*$ and θ_3^* , we rewrite (6.2.8) as

$$u(t) = \theta_1^{*T} \frac{a(s)}{\Lambda(s)} [u](t) + \bar{\theta}_{21}^{*T} \frac{A_{21}(s)}{\Lambda(s)} G_{n_0}(s) [u](t) + \bar{\theta}_{201}^{*T} G_{n_0}(s) [u](t) + \theta_3^* r(t), \quad (6.2.23)$$

with $z_{n_0}(t) = G_{n_0}(s)[u](t)$. This leads the plant y(t) = G(s)[u](t) to the closed-loop system with transfer function

$$G_c(s) = G(s)(1 - \theta_1^{*T} \frac{a(s)}{\Lambda(s)} - (\bar{\theta}_{21}^{*T} \frac{A_{21}(s)}{\Lambda(s)} + \bar{\theta}_{201}^{*T})G_{n_0}(s))^{-1}\theta_3^*$$
(6.2.24)

which has been made to match $W_m(s)$. From $G_c(s) = W_m(s)$, we obtain

$$1 - \theta_1^{*T} \frac{a(s)}{\Lambda(s)} - (\bar{\theta}_{21}^{*T} \frac{A_{21}(s)}{\Lambda(s)} + \bar{\theta}_{201}^{*T}) G_{n_0}(s) = \theta_3^{*} W_m^{-1}(s) G(s), \qquad (6.2.25)$$

which, for $G(s) = k_p \frac{Z(s)}{P(s)}$ and $G_{n_0}(s) = \frac{Z_0(s)}{P(s)}$, can be expressed as

$$\theta_1^{*T}a(s)P(s) + (\bar{\theta}_{21}^{*T}A_{21}(s) + \bar{\theta}_{201}^{*T}\Lambda(s))Z_0(s) = \Lambda(s)(P(s) - k_p\theta_3^*Z(s)P_m(s)).$$
(6.2.26)

The matching equation (6.2.26) is another approach to obtain the constant controller parameter $\theta_1^*, \theta_2^*, \theta_{20}^*$ and θ_3^* ensuring plant-model output matching. The result is summarized as follows.

Corollary 6.2.1. Constant parameter matrices $\theta_1^* \in \mathbb{R}^{n-n_0}$, $\theta_{21}^* \in \mathbb{R}^{n_0(n-n_0)}$, $\theta_{201}^* \in \mathbb{R}^{n_0}$ and $\theta_3^* \in \mathbb{R}$ exist such that the output matching equation (6.2.26) holds.

6.2.2.2 Output Matching for $n - \bar{n}_0$ $(n - \bar{n}_0 < n - n_0)$ Failures

When there are $n - \bar{n}_0$ sensor failures, we have total \bar{n}_0 ($\bar{n}_0 > n_0$) precisely measured sensor signals. To handle the matching for the cases of $n - \bar{n}_0$ ($n - \bar{n}_0 < n - n_0$) failures, one extreme method is to use arbitrary n_0 precisely measured sensor signals for plant-model matching with the corresponding controller parameters $\bar{\theta}_{21}^*$ and $\bar{\theta}_{201}^*$ used in the previous subsection, and compensate the other $n - n_0$ sensor output signals whether they are imprecisely measured or not. This method indicates that all the rest of extra $\bar{n}_0 - n_0$ precisely measured sensor output signals are treated as failed sensor output signals which are canceled in the control law by the corresponding zero controller parameters in (6.2.13).

However, as we all know, the more precisely measured sensor output signals we use, the better closed-loop system performance we may obtain. Thus, the abovementioned method is obviously inefficient. So the natural question here is: does the nominal control law (6.2.11) still have the capability of ensuring plant-model output matching while utilizing all the precisely measured sensor output signals to obtain a better performance? The answer is yes, and the affirmative answer is shown as below.

Theorem 6.2.2. There exist constant parameters $\theta_1^* \in \mathbb{R}^{n-n_0}, \theta_2^* \in \mathbb{R}^{n(n-n_0)}, \theta_{20}^* \in \mathbb{R}^n, \theta_3^* \in \mathbb{R}$ such that the controller structure (6.2.11) ensures closed-loop signal boundedness and plant-model output matching: $y(t) - y_m(t) = \varepsilon(t)$, for some exponentially decaying $\varepsilon(t)$ and $y_m(t)$ being the output of reference model (6.1.5), when there are $n - \bar{n}_0$ $(n - \bar{n}_0 < n - n_0)$ sensors being failed.

Pre-proof discussion. Recall the four steps in the proof of Theorem 6.2.1. In fact, Step 1, Step 3 and Step 4 in the proof of Theorem 6.2.1 are also applicable to the proof of output matching under the non-worst cases with $n - \bar{n}_0$ failures. The essential

difference between output matching under the worst cases and output matching under the general cases is Equation (6.2.16) (in the second step of the previous proof):

$$k_1^{*T}\hat{x}(t) = \theta_1^{*T} \frac{a(s)}{\Lambda(s)} [u](t) + \bar{\theta}_{21}^{*T} \frac{A_{21}(s)}{\Lambda(s)} [z_{n_0}](t) + \bar{\theta}_{201}^{*T} z_{n_0}(t) + \varepsilon_0(t)$$

Such an equation implies that only n_0 precisely measured signals are involved to ensure plant-model matching due to $z_{n_0}(t) \in \mathbb{R}^{n_0}$. Thus, to prove output matching by utilizing all $\bar{n}_0(\bar{n}_0 > n_0)$ precisely measured signals so as to obtain better transient performance for the non-worst cases with \bar{n}_0 being precisely measured signals, it is necessary to establish

$$k_1^{*T}\hat{x}(t) = \theta_1^{*T} \frac{a(s)}{\Lambda(s)} [u](t) + \bar{\theta}_{21}^{*T} \frac{\bar{A}_{21}(s)}{\Lambda(s)} [z_{\bar{n}_0}](t) + \bar{\theta}_{201}^{*T} z_{\bar{n}_0}(t), \qquad (6.2.27)$$

where $\theta_1^* \in \mathbb{R}^{n-n_0}$, $\bar{\theta}_{21}^* \in \mathbb{R}^{\bar{n}_0(n-\bar{n}_0)}$, $\bar{\theta}_{21}^* \in \mathbb{R}^{\bar{n}_0}$, $\bar{A}_{21}(s) = [I_{\bar{n}_0}, sI_{\bar{n}_0}, \dots, s^{n-\bar{n}_0-1}I_{\bar{n}_0}]^T$, and the healthy sensor output $z_{\bar{n}_0}(t) \in \mathbb{R}^{\bar{n}_0}$ collecting all \bar{n}_0 precisely measured state variables is state-observable. Here, since the cases with $n - n_0$ failures and the cases with $n - \bar{n}_0$ are exclusive, we keep using the same notations for the controller parameters for simplicity.

Proof of Theorem 6.2.2. The proof of output matching for $n - \bar{n}_0$ failures can also be divided into four steps. As we discussed before, all the steps in the proof of Theorem 6.2.1, except Step 2, are applicable for proving output matching with $n - \bar{n}_0$ failures.

<u>Step 1: Output matching by observer-based state feedback control.</u> For the sake of brevity, the technical details for this step will be omitted here, which can be found on Page 132.

<u>Step 2: Output matching by control with the healthy sensor output $z_{\bar{n}_0}(t)$.</u> We start the proof from the left hand side of (6.2.27), and introduce a weighted average estimate $\hat{x}(t)$ which is different from the state estimate in (6.2.5). <u>Step 2(a)</u>: weighted average estimate $\hat{x}(t)$. To make (6.2.27) satisfied, we first introduce a weighted average estimate $\hat{x}(t)$ which is generated by using all the \bar{n}_0 precisely measured signals.

For the non-worst cases there are \bar{n}_0 precisely measured signals, which implies that there are up to $\binom{\bar{n}_0}{n_0}$ of different healthy sensor output vectors $z_{n_0(i)}(t) \in \mathbb{R}^{n_0}$, $i = 1, 2, \ldots, p$ with some constant $p \leq C_{\bar{n}_0}^{n_0}$. For each of $z_{n_0(i)}(t)$, there may be an individual estimate $\hat{x}_{(i)}(t)$ of x(t) generated from a corresponding $n - n_0$ th state observer. The weighted average estimate $\hat{x}(t)$ is defined as a linear combination of all the p individual state estimates $\hat{x}_{(i)}(t)$. Such a state estimate $\hat{x}(t)$ is in the most general form, and can be proved to converge to the state x(t) exponentially as time goes to infinity.

Proposition 6.2.1. Assume there are p state estimates $\hat{x}_{(i)}(t)$ generating by p different $n - n_0 th$ reduced-order observers, respectively, such that $\lim_{t\to\infty} [\hat{x}_{(i)}(t) - x(t)] = 0, i = 1, 2, ..., p$. Then,

$$\hat{x}(t) = \alpha_1 \hat{x}_{(1)}(t) + \alpha_2 \hat{x}_{(2)}(t) + \ldots + \alpha_p \hat{x}_{(p)}(t), \qquad (6.2.28)$$

with $\alpha_1 + \alpha_2 + \ldots + \alpha_p = 1$, is an estimate of the state x(t) in a general form such that $\lim_{t\to\infty} [\hat{x}(t) - x(t)] = 0$.

Proof: According to the well-known observer theory [9], for each individual estimate $\hat{x}_{(i)}(t)$ of x(t), we have

$$\hat{x}_{(i)}(t) - x(t) = \bar{\varepsilon}_i(t), \quad i = 1, 2, \dots, p,$$
(6.2.29)

with some initial-condition related exponentially decaying terms $\bar{\varepsilon}_i(t)$.

Adding the p weighted estimates up with the corresponding parameters α_i , we obtain

$$\sum_{i=1}^{p} \alpha_i \hat{x}_{(i)}(t) - \sum_{i=1}^{p} \alpha_i x(t) = \sum_{i=1}^{p} \alpha_i \bar{\varepsilon}_i(t).$$
(6.2.30)



Figure 6.3: Schematic diagram of the weighted average estimate $\hat{x}(t)$.

From $\hat{x}(t) = \alpha_1 \hat{x}_{(1)}(t) + \alpha_2 \hat{x}_{(2)}(t) + \ldots + \alpha_p \hat{x}_{(p)}(t) = \sum_{i=1}^p \alpha_i \hat{x}_{(i)}(t)$ and $\sum_{i=1}^p \alpha_i x(t) = x(t)$ with $\alpha_1 + \alpha_2 + \ldots + \alpha_p = 1$, we have $\hat{x}(t) - x(t) = \bar{\varepsilon}(t)$, for an exponentially decaying term $\bar{\varepsilon}(t) = \sum_{i=1}^p \alpha_i \bar{\varepsilon}_i(t)$, that is, $\lim_{t \to \infty} [\hat{x}(t) - x(t)] = 0$.

The equation shows that the weighted average estimate $\hat{x}(t)$ will converge to the state x(t) exponentially as time goes to infinity. ∇

It is worth noting here that although each estimate $\hat{x}_{(i)}(t)$ is produced by an $n - n_0$ th order observer constructing by only n_0 precisely measured state variables, the general estimate $\hat{x}(t)$ is an estimate generating based on all the \bar{n}_0 precisely measured state variables, as Fig. 6.3 shows.

<u>Step 2(b): parametrization of $k_1^{*T} \hat{x}_{(i)}(t)$.</u> Using the weighted average estimate $\hat{x}(t)$, we first express the term $k_1^{*T} \hat{x}(t)$ by (6.2.28) as

$$k_1^{*T}\hat{x}(t) = k_1^{*T} \sum_{i=1}^p \alpha_i \hat{x}_{(i)}(t) = \sum_{i=1}^p \alpha_i (k_1^{*T} \hat{x}_{(i)}(t)).$$
(6.2.31)

To further process, with $P_{(i)} \in \mathbb{R}^{n \times n}$ being a corresponding transformation matrix, we express the *i*th individual state estimate $\hat{x}_{(i)}(t)$, $i = 1, 2, \ldots, p$, based on the observer theory [9], as

$$\hat{x}_{(i)}(t) = P_{(i)} \begin{bmatrix} z_{n_0(i)}(t) & w_{(i)}(t) + L_{r(i)} z_{n_0(i)}(t) \end{bmatrix}^T, \qquad (6.2.32)$$

where

$$w_{(i)}(t) = \frac{n_{1(i)}(s)}{\Lambda(s)} [u](t) + \frac{n_{2(i)}(s)}{\Lambda(s)} [z_{n_0(i)}](t) + \bar{\varepsilon}_{0(i)}(t)$$
(6.2.33)

with $\Lambda(s)$ being a stable polynomial of degree $n - n_0$ whose stability properties are prespecified by a corresponding constant matrix $L_{r(i)} \in \mathbb{R}^{(n-n_0) \times n_0}$, $n_{1(i)}(s)$ being an $(n - n_0) \times 1$ polynomial vector, $n_{2(i)}(s)$ being an $(n - n_0) \times n_0$ polynomial matrix whose maximum degrees are $n - n_0 - 1$ or less, and $\bar{\varepsilon}_{0(i)}(t)$ being an exponentially decaying term representing the effect of the initial condition $w_{(i)}(0)$.

Using (6.2.32) and (6.2.33) and ignoring the exponentially decaying term, we can express $k_1^{*T} \hat{x}_{(i)}(t)$ as

$$k_1^{*T}\hat{x}_{(i)}(t) = \theta_{1(i)}^{*T}\frac{a(s)}{\Lambda(s)}[u](t) + \bar{\theta}_{21(i)}^{*T}\frac{A_{21}(s)}{\Lambda(s)}[z_{n_0(i)}](t) + \bar{\theta}_{201(i)}^{*T}z_{n_0(i)}(t), \qquad (6.2.34)$$

where $\theta_{1(i)}^* \in \mathbb{R}^{n-n_0}$, $\bar{\theta}_{21(i)}^* \in \mathbb{R}^{n_0(n-n_0)}$ and $\bar{\theta}_{201(i)}^* \in \mathbb{R}^{n_0}$, such that $\bar{\theta}_{201(i)}^{*T} = k_{p1(i)}^{*T} + k_{p2(i)}^{*T} L_{r(i)}$, $k_{p2(i)}^{*T} n_{1(i)}(s) = \theta_{1(i)}^{*T} a(s)$ and $k_{p2(i)}^{*T} n_{2(i)}(s) = \bar{\theta}_{21(i)}^{*T} A_{21}(s)$, for $k_1^{*T} P_{(i)}^{-1} = [k_{p1(i)}^{*T}, k_{p2(i)}^{*T}]$ with $k_{p1(i)}^* \in \mathbb{R}^{n_0}$ and $k_{p2(i)}^* \in \mathbb{R}^{n-n_0}$, and $a_1(s) = [1, s, \dots, s^{n-n_0-1}]^T$, $A_{21}(s) = [I_{n_0}, sI_{n_0}, \dots, s^{n-n_0-1}I_{n_0}]^T$.

<u>Step 2(c)</u>: parametrization of $k_1^{*T} \hat{x}(t)$. Thus, from (6.2.31) and (6.2.34), the term $k_1^{*T} \hat{x}(t)$ in $u(t) = k_1^{*T} \hat{x}(t) + k_2^{*T} r(t)$ with $\hat{x}(t)$ being the weighted average estimate of the state x(t) becomes

$$k_{1}^{*T}\hat{x}(t) = \sum_{i=1}^{p} \alpha_{i}(k_{1}^{*T}\hat{x}_{(i)}(t))$$

=
$$\sum_{i=1}^{p} \alpha_{i}(\theta_{1(i)}^{*T}\frac{a(s)}{\Lambda(s)})[u](t) + \sum_{i=1}^{p} \alpha_{i}(\bar{\theta}_{21(i)}^{*T}\frac{A_{21}(s)}{\Lambda(s)}[z_{n_{0}(i)}](t)) + \sum_{i=1}^{p} \alpha_{i}(\bar{\theta}_{201(i)}^{*T}z_{n_{0}(i)}(t)).$$

(6.2.35)

It is worth noting that the total number of the sensor outputs including in p different signals $z_{n_0(i)}(t)$ is \bar{n}_0 ($\bar{n}_0 > n_0$). Thus, reparameterizing (6.2.35), we have

$$k_1^{*T}\hat{x}(t) = \theta_1^{*T} \frac{a(s)}{\Lambda(s)} [u](t) + \bar{\theta}_{21}^{*T} \frac{\bar{A}_{21}(s)}{\Lambda(s)} [z_{\bar{n}_0}](t) + \bar{\theta}_{201}^{*T} z_{\bar{n}_0}(t)$$
(6.2.36)

with $\theta_1^* \in \mathbb{R}^{n-n_0}, \, \bar{\theta}_{21}^* \in \mathbb{R}^{\bar{n}_0(n-\bar{n}_0)}, \, \bar{\theta}_{201}^* \in \mathbb{R}^{\bar{n}_0} \text{ and } \bar{A}_{21}(s) = [I_{\bar{n}_0}, sI_{\bar{n}_0}, \dots, s^{n-\bar{n}_0-1}I_{\bar{n}_0}]^T.$

So far, the desired equation (6.2.27) has been established, that is, output matching by the control law u(t), which is in the form of

$$u(t) = \theta_1^{*T} \frac{a(s)}{\Lambda(s)} [u](t) + \bar{\theta}_{21}^{*T} \frac{\bar{A}_{21}(s)}{\Lambda(s)} [z_{\bar{n}_0}](t) + \bar{\theta}_{201}^{*T} z_{\bar{n}_0}(t) + \theta_3^* r(t)$$
(6.2.37)

with $\bar{\theta}_{21}^* \in \mathbb{R}^{\bar{n}_0(n-\bar{n}_0)}, \ \bar{\theta}_{201}^* \in \mathbb{R}^{\bar{n}_0}$, has been proved.

<u>Step 3: Output matching by the nominal failure compensation control.</u> For the sake of brevity, the technical details for this step will be omitted here, which can be found on Page 134.

<u>Step 4: closed-loop signal boundedness.</u> For the sake of brevity, the technical details for this step will be omitted here, which can be found on Page 135. 6.2.1. ∇

Matching parameters flexibility. From the above proof, we also conclude that the nominal compensation control law (6.2.11) provides p degree-of-freedom for choosing the nominal controller parameters θ_1^* , θ_2^* and θ_{20}^* , since there are p of α_i to be chosen arbitrarily as long as the requirement $\sum_{i=1}^{p} \alpha_i = 1$ is satisfied. Furthermore, there exist infinite groups of the nominal controller parameters θ_1^* , θ_2^* and θ_{20}^* , since the choices of α_i , i = 1, 2, ..., p are infinite.

Remark 6.2.4. By the same approach of deriving the plant-model matching equation under the worst cases of $n - n_0$ failures, the matching equation under the non-worst cases of $n - \bar{n}_0$ failures can also be obtained.

6.2.2.3 Summary on Plant-Model Output Matching

From Theorem 6.2.1 and 6.2.2, we conclude that for the plant with a known failure pattern set and known system parameters, the nominal compensation control law (6.2.11), which is the compact version of (6.2.13),

- is able to achieve output matching (by the nominal controller parameters $\theta_1^*, \bar{\theta}_{21}^*, \bar{\theta}_{201}^*$ and θ_3^*) as long as there are up to $n - n_0$ failed sensors; and
- is able to compensate all the possible sensor failures of interest (by the nominal parameters $\bar{\theta}_{22}^*$ and $\bar{\theta}_{202}^*$) without using extra additional signal processing.

In addition, it is worth noting here although the existence of nominal constant parameters are guaranteed as long as there are up to $n - n_0$ sensor failures, the values of the controller parameters for different failure pattern $\sigma_{(k)}$, k = 0, 1, ..., N - 1, are different. See Remark 6.2.3 in Section 6.2.1.

Comparative advantages of the controller structure (6.2.11). Compared with an output feedback design for MRAC, the controller structure (6.2.11) with state sensor output feedback has the following comparative advantages:

- the order of the filter $\frac{1}{\Lambda(s)}$ is $n n_0$ which is less than the order of n 1 in an output feedback controller when $n_0 > 1$, which makes the signals $\omega_1(t)$ and $\omega_2(t)$ more responsive and less oscillating, and reduces the control implementation complexity caused by high-order filters.
- at least n_0 (or even \bar{n}_0) precisely measured state variables are used for plantmodel matching, which may help to obtain a better tracking performance.

These desired features, being important for some applications in practice, are inherited from the state feedback controller (6.1.3), which confirm our research motivations from the technical point view. In this section, the adaptive state sensor failure compensation scheme is developed for the case of the unknown plant parameters and the unknown state sensor failures (6.1.7). We will give an adaptive controller structure, chose a parameter adaption law for updating the controller parameters, and show the stability properties and tracking performance of the closed-loop system. For adaptive control, we assume: (A6.3) the sign of the high frequency gain k_p is known, for parameter adaptation; and (A6.4) the failed sensor measurements are bounded.

To handle the plant (6.1.1) with unknown (A, b, c) and unknown sensor failures pattern $\sigma_{(k)}$, we design the adaptive version of the controller structure (6.2.11) as

$$u(t) = \theta_1^T(t)\omega_1(t) + \theta_2^T(t)\omega_2(t) + \theta_{20}^T(t)z(t) + \theta_3(t)r(t), \qquad (6.3.1)$$

where $\theta_1(t) \in \mathbb{R}^{n-n_0}$, $\theta_2(t) \in \mathbb{R}^{n(n-n_0)}$, $\theta_{20}(t) \in \mathbb{R}^n$, $\theta_3(t) \in \mathbb{R}$ are the adaptive estimates of the unknown controller parameters θ_1^* , θ_2^* , θ_{20}^* , θ_3^* , respectively, and

$$\omega_1(t) = \frac{a(s)}{\Lambda(s)} [u](t), \ \omega_2(t) = \frac{A(s)}{\Lambda(s)} [z](t)$$
(6.3.2)

with $a(s) = [1, s, \dots, s^{n-n_0-1}]^T$, $A(s) = [I_n, sI_n, \dots, s^{n-n_0-1}I_n]^T$, and $\Lambda(s)$ being a monic stable polynomial of degree $n - n_0$.

6.3.1 Tracking Error Equation and Parameter Adaptation

In this subsection, we first derive the tracking error equation, and then chose an adaptive law to deal with the parameter uncertainties by making use of the tracking error equation to be developed.

6.3.1.1 Tracking Error Equation

6.3

In order to obtain the tracking error equation which is crucial for developing a stable adaptive law for updating the controller parameters $\theta_1(t), \theta_2(t), \theta_{20}(t)$ and $\theta_3(t)$, we first operate both sides of (6.2.26) on y(t) so that

$$\theta_1^{*T} a(s) P(s)[y](t) + (\bar{\theta}_{21}^{*T} A_{21}(s) + \bar{\theta}_{201}^{*T} \Lambda(s)) Z_0(s)[y](t)$$

= $\Lambda(s) (P(s) - k_p \theta_3^* Z(s) P_m(s))[y](t).$ (6.3.3)

Recall the relationship between $z_{n_0}(t)$ and y(t) obtained in Section 6.2.2.1:

$$Z_0(s)[y](t) = k_p Z(s)[z_{n_0}](t).$$
(6.3.4)

Substituting (6.3.4) and the plant: $P(s)[y](t) = k_p Z(s)[u](t)$, into (6.3.3), we have

$$\theta_1^{*T} a(s) k_p Z(s)[u](t) + (\bar{\theta}_{21}^{*T} A_{21}(s) + \bar{\theta}_{201}^{*T} \Lambda(s)) k_p Z(s)[z_{n_0}](t)$$

= $\Lambda(s) k_p Z(s)[u](t) - \Lambda(s) k_p \theta_3^* Z(s) P_m(s))[y](t).$ (6.3.5)

Because $\Lambda(s)$ and Z(s) are stable, (6.3.5) can be expressed as

$$u(t) = \theta_1^{*T} \frac{a_1(s)}{\Lambda(s)} [u](t) + \bar{\theta}_{21}^{*T} \frac{A_{21}(s)}{\Lambda(s)} [z_{n_0}](t) + \bar{\theta}_{201}^{*T} z_{n_0}(t) + \theta_3^* P_m[y](t) + \varepsilon_2(t) \quad (6.3.6)$$

for some initial condition-related exponentially decaying $\varepsilon_2(t)$.

With $\bar{\theta}_{22}^*$ and $\bar{\theta}_{202}^*$ in (6.2.13) being zero, we rewrite (6.3.6) as

$$u(t) = \theta_1^{*T} \frac{a_1(s)}{\Lambda(s)} [u](t) + \bar{\theta}_{21}^{*T} \frac{A_{21}(s)}{\Lambda(s)} [z_{n_0}](t) + \bar{\theta}_{22}^{*T} \frac{A_{22}(s)}{\Lambda(s)} [\bar{z}_{n_0}](t) + \bar{\theta}_{201}^{*T} z_{n_0}(t) + \bar{\theta}_{202}^{*T} \bar{z}_{n_0}(t) + \theta_3^* P_m(s) [y](t) + \varepsilon_2(t)$$
(6.3.7)

Reorganizing (6.3.7) and Ignoring the exponentially decaying term, we rewrite (6.3.7) as

$$u(t) = \theta_1^{*T} \frac{a(s)}{\Lambda(s)} [u](t) + \theta_2^{*T} \frac{A(s)}{\Lambda(s)} [z](t) + \theta_{20}^{*T} z(t) + \theta_3^{*} P_m(s) [y](t).$$
(6.3.8)

Substituting the expression (6.3.8) of u(t), which is derived from the matching condition, into the controller structure (6.3.1), we have the tracking error equation:

$$e(t) = y(t) - y_m(t) = \frac{k_p}{P_m(s)} [\tilde{\theta}^T \omega](t)$$
 (6.3.9)

where $\theta^* = [\theta_1^{*T}, \theta_2^{*T}, \theta_{20}^{*T}, \theta_3^{*T}]^T$, $\theta(t) = [\theta_1^T(t), \theta_2^T(t), \theta_{20}^T(t), \theta_3(t)]^T$, $\omega(t) = [\omega_1^T(t), \omega_2^T(t), \omega_2^T(t), \omega_2^T(t), \omega_2^T(t), \omega_2^T(t), \omega_2^T(t), \omega_3^T(t), \omega$

The tracking error (6.3.9) is in a desired linear and parameterized form which makes the design of parameter adaptation possible.

6.3.1.2 Parameter Adaptation

In this subsection, the estimation error model is derived, with which the parameter adaptation law is chosen based on the gradient method.

Estimation error. From the tracking error equation (6.3.9), we define the estimate error,

$$\epsilon(t) = e(t) + \rho(t)\xi(t) \tag{6.3.10}$$

for the estimates $\theta(t)$ and $\rho(t)$ of θ^* and $\rho^* = k_p$, where

$$\xi(t) = \theta^{T}(t)\zeta(t) - \frac{1}{P_{m}(s)}[\theta^{T}\omega](t), \zeta(t) = \frac{1}{P_{m}(s)}[\omega](t).$$
(6.3.11)

From (6.3.9) and (6.3.10), it follows that

$$\epsilon(t) = \rho^* \tilde{\theta}^T(t) \zeta(t) + \tilde{\rho}(t) \xi(t)$$
(6.3.12)

with $\tilde{\rho}(t) = \rho(t) - \rho^*$, which is in a desired linear form.

Adaptive law. Based on the desired estimation error form (6.3.12), we choose the gradient-type adaptive update laws for $\theta(t)$ and $\rho(t)$ as

$$\dot{\theta}(t) = -\frac{\operatorname{sign}(k_p)\Gamma\epsilon(t)\zeta(t)}{m_0^2(t)},\tag{6.3.13}$$

$$\dot{\rho}(t) = -\frac{\gamma \epsilon(t)\xi(t)}{m_0^2(t)},\tag{6.3.14}$$

with an adaptation gain matrix $\Gamma = \Gamma^T > 0$, an adaptation gain $\gamma > 0$, initial estimates $\theta(0)$ and $\rho(0)$ of θ^* and ρ^* , and $m_0(t) = \sqrt{1 + \zeta^T(t)\zeta(t) + \xi^2(t)}$.

6.3.2 Stability and Tracking Properties

The adaptive laws and the feedback control system have the following properties.

Lemma 6.3.1. The adaptive law (6.3.13)–(6.3.14) guarantees that $\theta(t) \in L^{\infty}$, $\rho(t) \in L^{\infty}$, and $\frac{\epsilon(t)}{m_0(t)} \in L^2 \bigcap L^{\infty}$, $\dot{\theta}(t) \in L^2 \bigcap L^{\infty}$ and $\dot{\rho}(t) \in L^2 \bigcap L^{\infty}$.

Proof: With (6.3.12), the time-derivative of the positive definite function

$$V(\tilde{\theta}, \tilde{\rho}) = |\rho^*| \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \gamma^{-1} \tilde{\rho}^2$$
(6.3.15)

along the trajectories of (6.3.13) and (6.3.14), satisfies $\dot{V}(t) = \frac{-2\epsilon^2(t)}{m_0^2(t)} \leq 0$. Hence, $\theta(t) \in L^{\infty}, \ \rho(t) \in L^{\infty} \text{ and } \frac{\epsilon(t)}{m_0(t)} \in L^2$, which, with (6.3.12), (6.3.13) and (6.3.14), in turn, implies $\frac{\epsilon(t)}{m_0(t)} \in L^{\infty}, \ \dot{\theta}(t) \in L^2 \cap L^{\infty}$ and $\dot{\rho}(t) \in L^2 \cap L^{\infty}$. ∇

Based on Lemma 6.3.1, the stability and tracking performance is guaranteed by the following theorem.

Theorem 6.3.1. The adaptive controller (6.3.1) with the adaptive laws (6.3.13) and (6.3.14), applied to the plant (6.1.1), guarantees the closed-loop signal boundedness and output tracking: $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} (y(t) - y_m(t)) = 0.$

Proof: <u>Step 1: introducing filtered signals for u(t) and y(t).</u> Introducing two fictitious signals

$$\eta_0(t) = \frac{1}{s+a_0}[u](t), \ \eta(t) = \frac{1}{s+a_0}[y](t), \tag{6.3.16}$$

and two fictitious filters $K_1(s)$ and K(s) as

$$sK_1(s) = 1 - K(s), \ K(s) = \frac{a^{n^*}}{(s+a)^{n^*}},$$
 (6.3.17)

where $a_0 > 0$ is arbitrary and a > 0 is to be specified, with $G(s) = k_p \frac{Z(s)}{P(s)}$, using the equality: $-a_0 K_1(s) + (s + a_0) K_1(s) = 1 - K(s)$, we obtain

$$\eta_0(t) + a_0 K_1(s)[\eta_0](t) - K_1(s)[u](t) = K(s)G^{-1}(s)[\eta](t).$$
(6.3.18)

For proving boundedness of the control signal u(t) generated from the adaptive control law (6.3.1), we first rewrite the adaptive controller structure as

$$u(t) = \theta_1^T(t) \frac{a(s)}{\Lambda(s)} + \bar{\theta}_{21}^T(t) \frac{A_{21}(s)}{\Lambda(s)} [z_{n_0}](t) + \bar{\theta}_{201}^T(t) z_{n_0}(t) + \theta_3(t) r(t) + \bar{\theta}_{22}^T(t) \frac{A_{22}(s)}{\Lambda(s)} [\bar{z}_{n_0}](t) + \bar{\theta}_{202}^T(t) \bar{z}_{n_0}(t),$$
(6.3.19)

through the relationship $\theta_2(t) = \mathcal{P}[\bar{\theta}_{21}^T(t), \bar{\theta}_{22}^T(t)]^T$ for $\bar{\theta}_{21}(t) \in \mathbb{R}^{n_0}, \bar{\theta}_{22}(t) \in \mathbb{R}^{n-n_0}$ and the relationship $\theta_{20}(t) = P^{-1}[\bar{\theta}_{201}^T(t), \bar{\theta}_{202}^T(t)]^T$ for $\bar{\theta}_{201}(t) \in \mathbb{R}^{n_0}$ and $\bar{\theta}_{202}(t) \in \mathbb{R}^{n-n_0}$ with the non-singular transfer matrices P and \mathcal{P} used in Section 3.

Hence the regrouped controller structure (6.3.19), operated by $K_1(s)$ on both sides, with the substitution of (6.3.16), gives the identity

$$K_{1}(s)[u](t) = K_{1}(s)\theta_{1}^{T}(\cdot)\frac{a(s)}{\Lambda(s)}(s+a_{0})[\eta_{0}](t) + K_{1}(s)\bar{\theta}_{21}^{T}(\cdot)\frac{A_{21}(s)}{\Lambda(s)}[z_{n_{0}}](t) + K_{1}(s)\bar{\theta}_{201}^{T}(\cdot)z_{n_{0}}(t) + K_{1}(s)\bar{\theta}_{201}^{T}(\cdot)(s_{n_{0}})[\bar{z}_{n_{0}}](t) + K_{1}(s)\bar{\theta}_{202}^{T}(t)\bar{z}_{n_{0}}(t).$$

$$(6.3.20)$$

<u>Step 2: expressing $z_{n_0}(t)$ by the filtered signals.</u> In order to process the healthy sensor output $z_{n_0}(t)$, we can first express the system state x(t), according to the state observer theory, for (A, c) detectable, as

$$\begin{aligned} x(t) &= (sI - A + Lc)^{-1} b[u](t) + (sI - A + Lc)^{-1} L[y](t) \\ &= \frac{G_1(s)}{\Lambda_0(s)} [u](t) + \frac{G_2(s)}{\Lambda_0(s)} [y](t), \end{aligned}$$
(6.3.21)

where the eigenvalues of the $n \times n$ matrix A - Lc are stable for some constant gain vector $L \in \mathbb{R}^{n \times 1}$, $\Lambda_0(s) = \det(sI - A + Lc)$ whose degree is n, L is a matrix such that and $G_1(s) = adj(sI - A + Lc)b$ and $G_2(s) = adj(sI - A + Lc)L$ are polynomial vectors whose maximum degrees are n - 1. With (6.3.21) and (6.3.16), $z_{n_0}(t) = C_0 x(t)$ can be expressed as

$$z_{n_0}(t) = C_0 \frac{G_1(s)}{\Lambda_0(s)} [u](t) + C_0 \frac{G_2(s)}{\Lambda_0(s)} [y](t)$$

= $C_0 \frac{G_1(s)}{\Lambda_0(s)} (s+a_0) [\eta_0](t) + C_0 \frac{G_2(s)}{\Lambda_0(s)} (s+a_0) [\eta](t).$ (6.3.22)

Step 3: establishing a relationship between the filtered u(t) and the filtered y(t). Using (6.3.20) and (6.3.22), we can express $K_1(s)[u](t)$ in (6.3.20) as

$$K_{1}(s)[u](t) = K_{1}(s)\theta_{1}^{T}(\cdot)\frac{a(s)}{\Lambda(s)}(s+a_{0})[\eta_{0}](t) + K_{1}(s)\bar{\theta}_{21}^{T}(\cdot)\frac{A_{21}(s)C_{0}G_{1}(s)}{\Lambda(s)\Lambda_{0}(s)}(s+a_{0})[\eta_{0}](t) + K_{1}(s)\bar{\theta}_{21}^{T}(\cdot)\frac{A_{21}(s)C_{0}G_{2}(s)}{\Lambda(s)\Lambda_{0}(s)}(s+a_{0})[\eta](t) + K_{1}(s)\bar{\theta}_{201}^{T}(\cdot)\frac{C_{0}G_{1}(s)}{\Lambda_{0}(s)}(s+a_{0})[\eta_{0}](t) + K_{1}(s)\bar{\theta}_{201}^{T}(\cdot)\frac{C_{0}G_{2}(s)}{\Lambda_{0}(s)}(s+a_{0})[\eta](t) + K_{1}(s)[\theta_{3}r](t) + K_{1}(s)\bar{\theta}_{22}^{T}(\cdot)\frac{A_{22}(s)}{\Lambda(s)}[\bar{z}_{n_{0}}](t) + K_{1}(s)\bar{\theta}_{202}^{T}(t)\bar{z}_{n_{0}}(t).$$

$$(6.3.23)$$

Substituting (6.3.23) into (6.3.18) and defining

$$P_{0}(s,\cdot) = 1 + K_{1}(s) \left(a_{0} - \theta_{1}^{T}(\cdot) \frac{a(s)}{\Lambda(s)} (s + a_{0}) - \bar{\theta}_{21}^{T}(\cdot) \frac{A_{21}(s)C_{0}G_{1}(s)}{\Lambda(s)\Lambda_{0}(s)} (s + a_{0}) - \bar{\theta}_{201}^{T}(\cdot) \frac{C_{0}G_{1}(s)}{\Lambda_{0}(s)} (s + a_{0}) \right),$$

$$(6.3.24)$$

we obtain

$$P_{0}(s)[\eta_{0}](t) = \left(K(s)G^{-1}(s) + K_{1}(s)\bar{\theta}_{21}^{T}(\cdot)\frac{A_{21}(s)C_{0}G_{2}(s)}{\Lambda(s)\Lambda_{0}(s)}(s+a_{0}) + \bar{\theta}_{201}^{T}(\cdot)\frac{C_{0}G_{2}(s)}{\Lambda_{0}(s)}(s+a_{0})\right)[\eta](t) + K_{1}(s)[\theta_{3}r](t) + b_{1}(t),$$

$$(6.3.25)$$

where $b_1(t) = K_1(s)\bar{\theta}_{22}^T(\cdot)\frac{A_{22}(s)}{\Lambda(s)}[\bar{z}_{n_0}](t) + K_1(s)\bar{\theta}_{202}^T(\cdot)[\bar{z}_{n_0}](t) \in L^{\infty}$, based on Lemma 6.3.1, $\bar{z}_{n_0}(t) \in L^{\infty}$ and an equivalent expression of $K_1(s)$:

$$K_1(s) = \frac{a}{s+a} \left(1 + \frac{a}{s+a} + \dots + \frac{a^{n^*-1}}{(s+a)^{a^{n^*-1}}} \right).$$
(6.3.26)

$$k_1(t) = \mathcal{L}^{-1}[K_1(s)] = e^{-at} \sum_{i=1}^{n^*} \frac{a^{n^*-i}}{(n^*-i)!} t^{n^*-i}, \qquad (6.3.27)$$

where $\mathcal{L}^{-1}[\cdot]$ is the inverse Laplace transform operator, which satisfies

$$||k_1(\cdot)||_1 = \int_0^\infty |k_1(t)| dt = \frac{n^*}{a}.$$
(6.3.28)

Hence there exists $a^0 > 0$ such that for any fixed $a > a^0$, the operator

$$T_0(s,\cdot) = (P_0(s,\cdot))^{-1} \tag{6.3.29}$$

is stable and proper.

Let $a > a^0$ be finite in K(s) and $K_1(s)$ so (6.3.25) implies that

$$\eta_0(t) = T_1(s, \cdot)[\eta](t) + b_0(t), \tag{6.3.30}$$

where $T_1(s, \cdot)$ is a stable and strictly proper operator, and $b_0(t) \in L^{\infty}$ due to $r(t) \in L^{\infty}$ and $b_1(t) \in L^{\infty}$.

<u>Step 4: formulating a closed-loop inequality of the filtered y(t).</u> Filtering both sides of (6.3.10) by $\frac{1}{s+a_0}$, we obtain

$$\eta(t) = \frac{1}{s+a_0} [y_m](t) + \frac{1}{s+a_0} [\epsilon - \rho \xi](t)$$

$$\leq \frac{1}{s+a_0} [|y_m|](t) + \frac{1}{s+a_0} [|\epsilon|](t) + \frac{1}{s+a_0} [|\rho \xi|](t).$$
(6.3.31)

Using the inequality that $m_0(t) \leq 1 + ||\zeta(t)||_1 + |\xi(t)|$, we obtain

$$|\epsilon(t)| \le \frac{|\epsilon(t)|}{m_0(t)} (1 + \|\zeta(t)\|_1 + |\xi(t)|).$$
(6.3.32)

From (6.3.31) and (6.3.32), we have

$$\eta(t) \le \frac{1}{s+a_0} [|y_m|](t) + \frac{1}{s+a_0} [|\rho\xi|](t) + \frac{1}{s+a_0} \left[\frac{|\epsilon|}{m_0}\right](t) + \frac{1}{s+a_0} \left[\frac{|\epsilon|\|\zeta\|}{m_0}\right](t)$$

$$+\frac{1}{s+a_0}\left[\frac{|\epsilon||\xi|}{m_0}\right](t). \tag{6.3.33}$$

Based on Lemma 4.1, we collect the bounded terms in (6.3.33) and express (6.3.33) as

$$\eta(t) \le x_0(t) + \frac{1}{s+a_0} [|\rho\xi|](t) + \frac{1}{s+a_0} \left[\frac{|\epsilon| ||\zeta||}{m_0}\right](t) + \frac{1}{s+a_0} \left[\frac{|\epsilon| |\xi|}{m_0}\right](t) \quad (6.3.34)$$

with $x_0(t) = \frac{1}{s+a_0} [|y_m|](t) + \frac{1}{s+a_0} \left[\frac{|\epsilon|}{m_0}\right] (t).$

For the second term of the right hand side of the inequality (6.3.34), we had $\zeta(t) = \frac{1}{P_m(s)}[\omega](t)$ and $\omega(t) = [\omega_1(t), \omega_2(t), z(t), r(t)]$ with $\omega_1(t) = \frac{a(s)}{\Lambda(s)}[u](t)$ and $\omega_2(t) = \frac{A(s)}{\Lambda(s)}[z](t)$. Since r(t) is bounded, we could conclude the boundedness of $\zeta(t)$ depends on the boundedness of $\frac{1}{s+a_0}[|u|](t)$ and the boundedness of $\frac{1}{s+a_0}[|z|](t)$, that is, the boundedness of $|\eta(t)|$ and the boundedness $|\eta_0(t)|$ based on (6.3.16) and (6.3.22). Furthermore, from (6.3.30), we conclude the boundedness of $|\zeta(t)|$ depends on the boundedness of $|\eta(t)| = \frac{1}{s+a_0}[|y|](t)$, which is crucial for the closed-loop stability proof.

For the second and the last term of (6.3.34), denoting $P_m(s) = s^{n^*} + a_{n^*-1}s^{n^*-1} + \cdots + a_1s + a_0$, we express $\xi(t)$ in (6.3.11) as

$$\xi(t) = \frac{s^{n^*-1} + a_{n^*-1}s^{n^*-2} + \dots + a_2s + a_1}{P_m(s)} \left[\dot{\theta}^T \frac{1}{P_m(s)}[\omega]\right](t)$$

$$+ \frac{s^{n^*-2} + a_{n^*-1}s^{n^*-3} + \dots + a_2}{P_m(s)} \left[\dot{\theta}^T \frac{s}{P_m(s)}[\omega]\right](t) + \dots$$

$$+ \frac{s + a_{n^*-1}}{P_m(s)} \left[\dot{\theta}^T \frac{s^{n^*-2}}{P_m(s)}[\omega]\right](t) + \frac{1}{P_m(s)} \left[\dot{\theta}^T \frac{s^{n^*-1}}{P_m(s)}[\omega]\right](t),$$
(6.3.35)

which shows the boundedness of $\xi(t)$ only depends on the boundedness of $\omega(t)$, that is, depends on the boundedness of $|\eta(t)|$.

Thus, Lemma 6.3.1, (6.3.30), (6.3.34) and (6.3.35), imply that

$$|\eta(t)| \le x_0(t) + T_2(s, \cdot)[x_1 T_3(s, \cdot)[|\eta|]](t)$$
(6.3.36)

for some $x_0(t) \in L^{\infty}$, $x_1(t) \in L^{\infty} \cap L^2$ with $x_1(t) \ge 0$ (which depends on $\frac{|\epsilon|(t)}{m_0(t)}$), some stable and strictly proper operator $T_2(s,t)$, and some stable and proper operator $T_3(s,t)$ with a non-negative impulse response.

<u>Step 5: applying Gronwall-Bellman Lemma for signal boundedness.</u> Introducing $\eta_1(t) = T_3(s, \cdot)[|\eta|](t)$, operating $T_3(s, t)$ on both sides of (6.3.36), noting that $T_3(s, t)$ has a non-negative impulse response, we have

$$\eta_1(t) \le b_1 + b_2 \int_0^t e^{-\alpha(t-\tau)} x_1(\tau) \eta_1(\tau) d\tau$$
(6.3.37)

for some α , b_1 , $b_2 > 0$. This inequality represents a feedback loop where $x_2(t) \in L^{\infty}$, leading to a small-gain structure. Applying the Gronwall Lemma to (6.3.37) with $x_1(t) \in L^2 \cap L^{\infty}$, we conclude that $\eta_1(t) \in L^{\infty}$, and so $\eta(t) \in L^{\infty}$ in (6.3.36). Hence, $\eta_0(t) \in L^{\infty}$ in (6.3.30), $\xi(t) \in L^{\infty}$ in (6.3.11), $\zeta(t) \in L^{\infty}$ in (6.3.11), $\epsilon(t) \in L^{\infty}$ in (6.3.12), $y(t) \in L^{\infty}$ in (6.3.10), $z(t) \in L^{\infty}$ in (6.2.21), and $u(t) \in L^{\infty}$ in (6.3.1).

From (6.3.12) we have $\dot{\epsilon}(t) \in L^{\infty}$, which, with $\epsilon(t) \in L^2 \cap L^{\infty}$, implies $\lim_{t\to\infty} \epsilon(t) = 0$ in (6.3.12), and $\lim_{t\to\infty} \dot{\theta}(t) = 0$ in (6.3.13). From this property and $\dot{\theta}(t) \in L^2$, it follows that $\xi(t) \in L^2$ and $\lim_{t\to\infty} \xi(t) = 0$ in (6.3.35). Finally, from (6.3.10), we have $e(t) = y(t) - y_m(t) \in L^2$ and $\dot{e}(t) \in L^{\infty}$, so that $\lim_{t\to\infty} e(t) = 0$. ∇

In summary, the main ideas of the developed adaptive control scheme for general plants are: the controller (6.3.1) with bounded parameters leads to the closed-loop inequality (6.3.36), the adaptive law (6.3.13)–(6.3.14), through the L^2 property of $\dot{\theta}(t)$ and $\frac{\epsilon(t)}{m_0(t)}$, ensures that the loop gain of (6.3.36) is small so that the signal boundedness is guaranteed, and the L^2 property and signal boundedness ensure that $\lim_{t\to\infty} e(t) = 0.$

Discussion: an application example of the adaptive control scheme. The proposed adaptive control scheme does not only has theoretical value, but also



Figure 6.4: An indoor UAV feedback control system using multi-camera as sensors.

has practical significance. Consider an indoor UAV feedback control system using multiple cameras for state sensors measurements as Fig. 6.4 shows, referring to [55]. Such a control system includes a multi-camera system and a ground computer. When the flight begins, the image of the entire environment including the UAV taken by the multi-camera system is transmitted into the ground computer [55]. By analyzing the obtained images, the computer finds the position, the velocity and the attitude of the UAV through image processing. For a such system, when MRAC is employed, the calculation of the adaptive laws should be done by the ground computer since its needs high computing performance. In order to reduce the computation cost and burden, the adaptive controller is preferred to be implemented by the microcontroller on the UAV, using the state information that sent from the ground computer. However, errors may occur during data transmission because of attenuation, distortion or noise, etc. Such errors may lead the situation that $z_i(t) = S_i(x_i)$ as shown in (6.1.7). Under such a situation, the traditional output feedback MRAC design is invalid, and even cause a break down of the control system. However, by using the proposed adaptive sensor failure compensation scheme in chapter, the desired stability and asymptotic tracking performance can still be achieved.

Remark 6.3.1. The results of Theorems 5.3.1, 5.3.2 and 5.4.1 can also be extended to systems with multiple inputs and multiple outputs. For multivariable systems, the system interactor matrix $\xi(s)$ represents the system infinite zero structure whose knowledge is used to construct a reference model system to generate the reference output signal $y_m(t)$. Associated with $\xi(s)$, the high-frequency gain matrix K_p is defined which is unknown due to the system uncertainties. To generate suitable reference models and ensure plant-model output tracking in the presence of system uncertainties and failure uncertainties in MIMO systems, special techniques for multivariable control systems are needed.

6.4 Simulation Study

In this section, we apply the adaptive state failure compensation design proposed in Section 4.1 to a linearized aircraft longitudinal dynamic model [84] for pitch angle control for verifying the effectiveness of the proposed control design.

6.4.1 Simulation System

For pitch angle control, the linearized longitudinal motion equation under landing scenario is:

$$\begin{bmatrix} \dot{U}_{b} \\ \dot{W}_{b} \\ \dot{Q}_{b} \\ \dot{\theta}_{0} \end{bmatrix} = \begin{bmatrix} -0.0264 & 0.1269 & -12.9260 & -32.1690 \\ -0.2501 & -0.8017 & 220.5500 & -0.1631 \\ 0.0002 & -0.0075 & -0.5510 & -0.0003 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} U_{b} \\ W_{b} \\ Q_{b} \\ \theta_{0} \end{bmatrix} + \begin{bmatrix} 0.0109 \\ -0.1858 \\ -0.0230 \\ 0 \end{bmatrix} \delta_{e}$$
$$y(t) = Cx(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x(t) = \theta_{0}(t).$$
(6.4.1)



Figure 6.5: System response of Case I without the adaptive compensation scheme.



Figure 6.6: System response of Case I with adaptive compensation scheme.



Figure 6.7: System response of Case II without adaptive compensation scheme.



Figure 6.8: System response of Case II with adaptive compensation scheme.



Figure 6.9: System response of Case III without adaptive compensation scheme.



Figure 6.10: System response of Case III with adaptive compensation scheme.



Figure 6.11: Sensor reading of the vertical velocity and the pitch rate in Case IV.



Figure 6.12: System response of Case IV with adaptive compensation scheme.

The four state variables are the forward velocity U_b (ft/s), vertical velocity W_b (ft/s), pitch rate Q_b (rad/s) and pitch angle θ_0 (rad) (the notation θ_0 is used to avoid possible confusion with $\theta(t)$ in the adaptive controller), the plant output y(t) is the pitch angle θ_0 , and the control input u(t) is the elevator angle position δ_e (degrees).

In the following simulation study, we use three cases to assess the validation of the adaptive control design.

Case I: Constant reference signal and constant sensor failures. The reference input signal r(t) is chosen as a constant signal. The sensor for measuring the vertical velocity is stuck at 11 ft/s after 240s and the sensor for measuring the pitch rate is stuck at 2 rad/s after 900s, that is, r(t) = 0.1, $z_2 = S_2(W_b) = \bar{s}_2 = 11$ ft/s for $t \ge 240s$, and $z_3 = S_3(Q_b) = \bar{s}_3 = 2$ rad/s for $t \ge 900s$.

Case II: Time-varying reference signal and constant sensor failures. The reference input signal r(t) is chosen as a time varying signal. The sensor for measuring the vertical velocity is stuck at 4 ft/s after 420s and the sensor for measuring the pitch rate is stuck at 1 rad/s after 800s, that is, $r(t) = 0.1 \sin(0.015t)$, $z_2 = S_2(W_b) = \bar{s}_2 = 4$ ft/s for $t \ge 420s$, and $z_3 = S_3(Q_b) = \bar{s}_3 = 1$ rad/s for $t \ge 800s$.

Case III: Time-varying reference signal and time-varying sensor failures. The reference input signal r(t) is chosen as a time varying signal. The sensor for measuring the pitch rate is stuck at 2 rad/s with some oscillation after 340s, and the sensor for measuring the vertical velocity is stuck around 5 ft/s with some oscillation after 870s and , that is, $r(t) = 0.1 \sin(0.015t)$, $z_3 = S_3(Q_b) = \bar{s}_3 = 2 + 0.5 \sin(0.01t) \operatorname{rad/s}$ for $t \geq 340s$, and $z_2 = S_2(W_b) = \bar{s}_2 = 5 + 0.2 \cos(0.1t) \operatorname{ft/s}$ for $t \geq 870s$.

Case IV: Constant reference signal and sensor noises. The reference input signal: r(t) = 0.1, is chosen as a constant signal. The state sensors for the vertical velocity and the pitch rate have additive white Gaussian noise. The sensor-noise-ratio

is 10dB (the sensor output signals of the vertical velocity and the pitch rate, with white Gaussian noises, are shown in Fig. 6.11).

In this simulation study, we assume the failure values are different from the state values as the failures occur. In addition, all the other state variables that do not mentioned are measured correctly.

6.4.2 Simulation Results

For all the adaptive control cases shown below, the adaptive controllers are implemented by (6.3.1) with the adaptive laws (6.3.13)–(6.3.14).

Simulation conditions. For simulation of Case I, $\Gamma = 7I$, $\gamma = 7$, $W_m(s) = 1/s^2 + 3s + 2$, $\Lambda(s) = s^2 + s + 2$, y(0) = 0.01, $y_m(0) = 0$. The initial conditions of the controller parameters are $\theta_1(0) = [1.28, -6.27]^T$, $\theta_2(0) = [4.58, -1.28, 0, 0.11, -4.06, 1.24, 0, 0.01]^T$, $\theta_{20}(0) = [-2.67, 1.13, 0.19, 2.69]^T$, and $\theta_3(0) = -40.98$. Simulation results of Case I, without and with adaptive compensation, are shown in Fig. 6.5 and Fig. 6.6, respectively.

For simulation of Case II and Case III, $\Gamma = 20I$, $\gamma = 20$, $W_m(s) = 1/s^2 + 0.7s + 1$, $\Lambda(s) = s^2 + 2s + 1$, y(0) = 0.04, $y_m(0) = 0$. The initial controller parameters are chosen as $\theta_1(0) = [0.03, 0.62]^T$, $\theta_2(0) = [0.75, -0.64, 0.29, 0.63, 0, 0.37, -0.03, 0.34]^T$, $\theta_{20}(0) = [-0.77, 0.41, 0.23, 1.34]^T$, and $\theta_3(0) = -26.56$. Simulation results of Case II, without and with adaptive compensation, are shown in Fig. 6.7 and Fig. 6.8, respectively. Simulation results of Case III, with and without adaptive compensation, are shown in Fig. 6.9 and Fig. 6.10, respectively.

System performance analysis. In Fig. 6.6(a), Fig. 6.8(a), Fig. 6.10(a) and Fig. 6.12, the dashed lines represent the reference pitch angle and the solid lines represent the aircraft outputs. The four figures verify the effectiveness of our adaptive

state sensor failure compensation scheme. In addition, compared the plots in Fig. 6.6(a), Fig. 6.8(a) and Fig. 6.10(a) generated with adaptation compensation to the plots in Fig. 6.5(a), Fig. 6.7(a) and Fig. 6.9(a) generated without adaptation compensation, it is obvious that the adaptive sensor failure compensation scheme has the capability to significantly improve tracking performance. All plots in Fig. 6.6(b), Fig. 6.8(b), Fig. 6.10(b) and Fig. 6.12 illustrate that the corresponding control signals stay in an acceptable range (less than 30 degree). Also, all signals in the closed-loop system are bounded whose plots are not shown due to the space limit.

Summary

In this chapter, we have investigated a new model reference adaptive control problem for output tracking with parametric uncertainties and sensor failure uncertainties. For the nominal control design, we proposed a new unified sensor failure compensation controller using the state sensor output directly, and proved the desired plant-model matching properties of the unified controller structure under different failure patterns. During the process, we developed a new approach for output matching proof based on the observer theory. The new approach provides additional degree-of-freedom to the controller parameters, which has not been reported before. For the adaptive control design, we derived a parameterized tracking error model, designed a parameter adaptation law, and conducted a complete stability analysis. It has been proved that without explicit fault diagnosis and isolation and additional fault estimation, such an adaptive sensor failure compensation scheme guarantees closed-loop signal boundedness as well as output tracking in the presence of bounded sensor failures. This works also reveals that the desired capacities of the proposed state sensor failure compensation controller are attributed to its similarity to state feedback controller which provides the redundant capacity for achieving desired control system performance. In addition, the effectiveness of the proposed adaptive sensor failure compensation scheme has been verified by a linearized airplane model.

Chapter 7

Sensor Failure Compensation for MIMO Systems With Application to UAVs

This chapter develops a multivariable model reference adaptive control (MRAC) scheme for sensor fault compensation with application to UAVs. A multivariable nominal controller structure for sensor fault compensation is proposed which is directly constructed by the state sensor output that may be subject to some sensor faults. A new adaptive control scheme is developed which guarantees asymptotic output tracking and closed-loop signal boundedness, in the presence of multiple kinds of uncertainties. The main technical contributions of this chapter include

- Development of a nonadaptive feedback controller structure with an offset compensation term, which ensures plant-model output matching, while the feedback signal may be subject to some sensor faults, for quadrotor systems working at non-equilibrium point;
- Development of a multivariable full-state feedback adaptive compensation scheme which ensures system output tracking in the presence of parameter uncertainties and sensor fault uncertainties, for quadrotor systems working at non-equilibrium

point; and

• Verification of the effectiveness of the adaptive sensor fault compensation designs by a simulation study.

7.1 Problem Statement

In this section, the linearized quadrotor model for control design is introduced first. With the knowledge of the quadrotor model, technical issues for quadrotor design are discussed. Then, the control problem is formulated regarding to the technical issues.

7.1.1 Description of An UAV System

As shown in Fig. 7.1, we investigate a standard quadrotor system in this chapter. The quadrotor system has four arms and each arm has a rotor attached at the end. For system analysis, two coordinate systems are defined in Fig. 7.1. One is the body frame (o_B, x_B, y_B, z_B) and the other one is the earth frame (o_E, x_E, y_E, z_E) . The origin of the body frame o_B is at the geometry center of the quadrotor which is assumed to be coincident with the mass center of the quadrotor. The x_B and y_B axes are set on the arms with rotor 1 and 2, and the z_B axis is determined by the right hand rule.

Nonlinear model of the quadrotor system. The dynamics of the quadrotor shown in Fig. 7.1 can be represented by the following differential equations:

$$\begin{aligned} \ddot{x}_E &= \left[(S_\phi S_\psi + C_\phi S_\Theta C_\psi) F_z - c_t \dot{x}_E \right] / m \\ \ddot{y}_E &= \left[(C_\phi S_\Theta S_\psi - S_\phi C_\psi) F_z - c_t \dot{y}_E \right] / m \\ \ddot{z}_E &= \left[C_\phi C_\Theta F_z - c_t \dot{z}_E \right] / m - g \\ \dot{p} &= \left[qr(J_y - J_z) - c_r p + T_x \right] / J_x \\ \dot{q} &= \left[pr(J_z - J_x) - c_r q + T_y \right] / J_y \end{aligned}$$



Figure 7.1: The structure and coordinate frames of a quadrotor.

$$\dot{r} = [pq(J_x - J_y) - c_r r + T_z]/J_z$$

$$\dot{\phi} = p + qS_{\phi}T_{\Theta} + rC_{\phi}T_{\Theta}$$

$$\dot{\Theta} = qC_{\phi} - rS_{\phi}$$

$$\dot{\psi} = [qS_{\phi} + rC_{\phi}]/C_{\Theta},$$
(7.1.1)

with x_E, y_E, z_E representing the positions of the quadrotor along the x, y and z directions in the earth frame respectively, ϕ, Θ, ψ representing the roll angle, the pitch angle, and the yaw angle in the earth frame respectively, p, q, r representing the angular velocities of the quadrotor around x_B, y_B and z_B axes in the body frame respectively, T_* representing the torque generated by the rotors on an axis in body frame, F_z representing the lifting force on the z_B axis, and C_* and S_* representing the cosine and sine function for certain attitude angle respectively.

Rewriting the differential equations into the state-space form, we obtain a nonlinear quadrotor model in the following form:

$$\dot{x}(t) = f(x(t), u(t))$$
(7.1.2)

where the state vector $x(t) = [x_E, y_E, z_E, \dot{x}_E, \dot{y}_E, \dot{z}_E, \phi, \Theta, \psi, p, q, r]^{\mathrm{T}}$, the control input

signal $u(t) = [F_z, T_x, T_y, T_z]^{\mathrm{T}}$, and $f(\dot{)}$ is ****.

Linearized model at an operating point. Considering an operating point (x_o, u_o) of the quadrotor system and applying the linearization method on the nonlinear quadrotor dynamics model (7.1.2) around the operating point, we have the linearized model of the quadrotor system, around the operating point (x_o, u_o) , as

$$\Delta \dot{x}(t) = A\Delta x(t) + B\Delta u(t) + f(x_o, u_o) + H.O.T., \qquad (7.1.3)$$

where $\Delta x(t) = x(t) - x_o$, $\Delta u(t) = u(t) - u_o$, $f(x_o, u_o)$ is the non-equilibrium point offset of the system at the operating point (x_o, u_o) , the constant parameters A and B are

$$A = \frac{\partial f}{\partial x}\Big|_{(x_o, u_o)} = \begin{bmatrix} 0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & -c_t I_{3 \times 3} & A_t & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} & A_s & A_w \\ 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & A_r \end{bmatrix}, \ B = \frac{\partial f}{\partial u}\Big|_{(x_o, u_o)} = \begin{bmatrix} 0_{3 \times 1} & 0_{3 \times 3} \\ B_t & 0_{3 \times 3} \\ 0_{3 \times 1} & 0_{3 \times 3} \\ 0_{3 \times 1} & B_r \end{bmatrix},$$

with the submatrices

$$A_{t} = \begin{bmatrix} C_{\phi}S_{\psi} - S_{\phi}S_{\Theta}C_{\psi} & C_{\phi}C_{\Theta}C_{\psi} & S_{\phi}C_{\psi} - C_{\phi}S_{\Theta}S_{\psi} \\ -C_{\Theta}C_{\psi} - S_{\phi}S_{\Theta}S_{\psi} & C_{\phi}C_{\Theta}S_{\psi} & S_{\phi}S_{\psi} + C_{\phi}S_{\Theta}C_{\psi} \end{bmatrix} \frac{F_{z}}{m},$$

$$A_{r} = \begin{bmatrix} -\frac{c_{r}}{J_{x}} & r\left(\frac{J_{y} - J_{z}}{J_{x}}\right) & q\left(\frac{J_{y} - J_{z}}{J_{x}}\right) \\ r\left(\frac{J_{z} - J_{x}}{J_{y}}\right) & -\frac{c_{r}}{J_{y}} & p\left(\frac{J_{z} - J_{x}}{J_{y}}\right) \\ q\left(\frac{J_{x} - J_{y}}{J_{z}}\right) & p\left(\frac{J_{x} - J_{y}}{J_{z}}\right) & -\frac{c_{r}}{J_{z}} \end{bmatrix} \end{bmatrix},$$

$$A_{s} = \begin{bmatrix} qC_{\phi}T_{\Theta} - rS_{\phi}T_{\Theta} & \frac{qS_{\phi} + rC_{\phi}}{C_{\Theta}^{2}} & 0 \\ -qS_{\phi} - rC_{\phi} & 0 & 0 \\ \frac{qC_{\phi} - rS_{\phi}}{C_{\Theta}} & \frac{S_{\Theta}}{C_{\Theta}^{2}}(qS_{\phi} + rC_{\phi}) & 0 \end{bmatrix},$$

$$A_w = \begin{bmatrix} 1 & S_{\phi} T_{\Theta} & C_{\phi} T_{\Theta} \\ 0 & C_{\phi} & -S_{\phi} \\ 0 & \frac{S_{\phi}}{C_{\Theta}} & \frac{C_{\phi}}{C_{\Theta}} \end{bmatrix}, B_t = \begin{bmatrix} S_{\phi} S_{\psi} + C_{\phi} S_{\Theta} C_{\psi} \\ -S_{\phi} C_{\psi} + C_{\phi} S_{\Theta} S_{\psi} \\ C_{\phi} C_{\Theta} \end{bmatrix} \frac{1}{m},$$

$$B_r = \operatorname{diag} \left\{ \frac{1}{J_x} \quad \frac{1}{J_y} \quad \frac{1}{J_z} \right\},\tag{7.1.4}$$

respectively.

Based on the different control objective, the output matrix C in the output equation: y(t) = Cx(t) can be chosen accordingly, as long as the detectability assumption of the system is satisfied. In the simulation study that shown in Section 7.5, we study yaw-position tracking, thus, the output vector y(t) = Cx(t) is chosen as $y(t) = [z_E, y_E, x_E, \psi]^{\mathrm{T}}$ with the output matrix

$$C = \begin{bmatrix} C_t & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{1\times3} & 0_{1\times3} & C_r & 0_{1\times3} \end{bmatrix}, C_t = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C_r = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \quad (7.1.5)$$

7.1.2 Technical Issues with Quadrotor Control

Regarding to the quadrotor model that has been introduced, two technical issues for quadrotor control are described in this section.

(a) Uncertain sensor faults. In practice, state feedback control design: $u(t) = K_1^{\mathrm{T}}(t)x(t) + K_2(t)r(t)$, is extensively used for quadrotor design. However, as we have mentioned in Section 1, due to inevitable temperature variation and vibrations, the sensor measurements of a quadrotor may be inaccurate.

Assume a quadrotor equips n sensors S_i , i = 1, 2, ..., n, which are used to measure the n state variables $x_i(t)$, receptively. In the presence of an unrecoverable fault at the *j*th sensor, the sensor output is described as

$$z_j = S_j(x_j) = \begin{cases} x_j & \text{with the healthy sensor } S_j \\ \bar{s}_j & \text{with the failed sensor } S_j \end{cases}$$
(7.1.6)

for some unknown bounded values \bar{s}_j with unknown indices $j \in \{1, 2, ..., n\}$. That is to say, only when the state sensor $S_i(\cdot)$ is healthy, $z_j(t) = x_j(t)$, otherwise, $z_j(t) \neq$
$x_j(t)$. In addition, for the state vector $x(t) = [x_1, x_2, \ldots, x_n]^T$, the sensor output vector with possible uncertain state sensor failures is $z(t) = [z_1, z_2, \ldots, z_n]^T$. Only when all the *n* state sensors $S_i(\cdot)$ are healthy, z(t) = x(t), otherwise, $z(t) \neq x(t)$.

Consequently, the sensor output vector z(t) is the actual signal feeding back for implementing the control signal: $u = K_1^{\mathrm{T}}(t)z(t) + K_2(t)r(t)$ (see Fig. 7.2). Thus, the traditional control feedback design will result in destruction of the feedback control system, if there exist unexpected sensor faults.

It is worth noting that the state sensor failures investigated in this chapter are uncertain, which means we do not know which sensors are failed, how much the failures are, and when the failures occur. Such uncertain state sensor failures require effective adaptive compensation in the design of a control scheme to guarantee desired system performance.



Figure 7.2: Control system under possible uncertain sensor failures.

(b) Uncertain dynamics off-set. Another technical challenge coming from the non-equilibrium off-set $f_o(x_o, u_o)$ in (7.1.3). Ignoring the H.O.T. and rewriting the linearized model given in (7.1.3), we have

$$\dot{x}(t) = Ax(t) + Bu(t) + f_o, \ y(t) = Cx(t), \tag{7.1.7}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times M}$ and $C \in \mathbb{R}^{M \times n}$ are constant parameter matrices, and $f_o \in \mathbb{R}^n$ is the non-equilibrium offset constant term. The input-output description of the system is

$$y(t) = G(s)[u](t) + \mathcal{L}^{-1}[G_f(s)\frac{f_o}{s}] = G(s)[u](t) + y_f(t),$$
(7.1.8)

where $G(s) = C(sI - A)^{-1}B$ and $G_f = C(sI - A)^{-1}$ and $y_f(t) = \mathcal{L}^{-1}[G_f(s)\frac{f_o}{s}].$

Due to the model inaccuracy and dynamics change, the parameters (A, B, C) and the non-equilibrium off-set term f_o are unknown. In addition, the non-equilibrium off-set term may also change during the flight. Thus, to make the plant-model output matching achievable in the presence of the system parameter uncertainties and the non-equilibrium off-set uncertainty, a new adaptive controller structure is needed for guaranteeing output tracking.

7.1.3 Control Problem

The control objective is to construct a feedback control law u(t), in the unknown linearized plant (7.1.7) with the non-equilibrium off-set term f_o , by using the sensor output z(t) being subject to the uncertain state sensor faults (2.3.1), such that all signals in the closed-loop system are bounded and the system output y(t) asymptotically tracks a given reference output signal $y_m(t)$ generated from a reference model system

$$y_m(t) = W_m(s)[r](t), \ W_m(s) = \xi_m^{-1}(s)$$
(7.1.9)

where $r(t) \in \mathbb{R}^M$ is a bounded reference input signal, and $\xi_m(s)$, defined in Lemma 1, is a modified left interactor matrix of the system transfer matrix $G(s) = C(sI - A)^{-1}B$, whose inverse matrix is stable, i.e., $W_m(s)$ is stable.

Plant assumptions. To achieve the control objectives, the following standard assumptions are assumed to be satisfied.

(A7.1) All zeros of $G(s) = C(sI - A)^{-1}B$ are stable, and (A, B, C) is stabilizable and detectable.

(A7.2) G(s) has full rank and its modified left interactor matrix $\xi_m(s)$ is known.

Assumption (A7.1) is for a stable plant-model output matching, and Assumption (A7.2) is for choosing a reference model system $W_m(s) = \xi_m^{-1}(s)$ suitable for plant-model output matching.

7.2 Nominal Compensation Design

In this section, we will address the plant-model output matching problem in the presence of sensor fault, for the case of known system parameters. Such a nominal compensation solution will provide a solid foundation for the adaptive compensation scheme to deal with both uncertain sensor faults for the case of unknown system parameters.

To achieve output tracking in the presence of some uncertain state sensor faults, we assume that

(A7.3) There exists a vector signal $z_{n_0}(t) = C_{n_0}x(t) \in \mathbb{R}^{n_0}$ being available for measurement from n_0 healthy sensors, with (A, C_{n_0}) observable for $C_{n_0} \in \mathbb{R}^{n_0 \times n}$ and rank $[C_{n_0}] = n_0$.

In order to achieve the control objective, it is obvious that at least one sensor measurement is correct, i.e., $n_0 \ge 1$. The relationship between the healthy sensor output $z_{n_0}(t)$ and the control input signal u(t) can be expressed as

$$z_{n_0}(t) = G_{n_0}(s)[u](t) + \mathcal{L}^{-1}\left[\frac{f_o}{s}G_{f_{n_0}}(s)\right] = G_{n_0}(s)[u](t) + y_{f_o}(t),$$
(7.2.1)

with $G_{n_0}(s) = C_{n_0}(sI-A)^{-1}B$ and $G_{f_{n_0}}(s) = C_{n_0}(sI-A)^{-1}$ and $y_{f_o}(t) = \mathcal{L}^{-1}[G_{f_{n_0}}(s)\frac{f_o}{s}]$.

7.2.1 Sensor Failure Compensation With Healthy Sensors $z_{n_0}(t)$

When the system parameters and the sensor faults are known, plant-model output matching can be achieved by a controller structure using only the healthy sensor output vector $z_{n_0}(t)$. Such a controller structure is in the form of

$$u(t) = \Theta_1^{*T} \omega_1(t) + \bar{\Theta}_{21}^{*T} \omega_2(t) + \bar{\Theta}_{201}^{*T} z_{n_0}(t) + \Theta_3^{*} r(t) + \Theta_4^{*}, \qquad (7.2.2)$$

where $\Theta_1^* \in \mathbb{R}^{M(n-n_0) \times M}$, $\overline{\Theta}_{21}^* \in \mathbb{R}^{n_0(n-n_0) \times M}$, $\overline{\Theta}_{201}^* \in \mathbb{R}^{n_0 \times M}$, $\Theta_3^* \in \mathbb{R}^{M \times M}$ are the constant nominal controller parameters , $\Theta_4^* \in \mathbb{R}^M$ is the constant compensation term to deal with the non-equilibrium off-set term f_o , and

$$\omega_1(t) = \frac{A_1(s)}{\Lambda(s)} [u](t), \\ \omega_2(t) = \frac{A_{21}(s)}{\Lambda(s)} [z_{n_0}](t),$$
(7.2.3)

with $A_1(s) = [1, s, \dots, s^{n-n_0-1}]^{\mathrm{T}}$, $A_{21}(s) = [I_{n_0}, sI_{n_0}, \dots, s^{n-n_0-1}I_{n_0}]^{\mathrm{T}}$, and $\Lambda(s)$ being the polynomial is a stable polynomial with degree $n - n_0$.

Output matching analysis. For plant-model output matching analysis, one of the keys is the following equation (7.2.4).

Lemma 7.2.1. [79] Constants Θ_1^* , $\bar{\Theta}_{21}^*$, $\bar{\Theta}_{201}^*$ and Θ_3^* exist such that

$$\Theta_1^{*T} A_1(s) P(s) + (\bar{\Theta}_{21}^{*T} A_{21}(s) + \bar{\Theta}_{201}^{*T} \Lambda(s)) Z_{n_0}(s)$$

= $\Lambda(s) (P(s) - K_p \Theta_3^* Z(s) P_m(s)),$ (7.2.4)

for some polynomial vector $Z_{n_0}(s)$ such that $P(s)[z_{n_0}(t)] = Z_{n_0}(s)[u](t)$.

The proof of this lemma can be found in our previous result [79], as the nominal solution to a partial-state feedback MRAC problem.

Next, we will present an analysis for showing the capability of the nominal controller structure with the correct measurement $z_{n_0(t)}$ in (7.2.2) for plant-output matching, in the presence of the non-equilibrium off-set term f_o .

Substituting (7.2.1) into the nominal controller structure (7.2.2), we have

$$u(t) = \Theta_1^{*T} F_1(s)[u](t) + \bar{\Theta}_{21}^{*T} F_2(s) G_{n_0}(s)[u](t) + \bar{\Theta}_{21}^{*T} F_2(s)[y_{f_o}](t) + \bar{\Theta}_{201}^{*T} G_{n_0}(s)[u](t) + \Theta_{201}^{*T} y_{f_o}(t) + \Theta_3^{*} r(t) + \Theta_4^{*},$$
(7.2.5)

with $F_1(s) = \frac{A_1(s)}{\Lambda(s)}$ and $F_2(s) = \frac{A_{21}(s)}{\Lambda(s)}$. This controller structure can be further expressed as

$$u(t) = (I - \Theta_1^{*T} F_1(s) - \bar{\Theta}_{21}^{*T} F_2(s) G_{n_0}(s) - \bar{\Theta}_{201}^{*T} G_{n_0}(s))^{-1} \\ \times (\bar{\Theta}_{21}^{*T} F_2(s) [y_{f_o}](t) + \bar{\Theta}_{201}^{*T} y_{f_o}(t) + \Theta_3^* r(t) + \Theta_4^*).$$
(7.2.6)

Hence, we could express the closed-loop signal y(t) as

$$y(t) = G(s)(I - \Theta_1^{*T}F_1(s) - \bar{\Theta}_{21}^{*T}F_2(s)G_{n_0}(s) - \bar{\Theta}_{201}^{*T}G_{n_0}(s))^{-1} \\ \times (\bar{\Theta}_2^{*T}F_2(s)[y_{f_o}](t) + \bar{\Theta}_{201}^{*T}y_{f_o}(t) + \Theta_3^{*}r(t) + \Theta_4^{*}) + y_f(t).$$

From Lemma 7.2.1, we can have the plant-model matching equation:

$$I - \Theta_1^{*T} F_1(s) - \bar{\Theta}_{21}^{*T} F_2(s) G_{n_0}(s) - \bar{\Theta}_{20}^{*T} G_{n_0}(s) = \Theta_3^{*} W_m^{-1}(s) G(s).$$
(7.2.7)

Thus, the closed-loop system becomes

$$y(t)$$

$$= G(s)[\Theta_{3}^{*}W_{m}^{-1}(s)G(s)]^{-1}(\bar{\Theta}_{21}^{*T}F_{2}(s)[y_{f_{o}}](t) + \bar{\Theta}_{201}^{*T}y_{f_{o}}(t) + \Theta_{3}^{*}r(t) + \Theta_{4}^{*}) + y_{f}(t)$$

$$= W_{m}(s)[r](t) + W_{m}(s)\Theta_{3}^{*-1}[\bar{\Theta}_{21}^{*T}F_{2}(s)[y_{f_{o}}](t) + \bar{\Theta}_{201}^{*T}y_{f_{o}}(t) + \Theta_{3}^{*}W_{m}^{-1}(s)[y_{f}](t) + \Theta_{4}^{*}](t)$$

$$= y_{m}(t) + \delta(t)$$
(7.2.8)

with $\delta(t) = W_m(s)\Theta_3^{*-1}[\bar{\Theta}_{21}^{*T}F_2(s)[y_{f_o}](t) + \bar{\Theta}_{201}^{*T}y_{f_o}(t) + \Theta_3^*W_m^{-1}(s)[y_f](t) + \Theta_4^*](t).$

Expressing $\delta(t)$ in the s-domain, we have

$$\delta(s) = W_m(s) K_p \bar{\Theta}_{21}^{*\mathrm{T}} F_2(s) G_{f_{n_0}}(s) \frac{f_o}{s} + \bar{\Theta}_{201}^{*\mathrm{T}} G_{f_{n_0}}(s) \frac{f_o}{s} + \Theta_3^* W_m^{-1}(s) G_f(s) \frac{f_o}{s} + \frac{\Theta_4^*}{s}.$$
(7.2.9)

Thus, applying the final value theorem, we have

$$\lim_{t \to \infty} (y(t) - y_m(t)) = \lim_{t \to \infty} \delta(t) = \lim_{s \to 0} s \delta(s)$$

=
$$\lim_{s \to 0} (W_m(s) K_p \bar{\Theta}_{21}^{*\mathrm{T}} F_2(s) G_{f_{n_0}}(s) + \bar{\Theta}_{201}^{*\mathrm{T}} G_{f_{n_0}}(s) + \Theta_3^* W_m^{-1}(s) G_{f_{n_0}}(s)) f_o + \Theta_4^*.$$

Hence, in order to make the plant-model output matching: $y(t) = y_m(t)$, the nominal value of Θ_4^* is chosen as

$$\Theta_4^* = Df_o \tag{7.2.10}$$

with $D = -\lim_{s \to 0} W_m(s) K_p \Theta_2^{*T} F_2(s) G_{f_{n_0}}(s) + \Theta_{201}^{*T} G_{f_{n_0}}(s) + \Theta_3^* W_m^{-1}(s) G_f(s).$

Summarizing the result we have derived, we present the desired plant-model matching property as follows.

Proposition 7.2.1. Constant parameters Θ_1^* , $\bar{\Theta}_{21}^*$, $\bar{\Theta}_{201}^*$, Θ_3^* satisfying (7.2.4) and Θ_4^* satisfying (7.2.10) exist such that the nominal compensation controller (7.2.2) ensures plant-model output matching: $y(t) - y_m(t) = \delta(t)$, when the healthy sensor output $z_{n_0}(t)$ exists and satisfies the assumption (A7.3).

Proposition 7.2.1 shows that when the plant parameters and the healthy sensor output $z_{n_0}(t)$ are known, the nominal parameters Θ_1^* , Θ_2^* , Θ_{20}^* , Θ_3^* and Θ_4^* exist for the nominal control law (7.2.2) to solve the MRAC problem in the presence of sensor faults.

Remark 7.2.1. Relative to the plant output y(t), the healthy sensor output $z_{n_0}(t)$ in (7.2.2) has four possibilities (cases):

- (1) z_{n_0} is a vector containing some or all elements of y;
- (2) z_{n_0} is a vector which does not contain any element of y;
- (3) z_{n_0} is a scalar as one element of y; and
- (4) z_{n_0} is a scalar not being any element of y.

It is worth noting that from Proposition 7.2.1, it is sufficient to use a scalar feedback signal $z_{n_0}(t) \in \mathbb{R}^{n_0}(n_0 = 1)$ for constructing the nominal compensation controller to make the *M*-output vector $y(t) \in \mathbb{R}^M$ ($M \ge 1$) to match the desired output. In other words, as long as there is only one state sensor measurement $z_i(t)$ is correct, the multivariable plant-model output matching is achievable.

7.2.2 Sensor Failure Compensation With Direct Sensor Output z(t)

In practice, the sensor faults are usually uncertain, in other words, we do not know which sensors are failed, how much the failures are, and when the failures occur. Thus, the controller structure (7.2.2) becomes ineffective, because of the unavailability of the healthy sensor output vector $z_{n_0(t)}$.

In this section, we will derive a new controller structure constructed by the direct sensor output signal z(t) which can guarantee output matching in the presence of possible sensor faults. We denote the sensor output vector as $z(t) = P^{-1}[z_{n_0}^T(t), \bar{z}_{n_0}^T(t)]^T$ with $P \in \mathbb{R}^{n \times n}$ being a transformation matrix such that $C_{n_0}P^{-1} = [I_{n_0}, 0]$. Without loss of generality, in the following nominal control design, we consider the case when $P = I_n$ due to the space limit (state transformation techniques are used for a general analysis). Controller structure. We first extend (7.2.2) as

$$u(t) = \Theta_1^{*T} \frac{A_1(s)}{\omega_1(t)} + \bar{\Theta}_{21}^{*T} \frac{A_{21}(s)}{\omega_2(t)} + \bar{\Theta}_{201}^{*T} z_{n_0}(t) + \Theta_3^* r(t) + \Theta_4^* + \bar{\Theta}_{22}^{*T} \frac{A_{22}(s)}{\Lambda(s)} [\bar{z}_{n_0}](t) + \bar{\Theta}_{202}^{*T} \bar{z}_{n_0}(t), \qquad (7.2.11)$$

where the controller parameters $\bar{\Theta}_{22}^* \in \mathbb{R}^{(n-n_0)(n-n_0)\times M}$, $\bar{\Theta}_{202}^* \in \mathbb{R}^{n-n_0\times M}$ are uniquely chosen as zero: $\bar{\Theta}_{22} = 0$, $\bar{\Theta}_{202}^* = 0$, with $A_{22}(s) = [I_{n-n_0}, sI_{n-n_0}, \dots, s^{n-n_0-1}I_{n-n_0}]^{\mathrm{T}}$.

The controller parameters Θ_1^* , $\bar{\Theta}_{21}^*$, $\bar{\Theta}_{201}^*$ and Θ_3^* , associating with the healthy sensor output $z_{n_0}(t)$, satisfy the plant-model matching equation (7.2.4) to ensure plantmodel output matching, the controller parameters Θ_4^* satisfy the compensation condition (7.2.10), and the controller parameters $\bar{\Theta}_{22}^*$ and $\bar{\Theta}_{202}^*$, associating with the failed sensor output \bar{z}_{n_0} , are set as zero to compensate the undesired effect from the failed sensor output.

Based on the relationship $z(t) = [z_{n_0(t)}^{\mathrm{T}}, \bar{z}_{n_0}^{\mathrm{T}}(t)]^{\mathrm{T}}$, we re-organize (7.2.11) and obtain the nominal sensor failure compensation controller constructed by z(t), in the form:

$$u(t) = \Theta_1^{*T} \omega_1(t) + \Theta_2^{*T} \omega_2(t) + \Theta_{20}^{*T} z(t) + \Theta_3^{*} r(t) + \Theta_4^{*}, \qquad (7.2.12)$$

where $\Theta_1^* \in \mathbb{R}^{M(n-n_0) \times M}$, $\Theta_2^* = [\bar{\Theta}_{21}^{*T}, \bar{\Theta}_{22}^{*T}]^T \in \mathbb{R}^{n(n-n_0) \times M}$, $\Theta_{20}^* = [\bar{\Theta}_{201}^{*T}, \bar{\Theta}_{202}^{*T}]^T \in \mathbb{R}^{n \times M}$, $\Theta_3^* \in \mathbb{R}^{M \times M}$, $\Theta_4^* \in \mathbb{R}^M$, and the regressors

$$\omega_1(t) = \frac{A_1(s)}{\Lambda(s)} [u](t), \quad \omega_2(t) = \frac{A_2(s)}{\Lambda(s)} [z](t), \tag{7.2.13}$$

with

$$A_1(s) = [I_M, sI_M, \dots, s^{n-n_0-1}I_M]^{\mathrm{T}}, A_2(s) = [I_n, sI_n, \dots, s^{n-n_0-1}I_n]^{\mathrm{T}},$$
(7.2.14)

and $\Lambda(s)$ is a stable and monic polynomial of degree $n - n_0$.

Remark 7.2.2. Recall the four possibilities of $z_{n_0}(t)$ we have listed in Section 7.2.1. and consider the development of the nominal controller structure (7.2.12) with z(t). It turns out that for guaranteeing the plant-model output matching, the correct measurement containing in z(t) could be

- (1) a vector containing some or all correct measurements of y;
- (2) a vector which does not contain any element of y;
- (3) a scalar as one element of y; and
- (4) a scalar not being any element of y.

From case (3) and case (4), we can conclude that by using the proposed nominal compensation controller structure (7.2.12), the system can tolerant at most n - 1 sensor faults. Such a result has not been reported in any literature yet.

Zero nominal controller parameters $\overline{\Theta}_{22}^*$ and $\overline{\Theta}_{202}^*$. In general, for guaranteeing that a basic MRAC problem is solvable, we need to find a set of constant controller parameters being used in a nominal controller structure for the purpose of making plant-model matching achievable. As long as such constant controller parameters exist, the adaptive system could use the estimation of such controller parameters, with the help of parameter adaption law, for achieving output tracking for the case of unknown system parameters.

Specifically, in this sensor fault compensation problem we find that with the nominal controller parameters $\bar{\Theta}_{22}^*$, $\bar{\Theta}_{202}$ chosen as zero, the nominal controller structure (7.2.12) with the direct sensor output z(t) becomes to the nominal controller structure (7.2.2) which is able to guarantee plant-model output matching. Such a specific choice for the parameters $\bar{\Theta}_{22}^*$ and Θ_{202}^* prevents the failed sensor $\bar{z}_{n_0}(t)$ to be used for feedback control while keeps the form of the nominal controller structure (7.2.12) constructed with the sensor output z(t). Such a controller structure helps

- to avoid explicit sensor failure detection and identification which requires sensor redundancy and increase system cost; and
- to avoid sensor fault estimation which requires additional signal processing which may cause feedback control system delay.

7.3 Adaptive Sensor Failure Compensation Design

In this section, we develop an adaptive feedback control design to deal with the plant parameter uncertainties and sensor fault uncertainty. For adaptive control, we assume (A7.4) all leading principle minors Δ_i , i = 1, 2, ..., M, of the high frequency matrix

 K_p of G(s) are nonzero and their signs are known.

Controller structure. To deal with the unknown (A, B, C), unknown offset term f_o in the plant (7.1.7) and the uncertain sensor fault, we design the adaptive version of the controller (7.2.12) as

$$u(t) = \Theta_1^{\mathrm{T}}(t)\omega_1(t) + \Theta_2^{\mathrm{T}}(t)\omega_2(t) + \Theta_{20}^{\mathrm{T}}(t)z(t) + \Theta_3(t)r(t) + \Theta_4(t),$$
(7.3.1)

where $\Theta_1(t) \in \mathbb{R}^{M(n-n_0) \times M}$, $\Theta_2(t) \in \mathbb{R}^{n(n-n_0) \times M}$, $\Theta_{20}(t) \in \mathbb{R}^{n \times M}$, $\Theta_3(t) \in \mathbb{R}^{M \times M}$, $\Theta_4(t) \in \mathbb{R}^M$ are the adaptive estimates of the unknown nominal parameters Θ_1^* , Θ_2^* , Θ_{20}^* , Θ_3^* and Θ_4^* , respectively, and $\omega_1(t)$ and $\omega_2(t)$ are the filters for u(t) and z(t)which have been given in (7.2.13).

Tracking error equation. To choose a stable adaptive law, we derive the following tracking error equation:

$$e(t) = y(t) - y_m(t) = W_m(s)K_p[\tilde{\Theta}^{\mathrm{T}}\omega](t),$$
 (7.3.2)

where $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$ with $\Theta^* = \left[\Theta_1^{*T}, \Theta_2^{*T}, \Theta_{20}^{*T}, \Theta_3^*, \Theta_4^*\right]^T$, $\Theta(t) = \left[\Theta_1^T(t), \Theta_2^T(t), \Theta_2^T(t), \Theta_2^T(t), \Theta_3^T(t), \Theta_4^T(t), \Theta_4^T(t),$

To deal with the uncertainty of K_p , the LDS decomposition: $K_p = L_s D_s S$, is to be used, where $S = S^T > 0$, L_s is a unit lower triangle matrix, and $D_s =$ $\operatorname{diag}\{s_1^*, s_2^*, \ldots, s_M^*\} = \operatorname{diag}\{\operatorname{sign}[d_1^*]\gamma_1, \ldots, \operatorname{sign}[d_M^*]\gamma_M\}$ with $\gamma_i > 0, i = 1, 2, \ldots, M$.

To proceed the output matching analysis, substituting the decomposition of K_p into the tracking error equation (7.3.2), we have

$$L_s^{-1}\xi_m(s)[e](t) = D_s S \tilde{\Theta}^{\mathrm{T}}(t)\omega(t).$$
(7.3.3)

To parameterize the unknown matrix L_s , we first introduce a constant matrix $\Theta_0^* = L_s^{-1} - I = \{\Theta_{ij}^*\}$ with $\Theta_{ij}^* = 0$ for i = 1, 2, ..., M and $j \ge i$, then we have

$$\xi_m(s)[e](t) + \Theta_0^* \xi_m(s)[e](t) = D_s S \tilde{\Theta}^{\mathrm{T}}(t) \omega(t).$$
(7.3.4)

To parameterize this tracking error equation, choosing a filter $h(s) = \frac{1}{f(s)}$, where f(s)is a stable and monic polynomial whose degree is equal to the maximum degree of the modified interactor matrix $\xi_m(s)$ and operating $h(s)I_M$ on both sides of (7.3.4), we have

$$\bar{e}(t) + \left[0, \Theta_2^{*T} \eta_2(t), \dots, \Theta_M^{*T} \eta_M(t)\right]^{\mathrm{T}} = D_s Sh(s) [\tilde{\Theta}^{\mathrm{T}} \omega](t), \qquad (7.3.5)$$

where $\bar{e}(t) = \xi_m(s)h(s)[e](t) = [\bar{e}_1(t), \dots, \bar{e}_M(t)]^{\mathrm{T}}, \eta_i(t) = [\bar{e}_1(t), \dots, \bar{e}_{i-1}(t)]^{\mathrm{T}} \in \mathbb{R}^{i-1},$ $i = 2, \dots, M$, and $\Theta_i^* = [\Theta_{i1}^*, \dots, \Theta_{ii-1}]^{\mathrm{T}}, i = 2, \dots, M.$

Estimation error model. Based on the tracking error equation (7.3.5), we introduce the estimation error signal:

$$\epsilon(t) = \left[0, \,\Theta_2^{\mathrm{T}}\eta_2(t), \,\Theta_3^{\mathrm{T}}\eta_3(t), \dots, \Theta_M^{\mathrm{T}}\eta_M(t)\right]^{\mathrm{T}} + \Psi(t)\xi(t) + \bar{e}(t), \tag{7.3.6}$$

where $\Theta_i(t)$, i = 2, ..., M are the estimates of Θ_i^* , and $\Psi(t)$ is the estimate of $\Psi^* = D_s S$, and

$$\xi(t) = \Theta^{\mathrm{T}}(t)\zeta(t) - h(s)[\Theta^{\mathrm{T}}\omega](t), \zeta(t) = h(s)[\omega](t).$$
(7.3.7)

From (7.3.5)-(7.3.7), we derive that

$$\epsilon(t) = \left[0, \,\tilde{\Theta}_2^{\mathrm{T}}\eta_2(t), \,\tilde{\Theta}_3^{\mathrm{T}}\eta_3(t), \dots, \tilde{\Theta}_M^{\mathrm{T}}\eta_M(t)\right]^{\mathrm{T}} + D_s S \tilde{\Theta}^{\mathrm{T}}(t)\zeta(t) + \tilde{\Psi}(t)\xi(t), \quad (7.3.8)$$

where $\tilde{\Theta}_i(t) = \Theta_i(t) - \Theta_i^*$, i = 2, ..., M, and $\tilde{\Psi}(t) = \Psi(t) - \Psi^*(t)$ are parameter errors.

Adaptive parameter update law. Based on the error model (7.3.8) which is linear in parameter errors, the adaptive laws are chosen as

$$\dot{\Theta}_i(t) = -\frac{\Gamma_{\Theta i}\epsilon_i(t)\eta_i(t)}{m^2(t)}, i = 2, 3, \dots, M$$
(7.3.9)

$$\dot{\Theta}^{\rm T}(t) = -\frac{D_s \epsilon(t) \zeta^{\rm T}(t)}{m^2(t)}, \ \dot{\Psi}(t) = -\frac{\Gamma \epsilon(t) \xi^{\rm T}(t)}{m^2(t)},$$
(7.3.10)

for updating parameter estimates, where the estimation error signal $\epsilon(t) = [\epsilon_1(t), \epsilon_2(t), \ldots, \epsilon_M(t)]^{\mathrm{T}}$ is computed from (7.3.6), $\Gamma_{\Theta i} = \Gamma_{\Theta i}^{\mathrm{T}} > 0$, $i = 2, 3, \ldots, M$ and $\Gamma = \Gamma^{\mathrm{T}} > 0$ are adaption gain matrices, and

$$m^{2}(t) = 1 + \zeta^{\mathrm{T}}(t)\zeta(t) + \xi^{\mathrm{T}}(t)\xi(t) + \sum_{i=2}^{M} \eta_{i}^{\mathrm{T}}(t)\eta_{i}(t).$$
(7.3.11)

Stability analysis and tracking performance. The adaptive law (7.3.9)– (7.3.10) ensures that (i) $\Theta_i(t) \in L^{\infty}$, i = 2, 3, ..., M, $\Theta(t) \in L^{\infty}$, $\Psi(t) \in L^{\infty}$, and $\frac{\epsilon(t)}{m(t)} \in L^2 \cap L^{\infty}$; and (ii) $\dot{\Theta}_i(t) \in L^2 \cap L^{\infty}$, i = 2, 3, ..., M, $\dot{\Theta}(t) \in L^2 \cap L^{\infty}$, and $\dot{\Psi}_i(t) \in L^2 \cap L^{\infty}$. The result can be obtained by constructing the positive definite function

$$V = \frac{1}{2} \left(\sum_{i=2}^{M} \tilde{\Theta}_{i}^{\mathrm{T}}(t) \Gamma_{\Theta i}^{-1} \tilde{\Theta}_{i} + \mathrm{tr}[\tilde{\Psi}^{\mathrm{T}} \Gamma^{-1} \tilde{\Psi}] + \mathrm{tr}[\tilde{\Theta} S \tilde{\Theta}^{\mathrm{T}}] \right).$$
(7.3.12)

Based on the above desired properties of the adaptive law (7.3.9)-(7.3.10), the following desired closed-loop system properties are established.

Theorem 7.3.1. The adaptive sensor fault compensator (7.3.1) with the adaptive law (7.3.9)-(7.3.10), when applied to the plant (7.1.7), guarantees the closed-loop

Parameter	Value	Unit
c_m	0.0025	m∙s
c_r	0.001	$N \cdot m \cdot s$
c_t	0.25	$N \cdot m \cdot s$
d	0.2	m
g	9.8	m/s^2
J_x	0.005	$kg \cdot s^2$
J_y	0.005	$kg \cdot s^2$
J_z	0.009	$kg \cdot s^2$
m	2	kg

Table 7.1: Parameter values of the quadrotor used for simulation study

signal boundedness and asymptotic output tracking: $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$, in the presence of possible sensor fault (7.1.6).

Together with the signal boundedness result, the proof of Theorem 7.3.1 can be completed in a similar way to that derived in Ch. 9, [82]. This proposed MRACbased sensor fault compensation scheme can guarantee asymptotic output tracking for both the cases of no sensor failures (i.e., z(t) = x(t)) and the case of uncertain state sensor failures (i.e., $z(t) \neq x(t)$).

7.4 Simulation Study

To evaluate the effectiveness of the proposed adaptive control designs, we will present a simulation result in this section.

Simulation systems. In this simulation study, we consider a quadrotor with system parameters given in Table 7.1, for position tacking and yaw tracking. Our goal is to let the quadrotor fly along x-direction uniformly with a speed of 12 m/s (with pitch angle $\theta = 0.19$ rad) and the yaw angle changing periodically for a searching task. so that the reference signal is chosen as $r(t) = [0, 0, 12t, \sin(0.1t)]^{T}$.

We linearize the quadrotor model under the high-speed cruise control condition.

The corresponding system parameters (A, B) of the linearized model are

For position tracking and yaw tracking, the output parameter C is chosen as

Simulation results. For simulation, the adaptation gains are chosen as $\Gamma = 25I_4$, $\Gamma_{\theta_1} = 10$, $\Gamma_{\theta_2} = 10I_2$, $\Gamma_{\theta_3} = 10I_3$, and the initial condition are chosen as



Figure 7.3: System response with the adaptive sensor failure compensation scheme.



Figure 7.4: System response without the adaptive sensor failure compensation scheme.

 $y(0) = [0.01, 0.01, 0, 0]^{\mathrm{T}}, y_m(0) = [0, 0, 0, 0]^{\mathrm{T}}$. The system responses, with and without adaptive compensation, are shown in Fig. 7.3 and Fig. 7.4, respectively. The simulation results support that the proposed multivariable sensor failure compensation scheme has the capability to ensure output tracking and signal boundedness in the presence of state sensor failures.

Summary

In this chapter, we have developed a multivariable adaptive sensor fault compensation control scheme for quadrotor systems. A nominal controller structure constructed by the sensor output vector which may be subject to sensor faults is proposed. It has been shown that the multivariable plant-model output matching is guaranteed by such a nominal controller, even only one sensor measurement is correct. To deal with uncertain sensor faults and uncertain system parameters, a new adaptive control scheme has been developed. The adaptive controller structure, with an additional compensation term for dealing with the non-equilibrium off-set term of the dynamics system, has been shown the capability of ensuring asymptotic output tracking and closed-loop signal boundedness, when applied to the control plant. The effectiveness of the developed control scheme has been assessed by a simulation study.

Chapter 8

Output Consensus of Multi-Agent System Using Partial-State Feedback

This chapter solves a partial-state feedback output consensus problem for uncertain multi-agent systems with relative degree one followers. Such a new distributed adaptive consensus scheme guarantees the desired leader-following output consensus and closed-loop signal boundedness. A simulation study on a multiple-aircraft system verifies the effectiveness of the proposed adaptive multi-agent output consensus scheme.

The new technical contributions of this work are:

- clarifying the output matching condition for leader-following output consensus of multi-agent systems;
- developing a new adaptive partial-state feedback control scheme for relativedegree-one followers, which avoids the restrictive matching conditions and complex controller structure as well; and
- analyzing system stability and consensus performance of the multi-agent system.

8.1 Problem Statement and Preliminaries

In this section, we will formulate the leader-following consensus control problem to be studied and give a brief introduction to graph theory which is a useful tool for describing the connection of the multi-agent system.

8.1.1 Problem Statement

Consider a multi-agent system including N followers and one virtual leader. The *i*th followers' dynamic equation is

$$\dot{x}_i(t) = A_i x_i(t) + b_i u_i(t), \ y_i(t) = b_i x_i(t), \ i = 1, \dots, N,$$

for the unknown parameter matrices $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times 1}$ and $C_i \in \mathbb{R}^{1 \times n}$, where $x_i(t) \in \mathbb{R}^n$ is the state vector of the *i*th follower, $u_i(t) \in \mathbb{R}$ is the control input of the *i*th follower, and $y_i(t) \in \mathbb{R}$ is the output of the *i*th follower. The input-output description of each follower is

$$y_i(t) = G_i(s)[u_i](t), G_i(s) = k_{pi} \frac{Z_i(s)}{P_i(s)}, i = 1, \dots, N$$
 (8.1.0)

where $k_{pi} \neq 0$, $P_i(s) = \det(sI - A_i) = s^n + p_{(n-1)i}s^{n-1}\cdots + p_{1i}s + p_{0i}$, and $Z(s) = s^m + \cdots + z_{1i}s + z_{0i}$ for some $m \geq 0$. The notation: y(t) = G(s)[u](t), is used to denote the output y(t) of a LTI system represented by a transfer function G(s) with input signal u(t).

The dynamic model of the virtual leader is given by

$$y_l(t) = W_l(s)[r](t), W_l(s) = \frac{1}{P_l(s)},$$
(8.1.0)

where $P_l(s)$ is a desired stable polynomial of degree $n^* = n - m$ (the followers' relative degree n^* is assumed to be known), and r(t) is a bounded piecewise continuous reference input signal. **Control objective.** For the multi-agent system consisting of (8.1.1) and (8.1.1), the control objective is to design a control protocol using the local partial-state vector $y_{0i}(t)$ to generate the control signal $u_i(t)$ in (8.1.1) for each follower such that all the signals in the multi-agent system are bounded and the output of all followers track the output $y_i(t)$ of the given leader asymptotically, i.e.,

$$\lim_{t \to \infty} (y_i(t) - y_l(t)) = 0, \ i = 1, \dots, N.$$
(8.1.0)

we first make the following basic assumptions:

(A8.1) A vector signal $y_{0i}(t) = C_{0i}x_i(t) \in \mathbb{R}^{n_{0i}}$ is available for measurement, with (A, C_{0i}) observable for $C_{0i} \in \mathbb{R}^{n_{0i} \times n}$ and $\operatorname{rank}[C_{0i}] = n_{0i}$; and

(A8.2) all zeros of $Z_i(s)$ are stable polynomials.

The assumption (A8.2) is for ensuring internal stability for output consensus [82].

Graph theory is used in this problem to model the information exchange between the agents, which is introduced in the next section.

8.1.2 Preliminaries

The information exchange among the N follower agents in this chapter is denoted by a undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with a set of nodes \mathcal{V} , a set of undirected edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the adjacency matrix of the graph $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$. The node v_i represents the *i*th follower agent. An unordered edge $(v_i, v_j) \in \mathcal{E}$ (or equivalently $(v_j, v_i) \in \mathcal{E}$) represents that the information exchanges between the agents v_i and v_j , and v_i and v_j are neighbors. In addition, $(v_i, v_j) \in \mathcal{E}$ follows that the adjacency element $a_{ij} = a_{ji} = 1$. A path is a sequence of unordered edges of the form $(v_{i1}, v_{i2}), (v_{i2}, v_{i3}), \ldots$, in a graph, where $v_{ij} \in \mathcal{V}$. If, for any two nodes $v_i, v_j \in \mathcal{V}$, there is a path between them, then \mathcal{G} is called a connected graph. To describe the information exchange from the leader to the followers, we denote the leader as v_0 . Let $\mathcal{V}_{\Sigma} = \{\mathcal{V}, v_0\}$ be the node set consisting of all the follower agents and the leader. Since the leader v_0 can not be affected by the followers v_i , the connection edges (v_i, v_0) between the leader v_0 and the *i*th agent v_i are directed which means that the follower v_i can obtain the information from v_0 , but not vice versa. Let \mathcal{E}_l be the edges set consisting of all edges (v_i, v_0) . Define $\mathcal{N}_i = \{v_j \in \mathcal{V}_{\Sigma} : (v_j, v_i) \in$ $\{\mathcal{E} \bigcup \mathcal{E}_l\}$ as the neighborhood of the *i*th follower and $\mathcal{N}_0 = \{v_j \in \mathcal{V} : (v_j, v_0) \in \mathcal{E}_l\}$ as the set of follower agents that are directly connected to the leader. Note that in this chapter, only part of the followers can connect to the leader.

For the N follower agents, the Laplacian matrix \mathcal{L} of undirected graph \mathcal{G} is defined by $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $D = \text{diag}\{d_1, d_2, \ldots, d_n\}$ and $d_i = \sum_{j=1}^N a_{ij}$. In addition, the connectivity between the followers and the leader is represented by $\mathcal{B} = \text{diag}\{b_1, b_2, \ldots, b_n\}$. If the follower v_i is connected to the leader v_0 , then $b_i = 1$, otherwise, $b_i = 0, i = 1, 2, \ldots, N$.

In order to make leader-following consensus realized, we make the following two assumptions on the graph:

(A8.3) The directed graph \mathcal{G} is connected.

(A8.4) At least one follower connects to the leader, i.e., $\mathcal{B} \neq 0$.

The satisfaction of these two assumptions leads the following lemma, which is important to deal with our output consensus control problem.

Lemma 8.1.1. [30] For the multi-agent system including (8.1.1) and (8.1.1), if Assumptions (A8.3) and (A8.4) hold, then $\mathcal{L} + \mathcal{B} > 0$.

From the properties of positive definite matrices, this lemma indicates that the

matrix $\mathcal{L} + \mathcal{B}$ is invertible as long as Assumptions (A8.3)–(A8.4) are satisfied, which is essential for stability analysis of our control problem.

8.2 Control Designs

In this section, a distributed adaptive consensus protocol is developed for the leaderfollowing multi-agent system consisting of (8.1.1) and (8.1.1) for dealing with the unknown followers' parameters. To analyze the consensus performance, before we start to present the control designs, we will introduce several different errors in the multi-agent system.

8.2.1 Different Errors in Multi-Agent Systems

For consensus performance analysis, we define

$$e_i(t) = y_i(t) - y_l(t), \quad i = 1, \dots, N,$$
(8.2.0)

for all the agents, which measures the disagreements between the leader and each agent v_i .

Also, we define

$$\bar{e}_{i,j}(t) = y_i(t) - y_j(t), \ i = 1, \dots, N, v_j \in \mathcal{N}_i$$
(8.2.0)

to measure the differences between any of two agents. From the definition of undirected graph, we know that $\bar{e}_{i,j}(t) = -\bar{e}_{j,i}(t)$.

Based on the above two definitions of $e_i(t)$ and $\bar{e}_{i,j}(t)$, we define the relative consensus error $\varepsilon_i(t)$ of v_i as

$$\varepsilon_i(t) = \sum_{j=1, j \neq i}^N a_{ij} \bar{e}_{ij}(t) + b_i e_i(t) = \sum_{v_j \in \mathcal{N}_i} (y_i - y_j).$$

From the definition of \mathcal{L} and \mathcal{B} and the definition of the $\varepsilon_i(t)$ and $e_i(t)$, we obtain

$$\varepsilon(t) = (\mathcal{L} + \mathcal{B})e(t) \tag{8.2.0}$$

for $\varepsilon(t) = [\varepsilon_1(t), \varepsilon_2(t), \dots, \varepsilon_N(t)]^{\mathrm{T}}$ and $e(t) = [e_1(t), e_2(t), \dots, e_N(t)]^{\mathrm{T}} = [y_1(t) - y_l(t), y_2(t) - y_l(t), \dots, y_N(t) - y_l(t)]^{\mathrm{T}}$. From Lemma 8.1.1, we conclude that as long as the relative consensus error vector $\varepsilon(t)$ go to zero as time goes to infinity, the output consensus: $\lim_{t\to\infty} e(t) = 0$, will be achieved.

8.2.2 Nominal Control Design

In this section, we first present a nominal controller structure when the followers' parameters (A_i, B_i, C_i) are known and clarify the output matching condition for output consensus, which provides *a priori* knowledge for the adaptive control design shown in Section 8.2.3.

Basic design idea. For general control systems, when the full-state $x_i(t)$ is available for measurement, the state feedback controller: $u(t) = k_1^{*T}x(t) + k_2^*r(t)$ with k_1^* and k_2^* bing the nominal controller parameters calculated by the plant-model matching condition is usually used; and when the full-state vector x(t) is not available for measurement, the observer-based state feedback controller: $u(t) = k_1^{*T}\hat{x}(t) + k_2^*r(t)$, with an estimate $\hat{x}(t)$ (generated from a state estimator or observer) of the state vector x(t) is generally used.

In partial-state feedback consensus control problem, partial-state vectors $y_{0i}(t)$ are measurable, we start the partial-state feedback controller derivation from developing a state feedback controller structure for multi-agent consensus control.

Controller structure by using state feedback. To achieve output consensus when the systems parameters (A_i, B_i, C_i) are known, we develop the distributed nominal control law for the ith follower as

$$u_i(t) = k_{1i}^{*\mathrm{T}} x_i(t) + k_{2i}^{*} r(t), \qquad (8.2.0)$$

with the nominal controller parameters $k_{1i}^* \in \mathbb{R}^n$ and $k_{2i}^* \in \mathbb{R}$ satisfying the

$$\det(sI - A_i - B_i k_{1i}^{*\mathrm{T}}) = P_l(s) Z_i(s) \frac{1}{k_{pi}}, k_{2i}^* = \frac{1}{k_{pi}}.$$
(8.2.0)

This matching condition is for ensuring output matching between the followers and the leader, which will be analyzed with details in the next part. The existence of the parameters k_{1i}^* and k_{2i}^* is guaranteed by Assumption (A8.1) [82].

Follower-leader output matching. With the nominal control law (8.2.2), the *i*th follower agent (8.1.1) becomes

$$\dot{x}_i(t) = A_i x_i(t) + B_i k_{1i}^{*\mathrm{T}} x_i(t) + B_i k_{2i}^{*} r(t), \ y_i(t) = C_i x_i(t).$$

From the matching condition (8.2.2), it follows that

$$C_i(sI - A_i - B_i k_{1i}^{*\mathrm{T}})^{-1} B_i k_{2i}^* = \frac{Z_i(s)k_{2i}^*}{\det(sI - A_i - B_i k_{1i}^{*\mathrm{T}})} = \frac{1}{P_l(s)} = W_l(s).$$

In view of (8.1.1), (8.2.2) and (8.2.2), we have

$$y_i(t) = C_i e^{(A_i + B_i k_{1i}^{*T})t} x_i(0) + y_l(t), i = 1, 2, \dots, N$$

Thus, the relative consensus error $\varepsilon_i(t)$ is

$$\varepsilon_i(t) = \sum_{v_j \in \mathcal{N}_i} (y_i - y_j) = n_i C_i e^{(A_i + B_i k_{1i}^{*\mathrm{T}})t} x_i(0) - C_j e^{(A_j + B_j k_{1j}^{*\mathrm{T}})t} x_j(0)$$

with n_i representing the number of agents in the neighborhood \mathcal{N}_i , which indicates that $\lim_{t\to\infty} \varepsilon_i(t) = 0, i = 1, 2, ..., N$. Hence, we have $\lim_{t\to\infty} \varepsilon(t) = 0$. Since Lemma 8.1.1 ensures that the matrix $\mathcal{L} + \mathcal{B}$ is invertible, it follows that $\lim_{t\to\infty} e(t) = 0$, namely, the output consensus: $\lim_{t\to\infty} (y_i(t) - y_l(t)) = 0, i = 1, ..., N$, is achieved. Controller structure by using partial-state feedback. From the partialstate observer technique shown in Section 3.3, we could obtain an observer to generate the state $x_i(t)$ for each follower as:

$$\hat{x}_i(t) = \left[y_{0i}^{\mathrm{T}}(t), (w_i(t) + L_{ri}y_{0i}(t))^{\mathrm{T}}\right]^{\mathrm{T}},$$
(8.2.0)

with

$$w_i(t) = \frac{n_{1i}(s)}{\Lambda_i(s)} [u_i](t) + \frac{n_{2i}(s)}{\Lambda_i(s)} [y_{0i}](t) + \epsilon_i(t).$$

where L_{ri} is a constant gain matrix, $\Lambda_i(s)$ is a stable polynomial with degree $n - n_{0i}$, $n_{1i}(s)$ is an $(n - n_{0i}) \times 1$ polynomial vector and $n_{2i}(s)$ is an $(n - n_{0i}) \times n_{0i}$ polynomial matrix, whose maximum degrees are $n - n_{0i} - 1$ or less. The state estimate $\hat{x}_i(t)$ generated from such a partial-state observer converges to the state vector $x_i(t)$ exponentially, as desired.

Substitute (8.2.2) into the nominal state feedback controller structure (8.2.2), a partial-state feedback controller structure is obtained:

$$u_i(t) = \theta_{1i}^{*\mathrm{T}} \omega_{1i}(t) + \theta_{2i}^{*\mathrm{T}} \omega_{2i}(t) + \theta_{20i}^{*\mathrm{T}} y_{0i}(t) + \theta_{3i}^{*} r(t), \qquad (8.2.0)$$

where

$$\omega_{1i}(t) = \frac{a_{1i}(s)}{\Lambda_i(s)} [u_i](t), \ \omega_{2i}(t) = \frac{A_{2i}(s)}{\Lambda_i(s)} [y_{0i}](t)$$
(8.2.0)

with $a_{1i}(s) = [1, s, \dots, s^{n-n_{0i}-1}]^{\mathrm{T}}$, $A_{2i}(s) = [I_{n_{0i}}, sI_{n_{0i}}, \dots, s^{n-n_{0i}-1}I_{n_{0i}}]^{\mathrm{T}}$ and $\Lambda_i(s)$ being a monic stable polynomial of degree $n - n_{0i}$, the nominal controller parameters $\theta_{1i}^* \in \mathbb{R}^{n-n_{0i}}, \ \theta_{2i}^* \in \mathbb{R}^{n_{0i} \times (n-n_{0i})}, \ \theta_{20i}^* \in \mathbb{R}^{n_{0i}}$ and $\theta_{3i}^* \in \mathbb{R}$ satisfy the plant-model matching condition:

$$\theta_{1i}^{*\mathrm{T}}a_{1i}(s)P_i(s) + (\theta_{2i}^{*\mathrm{T}}A_{2i}(s) + \theta_{20i}^{*\mathrm{T}}\Lambda_i(s))Z_{0i}(s) = \Lambda_i(s)(P_i(s) - k_{pi}\theta_{3i}^*Z_i(s)P_l(s)),$$

for a polynomial vector $Z_{0i}(s)$ in the relationship between $y_{0i}(t)$ and $u_i(t)$: $P_i(s)[y_{0i}](t)$ = $Z_{0i}(s)[u_i](t)$.

Based on the same technique that given in Section 3.3.3 and the follower-leader output matching result given in this section by the state feedback controller structure, we could conclude that the follower-leader output matching can be guaranteed by the proposed partial-state feedback controller structure (8.2.2).

Thus far, we clarify verify that the model reference control is able to solve the leader-following output consensus problem by both state-feedback and partial-state feedback, which provides a foundation for the following adaptive design for the case of unknown followers' parameters.

8.2.3 Adaptive Control Design

For the follower (8.1.1) with unknown (A_i, B_i, C_i) , the nominal controller parameters θ_{1i}^* , θ_{20i}^* and θ_{3i}^* in (8.2.2) depending on system parameters (A_i, B_i, C_i) can not be calculated so that the nominal state feedback control design can not be applied. In this section, we develop an adaptive partial-state feedback control design with a distributed adaptive law using only the relative output information to solve the leader-following output consensus problem, for dealing with the followers' parameter uncertainties. For adaptive control, we need the following assumption: (A8.4) the signs of the high frequency gains k_{pi} , $i = 1, \ldots, N$, are known.

In this section, we consider the multi-agent systems with relative-degree-one followers agents.

8.2.3.1 Adaptive Controller Structure

To handle the parametric uncertainties of the followers (8.1.1), we design the adaptive version of the controller (8.2.2) as

$$u_i(t) = \theta_{1i}^{\mathrm{T}}(t)\omega_{1i}(t) + \theta_{2i}^{\mathrm{T}}(t)\omega_{2i}(t) + \theta_{20i}^{\mathrm{T}}(t)y_{0i}(t) + \theta_{3i}(t)r(t), \quad i = 1, 2, \dots, N,$$

where $\theta_{1i}(t) \in \mathbb{R}^{n-n_{0i}}, \ \theta_{2i}(t) \in \mathbb{R}^{n_{0i} \times (n-n_{0i})}, \ \theta_{20i}(t) \in \mathbb{R}^{n_{0i}} \text{ and } \theta_{3i}(t) \in \mathbb{R} \text{ are the adaptive estimates of the unknown constant nominal parameters } \theta_{1i}^*, \ \theta_{2i}^*, \ \theta_{20i}^* \text{ and } \theta_{3i}^*,$ respectively, and

$$\omega_{1i}(t) = \frac{a_{1i}(s)}{\Lambda_i(s)} [u_i](t), \ \omega_{2i}(t) = \frac{A_{2i}(s)}{\Lambda_i(s)} [y_{0i}](t)$$
(8.2.0)

with $a_{1i}(s) = [1, s, \cdots, s^{n-n_{0i}-1}]^{\mathrm{T}}, A_{2i}(s) = [I_{n_{0i}}, sI_{n_{0i}}, \dots, s^{n-n_{0i}-1}I_{n_{0i}}]^{\mathrm{T}}.$

8.2.3.2 Design of Adaptive Law

For updating the adaptive controller parameters $\theta_{1i}(t)$, $\theta_{2i}(t)$, $\theta_{20i}(t)$ and $\theta_{3i}(t)$ for each follower, a distributed adaptive law is needed. Next, we analyze a multi-agent system with one leader and three followers, without loss of generality, to illustrate the complicated design process of the adaptive law.

An illustrative example. Consider the multi-agent system shown in Fig. 8.1, whose graphic structure is descried by the matrix $\mathcal{L} + \mathcal{B}$ as

$$\mathcal{L} + \mathcal{B} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Substituting the adaptive controller (8.2.3.1), we obtain the following error signals for the multi-agent system shown in Fig. 8.1:

$$e_{1}(t) = y_{1}(t) - y_{l}(t) = \frac{k_{p1}}{P_{l}(s)} [\tilde{\theta}_{1}^{\mathrm{T}} \omega_{1}](t),$$

$$e_{2}(t) = y_{2}(t) - y_{l}(t) = \frac{k_{p2}}{P_{l}(s)} [\tilde{\theta}_{2}^{\mathrm{T}} \omega_{2}](t),$$

$$\begin{split} \bar{e}_{3,1}(t) &= y_3(t) - y_1(t) = \frac{k_{p3}}{P_l(s)} [\tilde{\theta}_3^{\mathrm{T}} \omega_3](t) - \frac{k_{p1}}{P_l(s)} [\tilde{\theta}_1^{\mathrm{T}} \omega_1](t), \\ \bar{e}_{3,2}(t) &= y_3(t) - y_2(t) = \frac{k_{p3}}{P_l(s)} [\tilde{\theta}_3^{\mathrm{T}} \omega_3](t) - \frac{k_{p2}}{P_l(s)} [\tilde{\theta}_2^{\mathrm{T}} \omega_2](t), \\ \bar{e}_{1,3}(t) &= -\bar{e}_{3,1}(t), \quad \bar{e}_{2,3}(t) = -\bar{e}_{3,2}(t), \end{split}$$

with $\theta_i(t) = [\theta_{1i}^{\mathrm{T}}(t), \theta_{2i}^{\mathrm{T}}(t), \theta_{20i}^{\mathrm{T}}(t), \theta_{3i}(t)]^{\mathrm{T}}, \theta_i^* = [\theta_{1i}^{*\mathrm{T}}, \theta_{2i}^{*\mathrm{T}}, \theta_{20i}^{*\mathrm{T}}, \theta_{3i}^*]^{\mathrm{T}}, \omega_i(t) = [\omega_{1i}^{\mathrm{T}}(t), \omega_{2i}^{\mathrm{T}}(t), \omega_{2i}^{\mathrm{T}}(t$

For the followers with relative degree $n^* = 1$, we choose the characteristic polynomial of the leader as $P_l(s) = s + a_l$, $a_l > 0$, and the above error equations in (8.2.3.2) become

$$\begin{split} \dot{e}_{1}(t) &= -a_{0}e_{1}(t) + k_{p1}\tilde{\theta}_{1}^{\mathrm{T}}(t)\omega_{1}(t), \\ \dot{e}_{2}(t) &= -a_{0}e_{2}(t) + k_{p2}\tilde{\theta}_{2}^{\mathrm{T}}(t)\omega_{2}(t), \\ \dot{\bar{e}}_{3,1}(t) &= -a_{0}\bar{e}_{3,1}(t) + k_{p3}\tilde{\theta}_{3}^{\mathrm{T}}(t)\omega_{3}(t) - k_{p1}\tilde{\theta}_{1}^{\mathrm{T}}(t)\omega_{1}(t), \\ \dot{\bar{e}}_{3,2}(t) &= -a_{0}\bar{e}_{3,2}(t) + k_{p3}\tilde{\theta}_{3}^{\mathrm{T}}(t)\omega_{3}(t) - k_{p2}\tilde{\theta}_{2}^{\mathrm{T}}(t)\omega_{2}(t), \\ \dot{\bar{e}}_{1,3}(t) &= -\dot{\bar{e}}_{3,1}(t), \quad \dot{\bar{e}}_{2,3}(t) = -\dot{\bar{e}}_{3,2}(t). \end{split}$$

To choose a stable adaptive law, we then choose the following positive definite function $V = V_1 + V_2$ with

$$V_{1} = e_{1}^{2} + e_{2}^{2} + \frac{1}{2}\bar{e}_{3,1}^{2} + \frac{1}{2}\bar{e}_{3,2}^{2} + \frac{1}{2}\bar{e}_{1,3}^{2} + \frac{1}{2}\bar{e}_{2,3}^{2},$$

$$V_{2} = |k_{p1}|\tilde{\theta}_{1}^{T}\Gamma_{1}^{-1}\tilde{\theta}_{1} + |k_{p2}|\tilde{\theta}_{2}^{T}\Gamma_{2}^{-1}\tilde{\theta}_{2} + |k_{p3}|\tilde{\theta}_{3}^{T}\Gamma_{3}^{-1}\tilde{\theta}_{3},$$

with $\Gamma_i = \Gamma_i^{\mathrm{T}} > 0$, i = 1, 2, 3, being a constant matrix with an appropriate dimension.



Figure 8.1: The multi-agent network with a leader v_0 and three followers v_1-v_3 .

The time-derivative of V_1 is

$$\begin{split} \dot{V}_{1} &= 2e_{1}(t)\dot{e}_{1}(t) + 2e_{2}(t)\dot{e}_{2}(t) + \bar{e}_{3,1}(t)\dot{\bar{e}}_{3,1}(t) + \bar{e}_{3,2}(t)\dot{\bar{e}}_{3,2}(t) + \bar{e}_{1,3}(t)\dot{\bar{e}}_{1,3}(t) + \bar{e}_{2,3}(t)\dot{\bar{e}}_{2,3}(t) \\ &= -2a_{0}e_{1}^{2}(t) - 2a_{0}e_{2}^{2}(t) - 2a_{0}\bar{e}_{3,1}^{2}(t) - 2a_{0}\bar{e}_{3,2}^{2}(t) + 2k_{p1}\tilde{\theta}_{1}^{\mathrm{T}}(t)\omega_{1}(t)e_{1}(t) \\ &+ 2k_{p2}\tilde{\theta}_{2}^{\mathrm{T}}(t)\omega_{2}(t)e_{2}(t) + 2k_{p3}\tilde{\theta}_{3}^{\mathrm{T}}(t)\omega_{3}(t)\bar{e}_{3,1}(t) + 2k_{p1}\tilde{\theta}_{1}^{\mathrm{T}}(t)\omega_{1}(t)\bar{e}_{1,3}(t) \\ &+ 2k_{p3}\tilde{\theta}_{3}^{\mathrm{T}}(t)\omega_{3}(t)\bar{e}_{3,2}(t) + 2k_{p2}\tilde{\theta}_{2}^{\mathrm{T}}(t)\omega_{2}(t)\bar{e}_{2,3}(t), \end{split}$$

and the time-derivative of \mathcal{V}_2 is

$$\dot{V}_2 = 2|k_{p1}|\tilde{\theta}_1^{\mathrm{T}}(t)\Gamma_1^{-1}\dot{\tilde{\theta}}_1(t) + 2|k_{p2}|\tilde{\theta}_2^{\mathrm{T}}(t)\Gamma_2^{-1}\dot{\tilde{\theta}}_2(t) + 2|k_{p3}|\tilde{\theta}_3^{\mathrm{T}}(t)\Gamma_3^{-1}\dot{\tilde{\theta}}_3(t).$$

To make $\dot{V} = \dot{V}_1 + \dot{V}_2 < 0$, let

$$2|k_{p1}|\tilde{\theta}_{1}(t)\Gamma_{1}^{-1}\dot{\tilde{\theta}}_{1}(t) = -2k_{p1}\tilde{\theta}_{1}^{\mathrm{T}}(t)\omega_{1}(t)e_{1}(t) - 2k_{p1}\tilde{\theta}_{1}^{\mathrm{T}}(t)\omega_{1}(t)\bar{e}_{1,3}(t),$$

$$2|k_{p2}|\tilde{\theta}_{2}(t)\Gamma_{2}^{-1}\dot{\tilde{\theta}}_{2}(t) = -2k_{p2}\tilde{\theta}_{2}^{\mathrm{T}}(t)\omega_{2}(t)e_{2}(t) - 2k_{p2}\tilde{\theta}_{2}^{\mathrm{T}}(t)\omega_{2}(t)\bar{e}_{2,3}(t),$$

$$2|k_{p3}|\tilde{\theta}_{3}(t)\Gamma_{3}^{-1}\dot{\tilde{\theta}}_{3}(t) = -2k_{p3}\tilde{\theta}_{3}^{\mathrm{T}}(t)\omega_{3}(t)\bar{e}_{3,1}(t) - 2k_{p3}\tilde{\theta}_{3}^{\mathrm{T}}(t)\omega_{3}(t)\bar{e}_{3,2}(t),$$

which follows that

$$\dot{\tilde{\theta}}_{1}(t) = \dot{\theta}_{1}(t) = -\operatorname{sign}(k_{p1})\Gamma_{1}\omega_{1}(t)(e_{1}(t) + \bar{e}_{1,3}(t)),$$

$$\dot{\tilde{\theta}}_{2}(t) = \dot{\theta}_{2}(t) = -\operatorname{sign}(k_{p2})\Gamma_{2}\omega_{2}(t)(e_{2}(t) + \bar{e}_{2,3}(t)),$$

$$\dot{\tilde{\theta}}_{3}(t) = \dot{\theta}_{3}(t) = -\operatorname{sign}(k_{p3})\Gamma_{3}\omega_{3}(t)(\bar{e}_{3,1}(t) + \bar{e}_{3,2}(t)).$$

Thus, based on the definition of the relative consensus error $\varepsilon_i(t)$ in (8.2.1), we have derived

$$\dot{\theta}_1(t) = -\operatorname{sign}(k_{p1})\Gamma_1\omega_1(t)\varepsilon_1(t),$$

$$\dot{\theta}_2(t) = -\operatorname{sign}(k_{p2})\Gamma_2\omega_2(t)\varepsilon_2(t),$$

$$\dot{\theta}_3(t) = -\operatorname{sign}(k_{p3})\Gamma_3\omega_3(t)\varepsilon_3(t).$$

as the adaptive law for the agents $v_1 - v_3$.

With the choices of $\dot{\theta}_i(t)$, i = 1, 2, 3, the time-derivative of V becomes

$$\dot{V} = -2a_0e_1^2(t) - 2a_0e_2^2(t) - 2a_0\bar{e}_{3,1}^2(t) - 2a_0\bar{e}_{3,2}^2(t) \le 0,$$

which indicates that $e_1(t)$, $e_2(t)$, $\bar{e}_{1,3}(t)$, $\bar{e}_{2,3}(t)$, $\bar{e}_{1,3}(t)$, $\bar{e}_{1,3}(t)$, $\theta_i(t)$, i = 1, 2, 3, in the multi-agent system are bounded, so that all $y_i(t)$ are bounded. From the relationship: $y_{0i}(t) = \frac{1}{k_{pi}} Z_i^{-1}(s) Z_{0i}(s) [y_i](t) = \frac{Z_{0i}(s)}{k_{pi} P_l(s) Z_i(s)} P_l(s) [y_i](t)$, we have $y_{0i}(t) \in L^{\infty}$, so that $\omega_{2i}(t) = \frac{A_{2i}(s)}{\Lambda_i(s)} [y_{0i}](t) \in L^{\infty}$. Using the input-output relationship: $P_i(s) [y_i](t) = k_{pi} Z_i(s) [u_i](t)$, we have

$$\frac{s^{i}}{\Lambda_{i}(s)}[u_{i}](t) = \frac{P_{i}(s)}{k_{pi}Z_{i}(s)}\frac{s^{i}}{\Lambda_{i}(s)}[y_{i}](t), \qquad (8.2.-18)$$

which is bounded for $i = 0, 1, \dots, n - n_0 - 1$, because $\frac{P_i(s)}{k_{pi}Z_i(s)} \frac{s^i}{\Lambda_i(s)}$ is stable and proper and $y_i(t) \in L^{\infty}$. This implies $\omega_{1i}(t) \in L^{\infty}$, and so does $u_i(t)$. With the signal boundedness properties shown above, we have $\dot{e}_1(t) \in L^{\infty}$, $\dot{e}_2(t) \in L^{\infty}$, $\dot{\bar{e}}_{1,3}(t) \in$ L^{∞} , $\dot{\bar{e}}_{2,3}(t) \in L^{\infty}$, $\dot{\bar{e}}_{1,3}(t) \in L^{\infty}$, $\dot{\bar{e}}_{1,3}(t) \in L^{\infty}$ from (8.2.3.2). From (8.2.3.2), we also conclude that $e_1(t) \in L^2$, $e_2(t) \in L^2$, $\bar{e}_{1,3}(t) \in L^2$, $\bar{e}_{2,3}(t) \in L^2$, $\bar{e}_{1,3}(t) \in L^2$, $\bar{e}_{1,3}(t) \in L^2$. Therefore, using Barhalat Lemma and (8.2.1), we can conclude that $\lim_{t\to\infty} \varepsilon_i(t) = 0$. Then, according to Lemma 8.1.1, we have $\lim_{t\to\infty} (y_i(t) - y_l(t)) = 0$, for $i = 1, 2, \dots, N$, which means that the leader-following output consensus of the multi-agent system is achieved. ∇

In this part, we explained the design of adaptive law for updating the adaptive controller parameters in details with an example. Next, we will present the adaptive control law design for general multi-agent systems with N followers.

Adaptive laws for general cases. Generalizing the illustrative example, we conclude that the adaptive law can be chosen as

$$\dot{\theta}_i(t) = -\operatorname{sign}(k_{pi})\Gamma_i\omega_i(t)\varepsilon_i(t), i = 1, \dots, N,$$

with $\theta_i(t) = [\theta_{1i}^{\mathrm{T}}(t), \theta_{2i}^{\mathrm{T}}(t), \theta_{20i}^{\mathrm{T}}(t), \theta_{3i}(t)]^{\mathrm{T}}$, for updating the adaptive estimates $\theta_{1i}(t)$, $\theta_{2i}(t), \theta_{20i}(t)$ and $\theta_{3i}(t)$ in the adaptive controller (8.2.3.1):

$$u_{i}(t) = \theta_{1i}^{\mathrm{T}}(t)\omega_{1i}(t) + \theta_{2i}^{\mathrm{T}}(t)\omega_{2i}(t) + \theta_{20i}^{\mathrm{T}}(t)y_{0i}(t) + \theta_{3i}(t)r(t), i = 1, 2, \dots, N,$$

for leader-following output consensus of the multi-agent system with relative-degreeone followers.

Thus, for the multi-agent system including the N followers agents, the developed partial-state feedback law (8.2.3.1) and the corresponding adaptive law (8.2.3.2), we have the following theorem.

Theorem 8.2.1. The partial-state feedback adaptive controller (8.2.3.1) with the adaptive laws (8.2.3.2), applied to all the N uncertain relative-degree-one followers (8.1.1), guarantees that all the closed-loop signal in the multi-agent system are bounded and the leader-following output consensus: $\lim_{t\to\infty}(y_i(t) - y_l(t)) = 0$, is achieved.

In the stability proof, we take the Lyapunov candidate as

$$V = \sum_{v_i \in \mathcal{N}_0} e_i^2 + \frac{1}{2} \sum_{(v_i, v_j) \in \mathcal{E}} \bar{e}_{i,j}^2 + \sum_{i=1}^N |k_{pi}| \tilde{\theta}_i^{\mathrm{T}} \Gamma_i^{-1} \tilde{\theta}_i.$$
(8.2.-19)

Such a Lyapunov candidate measures all the closed-loop signals in the multi-agent system which can help us to establish the stability properties of the system. The complete proof of this theorem can be easily obtained by generalizing the stability analysis shown in the previous part, which is omitted here.

In the developed adaptive control scheme, the controller structure depends on the local partial-state information, and the adaptive law only requires for the relative output information from the neighborhood, which makes the adaptive control scheme completely distributed. **Remark 8.2.1.** In this chapter, we solve the output consensus problem for relativedegree-one followers by Lyapunov design. The adaptive design is inspired by the standard model reference adaptive control method. For higher relative degree followers, the estimation errors involving tracking error estimations and the unknown high-frequency gain estimations are required for the model reference adaptive control design, which challenges the corresponding adaptive control design for multi-agent systems. The distributed adaptive output consensus scheme for higher relative degree followers is currently under investigation.

8.3 Simulation Study

In this section, we use the lateral dynamic model of a Boeing 747 airplane [21] as the agents for verifying the effective of the developed adaptive consensus scheme.

The standard aircraft dynamic model. For yaw rate control, the linearized lateral motion equation is given as

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y = Cx(t), \quad (8.3.0)$$

,

with

$$A = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.598 & -0.115 & -0.0318 & 0 \\ -3.05 & 0.388 & -0.4650 & 0 \\ 0 & 0.0805 & 1 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0.00729 \\ -0.475 \\ 0.153 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

where the system state vector is chosen as $x = [\beta, r_0, p, \phi]^T$ with β (rad) being the side-slip angle, r_0 (rad/s) being the yaw rate, p(rad/s) being the roll rate and ϕ (rad) being the roll angle, the control input signal u(t) is the rudder angle δ_r , and the system output y(t) is the yaw rate $r_0(t)$. It can be verified that this is a relative degree one system model.

Multi-agent system description. In this simulation study, we study the consensus performance of three follower aircraft and one virtual leader, and their associated communication graph is shown in Fig. 8.1. It is well known that the system parameters may change due to wear and aging. Therefore, for the purpose of simulation, we choose the system parameters of the three followers, whose dynamics are given in (8.3), as $A_i = A + \Delta A_i$, $B_i = B + \Delta B_i$, $C_i = C$, for i = 1, 2, 3, with ΔA_i and ΔB_i being some perturbation constant matrices which are omitted here for space reasons. All the follower systems with (A_i, B_i, C_i) satisfy (A8.2) and are of relative-degree-one. The dynamics of the leader is given as $y_i(t) = \frac{1}{s+3}[r](t)$ with $r(t) = 0.5 \sin(0.08t)$.

Simulation results. For simulation, the adaptive control law (8.2.3.1) and the adaptive law (8.2.3.2) are adopted with the parameters $\Gamma_i = 0.5I$, i = 1, 2, 3. Initial conditions are chosen as $y_1(0) = 0$, $y_2(0) = -0.01$, $y_3(0) = 0.01$, $y_0(0) = 0$, and the initial controller parameters are chosen as 90% of the nominal controller parameters, respectively. The output trajectories of the leader and the followers are shown in Fig. 8.2. All the other closed-loop signals are also bounded which are omitted for brevity. The simulation results confirm the effectiveness of the consensus scheme developed in this chapter.

Summary

In this chapter, we have developed a new adaptive control scheme using local partialstate feedback for guaranteeing leader-following output consensus of multi-agent systems in the presence of followers' parametric uncertainties. The new adaptive control



Figure 8.2: Output trajectories of the followers and the leader.

scheme combines the advantages of the state feedback output consensus designs and the output feedback output consensus designs. This work clarifies the output matching conditions for leader-following output consensus. The system stability analysis has shown the feasibility of using model reference adaptive control for output consensus.

Chapter 9 Conclusions and Future Work

In this chapter, we give concluding remarks to this dissertation and suggest some research topics for future study in this area.

Conclusions. Given the importance of model reference adaptive control, we have developed a new partial-state feedback model reference adaptive control framework and explored two kinds of applications of the new framework in this research. The partial-state feedback MRAC schemes build up a bridge between the state feedback MRAC designs and the output feedback MRAC designs, and provides a manageable trade-off between the controller complexity and the number of required system measurements. It has been shown that with the use of partial-state signal for feedback, the feedback capability, the design flexibility, the system robustness of MRAC systems are increased while the controller structure keeps a relative simple form. Based on the unique features of the new control framework, the applications of this new control framework are explored. Inspired by the enhanced robustness of the partial-state feedback MRAC schemes, the sensor failure compensation schemes are developed. The control systems with the developed sensor failure compensation schemes not only enjoy the better transient performance as the traditional state feedback controller, but also has redundant capacity for achieving desired performance for output tracking which is desirable for tolerating uncertain sensor faults. Inspired by the additional feedback capacity and design flexibility that the partial-state feedback MRAC schemes provide, an output consensus control scheme via partial-state feedback is developed.

There are six control problems solved in this dissertation. The solutions of the first three control problems construct the theoretical foundation of the partial-state feedback MRAC framework, and the solutions of the other three control problems bring new control possibility to fault tolerant control and multi-agent control which reveals the potential of the new partial-state feedback MRAC framework. The detailed research topics are summarized as follows.

- A partial-state feedback MRAC scheme of SISO systems for output tracking (Chapter 3).
- A multivariable partial-state feedback MRAC scheme of MIMO systems for output tracking (Chapter 4).
- Higher-order tracking error convergence properties for multivariable model reference adaptive control systems (Chapter 5).
- An adaptive state feedback control design of SISO systems with sensor failure compensation for asymptotic output tracking (Chapter 6).
- An adaptive state feedback control design of MIMO systems with sensor failure compensation for asymptotic output tracking (with applications to quadrotor systems (Chapter 7).
- A distributed adaptive partial-state feedback control scheme for output consensus of multi-agent systems (Chapter 8).

Future topics. The studies in this dissertation give some encouraging results in using partial-state signal for model reference adaptive control. For the future study, the following research topics are listed as some possible directions in this area:

- Given the importance of nonlinear control systems, multivariable MRAC designs by partial-state feedback for nonlinear systems need to be studied;
- For safety-critically applications, partial-state feedback MRAC-based compensation designs for actuator failures and structural damage need to be studied;
- Considering the bright application prospect, the partial-state feedback output consensus control designs of multi-agent systems with (a) general relative degree agents; (b) multivariable agents; and (c) switching topology, need to be studied.
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